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COMPARISON OF ESTIMATION METHODS FOR THE KUMARASWAMY WEIBULL DISTRIBUTION

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Abstract. In this study, the performances of the different parameter estimation methods are compared for the Kumaraswamy Weibull distribution via Monte Carlo simulation study. Maximum Likelihood (ML), Least Squares (LS), Weighted Least Squares (WLS), Cramer-von Mises (CM) and Anderson Darling (AD) methods are used in the comparisons. The results of the Monte Carlo simulation study demonstrate that ML estimators for the parameters of the Kumaraswamy Weibull distribution are more efficient than the other estimators. It is followed by AD estimator. At the end of the study, a real data set taken from the literature is used to illustrate the applicability of the Kumaraswamy Weibull distribution.

1. Introduction

The Weibull is one of the most popular and widely used distribution in many fields of science such as engineering, reliability, biology, ecology and hydrology (see for example, Calabria and Pulcini [4], Keats et al. [16], Saeed et al. [20], Serban et al. [22]). However, the Weibull distribution does not provide a good fit to data sets with bathtub shaped or upside down bathtub shaped failure rates frequently encountered in engineering and reliability studies (see Cordeiro et al. [6], Akgül et al. [2], Maurya et al.[17]). Therefore, many generalized distributions have been developed for modeling these data sets (see, for example, Mudholkar and Srivastava [18], Sarhan and Zaindin [21], Elbatal et al. [8]). A new family of generalized Kumaraswamy (KwG) distributions obtained by combining the work of Eugene

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et al. [11] and Jones [14] is one of these generalized distributions, (see Cordeiro and Castro [5]). Probability density function (pdf) and the cumulative distribution function (cdf) of the KwG distribution for an arbitrary baseline pdf $g(x)$ and cdf $G(x)$ are given by

$$f(x) = abg(x)G(x)^{(a-1)}\{1 - G(x)^a\}^{(b-1)} \tag{1}$$

and

$$F(x) = 1 - [1 - G(x)^a]^b, \quad a, b > 0, x \in R, \tag{2}$$

respectively. Here, a and b are the shape parameters. KwG is a flexible distribution for modeling many different data sets including censored data therefore it is widely used in engineering and biology (see Gomes et al. [12], Elbatal and Elgarhy [9], Rocha et al.[19]).

The Kumaraswamy Weibull (KwWeibull) distribution is a special case of the KwG distribution obtained by inserting the pdf $g(x) = \frac{p}{\sigma^p}(x-\mu)^{p-1}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}$ and the cdf $G(x) = 1 - exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}$ of the well known Weibull distribution into (1). KwWeibull is a better alternative to Weibull distribution since it contains some well known distributions discussed in the literature as special cases such as the Weibull (see Cordeiro et al. [6]).

In this study, the estimators of the location and scale parameters of the KwWeibull distribution are obtained by using Maximum Likelihood (ML), Least Squares (LS), Weighted Least Squares (WLS), Cramer-von Mises (CM) and Anderson Darling (AD) estimation methods. Shape parameters are assumed to be known throughout the study. The most efficient estimators are identified by using an extensive Monte-Carlo simulation study for the different sample sizes and the parameter settings.

The remainder of this paper is organized as follows: In Section 2, a brief description of the KwWeibull distribution is given. In Section 3, the parameter estimation methods are presented. Results of the Monte-Carlo simulation study are given in Section 4. In Section 5, the KwWeibull distribution is used to model a real data set taken from the literature. Finally, the concluding remarks are given in Section 6.

2. Kumaraswamy Weibull Distribution

The pdf and cdf of KwWeibull distribution are given below:

$$f(x) = ab\frac{p}{\sigma^p}(x-\mu)^{p-1}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}\left[1-\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}\right]^{a-1}$$
$$\times\left\{1-\left[1-\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}\right]^a\right\}^{b-1} \quad \mu < x < \infty,\ \ \mu,\sigma > 0\ \ a,b,p > 0 \tag{3}$$

and

$$F(x) = 1-\left\{1-\left[1-\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}\right]^a\right\}^b, \tag{4}$$

respectively. Here, μ and σ represent the location (or threshold) and the scale parameters, respectively and a, b and p are the shape parameters. For different values of the shape parameters a, b and p, the plots of the pdf of KwWeibull distribution are shown in Figure 1.

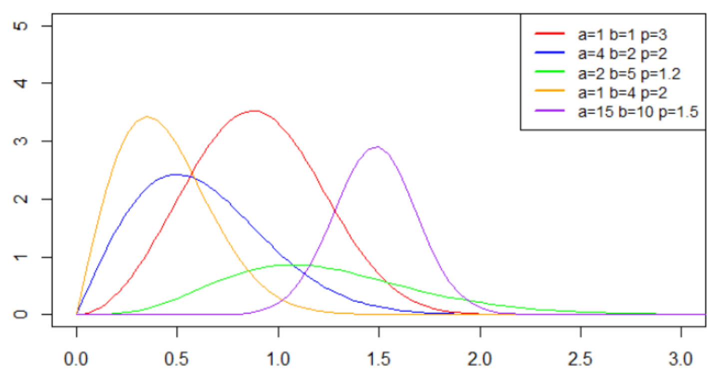

FIGURE 1. The pdf plots of the KwWeibull distribution

For better understanding the shape of the KwWeibull distribution, simulated skewness and kurtosis values of the KwWeibull distribution are given for different values of the shape parameters, see Table 1. It is clear from Table 1 that KwWeibull can be positively or negatively skewed depending on the values of the shape parameters. It can also be seen that kurtosis values can be less than or greater than that of Normal distribution subject to the values of shape parameters.

Table 1. Simulated skewness and kurtosis values for the KwWeibull distribution.

$a = b = 1$						
p=	1.5	2	2.5	3	4	6
Skewness	1.062	0.630	0.354	0.168	-0.088	-0.367
Kurtosis	4.368	3.219	2.843	2.722	2.734	2.998
$a = b = 2$						
p=	1.5	2	2.5	3	4	6
Skewness	0.709	0.381	0.178	0.041	-0.141	-0.336
Kurtosis	3.617	3.071	2.916	2.889	2.964	3.175
$a = 10$ and $b = 2$						
p=	1.5	2	2.5	3	4	6
Skewness	0.485	0.308	0.202	0.132	0.042	-0.046
Kurtosis	3.431	3.203	3.111	3.076	3.054	3.066
$a = 1$ and $b = 8$						
p=	1.5	2	2.5	3	4	6
Skewness	1.062	0.624	0.357	0.167	-0.087	-0.370
Kurtosis	4.348	3.226	2.843	2.720	2.739	3.022

3. Parameter Estimation Methods

Parameter estimation methods for estimating the location parameter μ and the scale parameter σ of KwWeibull distribution are described in the following subsections.

3.1. The Maximum Likelihood Method.

In this subsection, the ML estimators for the location and scale parameters of the KwWeibull distribution are obtained. Let $x_1, x_2, \ldots, x_n$ be a random sample from KwWeibull(a, b, p, μ, σ), then the log-likelihood ($\ln L$) function of the KwWeibull distribution is expressed as follows:

$$\begin{aligned}\ln L = {} & n(\ln a + \ln b + \ln p - \ln \sigma) + (p-1)\sum_{i=1}^{n} \ln\left(\tfrac{x_i-\mu}{\sigma}\right) - \sum_{i=1}^{n}\left(\tfrac{x_i-\mu}{\sigma}\right)^p \\ & + (a-1)\sum_{i=1}^{n} \ln\left(1-\exp\left\{-\left(\tfrac{x_i-\mu}{\sigma}\right)^p\right\}\right) \\ & + (b-1)\sum_{i=1}^{n} \ln\left(1-\left[1-\exp\left\{-\left(\tfrac{x_i-\mu}{\sigma}\right)^p\right\}\right]^a\right).\end{aligned} \tag{5}$$

$\ln L$ function is maximized with respect to the parameters of interest, i.e., μ and σ. By taking the derivatives of $\ln L$ with respect to the parameters μ and σ and equating them to zero, the following likelihood equations are obtained

$$
\begin{aligned}
\frac{\partial lnL}{\partial\mu} =& -\frac{(p-1)}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{-1}+\frac{p}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{p-1} \\
&-\frac{(a-1)p}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_i-\mu}{\sigma}\right)^{p-1}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}}{1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}} \\
&+\frac{a(b-1)p}{\sigma}\sum_{i=1}^{n}\frac{\left(1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\right)^{a-1}\left(\frac{x_i-\mu}{\sigma}\right)^{p-1}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}}{\left(1-\left[1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right)} \\
=&0
\end{aligned}
\tag{6}
$$

and

$$
\begin{aligned}
\frac{\partial\ln L}{\partial\sigma} =& -\frac{n}{\sigma}-\frac{n\,(p-1)}{\sigma}+\frac{p}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{p}-\frac{(a-1)\,p}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_i-\mu}{\sigma}\right)^{p}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}}{1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}} \\
&+\frac{a\,(b-1)\,p}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_i-\mu}{\sigma}\right)^{p}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\left(1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\right)^{a-1}}{\left(1-\left[1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right)} \\
=&0.
\end{aligned}
\tag{7}
$$

Solutions of these likelihood equations are called as the ML estimators of the parameters. When the likelihood equations for the location and scale parameters are examined, it is seen that the functions are not linear with respect to the parameters of interest. Therefore, numerical methods are needed for estimating the location and scale parameters.

3.2. **The Least Squares Method.** The LS estimators of the unknown parameters are obtained by minimizing the following equation

$$
S_{LS}=\sum_{i=1}^{n}\left(F\left(x_{(i)}\right)-\frac{i}{n+1}\right)^{2} \tag{8}
$$

with respect to the parameters of interest (see Swain [23]). Here and in the other subsections, $x_1, x_2, \ldots, x_n$ is a random sample from the distribution function $F(.)$, $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$ denotes the corresponding order statistics and $\frac{i}{n+1}$, $(i = 1, \ldots, n)$ are the expected values of $F\left(x_{(i)}\right)$. From Eq. (8), the LS estimators for the parameters of the KwWeibull distribution are obtained by minimizing the following equation with respect to the parameters μ and σ

$$S_{LS}(\mu,\sigma)=\sum_{i=1}^{n}\left(1-\left\{1-\left[1-\exp\left\{-\left(\frac{x_{(i)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}-\frac{i}{n+1}\right)^{2}. \tag{9}$$

3.3. **The Weighted Least Squares Method.** The WLS estimators of the unknown parameters are obtained by minimizing the following equation with respect to the parameters of interest (see Swain [23])

$$S_{WLS}=\sum_{i=1}^{n}w_i\left(F\left(x_{(i)}\right)-\frac{i}{n+1}\right)^{2} \tag{10}$$

where, $w_i=1/Var\left(F\left(x_{(i)}\right)\right)=(n+1)^2\,(n+2)\,/i(n-i+1),(i=1,2,...,n)$. From Eq.(10), the WLS estimators for the parameters of the KwWeibull distribution are obtained by minimizing the following equation with respect to the parameters μ and σ

$$S_{WLS}(\mu,\sigma)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1-\left\{1-\left[1-\exp\left\{-\left(\frac{x_{(i)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}-\frac{i}{n+1}\right)^{2}. \tag{11}$$

3.4. **The Cramér–Von Mises Method.** The CM estimators of the unknown parameters are obtained by minimizing the following equation

$$S_{CM}=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(x_{(i)})-\frac{2i-1}{2n}\right)^{2} \tag{12}$$

with respect to the parameters of interest (see Wolfowitz [24]). From Eq. (12), the CM estimators for the parameters of the KwWeibull distribution are obtained by minimizing the following equation with respect to the parameters μ and σ

$$S_{CM}(\mu,\sigma)=\frac{1}{12n}+\sum_{i=1}^{n}\left(1-\left\{1-\left[1-\ exp\left\{-\left(\frac{x_{(i)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}-\frac{2i-1}{2n}\right)^{2}. \tag{13}$$

3.5. **The Anderson-Darling Method.** The AD estimators of the unknown parameters are obtained by minimizing the following equation

$$S_{AD} = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\,log\left\{F\left(x_{(i)}\right)\left(1-F\left(x_{(n-i+1)}\right)\right)\right\} \tag{14}$$

with respect to the parameters of interest, (see Wolfowitz [25]). From Eq. (14), the AD estimators for the parameters of the KwWeibull distribution are obtained by minimizing the following equation with respect to the parameters μ and σ

$$\begin{aligned} S_{AD}(\mu,\sigma) = -n - \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(2i-1)log\left\{1-\left\{1-\left[1-\ exp\left\{-\left(\tfrac{x_{(i)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}\right.\\ \left.\times\left\{1-\left[1-exp\left\{-\left(\tfrac{x_{(n-i+1)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}\right\}. \end{aligned} \tag{15}$$

Here it should be noted that similar to ML estimates of parameters, LS, WLS, CM and AD estimates are obtained iteratively (see, Ergenç [10]).

4. Simulation Study

In this section, we perform an extensive Monte Carlo simulation study to compare the performances of the ML, LS, WLS, CM and AD estimators of the location parameter μ and scale parameter σ of the KwWeibull distribution. Without loss of generality, μ and σ are taken to be 0 and 1, respectively. All the simulations were conducted using R programming language for 10,000 Monte-Carlo runs. We use small ($n = 20$), moderate ($n = 50, 100$) and large ($n = 200, 500$) sample sizes. It is known that the estimation of the shape parameters along with the other parameters yields unreliable results when the sample size is not large enough (see, Bowman and Shenton [3], Kantar and Şenoğlu [15]). Therefore, it is assumed that the shape parameters a, b and p are known throughout the study. The performances of the estimators are compared with respect to the Bias, mean squares error (MSE) and Deficiency (Def) criteria, see the mathematical expressions given below

$$\begin{aligned} Bias &= \frac{1}{10,000}\sum_{i=1}^{10,000}\left(\hat{\theta}_i-\theta\right),\\ MSE &= \frac{1}{10,000}\sum_{i=1}^{10,000}\left(\hat{\theta}_i-\theta\right)^2 \end{aligned} \tag{16}$$

and

$$Def(\hat{\mu},\hat{\sigma}) = MSE(\hat{\mu}) + MSE(\hat{\sigma}). \tag{17}$$

Here, $\hat{\theta}_i$ is the ith simulated estimate of the parameter of interest (i.e. μ or σ) and θ is the true value of the parameter. Also, Def criterion is defined as the joint efficiencies of the estimators $\hat{\mu}$ and $\hat{\sigma}$. Simulated Bias, MSE and Def values for the ML, LS, WLS, CM and AD estimators for the location parameter μ and the scale parameter σ of the KwWeibull distribution are given in Table 2.

Table 2. The simulated Bias, MSE and Def values for the ML, LS, WLS, CM and AD estimators of the parameters μ and σ.

	$(a,b,p)=(1,1,1.5)$					$(a,b,p)=(1,1,3)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
$n=20$										
ML	0,061	0,011	-0,061	0,029	0,040	0,048	0,020	-0,052	0,022	0,043
LS	-0,034	0,017	0,056	0,045	0,062	-0,039	0,035	0,042	0,042	0,077
WLS	-0,019	0,012	0,036	0,037	0,049	-0,024	0,028	0,026	0,033	0,062
CM	0,010	0,013	-0,011	0,037	0,051	0,025	0,031	-0,033	0,037	0,068
AD	-0,005	0,011	0,009	0,033	0,043	-0,007	0,023	0,008	0,026	0,049
$n=50$										
ML	0,030	0,003	-0,030	0,011	0,014	0,021	0,007	-0,022	0,008	0,016
LS	-0,018	0,006	0,026	0,016	0,022	-0,015	0,013	0,016	0,015	0,028
WLS	-0,007	0,004	0,012	0,013	0,016	-0,006	0,010	0,005	0,011	0,021
CM	-0,002	0,005	0,004	0,015	0,019	0,010	0,012	-0,014	0,014	0,027
AD	-0,006	0,003	0,011	0,012	0,016	-0,004	0,009	0,004	0,010	0,019
$n=100$										
ML	0,017	0,001	-0,017	0,005	0,006	0,012	0,003	-0,013	0,004	0,007
LS	-0,012	0,003	0,017	0,008	0,010	-0,007	0,006	0,007	0,007	0,013
WLS	-0,004	0,001	0,006	0,006	0,008	-0,001	0,005	0,000	0,005	0,010
CM	-0,005	0,002	0,007	0,007	0,009	0,006	0,006	-0,008	0,007	0,013
AD	-0,006	0,002	0,008	0,006	0,008	-0,002	0,004	0,002	0,005	0,009
$n=200$										
ML	0,010	0,000	-0,010	0,002	0,003	0,006	0,002	-0,006	0,002	0,004
LS	-0,009	0,001	0,012	0,004	0,005	-0,005	0,003	0,004	0,004	0,007
WLS	-0,003	0,001	0,005	0,003	0,003	-0,001	0,002	0,000	0,003	0,005
CM	-0,005	0,001	0,007	0,004	0,005	0,002	0,003	-0,003	0,004	0,007
AD	-0,004	0,001	0,007	0,003	0,004	-0,002	0,002	0,002	0,002	0,005
$n=500$										
ML	0,005	0,000	-0,005	0,001	0,001	0,003	0,001	-0,003	0,001	0,001
LS	-0,006	0,000	0,007	0,001	0,002	-0,002	0,001	0,002	0,001	0,003
WLS	-0,002	0,000	0,002	0,001	0,001	0,000	0,001	0,000	0,001	0,002
CM	-0,004	0,000	0,005	0,001	0,002	0,000	0,001	-0,001	0,001	0,003
AD	-0,002	0,000	0,004	0,001	0,001	-0,001	0,001	0,001	0,001	0,002

Table 2. (continued)

	$(a,b,p)=(1,1,4)$					$(a,b,p)=(1,1,6)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,043	0,023	-0,045	0,023	0,046	0,041	0,027	-0,044	0,026	0,054
LS	-0,043	0,041	0,046	0,044	0,086	-0,041	0,047	0,042	0,047	0,093
WLS	-0,028	0,034	0,030	0,036	0,070	-0,027	0,039	0,027	0,038	0,077
CM	0,025	0,036	-0,030	0,039	0,075	0,032	0,042	-0,035	0,042	0,083
AD	-0,010	0,027	0,011	0,027	0,054	-0,008	0,032	0,007	0,031	0,062
					$n=50$					
ML	0,018	0,009	-0,019	0,009	0,018	0,015	0,010	-0,016	0,010	0,020
LS	-0,015	0,015	0,016	0,016	0,031	-0,019	0,017	0,019	0,017	0,033
WLS	-0,006	0,012	0,006	0,012	0,024	-0,008	0,013	0,008	0,013	0,026
CM	0,012	0,015	-0,014	0,015	0,030	0,010	0,016	-0,011	0,016	0,032
AD	-0,004	0,011	0,004	0,011	0,022	-0,006	0,012	0,006	0,012	0,024
					$n=100$					
ML	0,010	0,004	-0,010	0,004	0,009	0,009	0,005	-0,010	0,005	0,010
LS	-0,008	0,007	0,008	0,007	0,015	-0,007	0,008	0,008	0,008	0,016
WLS	-0,002	0,006	0,002	0,006	0,011	-0,001	0,006	0,001	0,006	0,012
CM	0,006	0,007	-0,007	0,007	0,014	0,007	0,008	-0,007	0,008	0,016
AD	-0,002	0,005	0,002	0,005	0,011	-0,001	0,006	0,001	0,006	0,012
					$n=200$					
ML	0,004	0,002	-0,004	0,002	0,004	0,004	0,002	-0,005	0,002	0,005
LS	-0,006	0,004	0,006	0,004	0,007	-0,004	0,004	0,004	0,004	0,008
WLS	-0,002	0,003	0,002	0,003	0,006	0,000	0,003	0,000	0,003	0,006
CM	0,001	0,003	-0,001	0,004	0,007	0,003	0,004	-0,003	0,004	0,008
AD	-0,003	0,003	0,003	0,003	0,005	-0,001	0,003	0,001	0,003	0,006
					$n=500$					
ML	0,002	0,001	-0,002	0,001	0,002	0,002	0,001	-0,002	0,001	0,003
LS	-0,002	0,001	0,002	0,001	0,003	-0,003	0,002	0,003	0,002	0,004
WLS	0,000	0,001	0,000	0,001	0,002	0,000	0,002	0,000	0,002	0,003
CM	0,001	0,001	-0,001	0,001	0,003	0,001	0,002	-0,001	0,002	0,004
AD	-0,001	0,001	0,001	0,001	0,002	-0,001	0,002	0,001	0,002	0,003

Table 2. (continued)

	$(a,b,p)=(1,2,1.5)$					$(a,b,p)=(1,2,3)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,039	0,004	-0,063	0,029	0,033	0,039	0,013	-0,052	0,022	0,035
LS	-0,021	0,007	0,054	0,046	0,053	-0,029	0,023	0,040	0,042	0,064
WLS	-0,012	0,005	0,034	0,038	0,043	-0,018	0,018	0,026	0,033	0,051
CM	0,006	0,005	-0,006	0,038	0,044	0,022	0,020	-0,034	0,037	0,057
AD	-0,003	0,004	0,015	0,033	0,037	-0,004	0,015	0,007	0,025	0,040
					$n=50$					
ML	0,019	0,001	-0,028	0,011	0,012	0,017	0,005	-0,023	0,008	0,013
LS	-0,011	0,002	0,029	0,016	0,018	-0,013	0,008	0,017	0,015	0,023
WLS	-0,004	0,001	0,014	0,013	0,014	-0,004	0,006	0,005	0,012	0,018
CM	-0,001	0,002	0,006	0,014	0,016	0,007	0,008	-0,013	0,015	0,022
AD	-0,004	0,001	0,013	0,012	0,014	-0,003	0,006	0,004	0,010	0,016
					$n=100$					
ML	0,011	0,000	-0,017	0,005	0,005	0,008	0,002	-0,011	0,004	0,006
LS	-0,007	0,001	0,017	0,008	0,008	-0,007	0,004	0,009	0,007	0,011
WLS	-0,002	0,001	0,005	0,006	0,006	-0,002	0,003	0,002	0,005	0,008
CM	-0,003	0,001	0,006	0,007	0,008	0,003	0,004	-0,006	0,007	0,011
AD	-0,003	0,001	0,008	0,006	0,006	-0,003	0,003	0,004	0,005	0,008
					$n=200$					
ML	0,007	0,000	-0,009	0,002	0,003	0,005	0,001	-0,006	0,002	0,003
LS	-0,005	0,000	0,013	0,004	0,004	-0,004	0,002	0,005	0,003	0,005
WLS	-0,002	0,000	0,005	0,003	0,003	-0,001	0,001	0,001	0,003	0,004
CM	-0,003	0,000	0,007	0,004	0,004	0,001	0,002	-0,002	0,003	0,005
AD	-0,003	0,000	0,007	0,003	0,003	-0,001	0,001	0,002	0,002	0,004
					$n=500$					
ML	0,003	0,000	-0,006	0,001	0,001	0,002	0,000	-0,003	0,001	0,001
LS	-0,004	0,000	0,007	0,001	0,002	-0,001	0,001	0,002	0,001	0,002
WLS	-0,001	0,000	0,002	0,001	0,001	0,000	0,001	0,000	0,001	0,002
CM	-0,003	0,000	0,005	0,001	0,002	0,001	0,001	-0,001	0,001	0,002
AD	-0,002	0,000	0,003	0,001	0,001	0,000	0,001	0,001	0,001	0,002

Table 2. (continued)

	$(a,b,p)=(1,2,4)$					$(a,b,p)=(1,2,6)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,036	0,016	-0,045	0,023	0,039	0,034	0,021	-0,041	0,025	0,045
LS	-0,035	0,028	0,045	0,043	0,071	-0,040	0,037	0,046	0,047	0,084
WLS	-0,024	0,023	0,030	0,034	0,057	-0,027	0,031	0,032	0,038	0,069
CM	0,022	0,025	-0,031	0,038	0,063	0,025	0,033	-0,031	0,041	0,074
AD	-0,008	0,019	0,010	0,026	0,045	-0,010	0,024	0,011	0,030	0,054
					$n=50$					
ML	0,013	0,006	-0,017	0,009	0,015	0,016	0,008	-0,018	0,010	0,018
LS	-0,015	0,010	0,019	0,015	0,026	-0,015	0,013	0,017	0,017	0,030
WLS	-0,007	0,008	0,009	0,012	0,020	-0,005	0,011	0,006	0,013	0,024
CM	0,008	0,010	-0,011	0,015	0,024	0,011	0,013	-0,013	0,016	0,029
AD	-0,005	0,008	0,006	0,011	0,018	-0,003	0,010	0,003	0,012	0,022
					$n=100$					
ML	0,008	0,003	-0,010	0,004	0,007	0,008	0,004	-0,009	0,005	0,009
LS	-0,007	0,005	0,009	0,008	0,013	-0,007	0,006	0,008	0,008	0,014
WLS	-0,002	0,004	0,002	0,006	0,010	-0,001	0,005	0,002	0,006	0,011
CM	0,005	0,005	-0,006	0,007	0,013	0,005	0,006	-0,007	0,008	0,014
AD	-0,002	0,004	0,002	0,005	0,009	-0,001	0,005	0,002	0,006	0,010
					$n=200$					
ML	0,004	0,001	-0,005	0,002	0,004	0,005	0,003	-0,006	0,004	0,007
LS	-0,003	0,003	0,004	0,004	0,006	-0,006	0,005	0,007	0,006	0,011
WLS	0,000	0,002	0,000	0,003	0,005	-0,001	0,004	0,001	0,005	0,008
CM	0,002	0,002	-0,003	0,004	0,006	0,003	0,005	-0,004	0,006	0,010
AD	-0,001	0,002	0,001	0,003	0,005	-0,002	0,004	0,002	0,004	0,008
					$n=500$					
ML	0,002	0,001	-0,002	0,001	0,001	0,005	0,002	-0,006	0,003	0,005
LS	-0,001	0,001	0,002	0,002	0,003	-0,004	0,004	0,005	0,005	0,009
WLS	0,000	0,001	0,000	0,001	0,002	-0,001	0,003	0,001	0,004	0,007
CM	0,001	0,001	-0,001	0,002	0,003	0,003	0,004	-0,004	0,005	0,008
AD	0,000	0,001	0,000	0,001	0,002	-0,001	0,003	0,001	0,004	0,006

TABLE 2. (continued)

	$(a,b,p)=(6,4.5,1.5)$					$(a,b,p)=(6,4.5,3)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,050	0,038	-0,042	0,027	0,064	0,046	0,033	-0,042	0,027	0,060
LS	-0,055	0,067	0,047	0,047	0,114	-0,046	0,056	0,043	0,046	0,102
WLS	-0,035	0,054	0,030	0,038	0,093	-0,030	0,046	0,027	0,038	0,084
CM	0,035	0,059	-0,030	0,041	0,100	0,037	0,050	-0,034	0,041	0,090
AD	-0,010	0,043	0,009	0,031	0,074	-0,006	0,037	0,005	0,030	0,068
					$n=50$					
ML	0,017	0,014	-0,014	0,010	0,025	0,017	0,013	-0,015	0,010	0,023
LS	-0,024	0,024	0,021	0,017	0,041	-0,020	0,020	0,018	0,016	0,036
WLS	-0,011	0,019	0,010	0,013	0,032	-0,009	0,016	0,008	0,013	0,029
CM	0,011	0,023	-0,009	0,016	0,038	0,013	0,019	-0,012	0,015	0,034
AD	-0,007	0,017	0,006	0,012	0,029	-0,004	0,014	0,004	0,012	0,026
					$n=100$					
ML	0,009	0,007	-0,008	0,005	0,012	0,007	0,006	-0,007	0,005	0,012
LS	-0,011	0,011	0,010	0,008	0,019	-0,011	0,010	0,010	0,008	0,018
WLS	-0,003	0,009	0,003	0,006	0,015	-0,003	0,008	0,003	0,006	0,014
CM	0,006	0,011	-0,005	0,008	0,019	0,006	0,010	-0,005	0,008	0,018
AD	-0,003	0,008	0,002	0,006	0,014	-0,003	0,008	0,002	0,006	0,014
					$n=200$					
ML	0,005	0,004	-0,004	0,002	0,006	0,004	0,003	-0,004	0,003	0,006
LS	-0,005	0,006	0,005	0,004	0,010	-0,006	0,005	0,005	0,004	0,009
WLS	0,000	0,004	0,000	0,003	0,007	-0,001	0,004	0,001	0,003	0,007
CM	0,003	0,006	-0,003	0,004	0,010	0,002	0,005	-0,002	0,004	0,009
AD	-0,001	0,004	0,001	0,003	0,007	-0,002	0,004	0,001	0,003	0,007
					$n=500$					
ML	0,002	0,001	-0,002	0,001	0,002	0,001	0,001	-0,001	0,001	0,002
LS	-0,002	0,002	0,001	0,002	0,004	-0,003	0,002	0,002	0,002	0,004
WLS	0,000	0,002	0,000	0,001	0,003	0,000	0,002	0,000	0,001	0,003
CM	0,002	0,002	-0,002	0,002	0,004	0,001	0,002	-0,001	0,002	0,004
AD	0,000	0,002	0,000	0,001	0,003	-0,001	0,002	0,001	0,001	0,003

TABLE 2. (continued)

	$(a,b,p)=(6,4.5,4)$					$(a,b,p)=(6,4.5,6)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
$n=20$										
ML	0,039	0,031	-0,037	0,027	0,058	0,038	0,031	-0,037	0,028	0,059
LS	-0,051	0,055	0,047	0,047	0,102	-0,051	0,054	0,049	0,049	0,103
WLS	-0,034	0,045	0,031	0,039	0,084	-0,035	0,045	0,033	0,040	0,085
CM	0,032	0,048	-0,030	0,041	0,089	0,029	0,047	-0,028	0,042	0,089
AD	-0,010	0,036	0,009	0,031	0,067	-0,011	0,036	0,010	0,032	0,069
$n=50$										
ML	0,017	0,012	-0,016	0,010	0,023	0,016	0,012	-0,015	0,011	0,022
LS	-0,019	0,019	0,018	0,017	0,036	-0,018	0,019	0,018	0,017	0,036
WLS	-0,007	0,015	0,007	0,013	0,029	-0,008	0,015	0,008	0,013	0,028
CM	0,013	0,018	-0,013	0,016	0,034	0,013	0,018	-0,013	0,016	0,034
AD	-0,003	0,014	0,003	0,012	0,026	-0,003	0,014	0,003	0,012	0,026
$n=100$										
ML	0,006	0,006	-0,006	0,005	0,011	0,008	0,006	-0,008	0,005	0,011
LS	-0,012	0,010	0,012	0,008	0,018	-0,009	0,009	0,009	0,008	0,017
WLS	-0,005	0,008	0,005	0,006	0,014	-0,002	0,007	0,002	0,006	0,014
CM	0,004	0,009	-0,004	0,008	0,017	0,007	0,009	-0,006	0,008	0,017
AD	-0,005	0,007	0,004	0,006	0,013	-0,002	0,007	0,001	0,006	0,013
$n=200$										
ML	0,004	0,003	-0,003	0,003	0,006	0,004	0,003	-0,004	0,003	0,006
LS	-0,006	0,005	0,006	0,004	0,009	-0,005	0,004	0,005	0,004	0,008
WLS	-0,001	0,004	0,001	0,003	0,007	-0,001	0,003	0,001	0,003	0,007
CM	0,002	0,005	-0,002	0,004	0,009	0,003	0,004	-0,003	0,004	0,008
AD	-0,002	0,004	0,002	0,003	0,007	-0,001	0,003	0,001	0,003	0,006
$n=500$										
ML	0,001	0,001	-0,001	0,001	0,002	0,002	0,001	-0,002	0,001	0,002
LS	-0,002	0,002	0,002	0,002	0,003	-0,001	0,002	0,001	0,002	0,003
WLS	0,000	0,001	0,000	0,001	0,003	0,001	0,001	-0,001	0,001	0,003
CM	0,001	0,002	-0,001	0,002	0,003	0,002	0,002	-0,002	0,002	0,003
AD	-0,001	0,001	0,001	0,001	0,003	0,000	0,001	0,000	0,001	0,003

Table 2. (continued)

	$(a,b,p)=(15,5,1.5)$					$(a,b,p)=(15,5,3)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
$n=20$										
ML	0,064	0,070	-0,039	0,026	0,097	0,046	0,044	-0,036	0,026	0,070
LS	-0,077	0,123	0,048	0,047	0,170	-0,066	0,078	0,052	0,048	0,126
WLS	-0,051	0,101	0,032	0,038	0,139	-0,045	0,064	0,036	0,039	0,103
CM	0,048	0,108	-0,029	0,041	0,148	0,033	0,067	-0,026	0,041	0,108
AD	-0,014	0,081	0,009	0,031	0,112	-0,015	0,051	0,012	0,031	0,082
$n=50$										
ML	0,024	0,028	-0,015	0,011	0,039	0,018	0,017	-0,014	0,010	0,028
LS	-0,030	0,045	0,018	0,017	0,063	-0,024	0,028	0,019	0,017	0,045
WLS	-0,013	0,036	0,008	0,014	0,050	-0,011	0,022	0,008	0,013	0,035
CM	0,019	0,043	-0,012	0,016	0,059	0,015	0,026	-0,012	0,016	0,042
AD	-0,007	0,033	0,004	0,012	0,045	-0,005	0,020	0,004	0,012	0,032
$n=100$										
ML	0,010	0,014	-0,006	0,005	0,019	0,008	0,008	-0,007	0,005	0,014
LS	-0,018	0,022	0,011	0,008	0,030	-0,013	0,013	0,011	0,008	0,021
WLS	-0,007	0,017	0,004	0,006	0,024	-0,005	0,010	0,004	0,006	0,016
CM	0,006	0,021	-0,004	0,008	0,029	0,006	0,013	-0,005	0,008	0,020
AD	-0,006	0,016	0,004	0,006	0,023	-0,004	0,010	0,003	0,006	0,016
$n=200$										
ML	0,006	0,007	-0,004	0,003	0,009	0,005	0,004	-0,004	0,003	0,007
LS	-0,008	0,010	0,005	0,004	0,014	-0,006	0,006	0,005	0,004	0,010
WLS	-0,001	0,008	0,001	0,003	0,011	-0,001	0,005	0,001	0,003	0,008
CM	0,004	0,010	-0,003	0,004	0,014	0,004	0,006	-0,003	0,004	0,010
AD	-0,002	0,008	0,001	0,003	0,011	-0,001	0,005	0,001	0,003	0,008
$n=500$										
ML	0,003	0,003	-0,002	0,001	0,004	0,001	0,002	-0,001	0,001	0,003
LS	-0,003	0,004	0,002	0,002	0,006	-0,003	0,003	0,002	0,002	0,004
WLS	0,000	0,003	0,000	0,001	0,004	-0,001	0,002	0,000	0,001	0,003
CM	0,002	0,004	-0,001	0,002	0,006	0,001	0,003	-0,001	0,002	0,004
AD	0,000	0,003	0,000	0,001	0,004	-0,001	0,002	0,001	0,001	0,003

Table 2. (continued)

	$(a,b,p)=(15,5,4)$					$(a,b,p)=(15,5,6)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,046	0,040	-0,038	0,027	0,067	0,044	0,035	-0,039	0,027	0,061
LS	-0,056	0,069	0,047	0,047	0,116	-0,054	0,060	0,048	0,047	0,107
WLS	-0,036	0,057	0,030	0,039	0,096	-0,035	0,050	0,031	0,039	0,089
CM	0,036	0,060	-0,030	0,041	0,101	0,033	0,053	-0,030	0,041	0,093
AD	-0,009	0,046	0,008	0,031	0,077	-0,010	0,040	0,008	0,031	0,070
					$n=50$					
ML	0,016	0,015	-0,014	0,011	0,026	0,017	0,014	-0,015	0,011	0,024
LS	-0,024	0,024	0,020	0,017	0,041	-0,022	0,021	0,019	0,017	0,038
WLS	-0,011	0,020	0,009	0,013	0,033	-0,010	0,017	0,009	0,013	0,031
CM	0,013	0,023	-0,011	0,016	0,039	0,013	0,020	-0,011	0,016	0,036
AD	-0,006	0,018	0,005	0,012	0,030	-0,004	0,016	0,004	0,012	0,028
					$n=100$					
ML	0,011	0,008	-0,009	0,005	0,013	0,009	0,007	-0,008	0,005	0,012
LS	-0,010	0,012	0,008	0,008	0,020	-0,010	0,011	0,008	0,008	0,019
WLS	-0,001	0,009	0,001	0,006	0,016	-0,002	0,008	0,002	0,007	0,015
CM	0,008	0,012	-0,007	0,008	0,019	0,008	0,010	-0,007	0,008	0,019
AD	-0,001	0,009	0,001	0,006	0,015	-0,001	0,008	0,001	0,006	0,014
					$n=200$					
ML	0,005	0,004	-0,004	0,003	0,006	0,004	0,003	-0,004	0,003	0,006
LS	-0,005	0,006	0,004	0,004	0,010	-0,005	0,005	0,004	0,004	0,009
WLS	0,000	0,005	0,000	0,003	0,008	-0,001	0,004	0,000	0,003	0,007
CM	0,004	0,006	-0,003	0,004	0,010	0,004	0,005	-0,003	0,004	0,009
AD	-0,001	0,004	0,001	0,003	0,007	-0,001	0,004	0,001	0,003	0,007
					$n=500$					
ML	0,002	0,002	-0,002	0,001	0,003	0,002	0,001	-0,002	0,001	0,002
LS	-0,002	0,002	0,002	0,002	0,004	-0,002	0,002	0,002	0,002	0,004
WLS	0,000	0,002	0,000	0,001	0,003	0,000	0,002	0,000	0,001	0,003
CM	0,001	0,002	-0,001	0,002	0,004	0,001	0,002	-0,001	0,002	0,004
AD	0,000	0,002	0,000	0,001	0,003	-0,001	0,002	0,000	0,001	0,003

4.1. **Comparisons for the biases.** In this subsection, the biases of the estimators $\hat{\mu}$ and $\hat{\sigma}$ obtained from the ML, LS, WLS, CM and AD methodologies are compared. For the estimators of the location parameter μ and the scale parameter σ, in general, the AD has the smallest bias among the other estimators for all values of the shape parameters and the sample sizes except for the sample size $n = 50$ and shape parameters $a = 1$, $b = 1$, $p = 1.5$ and $a = 1$, $b = 2$, $p = 1.5$ in which case CM provides the smallest bias. When the sample size n=100 and shape parameters $a = 1$, $b = 1$, $p = 1.5$, $a = 1$, $b = 1$, $p = 3$, $a = 1$, $b = 2$, $p = 1.5$ and $a = 1$,

$b = 2$, $p = 3$, WLS provides the smallest biases. AD is followed by the WLS and CM estimators for the small and moderate sample sizes in most of the cases. ML and LS estimators have larger biases than the other estimators for the small and moderate sample sizes. For the large sample sizes, all the estimators have negligible biases.

4.2. **Comparisons for the efficiencies.** Discussions about the efficiencies of the estimators of μ and σ with respect to the MSE criterion are given as follows. For the estimators of the location parameter μ, ML estimator shows the best performance among the others with respect to the MSE criterion in all cases. It is followed by the AD and WLS estimators for the sample sizes $n = 20$ and 50. It should also be pointed out that the LS estimator has shown the worst performance among the others for the sample sizes $n = 20$ and 50. For the sample sizes n $\geq$ 100, ML is the most efficient estimator among the others in general and it is followed by the AD and WLS estimators. For the estimators of the scale parameter σ, the ML is the most efficient among the others for all values of the shape parameters and the sample sizes. It is followed by the AD and WLS estimators for the small and moderate sample sizes. The LS estimator of σ shows the worst performance among the other estimators in almost all cases.

4.3. **Comparisons for the joint efficiencies.** According to the simulation results, the ML estimator shows the highest performance among the others for all values of the shapes parameters and the sample sizes. It is seen that the ML estimator is followed by AD estimator. On the other hand, the LS estimator has the worst performance among the other estimators in almost all cases.

5. Application

In this section, the KwWeibull distribution is used to model the relative humidity data set taken from Cortez and Morais [7]. Table 3 shows the descriptive statistics for the relative humidity data.

Table 3. Descriptive statistics for the relative humidity data.

n	Min	Max	Mean	Variance	Skewness	Kurtosis
517	15.0	100.0	44.29	266.3	0.85	2.59

Before analyzing the relative humidity data, profile likelihood method is used to identify the plausible values of the shape parameters a, b and p of the KwWeibull distribution (see for example, Islam and Tiku [13] and Acıtaş and Şenoğlu [1]). The steps of the profile likelihood procedure are given as follows:

Step 1. Calculate $\hat{\mu}$ and $\hat{\sigma}$ for the given a, b and p values.
Step 2. Calculate $\ln L$value by incorporating $\hat{\mu}$ and $\hat{\sigma}$ into lnL.
Step 3. Repeat *Steps* 1 and 2 for a serious values of a, b and p. Find a, b and p values maximizing the $\ln L$ function among the others and choose them as conceivable values of the shape parameters.

Following the steps of profile likelihood procedure, the values of shape parameters a, b and p are obtained as 5.637, 6.133 and 0.681, respectively. We also use QQ plot which is a well known and widely used graphical technique to identify the distribution of the relative humidity data set, see Figure 2. It can be seen from Figure 2 that KwWeibull distribution provides a good fit for the relative humidity data.

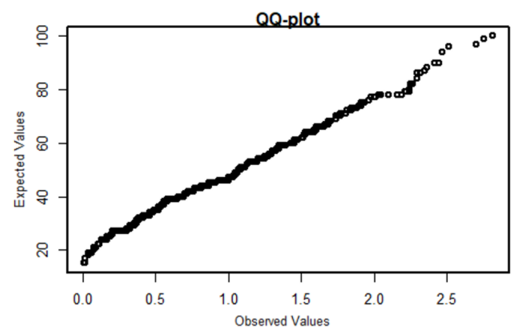

FIGURE 2. KwWeibull QQ plot for the relative humidity data

Based on the estimate values of the shape parameters, the ML estimates of location parameter μ and scale parameter σ are obtained as given in the Table 4. Estimates of the parameters μ, σ and p of Weibull distribution are also given for the relative humidity data to make the comparisons complete in Table 4. The Akaike information criterion (AIC), Bayesian information criterion (BIC) and Corrected AIC (AICc) values along with the Kolmogorov-Smirnov (KS) test statistic and associated $p-values$ are also given in Table 4.

The equalities for the AIC, BIC, AICc and KS are given by

$$\begin{aligned} AIC &= -2\ln L + 2k, \\ BIC &= -2\ln L + kln(n), \\ AICc &= AIC + (2k^2 + k)/(n-k-1) \end{aligned} \tag{18}$$

and

$$KS = \max \left| \hat{F}(X_{(i)}) - \tfrac{i}{n+1} \right|, \tag{19}$$

respectively. Here, $\hat{F}$ is the estimated cdf, $X_{(i)}$ is the $i-th$ order statistics, k is the number of the unknown parameters and n is the sample size.

Table 4. The estimates of the parameters of the KwWeibull and Weibull distributions for the relative humidity data

	$\hat{a}$	$\hat{b}$	$\hat{p}$	$\hat{\mu}$	$\hat{\sigma}$	KS	p-value	AIC	BIC	AICc
KwWeibull	5.637	6.133	0.681	25.466	11.763	0.043	0.273	4273.80	4295.05	4273.88
Weibull	-	-	1.924	33.662	14.485	0.097	0.063	4337.76	4346.27	4337.81

The smaller AIC, BIC and AICc values imply the better fitting performance. It is clear from Table 4 that the KwWeibull distribution is more preferable than the Weibull distribution in terms of these criteria. See also Figure 3 in which the histogram and the fitted densities based on the KwWeibull and the Weibull distributions are plotted. Here, it should be noted that the ML estimates of the parameters are used in obtaining the fitted densities.

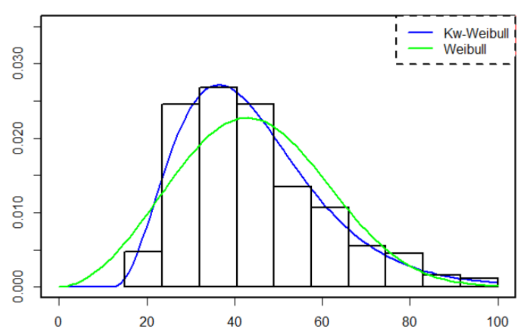

Figure 3. The histogram and the fitted densities based on the KwWeibull and Weibull distributions for the relative humidity data

It is seen from Figure 3 that KwWeibull distribution shows better fitting performance than the Weibull distribution. Then, we obtain the estimates of location parameter μ and scale parameter σ of the KwWeibull distribution when $\hat{a}$ =5.637, $\hat{b}$ =6.133 and $\hat{p}$=0.681 by using ML, LS, WLS, CM and AD methods to see the fitting performance of KwWeibull distribution for each estimation methods. Estimates of the location and scale parameters of KwWeibull distribution for each estimation methods are given in Table 5.

Table 5. Estimates of the location and scale parameters of the KwWeibull distribution for relative humidity data

Estimation Methods	$\hat{\mu}$	$\hat{\sigma}$	AIC	BIC	AICc
ML	25.466	11.763	4273.80	4295.05	4273.88
LS	19.712	14.327	4883.75	4904.99	4883.86
WLS	28.322	12.365	4646.50	4649.39	4646.53
CM	24.553	14.680	4675.32	4696.56	4675.43
AD	18.859	13.140	4622.33	4643.57	4622.44

The histogram and fitted densities based on different estimation methods are given in Figure 4 for the KwWeibull distribution.

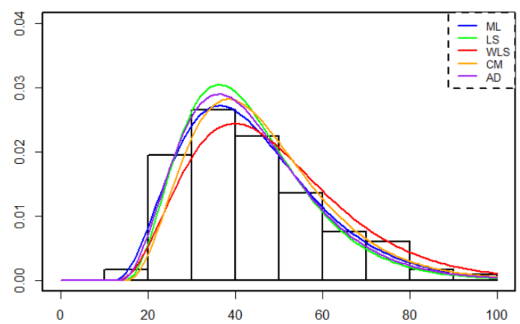

Figure 4. The histogram and the fitted densities based on ML, LS, WLS, CM and AD estimates for the KwWeibull distribution

It can easily be seen from both Table 5 and the Figure 4 that ML method shows the best performance among the others with respect to the fitting performance for the relative humidity data.

6. Conclusions

In this study, we obtain the estimators of location and scale parameters of KwWeibull distribution using the ML, LS, WLS, CM and AD methods. We perform an extensive Monte Carlo simulation study to compare the efficiencies of these estimators. It is concluded that ML estimator shows the best performance among the others and it is followed by AD estimator. The LS estimator demonstrates the worst performance in almost all cases. At the end of the study, we use relative humidity data taken from the literature. Modelling performances of the KwWeibull distribution and the well known and widely used Weibull distribution are compared for this data. It is concluded that KwWeibull distribution shows better fitting performance than the Weibull distribution for modeling the relative humidity data.

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

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CONSTRUCTING DIRECTED STRONGLY REGULAR GRAPHS BY USING SEMIDIRECT PRODUCTS AND SEMIDIHEDRAL GROUPS

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Abstract. In this paper, directed strongly regular graphs (DSRGs) are constructed by using semidirect products. The orbit condition in [3] has been weakened and this gives rise to the construction of DSRGs. Moreover, a different construction is given for DSRG by using semidihedral groups.

1. Introduction

Directed strongly regular graphs have attracted the attention of many mathematicians and many studies have been done on them. It was first discussed by Duval as the directed form of strongly regular graphs [2]. Duval also presented several construction methods in his work. The main problem today is to construct unknown ones by their parameters. For this purpose, many mathematical structures have been used. Some of these are designs [[5], [11]], coherent algebras [[5], [7], [10]], finite geometries [[4], [5], [6]], matrices [[2], [4], [6], [8]] and dihedral groups [10]. Some non-existence results are given by Jorgensen [9]. Duval [3] constructed directed strongly regular graphs by using semidirect products with an orbit condition. We change this condition with a weaker condition and give a construction of the directed strongly regular graphs. We also provide give a construction by using semidihedral groups. Our construction methods using semidirect product and semidihedral groups are new, however they do not give new parameters for small

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examples. Also, they are simple to use for finding larger parameters. Uniqueness and enumeration studies can be found in [1].

This paper is designed as follows. In Section 2, necessary background information on the graph is given and the notations we will use are introduced, in Section 3 the semidirect construction of DSRG of Cayley graphs are given, and finally, in Section 4, DSRG is constructed from semidihedral groups which is an example of semidirect products.

2. Preliminaries

A *directed graph* $\Gamma = (V, E)$ consists of a vertex set V and an edge set E, where an edge is an ordered pair of distinct vertices of Γ. Writing $(x, y) \in E$ means that there is a *directed edge* from x to y and that is shown by $x \to y$. Throughout the paper, the edges of the form (y, y) for some $y \in V$, i.e., loops, are not allowed. However, we allow *bidirected edge*, that is having edges $x \to y$ and $y \to x$ for the vertices x and y, simultaneously. The *indegree (outdegree)* of a vertex y in a directed graph Γ is the number of vertices x such that $x \to y$ $(y \to x)$, respectively. A graph Γ is called *k-regular* if every vertex in Γ has indegree and outdegree k. A *path* of length l from x to y is a sequence of $l + 1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. A directed graph Γ is called *directed strongly regular* with parameters (n, k, t, λ, μ) if it is k-regular and satisfies the following condition on the number of paths of length 2. The number of directed paths of length 2 between two vertices, say from x to y, of the graph Γ is λ if there is an edge from x to y, μ if there is not and t if $x = y$. Let G be a group and $S \subseteq G$ be a subset of G without the identity element. *Directed Cayley graph* $Cay(G, S)$ is a directed graph whose vertex set is G and for any two vertices x, y, there is a directed edge from x to y if $xy^{-1} \in S$.

Example 1. Let G be a symmetric group of order six with elements $\{e, a, a^2, b, ab, a^2b\}$ and the subset $S \subseteq G$ be the set $\{a^2, a^2b\}$. Then the directed graph $Cay(G, S)$ is shown as in Figure 1. The Cayley table of the elements of symmetric group of order 6 is shown as in Table 1.

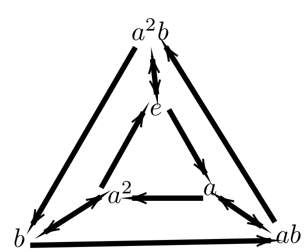

Figure 1. Cayley graph of symmetric group of order 6

$*$	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

TABLE 1. The Cayley table of the symmetric group of order 6

When studying directed strongly regular graphs adjacency matrix and group ring are advantageous tools. Let G be a finite group then the group ring $\mathbb{Z}[G]$ is a ring with identity element e and defined as the set of all formal sums of elements of G. The addition and multiplication are given by

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G}(a_g + b_g)g$$

and

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{r\in G} b_r r\right) = \sum_{g,r\in G} a_g b_r (g+r)$$

Let G be a group and $\mathbb{Z}[G] = \{\sum_{g\in G} a_g g | a_g \in \mathbb{Z}\}$. If $S \subset G$, the group ring element $\underline{S}$ will then be defined using the abuse of notation as $\underline{S} = \sum_{s\in S} s$. Furthermore, the group ring elements $\underline{S}^{(-1)}$ and $\underline{G}$ will be defined as $\underline{S}^{(-1)} := \sum_{s\in S} s^{-1}$ and $\underline{G} := \sum_{g\in G} g$.

Let S be a subset of a group G. In [2] they showed that $Cay(G, S)$ corresponds to a DSRG with parameters (n, k, t, λ, μ) if and only if $|S| = k$, $|G| = n$ and it satisfies the following group ring equation:

$$\underline{S}^2 = te + \lambda\underline{S} + \mu(\underline{G} - e - \underline{S}).$$

Let Γ be a directed graph with n vertices, then the adjacency matrix M of Γ is an $n \times n$ matrix with entries a_{ij} where $a_{ij} = 1$ if $v_i \to v_j$. Otherwise $a_{ij} = 0$. Since we disallow loops, the diagonal entries of M are all 0. Let I and J denote the $n \times n$ identity matrix and the all-one matrix, respectively. Then Γ is a directed strongly regular graph if and only if

i) $MJ = JM = kJ$
ii) $M^2 = tI + \lambda M + \mu(J - I - M)$.

3. SEMIDIRECT CONSTRUCTION OF CAYLEY DSRG

In this section, we give some definitions and lemmas related to the semidirect product of two groups. We will also proceed in a similar way to that of Duval and

Dmitri [3] by modifying the orbit setup they used. They proved that for a finite group A of order m and the cyclic group B of order q if some $\beta \in Aut(A)$ has the *q-orbit condition*, that is, each orbit of β contains only q elements, then the graph $Cay(A \ltimes_\theta B, A' \times B)$ is a DSRG with parameters

$$(mq, m-1, (m-1)/q, ((m-1)/q) - 1, (m-1)/q))$$

where $\theta : B \to Aut(A)$ by $\theta(b^r) = \beta^r$ and A' is the set of representatives of the nontrivial orbits of β.

Definition 1. (see [3]) Let A and B be two groups and $\theta : B \to Aut(A)$ be an action of B on A. Then the semidirect product $A \ltimes_\theta B$ for the set $\{(a, b) : a \in A \text{ and } b \in B\}$ is defined as follows:

$$(a, b)(a'b') = (a[\theta_b(a')], bb').$$

For groups A and B, $A \ltimes_\theta B$ forms a group of order $|A||B|$ with the identity element (e_A, e_B) and inverse $(a, b)^{-1} = (\theta_{b^{-1}}(a^{-1}), b^{-1})$.

Let A and B be the additive groups of finite fields $\mathbb{F}_{p^2}$ and $\mathbb{F}_2$ respectively, where p is a prime number. The Frobenius automorphism is defined as follows:

$$\beta : \mathbb{F}_{p^2} \to \mathbb{F}_{p^2}$$
$$\beta(x) = x^p$$

We will use the following notation in the rest of the paper: P is the set of elements of $\mathbb{F}_p$, R is the set of representatives of orbits of β and R^p is the set as $\{x^p : x \in R\}$.

The orbits of the action β on $\mathbb{F}_{p^2}$ consists of p orbits of size one and $\frac{p^2-p}{2}$ orbits of size two.

Let $A \times B$ be the direct product of the sets A and B and define the operation $\ltimes$ as the product of two elements as follows:

$$(a_1, b_1) \ltimes (a_2, b_2) = \begin{cases} (a_1 + a_2, b_2), & \text{if } b_1 = 0, \\ (a_1 + a_2^p, b_2 + 1), & \text{if } b_1 = 1. \end{cases}$$

Lemma 1. *$(G, \ltimes)$ forms a group of order $2p^2$ where $G = A \times B$.*

Proof. Let us start the proof by showing that G is closed under the operation $\ltimes$. For any $(a_1, b_1), (a_2, b_2) \in G$,

$$(a_1, b_1) \ltimes (a_2, b_2) = \begin{cases} (a_1 + a_2, b_2) \in G, & \text{if } b_1 = 0, \\ (a_1 + a_2^p, b_2 + 1) \in G, & \text{if } b_1 = 1. \end{cases}$$

Hence, G is closed under $\ltimes$. It is easy to see that $(0, 0)$ is the identity element of the group. Indeed for any element (a, b) the following is true,

$$(a, b) \ltimes (0, 0) = (0, 0) \ltimes (a, b) = (a, b).$$

Next, the inverse of any element $(a,b) \in G$ is given by

$$(a,b)^{-1} = \begin{cases} (-a,-b), & \text{if } b_1 = 0, \\ (-a^p,-b), & \text{if } b_1 = 1. \end{cases}$$

Finally, we will show the associative property. For $(a_1,b_1),(a_2,b_2),(a_3,b_3) \in G$ we have the following:

$$((a_1,b_1) \ltimes (a_2,b_2)) \ltimes (a_3,b_3) = \begin{cases} (a_1+a_2+a_3,b_3), & \text{if } b_1=0, b_2=0, \\ (a_1+a_2+a_3^p,b_3+1), & \text{if } b_1=0, b_2=1, \\ (a_1+a_2^p+a_3^p,b_3+1), & \text{if } b_1=1, b_2=0, \\ (a_1+a_2^p+a_3,b_3), & \text{if } b_1=1, b_2=1. \end{cases}$$

$$= (a_1,b_1) \ltimes ((a_2,b_2) \ltimes (a_3,b_3))$$

and we are done. □

We say that a group automorphism β has the *q-orbit condition* if each of its orbits contains either q elements or one element (including the trivial orbit that contains only identity element). We change (weakened) the q-orbit condition that is defined in [3]. Before giving our main theorem, we need the following lemma.

Lemma 2. *The following equations hold in the group ring* $\mathbb{Z}[G]$.

(a) $(\underline{P \times \{1\}})^2 = |P|(\underline{P \times \{0\}})$
(b) $(\underline{R \times B})(\underline{P \times \{1\}}) = |P|(\underline{R \times B})$
(c) $(\underline{P \times \{1\}})(\underline{R \times B}) = |P|(\underline{R^p \times B})$
(d) $(\underline{R \times B})^2 = \frac{p^2-3p}{2}(\underline{R \times B}) + \frac{p^2-p}{2}(\underline{R^p \times B}) + \frac{p^2-p}{2}(\underline{P \times B})$

Proof. We will only prove (b). We know that P is the set of elements of the obvious orbits of β which are in $\mathbb{F}_p$ and R is the set of representatives of orbits of β. We also know that $B = \mathbb{F}_2$. Then we have the following:

$$\begin{aligned}(\underline{R \times B})(\underline{P \times \{1\}}) &= ((\underline{R \times \{0\}}) + (\underline{R \times \{1\}}))(\underline{P \times \{1\}}) \\ &= (\underline{R \times \{0\}})(\underline{P \times \{1\}}) + (\underline{R \times \{1\}})(\underline{P \times \{1\}}) \\ &= (\underline{\{(\sigma+\gamma,1) : \sigma \in R \text{ and } \gamma \in P\}}) \\ &+ (\underline{\{(\sigma+\gamma^p,1) : \sigma \in R \text{ and } \gamma \in P\}}) \\ &= |P|(\underline{R \times \{0\}}) + |P|(\underline{R \times \{1\}}) \\ &= |P|(\underline{R \times B}).\end{aligned}$$

The proof of (a), (c) and (d) are similar. □

Theorem 1. *Let* $A = \mathbb{F}_{p^2}$ *and* $B = \mathbb{F}_2$ *be two additive finite fields where* p *is an odd prime. If some* $\beta \in Aut(A)$ *has the q-orbit condition (for instance, Frobenius automorphism), then we may construct a directed strongly regular graph with* *parameters*

$$(n = 2p^2, k = p^2, t = (p^2 + p)/2, \lambda = (p^2 - p)/2, \mu = (p^2 + p)/2)$$

as follows: Let us define $\theta : B \to Aut(A)$ *with* $\theta_0 = Id$ *and* $\theta_1 = \beta(x) = x^p$ *for the additive group* $B = \mathbb{F}_2$. *Let* R *be the set representatives of orbits with two elements and* P *be the set of orbits with one element (only base field elements). Note that* $R \cap -R = \emptyset$ *where* $-R = \{-r : r \in R\}$. *Then, for the set* $S = (R \times B) \cup (P \times \{1\})$ *the Cayley graph*

$$Cay(A \times_\theta B, S)$$

is a DSRG with parameters above.

Proof. Let the set S be $(R \times B) \cup (P \times \{1\})$. Then $|S| = k = 2|R| + |P| = 2 \cdot [(p^2 - p)/2] + p = p^2$. Our goal is to show that the graph $Cay(G, S)$ is a DSRG with parameters (n, k, t, λ, μ). So, we need to show that the summation $\underline{S} = \sum_{s \in S} s$ is valid in the following equation in $\mathbb{Z}[G]$,

$$\underline{S}^2 = te + \lambda \underline{S} + \mu(\underline{G} - e - \underline{S}).$$

To do that it will be enough to show that $\underline{S}$ satisfies the equation

$$\underline{S}^2 + |P|\underline{S} = \mu \underline{G}.$$

By Lemma 1 and Lemma 2 we get,

$$\begin{aligned}
\underline{S}^2 + |P|\underline{S} &= (\underline{(R \times B) \cup (P \times \{1\})})^2 + |P|(\underline{(R \times B) \cup (P \times \{1\})}) \\
&= \underline{(R \times B) \times_\theta (R \times B)} + \underline{(P \times \{1\}) \times_\theta (P \times \{1\})} + \underline{(R \times B) \times_\theta (P \times \{1\})} + \\
&\quad \underline{(P \times \{1\}) \times_\theta (R \times B)} + |P|\underline{(R \times B)} + |P|\underline{(P \times \{1\})} \\
&= ((p^2 - 3p)/2)\underline{(R \times B)} + ((p^2 - p)/2)\underline{(R^p \times B)} + ((p^2 - p)/2)\underline{(P \times B)} + \\
&\quad p\underline{(P \times \{0\})} + p\underline{(R \times B)} + p\underline{(R^p \times B)} + p\underline{(R \times B)} + p\underline{(P \times \{1\})} \\
&= p\underline{G} + ((p^2 - p)/2)\underline{G} \\
&= ((p^2 + p)/2)\underline{G} = \mu \underline{G}
\end{aligned}$$

as required. □

Example 2. Let $p = 3$, $A = \mathbb{F}_{p^2}$, $B = \mathbb{F}_2$. Consider the Frobenius automorphism

$$\beta : \mathbb{F}_{p^2} \to \mathbb{F}_{p^2}$$
$$\beta(x) = x^p.$$

For $G = A \times B$, $(G, \ltimes)$ forms a group of order $2p^2$. The product of (a_1, b_1) and (a_2, b_2) is given by

$$(a_1, b_1) \ltimes (a_2, b_2) = \begin{cases} (a_1 + a_2, b_2), & \text{if } b_1 = 0, \\ (a_1 + a_2^p, b_2 + 1), & \text{if } b_1 = 1. \end{cases}$$

Similarly, the inverse of (a, b) is given by

$$(a,b)^{-1} = \begin{cases} (-a,-b), & \text{if } b=0, \\ ((-a)^p,-b), & \text{if } b=1. \end{cases}$$

Thus the orbits of β are $\{0\},\{a,2a+1\},\{a+1,2a+2\},\{2\},\{a+2,2a\},\{1\}$.

From Theorem 1, multiplying one-element orbits by $\{1\}$ and two-element orbits by the set B, we construct the set $S=\{(a,0),(a,1),(a+1,0),(a+1,1),(a+2,0),(a+2,1),(0,1),(1,1),(2,1)\}$. Then the Cayley graph $Cay(A\times B,S)$ is a directed strongly regular graph with parameters $(18,9,6,3,6)$.

4. Semidihedral Construction of Cayley DSRG

In this section, we will construct directed strongly regular graphs from semidihedral groups by using Cayley graphs. The method of producing DSRG's using semidihedral groups in this section is different from the semidirect method given in Section 3. The choice of our generator set S here is independent of the q-orbit condition. A semidihedral group $SD(m)$ is also an example of the semidirect product of cyclic group C_2 with the dihedral group. But in this construction C_2 acts on $C_{2^{m-1}}$ by $x \rightarrowtail x^{2^{m-2}-1}$ instead of $x \rightarrowtail x^{-1}$. Before we give the main theorem, we need the following lemma.

Lemma 3. *Let $G = SD(m) = \langle a,x | a^{2^{m-1}} = x^2 = e, xax = a^{2^{m-2}-1}\rangle$ be the semidihedral group of order $m \geq 4$. Let $P = P_1 \cup P_2$ where $P_i = \{a^{i+4k} : k = 0,1,...,2^{m-3}-1\}$. Then*

$$xP = P'x \text{ where } P' = P_2 \cup P_3.$$

Proof. Let $P = P_1 \cup P_2$. By multiplying both sides of this equality by x, we get

$$\begin{aligned} xP &= xP_1 \cup xP_2 \\ &= \{xa^{1+4k} : k=0,1,...,2^{m-3}-1\} \cup \{xa^{2+4k} : k=0,1,...,2^{m-3}-1\} \\ &= \{a^{(1+4k)\cdot(2^{m-2}-1)}x : k=0,1,...,2^{m-3}-1\} \\ &\cup \{a^{(2+4k)\cdot(2^{m-2}-1)}x : k=0,1,...,2^{m-3}-1\}. \end{aligned} \tag{1}$$

Since the power of a in P_1 and P_2 is $1 \mod 4$, $2 \mod 4$ respectively and $m \geq 4$, if we multiply the powers of a by $2^{m-2}-1$ we will have

$$\begin{aligned} 1+4k &\equiv 1(\mod 4) \\ (1+4k)\cdot(2^{m-2}-1) &\equiv 2^{m-2}-1(\mod 4) \\ &\equiv -1(\mod 4) \\ &\equiv 3(\mod 4) \end{aligned} \tag{2}$$

and

$$\begin{aligned}2+4k &\equiv 2(\mod 4)\\ (2+4k)\cdot(2^{m-2}-1) &\equiv 2^{m-1}-2(\mod 4)\\ &\equiv -2(\mod 4)\\ &\equiv 2(\mod 4).\end{aligned} \tag{3}$$

Therefore, using Equations (2) and (3) in Equation (1), we will have the following

$$\begin{aligned}&\{a^{(1+4k)\cdot(2^{m-2}-1)}x : k=0,1,...,2^{m-3}-1\} \cup \{a^{(2+4k)\cdot(2^{m-2}-1)}x : k=0,1,...,2^{m-3}-1\}\\ &=\{a^{3+4k}x : k=0,1,...,2^{m-3}-1\} \cup \{a^{2+4k}x : k=0,1,...,2^{m-3}-1\}\\ &= P_3x \cup P_2x = (P_2 \cup P_3)x = P'x.\end{aligned}$$

This completes the proof. □

Note that we also have the equations $xP_1 = P_3x$ ($P_1x = xP_3$) and $xP_2 = P_2x$.

Theorem 2. *Let* $G = SD(m) = \langle a, x | a^{2^{m-1}} = x^2 = e, xax = a^{2^{m-2}-1}\rangle$ *be the semidihedral group of order* $m \geq 4$. *Let* $P = P_1 \cup P_2$ *where* $P_i = \{a^{i+4k} : k = 0, 1, ..., 2^{m-3}-1\}$. *Then* $Cay(G, P \cup xP)$ *is a DSRG with parameters*

$$(n = 2^m, k = 2^{m-1}, t = 3.2^{m-3}, \lambda = 2^{m-3}, \mu = 3.2^{m-3}).$$

Proof. Let $S = P \cup xP$. Then the parameter $k = |S| = 2|P| = 2 \cdot 2^{m-2} = 2^{m-1}$. Our goal is to show that $Cay(G, S)$ is a DSRG with parameters (n, k, t, λ, μ). Thus the formal sum $\underline{S} = \sum_{s \in S} s$ should satisfy the equation

$$\underline{S}^2 = te + \lambda\underline{S} + \mu(\underline{G} - e - \underline{S})$$

in the group ring $\mathbb{Z}[G]$. Therefore, we need to show that the equation

$$\underline{S}^2 + 2^{m-2}\underline{S} = 3 \cdot 2^{m-3}\underline{G}$$

holds. So,

$$\begin{aligned}\underline{S}^2 + 2^{m-2}\underline{S} &= (\underline{P} + \underline{xP})^2 + 2^{m-2}(\underline{P} + \underline{xP})\\ &= \underline{P}^2 + \underline{P}\cdot\underline{xP} + \underline{xP}\cdot\underline{P} + \underline{xP}\cdot\underline{xP} + 2^{m-2}(\underline{P} + \underline{xP})\\ &= \underline{P}^2 + \underline{xP'}\cdot\underline{P} + \underline{xP}\cdot\underline{P} + \underline{P'}\cdot\underline{P} + 2^{m-2}(\underline{P} + \underline{xP})\end{aligned} \tag{4}$$

where $P' = P_2 \cup P_3$ by Lemma 3.

In order to complete the proof let us compute P_iP_i and P_jP_j. Since P_0 is a subgroup of order 2^{m-3} and $P_1 = aP_0$, $P_2 = a^2P_0$ and $P_3 = a^3P_0$ are its cosets, we have

$$\begin{aligned}\underline{P_iP_i} &= \underline{a^{2i}P_0P_0} = |P_0|\underline{P_{2i}}\\ \underline{P_i}\underline{P_j} &= \underline{a^{i+j}P_0P_0} = |P_0|\underline{P_{i+j}}.\end{aligned}$$

It follows that

$$\begin{aligned}\underline{P}^2 &= \underline{P_1P_1} + \underline{P_1P_2} + \underline{P_2P_1} + \underline{P_2P_2} \\ &= |P_0|\underline{P_2} + |P_0|\underline{P_3} + |P_0|\underline{P_3} + |P_0|\underline{P_4} \\ &= |P_0|\underline{P_2} + 2\cdot|P_0|\underline{P_3} + |P_0|\underline{P_0},\end{aligned}$$

and

$$\begin{aligned}\underline{P'P} &= \underline{P_2P_1} + \underline{P_2P_2} + \underline{P_3P_1} + \underline{P_3P_2} \\ &= |P_0|\underline{P_3} + |P_0|\underline{P_0} + |P_0|\underline{P_0} + |P_0|\underline{P_1} \\ &= |P_0|\underline{P_3} + 2\cdot|P_0|\underline{P_0} + |P_0|\underline{P_1}.\end{aligned}$$

Now it only remains to write them in Equation (4) :

$$\begin{aligned}\underline{S}^2 + 2^{m-2}\underline{S} &= (\underline{P} + \underline{xP})^2 + 2^{m-2}(\underline{P} + \underline{xP}) \\ &= \underline{P}^2 + \underline{PxP} + \underline{xPP} + \underline{xPxP} + 2^{m-2}(\underline{P} + \underline{xP}) \\ &= \underline{P}^2 + \underline{xP'P} + \underline{xPP} + \underline{P'P} + 2^{m-2}(\underline{P} + \underline{xP}) \\ &= \underline{P}^2 + \underline{P'P} + (2\cdot|P_0|)\underline{P} + x(\underline{P}^2 + \underline{P'P} + (2\cdot|P_0|)\underline{P}) \\ &= |P_0|\underline{P_2} + 2\cdot|P_0|\underline{P_3} + |P_0|\underline{P_0} + |P_0|\underline{P_3} + 2\cdot|P_0|\underline{P_0} + |P_0|\underline{P_1} \\ &+ 2\cdot|P_0|\underline{P_1} + 2\cdot|P_0|\underline{P_2} + x(|P_0|\underline{P_2} + 2\cdot|P_0|\underline{P_3} + |P_0|\underline{P_0} + |P_0|\underline{P_3} \\ &+ 2\cdot|P_0|\underline{P_0} + |P_0|\underline{P_1} + 2\cdot|P_0|\underline{P_1} + 2\cdot|P_0|\underline{P_2}) \\ &= 3\cdot|P_0|(\underline{P_0} + \underline{P_1} + \underline{P_2} + \underline{P_3} + x\underline{P_0} + x\underline{P_1} + x\underline{P_2} + x\underline{P_3}) \\ &= 3\cdot(2^{m-3})\cdot\underline{G}.\end{aligned}$$

This completes the proof. □

Example 3. *Let $G = SD(4)$ be the semidihedral group of order 4 for $m = 4$ with elements $\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, x, xa, xa^2, xa^3, xa^4, xa^5, xa^6, xa^7\}$. Construct the subset S according to Theorem 2 as $\{P \cup xP\}$ where $P = \{a, a^2, a^5, a^6\}$. Then $Cay(G, S)$ is a DSRG with parameters $(16, 8, 6, 2, 6)$.*

Remark 1. *The directed strongly regular graph constructed in the Example 3 has already been presented in [2] by Duval. The author constructed the DSRG with parameters $(16, 8, 2, 6, 2)$ from a DSRG with parameters $(8, 4, 1, 3, 1)$ known to exist. This construction is specified as $T10$ in [1].*

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SOME REFINEMENTS OF BEREZIN NUMBER INEQUALITIES VIA CONVEX FUNCTIONS

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Abstract. The Berezin transform $\widetilde{A}$ and the Berezin number of an operator A on the reproducing kernel Hilbert space over some set Ω with normalized reproducing kernel $\widehat{k}_\lambda$ are defined, respectively, by $\widetilde{A}(\lambda) = \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$, $\lambda \in \Omega$ and $\mathrm{ber}(A) := \sup_{\lambda\in\Omega} \left|\widetilde{A}(\lambda)\right|$. A straightforward comparison between these characteristics yields the inequalities $\mathrm{ber}\,(A) \leq \frac{1}{2}\left(\|A\|_{\mathrm{ber}} + \|A^2\|_{\mathrm{ber}}^{1/2}\right)$. In this paper, we study further inequalities relating them. Namely, we obtained some refinements of Berezin number inequalities involving convex functions. In particular, for $A \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$ we show that

$$\mathrm{ber}^{2r}(A) \leq \frac{1}{4}\left(\|A^*A + AA^*\|_{\mathrm{ber}}^r + \|A^*A - AA^*\|_{\mathrm{ber}}^r\right) + \frac{1}{2}\mathrm{ber}^r\left(A^2\right).$$

1. Introduction and Preliminaries

Recall that the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ (shortly, RKHS) is the Hilbert space of complex-valued functions on some set Ω such that the evaluation functional $f \to f(\lambda)$ is bounded on $\mathcal{H}$ for every $\lambda \in \Omega$. Then, by Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique vector k_λ in $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The function k_λ is called the reproducing kernel of the space $\mathcal{H}$. It is well known that (see Aronzajn [2])

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

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for any orthonormal basis $\{e_n(z)\}_{n\geq 0}$ of the space $\mathcal{H}(\Omega)$. The normalized reproducing kernel is defined by $\widehat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}$. For a bounded linear operator A acting in the RKHS $\mathcal{H}$, its Berezin symbol $\widetilde{A}$ (see Berezin [7]) is defined by the formula

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \ (\lambda \in \Omega).$$

The Berezin symbol is a function that is bounded by norm of the operator. Karaev [19] defined the Berezin set and the Berezin number of operator A, respectively by

$$\mathrm{Ber}(A) := \mathrm{Range}\left(\widetilde{A}\right) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\}$$

and

$$\mathrm{ber}(A) := \sup_{\lambda\in\Omega} \left|\widetilde{A}(\lambda)\right|.$$

It is clear from definitions that $\widetilde{A}$ is a bounded function, $\mathrm{Ber}(A)$ lies in the numerical range $W(A)$, and so $\mathrm{ber}(A)$ does not exceed the numerical radius $w(A)$ of operator A. Recall that the numerical range and the numerical radius of operator A are defined, respectively, by

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

and

$$w(A) := \sup_{\|x\|=1} |\langle Ax, x\rangle|$$

(for more information, see [1, 9, 10, 15, 21, 22, 25–28, 31]). Berezin set and Berezin number of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [19].

Suppose that $B(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$. It is well-known that

$$\mathrm{ber}(A) \leq w(A) \leq \|A\| \tag{1}$$

and

$$\frac{\|A\|}{2} \leq w(A)$$

for any $A \in B(\mathcal{H})$. But, Karaev [20] showed that

$$\frac{\|A\|}{2} \leq \mathrm{ber}(A)$$

is not hold for every $A \in B(\mathcal{H})$. Also, Berezin number inequalities were given by using the other inequalities in [11, 13, 17, 20, 32].

Huban et al. [18, Theorem 2.14] improved the inequality (1) by proving that

$$\mathrm{ber}(A) \leq \frac{1}{2}\left(\|A\|_{\mathrm{ber}} + \left\|A^2\right\|_{\mathrm{ber}}^{1/2}\right) \tag{2}$$

for any $A \in \mathcal{B}(\mathcal{H})$.

It has been shown in [17] that if $A \in \mathcal{B}(\mathcal{H})$, then

$$\frac{1}{4}\left\| A^*A + AA^* \right\| \leq \mathrm{ber}^2(A) \leq \frac{1}{2}\left\| A^*A + AA^* \right\|. \tag{3}$$

The following estimate of the Berezin numbers has been given in [16],

$$\mathrm{ber}(A) \leq \frac{1}{2}\sqrt{\left\| AA^* + A^*A \right\|_{\mathrm{ber}} + 2\mathrm{ber}(A^2)} \leq \|A\|_{\mathrm{ber}}. \tag{4}$$

The inequality (4) also refines the inequality (2). This can be seen by using the fact that

$$\left\| AA^* + A^*A \right\|_{\mathrm{ber}} \leq \|A\|_{\mathrm{ber}}^2 + \left\| A^2 \right\|_{\mathrm{ber}}. \tag{5}$$

In this work, inspired by the numerical radius inequalities in [29], an extension of the inequality (3) is proved. In particular, for $A \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$ we prove that

$$\mathrm{ber}^{2r}(A) \leq \frac{1}{4}\left(\left\| A^*A + AA^* \right\|_{\mathrm{ber}}^r + \left\| A^*A - AA^* \right\|_{\mathrm{ber}}^r\right) + \frac{1}{2}\mathrm{ber}^r\left(A^2\right).$$

Other general related results are also established.

2. Main Results

In order to achieve our goal, we need the following series of corollaries.

Lemma 1. *([23]) Let A be an operator in $\mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors.*

(i) If $0 \leq \alpha \leq 1$, then $|\langle Ax, y\rangle|^2 \leq \left\langle |A|^{2\alpha} x, x \right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle$.

(ii) If f and g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, $(t \geq 0)$, then $|\langle Ax, y\rangle| \leq \|f(|A|)x\| \, \|g(|A^*|)y\|$.

Lemma 2. *([24]) Let A be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$ with the spectrum contained in the interval J, and let h be convex function on J. Then for any unit vector $x \in \mathcal{H}$,*

$$h(\langle Ax, x\rangle) \leq \langle h(A)x, x\rangle.$$

In [31, Lemma 2.4], the authors present an improvement of the Young inequality as follows:

Lemma 3. *Let $a, b > 0$ and $\min\{a, b\} \leq m \leq M \leq \max\{a, b\}$. Then*

$$\sqrt{ab} \leq \frac{2\sqrt{Mm}}{M+m}\frac{a+b}{2}. \tag{6}$$

In 1941, R.P. Boas [8] and in 1944, independently, R. Bellman [6] proved the following generalization of Bessel's inequality.

Lemma 4. *If $a, b_1, ..., b_n$ are elements of an inner product space $(\mathcal{H}, \langle ., . \rangle)$, then the following inequality holds:*

$$\sum_{i=1}^{n} |\langle a, b_i \rangle|^2 \leq \|a\|^2 \left(\max_{1 \leq i \leq n} \|b_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n}^{n} |\langle b_i, b_j \rangle|^2 \right)^{\frac{1}{2}} \right).$$

In particular, the case $n = 2$ in the above reduces to

$$|\langle a, b_1 \rangle|^2 + |\langle a, b_2 \rangle|^2 \leq \|a\|^2 \left(\max \left(\|b_1\|^2, \|b_2\|^2 \right) + |\langle b_1, b_2 \rangle| \right). \tag{7}$$

We recall the following refinement of the Cauchy-Schwarz inequality obtained by Dragomir in [9]. If a, b, e are vectors in $\mathcal{H}$ and $\|e\| = 1$, then we have

$$|\langle a, b \rangle| \leq |\langle a, e \rangle \langle e, b \rangle| + |\langle a, b \rangle - \langle a, e \rangle \langle e, b \rangle| \leq \|a\| \|b\|. \tag{8}$$

From the inequality (8) we deduce that

$$|\langle a, e \rangle \langle e, b \rangle| \leq \frac{1}{2} (\|a\| \|b\| + |\langle a, b \rangle|). \tag{9}$$

Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then, by taking $e = \widehat{k}_\lambda$, $a = A\widehat{k}_\lambda$ and $b = A^*\widehat{k}_\lambda$ in the inequality (9), we get

$$\left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \leq \frac{1}{2} \left(\left\| A\widehat{k}_\lambda \right\| \left\| A^*\widehat{k}_\lambda \right\| + \left| \left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \right) \tag{10}$$

and

$$\sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right|^2 \leq \sup_{\lambda \in \Omega} \frac{1}{2} \left(\left\| A\widehat{k}_\lambda \right\|^2 + \left| \left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \right)$$

which is equivalent to

$$\mathrm{ber}^2(A) \leq \frac{1}{2} \left(\|A\|_{\mathrm{Ber}}^2 + \mathrm{ber}\left(A^2\right) \right). \tag{11}$$

In addition to this, we have the following related inequality:

Theorem 1. *Let $A \in \mathcal{B}(\mathcal{H})$, f, g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)\, g(t) = t$, $(t \geq 0)$, and h be a non-negative increasing convex function on $[0, \infty)$. If*

$$0 < f^2 \left(\left| A^2 \right| \right) \leq m < M \leq g^2 \left(\left| \left(A^2\right)^* \right| \right),$$

or

$$0 < g^2 \left(\left| \left(A^2\right)^* \right| \right) \leq m < M \leq f^2 \left(\left| A^2 \right| \right),$$

then

$$h\left(\mathrm{ber}\left(A^2\right)\right) \leq \frac{2\sqrt{Mm}}{M+m} \left\| \frac{h\left(f^2\left(\left|A^2\right|\right)\right) + h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2} \right\|_{\mathrm{ber}}. \tag{12}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then, we have

$$h\left(\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)$$

$$\leq h\left(\sqrt{\left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}\right)$$

(by Lemma 1 (ii))

$$\leq h\left(\frac{2\sqrt{Mm}}{M+m}\left(\frac{\left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right)\right)$$

(by the inequality (6))

$$\leq \frac{2\sqrt{Mm}}{M+m}h\left(\frac{\left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right)$$

$$\leq \frac{2\sqrt{Mm}}{M+m}\left(\frac{h\left(\left\langle f^2\left(\left|A^2\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)+h\left(\left\langle g^2\left(\left|\left(A^2\right)^*\right|\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)}{2}\right)$$

$$\leq \frac{2\sqrt{Mm}}{M+m}\left(\frac{\left\langle h\left(f^2\left(\left|A^2\right|\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right)$$

(by Lemma 2)

$$= \frac{2\sqrt{Mm}}{M+m}\left\langle\frac{h\left(f^2\left(\left|A^2\right|\right)\right)+h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle.$$

Therefore,

$$h\left(\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)\leq \frac{2\sqrt{Mm}}{M+m}\left\langle\frac{h\left(f^2\left(\left|A^2\right|\right)\right)+h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle.$$

By taking the supremum over $\lambda\in\Omega$ above inequality, we deduce the desired result

$$h\left(\operatorname{ber}\left(A^2\right)\right)\leq \frac{2\sqrt{Mm}}{M+m}\left\|\frac{h\left(f^2\left(\left|A^2\right|\right)\right)+h\left(g^2\left(\left|\left(A^2\right)^*\right|\right)\right)}{2}\right\|_{\operatorname{ber}}.$$

This finalizes the proof. □

The following result may be stated as well.

Corollary 1. *Let $A\in\mathcal{B}(\mathcal{H})$, f,g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$, $(t\geq 0)$, and $r\geq 1$. If*

$$0<f^2\left(\left|A^2\right|\right)\leq m<M\leq g^2\left(\left|\left(A^2\right)^*\right|\right),$$

or

$$0 < g^2\left(\left|\left(A^2\right)^*\right|\right) \le m < M \le f^2\left(\left|A^2\right|\right),$$

then

$$\operatorname{ber}^r\left(A^2\right) \le \frac{2\sqrt{Mm}}{M+m}\left\|\frac{f^{2r}\left(\left|A^2\right|\right) + g^{2r}\left(\left|\left(A^2\right)^*\right|\right)}{2}\right\|_{\operatorname{ber}}.$$

Remark 1. *By taking $r = 1$ in Corollary 1, then it follows from the inequality (11) that*

$$\operatorname{ber}^2(A) \le \frac{1}{2}\left(\left\|A^2\right\|_{\operatorname{Ber}} + \frac{2\sqrt{Mm}}{M+m}\left\|\frac{f^2\left(\left|A^2\right|\right) + g^2\left(\left|\left(A^2\right)^*\right|\right)}{2}\right\|_{\operatorname{ber}}\right).$$

For various operators, the following conclusion is true.

Theorem 2. *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, $A, B \ge 0$, $0 \le \alpha \le 1$, and h be a non-negative increasing sub-multiplicative convex function on $[0, \infty)$. If*

$$0 < B^{2(1-\alpha)} \le m < M \le A^{2\alpha}$$

or

$$0 < A^{2\alpha} \le m < M \le B^{2(1-\alpha)},$$

then

$$h\left(\operatorname{ber}\left(A^\alpha C B^{1-\alpha}\right)\right) \le \frac{2\sqrt{Mm}}{M+m} h\left(\|C\|_{\operatorname{ber}}\right)\left\|\frac{h\left(B^{2(1-\alpha)}\right) + h\left(A^{2\alpha}\right)}{2}\right\|_{\operatorname{ber}}. \tag{13}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Then, by the Cauchy-Schwarz, we have

$$\begin{aligned}
& h\left(\left|\left\langle A^\alpha C B^{1-\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right) \\
& = h\left(\left|\left\langle C B^{1-\alpha}\widehat{k}_\lambda, A^\alpha\widehat{k}_\lambda\right\rangle\right|\right) \\
& \le h\left(\|C\|_{\operatorname{ber}}\left\|B^{1-\alpha}\widehat{k}_\lambda\right\|\left\|A^\alpha\widehat{k}_\lambda\right\|\right) \\
& \text{(by } h \text{ sub-multiplicativity)} \\
& = h\left(\|C\|_{\operatorname{ber}}\sqrt{\left\langle B^{1-\alpha}\widehat{k}_\lambda, B^{1-\alpha}\widehat{k}_\lambda\right\rangle\left\langle A^\alpha\widehat{k}_\lambda, A^\alpha\widehat{k}_\lambda\right\rangle}\right) \\
& \text{(by the inequality (6))} \\
& = h\left(\|C\|_{\operatorname{ber}}\sqrt{\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle A^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}\right) \\
& \le h\left(\|C\|_{\operatorname{ber}}\right) h\left(\sqrt{\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle A^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}\right)
\end{aligned}$$

$$\leq h\left(\|C\|_{\text{ber}}\right)h\left(\frac{2\sqrt{Mm}}{M+m}\left(\frac{\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle A^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right)\right)$$

$$\leq \frac{2\sqrt{Mm}}{M+m}h\left(\|C\|_{\text{ber}}\right)h\left(\frac{\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\langle A^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}\right)$$

(by Lemma 2)

$$\leq \frac{2\sqrt{Mm}}{M+m}h\left(\|C\|_{\text{ber}}\right)\frac{h\left(\left\langle B^{2(1-\alpha)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)+h\left(\left\langle A^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)}{2}$$

$$\leq \frac{2\sqrt{Mm}}{M+m}h\left(\|C\|_{\text{ber}}\right)\frac{\left\langle h\left(B^{2(1-\alpha)}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle h\left(A^{2\alpha}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle}{2}$$

$$= \frac{2\sqrt{Mm}}{M+m}h\left(\|C\|_{\text{ber}}\right)\left\langle\left(\frac{h\left(B^{2(1-\alpha)}\right)+h\left(A^{2\alpha}\right)}{2}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,$$

So,

$$h\left(\left|\left\langle A^\alpha CB^{1-\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)\leq \frac{2\sqrt{Mm}}{M+m}h\left(\|C\|_{\text{ber}}\right)\left\langle\left(\frac{h\left(B^{2(1-\alpha)}\right)+h\left(A^{2\alpha}\right)}{2}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,$$

and

$$\sup_{\lambda\in\Omega}h\left(\left|\left(A^\alpha\widetilde{CB^{1-\alpha}}\right)(\lambda)\right|\right)\leq \frac{2\sqrt{Mm}}{M+m}h\left(\|C\|_{\text{ber}}\right)\sup_{\lambda\in\Omega}\left\langle\left(\frac{h\left(B^{2(1-\alpha)}\right)+h\left(A^{2\alpha}\right)}{2}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle$$

which is equivalent to

$$h\left(\text{ber}\left(A^\alpha CB^{1-\alpha}\right)\right)\leq \frac{2\sqrt{Mm}}{M+m}h\left(\|C\|_{\text{ber}}\right)\left\|\frac{h\left(B^{2(1-\alpha)}\right)+h\left(A^{2\alpha}\right)}{2}\right\|_{\text{ber}},$$

which proves the desired inequalities. □

Corollary 2. *Let $A,B,C\in\mathcal{B}(\mathcal{H})$, $A,\ B\geq 0$, and $0\leq\alpha\leq 1$, and let $r\geq 1$. If*

$$0<B^{2(1-\alpha)}\leq m<M\leq A^{2\alpha},$$

or

$$0<A^{2\alpha}\leq m<M\leq B^{2(1-\alpha)},$$

then

$$\text{ber}^r\left(A^\alpha CB^{1-\alpha}\right)\leq \frac{2\sqrt{Mm}}{M+m}\|C\|_{\text{ber}}^r\left\|\frac{\left(A^{2r\alpha}\right)+\left(B^{2r(1-\alpha)}\right)}{2}\right\|_{\text{ber}}.$$

As a consequence of the above, we can present the following inequality.

Corollary 3. *Suppose that the assumptions of Corollary 2 are satisfied. Then*

$$\mathrm{ber}^r\left(A^{1/2}CB^{1/2}\right) \le \frac{2\sqrt{Mm}}{M+m}\|C\|_{\mathrm{ber}}^r\left\|\frac{A^r+B^r}{2}\right\|_{\mathrm{ber}}. \tag{14}$$

We can give the following corollary whose proof can be reached by using similar techniques from Theorem 3.4 and Lemma 3.5 in [30].

Corollary 4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be invertible self-adjoint operators and $C \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathrm{ber}^r\left(A^{1/2}CB^{1/2}\right) \le \|C\|_{\mathrm{ber}}^r\left\|\frac{A^r+B^r}{2}\right\|_{\mathrm{ber}}. \tag{15}$$

Remark 2. *Therefore, inequality (14) essentially gives a refinement of the inequality of (15) since $\frac{2\sqrt{Mm}}{M+m} \le 1$.*

The following result is of interest in itself.

Theorem 3. *Let $A \in \mathcal{B}(\mathcal{H})$, and let h be a non-negative increasing convex function on $[0,\infty)$.*

$$h\left(\mathrm{ber}^2(A)\right) \le \frac{1}{4}\left(h\left(\|A^*A+AA^*\|_{\mathrm{ber}}\right)+h\left(\|A^*A-AA^*\|_{\mathrm{ber}}\right)\right)+\frac{1}{2}h\left(\mathrm{ber}\left(A^2\right)\right).$$

In particular, for any $r \ge 1$,

$$\mathrm{ber}^{2r}(A) \le \frac{1}{4}\left(\|A^*A+AA^*\|_{\mathrm{ber}}^r+\|A^*A-AA^*\|_{\mathrm{ber}}^r\right)+\frac{1}{2}\mathrm{ber}^r\left(A^2\right).$$

Proof. Let $\lambda \in \Omega$ be an arbitrary. Put $b_1 = A\widehat{k}_\lambda$, $b_2 = A^*\widehat{k}_\lambda$, and $a = \widehat{k}_\lambda$ in the inequality (7). Since $\max(a,b) = \frac{|a+b|+|a-b|}{2}$, we get

$$\begin{aligned}
&\left|\left\langle \widehat{k}_\lambda, A\widehat{k}_\lambda\right\rangle\right|^2+\left|\left\langle \widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle\right|^2\\
&\le \max\left(\left\|A\widehat{k}_\lambda\right\|^2,\left\|A^*\widehat{k}_\lambda\right\|^2\right)+\left|\left\langle A\widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle\right| \qquad (16)\\
&= \frac{1}{2}\left(\left|\left\langle A^*A+AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|+\left|\left\langle A^*A-AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)+\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|.
\end{aligned}$$

Applying the AM-GM inequality for the left hand side of the above inequality, we get

$$\begin{aligned}
&\left|\left\langle A^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\\
&\le \frac{1}{4}\left(\left|\left\langle A^*A+AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|+\left|\left\langle A^*A-AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)+\frac{1}{2}\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|.
\end{aligned}$$

Whence,

$$\begin{aligned}
&h\left(\left|\left\langle A^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)\\
&\le h\left(\frac{1}{4}\left(\left|\left\langle A^*A+AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|+\left|\left\langle A^*A-AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)+\frac{1}{2}\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)
\end{aligned}$$

$$= h\left(\frac{\frac{1}{2}\left|\left\langle A^*A+AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|+\left|\left\langle A^*A-AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|+\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|}{2}\right)$$

$$\le \frac{1}{2}\left(h\left(\frac{\left|\left\langle A^*A+AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|+\left|\left\langle A^*A-AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|}{2}\right)+h\left(\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)\right)$$

$$\le \frac{1}{4}\left(h\left(\left|\left\langle A^*A+AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)+h\left(\left|\left\langle A^*A-AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)\right)+\frac{1}{2}h\left(\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right).$$

Therefore,

$$h\left(\left|\left\langle A^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)$$

$$\le \frac{1}{4}\left(h\left(\left|\left\langle A^*A+AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)+h\left(\left|\left\langle A^*A-AA^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)\right)+\frac{1}{2}h\left(\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right).$$

By taking the supremum over $\lambda\in\Omega$ above inequality, we have

$$h\left(\mathrm{ber}^2(A)\right)\le\frac{1}{4}\left(h\left(\|A^*A+AA^*\|_{\mathrm{ber}}\right)+h\left(\|A^*A-AA^*\|_{\mathrm{ber}}\right)\right)+\frac{1}{2}h\left(\mathrm{ber}\left(A^2\right)\right).$$

This completes the proof. □

Corollary 5. *Let $A\in\mathcal{B}(\mathcal{H})$ be an invertible operator. Then*

$$\mathrm{ber}(A)\le\sqrt{\frac{1}{2}\|A\|_{\mathrm{ber}}^2+\frac{3}{4}\|A^2\|_v-\frac{1}{4}\|A^{-1}\|_{\mathrm{ber}}^{-2}}.$$

Proof. By using similar techniques from [22], we get

$$\|A^*A-AA^*\|_{\mathrm{ber}}\le\|A\|_{\mathrm{ber}}^2-\left\|A^{-1}\right\|_{\mathrm{ber}}^{-2}. \tag{17}$$

On the other hand, from Theorem 3, we have

$$\mathrm{ber}^2(A)\le\frac{1}{4}\left(\|A^*A+AA^*\|_{\mathrm{ber}}+\|A^*A-AA^*\|_{\mathrm{ber}}\right)+\frac{1}{2}\mathrm{ber}(A^2).$$

Hence

$$\begin{aligned}\mathrm{ber}^2(A)&\le\frac{1}{4}\left(\|A^*A+AA^*\|_{\mathrm{ber}}+\|A^*A-AA^*\|_{\mathrm{ber}}\right)+\frac{1}{2}\mathrm{ber}(A^2)\\&\le\frac{1}{4}\left(\|A^*A+AA^*\|_{\mathrm{ber}}+\|A\|^2-\left\|A^{-1}\right\|_{\mathrm{ber}}^{-2}\right)+\frac{1}{2}\mathrm{ber}(A^2)\\&\quad\text{(by the inequality (17))}\\&\le\frac{1}{4}\left(2\|A\|_{\mathrm{ber}}^2+\left\|A^2\right\|_{\mathrm{ber}}-\left\|A^{-1}\right\|_{\mathrm{ber}}^{-2}\right)+\frac{1}{2}\mathrm{ber}(A^2)\\&\quad\text{(by the inequality (5))}\\&\le\frac{1}{2}\|A\|_{\mathrm{ber}}^2+\frac{3}{4}\left\|A^2\right\|_{\mathrm{ber}}-\frac{1}{4}\left\|A^{-1}\right\|_{\mathrm{ber}}^{-2}\\&\quad\text{(by the inequality (1))}\end{aligned}$$

as required. □

The following upper bound for the nonnegative difference $\text{ber}^2(A) - \text{ber}(A^2)$ can be obtained:

Corollary 6. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\text{ber}^2(A) - \text{ber}(A^2) \leq \frac{1}{4}\left(\left\| |A|^2 + |A^*|^2 \right\|_{\text{ber}} + \left\| |A|^2 - |A^*|^2 \right\|_{\text{ber}}\right).$$

For more recent results concerning Berezin radius inequalities for operators and other related results, we suggest [3–5, 12, 14, 16, 33].

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NEW INSIGHT INTO QUATERNIONS AND THEIR MATRICES

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Abstract. This paper aims to bring together quaternions and generalized complex numbers. Generalized quaternions with generalized complex number components are expressed and their algebraic structures are examined. Several matrix representations and computational results are introduced. An alternative approach for a generalized quaternion matrix with elliptic number entries has been developed as a crucial part.

1. Introduction

Hamilton introduced the Hamiltonian quaternions for representing vectors in the space, [1,2]. The real quaternion is written as $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ are components and $\mathbf{i}, \mathbf{j}, \mathbf{k} \notin \mathbb{R}$ are versors, [3]. The set of real quaternions, as an extension of complex numbers, is an associative but non-commutative Clifford algebra used in many fields of applied mathematics. The associative quaternions will be divided into two classes: in the first class, there are the non-commutative quaternions (Hamiltonian, hyperbolic, split, generalized quaternions [4–11] etc.), and in the second class, there are the commutative quaternions (generalized Segré quaternions [12,13], dual quaternions, [14–18] etc.).

The algebra of generalized quaternions as a non-commutative system, denoted by $\mathcal{Q}_{\alpha,\beta}$, includes a variety of well-known four-dimensional algebras as special cases.

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The conditions of the versors for them are given by:

$$\begin{array}{lll} \mathbf{i}^2=-\alpha, & \mathbf{j}^2=-\beta, & \mathbf{k}^2=-\alpha\beta, \\ \mathbf{ij}=-\mathbf{ji}=\mathbf{k}, & \mathbf{jk}=-\mathbf{kj}=\beta\mathbf{i}, & \mathbf{ki}=-\mathbf{ik}=\alpha\mathbf{j}, \end{array} \tag{1}$$

where $\alpha,\beta\in\mathbb{R}$. For $\alpha=\beta=1$ Hamiltonian quaternions, $\alpha=1,\beta=-1$ split quaternions, $\alpha=1,\beta=0$ semi-quaternions, $\alpha=-1,\beta=0$ split semi-quaternions, and $\alpha=\beta=0$ quasi-quaternions are obtained.

