MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

VOLUME XI - ISSUE I

2023-1

VOLUME XI ISSUE I
ISSN 2147-6268

March 2023
www.dergipark.org.tr/en/pub/mathenot

MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

Mohammed Mursaleen
Aligarh Muslim University,
India

Ioan Raşa
Technical University of Cluj-Napoca,
Romania

Reza Saadati
Iran University of Science and Technology,
Iran

Gökhan Coşkun
Düzce University,
Turkey

Murat Bekar
Gazi University,
Turkey

Contents

Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753/mathenot.1172408
11 (1) 1-13 (2023) - Research Article
ISSN: 2147-6268

A Soft Set Approach to Relations and Its Application to Decision Making

Kemal Taşköprü* and Elif Karaköse

Abstract
One of the most useful mathematical tools for examining the relationships among objects is the concept of relation. Besides, it can also be necessary to throw light on uncertainties in these relationships. Soft set theory, in which different approaches used in defining the notions bring about different applications in many areas, enables to overcome uncertainties. The purpose of this paper is to define soft relation in a different way and to give a decision making method using the concept of soft relation. For this purpose, firstly, the soft relations are defined on the collection of soft elements, unlike the previous ones. After their basic properties are provided, the correspondence between the soft and classical relations is investigated and some examples are given. Finally, an algorithm is proposed using the soft relation for solving decision making problems, where the decision is related to other circumstances, and given an illustrative example.

Keywords: *soft relation; closure of relations; equivalence; decision making.*

AMS Subject Classification (2020): *Primary: 03E20 ; Secondary: 03E72.*

***Corresponding author**

1. Introduction

The soft sets, introduced by Molodtsov [1], have enabled to be deal with uncertainties such as the fuzzy sets, vague sets, rough sets, intuitionistic fuzzy sets, and neutrosophic sets, which deal with uncertainties in different ways [2]. In substance, a soft set is considered as a parametrized set of alternatives and this parametrization allows the alternatives to be examined according to their properties. The soft set theory, with the integration of other set theories has been the subject of the various scientific fields of study containing vagueness, especially in decision making problems and many different mathematical structures [3–9]. After Maji et al. [10] and Ali et al. [11] laid the foundations of the soft set operations, different interpretations have emerged about extending mathematical structures to the soft set theory (See [12–18], and others in them). The soft elements and elementary (ε-) soft set

Received : *08-09-2022,* ***Accepted :*** *12-11-2022*

*(**Cite as** "K. Taşköprü, E. Karaköse, A Soft Set Approach to Relations and Its Application to Decision Making, Math. Sci. Appl. E-Notes, 11(1) (2023), 1-13")*

operations were brought forward by Das and Samanta [19], and some mathematical structures have been examined using these concepts by several authors [20–27].

The relations are fundamental concepts that have been used to classify, order or compare objects in many fields of research as a result of the fact that different structures can be related to each other as well. In order to apply the soft set theory to the relations, the relations in the soft sets were introduced to model fuzziness and hesitancy in the relationships between two objects [28–35]. Recently, Alcantud [39] introduced a new concept called softarison defined on a set of alternatives to make parametrized comparisons as the soft sets, which merged the soft sets with the relations and applied it to decision making problems. In addition, for the purpose of handling the decision making process in different ways in vagueness, the relations on the hybrid soft sets have been introduced and applied to decision making problems [40–46]. In the mentioned above studies and others discussing the relations in (hybrid) soft sets, a relation in these sets is actually described as corresponding to a (hybrid) soft set. Furthermore, in all the studies addressing decision making problems, the decision corresponds to an element that is generally determined from among the alternatives, according to the weights and attributes.

The alternatives may consist of certain factors and it may be desired to determine the factors that will form the preference according to the desired criteria. Concordantly, the soft elements correspond to single-element soft sets and a soft element provides a pattern that determines the appropriate alternative for each descriptive attribute from within the soft set. In this study, it is shown that the concepts of soft element and soft relation can be useful and applicable to investigate the factors that constitute the alternatives and the relationships between the alternatives. Unlike the previous studies about the relations in the soft sets, a novel approach is proposed to the soft relation using the concept of soft element and it is demonstrated that using this notion can be operable in decision making.

The paper is organised as follows: In Section 2, the basic information about soft sets and the notion of soft element are given and an overview of previous studies regarding the relations in the soft sets is presented. In Section 3, a soft relation is defined by using a collection of soft elements. After the definitions and properties of the soft relations are given via ε-soft set operations, the interactions of these relations and the classical relations are investigated. It is encountered that soft equivalence relations have different properties from the classical equivalences. In Section 4, a soft relation-based algorithm is proposed for handling decision making problems in which the decision is made as a soft element, that is, by determining the appropriate alternative corresponding to each parameter, and the decision is influenced by other factors. Then, an illustrative example is presented to choose an optimal system and to ensure optimal system integration. Finally, in the concluding section of the paper, various lines for further research on this topic are noted.

2. Preliminaries

Definition 2.1. [1] Let U be a universal set, P be a parameters set and $P(U)$ be the power set of U. A pair (G, P) is called a soft set on U, where $G : P \to P(U)$ is a mapping.

Definition 2.2. Let (G, P) and (H, P) be two soft sets on U. The soft set (G, P) is said to be a null soft set if $G(\alpha) = \emptyset$ and an absolute soft set if $G(\alpha) = U$ for each $\alpha \in P$, denoted by Φ and $\tilde{U}$, respectively. The soft set (G, P) is said to be a soft subset of (H, P) if $G(\alpha) \subset H(\alpha)$ for every $\alpha \in P$ and denoted by $(G, P)\tilde{\subset}(H, P)$. Also, $(G, P) = (H, P)$ if and only if $(G, P)\tilde{\subset}(H, P)$ and vice versa.

Definition 2.3. [33] Let (G, P) be a soft set on U and (H, P') be a soft set on U'. The Cartesian product of (G, P) and (H, P') is defined as $(F, P \times P') = (G, P) \times (H, P')$, where $H : P \times P' \to P(U \times U')$ is given by $H(\alpha, \alpha') = G(\alpha) \times H(\alpha')$ for each $(\alpha, \alpha') \in P \times P'$. Then, a soft set relation from (G, P) to (H, P') is a soft subset of $(G, P) \times (H, P')$ and a soft set relation on (G, P) is a soft subset of $(G, P) \times (G, P)$.

Before the above definition, the definition of Cartesian product and the soft set relation on the same universe U in [28] and it was studied based on this definition in [29–31]. Also, based on these studies, the soft set relations were merged with topology and transferred to the hybrid soft sets (See, [40–42, 45]).

Definition 2.4. [32] Let (ρ, P) be a soft set on $U \times U$, i.e. $\rho : P \to P(U \times U)$. Then, (ρ, P) is called a soft binary relation on U. Here, the soft binary relation is considered as a parametrized collection of binary relations on U.

Based on the above definition, the soft binary relations were merged with algebraic structures and hybrid soft sets (See, [35, 43] and others in them).

Apart from the above studies regarding the relations in the soft sets, by using the partial order relation on the universe U, and by examining the belonging relation of the elements to the set corresponding to the parameters, topological structures were studied on soft sets in [37, 38, 44].

Definition 2.5. [37] A soft point P_α^x of a soft set (G, P) on U is determined by the fact that $x \in G(\alpha)$ for the parameter $\alpha \in P$ and $x \in U$.

In the related studies in Definitions 2.3 and 2.4 and others, an element of a soft set is evaluated via Definition 2.5. Unlike this definition, Das and Samanta [19] introduced the soft element and gave elementary soft set operations. They consider a soft element to be about evaluating not just a single point for a single parameter, but the corresponding points for each parameter.

Definition 2.6. [19] A function $\varepsilon : P \to U$ is called a soft element of U and ε is said to be member of (G, P) if $\varepsilon(\alpha) \in G(\alpha)$ for each $\alpha \in P$. The class of soft elements of (G, P) are denoted by $SE(G, P)$ and the soft elements by $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, etc.