Additionally, the general bidimensional hypercomplex systems (namely generalized complex numbers ($\mathcal{GCN}$)) over the field of real numbers $\mathbb{R}$ are given by the ring ([19–24]):

$$\frac{\mathbb{R}[X]}{\langle h(X)\rangle}\cong\left\{z=x_1+x_2I:\ I^2=I\mathfrak{q}+\mathfrak{p},\quad \mathfrak{p},\mathfrak{q},x_1,x_2\in\mathbb{R},I\notin\mathbb{R}\right\},$$

where $h(X)=X^2-\mathfrak{q}X-\mathfrak{p}$ is monic quadratic. By denoting this set with $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, it is well known that the sign of $\Delta=\mathfrak{q}^2+4\mathfrak{p}$ determines the properties of the general bidimensional systems. These systems are ring isomorphic with one of the following three types:

- for $\Delta>0$ the *hyperbolic system*; the canonical system is the system of hyperbolic (double, split complex, perplex) numbers $\mathbb{H}\cong\mathbb{C}_{0,1}$ with $\mathfrak{p}=1$, $\mathfrak{q}=0$, [25–28],
- for $\Delta<0$ the *elliptic system*; the canonical system is the system of complex (ordinary) numbers $\mathbb{C}\cong\mathbb{C}_{0,-1}$ with $\mathfrak{p}=-1$, $\mathfrak{q}=0$, [28,29],
- for $\Delta=0$ the *parabolic system*; the canonical system is the system of dual numbers $\mathbb{D}\cong\mathbb{C}_{0,0}$ with $\mathfrak{p}=0$, $\mathfrak{q}=0$, [28,30,31].

Regarding the value $\mathcal{D}_z=z\overline{z}=(x_1+x_2I)(x_1-x_2I)={x_1}^2-\mathfrak{p}{x_2}^2+\mathfrak{q}x_1x_2$, which is called the characteristic determinant, $z\in\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ can be classified into three types, [20]. Hence $z\in\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ is called timelike, spacelike or null where $\mathcal{D}_z<0$, $\mathcal{D}_z>0$ and $\mathcal{D}_z=0$, respectively. Then all of the elements of the set $\mathbb{C}_{0,-1}$ are spacelike. For $\mathfrak{q}=0$, $I^2=\mathfrak{p}\in\mathbb{R}$, the generalized complex number system is denoted by $\mathbb{C}_{\mathfrak{p}}$ and called $\mathfrak{p}$-complex plane, [23].

In this paper, we aim to design generalized quaternions by taking the components as elements of $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$. Moreover, the algebraic structures and properties of these quaternions are investigated, and several types of matrix representations are introduced. Also, an alternative approach for the generalized quaternion matrix with elliptic number entries is considered as a further result.

2. Generalized Quaternions with *Gcn* Components

In this section, we present mathematical formulations of improved quaternions: *generalized quaternions with* $\mathcal{GCN}$ and examine special matrix correspondences.

Definition 1. *For $\alpha,\beta\in\mathbb{R}$, the set of generalized quaternions with $\mathcal{GCN}$ components are denoted by $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and the element of this set is defined as in the form:*

$$\widetilde{q}=a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},$$

where $a_0, a_1, a_2, a_3 \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ *and* $\mathbf{i}, \mathbf{j}, \mathbf{k} \notin \mathbb{R}$ *are generalized quaternion versors that satisfy the properties in equations* (1).

Axiomatically, the generalized complex unit I commutes with the three quaternion versors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, that is $\mathbf{i}I = I\mathbf{i}$, $\mathbf{j}I = I\mathbf{j}$ and $\mathbf{k}I = I\mathbf{k}$. It is obvious that for $\mathfrak{q} = 0, \mathfrak{p} = -1, \alpha = 1$, the usual complex operator is distinct from quaternion versor $\mathbf{i}$. Moreover $\mathbf{i}$ distinct from the usual hyperbolic unit for $\mathfrak{q} = 0, \mathfrak{p} = 1, \alpha = -1$ and distinct from the usual dual unit for $\mathfrak{q} = 0, \mathfrak{p} = 0, \alpha = 0$. This conditions can also be extended for the other versors.

Throughout this section, $\widetilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\widetilde{p} = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ are considered. Due to the generalized quaternions with $\mathcal{GCN}$ components are an extension of generalized quaternions, many properties of them are familiar. For any $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, $S_{\widetilde{q}} = a_0$ is the scalar part and $V_{\widetilde{q}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is the vector part. Equality of two improved quaternions is as follows: $\widetilde{p} = \widetilde{q} \Leftrightarrow S_{\widetilde{p}} = S_{\widetilde{q}}, V_{\widetilde{p}} = V_{\widetilde{q}}$. Addition (and hence subtraction) of $\widetilde{q}$ to another quaternion $\widetilde{p}$ acts in a componentwise way:

$$\begin{aligned} \widetilde{q} + \widetilde{p} &= (a_0 + b_0) + (a_1 + b_1)\,\mathbf{i} + (a_2 + b_2)\,\mathbf{j} + (a_3 + b_3)\,\mathbf{k} \\ &= S_{\widetilde{p}} + S_{\widetilde{q}} + V_{\widetilde{p}} + V_{\widetilde{q}}. \end{aligned} \tag{2}$$

The conjugate of $\widetilde{q}$ is the following quaternion:

$$\overline{\widetilde{q}} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k} = S_{\widetilde{q}} - V_{\widetilde{q}}. \tag{3}$$

The scalar multiplication of $\widetilde{q}$ with a scalar $c \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ gives another improved quaternion as:

$$c\widetilde{q} = ca_0 + ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k} = cS_{\widetilde{q}} + cV_{\widetilde{q}}. \tag{4}$$

Multiplication of the two quaternions is carried out as follows:

$$\begin{aligned} \widetilde{q}\widetilde{p} &= (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3) \\ &\quad + (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)\,\mathbf{i} \\ &\quad + (a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)\,\mathbf{j} \\ &\quad + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k}. \end{aligned} \tag{5}$$

Proposition 1. *$\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is a 4-dimensional module over $\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$ with base $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and is an 8-dimensional vector space over $\mathbb{R}$ with base $\{1, I, \mathbf{i}, I\mathbf{i}, \mathbf{j}, I\mathbf{j}, \mathbf{k}, I\mathbf{k}\}$.*

Definition 2. *For any $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, the scalar and vector products on $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ are, respectively, defined by:*

$$\begin{aligned} \langle \widetilde{q}, \widetilde{p} \rangle_g &= S_{\widetilde{q}}S_{\widetilde{p}} + \langle V_{\widetilde{q}}, V_{\widetilde{p}} \rangle_g = a_0b_0 + \alpha a_1b_1 + \beta a_2b_2 + \alpha\beta a_3b_3 = S_{\widetilde{q}\overline{\widetilde{p}}}, \\ \widetilde{q} \times_g \widetilde{p} &= S_{\widetilde{q}}V_{\overline{\widetilde{p}}} + S_{\overline{\widetilde{p}}}V_{\widetilde{q}} - V_{\widetilde{q}} \times_g V_{\widetilde{p}} = V_{\widetilde{q}\overline{\widetilde{p}}}, \end{aligned}$$

where $\langle , \rangle_g$ and $\times_g$ represent generalized scalar product and generalized vector products[1] *for $\alpha, \beta \in \mathbb{R}^+$, respectively.*

[1]For a more general description of the generalized inner and cross product, see [7].

Definition 3. *The norm of $\widetilde{q}$ is defined as:*

$$N_{\widetilde{q}} = \widetilde{q}\,\overline{\widetilde{q}} = \overline{\widetilde{q}}\,\widetilde{q} = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}. \tag{6}$$

Definition 4. *The inverse of $\widetilde{q}$ is calculated by:*

$$(\widetilde{q})^{-1} = \frac{\overline{\widetilde{q}}}{N_{\widetilde{q}}}$$

for non-null $N_{\widetilde{q}}$ that is $\mathcal{D}_{N_{\widetilde{q}}} \neq 0$.

Proposition 2. *For any $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $c_1, c_2 \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, the conjugate and norm hold the following properties:*

i. $\overline{\overline{\widetilde{q}}} = \widetilde{q}$,
ii. $\overline{c_1\widetilde{p} + c_2\widetilde{q}} = c_1\overline{\widetilde{p}} + c_2\overline{\widetilde{q}}$,
iii. $\overline{\widetilde{q}\,\widetilde{p}} = \overline{\widetilde{p}}\,\overline{\widetilde{q}}$,
iv. $N_{c_1\widetilde{q}} = c_1^2 N_{\widetilde{q}}$,
v. $N_{\widetilde{q}\,\widetilde{p}} = N_{\widetilde{q}} N_{\widetilde{p}}$.

Proof. Taking into account equations (2), (3) and (4), items i and ii are obvious.

iii. Considering the conjugate of equation (5), we have:

$$\begin{aligned}\overline{\widetilde{q}\widetilde{p}} &= (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3) \\ &\quad - (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)\,\mathbf{i} \\ &\quad - (a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)\,\mathbf{j} \\ &\quad - (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k}.\end{aligned}$$

Using equations (1), it is easy to check that

$$\overline{\widetilde{p}}\,\overline{\widetilde{q}} = (b_0 - b_1\mathbf{i} - b_2\mathbf{j} - b_3\mathbf{k})(a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}) = \overline{\widetilde{q}\,\widetilde{p}}.$$

iv. Having item ii and equation (6), we get: $N_{c_1\widetilde{q}} = (c_1\widetilde{q})\,\overline{(c_1\widetilde{q})} = c_1^2 N_{\widetilde{q}}$.

v. Using item iii and equation (6), we obtain:

$$N_{\widetilde{q}\,\widetilde{p}} = (\widetilde{q}\,\widetilde{p})\,\overline{(\widetilde{q}\,\widetilde{p})} = \widetilde{q}\,\widetilde{p}\overline{\widetilde{p}}\,\overline{\widetilde{q}} = N_{\widetilde{q}} N_{\widetilde{p}}.$$

□

Remark 1. *As an another perspective to $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, the following can be calculated:*

$$\begin{aligned}\widetilde{q} &= a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ &= (x_{01} + x_{02}I) + (x_{11} + x_{12}I)\,\mathbf{i} + (x_{21} + x_{22}I)\,\mathbf{j} + (x_{31} + x_{32}I)\,\mathbf{k} \\ &= q_0 + q_1 I,\end{aligned} \tag{7}$$

where $a_i = x_{i1} + x_{i2}I \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, $q_{j-1} = x_{0j} + x_{1j}\mathbf{i} + x_{2j}\mathbf{j} + x_{3j}\mathbf{k} \in \mathcal{Q}_{\alpha,\beta}$ for $0 \leq i \leq 3$, $1 \leq j \leq 2$. For $\widetilde{q} = q_0 + q_1 I$ and $\widetilde{p} = p_0 + p_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, if $\widetilde{p} = \widetilde{q}$, then $p_0 = q_0, p_1 = q_1$. The addition is $\widetilde{p} + \widetilde{q} = (p_0 + q_0) + (p_1 + q_1)\,I$. The conjugate and anti conjugate are $\widetilde{q}^{\dagger_1} = q_0 + \mathfrak{q}q_1 - q_1 I$ and $\widetilde{q}^{\dagger_2} = q_1 - q_0 I$, respectively. Additionally, $c\widetilde{q} = cq_0 + cq_1 I$, $c \in \mathbb{R}$ and

$$\widetilde{q}\,\widetilde{p} = (q_0p_0 + \mathfrak{p}q_1p_1) + (q_0p_1 + q_1p_0 + \mathfrak{q}q_1p_1)\,I.$$

It is worthy to note that $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is a 2-dimensional module over $\mathcal{Q}_{\alpha,\beta}$ (skew-field) with base $\{1, I\}$. The moduli is

$$N_{\widetilde{q}}^{\dagger_1} = \widetilde{q}\,\widetilde{q}^{\dagger_1} \tag{8}$$

and the inverse is $(\widetilde{q})^{-1} = \frac{\widetilde{q}^{\dagger_1}}{N_{\widetilde{q}}^{\dagger_1}}$ for non-null $N_{\widetilde{q}}^{\dagger_1}$. The analogue of the scalar product on $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ can also defined by as follows:

$$\langle \widetilde{q}, \widetilde{p} \rangle_g \quad = S_{q_0\overline{p}_0} + \mathfrak{p} S_{q_1\overline{p}_1} + \big(S_{q_0\overline{p}_1} + S_{q_1\overline{p}_0} + \mathfrak{q} S_{q_1\overline{p}_1}\big) I.$$

Proposition 3. *The followings hold for $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $c_1, c_2 \in \mathbb{R}$:*

i. $\left(\widetilde{q}^{\dagger_1}\right)^{\dagger_1} = \widetilde{q}$,
ii. $\left(\widetilde{q}^{\dagger_2}\right)^{\dagger_2} = -\widetilde{q}$,
iii. $(c_1\widetilde{q} \pm c_2\widetilde{p})^{\dagger_1} = c_1\widetilde{q}^{\dagger_1} \pm c_2\widetilde{p}^{\dagger_1}$,
iv. $(c_1\widetilde{q} \pm c_2\widetilde{p})^{\dagger_2} = c_1\widetilde{q}^{\dagger_2} \pm c_2\widetilde{p}^{\dagger_2}$,
v. $\widetilde{q} + \widetilde{q}^{\dagger_1} = 2q_0 + \mathfrak{q} q_1$,
vi. $(\widetilde{q}\,\widetilde{p})^{\dagger_1} \neq \widetilde{p}^{\dagger_1}\,\widetilde{q}^{\dagger_1}$,
vii. $N_{c_1\widetilde{q}}^{\dagger_1} = {c_1}^2 N_{\widetilde{q}}^{\dagger_1}$,
viii. $N_{\widetilde{q}\widetilde{p}}^{\dagger_1} \neq N_{\widetilde{q}}^{\dagger_1} N_{\widetilde{p}}^{\dagger_1}$.

Proof. vi. Let us consider $\widetilde{q} = (1+\mathbf{i})I$ and $\widetilde{p} = \mathbf{j} + I$. As it is seen the followings:

$$\widetilde{q}\,\widetilde{p} = \mathfrak{p}(1+\mathbf{i}) + (\mathfrak{q} + \mathfrak{q}\mathbf{i} + \mathbf{j} + \mathbf{k})I,$$

$$(\widetilde{q}\,\widetilde{p})^{\dagger_1} = \mathfrak{p}(1+\mathbf{i}) + \mathfrak{q}(\mathfrak{q} + \mathfrak{q}\mathbf{i} + \mathbf{j} + \mathbf{k}) - (\mathfrak{q} + \mathfrak{q}\mathbf{i} + \mathbf{j} + \mathbf{k})I,$$

and

$$\begin{aligned} \widetilde{p}^{\dagger_1}\,\widetilde{q}^{\dagger_1} &= (\mathbf{j} + \mathfrak{q} - I)(\mathfrak{q}(1+\mathbf{i}) - (1+\mathbf{i})I) \\ &= (\mathfrak{p} + \mathfrak{q}^2) + (\mathfrak{p} + \mathfrak{q}^2)\mathbf{i} + \mathfrak{q}\mathbf{j} - \mathfrak{q}\mathbf{k} - (\mathfrak{q} + \mathfrak{q}\mathbf{i} + \mathbf{j} - \mathbf{k})I. \end{aligned}$$

It follows that $(\widetilde{q}\,\widetilde{p})^{\dagger_1} \neq \widetilde{p}^{\dagger_1}\,\widetilde{q}^{\dagger_1}$.

viii. From equation (8), we have the following equations:

$$N_{\widetilde{q}\widetilde{p}}^{\dagger_1} = (\widetilde{q}\,\widetilde{p})(\widetilde{q}\,\widetilde{p})^{\dagger_1}$$

and

$$N_{\widetilde{q}}^{\dagger_1} N_{\widetilde{p}}^{\dagger_1} \quad = \left(\widetilde{q}\,\widetilde{q}^{\dagger_1}\right)\left(\widetilde{p}\,\widetilde{p}^{\dagger_1}\right).$$

On account of the generalized quaternions are non-commutative and item vi, we find $N_{\widetilde{q}\widetilde{p}}^{\dagger_1} \neq N_{\widetilde{q}}^{\dagger_1} N_{\widetilde{p}}^{\dagger_1}$. One can also see this inequality considering $\widetilde{q} = \mathbf{i}I$ and $\widetilde{p} = \mathbf{j}$ as $N_{\widetilde{q}\widetilde{p}}^{\dagger_1} = \mathfrak{p}\alpha\beta = -N_{\widetilde{q}}^{\dagger_1} N_{\widetilde{p}}^{\dagger_1}$.

The proof of the other items is a simple calculation considering Remark 1. □

2.1. **Matrix Correspondences.** In this subsection, we formulate 2×2, 4×4 and 8×8 matrix correspondences which provide an alternative formulation of multiplication.

Theorem 1. *Every generalized quaternion with $\mathcal{GCN}$ components can be represented by a 2×2 quaternionic matrix. $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_2(\widetilde{\mathcal{Q}}_{\alpha,\beta})$.*

Proof. For $\widetilde{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, $\mathcal{L} : \widetilde{\mathcal{Q}}_{\alpha,\beta} \to \mathcal{R}, \widetilde{q} \mapsto \mathcal{A}_{\widetilde{q}}$ is linear map, where

$$\mathcal{R} := \left\{ \mathcal{A}_{\widetilde{q}} \in \mathbb{M}_2(\widetilde{\mathcal{Q}}_{\alpha,\beta}) : \mathcal{A}_{\widetilde{q}} = \begin{bmatrix} a_0 + a_3\mathbf{k} & a_1\mathbf{i} + a_2\mathbf{j} \\ a_1\mathbf{i} + a_2\mathbf{j} & a_0 + a_3\mathbf{k} \end{bmatrix} \right\} \tag{9}$$

is a subset of $\mathbb{M}_2(\widetilde{\mathcal{Q}}_{\alpha,\beta})$. So there exists a correspondence between $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\mathcal{R}$ via the map $\mathcal{L}$. Hence, 2×2 quaternionic matrix representation of $\widetilde{q}$ is $\mathcal{A}_{\widetilde{q}}$.

□

Corollary 1. *$\mathcal{L}$ can be determined as the following representation:*

$$\mathcal{L}(a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = a_0 I_2 + a_1\mathsf{I} + a_2\mathsf{J} + a_3\mathsf{K}, \tag{10}$$

where

$$\mathsf{I} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \mathsf{J} = \begin{bmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{bmatrix}, \mathsf{K} = \begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix}.$$

Thus

$$\begin{array}{lll} \mathsf{I}^2 = -\alpha I_2, & \mathsf{J}^2 = -\beta I_2, & \mathsf{K}^2 = -\alpha\beta I_2, \\ \mathsf{IJ} = -\mathsf{JI} = \mathsf{K}, & \mathsf{JK} = -\mathsf{KJ} = -\beta\mathsf{I}, & \mathsf{KI} = -\mathsf{IK} = \alpha\mathsf{J}. \end{array}$$

Theorem 2. *For $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\lambda \in \mathbb{R}$, then the following identities hold:*

i. $\widetilde{q} = \widetilde{p} \Leftrightarrow \mathcal{A}_{\widetilde{q}} = \mathcal{A}_{\widetilde{p}}$,
ii. $\mathcal{A}_{\widetilde{q}+\widetilde{p}} = \mathcal{A}_{\widetilde{q}} + \mathcal{A}_{\widetilde{p}}$,
iii. $\mathcal{A}_{\lambda\widetilde{q}} = \lambda(\mathcal{A}_{\widetilde{q}})$,
iv. $\mathcal{A}_{\widetilde{q}\widetilde{p}} = \mathcal{A}_{\widetilde{q}}\mathcal{A}_{\widetilde{p}}$.

Proof. The proof is obvious considering the matrix form given in equation (9). However let us discuss the proof of the item iv for better understanding:

iv. Considering equation (5), we can write:

$$\mathcal{A}_{\widetilde{q}\widetilde{p}} = \begin{bmatrix} \begin{array}{c} a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3 \\ (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k} \end{array} & \begin{array}{c} (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)\,\mathbf{i} \\ +\,(a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)\,\mathbf{j} \end{array} \\ \\ \begin{array}{c} (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)\,\mathbf{i} \\ +\,(a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)\,\mathbf{j} \end{array} & \begin{array}{c} a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3 \\ +\,(a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\,\mathbf{k} \end{array} \end{bmatrix}. \tag{11}$$

Computing $\mathcal{A}_{\widetilde{q}}\mathcal{A}_{\widetilde{p}}$ as

$$\mathcal{A}_{\widetilde{q}}\mathcal{A}_{\widetilde{p}} = \begin{bmatrix} a_0 + a_3\mathbf{k} & a_1\mathbf{i} + a_2\mathbf{j} \\ a_1\mathbf{i} + a_2\mathbf{j} & a_0 + a_3\mathbf{k} \end{bmatrix} \begin{bmatrix} b_0 + b_3\mathbf{k} & b_1\mathbf{i} + b_2\mathbf{j} \\ b_1\mathbf{i} + b_2\mathbf{j} & b_0 + b_3\mathbf{k} \end{bmatrix}$$

gives equation (11) quickly. We thus get $\mathcal{A}_{\widetilde{q}\widetilde{p}} = \mathcal{A}_{\widetilde{q}}\mathcal{A}_{\widetilde{p}}$.

□

Theorem 3. *Every generalized quaternion with $\mathcal{GCN}$ components can be represented by a 4×4 generalized complex matrix. $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_4(\mathbb{C}_{\mathfrak{q},\mathfrak{p}})$.*

Proof. For $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, denote $\mathcal{K}$ as a subset of $\mathbb{M}(\mathbb{C}_{\mathfrak{q},\mathfrak{p}})$ given by:

$$\mathcal{K} := \left\{ \mathcal{B}^l_{\widetilde{q}} \in \mathbb{M}_4(\mathbb{C}_{\mathfrak{q},\mathfrak{p}}) : \mathcal{B}^l_{\widetilde{q}} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \right\} \tag{12}$$

and define linear the map $\mathcal{N} : \widetilde{\mathcal{Q}}_{\alpha,\beta} \to \mathcal{K}, \widetilde{q} \mapsto \mathcal{B}^l_{\widetilde{q}}$. There exists a correspondence between $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\mathcal{K}$ via the map $\mathcal{N}$. $\mathcal{B}^l_{\widetilde{q}}$ is the 4×4 left generalized complex matrix representation of $\widetilde{q}$ according to the standard base $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

4×4 right generalized complex matrix representation of $\widetilde{q}$ can be calculated similarly[2]. Throughout this paper $\mathcal{B}^l_{\widetilde{q}}$ will be considered. □

Corollary 2. *Considering the base $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, the column matrix representation of $\widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ is given by $\widetilde{p} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}^T$. Using $\mathcal{B}^l_{\widetilde{q}}$, the multiplication of $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ can also be written by: $\widetilde{q}\widetilde{p} = \mathcal{B}^l_{\widetilde{q}}\widetilde{p}$.*

Theorem 4. *Let $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. $\mathcal{B}^l_{\widetilde{q}}$ can be determined as:*

$$\mathcal{B}^l_{\widetilde{q}} = a_0 I_4 + a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K},$$

where

$$\mathbf{I} = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{J} = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & -\beta & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Undoubtedly, $\mathbf{I}, \mathbf{J}, \mathbf{K}$ satisfy the generalized quaternion versors conditions in equations (1).

Using $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ as $\widetilde{q} = (a_0 + a_1\mathbf{i}) + (a_2 + a_3\mathbf{i})\,\mathbf{j}$ and considering a different conjugate related to this form, we can write the following theorem:

Theorem 5. *Let $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$. Then, we have $\sigma\mathcal{B}^l_{\widetilde{q}}\sigma = \mathcal{B}^l_{\widetilde{q}^*}$, where $\sigma = \operatorname{diag}(1, 1, -1, -1)$ and $\widetilde{q}^* = (a_0 + a_1\mathbf{i}) - (a_2 + a_3\mathbf{i})\mathbf{j} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$.*

[2] 4×4 right generalized complex matrix representation of $\widetilde{q}$ is:

$$\mathcal{B}^r_{\widetilde{q}} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}.$$

Proof. An easy computation shows that

$$\begin{aligned}\sigma\mathcal{B}^{l}_{\widetilde{q}}\sigma &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a_0 & -\alpha a_1 & \beta a_2 & \alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ -a_2 & -\alpha a_3 & a_0 & -\alpha a_1 \\ -a_3 & a_2 & a_1 & a_0 \end{bmatrix}.\end{aligned}$$

Hence, one can see that the last matrix is $\mathcal{B}^{l}_{\widetilde{q}^*}$. □

Theorem 6. *Let $\widetilde{q}, \widetilde{p} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\lambda \in \mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, the following properties are satisfied:*

i. $\widetilde{q} = \widetilde{p} \Leftrightarrow \mathcal{B}^{l}_{\widetilde{q}} = \mathcal{B}^{l}_{\widetilde{p}}$,
ii. $\mathcal{B}^{l}_{\widetilde{q}+\widetilde{p}} = \mathcal{B}^{l}_{\widetilde{q}} + \mathcal{B}^{l}_{\widetilde{p}}$,
iii. $\mathcal{B}^{l}_{\lambda\widetilde{q}} = \lambda(\mathcal{B}^{l}_{\widetilde{q}})$,
iv. $\mathcal{B}^{l}_{\widetilde{q}\widetilde{p}} = \mathcal{B}^{l}_{\widetilde{q}}\mathcal{B}^{l}_{\widetilde{p}}$,
v. $\det(\mathcal{B}^{l}_{\widetilde{q}}) = N^2_{\widetilde{q}}$,
vi. $\operatorname{tr}(\mathcal{B}^{l}_{\widetilde{q}}) = 4S_{\widetilde{q}}$.

Proof. By considering the matrix form given in equation (12), the proof is clear. As well let us discuss the proof of the item iv for better understanding:

iv. Using equation (5), we obtain the following matrix for $\mathcal{B}^{l}_{\widetilde{q}\widetilde{p}}$:

$$\begin{bmatrix} \begin{matrix} a_0b_0 - \alpha a_1b_1 \\ -\beta a_2b_2 - \alpha\beta a_3b_3 \end{matrix} & \begin{matrix} -\alpha\,(a_0b_1 + a_1b_0 \\ +\beta a_2b_3 - \beta a_3b_2) \end{matrix} & \begin{matrix} -\beta\,(a_0b_2 - \alpha a_1b_3 \\ +a_2b_0 + \alpha a_3b_1) \end{matrix} & \begin{matrix} -\alpha\beta\,(a_0b_3 + a_1b_2 \\ -a_2b_1 + a_3b_0) \end{matrix} \\ \begin{matrix} a_0b_1 + a_1b_0 \\ +\beta a_2b_3 - \beta a_3b_2 \end{matrix} & \begin{matrix} a_0b_0 - \alpha a_1b_1 \\ -\beta a_2b_2 - \alpha\beta a_3b_3 \end{matrix} & \begin{matrix} -\beta\,(a_0b_3 + a_1b_2 \\ -a_2b_1 + a_3b_0) \end{matrix} & \begin{matrix} \beta\,(a_0b_2 - \alpha a_1b_3 \\ +a_2b_0 + \alpha a_3b_1) \end{matrix} \\ \begin{matrix} (a_0b_2 - \alpha a_1b_3 \\ +a_2b_0 + \alpha a_3b_1) \end{matrix} & \begin{matrix} \alpha\,(a_0b_3 + a_1b_2 \\ -a_2b_1 + a_3b_0) \end{matrix} & \begin{matrix} a_0b_0 - \alpha a_1b_1 \\ -\beta a_2b_2 - \alpha\beta a_3b_3 \end{matrix} & \begin{matrix} -\alpha\,(a_0b_1 + a_1b_0 \\ +\beta a_2b_3 - \beta a_3b_2) \end{matrix} \\ \begin{matrix} (a_0b_3 + a_1b_2 \\ -a_2b_1 + a_3b_0) \end{matrix} & \begin{matrix} -\,(a_0b_2 - \alpha a_1b_3 \\ +a_2b_0 + \alpha a_3b_1) \end{matrix} & \begin{matrix} a_0b_1 + a_1b_0 \\ +\beta a_2b_3 - \beta a_3b_2 \end{matrix} & \begin{matrix} a_0b_0 - \alpha a_1b_1 \\ -\beta a_2b_2 - \alpha\beta a_3b_3 \end{matrix} \end{bmatrix}. \tag{13}$$

Multiplying $\mathcal{B}^{l}_{\widetilde{q}}$ and $\mathcal{B}^{l}_{\widetilde{p}}$ as:

$$\mathcal{B}^{l}_{\widetilde{q}}\mathcal{B}^{l}_{\widetilde{p}} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}\begin{bmatrix} b_0 & -\alpha b_1 & -\beta b_2 & -\alpha\beta b_3 \\ b_1 & b_0 & -\beta b_3 & \beta b_2 \\ b_2 & \alpha b_3 & b_0 & -\alpha b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{bmatrix}$$

gives equation (13) quickly. Hence we get $\mathcal{B}^{l}_{\widetilde{q}\widetilde{p}} = \mathcal{B}^{l}_{\widetilde{q}}\mathcal{B}^{l}_{\widetilde{p}}$.

□

Theorem 7. *Let $\widetilde{q} \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\widetilde{q}^{-1}$ be the inverse of $\widetilde{q}$. Then,*

$$\mathcal{B}^{l}_{\widetilde{q}^{-1}} = \frac{1}{\sqrt{\det(\mathcal{B}^{l}_{\widetilde{q}})}}\mathcal{B}^{l}_{\overline{\widetilde{q}}}.$$

Proof. Taking into account Definition 4 and Theorem 6 items iii and v, the proof is obvious. □

Theorem 8. *Every $\mathcal{GCN}$ with generalized quaternion components can be represented by a 2×2 generalized quaternion matrix. $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_2(\mathcal{Q}_{\alpha,\beta})$.*

Proof. For $\widetilde{q} = q_0 + q_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$, denote $\mathcal{T}$ as a subset of $\mathbb{M}_2(\mathcal{Q}_{\alpha,\beta})$ given by:

$$\mathcal{T} := \left\{ \mathcal{D}_{\widetilde{q}} \in \mathbb{M}_2(\mathcal{Q}_{\alpha,\beta}) : \mathcal{D}_{\widetilde{q}} = \begin{bmatrix} q_0 & \mathfrak{p} q_1 \\ q_1 & q_0 + \mathfrak{q} q_1 \end{bmatrix} \right\}, \tag{14}$$

and define the linear map $\mathcal{M} : \widetilde{\mathcal{Q}}_{\alpha,\beta} \to \mathcal{T}$, $\widetilde{q} \mapsto \mathcal{D}_{\widetilde{q}}$. It can be concluded that there exists a correspondence between $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\mathcal{T}$ via the map $\mathcal{M}$. Hence, 2×2 generalized complex matrix representation of $\widetilde{q}$ with respect to the standard base $\{1, I\}$ is the matrix $\mathcal{D}_{\widetilde{q}}$. □

By using $\mathcal{D}_{\widetilde{q}}$ and $\widetilde{p} = \begin{bmatrix} p_0 & p_1 \end{bmatrix}^T$, we have: $\widetilde{q}\widetilde{p} = \mathcal{D}_{\widetilde{q}}\widetilde{p}$. Moreover, $\mathcal{D}_{\widetilde{q}}$ is also in the form $\mathcal{D}_{\widetilde{q}} = q_0 I_2 + q_1 \mathsf{I}$, where $\mathsf{I} = \begin{bmatrix} 0 & \mathfrak{p} \\ 1 & \mathfrak{q} \end{bmatrix}$ is the representation of I. It should be noted that there are many ways to choose I, for instance: $\mathsf{I} = \begin{bmatrix} \mathfrak{q} & 1 \\ \mathfrak{p} & 0 \end{bmatrix}$ (see in [32]).

Theorem 9. *For any $\widetilde{q} = q_0 + q_1 I$ and $\widetilde{p} = p_0 + p_1 I \in \widetilde{\mathcal{Q}}_{\alpha,\beta}$ and $\lambda \in \mathbb{R}$, the following properties are satisfied:*

i. $\widetilde{q} = \widetilde{p} \Leftrightarrow \mathcal{D}_{\widetilde{q}} = \mathcal{D}_{\widetilde{p}}$,
ii. $\mathcal{D}_{\widetilde{q}+\widetilde{p}} = \mathcal{D}_{\widetilde{q}} + \mathcal{D}_{\widetilde{p}}$,
iii. $\mathcal{D}_{\lambda\widetilde{q}} = \lambda(\mathcal{D}_{\widetilde{q}})$,
iv. $\mathcal{D}_{\widetilde{q}\widetilde{p}} = \mathcal{D}_{\widetilde{q}}\mathcal{D}_{\widetilde{p}}$,
v. $\mathsf{det}(\mathcal{D}_{\widetilde{q}}) = q_0^2 + \mathfrak{q} q_1 q_0 - \mathfrak{p} q_1^2$, *where the notation* det *represents the determinant of the quaternion matrix*[3].

Proof. The proof is obvious considering the matrix form given in equation (14).

iv. Using equation (1), we obtain:

$$\mathcal{D}_{\widetilde{q}\widetilde{p}} = \begin{bmatrix} q_0 p_0 + \mathfrak{p} q_1 p_1 & \mathfrak{p}(q_0 p_1 + q_1 p_0 + \mathfrak{q} q_1 p_1) \\ q_0 p_1 + q_1 p_0 + \mathfrak{q} q_1 p_1 & q_0 p_0 + \mathfrak{p} q_1 p_1 + \mathfrak{q}(q_0 p_1 + q_1 p_0 + \mathfrak{q} q_1 p_1) \end{bmatrix}. \tag{15}$$

Also, the computation of the following multiplication

$$\mathcal{D}_{\widetilde{q}}\mathcal{D}_{\widetilde{p}} = \begin{bmatrix} q_0 & \mathfrak{p} q_1 \\ q_1 & q_0 + \mathfrak{q} q_1 \end{bmatrix} \begin{bmatrix} p_0 & \mathfrak{p} p_1 \\ p_1 & p_0 + \mathfrak{q} p_1 \end{bmatrix}$$

gives equation (15). Hence we have $\mathcal{D}_{\widetilde{q}\widetilde{p}} = \mathcal{D}_{\widetilde{q}}\mathcal{D}_{\widetilde{p}}$.

[3] The determinant of an arbitrary 2×2 quaternion matrix is defined by $\mathsf{det}\left(\begin{bmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{bmatrix}\right) = \mathsf{da} - \mathsf{cb}$, [33].

□

Definition 5. *Let* $\widetilde{q}=q_0+q_1I\in\widetilde{\mathcal{Q}}_{\alpha,\beta}$. *The vector representation of* $\widetilde{q}$ *is defined as*

$$\overrightarrow{\widetilde{q}}=\begin{bmatrix}\overrightarrow{q_0}^T & \overrightarrow{q_1}^T\end{bmatrix}^T=\begin{bmatrix}\overrightarrow{q_0}\\ \overrightarrow{q_1}\end{bmatrix}\in\mathbb{M}_{8\times 1}(\mathbb{R}),$$

where $q_{j-1}=x_{0j}+x_{1j}\mathbf{i}+x_{2j}\mathbf{j}+x_{3j}\mathbf{k}\in\mathcal{Q}_{\alpha,\beta}$ *and*

$$\overrightarrow{q_{j-1}}=(x_{0j},x_{1j},x_{2j},x_{3j})^{\ T}=[x_{0j}\ x_{1j}\ x_{2j}\ x_{3j}]^T$$

are vectors (matrices) for $1\leq j\leq 2$.

Theorem 10. *Let* $\widetilde{q}=q_0+q_1I\in\widetilde{\mathcal{Q}}_{\alpha,\beta}$. *Then*

i. $\overrightarrow{\widetilde{q}^{\dagger_1}}=\mathcal{X}\overrightarrow{\widetilde{q}}$, *where* $\mathcal{X}=\begin{bmatrix}I_4 & \mathfrak{q}I_4\\ 0 & -I_4\end{bmatrix}\in\mathbb{M}_8(\mathbb{R})$.

ii. $\overrightarrow{\widetilde{q}^{\dagger_2}}=\mathcal{Y}\overrightarrow{\widetilde{q}}$, *where* $\mathcal{Y}=\begin{bmatrix}0 & I_4\\ -I_4 & 0\end{bmatrix}\in\mathbb{M}_8(\mathbb{R})$.

Proof.

i. Computing $\overrightarrow{\widetilde{q}^{\dagger_1}}$ and $\mathcal{X}\overrightarrow{\widetilde{q}}$ gives the equality as: $\overrightarrow{\widetilde{q}^{\dagger_1}}=\begin{bmatrix}\overrightarrow{q_0}+\mathfrak{q}\overrightarrow{q_1}\\ -\overrightarrow{q_1}\end{bmatrix}$ and

$$\mathcal{X}\overrightarrow{\widetilde{q}}=\begin{bmatrix}I_4 & \mathfrak{q}I_4\\ 0 & -I_4\end{bmatrix}\begin{bmatrix}\overrightarrow{q_0}\\ \overrightarrow{q_1}\end{bmatrix}=\begin{bmatrix}\overrightarrow{q_0}+\mathfrak{q}\overrightarrow{q_1}\\ -\overrightarrow{q_1}\end{bmatrix}.$$

With the same manner the other item can be proved. □

By applying the map $\Gamma(x_{i1}+x_{i2}I)=\begin{bmatrix}x_{i1} & \mathfrak{p}x_{i2}\\ x_{i2} & x_{i1}+\mathfrak{q}x_{i2}\end{bmatrix}$ to $\mathcal{B}^l_{\widetilde{q}}$, where $a_i=x_{i1}+x_{i2}I\in\mathbb{C}_{\mathfrak{q},\mathfrak{p}}$, for $0\leq i\leq 3$, the left real matrix representation $\mathcal{C}^l_{\widetilde{q}}$ of $\widetilde{q}$ (see in equation (7)) with respect to the base $\{1,I,\mathbf{i},I\mathbf{i},\mathbf{j},I\mathbf{j},\mathbf{k},I\mathbf{k}\}$ can be easily found. So, $\widetilde{\mathcal{Q}}_{\alpha,\beta}$ is the subset of $\mathbb{M}_8(\mathbb{R})$.

Example 1. *Take* $\widetilde{q}\in\widetilde{\mathcal{Q}}_{2,1}$ *with* $\mathcal{GCN}$ *components for* $\mathfrak{p}=-1$ *and* $\mathfrak{q}=1$:

$$\widetilde{q}=1+(-1+I)\,\mathbf{i}+I\mathbf{j}+(1+2I)\,\mathbf{k}.$$

Then,

$$\mathcal{A}_{\widetilde{q}}=\begin{bmatrix}1+(1+2I)\mathbf{k} & (-1+I)\,\mathbf{i}+I\mathbf{j}\\ (-1+I)\,\mathbf{i}+I\mathbf{j} & 1+(1+2I)\mathbf{k}\end{bmatrix},$$

$$\mathcal{B}^l_{\widetilde{q}}=\begin{bmatrix}1 & -2(-1+I) & -I & -2(1+2I)\\ -1+I & 1 & -1-2I & I\\ I & 2(1+2I) & 1 & -2(-1+I)\\ 1+2I & -I & -1+I & 1\end{bmatrix},$$

$$\mathcal{C}^l_{\widetilde{q}}=\begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 1 & -2 & 4 \\ 0 & 1 & -2 & 0 & -1 & -1 & -4 & -6 \\ -1 & -1 & 1 & 0 & -1 & 2 & 0 & -1 \\ 1 & 0 & 0 & 1 & -2 & -3 & 1 & 1 \\ 0 & -1 & 2 & -4 & 1 & 0 & 2 & 2 \\ 1 & 1 & 4 & 6 & 0 & 1 & -2 & 0 \\ 1 & -2 & 0 & 1 & -1 & -1 & 1 & 0 \\ 2 & 3 & -1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{D}_{\widetilde{q}}=\begin{bmatrix} 1-\mathbf{i}+\mathbf{k} & -\mathbf{i}-\mathbf{j}-2\mathbf{k} \\ \mathbf{i}+\mathbf{j}+2\mathbf{k} & 1+\mathbf{j}+3\mathbf{k} \end{bmatrix},$$

$$\mathcal{B}^l_{\widetilde{q}^{-1}}=\frac{1}{\sqrt{-189+45I}}\begin{bmatrix} 1 & 2(-1+I) & I & 2(1+2I) \\ 1-I & 1 & 1+2I & -I \\ -I & -2(1+2I) & 1 & 2(-1+I) \\ -1-2I & I & 1-I & 1 \end{bmatrix}.$$

Also, the vector representation of $\widetilde{q}^{\dagger_1}$ is computed by:

$$\begin{aligned}\overrightarrow{\widetilde{q}^{\dagger_1}}=\mathcal{X}\,\overrightarrow{\widetilde{q}} &= \begin{bmatrix} I_4 & I_4 \\ 0 & -I_4 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T \\ \begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}^T \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 & 3 & 0 & -1 & -1 & -2 \end{bmatrix}^T.\end{aligned}$$

3. Further Result: An Alternative Matrix Approach

The questions about numbers, hypercomplex numbers and quaternions included questions about their matrices. Inspired by matrix forms in the study [34], we give an answer for the question of the alternative representation of generalized quaternion matrix with elliptic number entries (see elliptic biquaternions in [35]). So this matrix is in the form:

$$\widetilde{Q}=A_0I_2+A_1\mathcal{I}+A_2\mathcal{J}+A_3\mathcal{K},$$

where $A_0, A_1, A_2, A_3 \in \mathbb{C}_{\mathfrak{p}}$ are elliptic numbers for $\mathfrak{p}<0$. The base elements can be defined as follows:

Case 1: For $\alpha,\beta\in\mathbb{R}^+$

$$\mathcal{I}=\begin{bmatrix} \sqrt{\frac{\alpha}{|\mathfrak{p}|}}I & 0 \\ 0 & -\sqrt{\frac{\alpha}{|\mathfrak{p}|}}I \end{bmatrix}, \mathcal{J}=\begin{bmatrix} 0 & \sqrt{\beta} \\ -\sqrt{\beta} & 0 \end{bmatrix}, \mathcal{K}=\begin{bmatrix} 0 & \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I \\ \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \end{bmatrix},$$

Case 2: For $\alpha\in\mathbb{R}^+$, $\beta\in\mathbb{R}^-$

$$\mathcal{I}=\begin{bmatrix} \sqrt{\frac{\alpha}{|\mathfrak{p}|}}I & 0 \\ 0 & -\sqrt{\frac{\alpha}{|\mathfrak{p}|}}I \end{bmatrix}, \mathcal{J}=\begin{bmatrix} 0 & \sqrt{-\beta} \\ \sqrt{-\beta} & 0 \end{bmatrix}, \mathcal{K}=\begin{bmatrix} 0 & \sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I \\ -\sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \end{bmatrix},$$

Case 3: For $\alpha \in \mathbb{R}^-$, $\beta \in \mathbb{R}^+$

$$\mathcal{I} = \begin{bmatrix} 0 & \sqrt{-\alpha} \\ \sqrt{-\alpha} & 0 \end{bmatrix}, \mathcal{J} = \begin{bmatrix} -\sqrt{\frac{\beta}{|\mathfrak{p}|}}I & 0 \\ 0 & \sqrt{\frac{\beta}{|\mathfrak{p}|}}I \end{bmatrix}, \mathcal{K} = \begin{bmatrix} 0 & \sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I \\ -\sqrt{\frac{-\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \end{bmatrix},$$

Case 4: For $\alpha, \beta \in \mathbb{R}^-$

$$\mathcal{I} = \begin{bmatrix} 0 & \sqrt{\frac{-\alpha}{|\mathfrak{p}|}}I \\ -\sqrt{\frac{-\alpha}{|\mathfrak{p}|}}I & 0 \end{bmatrix}, \mathcal{J} = \begin{bmatrix} 0 & \sqrt{-\beta} \\ \sqrt{-\beta} & 0 \end{bmatrix}, \mathcal{K} = \begin{bmatrix} \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I & 0 \\ 0 & -\sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ I \end{bmatrix}.$$

These elements satisfy the following conditions:

$$\begin{array}{ll} \mathcal{I}^2 = -\alpha I_2, & \mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I} = \mathcal{K}, \\ \mathcal{J}^2 = -\beta I_2, & \mathcal{J}\mathcal{K} = -\mathcal{K}\mathcal{J} = \beta\mathcal{I}, \\ \mathcal{K}^2 = -\alpha\beta I_2, & \mathcal{K}\mathcal{I} = -\mathcal{I}\mathcal{K} = \alpha\mathcal{J}. \end{array}$$

Taking into account Case 1, $\widetilde{Q}$ is rewritten as

$$\widetilde{Q} = \begin{bmatrix} A_0 + \sqrt{\frac{\alpha}{|\mathfrak{p}|}}IA_1 & \sqrt{\beta}A_2 + \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ IA_3 \\ -\sqrt{\beta}A_2 + \sqrt{\frac{\alpha\beta}{|\mathfrak{p}|}}\ IA_3 & A_0 - \sqrt{\frac{\alpha}{|\mathfrak{p}|}}IA_1 \end{bmatrix}.$$

One can see this matrix in Tian's paper [36] related to biquaternions (complexified quaternion) for $\alpha = \beta = 1$ and $\mathfrak{p} = -1$.

The conjugate (same as the adjoint), transpose, the elliptic conjugate, the total conjugate and determinant $\widetilde{Q}$ can be given as follows:

$$\begin{array}{ll} \overline{\widetilde{Q}} & = A_0I_2 - A_1\mathcal{I} - A_2\mathcal{J} - A_3\mathcal{K} = \operatorname{Adj}\widetilde{Q}, \\ \widetilde{Q}^T & = A_0I_2 + A_1\mathcal{I} - A_2\mathcal{J} + A_3\mathcal{K}, \\ \widetilde{Q}^{\mathbb{C}_\mathfrak{p}} & = A_0I_2 - A_1\mathcal{I} + A_2\mathcal{J} - A_3\mathcal{K} = \overline{\widetilde{Q}}^T, \\ \overline{\widetilde{Q}}^{\mathbb{C}_\mathfrak{p}} & = A_0I_2 + A_1\mathcal{I} - A_2\mathcal{J} + A_3\mathcal{K} = \overline{\left(\widetilde{Q}^{\mathbb{C}_\mathfrak{p}}\right)}, \end{array}$$

and

$$\begin{array}{ll} \det\widetilde{Q} & = A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 \\ & = A_0^2 + A_1^2 \det\mathcal{I} + A_2^2\det\mathcal{J} + A_3^2\det\mathcal{K}. \end{array}$$

For $\det\widetilde{Q} \neq 0$, the inverse of $\widetilde{Q}$ is defined by:

$$\widetilde{Q}^{-1} = \frac{1}{\det\widetilde{Q}}\overline{\widetilde{Q}} = \frac{1}{A_0^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}(A_0I_2 - A_1\mathcal{I} - A_2\mathcal{J} - A_3\mathcal{K}).$$

Similar calculations can be given for the other cases. Additionally, the relationships between the above operations and some properties of generalized quaternion matrices with elliptic number entries can be easily proved. We omit them for the sake of brevity. For $A_0, A_1, A_2, A_3 \in \mathbb{C}_{-1}$, we refer to [37] under the condition that $\alpha = \beta = 1$ and $\alpha = 1, \beta = -1$.

4. Concluding Remarks

Our paper is motivated by the question: What happens if the components of quaternions become $\mathcal{GCN}$? Based on this question, we develop the theory of generalized quaternions (non-commutative system) with $\mathcal{GCN}$ components $\mathfrak{p}, \mathfrak{q} \in \mathbb{R}$. Also, we investigate the algebraic structures and properties by considering them as a $\mathcal{GCN}$ and a quaternion. With specific values of α and β, we obtained different types of quaternions with $\mathcal{GCN}$ components in Section 2. Additionally, we establish matrix representations and give a numerical example. In Section 3, we also come up with a different way to deal with a generalized quaternion matrix with elliptic number entries.

The crucial part of this paper is that one can reduce the calculations to mentioned types of quaternions by considering hyperbolic, elliptic and parabolic number components for $\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$ (see Table 1). As a natural consequence of this situation, taking into account special conditions, the definition of special quaternions mentioned in the papers [38–47] are generalized via Definition 1, the papers [48–53] are generalized from the viewpoint of definition, algebraic properties, relations and matrix representations of quaternions and finally, different matrix forms in the papers [35–37] are generalized in Section 3. All of these situations can be examined in Table 2. For instance, all of the obtained calculations agree with complex quaternions for $\alpha = \beta = 1, \mathfrak{q} = 0, \mathfrak{p} = -1$.

With this unified method, we believe that these results give rise to ease of calculation via mathematical concordance, and in future studies, we intend to investigate other commutative and non-commutative quaternions created with $\mathcal{GCN}$ components in this manner. Now, the necessary and sufficient condition for similarity, co-similarity and semi-similarity for elements of the generalized quaternions with $\mathcal{GCN}$ components for $\mathfrak{p}, \mathfrak{q} \in \mathbb{R}$ is an open problem for researchers.

Table 1. Basic classification regarding components

$\Delta = \mathfrak{q}^2 + 4\mathfrak{p}$	**Type of components**	**References**
$\Delta < 0$	elliptic	biquaternion [35,50] (for $\mathfrak{q} = 0$)
$\Delta = 0$	parabolic	[41,51] (for $\mathfrak{q} = 0$)
$\Delta > 0$	hyperbolic	

TABLE 2. Classification considering components with regard to the value of $\mathfrak{p}, \mathfrak{q}, \alpha$ and β

Condition	α	β	**Component**	**Quaternion**	**Ref.**
$\mathfrak{q} = 0$, $\mathfrak{p} = -1$	1	1	complex	Hamiltonian	[14, 36, 44]
	1	-1	complex	split	[49, 53]
	1	0	complex	semi	[38, 39]
	-1	0	complex	split semi	
	0	0	complex	quasi	
$\mathfrak{q} = 0$, $\mathfrak{p} = 0$	1	1	dual	Hamiltonian	[45, 47, 54]
	1	-1	dual	split	[46]
	1	0	dual	semi	[42, 52]
	-1	0	dual	split semi	
	0	0	dual	quasi	
$\mathfrak{q} = 0$, $\mathfrak{p} = 1$	1	1	hyperbolic	Hamiltonian	[40]
	1	-1	hyperbolic	split	[43]
	1	0	hyperbolic	semi	
	-1	0	hyperbolic	split semi	[48]
	0	0	hyperbolic	quasi	

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UNIQUENESS OF THE SOLUTION TO THE INVERSE PROBLEM OF SCATTERING THEORY FOR SPECTRAL PARAMETER DEPENDENT KLEIN-GORDON S-WAVE EQUATION

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Abstract. In the present work, the inverse problem of the scattering theory for Klein-Gordon s-wave equation with a spectral parameter in the boundary condition is investigated. We define the scattering data set, and obtain the main equation of operator. Furthermore, the uniqueness of the solution of the inverse problem is proved.

1. Introduction

Scattering problems, which play a role in the structure of matter in Newtonian mechanics, are an important research topic of mathematical physics. Obtaining the scattering data by giving the potential function and investigating the properties of these scattering data is called the direct problem in scattering theory, while obtaining the potential function according to the scattering data is called the inverse problem. Therefore, the importance of inverse scattering problems in terms of natural sciences is an undeniable reality.

The inverse problem of scattering theory for the boundary value problem

$$-y'' + q(x)y = \lambda^2 y, \tag{1}$$

$$y(0) = 0 \tag{2}$$

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was studied in [13] and the author obtained that the Jost function of (1)-(2) defined by

$$e(\lambda) = 1 + \int_0^\infty K(0,t)e^{i\lambda t}dt \ , \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, \ \operatorname{Im}\lambda \geq 0\} \, .$$

has a finite number of simple zeros in $\mathbb{C}_+$. The scattering data of (1)-(2) is

$$\{S(\lambda), \lambda_k, m_k : k = 1, 2, ..., n\} \, ,$$

where λ_k are the zeros of Jost function, m_k^{-1} are the norm of the zeros of Jost function for $\lambda = \lambda_k$ in $L_2(0, \infty)$ and $S(\lambda)$ is scattering function of (1)-(2) given by

$$S(\lambda) := \frac{\overline{e(\lambda)}}{e(\lambda)} \ , \quad \lambda \in (-\infty, \infty).$$

As the potential function q is given, the problem of getting scattering data and investigating the properties of scattering data is called the direct problem for scattering theory. Oppositely, finding the potential function q according to the scattering data is known inverse problem of scattering theory. The direct and inverse scattering problems for a selfadjoint infinite system second-order difference equations with operator valued coefficients were considered in [11]. The uniqueness of the solution to the inverse problem of scattering theory for equation (1) with a spectral parameter in the boundary condition

$$y'(0) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y(0) = 0$$

was studied bh Kh. R. Mamedov ([12]). Also, the solution to the inverse problem of scattering theory for spectral parameter dependent Sturm-Liouville operator system was founded uniquely by G. Bascanbaz Tunca and E. Kir Arpat in [15], and the scattering analysis of a transmission boundary value problem which consists of a discrete Schrodinger equation and transmission conditions was investigated in [5]. Furthermore, the scattering theory of impulsive Sturm-Liouville equations, impulsive discrete Dirac systems, impulsive Sturm-Liouville equation in Quantum-Calculus and Dirac operator with impulsive condition on whole axis were investigated in [1,4,8,9]. The scattering function of impulsive matrix difference operators and scattering properties of eigenparameter dependent discrete impulsive Sturm-Liouville equations were studied in [2,3,6]. But scattering theory of Klein-Gordon s-wave equation with boundary condition depends on spectral parameter has not been investigated yet.

Let L_μ denotes the Klein-Gordon s-wave operator of second order with boundary condition generated by

$$y'' + [\lambda - q(x)]^2 \, y = 0 \ , \quad 0 \leq x < \infty \tag{3}$$

and

$$y'(0, \lambda) + (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y(0, \lambda) = 0,$$

where $\lambda = \mu^2$ is a complex spectral parameter, α_i are real numbers for $i = 0, 1, 2$, $\alpha_1 \leq 0$, $\alpha_2 > 0$, $(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2) \neq 0$ and q is a non-negative real valued function satisfying the following condition

$$\int_0^\infty x\left[|q(x)| + |q'(x)|\right] dx < \infty. \tag{4}$$

In this paper, we examine the inverse problem of scattering theory of L_μ under the condition (4).

2. Preliminaries

To be able to well defined mapping between λ and μ, we will study on the region $\operatorname{Re}\mu \geq 0$. If the condition (4) is satisfied, equation (3) has the following solutions

$$f^{(1)}(x,\mu) = f(x,\mu^2) = e^{i\left[\alpha(x)+\mu^2 x\right]} + \int_x^\infty K(x,t)e^{i\mu^2 t}dt, \tag{5}$$

$$\overline{f^{(1)}(x,\mu)} = \overline{f(x,\mu^2)} = e^{-i\left[\alpha(x)+\mu^2 x\right]} + \int_x^\infty K(x,t)e^{-i\mu^2 t}dt$$

for $\mu \in \mathbb{R}_1 := \{\mu : \operatorname{Re}\mu \geq 0,\ \operatorname{Im}\mu = 0\}$ and they have analytic continuation to $\overline{\mathbb{C}_1^+} := \{\mu \in \mathbb{C} : \operatorname{Re}\mu \geq 0,\ \operatorname{Im}\mu \geq 0\}$ and $\overline{\mathbb{C}_1^-} := \{\mu \in \mathbb{C} : \operatorname{Re}\mu \geq 0,\ \operatorname{Im}\mu \leq 0\}$, respectively where $\alpha(x) = \int_x^\infty q(t)dt$ and $K(x,t)$ is solution of integral equations of Volterra type which has continuous derivatives with respect to their arguments ([7]). Moreover, $K(x,t)$, $K_x(x,t)$, $K_t(x,t)$ satisfy the following inequalities

$$|K(x,t)| \leq c\omega\left(\tfrac{x+t}{2}\right)\exp\left(\gamma(x)\right),$$

$$|K_x(x,t)|, |K_t(x,t)| \leq c\left[\omega^2\left(\tfrac{x+t}{2}\right) + \theta\left(\tfrac{x+t}{2}\right)\right],$$

where

$$\omega(x) = \int_x^\infty \left[|q(t)|^2 + \left|q'(t)\right|\right] dt\ ,$$

$$\gamma(x) = \int_x^\infty \left[t\,|q(t)|^2 + 2\,|q(t)|\right] dt\ ,$$

$$\theta(x) = \tfrac{1}{4}\left[2\,|q(x)|^2 + \left|q'(x)\right|\right]$$

and $c>0$ is a constant. In addition, the function $K(x,t)$ and potential are related to

$$K(x,x)=2\int_{x}^{\infty}q(t)dt$$

([14]). Furthermore, $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ are respectively analytic in $\mathbb{C}_1^+ := \{\mu\in\mathbb{C} : \operatorname{Re}\mu>0,\ \operatorname{Im}\mu>0\}$ and $\mathbb{C}_1^- := \{\mu\in\mathbb{C} : \operatorname{Re}\mu>0,\ \operatorname{Im}\mu<0\}$ and they are continuous on real and imaginary axes with respect to μ. The solutions $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ are called Jost solutions of L_μ ([10]). From (5), $f^{(1)}(x,\mu)$ satisfies the asymptotic equalities

$$\begin{aligned} f^{(1)}(x,\mu)&=e^{i\mu^2 x}\left[1+o(1)\right]\ ,\ x\to\infty\ ,\\ f_x^{(1)}(x,\mu)&=e^{i\mu^2 x}\left[i\mu^2+o(1)\right]\ ,\ x\to\infty \end{aligned} \tag{6}$$

and

$$f^{(1)}(x,\mu)=e^{i\left[\alpha(x)+\mu^2 x\right]}+o(1)\ ,\ \ |\mu|\to\infty \tag{7}$$

([14]). From (6), the Wronskian of the solutions of $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ is

$$W\left[f^{(1)}(x,\mu),\overline{f^{(1)}(x,\mu)}\right]=\lim_{x\to\infty}W\left[f^{(1)}(x,\mu),\overline{f^{(1)}(x,\mu)}\right]=-2i\mu^2 \tag{8}$$

for $\mu\in\mathbb{R}_1$. Hence $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ are the fundamental solutions of (3) for $\mu\in\mathbb{R}_1^*=\mathbb{R}_1\setminus\{0\}$.

Let $\varphi^{(1)}(x,\mu)=\varphi(x,\mu^2)$ denotes the solution of (3) satisfying the initial conditions

$$\begin{aligned} \varphi^{(1)}(0,\mu) &= \varphi(0,\mu^2)=1,\\ \varphi_x^{(1)}(0,\mu) &= \varphi_x(0,\mu^2)-(\alpha_0+\alpha_1\mu^2+\alpha_2\mu^4). \end{aligned}$$

Definition 1.

$$\begin{aligned} W\left[\varphi^{(1)}(x,\mu),f^{(1)}(x,\mu)\right] &= \varphi^{(1)}(0,\mu)f_x^{(1)}(0,\mu)-\varphi_x^{(1)}(0,\mu)f^{(1)}(0,\mu)\\ &= f_x^{(1)}(0,\mu)+(\alpha_0+\alpha_1\mu^2+\alpha_2\mu^4)f^{(1)}(0,\mu)\\ &= F(\mu^2)=F_1(\mu) \end{aligned} \tag{9}$$

is called Jost function of L_μ ([10]).

Theorem 1. *Under the condition (4), Jost function has following asymptotic equality*

$$F_1(\mu)\approx\begin{cases} i\mu^2(1-i\alpha_1)e^{i\alpha(0)} & ,\ \ \alpha_1\neq 0,\ \ |\mu|\to\infty\\ \alpha_2\mu^4 & ,\ \ \alpha_1=0,\ \ |\mu|\to\infty \end{cases}, \tag{10}$$

where $\alpha_1\leq 0$ and $\alpha_2>0$.

Proof. This aymptotic equality can be seen smoothly from (7) and Definition 1. □

3. Main Equation of L_μ

Definition 2. *We can define scattering function using Jost function as follows for $\mu \in \mathbb{R}_1$:*

$$S_1(\mu) = S(\mu^2) = \frac{\overline{F(\mu^2)}}{F(\mu^2)} = \frac{\overline{F_1(\mu)}}{F_1(\mu)}. \tag{11}$$

Theorem 2. *Under the condition (4), the scattering function satisfies following asymptotic equality*

$$S_1(\mu) = 1 + O\left(\frac{1}{\mu^2}\right), \quad |\mu| \to \infty. \tag{12}$$

Proof. The proof can be easily attained using definition of scattering function and (7). □

Lemma 1. *Under the condition (4),*

$$F_1(\mu) = f_x^{(1)}(0,\mu) + (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4) f^{(1)}(0,\mu) \neq 0$$

for all $\mu \in \mathbb{R}_1^*$.

Proof. Let $F_1(\mu_0) = 0$ for any $\mu_0 \in \mathbb{R}_1^*$. Then, we obtain

$$f_x^{(1)}(0,\mu_0) = -(\alpha_0 + \alpha_1\mu_0 + \alpha_2\mu_0^4) f^{(1)}(0,\mu_0).$$

Also,

$$W\left[\overline{f^{(1)}(x,\mu)}, f^{(1)}(x,\mu)\right] = 2i\mu^2.$$

for all $\mu \in \mathbb{R}_1$. So,

$$f_x^{(1)}(0,\mu_0)\overline{f^{(1)}(0,\mu_0)} - f^{(1)}(0,\mu_0)\overline{f_x^{(1)}(0,\mu_0)} = 2i\mu_0^2$$

and, we get

$$-(\alpha_0+\alpha_1\mu_0^2+\alpha_2\mu_0^4) f^{(1)}(0,\mu_0)\overline{f^{(1)}(0,\mu_0)} + (\alpha_0+\alpha_1\mu_0^2+\alpha_2\mu_0^4)\overline{f^{(1)}(0,\mu_0)} f^{(1)}(0,\mu_0) = 2i\mu_0^2.$$

From last equation, we can write

$$2i\mu_0^2 = 0.$$

But this is a contradiction because of $\mu_0 \in \mathbb{R}_1^*$. □

Lemma 2. *The following equation*

$$\frac{2i\mu^2\varphi^{(1)}(x,\mu)}{f_x^{(1)}(0,\mu) + (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4) f^{(1)}(0,\mu)} = \overline{f^{(1)}(x,\mu)} - S_1(\mu) f^{(1)}(x,\mu) \tag{13}$$

holds. Furthermore, $\overline{S_1(\mu)} = [S_1(\mu)]^{-1}$.

Proof. Since $f^{(1)}(x,\mu)$ and $\overline{f^{(1)}(x,\mu)}$ are basic solutions of L_μ,

$$\varphi^{(1)}(x,\mu) = c_1 f^{(1)}(x,\mu) + c_2\overline{f^{(1)}(x,\mu)}. \tag{14}$$

From (14),

$$c_1(\mu)\, f^{(1)}(0,\mu) + c_2(\mu)\,\overline{f^{(1)}(0,\mu)} = 1$$

and

$$c_1 f_x^{(1)}(x,\mu) + c_2\overline{f_x^{(1)}(x,\mu)} = -(\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4).$$

By finding $c_1(\mu)$ and $c_2(\mu)$ from last two equations and using (8), we can obtain (13). In addition, we hold easily

$$\overline{S_1(\mu)} = \frac{F_1(\mu)}{\overline{F_1(\mu)}} = [S_1(\mu)]^{-1}$$

from (11). □

Lemma 3. *The all zeros of Jost function $F_1(\mu)$ are finite and on the imaginary axis. Also, they are simply on the upper imaginary axis.*

Proof. Using asymptotic equality (10), Lemma 1, uniqueness theorems for analytic functions and Bolzano-Weierstrass Theorem we can easily reach finiteness of the zeros of Jost function. Now, we will show that the zeros of $F_1(\mu)$ are on the imaginary axis. Let μ_0 be an arbitrary zero of $F_1(\mu)$. We can write

$$0 = F_1(\mu_0) = f_x^{(1)}(0,\mu_0) + (\alpha_0 + \alpha_1\mu_0^2 + \alpha_2\mu_0^4) f^{(1)}(0,\mu_0)$$

and

$$\begin{cases} f_{xx}^{(1)}(x,\mu_0) + \left[\mu_0^4 - 2\mu_0^2 q(x) + q^2(x)\right] f_x^{(1)}(x,\mu_0) = 0, \\ \overline{f_{xx}^{(1)}(x,\mu_0)} + \left[\overline{\mu_0^4} - 2\overline{\mu_0^2} q(x) + q^2(x)\right] \overline{f_x^{(1)}(x,\mu_0)} = 0 \end{cases}$$

from (3) and (9). By using the last equalities together the definition of Wronskian and the partial integration method, we find that

$$\begin{aligned} 0 &= \left(\mu_0^2 - \overline{\mu_0^2}\right)\Bigg\{\alpha_1\left|f^{(1)}(0,\mu_0)\right|^2 + \left(\mu_0^2 + \overline{\mu_0^2}\right)\left[\alpha_2 + \int_0^\infty \left|f^{(1)}(x,\mu_0)\right|^2 dx\right] \\ &\quad -2\int_0^\infty q(x)\left|f^{(1)}(x,\mu_0)\right|^2 dx\Bigg\} \end{aligned}$$

and then

$$\begin{aligned} 0 &= \left(\mu_0^2 - \overline{\mu_0^2}\right)\Bigg\{\alpha_1\left|f^{(1)}(0,\mu_0)\right|^2 + \left[(\operatorname{Re}\mu_0)^2 - (\operatorname{Im}\mu_0)^2\right]\left[\alpha_2 + \int_0^\infty \left|f^{(1)}(x,\mu_0)\right|^2 dx\right] \\ &\quad -2\int_0^\infty q(x)\left|f^{(1)}(x,\mu_0)\right|^2 dx\Bigg\}. \end{aligned}$$

The last equality is satisfied if $\mu_0^2 - \overline{\mu_0^2} = 0$ and $(\operatorname{Re}\mu_0)^2 = 0$, i.e. $\operatorname{Re}\mu_0 = 0$. So, all zeros of $F_1(\mu)$ are on the imaginary axis. Finally, to get the simplicity of any zero $\mu_0 = i\omega_0$, $\omega_0 > 0$, we need to prove that

$$\frac{\partial F_1(\mu_0)}{\partial \mu} \neq 0.$$

From equation (3), we have

$$\begin{aligned}
\overline{f_{xx}^{(1)}(x,\mu)} + q^2(x)\overline{f^{(1)}(x,\mu)} &= 2\mu^2 q(x)\overline{f^{(1)}(x,\mu)} - \mu^4 \overline{f^{(1)}(x,\mu)}\ , \\
\left(\dot{f}_{xx}^{(1)}\right)(x,\mu) + q^2(x)\left(\dot{f}^{(1)}\right)(x,\mu) &= 4\mu q(x) f^{(1)}(x,\mu) + 2\mu^2 q(x)\left(\dot{f}^{(1)}\right)(x,\mu) \\
&\quad -4\mu^3 f^{(1)}(x,\mu) - \mu^4\left(\dot{f}^{(1)}\right)(x,\mu)
\end{aligned}$$

and then

$$4\mu\int_0^\infty \left[q(x)-\mu^2\right]\left|f^{(1)}(x,\mu)\right|^2 dx = \left(\dot{f}^{(1)}\right)(0,\mu)\overline{f_x^{(1)}(0,\mu)} - \left(\dot{f}_x^{(1)}\right)(0,\mu)\overline{f^{(1)}(0,\mu)}\ ,$$

where $\left.\frac{\partial f^{(1)}(x,\mu)}{\partial \mu}\right|_{x=0} := \left(\dot{f}^{(1)}\right)(0,\mu)$ and $\mu = i\omega$, $\omega \geq 0$. Also, we find the following equation

$$\begin{aligned}
4i\omega\int_0^\infty \left[q(x)+\omega^2\right]\left|f^{(1)}(x,i\omega)\right|^2 dx &= \left(\dot{f}^{(1)}\right)(0,i\omega)\overline{f_x^{(1)}(0,i\omega)} \\
&\quad -\left(\dot{f}_x^{(1)}\right)(0,i\omega)\overline{f^{(1)}(0,i\omega)}. \qquad (15)
\end{aligned}$$

By the definition of $F_1(\mu)$, we hold

$$\begin{aligned}
f_x^{(1)}(0,\mu) &= F_1(\mu) - (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4) f^{(1)}(0,\mu)\ , \\
\left(\dot{f}_x^{(1)}\right)(0,\mu) &= (\dot{F}_1)(\mu) - \left(2\alpha_1\mu + 4\alpha_2\mu^3\right) f^{(1)}(0,\mu) - (\alpha_0 + \alpha_1\mu^2 + \alpha_2\mu^4)\left(\dot{f}^{(1)}\right)(0,\mu).
\end{aligned}$$

These derivatives are taken into account in the equation (15) with $\mu_0 = i\omega_0$, $\omega_0 > 0$,

$$\begin{aligned}
4i\omega_0\int_0^\infty \left[q(x)+\omega_0^2\right]\left|f^{(1)}(x,i\omega_0)\right|^2 dx &= -(\dot{F}_1)(i\omega_0)\overline{f^{(1)}(0,i\omega_0)} \\
&\quad + i\left(2\alpha_1\omega_0 - 4\alpha_2\omega_0^3\right)\left|f^{(1)}(0,i\omega_0)\right|^2 (16)
\end{aligned}$$

and from (3.6)

$$-(\dot{F}_1)(i\omega_0)\overline{f^{(1)}(0,i\omega_0)} = i\Bigg[\left(-2\alpha_1\omega_0 + 4\alpha_2\omega_0^3\right)\left|f^{(1)}(0,i\omega_0)\right|^2$$

$$+4\omega_0 \int_0^\infty \left[q(x)+\omega_0^2\right] \left|f^{(1)}(x,i\omega_0)\right|^2 dx \Bigg]. \quad (17)$$

If $f^{(1)}(0,i\omega_0)=0$ in (17), then it is occured that $f^{(1)}(x,i\omega_0)\equiv 0$ but this can not be. So, it is clear that the left side of (17) is nonzero. Therefore, it is attained that $(\dot{F_1})(\mu_0)\neq 0$ with $F_1(\mu_0)=0$. So, the zeros of Jost function are simply on the upper imaginary axis. □

Lemma 4. *If the function*

$$F_{S_1}(x)=\frac{1}{\pi}\int_0^\infty \mu\left[1-S_1(\mu)\right]e^{i\mu^2 x}d\mu \quad (18)$$

is Fourier transformation of $\mu\left[1-S_1(\mu)\right]$ *for all* $x\geq 0$*, it belongs to the* $L_2(0,\infty)$ *space.*

Proof. From (12), we can easily verify that

$$\mu\left[1-S_1(\mu)\right]\approx O\left(\frac{1}{\mu}\right)\ ,\quad |\mu|\to\infty.$$

It follows that $\mu\left[1-S_1(\mu)\right]\in L_2(0,\infty)$ and hence the function $F_{S_1}(x)$ also belongs to the space $L_2(0,\infty)$. □

Definition 3. *For* $k=1,2,...,n$,

$$m_k^{-1}=\frac{\left[f^{(1)}(0,\mu_k)\right]^2}{\mu_k^2}\left\{\frac{1}{\left|f^{(1)}(0,\mu_k)\right|^2}\int_0^\infty \left[q(x)-\mu_k^2\right]\left|f^{(1)}(x,\mu_k)\right|^2 dx-\frac{\alpha_1+2\alpha_2\mu_k^2}{2}\right\},$$

where μ_k *are zeros of Jost function on the upper imaginary axis.*

Lemma 5. *The kernel function* $K(x,t)$ *satisfies the main equation of* L_μ

$$e^{i\alpha(x)}G(x+y)+K(x,y)+\int_x^\infty K(x,t)G(t+y)dt=0\ ,\quad (x<y), \quad (19)$$

where

$$G(x)=\sum_{k=1}^n m_k e^{i\mu_k^2 x}+F_{S_1}(x). \quad (20)$$

Proof. Lets rewrite (13) as follows

$$\frac{2i\mu^2\varphi^{(1)}(x,\mu)}{F_1(\mu)}=\overline{f^{(1)}(x,\mu)}-S_1(\mu)f^{(1)}(x,\mu),$$

and substitute $f^{(1)}(x,\mu)$ in this by its expression (13), we get that

$$\begin{aligned}\frac{2i\mu^2\varphi^{(1)}(x,\mu)}{F_1(\mu)} &= e^{-i\left[\alpha(x)+\mu^2 x\right]}+\int\limits_x^\infty K(x,t)e^{-i\mu^2 t}dt \\ &- S_1(\mu)\left[e^{i\left[\alpha(x)+\mu^2 x\right]}+\int\limits_x^\infty K(x,t)e^{i\mu^2 t}dt\right].\end{aligned}$$

Also, by making the necessary arrangements and using (18), we reach

$$\frac{2i}{\pi}\int\limits_0^\infty \frac{\mu^3\varphi^{(1)}(x,\mu)e^{i\mu^2 y}}{F_1(\mu)}d\mu = e^{i\alpha(x)}F_{S_1}(x+y)+K(x,y)+\int\limits_x^\infty K(x,t)F_{S_1}(t+y)dt. \quad (21)$$

By using Jordan Lemma and Residue Theorem,

$$\begin{aligned}\frac{2i}{\pi}\int\limits_0^\infty \frac{\mu^3\varphi^{(1)}(x,\mu)e^{i\mu^2 y}}{F_1(\mu)}d\mu &= 2\pi i\frac{2i}{\pi}\sum_{k=1}^n \mathrm{Res}(F_1,\mu_k) \\ &= -\sum_{k=1}^n \frac{4\mu_k^3\varphi^{(1)}(x,\mu_k)e^{i\mu_k^2 y}}{(\dot{F}_1)(\mu_k)}\end{aligned}$$

and then

$$\frac{2i}{\pi}\int\limits_0^\infty \frac{\mu^3\varphi^{(1)}(x,\mu)}{F_1(\mu)}e^{i\mu^2 y}d\mu = \sum_{k=1}^n m_k f^{(1)}(x,\mu_k)e^{i\mu_k^2 y}$$

because of the fact that $\varphi^{(1)}(x,\mu_k)$ and $f^{(1)}(x,\mu_k)$ are linearly dependent with $\varphi^{(1)}(x,\mu_k)=\frac{f^{(1)}(x,\mu_k)}{f^{(1)}(0,\mu_k)}$ since $F_1(\mu_k)=0$. If we consider the last equation and (21) together, we get

$$\sum_{k=1}^n m_k\left[f^{(1)}(x,\mu_k)e^{i\mu_k^2 y}\right] = e^{i\alpha(x)}F_{S_1}(x+y)+K(x,y)+\int\limits_x^\infty K(x,t)F_{S_1}(t+y)dt,$$

and from (20), we obtain the main equation (19). □

Clearly, to form the main equation, it suffices to know the function $G(x)$. On the other hand, to find the function $G(x)$, it suffices to know only the set of values

$$\{S_1(\mu),(0<\mu<\infty);\mu_k;m_k,(k=1,2,...,n)\}.$$

which is called the scattering data for L_μ. Given the scattering data, we can use formula (20) to construct the function $G(x)$ and write out the main equation (19) for the unknown function $K(x,y)$. Solving this equation, we find the Kernel $K(x,y)$ of the transformation operator, and hence the potential

$$q(x)=-\frac{1}{2}\frac{d}{dx}K(x,x).$$

Theorem 3. *The equation (19) has a unique solution* $K(x,y) \in L_1[x,\infty)$.

Proof. We need to show that the homogeneous equation

$$\psi(y) + \int_x^\infty \psi(t)G(t+y)dt = 0 \tag{22}$$

has only the zero solution in $L_2(0,\infty)$.

We assume that (22) has a nonzero solution. Multiplying $\psi(y)$ both sides of (22) and integrating,

$$\int_x^\infty \psi^2(y)dy + \int_x^\infty \psi(y)\int_x^\infty \psi(t)G(t+y)dtdy = 0.$$

After that,

$$\begin{aligned} 0 &= \int_x^\infty \psi^2(y)dy + \int_x^\infty \psi(y)\int_x^\infty \psi(t)F_S(t+y)dtdy \\ &\quad + \int_x^\infty \psi(y)\int_x^\infty \psi(t)\sum_{k=1}^n m_k e^{i\mu_k^2(t+y)}dtdy \end{aligned}$$

from (20). Using (18) in last equation,

$$\begin{aligned} 0 &= \int_x^\infty \psi^2(y)dy + \int_x^\infty \psi(y)\int_x^\infty \psi(t)\sum_{k=1}^n m_k e^{i\mu_k^2(t+y)}dtdy \\ &\quad + \int_x^\infty \psi(y)\int_x^\infty \psi(t)\left[\frac{1}{\pi}\int_0^\infty \mu\left[1-S_1(\mu)\right]e^{i\mu^2(t+y)}d\mu\right]dtdy. \end{aligned} \tag{23}$$

In (23) interchanging integrals and using the uniform convergence of

$$\sum_{k=1}^n m_k e^{i\mu_k^2(t+y)}\psi(t),$$

(23) can be integrated by terms. So we obtain following equation

$$\begin{aligned} 0 &= \int_x^\infty \psi^2(y)dy + \sum_{k=1}^n m_k\left[\int_x^\infty \psi(t)e^{i\mu_k^2 t}dt\right]^2 \\ &\quad + \frac{1}{\pi}\int_0^\infty \mu\left[1-S_1(\mu)\right]\left[\int_x^\infty \psi(t)e^{i\mu^2 t}dt\right]^2 d\mu. \end{aligned} \tag{24}$$

On the other hand, by using Parseval equation of Fourier transformation in (24),

$$0 = \frac{1}{\pi}\int_0^\infty \mu \left|\Phi(\mu)\right|^2 d\mu + \sum_{k=1}^{n} m_k \left[\Phi(\mu_k)\right]^2 + \frac{1}{\pi}\int_0^\infty \mu \left[1 - S_1(\mu)\right] \left[\Phi(\mu)\right]^2 d\mu, \tag{25}$$

where Parseval equation of

$$\Phi(\mu) = \int_x^\infty \psi(t) e^{i\mu^2 t} dt$$

is

$$\int_x^\infty \psi^2(y) dy = \frac{1}{\pi}\int_0^\infty \mu \left|\Phi(\mu)\right|^2 d\mu.$$

Since

$$\arg\mu = 0,\ \arg(m_k) = \eta_1(\mu),\ \arg\left[\Phi(\mu)\right] = \eta_2(\mu) \text{ and } \arg\left[1 - S_1(\mu)\right] = \eta_3(\mu),$$

(25) rewrite as polar form

$$0 = \sum_{k=1}^{n} |m_k| \left|\Phi(\mu_k)\right|^2 e^{i[\eta_1(\mu_k)+2\eta_2(\mu_k)]} + \frac{1}{\pi}\int_{-\infty}^{\infty} |\mu| \left|\Phi(\mu)\right|^2 \left\{1 + |1 - S_1(\mu)|\, e^{i[2\eta_2(\mu)+\eta_3(\mu)]}\right\} d\mu. \tag{26}$$

Real part of (26) is

$$0 = \sum_{k=1}^{n} |m_k| \left|\Phi(\lambda_k)\right|^2 \cos\left(\eta_1(\mu_k) + 2\eta_2(\mu_k)\right) + \frac{1}{\pi}\int_{-\infty}^{\infty} |\mu| \left|\Phi(\mu)\right|^2 \left\{1 + |1 - S_1(\mu)| \cos\left[2\eta_2(\mu) + \eta_3(\mu)\right]\right\} d\mu.$$

Therefore, the last equation is equal to zero only situation is

$$\Phi(\mu) = 0 \text{ and so } \psi(t) = 0.$$

But this is a contradiction. So, the equation (19) has a unique solution for finite x. □

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FIXED-POINT THEOREMS IN EXTENDED FUZZY METRIC SPACES VIA α-ϕ-$\mathcal{M}^0$ AND β-ψ-$\mathcal{M}^0$ FUZZY CONTRACTIVE MAPPINGS

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Abstract. In this article we would like to present a new type of fuzzy contractive mappings which are called $\alpha - \phi - \mathcal{M}^0$ fuzzy contractive and $\beta - \psi - \mathcal{M}^0$ fuzzy contractive, and then we demonstrate two theorems which ensure the existence of a fixed point for these two types of mappings. And so we combine and generalize some existing notions in the literature ([5], [7]). Proved these theorems in the extended fuzzy metric spaces are in the more general version than the existing in the literature ones.

1. Introduction

The attention of fuzzy concept has been growing from the presented by Zadeh [20] in 1965. The concept of fuzzy was used a lot of fields such as mathematical analysis and general topology with many applications in economy and engineering. Recently, it is a paramount development that defining the concept of contractive mapping in fuzzy metric spaces. After the remarkable Banach [1] contraction principle, a large amount of mathematicians studied some contractive mappings to proof a fixed point exists. Afterwards, studies gained popularity with the notion of fuzzy metric space defined by Kramosil and Michalek [13], and then George and Veeramani [4] modified the concept of fuzzy metric space.

Contractivity's role in the fixed point theory is very important. There are a lot of studies in the literature regarding different versions contractive mappings in the different spaces ([2], [3], [5], [6], [8]- [17], [19]). Samet et al. [17] put forward new notions of contractive mapping and used these mappings to verify some fixed point theorems in metric spaces. Based on the same perspective, D. Gopal and C.

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Vetro [5] give some contractive mappings, which can be accepted generalizations of Samet et al. [17].

In this paper, we define new notions which are generalized versions of fuzzy contractive mappings introduced by D. Gopal and C.Vetro [5]. We study these contractions in extended fuzzy metric spaces introduced by V. Gregori et al. [7].

The new contractions are called $\alpha - \phi - \mathcal{M}^0$ fuzzy contractive mapping and $\beta - \psi - \mathcal{M}^0$ fuzzy contractive mapping. Moreover, we have proved some fixed point theorems with these mappings in this new space and so we got a generalized versions.

2. Preliminaries

Now in this section, we recall some definitions and results that will be used in the sequel.

Definition 1. *[18] A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is called a continuous triangular norm (t-norm) if the following conditions hold:*

T1 $*$ *is associative and commutative*;
T2 $*$ *is continuous*;
T3 $a * 1 = a$, *for all* $a \in [0,1]$;
T4 $a * b < c * d$, *whenever* $a < c$ *and* $b < d$, *for all* $a, b, c, d \in [0,1]$.

Kramosil and Michalek [13] generalized probabilistic metric space via concept of fuzzy metric. After then George and Veeramani [4] made slight modification in this fuzzy metric concept.

Definition 2. *[4], A fuzzy metric space is a triple* $(\mathcal{X}, \mathcal{M}, *)$*, where* $\mathcal{X}$ *is a non-empty set,* $*$ *is a continuous t-norm and* $\mathcal{M}$ *is a fuzzy set on* $\mathcal{X}^2 \times (0,\infty)$*, satisfying for all* $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$ *and for all* $\mathfrak{t}, \mathfrak{s} > 0$*, the following properties:*

(ÇV1) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) > 0$;
(ÇV2) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = 1$ *if and only if* $\mathfrak{x} = \mathfrak{y}$;
(ÇV3) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \mathcal{M}(\mathfrak{y}, \mathfrak{x}, \mathfrak{t})$;
(ÇV4) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) * \mathcal{M}(\mathfrak{y}, \mathfrak{z}, \mathfrak{s}) \leq \mathcal{M}(\mathfrak{x}, \mathfrak{z}, \mathfrak{t} + \mathfrak{s})$;
(ÇV5) $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \cdot) : (0,\infty) \to [0,1]$ *is continuous,*

$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})$ *could be considered as the degree of closeness between* x *and* y *with regard to* t*. In the above definition, if we replace (ÇV4) by (ÇV$_4^*$),* $\forall\ \mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$ *and* $\mathfrak{t}, \mathfrak{s} > 0$;

$$(ÇV_4^*) : \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) * \mathcal{M}(\mathfrak{y}, \mathfrak{z}, \mathfrak{s}) \leq \mathcal{M}(\mathfrak{x}, \mathfrak{z}, \max\{\mathfrak{t}, \mathfrak{s}\})$$

then the triple $(\mathcal{X}, \mathcal{M}, *)$ *is said to be non-Archimedean fuzzy metric space [14].*

Definition 3. *[8] A stationary fuzzy metric space is a triple* $(\mathcal{X}, \mathcal{M}, *)$ *such that* $\mathcal{X}$ *is a non-empty set,* $*$ *is a continuous t-norm and* $\mathcal{M}$ *is a fuzzy set on* $\mathcal{X}^2$ *satisfying the following conditions, for all* $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{X}$;

(S_1) $\mathcal{M}(\mathfrak{x},\mathfrak{y}) > 0$;
(S_2) $\mathcal{M}(\mathfrak{x},\mathfrak{y}) = 1$ *if and only if* $\mathfrak{x} = \mathfrak{y}$;
(S_3) $\mathcal{M}(\mathfrak{x},\mathfrak{y}) = \mathcal{M}(\mathfrak{y},\mathfrak{x})$;
(S_4) $\mathcal{M}(\mathfrak{x},\mathfrak{y}) * \mathcal{M}(\mathfrak{y},\mathfrak{z}) \leq \mathcal{M}(\mathfrak{x},\mathfrak{z})$.

In other words, a fuzzy metric space $(\mathcal{X},\mathcal{M},*)$ is said to be stationary if $\mathcal{M}$ does not depend on $\mathfrak{t}$.

A sequence $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ in a stationary fuzzy metric space $(\mathcal{X},\mathcal{M})$ is said to be Cauchy if $\lim\limits_{i,j\to\infty}\mathcal{M}(\mathfrak{x}_i,\mathfrak{x}_j) = 1$; a sequence $(\mathfrak{x}_i)_{i\in\mathbb{N}}$ in $\mathcal{X}$ converges to $\mathfrak{x}$ if $\lim\limits_{i\to\infty}\mathcal{M}(\mathfrak{x}_i,\mathfrak{x}) = 1$ [8].

Now we recall a kind of generalized fuzzy metric space introduced by V. Gregori, J-J Minana and D. Miravet [7]. They study those fuzzy metrics $\mathcal{M}$ on $\mathcal{X}$, in the George and Veeramani's sense, such that $\wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$.

Definition 4. *[7] The term $(\mathcal{X},\mathcal{M}^0,*)$ is called an extended fuzzy metric space if $\mathcal{X}$ is a (non-empty) set, $*$ is a continuous t-norm and $\mathcal{M}^0$ is a fuzzy set on $\mathcal{X}^2\times [0,\infty)$ satisfying the following conditions, for each $\mathfrak{x},\mathfrak{y},\mathfrak{z}\in\mathcal{X}$ and $\mathfrak{t},\mathfrak{s}\geq 0$;*
(EFM$_1$) $\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$;
(EFM$_2$) $\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = 1$ *if and only if* $\mathfrak{x} = \mathfrak{y}$;
(EFM$_3$) $\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = \mathcal{M}^0(\mathfrak{y},\mathfrak{x},\mathfrak{t})$;
(EFM$_4$) $\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) * \mathcal{M}^0(\mathfrak{y},\mathfrak{z},s) \leq \mathcal{M}^0(\mathfrak{x},\mathfrak{z},\mathfrak{t}+\mathfrak{s})$;
(EFM$_5$) $\mathcal{M}^0_{\mathfrak{x},\mathfrak{y}} : [0,\infty) \to (0,1]$ *is continuous, where* $\mathcal{M}^0_{\mathfrak{x},\mathfrak{y}}(\mathfrak{t}) = \mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t})$.

Theorem 1. *[7] Let M be a fuzzy set on $\mathcal{X}^2\times(0,\infty)$, and denote by $\mathcal{M}^0$ its extension to , $\mathcal{X}^2\times [0,\infty)$ given by*

$$\mathcal{M}^0(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = \mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) \text{ for all } \mathfrak{x},\mathfrak{y},\in\mathcal{X},\ \mathfrak{t}>0 \quad \text{and}$$

$$\mathcal{M}^0(\mathfrak{x},\mathfrak{y},0) = \wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}).$$

Then, $(\mathcal{X},\mathcal{M}^0,*)$ is an extended fuzzy metric space if and only if $(\mathcal{X}.\mathcal{M},*)$ is a fuzzy metric space satisfying for each $\mathfrak{x},\mathfrak{y}\in\mathcal{X}$ the condition $\wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$.

Proposition 1. *[7] Let $(\mathcal{X},\mathcal{M},*)$ be a fuzzy metric space. Define*

$$N_{\mathcal{M}}(\mathfrak{x},\mathfrak{y}) = \wedge_{t>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}).$$

Then, $(N_{\mathcal{M}},*)$ is a stationary fuzzy metric on $\mathcal{X}$ if and only if $\wedge_{t>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$ for all $\mathfrak{x},\mathfrak{y}\in\mathcal{X}$.

It is clear that

$$\mathcal{M}^0(\mathfrak{x},\mathfrak{y},0) = \wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) = N_{\mathcal{M}}(\mathfrak{x},\mathfrak{y}). \tag{1}$$

Definition 5. *[7] Let $(\mathcal{X},\mathcal{M},*)$ be a fuzzy metric space. $\mathcal{M}$ is called extendable if for each $\mathfrak{x},\mathfrak{y}\in\mathcal{X}$ the condition $\wedge_{\mathfrak{t}>0}\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}) > 0$ is satisfied. In such a case, we will say that $\mathcal{M}^0$ is the (fuzzy metric) extension of $\mathcal{M}$, and that $\mathcal{M}$ is the restriction of $\mathcal{M}^0$.*

Proposition 2. *[7] Let $(\mathcal{X},\mathcal{M}^0,*)$ is complete if and only if $(\mathcal{X},N_{\mathcal{M}},*)$ is complete.*

Samet et al. [17] introduced a new concept of $\alpha-\psi-$ *contractive* and $\alpha-$ *admissible* mappings in metric spaces. D. Gopal and C. Vetro [5] inspired from them [17] and introduced the notions of $\alpha-\phi-$ *fuzzy contractive* mapping and $\beta-\psi-$ *fuzzy contractive* mapping. We recall the notions as follows.

Remark 1. *[5] Denote by Φ the family of all right continuous functions $\phi:[0,\infty)\longrightarrow[0,\infty)$, with $\phi(r)<r$ for all $r>0$. Note that for every function $\phi\in\Phi$, $\lim_{n\to\infty}\phi^n(r)=0$ for each $r>0$, where $\phi^n(r)$ denotes the $n-th$ iterate of ϕ.*

Definition 6. *[5] Let $(\mathcal{X},\mathcal{M},*)$ be a fuzzy metric space. It is said that $\Im:\mathcal{X}\longrightarrow\mathcal{X}$ is an $\alpha-\phi-$ fuzzy contractive mapping if there exist two functions $\alpha:\mathcal{X}^2\times(0,\infty)\longrightarrow[0,\infty)$ and $\phi\in\Phi$ such that*

$$\alpha(\mathfrak{x},\mathfrak{y},\mathfrak{t})\left(\frac{1}{\mathcal{M}(\Im\mathfrak{x},\Im\mathfrak{y},\mathfrak{t})}-1\right)\leq\phi\left(\frac{1}{\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t})}-1\right)$$

for all $\mathfrak{x},\mathfrak{y}\in\mathcal{X}$ and $\mathfrak{t}>0$.

Definition 7. *[5] Let $(\mathcal{X},\mathcal{M},*)$ be a fuzzy metric space. It is said that $\Im:\mathcal{X}\longrightarrow\mathcal{X}$ is $\alpha-$ admissible if there exist a function $\alpha:\mathcal{X}^2\times(0,\infty)\longrightarrow[0,\infty)$ such that,*

$$\alpha(\mathfrak{x},\mathfrak{y},\mathfrak{t})\geq 1\Longrightarrow\alpha(\Im\mathfrak{x},\Im\mathfrak{y},\mathfrak{t})\geq 1$$

for all $\mathfrak{x},\mathfrak{y}\in\mathcal{X}$ and $\mathfrak{t}>0$.

Remark 2. *[5] Let Ψ be the class of all functions $\psi:[0,1]\longrightarrow[0,1]$ such that ψ is non-decreasing and left continuous and $\psi(r)>r$ for all $r\in(0,1)$. If $\psi\in\Psi$, then $\psi(1)=1$ and $\lim_{n\to\infty}\psi^n(r)=1$ for all $r\in(0,1]$.*

Definition 8. *[5] Let $(\mathcal{X},\mathcal{M},*)$ be a fuzzy metric space. It is said that $\Im:\mathcal{X}\longrightarrow\mathcal{X}$ is an $\beta-\psi-$ fuzzy contractive mapping if there exist two functions $\beta:\mathcal{X}^2\times(0,\infty)\longrightarrow(0,\infty)$ and $\psi\in\Psi$ such that,*

$$\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t})>0\Rightarrow\beta(\mathfrak{x},\mathfrak{y},\mathfrak{t})\mathcal{M}(\Im\mathfrak{x},\Im\mathfrak{y},\mathfrak{t})\geq\psi(\mathcal{M}(\mathfrak{x},\mathfrak{y},\mathfrak{t}))$$

for all $\mathfrak{x},\mathfrak{y}\in\mathcal{X}$ with $\mathfrak{x}\neq\mathfrak{y}$ and for all $\mathfrak{t}>0$.

Definition 9. *[5] Let $(\mathcal{X},\mathcal{M},*)$ be a fuzzy metric space. It is said that $\Im:\mathcal{X}\longrightarrow\mathcal{X}$ is a $\beta-$ admissible if there exist a function $\beta:\mathcal{X}^2\times(0,\infty)\longrightarrow(0,\infty)$ such that,*

$$\beta(\mathfrak{x},\mathfrak{y},\mathfrak{t})\leq 1\Longrightarrow\beta(\Im\mathfrak{x},\Im\mathfrak{y},\mathfrak{t})\leq 1 \text{ for all } \mathfrak{x},\mathfrak{y},\in\mathcal{X} \text{ and } \mathfrak{t}>0.$$

3. Main Result

3.1. **$\alpha-\phi-\mathcal{M}^0-$ fuzzy contractive mappings.** We are ready to introduce new definitions of $\alpha-\phi-\mathcal{M}^0-$ *fuzzy contractive* and $\alpha-\mathcal{M}^0-$ *admissible*. We would like to inform you that use these mappings in the new fuzzy metric space (introduced in [7]). Then, we prove the theorem (proved in [5]) but in the new fuzzy metric spaces. And so, we obtain new results that are generalizations of those in fuzzy metric spaces.

Definition 10. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable fuzzy metric space. $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is called $\alpha - \phi - \mathcal{M}^0 - fuzzy$ contractive mapping if*

$$\alpha(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})(\frac{1}{\mathcal{M}(\Im\mathfrak{x}, \Im\mathfrak{y}, \mathfrak{t})} - 1) \leq \phi(\frac{1}{\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})} - 1) \tag{2}$$

is ensured $\forall$ $\mathfrak{x}, \mathfrak{y}, \in \mathcal{X}$ and $\mathfrak{t} \geq 0$. Especially, $\Im$ is called $\alpha - \phi - 0 - fuzzy$ contractive if Equation (2) is ensured for $\mathfrak{t} = 0$.

Definition 11. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable fuzzy metric space. $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is called $\alpha - \mathcal{M}^0 -$ admissible mapping if*

$$\alpha(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) \geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}, \Im\mathfrak{y}, \mathfrak{t}) \geq 1 \tag{3}$$

is ensured $\forall$ $\mathfrak{x}, \mathfrak{y}, \in \mathcal{X}$ and $\mathfrak{t} \geq 0$. Especially, $\Im$ is called $\alpha - 0 -$admissible if Equation (3) is ensured for $\mathfrak{t} = 0$.

Theorem 2. *Let $(\mathcal{X}, \mathcal{M}, *)$ be a complete extendable fuzzy metric space and a mapping $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ be an $\alpha - \phi - \mathcal{M}^0 - fuzzy$ contractive ensuring the provisions given below:*
(i) $\Im$ is $\alpha - \mathcal{M}^0 -$ admissible;
(ii) $\exists$ $\mathfrak{x}_0 \in \mathcal{X}$ such that $\alpha(\mathfrak{x}_0, \Im\mathfrak{x}_0, \mathfrak{t}) \geq 1$, $\forall$ $\mathfrak{t} \geq 0$;
(iii) $\Im$ is continuous;
Then, $\Im$ has a fixed point.

Proof. We will examine the proof in two cases.
Case 1. $\mathfrak{t} > 0$;
In this case, since $\mathcal{M}^0(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})$ $\forall$ $\mathfrak{x}, \mathfrak{y} \in \mathcal{X}$, it is same situation in fuzzy metric spaces and introduced in the proof of the Theorem 3.5. [5].
Case 2. $\mathfrak{t} = 0$;
Let $\mathfrak{x}_0 \in \mathcal{X}$ such that $\alpha(\mathfrak{x}_0, \Im\mathfrak{x}_0, 0) \geq 1$.
Define the squence $\{\mathfrak{x}_n\}$ in $\mathcal{X}$ with $\mathfrak{x}_{n+1} = \Im\mathfrak{x}_n$, $\forall$ $n \in \mathbb{N}$.
Provided that $\mathfrak{x}_{n+1} = \mathfrak{x}_n$ for some $n \in \mathbb{N}$, then $\mathfrak{x}^* = \mathfrak{x}_n$ is a fixed point of $\Im$.
Presume that $\mathfrak{x}_n \neq \mathfrak{x}_{n+1}$, $\forall$ $n \in \mathbb{N}$.
From (ii),

$$\alpha(\mathfrak{x}_0, \mathfrak{x}_1, 0) = \alpha(\mathfrak{x}_0, \Im\mathfrak{x}_0, 0) \geq 1$$

and using (i), we have

$$\alpha(\mathfrak{x}_0, \Im\mathfrak{x}_0, 0) \geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}_0, \Im\mathfrak{x}_1, 0) \geq 1$$

By induction,

$$\begin{aligned}
\alpha(\Im\mathfrak{x}_0, \Im\mathfrak{x}_1, 0) &\geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}_1, \Im\mathfrak{x}_2, 0) \geq 1 \\
\alpha(\Im\mathfrak{x}_1, \Im\mathfrak{x}_2, 0) &\geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}_2, \Im\mathfrak{x}_3, 0) \geq 1 \\
&\ldots \\
\alpha(\Im\mathfrak{x}_{n-3}, \Im\mathfrak{x}_{n-2}, 0) &\geq 1 \Longrightarrow \alpha(\Im\mathfrak{x}_{n-2}, \Im\mathfrak{x}_{n-1}, 0) \geq 1
\end{aligned}$$

and so we get,

$$\alpha(\Im\mathfrak{x}_{n-2}, \Im\mathfrak{x}_{n-1}, 0) = \alpha(\mathfrak{x}_{n-1}, \mathfrak{x}_n, 0) \geq 1, \forall\ n \in \mathbb{N}. \tag{4}$$

Using (1), implementing (2) with $\mathfrak{x} = \mathfrak{x}_{n-1}$, $\mathfrak{y} = \mathfrak{x}_n$, $\mathfrak{t} = 0$ and using (4) respectively we obtain;

$$\begin{aligned} \frac{1}{\mathcal{M}^0(\mathfrak{x}_n, \mathfrak{x}_{n+1}, 0)} - 1 &= \frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1}, \Im\mathfrak{x}_n)} - 1 \\ &\leq \alpha(\mathfrak{x}_{n-1}, \mathfrak{x}_n, 0)(\frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1}, \Im\mathfrak{x}_n)} - 1) \\ &\leq \phi(\frac{1}{N_{\mathcal{M}}(\mathfrak{x}_{n-1}, \mathfrak{x}_n)} - 1) \\ &= \phi(\frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-2}, \Im\mathfrak{x}_{n-1})} - 1) \end{aligned}$$

This implies that,

$$\frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1}, \Im\mathfrak{x}_n)} - 1 \leq \phi^n(\frac{1}{N_{\mathcal{M}}(\mathfrak{x}_0, \mathfrak{x}_1)} - 1)$$

as $n \to \infty$

$$\lim_{n\to\infty}(\frac{1}{N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1}, \Im\mathfrak{x}_n)} - 1) \leq \lim_{n\to\infty}\phi^n(\frac{1}{N_{\mathcal{M}}(\mathfrak{x}_0, \mathfrak{x}_1)} - 1)$$

Since, as $n \to \infty$ and $\phi^n(r) \to 0$,

$$\lim_{n\to\infty}(\frac{1}{N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}_{n+1})} - 1) = 0$$

and so, we obtain that

$$\lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}_{n+1}) = 1.$$

which implies that for $n < m$ and using (1) with $\mathfrak{x} = \mathfrak{x}_n$, $\mathfrak{y} = \mathfrak{x}_m$, $\mathfrak{t} = 0$;

$$\mathcal{M}^0(\mathfrak{x}_n, \mathfrak{x}_m, 0) = \wedge_{t>0}\mathcal{M}(\mathfrak{x}_n, \mathfrak{x}_m, t) = N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}_m)$$

Using Definition 3,

$$N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}_m) \geq N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}_{n+1}) * N_{\mathcal{M}}(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}) * ... * N_{\mathcal{M}}(\mathfrak{x}_{m-1}, \mathfrak{x}_m)$$

and as $n \to \infty$,

$$\begin{aligned} \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}_m) &\geq \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}_{n+1}) * \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}) * ... * \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{m-1}, \mathfrak{x}_m) \\ &\geq 1 * 1 * ... * 1 \\ &\geq 1 \end{aligned}$$

We obtain,

$$\lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}_m) = 1$$

And so, we solve an important point of the proof that $\{\mathfrak{x}_n\}$ is a Cauchy squence. Since $\mathcal{X}$ is complete,

$$\exists\ \mathfrak{x}^* \in \mathcal{X}\ \ :\ \text{as } n \to \infty \text{ and } \mathfrak{x}_n \to \mathfrak{x}^*$$

Since $\Im$ is continuous, as $\mathfrak{x}_n \to \mathfrak{x}^*$we have $\Im\mathfrak{x}_n \to \Im\mathfrak{x}^*$ and using (1),

$$\mathcal{M}^0(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*, 0) = \wedge_{t>0}\mathcal{M}(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*, t) = N_{\mathcal{M}}(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*),\ \forall\ \mathfrak{x}_n \in \mathcal{X}.$$

And so we obtain,

$$\lim_{n\to\infty} N_{\mathcal{M}}(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*) = 1.$$

By the uniqueness of the limit, we get $\mathfrak{x}^* = \Im\mathfrak{x}^*$, that is, $\mathfrak{x}^*$ is a fixed point of $\Im$. □

3.2. **$\beta - \psi - \mathcal{M}^0-$ fuzzy contractive mappings.** We are ready to introduce new definitions of $\beta - \psi - \mathcal{M}^0 - fuzzy\ contractive$ and $\beta - \mathcal{M}^0-$ *admissible*. We would like to inform you that we use these mappings in the new fuzzy metric space (introduced in [7]). Then, we prove the theorem (proved in [5]) but in the new fuzzy metric spaces. And so, we obtain new results that are generalizations of those in fuzzy metric spaces.

Definition 12. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable fuzzy metric space. $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is called $\beta - \psi - \mathcal{M}^0 - fuzzy\ contractive$ mapping if*

$$\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) > 0 \Rightarrow \beta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}, \Im\mathfrak{y}, \mathfrak{t}) \geq \psi(\mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t})) \tag{5}$$

is ensured $\forall\ \mathfrak{x}, \mathfrak{y}, \in \mathcal{X}$ and $\mathfrak{t} \geq 0$. Especially, $\Im$ is called $\beta - \psi - 0 - fuzzy\ contractive$ if Equation (5) is ensured for $\mathfrak{t} = 0$.

Definition 13. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable fuzzy metric space. $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ is called $\beta - \mathcal{M}^0 - admissible$ mapping if*

$$\beta(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) \leq 1 \Longrightarrow \beta(\Im\mathfrak{x}, \Im\mathfrak{y}, \mathfrak{t}) \leq 1 \tag{6}$$

is ensured $\forall\ \mathfrak{x}, \mathfrak{y}, \in \mathcal{X}$ and $\mathfrak{t} \geq 0$. Especially, $\Im$ is called $\beta - 0 - admissible$ if Equation (6) is ensured for $\mathfrak{t} = 0$

By adding an additional condition, we prove a fixed point theorem introduced in [5] in extendable fuzzy metric space using these new mappings. This is a new context that using the new mappings in the extendable fuzzy metric space.

Theorem 3. *Let $(\mathcal{X}, \mathcal{M}, *)$ be an extendable complete non-Archimedean fuzzy metric space and a mapping $\Im : \mathcal{X} \longrightarrow \mathcal{X}$ be a $\beta - \psi - \mathcal{M}^0 - fuzzy\ contractive$ ensuring the provisions given below:*

(i) $\Im$ is $\beta - \mathcal{M}^0 - admissible$;

(ii) $\exists\ \mathfrak{x}_0 \in \mathcal{X}$ such that $\beta(\mathfrak{x}_0, \Im\mathfrak{x}_0, \mathfrak{t}) \leq 1\ \forall\ \mathfrak{t} \geq 0$;

(iii) for each sequence $\{\mathfrak{x}_n\}$ in $\mathcal{X}$ such that $\beta(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{t}) \leq 1\ \forall\ n \in \mathbb{N}$ and $\mathfrak{t} \geq 0$, $\exists\ k_0 \in N$ such that $\beta(\mathfrak{x}_{m+1}, \mathfrak{x}_{n+1}, \mathfrak{t}) \leq 1\ \forall\ m, n \in \mathbb{N}$ with $m > n \geq k_0$ and $\forall\ \mathfrak{t} \geq 0$;

(iv) if $\{\mathfrak{x}_n\}$ *is a sequence in* $\mathcal{X}$ *such that* $\beta(\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{t}) \leq 1 \ \forall \ n \in \mathbb{N}$ *and* $\mathfrak{t} \geq 0$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\beta(\mathfrak{x}_n, \mathfrak{x}, \mathfrak{t}) \leq 1 \ \forall \ n \in \mathbb{N}$ *and* $\forall \ \mathfrak{t} \geq 0$;

(v) $\forall \ \mathfrak{x}, \mathfrak{y} \in \mathcal{X}$ *and* $\forall \ \mathfrak{t} \geq 0, \exists \ \mathfrak{z} \in \mathcal{X}$ *such that* $\beta(\mathfrak{x}, \mathfrak{z}, \mathfrak{t}) \leq 1$ *and* $\beta(\mathfrak{y}, \mathfrak{z}, \mathfrak{t}) \leq 1$;

Then, $\Im$ *has a unique fixed point.*

Proof. We will examine the proof in two cases.

Case 1. $\mathfrak{t} > 0$;

In this case, since $\mathcal{M}^0(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}) = \mathcal{M}(\mathfrak{x}, \mathfrak{y}, \mathfrak{t}), \ \forall \, \mathfrak{x}, \mathfrak{y} \in \mathcal{X}$; it is same situation in fuzzy metric spaces and introduced in the proof of the Theorem 4.4 [5]. It is obtained that $\Im\mathfrak{x}^* = \mathfrak{x}^*$ in the [5].

Now we will show that uniqueness of the fixed point.

Presume that $\Im$ have two different fixed points; $\mathfrak{x}^*$and $\mathfrak{y}^*$.

Provided that $\beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) \leq 1$,then

$$\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) \geq \beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}^*, \Im\mathfrak{y}^*, \mathfrak{t}).$$

Since $\Im$ is $\beta - \psi - \mathcal{M}^0-$ fuzzy contractive, we have

$$\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) \geq \beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}^*, \Im\mathfrak{y}^*, \mathfrak{t}) \geq \psi(\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})).$$

Also, since $\psi(r) > r$, we obtain that

$$\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) \geq \beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}^*, \Im\mathfrak{y}^*, \mathfrak{t}) \geq \psi(\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})) > \mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}).$$

And so, we get

$$\mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) > \mathcal{M}(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t})$$

It is a contradiction.

That is, $\mathfrak{x}^*$and $\mathfrak{y}^*$ are not different points; $\mathfrak{x}^* = \mathfrak{y}^*$.

Presume that $\beta(\mathfrak{x}^*, \mathfrak{y}^*, \mathfrak{t}) > 1$,then from (v),

$$\exists \ \mathfrak{z} \in X : \ \beta(\mathfrak{x}^*, \mathfrak{z}, \mathfrak{t}) \leq 1 \text{and } \beta(\mathfrak{y}^*, \mathfrak{z}, \mathfrak{t}) \leq 1.$$

From (i), we obtain,

$$\begin{aligned} \beta(\mathfrak{x}^*, \mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}^*, \Im\mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{x}^*, \Im\mathfrak{z}, \mathfrak{t}) \leq 1 \\ \beta(\mathfrak{x}^*, \Im\mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}^*, \Im^2\mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{x}^*, \Im^2\mathfrak{z}, \mathfrak{t}) \leq 1 \\ &\quad ... \\ \beta(\mathfrak{x}^*, \Im^{n-1}\mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) \leq 1 \end{aligned}$$

and so we get,

$$\beta(\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) \leq 1, \forall \ n \in \mathbb{N} \text{ and } \forall \ \mathfrak{t} > 0. \tag{7}$$

Since $\Im$ is $\beta - \psi - \mathcal{M}^0-$ *fuzzy contractive*, using (7), we get,

$$\begin{aligned} \mathcal{M}^0(\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) &= \mathcal{M}(\mathfrak{x}^*, \Im^n\mathfrak{z}, \mathfrak{t}) = \mathcal{M}(\Im\mathfrak{x}^*, \Im(\Im^{n-1}\mathfrak{z}), \mathfrak{t}) \\ &\geq \beta(\mathfrak{x}^*, \Im^{n-1}\mathfrak{z}, \mathfrak{t})\mathcal{M}(\Im\mathfrak{x}^*, \Im(\Im^{n-1}\mathfrak{z}), \mathfrak{t}) \\ &\geq \psi(\mathcal{M}(\mathfrak{x}^*, \Im^{n-1}\mathfrak{z}, \mathfrak{t})) \\ &= \psi(\mathcal{M}(\Im\mathfrak{x}^*, \Im(\Im^{n-2}\mathfrak{z}), \mathfrak{t})) \end{aligned}$$

And by induction we have,

$$\mathcal{M}^0(\mathfrak{x}^*, \Im^n \mathfrak{z}, \mathfrak{t}) \geq \psi^n(\mathcal{M}^0(\mathfrak{x}^*, \mathfrak{z}, \mathfrak{t})), \ \forall \ n \in \mathbb{N}.$$

as $n \to \infty$

$$\lim_{n\to\infty} \mathcal{M}^0(\mathfrak{x}^*, \Im^n \mathfrak{z}, \mathfrak{t}) \geq \lim_{n\to\infty} \psi^n(\mathcal{M}^0(\mathfrak{x}^*, \mathfrak{z}, \mathfrak{t}))$$

Since $\psi^n(r) \to 1$,

$$\lim_{n\to\infty} \mathcal{M}^0(\mathfrak{x}^*, \Im^n \mathfrak{z}, \mathfrak{t}) = 1 \Rightarrow \Im^n \mathfrak{z} \to \mathfrak{x}^* \tag{8}$$

and by similar way, we get

$$\begin{aligned}
\beta(\mathfrak{y}^*, \mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im \mathfrak{y}^*, \Im \mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{y}^*, \Im \mathfrak{z}, \mathfrak{t}) \leq 1 \\
\beta(\mathfrak{y}^*, \Im \mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im \mathfrak{y}^*, \Im^2 \mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{y}^*, \Im^2 \mathfrak{z}, \mathfrak{t}) \leq 1 \\
& \ldots \\
\beta(\mathfrak{y}^*, \Im^{n-2} \mathfrak{z}, \mathfrak{t}) &\leq 1 \Rightarrow \beta(\Im \mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}) = \beta(\mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}) \leq 1
\end{aligned}$$

$$\beta(\mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}) \leq 1, \forall \ n \in \mathbb{N} \text{ and } \forall \ \mathfrak{t} > 0. \tag{9}$$

Since $\Im$ is $\beta - \psi - \mathcal{M}^0 -$ *fuzzy contractive*, using (9), we get,

$$\begin{aligned}
\mathcal{M}^0(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) &= \mathcal{M}(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) = \mathcal{M}(\Im \mathfrak{y}^*, \Im(\Im^{n-1} \mathfrak{z}), \mathfrak{t}) \\
&\geq \beta(\mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}) \mathcal{M}(\Im \mathfrak{y}^*, \Im(\Im^{n-1} \mathfrak{z}), \mathfrak{t}) \\
&\geq \psi(\mathcal{M}(\mathfrak{y}^*, \Im^{n-1} \mathfrak{z}, \mathfrak{t}))
\end{aligned}$$

And so, by induction we have,

$$\mathcal{M}^0(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) \geq \psi^n(\mathcal{M}^0(\mathfrak{y}^*, \mathfrak{z}, \mathfrak{t})), \forall \ n \in \mathbb{N}.$$

as $n \to \infty$

$$\lim_{n\to\infty} \mathcal{M}^0(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) \geq \lim_{n\to\infty} \psi^n(\mathcal{M}^0(\mathfrak{y}^*, \mathfrak{z}, \mathfrak{t}))$$

Since $\psi^n(r) \to 1$,

$$\lim_{n\to\infty} \mathcal{M}^0(\mathfrak{y}^*, \Im^n \mathfrak{z}, \mathfrak{t}) = 1 \Rightarrow \Im^n \mathfrak{z} \to \mathfrak{y}^* \tag{10}$$

From (8), (10) and the uniqueness of the limit $\mathfrak{x}^* = \mathfrak{y}^*$.

Case 2. $\mathfrak{t} = 0$;

Let $\mathfrak{x}_0 \in \mathcal{X}$ such that $\beta(\mathfrak{x}_0, \Im \mathfrak{x}_0, 0) \leq 1$.

Define the sequence $\mathfrak{x}_{n+1} = \Im \mathfrak{x}_n, \forall \ n \in \mathbb{N}$. If $\mathfrak{x}_{n+1} = \mathfrak{x}_n$ for some $n \in \mathbb{N}$,then $\mathfrak{x}^* = \mathfrak{x}_n$ is a fixed point of $\Im$.

Suppose $\mathfrak{x}_{n+1} \neq \mathfrak{x}_n, \forall \ n \in \mathbb{N}$.

From (ii),

$$\beta(\mathfrak{x}_0, \mathfrak{x}_1, 0) = \beta(\mathfrak{x}_0, \Im \mathfrak{x}_0, 0) \leq 1$$

and using (i), we obtain

$$\beta(\mathfrak{x}_0, \Im \mathfrak{x}_0, 0) \leq 1 \Rightarrow \beta(\Im \mathfrak{x}_0, \Im \mathfrak{x}_1, 0) \leq 1.$$

By induction,

$$\begin{aligned}\beta(\Im\mathfrak{x}_0,\Im\mathfrak{x}_{1,}0) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}_1,\Im\mathfrak{x}_{2,}0)\leq 1\\ \beta(\Im\mathfrak{x}_1,\Im\mathfrak{x}_{2,}0) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}_2,\Im\mathfrak{x}_{3,}0)\leq 1\\ &...\\ \beta(\Im\mathfrak{x}_{n-3},\Im\mathfrak{x}_{n-2,}0) &\leq 1 \Rightarrow \beta(\Im\mathfrak{x}_{n-2},\Im\mathfrak{x}_{n-1,}0)\leq 1\end{aligned}$$

and so we get,

$$\beta(\Im\mathfrak{x}_{n-2},\Im\mathfrak{x}_{n-1,}0)=\beta(\mathfrak{x}_{n-1},\mathfrak{x}_n,0)\leq 1,\ \forall\ n\in\mathbb{N}. \tag{11}$$

Implementing (5) with $\mathfrak{x}=\mathfrak{x}_{n-1}$, $\mathfrak{y}=\mathfrak{x}_n$, $\mathfrak{t}=0$ and using (11) respectively, we obtain;

$$\mathcal{M}^0(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n,0)\geq\beta(\mathfrak{x}_{n-1},\mathfrak{x}_n,0)\mathcal{M}^0(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n,0)\geq\psi(\mathcal{M}^0(\mathfrak{x}_{n-1},\mathfrak{x}_n,0))$$

Using (1), we get,

$$\begin{aligned}N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n) &\geq \beta(\mathfrak{x}_{n-1},\mathfrak{x}_n,0)(N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n))\\ &\geq \psi(N_{\mathcal{M}}(\mathfrak{x}_{n-1},\mathfrak{x}_n))\end{aligned}$$

And this implies that,

$$N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n)\geq\psi^n(N_{\mathcal{M}}(\mathfrak{x}_0,\mathfrak{x}_1)),\ \forall\ n\in\mathbb{N}.$$

as $n\to\infty$,

$$\lim_{n\to\infty}N_{\mathcal{M}}(\Im\mathfrak{x}_{n-1},\Im\mathfrak{x}_n)\geq\lim_{n\to\infty}\psi^n(N_{\mathcal{M}}(\mathfrak{x}_0,\mathfrak{x}_1))$$

Since $\psi^n(r)\to 1$,

$$\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_n,\mathfrak{x}_{n+1})=1.$$

The important point of the proof is setting that the sequence $\{\mathfrak{x}_n\}$ Cauchy in $\mathcal{X}$.

Suppose that it is false; there exists $0<\varepsilon<1$ and two subsequences $\{\mathfrak{x}_{p_n}\}$ and $\{\mathfrak{x}_{q_n}\}$ of $\{\mathfrak{x}_n\}$ such that q_n is the smallest index for which $p_n>q_n\geq n_0$, using (1)

$$\mathcal{M}^0(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n},0)=\wedge_{t>0}\mathcal{M}(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n},t)=N_{\mathcal{M}}(\mathfrak{x}_{p_n},x_{\mathfrak{x}q_n})\leq 1-\varepsilon$$

$$\mathcal{M}^0(\mathfrak{x}_{p_{n-1}},\mathfrak{x}_{q_n},0)=\wedge_{t>0}\mathcal{M}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{q_n},t)=N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{q_n})>1-\varepsilon$$

and by (iii); $n_0\in\mathbb{N}$ such that, for each $n\in\mathbb{N}$ with $n\geq n_0$,there exist $p_n,q_n\in\mathbb{N}$ $\beta(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n},0)\leq 1$.

And we get

$$1-\varepsilon\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n})\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{q_n})*N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{p_n})$$

as $n\to\infty$

$$\lim_{n\to\infty}(1-\varepsilon)\geq\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n})\geq\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{q_n})*\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{p_n})$$

Since $\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n-1},\mathfrak{x}_{p_n})=1$,

$$(1-\varepsilon)\geq\lim_{n\to\infty}N_{\mathcal{M}}(\mathfrak{x}_{p_n},\mathfrak{x}_{q_n})\geq(1-\varepsilon)$$

we obtain that

$$\lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n}) = (1-\varepsilon).$$

and similarly

$$\begin{aligned}
(1-\varepsilon) &\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n}) \\
&\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{p_n+1}) * N_{\mathcal{M}}(\mathfrak{x}_{p_n+1}, \mathfrak{x}_{q_n+1}) * N_{\mathcal{M}}(\mathfrak{x}_{q_n+1}, \mathfrak{x}_{q_n}) \\
&\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{p_n+1}) * \beta(\Im\mathfrak{x}_{p_n}, \Im\mathfrak{x}_{q_n}, 0) N_{\mathcal{M}}(\Im\mathfrak{x}_{p_n}, \Im\mathfrak{x}_{q_n}) * N_{\mathcal{M}}(\mathfrak{x}_{q_n+1}, \mathfrak{x}_{q_n}) \\
&\geq N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{p_n+1}) * \psi(N_M(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n})) * N_{\mathcal{M}}(\mathfrak{x}_{q_n}, \mathfrak{x}_{q_n+1}).
\end{aligned}$$

as $n \to \infty$, we get

$$\begin{aligned}
\lim_{n\to\infty}(1-\varepsilon) &\geq \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{p_n}, \mathfrak{x}_{p_n+1}) * \lim_{n\to\infty} \psi(N_M(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n})) * \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{q_n}, \mathfrak{x}_{q_n+1}) \\
(1-\varepsilon) &\geq \lim_{n\to\infty} \psi(N_M(\mathfrak{x}_{p_n}, \mathfrak{x}_{q_n})) \\
(1-\varepsilon) &\geq \psi(1-\varepsilon)
\end{aligned}$$

It is a contradiction, because of $\psi(r) > r$.

So we have obtained that $\{x_n\}$ is a Cauchy sequence. Since $\mathcal{X}$ is complete,

$$\exists\, \mathfrak{x}^* \in \mathcal{X} \ : \text{ as } n \to \infty \text{ and } \mathfrak{x}_n \to \mathfrak{x}^*$$

Using (11) and (iv);

$$\beta(\mathfrak{x}_n, \mathfrak{x}^*, 0) \leq 1, \forall n \in \mathbb{N}$$

from (5) with using (1) and S_4,

$$\begin{aligned}
N_{\mathcal{M}}(\Im\mathfrak{x}^*, \mathfrak{x}^*) &\geq N_{\mathcal{M}}(\Im\mathfrak{x}^*, \Im\mathfrak{x}_n) * N_{\mathcal{M}}(\Im\mathfrak{x}_n, \mathfrak{x}^*) \\
&\geq \beta(\mathfrak{x}_n, \mathfrak{x}^*, 0) N_{\mathcal{M}}(\Im\mathfrak{x}_n, \Im\mathfrak{x}^*) * N_{\mathcal{M}}(\mathfrak{x}_{n+1}, \mathfrak{x}^*) \\
&\geq \psi(N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}^*)) * N_{\mathcal{M}}(\mathfrak{x}_{n+1}, \mathfrak{x}^*)
\end{aligned}$$

as $n \to \infty$, $\psi(1) = 1$

$$\begin{aligned}
\lim_{n\to\infty} N_{\mathcal{M}}(\Im\mathfrak{x}^*, \mathfrak{x}^*) &\geq \lim_{n\to\infty} \psi(N_{\mathcal{M}}(\mathfrak{x}_n, \mathfrak{x}^*)) * \lim_{n\to\infty} N_{\mathcal{M}}(\mathfrak{x}_{n+1}, \mathfrak{x}^*) \\
&\geq \psi(1) * 1 = 1
\end{aligned}$$

and we obtain,

$$\lim_{n\to\infty} N_{\mathcal{M}}(\Im\mathfrak{x}^*, \mathfrak{x}^*) = 1.$$

And so, $\mathfrak{x}^* = \Im\mathfrak{x}^*$.That is, $\mathfrak{x}^*$ is a fixed point of $\Im$.

Now we will show that uniqueness of the fixed point.

Presume that $\Im$ have two different fixed points; $\mathfrak{x}^*$and $\mathfrak{y}^*$.

Provided that $\beta(\mathfrak{x}^*, \mathfrak{y}^*, 0) \leq 1$, then since $\Im$ is $\beta-\psi-0-$ fuzzy contractive, using (1) and $\psi(r) > r$, we have

$$\mathcal{M}^0(\mathfrak{x}^*, \mathfrak{y}^*, 0) \geq \beta(\mathfrak{x}^*, \mathfrak{y}^*, 0)\mathcal{M}^0(\Im\mathfrak{x}^*, \Im\mathfrak{y}^*, 0) \geq \psi(\mathcal{M}^0(\mathfrak{x}^*, \mathfrak{y}^*, 0)) > \mathcal{M}^0(\mathfrak{x}^*, \mathfrak{y}^*, 0)$$

$$N_{\mathcal{M}}(\mathfrak{x}^*, \mathfrak{y}^*) > N_{\mathcal{M}}(\mathfrak{x}^*, \mathfrak{y}^*)$$

it is a contradiction. That is, $\mathfrak{x}^* = \mathfrak{y}^*$.

Assume that $\beta(\mathfrak{x}^*, \mathfrak{y}^*, 0) > 1$,then from (v)

$$\exists\ \mathfrak{z} \in X :\ \beta(\mathfrak{x}^*, \mathfrak{z}, 0) \leq 1 \text{ and } \beta(\mathfrak{y}^*, \mathfrak{z}, 0) \leq 1.$$

From (i), we obtain,

$$\beta(\mathfrak{x}^*, \Im^n \mathfrak{z}, 0) \leq 1 \text{ and } \beta(\mathfrak{y}^*, \Im^n \mathfrak{z}, 0) \leq 1, \forall\ n \in \mathbb{N}. \tag{12}$$

Since $\Im$ is $\beta - \psi - 0-$ fuzzy contractive and using (10), we obtain

$$\begin{aligned} \mathcal{M}^0(\mathfrak{x}^*, \Im^n \mathfrak{z}, 0) &= \mathcal{M}(\mathfrak{x}^*, \Im^n \mathfrak{z}, 0) = \mathcal{M}(\Im \mathfrak{x}^*, \Im(\Im^{n-1} \mathfrak{z}), 0) = N_{\mathcal{M}}(\Im \mathfrak{x}^*, \Im(\Im^{n-1} \mathfrak{z})) \\ &\geq \beta(\mathfrak{x}^*, \Im^{n-1} \mathfrak{z}, 0) N_{\mathcal{M}}(\Im \mathfrak{x}^*, \Im(\Im^{n-1} \mathfrak{z})) \\ &\geq \psi(N_{\mathcal{M}}(\mathfrak{x}^*, \Im^{n-1} \mathfrak{z})) \end{aligned}$$

And by induction we obtain,

$$N_{\mathcal{M}}(\Im \mathfrak{x}^*, \Im(\Im^{n-1} \mathfrak{z})) \geq \psi^n(N_{\mathcal{M}}(\mathfrak{x}^*, \mathfrak{z})), \forall n \in \mathbb{N}.$$

As $n \to \infty$, we get $\Im^n \mathfrak{z} \to \mathfrak{x}^*$.

And by the similary way we obtain $\Im^n \mathfrak{z} \to \mathfrak{y}^*$.So the uniqueness of the limit $\mathfrak{x}^* = \mathfrak{y}^*$ □

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LAGRANGE STABILITY IN TERMS OF TWO MEASURES WITH INITIAL TIME DIFFERENCE FOR SET DIFFERENTIAL EQUATIONS INVOLVING CAUSAL OPERATORS

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Abstract. In this paper, we investigate generalized variational comparison results aimed to study the stability properties in terms of two measures for solutions of Set Differential Equations (SDEs) involving causal operators, taking into consideration the difference in initial conditions. Next, we employ these comparison results in proving the theorems that give sufficient conditions for equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures with initial time difference for the solutions of perturbed SDEs involving causal operators in regard to their unperturbed ones.

1. Introduction

Many researchers were interested in studying set differential equations (SDEs) in the recent decades [2,3,5,8–10,13,14,18,20,23,36,47] due to their unifying properties. Lakshmikantham et al. highlighted these properties in one of the most important resources on this topic [23]. The comprehensiveness of the SDEs is driven from the fact that they encompass the conventional differential and integral equations when the Hukuhara difference and integrals defined on the SDEs are restricted to $\mathbb{R}$; whereas they give us vector differential equations when the restriction is done to $\mathbb{R}^n$ [4,19,26].

On the other hand, many well-known differential equations such as integro differential equations [28], impulsive differential equations [22], and differential equations with delay [35], are examples of differential equations involving causal operators. Many research papers dealt with those types of equations. [1,7–10,21,43]

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SDEs with causal operators unifies the fundamental theory of SDEs, including various corresponding dynamical systems. Some relevant works can be found in [5,8–14,47]

Although it is never feasible to know the exact solutions of all dynamical systems in practice, their attributes may be determined through a variety of qualitative studies such as stability analysis [2–5,15,19,20,24,36], initial time difference (ITD) stability analysis [6,29,30,33,34,37,38,41–47], practical stability analysis [17,31,40,46], boundedness [2,6,11,16,32,37,38,40–42], etc.

Many techniques have been used in this process, including the Lyapunov second method [19,24,33,43,44], variation of parameters [25,32,33], "in terms of two measures" methodology [5,18,27,32,38,42,45,46], and so on.

In this manuscript, we develop generalized variational comparison results aimed to assess a combination of two concepts of stability and other qualitative aspects for SDEs with causal operators that unifies the conceptual framework behind SDEs. Furthermore, we give adequate criteria for equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures with ITD for the solutions of the perturbed forms of these types equations in comparison to their un-perturbed counterparts.

2. Preliminaries

In what follows, we denote the set of all compact non-empty subsets of $\mathbb{R}^n$ by $K(\mathbb{R}^n)$, and the set of all compact and convex non-empty subsets of $\mathbb{R}^n$ by $K_c(\mathbb{R}^n)$.

The Hausdorff metric between any bounded sets A and B in $\mathbb{R}^n$ is defined as

$$D(A,B) = \max\left[\sup_{x\in B} d(x,A)\ ,\ \sup_{y\in A} d(y,B)\right] \tag{1}$$

where

$$d(x,A) = \inf\ \{d(x,y)\ :\ y\in A\} \tag{2}$$

Each of $(K(\mathbb{R}^n), D)$ and $(K_c(\mathbb{R}^n), D)$ forms a complete metric space. The space $K_c(\mathbb{R}^n)$ equipped with the natural addition and non-negative scalar multiplication becomes a semi-linear metric space which can be embedded as a cone into a corresponding Banach space.

The Hausdorff metric satisfies the following properties:

$$\begin{aligned} &(1)\ D(A,B) = D(B,A) \\ &(2)\ D(A+C,B+C) = D(A,B) \\ &(3)\ D(kA,\ kB) = k\ D(A,B) \\ &(4)\ D(A,B) \le D(A,C) + D(C,B) \end{aligned} \tag{3}$$

for any $A,B,C\in K_c(\mathbb{R}^n)$ and $k\in\mathbb{R}_+$, where Minkowski addition of any two non-empty subsets A and B of $\mathbb{R}^n$ is defined by $A+B=\{a+b\ :\ a\in A,\ b\in B\}$ and where scalar multiplication of a value $k\in\mathbb{R}$ and a non-empty subset A of $\mathbb{R}^n$ is defined by $kA=\{ka\ :a\in A\}$. If $k=-1$, we get $-A=(-1)A=\{-a\ :a\in A\}$.

In general, $A+(-A)\neq\{0\}$ (unless $A=\{a\}$ is a singleton). To overcome with this implication of Minkowski difference, i.e.

$$A-B=A+(-1)\,B=\{a-b\ :a\in A,\quad b\in B\} \tag{4}$$

Hukuhara difference between two sets $A,B\in K_c\left(\mathbb{R}^n\right)$ is defined as follows:

If there exists a set $C\in K_c\left(\mathbb{R}^n\right)$ such that $C+B=A$, then Hukuhara difference exists and we denote it by $A\ominus B$, or simply $A-B$ when there is no confusion with Minkowski difference. i.e. $A\ominus B=C \Leftrightarrow C+B=A$.

An important property of Hukuhara difference is $A-A=\{0\}$ for $A\in K_c\left(\mathbb{R}^n\right)$.

Let $U:I\rightarrow K_c\left(\mathbb{R}^n\right)$ be a given multifunction, where I is an interval of real numbers. U is said to be Hukuhara differentiable at a point $t_0\in I$, if there exists an element $D_H U\left(t_0\right)\in K_c\left(\mathbb{R}^n\right)$ such that the limits

$$\lim_{h\to 0^+}\frac{U\left(t_0+h\right)-U\left(t_0\right)}{h}\quad\text{ and }\lim_{h\to 0^+}\frac{U\left(t_0\right)-U\left(t_0-h\right)}{h} \tag{5}$$

both exist in the topology of $K_c\left(\mathbb{R}^n\right)$ and are equal to $D_H U\left(t_0\right)$.

It is implicit in the definition of $D_H U\left(t_0\right)$ the exitance of the two differences $U\left(t_0+h\right)-U\left(t_0\right)$ and $U\left(t_0\right)-U\left(t_0-h\right)$, for sufficiently small $h>0$.

By embedding $K_c\left(\mathbb{R}^n\right)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, we find that if

$$G\left(t\right)=G\left(t_0\right)+\int_{t_0}^{t}F\left(s\right)ds,\quad t\in I \tag{6}$$

where $F:I\rightarrow K_c\left(\mathbb{R}^n\right)$ is integrable in the sense of Bochner, then G is Hukuhara differentiable, i. e. $D_H G\left(t\right)$ exits, and the equality $D_H G\left(t\right)=F\left(t\right)$, a. e. on I, holds.

Also, the Hukuhara integral

$$\int_I F\left(s\right)ds=\left[\int_I f\left(s\right)ds\ :\ f\textit{ is a continuous selector of }F\right] \tag{7}$$

for any compact set $I\subset\mathbb{R}_+$.

Let $E=C\left[\left[t_0,\infty\right),\,K_c\left(\mathbb{R}^n\right)\right]$ with norm

$$\sup_{t\in[t_0,\infty)}\frac{D\left[U\left(t\right),\theta\right]}{h\left(t\right)}\ <\infty \tag{8}$$

where $U\in E$, θ is the zero element of $\mathbb{R}^n$, which is regarded as a point set; and $h:\left[t_0,\infty\right)\rightarrow\mathbb{R}_+$ is a continuous map. E equipped with such a norm is a Banach space.

Let $Q{\in}C\left[E,E\right].$ Q is said to be a causal map if $U\left(s\right){=}V\left(s\right),\ \ t_0{\leq}s{\leq}t{<}\infty$, and $U,V{\in}E$ then

$$\left(QU\right)\left(s\right)=\left(QV\right)\left(s\right),\quad t_0\leq s\leq t<\infty. \tag{9}$$

Let us consider the following differential equations

$$D_HU\ =\left(QU\right)\left(t\right),\quad U\left(t_0\right)=U_0\ \ for\ \ U_0\in K_c\left(\mathbb{R}^n\right)\ \ and\ \ t\geq t_0\geq 0, \tag{10}$$

$$D_HU=\left(QU\right)\left(t\right),\quad U\left(\tau_0\right)=V_0\ \ for\ \ V_0\in K_c\left(\mathbb{R}^n\right)\ \ and\ \ t\geq\tau_0\geq 0 \tag{11}$$

$$D_HV=\left(PV\right)\left(t\right),\quad V\left(\tau_0\right)=V_0\ \ for\ \ V_0\in K_c\left(\mathbb{R}^n\right)\ \ and\ \ t\geq\tau_0 \tag{12}$$

$$D_HW=\left(SW\right)\left(t\right),\ W\left(\tau_0\right)=V_0-U_0\ \ for\ \ W\left(\tau_0\right)=W_0\in K_c\left(\mathbb{R}^n\right)\ \ and\ \ t\geq\tau_0 \tag{13}$$

where $Q,P,S:E{\to}E$ are causal operators, and satisfy a local Lipschitz condition on $\mathbb{R}_+\times S_\rho$ where $S_\rho{=}\left\{U{\in}K_c\left(\mathbb{R}^n\right)\ :D\left[U,\tilde{0}\right]{<}\rho{<}\infty\right\}.$

It is clear that (10) and (11) are different in the initial time and position. Moreover, if $\left(PV\right)\left(t\right)$ in (12) is written as $\left(PV\right)\left(t\right)=\left(QV\right)\left(t\right)+\left(RV\right)\left(t\right)$; Then, we consider (12) as the perturbed form corresponding to the unperturbed equation (11) with the perturbation term $\left(RV\right)\left(t\right)$.

Assuming that $\left(Q\tilde{0}\right)\left(t\right){\equiv}\tilde{0}$ for $t{\geq}0$, and assuming the necessary smoothness of P,Q and R to guarantee the existence and uniqueness of the solution $U\left(t\right){=}U\left(t,t_0,U_0\right)$ of (10) through $\left(t_0,U_0\right)$ for all $t{\geq}t_0$, and those of the solution $V\left(t\right){=}V\left(t,\tau_0,V_0\right)$ of (12) through $\left(\tau_0,V_0\right)$ for all $t{\geq}\tau_0$, in addition to their continuous dependence on the initial conditions.

If $U\in C^1\left[\,J_1,K_c\left(\mathbb{R}^n\right)\right]$ on $J_1=\left[t_0,t_0+T_1\right],$ then it is said to be a solution of (10) on J_1 if it satisfies (10) on J_1. If U,V and $W\in C^1\left[\,J_2,K_c\left(\mathbb{R}^n\right)\right]$ on $J_2=\left[t_0,t_0+T_2\right],$ then these are said to be solutions of (11), (12), (13) on J_2 provided that they satisfy (11), (12), (13) on J_2, respectively.

Now let us define a partial order in the metric space $\left(K_c\left(\mathbb{R}^n\right),\ D\right)$. First, we start by defining a cone in $K_c\left(\mathbb{R}^n\right)$.

Definition 1. *The subfamily $K\subset K_c\left(\mathbb{R}^n\right)$ is said to be a cone in $K_c\left(\mathbb{R}^n\right)$ if it consists of sets $U\in K_c\left(\mathbb{R}^n\right)$ such that any $u\in U$ is a non-negative n-component vector $u=\left(u_1,u_2,\ldots,u_n\right)$ satisfying $u_i\geq 0$ for $i=1\ldots n$. The subfamily $K^0\subset K_c\left(\mathbb{R}^n\right)$, that consists of sets $U\in K_c\left(\mathbb{R}^n\right)$ such that any $u\in U$ is a positive n-component vector $u=\left(u_1,u_2,\ldots,u_n\right)$ satisfying $u_i>0$ for $i=1\ldots n$, is the nonempty interior of the cone K.*

Definition 2. *For any $U,\ V\in K_c\left(\mathbb{R}^n\right),$ if there exists $Z\in K_c\left(\mathbb{R}^n\right)$ such that $Z\in K$ and $U=V+Z$ then we say that $U\geq V$ or $V\leq U$. Similarly, if there exists $Z\in K_c\left(\mathbb{R}^n\right)$ such that $Z\in K^0$ and $U=V+Z$ then we say that $U>V$ or $V<U$.*

We present below some needed classes to develop the stability results in terms of two measures.

$$\mathbb{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] \ : \ a(u) \ is \ strictly \ increasing \ in \ u \ and \ a(0) = 0\} \tag{14}$$

$$\mathbb{L} = \left\{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+] \ : \sigma(u) \ is \ strictly \ decreasing \ in \ u \ and \ \lim_{u\to\infty} \sigma(u) = 0 \ \right\} \tag{15}$$

$$\mathbb{CK} = \left\{ \begin{array}{c} a \in C\left[\mathbb{R}_+^2, \mathbb{R}_+\right] \ : \ a(t,s) \in \mathbb{K} \ for \ each \ t \\ and \ a(t,s) \ is \ continuous \ for \ each \ s \end{array} \right\} \tag{16}$$

$$\Gamma = \left\{h \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+] \ : \inf_{(t,U)} h(t,U) = 0 \ \right\} \tag{17}$$

$$\Gamma_0 = \left\{h \in \Gamma : \inf_U \ h(t,U) = 0 \ , \ for \ each \ t \in \mathbb{R}_+\right\} \tag{18}$$

Next, to introduce a Lyapunov-like function, we present some definitions needed in the qualitative analysis in terms of two measures.

Definition 3. *Let $L \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, then L is said to be*
(i) h-positive definite if there exists a $\rho > 0$ and a $b \in \mathbb{K}$ such that

$$h(t,U) < \rho \ \ implies \ \ b(h(t,U)) \le L(t,U) \tag{19}$$

(ii) h-decrescent if there exists a $\rho > 0$ and a function $a \in \mathbb{K}$ such that

$$h(t,U) < \rho \ \ implies \ \ L(t,U) \le a(h(t,U)) \tag{20}$$

(iii) h-weakly decrescent if there exists a $\rho > 0$ and a function $a \in \mathbb{CK}$ such that

$$h(t,U) < \rho \ \ implies \ \ L(t,U) \le a(t, \ h(t,U)) \tag{21}$$

Definition 4. *Let $h_0, h \in \Gamma$, then we say that h_0 is finer than h if there exists a $\rho > 0$ and a function $\phi \in \mathbb{CK}$ such that*

$$h_0(t,U) \le \rho \ \ implies \ \ h(t,U) \le \phi(t, h_0(t,U)) \tag{22}$$

h_0 is uniformly finer than h if the function ϕ in the above definition is independent of t.

Now, let us introduce the definitions of generalized Dini-like derivatives of L.

Definition 5. *We define the generalized derivative (Dini-like derivatives) for a real-valued function $L \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ as follows:*

$$D_*^+ L(t,s,U) = \lim_{h\to 0^+} \sup \frac{1}{h} \ \left[L\left(s+h, V\left(t, s+h, U + h\left(Q\tilde{U}\right)(s)\right)\right) - L(s, V(t,s,U))\right] \tag{23}$$

$$D_{*-}L(t,s,U) = \underset{h\to 0^-}{lim}\, inf \frac{1}{h} \left[L\left(s+h, V\left(t, s+h, U+h\left(Q\tilde{U}\right)(s)\right)\right) - L(s, V(t,s,U))\right] \quad (24)$$

for $t,s\in\mathbb{R}_+$ *and* $U\in K_c(\mathbb{R}^n)$.

Next, let us introduce the definitions of initial time difference (ITD) equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures, before proceeding with our main results.

Definition 6. *Let* $U(t,t_0,U_0)$ *be any solution of (10) for* $t \geq t_0 \geq 0$, *and let* $\tilde{U}(t,\tau_0,U_0) = U(t-\eta,t_0,U_0)$, *for* $\eta = \tau_0 - t_0$. *The solution* $V(t,\tau_0,V_0)$ *of (12) for* $t \geq \tau_0$ *is said to be*

(i) ITD (h_0,h)*-equi-bounded with respect to the solution* $\tilde{U}$, *if and only if given any* $\alpha > 0$ *and* $\tau_0 \in \mathbb{R}_+$, *there exists* $\beta = \beta(\alpha,\tau_0) > 0$ *such that* $h_0(\tau_0, V_0 - U_0) < \beta$ *implies*

$$h(t, V(t,\tau_0,V_0) - U(t-\eta,t_0,U_0)) < \alpha\ ,\quad t \geq \tau_0 \quad (25)$$

(ii) ITD (h_0,h)*-uniformly equi-bounded with respect to the solution* $\tilde{U}$ *if the previous implication in (i) holds for every* $\tau_0 \in \mathbb{R}_+$, *or in otherwords,* $\beta = \beta(\alpha,\tau_0) > 0$ *is independent of* τ_0.

It is worth pointing out that if β *in (ii) satisfy that* $\beta(\cdot,\tau_0) \in \mathbb{K}$, *then the solution* $V(t,\tau_0,V_0)$ *of (12) is ITD* (h_0,h)*-stable with respect to the solution* $\tilde{U}$. *In fact, for* $\varepsilon > 0$ *there exists a continuous function* $\delta = \delta(\varepsilon,\tau_0) > 0$ *in* τ_0, *such that whenever* $\alpha < \delta$, *we have* $\beta = \beta(\alpha,\tau_0) < \varepsilon$.

(iii) ITD (h_0,h)*-equi-attractive in the large with respect to the solution* $\tilde{U}$, *if and only if given any* ε, $\alpha > 0$ *and* $\tau_0 \in \mathbb{R}_+$, *there exists a* $T = T(\tau_0,\varepsilon,\alpha) > 0$ *such that* $h_0(\tau_0, V_0 - U_0) < \alpha$ *implies*

$$h(t, V(t,\tau_0,V_0) - U(t-\eta,t_0,U_0)) < \varepsilon\ ,\quad t \geq \tau_0 + T(\tau_0,\varepsilon,\alpha) \quad (26)$$

(iv) ITD (h_0,h)*-uniform equi-attractive in the large with respect to the solution* $\tilde{U}$, *if the previous implication in (iii) holds for every* $\tau_0 \in \mathbb{R}_+$, *or in otherwords,* $T = T(\tau_0,\varepsilon,\alpha) > 0$ *is independent of* τ_0.

(v) ITD (h_0,h)*-Lagrange stable with respect to the solution* $\tilde{U}$, *if and only if it is ITD* (h_0,h)*-equi-bounded and ITD* (h_0,h)*-equi-attractive in the large with respect to the solution* $\tilde{U}$.

(vi) ITD (h_0,h)*-uniform Lagrange stable with respect to the solution* $\tilde{U}$, *if and only if it is ITD* (h_0,h)*-Lagrange stable and both* $\beta = \beta(\alpha,\tau_0) > 0$ *in (i) and* $T = T(\tau_0,\varepsilon,\alpha) > 0$ *in (iii) are independent of* τ_0.

3. ITD Stability Results in Terms of Two Measures

3.1. **ITD Variational Comparison Results.** In what follows, let us present generalized variational comparison results aimed to study the stability properties in terms of two measures for solutions of SDEs involving causal operators, taking into consideration the difference in the initial conditions.

Before that, in order to study the stability properties for the SDEs with causal operators, let us assume that the solutions of the SDEs (10), (11), (12), and (13) exist and that they are unique; additionally, that all the Hukuhara differences exist, so the problem is well-posed.

Theorem 1. *Assume that (i) Both* $L(t,\Omega) \in C\left[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+^N\right]$ *and* $\|W(t,s,\Omega)\|$ *satisfy a local Lipschitz condition in* Ω *for any* t,s*; where* $W(t) = W(t,\tau_0,V_0 - U_0)$ *is the solution of (13) for* $t \geq \tau_0$*,* $\tilde{U}(t,\tau_0,U_0) = U(t-\eta,t_0,U_0)$*, for* $\eta = \tau_0 - t_0$*,* $U(t,t_0,U_0)$ *is any solution of (10) for* $t \geq t_0$*, and* $V(t) = V(t,\tau_0,V_0)$ *is the solution of (12) for* $t \geq \tau_0$*; and let* $\Omega(t) = V(t) - \tilde{U}(t)$*.*

(ii)

$$D_{*-}L(t,s,\Omega) \leq g(t,s,L(s,W(t,s,\Omega))) \tag{27}$$

where

$$\begin{aligned} &D_{*-}L(t,s,\Omega) \\ &= \lim_{\delta\to 0^-}\inf \frac{1}{\delta}\left(L\left(s+\delta, W\left(t,s+\delta,\Omega+\delta\left((PV)(s)-\left(Q\tilde{U}\right)(s)\right)\right)\right)\right. \\ &\left. - L(s,W(t,s,\Omega))\right) \end{aligned} \tag{28}$$

(iii) $g \in C\left[\mathbb{R}_+ \times \mathbb{R}_+^N, \mathbb{R}^N\right]$*,* $g(t,s,u)$ *is quasi-monotone non-decreasing in* u *for any* t,s*; [i.e., if* $u \leq v$*,* $u_i = v_i$ *for some* i *such that* $1 \leq i \leq N$*, then* $g_i(t,s,u) \leq g_i(t,s,v)$*, for* $t,s \in \mathbb{R}_+$ *(In this context, the inequality symbol used in the vectorial inequalities is understood to denote component-wise inequality [39])];*

and $r(t,s,\tau_0,V_0)$ *is the maximal solution of*

$$\frac{du(s)}{ds} = g(t,s,u(s)), \quad u(\tau_0) = u_0 \geq 0 \tag{29}$$

existing for $\tau_0 \leq s \leq t < \infty$*.*

Then, $L(\tau_0,W(t,\tau_0,V_0-U_0)) = u_0$ *implies*

$$L(t,\Omega(t,\tau_0,V_0-U_0)) \leq r_0(t,\tau_0,L(\tau_0,W(t,\tau_0,V_0-U_0))) \tag{30}$$

where $r_0(t,\tau_0,u_0) = r(t,t,\tau_0,u_0)$*.*

Proof. Let us set

$$m(t,s) = L(s,W(t,s,\Omega(s))) \quad for \quad \tau_0 \leq s \leq t. \tag{31}$$

Then, we have

$$\begin{aligned} m(t,\tau_0) &= L(\tau_0, W(t,\tau_0,\Omega(\tau_0))) = L\left(\tau_0, W\left(t,\tau_0, V(\tau_0) - \tilde{U}(\tau_0)\right)\right) \\ &= L(\tau_0, W(t,\tau_0, V_0 - U_0)) = u_0 \end{aligned} \tag{32}$$

For a sufficiently small positive value δ, we have

$$\begin{aligned} & m(t,s+\delta) - m(t,s) \\ & = L(s+\delta, W(t,s+\delta,\Omega(s+\delta))) - L(s, W(t,s,\Omega(s))) \\ & = L(s+\delta, W(t,s,\Omega(s)) + \delta(SW(t,s,\Omega(s)))(s) + \varepsilon(\delta)) - L(s, W(t,s,\Omega(s))) \end{aligned} \tag{33}$$

where ε stands for error and $\lim_{\delta\to 0^-} \frac{\varepsilon(\delta)}{\delta} = 0$.

Taking into consideration the assumptions in (i) regarding the locally Lipschitz property of $L(t,\Omega)$ and $\|W(t,s,\Omega)\|$ in Ω, it is seen that

$$\begin{aligned} m(t,s+\delta) - m(t,s) \le\ & k(\varepsilon_1(\delta) - \varepsilon_2(\delta)) \\ & + L\left(s+\delta, W\left(t,s,V(s) - \tilde{U}(s)\right) + \delta\left((PV)(s) - \left(Q\tilde{U}\right)(s)\right)\right) \\ & - L\left(s, W\left(t,s,V(s) - \tilde{U}(s)\right)\right) \end{aligned} \tag{34}$$

where $\varepsilon_1, \varepsilon_2$ stand for errors, k stands for Lipschitz constant.

The inequality in the assumption (ii) gives us the following estimation regarding the Dini derivative of $m(t,s)$

$$\begin{aligned} & D_{*-} m(t,s) \\ & \le \lim_{\delta\to 0^-} \inf \frac{1}{\delta} K(\varepsilon_1(\delta) - \varepsilon_2(\delta)) \\ & + \lim_{\delta\to 0^-} \inf \frac{1}{\delta} \quad L\left(s+\delta, W\left(t,s,V(s) - \tilde{U}(s)\right) + \delta\left((PV)(s) - \left(Q\tilde{U}\right)(s)\right)\right) \\ & - \lim_{\delta\to 0^-} \inf \frac{1}{\delta} \quad L\left(s, W\left(t,s,V(s) - \tilde{U}(s)\right)\right) \\ & \le g\left(t,s,L\left(s, W\left(t,s,V(s) - \tilde{U}(s)\right)\right)\right) \\ & = g(t,s,L(s,W(t,s,\Omega(s)))) = g(t,s,m(t,s)) \end{aligned} \tag{35}$$

for $\tau_0 \le s \le t < \infty$.

A comparison result [Theorem 1.7.1] from [26] gives us the following inequality

$$m(t,s) \le r(t,s,\tau_0, L(\tau_0, W(t,\tau_0,V_0 - U_0))) \quad for \quad \tau_0 \le s \le t. \tag{36}$$

Choosing $s = t$ in the right-hand side of the previous inequality, we get

$$\begin{aligned} m(t,s) &\le r(t,t,\tau_0, L(\tau_0, W(t,\tau_0,V_0-U_0))) \\ &= r_0(t,\tau_0, L(\tau_0, W(t,\tau_0,V_0-U_0))) \end{aligned} \tag{37}$$

which yields the desired estimation in (30) completing the proof. □

Theorem 2. *Under the assumptions of Theorem 1 with $N = 1$ and $g(t,s,u) \equiv 0$, we have*

$$L(t,\Omega(t,\tau_0,V_0-U_0)) \le L(\tau_0, W(t,\tau_0,V_0-U_0)), \quad t \ge \tau_0. \tag{38}$$

Furthermore, we assume

$$D_{*-}L(t,s,\Omega) \le -c(h(s,W(t,s,\Omega))), \quad \tau_0 \le s \le t < \infty \tag{39}$$

where $c \in \mathbb{K}$ and $h \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$.

Then, for $t \ge \tau_0$

$$L(t,\Omega(t,\tau_0,V_0-U_0)) \le L(\tau_0, W(t,\tau_0,V_0-U_0)) - \int_{\tau_0}^{t} c(h(s,W(t,s,\Omega(s))))\,ds. \tag{40}$$

Proof. Starting from the statement (35) in the proof of Theorem 1,

$$D_{*-}m(t,s) \le g(t,s,m(t,s)) \quad for \quad \tau_0 \le s \le t < \infty. \tag{41}$$

Then, since $g(t,s,u) \equiv 0$, we get by integrating the two sides of the previous inequality (41), for $s \in [\tau_0, t]$,

$$\int_{\tau_0}^{t} D_{*-}m(t,s)ds = L(t,W(t,t,\Omega(t))) - L(\tau_0, W(t,\tau_0,\Omega(\tau_0))) \le 0. \tag{42}$$

Hence, we have

$$L(t,\Omega(t,\tau_0,V_0-U_0)) \le L(\tau_0, W(t,\tau_0,\mathrm{V}_0-U_0)) \quad for \quad t \ge \tau_0. \tag{43}$$

Now, let us set

$$M(s,W(t,s,\Omega(s))) \equiv L(s,W(t,s,\Omega(s))) + \int_{\tau_0}^{s} c(h(\xi,W(t,\xi,\Omega(\xi))))d\xi. \tag{44}$$

Then, by taking Dini derivatives of both sides and by assumption (39), we have

$$\begin{aligned} D_{*-}M(t,s,\Omega(s)) &= D_{*-}L(t,s,\Omega(s)) + c(h(s,W(t,s,\Omega(s)))) \\ &\quad - c(h(\tau_0, W(t,\tau_0,\Omega(\tau_0)))) \\ &\le D_{*-}L(t,s,\Omega(s)) + c(h(s,W(t,s,\Omega(s)))) \\ &\le -c(h(s,W(t,s,\Omega(\mathrm{s})))) + c(h(s,W(t,s,\Omega(s)))) = 0. \end{aligned} \tag{45}$$

Thus, $D_{*-}M(t,s,\Omega(s)) \le 0$, in view of (43), gives us for $t \ge \tau_0$,

$$M(t,\Omega(t,\tau_0,V_0-U_0)) \le M(\tau_0, W(t,\tau_0,V_0-U_0)). \tag{46}$$

By the definition of M, this implies, for $t \geq \tau_0$,

$$\begin{aligned} & L\left(t, \Omega\left(t, \tau_0, V_0 - U_0\right)\right) + \int_{\tau_0}^{t} c\left(h\left(\xi, W\left(t, \xi, \Omega\left(\xi\right)\right)\right)\right) d\xi \\ & \leq L\left(\tau_0, W\left(t, \tau_0, V_0 - U_0\right)\right) + \int_{\tau_0}^{\tau_0} c\left(h\left(\xi, W\left(t, \xi, \Omega\left(\xi\right)\right)\right)\right) d\xi \end{aligned} \tag{47}$$

$$L\left(t, \Omega\left(t, \tau_0, V_0 - U_0\right)\right) + \int_{\tau_0}^{t} c\left(h\left(\xi, W\left(t, \xi, \Omega\left(\xi\right)\right)\right)\right) d\xi \leq L\left(\tau_0, W\left(t, \tau_0, V_0 - U_0\right)\right). \tag{48}$$

Moving the integral term to the right-hand side gives us the desired estimation (40) and this completes the proof. □

3.2. **Main ITD Stability Results in Terms of Two Measures.** Now, let us employ the comparison results in section 3.1 to prove the following theorems giving sufficient conditions for equi-boundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures for the solutions of perturbed SDEs involving causal operators in regard to their unperturbed ones.

The next theorem gives sufficient conditions to the ITD (h_0, h)-equi-boundedness of the solution $V(t, \tau_0, V_0)$ of (12) through (τ_0, V_0) for $t \geq \tau_0$ with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (10) through (t_0, U_0) for $t \geq t_0$; providing that the solution $V(t, \tau_0, V_0)$ of (12) is ITD (h_0, h_0)-equi-bounded with respect to $\tilde{U}$.

Theorem 3. *Assume that*

(i) Both $L(t, \Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ and $\|W(t, s, \Omega)\|$ satisfy a local Lipschitz condition in Ω for any t, s; where $W(t) = W(t, \tau_0, V_0 - U_0)$ is the solution of (13) for $t \geq \tau_0$ and

$$\Omega\left(t, \tau_0, V_0 - U_0\right) = V(t) - \tilde{U}(t) \quad for \quad t \geq \tau_0 \tag{49}$$

(ii)

$$D_{*-}L(t, s, \Omega) \leq -c\left(h\left(s, W\left(t, s, \Omega\left(\mathrm{s}\right)\right)\right)\right) \text{ in } S(h, M) \tag{50}$$

where

$$S(h, M) = \{(t, \Omega) : h(t, \Omega) < M \text{ for some } h \in \Gamma \text{ and } M > 0\} \tag{51}$$

and

$$\begin{aligned} & D_{*-}L(t, s, \Omega) \\ & = \lim_{\delta \to 0^-} \inf \frac{1}{\delta}\left(L\left(s + \delta, W\left(t, s + \delta, \Omega + \delta\left(\left(PV\right)(s) - \left(Q\tilde{U}\right)(s)\right)\right)\right)\right. \\ & \left. - L\left(s, W\left(t, s, \Omega\right)\right)\right) \end{aligned} \tag{52}$$

(iii) For $b \in \mathbb{K}$ *and* $a_1, a_0 \in \mathbb{CK}$,

$$\begin{aligned} & b\left(h\left(t,\Omega\right)\right)+\int_{\tau_0}^{t} c\left(h\left(s,W\left(t,s,\Omega\left(\mathrm{s}\right)\right)\right)\right)ds \leq L\left(t,\Omega\right) \text{ in } S\left(h,M\right) \quad \text{and} \\ & L\left(t,\Omega\right) \leq a_1\left(t,h\left(t,\Omega\right)\right)+a_0\left(t,h_0\left(t,\Omega\right)\right) \text{ in } S\left(h,M\right)\cap S\left(h_0,M\right) \end{aligned} \tag{53}$$

(iv) h_0 *is finer that* h*, that is, there exists a function* $\phi \in \mathbb{K}$ *such that*

$$h_0\left(t,\Omega\right) \leq M_0 \quad \text{implies} \quad h\left(t,\Omega\right) \leq \phi\left(h_0\left(t,\Omega\right)\right) \tag{54}$$

for some M_0 *with* $\phi\left(M_0\right) \leq M$;
(v) The solution $V\left(t,\tau_0,V_0\right)$ *of (12) for* $t \geq \tau_0$ *is ITD* (h_0,h_0)*-equi-bounded with respect to the solution* $\tilde{U}\left(t,\tau_0,U_0\right)=U\left(t-\eta,t_0,U_0\right),$ for $\eta=\tau_0-t_0$.

Then, this implies the ITD (h_0,h)*-equi-boundedness of the solution* $V\left(t,\tau_0,V_0\right)$ *of (12) for* $t \geq \tau_0$*, with respect to the solution* $\tilde{U}$

Proof. We shall show that the solution $V\left(t,\tau_0,V_0\right)$ of (12) for $t \geq \tau_0$ is ITD (h_0,h)-equi-bounded with respect to the solution $\tilde{U}$, that is, given any $\alpha>0$ and for some $\tau_0 \in \mathbb{R}_+$, there exists $\beta=\beta\left(\alpha,\ \tau_0\right)>0$ such that $h_0\left(\tau_0,\ V_0-U_0\right)<\beta$ implies

$$h\left(t,\ V\left(t,\ \tau_0,\ V_0\right)-U\left(t-\eta,\ t_0,\ U_0\right)\right)<\alpha \ for\ t\ \geq\ \tau_0 \tag{55}$$

Assume that (55) is not true, then there exist solutions $\tilde{U}\left(t\right)=U\left(t-\eta,t_0,U_0\right)$, where $U\left(t,t_0,U_0\right)$ is the solution of (10) for $t \geq t_0$; and $V\left(t\right)=V\left(t,\tau_0,V_0\right)$ of (12) for $t \geq \tau_0$, and $t_1>\tau_0$ such that

$$h_0\left(\tau_0,V_0-U_0\right)<\beta,\ h\left(t_1,\Omega\left(t_1\right)\right)=\alpha \ and\ h\left(t,\Omega\left(t\right)\right)\leq\alpha,\ for\ \tau_0\leq t\leq t_1 \tag{56}$$

where $\Omega\left(t\right)=V\left(t\right)-\tilde{U}\left(t\right)$ for $t \geq \tau_0$.

By Theorem 2, we have, for $\tau_0 \leq t \leq t_1$,

$$L\left(t,\Omega\left(t\right)\right) \leq L\left(\tau_0,W\left(t,\tau_0,V_0-U_0\right)\right)-\int_{\tau_0}^{t} c\left(h\left(s,W\left(t,s,\Omega\left(s\right)\right)\right)\right)ds \tag{57}$$

Then, using the assumptions (iii), (56) and (57), we obtain when $t = t_1$,

$$
\begin{aligned}
&b(\alpha) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(\mathrm{s})))) \, ds \\
&= b(h(t_1, \Omega(t_1))) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(\mathrm{s})))) \, ds \le L(t_1, \Omega(t_1)) \\
&\le L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \\
&\le L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \\
&\le a_1(\tau_0, h(\tau_0, W(t_1, \tau_0, V_0 - U_0))) + a_0(\tau_0, h_0(\tau_0, W(t_1, \tau_0, V_0 - U_0))) \\
&+ \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds
\end{aligned} \tag{58}
$$

We aim to reach a contradiction to conclude the proof of the theorem. We will use the assumption (v) for this purpose.

Given $0 < \alpha < M$ and that there exists a M_0 with $\phi(M_0) \le M$.

Choosing $N_1 = N_1(\tau_0, \alpha)$ such that $0 < N_1(\tau_0, \alpha) < M_0$, and

$$h_0(t, \Omega(t)) < N_1 \;\; \textit{implies} \;\; a_0(t, h_0(t, \Omega(t))) < \frac{b(\alpha)}{2} \;\; \textit{for } t \ge \tau_0 \tag{59}$$

By assumption (v), corresponding to this N_1, there exists a $\beta_1 = \beta_1(\tau_0, N_1) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \beta_1 \;\; \textit{implies} \;\; h_0(t, \Omega(t)) < N_1 \; \textit{for } t \ge \tau_0 \tag{60}$$

Thus (59) and (60) give us

$$h_0(\tau_0, V_0 - U_0) < \; \beta_1 \;\; \textit{implies} \;\; a_0(t, h_0(t, \Omega(t))) < \frac{b(\alpha)}{2} \;\; \textit{for } t \ge \tau_0 \tag{61}$$

Similarly, we choose $N_2 = N_2(\tau_0, \alpha)$ such that $0 < N_2(\tau_0, \alpha) < M_0$ and

$$h(t, \Omega(t)) < N_2 \;\; \textit{implies} \;\; a_1(t, h(t, \Omega(t))) < \frac{b(\alpha)}{2} \;\; \textit{for } t \ge \tau_0 \tag{62}$$

By the assumptions (iv) and (v) also, corresponding to $\phi^{-1}(N_2)$, there exists a $\beta_2 = \beta_2(\tau_0, N_2) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \beta_2 \;\; \textit{implies} \;\; h_0(t, \Omega(t)) < \phi^{-1}(N_2) \;\; \textit{for } t \ge \tau_0 \tag{63}$$

Since $\phi \in \mathbb{K}$ is strictly monotone increasing; then, we have by taking the composition of ϕ of both sides of the inequality $h_0(t, \Omega(t)) < \phi^{-1}(N_2)$ in (63), with considering (54),

$$h_0(\tau_0, V_0 - U_0) < \beta_2 \;\; implies$$
$$h(t, \Omega(\mathrm{t})) \le \phi(h_0(t, \Omega(t))) < \phi\left(\phi^{-1}(N_2)\right) = N_2 \;\; for \;\; t \ge \tau_0 \tag{64}$$

So, (62) and (64) give us, for $t \ge \tau_0$,

$$h_0(\tau_0, V_0 - U_0) < \beta_2 \;\; implies \;\; a_1(t, h(t, \Omega(t))) < \frac{b(\alpha)}{2} \tag{65}$$

Let $\beta = \min\{\beta_1, \beta_2\}$, then with this β the following statement holds.

$$\begin{aligned} & h_0(\tau_0, V_0 - U_0) < \beta \;\; implies \\ & a_0(t, h_0(t, \Omega(t))) < \frac{b(\alpha)}{2} \;\; and \;\; a_1(t, h(t, \Omega(t))) < \frac{b(\alpha)}{2} \;\; for \;\; t \ge \tau_0 \end{aligned} \tag{66}$$

Hence, when $t = t_1$, using (66), the statement (58) can be written as

$$\begin{aligned} & b(\alpha) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(\mathrm{s})))) \, ds \\ & = b(h(t_1, \Omega(t_1))) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(\mathrm{s})))) \, ds \le L(t_1, \Omega(t_1)) \\ & \le L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \\ & \le L(\tau_0, W(t_1, \tau_0, V_0 - U_0)) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \\ & \le a_1(\tau_0, h(\tau_0, W(t_1, \tau_0, V_0 - U_0))) + a_0(\tau_0, h_0(\tau_0, W(t_1, \tau_0, V_0 - U_0))) \\ & + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \\ & < \frac{b(\alpha)}{2} + \frac{b(\alpha)}{2} + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \\ & = b(\alpha) + \int_{\tau_0}^{t_1} c(h(s, W(t_1, s, \Omega(s)))) \, ds \end{aligned} \tag{67}$$

This contradiction proves that the solution $V(t, \tau_0, V_0)$ of (12) through (τ_0, V_0) for $t \ge \tau_0$ is ITD (h_0, h)-equi-bounded with respect to the solution $\tilde{U}$.

□

The next theorem gives sufficient conditions to the ITD equi-attractiveness in the large of the solution $V(t, \tau_0, V_0)$ of (12) through (τ_0, V_0) for $t \ge \tau_0$ with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (10) through (t_0, U_0) for $t \ge t_0$; providing that the solution $V(t,\tau_0,V_0)$ of (12) is ITD (h_0,h_0)- equi-attractive in the large with respect to $\tilde{U}$.

Theorem 4. *Assume that*

(i) Both $L(t,\Omega)\in C\left[\mathbb{R}_+\times K_c(\mathbb{R}^n),\mathbb{R}_+\right]$ *and* $\|W(t,s,\Omega)\|$ *satisfy a local Lipschitz condition in* Ω *for any* t,s*; where* $W(t)=W(t,\tau_0,V_0-U_0)$ *is the solution of (13) for* $t\geq\tau_0$ *and*

$$\Omega(t,\tau_0,V_0-U_0)=V(t)-\tilde{U}(t)\ \ \textit{for } t\geq\tau_0 \tag{68}$$

(ii)

$$D_{*-}L(t,s,\Omega)\leq -c\left(h\left(s,W\left(t,s,\Omega\left(\mathrm{s}\right)\right)\right)\right)\ \ \textit{in } S(h,M) \tag{69}$$

where

$$S(h,M)=\left\{(t,\Omega)\ :h(t,\Omega)<M \text{ for some } h\in\Gamma \text{ and } M>0\right\} \tag{70}$$

and

$$\begin{aligned} &D_{*-}L(t,s,\Omega)\\ &=\lim_{\delta\to 0^-}\inf\frac{1}{\delta}\left(L\left(s+\delta,W\left(t,s+\delta,\Omega+\delta\left((PV)(s)-\left(Q\tilde{U}\right)(s)\right)\right)\right)\right.\\ &\left.-L\left(s,W\left(t,s,\Omega\right)\right)\right) \end{aligned} \tag{71}$$

(iii) For $b\in\mathbb{K}$ *and* $a_1,a_0\in\mathbb{CK}$,

$$\begin{aligned} &b\left(h(t,\Omega)\right)+\int_{\tau_0}^{t}c\left(h\left(s,W\left(t,s,\Omega\left(\mathrm{s}\right)\right)\right)\right)ds\leq L(t,\Omega)\ \ \textit{in } S(h,M)\ \textit{and}\\ &L(t,\Omega)\leq a_1\left(t,h(t,\Omega)\right)+a_0\left(t,h_0(t,\Omega)\right)\ \ \textit{in } S(h,M)\cap S(h_0,M) \end{aligned} \tag{72}$$

(iv) h_0 *is finer that* h*, that is, there exists a function* $\phi\in\mathbb{K}$ *such that*

$$h_0(t,\Omega)\leq M_0\ \ \textit{implies}\ \ h(t,\Omega)\leq\phi\left(h_0(t,\Omega)\right) \tag{73}$$

for some M_0 *with* $\phi(M_0)\leq M$*;*

(v) The solution $V(t,\tau_0,V_0)$ *of (12) for* $t\geq\tau_0$ *is ITD* (h_0,h_0)*-equi-attractive in the large with respect to the solution* $\tilde{U}(t,\tau_0,U_0)=U(t-\eta,t_0,U_0)$, for $\eta=\tau_0-t_0$.

Then, this implies the ITD (h_0,h)*-equi-attractiveness in the large of the solution* $V(t,\tau_0,V_0)$ *of (12) with respect to the solution* $\tilde{U}$*.*

Proof. We shall show that the solution $V(t,\tau_0,V_0)$ of (12) for $t\geq\tau_0$ is ITD (h_0,h)-equi-attractive in the large with respect to the solution $\tilde{U}$, that is, given any $\varepsilon,\alpha>0$ and $\tau_0\in\mathbb{R}_+$, there exists a $T=T(\tau_0,\varepsilon,\alpha)>0$ such that $h_0(\tau_0,V_0-U_0)<\alpha$ implies

$$h\left(t,V(t,\tau_0,V_0)-U(t-\eta,t_0,U_0)\right)<\varepsilon\ ,\quad t\geq\tau_0+T(\tau_0,\varepsilon,\alpha) \tag{74}$$

Assume that (74) is not true, then there exist solutions $\tilde{U}(t) = U(t-\eta, t_0, U_0)$, where $U(t, t_0, U_0)$ is the solution of (10) for $t \geq t_0$; and $V(t) = V(t, \tau_0, V_0)$ of (12) for $t \geq \tau_0$, and a sequence $\{t_k\}$, $t_k \geq \tau_0 + T$ and $\lim_{k\to\infty} t_k = \infty$ such that

$$h_0(\tau_0, V_0 - U_0) < \alpha, \quad h(t_k, \Omega(t_k)) \geq \varepsilon \ for \ t_k \geq \tau_0 + T \tag{75}$$

where $\Omega(t) = V(t) - \tilde{U}(t)$ for $t \geq \tau_0$.

By Theorem 2, we have, for $t \geq \tau_0$,

$$L(t, \Omega(t)) \leq L(\tau_0, W(t, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t} c(h(s, W(t, s, \Omega(s))))\, ds \tag{76}$$

Then, using the assumptions (iii), (75) and (76), we obtain

$$\begin{aligned}
& b(\varepsilon) + \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(\mathrm{s}))))\, ds \\
& \leq b(h(t_k, \Omega(t_k))) + \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(\mathrm{s}))))\, ds \leq L(t_k, \Omega(t_k)) \\
& \leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) - \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(s))))\, ds \\
& \leq L(\tau_0, W(t_k, \tau_0, V_0 - U_0)) + \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(s))))\, ds \\
& \leq a_1(\tau_0, h(\tau_0, W(t_k, \tau_0, V_0 - U_0))) + a_0(\tau_0, h_0(\tau_0, W(t_k, \tau_0, V_0 - U_0))) \\
& + \int_{\tau_0}^{t} c(h(s, W(t_k, s, \Omega(s))))\, ds
\end{aligned} \tag{77}$$

We aim to reach a contradiction to conclude the proof of the theorem. We will use the assumption (v) for this purpose.

Given $0 < \varepsilon < M$ and that there exists a M_0 with $\phi(M_0) \leq M$.

Choosing $N_1 = N_1(\tau_0, \varepsilon)$ such that $0 < N_1(\tau_0, \varepsilon) < M_0$, and

$$h_0(t, \Omega(t)) < N_1 \ \ implies \ \ a_0(t, h_0(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \ for \ t \geq \tau_0 \tag{78}$$

By assumption (v), corresponding to this N_1, there exists a α_1 and a $T_1 = T_1(\tau_0, N_1, \alpha_1) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \alpha_1 \ \ implies \ \ h_0(t, \Omega(t)) < N_1 \ for \ t \geq \tau_0 + T_1 \tag{79}$$

Thus (78) and (79) give us

$$h_0(\tau_0, V_0 - U_0) < \alpha_1 \ \ implies \ \ a_0(t, h_0(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \ for \ t \geq \tau_0 + T_1 \tag{80}$$

Similarly, we choose $N_2 = N_2(\tau_0, \varepsilon)$ such that $0 < N_2(\tau_0, \varepsilon) < M_0$ and

$$h(t, \Omega(t)) < N_2 \;\; implies \;\; a_1(t, h(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \;\; for \; t \geq \tau_0 \tag{81}$$

By the assumptions (iv) and (v) also, corresponding to $\phi^{-1}(N_2)$, there exists a α_2 and a $T_2 = T_2(\tau_0, N_2, \alpha_2) > 0$ such that

$$h_0(\tau_0, V_0 - U_0) < \alpha_2 \;\; implies \;\; h_0(t, \Omega(t)) < \phi^{-1}(N_2) \;\; for \; t \geq \tau_0 + T_2 \tag{82}$$

Since $\phi \in \mathbb{K}$ is strictly monotone increasing; then, we have by taking the composition of ϕ of both sides of the inequality $h_0(t, \Omega(t)) < \phi^{-1}(N_2)$ in (82), with considering (73),

$$\begin{aligned} &h_0(\tau_0, V_0 - U_0) < \alpha_2 \;\; implies \\ &h(t, \Omega(\mathrm{t})) \leq \phi(h_0(t, \Omega(t))) < \phi\left(\phi^{-1}(N_2)\right) = N_2 \; for \; t \geq \tau_0 + T_2 \end{aligned} \tag{83}$$

So, (81) and (83) give us, for $t \geq \tau_0 + T_2$,

$$h_0(\tau_0, V_0 - U_0) < \alpha_2 \;\; implies \;\; a_1(t, h(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \tag{84}$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$, and $T = \max\{T_1, T_2\}$, then,

$$T{=}T(T_1, T_2) = T(\tau_0, N_1, \alpha_1, N_2, \alpha_2) = T(\tau_0, \varepsilon, \alpha) \tag{85}$$

Therefore, with these α, T the following statement holds.

$$\begin{aligned} &h_0(\tau_0, V_0 - U_0) < \alpha \;\; implies \\ &a_0(t, h_0(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \; and \; a_1(t, h(t, \Omega(t))) < \frac{b(\varepsilon)}{2} \;\; for \; t \geq \tau_0 + T \end{aligned} \tag{86}$$

Hence, when $t = t_1$, using (86), the statement (77) can be written as

$$\begin{aligned}
&b(\varepsilon) + \int_{\tau_0}^{t_1} c\left(h\left(s, W\left(t_k, s, \Omega\left(\mathrm{s}\right)\right)\right)\right) ds \\
&\leq b\left(h\left(t_k, \Omega\left(t_k\right)\right)\right) + \int_{\tau_0}^{t_1} c\left(h\left(s, W\left(t_k, s, \Omega\left(\mathrm{s}\right)\right)\right)\right) ds \leq L\left(t_k, \Omega\left(t_k\right)\right) \\
&\leq L\left(\tau_0, W\left(t_k, \tau_0, V_0 - U_0\right)\right) - \int_{\tau_0}^{t_1} c\left(h\left(s, W\left(t_k, s, \Omega\left(s\right)\right)\right)\right) ds \\
&\leq L\left(\tau_0, W\left(t_k, \tau_0, V_0 - U_0\right)\right) + \int_{\tau_0}^{t_1} c\left(h\left(s, W\left(t_k, s, \Omega\left(s\right)\right)\right)\right) ds \\
&\leq a_1\left(\tau_0, h\left(\tau_0, W\left(t_k, \tau_0, V_0 - U_0\right)\right)\right) + a_0\left(\tau_0, h_0\left(\tau_0, W\left(t_k, \tau_0, V_0 - U_0\right)\right)\right) \\
&+ \int_{\tau_0}^{t_1} c\left(h\left(s, W\left(t_k, s, \Omega\left(s\right)\right)\right)\right) ds \\
&< \frac{b(\varepsilon)}{2} + \frac{b(\varepsilon)}{2} + \int_{\tau_0}^{t_1} c\left(h\left(s, W\left(t_k, s, \Omega\left(s\right)\right)\right)\right) ds \\
&= b(\varepsilon) + \int_{\tau_0}^{t_1} c\left(h\left(s, W\left(t_k, s, \Omega\left(s\right)\right)\right)\right) ds
\end{aligned} \tag{87}$$

This contradiction proves the ITD (h_0, h)-equi-attractiveness in the large of the solution $V(t, \tau_0, V_0)$ of (12) for $t \geq \tau_0 + T(\tau_0, \varepsilon, \alpha)$ with respect to the solution $\tilde{U}$. □

The next theorem gives sufficient conditions to the ITD (h_0, h)-Lagrange stability of the solution $V(t, \tau_0, V_0)$ of (12) through (τ_0, V_0) for $t \geq \tau_0$ with respect to the solution $\tilde{U}(t, \tau_0, U_0) = U(t - \eta, t_0, U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (10) through (t_0, U_0) for $t \geq t_0$; providing that the solution $V(t, \tau_0, V_0)$ of (12) is ITD (h_0, h_0)- Lagrange stable with respect to $\tilde{U}$.

Theorem 5. *Assume that*

(i) Both $L(t, \Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ *and* $\|W(t, s, \Omega)\|$ *satisfy a local Lipschitz condition in* Ω *for any* t, s*; where* $W(t) = W(t, \tau_0, V_0 - U_0)$ *is the solution of (13) for* $t \geq \tau_0$ *and*

$$\Omega(t, \tau_0, V_0 - U_0) = V(t) - \tilde{U}(t) \ \ for\ t \geq \tau_0 \tag{88}$$

(ii)

$$D_{*-}L(t, s, \Omega) \leq -c\left(h\left(s, W\left(t, s, \Omega\left(\mathrm{s}\right)\right)\right)\right) \ \ in\ S(h, M) \tag{89}$$

where

$$S(h, M) = \{(t, \Omega)\ : h(t, \Omega) < M \text{ for some } h \in \Gamma \text{ and } M > 0\} \tag{90}$$

and

$$\begin{aligned} &D_{*-}L(t,s,\Omega) \\ &= \lim_{\delta\to 0^-}\inf\frac{1}{\delta}\left(L\left(s+\delta, W\left(t, s+\delta, \Omega+\delta\left((PV)(s)-\left(Q\tilde{U}\right)(s)\right)\right)\right)\right. \\ &\left. - L(s, W(t,s,\Omega))\right) \end{aligned} \tag{91}$$

(iii) For $b\in\mathbb{K}$ and $a_1, a_0\in\mathbb{CK}$,

$$\begin{aligned} &b(h(t,\Omega)) + \int_{\tau_0}^{t} c(h(s, W(t,s,\Omega(\mathrm{s}))))\,ds \le L(t,\Omega) \ \ \textit{in } S(h,M)\,\textit{and} \\ &L(t,\Omega) \le a_1(t, h(t,\Omega)) + a_0(t, h_0(t,\Omega)) \ \ \textit{in } S(h,M)\cap S(h_0,M) \end{aligned} \tag{92}$$

(iv) h_0 is finer that h, that is, there exists a function $\phi\in\mathbb{K}$ such that

$$h_0(t,\Omega)\le M_0 \quad \textit{implies} \quad h(t,\Omega)\le\phi(h_0(t,\Omega)) \tag{93}$$

for some M_0 with $\phi(M_0)\le M$;

(v) The solution $V(t,\tau_0,V_0)$ of (12) for $t\ge\tau_0$ is ITD (h_0,h_0)-Lagrange stable with respect to the solution $\tilde{U}(t,\tau_0,U_0)=U(t-\eta,t_0,U_0)$, for $\eta=\tau_0-t_0$.

Then, this implies the ITD (h_0,h)-Lagrange stability of the solution $V(t,\tau_0,V_0)$ of (12) for $t\ge\tau_0$ with respect to the solution $\tilde{U}$.

Proof. The ITD (h_0,h_0)-Lagrange stability of the solution $V(t,\tau_0,V_0)$ of (12) for $t\ge\tau_0$ with respect to the solution $\tilde{U}$ gives us by definition the ITD (h_0,h_0)-equi-boundedness and the ITD (h_0,h_0)-equi-attractiveness in the large of the solution $V(t,\tau_0,V_0)$ of (12) for $t\ge\tau_0$ with respect to the solution $\tilde{U}$. Hence, by applying Theorem 3 and Theorem 4 respectively, we obtain the ITD (h_0,h)-equi-boundedness and the ITD (h_0,h)-equi-attractiveness in the large of the solution $V(t,\tau_0,V_0)$ of (12) with respect to the solution $\tilde{U}$. That is to say it is ITD (h_0,h)-Lagrange stable with respect to the solution $\tilde{U}$, by definition. □

4. Conclusions

In this manuscript, we have presented sufficient conditions for ITD equiboundedness, equi-attractiveness in the large, and Lagrange stability in terms of two measures for the solutions of perturbed SDEs involving causal operators in regard to their unperturbed ones, and proved the sufficiency of these conditions using ITD variational comparison results.

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HADAMARD PRODUCT OF HOLOMORPHIC MAPPINGS ASSOCIATED WITH THE CONIC SHAPED DOMAIN

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Abstract. We define certain subclasses $\delta\text{–}\mathcal{UM}(\ell,\eta_1,\eta_2)$ and $\delta\text{–}\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ of holomorphic mappings involving some differential inequalities. These functions are actually generalizations of some basic families of starlike and convex mappings. We study sufficient conditions for $f\in\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$. We also discuss the characterization for $f\in\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ along with the coefficient bounds and other problems. Using certain conditions for functions in the class $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$, we also define another class $\delta-\mathcal{UM}^*(\ell,\eta_1,\eta_2)$ and study some subordination related result.

1. Introductory Concept

Let $\mathcal{H}=\mathcal{H}(\mathbb{U})$ denote the family of mappings f holomorphic in the open unit disc $\mathbb{U}:=\{z\in\mathbb{C} \text{ and } |z|<1\}$. For $m\in\mathbb{N}$ and $\alpha\in\mathbb{C}$, let $f\in\mathcal{H}[\alpha,m]\subset\mathcal{H}: f(z)=\alpha+\sum_{m=1}^{\infty}\alpha_m z^m$ and $f\in\mathcal{A}\subset\mathcal{H}[\alpha,m]$:

$$f(z)=z+\sum_{m=2}^{\infty}\alpha_m z^m, z\in\mathbb{U}. \tag{1}$$

Let $\mathcal{P}$ denote the family of Carathéodory mappings q with $\Re(q(z))>0$ and

$$q(z)=1+\sum_{m=1}^{\infty}q_m z^m, z\in\mathbb{U}.$$

The Möbius transformation $l_0(z)=\frac{1+z}{1-z}, z\in\mathbb{U}$ is an extremal mapping for the family $\mathcal{P}$. For $f,\ell\in\mathcal{H}$, we say the mapping f *is subordinate to* ℓ and mathematically write $f(z)\prec\ell(z)$, if for $w\in\mathcal{H}(\mathbb{U}): w(0)=0$ and $|w(z)|<1$, we have

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$f(z) = \ell\big(w(z)\big)$. For reference, see [10]. Applying subordination, Janowski [8] introduced the family $\mathcal{P}[\eta_1, \eta_2]$ for $-1 \leq \eta_2 < \eta_1 \leq 1$. A mapping $q \in \mathcal{P}[\eta_1, \eta_2]$, if

$$q(z) \prec \frac{1+\eta_1 z}{1+\eta_2 z} \text{ or } q(z) = \frac{1+\eta_1 w(z)}{1+\eta_2 w(z)}, z \in \mathbb{U},$$

where w is a *Schwarz mapping*. For detail of some work related to subordination, we refer, [2–6, 8, 10]. Clearly, $\mathcal{P}[\eta_1, \eta_2]$ is contained in $\mathcal{P}\left(\frac{1-\eta_1}{1-\eta_2}\right)$. This family is related with the class $\mathcal{P}$. A mappings $q \in \mathcal{P}$ iff

$$\frac{(\eta_1+1)\, q(z) - (\eta_1 - 1)}{(\eta_2+1)\, q(z) - (\eta_2 - 1)} \in \mathcal{P}[\eta_1, \eta_2].$$

The simplest representation of a conic domain Δ_δ, $\delta \geq 0$ is given in the following:

$$\Delta_\delta = \left\{ w = u + iv : u > \delta\sqrt{(u-1)^2 + v^2} \right\}.$$

A mapping $f \in \delta - \mathcal{US}(\beta)$ if the following inequality holds:

$$\Re\left\{ \frac{zf'(z)}{f(z)} - \beta \right\} > \delta \left| \frac{zf'(z)}{f(z)} - 1 \right| \qquad (z \in \mathbb{U}), \tag{2}$$

where $-1 \leq \beta < 1$ and $\delta \geq 0$.

A mapping $f \in \delta - \mathcal{UC}(\beta)$ iff $zf' \in \delta - \mathcal{US}(\beta)$.

The above families are studied by Goodman [7] and Rönning [13]. For mappings $f, \ell \in \mathcal{A}$, the convolution $f * \ell$ is defined by

$$f(z) * \ell(z) = z + \sum_{m=2}^{\infty} \alpha_m \gamma_m z^m = \ell(z) * f(z) \qquad (z \in \mathbb{U}),$$

where the mapping f is given by (1) and

$$\ell(z) = z + \sum_{m=2}^{\infty} \gamma_m z^m \qquad (z \in \mathbb{U}). \tag{3}$$

In 2008, Raina [12] introduced the family $\delta - \mathcal{US}(\ell, \beta)$ which may be defined as follows:

Definition 1. *Let ℓ be given by (3) with $\gamma_m \geq 0$, we say that $f \in \delta - \mathcal{US}(\ell, \beta)$ if $f(z) * \ell(z) \neq 0$ and*

$$\Re\left\{ \frac{z\,(f*\ell)'(z)}{f(z) * \ell(z)} - \beta \right\} > \delta \left| \frac{z\,(f*\ell)'(z)}{f(z) * \ell(z)} - 1 \right| \qquad (z \in \mathbb{U}),$$

where

$$(f * \ell)(z) = z + \sum_{m=2}^{\infty} \alpha_m \gamma_m z^m \qquad (\gamma_m \geq 0, z \in \mathbb{U}), \tag{4}$$

$-1 \leq \beta < 1$ *and* $\delta \geq 0$.

Generally this family consists of uniformly δ-starlike mappings $f * \ell$ of order β in $\mathbb{U}$.

In 2011, Noor and Malik [11] introduced the family $\delta - \mathcal{UM}(\eta_1, \eta_2)$ which is defined as:

Definition 2. *A mapping $f \in \mathcal{A}$ given by (1), is in the family $\delta - \mathcal{UM}(\eta_1, \eta_2)$ provided that $f(z) \neq 0$ and*

$$\Re\left\{\frac{(\eta_2 - 1)\frac{zf'(z)}{f(z)} - (\eta_1 - 1)}{(\eta_2 + 1)\frac{zf'(z)}{f(z)} + (\eta_1 - 1)}\right\} > \delta\left|\frac{(\eta_2 - 1)\frac{zf'(z)}{f(z)} - (\eta_1 - 1)}{(\eta_2 + 1)\frac{zf'(z)}{f(z)} + (\eta_1 - 1)} - 1\right| \qquad (z \in \mathbb{U}),$$

where $-1 \leq \eta_2 < \eta_1 < 1$ and $\delta \geq 0$.

This family consists of mappings f which are associated with uniformly δ-starlike mappings in $\mathbb{U}$. Extending the idea of Noor and Malik [11], we define a new family $\delta - \mathcal{UM}(\ell, \eta_1, \eta_2)$ of holomorphic mappings.

Definition 3. *Let $f \in \mathcal{A}$. Then $f \in \delta - \mathcal{UM}(\ell, \eta_1, \eta_2)$, if it satisfies the condition:*

$$\Re\left\{\frac{(\eta_2 - 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} - (\eta_1 - 1)}{(\eta_2 + 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} + (\eta_1 - 1)}\right\} > \delta\left|\frac{(\eta_2 - 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} - (\eta_1 - 1)}{(\eta_2 + 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} + (\eta_1 - 1)}\right|, (z \in \mathbb{U}), \tag{5}$$

where $f * \ell$ is given by (4), $-1 \leq \eta_2 < \eta_1 < 1$ and $\delta \geq 0$.

The mapping $f * \ell$ converges as a convolution of holomorphic mappings defined in $\mathbb{U}$. Clearly $f * \ell$ is associated with uniformly δ-starlike mappings in $\mathbb{U}$.

Let $\Im$ be the family of holomorphic mappings f of positive coefficients and having the series representation of the form:

$$f(z) = z - \sum_{m=2}^{\infty} \alpha_m z^m, \ \alpha_m \geq 0, z \in \mathbb{U}. \tag{6}$$

For details of this family, we refer [14].

Let f be given by (1). Then $f \in \delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$, if and only if

$$f \in \delta - \mathcal{UM}(\ell, \eta_1, \eta_2) \cap \Im,$$

where $-1 \leq \eta_2 < \eta_1 < 1, \delta \geq 0$ and $\Im$ is given by (6).

For some special choices, we obtain the following known classes:

i. $\delta - \mathcal{UM}\left(\frac{z}{1-z}, \eta_1, \eta_2\right) = \delta - \mathcal{US}(\eta_1, \eta_2)$ and $\delta - \mathcal{UM}\left(\frac{z}{(1-z)^2}, 1, \eta_1, \eta_2\right) = \delta - \mathcal{UC}(\eta_1, \eta_2)$.

ii. $\delta - \mathcal{UM}\left(\frac{z}{1-z}, 1, -1\right) = \delta - \mathcal{US}$ and $\delta - \mathcal{UM}\left(\frac{z}{(1-z)^2}, 1, -1\right) = \delta - \mathcal{UC}$..

iii. $\delta - \mathcal{UM}\left(\frac{z}{1-z}, 1 - 2\beta, -1\right) = \delta - \mathcal{US}(\beta)$ and $\delta - \mathcal{UM}\left(\frac{z}{(1-z)^2}, 1 - 2\beta, -1\right) = \delta - \mathcal{UC}(\beta)$.

iv. $0 - \mathcal{UM}\left(\frac{z}{1-z}, \eta_1, \eta_2\right) = \mathcal{S}^*(\eta_1, \eta_2)$ and $0 - \mathcal{UM}\left(\frac{z}{(1-z)^2}, \eta_1, \eta_2\right) = \mathcal{C}(\eta_1, \eta_2)$.

The class $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$ also reduces to the families mentioned in (2), see [13]. For detail of the above classes and various other cases related to the earlier contributions, see [1,3,8,9,11,15] with references therein.

2. Preliminaries

Subsequently, we define the subordinating factor sequence.

Definition 4. *A sequence $\langle c_m : m=1,2,3,...\rangle$ is termed as a subordinating factor sequence for some mappings in $\mathcal{C}$, if for each $f\in\mathcal{C}$, we have*

$$\sum_{m=1}^{\infty}\alpha_m c_m z^m \prec f(z) \qquad (\alpha_1=1, z\in\mathbb{U}). \tag{7}$$

Lemma 1. *The sequence $\langle c_m : m=1,2,3,...\rangle$ is a subordinating factor sequence, iff*

$$\Re\left\{1-2\sum_{m=2}^{\infty}c_m z^m\right\}>0.$$

For detail, see [9,16]. Throughout, we assume $\delta\geq 0$ and $-1\leq\eta_2<\eta_1\leq 1$.

3. Main Discussion

Theorem 1. *For a given mapping ℓ defined by (3) with $\gamma_m\geq 0$, if a mapping $f\in\mathcal{A}$ satisfies the inequality*

$$\sum_{m=2}^{\infty}\left[\{3+2\delta+\eta_2\}(m-1)+\eta_2-\eta_1\right]|\alpha_m|\gamma_m\leq\eta_1-\eta_2, \tag{8}$$

then $f\in\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$, where $m\geq\frac{1+\eta_1}{1+\eta_2}$ for $-1\leq\eta_2<\eta_1\leq 1$ and $\delta\geq 0$.

Proof. To have the desired proof, we only show that

$$\delta\left|\frac{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2+1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}+(\eta_1-1)}-1\right|-\Re\left\{\frac{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2+1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}+(\eta_1-1)}-1\right\}\leq 1$$

where $f*\ell$ is given by (4), $-1\leq\eta_2<\eta_1<1$ and $\delta\geq 0$. For $f*\ell$ given by (4), we see that

$$z\left(f\left(z\right)*\ell\left(z\right)\right)'=z+\sum_{m=2}^{\infty}m\left|\alpha_m\right|\gamma_m z^m.$$

Consider that

$$\delta\left|\frac{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2+1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1+1)}-1\right|-\Re\left\{\frac{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2+1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1+1)}-1\right\}$$
$$\leq(1+\delta)\left|\frac{(\eta_2-1)\,z\left(f\left(z\right)*\ell\left(z\right)\right)'-(\eta_1-1)\,f\left(z\right)*\ell\left(z\right)}{(\eta_2+1)\,z\left(f\left(z\right)*\ell\left(z\right)\right)'-(\eta_1+1)\,f\left(z\right)*\ell\left(z\right)}-1\right|$$

$$= 2(1+\delta)\left|\frac{z\left(f\left(z\right)*\ell\left(z\right)\right)' - f\left(z\right)*\ell\left(z\right)}{\left(\eta_2+1\right)z\left(f\left(z\right)*\ell\left(z\right)\right)' - \left(\eta_1+1\right)f\left(z\right)*\ell\left(z\right)}\right|$$

$$\leq \frac{2\sum\limits_{m=2}^{\infty}\left(1+\delta\right)\left(m-1\right)\left|\alpha_m\right|\gamma_m}{\eta_1-\eta_2-\sum\limits_{m=2}^{\infty}\left\{m\eta_2-\eta_1+m-1\right\}\left|\alpha_m\right|\gamma_m} \qquad \left(m \geq \frac{1+\eta_1}{1+\eta_2}\right).$$

The last expression is bounded by 1 if

$$\sum_{m=2}^{\infty}\left[\left(3+2\delta+\eta_2\right)\left(m-1\right)+\eta_2-\eta_1\right]\left|\alpha_m\right|\gamma_m \leq \eta_1-\eta_2.$$

□

We next prove the characterization of the mapping f as below.

Theorem 2. *A mapping f given by (6) belongs to the family $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ if and only if*

$$\sum_{m=2}^{\infty}\left\{\left(m-1\right)\left(1+2\delta-\eta_2\right)+\eta_1-\eta_2\right\}\alpha_m\gamma_m \leq \eta_1-\eta_2, \tag{9}$$

where $-1 \leq \eta_2 < \eta_1 \leq 1, \gamma_m > 0$ and $\delta \geq 0$.