Throughout the work, the soft sets (G, P) on U such that $G(\alpha) \neq \emptyset$ for every $\alpha \in P$ and the null soft set Φ will be considered. The class of these soft sets is denoted by $S(\tilde{U})$ and $S_P(U)$ represents the set of all soft sets over U with parameters P.

The soft set $SS(\mathfrak{B})$ produced by the class of soft elements $\mathfrak{B}$ is defined by

$$(G, P) = SS(\mathfrak{B}) = \{(\alpha, G(\alpha)) : \forall \alpha \in P,\ G(\alpha) = \bigcup_{\tilde{x} \in \mathfrak{B}} \{\tilde{x}(\alpha)\}\}.$$

The ε-union and ε-intersection of (G, P),$(H, P) \in S(\tilde{U})$ are defined by

$$(G, P) \Cup (H, P) = SS(SE(G, P) \cup SE(H, P))$$

and

$$(G, P) \Cap (H, P) = SS(SE(G, P) \cap SE(H, P)),$$

respectively. The ε-complement of (G, P) is defined $(G, P)^{\mathbb{C}} = SS(SE(G, P)^c)$, where $(G, P)^c = (G^c, P)$ is soft complement of (G, P) and $G^c : P \to P(U)$ is a mapping given by $G^c(\alpha) = U \backslash G(\alpha)$, $\forall \alpha \in P$. (For details, see [25]).

From now on, the notation of a soft set is used as G instead of (G, P) for simplicity and $SE(\tilde{U})$ denotes the set of all soft elements over U with parameters set P.

3. Soft relations

This section proposes a novel approach to the relations in the soft sets. The relations in the soft sets are actually referred to as a soft set in Definitions 2.3 and 2.4, whereas a soft relation based on the concept of soft element is defined to be a subclass of the Cartesian product of any two collections of soft elements.

Definition 3.1. Let U and U' be two universal sets and P be a parameters set. A soft relation $\mathcal{R}$ from $\tilde{U}$ to $\tilde{U}'$ is defined as a subclass of $SE(\tilde{U}) \times SE(\tilde{U}')$ and then a soft relation $\mathcal{R}$ on $\tilde{U}$ is denoted by

$$\mathcal{R} = \{(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \in \mathfrak{B} \subset SE(\tilde{U})\} \subseteq SE(\tilde{U}) \times SE(\tilde{U}).$$

All the properties of soft relations can be defined similarly to those of classical relations and some situations that make a difference are given as follows.

Definition 3.2. Let $\tilde{U}$ be an absolute soft set with parameter set P having a soft relation $\mathcal{R}$. The soft relation $\mathcal{R}$ is called

- *reflexive* if $\tilde{x}\mathcal{R}\tilde{x}$ for each $\tilde{x} \in SE(\tilde{U})$,
- *irreflexive* if $\neg\tilde{x}\mathcal{R}\tilde{x}$ for each $\tilde{x} \in SE(\tilde{U})$,
- *symmetric* if $\tilde{x}\mathcal{R}\tilde{y} \Rightarrow \tilde{y}\mathcal{R}\tilde{x}$ for each $\tilde{x}, \tilde{y} \in SE(\tilde{U})$,
- *asymmetric* if $\tilde{x}\mathcal{R}\tilde{y} \Rightarrow \neg\tilde{y}\mathcal{R}\tilde{x}$ for each $\tilde{x}, \tilde{y} \in SE(\tilde{U})$,
- *antisymmetric* if $\tilde{x}\mathcal{R}\tilde{y}$ and $\tilde{y}\mathcal{R}\tilde{x} \Rightarrow \tilde{x} = \tilde{y}$ for each $\tilde{x}, \tilde{y} \in SE(\tilde{U})$,
- *transitive* if $\tilde{x}\mathcal{R}\tilde{y}$ and $\tilde{y}\mathcal{R}\tilde{z} \Rightarrow \tilde{x}\mathcal{R}\tilde{z}$ for each $\tilde{x}, \tilde{y}, \tilde{z} \in SE(\tilde{U})$,

- *total (complete, connected, comparable or connex)* if $\tilde{x}\mathcal{R}\tilde{y}$ or $\tilde{y}\mathcal{R}\tilde{x}$ for each $\tilde{x}, \tilde{y} \in SE(\tilde{U})$,

where $\tilde{x}R\tilde{y}$ means that $(\tilde{x}, \tilde{y}) \in \mathcal{R}$. Also, a soft relation $\mathcal{R}$ is called

- **pre-order** if it is reflexive and transitive,
- **total pre-order (weak order)** if it is reflexive, total and transitive,
- **partial order** if it is reflexive, antisymmetric and transitive,
- **strict partial order** if it is irreflexive, asymmetric and transitive,
- **total order (complete order, linear order)** if it is reflexive, antisymmetric, total and transitive,
- **equivalence relation** if it is reflexive, symmetric and transitive.

Proposition 3.1. *Each parametrized family of classical relations can be considered as a soft relation and every soft relation on can be considered as a parametrized family of classical relations.*

Proof. If $\{R_\alpha : \alpha \in P\}$ is a family of classical relations on U with parameter set P then $\mathcal{R}$ is a soft relation on $\tilde{U}$ such that $\mathcal{R}(\alpha) = \mathcal{R}_\alpha = \{(\tilde{x}, \tilde{y})(\alpha) = (\tilde{x}(\alpha), \tilde{y}(\alpha)) \in R_\alpha : \tilde{x}, \tilde{y} \in SE(\tilde{U})\}$ for all $\alpha \in P$. Conversely, if $\mathcal{R}$ is a soft relation on $\tilde{U}$ then for each $\alpha \in P$, $\mathcal{R}_\alpha$ is a classical relation on U. Hence, each parametrized family of classical relations can be considered as a soft relation, and vice versa. □

Suppose that R is a relation on U. Then, $\mathcal{R}$ is a soft relation on $\tilde{U}$ such that $\mathcal{R}_\alpha = R$ for all parameters $\alpha \in P$. So, the soft relation $\mathcal{R}$ determined by using the classical relation R is called a soft relation produced by R.

Remark 3.1. Let $G \in S(\tilde{U})$ be a soft set with parameters set P. If $\mathcal{R}$ is a soft relation on G the family $\{R_\alpha : \alpha \in P\}$ is obtained as a parametrized family of classical relations on U in a similar way to Proposition 3.1. But, if a parametrized family of classical relations $\{R_\alpha : \alpha \in P\}$ on U is given, there can be some $\tilde{x}, \tilde{y} \in SE(G)$ and some $\alpha \in U$ such that $(\tilde{x}, \tilde{y})(\alpha) = (\tilde{x}(\alpha), \tilde{y}(\alpha)) \notin R_\alpha$. Hence, it is not obtained a soft relation on G. In such a case, since each soft element $\tilde{x} \in SE(G)$ is a function from P to U, if the family $\{R_\alpha : \alpha \in P\}$ is given such that

$$R_\alpha \subset \bigcap_{\alpha \in P} G(\alpha) \times \bigcap_{\alpha \in P} G(\alpha)$$

for each $\alpha \in P$, it is obtained a soft relation $\mathcal{R}$ on G in a similar way to Proposition 3.1. An example of such a case is given below in Example 3.1.

Proposition 3.2. *Every parametrized family of classical reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and transitive relations can be considered as a reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and transitive soft relation, respectively.*