Proof. Suppose that $f \in \delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$. Then, making use of the fact that

$$\Re w > \delta\left|w-1\right| \Leftrightarrow \Re\left\{w(1+\delta e^{i\theta})-\delta e^{i\theta}\right\} > 0$$

and taking

$$w\left(z\right) = \frac{\left(\eta_2-1\right)\frac{z\left(f(z)*\ell(z)\right)'}{f(z)*\ell(z)} - \left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z\left(f(z)*\ell(z)\right)'}{f(z)*\ell(z)} - \left(\eta_1+1\right)},$$

where $f*\ell$ is given by (4) with $\alpha_m \geq 0, -1 \leq \eta_2 < \eta_1 < 1$, and $\delta \geq 0$ in (5), we obtain

$$\Re\left\{\left(1+\delta e^{i\theta}\right)\frac{\left(\eta_2-1\right)\frac{z\left(f(z)*\ell(z)\right)'}{f(z)*\ell(z)} - \left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z\left(f(z)*\ell(z)\right)'}{f(z)*\ell(z)} - \left(\eta_1+1\right)} - \delta e^{i\theta}\right\} > 0,$$

or equivalently

$$\Re\left\{\left(1+\delta e^{i\theta}\right)\frac{\left(\eta_2-1\right)z\left(f\left(z\right)*\ell\left(z\right)\right)' - \left(\eta_1-1\right)f\left(z\right)*\ell\left(z\right)}{\left(\eta_2+1\right)z\left(f\left(z\right)*\ell\left(z\right)\right)' - \left(\eta_1+1\right)f\left(z\right)*\ell\left(z\right)} - \delta e^{i\theta}\right\} > 0,$$

which on simple manipulation yields

$$\Re\left\{\frac{\left(\eta_1-\eta_2\right)+\sum\limits_{m=2}^{\infty}\left\{m\eta_2-m-2\delta m e^{i\theta}+1-\eta_1+2\delta e^{i\theta}\right\}\alpha_m\gamma_m z^{m-1}}{\left(\eta_1-\eta_2\right)+\sum\limits_{m=2}^{\infty}\left\{m\left(\eta_2+1\right)-1-\eta_1\right\}\alpha_m\gamma_m z^{m-1}}\right\} > 0.$$

This result holds true for all $z \in \mathbb{U}$. Taking the limit $z \to 1^-$ through real values, we thus obtain that

$$\Re\left\{\frac{(\eta_1-\eta_2)+\sum\limits_{m=2}^{\infty}\left\{m\eta_2-m-2\delta me^{i\theta}+1-\eta_1+2\delta e^{i\theta}\right\}\alpha_m\gamma_m}{(\eta_1-\eta_2)+\sum\limits_{m=2}^{\infty}\left\{m\left(\eta_2+1\right)-1-\eta_1\right\}\alpha_m\gamma_m}\right\}>0,$$

which further implies that

$$\left\{\eta_1-\eta_2-\sum_{m=2}^{\infty}\left\{\left(1+2\delta-\eta_2\right)(m-1)+\eta_1-\eta_2\right\}\alpha_m\gamma_m\right\}>0,$$

so we have

$$\sum_{m=2}^{\infty}\left\{\left(1+2\delta-\eta_2\right)(m-1)+\eta_1-\eta_2\right\}\alpha_m\gamma_m<\eta_1-\eta_2.$$

Conversely, we let the inequality (9) hold true. Then, in view of the fact that $\Re\left(w\left(z\right)\right)>0$ if and only if $|w(z)-1|<|w(z)+1|$, where

$$w\left(z\right)=\frac{\left(\eta_2-1\right)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-\left(\eta_1+1\right)}-\delta\left|\frac{\left(\eta_2-1\right)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-\left(\eta_1+1\right)}-1\right|. \tag{10}$$

we consider

$$\begin{aligned}
&|w(z)+1| \\
&=\left|\frac{\left(\eta_2-1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1+1\right)}-\delta\left|\frac{\left(\eta_2-1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1+1\right)}-1\right|+1\right| \\
&=\frac{2\left|z\right|}{|G|}\left|\eta_1-\eta_2+\sum_{m=2}^{\infty}\left\{m\eta_2-\eta_1+\delta m-\delta\right\}\alpha_m\gamma_m z^{m-1}\right| \\
&>\frac{2}{|G|}\left[\eta_1-\eta_2-\sum_{m=2}^{\infty}\left\{m\eta_2-\eta_1+\delta m-\delta\right\}\alpha_m\gamma_m\right],
\end{aligned} \tag{11}$$

where $G=\left(\eta_2+1\right)z\left(f*\ell\right)'(z)-\left(\eta_1+1\right)f\left(z\right)*\ell\left(z\right)$. Also for $|w(z)-1|=W$

$$\begin{aligned}
W&=\left|\frac{\left(\eta_2-1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\eta_1\text{-}1}{\left(\eta_2+1\right)\frac{\frac{z(f*\ell)'(z)}{z(f*\ell)'(z)}}{(f*\ell)(z)}-\eta_1-1}-1-\delta\left|\frac{\left(\eta_2-1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\eta_1-1}-1\right|\right| \\
&=2\left|\frac{-\frac{z(f*\ell)'(z)}{(f*\ell)(z)}+1}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\eta_1-1}-\delta\left|\frac{-\frac{z(f*\ell)'(z)}{(f*\ell)(z)}+1}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\eta_1-1}\right|\right|
\end{aligned}$$

$$< \frac{2\,|z|}{|G|}\sum_{m=2}^{\infty}(m+m\delta-1-\delta)\,\alpha_m\gamma_m. \tag{12}$$

where $G=(\eta_2+1)\,z\,(f(z)*\ell(z))'-(\eta_1+1)\,f(z)*\ell(z)$. From the condition (9) and the inequalities (11) and (12), we deduce that

$$|w(z)+1|-|w(z)-1|>0,$$

where w is defined by (10). This completes the proof of Theorem 2. □

We next provide coefficient bound for a given mapping f to belong to the family $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$.

Corollary 1. *A mapping f belongs to the family $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ if*

$$\sum_{m=2}^{\infty}\alpha_m<\frac{\eta_1-\eta_2}{\{1+2\delta-2\eta_2+\eta_1\}\,\gamma_2},\gamma_2>0.$$

where $-1\le\eta_2<\eta_1<1$, and $\delta\ge0$.

Corollary 2. *For a mapping f belonging to the family $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$, we have*

$$\alpha_m<\frac{\eta_1-\eta_2}{\{1+2\delta-2\eta_2+\eta_1\}\,\gamma_2},\gamma_2>0.$$

where $-1\le\eta_2<\eta_1<1$ and $\delta\ge0$.

The subsequent theorem deals with the integral representation for a given mapping $f\in\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$.

Theorem 3. *If a mapping f given by (6) belongs to the family $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$, then f has the following representation:*

$$f(z)=\ell^{(-1)}(z)*\exp\left(\int_0^z\frac{2\delta\eta_1-Q(t)(\eta_1-1)}{t\,\{2\delta+Q(t)(\eta_2-1)\}}\mathrm{d}t\right),$$

where $-1\le\eta_2<\eta_1<1$ and $\delta\ge0$.

Proof. For $\delta=0$, the assertion of the Theorem 3 is obvious. Let $\delta>0$. Then, for $f\in\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ and

$$w(z)=\frac{(\eta_2-1)\,\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2+1)\,\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}+(\eta_1-1)}$$

we have

$$Re(w)>\delta|w-1|,$$

which implies that

$$|\frac{w-1}{w}|<\frac{1}{\delta}.$$

We suppose that

$$\frac{w-1}{w}=\frac{Q(z)}{\delta}$$

and

$$w(z) = \frac{\delta}{\delta - Q(z)},$$

which yields

$$\frac{(\eta_2 - 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} - (\eta_1 - 1)}{(\eta_2 - 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} - (\eta_1 + 1)} = \frac{\delta}{\delta - Q(z)}.$$

Thus on simplification, we have

$$\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} = \frac{2\delta\eta_1 - Q(z)(\eta_1 - 1)}{2\delta + Q(z)(\eta_2 - 1)}.$$

which proves that

$$f(z)*\ell(z) = \exp\left(\int_0^z \frac{2\delta\eta_1 - Q(t)(\eta_1 - 1)}{t\{2\delta + Q(t)(\eta_2 - 1)\}}\mathrm{d}t\right)$$

or

$$f(z) = \ell^{(-1)}(z) * \exp\left(\int_0^z \frac{2\delta\eta_1 - Q(t)(\eta_1 - 1)}{t\{2\delta + Q(t)(\eta_2 - 1)\}}\mathrm{d}t\right).$$

This finishes the proof of Theorem 3. □

Theorem 4. *If f_j is such that*

$$f_j(z) = z - \sum_{m=2}^{\infty} \alpha_{m,j} z^m \in \delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2), \qquad (j = 1, 2, z \in \mathbb{U}),$$

then

$$f(z) = (1 - \lambda) f_1(z) + \lambda f_2(z) \in \delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2), \qquad (0 \le \lambda \le 1, z \in \mathbb{U}).$$

Proof. For the mappings f_j such that $f_j(z) = z - \sum_{m=2}^{\infty} \alpha_{m,j} z^m \in \delta\text{–}\mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$, by using Theorem 2, we write

$$\sum_{m=2}^{\infty} \{(1 + 2\delta - \eta_2)(m - 1) + \eta_1 - \eta_2\} \alpha_{m,1}\gamma_m \le \eta_1 - \eta_2 \qquad (13)$$

and

$$\sum_{m=2}^{\infty} \{(1 + 2\delta - \eta_2)(m - 1) + \eta_1 - \eta_2\} \alpha_{m,2}\gamma_m \le \eta_1 - \eta_2. \qquad (14)$$

In view of (13) and (14), we have

$$\begin{aligned}
&(1 - \lambda) \sum_{m=2}^{\infty} \{(1 + 2\delta - \eta_2)(m - 1) + \eta_1 - \eta_2\} \alpha_{m,1}\gamma_m \\
&+ \lambda \sum_{m=2}^{\infty} \{(1 + 2\delta - \eta_2)(m - 1) + \eta_1 - \eta_2\} \alpha_{m,2}\gamma_m \\
&\le (1 - \lambda)(\eta_1 - \eta_2) + \lambda(\eta_1 - \eta_2) = \eta_1 - \eta_2.
\end{aligned}$$

Again by using Theorem 2, we reach the conclusion. □

In the following, we define the family $\delta\text{–}\mathcal{UM}^*(\ell, \eta_1, \eta_2)$ of holomorphic mappings f satisfying the coefficient conditions (8). Assume that

$$f(z) = z + \sum_{m=2}^{\infty} \alpha_m z^m \in \mathcal{A}.$$

Then $f \in \delta - \mathcal{UM}^*(\ell, \eta_1, \eta_2)$, if it satisfies the condition:

$$\sum_{m=2}^{\infty} \left[(3 + 2\delta + \eta_2)(m-1) + \eta_2 - \eta_1 \right] |\alpha_m| \gamma_m \leq \eta_1 - \eta_2,$$

for some $\gamma_m \geq 0, \delta \geq 0$ and $-1 \leq \eta_2 < \eta_1 \leq 1$.

For special choices of η_1, η_2, δ and the mapping ℓ, we refer the study of Aouf and Mostafa [2] and others. Clearly

$$\delta - \mathcal{UM}^*(\ell, \eta_1, \eta_2) \subset \delta - \mathcal{UM}(\ell, \eta_1, \eta_2).$$

Adopting the required procedure found in [2,3,15], we have:

Theorem 5. *If $f \in \delta - \mathcal{UM}^*(\ell, \eta_1, \eta_2)$ and*

$$\Re(f(z)) > -\frac{\eta_1 - \eta_2 + (3 + 2\delta + 2\eta_2 - \eta_1)\gamma_2}{(3 + 2\delta + 2\eta_2 - \eta_1)\gamma_2} \qquad (z \in \mathbb{U}), \tag{15}$$

then

$$\frac{(1 + 2\delta - 2\eta_2 + \eta_1)\gamma_2}{2\left[\eta_1 - \eta_2 + (1 + 2\delta - 2\eta_2 + \eta_1)\right]\gamma_2} f(z) * h(z) \prec h(z) \qquad (z \in \mathbb{U}), \tag{16}$$

for all $h \in \mathcal{C}$. The constant factor $\frac{(1+2\delta-2\eta_2+\eta_1)\gamma_2}{2[\eta_1-\eta_2+(1+2\delta-2\eta_2+\eta_1)]\gamma_2}$ in (16) cannot be replaced by a larger one.

Proof. Let $f \in \delta - \mathcal{UM}^*(\ell, \eta_1, \eta_2)$ and let $h(z) = z + \sum_{m=2}^{\infty} c_m z^m$. Then

$$\frac{(3 + 2\delta + 2\eta_2 - \eta_1)\gamma_2 f(z) * h(z)}{2\left[\eta_1 - \eta_2 + (3 + 2\delta + 2\eta_2 - \eta_1)\right]\gamma_2} = \frac{(3 + 2\delta + 2\eta_2 - \eta_1)\gamma_2 \left(z + \sum_{m=2}^{\infty} \alpha_m c_m z^m \right)}{2\left[\eta_1 - \eta_2 + (3 + 2\delta + 2\eta_2 - \eta_1)\right]\gamma_2}$$

In view of Definition 4 and Lemma 1, (16) will hold true if

$$\left\langle \frac{(3 + 2\delta + 2\eta_2 - \eta_1)\gamma_2 \alpha_m}{2\left[\eta_1 - \eta_2 + (3 + 2\delta + 2\eta_2 - \eta_1)\gamma_2\right]}, m = 1, 2, \ldots \right\rangle, \alpha_1 = 1 \tag{17}$$

is a subordinating factor sequence. Using Lemma 1, we observe that (17) is equivalent to

$$\Re\left\{ 1 + \sum_{m=1}^{\infty} \frac{(3 + 2\delta + 2\eta_2 - \eta_1)\gamma_2 \alpha_m z^m}{\left[\eta_1 - \eta_2 + (3 + 2\delta + 2\eta_2 - \eta_1)\gamma_2\right]} \right\} > 0. \tag{18}$$

The mapping

$$\varphi(m) = \left\{ (3 + 2\delta + \eta_2)(m-1) + \eta_2 - \eta_1 \right\} \gamma_m, \gamma_m \geq \gamma_2 > 0.$$

is an increasing mapping for $m \geq 2$. Considering this fact along with (18), we can write

$$\Re\left\{1+\sum_{m=1}^{\infty}\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\alpha_m z^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\right\}$$
$$=\Re\left\{1+\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 z}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}+\sum_{m=2}^{\infty}\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\alpha_m z^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\right\}$$
$$\geq 1-\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\,|z|}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}-\frac{\sum\limits_{m=2}^{\infty}(3+2\delta+2\eta_2-\eta_1)\gamma_2\,|\alpha_m|\,|z|^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}$$
$$\geq 1-\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}-\frac{\sum\limits_{m=2}^{\infty}(3+2\delta+2\eta_2-\eta_1)\gamma_2\,|\alpha_m|\,r^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}$$
$$\geq 1-\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\eta_1)\gamma_2]}-\frac{\sum\limits_{m=2}^{\infty}(3+2\delta+2\eta_2-\eta_1)\gamma_m\,|\alpha_m|\,r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}$$

On using (8), we see that

$$\Re\left\{1+\sum_{m=1}^{\infty}\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\,|\alpha_m|\,z^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\right\}$$
$$\geq 1-\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}-\frac{(\eta_1-\eta_2)\,r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}.$$
$$=1-r>0, r\to 1.$$

This leads to (18). Thus we have (16). Also (15) is obtained from (16) for the mapping

$$h\,(z)=\frac{z}{1-z},\qquad (z\in\mathbb{U}).$$

For the sharpness of

$$\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[(\eta_1-\eta_2)+(3+2\delta+2\eta_2-\eta_1)\gamma_2]},$$

we consider the mapping f_0 such that

$$f_0(z)=z-\frac{(\eta_1-\eta_2)}{(3+2\delta+2\eta_2-\eta_1)}z^2. \tag{19}$$

Combining (16) and (19), we write

$$\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}f_0(z)\prec\frac{z}{1-z},\ z\in\mathbb{U}.$$

Consider

$$\begin{aligned}
&\Re\left\{\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}f_0(z)\right\} \\
&= \frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\Re\left(f_0(z)\right) \\
&\geq -\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2)}{2[(\eta_1-\eta_2)+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\left(\frac{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1))\gamma_2]}{(3+2\delta+2\eta_2-\eta_1)\gamma_2}\right).
\end{aligned}$$

Thus, we have

$$\min_{|z|\leq r}\Re\left\{\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}f_0(z)\right\} = -\frac{1}{2}.$$

This proves that the constant $\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}$ is the best possible. □

4. Concluding Remarks

In this research, we have used convolution between holomorphic mappings in defining some subfamilies $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$ and $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ of holomorphic mappings involving starlike and convex mappings and associated with the conic domains. We derived sufficient conditions for the mappings to be in the family $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$. We also discussed the characterization of mappings in the family $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ along with the coefficient bounds, integral representation and convex combination. Using the sufficient conditions for mappings belonging to the family $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$, we also defined a family $\delta-\mathcal{UM}^*(\ell,\eta_1,\eta_2)$ and then making use of a particular sequence, we discussed some subordination result. Our findings can be related with the existing known results.

5. Research Background and Significance

Goodman studied the uniformly convex and starlike functions, whereas, Kanas and Wisniowska explored k-uniformly convex and k-uniformly starlike functions. While using the convolution technique, Raina introduced the similar family of analytic functions. In view of Janowski functions, Noor and Malik extended their results for the petal like domains. Using Hadamard product used by Raina and in contaxt of Noor and Malik work, we defined new classes of analytic functions and studied them in various aspects.

Functions with positive real part as well as function with certain assumptions on the arguments are of funcdamental importance in the study of starlike, convex, close-to-convex and Bazilevic functions which are related with the Kufarev differential equation. We study the characterization and bounds on the functions from the differential and integral ineqalities. Same study for the complex valued function is carried out using the idea of differential subordination. The study of the geometric properties of various types of image domains is still a prime focus of the theorists. Techniques of convolutions and other classical methods are still in progress in studying these images of compex analytic univalent and multivalent functions In this research, we have used convolution between holomorphic mappings in defining some subfamilies $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$ and $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ of holomorphic mappings involving starlike and convex mappings and associated with the conic domains. We derived sufficient conditions for the mappings to be in the family $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$. We also discussed the characterization of mappings in the family $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ along with the coefficient bounds, integral representation and convex combination. Using the sufficient conditions for mappings belonging to the family $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$, we also defined a family $\delta-\mathcal{UM}^*(\ell,\eta_1,\eta_2)$ and then making use of a subordinating factor sequence, we discuss some subordination result. Our findings can be related with the existing literature of subject. Various problems like radius of convexity, starlikeness and close-to-convexity are still open.

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Declaration of Competing Interests The authors declare that the research is original and have not submitted anywhere in any form. Authors have no competing interest to disclose. The authors accept any sort of ethical responsibility regarding the manuscript publication.

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A MATHEMATICAL ANALYSIS OF COOPERATIVITY AND FRACTIONAL SATURATION OF OXYGEN IN HEMOGLOBIN

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Abstract. Hemoglobin (Hb) possesses good properties of cooperative system and it normally executes oxygen and other essential items via erythrocytes in the body. The chemical action of Hb is to combine with oxygen (O_2) in the lungs to form oxyhemoglobin (HbO_2). Binding of oxygen with a hemoglobin is one of the important cooperative mechanism and is an emerging mathematical research area with wide range of applications in biomedical engineering and medical physiology. To this end, a mathematical model is proposed to study the fractional saturation of oxygen in hemoglobin to understand the binding effect and its stability at various stages. The mathematical formulation is based on the system of ordinary differential equations together with rate equations under different association and dissociation rate constants. The five states of the cooperative systems $Hb, HbO_2, Hb(O_2)_2, Hb(O_2)_3$ and $Hb(O_2)_4$ are modelled and the Hill's function has been used to approximate the binding effect and saturation of ligand (O_2) with respect to various rate constants. Also, the Adair equation has been employed to interpret the saturation concentrations of oxygen in hemoglobin.

1. Introduction

In a biological system the interaction and intricate association between macromolecules is an established fact. This meticulous interaction between different ligands with their respective receptor molecules determine the fate of most cellular processes which decides reaction, adjustment and conduct of basic functions in all living organisms [12, 13]. The well known example in biological system is interaction between hemoglobin with its four binding sites for oxygen [2, 3, 17]. The ligand oxygen binds to the four binding sites of hemoglobin molecule and the interaction can be seen on the overall binding curve [18]. This is a striking example of allosteric

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binding. The binding curve of hemoglobin when associated with oxygen molecules gives sharp sigmoidal curve which indicates influence of oxygen molecules over the functionality of hemoglobin. The inference which could be drawn from the sigmoid binding curve indicates that the extremal states of full and zero saturation are more stable than the intermediate states of partial saturation [2, 17, 31]. A general kinetic model, presages an unanticipated multiplicative boost in affinity as a function of ligand sites. Modeling of this interaction by a foremost depart time approach denotes that the probability of ligand rebinding increases exponentially with the number of sites [30]. A few small single-dentate molecules when bind to a large polydentate molecule such that affinity of oxygen for binding interactions increases, it arises the cooperativity [14,23]. In cooperative enzymes, low and high affinity substrate binding sites are present, and the cooperative binding of substrate to enzyme can take place. The binding of one substrate molecule induces structural and/or electronic changes that result in altered substance binding affinities in the remaining vacant site. As there is no straightforward relationship between macroscopic and microscopic binding behavior, a mathematical model has been developed to create a bridge between them. The model considered the minimal interaction essential to produce fixed overall binding curve [20]. The whole binding cascade of human hemoglobin corresponds of a series of partly ligated intermediates. The discrete intervening constants cannot be differentiated in O_2 binding curves. The characterization of these O_2 binding constants has shown the Hb cascade to be unbalanced in nature, with binding dependent upon the particular distribution of O_2 among the four heme sites. The kinetic constant noticed for the dissociation of this intervening O_2 binding constant confirms the value for its equilibrium [19]. The rationale behind the current study was primarily to understand the fractional saturation of oxygen in hemoglobin under various rate constants. Moreover, the cooperative property of hemoglobin has been exhaustively discussed using basic mathematical tools.

2. Materials and Methods

Cooperativity is a fundamental specificity of various biochemical systems [24]. It was Archibald Hill [1,16] who first analysed the cooperativity binding of oxygen by hemoglobin and postulated that several (n) oxygen molecules bind simultaneously to a hemoglobin molecule:

$$Hb_4 + nO_2 \rightleftharpoons Hb_4O_{2n} \tag{1}$$

The expression for the association constant becomes

$$K_a = \frac{[Hb_4O_{2n}]}{[Hb_4][O_2]^n}, \tag{2}$$

and the binding equilibrium from the stand point of the fraction, Y, of oxygen binding sites on the hemoglobin that are occupied by ligand:

$$Y = \frac{[O_2]^n}{[O_2]^n + K_d}. \tag{3}$$

This Hill equation delineates the sigmoidal binding curves for hemoglobin as shown in Figure 1. The value of n is known as the Hill coefficient. The value of n is not always an integer. Among binding sites, for cooperative binding, the Hill coefficients settled provide a fertile measure of the Gibbs free energy of interaction and their values are independent of the free energy of association for empty sites [5]. The values of the Hill equation parameters also depend on hemoglobin concentration and shows that at high concentration of hemoglobin, the visible Hill coefficient, n, decreases and the binding affinity, k, increases [27]. The Hill's equation has been used to approximate binding effect and saturation of oxygen under various rate constants. The utility of Adair equation helped us to illustrate the saturation of oxygen concentration in hemoglobin. Many enzymes are composed of distinct subunits (oligomers), each bearing an equivalent catalytic site. If the sites are identical and dependent of each other, the presence of substrate at one site effects on substrate binding and catalytic properties at other sites, this process is known as cooperative system. [21]. We considered an oligomeric cooperative system, a cooperative tetramer (hemoglobin) in which we have discussed fractional saturation of hemoglobin at various states under variable rate constants.

2.1. **Some Basic Definitions.**

Definition 1. *A ligand is a substance that binds to a target molecule to serve a given purpose.*

Definition 2. *Allosteric enzymes are the enzymes that change their conformational ensemble upon binding of an effector which results in an apparent change in binding affinity at a different ligand binding site.*

Definition 3. *An oligomer is a protein consisting of many sub-units. It may be dimer, trimer, tetramer and so on, according to the number of subunits.*

Definition 4. *The fractional saturation of O_2 is defined as $Y(O_2)$ = number of occupied binding sites/total number of binding sites.*

2.2. **Mathematical Formulation.**

We shall first consider the theory for a hemoglobin molecule consisting of four protomers, each containing one active centre. Active sites are assumed to be independent of each other in their interaction with the molecule of an oxygen (substrate). The individual reactions of oxygen with hemoglobin are as follows,

$$o_2 + c_j \underset{k_{-1}}{\overset{k_{+1}}{\rightleftharpoons}} c_{j+1}; \quad j = 0, 1, 2, 3. \tag{4}$$

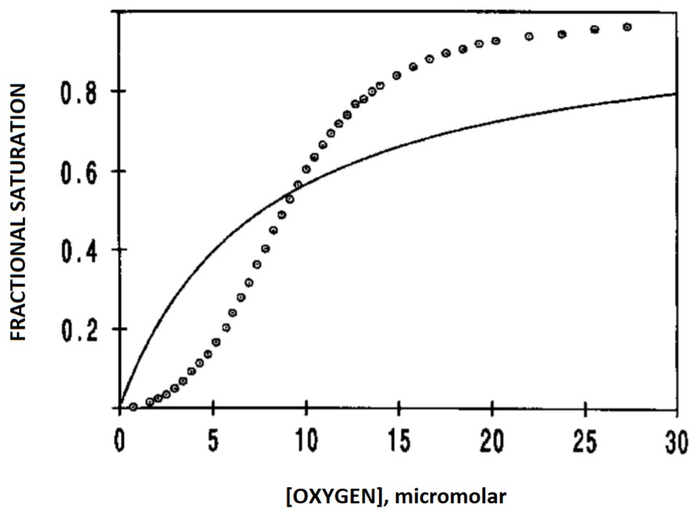

FIGURE 1. Sigmoid- Shaped hemoglobin oxygen- binding curve. These data were measured at pH 7.4 , 21.5^0 , 0.1 M NaCl, 0.1 M Tris, 1.0 mM Na_2EDTA, and a hemoglobin A concentration of 382.5μ M (heme). Also shown is the least-squares estimated hyperbola based on the Hufer model.

where c_j are the complex of the hemoglobin combined with oxygen molecules (j runs from 0 to 4) and the rate constant for binding the oxygen to a particular site of the hemoglobin are denoted by k_{+i} for association and k_{-i} for dissociation, i = 1,2,3,4.

Alternative representation of (4) of the reactions [21], we shall introduce Figure 2 below which has a one - one correspondence with the rate equations for the concentrations of c_j.

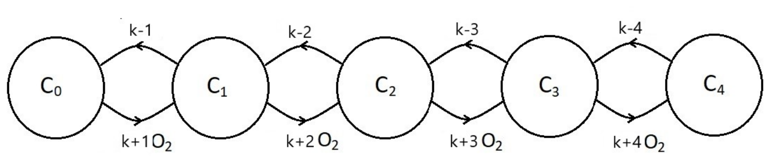

FIGURE 2. Schematic graphical representation of hemoglobin states.

From the Figure 2, it is obvious to see that, there are four unoccupied sites in the state c_0 and the rate constant from c_0 to c_1 is k_{+1}. Also from c_1 to c_0 , dissociation rate constant is k_{-1}. Similarly, the rate constant from c_3 to c_4 is k_{+4} and from c_4 to c_3, the dissociation rate constant is k_{-4}.

By deriving the above process, the rate equations (in lower case letters) are defined as:

$$o_2 + c_0 \underset{k_{-1}}{\overset{k_{+1}}{\rightleftharpoons}} c_1, \tag{5}$$

$$o_2 + c_1 \underset{k_{-2}}{\overset{k_{+2}}{\rightleftharpoons}} c_2, \tag{6}$$

$$o_2 + c_2 \underset{k_{-3}}{\overset{k_{+3}}{\rightleftharpoons}} c_3, \tag{7}$$

$$o_2 + c_3 \underset{k_{-4}}{\overset{k_{+4}}{\rightleftharpoons}} c_4. \tag{8}$$

The differential equations corresponding to the above reactions are as follows:

$$\frac{dc_0}{dt} = -k_{+1}o_2c_0 + k_{-1}c_1, \tag{9}$$

$$\frac{dc_1}{dt} = k_{+1}o_2c_0 - k_{-1}c_1 - k_{+2}o_2c_1 + k_{-2}c_2, \tag{10}$$

$$\frac{dc_2}{dt} = k_{+2}o_2c_1 - k_{-2}c_2 - k_{+3}o_2c_2 + k_{-3}c_3, \tag{11}$$

$$\frac{dc_3}{dt} = k_{+3}o_2c_2 - k_{-3}c_3 - k_{+4}o_2c_3 + k_{-4}c_4, \tag{12}$$

$$\frac{dc_4}{dt} = k_{+4}o_2c_3 - k_{-4}c_4. \tag{13}$$

2.3. **Solution of the Model.**

To estimate the concentration of O_2 at various states and subsequent changes of complexes in different states, it is important to compute the values of Eq.(9) - Eq.(13) at steady state points. Now the steady state values of Eq.(13) are given by

$$\frac{dc_4}{dt} = 0.$$

In the equilibrium model, the substrate-binding step is considered to be rapid comparative to the rate of breakdown of the ES complex. Therefore, the substrate binding reaction is considered to be at equilibrium and depends on rate constants [22, 25]. Similarly, in this case, oxygen binding reaction is assumed to be at equilibrium.

Define the equilibrium constant as $k_{di} = \frac{k_{-i}}{k_{+i}}$, the dissociation constant.

It follows that

$$k_{+4}o_2c_3 = k_{-4}c_4,$$

$$\Rightarrow c_4 = \frac{k_4}{k_{-4}}o_2c_3. \tag{14}$$

Note that k_{di} has dimensions of oxygen concentration, so $\frac{o_2}{k_{di}}$ is dimensionless. Going next to (12) and setting $\frac{dc_3}{dt} = 0$, it follows that

$$c_3 = \frac{k_3}{k_{-3}}o_2c_2. \tag{15}$$

Proceeding in this fashion, we find that

$$c_2 = \frac{k_2}{k_{-2}}o_2c_1, \tag{16}$$

and

$$c_1 = \frac{k_1}{k_{-1}}o_2c_0. \tag{17}$$

By combining the equations(14), (15), (16)and (17) we see that all the equilibrium values of $c_j; j \geq 1$ may be expressed in terms of c_0 in a regular fashion.
Thus,

$$c_1 = \frac{1}{k_{d1}}[o_2][c_0], \tag{18}$$

$$c_2 = \frac{1}{k_{d1}k_{d2}}[o_2]^2[c_0], \tag{19}$$

$$c_3 = \frac{1}{k_{d1}k_{d2}k_{d3}}[o_2]^3[c_0], \tag{20}$$

$$c_4 = \frac{1}{k_{d1}k_{d2}k_{d3}k_{d4}}[o_2]^4[c_0]. \tag{21}$$

The complexes c_1, c_2, c_3 and c_4 occupy oxygen sites partially with an ascending behaviour until they fully saturate, therefore the saturation function $Y(o_2)$ can be computed as [21,25]:

$$Y(o_2) = \frac{c_1 + 2c_2 + 3c_3 + 4c_4}{4(c_0 + c_1 + c_2 + c_3 + c_4)} \tag{22}$$

using the concentration levels obtained in equations (18)-(21) and from Eq.(22), we have

$$Y(o_2) = \frac{\frac{[o_2]}{k_{d1}} + 2\frac{[o_2]^2}{k_{d1}k_{d2}} + 3\frac{[o_2]^3}{k_{d1}k_{d2}k_{d3}} + 4\frac{[o_2]^4}{k_{d1}k_{d2}k_{d3}k_{d4}}}{4(1 + \frac{[o_2]}{k_{d1}} + \frac{[o_2]^2}{k_{d1}k_{d2}} + \frac{[o_2]^3}{k_{d1}k_{d2}k_{d3}} + \frac{[o_2]^4}{k_{d1}k_{d2}k_{d3}k_{d4}})} \tag{23}$$

Eq.(23) is known as Adair equation for 4 sites.

The graphs of Y with respect to O_2 gives a sharp sigmoidal curve (see Fig.3) which indicates influence of oxygen molecules over the functionality of hemoglobin i.e; it increases the affinity for oxygen molecules. When the affinities of the later binding events are fundamentally greater than those of the previous events. This is called as positive cooperativity. For positive cooperativity ($k_4 \ll k_3, k_2, k_1$), the concentration of c_1, c_2, c_3 are small compared to the concentration of c_4. Thus, if these terms are omitted from Eq.(22), so Eq.(23) becomes,

$$Y(o_2) = \frac{4\frac{[o_2]^4}{k_{d1}k_{d2}k_{d3}k_{d4}}}{4(1 + \frac{[o_2]^4}{k_{d1}k_{d2}k_{d3}k_{d4}})} \tag{24}$$

$$= \frac{[o_2]^4}{\alpha^4 + [o_2]^4} \tag{25}$$

where $\alpha^4 = k_{d1}k_{d2}k_{d3}k_{d4}$

The Eq.(25) is formalised in the Hill function as in Eq.(3),

$$Y = \frac{[o_2]^n}{\alpha^n + [o_2]^n} \tag{26}$$

Eq.(26) is utilized to depict measures that include multiple near-simultaneous binding events. The constant α is the half-saturating concentration of ligand, and thus can be interpreted as an averaged dissociation constant.

It is trivial that in hemoglobin, the partial pressure of oxygen, pO_2 is the concentration of free oxygen $[O_2]$. Then $\alpha^4 = (p_{50})^4$, where p_{50} is the value of po_2 when half of the oxygen-binding sites are occupied. Making these replacements and taking logarithms, Eq.(25) may be revised to yield,

$$\log \frac{Y(o_2)}{1 - Y(o_2)} = 4\log[po_2] - 4\log p_{50}. \tag{27}$$

So, if hemoglobin bound all four oxygen molecules in a single step, then a plot of $\log \frac{Y}{1-Y}$ versus $\log[po_2]$ would be a straight line with a slope of n = 4 (called a Hill plot) and an intercept on the $\log[po_2]$ axis of $\log p_{50}$ (see Figure 4). We can also write, if $n = 1$ in Eq. (26), then it becomes the same result, if the protein is a monomer and Hill function reduces to hyperbolic function [4]. There are few

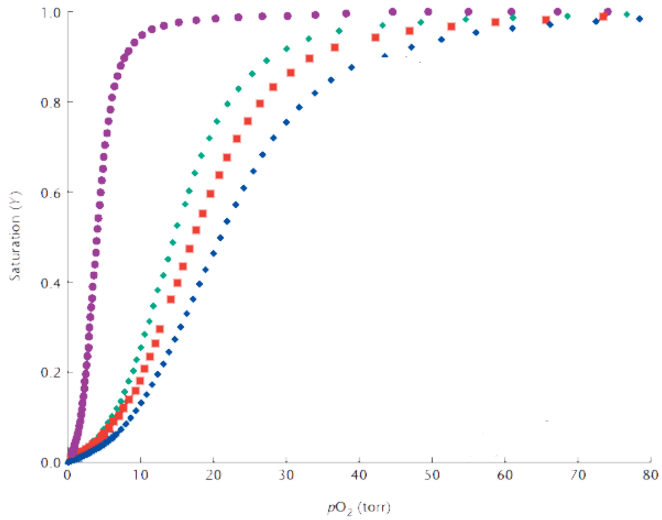

FIGURE 3. Relationship between the partial pressure of oxygen (pO_2) and percentage saturation of the hemoglobin with oxygen.

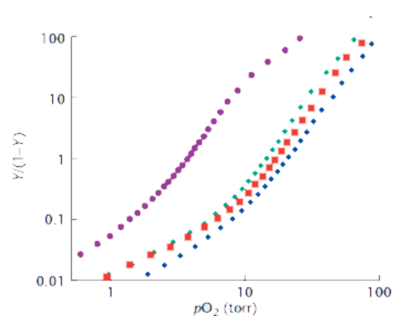

FIGURE 4. Lineweaver Burk plot of Hemoglobin.

examples like Ribonuclease, hexokinase, glucokinase in which a single monomeric enzyme show sigmoidal behavior, but cooperativity of these enzymes could not have been generated by the interaction of subunits. This mechanism is known as Kinetic cooperativity [15]. However, the Hill plot exhibits a sigmoidal shape (as shown in Fig.3), indicating that binding occurs in stages, such that the oxygen affinity of hemoglobin depends on the number of subunits in the tetramer that are already oxygenated.

3. Discussion and Conclusion

A mathematical model has been established that describes the interaction between an oxygen molecule and the hemoglobin molecule. Hemoglobin is a tetrameric protein and shows cooperativity i.e; consisting of four subunits and each subunit binds with one oxygen molecule. Cooperativity is a fundamental specificity of various biochemical systems [24]. Different biochemical mechanisms can create ultra sensitivity, including zero-order kinetics, second- and higher-order dependence on enzyme concentration, positive feedback and protein translocation [9, 10]. In this work, we have more focused on the role of mathematics on fractional saturation of oxygen in hemoglobin. A number of simplifying hypothesis are available to explore binding processes. In this case, we have introduced steady state hypothesis and then setting time derivatives (eq.13) equal to zero and solve for the steady state process. Although, this model provides a basis for understanding the S-shaped or sigmoidal character binding curve but it also confirms that the nature of individual binding sites (five binding states discussed above) does not account for sigmoid behaviour. In Figure 3 plots of the free oxygen concentration (partial pressure of oxygen) versus saturation (Y) exhibit a sigmoidal character and indicates that the affinity of hemoglobin for the first oxygen molecule is less than the subsequent ones. Figure 4 represents the Lineweaver Burk plot of hemoglobin. Furthermore, our theoretical results are a step towards understanding the role of differential equations in cooperativity in relation with empirical models. A simple, straightforward and a new method of estimating the fractional saturation of oxygen in haemoglobin is derived in this paper. The role of mathematical tools have made the estimation of oxygen transition from one state to another more realistic and reasonable. The proposed model can be further extended by incorporating other parameters like temperature dependent rate constants, flux of diffusion at various binding sites etc.

Author Contribution Statements Both the authors contributed equally to formulate, evaluate and interpret the model and its analysis proposed in this paper. All authors read and approved the final manuscript.

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FREE RESOLUTIONS FOR THE TANGENT CONES OF SOME HOMOGENEOUS PSEUDO SYMMETRIC MONOMIAL CURVES

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Abstract. In this article, we study minimal graded free resolutions of Cohen -Macaulay tangent cones of some monomial curves associated to 4-generated pseudo symmetric numerical semigroups. We explicitly give the matrices in these minimal free resolutions.

1. Introduction

Minimal graded free resolutions are very nice objects to study the modules over finitely generated graded algebras . It carries out the information about the Hilbert series, the Castelnuovo-Mumford regularity and many other geometrical invariants of the module, which makes these resolutions very important for algebrebraic geometry and commutative algebra. Construction of an explicit minimal free resolution of a finitely generated algebra is a difficult problem in general. This problem has been studied by many mathematicans, in particular for the homogeneous coordinate ring of an affine monomial curve in [1,5,6,11–13,15].

"Describing the Betti numbers and the minimal resolution of the tangent cone of S when S is a 4-generated semigroup which is (almost) symmetric or nearly Gorenstein " was an open problem (See [16], Problem 9.9). Symmetric numerical semigroup case is studied by Mete and Zengin in [11] and in [12]. They computed the Betti numbers by explicitly computing the minimal graded free resolution. Pseudo symmetric semigroup case is studied in [15] by showing that being homogeneous and being homogeneous type are equivalent for 4 generated pseudo symmetric monomial curves with Cohen-Macaulay tangent cones by computing the Betti sequences for nonhomogeneous case. Though in the homogeneous case the Betti sequence is

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.

already known as $(1, 5, 6, 2)$, an explicit computation of minimal graded free resolutions were not given. In this paper, we focus on 4 generated pseudo symmetric semigroups with Cohen-Macaulay tangent cones that are homogeneous (and hence homogeneous type) and calculate the explicit minimal graded free resolutions when n_1 is the smallest among n_1, n_2, n_3, n_4.

2. Preliminaries

Let n_1, n_2, n_3, n_4 be positive integers with $\gcd(n_1, ..., n_k) = 1$. Consider the numerical semigroup $S =< n_1, n_2, ..., n_k >= \{\sum_{i=1}^{k} u_i n_i | u_i \in \mathbb{N}\}$. Let $A = K[X_1, X_2, ..., X_k]$ be the coordinate ring over the field K and $K[S]$ be the semigroup ring $K[t^{n_1}, t^{n_2}, ..., t^{n_k}]$ of S. If we denote the kernel of the surjection

$$\begin{aligned} \phi_0 \colon A &\to K[S] \\ X_i &\mapsto t^{n_i} \end{aligned}$$

by I_S, then $K[S] \simeq A/I_S$. If we denote the affine curve with parametrization

$$X_1 = t^{n_1}, \quad X_2 = t^{n_2}, \; ...., \; X_k = t^{n_k}$$

corresponding to S by C_S, then the local ring corresponding to S is $R_S = K[[t^{n_1}, ..., t^{n_k}]]$. The Hilbert function of the local ring R_S is the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(R_S) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. It is known that

$$gr_{\mathfrak{m}}(R_S) \cong K[S]/I_{S*}$$

where $I_{S*} =< f_* | f \in I_S >$ is the defining ideal of the tangent cone with f_* denoting the initial form of f.

s being an element of the semi-group S, the apery set of S with respect to s is defined to be $AP(S, s) = \{x \in S \mid x - s \notin S\}$ and the set of lengths of s in S is $L(s) = \{\sum_{i=1}^{k} u_i | s = \sum_{i=1}^{k} u_i n_i, u_i \geq 0\}$. A subset $T \subset S$ is said to be homogeneous if either it is empty or $L(s)$ is a singleton for all $0 \neq s \in T$. n_i being the smallest among $n_1, n_2, ..., n_k$, the numerical semigroup S is said to be homogeneous if the apery set $AP(S, n_i)$ is homogeneous. It has been shown in [9] that $AP(S, n_i)$ is homogeneous if and only if there is a minimal set of generators G of I_S such that X_i belongs to the support of all nonhomogeneous elements of E.

A semigroup S is said to be of homogeneous type if the Betti numbers of the semigroup ring $K[S]$ and the Betti numbers of the associated graded ring (tangent cone) coincide, [8]. It is known that if a semigroup is of homogeneous type then the corresponding tangent cone is Cohen-Macaulay. Furthermore, if the semigroup S is homogeneous and the tangent cone is Cohen-Macaulay then S is also of homogeneous type. Converse is not true in general: there are numerical semigroups which are of homogeneous type but not homogeneous. Some counter examples are given in embedding dimension 4 , see [9].

In [10] the generators of I_S corresponding to a 4-generated pseudo symmetric numerical semigroup are given as $< f_1, f_2, f_3, f_4, f_5 >$ where

$$f_1 = X_1^{\alpha_1} - X_3X_4^{\alpha_4-1}, \quad f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}}X_4, \quad f_3 = X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2,$$
$$f_4 = X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}, \quad f_5 = X_1^{\alpha_{21}+1}X_3^{\alpha_3-1} - X_2X_4^{\alpha_4-1}$$

where here $\alpha_i > 1$, $1 \le i \le 4$, and $0 < \alpha_{21} < \alpha_1$, such that $n_1 = \alpha_2\alpha_3(\alpha_4 - 1) + 1$, $n_2 = \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3$, $n_3 = \alpha_1\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_2 - 1)(\alpha_4 - 1) - \alpha_4 + 1$, $n_4 = \alpha_1\alpha_2(\alpha_3 - 1) + \alpha_{21}(\alpha_2 - 1) + \alpha_2$.

Barucci, Fröberg and Şahin in [1] showed that the Betti sequence of $K[S]$ is $(1, 5, 6, 2)$ for 4 generated pseudo symmetric monomial curves but I_{S_*} or the Betti numbers of the tangent cone were not known. In [14], we described the Cohen–Macaulay property of the tangent cone in terms of Komeda's parametrization for 4-generated pseudo symmetric monomial curves.

3. Free Resolutions

When n_1 is the smallest among $\{n_1, n_2, n_3, n_4\}$, since the semigroup is always homogeneous, it is known that the Betti sequence is $(1, 5, 6, 2)$. It is also known from [14] that the tangent cone is Cohen-Macaulay iff $\alpha_4 \le \alpha_2 + \alpha_3 \le \alpha_{21} + \alpha_3 - 1 \le \alpha_1$. To compute these homogeneous summands, we will use:

Lemma 1 ([14], page 16)**.** *When n_1 is the smallest and the tangent cone is Cohen-Macaulay, $\{f_1, f_2, f_3, f_4, f_5\}$ forms a standard basis for I_S.*

Since the homogeneous summands change when there are equalities, there are 8 different possibilities for the tangent cone that should be considered:

(1) $\alpha_4 < \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$
(2) $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$
(3) $\alpha_4 < \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$
(4) $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$
(5) $\alpha_4 < \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 = \alpha_1$
(6) $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 = \alpha_1$
(7) $\alpha_4 < \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 = \alpha_1$
(8) $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 = \alpha_1$

However, case 8 is irredundant as can be seen from the next proposition.

Proposition 1. *If $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 = \alpha_1$ then $n_1 = n_2$*

Proof. $n_1 = \alpha_2\alpha_3(\alpha_4 - 1) + 1 = (\alpha_{21} + 1)\alpha_3(\alpha_{21} + \alpha_3 - 1) + 1 = \alpha_{21}\alpha_3(\alpha_{21} + \alpha_3 - 1) + \alpha_3(\alpha_{21} + \alpha_3 - 1) + 1$.

On the other hand,

$n_2 = \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3 = \alpha_{21}\alpha_3(\alpha_{21} + \alpha_3) + (\alpha_3 - 1)(\alpha_3 - 1) + \alpha_3 = \alpha_{21}\alpha_3(\alpha_{21} + \alpha_3 - 1) + \alpha_{21}\alpha_3 + (\alpha_3 - 1)^2 + \alpha_3 = \alpha_{21}\alpha_3(\alpha_{21} + \alpha_3 - 1) + \alpha_3(\alpha_{21} + \alpha_3 - 1) + 1 = n_1$ □

There is a general form of the minimal graded free resolution of the tangent cone in possibilities (1) and (3), (2) and (4), (5) and (7). We will list these and the minimal graded free resolution in case (6) respectively.

The content of the rest of the paper will be as follows: for each of these four possibilities, we will give the generators of I_{S*} as a corollary of lemma 3.4 of [14] and give our main theorems to compute the minimal graded free resolutions. To find the generators of I_{S*}, since we only take the homogeneous summands of the elements in G in Lemma 1 in respective cases, we will not write the proofs of corollaries. To prove the given complexes in our theorems are exact, we will use Buchsbaum-Eisenbud criterion, see [2] for the details. θ being a matrix, we will denote the minor obtained from θ by erasing its ith row , jth column with $[\theta]_{r_i,c_j}$.

3.1. **If** $\alpha_4 < \alpha_2 + \alpha_3 - 1 \leq \alpha_{21} + \alpha_3 < \alpha_1$.

Corollary 1. *I_{S*} is generated by $G_* = \{X_3X_4^{\alpha_4-1}, X, X_3^{\alpha_3}, X_4^{\alpha_4}, X_2X_4^{\alpha_4-1}\}$ where $X = X_2^{\alpha_2}$ if $\alpha_4 < \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$ and $X = f_2$ if $\alpha_4 < \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$.*

Theorem 1. *If S is a 4-generated pseudo symmetric semigroup, then minimal graded free resolution of the tangent cone is*

$$0 \longrightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where

$$\phi_1 = \begin{bmatrix} X_3X_4^{\alpha_4-1} & X & X_3^{\alpha_3} & X_4^{\alpha_4} & X_2X_4^{\alpha_4-1} \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} -X_2 & 0 & -X_3^{\alpha_3-1} & 0 & X_4 & 0 \\ 0 & -X_3^{\alpha_3} & 0 & 0 & 0 & -X_4^{\alpha_4-1} \\ 0 & X & X_4^{\alpha_4-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -X_2 & -X_3 & Y \\ X_3 & 0 & 0 & X_4 & 0 & X_2^{\alpha_2-1} \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} X_4 & X_2^{\alpha_2-1}X_3^{\alpha_3-1} \\ 0 & X_4^{\alpha_4-1} \\ 0 & -X \\ -X_3 & 0 \\ X_2 & Z \\ 0 & -X_3^{\alpha_3} \end{bmatrix}$$

with $(X,Y,Z) = (X_2^{\alpha_2}, 0, 0)$ if $\alpha_2 \neq \alpha_{21}+1$ and $(X,Y,Z) = (f_2, -X_1^{\alpha_{21}}, X_1^{\alpha_{21}}X_3^{\alpha_3-1})$ if $\alpha_2 = \alpha_{21}+1$.

Proof. It is easy to see that $\phi_1\phi_2 = \phi_2\phi_3$ so that we have a complex. To show the complex is exact, rank$\phi_1 = 1$, rank$\phi_2 = 4$ and rank$\phi_3 = 2$ and hence rankϕ_1 + rankϕ_2 = rankA^5, rankϕ_2 + rankϕ_3 = rankA^6. Then by Buchsbaum- Eisenbud criterion, it is enough to check that $I(\phi_i)$ has a regular sequence of length i for $i = 1, 2, 3$. There is nothing to show for $i = 1$. A regular sequence of length 2 can be obtained as $-X_3^{2\alpha_3+1}$ from the minor $[\phi_2]_{r_3,c_4,c_6}$ and X_2X^2 from the minor

$[\phi_2]_{r_2,c_3,c_5}$ for $I(\phi_2)$. A regular sequence of length 3 can be obtained for ϕ_3 as $X_4^{\alpha_4}$ from the minor $[\phi_3]_{r_3,r_4,r_5,r_6}$, $X_3^{\alpha_3+1}$ from the minor $[\phi_3]_{r_1,r_2,r_3,r_5}$ and $-X_2X$ from the minor $[\phi_3]_{r_1,r_2,r_4,r_6}$. □

3.2. **If** $\alpha_4 = \alpha_2 + \alpha_3 - 1 \leq \alpha_{21} + \alpha_3 < \alpha_1$.

Corollary 2. *I_{S*} is generated by*

$$G_* = \{X_3X_4^{\alpha_4-1}, X, X_3^{\alpha_3}, X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}, Y\}$$

where $(X,Y) = (f_2, f_5)$ *if* $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$,
$(X,Y) = (X_2^{\alpha_2}, -X_2X_4^{\alpha_4-1})$ *if* $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$

Theorem 2. *In this case, minimal graded free resolution of the tangent cone is*

$$0 \longrightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where

$$\phi_1 = \begin{bmatrix} X_3X_4^{\alpha_4-1} & X & X_3^{\alpha_3} & X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1} & Y \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} X_2 & 0 & X_4 & X_3^{\alpha_3-1} & 0 & 0 \\ 0 & -X_1X_3^{\alpha_3-1} & 0 & 0 & -X_4^{\alpha_4-1} & X_3^{\alpha_3} \\ -X_1Z & 0 & -X_1X_2^{\alpha_2-1} & -X_4^{\alpha_4-1} & 0 & -X \\ 0 & -X_2 & -X_3 & 0 & -Z & 0 \\ X_3 & -X_4 & 0 & 0 & -X_2^{\alpha_2-1} & 0 \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} X_4 & X_2^{\alpha_2-1}X_3^{\alpha_3-1} \\ X_3 & 0 \\ -X_2 & (-X_2X_4^{\alpha_4-1} - Y)/X_1 \\ 0 & -X \\ 0 & X_3^{\alpha_3} \\ X_1 & X_4^{\alpha_4-1} \end{bmatrix}$$

where
$(X,Y,Z) = (f_2, f_5, X_1^{\alpha_{21}})$ *if* $\alpha_4 = \alpha_2 + \alpha_3 - 1 = \alpha_{21} + \alpha_3 < \alpha_1$
$(X,Y,Z) = (X_2^{\alpha_2}, -X_2X_4^{\alpha_4-1}, 0)$ *if* $\alpha_4 = \alpha_2 + \alpha_3 - 1 < \alpha_{21} + \alpha_3 < \alpha_1$

Proof. $\phi_1\phi_2 = \phi_2\phi_3$ so that we have a complex. Similarly to the previous case, it is easy to see that rank$\phi_1 = 1$, rank$\phi_2 = 4$ and rank$\phi_3 = 2$ and hence rankϕ_1 + rankϕ_2 = rankA^5, rankϕ_2 + rankϕ_3 = rankA^6. A regular sequence of length 2 is $X_3^{2\alpha_3+1}$ from the minor $[\phi_2]_{r_3,c_2,c_5}$, $X_2^{2\alpha_2+1}$ if $\alpha_2 - 1 < \alpha_{21} + \alpha_3$ and $-X_2f_2^2$ if $\alpha_2 - 1 = \alpha_{21} + \alpha_3$ from the minor $[\phi_2]_{r_3,c_3,c_4}$ for $I(\phi_2)$. A regular sequence of length 3 can be obtained as f_4 from the minor $[\phi_3]_{r_2,r_3,r_4,r_5}$, $X_3^{\alpha_3+1}$ from the minor $[\phi_3]_{r_1,r_3,r_4,r_6}$, $X_2^{\alpha_2+1}$ if $\alpha_2 - 1 < \alpha_{21} + \alpha_3$ and X_2f_2 if $\alpha_2 - 1 = \alpha_{21} + \alpha_3$ from the minor $[\phi_3]_{r_1,r_2,r_5,r_6}$ for $I(\phi_3)$. □

3.3. **If** $\alpha_4 < \alpha_2 + \alpha_3 - 1 \leq \alpha_{21} + \alpha_3 = \alpha_1$.

Corollary 3. *I_{S*} is generated by*

$$G_* = \{X_3X_4^{\alpha_4-1}, X, X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2, X_4^{\alpha_4}, X_2X_4^{\alpha_4-1}\}$$

where $X = f_2$ if $\alpha_4 < \alpha_2+\alpha_3-1 = \alpha_{21}+\alpha_3 = \alpha_1$ and $X_2^{\alpha_2}$ if $\alpha_4 < \alpha_2+\alpha_3-1 < \alpha_{21}+\alpha_3 = \alpha_1$.

Theorem 3. *In this case, minimal graded free resolution of the tangent cone is*

$$0 \longrightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where

$$\phi_1 = \begin{bmatrix} X_3X_4^{\alpha_4-1} & X & X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2 & X_4^{\alpha_4} & X_2X_4^{\alpha_4-1} \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} -X_3^{\alpha_3-1} & 0 & X_2 & -X_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -f_3 & X_4^{\alpha_4-1} \\ X_4^{\alpha_4-1} & 0 & 0 & 0 & X & 0 \\ 0 & X_2 & 0 & X_3 & 0 & Y \\ X_1^{\alpha_1-\alpha_{21}-1} & -X_4 & -X_3 & 0 & 0 & -X_2^{\alpha_2-1} \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} 0 & X \\ -X_3 & Z \\ X_4 & X_2^{\alpha_2-1}X_3^{\alpha_3-1} \\ X_2 & YX_3^{\alpha_3-1} \\ 0 & -X_4^{\alpha_4-1} \\ 0 & -f_3 \end{bmatrix}$$

where (X,Y,Z) equals to $(f_2, X_1^{\alpha_{21}}, -X_1^{\alpha_1-1})$ if $\alpha_4 < \alpha_2+\alpha_3-1 = \alpha_{21}+\alpha_3 = \alpha_1$ and $(X_2^{\alpha_2}, 0, 0)$ if $\alpha_4 < \alpha_2+\alpha_3-1 < \alpha_{21}+\alpha_3 = \alpha_1$

Proof. $\phi_1\phi_2 = \phi_2\phi_3$ is obvious and rank$\phi_1 = 1$, rank$\phi_2 = 4$ and rank$\phi_3 = 2$ and hence rankϕ_1 + rankϕ_2 = rankA^5, rankϕ_2 + rankϕ_3 = rankA^6. $I(\phi_2)$ has a regular sequence of length 2 as $X_4^{2\alpha_4}$ from the minor $[\phi_2]_{r_4,r_3,c_5}$, X_2X^2 from the minor $[\phi_2]_{r_2,c_1,c_4}$. A regular sequence of length 3 can be obtained as X_3f_3 from the minor $[\phi_3]_{r_1,r_3,r_4,r_5}$, $-X_4^{\alpha_4}$ from the minor $[\phi_3]_{r_1,r_2,r_4,r_6}$, $-X_2X$ from the minor $[\phi_3]_{r_2,r_3,r_5,r_6}$ for $I(\phi_3)$. □

Finally, minimal graded free resolution of the tangent cone in (6) is:

3.4. **If** $\alpha_4 = \alpha_2+\alpha_3-1 < \alpha_{21}+\alpha_3 = \alpha_1$.

Corollary 4. *In this case I_{S_*} is generated by*

$$G_* = \{X_1^{\alpha_1-\alpha_{21}-1}X_2 - X_3^{\alpha_3}, X_2^{\alpha_2}, X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1} - X_4^{\alpha_4}, X_2X_4^{\alpha_4-1}, X_3X_4^{\alpha_4-1}\}$$

Theorem 4. *If $\alpha_4 = \alpha_2+\alpha_3-1 < \alpha_{21}+\alpha_3 = \alpha_1$, then minimal graded free resolution of the tangent cone is*

$$0 \longrightarrow A^2 \xrightarrow{\phi_3} A^6 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where

$$\phi_1 = \begin{bmatrix} X_1^{\alpha_1-\alpha_{21}-1}X_2 - X_3^{\alpha_3} & X_2^{\alpha_2} & X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1} - X_4^{\alpha_4} & X_2X_4^{\alpha_4-1} & X_3X_4^{\alpha_4-1} \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} 0 & 0 & X_1X_2^{\alpha_2-1} & -X_4^{\alpha_4-1} & -X_2^{\alpha_2} & 0 \\ 0 & -X_1X_3^{\alpha_3-1} & -X_1^{\alpha_1-\alpha_{21}} & 0 & -f_3 & -X_4^{\alpha_4-1} \\ 0 & X_2 & X_3 & 0 & 0 & 0 \\ -X_3 & X_4 & 0 & X_1^{\alpha_1-\alpha_{21}-1} & 0 & X_2^{\alpha_2-1} \\ X_2 & 0 & X_4 & -X_3^{\alpha_3-1} & 0 & 0 \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} -X_4 & -X_2^{\alpha_2-1}X_3^{\alpha_3-1} \\ -X_3 & 0 \\ X_2 & 0 \\ 0 & -X_2^{\alpha_2} \\ X_1 & X_4^{\alpha_4-1} \\ 0 & -f_3 \end{bmatrix}$$

Proof. $\phi_1\phi_2 = \phi_2\phi_3$ and rank$\phi_1 = 1$, rank$\phi_2 = 4$, rank$\phi_3 = 2$. Thus, rankϕ_1 + rankϕ_2 = rankA^5, rankϕ_2 + rankϕ_3 = rankA^6. $I(\phi_2)$ has a regular sequence of length 2 as $X_3^2X_4^{\alpha_4-1}f_3$ from the minor $[\phi_2]_{r_5,c_2,c_6}$, $-X_2^{2\alpha_2+1}$ from the minor $[\phi_2]_{r_2,c_3,c_4}$ for $I(\phi_2)$. A regular sequence of length 3 can be obtained as f_4 from the minor $[\phi_3]_{r_2,r_3,r_4,r_6}$, X_3f_3 from the minor $[\phi_3]_{r_1,r_3,r_4,r_5}$, $X_2^{\alpha_2+1}$ from the minor $[\phi_3]_{r_1,r_2,r_5,r_6}$ for $I(\phi_3)$. □

4. Conclusion

Since we investigated 4-generated pseudo symmetric semigroups that are homogeneous with Cohen-Macaulay tangent cones when n_1 is the smallest, and since these semigroups are of homogeneous type automatically, in addition to the known Betti sequence $(1, 5, 6, 2)$ of the tangent cone, which Barucci, Fröberg and Şahin obtained in [1], using the standard basis found in [14], we computed the generators of I_{S_*} and we have given a complete characterization to the minimal graded-free resolution of the tangent cone in all possible situations.

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INVARIANTS OF A MAPPING OF A SET TO THE TWO-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. Let E_2 be the 2-dimensional Euclidean space and T be a set such that it has at least two elements. A mapping $\alpha : T \to E_2$ will be called a T-figure in E_2. Let $\mathbb{R}$ be the field of real numbers and $O(2,\mathbb{R})$ be the group of all orthogonal transformations of E_2. Put $SO(2,\mathbb{R}) = \{g \in O(2,\mathbb{R}) | detg = 1\}$, $MO(2,\mathbb{R}) = \{F : E_2 \to E_2 \mid Fx = gx + b, g \in O(2,\mathbb{R}), b \in E_2\}$, $MSO(2,\mathbb{R}) = \{F \in MO(2,\mathbb{R}) | detg = 1\}$. The present paper is devoted to solutions of problems of G-equivalence of T-figures in E_2 for groups $G = O(2,\mathbb{R}), SO(2,\mathbb{R}), MO(2,\mathbb{R}), MSO(2,\mathbb{R})$. Complete systems of G-invariants of T-figures in E_2 for these groups are obtained. Complete systems of relations between elements of the obtained complete systems of G-invariants are given for these groups.

1. Introduction

Let $\mathbb{R}$ be the field of real numbers, and let E_2 be the 2-dimensional Euclidean space.

The present paper is devoted to solution of problems of G-equivalence of T-figures in E_2 for groups $G = O(2,\mathbb{R}), SO(2,\mathbb{R}), MO(2,\mathbb{R}), MSO(2,\mathbb{R})$ in terms of G-invariants of a T-figure. We have obtain complete systems of G-invariants of T-figures for these groups and describe complete systems of relations between elements of the obtained complete systems of G-invariants.

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Let V be a finite dimensional vector space over a field K and β be a non-degenerate bilinear form on V. Denote by $O(\beta, K)$ the group of all β-orthogonal (that is the form β preserving) transformations of V. Let $MO(\beta, K)$ be the group generated by the group $O(\beta, K)$ and all translations of V. In the paper [6], for the orthogonal group $O(\beta, K)$ in the Euclidean, spherical, hyperbolic and de-Sitter geometries, the orbit of m vectors is characterized by their Gram matrix and an additional subspace. In the book [2, Proposition 9.7.1], for the group $MO(\beta, K)$ in the Euclidean geometry, the orbit of m-vectors is characterized by distances between m-vectors. A complete system of relations between elements of this complete system is also given in [2, Theorem 9.7.3.4]. In the paper [13], a complete system of invariants of m-tuples in the two-dimensional pseudo-Euclidean geometry of index 1 and a complete system relations between the obtained complete system of invariants are given. In the paper [15], a complete system of invariants of m-tuples in the one-dimensional projective space and a complete system relations between the obtained complete system of invariants are given. Invariants of m-points in Lorentzian geometry investigated in the paper [23]. Invariants of m-points appear also in the theory of invariants of Bezier curves ([5, 22]), in Computer vision theory ([19, 27]), in Computational Geometry ([21]). General theory of m-point invariants considered in the invariant theory (see [3, 8, 20, 30, 31]).

Complete systems of global invariants of paths and curves are investigated in papers [1, 7–9, 12, 14, 24–26]. Complete systems of global invariants of surfaces and vector fields are investigated in papers [10, 11, 28]. Complete systems of global invariants of T-figures in the affine geometry are investigated in the paper [17, 18].

This paper is organized as follows. In Section 1, some known results (Propositions 1-4) on the linear representation of the field of complex numbers in two-dimensional real space are given. Definitions of T-figures in the field $\mathbb{C}$ of complex numbers and in the two-dimensional linear space $\mathbb{R}^2$ are given. Put $S(\mathbb{C}^*) = \{z \in \mathbb{C} | |z| = 1\}$. A definition of $S(\mathbb{C}^*)$-equivalence of T-figures in $\mathbb{C}$ with respect to the group $S(\mathbb{C}^*)$ is given. A definition of $\Lambda(S(\mathbb{C}^*))$-equivalence of T-figures in $\mathbb{R}^2$ with respect to the group $\Lambda(S(\mathbb{C}^*))$ of linear transformation of $\mathbb{R}^2$ is given. It is proved Theorem 1 on a relation between the $S(\mathbb{C}^*)$-equivalence of T-figures in $\mathbb{C}$ and $\Lambda(S(\mathbb{C}^*))$-equivalence of T-figures in $\mathbb{R}^2$. In Section 2, evident forms of elements of groups $SO(2, \mathbb{R})$ and $O(2, \mathbb{R})$ are given. In Section 3, a complete system of G-invariants of a T-figure in the two-dimensional linear space $\mathbb{R}^2$ over the field $\mathbb{R}$ of real numbers for the group $G = SO(2, \mathbb{R})$ is given. A complete system of relations between elements of the obtained complete system of invariants are given. In Section 4, a complete system of G-invariants of a T-figure in $\mathbb{R}^2$ for the group $G = O(2, \mathbb{R})$ is given. A complete system of relations between elements of the obtained complete system of G-invariants is given. In Section 5, a complete system of G-invariants of a T-figure in $\mathbb{R}^2$ for the group $G = MSO(2, \mathbb{R})$ is given. A complete system of relations between elements of the obtained complete system of G-invariants is given. In Section 6, a complete system of G-invariants of a T-figure in $\mathbb{R}^2$ for the group $G = MO(2, \mathbb{R})$ is given. A complete system of relations between elements of the obtained complete system of G-invariants is given.

2. Some properties of a linear representation of the field of complex numbers in two-dimensional real space

A part of results of this section is known (see [16]).
Denote the field of complex numbers by $\mathbb{C}$. Let $c = c_1 + ic_2 \in \mathbb{C}$. Denote by Λ_c the matrix of the form $\begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix}$. Denote by $\Lambda(\mathbb{C})$ the set $\{\Lambda_c | c \in \mathbb{C}\}$. We consider on the set $\Lambda(\mathbb{C})$ following matrix operations: the component-wise addition and the multiplication of matrices. Then $\Lambda(\mathbb{C})$ is a field with respect to these operations. In it the unit element is the unit matrix.

Proposition 1. *The mapping $\Lambda : \mathbb{C} \to \Lambda(\mathbb{C})$, where $\Lambda : c \to \Lambda_c, \forall c \in \mathbb{C}$, is an isomorphism of the fields $\mathbb{C}$ and $\Lambda(\mathbb{C})$.*

Proof. It is obvious. □

Let $a = a_1 + ia_2 \in \mathbb{C}, b = b_1 + ib_2 \in \mathbb{C}$. Put $\langle a, b\rangle = a_1b_1 + a_2b_2$. Then $\langle a, b\rangle$ is a bilinear form on $\mathbb{R}^2$ and $\langle a, a\rangle = a_1^2 + a_2^2$ is a quadratic form on $\mathbb{R}^2$. For convenience, we denote by $Q(a)$ the quadratic form $\langle a, a\rangle$.

The following propositions 2, 3 and 4 are known.

Proposition 2. *The following equalities $Q(x) = det(\Lambda_x)$ and $Q(xy) = Q(x)Q(y)$ hold for all $x = x_1 + ix_2, y = y_1 + iy_2 \in \mathbb{C}$.*

For $x = x_1 + ix_2 \in \mathbb{C}$, we set $\overline{x} = x_1 - ix_2$.

Proposition 3. *The mapping $x \to \overline{x}$ is an involution of the field $\mathbb{C}$ and the following equalities $x + \overline{x} = 2x_1, \langle x, x\rangle = x\overline{x} = \overline{x}x = x_1^2 + x_2^2$, $Q(x) = Q(\overline{x})$ hold for all $x = x_1 + ix_2 \in \mathbb{C}$.*

Proposition 4. *Let $x \in \mathbb{C}$. Then the element x^{-1} exists if and only if $Q(x) \neq 0$. In the case $Q(x) \neq 0$, the equalities $x^{-1} = \frac{\overline{x}}{Q(x)}$ and $Q(x^{-1}) = \frac{1}{Q(x)}$ hold.*

Put $\mathbb{C}^* = \{x \in \mathbb{C} \mid Q(x) \neq 0\}$. $\mathbb{C}^*$ is a group with respect to the multiplication operation in the field $\mathbb{C}$. Denote by $\Lambda(\mathbb{C}^*)$ the set of all matrices Λ_a, where $a \in \mathbb{C}^*$. For $a \in \mathbb{C}^*$, we have $Q(a) = a_1^2 + a_2^2 \neq 0$ and $Q(a) = det(\Lambda_a) \neq 0$.

Below everywhere we will consider every element $x \in \mathbb{R}^2$ and $x \in E_2$ as a column vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Denote by Γ the following mapping $\Gamma : \mathbb{C} \to \mathbb{R}^2$, where $\Gamma(x_1 + ix_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. It is obvious that the mapping Γ is an isomorphism of linear spaces $\mathbb{C}$ and $\mathbb{R}^2$. Hence there exists the converse isomorphism $\Gamma^{-1}$of Γ and $\Gamma^{-1}(x) = x_1 + ix_2, \forall x \in \mathbb{R}^2$.

Denote by W the following matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Denote by L_a the following linear operator on $\mathbb{C}$: $L_a(x) = a \cdot x, \forall x \in \mathbb{C}$, $a \in \mathbb{C}^*$. Then the following equalities are obvious:

$\Gamma(a_1 + ia_2) = W\Gamma(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ -a_2 \end{pmatrix} = \Gamma(\overline{a}), \forall a = a_1 + ia_2 \in \mathbb{C}^*$.

$$\Gamma(L_a(x)) = \Gamma(a \cdot x) = \begin{pmatrix} a_1x_1 - a_2x_2 \\ a_1x_2 + a_2x_1 \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Lambda_a \cdot \Gamma(x), \tag{1}$$

$\forall a \in \mathbb{C}^*, \forall x \in \mathbb{C}$, where $\Lambda_a \cdot \Gamma(x)$ is the multiplication of matrices Λ_a and $\Gamma(x)$.

Hence $\Lambda_a \in \Lambda(\mathbb{C}^*)$ and the mapping $\Lambda : \mathbb{C}^* \to \Lambda(\mathbb{C}^*)$, where $\Lambda(a) = \Lambda_a$, is a linear representation of the groups.

Put $S(\mathbb{C}^*) = \{x \in \mathbb{C} \mid Q(x) = 1\}$. It is a subgroup of the group $\mathbb{C}^*$. $\Lambda(S(\mathbb{C}^*))$ is a subgroup of the group $\Lambda(\mathbb{C}^*)$ and the mapping $\Lambda : S(\mathbb{C}^*) \to \Lambda(\mathbb{C}^*)$, where $\Lambda(a) = \Lambda_a$, is a linear representation of the group $S(\mathbb{C}^*)$ in $\mathbb{R}^2$. $\Lambda(\mathbb{C}^*)$ is a group with respect to the multiplication of matrices. Let T be a set such that it has at least two elements. Denote by $\mathbb{C}^T$ the set of all mappings of the set T to the field $\mathbb{C}$. An element of $\alpha \in \mathbb{C}^T$ will be called a T-figure in the field $\mathbb{C}$. For the figure α, we also use the notation $\alpha(t)$, considering α as a function on T with values in $\mathbb{C}$. Denote by E_2^T the set of all mappings of the set T to E_2. An element $\gamma \in E_2^T$ will be called a T-figure in the space E_2. For the figure γ, we also use the notation $\gamma(t)$, considering γ as a function on T with values in E_2.

Let G be a subgroup of the group $\mathbb{C}^*$.

Definition 1. *Two T-figures $\alpha \in \mathbb{C}^T$ and $\beta \in \mathbb{C}^T$ is called G-equivalent if there exists $g \in G$ such that $\beta(t) = g \cdot \alpha(t), \forall t \in T$. In this case, we also write as follows: $\alpha \overset{G}{\sim} \beta$ or $\alpha(t) \overset{G}{\sim} \beta(t), \forall t \in T$.*

Let G be a subgroup of the group $\mathbb{C}^*$.

Definition 2. *Two T-figures $\gamma \in E_2^T$ and $\eta \in E_2^T$ is called $\Lambda(G)$-equivalent if there exists $a \in G$ such that $\eta(t) = \Lambda_a\gamma(t), \forall t \in T$. In this case, we also write as follows: $\gamma \overset{\Lambda(G)}{\sim} \eta$ or $\gamma(t) \overset{\Lambda(G)}{\sim} \eta(t), \forall t \in T$.*

Theorem 1. *Let $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ be two T-figures in $\mathbb{C}$. Then T-figures $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ are $S(\mathbb{C}^*)$-equivalent if and only if T-figures $\Gamma(\alpha(t))$ and $\Gamma(\beta(t))$ in E_2 are $\Lambda(S(\mathbb{C}^*))$-equivalent.*

Proof. Assume that T-figures $\alpha(t) = \alpha_1 + i\alpha_2(t)$ and $\beta(t) = \alpha_1 + i\beta_2(t)$ are $S(\mathbb{C}^*)$-equivalent. Then there exists $a = a_1 + ia_2 \in S(\mathbb{C}^*)$ such that $\beta(t) = a \cdot \alpha(t), \forall t \in T$.

Using this equality and the equality (1), we obtain following equality:

$$\Gamma(\beta(t)) = \Gamma(a \cdot \alpha(t)) = \begin{pmatrix} a_1\alpha_1(t) - a_2\alpha_2(t) \\ a_1\alpha_2(t) + a_2\alpha_1(t) \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}$$
$$= \Lambda_a \Gamma(\alpha(t)), \forall t \in T.$$

This equality means that T-figures $\Gamma(\alpha(t))$ and $\Gamma(\beta(t))$ are $\Lambda(S(\mathbb{C}^*))$-equivalent .

Conversely, assume that T-figures $\Gamma(\alpha(t))$ and $\Gamma(\beta(t))$ are $\Lambda(S(\mathbb{C}^*))$-equivalent. Since Γ is an isomorphism, Γ^{-1} exists. Then the above equality implies that $\beta(t) = \Gamma^{-1}(\Gamma(\beta(t))) = \Gamma^{-1}(\Gamma(a \cdot \alpha(t))) = a \cdot \alpha(t), \forall t \in T$. Hence T-figures $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ are $S(\mathbb{C}^*)$-equivalent. □

3. Fundamental groups of transformations of the 2-dimensional Euclidean space

Let E_2 be the 2-dimensional Euclidean space with the scalar product $\langle a, b\rangle = a_1b_1 + a_2b_2$, where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in E_2$.

Definition 3. *A mapping $F : E_2 \to E_2$ is called orthogonal if $\langle Fx, Fy\rangle = \langle x, y\rangle$ for all $x, y \in E_2$.*

Denote the set of all orthogonal transformations of E_2 by $O(2, \mathbb{R})$.

The following propositions 5-7 are well known.

Proposition 5. *([4], p.221) Every orthogonal transformation of E_2 is linear.*

Proposition 6. *$O(2, \mathbb{R})$ is a group with respect to the multiplication operation of matrices.*

Let $a = a_1 + ia_2, b = b_1 + ib_2 \in \mathbb{C}$. Denote the identity matrix of the bilinear form $\langle a, b\rangle = a_1b_1 + a_2b_2$ by $I = \|\delta_{ij}\|_{i,j=1,2}$, where $\delta_{11} = \delta_{22} = 1, \delta_{12} = \delta_{21} = 0$. By Proposition 5, we can consider every element of $O(2, \mathbb{R})$ as a 2×2-matrix. Let $H \in O(2, \mathbb{R})$, where $H = \|h_{ij}\|_{i,j=1,2}$. Let H^T be the transpose matrix of H. It is known that the equality $\langle Hx, Hy\rangle = \langle x, y\rangle$ for all $x, y \in E_2$ is equivalent to the equality

$$H^T H = I. \tag{2}$$

This equality implies the following

Proposition 7. *Let $H \in O(2, \mathbb{R})$. Then $det(H) = 1$ or $det(H) = -1$.*

We denote by $SO(2, \mathbb{R})$ the set $\{H \in O(2, \mathbb{R}) : det(H) = 1\}$. $SO(2, \mathbb{R})$ is a subgroup of $O(2, \mathbb{R})$. $O(2, \mathbb{R}) = SO(2, \mathbb{R}) \cup \{HW \mid H \in SO(2, \mathbb{R})\}$, where HW is the multiplication of matrices H and W, where $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Theorem 2. *The equality $SO(2, \mathbb{R}) = \Lambda(S(\mathbb{C}^*))$ holds.*

Proof. $\Leftarrow$. We assume that $H \in \Lambda(S(\mathbb{C}^*))$. Then it has the following form $H = \|h_{ij}\|_{i,j=1,2}$, where $h_{11} = h_{22} = c, h_{21} = d, h_{12} = -d$, $c, d \in \mathbb{R}$ and $det(H) = c^2 + d^2 = 1$. We prove that $H \in SO(2, \mathbb{R})$. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in E_2$. We have

$$H(x) = \begin{pmatrix} cx_1 - dx_2 \\ dx_1 + cx_2 \end{pmatrix}, H(y) = \begin{pmatrix} cy_1 - dy_2 \\ dy_1 + cy_2 \end{pmatrix}.$$

Using the equality $c^2 + d^2 = 1$, we obtain

$$\langle H(x), H(y)\rangle = (cx_1 - dx_2)(cy_1 - dy_2) + (dx_1 + cx_2)(dy_1 + cy_2) = (c^2 + d^2)(x_1y_1 + x_2y_2) = \langle x, y\rangle .$$

Hence $H \in SO(2, \mathbb{R})$.

$\Rightarrow$. We assume that $H \in SO(2, \mathbb{R})$, where $H = \|h_{ij}\|_{i,j=1,2}$. Then $det(H) = h_{11}h_{22} - h_{12}h_{21} = 1$ and the equality (2) holds. These equalities imply the following system of equalities

$$h_{11}^2 + h_{21}^2 = 1 \tag{3}$$
$$h_{11}h_{12} + h_{21}h_{22} = 0 \tag{4}$$
$$h_{12}^2 + h_{22}^2 = 1 \tag{5}$$
$$h_{11}h_{22} - h_{12}h_{21} = 1 \tag{6}$$

We consider two cases $h_{12} = 0$ and $h_{12} \neq 0$.

Let $h_{12} = 0$. Then (5) implies $h_{22}^2 = 1$. Hence $h_{22} = 1$ or $h_{22} = -1$. Let $h_{22} = 1$. Then the equalities $h_{22} = 1, h_{12} = 0$ and (4) imply $h_{21} = 0$. Using equalities $h_{21} = 0$ and (3), we obtain $h_{11}^2 = 1$. Hence $h_{11} = 1$ or $h_{11} = -1$. Thus, in the case $h_{12} = 0$ and $h_{22} = 1$, we obtain $h_{21} = 0$ and $h_{11} = 1$ or $h_{11} = -1$. Hence, in this case, we obtain only the following two matrices:

$$A_1 = \{h_{11} = h_{22} = 1, h_{12} = h_{21} = 0\}, A_2 = \{h_{11} = -1, h_{12} = h_{21} = 0, h_{22} = 1\}.$$

It is obviously that $A_1 \in \Lambda(S(\mathbb{C}^*))$ and $A_2 \notin SO(2, \mathbb{R})$.

Let $h_{22} = -1$. Then the equalities $h_{22} = -1, h_{12} = 0$ and (4) imply $h_{21} = 0$. Using equalities $h_{21} = 0$ and (3), we obtain $h_{11}^2 = 1$. Hence $h_{11} = 1$ or $h_{11} = -1$. Thus, in the case $h_{12} = 0$ and $h_{22} = -1$, we obtain $h_{21} = 0$ and $h_{11} = 1$ or $h_{11} = -1$. Hence, in this case, we obtain only the following two matrices:

$$A_3 = \{h_{11} = 1, h_{12} = h_{21} = 0, h_{22} = -1\}, A_4 = \{h_{11} = h_{22} = -1, h_{12} = h_{21} = 0\}.$$

It is obviously that $A_4 \in \Lambda(S(\mathbb{C}^*))$ and $A_3 \notin SO(2, \mathbb{R})$.
Let $h_{12} \neq 0$. Using (4), we obtain

$$h_{11} = -\frac{h_{21}h_{22}}{h_{12}}.$$

Using this equality and equalities (3), (5), we obtain:

$$(-\frac{h_{21}h_{22}}{h_{12}})^2 + h_{21}^2 = 1 \Rightarrow h_{21}^2 h_{22}^2 + h_{12}^2 h_{21}^2 = h_{12}^2 \Rightarrow h_{21}^2(h_{22}^2 + h_{12}^2) = h_{12}^2 \Rightarrow h_{21}^2 = h_{12}^2 \Rightarrow h_{12}^2 - h_{21}^2 = 0.$$

Hence we obtain $h_{12} - h_{21} = 0$ or $h_{12} + h_{21} = 0$. We consider two cases $h_{12} - h_{21} = 0$ and $h_{12} + h_{21} = 0$.

Let $h_{12} - h_{21} = 0$. Then $h_{12} = h_{21}$. Since $h_{12} \neq 0$, we obtain $h_{21} \neq 0$. Using the equality $h_{12} = h_{21}$ and (4), we obtain $h_{11}h_{21} - h_{21}h_{22} = 0$. Hence $h_{21}(h_{11} + h_{22}) = 0$. Since $h_{21} \neq 0$, this equality implies $h_{11} = -h_{22}$. Thus we have obtained the following equalities: $h_{12} = h_{21}$ and $h_{11} = -h_{22}$. Using (6), we obtain $-h_{11}^2 - h_{12}^2 = 1$. Since $h_{12} \neq 0$ and $-(h_{11}^2 + h_{12}^2) = 1$, we have a contradiction. Hence this case is not possible.

Consider the case $h_{12} + h_{21} = 0$. This equality implies the equality $h_{12} = -h_{21}$. Using this equality and the equality (4) : $h_{11}h_{12} + h_{21}h_{22} = 0$, we obtain $h_{11}h_{12} - h_{12}h_{22} = 0$. Hence $h_{12}(h_{11} - h_{22}) = 0$. Since $h_{12} \neq 0$, this equality implies $h_{11} = h_{22}$. Hence the equalities $h_{12} = -h_{21}$, $h_{11} = h_{22}$ hold. These equalities and (3) imply that the matrix H has the form $\begin{pmatrix} h_{11} & -h_{21} \\ h_{21} & h_{11} \end{pmatrix}$, where $det(H) = 1$. Hence $H \in \Lambda(S(\mathbb{C}^*))$. □

Corollary 1. *Let $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ be T-figures in $\mathbb{C}$. Then T-figures $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ and $\beta(t) = \beta_1(t) + i\beta_2(t)$ are $S(\mathbb{C}^*)$-equivalent if and only if T-figures $\Gamma(\alpha(t))$ and $\Gamma(\beta(t))$ in E_2 are $SO(2,\mathbb{R})$-equivalent.*

Proof. It follows from Theorems 1 and 2. □

Denote by $MO(2,\mathbb{R})$ the group of all transformations of E_2 generated by the group $O(2,\mathbb{R})$ and all translations of E_2. Elements of the group $MO(2,\mathbb{R})$ has the following form $F : E_2 \to E_2$, where $F(x) = g(x) + a$, $g \in O(2,\mathbb{R})$, $a \in \mathbb{R}^2$. Denote by $MSO(2,\mathbb{R})$ the group of all transformations of E_2 generated by the group $SO(2,\mathbb{R})$ and all translations of E_2. Elements of the group $MSO(2,\mathbb{R})$ has the following form $F : E_2 \to E_2$, where $F(x) = g(x) + a$, $g \in SO(2,\mathbb{R})$, $a \in \mathbb{R}^2$.

4. Complete systems of G-invariants of a T-figure in E_2 for the group $G = SO(2,\mathbb{R})$

Let G be a subgroup of the group $MO(2,\mathbb{R})$.

Definition 4. *Two T-figures α and β in E_2 are called G-equivalent if there exists $g \in G$ such that $\alpha = g\beta$. In this case, we also write as follows: $\alpha \overset{G}{\sim} \beta$ or $\alpha(t) \overset{G}{\sim} \beta(t), \forall t \in T$.*

Definition 5. *A function $f(\alpha(t), \beta(t), \ldots, \gamma(t))$ of a finite number of T-figures $\alpha(t), \beta(t), \ldots, \gamma(t)$ is called G-invariant function if*

$f(F\alpha(t), F\beta(t), \ldots, F\gamma(t)) = f(\alpha(t), \beta(t), \ldots, \gamma(t))$ for all $F \in G$, all T-figures $\alpha(t), \beta(t), \ldots, \gamma(t)$ and all $t \in T$.

Example 1. *By the definitions of the groups $O(2,\mathbb{R})$ and $SO(2,\mathbb{R})$, we obtain that the quadratic form $Q : E_2 \to \mathbb{R}$,$Q(x) = \langle x, x\rangle$ is $O(2,\mathbb{R})$-invariant function on E_2 and the bilinear form $f : E_2 \times E_2 \to \mathbb{R}$, $f(x, y) = \langle x, y\rangle$ are $O(2,\mathbb{R})$-invariant functions on the set $E_2 \times E_2$.*

Example 2. *Denote by $[x\,y]$ the determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ of $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in E_2$. Consider the function $h : E_2 \times E_2 \to \mathbb{R}$, $h(x, y) = [x\,y]$. Using the equality $det(g) = 1, \forall g \in SO(2,\mathbb{R})$, we obtain $[(gx)\,(gy)] = det(g)[x\,y] = [x\,y], \forall g \in SO(2,\mathbb{R}), \forall x, y \in E_2$. This means that $[x\,y]$ is an $SO(2,\mathbb{R})$-invariant function on the set $E_2 \times E_2$. Clearly, $h(x, y)$ is not an $O(2,\mathbb{R})$-invariant function on the set $E_2 \times E_2$.*

Example 3. *By definitions of the groups $G = MO(2,\mathbb{R}), MSO(2,\mathbb{R})$ we obtain that function $f : E_2 \times E_2 \to \mathbb{R}$, $f(x, y) = \langle x - y, x - y\rangle$ is an G-invariant function on the set $E_2 \times E_2$.*

Definition 6. *A system $\{f_1, f_2, \ldots, f_m\}$ of G-invariant functions $f_1, f_2, \ldots, f_m$ of a T-figure α in E_2^T will be called a complete system of G-invariant functions of T-figure if equalities $f_j(\alpha) = f_j(\beta), \forall j \in 1, 2, \ldots, m$ imply $\alpha \overset{G}{\sim} \beta$.*

Denote by θ the vector $\theta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in E_2$. Let α be a T-figure in E_2. Denote by $Z(\alpha)$ the set $\{t \in T | \alpha(t) = \theta\}$. Denote by $\theta_T(t)$ the T-figure such that $\theta_T(t) = \theta, \forall t \in T$.

Denote by 2^T the set of all subsets of the set T.

Proposition 8. (1) *Let G be a subgroup of $\mathbb{C}^*$. Assume that $\alpha, \beta \in \mathbb{C}^T$ such that $\alpha \overset{G}{\sim} \beta$. Then $Z(\alpha) = Z(\beta)$. This means that the function $Z : \mathbb{C}^T \to 2^T$ is a G-invariant function on $\mathbb{C}^T$.*

(2) *Let G be a subgroup of $O(2,\mathbb{R})$. Assume that $\alpha, \beta \in E_2^T$ such that $\alpha \overset{G}{\sim} \beta$. Then $Z(\alpha) = Z(\beta)$ that is the function $Z : E_2^T \to 2^T$ is a G-invariant function on E_2^T.*

Proof. It is obvious. □

Proposition 9. *Let $\mathbb{C}$ be the field of complex numbers and $x = x_1 + ix_2, y = y_1 + iy_2 \in \mathbb{C}$ such that $x \neq 0$. Then,*

(1) *the element* yx^{-1} *exists, the equality* $yx^{-1} = \frac{\langle x,y\rangle}{Q(x)} + i\frac{[x\,y]}{Q(x)}$ *and the following equality hold*

$$\Lambda_{yx^{-1}} = \begin{pmatrix} \frac{\langle x,y\rangle}{Q(x)} & -\frac{[x\,y]}{Q(x)} \\ \frac{[x\,y]}{Q(x)} & \frac{\langle x,y\rangle}{Q(x)} \end{pmatrix} \tag{7}$$

where $\langle x,y\rangle = x_1y_1 + x_2y_2$ *and* $[x\,y] = x_1y_2 - x_2y_1$.

(2) $\det(\Lambda_{yx^{-1}}) \neq 0$ *if and only if* $Q(y) \neq 0$.

Proof. It is given in [16, Proposition 4. 9]. □

Now we consider the G-equivalence problem of T-figures in the field $\mathbb{C}$ for the group $S(\mathbb{C}^*)$.

Let α and β be T-figures in $\mathbb{C}$ such that $\alpha(t) = \beta(t) = 0, \forall t \in T$, that is $Z(\alpha) = Z(\beta) = T$. In this case, it is obvious that $\alpha \overset{S(\mathbb{C}^*)}{\sim} \beta$.

Theorem 3. *Let* α *be a* T*-figure in the field* $\mathbb{C}$ *such that* $Z(\alpha) \neq T$, *and* $t_0 \in T \setminus Z(\alpha)$.

(i) *Suppose that a* T*-figure* β *in* $\mathbb{C}$ *such that* $\alpha \overset{S(\mathbb{C}^*)}{\sim} \beta$. *Then the following equalities hold:*

$$\begin{cases} Z(\alpha) = Z(\beta) \\ \langle \alpha(t_0), \alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ [\alpha(t_0)\alpha(t)] = [\beta(t_0)\,\beta(t)]\,, \forall t \in T \setminus Z(\alpha). \end{cases} \tag{8}$$

(ii) *Conversely, assume that a* T*-figure* β *in* $\mathbb{C}$ *such that the equalities* (8) *hold. Then there exists a single element* $g \in S(\mathbb{C}^*)$ *such that* $\beta = g \cdot \alpha$. *In this case, it has the following form* $g = \beta(t_0)(\alpha(t_0))^{-1}$.

Proof. Assume that $\alpha \overset{S(\mathbb{C}^*)}{\sim} \beta$. Then there exists $a \in S(\mathbb{C}^*)$ such that $\beta(t) = a \cdot \alpha(t), \forall t \in T$. By Proposition 8-(1), we obtain the equality $Z(\alpha) = Z(\beta)$. Hence the equality $Z(\alpha) = Z(\beta)$ in (8) is proved.

The equality $Z(\alpha) = Z(\beta)$ and the inequality $Z(\alpha) \neq T$ imply inequality $Z(\beta) \neq T$. Since $t_0 \in T \setminus Z(\alpha) = T \setminus Z(\beta)$, we obtain that $\alpha(t_0) \neq 0$ and $\beta(t_0) \neq 0$. The inequality $\alpha(t_0) \neq 0$ implies an existence of $(\alpha(t_0))^{-1}$. Consider following functions $\alpha(t) \cdot (\alpha(t_0))^{-1}$ and $\beta(t) \cdot (\beta(t_0))^{-1}$ on T. The above equality $\beta(t) = a \cdot \alpha(t), \forall t \in T$, implies following equality: $\beta(t) \cdot (\beta(t_0))^{-1} = a \cdot \alpha(t) \cdot (a \cdot \alpha(t_0))^{-1} = (a \cdot a^{-1}) \cdot \alpha(t) \cdot (\alpha(t_0))^{-1} = \alpha(t) \cdot (\alpha(t_0))^{-1}, \forall t \in T$. Hence following equality holds: $\beta(t) \cdot (\beta(t_0))^{-1} = \alpha(t) \cdot (\alpha(t_0))^{-1}, \forall t \in T$. Using Proposition 9, we obtain following equalities:
$\alpha(t) \cdot (\alpha(t_0))^{-1} = \frac{\langle \alpha(t_0), \alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))}, \beta(t) \cdot (\beta(t_0))^{-1} = \frac{\langle \beta(t_0), \beta(t)\rangle}{Q(\beta(t_0))} + i\frac{[\beta(t_0)\,\beta(t)]}{Q(\beta(t_0))}$.
These equalities and the equality $\beta(t) \cdot (\beta(t_0))^{-1} = \alpha(t) \cdot (\alpha(t_0))^{-1}, \forall t \in T$, imply following equality: $\frac{\langle \alpha(t_0), \alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0), \beta(t)\rangle}{Q(\beta(t_0))} + i\frac{[\beta(t_0)\,\beta(t)]}{Q(\beta(t_0))}, \forall t \in T$. This

equality imply following equalities:

$$\begin{cases} \frac{\langle \alpha(t_0),\alpha(t)\rangle}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0),\beta(t)\rangle}{Q(\beta(t_0))}, \forall t \in T \\ \frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))} = \frac{[\beta(t_0)\,\beta(t)]}{Q(\beta(t_0))}, \forall t \in T. \end{cases} \tag{9}$$

The equality $\beta(t) = a \cdot \alpha(t), \forall t \in T$, implies following equality $Q(\beta(t_0)) = Q(a \cdot \alpha(t_0))$. Using Proposition 2, we obtain following equality $Q(\beta(t_0)) = Q(a) \cdot Q(\alpha(t_0))$. Since $a \in S(\mathbb{C}^*)$, we have $Q(a) = 1$. This equality and previous equality $Q(\beta(t_0)) = Q(a) \cdot Q(\alpha(t_0))$ imply following equality $Q(\beta(t_0)) = Q(\alpha(t_0))$. This equality and (9) imply following equalities:

$$\begin{cases} \frac{\langle \alpha(t_0),\alpha(t)\rangle}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0),\beta(t)\rangle}{Q(\alpha(t_0))}, \forall t \in T \\ \frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))} = \frac{[\beta(t_0)\,\beta(t)]}{Q(\alpha(t_0))}, \forall t \in T. \end{cases}$$

These equalities imply following equalities in (8):

$$\begin{cases} \langle \alpha(t_0),\alpha(t)\rangle = \langle \beta(t_0),\beta(t)\rangle, \forall t \in T \\ [\alpha(t_0)\,\alpha(t)] = [\beta(t_0)\,\beta(t)], \forall t \in T. \end{cases}$$

Hence equalities (8) is proved.

Conversely, assume that T-figures α and β in $\mathbb{C}$ such that the equalities (8) hold. By the supposition in the present theorem $t_0 \in T \setminus Z(\alpha(t))$. This implies $\alpha(t_0) \neq 0$. This inequality and the equality $Z(\alpha(t)) = Z(\beta(t))$ in (8) imply the inequality $\beta(t_0) \neq 0$. In the equality $\langle \alpha(t_0),\alpha(t)\rangle = \langle \beta(t_0),\beta(t)\rangle, \forall t \in T$, in (8) we put $t = t_0$. Then we obtain following equality $\langle \alpha(t_0),\alpha(t_0)\rangle = \langle \beta(t_0),\beta(t_0)\rangle$. This equality and the following equalities $Q(\alpha(t_0)) = \langle \alpha(t_0),\alpha(t_0)\rangle$, $Q(\beta(t_0)) = \langle \beta(t_0),\beta(t_0)\rangle$ imply following equality $Q(\alpha(t_0)) = Q(\beta(t_0))$. The inequality $\alpha(t_0) \neq 0$ implies following inequality $Q(\alpha(t_0)) \neq 0$. This inequality, the equality $Q(\alpha(t_0)) = Q(\beta(t_0))$ and the equalities in (8) imply following equality:

$$\begin{cases} \frac{\langle \alpha(t_0),\alpha(t)\rangle}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0),\beta(t)\rangle}{Q(\beta(t_0))}, \forall t \in T \\ \frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))} = \frac{[\beta(t_0)\,\beta(t)]}{Q(\beta(t_0))}, \forall t \in T. \end{cases}$$

These equalities imply following equalities:

$$\frac{\langle \alpha(t_0),\alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))} = \frac{\langle \beta(t_0),\beta(t)\rangle}{Q(\beta(t_0))} + i\frac{[\beta(t_0)\,\beta(t)]}{Q(\beta(t_0))}, \forall t \in T. \tag{10}$$

By Proposition 9, we obtain following equalities:

$$\alpha(t) \cdot (\alpha(t_0))^{-1} = \frac{\langle \alpha(t_0),\alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))}, \tag{11}$$

$$\beta(t) \cdot (\beta(t_0))^{-1} = \frac{\langle \beta(t_0),\beta(t)\rangle}{Q(\beta(t_0))} + i\frac{[\beta(t_0)\,\beta(t)]}{Q(\beta(t_0))}, \forall t \in T. \tag{12}$$

Equalities (10), (11) and (12) imply following equality:

$$\beta(t)\cdot(\beta(t_0))^{-1}=\alpha(t)\cdot(\alpha(t_0))^{-1},\forall t\in T. \quad (13)$$

This equality implies following equality:

$$\beta(t)=\beta(t_0)\cdot(\alpha(t_0))^{-1}\cdot\alpha(t),\forall t\in T. \quad (14)$$

Since $Q(\alpha(t_0))=Q(\beta(t_0))$, using this equality and Propositions 2, 4, we obtain following equality: $Q(\beta(t_0)\cdot(\alpha(t_0))^{-1})=Q(\beta(t_0))\cdot(Q(\alpha(t_0)))^{-1}=Q(\beta(t_0))\cdot(Q(\beta(t_0)))^{-1}=1$. This means that $\beta(t_0)(\alpha(t_0))^{-1}\in S(\mathbb{C}^*)$. Hence (14) implies that $\alpha(t)\overset{S(\mathbb{C}^*)}{\sim}\beta(t),\forall t\in T$.

Prove the uniqueness of $h\in S(\mathbb{C}^*)$ satisfying the conditions $\beta(t)=h\alpha(t),\forall t\in T$. Assume that $h\in S(\mathbb{C}^*)$ such that $\beta(t)=h\alpha(t),\forall t\in T$. In particularly, for $t=t_0$, the equality $\beta(t)=h\alpha(t)$ implies following equality: $\beta(t_0)=h\alpha(t_0)$. This equality and the inequality $\alpha(t_0)\neq 0$ imply following equality $\beta(t_0)(\alpha(t_0))^{-1}=h$. Hence the uniqueness of h is proved. □

Theorem 4. *Let α be a T-figure in E_2 such that $Z(\alpha)\neq T$, and $t_0\in T\setminus Z(\alpha)$.*

(*i*) *Suppose that a T-figure β in E_2 such that $\alpha\overset{SO(2,\mathbb{R})}{\sim}\beta$. Then the following equalities hold:*

$$\begin{cases} Z(\alpha)=Z(\beta)\\ \langle\alpha(t_0),\alpha(t)\rangle=\langle\beta(t_0),\beta(t)\rangle,\forall t\in T\setminus Z(\alpha)\\ [\alpha(t_0)\alpha(t)]=[\beta(t_0)\,\beta(t)]\,,\forall t\in T\setminus Z(\alpha).\end{cases} \quad (15)$$

(*ii*) *Conversely, assume that a T-figure β in E_2 such that the equalities (15) hold. Then there exists a single matrix $H\in SO(2,\mathbb{R})$ such that $\beta=H\alpha$. In this case, H has the following form*

$$H=\begin{pmatrix}\frac{\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0),\rangle} & -\frac{[\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle}\\ \frac{[\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle} & \frac{\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle}\end{pmatrix}, \quad (16)$$

where $\det(H)=(\frac{\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2+(\frac{[\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2=1$.

Proof. We consider T-figures α and β in E_2 as column vector functions: $\alpha(t)=\begin{pmatrix}\alpha_1(t)\\ \alpha_2(t)\end{pmatrix}$, $\beta(t)=\begin{pmatrix}\beta_1(t)\\ \beta_2(t)\end{pmatrix}$. Assume that $\alpha\overset{SO(2,\mathbb{R})}{\sim}\beta$. Then, by Proposition 8-(2), $Z(\alpha)=Z(\beta)$. This equality and the inequality $Z(\alpha)\neq T$ imply inequality $Z(\beta)\neq T$. Since functions $\langle\alpha(t_0),\alpha(t)\rangle$ and $[\alpha(t_0)\alpha(t)]$ are $SO(2,\mathbb{R})$-invariant, the $SO(2,\mathbb{R})$-equivalence $\alpha\overset{SO(2,\mathbb{R})}{\sim}\beta$, and the equality $Z(\alpha)=Z(\beta)$ imply equalities (15).

Conversely, assume that a T-figures α and β in E_2 such that the equalities (15) hold. Consider following T-figures in the field $\mathbb{C}$: $\Gamma^{-1}(\alpha(t))=\alpha_1(t)+i\alpha_2(t),\forall t\in T$, $\Gamma^{-1}(\beta(t))=\beta_1(t)+i\beta_2(t),\forall t\in T$. For these T-figures in $\mathbb{C}$ the equalities (15) also hold. Then, by Theorem 3, these T-figures are $S(\mathbb{C}^*)$-equivalent and there exists a single element $g\in S(\mathbb{C}^*)$ such that $\beta_1(t)+i\beta_2(t)=g\cdot(\alpha_1(t)+i\alpha_2(t)),\forall t\in T$. In

this case, by Theorem 3, g has the following form:
$g = \frac{\beta_1(t_0)+i\beta_2(t_0)}{\alpha_1(t_0)+i\alpha_2(t_0)} = \frac{(\beta_1(t_0)+i\beta_2(t_0))\cdot(\alpha_1(t_0)-i\alpha_2(t_0))}{(\alpha_1(t_0)+i\alpha_2(t_0))\cdot(\alpha_1(t_0)-i\alpha_2(t_0))}$
$= \frac{(\alpha_1(t_0)\beta_1(t_0)+\alpha_2(t_0)\beta_2(t_0))+i(\alpha_1(t_0)\beta_2(t_0)-\alpha_2(t_0)\beta_1(t_0))}{(\alpha_1(t_0))^2+(\alpha_2(t_0))^2} = \frac{\langle\alpha(t_0),\beta(t)\rangle+i[\alpha(t_0)\,\beta(t_0)]}{Q(\alpha(t_0))}$.

The $S(\mathbb{C}^*)$-equivalence of the T-figures $\Gamma^{-1}(\alpha)$, and $\Gamma^{-1}(\beta(t)) = \beta_1(t)+i\beta_2(t), \forall t \in T$ in $\mathbb{C}$, by Theorem 3, implies $SO(2,\mathbb{R})$-equivalence of T-figures α and β in E_2. In this case there exists a single element $H \in SO(2,\mathbb{R})$ such that $H = \Lambda_g$ and $\beta(t) = H \cdot \alpha(t), \forall t \in T$. By Proposition 9, the above form of $g = \frac{\langle\alpha(t_0),\beta(t)\rangle+i[\alpha(t_0)\,\beta(t_0)]}{Q(\alpha(t_0))}$ implies that H has the form (16), where $det(H) = 1$. □

Remark 1. *Assume that T be a set such that it has at least two elements. By Theorem 4, the system*

$$\{Z(\alpha), \langle\alpha(t_0), \alpha(t)\rangle, [\alpha(t_0)\,\alpha(t)]\} \tag{17}$$

is a complete system of $SO(2,\mathbb{R})$-invariant functions on the set of all T-figures α in E_2 such that $Z(\alpha) \neq T$, and $t_0 \in T \setminus Z(\alpha)$.

Now let us find a complete system of relations between elements of this complete system.

Theorem 5. *Let (17) be the complete system of $SO(2,\mathbb{R})$-invariants of a T-figure α in E_2. Assume that:*
(1.1) U is a subset of T such that $U \neq T$
(1.2) $t_0 \in T \setminus U$
(1.3) r be a real number such that $r > 0$
(1.4) $a(t) = (a_1(t), a_2(t))$ be a mapping $a : T \to E_2$ such that following two properties hold:
(1.4.1) $a_1(t) = 0, \forall t \in U,$ and $a_1(t_0) = r$
(1.4.2) $a_2(t) = 0, \forall t \in U,$ and $a_2(t_0) = 0$.
Then there exists a T-figure α in E_2 such that following equalities hold:
(2.1) $Z(\alpha) = U$
(2.2) $\langle\alpha(t_0), \alpha(t)\rangle = a_1(t), \forall t \in T$
(2.3) $[\alpha(t_0)\,\alpha(t)] = a_2(t), \forall t \in T$.

Proof. Assume that α is a T-figure in E_2 such that $Z(\alpha) \neq T$ and $t_0 \in T \setminus Z(\alpha)$.

(2.1) – (2.3) We choose a T-figure α as follows. Put $\alpha(t_0) = (\sqrt{r}, 0)$. Then we obtain $\langle\alpha(t_0), \alpha(t_0)\rangle = r$. This equality implies $Q(\alpha(t_0)) = \langle\alpha(t_0), \alpha(t_0)\rangle = r$. Hence $\langle\alpha(t_0), \alpha(t_0)\rangle = a_1(t_0) = r$. We choose α on the set U as follows. We put $\alpha(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \forall t \in U$. This equality implies $\langle\alpha(t), \alpha(t)\rangle = a(t) = 0, \forall t \in U$.

For fixed $t \in T$, we consider $a(t)$ and $\alpha(t)$ as elements of the field $\mathbb{C}$ of complex numbers: $a(t) = a_1(t) + ia_2(t)$, $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$. We put $\alpha(t) = \frac{a(t)\alpha(t_0)}{r}, \forall t \in T \setminus (U \cup \{t_0\})$. Since $\alpha(t_0) = \sqrt{r} \neq 0$, $(\alpha(t_0))^{-1}$ exists. Then the equalities $\alpha(t) = \frac{a(t)\alpha(t_0)}{r}, \forall t \in T \setminus (U \cup \{t_0\})$, imply equalities $(\alpha(t_0))^{-1}\alpha(t) = \frac{a(t)}{r}, \forall t \in$

$T \setminus (U \cup \{t_0\})$. By Proposition 9, $(\alpha(t_0))^{-1}\alpha(t) = \frac{\langle \alpha(t_0), \alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))}, \forall t \in T$. The equality $Q(\alpha(t_0)) = \langle \alpha(t_0), \alpha(t_0)\rangle = r$, the last two equalities $(\alpha(t_0))^{-1}\alpha(t) = \frac{a(t)}{r}, \forall t \in T \setminus (U \cup \{t_0\})$, $(\alpha(t_0))^{-1}\alpha(t) = \frac{\langle \alpha(t_0), \alpha(t)\rangle}{Q(\alpha(t_0))} + i\frac{[\alpha(t_0)\,\alpha(t)]}{Q(\alpha(t_0))}, \forall t \in T$, and equalities $\langle \alpha(t), \alpha(t)\rangle = a(t) = 0, \forall t \in U$, imply equalities $\frac{\langle \alpha(t_0), \alpha(t)\rangle}{r} + i\frac{[\alpha(t_0)\,\alpha(t)]}{r} = \frac{a(t)}{r}, \forall t \in T$. These equalities imply $Z(\alpha) = U$, $\langle \alpha(t_0), \alpha(t)\rangle = a_1(t), \forall t \in T$, and $[\alpha(t_0)\,\alpha(t)] = a_2(t)$, $\forall t \in T$. The statements (2.1)-(2.3) are proved. □

5. Complete systems of G-invariants of a T-figure in E_2 for the group $G = O(2,\mathbb{R})$

By Proposition 7, the following equality holds:
$O(2,\mathbb{R}) = SO(2,\mathbb{R}) \cup \{HW \mid H \in SO(2,\mathbb{R})\}$, where HW is the multiplication of matrices H and W, where $W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For shortness, denote the set $\{HW \mid H \in SO(2,\mathbb{R})\}$ by $SO(2,\mathbb{R})\cdot W$. We note that $SO(2,\mathbb{R}) \cap SO(2,\mathbb{R})\cdot W = \emptyset$.

Let α and β be T-figures in E_2. Assume that $\alpha \overset{O(2,\mathbb{R})}{\sim} \beta$. Then there exists $F \in O(2,\mathbb{R})$ such that $\beta(t) = F\alpha(t), \forall t \in T$. Denote by $Equ(\alpha,\beta)$ the set of all $F \in O(2,\mathbb{R})$ such that $\beta(t) = F\alpha(t), \forall t \in T$.

Proposition 10. *Let α and β be T-figures in E_2 such that $\alpha \overset{O(2,\mathbb{R})}{\sim} \beta$. Then there exist only following three possibilities for the set $Equ(\alpha,\beta)$:*
(I) $Equ(\alpha,\beta) = \{F\}$, where $F \in SO(2,\mathbb{R})$.
(II) $Equ(\alpha,\beta) = \{F\}$, where $F \in SO(2,\mathbb{R})\cdot W$.
(III) $Equ(\alpha,\beta) = \{F_1, F_2\}$, where $F_1 \in SO(2,\mathbb{R})$, $F_2 \in SO(2,\mathbb{R})\cdot W$.

Proof. Assume that $\alpha \overset{O(2,\mathbb{R})}{\sim} \beta$. Then there exists $F \in O(2,\mathbb{R})$ such that $F \in Equ(\alpha,\beta)$. Since $F \in O(2,\mathbb{R})$ and $F \in O(2,\mathbb{R}) = SO(2,\mathbb{R}) \cup \{HW \mid H \in SO(2,\mathbb{R})\}$, then $F \in SO(2,\mathbb{R})$ or $F \in \{HW \mid H \in SO(2,\mathbb{R})\}$.

- (I) Let $F \in Equ(\alpha,\beta)$, where $F \in SO(2,\mathbb{R})$. By Theorem 4, in this case there exists only one $F \in SO(2,\mathbb{R})$ such that following equalities $\beta(t) = F\alpha(t), \forall t \in T$, hold. Hence, in this case, the set $Equ(\alpha,\beta)$ has a only one element of $SO(2,\mathbb{R})$. Assume that the set $Equ(\alpha,\beta)$ has not elements of $SO(2,\mathbb{R})\cdot W$. Then, in this case, the set $Equ(\alpha,\beta)$ has only a single element $F \in O(2,\mathbb{R})$ and it is such that $F \in SO(2,\mathbb{R})$.
- (II) Let $F \in Equ(\alpha,\beta)$, where $F \in \{HW \mid H \in SO(2,\mathbb{R})\}$. Then following equality $\beta(t) = F\alpha(t), \forall t \in T$, holds. Since $F \in \{HW \mid H \in SO(2,\mathbb{R})\}$, there exists $H \in SO(2,\mathbb{R})$ such that $F = HW$. Then we have following equality $\beta(t) = HW\alpha(t), \forall t \in T$. By Theorem 4, in this case there exists only one $H \in SO(2,\mathbb{R})$ such that following equalities $\beta(t) = HW\alpha(t), \forall t \in T$, hold. Hence, in this case, the set $Equ(\alpha,\beta)$ has only one element of $\{HW \mid H \in SO(2,\mathbb{R})\}$. Assume that the set $Equ(\alpha,\beta)$ has not elements of $SO(2,\mathbb{R})$. Then, in this case, the set $Equ(\alpha,\beta)$ has only one

element of $\{HW \mid H \in SO(2,\mathbb{R})\}$ such that $Equ(\alpha,\beta) = \{F\}$, where $F \in \{HW \mid H \in SO(2,\mathbb{R})\}$.

(III) Let $Equ(\alpha,\beta)$ be such that $F_1 \in Equ(\alpha,\beta)$ and $F_2 \in Equ(\alpha,\beta)$, where $F_1 \in SO(2,\mathbb{R})$ and $F_2 \in \{HW \mid H \in SO(2,\mathbb{R})\}$. Then following equalities hold: $\beta(t) = F_1\alpha(t), \forall t \in T$, and $\beta(t) = F_2\alpha(t) = HW\alpha(t), \forall t \in T$, where $H \in SO(2,\mathbb{R})$. By Theorem 4, in the case $\beta(t) = F_1\alpha(t), \forall t \in T$, there exists only one $F_1 \in SO(2,\mathbb{R})$ such that following equalities $\beta(t) = F_1\alpha(t), \forall t \in T$, hold. Hence, in this case, the set $Equ(\alpha,\beta)$ has only one element of $SO(2,\mathbb{R})$. By Theorem 4, in the case $\beta(t) = F_2\alpha(t) = HW\alpha(t), \forall t \in T$, where $H \in SO(2,\mathbb{R})$, there exists only one element $F_2 \in \{HW \mid H \in SO(2,\mathbb{R})\}$ such that following equalities $\beta(t) = F_2\alpha(t) = HW\alpha(t), \forall t \in T$ hold, where $H \in SO(2,\mathbb{R})$. Then, in this case, the set $Equ(\alpha,\beta)$ have only two elements: only one element of $SO(2,\mathbb{R})$ and only one element of $SO(2,\mathbb{R})\cdot W$.

□

Theorem 6. *Let α be a T-figure in E_2 such that $Z(\alpha) \neq T$ and $t_0 \in T \setminus Z(\alpha)$.*

(i) *Suppose that a T-figure β in E_2 such that the following equalities $\beta(t) = HW\alpha(t), \forall t \in T$, hold for some $H \in SO(2,\mathbb{R})$. Then following equalities hold:*

$$\begin{cases} Z(\alpha) = Z(\beta) \\ \langle \alpha(t_0), \alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ -\left[\alpha(t_0)\alpha(t)\right] = \left[\beta(t_0)\,\beta(t)\right], \forall T \setminus Z(\alpha). \end{cases} \tag{18}$$

(ii) *Conversely, assume that a T-figure β in E_2 such that the equalities (18) hold. Then there exists only one matrix $U \in SO(2,\mathbb{R})$ such that $\beta(t) = UW\alpha(t), \forall t \in T$. In this case, U has the following form*

$$U = \begin{pmatrix} \frac{\langle W\alpha(t_0), \beta(t_0)\rangle}{\langle \alpha(t_0), \alpha(t_0)\rangle} & -\frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0)\rangle} \\ \frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0)\rangle} & \frac{\langle W\alpha(t_0), \beta(t_0)\rangle}{\langle \alpha(t_0), \alpha(t_0)\rangle} \end{pmatrix}, \tag{19}$$

where $\det(U) = (\frac{\langle W\alpha(t_0), \beta(t_0)\rangle}{\langle \alpha(t_0), \alpha(t_0)\rangle})^2 + (\frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0)\rangle})^2 = 1.$

Proof. Suppose that a T-figure β in E_2 such that the following equalities $\beta(t) = HW\alpha(t), \forall t \in T$, hold for some $H \in SO(2,\mathbb{R})$. This means T-figures $W\alpha$ and β are $SO(2,\mathbb{R})$-equivalent. Then, by Theorem 4, we obtain following equalities:

$$\begin{cases} Z(W\alpha) = Z(\beta) \\ \langle W\alpha(t_0), W\alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ \left[W\alpha(t_0)W\alpha(t)\right] = \left[\beta(t_0)\,\beta(t)\right], \forall t \in T \setminus Z(\alpha). \end{cases} \tag{20}$$

These equalities and equalities $Z(W\alpha) = Z(\alpha)$, $\langle W\alpha(t_0), W\alpha(t)\rangle = \langle \alpha(t_0), \alpha(t)\rangle$, $[W\alpha(t_0)W\alpha(t)] = -\left[\alpha(t_0)\alpha(t)\right]$ imply equalities (18).

Conversely, assume that a T-figure β in E_2 such that the equalities (18) hold. Then equalities (18) and equalities $Z(W\alpha) = Z(\alpha)$, $\langle W\alpha(t_0), W\alpha(t)\rangle = \langle \alpha(t_0), \alpha(t)\rangle$,

$[W\alpha(t_0)W\alpha(t)] = -[\alpha(t_0)\alpha(t)]$ imply equalities (20). By Theorem 4, equalities (20) and Proposition 10 imply an existence of only one $U \in SO(2,\mathbb{R})$ such that following equalities $\beta(t) = UW\alpha(t), \forall t \in T$, hold. By Theorem 4, the matrix U has the form (19). □

Remark 2. *Assume that T be a set such that it has at least two elements. By Theorem 6, the system $\{Z(\alpha), \langle\alpha(t_0), \alpha(t)\rangle, [W\alpha(t_0)\,W\alpha(t)]\}$ is a complete system of $SO(2,\mathbb{R})$-invariant functions on the set of all T-figures $W\alpha$ such that $Z(\alpha) \neq T$, and $t_0 \in T \setminus Z(\alpha)$. Complete system of relations between elements of this system follows easy from Theorem 5.*

Theorem 7. *Let α and β be T-figures in E_2. Assume that $Z(\alpha) \neq T$ and $t_0 \in T \setminus Z(\alpha)$.*

(*i*) *Suppose that matrices $H_1, H_2 \in SO(2,\mathbb{R})$ exist such that $\beta(t) = H_1\alpha(t), \forall t \in T$, and $\beta(t) = H_2W\alpha(t), \forall t \in T$. Then following equalities hold:*

$$\begin{cases} Z(\alpha) = Z(\beta) \\ \langle\alpha(t_0), \alpha(t)\rangle = \langle\beta(t_0), \beta(t)\rangle \\ rank(\alpha) = rank(\beta) = 1 \end{cases} \tag{21}$$

for all $t \in T \setminus Z(\alpha(t))$.

(*ii*) *Conversely, assume that the equalities (21) hold. Then only two matrices $H_1 \in SO(2,\mathbb{R})$ and $H_2 \in SO(2,\mathbb{R})$ exist such that following equalities $\beta(t) = H_1\alpha(t), \forall t \in T$, $\beta(t) = H_2W\alpha(t), \forall t \in T$, hold. Here the matrix H_1 has the following form:*

$$H_1 = \begin{pmatrix} \frac{\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle} & -\frac{[\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle} \\ \frac{[\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle} & \frac{\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle} \end{pmatrix}, \tag{22}$$

where $\det(H_1) = (\frac{\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2 + (\frac{[\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2 = 1.$

Here the matrix $H_2 \in SO(2,\mathbb{R})$ has the following form

$$H_2 = \begin{pmatrix} \frac{\langle W\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle} & -\frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle} \\ \frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle} & \frac{\langle W\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle} \end{pmatrix}, \tag{23}$$

where $\det(H_2) = (\frac{W\langle\alpha(t_0),\beta(t_0)\rangle}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2 + (\frac{[W\alpha(t_0)\,\beta(t_0)]}{\langle\alpha(t_0),\alpha(t_0)\rangle})^2 = 1.$

Proof. (*i*) Suppose that there exist $H_1 \in SO(2,\mathbb{R})$ such that $\beta(t) = H_1\alpha(t), \forall t \in T$. Then, by Theorem 4 the following equalities hold:

$$\begin{cases} Z(\alpha) = Z(\beta) \\ \langle\alpha(t_0), \alpha(t)\rangle = \langle\beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ [\alpha(t_0)\alpha(t)] = [\beta(t_0)\,\beta(t)], \forall T \setminus Z(\alpha). \end{cases} \tag{24}$$

Suppose that there exist $H_2 \in SO(2,\mathbb{R})$ such that $\beta(t) = H_2 W\alpha(t), \forall t \in T$. Then, by Theorem 6, the following equalities hold:

$$\begin{cases} Z(\alpha) = Z(\beta) \\ \langle \alpha(t_0), \alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha) \\ [\alpha(t_0)\alpha(t)] = -\left[\beta(t_0)\,\beta(t)\right], \forall T \setminus Z(\alpha). \end{cases} \tag{25}$$

Equalities (24) and (25)imply the following equalities:

$$\begin{cases} Z(\alpha) = Z(\beta) \\ \langle \alpha(t_0), \alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha). \end{cases} \tag{26}$$

Equalities (24) implies the following equalities:

$$[\alpha(t_0)\alpha(t)] = \left[\beta(t_0)\,\beta(t)\right], \forall T \setminus Z(\alpha). \tag{27}$$

Equalities (25) implies the following equalities:

$$[\alpha(t_0)\alpha(t)] = -\left[\beta(t_0)\,\beta(t)\right], \forall T \setminus Z(\alpha). \tag{28}$$

Equalities (27)and (28) imply following equalities:

$$\left[\beta(t_0)\,\beta(t)\right] = -\left[\beta(t_0)\,\beta(t)\right], \forall T \setminus Z(\alpha). \tag{29}$$

These equalities imply following equalities:

$$\left[\beta(t_0)\,\beta(t)\right] = 0, \forall T \setminus Z(\alpha). \tag{30}$$

These equalities and the equalities (27) imply following equalities

$$[\alpha(t_0)\alpha(t)] = 0, \forall T \setminus Z(\alpha). \tag{31}$$

The equalities (31) imply that there exists a real function $a(t)$ on T such that $a(t) = 0, \forall t \in Z(\alpha)$, $a(t) \neq 0, \forall T \setminus Z(\alpha)$ and equalities $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$ hold.

Similarly, equalities (30) imply that there exists a real function $b(t)$ on T such that $b(t) = 0, \forall t \in Z(\alpha)$, $b(t) \neq 0, \forall T \setminus Z(\alpha)$ and equalities $\beta(t) = b(t)\beta(t_0), \forall t \in T$ hold.

The above equalities $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$ and $\beta(t) = b(t)\beta(t_0), \forall t \in T$ imply the equality $rank(\alpha) = rank(\beta) = 1$ in the equalities (21). This equality and the equalities (24)imply the equalities (21).

Conversely, assume that the equalities (21) hold. Then the equality $rank(\alpha) = 1$ in (21) implies an existence of a real function $a(t)$ on T such that $a(t) = 0, \forall t \in Z(\alpha)$, $a(t) \neq 0, \forall T \setminus Z(\alpha)$ and $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$.

Similarly, the equality $rank(\beta) = 1$ in (21) implies an existence of a real function $b(t)$ on T such that $b(t) = 0, \forall t \in Z(\alpha)$, $b(t) \neq 0, \forall T \setminus Z(\alpha)$, and $\beta(t) = b(t)\beta(t_0), \forall t \in T$. The equalities $Z(\alpha) = Z(\beta)$, and $\langle \alpha(t_0), \alpha(t)\rangle = \langle \beta(t_0), \beta(t)\rangle, \forall t \in T \setminus Z(\alpha)$, imply following equality $a(t) = b(t), \forall t \in T$. Hence we obtain following equalities $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$, and $\beta(t) = a(t)\beta(t_0), \forall t \in T$.

Since $t_0 \in T \setminus Z(\alpha)$, we have $a(t_0) \neq 0$. By the equality $Z(\alpha) = Z(\beta)$, we obtain $\beta(t_0) \neq 0$. By [16, Theorem 5.1], only two matrices $H_1 \in SO(2,\mathbb{R})$ and

$H_2 \in SO(2,\mathbb{R})$ exist such that $\beta(t_0) = H_1\alpha(t_0)$ and $\beta(t_0) = H_2W\alpha(t_0)$. By [16, Theorem 5.1.], H_1 has the form (23) and H_2 has the form (24).

The above equalities $\beta(t) = a(t)\beta(t_0), \forall t \in T$, $\beta(t_0) = H_1\alpha(t_0)$, $\beta(t_0) = H_2W\alpha(t_0)$ imply following equalities: $\beta(t) = H_1\alpha(t), \forall t \in T$, and $\beta(t) = H_2W\alpha(t), \forall t \in T$. □

Remark 3. *Assume that T be a set such that it has at least two elements. By Theorem 7, the system $\{Z(\alpha), \langle\alpha(t_0), \alpha(t)\rangle, rank(\alpha)\}$ is a complete system of $SO(2,\mathbb{R})$-invariant functions on the set of all T-figures α such that $Z(\alpha) \neq T$, $rank(\alpha) = 1$ and $t_0 \in T \setminus Z(\alpha)$. Complete system of relations between elements of this system follows easy from Theorem 5.*

Corollary 2. *Let α and β be a T-figures in E_2 such that $Z(\alpha) \neq T$ and $Z(\beta) \neq T$. Assume that there exists a single matrix $F \in O(2,\mathbb{R})$ such that $\beta(t) = F\alpha(t), \forall t \in T$. Then $rank(\alpha) = rank(\beta) = 2$.*

Conversely, assume that $\alpha \overset{O(2,\mathbb{R})}{\sim} \beta$, and $rank(\alpha) = rank(\beta) = 2$. Then there exists a single matrix $F \in O(2,\mathbb{R})$such that $\beta(t) = F\alpha(t), \forall t \in T$.

Proof. It follows from Theorems 4,6 and 7. □

6. Complete systems of invariants of a T-figure in E_2 for the group $MSO(2,\mathbb{R})$

Let $G = O(2,\mathbb{R})$ or $G = SO(2,\mathbb{R})$. Denote by $G \ltimes Tr(2,\mathbb{R})$ the group of all transformations of E_2 generated by elements of G and all translations of E_2. In particularly, $MO(2,\mathbb{R}) = O(2,\mathbb{R}) \ltimes Tr(2,\mathbb{R})$ and $MSO(2,\mathbb{R}) = SO(2,\mathbb{R}) \ltimes Tr(2,\mathbb{R})$.

Assume that the set T has only one element. Let α and β be T-figures. Then they are $Tr(2,\mathbb{R})$-equivalent. Hence they are $G \ltimes Tr(2,\mathbb{R})$-equivalent. Below we assume that T has at last two elements.

Proposition 11. *Let $G = O(2,\mathbb{R})$ or $G = SO(2,\mathbb{R})$ and T be a set such that it has at last two elements.*

(1) *Assume that $\alpha \overset{G \ltimes Tr(2,\mathbb{R})}{\sim} \beta$, and t_0 is a fixed element of T. Then $(\alpha(t) - \alpha(t_0)) \overset{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$.*

(2) *Assume that $(\alpha(t) - \alpha(t_0)) \overset{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$, for some element $t_0 \in T$. Then $\alpha \overset{G \ltimes Tr(2,\mathbb{R})}{\sim} \beta$.*

Proof. $\Rightarrow$ Assume that $\alpha \overset{G \ltimes Tr(2,\mathbb{R})}{\sim} \beta$. Then there exists $F \in G$ and $a \in E_2$ such that $\beta(t) = F\alpha(t) + a, \forall t \in T$. In particularly, for $t = t_0$, we have $\beta(t_0) = F\alpha(t_0) + a$. This equality implies $a = \beta(t_0) - F\alpha(t_0)$. This equality and equalities $\beta(t) = F\alpha(t) + a, \forall t \in T$, imply equalities $\beta(t) = F\alpha(t) + \beta(t_0) - F\alpha(t_0), \forall t \in T$. These equalities imply equalities $\beta(t) - \beta(t_0) = F(\alpha(t) - \alpha(t_0)), \forall t \in T$, that is $(\alpha(t) - \alpha(t_0)) \overset{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$.

$\Leftarrow$ Assume that $(\alpha(t) - \alpha(t_0)) \overset{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$. Then there exists $F \in G$ such that $\beta(t) - \beta(t_0) = F(\alpha(t) - \alpha(t_0)), \forall t \in T$. Put $a = \beta(t_0) - F\alpha(t_0)$.

This equality implies $\beta(t_0) = F\alpha(t_0) + a$. The equality $a = \beta(t_0) - F\alpha(t_0)$ and equalities $\beta(t)-\beta(t_0) = F(\alpha(t)-\alpha(t_0)), \forall t \in T$, $\beta(t_0) = F\alpha(t_0)+a$ imply equalities $\beta(t) = F\alpha(t) + a, \forall t \in T$. Hence $\alpha \overset{G\ltimes Tr(2,\mathbb{R})}{\sim} \beta$. □

Proposition 12. *Let $G = SO(2,\mathbb{R})$ or $G = O(2,\mathbb{R})$. Assume that α and β are T-figures such that $\alpha \overset{G\ltimes Tr(2,\mathbb{R})}{\sim} \beta$ and $t_0 \in T$. Then $Z(\alpha(t)-\alpha(t_0)) = Z(\beta(t)-\beta(t_0))$.*

Proof. This statement follows from Propositions 8 and 11. □

This proposition means that the function $Z(\alpha(t) - \alpha(t_0))$ is a $G \ltimes Tr(2,\mathbb{R})$-invariant function of a T-figure $\alpha(t)$ for any $t_0 \in T$.

Proposition 13. *Let $G = SO(2,\mathbb{R})$ or $G = O(2,\mathbb{R})$. Assume that $t_0 \in T$ and $Z(\alpha(t) - \alpha(t_0)) = Z(\beta(t) - \beta(t_0) = T$. Then $\alpha \overset{G\ltimes Tr(2,\mathbb{R})}{\sim} \beta$.*

Proof. In this case, we have $\alpha(t) = \alpha(t_0), \forall t \in T$, and $\beta(t) = \beta(t_0), \forall t \in T$. These equalities imply $\beta(t) = \alpha(t) + (\beta(t_0) - \alpha(t_0)), \forall t \in T$. Hence T-figures α and β are $G \ltimes Tr(2,\mathbb{R})$-equivalent. □

Theorem 8. *Let $t_0 \in T$, α be a T-figure in E_2 such that $Z(\alpha(t)-\alpha(t_0)) \neq T$, and $t_1 \in T \setminus Z(\alpha(t) - \alpha(t_0))$ be fixed.*

(*i*) *Suppose that a T-figure β in E_2 such that $\alpha \overset{MSO(2,\mathbb{R})}{\sim} \beta$. Then following equalities hold:*

$$\begin{cases} Z(\alpha(t) - \alpha(t_0)) = Z(\beta(t) - \beta(t_0) \\ \langle\alpha(t_1) - \alpha(t_0), \alpha(t) - \alpha(t_0)\rangle = \langle\beta(t_1) - \beta(t_0), \beta(t) - \beta(t_0)\rangle \\ [(\alpha(t_1) - \alpha(t_0))\,(\alpha(t) - \alpha(t_0))] = [(\beta(t_1) - \beta(t_0))\,(\beta(t) - \beta(t_0)] \end{cases} \tag{32}$$

for all $t \in T \setminus Z(\alpha(t) - \alpha(t_0))$.

(*ii*) *Conversely, assume that a T-figure β in E_2 such that the equalities (32) hold. Then there exists only one element $F \in MSO(2,\mathbb{R})$ such that $\beta = F\alpha$. The evident form of F as follows:$F\alpha(t) = H\alpha(t) + a, \forall t \in T$, where $H \in SO(2,\mathbb{R})$, $a \in E_2$. Here evident form of H as follows*

$$H = \begin{pmatrix} \frac{\langle\alpha(t_1)-\alpha(t_0),\,\beta(t_1)-\beta(t_0)\rangle}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle} & -\frac{[(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle} \\ \frac{[(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle} & \frac{\langle\alpha(t_1)-\alpha(t_0),\,\beta(t_1)-\beta(t_0)\rangle}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle} \end{pmatrix}, \tag{33}$$

where $\det(H) = (\frac{\langle\alpha(t_1)-\alpha(t_0),\,\beta(t_1)-\beta(t_0)\rangle}{\langle\alpha(t_1)-\alpha(t_0),\alpha(t_1)-\alpha(t_0)\rangle})^2 + (\frac{[(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle\alpha(t_1)-\alpha(t_0),\,\alpha(t_1)-\alpha(t_0)\rangle})^2 = 1$. *The element a has the following form: $a = \beta(t_0) - H\alpha(t_0)$.*

Proof. It follows from Proposition 11 and Theorem 4 □

Corollary 3. *Let α and β be T-figures in E_2. Assume that α and $t_0 \in T$ are such that $Z(\alpha(t) - \alpha(t_0)) \neq T$. Assume that $F_1 \in SO(2,\mathbb{R})$, $a_1 \in E_2$, $F_2 \in SO(2,\mathbb{R})$, $a_2 \in E_2$ such that:*
1) $\beta(t) = F_1\alpha(t) + a_1, \forall t \in T$,
2) $\beta(t) = F_2\alpha(t) + a_2, \forall t \in T$.

Then $F_1 = F_2, a_1 = a_2$.

Proof. It follows easy from Proposition 11and Theorem 8. □

Remark 4. *Let* $t_0 \in T$. *By Theorem 8, the system*
$\{Z(\alpha(t)-\alpha(t_0)), \langle\alpha(t_1)-\alpha(t_0), \alpha(t)-\alpha(t_0)\rangle, [(\alpha(t_1)-\alpha(t_0))\,(\alpha(t)-\alpha(t_0))]\}$
is a complete system of $MSO(2,\mathbb{R})$*-invariant functions on the set of all* T*-figures* α *in* E_2 *such that* $Z(\alpha(t)-\alpha(t_0)) \neq T$, *where* $t_1 \in T \setminus Z(\alpha(t)-\alpha(t_0))$ *be fixed. A complete system of relations between elements of this complete system is obtained as in Theorem 5.*

7. Complete systems of invariants of a T-figure in E_2 for the group $MO(2,\mathbb{R})$

Let α and β be T-figures in E_2. Assume that α and $t_0 \in T$ such that $Z(\alpha(t)-\alpha(t_0)) \neq T$. Then, by Proposition 11 $\alpha \overset{MO(2,\mathbb{R})}{\sim} \beta$ if and only if $(\alpha(t)-\alpha(t_0)) \overset{O(2,\mathbb{R})}{\sim} (\beta(t)-\beta(t_0), \forall t \in T$. In this case, by Proposition 10, there exist only three following possibilities for the set $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$:
(I) $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$ has only one element F, where $F \in SO(2,\mathbb{R})$.
(II) $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$ has only one element F, where $F \in SO(2,\mathbb{R})\cdot W$.
(III) $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$ has only two elements F_1 and F_2, where $F_1 \in SO(2,\mathbb{R})$ and $F_2 \in SO(2,\mathbb{R})\cdot W$.

A description of the set $Equ(\alpha(t)-\alpha(t_0), \beta(t)-\beta(t_0))$ and a complete system of invariants of a T-figure in E_2 in the case (I) are given in Section 5.

Consider the case (II).

Theorem 9. *Let* α *be a* T*-figure in* E_2 *such that* $Z(\alpha(t)-\alpha(t_0)) \neq T$ *for some* $t_0 \in T$ *and* $t_1 \in T \setminus Z(\alpha(t)-\alpha(t_0))$ *be fixed.*

- (i) *Suppose that a* T*-figure* β *such that the following equalities* $\beta(t) = HW\alpha(t) + d, \forall t \in T$, *hold for some* $H \in SO(2,\mathbb{R})$ *and some* $d \in E_2$. *Then following equalities hold:*
$$\begin{cases} Z(\alpha(t)-\alpha(t_0)) = Z(\beta(t)-\beta(t_0)) \\ \langle\alpha(t_1)-\alpha(t_0), \alpha(t)-\alpha(t_0)\rangle = \langle\beta(t_1)-\beta(t_0), \beta(t)-\beta(t_0)\rangle \\ -[\alpha(t_1)-\alpha(t_0)\,\alpha(t)-\alpha(t_0)] = [\beta(t_1)-\beta(t_0)\,\beta(t)-\beta(t_0)]\,. \end{cases} \tag{34}$$
for all $t \in T \setminus Z(\alpha(t)-\alpha(t_0))$.
- (ii) *Conversely, assume that a* T*-figure* β *in* E_2 *such that the equalities (34) hold. Then a single matrix* $U \in SO(2,\mathbb{R})$ *and a single* $d \in E_2$ *exist such that* $\beta(t) = UW\alpha(t) + d, \forall t \in T$. *In this case,* U *has following form*
$$U = \begin{pmatrix} \frac{\langle W(\alpha(t_1)-\alpha(t_0)), (\beta(t_1)-\beta(t_0))\rangle}{\langle(\alpha(t_1)-\alpha(t_0)), (\alpha(t_1)-\alpha(t_0))\rangle} & -\frac{[W(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle(\alpha(t_1)-\alpha(t_0)), (\alpha(t_1)-\alpha(t_0))\rangle} \\ \frac{[W(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle(\alpha(t_1)-\alpha(t_0)), (\alpha(t_1)-\alpha(t_0))\rangle} & \frac{\langle W(\alpha(t_1)-\alpha(t_0)), (\beta(t_1)-\beta(t_0))\rangle}{\langle(\alpha(t_1)-\alpha(t_0)), (\alpha(t_1)-\alpha(t_0))\rangle} \end{pmatrix}, \tag{35}$$
where
$\det(U) = (\frac{\langle W(\alpha(t_1)-\alpha(t_0)), (\beta(t_1)-\beta(t_0))\rangle}{\langle(\alpha(t_1)-\alpha(t_0)), (\alpha(t_1)-\alpha(t_0))\rangle})^2 + (\frac{[W(\alpha(t_1)-\alpha(t_0))\,(\beta(t_1)-\beta(t_0))]}{\langle(\alpha(t_1)-\alpha(t_0)), (\alpha(t_1)-\alpha(t_0))\rangle})^2 = 1$. *The element* d *has following form:* $d = \beta(t_0) - UW\alpha(t_0)$.

Proof. It follows easy from Proposition 11 and Theorem 6 □

Consider the case (III).

Theorem 10. *Let α be a T-figure in E_2 such that $Z(\alpha(t)-\alpha(t_0)) \neq T$ for some $t_0 \in T$ and $t_1 \in T \setminus Z(\alpha(t)-\alpha(t_0))$ be fixed.*

(i) *Suppose that matrices $F_1 \in SO(2,\mathbb{R})$, $F_2 \in SO(2,\mathbb{R})$ and vectors $d_1 \in E_2, d_2 \in E_2$ exist such that $\beta(t) = F_1\alpha(t) + d_1, \forall t \in T$, and $\beta(t) = F_2W\alpha(t) + d_2, \forall t \in T$. Then following equalities hold:*

$$\begin{cases} Z(\alpha(t)-\alpha(t_0)) = Z(\beta(t)-\beta(t_0)) \\ \langle \alpha(t_1)-\alpha(t_0), \alpha(t)-\alpha(t_0)\rangle = \langle \beta(t_1)-\beta(t_0), \beta(t)-\beta(t_0)\rangle \\ rank(\alpha(t)-\alpha(t_0)) = rank(\beta(t)-\beta(t_0)) = 1, \end{cases} \quad (36)$$

for all $t \in T \setminus Z(\alpha(t)-\alpha(t_0))$.

(ii) *Conversely, assume that the equalities (36) hold. Then only two matrices $H_1 \in SO(2,\mathbb{R})$, $H_2 \in SO(2,\mathbb{R})$ and only two vectors $d_1 \in E_2, d_2 \in E_2$ exist such that following equalities $\beta(t) = H_1\alpha(t) + d_1, \forall t \in T$, $\beta(t) = H_2W\alpha(t) + d_2, \forall t \in T$, hold. Here the matrix H_1 has following form:*

$$H_1 = \begin{pmatrix} \frac{\langle \alpha(t_1)-\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} & -\frac{[\alpha(t_1)-\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} \\ \frac{[\alpha(t_1)-\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} & \frac{\langle \alpha(t_1)-\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} \end{pmatrix}, \quad (37)$$

where $\det(H_1) = (\frac{\langle \alpha(t_1)-\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle})^2 + (\frac{[\alpha(t_1)-\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle})^2 = 1$. *Vector d_1 has following form $d_1 = \beta(t_0) - H_1\alpha(t_0)$.*

Here the matrix $H_2 \in SO(2,\mathbb{R})$ has following form

$$H_2 = \begin{pmatrix} \frac{\langle W\alpha(t_1)-W\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} & -\frac{[W\alpha(t_1)-W\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} \\ \frac{[W\alpha(t_1)-W\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} & \frac{\langle W\alpha(t_1)-W\alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle} \end{pmatrix}, \quad (38)$$

where
$\det(H_2) = (\frac{W\alpha(t_1)-W\langle \alpha(t_0), \beta(t_1)-\beta(t_0)\rangle}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle})^2 + (\frac{[W\alpha(t_1)-W\alpha(t_0)\ \beta(t_1)-\beta(t_0)]}{\langle \alpha(t_1)-\alpha(t_0), \alpha(t_1)-\alpha(t_0)\rangle})^2 = 1$. *Vector d_2 has following form $d_2 = \beta(t_0) - H_2W\alpha(t_0)$.*

Proof. It follows easy from Proposition 11and Theorem 7 □

8. Conclusion

Results and methods of the present paper are useful in the theory of G-invariants of systems of points, curves, vector fields, topological figures and polynomial figures in the two-dimensional Euclidean space E_2 for groups $G = SO(2,\mathbb{R})$, $O(2,\mathbb{R})$, $MSO(2,\mathbb{R})$ and $MO(2,\mathbb{R})$. Results and methods of the present paper are also useful in the theory of G-invariants of mechanical figures in the two-dimensional Euclidean space E_2 for Galilei groups.

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B-LIFT CURVES AND ITS RULED SURFACES

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Abstract. In this paper, we have described the B-Lift curve in Euclidean space as a curve obtained by combining the endpoints of the binormal vector of a unit speed curve. Subsequently, we have explored the Frenet frames of the B-Lift curves. Moreover, we have introduced the tangent, normal and binormal surfaces of the B-Lift curve and examined the geometric invariants of these surfaces. Finally, we have investigated the singularities of these surface and visualized the surfaces with MATLAB program.

1. Introduction

Ruled surfaces have important applications in kinematics, computer science, physics, etc. A ruled surface is defined by a straight line that is moving along a curve [1]. Many mathematicians have studied the ruled surfaces [2–8]. E. Ergün and M. Çalışkan [2] created ruled surfaces by accepting the natural lift of a curve as the base curve and they characterized these surfaces. The natural lift curve is described in an example in Thorpe's book. Generally, the natural lift curve is defined as the curve formed by combining the end points of the tangent vectors of the curve [9].

One of the main purposes of classical differential geometry is to investigate some classes of surfaces such as developable surfaces and minimal surfaces. Ruled surfaces are developable surfaces with zero Gaussian curvature suach that these surfaces are called minimal surfaces [10]. S. Izumiya and N. Takeuchi presented new results for the Gaussian curvature and the main curvature of the ruled surface [3].

A point is called the singular point of the surface if the tangent vector at any point does not lie in a plane. At the singular point, the surface intersects itself. If

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all points of a curve on a surface are singular, this curve is called a singular curve [1]. Recently, many studies have been done on the singularity of curves [11–16].

In this study, we define a new curve which is called B-Lift curve and we calculate its Frenet vectors. Furthermore, we examine the integral invariants of the tangent, principal normal and binormal surfaces of the B-Lift curve. Also, we study the singular points of the ruled surfaces of the B-Lift curve. Finally, we give examples of these situations and drawn our surfaces.

2. Preliminaries

Let a vector $\vec{x} = (x_1, x_2, x_3)$ be given in $\mathbb{R}^3$. The norm of $\vec{x} = (x_1, x_2, x_3)$ is defined by

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

A vector which its norm is 1 is called a unit vector. For the vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, the inner product $<,>: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined as

$$< \vec{x}, \vec{y} >= x_1 y_1 + x_2 y_2 + x_3 y_3$$

which is called Euclidean inner product. If $\gamma^{'}(s) \neq 0$, $\gamma : \mathrm{I} \to \mathbb{R}^3$ is called regular curve , for all s $\in$ I. Let $\gamma : \mathrm{I} \to \mathbb{R}^3$ be a curve, if $\|\gamma^{'}(s)\| = 1$ then the curve is called unit speed curve [1].

A curve α is called general helix in $\mathbb{R}^3$ if tangent vector of the curve makes a constant angle with a fixed straight line. M. A. Lancret discovered that the ratio of curvatures of the general helix is constant in 1802 [17].

Let γ be a regular curve in $\mathbb{R}^3$. The set $\{T(s), N(s), B(s)\}$ is called Frenet frame given by tangent, principal normal and binormal vectors, respectively.

$$T(s) = \gamma^{'}(s),$$
$$N(s) = \frac{\gamma^{''}(s)}{\|\gamma^{''}(s)\|},$$
$$B(s) = T(s) \times N(s),$$

Here $T(s)$, $N(s)$ and $B(s)$ are the unit tangent, principal normal and binormal vectors of $\gamma(s)$, respectively. Frenet-Serret formulas are following as [10]:

$$T^{'}(s) = \kappa(s) N(s),$$
$$N^{'}(s) = -\kappa(s) T(s) + \tau(s) B(s),$$
$$B^{'}(s) = -\tau(s) N(s).$$

When a point moves along a curve with unit speed, the rotation is determined by an angular velocity vector W that is called Darboux vector. The Darboux vector W is presented as $W = \tau T + \kappa B$. Moreover, $\kappa = \|W\| \cos\varphi$ and $\tau = \|W\| \sin\varphi$ are written. Here φ is the angle between Darboux vector and binormal vector of $\gamma(s)$ [10].

Let γ be a regular curve and ω be unit direction of a straight line in $\mathbb{R}^3$, then the ruled surface ϕ is the surface formed by the continuous moving of ω based on the curve γ. The parametric representation of the ruled surface ϕ is given as follows [10]:

$$\phi(s,v)=\gamma(s)+v\omega(s).$$

For the ruled surface ϕ(s,v), we can write

$$\phi_s\times\phi_v=\gamma^{'}(s)\times\omega(s)+v\omega^{'}(s)\times\omega(s).$$

Hence (s_0,v_0) is a singular point of ϕ(s,v) if and only if $\gamma^{'}(s_0)\times\omega(s_0)+v_0\omega^{'}(s_0)\times\omega(s_0)=0$. If $\omega^{'}(s)\times\omega(s)=0$, the ruled surface ϕ(s,v) is called a cylindrical surface. Therefore, if $\omega^{'}(s)\times\omega(s)\neq 0$ the ruled surface ϕ(s,v) is called non-cylindrical surface [10].

The foot of the common normal between two consecutive generators is called the striction point on a ruled surface. The striction curve formed by the set of striction points is as follows [10]:

$$b(s)=\gamma(s)-\frac{<\gamma^{'}(s),\omega^{'}(s)>}{<\omega^{'}(s),\omega^{'}(s)>}\omega(s).$$

The distribution parameter for a ruled surface is described as follows [10]:

$$P_w=\frac{det(\gamma^{'},\omega,\omega^{'})}{||\omega^{'}||^2}.$$

A ruled surface ϕ is developable if and only if $P_w=0$ [10].

Let ϕ(s,v) be a ruled surface. Then the Gaussian curvature of ϕ(s,v) is given by

$$K(s,v)=-\frac{(det(\gamma^{'}(s),\omega(s),\omega^{'}(s)))^2}{(EG-F^2)^2}$$

and mean curvature of ϕ(s,v) given by

$$H(s,v)=\frac{-2<\gamma^{'}(s),\omega(s)>det(\gamma^{'}(s),\omega(s),\omega^{'}(s))+det(\gamma^{''}(s)+v\omega^{''}(s),\gamma^{'}(s)+v\omega^{'}(s),\omega(s))}{2(EG-F^2)^{3/2}}$$

where $E=E(s,v)=||\gamma^{'}(s)+v\omega^{'}(s)||^2, F=F(s,v)=<\gamma^{'}(s),\omega(s)>$,
$G=G(s,v)=1$ [3].

Let γ be a regular curve in $\mathbb{R}^3$ and the set $\{T(s),N(s),B(s)\}$ be the Frenet vectors of the curve γ. Then the tangent, principal normal and binormal surfaces of the curve γ are given in the following equalities [3]:

$$\begin{aligned}\phi_T(s,v)&=\gamma(s)+vT(s)\\ \phi_N(s,v)&=\gamma(s)+vN(s)\\ \phi_B(s,v)&=\gamma(s)+vB(s).\end{aligned}$$

3. B-Lift Curves and its Ruled Surfaces

Definition 1. *Let $\gamma : I \to M \subset \mathbb{R}^3$ be a unit speed curve, then $\gamma_B : I \to TM$ is called the B-Lift curve and ensures the following equation:*

$$\gamma_B(s) = (\gamma(s), B(s)) = B(s)|_{\gamma(s)}. \tag{1}$$

Proposition 1. *Assume that γ_B is the B-Lift curve of a unit speed curve γ. Thus, the following equations are provided:*

$$\begin{aligned} T_B(s) &= -N(s), \\ N_B(s) &= \frac{\kappa(s)}{||W(s)||}T(s) - \frac{\tau(s)}{||W(s)||}B(s), \\ B_B(s) &= \frac{\tau(s)}{||W(s)||}T(s) + \frac{\kappa(s)}{||W(s)||}B(s) \end{aligned}$$

where $\{T(s), N(s), B(s)\}$ and $\{T_B(s), N_B(s), B_B(s)\}$ are the Frenet vectors of the curve γ and γ_B, respectively. (In particular, the torsion will be considered greater than zero.)

(i) Let γ_B be B-Lift curve of the regular curve γ. Then the tangent surface of B-Lift curve is given as follows:

$$\phi_{T_B}(s, v) = \gamma_B(s) + vT_B(s). \tag{2}$$

From (1) and Proposition 1, we have

$$\phi_{T_B}(s, v) = B(s) + v(-N(s)). \tag{3}$$

Now, we investigate the singular point of the ruled surface ϕ_{T_B}

$$\begin{aligned} (\phi_{T_B})_s \times (\phi_{T_B})_v &= (B^{'}(s) \times (-N(s)) + v(\kappa(s)T(s) - \tau(s)N(s)) \times -N(s) \\ &= -v\kappa(s)B(s). \end{aligned}$$

Since for every $(s_0, v_0) \in I \times (\mathbb{R} - \{0\})$, $(\phi_{T_B})_{s_0} \times (\phi_{T_B})_{v_0} = -v_0\kappa(s_0)B(s_0) \neq 0$, the ruled surface ϕ_{T_B} has no singular point. Since for every $(s_0, v_0) \in I \times (\mathbb{R} - \{0\})$, $\omega^{'}(s_0) \times \omega(s_0) = \kappa(s_0)B(s_0) \neq 0$, the ruled surface ϕ_{T_B} is non-cylindrical surface. The distribution parameter of the tangent surface ϕ_{T_B} is

$$P_{T_B} = \frac{det(B^{'}, -N, -N^{'})}{|| - N^{'}||^2} = 0.$$

The striction curve of the ruled surface ϕ_{T_B} is

$$\begin{aligned} b_{T_B}(s) &= \gamma_B(s) - \frac{< \gamma_B^{'}(s), T_B^{'}(s) >}{< T_B^{'}(s), T_B^{'}(s) >}T_B(s) \\ &= B(s) - \frac{< -\tau N, \kappa T - \tau B >}{< \kappa T - \tau B, \kappa T - \tau B >}(\kappa T - \tau B) \\ &= B(s). \end{aligned}$$

The Gaussian curvature of the ruled surface ϕ_{T_B} is

$$K_{T_B}(s,v) = -\frac{(det(-\tau N, -N, \kappa T - \tau B))^2}{(EG - F^2)^2} = 0.$$

The mean curvature of the ruled surface ϕ_{T_B} is

$$\begin{aligned} H_{T_B}(s,v) &= \frac{det(\kappa\tau T - \tau^{'} N - \tau^2 B + v(\kappa^{'} T + (\kappa^2+\tau^2)N - \tau^{'} B, -\tau N + v(\kappa T - \tau B), -N)}{2(EG-F^2)^{3/2}} \\ &= \frac{v^2(\frac{\tau}{\kappa})^{'}\kappa^2}{2(EG-F^2)^{3/2}}. \end{aligned}$$

Corollary 1. *The ruled surface ϕ_{T_B} is developable.*

Corollary 2. *Let the curve $\gamma : I \to \mathbb{R}^3$ be a general helix curve. Then the ruled surface ϕ_{T_B} is a minimal surface.*

(ii) Let γ_B be B-Lift curve of the regular curve γ. Then the principal normal surface of B-Lift curve is given as

$$\phi_{N_B}(s,v) = \gamma_B(s) + vN_B(s). \quad (4)$$

From (1) and Proposition 1, we get

$$\phi_{N_B}(s,v) = B(s) + v(\frac{\kappa(s)}{||W(s)||}T(s) - \frac{\tau(s)}{||W(s)||}B(s)). \quad (5)$$

$$(\phi_{N_B})_s \times (\phi_{N_B})_v = (-\tau + \frac{\tau^2}{||W||}, v(\frac{\kappa^{'}\tau - \kappa\tau^{'}}{||W||^2}), -\kappa + \frac{\kappa\tau}{||W||}). \quad (6)$$

The distribution parameter of the principal normal surface of the curve γ_B is

$$\begin{aligned} P_{N_B} &= \frac{det(B^{'}, N_B, N_B^{'})}{||N_B^{'}||^2} \\ &= \frac{\tau(-\frac{\kappa\tau^{'}}{||W||^2} + \frac{\kappa^{'}\tau}{||W||^2})}{(\frac{\kappa^{'}}{||W||})^2 + (\frac{\kappa^2+\tau^2}{||W||})^2 + (\frac{\tau^{'}}{||W||})^2}. \end{aligned}$$

The striction curve of the ruled surface ϕ_{N_B} is

$$\begin{aligned} b_{N_B}(s) &= \gamma_B(s) - \frac{< \gamma_B^{'}(s), N_B^{'}(s) >}{< N_B^{'}(s), N_B^{'}(s) >} N_B(s) \\ &= B(s) - \frac{< -\tau N, \frac{\kappa^{'}}{||W||}T + \frac{\kappa^2+\tau^2}{||W||}N - \frac{\tau^{'}}{||W||}B >}{< \frac{\kappa^{'}}{||W||}T + \frac{\kappa^2+\tau^2}{||W||}N - \frac{\tau^{'}}{||W||}B, \frac{\kappa^{'}}{||W||}T + \frac{\kappa^2+\tau^2}{||W||}N - \frac{\tau^{'}}{||W||}B >}(\frac{\kappa}{||W||}T - \frac{\tau}{||W||}B). \end{aligned}$$

The Gaussian curvature of the ruled surface ϕ_{N_B} is

$$K_{N_B}(s,v) = -\frac{(det(\gamma_B^{'}, N_B, N_B^{'}))^2}{(EG-F^2)^2}$$

$$= \frac{\tau(-\frac{\kappa\tau'}{||W||^2} + \frac{\kappa'\tau}{||W||^2})}{(EG - F^2)^2}.$$

The mean curvature of the ruled surface ϕ_{N_B} is

$$\begin{aligned} H_{N_B}(s,v) &= \frac{det(\gamma_B'' + vN_B'', \gamma_B' + vN_B', N_B)}{2(EG - F^2)^{3/2}} \\ &= \frac{v^2(\frac{3\kappa\kappa' + 3\tau\tau'}{||W||^3})(\kappa'\tau - \kappa\tau') + v^2(\frac{\kappa^2+\tau^2}{||W||^3})(\kappa\tau'' - \kappa''\tau) + v\tau'(\frac{\kappa\tau' - \tau\kappa'}{||W||^2}) + v\tau(\frac{\kappa''\tau - \tau''\kappa}{||W||^2})}{2(EG - F^2)^{3/2}}. \end{aligned}$$

Corollary 3. *Assume that $\gamma : I \to \mathbb{R}^3$ is a general helix curve. Hence the ruled surface ϕ_{N_B} is a developable surface.*

Corollary 4. *Let $\gamma : I \to \mathbb{R}^3$ be a general helix curve. Then the ruled surface ϕ_{N_B} is a minimal surface.*

(iii) Let γ_B be B-Lift curve of the regular curve γ. Then the binormal surface of B-Lift curve is given by

$$\phi_{B_B}(s,v) = \gamma_B(s) + vB_B(s). \tag{7}$$

From (1) and Proposition 1, we know

$$\phi_{B_B}(s,v) = B(s) + v(\frac{\tau(s)}{||W(s)||}T(s) + \frac{\kappa(s)}{||W(s)||}B(s)). \tag{8}$$

$$(\phi_{B_B})_s \times (\phi_{B_B})_v = (-\frac{\kappa\tau}{||W||}, v(\frac{\kappa'\tau - \kappa\tau'}{||W||^2}), \frac{\tau^2}{||W||}). \tag{9}$$

From (9), the ruled surface ϕ_{B_B} has no singular point and since $B_B \times B_B' \neq 0$, ϕ_{B_B} is non-cylindrical surface.
The distribution parameter of the ruled surface ϕ_{B_B} is

$$\begin{aligned} P_{B_B} &= \frac{det(B', B_B, B_B')}{||B_B'||^2} \\ &= \frac{\tau(-\frac{\kappa\tau'}{||W||^2} + \frac{\kappa'\tau}{||W||^2})}{(\tau')^2 + (\kappa')^2}. \end{aligned}$$

The striction curve of the ruled surface ϕ_{B_B} is

$$\begin{aligned} b_{B_B}(s) &= \gamma_B(s) - \frac{< \gamma_B'(s), B_B'(s) >}{< B_B'(s), B_B'(s) >}B_B(s) \\ &= B(s). \end{aligned}$$

The Gaussian curvature of the ruled surface ϕ_{B_B} is

$$K_{B_B}(s,v) = -\frac{(det(\gamma_B', B_B, B_B'))^2}{(EG - F^2)^2}$$

$$= \frac{\tau(-\frac{\kappa\tau'}{||W||^2} + \frac{\kappa'\tau}{||W||^2})}{(EG - F^2)^2}.$$

The mean curvature of the ruled surface ϕ_{B_B} is

$$\begin{aligned} H_{B_B}(s,v) &= \frac{det(\gamma_B'' + vB_B'', \gamma_B' + vB_B', B_B)}{2(EG - F^2)^{3/2}} \\ &= \frac{\frac{\tau v}{||W||^2}(-\kappa'\tau + \tau'\kappa + \kappa''\tau - \kappa\tau'') - \frac{\tau^2}{||W||}(\kappa^2 + \tau^2) - \frac{v^2}{||W||^3}(\tau'\kappa - \kappa'\tau)^2}{2(EG - F^2)^{3/2}}. \end{aligned}$$

Corollary 5. *Let $\gamma : I \to \mathbb{R}^3$ be a general helix curve. Then the ruled surface ϕ_{B_B} is a developable surface.*

Example 1. *Let us consider the unit speed general helix curve that is given as following equality:*

$$\gamma(s) = (\frac{\sqrt{3}}{3}s^{3/2}, \frac{\sqrt{3}}{3}(1-s)^{3/2}, \frac{s}{2}).$$

Then the curve γ_B is given as follows:

$$\gamma_B(s) = (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}).$$

The Frenet vectors of the curve γ_B can be calculated by

$$\begin{aligned} T_B(s) &= (-(1-s)^{1/2}, -s^{1/2}, 0), \\ N_B(s) &= (s^{1/2}, -(1-s)^{1/2}, 0), \\ B_B(s) &= (0, 0, 1). \end{aligned}$$

From (3), (5) and (8), the tangent, normal and binormal surfaces are calculated as follows:

$$\begin{aligned} \phi_{T_B}(s,v) &= \gamma_B(s) + vT_B(s) \\ &= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}) + v(-(1-s)^{1/2}, -s^{1/2}, 0) \\ \phi_{N_B}(s,v) &= \gamma_B(s) + vN_B(s) \\ &= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}) + v(s^{1/2}, -(1-s)^{1/2}, 0) \\ \phi_{B_B}(s,v) &= \gamma_B(s) + vB_B(s) \\ &= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}) + v(0, 0, 1). \end{aligned}$$

The distrubition parameters of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are

$$P_{T_B} = \frac{det(B', T_B, T_B')}{||T_B'||^2} = 0,$$

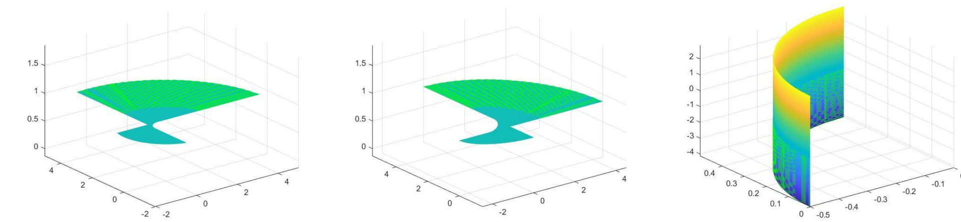

FIGURE 1. Illustration of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B}, respectively.

$$\begin{aligned} P_{N_B} &= \frac{det(B', N_B, N_B')}{||N_B'||^2} = 0, \\ P_{B_B} &= \frac{det(B', B_B, B_B')}{||B_B'||^2} = 0. \end{aligned}$$

Since $P_{T_B} = P_{N_B} = P_{B_B} = 0$, the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are developable. The striction lines of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are given by

$$\begin{aligned} b_{T_B}(s) &= \gamma_B(s) - \frac{< \gamma_B'(s), T_B'(s) >}{< T_B'(s), T_B'(s) >} T_B(s) \\ &= B(s) \\ &= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}). \\ b_{N_B}(s) &= \gamma_B(s) - \frac{< \gamma_B'(s), N_B'(s) >}{< N_B'(s), N_B'(s) >} N_B(s) \\ &= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}) + \frac{1}{2}(s^{1/2}, -(1-s)^{1/2}, 0) \\ &= (0, 0, \frac{\sqrt{3}}{2}). \\ b_{B_B}(s) &= \gamma_B(s) - \frac{< \gamma_B'(s), B_B'(s) >}{< B_B'(s), B_B'(s) >} B_B(s) \\ &= B(s) \\ &= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}). \end{aligned}$$

Gaussian curvatures of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are given as follows:

$$K_{T_B}(s, v) = -\frac{(det(\gamma_B', T_B, T_B'))^2}{(EG - F^2)^2}$$

$$
\begin{aligned}
&= 0 \\
K_{N_B}(s,v) &= -\frac{(det(\gamma_B^{'}, N_B, N_B^{'}))^2}{(EG-F^2)^2} \\
&= 0 \\
K_{B_B}(s,v) &= -\frac{(det(\gamma_B^{'}, B_B, B_B^{'}))^2}{(EG-F^2)^2} \\
&= 0.
\end{aligned}
$$

Mean curvatures of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} *and* ϕ_{B_B} *are calculated as*

$$
\begin{aligned}
H_{T_B}(s,v) &= \frac{-2 < \gamma^{'}(s), T_B(s) > det(\gamma^{'}(s), T_B(s), T_B^{'}(s))}{2(EG-F^2)^{3/2}} \\
&+ \frac{det(\gamma^{''}(s) + vT_B^{''}(s), \gamma^{'}(s) + vT_B^{'}(s), T_B(s))}{2(EG-F^2)^{3/2}} \\
&= 0 \\
H_{N_B}(s,v) &= \frac{-2 < \gamma^{'}(s), N_B(s) > det(\gamma^{'}(s), N_B(s), N_B^{'}(s))}{2(EG-F^2)^{3/2}} \\
&+ \frac{det(\gamma^{''}(s) + vN_B^{''}(s), \gamma^{'}(s) + vN_B^{'}(s), N_B(s))}{2(EG-F^2)^{3/2}} \\
&= 0 \\
H_{B_B}(s,v) &= \frac{-2 < \gamma^{'}(s), B_B(s) > det(\gamma^{'}(s), B_B(s), B_B^{'}(s))}{2(EG-F^2)^{3/2}} \\
&+ \frac{det(\gamma^{''}(s) + vB_B^{''}(s), \gamma^{'}(s) + vB_B^{'}(s), B_B(s))}{2(EG-F^2)^{3/2}} \\
&= 0.
\end{aligned}
$$

Since $H_{T_B}(s,v) = H_{N_B}(s,v) = H_{B_B}(s,v) = 0$, *the ruled surfaces* ϕ_{T_B}, ϕ_{N_B} *and* ϕ_{B_B} *are minimal surfaces.*

4. Conclusion

In this study, based on Thorpe's definition [9], we have introduced the B-lift curve and calculated the Frenet vectors of the B-Lift curves. Furthermore, we have given the tangent, normal, and binormal surfaces of the B-Lift curves and calculated the integral invariants of these surfaces.

Author Contribution Statements The authors jointly worked on the results and they read and approved the final manuscript.

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THE LINEAR ALGEBRA OF A GENERALIZED TRIBONACCI MATRIX

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Abstract. In this paper, we consider a generalization of a regular Tribonacci matrix for two variables and show that it can be factorized by some special matrices. We produce several new interesting identities and find an explicit formula for the inverse and $k-$th power. We also give a relation between the matrix and a matrix exponential of a special matrix.

1. Introduction

Integer sequences are widely used in many areas such as physics, engineering, arts and nature. There have been several studies in the literature that concern about the second order integer sequences and their generalizations such as Fibonacci, Lucas, Pell and Jacobsthal, see [8,9,11–13,17]. Horadam interested in the generalized Fibonacci sequence $\{W_n(a,b;p,q)\}_{n\geq 0}$, where a,b are nonnegative integers and p,q are arbitrary integers, and studied some properties of the sequence, see [11,12]. Another generalization of the Fibonacci sequence is called as the Tribonacci sequence. The Tribonacci sequence is the most familiar series of numbers obtained by generalizing Fibonacci sequence as orders.

For $n \geq 0$, we use the following definition of the sequence of Tribonacci numbers which is given by third order recurrence relation

$$t_{n+3} = t_{n+2} + t_{n+1} + t_n$$

with initial conditions

$$t_0 = t_1 = 1, \quad t_2 = 2.$$

The first few terms of the Tribonacci numbers are given in Table 1.

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n	0	1	2	3	4	5	6	7	8	9	10	11	12
t_n	1	1	2	4	7	13	24	44	81	149	274	504	927

TABLE 1. The first few terms of the Tribonacci sequence

The characteristic polynomial $x^3 - x^2 - x - 1 = 0$ of the third order Tribonacci recurrence has a unique real root of maximum modulus and this is

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} \approx 1.83929,$$

the Tribonacci constant, see [21]. Many researchers studied some properties of the Tribonacci sequence, see [4–6, 10, 15, 20, 22, 23, 25].

A matrix T_n of order $n+1$ with entries

$$t_{i,j} = \begin{cases} \frac{2t_j}{t_{i+2}+t_i-1}, & \text{if } 0 \le j \le i, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

is defined in [26] and the Tribonacci space sequences $\ell_p(T)$ are introduced. For $n = 4$, the matrix T_4 will look as follows

$$T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{15} & \frac{1}{15} & \frac{2}{15} & \frac{4}{15} & \frac{7}{15} \end{bmatrix}.$$

Definition 1. *A square matrix R is regular if and only if R is a stochastic matrix and some power R^k, for $k \ge 1$, has all entries nonzero.*

Thus from the definition of the regular matrix, we obtain that the matrix defined in (1) is a regular matrix.

Inspiring by this study, we define a two variable generalization of the matrix given in (1) and obtain several interesting new properties. We are also interested in matrix factorization of the defined matrix which is a method of representing a matrix as a product of some matrices. There are various types of matrix factorizations such as singular value decomposition, LU factorization, Cholesky factorization, etc. This method is used to simplify calculations, especially in solving a problem that is difficult to solve in its original form. Several authors are interested in matrix factorizations of some special matrices, see [1, 2, 7, 18, 19, 27].

2. A Generalization of the Regular Tribonacci Matrix

In this section, we give a generalization of the matrix defined in (1). We define a matrix $T_n(x,y) = [t_{i,j}(x,y)]$ of order $n+1$ with entries

$$t_{i,j}(x,y) = \begin{cases} \frac{2t_j}{t_{i+2}+t_i-1}x^{i-j}y^j, & \text{if } 0 \le j \le i, \\ 0, & \text{otherwise.} \end{cases}$$

Thus for $n=4$, the matrix will look as follows

$$T_4(x,y) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 \\ \frac{1}{4}x^2 & \frac{1}{4}xy & \frac{1}{2}y^2 & 0 & 0 \\ \frac{1}{8}x^3 & \frac{1}{8}x^2y & \frac{1}{4}xy^2 & \frac{1}{2}y^3 & 0 \\ \frac{1}{15}x^4 & \frac{1}{15}x^3y & \frac{2}{15}x^2y^2 & \frac{4}{15}xy^3 & \frac{7}{15}y^4 \end{bmatrix}.$$

We will denote the (i,j) entry of this matrix as $(T_n(x,y))_{i,j}$. It is easy to see that when x or y is zero, $t_{i,j}(x,y)$ will be trivial. Therefore we generally assume that x and y in $T_n(x,y)$ are non-zero real numbers. It is clear that for $x=y=1$ we have

$$t_{i,j}(1,1) = t_{i,j}$$

and so in this case we obtain the regular Tribonacci matrix (1).

2.1. Multiplication of two Tribonacci matrices. The Tribonacci matrix $T_n(x,y)$ has some interesting properties and applications. Thus we give some of these properties now. For $n, j \in \mathbb{N}$, we define

$$(x \oplus y)_j^n = \sum_{k=0}^{n} t_{k+j,k+j} x^{n-k} y^k.$$

Theorem 1. *For any positive integer n and any real numbers x, y, z and w, we have*

$$\Big(T_n(x,y)T_n(w,z)\Big)_{i,j} = \Big(T_n\big((x \oplus yw)_j, yz\big)\Big)_{i,j}. \tag{2}$$

Proof. From the definition of the matrix $T_n(x,y)$ and the rules of the matrix multiplication, the (i,j) entry of $T_n(x,y)T_n(w,z)$ is 0 for $j > i$. For $j \le i$ it can be obtained as

$$\begin{aligned} \sum_{k=j}^{i} t_{i,k}(x,y)t_{k,j}(w,z) &= \sum_{k=j}^{i} \frac{2t_k}{t_{i+2}+t_i-1}x^{i-k}y^k \frac{2t_j}{t_{k+2}+t_k-1}w^{k-j}z^j \\ &= \frac{2t_j}{t_{i+2}+t_i-1} \sum_{k=j}^{i} \frac{2t_k}{t_{k+2}+t_k-1} x^{i-k}y^k w^{k-j} z^j \\ &= t_{i,j} \sum_{k=j}^{i} t_{k,k} x^{i-k} y^k w^{k-j} z^j \end{aligned}$$

$$
\begin{aligned}
&= t_{i,j}\sum_{k=0}^{i-j} t_{k+j,k+j}x^{i-j-k}y^{k+j}w^k z^j \\
&= t_{i,j}(yz)^j\sum_{k=0}^{i-j} t_{k+j,k+j}x^{i-j-k}(yw)^k \\
&= t_{i,j}(x\oplus yw)_j^{i-j}(yz)^j
\end{aligned}
$$

This is also the (i,j) entry of $T_n\big((x\oplus yw)_j, yz\big)$, so equation (2) holds. □

For $w=x$ and $z=y$ in (2), we

$$(T_n^2(x,y))_{i,j} = T_n\left(x(1\oplus y)_j, y^2\right)_{i,j}.$$

Using formula (2) again, multiplying $T_n^2(x,y)$ and $T_n(x,y)$, we get

$$(T_n^3(x,y))_{i,j} = T_n\left(x(1\oplus y\oplus y^2)_j, y^3\right)_{i,j}.$$

Then using the mathematical induction method, the following results can be obtained.

$$(T_n^k(x,y))_{i,j} = T_n\left(x(1\oplus y\oplus\cdots\oplus y^{k-1})_j, y^k\right)_{i,j}.$$

2.2. **The inverse of the matrix $T_n(x,y)$.** The inverse of the Tribonacci matrix $T_n(x,y)$ is given by the following theorem.

Theorem 2. *The $(i,j)-$entry of the inverse of the matrix $T_n(x,y)$ is*

$$(T_n(x,y)^{-1})_{i,j} = \begin{cases} \frac{t_{i+2}+t_i-1}{2t_iy^i}, & \text{if } i=j, \\ -\frac{(t_{i+2}+t_i-1-2t_i)x}{2t_iy^i}, & \text{if } i=j+1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By straightforward computation of matrix multiplication, we get the desired result. □

2.3. **The factorization of the Tribonacci matrix.** We define the matrices of order $n+1$ with the following entries

$$
\begin{aligned}
(S_n(x,y))_{i,j} &= \begin{cases} t_{i,j}(x,y)t^{-1}_{j-1,j-1}(x,y) + t_{i,j+1}(x,y)t^{-1}_{j,j-1}(x,y) & i\ge j, \\ 0 & i<j, \end{cases} \\
\overline{T}_{n-1}(x,y) &= \begin{bmatrix} 1 & 0 \\ 0 & T_{n-1}(x,y) \end{bmatrix}, \quad n\ge 1, \\
G_n &= S_n, \quad G_k(x,y) = \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & S_k(x,y) \end{bmatrix}, \quad 1\le k\le n-1.
\end{aligned}
$$

Let us consider the product of the matrices $T_n(x,y)$ and $\overline{T}^{-1}_{n-1}(x,y)$. Here we represent the (i,j) entry of the matrices $T_n^{-1}(x,y)$ and $\overline{T}^{-1}_{n-1}(x,y)$ as $t^{-1}_{i,j}(x,y)$ and

$\overline{t}_{i,j}^{-1}(x,y)$, respectively. From the definitions of the matrices, the (i,j) entry of $T_n(x,y)\overline{T}_{n-1}^{-1}(x,y)$ for $i<j$ equals 0 and for $i \geq j$, we have

$$\sum_{k=j}^{i} t_{i,k}(x,y)\overline{t}_{k,j}^{-1}(x,y) \quad = \quad \sum_{k=j}^{i} t_{i,k}(x,y)t_{k-1,j-1}^{-1}(x,y). \tag{3}$$

Then it can be seen that the term of the sum (3) is nonzero only for $k-1=j-1$ and $k-1=j$, that is, for $k=j$ and $k=j+1$. Thus

$$\sum_{k=j}^{i} t_{i,k}(x,y)t_{k-1,j-1}^{-1}(x,y) \quad = \quad t_{i,j}(x,y)t_{j-1,j-1}^{-1}(x,y)+t_{i,j+1}(x,y)t_{j,j-1}^{-1}(x,y).$$

Therefore we obtained the following result.

Lemma 1. *For any positive integer n and any real numbers x and y, we have*

$$T_n(x,y)=S_n(x,y)\overline{T}_{n-1}(x,y).$$

Example 1.

$$\begin{aligned}
&S_5(x,y)\overline{T}_4(x,y)\\
&=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 & 0\\ \frac{1}{4}x^2 & -\frac{1}{4}xy & y & 0 & 0 & 0\\ \frac{1}{8}x^3 & -\frac{1}{8}x^2y & 0 & y & 0 & 0\\ \frac{1}{15}x^4 & -\frac{1}{15}x^3y & 0 & \frac{1}{15}xy & \frac{14}{15}y & 0\\ \frac{1}{28}x^5 & -\frac{1}{28}x^4y & 0 & \frac{1}{28}x^2y & -\frac{3}{98}xy & \frac{195}{196}y \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0\\ 0 & \frac{1}{4}x^2 & \frac{1}{4}xy & \frac{1}{2}y^2 & 0 & 0\\ 0 & \frac{1}{8}x^3 & \frac{1}{8}x^2y & \frac{1}{4}xy^2 & \frac{1}{2}y^3 & 0\\ 0 & \frac{1}{15}x^4 & \frac{1}{15}x^3y & \frac{2}{15}x^2y^2 & \frac{4}{15}xy^3 & \frac{7}{15}y^4 \end{bmatrix}\\
&=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 & 0\\ \frac{1}{4}x^2 & \frac{1}{4}xy & \frac{1}{2}y^2 & 0 & 0 & 0\\ \frac{1}{8}x^3 & \frac{1}{8}x^2y & \frac{1}{4}xy^2 & \frac{1}{2}y^3 & 0 & 0\\ \frac{1}{15}x^4 & \frac{1}{15}x^3y & \frac{2}{15}x^2y^2 & \frac{4}{15}xy^3 & \frac{7}{15}y^4 & 0\\ \frac{1}{28}x^5 & \frac{1}{28}x^4y & \frac{1}{14}x^3y^2 & \frac{1}{7}x^2y^3 & \frac{1}{4}xy^4 & \frac{13}{28}y^5 \end{bmatrix}\\
&=T_5(x,y).
\end{aligned}$$

Using Lemma 1 and the definition of the matrices $G_k(x,y)$, we present the decomposition of $T_n(x,y)$ in the following.

Theorem 3. *The matrix $T_n(x,y)$ can be factorized as*

$$T_n(x,y)=G_n(x,y)G_{n-1}(x,y)\cdots G_1(x,y).$$

In particular,

$$T_n=G_nG_{n-1}\cdots G_1,$$

where $T_n:=T_n(1,1)$, $G_k:=G_k(1,1)$, $k=1,2,\ldots,n$.

For the inverse of the matrix $T_n(x,y)$, we get

$$T_n^{-1}(x,y)=G_1^{-1}(x,y)G_2^{-1}(x,y)\cdots G_n^{-1}(x,y).$$

Example 2. *Since*

$$T_5(x,y)=\begin{bmatrix}1&0&0&0&0&0\\ \frac{1}{2}x&\frac{1}{2}y&0&0&0&0\\ \frac{1}{4}x^2&\frac{1}{4}xy&\frac{1}{2}y^2&0&0&0\\ \frac{1}{8}x^3&\frac{1}{8}x^2y&\frac{1}{4}xy^2&\frac{1}{2}y^3&0&0\\ \frac{1}{15}x^4&\frac{1}{15}x^3y&\frac{2}{15}x^2y^2&\frac{4}{15}xy^3&\frac{7}{15}y^4&0\\ \frac{1}{28}x^5&\frac{1}{28}x^4y&\frac{1}{14}x^3y^2&\frac{1}{7}x^2y^3&\frac{1}{4}xy^4&\frac{13}{28}y^5\end{bmatrix},$$

we can factorize this matrix as

$G_5(x,y)G_4(x,y)G_3(x,y)G_2(x,y)G_1(x,y)=$

$$\begin{bmatrix}1&0&0&0&0&0\\ \frac{1}{2}x&\frac{1}{2}y&0&0&0&0\\ \frac{1}{4}x^2&-\frac{1}{4}xy&y&0&0&0\\ \frac{1}{8}x^3&-\frac{1}{8}x^2y&0&y&0&0\\ \frac{1}{15}x^4&-\frac{1}{15}x^3y&0&\frac{1}{15}xy&\frac{14}{15}y&0\\ \frac{1}{28}x^5&-\frac{1}{28}x^4y&0&\frac{1}{28}x^2y&-\frac{3}{98}xy&\frac{195}{196}y\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&\frac{1}{2}x&\frac{1}{2}y&0&0&0\\ 0&\frac{1}{4}x^2&-\frac{1}{4}xy&y&0&0\\ 0&\frac{1}{8}x^3&-\frac{1}{8}x^2y&0&y&0\\ 0&\frac{1}{15}x^4&-\frac{1}{15}x^3y&0&\frac{1}{15}xy&\frac{14}{15}y\end{bmatrix}\times$$

$$\begin{bmatrix}1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&\frac{1}{2}x&\frac{1}{2}y&0&0\\ 0&0&\frac{1}{4}x^2&-\frac{1}{4}xy&y&0\\ 0&0&\frac{1}{8}x^3&-\frac{1}{8}x^2y&0&y\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&\frac{1}{2}x&\frac{1}{2}y&0\\ 0&0&0&\frac{1}{4}x^2&-\frac{1}{4}xy&y\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&\frac{1}{2}x&\frac{1}{2}y\end{bmatrix}.$$

3. Some Applications of the Matrix $T_n(x,y)$

In this section, we give some applications of the defined matrix $T_n(x,y)$. Firstly, we present a relation between the matrices $T_n(x,ay)$ and $T_n(x,-y)$ for a nonzero real number a.

Theorem 4. *For a nonzero real number a, the matrices $T_n(x,ay)$ and $T_n(x,-y)$ satisfy the following*

$$T_n\Big(x,\frac{y}{a}\Big)^{-1}=T_n(x,-y)^{-1}T_n(x,ay)T_n(x,-y)^{-1}.$$

Proof. The proof can be done easily by definition of the matrices and matrix multiplication. □

We give another factorization of the matrices $T_n(x,y)$ and $T_n(-x,y)$ where the variables x and y are separated from these matrices.

Theorem 5. *Let $D_n(x):=\operatorname{diag}\{1,x,x^2,\dots,x^n\}$ be a diagonal matrix. For any positive integer k and any non-zero real numbers x and y, we have*

$$T_k(x,y)=D_k(x)T_k(1,1)D_k^{-1}(x/y),$$
$$T_k(-x,y)=D_k(x)T_k(-1,1)D_k^{-1}(x/y).$$

Remark 1. *The entries of the matrix $T_n(x,y)$ can be separated by the indices, that is for $i \geq j$*

$$\Big(T_n(x,y)\Big)_{i,j} = \frac{2t_j}{t_{i+2}+t_i-1}x^{i-j}y^j = \frac{x^i}{t_{i+2}+t_i-1}2t_j\left(\frac{y}{x}\right)^j = a_ib_j$$

where

$$a_i = \frac{x^i}{t_{i+2}+t_i-1} \text{ and } b_j = 2t_j\left(\frac{y}{x}\right)^j.$$

In [19], the authors give some properties of such matrices. The related results provide the alternative proofs for Theorem 2 and Theorem 5.

Theorem 6. *Let $K_n(x,y) = [k_{i,j}]$ be a matrix with entries $k_{i,j} = t_jx^{i-j}y^j$ and $D_n^{'}$ be a diagonal matrix with diagonal entries $\{1, \frac{1}{2}, \cdots, \frac{2}{t_{i+2}+t_i-1}, \cdots, \frac{2}{t_{n+2}+t_n-1}\}$. Then we have*

$$T_n(x,y) = D_n^{'}K_n(x,y).$$

Proof. Multiplying $T_n(x,y)$ from the left with the diagonal matrix with entries $\{1, 2, \cdots, \frac{t_{i+2}+t_i-1}{2}, \cdots, \frac{t_{n+2}+t_n-1}{2}\}$, we get clearly the matrix $K_n(x,y)$. Hence the result follows. □

Example 3. *For $n = 5$, we have*

$$\begin{aligned} T_5(x,y) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}x & \frac{1}{2}y & 0 & 0 & 0 & 0 \\ \frac{1}{4}x^2 & \frac{1}{4}xy & \frac{1}{2}y^2 & 0 & 0 & 0 \\ \frac{1}{8}x^3 & \frac{1}{8}x^2y & \frac{1}{4}xy^2 & \frac{1}{2}y^3 & 0 & 0 \\ \frac{1}{15}x^4 & \frac{1}{15}x^3y & \frac{2}{15}x^2y^2 & \frac{4}{15}xy^3 & \frac{7}{15}y^4 & 0 \\ \frac{1}{28}x^5 & \frac{1}{28}x^4y & \frac{1}{14}x^3y^2 & \frac{1}{7}x^2y^3 & \frac{1}{4}xy^4 & \frac{13}{28}y^5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{28} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x & y & 0 & 0 & 0 & 0 \\ x^2 & xy & 2y^2 & 0 & 0 & 0 \\ x^3 & x^2y & 2xy^2 & 4y^3 & 0 & 0 \\ x^4 & x^3y & 2x^2y^2 & 4xy^3 & 7y^4 & 0 \\ x^5 & x^4y & 2x^3y^2 & 4x^2y^3 & 7xy^4 & 13y^5 \end{bmatrix} \\ &= D_5^{'}K_5(x,y). \end{aligned}$$

Now, we present a matrix whose Cholesky factorization includes the matrix $T_n(1,1)$. First, we need the following result.

Lemma 2 ([16]). *For $n \geq 0$, the Tribonacci numbers t_n satisfy*

$$\sum_{k=1}^{n} t_k^2 = \frac{4t_nt_{n+1} - (t_{n+1}-t_{n-1})^2 + 1}{4}. \tag{4}$$

Theorem 7. *A matrix $Q_n = [c_{i,j}]$ with entries*

$$c_{i,j} = \frac{4t_k t_{k+1} - (t_{k+1} - t_{k-1})^2 + 1}{(t_{i+2} + t_i - 1)(t_{j+2} + t_j - 1)}$$

where $k = \min\{i, j\}$, is a symmetric matrix and its Cholesky factorization is $T_n(1,1)T_n(1,1)^T$.

Proof. Since

$$c_{i,j} = \tfrac{4t_k t_{k+1} - (t_{k+1} - t_{k-1})^2 + 1}{(t_{i+2} + t_i - 1)(t_{j+2} + t_j - 1)} = c_{j,i},$$

Q_n is symmetric. Now, we will show that $Q_n = T_n(1,1)T_n(1,1)^T$. By matrix multiplication,

$$\begin{aligned} T_n(1,1)T_n(1,1)^T = \sum_{k=0}^{n} t_{i,k} t_{j,k} &= \sum_{k=0}^{n} \frac{2t_k}{t_{i+2} + t_i - 1} \frac{2t_k}{t_{j+2} + t_j - 1} \\ &= \frac{4}{(t_{i+2} + t_i - 1)(t_{j+2} + t_j - 1)} \sum_{k=0}^{n} t_k^2. \end{aligned}$$

The proof is completed by substituting (4) in the last equation. □

In the last part of this section, we will give a relation between the matrix $T_n(x, y)$ and the exponential of a special matrix. Matrix exponentials are defined by simply plugging matrices into the usual Maclaurin series for the exponential function. In other words, for any square matrix M, the exponential of M is defined to be the matrix

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots + \frac{M^k}{k!} + \cdots .$$

For any square matrix M, we have the following result:

Theorem 8 ([3,24]).

(i) *For any numbers r and s, we have $e^{(r+s)M} = e^{rM} e^{sM}$.*
(ii) $(e^M)^{-1} = e^{-M}$.
(iii) *By taking the derivative with respect to x of each entry of e^{Mx}, we get the matrix $\frac{d}{dx} e^{Mx} = M e^{Mx}$.*

Definition 2. *The matrix $M_n = [m_{i,j}]$ is defined by*

$$m_{i,j} = \begin{cases} \frac{t_j}{t_i}, & \text{if } i = j + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

We want to obtain a relation between $T_n(x, y)$ and $e^{M_n x}$, so we prove the following auxiliary result.

Lemma 3. *For every nonnegative integer k, the entries of the matrix M_n^k are given by*

$$(M_n^k)_{i,j} = \begin{cases} \frac{t_j}{t_i}, & \text{if } i = j + k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof will be done by induction on k. The case $k = 0$ follows straightforward. Let us assume the inductive hypothesis on $M_n^{k+1} = M_n M_n^k$. It is not hard to see for $i \neq j + k + 1$, $(M_n^{k+1})_{i,j} = 0$. For $i = j + k + 1$, we have

$$(M_n^{k+1})_{i,j} = \frac{t_{i-1}}{t_i}\frac{t_j}{t_{j+k}} = \frac{t_{j+k}}{t_{j+k+1}}\frac{t_j}{t_{j+k}} = \frac{t_j}{t_{j+k+1}}.$$

□

Theorem 9. *For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have*

$$(T_n(0,1)^{-1}T_n(x,1))_{i,j} = (i-j)!(e^{M_n x})_{i,j}.$$

Proof. Suppose that there is a matrix L_n such that $(T_n(0,1)^{-1}T_n(x,1))_{i,j} = (i-j)!(e^{L_n x})_{i,j}$. Then we have

$$\frac{d}{dx}(T_n(0,1)^{-1}T_n(x,1))_{i,j} = L_n(i-j)!(e^{L_n x})_{i,j} = L_n(T_n(0,1)^{-1}T_n(x,1))_{i,j}$$

and so

$$\frac{d}{dx}(T_n(0,1)^{-1}T_n(x,1))_{i,j}\ |_{x=0} = L_n.$$

Thus there is at most one matrix L_n such that $(T_n(0,1)^{-1}T_n(x,1))_{i,j} = (i-j)!(e^{L_n x})_{i,j}$. It can be easily seen that $L = M_n$, where M_n is the matrix given in Definition 2, by calculating $\frac{d}{dx}(T_n(0,1)^{-1}T_n(x,1))_{i,j}\ |_{x=0}$. We conclude that $M_n^k = 0$ for $k \geq n+1$, thus

$$e^{M_n x} = \sum_{k=0}^{n} M_n^k \frac{x^k}{k!}.$$

For $i < j$, we see that $(e^{M_n x})_{i,j} = 0$ and we also have $(e^{M_n x})_{i,i} = 1$. Now, suppose that $i > j$ and let $i = j + k$.

$$(e^{M_n x})_{i,j} = (M_n^k)_{i,j}\frac{x^k}{k!} = \frac{t_j}{t_{j+k}}\frac{x^k}{k!} = \frac{1}{k!}(T_n(0,1)^{-1}T_n(x,1))_{i,j}.$$

Hence the proof is completed. □

Example 4. *We obtain the matrix* $\frac{d}{dx}T_5(0,1)^{-1}T_5(x,1)$ *by taking the derivative of each entry of the matrix* $T_5(0,1)^{-1}T_5(x,1)$ *with respect to* x*. Thus*

$$\frac{d}{dx}T_5(0,1)^{-1}T_5(x,1)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2}x & \frac{1}{4} & 0 & 0 & 0 & 0\\ \frac{3}{8}x^2 & \frac{1}{4}x & \frac{1}{4} & 0 & 0 & 0\\ \frac{4}{5}x^3 & \frac{1}{5}x^2 & \frac{4}{15}x & \frac{4}{15} & 0 & 0\\ \frac{5}{28}x^4 & \frac{1}{7}x^3 & \frac{3}{14}x^2 & \frac{2}{7}x & \frac{1}{4} & 0 \end{bmatrix}.$$

Hence we have

$$M_5=T_5(0,1)^{-1}\frac{d}{dx}T_5(x,1)\mid_{x=0}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{4}{7} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{7}{13} & 0 \end{bmatrix}$$

and

$$M_5^2=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1\times\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}\times\frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2}\times\frac{4}{7} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{4}{7}\times\frac{7}{13} & 0 & 0 \end{bmatrix},$$

$$M_5^3=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1\times\frac{1}{2}\times\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}\times\frac{1}{2}\times\frac{4}{7} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2}\times\frac{4}{7}\times\frac{7}{13} & 0 & 0 & 0 \end{bmatrix},$$

$$M_5^4=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1\times\frac{1}{2}\times\frac{1}{2}\times\frac{4}{7} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}\times\frac{1}{2}\times\frac{4}{7}\times\frac{7}{13} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_5^5=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1\times\frac{1}{2}\times\frac{1}{2}\times\frac{4}{7}\times\frac{7}{13} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let M_n be the matrix defined in (5) and $U_n(x) = e^{M_n x}$. At the end of this section, we will find the explicit inverse of the matrix $R_n(x) = [I_n - \lambda U_n(x)]^{-1}$ for real number λ such that $\mid \lambda \mid < 1$. To achieve this, we need the following result.

Lemma 4 ([14], Corollary 5.6.16)**.** *A matrix A of order n is nonsingular if there is a matrix norm $\|\cdot\|$ such that $\|I - A\| < 1$. If this condition is satisfied,*

$$A^{-1} = \sum_{k=0}^{\infty} (I - A)^k.$$

Theorem 10. *The matrix $R_n(x)$ is defined for real number λ such that $\mid \lambda \mid < 1$. The entries of the matrix are*

$$(R_n(x))_{i,i} = \frac{1}{1-\lambda}$$

and

$$(R_n(x))_{i,j} = (U_n(x))_{i,j} \mathfrak{L}i_{j-i}(\lambda)$$

for $i > j$, where $\mathfrak{L}i_n(z)$ is the polylogarithm function

$$\mathfrak{L}i_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

Proof. The statement in Lemma 4 is equivalent to: If $\|\cdot\|$ is a matrix norm and if $\|A\| < 1$ for a square matrix of order n, then $I - A$ is invertible and $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$. Then for $|\lambda| < 1$, we can write

$$(R_n(x))_{i,j} = \sum_{k=0}^{\infty} (U_n(x))^k \lambda^k = \sum_{k=0}^{\infty} (U_n(xk))_{i,j} \lambda^k = (U_n(x))_{i,j} \sum_{k=0}^{\infty} \lambda^k k^{i-j}.$$

We obtain the desired result by writing the sum for $i = j$ and $i > j$. □

Example 5.

$$\begin{aligned} I_4 - \lambda U_4(x) &= I_4 - \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ x\lambda & \lambda & 0 & 0 & 0 \\ \frac{1}{4}\lambda x^2 & \frac{1}{2}\lambda x & \lambda & 0 & 0 \\ \frac{1}{24}\lambda x^3 & \frac{1}{8}\lambda x^2 & \frac{1}{2}\lambda x & \lambda & 0 \\ \frac{1}{168}\lambda x^4 & \frac{1}{42}\lambda x^3 & \frac{1}{7}\lambda x^2 & \frac{4}{7}\lambda x & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ -x\lambda & 1-\lambda & 0 & 0 & 0 \\ -\frac{1}{4}\lambda x^2 & -\frac{1}{2}\lambda x & 1-\lambda & 0 & 0 \\ -\frac{1}{24}\lambda x^3 & -\frac{1}{8}\lambda x^2 & -\frac{1}{2}\lambda x & 1-\lambda & 0 \\ -\frac{1}{168}\lambda x^4 & -\frac{1}{42}\lambda x^3 & -\frac{1}{7}\lambda x^2 & -\frac{4}{7}\lambda x & 1-\lambda \end{bmatrix}. \end{aligned}$$

The inverse of this matrix equals

$$\begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 \\ \frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} & 0 & 0 & 0 \\ \frac{1}{4}\frac{\lambda^2+\lambda}{(1-\lambda)^3}x^2 & \frac{1}{2}\frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} & 0 & 0 \\ \frac{1}{24}\frac{\lambda^3+4\lambda^2+\lambda}{(1-\lambda)^4}x^3 & \frac{1}{8}\frac{\lambda^2+\lambda}{(1-\lambda)^3}x^2 & \frac{1}{2}\frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} & 0 \\ \frac{1}{168}\frac{\lambda^4+11\lambda^3+11\lambda^2+\lambda}{(1-\lambda)^5}x^4 & \frac{1}{42}\frac{\lambda^3+4\lambda^2+\lambda}{(1-\lambda)^4}x^3 & \frac{1}{7}\frac{\lambda^2+\lambda}{(1-\lambda)^3}x^2 & \frac{4}{7}\frac{\lambda}{(1-\lambda)^2}x & \frac{1}{1-\lambda} \end{bmatrix}.$$

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A-DAVIS-WIELANDT-BEREZIN RADIUS INEQUALITIES

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Abstract. We consider operator V on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ over some set Ω with the reproducing kernel $K_{\mathcal{H},\lambda}(z) = K(z, \lambda)$ and define A-Davis-Wielandt-Berezin radius $\eta_A(V)$ by the formula

$$\eta_A(V) := \sup\left\{\sqrt{\left|\left\langle V k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \left\|V k_{\mathcal{H},\lambda}\right\|_A^4} : \lambda \in \Omega\right\}$$

and $\widetilde{V}$ is the Berezin symbol of V where any positive operator A-induces a semi-inner product on $\mathcal{H}$ is defined by $\langle x, y\rangle_A = \langle Ax, y\rangle$ for $x, y \in \mathcal{H}$. We study equality of the lower bounds for A-Wielandt-Berezin radius mentioned above. We establish some lower and upper bounds for the A-Davis-Wielandt-Berezin radius of reproducing kernel Hilbert space operators. In addition, we get an upper bound for the A-Davis-Wielandt-Berezin radius of sum of two bounded linear operators.

1. Introduction

Many researchers in mathematics and mathematical physics are interested in the Berezin symbol of an operator defined with the aid of a reproducing kernel Hilbert space. In this context, several mathematicians have conducted substantial research on the Berezin radius inequality (see [4, 14, 16, 20, 21]). In fact, it is of interest to academics to get refinements and extensions of this disparity. We show various inequalities for the A-Davis-Wielandt-Berezin radius of operators on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω in this study. By using A-Berezin transforms, we study some lower and upper bounds for the A-Davis-Wielandt-Berezin radius of some operators. In addition, we get an upper bound for the A-Davis-Wielandt-Berezin radius of sum of two bounded linear operators.

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Keywords. Berezin symbol, A-Davis-Wielandt-Berezin radius, A-Berezin number, A-Berezin norm, semi inner product, reproducing kernel Hilbert spaces.

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We will now outline the preliminary concepts needed to proceed with the findings of this investigation.

Remember that a reproducing kernel Hilbert space (abbreviated RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω in which:

(a) the evaluation functionals

$$\varphi_\lambda(f) = f(\lambda),\ \lambda \in \Omega,$$

are continuous on $\mathcal{H}$;

(b) for every $\lambda \in \Omega$ there exists a function $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$.

Then, via the classical Riesz representation theorem, we know if $\mathcal{H}$ is an RKHS on Ω, there is a unique element $K_{\mathcal{H},\lambda} \in \mathcal{H}$ such that $h(\lambda) = \langle h, K_{\mathcal{H},\lambda}\rangle$ for every $\lambda \in \Omega$ and all $h \in \mathcal{H}$. The reproducing kernel at λ is denoted by the element $K_{\mathcal{H},\lambda}$. Further, we will denote the normalized reproducing kernel at λ as $k_{\mathcal{H},\lambda} := \frac{K_{\mathcal{H},\lambda}}{\|K_{\mathcal{H},\lambda}\|}$. Let $\mathcal{L}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ including the identity operator $1_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{H})$.

Linear operators induced by functions are frequently encountered in functional analysis; they include Hankel operators, composition operators, and Toeplitz operators. The inducing function is sometimes referred to as the symbol of the resultant operator. In many circumstances, a linear operator on a Hilbert space $\mathcal{H}$ also gives rise to a function on Ω. Hence, we frequently examine operators induced by functions, and we may similarly research functions induced by operators. The Berezin symbol is an outstanding exemplar of an operator-function link. More accurately, for an operator $V \in \mathcal{L}(\mathcal{H})$, the Berezin symbol (transform) of V, denoted by $\widetilde{V}$, is the complex-valued function on Ω defined by

$$\widetilde{V}(\lambda) := \langle V k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle .$$

For each bounded operator V on $\mathcal{H}$, the Berezin symbol $\widetilde{V}$ is a bounded real-analytic function on Ω. Features of the operator V, are often seen in the features of the Berezin transform $\widetilde{V}$. F. Berezin proposed the Berezin transform in [8] and it has proven to be a fundamental tool in operator theory, since many essential features of significant operators are contained in their Berezin transforms.

The Berezin radius (number) of operator V is defined by

$$\operatorname{ber}(V) := \sup_{\lambda\in\Omega} \left|\widetilde{V}(\lambda)\right| .$$

The Berezin set and the Berezin norm of operator are defined, respectively, by

$$\operatorname{Ber}(V) := \operatorname{Range}\left(\widetilde{V}\right) \text{ and } \|V\|_{\mathrm{Ber}} := \sup_{\lambda\in\Omega} \|V k_{\mathcal{H},\lambda}\| .$$

The Berezin transform and Berezin radius have been studied by many mathematicians over the years (see [3, 4, 14, 26]).