Proof. It is easily seen that the proof follows from Proposition 3.1. □

Example 3.1. Let $P = \{\alpha, \beta\}$ and $U = \{u, v, w\}$, then

$$SE(\tilde{U}) = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5, \tilde{e}_6, \tilde{e}_7, \tilde{e}_8, \tilde{e}_9\},$$

where

$$\begin{array}{lll} \tilde{e}_1 = \{(\alpha, u), (\beta, u)\}, & \tilde{e}_4 = \{(\alpha, v), (\beta, u)\}, & \tilde{e}_7 = \{(\alpha, w), (\beta, u)\}, \\ \tilde{e}_2 = \{(\alpha, u), (\beta, v)\}, & \tilde{e}_5 = \{(\alpha, v), (\beta, v)\}, & \tilde{e}_8 = \{(\alpha, w), (\beta, v)\}, \\ \tilde{e}_3 = \{(\alpha, u), (\beta, w)\}, & \tilde{e}_6 = \{(\alpha, v), (\beta, w)\}, & \tilde{e}_9 = \{(\alpha, w), (\beta, w)\}. \end{array}$$

Suppose that the parametrized classical relations are defined as

$$\begin{aligned} R_1 &= \{(u, u), (v, v), (w, w), (u, v)\}, \\ R_2 &= \{(u, u), (v, v), (w, w), (u, v), (v, u)\}. \end{aligned}$$

The properties of these relations are as follows:

	Reflexive	Irreflexive	Symmetric	Antisymmetric	Asymmetric	Total	Transitive
R_1	✓	✗	✗	✓	✗	✗	✓
R_2	✓	✗	✓	✗	✗	✗	✓

If the parametrized family of classical relations $\{R_1, R_2\}$ is considered, the soft relations are obtained as

$$\begin{aligned}\mathcal{R}^{(12)} = \{&(\tilde{e}_1,\tilde{e}_1),\ldots,(\tilde{e}_9,\tilde{e}_9),(\tilde{e}_1,\tilde{e}_2),(\tilde{e}_1,\tilde{e}_4),(\tilde{e}_1,\tilde{e}_5),(\tilde{e}_2,\tilde{e}_1),(\tilde{e}_2,\tilde{e}_4),(\tilde{e}_2,\tilde{e}_5),(\tilde{e}_3,\tilde{e}_6),\\&(\tilde{e}_4,\tilde{e}_5),(\tilde{e}_5,\tilde{e}_4),(\tilde{e}_7,\tilde{e}_8),(\tilde{e}_8,\tilde{e}_7)\},\end{aligned}$$

$$\begin{aligned}\mathcal{R}^{(21)} = \{&(\tilde{e}_1,\tilde{e}_1),\ldots,(\tilde{e}_9,\tilde{e}_9),(\tilde{e}_1,\tilde{e}_2),(\tilde{e}_1,\tilde{e}_4),(\tilde{e}_1,\tilde{e}_5),(\tilde{e}_2,\tilde{e}_5),(\tilde{e}_3,\tilde{e}_6),(\tilde{e}_4,\tilde{e}_1),(\tilde{e}_4,\tilde{e}_2),\\&(\tilde{e}_4,\tilde{e}_5),(\tilde{e}_5,\tilde{e}_2),(\tilde{e}_6,\tilde{e}_3),(\tilde{e}_7,\tilde{e}_8)\}.\end{aligned}$$

Also, the soft relations produced by the classical relations R_1 and R_2 are obtained as

$$\mathcal{R}^{(1)} = \{(\tilde{e}_1,\tilde{e}_1),\ldots,(\tilde{e}_9,\tilde{e}_9),(\tilde{e}_1,\tilde{e}_2),(\tilde{e}_1,\tilde{e}_4),(\tilde{e}_1,\tilde{e}_5),(\tilde{e}_2,\tilde{e}_5),(\tilde{e}_3,\tilde{e}_6),(\tilde{e}_4,\tilde{e}_5),(\tilde{e}_7,\tilde{e}_8)\},$$

$$\begin{aligned}\mathcal{R}^{(2)} = \{&(\tilde{e}_1,\tilde{e}_1),\ldots,(\tilde{e}_9,\tilde{e}_9),(\tilde{e}_1,\tilde{e}_2),(\tilde{e}_1,\tilde{e}_4),(\tilde{e}_1,\tilde{e}_5),(\tilde{e}_2,\tilde{e}_1),(\tilde{e}_2,\tilde{e}_4),(\tilde{e}_2,\tilde{e}_5),(\tilde{e}_3,\tilde{e}_6),\\&(\tilde{e}_4,\tilde{e}_1),(\tilde{e}_4,\tilde{e}_2),(\tilde{e}_4,\tilde{e}_5),(\tilde{e}_5,\tilde{e}_1),(\tilde{e}_5,\tilde{e}_2),(\tilde{e}_5,\tilde{e}_4),(\tilde{e}_6,\tilde{e}_3),(\tilde{e}_7,\tilde{e}_8),(\tilde{e}_8,\tilde{e}_7)\}.\end{aligned}$$