Recall that the Berezin range of an operator V is a subset of the numerical range of V,

$$W(V) = \{\langle Vu, u\rangle : \|u\| = 1\}.$$

It is well knowledge that $\operatorname{Ber}(V) \subseteq W(V)$, $\operatorname{ber}(V) \leq w(V)$ (numerical radius) and $\operatorname{ber}(V) \leq \|V\|_{\text{Ber}}$. See [5, 9, 18, 22, 24, 27] for further details. Two of these generalizations are the Davis-Wielandt radius $dw(V)$ and Davis-Wielandt shell $DW(V)$ of $V \in \mathcal{L}(\mathcal{H})$ defined by

$$dw(V) := \sup\left\{\sqrt{|\langle Vu, u\rangle|^2 + \|Vu\|^4} : u \in \mathcal{H} \text{ and } \|u\| = 1\right\};$$

and

$$DW(V) := \left\{\left(\langle Vu, u\rangle, \|Vu\|^2\right) : u \in \mathcal{H} \text{ and } \|u\| = 1\right\} \subseteq \mathbb{C} \times \mathbb{R}$$

see [5, 10, 25, 28].

$\mathcal{N}(V)$, its range by $\mathcal{R}(V)$ and adjoint of V by V^* denote the null space of every operator V. If U is a linear subspace of $\mathcal{H}$, then $\overline{U}$ stands for its closure in the norm topology of $\mathcal{H}$. An operator $A \in \mathcal{L}(\mathcal{H})$ is called positive, denoted by $A \geq 0$, if $\langle Au, u\rangle \geq 0$ for all $u \in \mathcal{H}$. For $V \in \mathcal{L}(\mathcal{H})$, the absolute value of V, denoted by $|V|$, is defined as $|V| = (V^*V)^{1/2}$. Along with the article, A denotes a non-zero positive operator on $\mathcal{H}$. Notice that any positive operator A induces a semi-inner product on $\mathcal{H}$ defined by

$$\langle u, v\rangle_A := \langle Au, v\rangle_{\mathcal{H}}, \; \forall u, v \in \mathcal{H}.$$

The seminorm induced by $\langle ., .\rangle_A$ is given by $\|u\|_A = \sqrt{\langle u, u\rangle_A} = \left\|A^{1/2}u\right\|$ for all $u \in \mathcal{H}$.

It can be easily verified that $\|.\|_A$ is norm if and only if A is injective and that the seminormed space $(\mathcal{H}, \|.\|_A)$ which is complete if and only if $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$.

Definition 1. *For $V \in \mathcal{L}(\mathcal{H})$, the A-Berezin set of $\langle Vk_\lambda, k_\lambda\rangle_A$ is defined by*

$$\operatorname{Ber}_A(V) := \{\langle Vk_\lambda, k_\lambda\rangle_A : \lambda \in \Omega\}.$$

$\operatorname{Ber}_A(V)$ is a nonempty subset of $\mathbb{C}$ and it is in general not closed even if $\mathcal{H}$ is finite dimensional are important to be significant.

Definition 2. *(i) A-Berezin transform (also called A-Berezin symbol) $\widetilde{V}^A$ is defined on Ω by*

$$\widetilde{V}^A(\lambda) := \langle Vk_\lambda, k_\lambda\rangle_A \;\; (\lambda \in \Omega),$$

(ii) The supremum modulus of $\operatorname{Ber}_A(V)$*, denoted by* $\operatorname{ber}_A(V)$*, is referred to as the A-Berezin number of V, i.e.,*

$$\operatorname{ber}_A(V) := \sup_{\lambda \in \Omega} |\langle Vk_\lambda, k_\lambda\rangle_A|,$$

(iii) A-Berezin norm of operators $V \in \mathcal{L}(\mathcal{H}(\Omega))$ is defined by

$$\|V\|_{A-\text{Ber}} := \sup_{\lambda \in \Omega} \|AVk_\lambda\|_{\mathcal{H}}.$$

We get the Berezin number if $A = I$. As a result of this new idea, the Berezin number of reproducing kernel Hilbert space operators and the Berezin norm of operators become more generic. See [15,19] for further information on A-Berezin number inequalities.

Definition 3. *([12]) Let $V \in \mathcal{L}(\mathcal{H})$. An operator $U \in \mathcal{L}(\mathcal{H})$ is called an A-adjoint of V if for every $\lambda, \mu \in \Omega$, identity $\langle Vk_\lambda, k_\mu \rangle_A = \langle k_\lambda, Uk_\mu \rangle_A$ holds.*

Definition 4. *Let $V \in \mathcal{L}(\mathcal{H}(\Omega))$. An operator $U \in \mathcal{L}(\mathcal{H}(\Omega))$ is called (A, r)-adjoint of V if for every $\lambda, \mu \in \Omega$, the identity $\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \rangle_A = \langle k_{\mathcal{H},\lambda}, Uk_{\mathcal{H},\lambda} \rangle_A$ holds.*

Following [12,13], notice that the existence of an A-adjoint of V is identical to the existence of a solution of the equation $AX = V^*A$. Thanks to the Douglas theorem, these types of equations can be studied and the readers can consult to Moslehian et al. [23]. In summary, Douglas theorem states unequivocally that the operator equation $VX = U$ has a bounded linear solution X if and only if $\mathcal{R}(U) \subseteq \mathcal{R}(V)$. Furthermore, it has just one solution, represented by Q, that satisfies $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(V^*)}$ among its numerous solutions. This type of Q is known as the reduced solution or Douglas solution of $VX = U$. $\mathcal{L}_A(\mathcal{H})$ denotes the set of all operators in $\mathcal{L}(\mathcal{H})$ that admit A-adjoint. According to the Douglas theorem,

$$\mathcal{L}_A(\mathcal{H}) = \{V \in \mathcal{L}(\mathcal{H}) : \mathcal{R}(V^*A) \subseteq \mathcal{R}(A)\}.$$

Moreover, $\mathcal{L}_{A^{1/2}}(\mathcal{H})$ denotes the set all operators admitting $A^{1/2}$-adjoints. When we use the Douglas theorem, we get

$$\mathcal{L}_{A^{1/2}}(\mathcal{H}) = \{V \in \mathcal{L}(\mathcal{H}) : \exists \lambda > 0, \ \|Vu\|_A \leq \lambda \|u\|_A, \forall u \in \mathcal{H}\}.$$

A-bounded refers to the operator in $\mathcal{L}_{A^{1/2}}(\mathcal{H})$.

If $V \in \mathcal{L}_A(\mathcal{H})$, then the reduced solution (or Douglas solution) to the equation $AX = V^*A$ is a well-known A-adjoint operator of V, which is represented by V^{*_A}. We observe that

$$V^{*_A} = A^\dagger V^* A,$$

where $A^\dagger$ is the Moore-Penrose inverse of A (see [1,2]). It is commonly known that the operator V^{*_A} satisfies

$$AV^{*_A} = V^*A, \ \mathcal{R}(V^{*_A}) \subseteq \overline{\mathcal{R}(A)} \text{ and } \mathcal{N}(V^{*_A}) = \mathcal{N}(V^*A).$$

Also, note that if $V \in \mathcal{L}_A(\mathcal{H})$, then $V^{*_A} \in \mathcal{L}_A(\mathcal{H})$ and $(V^{*_A})^{*_A} = P_A V P_A$, where P_A represents the ortogonal projection onto $\overline{\mathcal{R}(A)}$. Furthermore, if $V \in \mathcal{L}_A(\mathcal{H})$, then $\|V^{*_A}\| = \|V\|_A$. In order to reach more results and proofs related to these classes of operators, the researchers may want to overview [1,2].

If AV is selfadjoint, that is, $AV = V^*A$, then an operator $V \in \mathcal{L}(\mathcal{H})$ is called to be A-selfadjoint. Furthermore, an operator V is said to be A-positive if $AV \geq 0$ and we write $V \geq_A 0$.

The Hilbert space $\left(\mathcal{R}\left(A^{1/2}\right), \langle .,. \rangle_{\mathbb{R}\left(A^{1/2}\right)}\right)$ shall be designated simply by $\mathbb{R}\left(A^{1/2}\right)$ in the sequal.

Feki in [12] has found some upper bounds for the A-Davis-Wielandt radius of operators in $\mathcal{L}_A(\mathcal{H})$.

Definition 5. *For any $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$, we define its A-Davis-Wielandt-Berezin shell and A-Davis-Wielandt-Berezin radius, respectively, by the formulas*

$$\mathbf{H}_A(V) := \left\{ \left(\langle V k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \rangle_A, \|A k_{\mathcal{H},\lambda}\|_A^2 \right),\ \lambda \in \Omega \right\}$$

and

$$\eta_A(V) := \sup_{\lambda \in \Omega} \sqrt{\left| \widetilde{V}^A(\lambda) \right|^2 + \|V k_{\mathcal{H},\lambda}\|_A^4}$$

It is apparent that $\eta_A(V) \leq dw_A(V)$. For $V, U \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$ one has
(i) $\eta_A(V) \geq 0$ and $\eta_A(V) = 0$ if and only if $V = 0$;
(ii) $\eta_A(\alpha V) \begin{cases} \geq |\alpha|\, \eta_A(V) & \text{if } |\alpha| > 1 \\ = |\alpha|\, \eta_A(V) & \text{if } |\alpha| = 1 \\ \leq |\alpha|\, \eta_A(V) & \text{if } |\alpha| < 1. \end{cases}$
(iii) $\eta_A(V+U) \leq \sqrt{2\left(\eta_A(V) + \eta_A(U) + 4\left(\eta_A(V) + \eta_A(U)\right)^2\right)}$;
therefore $\eta_A(\cdot)$ cannot be a norm on $\mathcal{L}(\mathcal{H}(\Omega))$. The following property of $\eta_A(\cdot)$ is immediate:

$$\max\left\{ \operatorname{ber}_A(V), \|V\|_{A-\mathrm{ber}}^2 \right\} \leq \eta_A(V) \leq \sqrt{\operatorname{ber}_A^2(V) + \|V\|_{A-\mathrm{ber}}^4}\ (V \in \mathcal{L}_{A,r}(\mathcal{H})). \tag{1}$$

Recently, Bhanja et al. in [6] have reached some upper bounds for the A-Davis-Wielandt radius of operators in $\mathcal{L}_A(\mathcal{H}(\Omega))$. The purpose of this article is to find out some lower and upper bounds for the A-Davis-Wielandt-Berezin radius of reproducing kernel Hilbert space operators. For this aim, we employ some well-known inequalities for vectors in inner product spaces (see [6,7,11]). We also get an upper bound for the A-Davis-Wielandt-Berezin radius of sum of two bounded linear operators.

In particular, for $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$ we prove that

$$\eta_A^2(V) \leq \sup_{\theta \in \mathbb{R}} \operatorname{ber}_A^2\left(e^{i\theta} V + V^{*A} V\right) - 2\widetilde{c}_A(V)\, m_{A-\mathrm{ber}}^2(V)$$

and

$$\eta_A^2(V) \leq \inf_{z \in \mathbb{C}} \left\{ \left(2\left\| \operatorname{Re}(z) \operatorname{Re}_A(V) + \operatorname{Im}(z) \operatorname{Im}_A(V) \right\|_{A-\mathrm{ber}} + \left\| V^{*A} V - 2\operatorname{Re}(\overline{z} V) \right\|_{A-\mathrm{ber}} \right)^2 \right.$$
$$\left. + 2\left\| \operatorname{Re}(\overline{z} V) \right\|_{A-\mathrm{ber}} - |z|^2 + \operatorname{ber}_A^2(V - zI) \right\}.$$

2. Prerequisites

In the present section, we need some auxiliary lemmas including Buzano [7] inequality, Dragomir [11] inequality and Bhanja et al. [6] inequality in order to prove our results.

Buzano [7] made an extension of the Cauchy-Schwarz inequality which states that for any $a_1, a_2, a_3 \in \mathcal{H}$ with $\|a_3\| = 1$

$$|\langle a_1, a_3\rangle \langle a_3, a_2\rangle| \leq \frac{1}{2}\left(|\langle a_1, a_2\rangle| + \|a_1\| \|a_2\|\right). \tag{2}$$

Dragomir [11] proved the following inequalities.

Lemma 1. *Let u_1, $u_2 \in \mathcal{H}$ and $z \in \mathbb{C}$. Then the following equality holds:*

$$\|u_1\|^2 \|u_2\|^2 - |\langle u_1, u_2\rangle|^2 = \|u_1 - zu_2\|^2 \|u_2\|^2 - |\langle u_1 - zu_2, u_2\rangle|^2.$$

We need the following lemmas, given in [6].

Lemma 2. *Let $u_1, u_2, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then*

$$|\langle u_1, e\rangle_A \langle e, u_2\rangle_A| \leq \frac{1}{2}\left(|\langle u_1, u_2\rangle_A| + \|u_1\|_A \|u_2\|_A\right). \tag{3}$$

Lemma 3. *Let $u_1, u_2, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then*

$$\|u_1\|_A^2 \|u_2\|_A^2 - |\langle u_1, u_2\rangle_A|^2 \geq 2\,|\langle u_1, e\rangle_A \langle e, u_2\rangle_A| \left(\|u_1\|_A \|u_2\|_A - |\langle u_1, u_2\rangle_A|\right).$$

Lemma 4. *Let $u_1, u_2, e \in \mathcal{H}$ and $z \in \mathbb{C}$. Then we have the following equality:*

$$\|u_1\|_A^2 \|u_2\|_A^2 - |\langle u_1, u_2\rangle_A|^2 = \|u_1 - zu_2\|_A^2 \|u_2\|_A^2 - |\langle u_1 - zu_2, u_2\rangle_A|^2.$$

3. Main results

We use the lemmas from the preceding section to derive additional inequalities for the A-Davis-Wielandt-Berezin radius of operators on $\mathcal{H} = \mathcal{H}(\Omega)$.

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS. The A-Berezin symbol of operator $V \in \mathcal{L}(\mathcal{H}(\Omega))$ is naturally defined the by the formula

$$\widetilde{V}^A(\lambda) := \langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A = \langle AVk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle, \ \lambda \in \Omega.$$

Therefore, $\mathcal{L}_{A,r}(\mathcal{H}) := \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$ denotes the set of all operators in $\mathcal{L}(\mathcal{H}(\Omega))$ admitting (A, r)-adjoints.

For $V \in \mathcal{L}_{A,r}(\mathcal{H})$, its Crawford number $c_A(V)$ is defined by

$$c_A(V) := \inf\{|\langle Vu, u\rangle_A| : u \in \mathcal{H}, \ \|u\|_A = 1\}$$

(see [27]). We also introduce the number $\widetilde{c}_A(V) := \inf_{\lambda\in\Omega} \left|\widetilde{V}^A(\lambda)\right|$. It is clear that

$$c_A(V) \leq \widetilde{c}_A(V) \leq \mathrm{ber}_A(V).$$

Our first result in this paper reads as follows.

Theorem 1. *Let $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. Then, the following inequalities hold.*

(i) $\eta_A^2(V) \geq \max\left\{\mathrm{ber}_A^2(V) + \widetilde{c}_A^2\left(V^{*A}V\right), \|V\|_{A\text{-Ber}}^4 + \widetilde{c}_A^2(V)\right\}$,

(ii) $\eta_A^2(V) \geq 2\max\left\{\mathrm{ber}_A(V)\,\widetilde{c}_A\left(V^{*A}V\right), \widetilde{c}_A(V)\|V\|_{A-\mathrm{Ber}}^2\right\}$.

Proof. For any $\lambda \in \Omega$, we have

$$\begin{aligned}\eta_A^2(V) &\geq \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\ &= \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A^2 \\ &\geq \left|\widetilde{V}^A(\lambda)\right|^2 + \inf_{\lambda\in\Omega}\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A^2,\end{aligned}$$

hence, taking supremum over $\lambda \in \Omega$ gives

$$\eta_A^2(V) \geq \mathrm{ber}_A^2(V) + \widetilde{c}_A^2\left(V^{*A}V\right).$$

Moreover, by taking into consideration $\eta_A^2(V) \geq \left|\widetilde{V}^A(\lambda)\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4$, we see that

$$\eta_A^2(V) \geq \widetilde{c}_A^2(V) + \|Vk_{\mathcal{H},\lambda}\|_A^4.$$

Hence, on taking the supremum over $\lambda \in \Omega$, we obtain

$$\eta_A^2(V) \geq \widetilde{c}_A^2(V) + \|V\|_{A-\mathrm{Ber}}^4,$$

which proves (i).

Let $\lambda \in \Omega$ be arbitrary. It can be observed that

$$\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \geq 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\|Vk_{\mathcal{H},\lambda}\|_A^2 \tag{4}$$

and

$$\begin{aligned}\eta_A^2(V) &\geq 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\ &\geq 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\inf_{\lambda\in\Omega}\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\ &= 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\widetilde{c}_A\left(V^{*A}V\right).\end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we thus have

$$\eta_A^2(V) \geq 2\mathrm{ber}_A(V)\,\widetilde{c}_A\left(V^{*A}V\right).$$

From the inequality (4), we get

$$\eta_A^2(V) \geq 2\widetilde{c}_A(V)\|Vk_{\mathcal{H},\lambda}\|_A^2.$$

Taking supremum over all $\lambda \in \Omega$, we thus have

$$\eta_A^2(V) \geq 2\widetilde{c}_A(V)\|V\|_{A\text{-Ber}}^2.$$

Hence the proof is complete. □

Remark 1. *It is clear that the lower bound obtained in Theorem 1 (i) is more solid than that in (1). Also, both of inequalities in ([17], Th. 1) follow from Theorem 1 by considering $A = I$.*

For $A \in \mathcal{L}(\mathcal{H}(\Omega))$, we define

$$m_{A-\mathrm{ber}}^{2}(V) := \inf_{\lambda\in\Omega} \|Vk_{\mathcal{H},\lambda}\|_{A}^{2}.$$

We get an upper bound for the A-Davis-Wielandt-Berezin radius of bounded linear operators on RKHS in the following result.

Theorem 2. *Let $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. Then*

$$\eta_{A}^{2}(V) \leq \sup_{\theta\in\mathbb{R}} \mathrm{ber}_{A}^{2}\left(e^{i\theta}V + V^{*A}V\right) - 2\widetilde{c}_{A}(V)\, m_{A-\mathrm{ber}}^{2}(V).$$

Proof. Let $\lambda \in \Omega$ be arbitrary. Then there exists $\theta \in \mathbb{R}$ such that

$$\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_{A}\right| = e^{i\theta}\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_{A}.$$

Now,

$$\begin{aligned}
&\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_{A}\right|^{2} + \|Vk_{\mathcal{H},\lambda}\|_{A}^{4}\\
&= \left\langle e^{i\theta}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A}^{2} + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A}^{2}\\
&= \left(\left\langle e^{i\theta}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A} + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A}\right)^{2}\\
&\quad - 2\left\langle e^{i\theta}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A}\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&2\left\langle e^{i\theta}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A}\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A} + \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_{A}\right|^{2} + \|Vk_{\mathcal{H},\lambda}\|_{A}^{4}\\
&= \left\langle \left(e^{i\theta}V + V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A}^{2}\\
&\leq \mathrm{ber}_{A}^{2}\left(e^{i\theta}V + V^{*A}V\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_{A}\right|\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_{A} + \left|\langle V_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_{A}\right|^{2} + \|Vk_{\mathcal{H},\lambda}\|^{4}\\
&\leq \sup_{\theta\in\mathbb{R}} \mathrm{ber}_{A}^{2}\left(e^{i\theta}V + V^{*A}V\right)
\end{aligned}$$

and so,

$$2\widetilde{c}_{A}(V)\, m_{A-\mathrm{ber}}^{2}(V) + \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_{A}\right|^{2} + \|Vk_{\mathcal{H},\lambda}\|_{A}^{4} \leq \sup_{\theta\in\mathbb{R}} \mathrm{ber}_{A}^{2}\left(e^{i\theta}V + |V|_{A}^{2}\right).$$

Hence, taking supremum over $\lambda \in \Omega$ gives

$$\eta_{A}^{2}(V) \leq \sup_{\theta\in\mathbb{R}} \mathrm{ber}_{A}^{2}\left(e^{i\theta}V + V^{*A}V\right) - 2\widetilde{c}_{A}(V)\, m_{A-\mathrm{ber}}^{2}(V).$$

This completes the proof. □

Remark 2. *According to the inequality in ([17], Th. 2),*

$$\eta^{2}(V) \leq \sup_{\theta\in\mathbb{R}} \mathrm{ber}^{2}\left(e^{i\theta}V + V^{*}V\right) - 2\widetilde{c}(V)\, m_{\mathrm{ber}}^{2}(V).$$

This shows that the inequality in ([17], Th. 2) follows from Theorem 2 by considering $A = I$.

We can now show the following inequality for the A-Davis-Wielandt-Berezin radius of bounded linear operators.

Theorem 3. *Let* $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. *Then*

$$\frac{1}{2}\left\{\mathrm{ber}_A^2\left(V+V^{*A}V\right)+\widetilde{c}_A^2\left(V-V^{*A}V\right)\right\}\leq \eta_A^2(V)$$
$$\leq \frac{1}{2}\left\{\mathrm{ber}_A^2\left(V+V^{*A}V\right)+\mathrm{ber}_A^2\left(V-V^{*A}V\right)\right\}.$$

Proof. Let $\lambda\in\Omega$ be arbitrary. Then

$$\begin{aligned}
&\left|\langle Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\rangle_A\right|^2+\|Vk_{\mathcal{H},\lambda}\|_A^4\\
&=\frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\rangle_A+\langle Vk_{\mathcal{H},\lambda},Vk_{\mathcal{H},\lambda}\rangle_A\right|^2\\
&+\frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\rangle_A-\langle Vk_{\mathcal{H},\lambda},Vk_{\mathcal{H},\lambda}\rangle_A\right|^2\\
&=\frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\rangle_A+\left\langle V^{*A}Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\right|^2\\
&+\frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\rangle_A-\left\langle V^{*A}Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\right|^2\\
&=\frac{1}{2}\left|\left\langle\left(V+V^{*A}V\right)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\right|^2+\frac{1}{2}\left|\left\langle\left(V-V^{*A}V\right)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\right|^2\\
&\geq\frac{1}{2}\left\{\left|\left\langle\left(V+V^{*A}V\right)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\right|^2+\widetilde{c}_A^2\left(V-V^{*A}V\right)\right\}
\end{aligned}$$

Therefore, taking supremum over $\lambda\in\Omega$, we get

$$\eta_A^2(V)\geq\frac{1}{2}\left\{\mathrm{ber}_A^2\left(V+V^{*A}V\right)+\widetilde{c}_A^2\left(V-V^{*A}V\right)\right\}.$$

Similarly,

$$\begin{aligned}
&\left|\langle Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\rangle_A\right|^2+\|Vk_{\mathcal{H},\lambda}\|_A^4\\
&=\frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\rangle_A+\langle Vk_{\mathcal{H},\lambda},Vk_{\mathcal{H},\lambda}\rangle_A\right|^2\\
&+\frac{1}{2}\left|\langle Vk_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\rangle_A-\langle Vk_{\mathcal{H},\lambda},Vk_{\mathcal{H},\lambda}\rangle_A\right|^2\\
&=\frac{1}{2}\left|\left\langle\left(V+V^{*A}V\right)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\right|^2+\frac{1}{2}\left|\left\langle\left(V-V^{*A}V\right)k_{\mathcal{H},\lambda},k_{\mathcal{H},\lambda}\right\rangle_A\right|^2.
\end{aligned}$$

Therefore, taking supremum over $\lambda\in\Omega$, we get

$$\eta_A^2(V)\leq\frac{1}{2}\left\{\mathrm{ber}_A^2\left(V+V^{*A}V\right)+\mathrm{ber}_A^2\left(V-V^{*A}V\right)\right\}.$$

Hence completes the proof. □

Now we give upper bounds for the A-Davis-Wielandt-Berezin radius of $V\in\mathcal{L}_{A,r}(\mathcal{H})$.

Theorem 4. *Let $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. Then the inequalities listed below are true.*
(i) $\eta_A^2(V) \leq \left\| V^{*A}V + \left(V^{*A}V\right)^{*A} V^{*A}V \right\|_{A-\mathrm{ber}}$,
(ii) $\eta_A^2(V) \leq \frac{1}{2}\left(\mathrm{ber}_A\left(V^2\right) + \|V\|_A^2\right) + \|V\|_{A-\mathrm{Ber}}^4$.

Proof. Let $\lambda \in \Omega$ be arbitrary. Applying (3) for $u_1 = Vk_{\mathcal{H},\lambda}$, $e = k_{\mathcal{H},\lambda}$ and $u_2 = Vk_{\mathcal{H},\lambda}$, we have that

$$\begin{aligned}\left|\widetilde{V}(\lambda)\right|_A^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 &= \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \langle k_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right| \\ &\quad + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \left\langle k_{\mathcal{H},\lambda}, V^{*A}Vk_{\mathcal{H},\lambda}\right\rangle_A \\ &\leq \frac{1}{2}\left(\|Vk_{\mathcal{H},\lambda}\|_A^2 + \langle Vk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right) \\ &\quad + \frac{1}{2}\left(\left\|V^{*A}Vk_{\mathcal{H},\lambda}\right\|_A^2 + \left\langle V^{*A}Vk_{\mathcal{H},\lambda}, V^{*A}Vk_{\mathcal{H},\lambda}\right\rangle_A\right) \\ &= \left\langle \left(V^{*A}V + \left(V^{*A}V\right)^{*A} V^{*A}V\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A.\end{aligned}$$

taking the supremum over $\lambda \in \Omega$, we have

$$\sup_{\lambda\in\Omega}\left\{\left|\widetilde{V}(\lambda)\right|_A^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4\right\} \leq \sup_{\lambda\in\Omega}\left\langle \left(V^{*A}V + \left(V^{*A}V\right)^{*A} V^{*A}V\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A.$$

This proves (i). The proof of (ii) is immediate from

$$\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 = \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \left\langle k_{\mathcal{H},\lambda}, V^{*A}k_{\mathcal{H},\lambda}\right\rangle_A\right| \tag{5}$$

by applying (3) for $u = Vk_{\mathcal{H},\lambda}$, $e = k_{\mathcal{H},\lambda}$, $v = V^{*}k_{\mathcal{H},\lambda}$ in (5). The theorem is proved. □

It is widely known that if V is A-normaloid then $\left\|V^2\right\|_A = \|V\|_A^2$. Hence, both the inequalities in Theorem 4 becomes equality if V is A-normaloid can be observed easily.

We now obtain another upper bounds for the Davis-Wielandt-Berezin radius of bounded linear operators.

Theorem 5. *If $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$, then we have*

$$\begin{aligned}\eta_A^2(V) \leq 3\left\|\left(V^{*A}V\right)^{*A} V^{*A}V + V^{*A}V\right\|_{A-\mathrm{ber}} - \widetilde{c}_A\left(V^{*A}V + V\right) m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\ - \widetilde{c}_A\left(V^{*A}V - V\right) m_{A-\mathrm{ber}}\left(V^{*A}V - V\right).\end{aligned} \tag{6}$$

Proof. Let $\lambda \in \Omega$ be arbitrary. It follows from Lemmas 2-3 that

$$\begin{aligned}&\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 \\ &\leq \|Vk_{\mathcal{H},\lambda}\|_A^2 \|k_{\mathcal{H},\lambda}\|_A^2 \\ &- 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \langle k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| \left(\|Vk_{\mathcal{H},\lambda}\|_A \|k_{\mathcal{H},\lambda}\|_A - \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|\right)\end{aligned}$$

$$= \|Vk_{\mathcal{H},\lambda}\|_A^2 + 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| \left|\langle k_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A\right| - 2\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| \|Vk_{\mathcal{H},\lambda}\|_A$$
$$\le \|Vk_{\mathcal{H},\lambda}\|_A^2 + \|Vk_{\mathcal{H},\lambda}\|_A^2 + \langle Vk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda}\rangle_A - 2\widetilde{c}_A(V)\|Vk_{\mathcal{H},\lambda}\|_A$$
$$\le 3\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A - 2\widetilde{c}_A(V)\, m_{A-\mathrm{ber}}(V).$$

Therefore, we get

$$\begin{aligned}
&\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\
&= \frac{1}{2}\left(\left|\|Vk_{\mathcal{H},\lambda}\|_A^2 + \langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 + \left|\|Vk_{\mathcal{H},\lambda}\|_A^2 - \langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2\right) \\
&= \frac{1}{2}\left(\left|\left\langle\left(V^{*A}V + V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \left|\left\langle\left(V^{*A}V - V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2\right) \\
&\le \frac{1}{2}\Big(3\left\langle\left|V^{*A}V + V\right|_A^2 k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A - 2\widetilde{c}_A\left(V^{*A}V + V\right) m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\
&\quad + 3\left\langle\left|V^{*A}V - V\right|_A^2 k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A - 2\widetilde{c}_A\left(V^{*A}V - V\right) m_{A-\mathrm{ber}}\left(V^{*A}V - V\right)\Big) \\
&= \frac{3}{2}\left\langle\left(\left|V^{*A}V + V\right|_A^2 + \left|V^{*A}V - V\right|_A^2\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad - \widetilde{c}_A\left(V^{*A}V + V\right) m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\
&\quad - \widetilde{c}_A\left(V^{*A}V - V\right) m_{A-\mathrm{ber}}\left(V^{*A}V - V\right) \\
&= 3\left\langle\left(\left(V^{*A}V\right)^{*A} V^{*A}V + V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad - \widetilde{c}_A\left(V^{*A}V + V\right) m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) - \widetilde{c}_A\left(V^{*A}V - V\right) m_{A-\mathrm{ber}}\left(V^{*A}V - V\right).
\end{aligned}$$

Thus, by taking supremum over $\lambda \in \Omega$, we obtain

$$\begin{aligned}
\sup_{\lambda\in\Omega}\left(\left|\widetilde{V}^A(\lambda)\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4\right) &\le 3\sup_{\lambda\in\Omega}\left\langle\left(\left(V^{*A}V\right)^{*A} V^{*A}V + V^{*A}V\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad - \sup_{\lambda\in\Omega}\widetilde{c}_A\left(V^{*A}V + V\right) m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\
&\quad - \widetilde{c}_A\left(V^{*A}V - V\right) m_{A-\mathrm{ber}}\left(V^{*A}V - V\right)
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\eta_A^2(V) &\le 3\left\|\left(V^{*A}V\right)^{*A} V^{*A}V + V^{*A}V\right\|_{A-\mathrm{ber}} \\
&\quad - \widetilde{c}_A\left(V^{*A}V + V\right) m_{A-\mathrm{ber}}\left(V^{*A}V + V\right) \\
&\quad - \widetilde{c}_A\left(V^{*A}V - V\right) m_{A-\mathrm{ber}}\left(V^{*A}V - V\right).
\end{aligned}$$

This immediately proves (6) as required. □

We are now able to establish the following theorem.

Theorem 6. *Let $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. Then the inequalities listed below are true.*

(i)

$$\eta_A^2(V) \le \inf_{r\in\mathbb{R}} \sup_{\theta\in\mathbb{R}} \Big\{ 2|r| \left\| \cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - rI \right\|_A + \frac{1}{2} \left\| \cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI \right\|_A^2 + \frac{1}{2} \left\| \cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V) \right\|_A^2 \Big\}.$$

(ii)

$$\eta_A^2(V) \le \frac{1}{2} \sup_{\theta\in\mathbb{R}} \Big\{ \left\| \cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) \right\|_A^2 + \left\| \cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V) \right\|_A^2 \Big\}.$$

Proof. (i) Let $\lambda \in \Omega$ be arbitrary. Then there exists $\theta \in \mathbb{R}$ such that $\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| = e^{-i\theta} \langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A$. By applying the Cartesian decomposition of V, we see that

$$\begin{aligned}
\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| &= \left\langle e^{-i\theta} Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&= \langle ((\cos\theta - i\sin\theta)(\operatorname{Re}_A(V) + i\operatorname{Im}_A(V)))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&= \langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&\quad + i\,\langle (\cos\theta \operatorname{Im}_A(V) - \sin\theta \operatorname{Re}_A(V))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A .
\end{aligned}$$

So, by $\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| \in \mathbb{R}$ we get

$$\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right| = \langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A .$$

Thus, by using Lemma 4, we get for any $r \in \mathbb{R}$,

$$\begin{aligned}
\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 &= \left|\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&= \|(\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda}\|_A^2 \\
&\quad - \|(\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda} - rk_{\mathcal{H},\lambda}\|_A^2 \\
&\quad + \left|\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))\, k_{\mathcal{H},\lambda} - rk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|_A^2 \\
&= \left\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))^2 k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad - \left\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI)^2 k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A \\
&\quad + \left|\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI)\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&= \Big\langle \Big\{ (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V))^2 \\
&\quad - (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI)^2 \Big\} k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \Big\rangle_A \\
&\quad + \left|\langle (\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI)\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2
\end{aligned}$$

$$= \left\langle \left(2r\left(\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V)\right) - r^2 I\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A + \left|\left\langle\left(\cos\theta \operatorname{Re}_A(V) + \sin\theta \operatorname{Im}_A(V) - rI\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2.$$

By using Lemma 4, we obtain

$$\begin{aligned}\|Vk_{\mathcal{H},\lambda}\|_A^4 &= \left|\left\langle V^{*A}Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\ &= \left\langle\left(2rV^{*A}V - r^2 I\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A + \left|\left\langle\left(V^{*A}V - rI\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2.\end{aligned}$$

Now,

$$\begin{aligned}&\left|\left\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\ &= \left\langle 2r\left\{\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\} k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A - 2r^2 \\ &\quad + \frac{1}{2}\left|\left\langle\left(\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\ &\quad + \frac{1}{2}\left|\left\langle\left(\cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right|^2 \\ &\le 2|r|\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - rI\right\|_A \\ &\quad + \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI\right\|_A^2 \\ &\quad + \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) - |V|_A^2 + \sin\theta \operatorname{Im}_A(V)\right\|_A^2 \\ &\le \sup_{\theta\in\mathbb{R}}\left\{2|r|\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - rI\right\|_A\right. \\ &\quad + \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI\right\|_A^2 \\ &\quad \left.+ \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\|_A^2\right\}.\end{aligned}$$

Therefore, taking supremum over all $\lambda \in \Omega$, we get

$$\begin{aligned}\eta_A^2(V) \le \sup_{\theta\in\mathbb{R}}&\left\{2|r|\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - rI\right\|_A\right. \\ &+ \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V) - 2rI\right\|_A^2 \\ &\left.+ \frac{1}{2}\left\|\cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\|_A^2\right\}.\end{aligned}$$

Because this inequality holds for every $r \in \mathbb{R}$, we have the required inequality.

(ii) If we pick $r = 0$, for example,

$$\begin{aligned}\eta_A^2(V) \le \frac{1}{2}\sup_{\theta\in\mathbb{R}}&\left\{\left\|\cos\theta \operatorname{Re}_A(V) + V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\|_A^2\right. \\ &\left.+ \left\|\cos\theta \operatorname{Re}_A(V) - V^{*A}V + \sin\theta \operatorname{Im}_A(V)\right\|_A^2\right\}.\end{aligned}$$

□

Following so, we find the inequality shown below.

Theorem 7. *Let $V \in \mathcal{L}_{A,r}(\mathcal{H}(\Omega))$. Then the inequalities listed below are true.*
(i)

$$\eta_A^2(V) \le \inf_{z\in\mathbb{C}} \Big\{ \Big(2\left\|\operatorname{Re}(z)\operatorname{Re}_A(V) + \operatorname{Im}(z)\operatorname{Im}_A(V)\right\|_{A-\text{ber}} + \left\|V^{*A}V - 2\operatorname{Re}(\overline{z}V)\right\|_{A-\text{ber}}\Big)^2 + 2\left\|\operatorname{Re}(\overline{z}V)\right\|_{A-\text{ber}} - |z|^2 + \operatorname{ber}_A^2(V - zI)\Big\}.$$

(ii) $\eta_A^2(V) \le \operatorname{ber}_A^2(V) + \|V\|_{A-\text{ber}}^4$.

Proof. Let $z \in \mathbb{C}$. Choosing in Lemma 4 $u_1 = Vk_{\mathcal{H},\lambda}$ and $u_2 = k_{\mathcal{H},\lambda}$, we have for all $\lambda \in \Omega$

$$\|Vk_{\mathcal{H},\lambda}\|_A^2\|k_{\mathcal{H},\lambda}\|_A^2 - \left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 = \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2\|k_{\mathcal{H},\lambda}\|_A^2 - \left|\langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2.$$

Then by using the Cartesian decomposition of V we have that

$$\begin{aligned}
\|Vk_{\mathcal{H},\lambda}\|_A^2 &= \left(\langle \operatorname{Re}_A(V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right)^2 - \left(\langle \operatorname{Re}_A(V - zI)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right)^2 \\
&\quad + \left(\langle \operatorname{Im}_A(V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right)^2 - \left(\langle \operatorname{Im}_A(V - zI)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right)^2 \\
&\quad + \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2 \\
&= \langle (2\operatorname{Re}_A(V) - \operatorname{Re}(z)I)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \langle \operatorname{Re}(z)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&\quad + \langle (2\operatorname{Im}_A(V) - \operatorname{Im}(z)I)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \langle \operatorname{Im}(z)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&\quad + \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2 \\
&= 2\operatorname{Re}(z)\langle \operatorname{Re}_A(V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A + 2\operatorname{Im}(z)\langle \operatorname{Im}_A(V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A \\
&\quad - (\operatorname{Re}(z))^2 - (\operatorname{Im}(z))^2 + \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2 \\
&= 2\left(\operatorname{Re}(z)\langle \operatorname{Re}_A(V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A + \operatorname{Im}(z)\langle \operatorname{Im}_A(V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right) \\
&\quad - |z|^2 + \langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\rangle_A \\
&= 2\left(\operatorname{Re}(z)\langle \operatorname{Re}_A(V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A + \operatorname{Im}(z)\langle \operatorname{Im}_A(V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right) \\
&\quad + \left\langle \left(V^{*A}V - 2\operatorname{Re}_A(\overline{z}V)\right)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A.
\end{aligned}$$

Again by using Lemma 4, we get

$$\begin{aligned}
\left|\langle Vk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 &= \|Vk_{\mathcal{H},\lambda}\|_A^2 - \|Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}\|_A^2 + \left|\langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2 \\
&= 2\langle \operatorname{Re}(\overline{z}V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A - |z|^2 + \left|\langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2.
\end{aligned}$$

So, we deduce that

$$\begin{aligned}
&\left|\widetilde{V}^A(z)\right|^2 + \|Vk_{\mathcal{H},\lambda}\|_A^4 \\
&\le 2\langle \operatorname{Re}(\overline{z}V)k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A - |z|^2 + \left|\langle Vk_{\mathcal{H},\lambda} - zk_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A\right|^2
\end{aligned}$$

$$+ 2 \langle (\mathrm{Re}\,(z)\,\mathrm{Re}_A\,(V) + \mathrm{Im}\,(z)\,\mathrm{Im}_A\,(V))\, k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \rangle_A$$
$$+ \left\langle \left(V^{*A}V - 2\,\mathrm{Re}_A\,(\overline{z}V)\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda} \right\rangle_A)^2.$$

for all $\lambda \in \Omega$. Hence, taking supremum over $\lambda \in \Omega$, and infimum over all $z \in \mathbb{C}$, we have

$$\eta_A^2\,(V) \leq \inf_{z\in\mathbb{C}} \left\{ \left(2\,\|\mathrm{Re}\,(z)\,\mathrm{Re}_A\,(V) + \mathrm{Im}\,(z)\,\mathrm{Im}_A\,(V)\|_{A-\mathrm{ber}} + \left\|V^{*A}V - 2\,\mathrm{Re}_A\,(\overline{z}V)\right\|_{A-\mathrm{ber}} \right)^2 \right.$$
$$\left. +2\,\|\mathrm{Re}_A\,(\overline{z}V)\|_{A-\mathrm{ber}} - |z|^2 + \mathrm{ber}_A^2\,(V - zI) \right\}.$$

(ii) Taking $z = 0$, we get $\eta_A^2\,(V) \leq \mathrm{ber}_A^2\,(V) + \|V\|_{A-\mathrm{ber}}^4$. This proves the required result. □

Then, we have an upper bound on the A-Davis-Wielandt-Berezin radius of sum of two bounded linear operators.

Theorem 8. *Let U, $V \in \mathcal{L}_{A,r}\,(\mathcal{H}\,(\Omega))$. Then the inequalities listed below are true.*
(i) $\eta_A\,(U + V) \leq \eta_A\,(U) + \eta_A\,(V) + \mathrm{ber}_A\left(U^{*A}V + V^{*A}U\right)$;
(ii) If $U^{*A}V + V^{*A}U = 0$, *then* $\eta_A\,(U + V) \leq \eta_A\,(U) + \eta\,(V)$.

Proof. (i) It follows from Definition 5 that

$$\begin{aligned}\mathbf{H}_A\,(U+V) &= \left\{ \left(\langle (U+V)\,k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A, \langle (U+V)\,k_{\mathcal{H},\lambda}, (U+V)\,k_{\mathcal{H},\lambda}\rangle_A\right),\ \lambda\in\Omega \right\} \\ &= \left\{ \left(\langle U k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A, \langle U k_{\mathcal{H},\lambda}, U k_{\mathcal{H},\lambda}\rangle_A\right) \right. \\ &\quad + \left(\langle V k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\rangle_A, \langle V k_{\mathcal{H},\lambda}, V k_{\mathcal{H},\lambda}\rangle_A\right) \\ &\quad \left. + \left(0, \left\langle \left(U^{*A}V + V^{*A}U\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right) : \lambda\in\Omega \right\}.\end{aligned}$$

So, $\mathbf{H}_A\,(U+V) \subseteq \mathbf{H}_A\,(U) + \mathbf{H}_A\,(V) + X$, where

$$X = \left\{ \left(0, \left\langle \left(U^{*A}V + V^{*A}U\right) k_{\mathcal{H},\lambda}, k_{\mathcal{H},\lambda}\right\rangle_A\right) : \lambda\in\Omega \right\}.$$

This demonstrates (i). The evidence of (ii) is obvious from (i) and $A\left(U^{*A}V + V^{*A}U\right) = O$, and the proof of theorem is completed. □

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S-n-IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity and S a multiplicatively closed subset of R. This paper aims to introduce the concept of S-n-ideals as a generalization of n-ideals. An ideal I of R disjoint with S is called an S-n-ideal if there exists $s \in S$ such that whenever $ab \in I$ for $a, b \in R$, then $sa \in \sqrt{0}$ or $sb \in I$. The relationships among S-n-ideals, n-ideals, S-prime and S-primary ideals are clarified. Besides several properties, characterizations and examples of this concept, S-n-ideals under various contexts of constructions including direct products, localizations and homomorphic images are given. For some particular S and $m \in \mathbb{N}$, all S-n-ideals of the ring $\mathbb{Z}_m$ are completely determined. Furthermore, S-n-ideals of the idealization ring and amalgamated algebra are investigated.

1. Introduction

Throughout this paper, we assume that all rings are commutative with non-zero identity. For a ring R, we will denote by $U(R)$, $reg(R)$ and $Z(R)$, the set of unit elements, regular elements and zero-divisor elements of R, respectively. For an ideal I of R, the radical of I denoted by $\sqrt{I}$ is the ideal $\{a \in R : a^n \in I$ for some positive integer $n\}$ of R. In particular, $\sqrt{0}$ denotes the set of all nilpotent elements of R. We recall that a proper ideal I of a ring R is called prime (primary) if for $a, b \in R$, $ab \in I$ implies $a \in I$ or $b \in I$ ($b \in \sqrt{I}$). Several generalizations of prime and primary ideals were introduced and studied, (see for example [2]- [4], [6], [17]).

Let S be a multiplicatively closed subset of a ring R and I an ideal of R disjoint with S. Recently, Hamed and Malek [12] used a new approach to generalize prime ideals by defining S-prime ideals. I is called an S-prime ideal of R if there exists

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an $s \in S$ such that for all $a, b \in R$ whenever $ab \in I$, then $sa \in I$ or $sb \in I$. Then analogously, Visweswaran [16] introduced the notion of S-primary ideals. I is called an S-primary ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in I$ or $sb \in \sqrt{I}$. Many other generalizations of S-prime and S-primary ideals have been studied. For example, in [1], the authors defined I to be a weakly S-prime ideal if there exists an $s \in S$ such that for all $a, b \in R$ if $0 \neq ab \in I$, then $sa \in I$ or $sb \in I$. In 2015, Mohamadian [14] defined a new type of ideals called r-ideals. An ideal I of a ring R is said to be r -ideal, if $ab \in I$ and $a \notin Z(R)$ imply that $b \in I$ for each $a, b \in R$. Generalizing this concept, in 2017 the notion of n-ideals was first introduced and studied [15]. The authors called a proper ideal I of R an n-ideal if $ab \in I$ and $a \notin \sqrt{0}$ imply that $b \in I$ for each $a, b \in R$. Many other generalizations of n-ideals have been introduced recently, see for example [13] and [18]. Motivated and inspired by these studies, in this article, we study the S-version of the class of n-ideals by determining the structure of S-n-ideals of a ring. We call I an S-n-ideal of a ring R if there exists an (fixed) $s \in S$ such that for all $a, b \in R$ if $ab \in I$ and $sa \notin \sqrt{0}$, then $sb \in I$. We call this fixed element $s \in S$ an S-element of I. Clearly, for any multiplicatively closed subset S of R, every n-ideal is an S-n-ideal and the classes of n-ideals and S-n-ideals coincide if $S \subseteq U(R)$. However, this generalization of n-ideals is proper as we can see in Example 1. In Section 2, we start by giving an example of an S-n-ideal of a ring R that is not an n-ideal. Then we give many properties of S-n-ideals and show that S-n-ideals enjoy analogs of many of the properties of n-ideals. Also we discuss the relationship among S-n-ideals, n-ideals, S-prime and S-primary ideals, (Propositions 1, 6 and Examples 1, 2). In Theorems 1 and 2, we present some characterizations for S-n-ideals of a general commutative ring. Moreover, we investigate some conditions under which $(I :_R s)$ is an S-n-ideal of R for an S-n-ideal I of R and an S-element s of I, (Propositions 2, 3 and Example 3). For a particular case that $S \subseteq reg(R)$, we justify some other results. For example, in this case, we prove that a maximal S-n-ideal of R is S-prime, (Proposition 6). In addition, we show in Proposition 4 that every proper ideal of a ring R is an S-n-ideal if and only if R is a UN-ring (a ring for which every nonunit element is a product of a unit and a nilpotent). Let $n \in \mathbb{N}$, say, $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $p_1, p_2, ..., p_k$ are distinct prime integers and $r_i \geq 1$ for all i. Then for all $2 \leq i \leq k-1$, $S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k} = \left\{ \bar{p}_1^{m_1} \bar{p}_2^{m_2} ... \bar{p}_{i-1}^{m_{i-1}} \bar{p}_{i+1}^{m_{i+1}} ... \bar{p}_{k-1}^{m_{k-1}} : m_j \in \mathbb{N} \cup \{0\} \right\}$ is a multiplicatively closed subset of $\mathbb{Z}_n$. In Theorem 4, we determine all $S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k}$-$n$-ideals of $\mathbb{Z}_n$ for all i. In particular, we determine all S_p-n-ideals of $\mathbb{Z}_n$ where $S_p = \left\{ 1, \bar{p}, \bar{p}^2, \bar{p}^3, ... \right\}$ for any prime integer p dividing n, (Theorem 3). Furthermore, we study the stability of S-n-ideals with respect to various ring theoretic constructions such as localization, factor rings and direct product of rings, (Propositions 11, 12 and 14). Let R be a ring and M be an R-module. For a multiplicatively closed subset S of R, the set $S(+)M = \{(s, m) : s \in S,\ m \in M\}$ is clearly a multiplicatively closed subset of the idealization ring $R(+)M$. In Section 3, first, we clarify the relation between the S-n-ideals of a ring R and the $S(+)M$-n-ideals $R(+)M$, (Proposition 17). For rings R and R', an ideal J of R' and a ring homomorphism $f : R \to R'$, the amalgamation of R and R' along J with respect to f is the subring $R \bowtie^f J = \{(r, f(r) + j) : r \in R, j \in J\}$ of $R \times R'$. Clearly, the set $S \bowtie^f J = \{(s, f(s) + j) : s \in S, j \in J\}$ is a multiplicatively closed subset of $R \bowtie^f J$ whenever S is a multiplicatively closed subset of R. We finally determine when the ideals $I \bowtie^f J = \{(i, f(i) + j) : i \in I, j \in J\}$ and $\bar{K}^f = \{(a, f(a) + j) : a \in R, j \in J, f(a) + j \in K\}$ of $R \bowtie^f J$ are $(S \bowtie^f J)$-n-ideals, (Theorems 5 and 6).

2. Properties of S-n-ideals

Definition 1. *Let R be a ring, S be a multiplicatively closed subset of R and I be an ideal of R disjoint with S. We call I an S-n-ideal of R if there exists an (fixed) $s \in S$ such that for all $a, b \in R$ if $ab \in I$ and $sa \notin \sqrt{0}$, then $sb \in I$. This fixed element $s \in S$ is called an S-element of I.*

Let I be an ideal of a ring R. If I is an n-ideal of R, then clearly I is an S-n-ideal for any multiplicatively closed subset of R disjoint with I. However, it is clear that the classes of n-ideals and S-n-ideals coincide if $S \subseteq U(R)$. Moreover, obviously any S-n-ideal is an S-primary ideal and the two concepts coincide if the ideal is contained in $\sqrt{0}$. However, the converses of these implications are not true in general as we can see in the following examples.

Example 1. *Let $R = \mathbb{Z}_{12}$, $S = \{\overline{1}, \overline{3}, \overline{9}\}$ and consider the ideal $I =< \overline{4} >$. Choose $s = \overline{3} \in S$ and let $a, b \in R$ with $ab \in I$ but $3b \notin I$. Now, $ab \in< \overline{2} >$ implies $a \in< \overline{2} >$ or $b \in< \overline{2} >$. Assume that $a \notin< \overline{2} >$ and $b \in< \overline{2} >$. Since $a \notin< \overline{2} >$, then $a \in \{\overline{1}, \overline{3}, \overline{5}, \overline{7}, \overline{9}, \overline{11}\}$ and since $3b \notin I$, we have $b \in \{\overline{2}, \overline{6}, \overline{10}\}$. Thus, in each case $ab \notin I$, a contradiction. Hence, we must have $a \in< \overline{2} >$ and so $\overline{3}a \in< \overline{6} >= \sqrt{0}$. On the other hand, I is not an n-ideal as $\overline{2} \cdot \overline{2} \in I$ but neither $\overline{2} \in \sqrt{0}$ nor $\overline{2} \in I$.*

A (prime) primary ideal of a ring R that is not an n-ideal is a direct example of an (S-prime) S-primary ideal that is not an S-n-ideal where $S = \{1\}$. For a less trivial example, we have the following.

Example 2. *Let $R = \mathbb{Z}[X]$ and let $I = \langle 4x \rangle$. consider the multiplicatively closed subset $S = \{4^m : m \in \mathbb{N} \cup \{0\}\}$ of R. Then I is an S-prime (and so S-primary) ideal of R, [16, Example 2.3]. However, I is not an S-n-ideal since for all $s = 4^m \in S$, we have $(2x)(2) \in I$ but $s(2x) \notin \sqrt{0_{\mathbb{Z}[x]}}$ and $s(2) \notin I$.*

Proposition 1. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S.*

1. If I is an S-n-ideal, then $sI \subseteq \sqrt{0}$ for some $s \in S$. If moreover, $S \subseteq reg(R)$, then $I \subseteq \sqrt{0}$.
2. $\sqrt{0}$ is an S-n-ideal of R if and only if $\sqrt{0}$ is an S-prime ideal of R.
3. Let $S \subseteq reg(R)$. Then 0 is an S-n-ideal of R if and only if 0 is an n-ideal.

Proof. (1) Let $a \in I$. Since $I \cap S = \emptyset$, $s \cdot 1 \notin I$ for all $s \in S$. Hence, $a \cdot 1 \in I$ implies that there exists an $s \in S$ such that $sa \in \sqrt{0}$. Thus, $sI \subseteq \sqrt{0}$ as desired. Moreover, if $S \subseteq reg(R)$, then clearly $I \subseteq \sqrt{0}$.

(2) Clear.

(3) Suppose s is an S-element of 0 and $ab = 0$ for some $a, b \in R$. Then $sa \in \sqrt{0}$ or $sb = 0$ which implies $s^n a^n = 0$ for some positive integer n or $sb = 0$. Since $S \subseteq reg(R)$, we have $a^n = 0$ or $b = 0$, as needed. □

Next, we characterize S-n-ideals of rings by the following.

Theorem 1. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. The following statements are equivalent.*

(1) I is an S-n-ideal of R.
(2) There exists an $s \in S$ such that for any two ideals J, K of R, if $JK \subseteq I$, then $sJ \subseteq \sqrt{0}$ or $sK \subseteq I$.

Proof. (1)⇒(2). Suppose I is an S-n-ideal of R. Assume on the contrary that for each $s \in S$, there exist two ideals J', K' of R such that $J'K' \subseteq I$ but $sJ' \nsubseteq \sqrt{0}$ and $sK' \nsubseteq I$. Then, for each $s \in S$, we can find two elements $a \in J'$ and $b \in K'$ such that $ab \in I$ but neither $sa \in \sqrt{0}$ nor $sb \in I$. By this contradiction, we are done.

(2)⇒(1). Let $a, b \in R$ with $ab \in I$. Taking $J =< a >$ and $K =< b >$ in (2), we get the result. □

Theorem 2. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. If $\sqrt{0}$ is an S-n-ideal of R, then the following are equivalent.*

(1) I is an S-n-ideal of R.
(2) There exists $s \in S$ such that for ideals $I_1, I_2, ..., I_n$ of R, if $I_1 I_2 \cdots I_n \subseteq I$, then $sI_j \subseteq \sqrt{0}$ or $sI_k \subseteq I$ for some $j, k \in \{1, ..., n\}$.
(3) There exists $s \in S$ such that for elements $a_1, a_2, ..., a_n$ of R, if $a_1 a_2 \cdots a_n \in I$, then $sa_j \in \sqrt{0}$ or $sa_k \in I$ for some $j, k \in \{1, ..., n\}$.

Proof. (1)⇒(2). Let $s_1 \in S$ be an S-element of I. To prove the claim, we use mathematical induction on n. If $n = 2$, then the result is clear by Theorem 1. Suppose $n \geq 3$ and the claim holds for $n - 1$. Let $I_1, I_2, ..., I_n$ be ideals of R with $I_1 I_2 \cdots I_n \subseteq I$. Then by Theorem 1, we conclude that either $s_1 I_1 \subseteq \sqrt{0}$ or $s_1 I_2 \cdots I_n \subseteq I$. Assume $(s_1 I_2) \cdots I_n \subseteq I$. By the induction hypothesis, we have either, say, $s_1^2 I_2 \subseteq \sqrt{0}$ or $s_1 I_k \subseteq I$ for some $k \in \{3, ..., n\}$. Assume $s_1^2 I_2 \subseteq \sqrt{0}$ and choose an S-element $s_2 \in S$ of $\sqrt{0}$. If $s_2(s_1^2 R) \subseteq \sqrt{0} \cap S$, we get a contradiction. Thus, $s_2 I_2 \subseteq \sqrt{0}$. By choosing $s = s_1 s_2$, we get $sI_j \subseteq \sqrt{0}$ or $sI_k \subseteq I$ for some $j, k \in \{1, ..., n\}$, as needed.

(2)⇒(3). This is a particular case of (2) by taking $I_j :=< a_j >$ for all $j \in \{1, ..., n\}$.

(3)⇒(1). Clear by choosing $n = 2$ in (3). □

Proposition 2. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. Then*

(1) If $(I : s)$ is an n-ideal of R for some $s \in S$, then I is an S-n-ideal.
(2) If I is an S-n-ideal and $(\sqrt{0} : s)$ is an n-ideal where $s \in S$ is an S-element of I, then $(I : s)$ is an n-ideal of R.
(3) If I is an S-n-ideal and $S \subseteq reg(R)$, then $(I : s)$ is an n-ideal of R for any S-element s of I.

Proof. (1) Suppose that $(I : s)$ is an n-ideal of R for some $s \in S$. We show that s is an S-element of I. Let $a, b \in R$ with $ab \in I$ and $sa \notin \sqrt{0}$. Then $ab \in (I : s)$ and $a \notin \sqrt{0}$ imply that $b \in (I : s)$.Thus, $sb \in I$ and I is an S-n-ideal.

(2) Suppose $a, b \in R$ with $ab \in (I : s)$. Then $a(sb) \in I$ which implies $sa \in \sqrt{0}$ or $s^2 b \in I$. Suppose $sa \in \sqrt{0}$. Since $(\sqrt{0} : s)$ is an n-ideal, $(\sqrt{0} : s) = \sqrt{0}$ by [15, Proposition 2.3] and so $a \in \sqrt{0}$. Now, suppose $s^2 b \in I$. If $sb \notin I$, then since I is an S-n-ideal, $s^3 \in \sqrt{0}$ and so $s \in \sqrt{0}$ which contradicts the assumption that $(\sqrt{0} : s)$ is proper. Thus, $sb \in I$ and $b \in (I : s)$ as needed.

(3) Suppose $S \subseteq reg(R)$ and I is an S-n-ideal. Let $a, b \in R$ with $ab \in (I : s)$ so that $a(sb) \in I$. If $sa \in \sqrt{0}$, then $s^m a^m = 0$ for some integer m. Since $S \subseteq reg(R)$, we get $a^m = 0$ and so $a \in \sqrt{0}$. If $s^2 b \in I$, then similar to the proof of (2) we conclude that $b \in (I : s)$. □

Note that the conditions that $(\sqrt{0} : s)$ is an n-ideal in (2) and $S \subseteq reg(R)$ in (3) of Proposition 2 are crucial. Indeed, consider $R = \mathbb{Z}_{12}$, $S = \{\overline{1}, \overline{3}, \overline{9}\}$. We showed in Example 1 that $I =< \overline{4} >$ is an S-n-ideal which is not an n-ideal, and so $(I : \overline{3}) = I$ is not an n-ideal. Here, observe that $S \nsubseteq reg(R)$ and $(\sqrt{0} : 3) =< \overline{2} >$ is not an n-ideal of $\mathbb{Z}_{12}$.

Proposition 3. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R and I be an S-prime ideal of R. Then I is an S-n-ideal if and only if $(I : s) = \sqrt{0}$ for some $s \in S$.*

Proof. Suppose I is an S-n-ideal of R and s_1 be an S-element of I. Then $(I : s_1)$ is an n-ideal of R by Proposition 2. Moreover, $(I : ts_1)$ is an n-ideal for all $t \in S$. Indeed, if $ab \in (I : ts_1)$ for $a, b \in R$, then $abts_1 \in I$ and so either $s_1^2 a \in \sqrt{0}$ or $s_1 tb \in I$. If $s_1^2 a \in \sqrt{0}$, then $a \in \sqrt{0}$ as $S \subseteq reg(R)$. Otherwise, we have $b \in (I : ts_1)$ as needed. Since I is an S-prime ideal of R, $(I : s_2)$ is a prime ideal of R where $s_2 \in S$ such that whenever $ab \in I$ for $a, b \in R$, either $s_2 a \in I$ or $s_2 b \in I$, [12, Proposition 1]. Similar to the above argument, we can also conclude that $(I : ts_2)$ is a prime ideal for all $t \in S$. Now, choose $s = s_1 s_2$. Then $(I : s)$ is both a prime and an n-ideal of R and so $(I : s) = \sqrt{0}$ by [15, Proposition 2.8]. Conversely, suppose $(I : s) = \sqrt{0}$ for some $s \in S$. Since I is an S-prime ideal, $(I : s')$ is a prime ideal of R for some $s' \in S$. Moreover, if $a \in (I : s')$, then $as' \in I \subseteq (I : s) \subseteq \sqrt{0}$ and so $a \in \sqrt{0}$ as $S \subseteq reg(R)$. Thus, $(I : s') = \sqrt{0}$ is a prime ideal and so it an n-ideal again by [15, Proposition 2.8]. Therefore, I is an S-n-ideal by Proposition 2. □

In the following example we justify that the condition $S \subseteq reg(R)$ can not be omitted in Proposition 3.

Example 3. *The ideal $I =< \overline{2} >$ of $\mathbb{Z}_{12}$ is prime and so S-prime for $S = \{\overline{1}, \overline{3}, \overline{9}\} \nsubseteq reg(\mathbb{Z}_{12})$. Moreover, one can directly see that $s = 3$ is an S-element of I and so I is also an S-n-ideal of $\mathbb{Z}_{12}$. But $(I : s) = I \neq \sqrt{0}$ for all $s \in S$.*

A ring R is said to be a UN-ring if every nonunit element is a product of a unit and a nilpotent. Next, we obtain a characterization for rings in which every proper ideal is an S-n-ideal where $S \subseteq reg(R)$.

Proposition 4. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R. The following are equivalent.*

(1) Every proper ideal of R is an n-ideal.
(2) Every proper ideal of R is an S-n-ideal.
(3) R is a UN-ring.

Proof. Since (1)⇒(2) is straightforward and (3)⇒(1) is clear by [15, Proposition 2.25], we only need to prove (2)⇒(3).

(2)⇒(3). Let I be a prime ideal of R. Then I is an S-prime and from our assumption, it is also an S-n-ideal. Thus $I \subseteq (I : s) = \sqrt{0}$ is a prime ideal of R by Proposition 3. Thus $\sqrt{0}$ is the unique prime ideal of R and so R is a UN-ring by [7, Proposition 2 (3)]. □

The equivalence of (1) and (2) in Proposition 4 need not be true if $S \nsubseteq reg(R)$.

Example 4. *Consider the ring $\mathbb{Z}_6$ and let $S = \{1, 3\}$. If $I = \langle\bar{0}\rangle$ or $\langle\bar{2}\rangle$, then a simple computations can show that I is an S-n-ideal of $\mathbb{Z}_6$. However, $\mathbb{Z}_6$ has no proper n-ideals, [15, Example 2.2].*

A ring R is said to be von Neumann regular if for all $a \in R$, there exists an element $b \in R$ such that $a = a^2 b$.

Proposition 5. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R.*

(1) Let R be a reduced ring. Then R is an integral domain if and only if there exists an S-prime ideal of R which is also an S-n-ideal
(2) R is a field if and only if R is von Neumann regular and 0 is an S-n-ideal of R.

Proof. (1) Let R be an integral domain. Since $0 = \sqrt{0}$ is prime, it is also an n-ideal again by [15, Corollary 2.9]. Thus $\sqrt{0}$ is both S-prime and S-n-ideal of R, as required. Conversely, suppose I is both S-prime and S-n-ideal of R. Hence, from Proposition 3 we conclude $(I : s) = \sqrt{0}$ which is an n-ideal by Proposition

2. $\sqrt{0} = 0$ is also a prime ideal by [15, Corollary 2.9], and thus R is an integral domain.

(2) Since $S \subseteq reg(R)$, from Proposition 1, 0 is an S-n-ideal of R if and only if 0 is an n-ideal. Thus, the claim is clear by [15, Theorem 2.15]. □

Let $n \in \mathbb{N}$. For any prime p dividing n, we denote the multiplicatively closed subset $\{1, \bar{p}, \bar{p}^2, \bar{p}^3, ...\}$ of $\mathbb{Z}_n$ by S_p. Next, for any p dividing n, we clarify all S_p-n-ideals of $\mathbb{Z}_n$.

Theorem 3. *Let* $n \in \mathbb{N}$.

(1) If $n = p^r$ for some prime integer p and $r \geq 1$, then $\mathbb{Z}_n$ has no S_p-n-ideals.
(2) If $n = p_1^{r_1} p_2^{r_2}$ where p_1 and p_2 are distinct prime integers and $r_1, r_2 \geq 1$, then for all $i = 1, 2$, every ideal of $\mathbb{Z}_n$ disjoint with S_{p_i} is an S_{p_i}-n-ideal.
(3) If $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $p_1, p_2, ..., p_k$ are distinct prime integers and $k \geq 3$, then for all $i = 1, 2, ..., k$, $\mathbb{Z}_n$ has no S_{p_i}-n-ideals.

Proof. (1) Clear since $I \cap S_p \neq \phi$ for any ideal I of $\mathbb{Z}_n$.

(2) Let $I = \langle \bar{p}_1^{t_1} \bar{p}_2^{t_2} \rangle$ be an ideal of $\mathbb{Z}_n$ distinct with S_{p_1}. Then we must have $t_2 \geq 1$. Choose $s = \bar{p}_1^{t_1} \in S_{p_1}$ and let $ab \in I$ for $a, b \in \mathbb{Z}_n$. If $a \in \langle \bar{p}_2 \rangle$, then $sa \in \langle \bar{p}_1 \bar{p}_2 \rangle = \sqrt{0}$. If $a \notin \langle \bar{p}_2 \rangle$, then clearly $b \in \langle \bar{p}_2^{t_2} \rangle$ and so $sb \in I$. Therefore, I is an S_{p_1}-n-ideal of $\mathbb{Z}_n$. By a similar argument, we can show that every ideal of $\mathbb{Z}_n$ distinct with S_{p_2} is an S_{p_2}-n-ideal.

(3) Let $I = \langle \bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_k^{t_k} \rangle$ be an ideal of $\mathbb{Z}_n$ distinct with S_{p_1}. Then there exists $j \neq 1$ such that $t_j \geq 1$, say, $j = k$. Thus, $\bar{p}_k^{t_k} (\bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_{k-1}^{t_{k-1}}) \in I$ but $s \bar{p}_k^{t_k} \notin \sqrt{0}$ and $s(\bar{p}_1^{t_1} \bar{p}_2^{t_2} ... \bar{p}_{k-1}^{t_{k-1}}) \notin I$ for all $s \in S_{p_1}$. Therefore, I is not an S_{p_1}-n-ideal of $\mathbb{Z}_n$. Similarly, I is not an S_{p_i}-n-ideal of $\mathbb{Z}_n$ for all $i = 1, 2, ..., k$. □

Corollary 1. *Let* $n \in \mathbb{N}$. *Then for any prime* p *dividing* n, *either* $\mathbb{Z}_n$ *has no* S_p-n-*ideals or every ideal of* $\mathbb{Z}_n$ *disjoint with* S_p *is an* S_p-n-*ideal.*

In general if $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $r_i \geq 1$ for all i, then

$$S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k} = \{\bar{p}_1^{m_1} \bar{p}_2^{m_2} ... \bar{p}_{i-1}^{m_{i-1}} \bar{p}_{i+1}^{m_{i+1}} ... \bar{p}_k^{m_k} : m_j \in \mathbb{N} \cup \{0\}\}$$

is also a multiplicatively closed subset of $\mathbb{Z}_n$ for all i. Next, we generalize Theorem 3.

Theorem 4. *Let* $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ *where* $p_1, p_2, ..., p_k$ *are distinct prime integers and* $r_i \geq 1$ *for all* i.

(1) $\mathbb{Z}_n$ has no $S_{p_1 p_2 ... p_k}$-n-ideals.
(2) For $i = 1, 2, ..., k$, every ideal of $\mathbb{Z}_n$ disjoint with $S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k}$ is an $S_{p_1 p_2 ... p_{i-1} p_{i+1} ... p_k}$-$n$-ideal.
(3) Let $k \geq 3$. If $m \leq k - 2$, then $\mathbb{Z}_n$ has no $S_{p_1 p_2 ... p_m}$-n-ideals.

Proof. (1) This is clear since $I \cap S_{p_1p_2...p_k} \neq \phi$ for any ideal I of $\mathbb{Z}_n$.

(2) With no loss of generality, we may choose $i = k$. Let $I = \langle \bar{p}_1^{t_1}\bar{p}_2^{t_2}...\bar{p}_k^{t_k} \rangle$ be an ideal of $\mathbb{Z}_n$ disjoint with $S_{p_1p_2...p_{k-1}}$. Then we must have $t_k \geq 1$. Choose $s = \bar{p}_1^{t_1}\bar{p}_2^{t_2}...\bar{p}_{k-1}^{t_{k-1}} \in S_{p_1p_2...p_{k-1}}$ and let $a, b \in \mathbb{Z}_n$ such that $ab \in I$. If $a \in \langle \bar{p}_k \rangle$, then $sa \in \langle \bar{p}_1\bar{p}_2...\bar{p}_k \rangle = \sqrt{0}$. If $a \notin \langle \bar{p}_k \rangle$, then we must have $b \in \langle \bar{p}_k^{t_k} \rangle$. Thus, $sb \in I$ and I is an $S_{p_1p_2...p_{k-1}}$-n-ideal of $\mathbb{Z}_n$.

(3) Assume $m = k - 2$ and let $I = \langle \bar{p}_1^{t_1}\bar{p}_2^{t_2}...\bar{p}_k^{t_k} \rangle$ be an ideal of $\mathbb{Z}_n$ disjoint with $S_{p_1p_2...p_{k-2}}$. Then at least one of t_{k-1} and t_k is nonzero, say, $t_k \gneqq 0$. Hence, $\bar{p}_k^{t_k}(\bar{p}_1^{t_1}\bar{p}_2^{t_2}...\bar{p}_{k-1}^{t_{k-1}}) \in I$ but clearly $s\bar{p}_k^{t_k} \notin \sqrt{0}$ and $s(\bar{p}_1^{t_1}\bar{p}_2^{t_2}...\bar{p}_{k-1}^{t_{k-1}}) \notin I$ for all $s \in S_{p_1p_2...p_{k-2}}$. Therefore, $\mathbb{Z}_n$ has no $S_{p_1p_2...p_{k-2}}$-n-ideals. A similar proof can be used if $1 \leq m \lneqq k - 2$. □

An ideal I of a ring R is called a maximal S-n-ideal if there is no S-n-ideal of R that contains I properly. In the following proposition, we observe the relationship between maximal S-n-ideals and S-prime ideals.

Proposition 6. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R. If I is a maximal S-n-ideal of R, then I is S-prime (and so $(I : s) = \sqrt{0}$ for some $s \in S$).*

Proof. Suppose I is a maximal S-n-ideal of R and $s \in S$ is an S-element of I. Then $(I : s)$ is an n-ideal of R by Proposition 2. Moreover, $(I : s)$ is a maximal n-ideal of R. Indeed, if $(I : s) \subsetneq J$ for some n-ideal (and so S-n-ideal) J of R, then $I \subseteq (I : s) \subsetneq J$ which is a contradiction. By [15, Theorem 2.11], $(I : s) = \sqrt{0}$ is a prime ideal of R and so I is an S-prime ideal by [12, Proposition 1]. □

Proposition 7. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. If I is an S-n-ideal, and J is an ideal of R with $J \cap S \neq \emptyset$, then IJ and $I \cap J$ are S-n-ideals of R.*

Proof. Let $s' \in J \cap S$. Let $a, b \in R$ with $ab \in IJ$. Since $ab \in I$, we have $sa \in \sqrt{0}$ or $sb \in I$ where s is an S-element of I. Hence, $(s's)a \in J\sqrt{0} \subseteq \sqrt{0}$ or $(s's)b \in IJ$. Thus, IJ is an S-n-ideal of R. The proof that $I \cap J$ is an S-n-ideal is similar. □

Proposition 8. *Let S be a multiplicatively closed subset of a ring R and I_1, I_2,..., I_n be proper ideals of R.*

(1) If I_i is an S-n-ideal of R for all $i = 1, ..., n$, then $\bigcap_{i=1}^{n} I_i$ is an S-n-ideal of R.
(2) If $\left(\bigcap_{j \in \Omega} I_j \right) \cap S \neq \emptyset$ for $\Omega \subseteq \{1, ..., n\}$ and I_k is an S-n-ideal of R for all $k \in \{1, ..., n\} - \Omega$, then $\bigcap_{i=1}^{n} I_i$ is an S-n-ideal of R.

Proof. (1) Suppose that for all $i = 1, ..., n$, I_i is an S-n-ideal of R and note that $\left(\bigcap_{i=1}^{n} I_i\right) \cap S = \emptyset$. For all $i = 1, ..., n$, choose $s_i \in S$ such that whenever $a, b \in R$ such that $ab \in I_i$, then $s_i a \in \sqrt{0}$ or $s_i b \in I_i$. Let $a, b \in R$ such that $ab \in \bigcap_{i=1}^{n} I_i$. Then $ab \in I_i$ for all $i = 1, ..., n$. If we let $s = \prod_{i=1}^{n} s_i \in S$, then clearly $sa \in \sqrt{0}$ or $sb \in \bigcap_{i=1}^{n} I_i$ and the result follows.

(2) Choose $s' \in \left(\bigcap_{j\in\Omega} I_j\right) \cap S$. Let $a, b \in R$ with $ab \in \bigcap_{i=1}^{n} I_i$. Then for all $k \in \{1, ..., n\} - \Omega$, $ab \in I_k$ and so $s_k a \in \sqrt{0}$ or $s_k b \in I_j$ for some S-element s_k of I_k. Hence, $(s' \prod_{k\in\{1,...,n\}-\Omega} s_k)a \in \sqrt{0}$ or $(s' \prod_{k\in\{1,...,n\}-\Omega} s_k)b \in \bigcap_{i=1}^{n} I_i$ and so $\bigcap_{i=1}^{n} I_i$ is an S-n-ideal of R. □

Let S and T be two multiplicatively closed subsets of a ring R with $S \subseteq T$. Let I be an ideal disjoint with T. It is clear that if I is a S-n-ideal, then it is T-n-ideal. The converse is not true since while $I =< \overline{4} >$ is an S-n-ideal of $\mathbb{Z}_{12}$ for $S = \{\overline{1}, \overline{3}, \overline{9}\}$, it is not a T-n-ideal for $T = \{\overline{1}\} \subseteq S$.

Proposition 9. *Let S and T be two multiplicatively closed subsets of a ring R with $S \subseteq T$ such that for each $t \in T$, there is an element $t' \in T$ such that $tt' \in S$. If I is a T-n-ideal of R, then I is an S-n-ideal of R.*

Proof. Suppose $ab \in I$. Then there is a T-element $t \in T$ of I satisfying $ta \in \sqrt{0}$ or $tb \in I$. Hence there exists some $t' \in T$ with $s = tt' \in S$, and thus $sa \in \sqrt{0}$ or $sb \in I$. □

Let S be a multiplicatively closed subset of a ring R. The saturation of S is the set $S^* = \{r \in R : \frac{r}{1}$ is a unit in $S^{-1}R\}$. It is clear that S^* is a multiplicatively closed subset of R and that $S \subseteq S^*$. Moreover, it is well known that $S^* = \{x \in R : xy \in S$ for some $y \in R\}$, see [11]. The set S is called saturated if $S^* = S$.

Proposition 10. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. Then I is an S-n-ideal of R if and only if I is an S^*-n-ideal of R.*

Proof. Suppose I is an S^*-n-ideal of R. By Proposition 9, it is enough to prove that for each $t \in S^*$, there is an element $t' \in S^*$ such that $tt' \in S$. Let $t \in S^*$ and choose $t' \in R$ such that $ty \in S$. Then $t' \in S^*$ and $tt' \in S$ as required. The converse is obvious. □

Let S and T be multiplicatively closed subsets of a ring R with $S \subseteq T$. Then clearly, $T^{-1}S = \left\{\frac{s}{t} : t \in T,\ s \in S\right\}$ is a multiplicatively closed subset of $T^{-1}R$.

Proposition 11. *Let S, T be multiplicatively closed subsets of a ring R with $S \subseteq T$ and I be an ideal of R disjoint with T. If I is an S-n-ideal of R, then $T^{-1}I$ is an $T^{-1}S$-n-ideal of $T^{-1}R$. Moreover, we have $T^{-1}I \cap R = (I : u)$ for some S-element u of I.*

Proof. Suppose I is an S-n-ideal. Suppose $T^{-1}S \cap T^{-1}I \neq \phi$, say, $\frac{a}{t} \in T^{-1}S \cap T^{-1}I$. Then $a \in S$ and $ta \in I$ for some $t \in T$. Since $S \subseteq T$, then $ta \in T \cap I$, a contradiction. Thus, $T^{-1}I$ is proper in $T^{-1}R$ and $T^{-1}S \cap T^{-1}I = \phi$. Let $s \in S$ be an S-element of I and choose $\frac{s}{1} \in T^{-1}S$. Suppose $a, b \in R$ and $t_1, t_2 \in T$ with $\frac{a}{t_1}\frac{b}{t_2} \in T^{-1}I$ and $\frac{s}{1}\frac{a}{t_1} \notin \sqrt{0_{T^{-1}R}}$. Then $tab \in I$ for some $t \in T$ and $sa \notin \sqrt{0}$. Since I is an S-n-ideal, we must have $stb \in I$. Thus, $\frac{s}{1}\frac{b}{t_2} = \frac{stb}{tt_2} \in T^{-1}I$ as needed. Now, let $r \in T^{-1}I \cap R$ and choose $i \in I$, $t \in T$ such that $\frac{r}{1} = \frac{i}{t}$. Then $vr \in I$ for some $v \in T$. Since I is an S-n-ideal, then there exists $u \in S \subseteq T$ such that $uv \in \sqrt{0}$ or $ur \in I$. But $uv \notin \sqrt{0}$ as $T \cap \sqrt{0} = \phi$ and so $ur \in I$. It follows that $r \in (I : u)$ for some S-element u of I. Since clearly $(I : u) \subseteq T^{-1}I \cap R$ for all $u \in T$, the proof is completed. □

In particular, if $S = T$, then all elements of $T^{-1}S$ are units in $T^{-1}R$. As a special case of of Proposition 11, we have the following.

Corollary 2. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. If I is an S-n-ideal of R, then $S^{-1}I$ is an n-ideal of $S^{-1}R$. Moreover, we have $S^{-1}I \cap R = (I : s)$ for some S-element s of I.*

Proof. Suppose I is an S-n-ideal. Then $S^{-1}I$ is an $S^{-1}S$-n-ideal of $S^{-1}R$ by Proposition 11. Let $a, b \in R$, $s_1, s_2 \in S$ with $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}I$. Then by assumption, $\frac{s}{t}\frac{a}{s_1} \in \sqrt{0_{S^{-1}R}}$ or $\frac{s}{t}\frac{b}{s_2} \in S^{-1}I$ for some $S^{-1}S$-element $\frac{s}{t}$ of $S^{-1}I$. Since $\frac{s}{t}$ is a unit in $S^{-1}R$, then $S^{-1}I$ is an n-ideal of $S^{-1}R$ as required. The other part follows directly by Proposition 11. □

Corollary 3. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. Then I is an S-n-ideal of R if and only if $S^{-1}I$ is an n-ideal of $S^{-1}R$, $S^{-1}I \cap R = (I : s)$ and $S^{-1}\sqrt{0} \cap R = (\sqrt{0} : t)$ for some $s, t \in S$.*

Proof. ⇒) Suppose I is an S-n-ideal of R. Then $S^{-1}I$ is an n-ideal of $S^{-1}R$ by Corollary 2. The other part of the implication follows by using a similar approach to that used in the proof of Proposition 11.

⇐) Suppose $S^{-1}I$ is an n-ideal of $S^{-1}R$, $S^{-1}I \cap R = (I : s)$ and $S^{-1}\sqrt{0} \cap R = (\sqrt{0} : t)$ for some $s, t \in S$. Choose $u = st \in S$ and let $a, b \in R$ such that $ab \in I$. Then $\frac{a}{1}\frac{b}{1} \in S^{-1}I$ and so $\frac{a}{1} \in \sqrt{S^{-1}0} = S^{-1}\sqrt{0}$ or $\frac{b}{1} \in S^{-1}I$. If $\frac{a}{1} \in \sqrt{S^{-1}0}$, then there is $w \in S$ such that $wa \in \sqrt{0}$. Thus, $a = \frac{wa}{w} \in S^{-1}\sqrt{0} \cap R = (\sqrt{0} : t)$. Hence, $ta \in \sqrt{0}$ and so $ua = sta \in \sqrt{0}$. If $\frac{b}{1} \in S^{-1}I$, then there is $v \in S$ such that $vb \in I$ and so $b = \frac{vb}{v} \in S^{-1}I \cap R = (I : s)$. Therefore, $ub = tsb \in I$ and I is an S-n-ideal of R. □

Proposition 12. *Let $f : R_1 \to R_2$ be a ring homomorphism and S be a multiplicatively closed subset of R_1. Then the following statements hold.*

(1) If f is an epimorphism and I is an S-n-ideal of R_1 containing $Ker(f)$, then $f(I)$ is an $f(S)$-n-ideal of R_2.
(2) If $Ker(f) \subseteq \sqrt{0_{R_1}}$ and J is an $f(S)$-n-ideal of R_2, then $f^{-1}(J)$ is an S-n-ideal of R_1.

Proof. First we show that $f(I) \cap f(S) = \emptyset$. Otherwise, there is $t \in f(I) \cap f(S)$ which implies $t = f(x) = f(s)$ for some $x \in I$ and $s \in S$. Hence, $x - s \in Ker(f) \subseteq I$ and $s \in I$, a contradiction.

(1) Let $a, b \in R_2$ and $ab \in f(I)$. Since f is onto, $a = f(x)$ and $b = f(y)$ for some $x, y \in R_1$. Since $f(x)f(y) \in f(I)$ and $Ker(f) \subseteq I$, we have $xy \in I$ and so there exists an $s \in S$ such that $sx \in \sqrt{0_{R_1}}$ or $sy \in I$. Thus, $f(s)a \in \sqrt{0_{R_2}}$ or $f(s)b \in f(I)$, as needed.

(2) Let $a, b \in R_1$ with $ab \in f^{-1}(J)$. Then $f(ab) = f(a)f(b) \in J$ and since J is an $f(S)$-n-ideal of R_2, there exists $f(s) \in f(S)$ such that $f(s)f(a) \in \sqrt{0_{R_2}}$ or $f(s)f(b) \in J$. Thus, $sa \in \sqrt{0_{R_1}}$ (as $Ker(f) \subseteq \sqrt{0_{R_1}}$) or $sb \in f^{-1}(J)$. □

Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. If we denote $r + I \in R/I$ by $\bar{r}$, then clearly the set $\bar{S} = \{\bar{s} : s \in S\}$ is a multiplicatively closed subset of R/I. In view of Proposition 12, we conclude the following result for $\overline{S}$-n-ideals of R/I.

Corollary 4. *Let S be a multiplicatively closed subset of a ring R and I, J are two ideals of R with $I \subseteq J$.*

(1) If J is an S-n-ideal of R, then J/I is an $\overline{S}$-n-ideal of R/I. Moreover, the converse is true if $I \subseteq \sqrt{0}$.
(2) If R is a subring of R' and I' is an S-n-ideal of R', then $I' \cap R$ is an S-n-ideal of R.

Proof. (1) Note that $(J/I) \cap \overline{S} = \phi$ if and only if $I \cap S = \phi$. Now, we apply the canonical epimorphism $\pi : R \to R/I$ in Proposition 12.

(2) Apply the natural injection $i : R \to R'$ in Proposition 12 (2). □

We recall that a proper ideal I of a ring R is called superfluous if whenever $I + J = R$ for some ideal J of R, then $J = R$.

Proposition 13. *Let $S \subseteq reg(R)$ be a multiplicatively closed subset of a ring R.*

(1) If I is an S-n-ideal of R, then it is superfluous.
(2) If I and J are S-n-ideals of R, then $I + J$ is an S-n-ideal.

Proof. (1) Suppose $I + J = R$ for some ideal J of R and let $j \in J$. Then $1 - j \in I \subseteq \sqrt{0} \subseteq J(R)$ by (1) of Proposition 1. Thus, $j \in U(R)$ and $J = R$ as needed.

(2) Suppose I and J are S-n-ideals of R. Since $I, J \subseteq \sqrt{0}$, $I + J \subseteq \sqrt{0}$ and so $(I + J) \cap S = \phi$. Now, $I/(I \cap J)$ is an $\overline{S}_1$-n-ideal of $R/(I \cap J)$ by (1) of Corollary

4 where $\bar{S}_1 = \{s + (I \cap J) : s \in S\}$. If $\bar{S}_2 = \{s + J : s \in S\}$, then clearly $\bar{S}_1 \subseteq \bar{S}_2$ and so $I/(I \cap J)$ is also an $\overline{S}_2$-n-ideal of $R/(I \cap J)$. By the isomorphism $(I + J)/J \cong I/(I \cap J)$, we conclude that $(I + J)/J$ is an $\overline{S}_2$-n-ideal of R/J. Now, the result follows again by (1) of Corollary 4. □

Proposition 14. *Let R and R' be two rings, $I \trianglelefteq R$ and $I' \trianglelefteq R'$. If S and S' are multiplicatively closed subsets of R and R', respectively, then*

1. $I \times R'$ is an $(S \times S')$-n-ideal of $R \times R'$ if and only if I is an S-n-ideal of R and $S' \cap \sqrt{0_{R'}} \neq \phi$.
2. $R \times I'$ is an $(S \times S')$-n-ideal of $R \times R'$ if and only if I' is an S'-n-ideal of R' and $S \cap \sqrt{0_R} \neq \phi$.

Proof. It is clear that $(I \times R') \cap (S \times S') = \emptyset$ if and only if $I \cap S = \emptyset$ and $(R \times I') \cap (S \times S') = \emptyset$ if and only if $I' \cap S' = \emptyset$.

(1) Let $a, b \in R$ with $ab \in I$. Choose an $(S \times S')$-element (s, s') of $I \times R'$. If $sb \notin I$, then $(a, 1)(b, 1) \in I \times R'$ with $(s, s')(b, 1) \notin I \times R'$. Since $I \times R'$ is an $(S \times S')$-n-ideal, then $(s, s')(a, 1) \in \sqrt{0_{R \times R'}} = \sqrt{0_R} \times \sqrt{0_{R'}}$. Thus, $sa \in \sqrt{0_R}$ and $s' \in S' \cap \sqrt{0_{R'}}$ I. If $sb \in I$, then $(b, 1)(s, s') \in I \times R'$and so $(s, s')(b, 1) \in \sqrt{0_{R \times R'}} = \sqrt{0_R} \times \sqrt{0_{R'}}$ as $(s, s')^2 \notin I \times R'$. In both cases, we conclude that I is an S-n-ideal of R and $S' \cap \sqrt{0_{R'}} \neq \phi$. Conversely, suppose I is an S-n-ideal of R, s is some S-element of I and $s' \in S' \cap \sqrt{0_{R'}}$. Let $(a, a')(b, b') \in I \times R'$ for $(a, a'), (b, b') \in R \times R'$. Then $ab \in I$ which implies $sa \in \sqrt{0_R}$ or $sb \in I$. Hence, we have either $(s, s')(a, a') \in \sqrt{0_R} \times \sqrt{0_{R'}}$ or $(s, s')(b, b') \in I \times R'$. Therefore, (s, s') is an $S \times S'$-element of $I \times R'$ as needed.

(2) Similar to (1). □

The assumptions $S' \cap \sqrt{0_{R'}} \neq \phi$ and $S \cap \sqrt{0_R} \neq \phi$ in Proposition 14 are crucial. Indeed, let $R = R' = \mathbb{Z}_{12}$, $S = S' = \{\overline{1}, \overline{3}, \overline{9}\}$ and $I =< \overline{4} >$. It is shown in Example 1 that I is an S-n-ideal of R while $I \times R'$ is not an $(S \times S')$-n-ideal of $R \times R'$ as $(\overline{2}, \overline{1})(\overline{2}, \overline{1}) \in I \times R'$ but for all $(s, s') \in S \times S$, neither $(s, s')(\overline{2}, \overline{1}) \in I \times R'$ nor $(s, s')(\overline{2}, \overline{1}) \in \sqrt{0_{R \times R'}}$.

Remark 1. *Let S and S' be multiplicatively closed subsets of the rings R and R', respectively. If I and I' are proper ideals of R and R' disjoint with S, S', respectively, then $I \times I'$ is not an $(S \times S')$-n-ideal of $R \times R'$.*

Proof. First, note that $S \cap \sqrt{0_R} = S' \cap \sqrt{0_{R'}} = \emptyset$. Assume on the contrary that $I \times I'$ is an $(S \times S')$-n-ideal of $R \times R'$ and (s, s') is an $(S \times S')$-element of $I \times I'$. Since $(1, 0)(0, 1) \in I \times I'$, we conclude either $(s, s')(1, 0) \in \sqrt{0_R} \times \sqrt{0_{R'}}$ or $(s, s')(0, 1) \in I \times I'$ which implies $s \in \sqrt{0_R}$ or $s' \in I'$, a contradiction. □

Proposition 15. *Let R and R' be two rings, S and S' be multiplicatively closed subsets of R and R', respectively. If I and I' are proper ideals of R, R', respectively then $I \times I'$ is an $(S \times S')$-n-ideal of $R \times R'$ if one of the following statements holds.*

1. I is an S-n-ideal of R and $S' \cap \sqrt{0_{R'}} \neq \phi$.
2. I' is an S'-n-ideal of R' and $S \cap \sqrt{0_R} \neq \phi$.

Proof. Clearly $(I \times I') \cap (S \times S') = \emptyset$ if and only if $I \cap S = \emptyset$ or $I' \cap S' = \emptyset$. Suppose I is an S-n-ideal of R and $S' \cap \sqrt{0_{R'}} \neq \phi$. Then $I \cap S = \emptyset$ and $0_{R'} \in I' \cap S' \neq \emptyset$. Choose an S-element s of I and let $(a, a')(b, b') \in I \times I'$ for $(a, a'), (b, b') \in R \times R'$. Then $ab \in I$ which implies $sa \in \sqrt{0_R}$ or $sb \in I$. Hence, we have either $(s, 0)(a, a') \in \sqrt{0_R} \times \sqrt{0_{R'}}$ or $(s, 0)(b, b') \in I \times I'$. Therefore, $(s, 0)$ is an $S \times S'$-element of $I \times I'$. Similarly, if I' is an S'-n-ideal of R' and $S \cap \sqrt{0_R} \neq \phi$, then also $I \times I'$ is an $(S \times S')$-n-ideal of $R \times R'$. □

3. S-n-ideals of Idealizations and Amalgamations

Recall that the idealization of an R-module M denoted by $R(+)M$ is the commutative ring $R \times M$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. It is well known that if $I(+)N$ is an ideal of $R(+)M$, then $\sqrt{I(+)N} = \sqrt{I}(+)M$ and in particular, $\sqrt{0_{R(+)M}} = \sqrt{0}(+)M$. If S is a multiplicatively closed subset of R, then clearly the sets $S(+)M = \{(s, m) : s \in S, m \in M\}$ and $S(+)0 = \{(s, 0) : s \in S\}$ are multiplicatively closed subsets of the ring $R(+)M$.

Next, we determine the relation between S-n-ideals of R and $S(+)M$-n-ideals of the $R(+)M$.

Proposition 16. *Let N be a submodule of an R-module M, S be a multiplicatively closed subset of R and I be an ideal of R where $IM \subseteq N$. If $I(+)N$ is an $S(+)M$-n-ideal of $R(+)M$, then I is an S-n-ideal of R.*

Proof. Clearly, $S \cap I = \phi$. Choose an $S(+)M$-element (s, m) of $I(+)N$ and let $a, b \in R$ such that $ab \in I$. Then $(a, 0)(b, 0) \in I(+)N$ and so $(s, m)(a, 0) \in \sqrt{0}(+)M$ or $(s, m)(b, 0) \in I(+)N$. Hence, $sa \in \sqrt{0}$ or $sb \in I$ and I is an S-n-ideal of R □

Proposition 17. *Let S be a multiplicatively closed subset of a ring R, I be an ideal of R disjoint with S and M be an R-module. The following are equivalent.*

1. I is an S-n-ideal of R.
2. $I(+)M$ is an $S(+)0$-n-ideal of $R(+)M$.
3. $I(+)M$ is an $S(+)M$-n-ideal of $R(+)M$.

Proof. (1)⇒(2). Suppose I is an S-n-ideal of R, s is an S-element of I and note that $S(+)0 \cap I(+)M = \phi$. Choose $(s, 0) \in S(+)0$ and let $(a, m_1), (b, m_2) \in R(+)M$ such that $(a, m_1)(b, m_2) \in I(+)M$. Then $ab \in I$ and so either $sa \in \sqrt{0}$ or $sb \in I$. It follows that $(s, 0)(a, m_1) \in \sqrt{0}(+)M = \sqrt{0_{R(+)M}}$ or $(s, 0)(b, m_2) \in I(+)M$. Thus, $I(+)M$ is an $S(+)0$-n-ideal of $R(+)M$.

(2)⇒(3). Clear since $S(+)0 \subseteq S(+)M$.

(3)⇒(1). Proposition 16. □

Remark 2. *The converse of Proposition 16 is not true in general. For example, if $S = \{1, -1\}$, then 0 is an S-n-ideal of $\mathbb{Z}$ but $0(+)\bar{0}$ is not an $(S(+)\mathbb{Z}_6)$-n-ideal* *of* $\mathbb{Z}(+)\mathbb{Z}_6$*. For example,* $(2,\bar{0})(0,\bar{3}) \in 0(+)\bar{0}$ *but clearly* $(s,m)(2,\bar{0}) \notin \sqrt{0}(+)\mathbb{Z}_6 = \sqrt{0_{\mathbb{Z}(+)\mathbb{Z}_6}}$ *and* $(s,m)(0,\bar{3}) \notin 0(+)\bar{0}$ *for all* $(s,m) \in S(+)\mathbb{Z}_6$*.*

Let R and R' be two rings, J be an ideal of R' and $f : R \to R'$ be a ring homomorphism. The set $R \bowtie^f J = \{(r, f(r)+j) : r \in R, j \in J\}$ is a subring of $R \times R'$ called the amalgamation of R and R' along J with respect to f. In particular, if $Id_R : R \to R$ is the identity homomorphism on R, then $R \bowtie J = R \bowtie^{Id_R} J = \{(r, r+j) : r \in R, j \in J\}$ is the amalgamated duplication of a ring along an ideal J. Many properties of this ring have been investigated and analyzed over the last two decades, see for example [9], [10].

Let I be an ideal of R and K be an ideal of $f(R)+J$. Then $I \bowtie^f J = \{(i, f(i)+j) : i \in I, j \in J\}$ and $\bar{K}^f = \{(a, f(a)+j) : a \in R, j \in J, f(a)+j \in K\}$ are ideals of $R \bowtie^f J$, [10]. For a multiplicatively closed subset S of R, one can easily verify that $S \bowtie^f J = \{(s, f(s)+j) : s \in S, j \in J\}$ and $W = \{(s, f(s)) : s \in S\}$ are multiplicatively closed subsets of $R \bowtie^f J$. If $J \subseteq \sqrt{0_{R'}}$, then one can easily see that $\sqrt{0_{R \bowtie^f J}} = \sqrt{0_R} \bowtie^f J$.

Next, we determine when the ideal $I \bowtie^f J$ is $(S \bowtie^f J)$-n-ideal in $R \bowtie^f J$.

Theorem 5. *Consider the amalgamation of rings* R *and* R' *along the ideals* J *of* R' *with respect to a homomorphism* f*. Let* S *be a multiplicatively closed subset of* R *and* I *be an ideal of* R *disjoint with* S*. Consider the following statements:*

(1) $I \bowtie^f J$ *is a* W*-*n*-ideal of* $R \bowtie^f J$*.*

(2) $I \bowtie^f J$ *is a* $(S \bowtie^f J)$*-*n*-ideal of* $R \bowtie^f J$*.*

(3) I *is a* S*-*n*-ideal of* R*.*

Then (1) $\Rightarrow$ *(2)* $\Rightarrow$ *(3). Moreover, if* $J \subseteq \sqrt{0_{R'}}$*, then the statements are equivalent.*

Proof. (1)$\Rightarrow$(2). Clear, as $W \subseteq S \bowtie^f J$.

(2)$\Rightarrow$(3). First note that $(S \bowtie^f J) \cap (I \bowtie^f J) = \emptyset$ if and only if $S \cap I = \emptyset$. Suppose $I \bowtie^f J$ is an $(S \bowtie^f J)$-n-ideal of $R \bowtie^f J$. Choose an $(S \bowtie^f J)$-element $(s, f(s))$ of $I \bowtie^f J$. Let $a, b \in R$ such that $ab \in I$ and $sa \notin \sqrt{0_R}$. Then $(a, f(a))(b, f(b)) \in I \bowtie^f J$ and clearly $(s, f(s))(a, f(a)) \notin \sqrt{0_{R \bowtie^f J}}$. Hence, $(s, f(s))(b, f(b)) \in I \bowtie^f J$ and so $sb \in I$. Thus, s is an S-element of I and I is an S-n-ideal of R.

Now, suppose $J \subseteq \sqrt{0_{R'}}$. We prove (3)$\Rightarrow$(1). Suppose s is an S-element of I and let $(a, f(a)+j_1)(b, f(b)+j_2) = (ab, (f(a)+j_1)(f(b)+j_2)) \in I \bowtie^f J$ for $(a, f(a)+j_1), (b, f(b)+j_1) \in R \bowtie^f J$. If $(s, f(s))(a, f(a)+j_1) \notin \sqrt{0_{R \bowtie^f J}} = \sqrt{0_R} \bowtie^f J$, then $sa \notin \sqrt{0_R}$. Since $ab \in I$, we conclude that $sb \in I$ and so $(s, f(s))(b, f(b)+j_2) \in I \bowtie^f J$. Thus, $(s, f(s))$ is a W-element of $I \bowtie^f J$ and $I \bowtie^f J$ is a W-n-ideal of $R \bowtie^f J$. □

Corollary 5. *Consider the amalgamation of rings* R *and* R' *along the ideal* $J \subseteq \sqrt{0_{R'}}$ *of* R' *with respect to a homomorphism* f*. Let* S *be a multiplicatively closed subset of* R*. The* $(S \bowtie^f J)$*-*n*-ideals of* $R \bowtie^f J$ *containing* $\{0\} \times J$ *are of the form* $I \bowtie^f J$ *where* I *is a* S*–*n*-ideal of* R*.*

Proof. From Theorem 5, $I \bowtie^f J$ is a $(S \bowtie^f J)$-n-ideal of $R \bowtie^f J$ for any S-n-ideal I of R. Let K be a $(S \bowtie^f J)$-n-ideal of $R \bowtie^f J$ containing $\{0\} \times J$. Consider the surjective homomorphism $\varphi : R \bowtie^f J \to R$ defined by $\varphi(a, f(a) + j) = a$ for all $(a, f(a) + j) \in R \bowtie^f J$. Since $Ker(\varphi) = \{0\} \times J \subseteq K$, $I := \varphi(K)$ is a S-n-ideal of R by Proposition 12. Since $\{0\} \times J \subseteq K$, we conclude that $K = I \bowtie^f J$. □

Let T be a multiplicatively closed subset of R'. Then clearly, the set $\bar{T}^f = \{(s, f(s) + j) : s \in R,\ j \in J,\ f(s) + j \in T\}$ is a multiplicatively closed subset of $R \bowtie^f J$.

Theorem 6. *Consider the amalgamation of rings R and R' along the ideals J of R' with respect to an epimorphism f. Let K be an ideal of R' and T be a multiplicatively closed subset of R' disjoint with K. If $\bar{K}^f$ is a $\bar{T}^f$-n-ideal of $R \bowtie^f J$, then K is a T-n-ideal of R'. The converse is true if $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$.*

Proof. First, note that $T \cap K = \phi$ if and only if $\bar{T}^f \cap \bar{K}^f = \phi$. Suppose $\bar{K}^f$ is a $\bar{T}^f$-n-ideal of $R \bowtie^f J$ and $(s, f(s) + j)$ is some $\bar{T}^f$-element of $\bar{K}^f$. Let a',$b' \in R'$ such that $a'b' \in K$ and choose $a, b \in R$ where $f(a) = a'$ and $b = f(b')$. Then $(a, f(a)), (b, f(b)) \in R \bowtie^f J$ with $(a, f(a))(b, f(b)) = (ab, f(ab)) \in \bar{K}^f$. By assumption, we have either $(s, f(s) + j)(a, f(a)) = (sa, (f(s) + j)f(a)) \in \sqrt{0_{R \bowtie^f J}}$ or $(s, f(s) + j)(b, f(b)) = (sb, (f(s) + j)f(b)) \in \bar{K}^f$. Thus, $f(s) + j \in T$ and clearly, $(f(s)+j)f(a) \in \sqrt{0_{R'}}$ or $(f(s)+j)f(b) \in K$. It follows that K is a T-n-ideal of R'. Now, suppose K is a T-n-ideal of R', $t = f(s)$ is a T-element of K, $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$. Let $(a, f(a) + j_1)(b, f(b) + j_2) = (ab, (f(a) + j_1)(f(b) + j_2)) \in \bar{K}^f$ for $(a, f(a) + j_1), (b, f(b) + j_2) \in R \bowtie^f J$. Then $(f(a) + j_1)(f(b) + j_2) \in K$ and so $f(s)(f(a) + j_1) \in \sqrt{0_{R'}}$ or $f(s)(f(b) + j_2) \in K$. Suppose $f(s)(f(a) + j_1) \in \sqrt{0_{R'}}$. Since $J \subseteq \sqrt{0_{R'}}$, then $f(sa) \in \sqrt{0_{R'}}$ and so $(sa)^m \in Ker(f) \subseteq \sqrt{0_R}$ for some integer m. Hence, $sa \in \sqrt{0_R}$ and $(s, f(s))(a, f(a) + j_1) \in \sqrt{0_{R \bowtie^f J}}$. If $f(s)(f(b) + j_2) \in K$, then clearly, $(s, f(s))(b, f(b) + j_2) \in \bar{K}^f$. Therefore, $\bar{K}^f$ is a $\bar{T}^f$-n-ideal of $R \bowtie^f J$ as needed. □

In particular, $S \times f(S)$ is a multiplicatively closed subset of $R \bowtie^f J$ for any multiplicatively closed subset S of R. Hence, we have the following corollary of Theorem 6.

Corollary 6. *Let R, R', J, S and f be as in Theorem 5. Let K be an ideal of R' and $T = f(S)$. Consider the following statements.*

(1) $\bar{K}^f$ is a $(S \times T)$-n-ideal of $R \bowtie^f J$.
(2) $\bar{K}^f$ is a $\bar{T}^f$-n-ideal of $R \bowtie^f J$.
(3) K is a T-n-ideal of R.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Moreover, if $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$, then the statements are equivalent.

We note that if $J \nsubseteq \sqrt{0_{R'}}$, then the equivalences in Theorems 5 and 6 are not true in general.

Example 5. *Let* $R = \mathbb{Z}$, $I = \langle 0 \rangle = K$, $J = \langle 3 \rangle \nsubseteq \sqrt{0_{\mathbb{Z}}}$ *and* $S = \{1\} = T$. *We have* $I \bowtie J = \{(0, 3n) : n \in \mathbb{Z}\}$, $\bar{K} = \{(3n, 0) : n \in \mathbb{Z}\}$, $S \bowtie J = \{(1, 3n+1) : n \in \mathbb{Z}\}$, $\bar{T} = \{(1-3n, 1) : n \in \mathbb{Z}\}$ *and* $\sqrt{0_{R \bowtie J}} = \{(0, 0)\}$.

1. (1) I is a S-n-ideal of R but $I \bowtie J$ is not a $(S \bowtie J)$-n-ideal of $R \bowtie J$. Indeed, we have $(0,3), (1,4) \in R \bowtie J$ with $(0,3)(1,4) = (0,12) \in I \bowtie J$. But $(1, 3n+1)(0,3) \notin \sqrt{0_{R \bowtie J}}$ and $(1, 3n+1)(1,4) \notin I \bowtie J$ for all $n \in \mathbb{Z}$.
2. (2) K is a T-n-ideal of R but $\bar{K}$ is not a $\bar{T}$-n-ideal of $R \bowtie J$. For example, $(-3,0), (-4,-1) \in R \bowtie J$ with $(-3,0)(-4,-1) = (12,0) \in \bar{K}$. However, $(1-3n, 1)(-3,0) \notin \sqrt{0_{R \bowtie J}}$ and $(1-3n,1)(-4,-1) \notin \bar{K}$ for all $n \in \mathbb{Z}$.

By taking $S = \{1\}$ in Theorem 5 and Corollary 6, we get the following particular case.

Corollary 7. *Let* R, R', J, I, K *and* f *be as in Theorems 5 and 6.*

1. (1) If $I \bowtie^f J$ is an n-ideal of $R \bowtie^f J$, then I is an n-ideal of R. Moreover, the converse is true if $J \subseteq \sqrt{0_{R'}}$.
2. (2) If $\bar{K}^f$ is an n-ideal of $R \bowtie^f J$, then K is an n-ideal of R'. Moreover, the converse is true if $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$.

Corollary 8. *Let* R, R',I, J, K, S *and* T *be as in Theorems 5 and 6.*

1. (1) If $I \bowtie J$ is a $(S \bowtie J)$-n-ideal of $R \bowtie J$, then I is a S–n-ideal of R. Moreover, the converse is true if $J \subseteq \sqrt{0_{R'}}$.
2. (2) If $\bar{K}$ is a $\bar{T}$-n-ideal of $R \bowtie J$, then K is a T-n-ideal of R'. The converse is true if $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$.

As a generalization of S-n-ideals to modules, in the following we define the notion of S -n-submodules which may inspire the reader for the other work.

Definition 2. *Let* S *be a multiplicatively closed subset of a ring* R, *and let* M *be a unital* R*-module. A submodule* N *of* M *with* $(N :_R M) \cap S = \emptyset$ *is called an* S *-n-submodule if there is an* $s \in S$ *such that* $am \in N$ *implies* $sa \in \sqrt{(0 :_R M)}$ *or* $sm \in N$ *for all* $a \in R$ *and* $m \in M$.

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BERTRAND PARTNER P-TRAJECTORIES IN THE EUCLIDEAN 3-SPACE E^3

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Abstract. The concept of a pair of curves, called as Bertrand partner curves, was introduced by Bertrand in 1850. Bertrand partner curves have been studied widely in the literature from past to present. In this study, we take into account of the concept of Bertrand partner trajectories according to Positional Adapted Frame (PAF) for the particles moving in 3-dimensional Euclidean space. Some characterizations are given for these trajectories with the aid of the PAF elements. Then, we obtain some special cases of these trajectories. Moreover, we provide a numerical example.

1. Introduction

The theory of curves is one of the extensive fields of study for especially differential geometry, and in the existing literature, a great number of studies have been done because of the fact that this topic is attached to the attention of a great deal of researchers. This theory investigates the geometric property of the plane and space curves by means of calculus methods. The moving frames can be seen most important structures in analyzing the calculus of curves.

Until today, many authors have been used the moving frames to investigate many special curves. For example spherical curves, Mannheim curve couple, Bertrand curve couple, involute-evolute curve couple are discussed by using the moving frames. One of these moving frames called as Positional Adapted Frame (PAF) was introduced by Özen and Tosun in 2021. The authors defined this moving

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frame for the trajectories having non-vanishing angular momentum in Euclidean 3-space [15]. There can be found some other studies [11, 16, 19] which are performed by considering this frame.

Bertrand curve couple is one of the most popular special type curve couples. The principal normal line of one of these partner curves coincides with the principal normal line of the other partner curve at the corresponding points of these curves. This definition was given by French mathematician Joseph Louis François Bertrand in 1850 [1]. In this study, Bertrand also characterized this curve with respect to its curvature and torsion. By following the steps similar to those of Bertrand, this topic was expanded to different moving frames. For example, the studies [21], [14] and [7] expanded this topic to the type-2 Bishop frame, Darboux frame and q-frame, respectively. Also, many mathematicians presented various studies about the concept of Bertrand curve couple with different perspectives. Some of them can be found in [3, 10, 12, 17, 20]. In this study, we will consider this topic with respect to the Positional Adapted Frame.

This study is organized as follows. In Section 2, we review some required information to understand the ensuing section. In Section 3, we deal with Bertrand partner trajectories according to Positional Adapted Frame in 3-dimensional Euclidean space. We call these trajectories as Bertrand partner P-trajectories. We examine the relationships between the PAF elements of the aforesaid partners. Also, we give the relations between the Serret-Frenet basis vectors of Bertrand partner P-trajectories. Moreover, we get the necessary conditions in terms of the PAF curvatures of other to be an osculating curve for one of these partners. Lastly, we provide a numerical example so that the readers can visualize the Bertrand partner P-trajectories.

2. Basic Concepts

In this section, we have reviewed some required and fundamental concepts to disambiguate the ensuing section of the paper.

In Euclidean 3-space E^3, let $\boldsymbol{U} = (u_1, u_2, u_3), \boldsymbol{V} = (v_1, v_2, v_3) \in \mathbb{R}^3$ be given. The standard dot product of these vectors and the norm of $\boldsymbol{U}$ are given as $\langle \boldsymbol{U}, \boldsymbol{V} \rangle = u_1v_1 + u_2v_2 + u_3v_3$ and $\|\boldsymbol{U}\| = \sqrt{\langle \boldsymbol{U}, \boldsymbol{U} \rangle}$, respectively. A differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$ is called as a unit speed curve if $\left\| \frac{d\alpha}{ds} \right\| = 1$ holds for each $s \in I$. In that case, s is called as arc-length parameter of the curve α. If the derivative of a differentiable curve never vanishes along this curve, it is said to be a regular curve. Any regular curve always has a parameterization such that it will be a unit speed curve [18]. Note that the symbol prime "$\prime$" will be used to indicate the differentiation according to the arc-length parameter s in the rest of the paper.

Let us take into consideration a point particle P of a constant mass moves on a unit speed regular curve $\alpha = \alpha(s)$. The base vectors of the Serret-Frenet frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ of α are defined by the equations $\mathbf{T}(s) = \alpha'(s)$,

$\mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$, $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$. The base vectors $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are called as unit tangent vector, principal normal vector and binormal vector, respectively. The Serret-Frenet derivative formulas are expressed as in the following:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} \tag{1}$$

where $\kappa(s) = \|\mathbf{T}'(s)\|$ is the curvature and $\tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle$ is the torsion [18]. We must emphasize that the curvature κ never vanishes for the curves we will consider in this paper.

On the other hand, it is well known that the vector product of the position vector $\mathbf{x} = \langle \alpha(s), \mathbf{T}(s) \rangle \mathbf{T}(s) + \langle \alpha(s), \mathbf{N}(s) \rangle \mathbf{N}(s) + \langle \alpha(s), \mathbf{B}(s) \rangle \mathbf{B}(s)$ and the linear momentum vector $\mathbf{p}(t) = m\left(\frac{ds}{dt}\right)\mathbf{T}(s)$ of the particle P yields the angular momentum vector of P about the origin as $\mathbf{H}^O = m\langle \alpha(s), \mathbf{B}(s) \rangle \left(\frac{ds}{dt}\right)\mathbf{N}(s) - m\langle \alpha(s), \mathbf{N}(s) \rangle \left(\frac{ds}{dt}\right)\mathbf{B}(s)$. Here m and t denote the constant mass of P and the time [4,9]. Let this vector not equal to zero vector during the motion of P. Making this supposition assures that the coefficient functions $\langle \alpha(s), \mathbf{N}(s) \rangle$ and $\langle \alpha(s), \mathbf{B}(s) \rangle$ of the position vector $\mathbf{x}$ do not equal to zero at the same time. Then, one can easily say that the tangent line of $\alpha = \alpha(s)$ does not pass through the origin along the trajectory of P. Take into account of the vector whose initial point is the foot of the perpendicular (from origin to instantaneous rectifying plane $Sp\{\mathbf{T}(s), \mathbf{B}(s)\}$) and endpoint is the foot of the perpendicular (from origin to instantaneous osculating plane $Sp\{\mathbf{T}(s), \mathbf{N}(s)\}$). The unit vector in direction of the equivalent of the aforementioned vector at the point $\alpha(s)$ determines the PAF basis vector $\mathbf{Y}(s)$. The other PAF basis vector $\mathbf{M}(s)$ is obtained by the vector product $\mathbf{Y}(s) \wedge \mathbf{T}(s)$. Consequently, the vectors

$$\begin{aligned}
\mathbf{T}(s) &= \mathbf{T}(s), \\
\mathbf{M}(s) &= \frac{\langle \alpha(s), \mathbf{B}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}}\mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{N}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}}\mathbf{B}(s), \\
\mathbf{Y}(s) &= \frac{\langle -\alpha(s), \mathbf{N}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}}\mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{B}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}}\mathbf{B}(s),
\end{aligned}$$

form the Positional Adapted Frame $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s)\}$ (see [15] for more details on PAF).

The relation between the Serret-Frenet frame and PAF is as in the following:

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{M}(s) \\ \mathbf{Y}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega(s) & -\sin\Omega(s) \\ 0 & \sin\Omega(s) & \cos\Omega(s) \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} \tag{2}$$

where $\Omega(s)$ is the angle between the vector $\mathbf{B}(s)$ and the vector $\mathbf{Y}(s)$ which is positively oriented from the vector $\mathbf{B}(s)$ to vector $\mathbf{Y}(s)$. On the other hand, the derivative formulas of PAF are presented as follows [15]:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{M}'(s) \\ \mathbf{Y}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & k_3(s) \\ -k_2(s) & -k_3(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{M}(s) \\ \mathbf{Y}(s) \end{pmatrix} \tag{3}$$

where

$$\begin{aligned} k_1(s) &= \kappa(s)\cos\Omega(s), \\ k_2(s) &= \kappa(s)\sin\Omega(s), \\ k_3(s) &= \tau(s) - \Omega'(s). \end{aligned}$$

Here, the rotation angle $\Omega(s)$ is determined by means of the following equation:

$$\Omega(s) = \begin{cases} \arctan\left(-\frac{\langle \alpha(s), \mathbf{N}(s)\rangle}{\langle \alpha(s), \mathbf{B}(s)\rangle}\right) \;\; if \;\; \langle \alpha(s), \mathbf{B}(s)\rangle > 0, \\ \arctan\left(-\frac{\langle \alpha(s), \mathbf{N}(s)\rangle}{\langle \alpha(s), \mathbf{B}(s)\rangle}\right) + \pi \;\; if \;\; \langle \alpha(s), \mathbf{B}(s)\rangle < 0, \\ -\frac{\pi}{2} \;\; if \;\; \langle \alpha(s), \mathbf{B}(s)\rangle = 0\,, \;\; \langle \alpha(s), \mathbf{N}(s)\rangle > 0, \\ \frac{\pi}{2} \;\; if \;\; \langle \alpha(s), \mathbf{B}(s)\rangle = 0\,, \;\; \langle \alpha(s), \mathbf{N}(s)\rangle < 0. \end{cases}$$

The elements of the set $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s), k_1(s), k_2(s), k_3(s)\}$ are called as PAF apparatuses of $\alpha = \alpha\,(s)$ [15].

Note that PAF is a generic adapted moving frame just like Bishop frame [2], Darboux frame [6], B-Darboux frame [8] etc. Generic adapted moving frames are obtained from Serret-Frenet frame by a rotation (see [5] for more details on generic adapted moving frame). Since the analytical approach is used to determine the rotation angle in PAF, the rotation angle can be easily determined, while in many other moving frames, the determination of the angle is based on integral calculations. These calculations often cause difficulties for researchers. Also, PAF enables the researchers to study the kinematics of a moving particle and the differential geometry of this particle at the same time. Moreover, PAF contains information about the position vector of the moving particle. When viewed from this aspect, it is a useful tool for the researchers studying on kinematics and inverse kinematics.

Now we give the definition of the osculating curve in 3-dimensional Euclidean space since we will discuss this topic in the next section. A curve $\beta = \beta(s)$ is called as osculating curve if its position vector always lies in its osculating plane. One can find more details on this topic in [13].

Theorem 1. *[15] Let $\alpha = \alpha(s)$ be the unit speed parameterization of the trajectory. Then, α is an osculating curve if and only if $k_1 = 0$.*

More details can be found in the studies [11, 15, 16, 19] for Positional Adapted Frame (PAF).

3. Bertrand Partner P-Trajectories

In this section, we introduce the Bertrand partner P-trajectories and give some characterizations of them. Furthermore, we provide an example in order to illustrate this topic.

Definition 1. *Let Q and $\breve{Q}$ be the moving point particles of constant masses in the Euclidean 3-space. Show the unit speed parameterization of the trajectories of Q and $\breve{Q}$ with $\alpha = \alpha(s)$ and $\breve{\alpha} = \breve{\alpha}(\breve{s})$, respectively. Let the PAF apparatus of the trajectories α and $\breve{\alpha}$ be represented by $\{\mathbf{T}, \mathbf{M}, \mathbf{Y}, k_1, k_2, k_3\}$ and $\left\{\breve{\mathbf{T}}, \breve{\mathbf{M}}, \breve{\mathbf{Y}}, \breve{k}_1, \breve{k}_2, \breve{k}_3\right\}$, respectively. If the PAF base vector $\mathbf{M}$ coincides with the PAF base vector $\breve{\mathbf{M}}$ at the corresponding points of the trajectories α and $\breve{\alpha}$, in this case α is said to be a Bertrand partner P-trajectory of $\breve{\alpha}$. Moreover, the pair $\{\alpha, \breve{\alpha}\}$ is called as a Bertrand P-pair.*

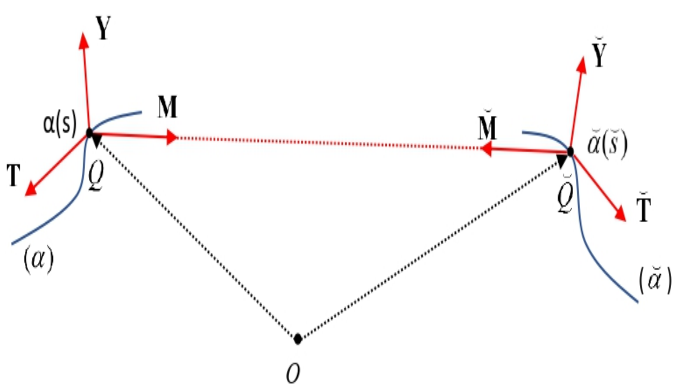

Figure 1. Bertrand partner P-trajectories

According to the definition of Bertrand P-pair, we get the following matrix equation

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{M} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \breve{\mathbf{T}} \\ \breve{\mathbf{M}} \\ \breve{\mathbf{Y}} \end{pmatrix} \tag{4}$$

where ϕ is the angle between the tangent vectors $\mathbf{T}$ and $\breve{\mathbf{T}}$.

Theorem 2. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be any Bertrand P-pair in E^3. In that case, the distance between the corresponding points of α and $\breve{\alpha}$ is constant.*

Proof. By the definition of Bertrand P-trajectories, we can write:

$$\alpha(s) = \breve{\alpha}(\breve{s}) + \psi(\breve{s})\,\breve{\mathbf{M}}(\breve{s}) \tag{5}$$

where ψ is a real valued smooth function of $\breve{s}$ (see Figure 1). By taking the derivative of the equation (5) with respect to $\breve{s}$ and considering the PAF derivative formulas (3), we get:

$$\mathbf{T}\frac{ds}{d\breve{s}} = \left(1 - \psi\breve{k}_1\right)\breve{\mathbf{T}} + \psi'\breve{\mathbf{M}} + \psi\breve{k}_3\breve{\mathbf{Y}}. \tag{6}$$

Since $\mathbf{T}$, $\breve{\mathbf{T}}$ and $\breve{\mathbf{Y}}$ are orthogonal to $\breve{\mathbf{M}}$, and also $\breve{\mathbf{M}}$ is a unit vector, we have $\psi' = 0$ with the help of the inner product. Therefore, ψ is a non-zero constant and the equation (6) becomes:

$$\mathbf{T}\frac{ds}{d\breve{s}} = \left(1 - \psi\breve{k}_1\right)\breve{\mathbf{T}} + \psi\breve{k}_3\breve{\mathbf{Y}}. \tag{7}$$

In the light of these results, the distance between the corresponding points of the trajectories can be given as:

$$d(\alpha(s), \breve{\alpha}(\breve{s})) = \|\alpha(s) - \breve{\alpha}(\breve{s})\| = \left\|\psi\breve{\mathbf{M}}\right\| = \mid \psi \mid .$$

Therefore, we can say that the distance between each corresponding points of α and $\breve{\alpha}$ is constant. □

Theorem 3. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be any Bertrand P-pair in E^3. Then, the equation*

$$\frac{d}{ds}(\cos\phi) = k_2\left\langle \mathbf{Y}, \breve{\mathbf{T}}\right\rangle + \breve{k}_2\frac{d\breve{s}}{ds}\left\langle \mathbf{T}, \breve{\mathbf{Y}}\right\rangle$$

is satisfied.

Proof. Since ϕ is the angle between the tangent vectors $\mathbf{T}$ and $\breve{\mathbf{T}}$, one can easily write $\left\langle \mathbf{T}, \breve{\mathbf{T}}\right\rangle = \|\mathbf{T}\|\left\|\breve{\mathbf{T}}\right\|\cos\phi = \cos\phi$. Let us differentiate this equation with respect to s. Thus, we get:

$$\begin{aligned}\frac{d}{ds}(\cos\phi) &= \frac{d}{ds}\left\langle \mathbf{T}, \breve{\mathbf{T}}\right\rangle \\ &= \left\langle k_1\mathbf{M} + k_2\mathbf{Y}, \breve{\mathbf{T}}\right\rangle + \left\langle \mathbf{T}, (\breve{k}_1\breve{\mathbf{M}} + \breve{k}_2\breve{\mathbf{Y}})\frac{d\breve{s}}{ds}\right\rangle.\end{aligned}$$

This equation gives us the desired result. □

Corollary 1. *The angles between the tangent vectors at the corresponding points of a Bertrand P-pair is generally not constant.*

Theorem 4. *Let $\{\alpha = \alpha(s),\ \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, the following relations*

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{M} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} \left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds} & 0 & \psi\breve{k}_3\frac{d\breve{s}}{ds} \\ 0 & 1 & 0 \\ -\psi\breve{k}_3\frac{d\breve{s}}{ds} & 0 & \left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds} \end{pmatrix} \begin{pmatrix} \breve{\mathbf{T}} \\ \breve{\mathbf{M}} \\ \breve{\mathbf{Y}} \end{pmatrix} \tag{8}$$

are satisfied between the PAF vectors of α and $\breve{\alpha}$.

Proof. Suppose that $\{\alpha, \breve{\alpha}\}$ is a Bertrand P-pair in E^3. By using the equations (4) and (7), we get:

$$\cos\phi\frac{ds}{d\breve{s}}\breve{\mathbf{T}} - \sin\phi\frac{ds}{d\breve{s}}\breve{\mathbf{Y}} = \left(1-\psi\breve{k}_1\right)\breve{\mathbf{T}} + \psi\breve{k}_3\breve{\mathbf{Y}}.$$

The last equation gives us the following:

$$\begin{cases} \cos\phi = \left(1-\psi\breve{k}_1\right)\dfrac{d\breve{s}}{ds}, \\ \sin\phi = -\psi\breve{k}_3\dfrac{d\breve{s}}{ds}. \end{cases} \tag{9}$$

If we substitute the equation (9) in the equation (4), we obtain the desired result. □

Corollary 2. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, we have:*

$$\tan\phi = \frac{-\psi\breve{k}_3}{1-\psi\breve{k}_1} \tag{10}$$

where ϕ is the angle between $\mathbf{T}$ and $\breve{\mathbf{T}}$.

Corollary 3. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then,*

$$\int \cos\phi ds + \psi \int \breve{k}_1 d\breve{s} = \breve{s} + c_1$$

where c_1 denotes the integration constant.

Corollary 4. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, the equality*

$$\int \sin\phi ds + \psi \int \breve{k}_3 d\breve{s} = 0$$

holds.

Theorem 5. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3 and their Serret-Frenet apparatuses be $\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau\}$ and $\left\{\breve{\mathbf{T}}, \breve{\mathbf{N}}, \breve{\mathbf{B}}, \breve{\kappa}, \breve{\tau}\right\}$, respectively.*

Then, the relations between the Serret-Frenet vectors of this pair are given as:

$$\begin{aligned}
\breve{\mathbf{T}} &= \left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds}\mathbf{T} - \psi\breve{k}_3\sin\Omega\frac{d\breve{s}}{ds}\mathbf{N} - \psi\breve{k}_3\cos\Omega\frac{d\breve{s}}{ds}\mathbf{B},\\
\breve{\mathbf{N}} &= \psi\breve{k}_3\sin\breve{\Omega}\frac{d\breve{s}}{ds}\mathbf{T} + \left(\cos\breve{\Omega}\cos\Omega + \left(1-\psi\breve{k}_1\right)\sin\breve{\Omega}\sin\Omega\frac{d\breve{s}}{ds}\right)\mathbf{N}\\
&\quad + \left(-\cos\breve{\Omega}\sin\Omega + \left(1-\psi\breve{k}_1\right)\sin\breve{\Omega}\cos\Omega\frac{d\breve{s}}{ds}\right)\mathbf{B},\\
\breve{\mathbf{B}} &= \psi\breve{k}_3\cos\breve{\Omega}\frac{d\breve{s}}{ds}\mathbf{T} + \left(-\sin\breve{\Omega}\cos\Omega + \left(1-\psi\breve{k}_1\right)\cos\breve{\Omega}\sin\Omega\frac{d\breve{s}}{ds}\right)\mathbf{N}\\
&\quad + \left(\sin\breve{\Omega}\sin\Omega + \left(1-\psi\breve{k}_1\right)\cos\breve{\Omega}\cos\Omega\frac{d\breve{s}}{ds}\right)\mathbf{B},
\end{aligned}$$

where Ω is the angle between the vectors $\mathbf{B}$ and $\mathbf{Y}$ and also, $\breve{\Omega}$ is the angle between the vectors $\breve{\mathbf{B}}$ and $\breve{\mathbf{Y}}$.

Proof. Using the equation (2), we can write:

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{M}\\ \mathbf{Y}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\Omega & -\sin\Omega\\ 0 & \sin\Omega & \cos\Omega\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix} \tag{11}$$

and also

$$\begin{pmatrix}\breve{\mathbf{T}}\\ \breve{\mathbf{N}}\\ \breve{\mathbf{B}}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\breve{\Omega} & \sin\breve{\Omega}\\ 0 & -\sin\breve{\Omega} & \cos\breve{\Omega}\end{pmatrix}\begin{pmatrix}\breve{\mathbf{T}}\\ \breve{\mathbf{M}}\\ \breve{\mathbf{Y}}\end{pmatrix}. \tag{12}$$

On the other hand, by using the equation (8), we get:

$$\begin{pmatrix}\breve{\mathbf{T}}\\ \breve{\mathbf{M}}\\ \breve{\mathbf{Y}}\end{pmatrix} = \begin{pmatrix}\left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds} & 0 & -\psi\breve{k}_3\frac{d\breve{s}}{ds}\\ 0 & 1 & 0\\ \psi\breve{k}_3\frac{d\breve{s}}{ds} & 0 & \left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{M}\\ \mathbf{Y}\end{pmatrix}. \tag{13}$$

If the equation (13) is substituted into the equation (12), then

$$\begin{pmatrix}\breve{\mathbf{T}}\\ \breve{\mathbf{N}}\\ \breve{\mathbf{B}}\end{pmatrix} = \begin{pmatrix}\left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds} & 0 & -\psi\breve{k}_3\frac{d\breve{s}}{ds}\\ \psi\breve{k}_3\sin\breve{\Omega}\frac{d\breve{s}}{ds} & \cos\breve{\Omega} & \left(1-\psi\breve{k}_1\right)\sin\breve{\Omega}\frac{d\breve{s}}{ds}\\ \psi\breve{k}_3\cos\breve{\Omega}\frac{d\breve{s}}{ds} & -\sin\breve{\Omega} & \left(1-\psi\breve{k}_1\right)\cos\breve{\Omega}\frac{d\breve{s}}{ds}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{M}\\ \mathbf{Y}\end{pmatrix} \tag{14}$$

is found. By using the equation (11) in the equation (14), we complete the proof. □

Theorem 6. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, the following relations*

(1) $k_1 = \dfrac{\breve{k}_1 - \psi\breve{k}_1^{\,2} - \psi\breve{k}_3^{\,2}}{1 - 2\psi\breve{k}_1 + \psi^2\left(\breve{k}_1^{\,2} + \breve{k}_3^{\,2}\right)}$,

(2) $\breve{k}_1 = \dfrac{k_1 - \eta k_1^2 - \eta k_3^2}{1 - 2\eta k_1 + \eta^2\left(k_1^2 + k_3^2\right)}$,

are satisfied between k_1*,* k_3*,* $\breve{k}_1$ *and* $\breve{k}_3$*. Here* η *is a constant satisfying* $\mid \eta \mid = \mid \psi \mid$*.*

Proof. (1) Assume that $\{\alpha, \breve{\alpha}\}$ is a Bertrand P-pair in E^3. Via the equation (9) and the equality $\cos^2\phi + \sin^2\phi = 1$, we get:

$$\left(\frac{d\breve{s}}{ds}\right)^2\left(\left(1 - \psi\breve{k}_1\right)^2 + \psi^2\breve{k}_3^{\,2}\right) = 1.$$

Hence, we have:

$$\left(\frac{ds}{d\breve{s}}\right)^2 = 1 - 2\psi\breve{k}_1 + \psi^2\left(\breve{k}_1^{\,2} + \breve{k}_3^{\,2}\right). \tag{15}$$

On the other hand, if we differentiate the equation (7) with respect to $\breve{s}$ and use the PAF derivative formulas, we obtain:

$$\begin{aligned}\frac{d^2s}{d\breve{s}^2}\mathbf{T} + k_1\left(\frac{ds}{d\breve{s}}\right)^2\mathbf{M} + k_2\left(\frac{ds}{d\breve{s}}\right)^2\mathbf{Y} = &\left(-\psi\breve{k}_1{}' - \psi\breve{k}_2\breve{k}_3\right)\breve{\mathbf{T}}\\ &+ \left(\breve{k}_1\left(1 - \psi\breve{k}_1\right) - \psi\breve{k}_3^{\,2}\right)\breve{\mathbf{M}}\\ &+ \left(\breve{k}_2\left(1 - \psi\breve{k}_1\right) + \psi\breve{k}_3{}'\right)\breve{\mathbf{Y}}.\end{aligned} \tag{16}$$

By taking into consideration the equation (16) and utilizing the definition of Bertrand P-pair, we get:

$$k_1\left(\frac{ds}{d\breve{s}}\right)^2 = \left(1 - \psi\breve{k}_1\right)\breve{k}_1 - \psi\breve{k}_3^{\,2}. \tag{17}$$

From the equations (15) and (17), one can easily see the desired result.

(2) According to the definition of the Bertrand P-pair, we can write:

$$\breve{\alpha}\left(\breve{s}\right) = \alpha\left(s\right) + \eta\mathbf{M}\left(s\right)$$

where η is a constant satisfying $\mid \eta \mid = \mid \psi \mid$ (see Figure 1). Let us take the derivative of this equation with respect to s twice. In that case, we obtain:

$$\breve{\mathbf{T}}\frac{d\breve{s}}{ds} = \left(1 - \eta k_1\right)\mathbf{T} + \eta k_3\mathbf{Y} \tag{18}$$

and

$$\begin{aligned}\frac{d^2\breve{s}}{ds^2}\breve{\mathbf{T}} + \breve{k}_1\left(\frac{d\breve{s}}{ds}\right)^2\breve{\mathbf{M}} + \breve{k}_2\left(\frac{d\breve{s}}{ds}\right)^2\breve{\mathbf{Y}} = &\left(-\eta k_1' - \eta k_2 k_3\right)\mathbf{T}\\ &+ \left(k_1\left(1 - \eta k_1\right) - \eta k_3^2\right)\mathbf{M}\\ &+ \left(k_2\left(1 - \eta k_1\right) + \eta k_3'\right)\mathbf{Y}.\end{aligned} \tag{19}$$

On the other hand, we can write $\breve{\mathbf{T}} = \cos\phi\,\mathbf{T} + \sin\phi\,\mathbf{Y}$ by the equation (4). Then, by using the equation (18), we find:

$$\cos\phi\frac{d\breve{s}}{ds}\mathbf{T} + \sin\phi\frac{d\breve{s}}{ds}\mathbf{Y} = (1-\eta k_1)\,\mathbf{T} + \eta k_3\mathbf{Y}.$$

Hence, we get the equations $\cos\phi\frac{d\breve{s}}{ds} = 1-\eta k_1$ and $\sin\phi\frac{d\breve{s}}{ds} = \eta k_3$. These equations give us the equation:

$$\left(\frac{d\breve{s}}{ds}\right)^2 = 1 - 2\eta k_1 + \eta^2\left(k_1^2 + k_3^2\right). \tag{20}$$

Moreover, by taking the inner product of the vectors at the right and left sides of the equation (19) with the vector $\mathbf{M}$, we have:

$$\breve{k}_1\left(\frac{d\breve{s}}{ds}\right)^2 = k_1 - \eta k_1^2 - \eta k_3^2. \tag{21}$$

Therefore, we obtain the desired result by using the equation (20).

□

Thanks to the Theorem 1 and Theorem 6, we can attain the following corollaries.

Corollary 5. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in Euclidean 3-space E^3. If $\breve{k}_1 = \breve{k}_3 = 0$, then $k_1 = 0$.*

Corollary 6. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in Euclidean 3-space E^3. If $k_1 = k_3 = 0$, then $\breve{k}_1 = 0$.*

Corollary 7. *Let $\{\alpha = \alpha(s),\ \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, the following mathematical expressions hold:*

(1) *α is an osculating curve if and only if* $\frac{\breve{k}_1 - \psi\breve{k}_1{}^2 - \psi\breve{k}_3{}^2}{1-2\psi\breve{k}_1+\psi^2\left(\breve{k}_1{}^2+\breve{k}_3{}^2\right)} = 0,$

(2) *$\breve{\alpha}$ is an osculating curve if and only if* $\frac{k_1-\eta k_1^2-\eta k_3^2}{1-2\eta k_1+\eta^2\left(k_1^2+k_3^2\right)} = 0.$

Example 1. *In the Euclidean 3-space, suppose that a point particle Q moves on the trajectory*

$$\begin{aligned}&\alpha : (0,\pi/2) \to E^3\\ &\qquad s \mapsto \alpha(s) = \left(\frac{8}{17}\cos 2s, \frac{12}{17} - \sin 2s, -\frac{15}{17}\cos 2s\right).\end{aligned} \tag{22}$$

By straightforward calculations, we get the following Serret-Frenet apparatus:

$$\begin{cases} \mathbf{T}(s) = \left(-\frac{8}{17}\sin 2s, -\cos s, \frac{15}{17}\sin 2s\right) \\ \mathbf{N}(s) = \left(-\frac{8}{17}\cos 2s, \sin 2s, \frac{15}{17}\cos 2s\right) \\ \mathbf{B}(s) = \left(-\frac{15}{17}, 0, -\frac{8}{17}\right) \end{cases} \quad \textit{and} \quad \begin{cases} \kappa(s) = 1 \\ \tau(s) = 0. \end{cases}$$

Since $\langle \alpha(s), \mathbf{B}(s)\rangle = 0$ *and* $\langle \alpha(s), \mathbf{N}(s)\rangle = -1 + \frac{12}{17}\sin 2s < 0$, *we get* $\Omega = \frac{\pi}{2}$. *Then, the elements of PAF are found as:*

$$\begin{cases} \mathbf{T}(s) = \left(-\frac{8}{17}\sin 2s, -\cos 2s, \frac{15}{17}\sin 2s\right) \\ \mathbf{M}(s) = \left(\frac{15}{17}, 0, \frac{8}{17}\right) \\ \mathbf{Y}(s) = \left(-\frac{8}{17}\cos 2s, \sin 2s, \frac{15}{17}\cos 2s\right) \end{cases} \quad \textit{and} \quad \begin{cases} k_1(s) = 0 \\ k_2(s) = 1 \\ k_3(s) = 0. \end{cases}$$

Therefore, Bertrand partner P-trajectory of α *can be given as:*

$$\breve{\alpha}(s) = \left(\frac{8}{17}\cos 2s + \eta\frac{15}{17}, \frac{12}{17} - \sin 2s, -\frac{15}{17}\cos 2s + \eta\frac{8}{17}\right) \tag{23}$$

by means of the equality $\breve{\alpha}(s) = \alpha(s) + \eta\mathbf{M}(s)$.

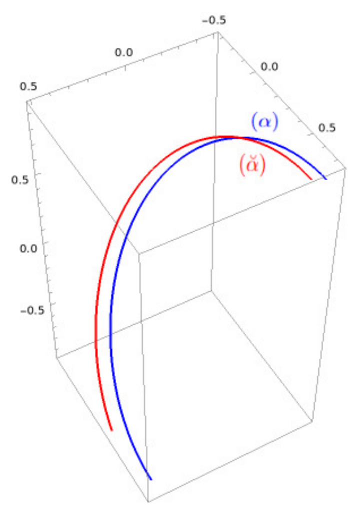

FIGURE 2. The trajectories α and $\breve{\alpha}$ given in (22) and (23)

In the Figure 2, the trajectories $\alpha = \alpha(s)$ (blue) and $\breve{\alpha} = \breve{\alpha}(s)$ (red) can be seen. Here we take $\eta = 0.1$.

On the other hand, by using the Theorem 6 and Corollary 7, we get $\breve{k}_1 = 0$. So, we can conclude that the trajectory $\breve{\alpha}$ is an osculating curve. It should be noted that the Figure 2 is drawn by utilizing the website Wolfram Mathematica (Wolfram Cloud).

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COMPARISON OF SOME DYNAMICAL SYSTEMS ON THE QUOTIENT SPACE OF THE SIERPINSKI TETRAHEDRON

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Abstract. In this paper, it is aimed to construct two different dynamical systems on the Sierpinski tetrahedron. To this end, we consider the dynamical systems on a quotient space of $\{0, 1, 2, 3\}^{\mathbb{N}}$ by using the code representations of the points on the Sierpinski tetrahedron. Finally, we compare the periodic points to investigate topological conjugacy of these dynamical systems and we conclude that they are not topologically equivalent.

1. Introduction

In the literature, there are many works to analyze the structures on the fractals [1–17]. Defining different dynamical systems on the fractals is one of these studies [3,4,8,17]. With the method given in [4], dynamical systems are naturally constructed on the self-similar sets using their iterated function systems. Moreover, there are different ways to define the dynamical systems on these sets considering their structures. With the help of the folding, expanding, translation and rotation mappings, many dynamical systems can also be obtained on the fractals as given in [17]. On the other hand, expressing the dynamical systems using the code representations of the points can provide many advantages. The utility of this situation can be seen while showing whether these systems are chaotic or not [3,17]. For this purpose, we also need to use the intrinsic metrics which are defined by means of the code representations on the related fractals. For instance, the intrinsic metric on the Sierpinski tetrahedron (ST) (see Theorem 1) is required to prove that the dynamical system, defined on the code set of ST, is chaotic [3], and it is also used to show some geometrical properties such as number of the geodesics in [9].

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In this paper, we first focus on the quotient space of the Sierpinski tetrahedron $\{0,1,2,3\}^{\mathbb{N}}/_{\sim}$. On this space, we define two dynamical systems $\{ST;G\}$ and $\{ST;T\}$ in Proposition 3 and Proposition 5 respectively. Then we compare their fixed points and deduce that they are not topologically equivalent in Remark 2. On the other hand, in Proposition 4 and Remark 1, we show that $\{ST;G\}$ is topologically equivalent to $\{ST;F\}$ which is given in [3] (see Proposition 1). Hence, we also conclude that $\{ST;G\}$ is chaotic in the sense of Devaney by the help of the topological conjugacy H.

We now recall some basic notions in the following section:

2. Preliminaries

As a fractal, the Sierpinski tetrahedron with vertices are $P_0 = (0,0,0)$, $P_1 = (1,0,0)$, $P_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $P_3 = (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3})$ is the attractor of the iterated function system (IFS) $\{\mathbb{R}^3; f_0, f_1, f_2, f_3\}$ where

$$\begin{aligned} f_0(x,y,z) &= \left(\frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z\right), \\ f_1(x,y,z) &= \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y, \frac{1}{2}z\right), \\ f_2(x,y,z) &= \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}, \frac{1}{2}z\right), \\ f_3(x,y,z) &= \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{12}, \frac{1}{2}z + \frac{\sqrt{6}}{6}\right). \end{aligned}$$

Let $ST_i = f_i(ST)$ for $i = 0,1,2,3$. It is obvious that $ST_i \cap ST_j \neq \emptyset$ for $i \neq j$ where $i,j = 0,1,2,3$ and $\bigcup_{i=0}^{3} ST_i = ST$. Suppose that σ is a word of length $k-1$ on the set $\{0,1,2,3\}$ such as $\sigma = a_1a_2a_3\ldots a_{k-1}$ where $a_i \in \{0,1,2,3\}$. Similarly, we get $ST_\sigma = f_{a_{k-1}} \circ f_{a_{k-2}} \circ \cdots \circ f_{a_1} \circ f_{a_0}(ST)$. In the Figure 1, one can see that the sub-tetrahedron ST_{313} of ST for $\sigma = 313$. Since $ST_{a_1}, ST_{a_1a_2}, ST_{a_1a_2a_3}, \ldots$ is a sequence of the nested sets such that

$$ST_{a_1} \supset ST_{a_1a_2} \supset ST_{a_1a_2a_3} \supset \ldots \supset ST_{a_1a_2\ldots a_n} \supset \ldots,$$

$\bigcap_{k=1}^{\infty} ST_\sigma$ indicates a singleton, A, from the Cantor intersection theorem. The code representations of A is the sequence $a_1a_2a_3\ldots$ where $a_i \in \{0,1,2,3\}$.

On the other hand, the intersection of the sequences $ST_\sigma, ST_{\sigma\alpha}, ST_{\sigma\alpha\beta}, ST_{\sigma\alpha\beta\beta}, \ldots$ and $ST_\sigma, ST_{\sigma\beta}, ST_{\sigma\beta\alpha}, ST_{\sigma\beta\alpha\alpha}, \ldots$ satisfying

$$ST_\sigma \supset ST_{\sigma\alpha} \supset ST_{\sigma\alpha\beta} \supset ST_{\sigma\alpha\beta\beta} \supset \ldots$$

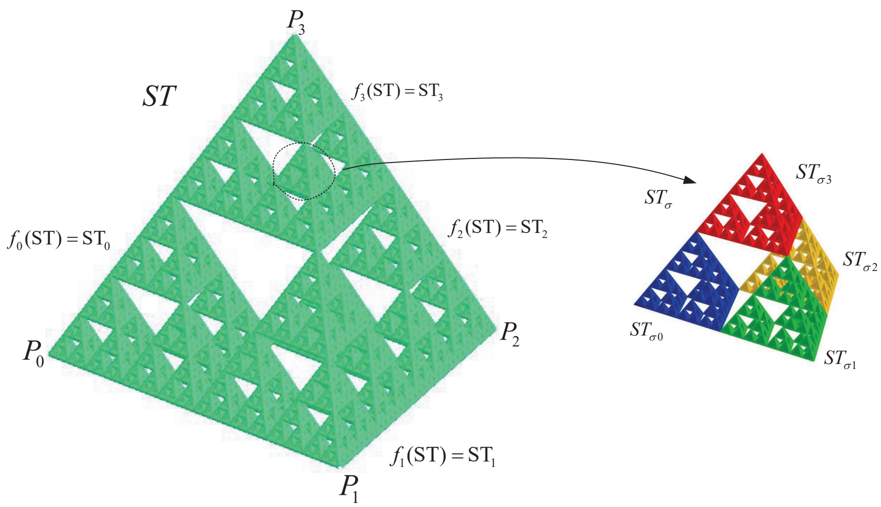

FIGURE 1. The Sierpinski tetrahedron and a small piece ST_σ of ST

and

$$ST_\sigma \supset ST_{\sigma\beta} \supset ST_{\sigma\beta\alpha} \supset ST_{\sigma\beta\alpha\alpha} \supset \ldots$$

represents the same point on ST and the code representations of these points are $\sigma\alpha\beta\beta\beta\ldots$ and $\sigma\beta\alpha\alpha\alpha\ldots$. Therefore, ST can be defined as the quotient space $\{0,1,2,3\}^{\mathbb{N}}/_\sim$ where

$c' \sim c'' \Leftrightarrow c' = c''$ or there are $c_i, \alpha, \beta \in \{0,1,2,3\}$ such that

$c' = c_1c_2\ldots c_n\alpha\beta\beta\beta\ldots, c'' = c_1c_2\ldots c_n\beta\alpha\alpha\alpha\ldots$ for an integer n.

The dynamical system, defined in [3] on this quotient space, is given with the following proposition:

Proposition 1. *Let the code representations of points X and Y of the Sierpinski tetrahedron be $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ respectively. The function $F : ST \to ST$, $F(X) = Y$ such that*

$$y_i \equiv x_{i+1} + x_1 \pmod 4 \tag{1}$$

where $x_i, y_i \in \{0,1,2,3\}$ and $i = 1,2,3,\ldots$ is a dynamical system on the code sets of the Sierpinski tetrahedon.

We also give two chaotic dynamical systems on the quotient space of the Sierpinski tetrahedron and we investigate these dynamical systems in terms of topological conjugacy.

Definition 1. *Let $\{X_1; f_1\}$ and $\{X_2; f_2\}$ be two dynamical systems. If there is a homeomorphism $\theta : X_1 \to X_2$ such that $f_2 = \theta \circ f_1 \circ \theta^{-1}$ (or that means $\forall x \in$*

X_1, $\theta(f_1(x)) = f_2(\theta(x)))$, these dynamical systems are equivalent or topologically conjugate. θ is called a topological conjugacy (see [4]).

Proposition 2. *If the dynamical systems $\{X_1; f_1\}$ and $\{X_2; f_2\}$ have the different number of $n-$periodic points for at least $n \in \mathbb{N}$, then they are not topologically conjugate (see [10]).*

Definition 2. *A dynamical system $\{X; f\}$ is chaotic in the sense of Devaney if it is sensitivite dependence on the initial condition, topologically transitive and it has density of periodic points (see [6]).*

We need a useful metric in order to investigate the dynamical systems are chaotic or not. The intrinsic metric on the quotient space of the Sierpinski tetrahedron is formulated with the following theorem:

Theorem 1. *If $a_1a_2\dots a_{k-1}a_ka_{k+1}\dots$ and $b_1b_2\dots b_{k-1}b_kb_{k+1}\dots$ are two representations of the points A and B respectively on the Sierpinski tetrahedron such that $a_i = b_i$ for $i = 1, 2, \dots, k-1$ and $a_k \neq b_k$, then the formula*

$$d(A,B) = \min\left\{ \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i}, \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\phi_i + \varphi_i}{2^i} \right\} \tag{2}$$

such that

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \qquad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases},$$

$$\gamma_i = \begin{cases} 0, & a_i = c_k \\ 1, & a_i \neq c_k \end{cases}, \qquad \delta_i = \begin{cases} 0, & b_i = c_k \\ 1, & b_i \neq c_k \end{cases},$$

$$\phi_i = \begin{cases} 0, & a_i = d_k \\ 1, & a_i \neq d_k \end{cases}, \qquad \varphi_i = \begin{cases} 0, & b_i = d_k \\ 1, & b_i \neq d_k \end{cases}$$

where $a_k \neq c_k \neq b_k$ and $a_k \neq d_k \neq b_k$ and $c_k \neq d_k$ $(a_i, b_i, c_k, d_k \in \{0, 1, 2, 3\},\ i = 1, 2, 3, \dots)$ gives the distance $d(A, B)$ between the points A and B.

This metric gives the distance of the shortest path between any points on ST.

3. A Chaotic Dynamical System on the Sierpinski Tetrahedron $\{ST; G\}$

In this section, we construct a dynamical system which is different from (1) on ST and we investigate some periodic points of this dynamical system.

Proposition 3. *Let the code representations of $X, Y \in ST$ be $x_1x_2x_3\dots$ and $y_1y_2y_3\dots$ respectively where $i = 1, 2, 3, \dots$ and $x_i, y_i \in \{0, 1, 2, 3\}$. Suppose that the function $G : ST \to ST$ is defined according to four different situations of x_1 :*

$$G(0x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 1 \\ 1, & x_{i+1} = 2 \\ 2, & x_{i+1} = 3 \\ 3, & x_{i+1} = 0 \end{cases} \quad (i \geq 1)$$

$$G(1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 1 \\ 2, & x_{i+1} = 2 \\ 3, & x_{i+1} = 3 \end{cases} \quad (i \geq 1)$$

$$G(2x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 3 \\ 1, & x_{i+1} = 0 \\ 2, & x_{i+1} = 1 \\ 3, & x_{i+1} = 2 \end{cases} \quad (i \geq 1)$$

$$G(3x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 2 \\ 1, & x_{i+1} = 3 \\ 2, & x_{i+1} = 0 \\ 3, & x_{i+1} = 1 \end{cases} \quad (i \geq 1).$$

In this case, $\{ST; G\}$ *states a dynamical system.*

Proof. We know from the hypothesis, there are four different rules in regard to the cases of x_1. If X has a unique code representation, then it is obvious that $G(X)$ also has a unique code representation. For $\alpha, \beta \in \{0,1,2,3\}$ and $\alpha \neq \beta$, let $x_1x_2x_3\ldots x_n\alpha\beta\beta\beta\ldots$ and $x_1x_2x_3\ldots x_n\beta\alpha\alpha\alpha\ldots$ be two different code representations of X then we have

$$G(x_1x_2x_3\ldots x_n\alpha\beta\beta\beta\ldots) = y_1y_2y_3\ldots y_ny_{n+1}y_{n+2}\ldots$$

$$G(x_1x_2x_3\ldots x_n\beta\alpha\alpha\alpha\ldots) = z_1z_2z_3\ldots z_nz_{n+1}z_{n+2}\ldots$$

where $y_i, z_i \in \{0,1,2,3\}$. Therefore, we must show that $y_1y_2y_3\ldots y_ny_{n+1}y_{n+2}\ldots$ and $z_1z_2z_3\ldots z_nz_{n+1}z_{n+2}\ldots$ are different code representations of $G(X)$.
If $x_1 = 0$, then we get

$$y_i \equiv z_i \equiv x_{i+1} + 3 \pmod{4}$$

for $i = 1,2,3,\ldots,n-1$ because of the definition of G. As well, for $i = 1,2,3,\ldots$

$$y_n \equiv \alpha + 3 \pmod{4},$$

$$y_{n+i} \equiv \beta + 3 \pmod{4},$$

$$z_n \equiv \beta + 3 \pmod{4},$$

$$z_{n+i} \equiv \alpha + 3 \pmod{4}$$

are obtained. Let us define $s_i \equiv x_{i+1} + 3 \pmod{4}$ and $\alpha + 3 \equiv \gamma \pmod{4}$, $\beta + 3 \equiv \delta \pmod{4}$ for $i = 1,2,3,\ldots,n-1$. Thus, we get $\gamma \neq \delta$

$$y_1y_2y_3\ldots y_ny_{n+1}y_{n+2}\ldots = s_1s_2s_3\ldots s_{n-1}\gamma\delta\delta\delta\ldots$$

and

$$z_1z_2z_3\dots z_nz_{n+1}z_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\delta\gamma\gamma\gamma\dots.$$

For the case $x_1 = 1$, we obtain $y_i = z_i = x_{i+1}$ for $i = 1, 2, 3, \dots, n-1$. What's more, for $i = 1, 2, 3, \dots$

$$\begin{aligned} y_n &= \alpha, \\ y_{n+i} &= \beta, \\ z_n &= \beta, \\ z_{n+i} &= \alpha \end{aligned}$$

are computed. So, we obtain the following results

$$y_1y_2y_3\dots y_ny_{n+1}y_{n+2}\dots = x_2x_3x_4\dots x_{n-1}\alpha\beta\beta\beta\dots$$

and

$$z_1z_2z_3\dots z_nz_{n+1}z_{n+2}\dots = x_2x_3x_4\dots x_{n-1}\beta\alpha\alpha\alpha\dots.$$

If $x_1 = 2$, then

$$y_i \equiv z_i \equiv x_{i+1} + 1 \pmod 4$$

where $i = 1, 2, 3, \dots, n-1$. Moreover, for $i = 1, 2, 3, \dots$, we have

$$\begin{aligned} y_n &\equiv \alpha + 1 \pmod 4, \\ y_{n+i} &\equiv \beta + 1 \pmod 4, \\ z_n &\equiv \beta + 1 \pmod 4, \\ z_{n+i} &\equiv \alpha + 1 \pmod 4. \end{aligned}$$

Hence, we observe that

$$y_1y_2y_3\dots y_ny_{n+1}y_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\gamma\delta\delta\delta\dots$$

and

$$z_1z_2z_3\dots z_nz_{n+1}z_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\delta\gamma\gamma\gamma\dots$$

for $i = 1, 2, 3, \dots, n-1$ where $s_i \equiv x_{i+1} + 1 \pmod 4$ and $\alpha + 1 \equiv \gamma \pmod 4$, $\beta + 1 \equiv \delta \pmod 4$.

If $x_1 = 3$, then for $i = 1, 2, 3, \dots, n-1$, we get

$$y_i \equiv z_i \equiv x_{i+1} + 2 \pmod 4.$$

In addition, for $i = 1, 2, 3, \dots$,

$$\begin{aligned} y_n &\equiv \alpha + 2 \pmod 4, \\ y_{n+i} &\equiv \beta + 2 \pmod 4, \\ z_n &\equiv \beta + 2 \pmod 4, \\ z_{n+i} &\equiv \alpha + 2 \pmod 4 \end{aligned}$$

are satisfied. Here, for $i = 1, 2, 3, \dots, n-1$, $s_i \equiv x_{i+1} + 2 \pmod 4$ and $\alpha + 2 \equiv \gamma \pmod 4$ and $\beta + 2 \equiv \delta \pmod 4$. Since, $\gamma \neq \delta$

$$y_1y_2y_3\dots y_ny_{n+1}y_{n+2}\dots = s_1s_2s_3\dots s_{n-1}\gamma\delta\delta\delta\dots$$

and

$$z_1z_2z_3\ldots z_nz_{n+1}z_{n+2}\ldots = s_1s_2s_3\ldots s_{n-1}\delta\gamma\gamma\gamma\ldots$$

are the different code representations of the point $G(X)$. This shows that G is well-defined on the quotient space of ST. Thus, the proof is completed. □

Proposition 4. *Suppose that the code representations of the points $X,\ X' \in ST$ are $x_1x_2x_3\ldots$ and $x'_1x'_2x'_3\ldots$ respectively where $x_i,\ x'_i \in \{0,1,2,3\}$ for all $i \in \mathbb{N}$.*

There is a function $H : ST \to ST$ such that

$$H(X) = X',\ x'_i = \begin{cases} 0, & x_i = 3 \\ 1, & x_i = 0 \\ 2, & x_i = 1 \\ 3, & x_i = 2 \end{cases} \tag{3}$$

which satisfies $H(F(X)) = G(H(X))$ is a homoemorphism, where F is defined in Proposition 1.

Proof. It is obvious that H is surjective function and $d(H(X), H(Y)) = d(X, Y)$ for all $X, Y \in ST$. So, we conclude that H is a homeomorphism. One can obtain that $H(F(X)) = G(H(X))$ for all $X \in ST$ with easy computations. □

Remark 1. *Since the function $H : ST \to ST$ defined in (3) is a homeomorphism for $\forall X \in ST$, the dynamical systems $\{ST; F\}$ and $\{ST; G\}$ are topologically conjugate. Therefore, $\{ST; G\}$ is also chaotic since $\{ST; F\}$ is chaotic and $\{ST, d\}$ is compact.*

According to Remark 1, the dynamical systems $\{ST; F\}$ and $\{ST; G\}$ are topologically conjugate. In consequence, the number of periodic points of these systems are equal.

While the periodic points of F are known, the periodic points of G can be found with the help of the homeomorphism H in (3). We have the fixed points and 2−periodic points of F from [3]. Because of the fixed points of F, which are

$$\bullet\overline{0} = 000\ldots,\quad \bullet\overline{1032} = 10321032\ldots,\quad \bullet\overline{20} = 202020\ldots,\quad \bullet\overline{3012} = 30123012\ldots$$

the fixed points of G are obtained as follows

$$\bullet H(\overline{0}) = \overline{1},\quad \bullet H(\overline{1032}) = \overline{2103},\quad \bullet H(\overline{20}) = \overline{31},\quad \bullet H(\overline{3012}) = \overline{0123}.$$

Similarly, the 2− periodic points of G are

$$\bullet H(\overline{13023120}) = \overline{20130231},\quad \bullet H(\overline{0220}) = \overline{1331},\quad \bullet H(\overline{01302312}) = \overline{12013023}$$

$$\bullet H(\overline{03102132}) = \overline{10213203},\quad \bullet H(\overline{12}) = \overline{23},\quad \bullet H(\overline{11223300}) = \overline{22330011}$$

$$\bullet H(\overline{2200}) = \overline{3311},\quad \bullet H(\overline{21100332}) = \overline{32211003},\quad \bullet H(\overline{23300112}) = \overline{30011223}$$

$$\bullet H(\overline{31021320}) = \overline{02132031},\quad \bullet H(\overline{32}) = \overline{03},\quad \bullet H(\overline{33221100}) = \overline{00332211}.$$

4. A Dynamical System on the Sierpinski Tetrahedron $\{ST;T\}$

We now define a new dynamical system which is not topologically conjugate with $\{ST;G\}$ and automatically with $\{ST;F\}$.

Proposition 5. *The code representations of $X,Y\in ST$ are $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ respectively. The function $T:ST\to ST$ are defined for $i=1,2,3,\ldots$ and $x_i,y_i\in\{0,1,2,3\}$ as follows*

$$T(0x_2x_3\ldots)=x_2x_3x_4\ldots$$

$$T(1x_2x_3\ldots)=y_1y_2y_3\ldots,\quad y_i=\begin{cases}0, & x_{i+1}=3\\ 1, & x_{i+1}=0\\ 2, & x_{i+1}=2\\ 3, & x_{i+1}=1\end{cases}\ (i\geq 1).$$

If $x_1=2$, there are four situations:

Case 1:

$$T(222\ldots 20x_{k+1}x_{k+2}x_{k+3}\ldots)=y_1y_2y_3\ldots,\quad y_i=\begin{cases}0, & x_{i+1}=2\\ 1, & x_{i+1}=3\\ 2, & x_{i+1}=0\\ 3, & x_{i+1}=1\end{cases}\ (i\geq 1)$$

Case 2:

$$T(222\ldots 21x_{k+1}x_{k+2}x_{k+3}\ldots)=y_1y_2y_3\ldots,\quad y_i=\begin{cases}0, & x_{i+1}=2\\ 1, & x_{i+1}=3\\ 2, & x_{i+1}=1\\ 3, & x_{i+1}=0\end{cases}\ (i\geq 1)$$

Case 3:

$$T(22\ldots 23x_s\ldots 0x_{k+1}x_{k+2}x_{k+3}\ldots)=y_1y_2y_3\ldots,$$

where $x_s\in\{2,3\}$ for $s<k$

$$y_i=\begin{cases}0, & x_{i+1}=2\\ 1, & x_{i+1}=0\\ 2, & x_{i+1}=3\\ 3, & x_{i+1}=1\end{cases}\ (i\geq 1)$$

Case 4:

$$T(22\ldots 23x_s\ldots 1x_{k+1}x_{k+2}x_{k+3}\ldots)=y_1y_2y_3\ldots,$$

where $x_s\in\{2,3\}$ for $s<k$

$$y_i=\begin{cases}0, & x_{i+1}=2\\ 1, & x_{i+1}=1\\ 2, & x_{i+1}=3\\ 3, & x_{i+1}=0\end{cases}\ (i\geq 1).$$

(Note that, due to above rules $T(\overline{2}) = \overline{0}$, $T(2\overline{3}) = \overline{2}$ and $T(23\overline{2}) = 2\overline{0}$ are obtained.)
If $x_1 = 3$, then

$$T(3x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 1 \\ 1, & x_{i+1} = 3 \\ 2, & x_{i+1} = 2 \\ 3, & x_{i+1} = 0 \end{cases} \ (i \geq 1).$$

Then, $\{ST; T\}$ is a dynamical system.

Proof. To state that $\{ST; T\}$ is a dynamical system, the images of the points expressed by two different code representations must indicate the same point. For example, $0\overline{1}$ and $1\overline{0}$ or $23\overline{0}$ and $20\overline{3}$ indicates the same point on ST. Thus, we investigate the images of following points $0\overline{1}, 0\overline{2}, 0\overline{3}, 1\overline{0}, 1\overline{2}, 1\overline{3}, 2\overline{0}, 2\overline{1}, 2\overline{3}, 3\overline{0}, 3\overline{1}, 3\overline{2}, 00\overline{1}, 01\overline{0}, 00\overline{2}, 02\overline{0}, 00\overline{3}, 03\overline{0}, 01\overline{2}, 02\overline{1}, 01\overline{3}, 03\overline{1}, 02\overline{3}, 03\overline{2}$ $11\overline{0}, 10\overline{1}, 10\overline{2}, 12\overline{0}, 10\overline{3}, 13\overline{0}, 12\overline{1}, 11\overline{2}, 11\overline{3}, 13\overline{1}, 12\overline{3}, 13\overline{2}, 20\overline{1}, 21\overline{0}, 20\overline{2}, 22\overline{0}, 20\overline{3}, 23\overline{0}, 21\overline{2}, 22\overline{1}, 21\overline{3}, 23\overline{1}, 23\overline{2}, 22\overline{3}$ and $30\overline{1}, 31\overline{0}, 30\overline{2}, 32\overline{0}, 30\overline{3}, 33\overline{0}$, $31\overline{2}, 32\overline{1}, 31\overline{3}, 33\overline{1}, 32\overline{3}, 33\overline{2}$. So, we get the following results,

$$\begin{array}{lll}
T(0\overline{1}) = \ \overline{1}, & T(0\overline{2}) = \ \overline{2}, & T(0\overline{3}) = \ \overline{3}, \\
T(1\overline{0}) = \ \overline{1}, & T(2\overline{0}) = \ \overline{2}, & T(3\overline{0}) = \ \overline{3}, \\
\\
T(1\overline{2}) = \ \overline{2}, & T(1\overline{3}) = \ \overline{0}, & T(2\overline{3}) = \ \overline{2}, \\
T(2\overline{1}) = \ \overline{2}, & T(3\overline{1}) = \ \overline{0}, & T(3\overline{2}) = \ \overline{2},
\end{array}$$

$$\begin{array}{lll}
T(00\overline{1}) = \ 0\overline{1}, & T(00\overline{2}) = \ 0\overline{2}, & T(00\overline{3}) = \ 0\overline{3}, \\
T(01\overline{0}) = \ 1\overline{0}, & T(02\overline{0}) = \ 2\overline{0}, & T(03\overline{0}) = \ 3\overline{0}, \\
\\
T(01\overline{2}) = \ 1\overline{2}, & T(01\overline{3}) = \ 1\overline{3}, & T(02\overline{3}) = \ 2\overline{3}, \\
T(02\overline{1}) = \ 2\overline{1}, & T(03\overline{1}) = \ 3\overline{1}, & T(03\overline{2}) = \ 3\overline{2}, \\
\\
T(10\overline{1}) = \ 1\overline{3}, & T(10\overline{2}) = \ 1\overline{2}, & T(10\overline{3}) = \ 1\overline{0}, \\
T(11\overline{0}) = \ 3\overline{1}, & T(12\overline{0}) = \ 2\overline{1}, & T(13\overline{0}) = \ 0\overline{1}, \\
\\
T(11\overline{2}) = \ 3\overline{2}, & T(11\overline{3}) = \ 3\overline{0}, & T(12\overline{3}) = \ 2\overline{0}, \\
T(12\overline{1}) = \ 2\overline{3}, & T(13\overline{1}) = \ 0\overline{3}, & T(13\overline{2}) = \ 0\overline{2}, \\
\\
T(20\overline{1}) = \ 2\overline{3}, & T(20\overline{2}) = \ 2\overline{0}, & T(20\overline{3}) = \ 2\overline{1}, \\
T(21\overline{0}) = \ 2\overline{3}, & T(22\overline{0}) = \ 0\overline{2}, & T(23\overline{0}) = \ 2\overline{1}, \\
\\
T(21\overline{2}) = \ 2\overline{0}, & T(21\overline{3}) = \ 2\overline{1}, & T(22\overline{3}) = \ 0\overline{2}, \\
T(22\overline{1}) = \ 0\overline{2}, & T(23\overline{1}) = \ 2\overline{1}, & T(23\overline{2}) = \ 2\overline{0}, \\
\\
T(30\overline{1}) = \ 3\overline{0}, & T(30\overline{2}) = \ 3\overline{2}, & T(30\overline{3}) = \ 3\overline{1}, \\
T(31\overline{0}) = \ 0\overline{3}, & T(32\overline{0}) = \ 2\overline{3}, & T(33\overline{0}) = \ 1\overline{3}, \\
\\
T(31\overline{2}) = \ 0\overline{2}, & T(31\overline{3}) = \ 0\overline{1}, & T(32\overline{3}) = \ 2\overline{1} \\
T(32\overline{1}) = \ 2\overline{0}, & T(33\overline{1}) = \ 1\overline{0}, & T(33\overline{2}) = \ 1\overline{2}
\end{array}.$$

As seen from above, the image of the different code representations of the same points state the same addresses.

In general, if $\sigma = x_1x_2x_3\ldots x_n$ then $\sigma0\overline{1}$ and $\sigma1\overline{0}$, $\sigma1\overline{2}$ and $\sigma2\overline{1}$, $\sigma0\overline{2}$ and $\sigma2\overline{0}$, $\sigma0\overline{3}$ and $\sigma3\overline{0}$, $\sigma1\overline{3}$ and $\sigma3\overline{1}$, $\sigma3\overline{2}$ and $\sigma2\overline{3}$, $\sigma00\overline{1}$ and $\sigma01\overline{0}$, $\sigma00\overline{2}$ and $\sigma02\overline{0}$, $\sigma00\overline{3}$ and $\sigma03\overline{0}$, $\sigma01\overline{2}$ and $\sigma02\overline{1}$, $\sigma01\overline{3}$ and $\sigma03\overline{1}$, $\sigma02\overline{3}$ and $\sigma03\overline{2}$, $\sigma11\overline{0}$ and $\sigma10\overline{1}$, $\sigma10\overline{2}$ and $\sigma12\overline{0}$, $\sigma10\overline{3}$ and $\sigma13\overline{0}$, $\sigma12\overline{1}$ and $\sigma11\overline{2}$, $\sigma11\overline{3}$ and $\sigma13\overline{1}$, $\sigma12\overline{3}$ and $\sigma13\overline{2}$, $\sigma20\overline{1}$ and $\sigma21\overline{0}$, $\sigma20\overline{2}$ and $\sigma22\overline{0}$, $\sigma20\overline{3}$ and $\sigma23\overline{0}$, $\sigma21\overline{2}$ and $\sigma22\overline{1}$, $\sigma21\overline{3}$ and $\sigma23\overline{1}$, $\sigma22\overline{3}$ and $\sigma23\overline{2}$, $\sigma30\overline{1}$ and $\sigma31\overline{0}$, $\sigma30\overline{2}$ and $\sigma32\overline{0}$, $\sigma30\overline{3}$ and $\sigma33\overline{0}$, $\sigma31\overline{2}$ and $\sigma32\overline{1}$, $\sigma31\overline{3}$ and $\sigma33\overline{1}$, $\sigma32\overline{3}$ and $\sigma33\overline{2}$ are different representations of same points and the image of these sequences represents the same addresses independently of σ. This shows that T is well-defined on ST. □

We can compute the $n-$ periodic points of T by using the equation

$$T^n(x_1x_2x_3\ldots) = x_1x_2x_3\ldots.$$

Since $T(\overline{0}) = \overline{0}$, $T(\overline{103}) = \overline{103}$, $T(\overline{301}) = \overline{301}$, $T(\overline{20}) = \overline{20}$ and $T(\overline{2130}) = \overline{2130}$,

$$\bullet\overline{0} = 00\ldots, \quad \bullet\overline{103} = 103103\ldots, \quad \bullet\overline{301} = 301301\ldots,$$

$$\bullet\overline{20} = 2020\ldots, \quad \bullet\overline{2130} = 21302130\ldots$$

are the fixed points of T.

Moreover,

$$\bullet\overline{013} = 013013\ldots, \quad \bullet\overline{031} = 031031\ldots, \quad \bullet\overline{0220} = 02200220\ldots$$

$$\bullet\overline{02211330} = 0221133002211330\ldots, \quad \bullet\overline{1} = 111\ldots, \quad \bullet\overline{130} = 130130\ldots$$

$$\bullet\overline{2010} = 20102010\ldots, \quad \bullet\overline{201030} = 201030201030\ldots$$

$$\bullet\overline{2200} = 22002200\ldots, \quad \bullet\overline{22113300} = 2211330022113300\ldots, \quad \bullet\overline{2320} = 23202320\ldots$$

$$\bullet\overline{232120} = 232120232120\ldots, \quad \bullet\overline{2120} = 21202120\ldots, \quad \bullet\overline{2030} = 20302030\ldots$$

$$\bullet\overline{210} = 210210\ldots, \quad \bullet\overline{230} = 230230\ldots, \quad \bullet\overline{21031230} = 2103123021031230\ldots$$

$$\bullet\overline{23120130} = 2312013023120130\ldots, \quad \bullet\overline{310} = 310310\ldots$$

are $2-$ periodic points of T.

Remark 2. *Since $\{ST;G\}$ and $\{ST;T\}$ have the different number of fixed points, they are not topologically conjugate (see Proposition 2).*

5. Conclusion

This paper gives a way to define different dynamical systems on the Sierpinski tetrahedron. This method can be also used for the other fractals which have the intrinsic metrics defined by using the code representations of the points.

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HYBRINOMIALS RELATED TO HYPER-LEONARDO NUMBERS

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Abstract. In this paper, we define hybrinomials related to hyper-Leonardo numbers. We study some of their properties such as the recurrence relation and summation formulas. In addition, we introduce hybrid hyper-Leonardo numbers.

1. Introduction

Integer sequences are the subject of many studies which are shown in recent literature [1–8]. The most famous integer sequence is called Fibonacci sequence and is defined by the following recurrence relation $(n \geq 1)$ [1]:

$$F_{n+1} = F_n + F_{n-1} \quad \text{with} \quad F_0 = 0, \quad F_1 = 1.$$

Leonardo sequence, which has similar properties to the Fibonacci sequence, is defined by Catarino and Borges [5], as follows:

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \qquad (n \geq 2),$$

with the initial conditions $Le_0 = Le_1 = 1$. Although commonly called "Leonardo numbers" in the literature, Kürüz et al. [9] preferred to call them "Leonardo Pisano numbers" and introduced Leonardo Pisano polynomials as

$$Le_n(x) = \begin{cases} 1, & n = 0, 1 \\ x + 2, & n = 2 \\ 2xLe_{n-1}(x) - Le_{n-3}(x), & n \geq 3. \end{cases}$$

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.

Hyper Leonardo numbers $Le_n^{(r)}$ are defined as a generalization of the Leonardo numbers by the formula

$$Le_n^{(r)} = \sum_{s=0}^{n} Le_s^{(r-1)} \quad \text{with} \quad Le_n^{(0)} = Le_n, \quad Le_0^{(r)} = Le_0 \quad \text{and} \quad Le_1^{(r)} = r+1,$$

where r is a positive integer [10]. The hyper-Leonardo numbers have the following recurrence relation for $n \geq 1$ and $r \geq 1$ [10]:

$$Le_n^{(r)} = Le_{n-1}^{(r)} + Le_n^{(r-1)}.$$

Hyper-Leonardo polynomials are defined as:

$$Le_n^{(r)}(x) = \sum_{s=0}^{n} Le_s^{(r-1)}(x)$$

with the initial conditions $Le_n^{(0)}(x) = Le_n(x)$, $Le_0^{(r)}(x) = 1$ and $Le_1^{(r)}(x) = r+1$ [11]. Note that, for $x = 1$, hyper-Leonardo polynomials $Le_n^{(r)}(x)$ give the hyper-Leonardo numbers $Le_n^{(r)}$. Hyper-Leonardo polynomials have the following recurrence relation for $n \geq 1$ and $r \geq 1$ [11]:

$$Le_n^{(r)}(x) = Le_{n-1}^{(r)}(x) + Le_n^{(r-1)}(x). \tag{1}$$

For $n \geq 3$ and $r \geq 1$, there is also the recurrence relation for hyper-Leonardo polynomials [11]:

$$\begin{aligned} Le_n^{(r)}(x) =& 2xLe_{n-1}^{(r)}(x) - Le_{n-3}^{(r)}(x) + \binom{n+r-1}{r-1} \\ &- \binom{n+r-2}{r-1}(2x-1) - \binom{n+r-3}{r-1}(x-2). \end{aligned} \tag{2}$$

If $n \geq 2$ and $r \geq 1$, then there is the summation formula for hyper-Leonardo polynomials [11]:

$$\sum_{s=0}^{r} Le_n^{(s)}(x) = Le_{n+1}^{(r)}(x) + (1-2x)Le_n(x) + Le_{n-2}(x). \tag{3}$$

In recent years, hybrid numbers have been the subject of research [12–19]. Özdemir [19] introduced hybrid numbers, as a generalization of complex, hyperbolic and dual numbers, sets by:

$$\mathbb{K} = \{a + bi + c\epsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = hi = \epsilon + i\}.$$

Szynal-Liana and Wloch [12] defined the n-th Fibonacci hybrid number as

$$HF_n = F_n + iF_{n+1} + \epsilon F_{n+2} + hF_{n+3}.$$

Alp and Koçer [18] defined hybrid-Leonardo numbers by using the Leonardo numbers as:

$$HLe_n = Le_n + Le_{n+1}i + Le_{n+2}\epsilon + Le_{n+3}h.$$

The authors also obtained some identities for the hybrid-Leonardo numbers such as [18]:

$$HLe_n = HLe_{n-1} + HLe_{n-2} + (1 + i + \epsilon + h), \quad (n \geq 2),$$
$$HLe_n = 2HF_{n+1} - (1 + i + \epsilon + h), \quad (n \geq 0),$$
$$HLe_{n+1} = 2HLe_n - HLe_{n-2}, \quad (n \geq 2).$$

Kürüz et al. [9] defined Leonardo Pisano hybrinomials, by using the Leonardo Pisano polynomials, as follows:

$$Le_n^{[H]}(x) = Le_n(x) + iLe_{n+1}(x) + \epsilon Le_{n+2}(x) + hLe_{n+3}(x).$$

The Leonardo Pisano hybrinomials have the following recurrence relation [9]:

$$Le_n^{[H]}(x) = 2xLe_{n-1}^{[H]}(x) - Le_{n-3}^{[H]}(x).$$

Motivated by the above papers, we define hybrinomials related to hyper-Leonardo numbers. We also define hybrid hyper-Leonardo numbers by using the newly defined hybrinomials. Then, we investigate some of their properties such as the recurrence relations and summation formulas.

2. Main Results

Definition 1. *Hybrinomials related to hyper-Leonardo numbers are defined as*

$$LeH_n^{(r)}(x) = Le_n^{(r)}(x) + Le_{n+1}^{(r)}(x)\, i + Le_{n+2}^{(r)}(x)\, \epsilon + Le_{n+3}^{(r)}(x)\, h,$$

where $Le_n^{(r)}(x)$ are the ordinary hyper-Leonardo polynomials.

The first few hybrinomials related to the hyper-Leonardo numbers are

$$\begin{aligned}
LeH_0^{(1)}(x) &= 1 + 2i + \epsilon(x+4) + h\left(2x^2 + 5x + 3\right), \\
LeH_1^{(1)}(x) &= 2 + i(x+4) + \epsilon\left(2x^2 + 5x + 3\right) + h\left(4x^3 + 10x^2 + 3x + 2\right), \\
LeH_2^{(1)}(x) &= (x+4) + i\left(2x^2 + 5x + 3\right) + \epsilon\left(4x^3 + 10x^2 + 3x + 2\right) \\
&\quad + h\left(8x^4 + 20x^3 + 6x^2\right)
\end{aligned}$$

and

$$\begin{aligned}
LeH_0^{(2)}(x) &= 1 + 3i + \epsilon(x+7) + h\left(2x^2 + 6x + 10\right), \\
LeH_1^{(2)}(x) &= 3 + i(x+7) + \epsilon\left(2x^2 + 6x + 10\right) + h\left(4x^3 + 12x^2 + 9x + 12\right), \\
LeH_2^{(2)}(x) &= (x+7) + i\left(2x^2 + 6x + 10\right) + \epsilon\left(4x^3 + 12x^2 + 9x + 12\right) \\
&\quad + h\left(8x^4 + 24x^3 + 18x^2 + 9x + 12\right).
\end{aligned}$$

For $x = 1$, the hybrinomials defined in Definition 1 give the hybrid numbers in the following definition:

Definition 2. *The n-th hybrid hyper-Leonardo number $LeH_n^{(r)}$ is defined as*

$$LeH_n^{(r)} = Le_n^{(r)} + iLe_{n+1}^{(r)} + \epsilon Le_{n+2}^{(r)} + hLe_{n+3}^{(r)},$$

where $Le_n^{(r)}$ is the n-th hyper-Leonardo numbers.

This table contains the values of the hybrid hyper-Leonardo numbers.

	$r=0$	$r=1$	$r=2$	$r=3$
n=0	$1+i+3\epsilon+5h$	$1+2i+5\epsilon+10h$	$1+3i+8\epsilon+18h$	$1+4i+12\epsilon+30h$
n=1	$1+3i+5\epsilon+9h$	$2+5i+10\epsilon+19h$	$3+8i+18\epsilon+37h$	$4+12i+30\epsilon+67h$
n=2	$3+5i+9\epsilon+15h$	$5+10i+19\epsilon+34h$	$8+18i+37\epsilon+71h$	$12+30i+67\epsilon+138h$
n=3	$5+9i+15\epsilon+25h$	$10+19i+34\epsilon+59h$	$18+37i+71\epsilon+130h$	$30+67i+138\epsilon+268h$
n=4	$9+15i+25\epsilon+41h$	$19+34i+59\epsilon+100h$	$37+71i+130\epsilon+230h$	$67+1381i+268\epsilon+498h$

TABLE 1. The first few hybrid hyper-Leonardo numbers $LeH_n^{(r)}$.

Theorem 1. *$LeH_n^{(r)}(x)$ has the recurrence relation for $n \geq 1$ and $r \geq 1$:*

$$LeH_n^{(r)}(x) = LeH_{n-1}^{(r)}(x) + LeH_n^{(r-1)}(x). \tag{4}$$

Proof. By using Definition 1 and the recurrence relation in equation (1), we have

$$\begin{aligned}
&LeH_{n-1}^{(r)}(x) + LeH_n^{(r-1)}(x)\\
&= \left(Le_{n-1}^{(r)}(x) + iLe_n^{(r)}(x) + \epsilon Le_{n+1}^{(r)}(x) + hLe_{n+2}^{(r)}(x)\right)\\
&\quad + \left(Le_n^{(r-1)}(x) + iLe_{n+1}^{(r-1)}(x) + \epsilon Le_{n+2}^{(r-1)}(x) + hLe_{n+3}^{(r-1)}(x)\right)\\
&= Le_{n-1}^{(r)}(x) + Le_n^{(r-1)}(x) + i\left(Le_n^{(r)}(x) + Le_{n+1}^{(r-1)}(x)\right)\\
&\quad + \epsilon\left(Le_{n+1}^{(r)}(x) + Le_{n+2}^{(r-1)}(x)\right) + h\left(Le_{n+2}^{(r)}(x) + Le_{n+1}^{(r-1)}(x)\right)\\
&= Le_n^{(r)}(x) + iLe_{n+1}^{(r)}(x) + \epsilon Le_{n+2}^{(r)}(x) + hLe_{n+3}^{(r)}(x)\\
&= LeH_n^{(r)}(x).
\end{aligned}$$

□

Corollary 1. *The hybrid hyper-Leonardo numbers have the recurrence relation for $n \geq 1$ and $r \geq 1$:*

$$LeH_n^{(r)} = LeH_{n-1}^{(r)} + LeH_n^{(r-1)}.$$

Theorem 2. *$LeH_n^{(r)}(x)$ has the summation formula:*

$$\sum_{s=0}^{n} LeH_s^{(r)}(x) = LeH_n^{(r+1)}(x) - \left(iLe_0^{(r+1)}(x) + \epsilon Le_1^{(r+1)}(x) + hLe_2^{(r+1)}(x)\right).$$

Proof. We use the induction method on n. Since,

$$\begin{aligned}
&LeH_0^{(r+1)}(x) - \left(iLe_0^{(r+1)}(x) + \epsilon Le_1^{(r+1)}(x) + hLe_2^{(r+1)}(x)\right) \\
&= Le_0^{(r+1)}(x) + iLe_1^{(r+1)}(x) + \epsilon Le_2^{(r+1)}(x) + hLe_3^{(r+1)}(x) \\
&\quad - \left(iLe_0^{(r+1)}(x) + \epsilon Le_1^{(r+1)}(x) + hLe_2^{(r+1)}(x)\right) \\
&= Le_0^{(r+1)}(x) + i\left(Le_1^{(r+1)}(x) - Le_0^{(r+1)}(x)\right) + \epsilon\left(Le_2^{(r+1)}(x) - Le_1^{(r+1)}(x)\right) \\
&\quad + h\left(Le_3^{(r+1)}(x) - Le_2^{(r+1)}(x)\right) \\
&= Le_0^{(r)}(x) + iLe_1^{(r)}(x) + \epsilon Le_2^{(r)}(x) + hLe_3^{(r)}(x) \\
&= LeH_0^{(r)}(x),
\end{aligned}$$

the result is true for $n = 0$. Assume that the result is true for $n = k$. Then,

$$\sum_{s=0}^{k} LeH_s^{(r)}(x) = LeH_k^{(r+1)}(x) - \left(iLe_0^{(r+1)}(x) + \epsilon Le_1^{(r+1)}(x) + hLe_2^{(r+1)}(x)\right).$$

Now, we must show that the result is true for $n = k+1$. Considering the recurrence relation in equation (4), we get

$$\begin{aligned}
\sum_{s=0}^{k+1} LeH_s^{(r)}(x) &= \sum_{s=0}^{k} LeH_s^{(r)}(x) + LeH_{k+1}^{(r)}(x) \\
&= LeH_k^{(r+1)}(x) - \left(iLe_0^{(r+1)}(x) + \epsilon Le_1^{(r+1)}(x) + hLe_2^{(r+1)}(x)\right) \\
&\quad + LeH_{k+1}^{(r)}(x) \\
&= LeH_{k+1}^{(r+1)}(x) - \left(iLe_0^{(r+1)}(x) + \epsilon Le_1^{(r+1)}(x) + hLe_2^{(r+1)}(x)\right).
\end{aligned}$$

□

Corollary 2. *The hybrid hyper-Leonardo numbers have the summation formula:*

$$\sum_{s=0}^{n} LeH_s^{(r)} = LeH_n^{(r+1)} - \left(iLe_0^{(r+1)} + \epsilon Le_1^{(r+1)} + hLe_2^{(r+1)}\right).$$

Theorem 3. *For $n \geq 3$ and $r \geq 1$, the recurrence relation*

$$\begin{aligned}
LeH_n^{(r)}(x) &= 2xLeH_{n-1}^{(r)}(x) - LeH_{n-3}^{(r)}(x) \\
&\quad + \binom{n+r-1}{r-1} - \binom{n+r-2}{r-1}(2x-1) - \binom{n+r-3}{r-1}(x-2) \\
&\quad + i\left[\binom{n+r}{r-1} - \binom{n+r-1}{r-1}(2x-1) - \binom{n+r-2}{r-1}(x-2)\right] \\
&\quad + \epsilon\left[\binom{n+r+1}{r-1} - \binom{n+r}{r-1}(2x-1) - \binom{n+r-1}{r-1}(x-2)\right] \\
&\quad + h\left[\binom{n+r+2}{r-1} - \binom{n+r+1}{r-1}(2x-1) - \binom{n+r}{r-1}(x-2)\right]
\end{aligned}$$

is true.

Proof. Considering Definition 1 and equation (2), the proof is clear. □

Corollary 3. *For $n \geq 3$ and $r \geq 1$, the hybrid hyper-Leonardo numbers have the recurrence relation:*

$$\begin{aligned} LeH_n^{(r)} &= 2LeH_{n-1}^{(r)} - LeH_{n-3}^{(r)} + \binom{n+r-1}{r-1} - \binom{n+r-2}{r-1} + \binom{n+r-3}{r-1} \\ &+i\left[\binom{n+r}{r-1} - \binom{n+r-1}{r-1} + \binom{n+r-2}{r-1}\right] \\ &+\epsilon\left[\binom{n+r+1}{r-1} - \binom{n+r}{r-1} + \binom{n+r-1}{r-1}\right] \\ &+h\left[\binom{n+r+2}{r-1} - \binom{n+r+1}{r-1} + \binom{n+r}{r-1}\right]. \end{aligned}$$

Theorem 4. *If $n \geq 2$ and $r \geq 1$, then the summation formula*

$$\sum_{s=0}^{r} LeH_n^{(s)}(x) = LeH_{n+1}^{(r)}(x) + (1-2x)\, LeH_n(x) + LeH_{n-2}(x)$$

is true.

Proof. By considering equation (3), we get

$$\begin{aligned} \sum_{s=0}^{r} LeH_n^{(s)}(x) &= \sum_{s=0}^{r}\left(Le_n^{(s)}(x) + iLe_{n+1}^{(s)}(x) + \epsilon Le_{n+2}^{(s)}(x) + hLe_{n+3}^{(s)}(x)\right) \\ &= \sum_{s=0}^{r} Le_n^{(s)}(x) + i\sum_{s=0}^{r} Le_{n+1}^{(s)}(x) + \epsilon\sum_{s=0}^{r} Le_{n+2}^{(s)}(x) \\ &\quad +h\sum_{s=0}^{r} Le_{n+3}^{(s)}(x) \\ &= Le_{n+1}^{(r)}(x) + (1-2x)\, Le_n(x) + Le_{n-2}(x) \\ &\quad +i\left(Le_{n+2}^{(r)}(x) + (1-2x)\, Le_{n+1}(x) + Le_{n-1}(x)\right) \\ &\quad +\epsilon\left(Le_{n+3}^{(r)}(x) + (1-2x)\, Le_{n+2}(x) + Le_n(x)\right) \\ &\quad +h\left(Le_{n+4}^{(r)}(x) + (1-2x)\, Le_{n+3}(x) + Le_{n+1}(x)\right) \\ &= LeH_{n+1}^{(r)}(x) + (1-2x)\, LeH_n(x) + LeH_{n-2}(x). \end{aligned}$$

□

Corollary 4. *If $n \geq 1$ and $r \geq 1$, then there is the relation between the hybrid hyper-Leonardo numbers and Fibonacci hybrid numbers:*

$$\sum_{s=0}^{r} LeH_n^{(s)} = LeH_{n+1}^{(r)} - 2HF_n.$$

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q-DIFFERENCE OPERATOR ON $L_q^2(0,+\infty)$

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Abstract. In this research, the minimal and maximal operators defined by q- difference expression are given in the Hilbert space $L_q^2(0,\infty)$. The existence problem of a q^{-1}-normal extension for the minimal operator is mentioned. In addition, the sets of the minimal operator spectrum and the maximal operator spectrum are examined.

1. Introduction

The q-analysis first appeared in the 1740s, when Euler launched the division theory, also called the total analytic number theory, Euler wrote and compiled works in the early 1800s [4]. The advancement of q-calculus continued in 1813 under the study of Gauss, who gave the hypergeometric series and their interrelationships [5].

The study of quantum calculus, or q-calculus, which has been going on for 300 years since Euler, has often been regarded as one of the most difficult topics to deal with in mathematics. Today, due to its use in a variety of areas, such as mathematics, physics, rapid progress is being made in studies in the field of q-calculus. The working history of q- analysis, quantum mechanics, theta functions, hypergeometric functions, analytic number theory, finite difference theory, Mock theta functions, Bernoulli and Euler polynomials, gamma function theory has a wide variety of applications in combinatorics. Moreover, there is the application of the q-difference operator to thermodynamics. It has been demonstrated that the formalization of the q-calculus may be used to realize the thermodynamics of

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the q-deformed algebra. It is found that if it is used a suitable Jackson derivative instead of the ordinary thermodynamic derivative, then the entire structure of thermodynamics is maintained [9]. For some numerous contributions the history of q-calculus, fundamental principles, and fundamentals of q-differential equations, the key books [3], [8] and [1] can be cited.

Moreover, a closed linear operator T with dense domain on any Hilbert space is said formally q-normal operator iff $D(T) \subset D(T^*)$ and

$$TT^* = qT^*T.$$

When $D(T) = D(T^*)$ is satisfied for a formally q-normal operator, then T said a q-normal operator. Moreover, q-normal operators appear in quantum group theory in the study of the hermitean quantum plane and of quantum groups. For instance, the q-deformed quantum plane C_q^1 is a $*$-algebra with one generator T such that $TT^* = qT^*T$ [10]. Definitions of these and other classes which are called q-deformed operators was given and investigated by Ota [10], for detail analysis see [2, 11–14].

2. The Minimal and Maximal Operators $L_q^2(0,+\infty)$

Suppose that $L_q^2(0,+\infty)$ is defined as

$$L_q^2(0,+\infty) = \left\{ u : [0,+\infty) \to \mathbb{C} \ : \ \int\limits_0^{+\infty} |u(t)|^2 d_q t = (1-q) \sum_{k=-\infty}^{+\infty} q^k \left| u\left(q^k\right)\right|^2 < +\infty \right\}.$$

$L_q^2(0,+\infty)$ is a linear vector space with equivalent classes, which are defined for two functions u and v in the same equivalent class iff $u\left(q^k\right) = v\left(q^k\right),\ k \in \mathbb{Z}$. Also $L_q^2(0,+\infty)$ is separable and its the inner product is follows [1]

$$(u,v)_{L_q^2(0,+\infty)} := \int\limits_0^{+\infty} u(t)\overline{v(t)} d_q t\ ,\ u,v \in L_q^2(0,+\infty).$$

In addition, Jackson reintroduced the q-difference operator [7] and he defined as

$$D_q u(t) = \frac{u(t) - u(qt)}{(1-q)t}, \quad t \neq 0$$

and also the q-derivative for $t=0$ is defined for $|q|<1$ as

$$D_q u(0) = \lim_{n\to+\infty} \frac{u(tq^n) - u(0)}{tq^n},\ t = 0,$$

if there is the limit and it is independent of t.

Note that we have assume $0<q<1$ for this paper.

Corollary 1. *If $u \in L_q^2(0,+\infty)$, then $\lim\limits_{n\to+\infty} u\left(\frac{1}{q^n}\right) = 0$.*

Proposition 1. *If $D_q u(t) \in L_q^2(0,+\infty)$, then the limit $\lim\limits_{n\to+\infty} u(q^n)$ exists.*

Proof. Let $D_q u(t)$ be in $L_q^2(0,+\infty)$. Because the characteristic function $\chi_{[0,1]} \in L_q^2(0,+\infty)$ and

$$\begin{aligned}\left(D_q u, \chi_{[0,1]}\right)_{L_q^2(0,+\infty)} &= \int_0^{+\infty} \chi_{[0,1]}(t)\, D_q u(t)\, d_q t \\ &= (1-q)\sum_{k=0}^{+\infty} q^k \frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{(1-q)\,q^k} \\ &= \sum_{k=0}^{+\infty} u\left(q^k\right)-u\left(q^{k+1}\right) \\ &= \lim_{n\to+\infty} \sum_{k=0}^{n} u\left(q^k\right)-u\left(q^{k+1}\right) \\ &= u(1)-\lim_{n\to+\infty} u\left(q^n\right),\end{aligned}$$

are true, the limit $\lim\limits_{n\to+\infty} u\left(q^n\right)$ exists. □

First of all, we give the abstract definition of maximal and minimal operators for differential operators [6]. Suppose that Ω is an n-dimensional infinitely differentiable manifold and a differential expression

$$p(.) = \sum_{|\alpha|\leqslant m} a_\alpha D^\alpha,$$

where the coefficients a_α are infinitely differentiable functions of $x=(x_1,\dots,x_n)$. Also, $\alpha\in\mathbb{C}^n$, $|\alpha|=\alpha_1+\ldots+\alpha_n$, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}\ldots D_n^{\alpha_n}$ and $D_k=\frac{1}{i}\frac{\partial}{\partial x_k}$ are denoted. The formal adjoint of the expression $p(.)$ is the form $p^+(.) = \sum\limits_{|\alpha|\leqslant m}(-1)^{|\alpha|}\overline{a_\alpha} D^\alpha$ in $L^2(\Omega)$. In this case, two operators

$$\begin{aligned}&{P_0}'u = p(u),\;\; {P_0}' : C_0^\infty(\Omega)\subset L^2(\Omega)\to L^2(\Omega),\\ &{P_0^+}'u = p^+(u),\;\; {P_0^+}' : C_0^\infty(\Omega)\subset L^2(\Omega)\to L^2(\Omega)\end{aligned}$$

have closures in $L^2(\Omega)$ and these closures are denoted by P_0 and P_0^+ respectively. The operator P_0 is said as the minimal operator defined by the expression p. Similarly, P_0^+ is called the minimal operator defined by the differential expression p^+. The adjoint P of P_0^+ is said the maximal operator generated by p. It is easy seen that $D(P_0)=D(P^+)$ and $D(P)=D(P_0^+)$.

The q-derivative for multiplication of two functions $u(t)$ and $v(t)$ defined on $[0,+\infty)$ is follows for all $t\in(0,+\infty)$

$$D_q(uv)(t) = v(t)\, D_q u(t) + u(qt)\, D_q v(t)\,.$$

This relation said q-product rule. It is obtain that

$$
\begin{aligned}
\int\limits_0^{+\infty} D_q(uv)(t)\,d_qt &= (1-q)\sum_{k=-\infty}^{+\infty} q^k\left(\frac{u\left(q^k\right)v\left(q^k\right)-u\left(q^{k+1}\right)v\left(q^{k+1}\right)}{(1-q)\,q^k}\right)\\
&= \sum_{k=-\infty}^{+\infty} u\left(q^k\right)v\left(q^k\right)-u\left(q^{k+1}\right)v\left(q^{k+1}\right)\\
&= \lim_{n,m\to+\infty}\sum_{k=-m}^{n} u\left(q^k\right)v\left(q^k\right)-u\left(q^{k+1}\right)v\left(q^{k+1}\right)\\
&= \lim_{n,m\to+\infty} u\left(q^{-m}\right)v\left(q^{-m}\right)-u\left(q^{-m+1}\right)v\left(q^{-m+1}\right)\\
&\quad +u\left(q^{-m+1}\right)v\left(q^{-m+1}\right)-u\left(q^{-m+2}\right)v\left(q^{-m+2}\right)\\
&\quad +u\left(q^{-m+2}\right)v\left(q^{-m+2}\right)-\ldots+u\left(q^{-1}\right)v\left(q^{-1}\right)\\
&\quad -u(1)\,v(1)+u(q)\,v(q)+\ldots+u(q^n)\,v(q^n)-u\left(q^{n+1}\right)v\left(q^{n+1}\right)\\
&= \lim_{n,m\to+\infty} u\left(q^{-m}\right)v\left(q^{-m}\right)-u(q^n)\,v(q^n)\\
&= -\lim_{n\to+\infty} u(q^n)\,v(q^n)
\end{aligned}
$$

is finite for any $u(t), v(t), D_qu(t), D_qv(t) \in L_q^2((0,+\infty))$. Because

$$
\begin{aligned}
(D_qu,v)_{L_q^2(0,+\infty)} &= \int\limits_0^{+\infty} D_qu(t)\,\overline{v(t)}d_qt \qquad (1)\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}-\int\limits_0^{+\infty} u(t)\,\overline{D_qu(t)}d_qt\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}-(1-q)\sum_{k=-\infty}^{+\infty} q^ku\left(q^{k+1}\right)\frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{(1-q)\,q^k}\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}+(1-q)\sum_{k=-\infty}^{+\infty} q^{k+1}u\left(q^{k+1}\right)\frac{u\left(q^{k+1}\right)-u\left(q^k\right)}{(1-q)\,q^{k+1}}\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}-(1-q)\sum_{k=-\infty}^{+\infty} q^ku\left(q^k\right)\frac{1}{q}D_{q^{-1}}u(t)\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}+\int\limits_0^{+\infty} u(t)\,\overline{-\frac{1}{q}D_{q^{-1}}v(t)}d_qt\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}+\left(u,-\frac{1}{q}D_{q^{-1}}v\right)_{L_q^2(0,+\infty)}, \qquad (2)
\end{aligned}
$$

the formal adjoint expression of the expression D_q is $-\frac{1}{q}D_{q^{-1}}$ on $L_q^2(0,+\infty)$.

Now, let's define the linear operators $L_0 : D_0 \subset L_q^2(0,+\infty) \to L_q^2(0,+\infty)$ of the form $L_0u(t) = D_q u(t)$ where its domain is

$$D_0 = \left\{ u \in L_q^2(0,+\infty) \ : \ D_q u(t) \in L_q^2(0,+\infty) \ \ and \ \lim_{k\to+\infty} u\left(q^k\right) = 0 \right\}$$

and $L : D \subset L_q^2(0,+\infty) \to L_q^2(0,+\infty)$ of the form $L_0u(t) = D_q u(t)$ where

$$D = \left\{ u \in L_q^2(0,+\infty) \ : \ D_q u(t) \in L_q^2(0,+\infty) \right\}.$$

We say that these operators are the minimal operator and the maximal operator generated by the q-difference expression, respectively. Moreover, $L_0 \subset L$ is obvious, i.e. the maximal operator L is an extension of the minimal operator L_0.

Theorem 1. *The operator L_0 is a formally q^{-1}-normal operator on $L_q^2(0,+\infty)$.*

Proof. The set of functions

$$\varphi_m(t) := \begin{cases} \dfrac{1}{q^{\frac{m}{2}}\sqrt{1-q}}, & t = q^m \\ 0, & ,otherwise \end{cases}, \quad m \in \mathbb{Z}$$

is an orthogonal basis of $L_q^2(0,+\infty)$ and this basis is clearly contained in D_0. Therefore, the minimal linear operator L_0 has dense domain.

Now let's show that the minimal operator is closed. Suppose that any sequence $\{u_n\} \subset D_0$ such that $u_n \xrightarrow[n\to\infty]{} u$ and $L_0u_n \xrightarrow[n\to\infty]{} f$. In this case,

$$\|u_n - u\|^2_{L_q^2(0,+\infty)} = (1-q) \sum_{k=-\infty}^{+\infty} q^k \left|u_n\left(q^k\right) - u\left(q^k\right)\right|^2 \xrightarrow[n\to\infty]{} 0.$$

Because of the last relation, we have

$$\lim_{n\to\infty} u_n\left(q^k\right) = u\left(q^k\right) \tag{3}$$

From this relation,

$$\lim_{n\to+\infty} \frac{u_n\left(q^k\right) - u_n\left(q^{k+1}\right)}{(1-q)\,q^k} = \frac{u\left(q^k\right) - u\left(q^{k+1}\right)}{(1-q)\,q^k} = f\left(q^k\right),\ k \in \mathbb{Z}.$$

is attained. Also, from (3) and the boundary condition at $t = 0$

$$\left|u\left(q^k\right)\right| \leqslant \left|u_n\left(q^k\right) - u\left(q^k\right)\right| + \left|u_n\left(q^k\right)\right| \xrightarrow[n,k\to+\infty]{} 0$$

is true. This means that $u \in D(L_0)$ and $Lu(t) = f$. Therefore, the minimal linear operator L_0 is closed. On the other hand, $D(L_0^*) = D$ and the following equations can be easily obtained

$$\|L_0u(t)\|^2_{L_q^2(0,+\infty)} = \int_0^{+\infty} |D_q u(t)|^2 d_q t$$

$$= (1-q)\sum_{k=-\infty}^{+\infty} q^k \left|D_q u\left(q^k\right)\right|^2$$
$$= (1-q)\sum_{k=-\infty}^{+\infty} q^k \left|\frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{(1-q)\,q^k}\right|^2$$

for any $u \in D(L_0)$. Also,

$$\begin{aligned}
\|L_0^* u(t)\|^2_{L_q^2(0,+\infty)} &= \int_0^{+\infty} \left|-\frac{1}{q}D_{q^{-1}}u(t)\right|^2 d_q t \\
&= (1-q)\sum_{k=-\infty}^{+\infty} q^k \left|-\frac{1}{q}D_{q^{-1}}u\left(q^k\right)\right|^2 \\
&= \frac{1}{q^2}(1-q)\sum_{k=-\infty}^{+\infty} q^k \left|\frac{u\left(q^k\right)-u\left(q^{k-1}\right)}{\left(1-\frac{1}{q}\right)q^k}\right|^2 \\
&= \frac{1}{q}(1-q)\sum_{k=-\infty}^{+\infty} q^{k-1} \left|\frac{u\left(q^{k-1}\right)-u\left(q^k\right)}{(1-q)\,q^{k-1}}\right|^2 \\
&= \frac{1}{q}(1-q)\sum_{k=-\infty}^{+\infty} q^k \left|\frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{(1-q)\,q^k}\right|^2.
\end{aligned}$$

is hold. As a result of the last equations for all $u \in D(L_0) \subset D(L_0^*)$

$$\|L_0^* u\| = \sqrt{q^{-1}}\,\|L_0 u\|$$

is seen. This is completed the proof. □

Corollary 2. *The minimal operator L_0 is a maximal formally q-normal in $L_q^2(0,+\infty)$.*

Proof. Assume that $\tilde{L_0}$ is a q-normal extension of L_0, i.e. $L_0 \subset \tilde{L_0}$. Therefore, for all $u \in D(\tilde{L_0}) = D(\tilde{L_0}^*)$

$$\begin{aligned}
\left(\tilde{L_0}u, u\right)_{L_q^2(0,+\infty)} - \left(u, \tilde{L_0}^* u\right)_{L_q^2(0,+\infty)} &= (D_q u, u)_{L_q^2(0,+\infty)} - \left(u, -\frac{1}{q}D_{q^{-1}}u\right)_{L_q^2(0,+\infty)} \\
&= -\lim_{k\to+\infty} |u\left(q^k\right)|^2 = 0
\end{aligned}$$

is obtained from the equation (1). This means that $D(\tilde{L_0}) = D(L_0)$ and $\tilde{L_0} = L_0$. However this is a contradiction. According to this result and Theorem 2.1, the minimal operator L_0 is a maximal formally q-normal operator in $L_q^2(0,+\infty)$. □

3. Spectrum Sets of the operators L_0 and L

Theorem 2. *The point spectrum set of* L_0 *is*

$$\sigma_p(L_0)=\left\{\frac{q^m}{1-q}:\ m\in\mathbb{Z}\right\}.$$

Proof. Suppose that a complex number λ is an element of the point spectrum of L_0. Therefore, there is a non-zero element $u(t)$ corresponding to a complex number λ in $D(L_0)$, which that satisfies the following equation

$$\frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{(1-q)\,q^k}=\lambda u\left(q^k\right),\quad k\in\mathbb{Z}.$$

We gain that

$$u\left(q^{k+1}\right)=\left(1-\lambda\,(1-q)\,q^k\right)u\left(q^k\right) \tag{4}$$

for all $k\in\mathbb{Z}$. If $\lambda=\frac{1}{(1-q)q^m}$ for any $m\in\mathbb{Z}$ is true, then the eigenvector $u(t)$ should be defined as

$$\begin{aligned} &u\left(q^k\right)=0,\quad k\geqslant m+1\\ &u\left(q^k\right)=\left(\prod_{i=k-m}^{-1}\frac{1}{1-q^i}\right)u\left(q^m\right),\quad k\leqslant m-1. \end{aligned}$$

Since $0<q<1$ and the limit

$$\lim_{k\to-\infty}\left|1-q^k\right|=+\infty$$

is true, a negative integer k_0 is exist such that

$$\prod_{n=k_0+1-m}^{-1}\frac{1}{|1-q^n|}\leqslant 1.$$

From this result and $0<q<1$ it is get that

$$\begin{aligned} \|u\|^2_{L_q^2(0,+\infty)} &= \sum_{k=-\infty}^{+\infty} q^k\left|u\left(q^k\right)\right|^2\\ &= \sum_{k=k_0}^{m} q^k\left|u\left(q^k\right)\right|^2+\sum_{k=-\infty}^{k_0-1} q^k\left|u\left(q^k\right)\right|^2\\ &= \sum_{k=k_0}^{m} q^k\left|u\left(q^k\right)\right|^2+\sum_{k=-\infty}^{k_0-1} q^k\left(\prod_{i=k-m}^{-1}\left|\frac{1}{1-q^i}\right|^2\right)\left|u\left(q^m\right)\right|^2\\ &\leqslant \sum_{k=k_0}^{m} q^k\left|u\left(q^k\right)\right|^2+\sum_{k=-\infty}^{k_0-1} q^k\left|\frac{1}{1-q^{k-m}}\right|^2\left|u\left(q^m\right)\right|^2 \end{aligned}$$

$$= \sum_{k=k_0}^{m} q^k \left|u\left(q^k\right)\right|^2 + \sum_{k=-\infty}^{k_0-1} q^k \left|\frac{q^{-k}}{q^{-k}-q^{-m}}\right|^2 \left|u\left(q^m\right)\right|^2$$
$$= \sum_{k=k_0}^{m} q^k \left|u\left(q^k\right)\right|^2 + \sum_{k=-\infty}^{k_0-1} q^{-k} \left|\frac{1}{q^{-k}-q^{-m}}\right|^2 \left|u\left(q^m\right)\right|^2 < +\infty.$$

These prove that $u(t)$ is an eigenvector corresponding to $\frac{q^m}{1-q}$ for $m \in \mathbb{Z}$.

On the other hand, λ is different from $\frac{1}{(1-q)q^m}$ for any $m \in \mathbb{Z}$, then

$$u\left(q^k\right) = \left(\prod_{i=0}^{k-1} \left(1-\lambda\left(1-q\right)q^i\right)\right) u\left(1\right), \quad k \in \mathbb{N}.$$

Hence, $u \in D_0$, $u\left(q^k\right) \xrightarrow[k\to+\infty]{} 0$ iff there exists $m \in \mathbb{N}$ satisfied the following equality

$$1-\lambda\left(1-q\right)q^m = 0$$

must be supplied [15] or $u(1) = 0$. In this case, $u(1) = 0$ and so $u = 0$ is obtained from the equation (4). These results imply that $\sigma_r(L_0) = \left\{\frac{q^m}{1-q} : \ m \in \mathbb{Z}\right\}$. □

Theorem 3. *The set of L_0 residual spectrum is empty.*

Proof. Assume that $\lambda \in \mathbb{C}$ is in $\sigma_r\left(L_0\right)$. Since $L_q^2\left(0,+\infty\right) = \overline{R(L_0-\lambda E)} \oplus Ker\left(L_0^*-\bar{\lambda}E\right)$ is provided, where E is the identity operator in $L_q^2\left(0,+\infty\right)$, it is clear that $\bar{\lambda} \in \sigma_p\left(L_0^*\right)$. Therefore, there exists an element $u \in L_q^2\left(0,+\infty\right), u \neq 0$ and

$$L_0^* u\left(t\right) = \bar{\lambda} u\left(t\right).$$

Therefore, we have

$$-\frac{1}{q}\frac{u\left(q^k\right)-u\left(q^{k-1}\right)}{\left(1-\frac{1}{q}\right)q^k} = \frac{u\left(q^k\right)-u\left(q^{k-1}\right)}{\left(1-q\right)q^k} = \bar{\lambda}u\left(q^k\right)$$

for all $k \in \mathbb{Z}$. The following equation is obtained from this equation

$$u\left(q^{k-1}\right) = \left(1-\bar{\lambda}\left(1-q\right)q^k\right)u\left(q^k\right)$$

for all $k \in \mathbb{Z}$. If $\bar{\lambda}$ is equal to $\frac{1}{(1-q)q^m}$ for any $m \in \mathbb{Z}$, then

$$u\left(q^k\right) = 0, \quad k \leqslant m-1$$
$$u\left(q^k\right) = \left(\prod_{i=m+1}^{k} \frac{1}{1-q^{i-m}}\right) u\left(q^m\right), \quad k \geqslant m+1$$

is holds. Because $\sum\limits_{k=m+1}^{+\infty} q^k \left(\prod\limits_{i=m+1}^{k} \frac{1}{1-q^{i-m}} \right)^2$ converges to a complex number, the function $u(t)$ defined as above is an element of $L_q^2(0,+\infty)$.

Otherwise, if $\bar{\lambda}$ is not equal to $\frac{1}{(1-q)q^m}$ for any $m \in \mathbb{Z}$, then it must be $u(1) \neq 0$ and

$$u\left(q^k\right) = \left(\prod_{i=0}^{-k} \left(1 - \bar{\lambda}\left(1-q\right) q^{-i}\right) \right) u(1), \quad k \leqslant 0.$$

But the limit $\lim\limits_{k \to -\infty} u\left(q^k\right)$ does not exist when $\bar{\lambda}$ is not equal to $\frac{1}{(1-q)q^m}$ for any $m \in \mathbb{Z}$. As a result of these,

$$\sigma_r\left(L_0\right) = \emptyset$$

is obtained.

□

Corollary 3. *It is held that* $0 \in \sigma_c(L_0)$ *for the minimal operator* L_0*.*

Corollary 4. *The point spectrum and residual spectrum of* L_0^* *are as follows*

$$\sigma_p(L_0^*) = \left\{ \frac{q^m}{1-q} : \ m \in \mathbb{Z} \right\} \quad \text{and} \quad \sigma_r(L_0^*) = \emptyset.$$

Theorem 4. *The point and continuous spectrum sets of the maximal operator are in the form*

$$\sigma_p\left(L\right) = \mathbb{C} \backslash \left\{0\right\} \quad \text{and} \quad \sigma_c\left(L\right) = \left\{0\right\}.$$

Proof. Suppose that λ is a nonzero complex number. We deal with the solution of following problem

$$\left(L - \lambda E\right) u\left(q^k\right) = 0, \ \ k \in \mathbb{Z}.$$

It is written for any $k \in \mathbb{Z}$

$$u\left(q^{k+1}\right) = \left(1 - \lambda\left(1-q\right) q^k\right) u\left(q^k\right). \tag{5}$$

If $u\left(q^k\right)$ are different from zero for all $k \in \mathbb{Z}$, then we have

$$u\left(q^{k+1}\right) = \left(\prod_{n=0}^{k} \left(1 - \lambda\left(1-q\right) q^n\right) \right) u\left(1\right)$$

for all positive integer k. Since the infinite product $\prod\limits_{k=0}^{+\infty} \left(1 - \lambda\left(1-q\right) q^k\right)$ converges, the sequence $\left\{u\left(q^k\right)\right\}_{k \in \mathbb{N}}$ is bounded. From this result the series $\sum\limits_{k=0}^{+\infty} q^k \left|u\left(q^k\right)\right|^2$ is finite.

In the case of negative integers, we gain

$$u\left(q^{k}\right)=\left(\prod_{n=k}^{-1}\left(1-\lambda\left(1-q\right)q^{k}\right)^{-1}\right)u\left(1\right)$$

for all $k\leqslant -1$. Because the limit

$$\lim_{k\to-\infty}\left|1-\lambda\left(1-q\right)q^{k}\right|=+\infty \tag{6}$$

is true, it is clear that

$$\prod_{n=k-1}^{-1}\frac{1}{\left|1-\lambda\left(1-q\right)q^{n}\right|}\leqslant 1$$

for small enough negative integers k. This result give us the following inequality

$$\begin{aligned}
q^{k}\left|u\left(q^{k}\right)\right|^{2} &= q^{k}\left(\prod_{n=k}^{-1}\frac{1}{\left|1-\lambda\left(1-q\right)q^{n}\right|^{2}}\right)\left|u\left(1\right)\right|^{2}\\
&\leqslant q^{k}\frac{1}{\left|1-\lambda\left(1-q\right)q^{k}\right|^{2}}\left|u\left(1\right)\right|^{2}\\
&= q^{k}\frac{q^{-2k}}{\left|q^{-k}-\lambda\left(1-q\right)\right|^{2}}\left|u\left(1\right)\right|^{2}\\
&= \frac{q^{-k}}{\left|q^{-k}-\lambda\left(1-q\right)\right|^{2}}\left|u\left(1\right)\right|^{2}
\end{aligned}$$

for small enough negative integers k. Because of the limit (6) and the fact that the geometric series $\sum_{k=-\infty}^{0}\alpha q^{-k}$ converges for $0<q<1$, these results allow us that the series $\sum_{k=-\infty}^{0}q^{k}\left|u\left(q^{k}\right)\right|^{2}$ converges absolutely. These show us to conclude that $\sum_{k=-\infty}^{+\infty}q^{k}\left|u\left(q^{k}\right)\right|^{2}$ is convergent.

When $u(q^{m+1})$ is equal to zero for an integer $m\in\mathbb{Z}$, it is obtained that $u(q^{k})=0$ for all $k\geqslant m+1$. We note that this condition includes the case of $\lambda=\frac{q^{-m}}{1-q},\ \ m\in\mathbb{Z}$. Moreover, the equation

$$u\left(q^{k}\right)=\left(\prod_{n=k}^{m-1}\left(1-\lambda\left(1-q\right)q^{n}\right)^{-1}\right)u\left(q^{m}\right)$$

is easily checked for all $k<m$. We already know that

$$\sum_{k=-\infty}^{m-1}q^{k}\left(\prod_{n=k}^{m-1}\left|\left(1-\lambda\left(1-q\right)q^{n}\right)^{-1}\right|^{2}\right)\left|u\left(q^{m}\right)\right|^{2}$$

is convergent. Because of all these reasons, we get that $u(t)$ is an eigenvector of the maximal operator L for $\lambda \in \mathbb{C}\setminus\{0\}$.

If $\lambda = 0$, then returning to the equation (5) it must be $u(t) = 0$. This means that zero is not an eigenvalue. Also, if $0 \in \sigma_r(L)$, then it must be $0 \in \sigma_p(L^*)$ because of $L_q^2(0,+\infty) = \overline{R(L)} \oplus Ker(L^*)$. But, it can be easily proved that $0 \notin \sigma_p(L^*)$. Therefore, it must be $\sigma_c(L) = \{0\}$ from the fact of the closeness of the spectrum. □

Remark 1. *It can be defined the two operators P_0 and P defined by $p(.) = \frac{d}{dt}$ in $L^2(0,+\infty)$ and these operators are called the minimal and maximal operators, respectively. Also, their domains are as follows*

$$\begin{aligned} D(P_0) &= \left\{u \in L^2(0,+\infty): \ u' \in L^2(0,+\infty) \ \text{ and } \ u(0) = 0\right\}, \\ D(P) &= \left\{u \in L^2(0,+\infty): \ u' \in L^2(0,+\infty)\right\}. \end{aligned}$$

The operator P_0 is maximal formal normal. It means that there is not any normal extension of L_0. Moreover, the point and residual spectrum sets of P_0 are $\sigma_p(P_0) = \emptyset$ and $\sigma_r(P_0) = \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$ and the spectrum parts of the maximal operator P are $\sigma_p(P) = \{\lambda \in \mathbb{C} : Re(\lambda) < 0\}$, $\sigma_r(P) = \emptyset$ and $\sigma_c(P) = \{\lambda \in \mathbb{C} : Re(\lambda) = 0\}$.

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NEW TYPES OF CONNECTEDNESS AND INTERMEDIATE VALUE THEOREM IN IDEAL TOPOLOGICAL SPACES

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Abstract. The definitions of new type separated subsets are given in ideal topological spaces. By using these definitions, we introduce new types of connectedness. It is shown that these new types of connectedness are more general than some previously defined concepts of connectedness in ideal topological spaces. The new types of connectedness are compared with well-known connectedness in point-set topology. Then, the intermediate value theorem for ideal topological spaces is given. Also, for some special cases, it is shown that the intermediate value theorem in ideal topological spaces and the intermediate value theorem in topological spaces coincide.

1. Introduction

The concept of ideal in topological spaces was first studied by Kuratowski [16] and Vaidyanathswamy [33]. More properties are given for ideal topological spaces in [10]. In [10,33], it is shown that the local function of a set is a generalization of the concepts of closure point, ω-accumulation point and condensation point of that set. The concept of ideal was applied not only to topology but also to different areas of mathematics. For example, the Cantor-Bendixson Theorem is generalized in [6]. New special spaces such as $\mathcal{I}$-Rothberger [7], $\mathcal{I}$-Baire [17], $\mathcal{I}$-Resolvable and $\mathcal{I}$-Hyperconnected [3],$\mathcal{I}$-Extremally Disconnected [12], $\mathcal{I}$-Alexandroff and $\mathcal{I}_g$-Alexandroff [4] are defined by using ideal. In addition, the concepts of ideal and local function are studied in fuzzy set theory [28], soft set theory [11] and ditopological texture spaces [15].

Connectedness is a topological invariant. So, the concept of connectedness has an important role in general topology. The intermediate value theorem in calculus

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was generalized by means of connectedness in topological spaces [25]. Many types of connectedness are defined by using the local function in [20,31] and these connectedness are stronger connectedness. The generalization of connectedness has been defined in [18,26] More features of connectedness types given in [20] were examined in [14]. In addition, many operators such as local closure function [1], semi-closure local function [9] , weak semi-local function [35,36], semi-local function [13], a-local function [2,21], $\mathcal{M}$-local function [22], c^*-local function [29], Ω-operator [19] and ψ^*-operator [23] are defined in recent years. In this study, we define new types of connectedness by using local functions and local closure functions. In this way, we generalize all connectedness types in [20]. After that, new types of connectedness are compared with well-known connectedness. Also, we define new components with the help of new types of connectedness. In the last section, we give the intermediate value theorem in ideal topological spaces. For the minimal ideal $\mathcal{I} = \{\emptyset\}$, we show that the intermediate value theorem in general topological spaces and the intermediate value theorem in ideal topological spaces coincide.

2. Preliminaries

In any topological space (U, τ), we denote the interior and the closure of the subset M as $Int(M)$ and $Cl(M)$, respectively. The power set of U is denoted by $\mathcal{P}(U)$. Both open and closed subsets are called clopen. The collection of all open neighborhoods of the point x is denoted by $\tau(x)$.

Definition 1. *[16] Let U be nonempty set and $\mathcal{I} \subseteq \mathcal{P}(U)$. If the following conditions are satisfied:*

(1) $\emptyset \in \mathcal{I}$.
(2) *If $M \in \mathcal{I}$ and $K \subseteq M$, then $K \in \mathcal{I}$.*
(3) *If $M, K \in \mathcal{I}$, then $M \cup K \in \mathcal{I}$.*

then the collection $\mathcal{I}$ is called an ideal on U.

The ideal $\mathcal{I} = \{\emptyset\}$ is called minimal ideal and the ideal $\mathcal{I} = \mathcal{P}(U)$ is called maximal ideal. Although the topology is not needed to define an ideal, some collections of sets in the topological spaces form ideals. In any topological space (U, τ), a subset M is called nowhere dense, if $Int(Cl(M)) = \emptyset$. The subset M is called discrete set if $M \cap M^d = \emptyset$ (where M^d is derived set of M). A subset of U is called meager (or set of first category) if it can be written as a countable union of nowhere dense subsets of U. A subset of U is called relatively compact if its closure is compact. The collection of all nowhere dense subsets $\mathcal{I}_{nw} = \{M \subseteq U : M \text{ is nowhere dense}\}$, the collection of all closed-discrete subsets $\mathcal{I}_{cd} = \{M \subseteq U : M \text{ is closed and discrete}\}$, the collection of all meager subsets $\mathcal{I}_{mg} = \{M \subseteq U : M \text{ is meager set}\}$, the collection of all relatively compact subsets $\mathcal{I}_K = \{M \subseteq U : M \text{ is relatively compact}\}$ and $\mathcal{I}_{f \circ g} = \{A \subseteq U : f \circ g(A) = \emptyset\}$, where $f \sim^U g$ are ideals on U [16,24,33].

If (U, τ) is a topological space with an ideal $\mathcal{I}$ on U, this space is called an ideal topological space or briefly $\mathcal{I}$-space. Sometimes we denote this case with the triple $(U, \tau, \mathcal{I})$.

Definition 2. *[16] In any $\mathcal{I}$-space (U, τ), a function $(.)^* : \mathcal{P}(U) \to \mathcal{P}(U)$ is defined by*

$$M^*(\mathcal{I}, \tau) = \{x \in U : O \cap M \notin \mathcal{I} \text{ for every } O \in \tau(x)\}$$

is called the local function of a subset M.

Sometimes we write briefly $M^*(\mathcal{I})$ or M^* instead of $M^*(\mathcal{I}, \tau)$. $M \cup M^* = Cl^*(M)$ is a Kuratowski closure operator. So this operator generates a topology on U. This topology is denoted by τ^* and defined as $\tau^* = \{M \subseteq U : Cl^*(U \setminus M) = (U \setminus M)\}$. Moreover $\tau \subseteq \tau^*$ and so $M \subseteq Cl^*(M) \subseteq Cl(M)$. Elements of τ^* are called $*$-open. The complement of a $*$-open subset is called $*$-closed.

Proposition 1. *[10, 16, 33] Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$.*

(1) *If $M \subseteq K$, then $M^* \subseteq K^*$.*
(2) *$M^* = Cl(M^*) \subseteq Cl(M)$. That is, M^* is closed set.*
(3) *$(M \cup K)^* = M^* \cup K^*$.*
(4) *If $\mathcal{I} = \{\emptyset\}$, then $M^*(\{\emptyset\}) = Cl(M)$.*
(5) *If $\mathcal{I} = \mathcal{P}(U)$, then $M^*(\mathcal{P}(U)) = \emptyset$.*

Definition 3. *[1] In any $\mathcal{I}$-space (U, τ), a function $\Gamma(.) : \mathcal{P}(U) \to \mathcal{P}(U)$ defined by*

$$\Gamma(M)(\mathcal{I}, \tau) = \{x \in U : Cl(O) \cap M \notin \mathcal{I} \text{ for every } O \in \tau(x)\}$$

is called the local closure function of the subset M.

Sometimes we write briefly $\Gamma(M)(\mathcal{I})$ or $\Gamma(M)$ instead of $\Gamma(M)(\mathcal{I}, \tau)$.

The θ-closure of any subset M is defined in [34] as $Cl_\theta(M) = \{x \in U : Cl(O) \cap M \neq \emptyset$ for every $O \in \tau(x)\}$.

Proposition 2. *[1] Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$.*

(1) *If $M \subseteq K$, then $\Gamma(M) \subseteq \Gamma(K)$.*
(2) *$\Gamma(M) = Cl(\Gamma(M)) \subseteq Cl_\theta(M)$. That is $\Gamma(M)$ is closed set.*
(3) *$\Gamma(M \cup K) = \Gamma(M) \cup \Gamma(K)$*
(4) *If $\mathcal{I} = \{\emptyset\}$, then $\Gamma(M)(\{\emptyset\}) = Cl_\theta(M)$.*
(5) *If $\mathcal{I} = \mathcal{P}(U)$, then $\Gamma(M)(\mathcal{P}(U)) = \emptyset$.*

Lemma 1. *[1] In any $\mathcal{I}$-space (U, τ), $M^*(\mathcal{I}, \tau) \subseteq \Gamma(M)(\mathcal{I}, \tau)$.*

Definition 4. *[30] Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The subset M is called Γ-dense-in-itself if $M \subseteq \Gamma(M)$.*

Definition 5. *[8] Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The subset M is called $*$-dense-in-itself if $M \subseteq M^*$.*

Nonempty subsets M, K of a topological space (U, τ) are called separated if $Cl(M) \cap K = M \cap Cl(K) = \emptyset$. The topological space (U, τ) is called connected if U is not the union of two separated subsets. The subset M in a topological space is connected if and only if M is not the union of separated subsets in the subspace (M, τ_M) or equivalently M is not the union of two separated subsets in (U, τ). There are many expressions equivalent to definition of connectedness in the literature [5, 25, 32]. We say that the subsets M, K are τ-separated if they are separated subsets in (U, τ). We say that the subset M is τ-connected if it is a connected subset in (U, τ). That an $\mathcal{I}$-space (U, τ) is τ-connected means that the topological space (U, τ) is τ-connected.

Definition 6. *[20] Let (U, τ) be an $\mathcal{I}$-space and M, K be nonempty subsets in this space. These subsets are called $*_*$-separated (resp. $*$-Cl^*-separated, $*$-Cl-separated), if $M^* \cap K = M \cap K^* = M \cap K = \emptyset$ (resp. $M^* \cap Cl^*(K) = Cl^*(M) \cap K^* = M \cap K = \emptyset$, $M^* \cap Cl(K) = Cl(M) \cap K^* = M \cap K = \emptyset$).*

Definition 7. *[20] Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. The subset M is called $*_*$-connected (resp. $*$-Cl^*-connected, $*$-Cl-connected) if it is not the union of two $*_*$-separated (resp. $*$-Cl^*-separated, $*$-Cl-separated) subsets.*

From these definitions, the following diagrams are obtained in [20].

$*$-Cl-separated $\Longrightarrow$ $*$-Cl^*-separated $\Longrightarrow$ $*_*$-separated $\Longleftrightarrow$ τ^*-separated

FIGURE 1. Relations among types of separated subsets which are defined via local function

τ^*-connected $\Longleftrightarrow$ $*_*$-connected $\Longrightarrow$ $*$-Cl^*-connected $\Longrightarrow$ $*$-Cl-connected

FIGURE 2. Relations among types of connectedness which are defined via local function

3. New Types of Separated Subsets via Local Closure

Definition 8. *Let (U, τ) be an $\mathcal{I}$-space and M,K be nonempty subsets of U. These subsets are called*

(1) *Γ-Cl-separated if $\Gamma(M) \cap Cl(K) = Cl(M) \cap \Gamma(K) = M \cap K = \emptyset$.*
(2) *Γ-Cl^*-separated if $\Gamma(M) \cap Cl^*(K) = Cl^*(M) \cap \Gamma(K) = M \cap K = \emptyset$.*
(3) *Γ-separated if $\Gamma(M) \cap K = M \cap \Gamma(K) = M \cap K = \emptyset$.*
(4) *Γ-$*$-separated if $\Gamma(M) \cap K^* = M^* \cap \Gamma(K) = M \cap K = \emptyset$.*
(5) *2^*-separated if $M^* \cap K^* = M \cap K = \emptyset$.*

Theorem 1. *Let (U, τ) be an $\mathcal{I}$-space and M,K be nonempty subsets of U.*

(1) *If M,K are Γ-Cl-separated, then they are Γ-Cl^*-separated subsets.*

(2) *If M,K are Γ-Cl-separated, then they are $*$-Cl-separated subsets.*
(3) *If M,K are Γ-Cl^*-separated, then they are Γ-separated subsets.*
(4) *If M,K are Γ-Cl^*-separated, then they are $*$-Cl^*-separated subsets.*
(5) *If M,K are Γ-separated, then they are $*_*$-separated subsets.*
(6) *If M,K are Γ-Cl^*-separated, then they are Γ-$*$-separated subsets.*
(7) *If M,K are Γ-$*$-separated, then they are 2^*-separated subsets.*
(8) *If M,K are $*$-Cl^*-separated, then they are 2^*-separated subsets.*

Proof. Since $M \subseteq Cl^*(M) \subseteq Cl(M)$, $K \subseteq Cl^*(K) \subseteq Cl(K)$ and Definition 8 , (1)-(3)-(6)-(8) are obtained. By using Lemma 1 and Definition 8, (2)-(4)-(5)-(7) are obtained. □

In addition to this theorem, since $\tau \subseteq \tau^*$, τ-separated subsets are τ^*-separated. From Theorem 1 and Figure 1, we obtain the following diagram:

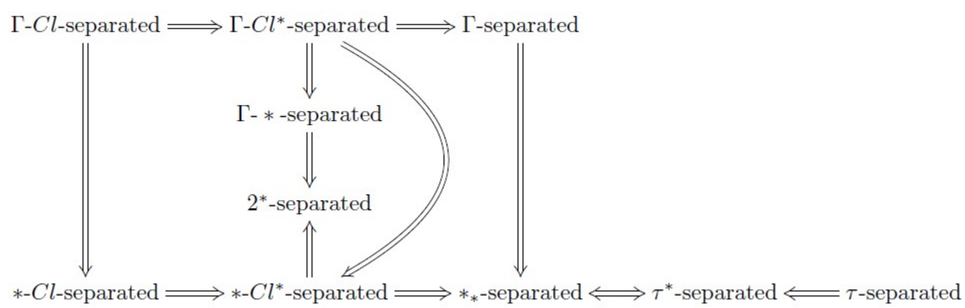

FIGURE 3. Relations among new types of separated subsets

For this diagram, counterexamples and independent concepts are shown in Example 1 and Example 2.

Example 1. *Let $\tau = \{\emptyset, U, \{x\}, \{d\}, \{x,y\}, \{x,z\}, \{a,c\}, \{x,d\}, \{x,y,z\}, \{a,c,d\}, \{x,a,c\}, \{x,z,d\}, \{x,y,d\}, \{a,b,c,d\}, \{x,a,c,d\}, \{x,y,a,c\}, \{x,z,a,c\}, \{x,y,z,d\}, \{x,y,z,a,c\}, \{x,a,b,c,d\}, \{x,y,a,c,d\}, \{x,z,a,c,d\}, \{x,z,a,b,c,d\}, \{x,y,a,b,c,d\}, \{x,y,z,a,c,d\}\}$ be a topology on $U = \{a,b,c,d,x,y,z\}$ and let $\mathcal{I} = \{\emptyset, \{x\}, \{a\}, \{a,x\}\}$ be an ideal on U. The following table gives information about some subsets of this ideal topological space.*

According to Table:

(1) *C and E are Γ-Cl^*-separated subsets but not Γ-Cl-separated.*
(2) *D and G are $*$-Cl-separated subsets but not Γ-Cl-separated.*
(3) *D and G are $*$-Cl^*-separated but not Γ-Cl^*-separated.*
(4) *C and H are Γ-separated subsets but not Γ-Cl^*-separated.*
(5) *D and G are $*_*$-separated subsets but not Γ-separated.*
(6) *E and F are Γ-$*$-separated subsets but not Γ-Cl^*-separated.*

TABLE 1. Information about some subsets according to the given $\mathcal{I}$-space

$A=\{b\}$	$A^*=\{b\}$	$\Gamma(A)=\{a,b,c,d\}$	$Cl^*(A)=\{b\}$	$Cl(A)=\{b\}$
$B=\{c\}$	$B^*=\{a,b,c\}$	$\Gamma(B)=\{a,b,c\}$	$Cl^*(B)=\{a,b,c\}$	$Cl(B)=\{a,b,c\}$
$C=\{d\}$	$C^*=\{b,d\}$	$\Gamma(C)=\{b,d\}$	$Cl^*(C)=\{b,d\}$	$Cl(C)=\{b,d\}$
$D=\{z\}$	$D^*=\{z\}$	$\Gamma(D)=\{x,y,z\}$	$Cl^*(D)=\{z\}$	$Cl(D)=\{z\}$
$E=\{a,y\}$	$E^*=\{y\}$	$\Gamma(E)=\{x,y,z\}$	$Cl^*(E)=\{a,y\}$	$Cl(E)=\{a,b,c,y\}$
$F=\{b,c\}$	$F^*=\{a,b,c\}$	$\Gamma(F)=\{a,b,c,d\}$	$Cl^*(F)=\{a,b,c\}$	$Cl(F)=\{a,b,c\}$
$G=\{b,y\}$	$G^*=\{b,y\}$	$\Gamma(G)=U$	$Cl^*(G)=\{b,y\}$	$Cl(G)=\{b,y\}$
$H=\{c,y\}$	$H^*=\{a,b,c,y\}$	$\Gamma(H)=\{a,b,c,x,y,z\}$	$Cl^*(H)=\{a,b,c,y\}$	$Cl(H)=\{a,b,c,y\}$
$K=\{d,x\}$	$K^*=\{b,d\}$	$\Gamma(K)=\{b,d\}$	$Cl^*(K)=\{b,d,x\}$	$Cl(K)=\{b,d,x,y,z\}$
$L=\{d,y\}$	$L^*=\{b,d,y\}$	$\Gamma(L)=\{b,d,x,y,z\}$	$Cl^*(L)=\{b,d,y\}$	$Cl(L)=\{b,d,y\}$
$M=\{x,z\}$	$M^*=\{z\}$	$\Gamma(M)=\{x,y,z\}$	$Cl^*(M)=\{x,z\}$	$Cl(M)=\{x,y,z\}$

(7) *G and M are 2^*-separated subsets but not Γ-$*$-separated.*
(8) *E and F are 2^*-separated subsets but not $*$-Cl^*-separated.*
(9) *D and G are $*$-Cl-separated subsets but not Γ-Cl^*-separated. C and E are Γ-Cl^*-separated subsets but not $*$-Cl-separated. That is, the concepts of $*$-Cl-separated and Γ-Cl^*-separated are independent of each other.*
(10) *E and F are Γ-$*$-separated subsets but not $*$-Cl-separated. D and G are $*$-Cl-separated subsets but not Γ-$*$-separated That is, the concepts of $*$-Cl-separated and Γ-$*$-separated are independent of each other.*
(11) *E and F are Γ-$*$-separated subsets but not $*$-Cl^*-separated. D and G are $*$-Cl^*-separated subsets but not Γ-$*$-separated. That is, the concepts of Γ-$*$-separated and $*$-Cl^*-separated are independent of each other.*
(12) *A and E are Γ-$*$-separated subsets but not Γ-separated. C and H are Γ-separated subsets but not Γ-$*$-separated. That is, the concepts of Γ-$*$-separated and Γ-separated are independent of each other.*
(13) *E and F are Γ-$*$-separated subsets but not $*_*$-separated. D and G are $*_*$-separated subsets but not Γ-$*$-separated. That is, the concepts of Γ-$*$-separated and $*_*$-separated are independent of each other.*
(14) *E and F are 2^*-separated subsets but not Γ-separated. C and H are Γ-separated subsets but not 2^*-separated. That is, the concepts of 2^*-separated and Γ-separated are independent of each other.*
(15) *H and K are $*_*$-separated subsets but not 2^*-separated. E and F are 2^*-separated subsets but not $*_*$-separated. That is, the concepts of 2^*-separated and $*_*$-separated are independent of each other.*

(16) *D and G are $*$-Cl-separated subsets but not Γ-separated. C and H are Γ-separated subsets but not $*$-Cl-separated. So, the concepts of $*$-Cl-separated and Γ-separated are independent of each other.*

(17) *D and G are $*$-Cl^*-separated subsets but not Γ-separated. B and L are Γ-separated subsets but not $*$-Cl^*-separated. So, the concepts of $*$-Cl^*-separated and Γ-separated are independent of each other.*

Lemma 2. *Let (U, τ) be $\mathcal{P}(U)$-space and M, K be nonempty subsets of U such that $M \cap K = \emptyset$. Then, the subsets M and K are Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*,2^*, $*_*$)-separated.*

Proof. In this space, since $\Gamma(M) = \Gamma(K) = M^* = K^* = \emptyset$, these subsets are Γ-Cl (Γ-Cl^*,$*$-Cl, $*$-Cl^*, Γ, Γ-$*$,2^*, $*_*$)-separated. □

Example 2. Let $(\mathbb{R}, \tau_L)$ be $\mathcal{P}(\mathbb{R})$-space, where $\mathbb{R}$ is the set of real numbers with left-ray topology τ_L i.e. $\tau_L = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Consider the subsets $M = (-\infty, 3)$ and $K = (3, 5)$. Since $Cl(M) = \mathbb{R}$ and $Cl(K) = [3, +\infty)$, these subsets are not τ-separated. But M and K are Γ-Cl ($*$-Cl, $*$-Cl^*, Γ, Γ-$*$, Γ-Cl^*,2^*)-separated subsets from Lemma 2.

In Example 1, D and G are τ-separated subsets but not Γ-Cl(Γ, Γ-$*$, Γ-Cl^*)-separated. Moreover, B and L are τ-separated subsets but not $*$-Cl ($*$-Cl^*, 2^*) separated.

Consequently, the concepts of Γ-Cl ($*$-Cl, $*$-Cl^*,Γ, Γ-$*$, Γ-Cl^*,2^*)-separated and τ-separated are independent of each other.

Theorem 2. *[27] In any $\mathcal{I}$-space (U, τ), each of the following conditions implies that $M^* = \Gamma(M)$ for any subset M of U:*

(1) *τ has a clopen base.*
(2) *τ is a T_3-space on U.*
(3) *$\mathcal{I} = \mathcal{I}_{cd}$.*
(4) *$\mathcal{I} = \mathcal{I}_K$.*
(5) *$\mathcal{I}_{nw} \subseteq \mathcal{I}$.*
(6) *$\mathcal{I} = \mathcal{I}_{mg}$.*

Corollary 1. *Assume that any of the conditions in Theorem 2 is satisfied and M, K are the subsets in any $\mathcal{I}$-space (U, τ). Then,*

(1) *The subsets M and K are Γ-Cl-separated if and only if they are $*$-Cl-separated.*
(2) *The subsets M and K are Γ-Cl^*-separated if and only if they are $*$-Cl^*-separated.*
(3) *The subsets M and K are Γ-separated if and only if they are $*_*$-separated.*
(4) *The subsets M and K are 2^*-separated if and only if they are Γ-$*$-separated.*

Proof. It is obvious from Definition 8 and Theorem 2. □

Theorem 3. Let (U,τ) be an $\mathcal{I}$-space and $M,K \subseteq U$. Subsets M and K are both Γ-separated and Γ-$*$-separated if and only if they are Γ-Cl^*-separated.

Proof. Since M and K are both Γ-separated and Γ-$*$-separated,

$$\begin{aligned}\Gamma(M)\cap Cl^*(K) &= \Gamma(M)\cap(K\cup K^*)\\ &= (\Gamma(M)\cap K)\cup(\Gamma(M)\cap K^*)\\ &= \emptyset\end{aligned}$$

$$\begin{aligned}Cl^*(M)\cap\Gamma(K) &= (M\cup M^*)\cap\Gamma(K)\\ &= (M\cap\Gamma(K))\cup(M^*\cap\Gamma(K))\\ &= \emptyset\end{aligned}$$

and $M\cap K=\emptyset$. So, M and K are Γ-Cl^*-separated subsets.

Conversely, let M and K be Γ-Cl^*-separated subsets. From Figure 3, these subsets are both Γ-separated and Γ-$*$-separated. □

Theorem 4. *Let (U,τ) be an $\mathcal{I}$-space and $M,K \subseteq U$. Subsets M and K are both $*_*$-separated and 2^*-separated if and only if these subsets are $*$-Cl^*-separated.*

Proof. Since M and K are both $*_*$-separated and 2^*-separated,

$$\begin{aligned}M^*\cap Cl^*(K) &= M^*\cap(K\cup K^*)\\ &= (M^*\cap K)\cup(M^*\cap K^*)\\ &= \emptyset\end{aligned}$$

$$\begin{aligned}Cl^*(M)\cap K^* &= (M\cup M^*)\cap K^*\\ &= (M\cap K^*)\cup(M^*\cap K^*)\\ &= \emptyset\end{aligned}$$

and $M\cap K=\emptyset$. So, M and K are $*$-Cl^*-separated subsets.

Conversely, let M and K be $*$-Cl^*-separated subsets. From Figure 3, these subsets are both $*_*$-separated and 2^*-separated. □

Theorem 5. *Let (U,τ) be an $\mathcal{I}$-space and $M,K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M,K are Γ-separated.*
(2) *The subsets M,K are Γ-dense-in-itself.*
(3) $M\cup K\in\tau$.

then $M\in\tau$ and $K\in\tau$.

Proof. Since the subsets M,K are Γ-separated, $M\cap\Gamma(K)=\emptyset$. So, $M\subseteq(U\setminus\Gamma(K))$. From Proposition 2-(2), $U\setminus\Gamma(K)$ is open set and hence $(M\cup K)\cap(U\setminus\Gamma(K))=M$ is an open subset. Similarly, it can be showed that the subset K is open. □

Corollary 2. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-Cl (Γ-Cl^*)-separated.*
(2) *The subsets M, K are Γ-dense-in-itself.*
(3) $M \cup K \in \tau$.

then $M \in \tau$ and $K \in \tau$.

Proof. From Figure 3 and Theorem 5 , it is obtained. □

Theorem 6. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-separated.*
(2) *The subsets M, K are Γ-dense-in-itself.*
(3) $M \cup K \in \tau^*$.

then $M \in \tau^*$ and $K \in \tau^*$.

Proof. Since the subsets M, K are Γ-separated, $M \cap \Gamma(K) = \emptyset$. So, $M \subseteq (U \setminus \Gamma(K))$. From Proposition 2-(2), $U \setminus \Gamma(K)$ is open set. Since $\tau \subseteq \tau^*$, $U \setminus \Gamma(K) \in \tau^*$ and hence $(M \cup K) \cap (U \setminus \Gamma(K)) = M$ is in τ^*. Similarly, it can be showed that the subset K is in τ^*. □

Corollary 3. *Let (U, τ) be an $\mathcal{I}$-space space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-Cl (Γ-Cl^*)-separated.*
(2) *The subsets M, K are Γ-dense-in-itself.*
(3) $M \cup K \in \tau^*$.

then $M \in \tau^*$ and $K \in \tau^*$.

Proof. It is obtained from Figure 3 and Theorem 6. □

Theorem 7. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets M, K are Γ-$*$-separated.*
(2) *The subsets M, K are $*$-dense-itself.*
(3) $M \cup \Gamma(K) \in \tau$ *and* $\Gamma(M) \cup K \in \tau$.

then $\Gamma(M)$ and $\Gamma(K)$ are clopen subsets.

Proof. From Proposition 2-(2), $\Gamma(M)$ and $\Gamma(K)$ are closed subsets. We only show that they are open subsets. Since the subsets M, K are Γ-$*$-separated, $\Gamma(M) \cap K^* = \emptyset$. So, $\Gamma(M) \subseteq (U \setminus K^*)$. From Proposition 1-(2), $U \setminus K^*$ is open set and hence $(\Gamma(M) \cup K) \cap (U \setminus K^*) = \Gamma(M)$ is open. Similarly, it can be showed that the subset $\Gamma(K)$ is open. □

Corollary 4. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* Γ-Cl *(* Γ-Cl^**)-separated.*
(2) *The subsets* M, K *are* $*$*-dense-itself.*
(3) $M \cup \Gamma(K) \in \tau$ *and* $\Gamma(M) \cup K \in \tau$.

then $\Gamma(M)$ *and* $\Gamma(K)$ *are clopen subsets.*

Proof. It is obtained from Figure 3 and Theorem 7. □

Theorem 8. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$*. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* Γ-$*$*-separated.*
(2) *The subsets* M, K *are* Γ*-dense-itself.*
(3) $M^* \cup K \in \tau$ *and* $M \cup K^* \in \tau$.

then M^* *and* K^* *are clopen subsets.*

Proof. From Proposition 1-(2), M^* and K^* are closed subsets. We must show that they are open subsets. Since the subsets M, K are Γ-$*$-separated, $M^* \cap \Gamma(K) = \emptyset$. So $M^* \subseteq U \setminus \Gamma(K)$. Since $U \setminus \Gamma(K)$ is open subset, $(M^* \cup K) \cap (U \setminus \Gamma(K)) = M^* \in \tau$. Similarly, it can be showed that the subset K^* is open.

□

Corollary 5. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$*. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* Γ-Cl *(* Γ-Cl^**)-separated.*
(2) *The subsets* M, K *are* Γ*-dense-itself.*
(3) $M^* \cup K \in \tau$ *and* $M \cup K^* \in \tau$.

then M^* *and* K^* *are clopen subsets.*

Proof. It is obtained from Figure 3 and Theorem 8. □

Theorem 9. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$*. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* 2^**-separated.*
(2) *The subsets* M, K *are* $*$*-dense-itself.*
(3) $M^* \cup K \in \tau$ *and* $M \cup K^* \in \tau$.

then M^* *and* K^* *are clopen subsets.*

Proof. From Proposition 1-(2), M^* and K^* are closed subsets. We must show that they are open subsets. Since the subsets M, K are 2^*-separated, $M^* \cap K^* = \emptyset$. So, $M^* \subseteq U \setminus K^*$. Since $U \setminus K^*$ is open, $(M^* \cup K) \cap (U \setminus K^*) = M^* \in \tau$. Similarly, it can be showed that the subset K^* is open. □

Corollary 6. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$*. If the following conditions are satisfied:*

(1) *The subsets* M, K *are* Γ-Cl *(* Γ-Cl^**,* Γ-$*$*)-separated.*
(2) *The subsets* M, K *are* $*$*-dense-itself.*

(3) $M^* \cup K \in \tau$ *and* $M \cup K^* \in \tau$.

then M^* *and* K^* *are clopen subsets.*

Proof. From Figure 3 and Theorem 9, it is obtained. □

Theorem 10. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$. *If the following conditions are satisfied:*

(1) *The subsets* M, K *are* $*_*$*-separated.*
(2) *The subsets* M, K *are* $*$*-dense-itself.*
(3) $M \cup K \in \tau$.

then M *and* K *are open subsets.*

Proof. Since the subsets M, K are $*_*$-separated, $M \cap K^* = \emptyset$. So $M \subseteq U \setminus K^*$. Since $U \setminus K^*$ is open subset, $(M \cup K) \cap (U \setminus K^*) = M$ is in τ. Similarly, it can show that the subset K is open. □

Corollary 7. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $M, K \subseteq U$. *If the following conditions are satisfied:*

(1) *The subsets* M, K *are* Γ*-*Cl *(* Γ*-*Cl^**,* Γ*,* $*$*-*Cl*,* $*$*-*Cl^**,* τ*)-separated.*
(2) *The subsets* M, K *are* $*$*-dense-itself.*
(3) $M \cup K \in \tau$.

then M *and* K *are open subsets.*

Proof. From Figure 3 and Theorem 10, it is obtained. □

Theorem 11. *Let* (U, τ) *be* $\{\emptyset\}$*-space and* $M, K \subseteq U$. *Then the following statements are equivalent:*

(1) *The subsets* M *and* K *are* $*_*$*-separated.*
(2) *The subsets* M *and* K *are* τ*-separated.*

Proof. Since $M^*(\{\emptyset\}) = Cl(M)$, $K^*(\{\emptyset\}) = Cl(K)$, these expressions are equivalent. □

Theorem 12. *Let* (U, τ) *be* $\{\emptyset\}$*-space and* $M, K \subseteq U$. *Then the following statements are equivalent:*

(1) *The subsets* M *and* K *are* 2^**-separated.*
(2) *The subsets* M *and* K *are* $*$*-*Cl^**-separated.*
(3) *The subsets* M *and* K *are* $*$*-*Cl*-separated.*

Proof. Since $M^*(\{\emptyset\}) = Cl^*(M) = Cl(M)$ and $K^*(\{\emptyset\}) = Cl^*(K) = Cl(K)$, these expressions are equivalent. □

Theorem 13. *Let* (U, τ) *be* $\{\emptyset\}$*-space and* $M, K \subseteq U$. *Then the following statements are equivalent:*

(1) *The subsets* M *and* K *are* Γ*-*$*$*-separated.*
(2) *The subsets* M *and* K *are* Γ*-*Cl^**-separated.*

(3) *The subsets M and K are Γ-Cl-separated.*

Proof. Since $M^*(\{\emptyset\}) = Cl^*(M) = Cl(M)$ and $K^*(\{\emptyset\}) = Cl^*(K) = Cl(K)$, these expressions are equivalent. □

4. New Types of Connectedness via Local Closure

Definition 9. *Let (U,τ) be an $\mathcal{I}$-space and $M \subseteq U$. The subset M is called Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected if it is not the union of two Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-separated subsets in $\mathcal{I}$-space (U,τ). Otherwise, the subset M is called not Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected . Particularly, if U is Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected, the $\mathcal{I}$-space (U,τ) is called Γ-Cl (resp. Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected $\mathcal{I}$-space.*

Theorem 14. *In any $\mathcal{I}$-space,*

(1) *Every Γ-Cl^*-connected subset is Γ-Cl-connected.*
(2) *Every $*$-Cl-connected subset is Γ-Cl-connected.*
(3) *Every Γ-connected subset is Γ-Cl^*-connected.*
(4) *Every $*$-Cl^*-connected subset is Γ-Cl^*-connected.*
(5) *Every $*_*$-connected subset is Γ-connected.*
(6) *Every Γ-$*$-connected subset is Γ-Cl^*-connected.*
(7) *Every 2^*-connected subset is Γ-$*$-connected.*
(8) *Every 2^*-connected subset is $*$-Cl^*-connected.*

Proof. (1) Let M be Γ-Cl^*-connected subset. Suppose that it is not Γ-Cl-connected. So, there are subsets K, S which are Γ-Cl-separated and $K \cup S = M$. From Theorem 1-(1), the subsets K and S are Γ-Cl^*-separated. Hence, the subset M is not Γ-Cl^*-connected. This is a contradiction. Consequently, the subset M is Γ-Cl-connected.

By using Theorem 1 (or Figure 3), other proofs are obtained similarly. □

The following diagram is obtained by Theorem 14 and Figure 2.

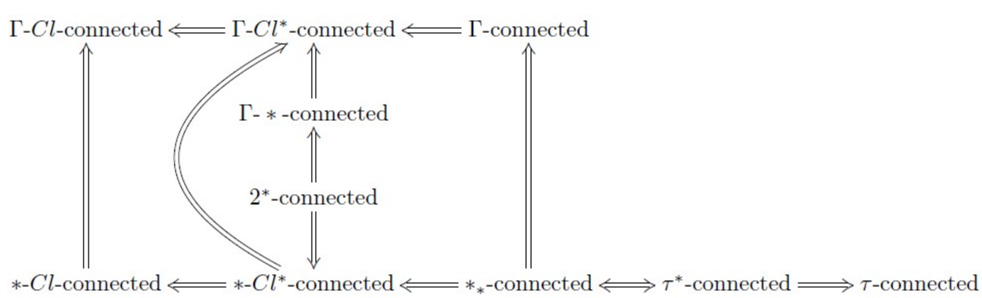

FIGURE 4. Relations among new types of connectedness

For this diagram, counterexamples and independent concepts are shown in Example 3 and Example 4.

Example 3. *Consider the $\mathcal{I}$-space in Example 1.*

(1) *The subset $P = \{y, z\}$ is Γ (resp. Γ-Cl^*, Γ-Cl, Γ-$*$)-connected but not $*_*$ (resp. $*$-Cl^*, $*$-Cl, 2^*)-connected.*
(2) *The subset $R = \{a, d\}$ is Γ-Cl-connected but not Γ-Cl^*-connected.*
(3) *The subset $S = \{c, d\}$ is Γ-Cl^*-connected but not Γ-connected.*
(4) *The subset $T = \{a, c\}$ is $*$-Cl^* (resp. Γ-Cl^*)-connected but not 2^* (resp. Γ-$*$)-connected.*
(5) *The subset $P = \{y, z\}$ is Γ-Cl^*-connected but not $*$-Cl-connected. The subset $R = \{a, d\}$ is $*$-Cl-connected but not Γ-Cl^*-connected. That is, the concepts of $*$-Cl-connected and Γ-Cl^*-connected are independent of each other.*
(6) *The subset $P = \{y, z\}$ is Γ-connected but not $*$-Cl-connected. The subset $R = \{a, d\}$ is $*$-Cl-connected but not Γ-connected. That is, the concepts of Γ-connected and $*$-Cl-connected are independent of each other.*
(7) *The subset $P = \{y, z\}$ is Γ-$*$-connected but not $*$-Cl-connected. The subset $R = \{a, d\}$ is $*$-Cl-connected but not Γ-$*$-connected. That is, the concepts of Γ-$*$-connected and $*$-Cl-connected are independent of each other.*
(8) *The subset $P = \{y, z\}$ is Γ-$*$-connected but not $*$-Cl^*-connected. The subset $T = \{a, c\}$ is $*$-Cl^*-connected but not Γ-$*$-connected. That is, the concepts of Γ-$*$-connected and $*$-Cl^*-connected are independent of each other.*
(9) *The subset $P = \{y, z\}$ is Γ-connected but not 2^*-connected. The subset $S = \{c, d\}$ is 2^*-connected but not Γ-connected. That is, the concepts of Γ-connected and 2^*-connected are independent of each other.*
(10) *The subset $P = \{y, z\}$ is Γ-connected but not $*$-Cl^*-connected. The subset $S = \{c, d\}$ is $*$-Cl^*-connected but not Γ-connected. That is, the concepts of Γ-connected and $*$-Cl^*-connected are independent of each other.*
(11) *The subset $S = \{c, d\}$ is Γ-$*$-connected but not Γ-connected. The subset $T = \{a, c\}$ is Γ-connected but not Γ-$*$-connected. That is, the concepts of Γ-$*$-connected and Γ-connected are independent of each other.*
(12) *The subset $S = \{c, d\}$ is Γ-$*$-connected but not $*_*$-connected. The subset $T = \{a, c\}$ is $*_*$-connected but not Γ-$*$-connected. That is, the concepts of Γ-$*$-connected and $*_*$-connected are independent of each other.*
(13) *The subset $S = \{c, d\}$ is 2^*-connected but not $*_*$-connected. The subset $T = \{a, c\}$ is $*_*$-connected but not 2^*-connected. That is, the concepts of 2^*-connected and $*_*$-connected are independent of each other.*

Lemma 3. *Let (U, τ) be $\mathcal{P}(U)$-space and M be a subset of U. If the subset M has more than one element, it is not Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*,2^*, $*_*$)-connected.*

Proof. Let K,S be nonempty subsets such that $M = K \cup S$ and $K \cap S = \emptyset$. From Lemma 2, the subsets K and S are Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*,2^*, $*_*$)-separated. So, M is not Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*,2^*, $*_*$)-connected. □

Example 4. *Consider the $\mathcal{P}(\mathbb{R})$-space in Example 2. The subset $M = (-\infty, 3)$ is τ_L-connected but not Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected from Lemma 3.*

According to the $\mathcal{I}$-space given in Example 1, $S = \{c, d\}$ is Γ-Cl ($*$-Cl^*, $*$-Cl, Γ-$*$, Γ-Cl^*, 2^*)-connected but not τ-connected. Moreover, the subset $P = \{y, z\}$ is Γ-connected but not τ-connected.

Consequently, the concepts of Γ-Cl ($*$-Cl^*, $*$-Cl, Γ, Γ-$*$, Γ-Cl^*, 2^*)-connected and τ-connected are independent of each other.

Lemma 4. *[1] Let (U, τ) be a topological space and $M \subseteq U$. If the subset M is open, $Cl(M) = Cl_\theta(M)$.*

Lemma 5. *If the subset M is clopen in any $\mathcal{I}$-space,*

$$M^* \subseteq \Gamma(M) \subseteq M = Cl(M) = Cl_\theta(M).$$

Proof. It is obtained by Lemma 4, Lemma 1 and Proposition 2-(2). □

Theorem 15. *If any $\mathcal{I}$-space (U, τ) is Γ-Cl-connected, then it is τ-connected. That is, if the set U is Γ-Cl-connected, then U is τ-connected.*

Proof. Suppose that U is Γ-Cl-connected but not τ-connected. So, there is a clopen proper subset M in this space. From Lemma 5,

$$\Gamma(M) \cap Cl(U \setminus M) \subseteq M \cap (U \setminus M) = \emptyset$$
$$Cl(M) \cap \Gamma(U \setminus M) \subseteq M \cap (U \setminus M) = \emptyset$$

and $M \cap (U \setminus M) = \emptyset$. So, the subsets M and $(U \setminus M)$ are Γ-Cl-separated. Since $M \cup (U \setminus M) = U$, U is not Γ-Cl-connected. This is a contradiction. As a result, U is τ-connected. □

Theorem 16. *If any $\mathcal{I}$-space (U, τ) is Γ-Cl^* (Γ, Γ-$*$, 2^*, $*$-Cl, $*$-Cl^*, $*_*$)-connected, then it is τ-connected.*

Proof. The proof is obtained by Figure 4 and Theorem 15. □

Corollary 8. *Suppose that any of the conditions in Theorem 2 is satisfied and let M be subsets in any $\mathcal{I}$-space (U, τ). Then,*

(1) *The subset M is Γ-Cl-connected if and only if it is $*$-Cl-connected.*
(2) *The subset M is Γ-Cl^*-connected if and only if it is $*$-Cl^*-connected.*
(3) *The subset M is Γ-connected if and only if it is $*_*$-connected.*
(4) *The subset M is 2^*-connected if and only if it is Γ-$*$-connected.*

Proof. It is obvious from Definition 9 and Theorem 2. □

Corollary 9. *Let (U, τ) be an $\mathcal{I}$-space and M, K be subsets of U .*

(1) *If the subsets M,K are both Γ-separated, Γ-$*$-separated subsets and $S = M \cup K$, then S is not Γ-Cl^*-connected subset.*
(2) *If the subset S is not Γ-Cl^*-connected, there are both Γ-separated and Γ-$*$-separated subsets M,K such that $M \cup K = S$.*
(3) *If the subsets M,K are both 2^*-separated, $*_*$-separated subsets and $S = M \cup K$, then S is not $*$-Cl^*-connected subset.*
(4) *If the subset S is not $*$-Cl^*-connected, there are both 2^*-separated and $*_*$-separated subsets M,K such that $M \cup K = S$.*

Proof. It is obtained from Theorem 3 and Theorem 4. □

The following corollaries are obtained from Theorem 11, Theorem 12 and Theorem 13, respectively.

Corollary 10. *Let (U, τ) be $\{\emptyset\}$-space and $M \subseteq U$. Then the following statements are equivalent:*

(1) *The subset M is $*_*$-connected.*
(2) *The subset M is τ-connected.*

Corollary 11. *Let (U, τ) be $\{\emptyset\}$-space and $M \subseteq U$. Then the following statements are equivalent:*

(1) *The subset M is 2^*-connected.*
(2) *The subset M is $*$-Cl^*-connected.*
(3) *The subset M is $*$-Cl-connected.*

Corollary 12. *Let (U, τ) be $\{\emptyset\}$-space and $M \subseteq U$. Then the following statements are equivalent:*

(1) *The subset M is Γ-$*$-connected.*
(2) *The subset M is Γ-Cl^*-connected.*
(3) *The subset M is Γ-Cl-connected.*

Theorem 17. *Let (U, τ) be $\{\emptyset\}$-space and $M \subseteq U$. If the subset M is τ-connected, then it is Γ (Γ-Cl^*, Γ-Cl, $*$-Cl, $*$-Cl^*, Γ-$*$, 2^*)-connected.*

Proof. Let the subset M be τ-connected. From Corollary 10, M is $*_*$-connected. So, it is Γ (Γ-Cl^*, Γ-Cl, $*$-Cl, $*$-Cl^*)-connected by Figure 4. Moreover M is 2^*-connected and Γ-$*$-connected by Corollary 11 and Corollary 12, respectively. □

Considering $\{\emptyset\}$-space (U, τ) given in Theorem 17, it is seen that Γ (Γ-Cl^*, Γ-Cl, $*$-Cl, $*$-Cl^*, Γ-$*$, 2^*)-connectedness is more general concept than the well-known τ-connectedness. Moreover, in this space, $*_*$-connectedness and τ-connectedness are coincident concepts from Corollary 10. However, in any $\mathcal{I}$-space (U, τ), when τ-connectedness of only the set U is considered in Theorem 15 and Theorem 16, it is seen that the concept of τ-connectedness is more general than the concept of Γ (Γ-Cl^*, Γ-Cl, $*$-Cl, $*$-Cl^*, Γ-$*$, 2^*)-connectedness. So the following result is easily obtained.

Corollary 13. *Let (U, τ) be $\{\emptyset\}$-space. The following statements are equivalent:*

(1) *The set U is Γ-Cl-connected.*
(2) *The set U is Γ-Cl^*-connected.*
(3) *The set U is Γ-$*$-connected.*
(4) *The set U is 2^*-connected.*
(5) *The set U is $*$-Cl-connected.*
(6) *The set U is $*$-Cl^*-connected.*
(7) *The set U is $*_*$-connected.*
(8) *The set U is τ-connected.*
(9) *The set U is Γ-connected.*

Proof. It is obtained by Theorem 15, Theorem 16 and Theorem 17. □

5. Theorems on New Types of Connectedness via Local Closure

Theorem 18. *Let (U, τ) be an $\mathcal{I}$-space. If M is Γ-Cl-connected subset of U and S, T are Γ-Cl-separated subsets such that $M \subseteq S \cup T$, then either $M \subseteq S$ or $M \subseteq T$.*

Proof. Since $M = (M \cap S) \cup (M \cap T)$ and the subsets S, T are Γ-Cl-separated,

$$\Gamma(M \cap S) \cap Cl(M \cap T) \subseteq \Gamma(S) \cap Cl(T) = \emptyset$$
$$Cl(M \cap S) \cap \Gamma(M \cap T) \subseteq Cl(S) \cap \Gamma(T) = \emptyset$$

and $(M \cap S) \cap (M \cap T) \subseteq S \cap T = \emptyset$. If $(M \cap S)$ and $(M \cap T)$ are nonempty subsets, the subset M is not Γ-Cl-connected. This is a contradiction. So, either $(M \cap S) = \emptyset$ or $(M \cap T) = \emptyset$. Since $M \subseteq S \cup T$, either $M \subseteq S$ or $M \subseteq T$. □

Theorem 19. *Let (U, τ) be an $\mathcal{I}$-space. If M is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset of U and S, T are Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-separated subsets such that $M \subseteq S \cup T$, then either $M \subseteq S$ or $M \subseteq T$.*

Proof. It is obtained similar to the proof of Theorem 18. □

Theorem 20. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If M is Γ-Cl-connected subset and $M \subseteq K \subseteq \Gamma(M)$, then K is Γ-Cl-connected subset.*

Proof. Suppose that the subset K is not Γ-Cl-connected. Then, there exist Γ-Cl-separated nonempty subsets T, S such that $T \cup S = K$. Since the subsets S and T are Γ-Cl-separated and $M \subseteq K = S \cup T$, by using Theorem 18, we have $M \subseteq S$ or $M \subseteq T$. Suppose that $M \subseteq S$. Then, from Proposition 2-(1), $\Gamma(M) \subseteq \Gamma(S)$. From the hypothesis, $T \subseteq K \subseteq \Gamma(M) \subseteq \Gamma(S)$. Since $\Gamma(M), \Gamma(S)$ are closed subsets by Proposition 2-(2), $Cl(T) \subseteq \Gamma(M) \subseteq \Gamma(S)$, and since the subsets S and T are Γ-Cl-separated, $Cl(T) = Cl(T) \cap \Gamma(M) \subseteq Cl(T) \cap \Gamma(S) = \emptyset$. That is, $T = \emptyset$. This is a contradiction. Similarly, a contradiction is obtained if $M \subseteq T$. Consequently, the subset K is Γ-Cl-connected. □

Theorem 21. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If M is Γ-Cl^*(resp. Γ)-connected subset of U and $M \subseteq K \subseteq \Gamma(M)$, then K is Γ-Cl^*(resp. Γ)-connected subset.*

Proof. It is obtained similar to the proof of Theorem 20. □

Corollary 14. *Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$.*

(1) *If M is both $*$-dense-in-itself and Γ-Cl-connected subset, then M^* is Γ-Cl-connected.*
(2) *If M is both $*$-dense-in-itself and Γ-Cl^*(resp. Γ)-connected subset, then M^* is Γ-Cl^*(resp. Γ)-connected.*
(3) *If M is both Γ-dense-in-itself and Γ-Cl-connected subset, then $\Gamma(M)$ is Γ-Cl-connected.*
(4) *If M is both Γ-dense-in-itself and Γ-Cl^*(resp. Γ)-connected subset, then $\Gamma(M)$ is Γ-Cl^*(resp. Γ)-connected.*
(5) *If M is both Γ-dense-in-itself and Γ-Cl-connected subset, then $Cl(M)$ is Γ-Cl-connected.*
(6) *If M is both Γ-dense-in-itself and Γ-Cl^*(resp. Γ)-connected subset, then $Cl(M)$ is Γ-Cl^*(resp. Γ)-connected.*

Proof. (1) Since M is $*$-dense-in-itself and by Lemma 1, $M \subseteq M^* \subseteq \Gamma(M)$. From Theorem 20 , M^* is Γ-Cl-connected subset.
(2) By using Theorem 21, it is obtained similar to the proof of (1).
(3) Since M is Γ-dense-in-itself, we have $M \subseteq \Gamma(M) \subseteq \Gamma(M)$. From Theorem 20, $\Gamma(M)$ is Γ-Cl-connected subset.
(4) By using Theorem 21, it is obtained similar to the proof of (3).
(5) Since M is Γ-dense-in-itself, $M \subseteq \Gamma(M)$ and so $M \subseteq Cl(M) \subseteq Cl(\Gamma(M))$. Since $\Gamma(M)$ is closed subset from Proposition 2-(2), $M \subseteq Cl(M) \subseteq Cl(\Gamma(M)) = \Gamma(M)$. That is, $M \subseteq Cl(M) \subseteq \Gamma(M)$ and M is Γ-Cl-connected from the hypothesis. Using Theorem 20, we obtain that $Cl(M)$ is Γ-Cl-connected subset.
(6) Since M is Γ-dense-in-itself, $M \subseteq Cl(M) \subseteq \Gamma(M)$ is obtained as in the proof of (5). M is Γ-Cl^*(resp. Γ)-connected from the hypothesis. By using Theorem 21, we obtain that $Cl(M)$ is Γ-Cl^*(resp. Γ)-connected subset.

□

Theorem 22. *Let (U, τ) be an $\mathcal{I}$-space and $\{N_k : k \in \Delta\}$ be a nonempty collection of Γ-Cl-connected subsets of U (where Δ is arbitrary index set). If $\bigcap_{k\in\Delta} N_k \neq \emptyset$, then $\bigcup_{k\in\Delta} N_k$ is Γ-Cl-connected.*

Proof. Suppose that $\bigcup_{k\in\Delta} N_k$ is not Γ-Cl-connected. Then, there exist Γ-Cl-separated nonempty subsets T, S such that $T \cup S = \bigcup_{k\in\Delta} N_k$. Since $\bigcap_{k\in\Delta} N_k \neq \emptyset$, there exists a point $x \in N_k$ for every $k \in \Delta$. Since T, S are Γ-Cl-separated and $x \in \bigcup_{k\in\Delta} N_k$, we have $x \in T$ or $x \in S$. Suppose now that $x \in S$. So, $N_k \cap S \neq \emptyset$ for every $k \in \Delta$. Then, by Theorem 18, $N_k \subseteq S$ for every $k \in \Delta$. Therefore, we obtain $\bigcup_{k\in\Delta} N_k \subseteq S$. That is, $T = \emptyset$. This is a contradiction. Similarly, a contradiction is also obtained if we suppose that $x \in T$. Consequently, $\bigcup_{k\in\Delta} N_k$ is Γ-Cl-connected. □

Theorem 23. *Let (U, τ) be an $\mathcal{I}$-space and $\{N_k : k \in \Delta\}$ be a nonempty collection of Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subsets of U. If $\bigcap_{k\in\Delta} N_k \neq \emptyset$, then $\bigcup_{k\in\Delta} N_k$ is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected.*

Proof. By using Theorem 19, it is obtained similar to the proof of Theorem 22. □

Theorem 24. *Let (U, τ) be an $\mathcal{I}$-space, $\{N_k : k \in \Delta\}$ be a nonempty collection of Γ-Cl-connected subsets and M be Γ-Cl-connected subset. If $M \cap N_k \neq \emptyset$ for every $k \in \Delta$, then $M \cup (\bigcup_{k\in\Delta} N_k)$ is a Γ-Cl-connected subset.*

Proof. For every $k \in \Delta$, since N_k and M are Γ-Cl-connected subsets such that $M \cap N_k \neq \emptyset$, by using Theorem 22, we obtain that the subset $M \cup N_k$ are Γ-Cl-connected for every $k \in \Delta$. Since $M \subseteq M \cup N_k$ for every $k \in \Delta$, $M \subseteq \bigcap_{k\in\Delta}(M \cup N_k) \neq \emptyset$. From Theorem 22, $\bigcup_{k\in\Delta}(M \cup N_k) = M \cup (\bigcup_{k\in\Delta} N_k)$ is a Γ-Cl-connected subset. □

Theorem 25. *Let (U, τ) be an $\mathcal{I}$-space, $\{N_k : k \in \Delta\}$ be a nonempty collection of Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subsets and M be Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset. If $M \cap N_k \neq \emptyset$ for every $k \in \Delta$, then $M \cup (\bigcup_{k\in\Delta} N_k)$ is a Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset.*

Proof. By using Theorem 23, it is obtained similar to the proof of Theorem 24. □

Theorem 26. *Let (U, τ) be an $\mathcal{I}$-space and $\{N_k : k \in \mathbb{N}\}$ be a nonempty collection of Γ-Cl-connected subsets such that $N_k \cap N_{k+1} \neq \emptyset$ for every $k \in \mathbb{N}$. Then $\bigcup_{k\in\mathbb{N}} N_k$ is a Γ-Cl-connected subset.*

Proof. We can use induction method. Firstly, N_1 is Γ-Cl-connected. Now assume that the theorem is true for $k-1$. That is, $N_1 \cup N_2 \cup ... \cup N_{k-1}$ is Γ-Cl-connected. From Theorem 22, $M_k = N_1 \cup N_2 \cup ... \cup N_k$ is Γ-Cl-connected and $\bigcap_{k\in\mathbb{N}} M_k = N_1 \neq \emptyset$. Again from Theorem 22, $\bigcup_{k\in\mathbb{N}} M_k = \bigcup_{k\in\mathbb{N}} N_k$ is a Γ-Cl-connected subset. □

Theorem 27. *Let (U, τ) be an $\mathcal{I}$-space and $\{N_k : k \in \mathbb{N}\}$ be a nonempty collection of Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subsets such that $N_k \cap N_{k+1} \neq \emptyset$ for every $k \in \mathbb{N}$. Then $\bigcup_{k\in\mathbb{N}} N_k$ is a Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset.*

Proof. By using Theorem 23, it is obtained similar to the proof of Theorem 26. □

Theorem 28. *Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. If for each distinct pair of points $a, b \in M$ there is a Γ-Cl-connected subset E such that $a, b \in E \subseteq M$, then M is Γ-Cl-connected subset.*

Proof. Suppose that the subset M is not Γ-Cl-connected. Then there are Γ-Cl-separated nonempty subsets S, K such that $S \cup K = M$. Let $a \in S$ and $b \in K$. By hypothesis, there is Γ-Cl-connected subset E such that $a, b \in E \subseteq M$. Since $E \subseteq S \cup K$, $E \subseteq S$ or $E \subseteq K$ by Theorem 18. Suppose that $E \subseteq S$. So, $b \in S \cap K \neq \emptyset$. This is a contradiction. Similarly, a contradiction is obtained if we suppose that $E \subseteq K$. □

Theorem 29. *Let (U, τ) be an $\mathcal{I}$-space and $M \subseteq U$. If for each distinct pair of points $a, b \in M$ there is a Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset E such that $a, b \in E \subseteq M$, then M is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset.*

Proof. By using Theorem 19, it is obtained similar to the proof of Theorem 28. □

Theorem 30. *Let (U, τ) be Γ-Cl-connected $\mathcal{I}$-space, M be Γ-Cl-connected subset and K, C be Γ-Cl-separated subsets. If $U \setminus M = K \cup C$, then both $M \cup K$ and $M \cup C$ are Γ-Cl-connected subsets.*

Proof. Suppose that $M \cup K$ is not Γ-Cl-connected. There are Γ-Cl-separated nonempty subsets S, T such that $S \cup T = M \cup K$. Since $M \subseteq S \cup T = M \cup K$ and M is a Γ-Cl-connected subset, $M \subseteq S$ or $M \subseteq T$, by Theorem 18. Suppose that $M \subseteq T$. Then, $S \cup T = M \cup K \subseteq T \cup K$, and so $S \subseteq K$. Since K and C are Γ-Cl-separated subsets, S and C are Γ-Cl-separated subsets. So,

$$\Gamma(S) \cap Cl(T \cup C) = [\Gamma(S) \cap Cl(T)] \cup [\Gamma(S) \cap Cl(C)] = \emptyset$$
$$Cl(S) \cap \Gamma(T \cup C) = [Cl(S) \cap \Gamma(T)] \cup [Cl(S) \cap \Gamma(C)] = \emptyset$$

and $S \cap (T \cup C) = (S \cap T) \cup (S \cap C) = \emptyset$. As a result, S and $T \cup C$ are Γ-Cl-separated subsets. Since $U \setminus M = K \cup C$, we have $U = M \cup (K \cup C) = S \cup (T \cup C)$. This contradicts with the fact that (U, τ) is an Γ-Cl-connected $\mathcal{I}$-space. Consequently, the subset $M \cup K$ is Γ-Cl-connected.

If $M \subseteq S$, a contradiction can be obtained again in this way. Similarly, it can be proved that $M \cup C$ is Γ-Cl-connected subset. □

Theorem 31. *Let (U, τ) be Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected $\mathcal{I}$-space, M be a Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset and K, C be Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-separated subsets. If $U \setminus M = K \cup C$, then $M \cup K$ and $M \cup C$ are Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subsets.*

Proof. By using Theorem 19, it is obtained similar to the proof of Theorem 30. □

Theorem 32. *Let (U, τ) be an $\mathcal{I}$-space and M, K be Γ-Cl-connected subsets of U. If these subsets are not Γ-Cl-separated, then $M \cup K$ is Γ-Cl-connected subset.*

Proof. Suppose that $M \cup K$ is not Γ-Cl-connected subset. So, there are Γ-Cl-separated nonempty subsets S, T such that $S \cup T = M \cup K$. Then, we have $M \subseteq S \cup T$ and $K \subseteq S \cup T$. From Theorem 18, there are four cases to be considered:

(1) $M \subseteq S$ and $K \subseteq S$

(2) $M \subseteq S$ and $K \subseteq T$
(3) $M \subseteq T$ and $K \subseteq T$
(4) $M \subseteq T$ and $K \subseteq S$

If case (1) or case (3) is satisfied, then $T = \emptyset$ or $S = \emptyset$, respectively. Both are contradiction.

Suppose that case (2) is satisfied. If $M = S$ and $K = T$, then the subsets M and K are Γ-Cl-separated. This is a contradiction. If $M \subsetneqq S$, then $T \subsetneqq K$ due to $S \cup T = M \cup K$. Similarly, if $K \subsetneqq T$, then $S \subsetneqq M$. These contradict with case (2). Additionally, for case (4), we obtain similar contradictions. Consequently, $M \cup K$ is Γ-Cl-connected subset. □

Theorem 33. *Let (U, τ) be an $\mathcal{I}$-space and $M, K \subseteq U$. If these subsets are not Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-separated, then $M \cup K$ is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected subset.*

Proof. By using Theorem 19, it is obtained similar to the proof of Theorem 32. □

Lemma 6. *Let (U, τ) be an $\mathcal{I}$-space and M, K be subsets of U. Then*

$$\Gamma(M \cap K) \subseteq \Gamma(M) \cap \Gamma(K).$$

Proof. Let $x \in \Gamma(M \cap K)$. Then, $[Cl(O) \cap (M \cap K)] \notin \mathcal{I}$ for every $O \in \tau(x)$. Because of the definition of ideal, $Cl(O) \cap M \notin \mathcal{I}$ and $Cl(O) \cap K \notin \mathcal{I}$. So, $x \in \Gamma(M)$ and $x \in \Gamma(K)$. That is, $x \in \Gamma(M) \cap \Gamma(K)$. □

In the following example, we show that the inclusion $\Gamma(M \cap K) \subseteq \Gamma(M) \cap \Gamma(K)$ strictly hold.

Example 5. *Consider the $\mathcal{I}$-space in Example 1. In Table 1, $\Gamma(A \cap B) = \emptyset \subsetneqq \{a, b, c\} = \Gamma(A) \cap \Gamma(B)$.*

Theorem 34. *Let (U, τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

(1) *The subset K is both Γ-Cl-connected and closed.*
(2) *$\Gamma(M) \subseteq Cl(M)$ and $\Gamma(U \setminus M) \subseteq Cl(U \setminus M)$.*
(3) *$K \cap M \neq \emptyset$ and $K \cap (U \setminus M) \neq \emptyset$.*

then $K \cap Bd(M) \neq \emptyset$ where $Bd(M)$ is boundary of the subset M.

Proof. Suppose that $K \cap Bd(M) = \emptyset$. So, $K \cap (Cl(M) \cap Cl(U \setminus M)) = \emptyset$. The subset K can be expressed as $K = U \cap K = (M \cup (U \setminus M)) \cap K = (M \cap K) \cup ((U \setminus M) \cap K)$. Then, by using Lemma 6,

$$\begin{aligned}\Gamma(M \cap K) \cap Cl((U \setminus M) \cap K) &\subseteq \Gamma(M) \cap \Gamma(K) \cap [Cl(U \setminus M) \cap Cl(K)] \\ &\subseteq Cl(M) \cap \Gamma(K) \cap Cl(U \setminus M) \cap K = \emptyset\end{aligned}$$

$$\begin{aligned}Cl(M \cap K) \cap \Gamma((U \setminus M) \cap K) &\subseteq Cl(M) \cap Cl(K) \cap [\Gamma(U \setminus M) \cap \Gamma(K)] \\ &\subseteq Cl(M) \cap K \cap Cl(U \setminus M) \cap \Gamma(K) = \emptyset\end{aligned}$$

and $(M\cap K)\cap((U\setminus M)\cap K)=\emptyset$. Therefore, the subset K is not Γ-Cl-connected. This is a contradiction. Consequently, $K\cap Bd(M)\neq\emptyset$. □

Theorem 35. *Let (U,τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

(1) *The subset K is Γ-connected.*
(2) $\Gamma(M)\subseteq Cl(M)$ *and* $\Gamma(U\setminus M)\subseteq Cl(U\setminus M)$.
(3) $K\cap M\neq\emptyset$ *and* $K\cap(U\setminus M)\neq\emptyset$.

then $K\cap Bd(M)\neq\emptyset$.

Proof. Suppose that $K\cap Bd(M)=\emptyset$. So, $K\cap(Cl(M)\cap Cl(U\setminus M))=\emptyset$. The subset K can be expressed as $K=U\cap K=(M\cup(U\setminus M))\cap K=(M\cap K)\cup((U\setminus M)\cap K)$. Then, by using Lemma 6,

$$\begin{aligned}\Gamma(M\cap K)\cap((U\setminus M)\cap K)&\subseteq\Gamma(M)\cap\Gamma(K)\cap(U\setminus M)\cap K\\&\subseteq Cl(M)\cap\Gamma(K)\cap Cl(U\setminus M)\cap K=\emptyset\end{aligned}$$

$$\begin{aligned}(M\cap K)\cap\Gamma((U\setminus M)\cap K)&\subseteq M\cap K\cap\Gamma(U\setminus M)\cap\Gamma(K)\\&\subseteq Cl(M)\cap K\cap Cl(U\setminus M)\cap\Gamma(K)=\emptyset\end{aligned}$$

and $(M\cap K)\cap((U\setminus M)\cap K)=\emptyset$. Therefore, the subset K is not Γ-connected. This is a contradiction. Consequently, $K\cap Bd(M)\neq\emptyset$. □

Theorem 36. *Let (U,τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

(1) *The subset K is both Γ-Cl^*-connected and $*$-closed.*
(2) $\Gamma(M)\subseteq Cl^*(M)$ *and* $\Gamma(U\setminus M)\subseteq Cl^*(U\setminus M)$.
(3) $K\cap M\neq\emptyset$ *and* $K\cap(U\setminus M)\neq\emptyset$.

then $K\cap Bd^*(M)\neq\emptyset$ where $Bd^*(M)$ is boundary of the subset M with respect to τ^*.

Proof. Suppose that $K\cap Bd^*(M)=\emptyset$. So, $K\cap(Cl^*(M)\cap Cl^*(U\setminus M))=\emptyset$. The subset K can be expressed as $K=U\cap K=(M\cup(U\setminus M))\cap K=(M\cap K)\cup((U\setminus M)\cap K)$. Then,

$$\begin{aligned}\Gamma(M\cap K)\cap Cl^*((U\setminus M)\cap K)&\subseteq\Gamma(M)\cap\Gamma(K)\cap[Cl^*(U\setminus M)\cap Cl^*(K)]\\&\subseteq Cl^*(M)\cap\Gamma(K)\cap Cl^*(U\setminus M)\cap K=\emptyset\end{aligned}$$

$$\begin{aligned}Cl^*(M\cap K)\cap\Gamma((U\setminus M)\cap K)&\subseteq Cl^*(M)\cap Cl^*(K)\cap[\Gamma(U\setminus M)\cap\Gamma(K)]\\&\subseteq Cl^*(M)\cap K\cap Cl^*(U\setminus M)\cap\Gamma(K)=\emptyset\end{aligned}$$

and $(M\cap K)\cap(K\cap(U\setminus M))=\emptyset$. Therefore, the subset K is not Γ-Cl^*-connected. This is a contradiction. So, $K\cap Bd^*(M)\neq\emptyset$.

□

Corollary 15. *Let (U,τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

1. *The subset K is Γ-$*$ (2^*)-connected and $*$-closed.*
2. $\Gamma(M) \subseteq Cl^*(M)$ *and* $\Gamma(U \setminus M) \subseteq Cl^*(U \setminus M)$.
3. $K \cap M \neq \emptyset$ *and* $K \cap (U \setminus M) \neq \emptyset$.

then $K \cap Bd^*(M) \neq \emptyset$.

Proof. It is obvious from Figure 4 and Theorem 36. □

Theorem 37. *Let (U,τ) be an $\mathcal{I}$-space. If the following conditions are satisfied for the subsets M and K:*

1. *The subset K is Γ-connected.*
2. $\Gamma(M) \subseteq Cl^*(M)$ *and* $\Gamma(U \setminus M) \subseteq Cl^*(U \setminus M)$.
3. $K \cap M \neq \emptyset$ *and* $K \cap (U \setminus M) \neq \emptyset$.

then $K \cap Bd^*(M) \neq \emptyset$.

Proof. Suppose that $K \cap Bd^*(M) = \emptyset$. So, $K \cap (Cl^*(M) \cap Cl^*(U \setminus M)) = \emptyset$. The subset K can be expressed as $K = U \cap K = (M \cup (U \setminus M)) \cap K = (M \cap K) \cup ((U \setminus M) \cap K)$. Then

$$\begin{aligned}\Gamma(M \cap K) \cap ((U \setminus M) \cap K) &\subseteq \Gamma(M) \cap \Gamma(K) \cap (U \setminus M) \cap K\\ &\subseteq Cl^*(M) \cap \Gamma(K) \cap Cl^*(U \setminus M) \cap K = \emptyset\end{aligned}$$

$$\begin{aligned}(M \cap K) \cap \Gamma((U \setminus M) \cap K) &\subseteq M \cap K \cap [\Gamma(U \setminus M) \cap \Gamma(K)]\\ &\subseteq Cl^*(M) \cap K \cap Cl^*(U \setminus M) \cap \Gamma(K) = \emptyset\end{aligned}$$

and $(M \cap K) \cap (K \cap (U \setminus M)) = \emptyset$. Therefore, the subset K is not Γ-Cl^*-connected. This is a contradiction. Finally, $K \cap Bd^*(M) \neq \emptyset$.

□

6. New Type Components via Local Closure

Definition 10. *Let (U,τ) be an $\mathcal{I}$-space and x be a point of U. The union of all Γ-Cl(resp. Γ-Cl^*, Γ, Γ-$*$, 2^*)-connected subsets that contain the point x is called Γ-Cl(resp. Γ-Cl^*, Γ, Γ-$*$, 2^*)-component of U containing x. That is, we define a Γ-Cl(resp. Γ-Cl^*, Γ, Γ-$*$, 2^*)-component of the point x as follows:*

1. *The subset* $\mathcal{C}_{\Gamma\text{-}Cl}(x) = \bigcup\{M \subseteq U : M \text{ is } \Gamma\text{-}Cl\text{-connected and } x \in M\}$ *is called Γ-Cl-component of the point x.*
2. *The subset* $\mathcal{C}_{\Gamma\text{-}Cl^*}(x) = \bigcup\{M \subseteq U : M \text{ is } \Gamma\text{-}Cl^*\text{-connected and } x \in M\}$ *is called Γ-Cl^*-component of the point x.*
3. *The subset* $\mathcal{C}_{\Gamma}(x) = \bigcup\{M \subseteq U : M \text{ is } \Gamma\text{-connected and } x \in M\}$ *is called Γ-component of the point x.*
4. *The subset* $\mathcal{C}_{\Gamma\text{-}*}(x) = \bigcup\{M \subseteq U : M \text{ is } \Gamma\text{-}*\text{-connected and } x \in M\}$ *is called Γ-$*$-component of the point x.*

(5) *The subset* $\mathcal{C}_{2^*}(x) = \bigcup\{M \subseteq U : M$ *is* 2^**-connected and* $x \in M\}$ *is called* 2^**-component of the point* x.

Theorem 38. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* x *be a point of* U.

(1) *The subset* $\mathcal{C}_{\Gamma\text{-}Cl}(x)$ *is* Γ*-*Cl*-connected subset which contains* x.
(2) *The subset* $\mathcal{C}_{\Gamma\text{-}Cl}(x)$ *is maximal* Γ*-*Cl*-connected subset which contains* x .

Proof. (1) Since $x \in \bigcap\{M \subseteq U : M$ is Γ-Cl-connected and $x \in M\} \neq \emptyset$, $\mathcal{C}_{\Gamma\text{-}Cl}(x) = \bigcup\{M \subseteq U : M$ is Γ-Cl-connected and $x \in M\}$ is Γ-Cl-connected by Theorem 22.

(2) It is obvious from Definition 10 and (1).

□

Theorem 39. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* x *be a point of* U.

(1) *The subset* $\mathcal{C}_{\Gamma\text{-}Cl^*}(x)$*(resp.* $\mathcal{C}_{\Gamma}(x)$, $\mathcal{C}_{\Gamma\text{-}*}(x)$, $\mathcal{C}_{2^*}(x)$*) is* Γ*-*Cl^**(resp.* Γ, Γ*-*$*$, 2^**)-connected subset which contains* x.
(2) *The subset* $\mathcal{C}_{\Gamma\text{-}Cl^*}(x)$*(resp.* $\mathcal{C}_{\Gamma}(x)$, $\mathcal{C}_{\Gamma\text{-}*}(x)$, $\mathcal{C}_{2^*}(x)$*) is maximal* Γ*-*Cl^**(resp.* Γ, Γ*-*$*$, 2^**)-connected subset which contains* x .

Proof. By using Theorem 23 and Definition 10 , it is obtained similar to the proof of Theorem 38 . □

Theorem 40. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $x, y \in U$. *Then*

(1) $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cap \mathcal{C}_{\Gamma\text{-}Cl}(y) = \emptyset$ *or* $\mathcal{C}_{\Gamma\text{-}Cl}(x) = \mathcal{C}_{\Gamma\text{-}Cl}(y)$.
(2) *The set of all distinct* Γ*-*Cl*-components forms a partition of* U.

Proof. (1) Let $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cap \mathcal{C}_{\Gamma\text{-}Cl}(y) \neq \emptyset$. From Theorem 38-(1) and Theorem 22, $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y)$ is Γ-Cl-connected. We have $\mathcal{C}_{\Gamma\text{-}Cl}(x) \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y)$ and $\mathcal{C}_{\Gamma\text{-}Cl}(y) \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y)$. From Theorem 38-(2), $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y) \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(x)$ and $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y) \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(y)$. So, $\mathcal{C}_{\Gamma\text{-}Cl}(x) \cup \mathcal{C}_{\Gamma\text{-}Cl}(y) = \mathcal{C}_{\Gamma\text{-}Cl}(x) = \mathcal{C}_{\Gamma\text{-}Cl}(y)$.

(2) Since $\bigcup_{x \in U} \mathcal{C}_{\Gamma\text{-}Cl}(x) = U$, it is obvious from (1).

□

Theorem 41. *Let* (U, τ) *be an* $\mathcal{I}$*-space and* $x, y \in U$. *Then,*

(1) $\mathcal{C}_{\Gamma\text{-}Cl^*}(x) \cap \mathcal{C}_{\Gamma\text{-}Cl^*}(y) = \emptyset$ *or* $\mathcal{C}_{\Gamma\text{-}Cl^*}(x) = \mathcal{C}_{\Gamma\text{-}Cl^*}(y)$.
(2) $\mathcal{C}_{\Gamma}(x) \cap \mathcal{C}_{\Gamma}(y) = \emptyset$ *or* $\mathcal{C}_{\Gamma}(x) = \mathcal{C}_{\Gamma}(y)$.
(3) $\mathcal{C}_{\Gamma\text{-}*}(x) \cap \mathcal{C}_{\Gamma\text{-}*}(y) = \emptyset$ *or* $\mathcal{C}_{\Gamma\text{-}*}(x) = \mathcal{C}_{\Gamma\text{-}*}(y)$.
(4) $\mathcal{C}_{2^*}(x) \cap \mathcal{C}_{2^*}(y) = \emptyset$ *or* $\mathcal{C}_{2^*}(x) = \mathcal{C}_{2^*}(y)$.
(5) *The set of all distinct* $\mathcal{C}_{\Gamma\text{-}Cl^*}(x)$*(resp.* $\mathcal{C}_{\Gamma}(x)$, $\mathcal{C}_{\Gamma\text{-}*}(x)$, $\mathcal{C}_{2^*}(x)$*)-components forms a partition of* U.

Proof. By using Theorem 39 and Theorem 23, all statements above are obtained similar to the proof of Theorem 40. □

Theorem 42. *Let* (U, τ) *be an* $\mathcal{I}$*-space. If* M *is* Γ*-*Cl*-connected and nonempty clopen subset of* U*, then* M *is* Γ*-*Cl*-component.*

Proof. Let $\mathcal{C}_{\Gamma\text{-}Cl}(x)$ be Γ-Cl-component of the point $x \in M$. From Theorem 38-(2), $M \subseteq \mathcal{C}_{\Gamma\text{-}Cl}(x)$. Suppose that $M \subsetneq \mathcal{C}_{\Gamma\text{-}Cl}(x)$. Then, $(M \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) \cap [(U \setminus M) \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)] = \emptyset$ and $(M \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) \cup [(U \setminus M) \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)] = \mathcal{C}_{\Gamma\text{-}Cl}(x)$. From Lemma 5,

$$\Gamma(M) \cap Cl(U \setminus M) \subseteq Cl(M) \cap (U \setminus M) = M \cap (U \setminus M) = \emptyset$$
$$Cl(M) \cap \Gamma(U \setminus M) \subseteq M \cap Cl(U \setminus M) = M \cap (U \setminus M) = \emptyset$$

These imply that

$$\Gamma(M \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) \cap Cl((U \setminus M) \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) = \emptyset$$
$$Cl(M \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) \cap \Gamma((U \setminus M) \cap \mathcal{C}_{\Gamma\text{-}Cl}(x)) = \emptyset$$

So, $\mathcal{C}_{\Gamma\text{-}Cl}(x)$ is not Γ-Cl-connected. This is a contradiction. Consequently, $M = \mathcal{C}_{\Gamma\text{-}Cl}(x)$. That is, M is Γ-Cl-component. □

Theorem 43. *Let (U, τ) be an $\mathcal{I}$-space. If M is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-connected and nonempty clopen subset of U, then M is Γ-Cl^*(resp. Γ, Γ-$*$, 2^*)-component.*

Proof. By using Lemma 5, it is obtained similar to the proof of Theorem 42. □

7. The Image of New Types of Connectedness Under a Continuous Map in Ideal Topological Spaces

$f : (U, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ is continuous map means that $f : (U, \tau_1) \to (Y, \tau_2)$ is continuous.

Theorem 44. *Let (U, τ_1) be Γ-Cl-connected $\mathcal{I}$-space and (Y, τ_2) be any topological space. If $f : (U, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ is a continuous map, then $f(U)$ is τ_2-connected.*

Proof. From Theorem 15, the set U is τ_1-connected. Since the image of a connected space under a continuous map is connected, $f(U)$ is τ_2-connected. □

Corollary 16. *Let (U, τ_1) be Γ-Cl^* (Γ, Γ-$*$, 2^*, $*$-Cl, $*$-Cl^*, $*_*$)-connected $\mathcal{I}$-space and (Y, τ_2) be any topological space. If $f : (U, \tau_1, I) \to (Y, \tau_2)$ is a continuous map, then $f(U)$ is τ_2-connected.*

Proof. It is obvious from Theorem 44 and Figure 4. □

Corollary 17. *Let $f : (U, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ be continuous and surjective function. If U is Γ-Cl (Γ-Cl^*, Γ, Γ-$*$, 2^*)-connected, then Y is τ-connected.*

Proof. It is obvious from Theorem 44 and Corollary 16. □

It is shown in [14] that Corollary 17 is also satisfied for $*$-Cl ($*$-Cl^*, $*_*$)-connectedness. This is clear from Theorem 44 and Corollary 16. Because Γ-Cl-connectedness is more general than $*$-Cl ($*$-Cl^*, $*_*$)-connectedness.

Theorem 45. *[25](Intermediate Value Theorem) Let $f : (U, \tau_1) \to (Y, \tau_2)$ be continuous map, where (U, τ_1) is a τ_1-connected topological space, Y is an ordered set with " $<$ " and τ_2 is order topology on Y. If $a, b \in U$ and $f(a) < r < f(b)$, then there exists a point $c \in U$ such that $f(c) = r$.*

Now, we give the intermediate value theorem for the ideal topological spaces.

Theorem 46. *Let $f : (U, \tau_1, \mathcal{I}) \to (Y, \tau_2)$ be continuous map, where (U, τ_1) is a Γ-Cl (Γ-Cl^*, Γ, Γ-$*$, 2^*, $*$-Cl, $*$-Cl^*, $*_*$)-connected $\mathcal{I}$-space, Y is an ordered set with " $<$ " and τ_2 is order topology on Y. If $a, b \in U$ and $f(a) < r < f(b)$, then there exists a point $c \in U$ such that $f(c) = r$.*

Proof. From Theorem 15 (and Corollary 16), the set U is τ_1-connected. That is, (U, τ_1) is connected space. Then, the claim is obtained by Theorem 45. □

Specially, if we choose the minimal ideal $\mathcal{I} = \{\emptyset\}$ in Theorem 46, by using Corollary 13, we obtain the intermediate value theorem. That is, a special case of Theorem 46 gives the intermediate value theorem.

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