Here, the notation $\mathcal{R}^{(12)}$ ($\mathcal{R}^{(21)}$) refers to the soft relation generated by R_1 (R_2) and R_2 (R_1) for the α and β parameters, respectively. In addition, the notation $\mathcal{R}^{(1)}$ ($\mathcal{R}^{(2)}$) refers to the soft relation generated by R_1 (R_2) for both the α and β parameters, respectively. Then, the properties of these soft relations are as follows:

	Reflexive	Irreflexive	Symmetric	Antisymmetric	Asymmetric	Total	Transitive
$\mathcal{R}^{(12)}$	✓	✗	✗	✗	✗	✗	✓
$\mathcal{R}^{(21)}$	✓	✗	✗	✗	✗	✗	✓
$\mathcal{R}^{(1)}$	✓	✗	✗	✓	✗	✗	✓
$\mathcal{R}^{(2)}$	✓	✗	✓	✗	✗	✗	✓

In addition, let $G = \{(\alpha,\{v,w\}),(\beta,\{u,w\})\}$ be a soft set on U. Then, $SE(G) = \{\tilde{e}_4,\tilde{e}_6,\tilde{e}_7,\tilde{e}_9\}$. Hence, one can obtain that $(\tilde{e}_4,\tilde{e}_9)(\alpha) = (v,w) \notin R_1, R_2$ and $(\tilde{e}_4,\tilde{e}_9)(\beta) = (u,w) \notin R_1, R_2$. Thus, a soft relation on G having the pair $(\tilde{e}_4,\tilde{e}_9)$ cannot be generated by the classical relations R_1 and R_2.

Suppose that another parametrized classical relations are defined as

$$\begin{aligned}R_3 &= \{(u,u),(v,v),(w,w),(v,u),(v,w),(u,w)\},\\R_4 &= \{(u,v),(v,w),(w,u)\}.\end{aligned}$$

The properties of these relations are as follows:

	Reflexive	Irreflexive	Symmetric	Antisymmetric	Asymmetric	Total	Transitive
R_3	✓	✗	✗	✓	✗	✓	✓
R_4	✗	✓	✗	✓	✓	✗	✗

If the parametrized family of classical relations $\{R_3, R_4\}$ is considered, the soft relations are obtained as

$$\begin{aligned}\mathcal{R}^{(34)} = \{&(\tilde{e}_1,\tilde{e}_2),(\tilde{e}_1,\tilde{e}_8),(\tilde{e}_2,\tilde{e}_3),(\tilde{e}_2,\tilde{e}_9),(\tilde{e}_3,\tilde{e}_1),(\tilde{e}_3,\tilde{e}_7),(\tilde{e}_4,\tilde{e}_2),(\tilde{e}_4,\tilde{e}_5),(\tilde{e}_4,\tilde{e}_8),(\tilde{e}_5,\tilde{e}_3),\\&(\tilde{e}_5,\tilde{e}_6),(\tilde{e}_5,\tilde{e}_9),(\tilde{e}_6,\tilde{e}_1),(\tilde{e}_6,\tilde{e}_4),(\tilde{e}_6,\tilde{e}_7),(\tilde{e}_7,\tilde{e}_8),(\tilde{e}_8,\tilde{e}_9),(\tilde{e}_9,\tilde{e}_7)\},\end{aligned}$$

$$\begin{aligned}\mathcal{R}^{(43)} = \{&(\tilde{e}_1,\tilde{e}_4),(\tilde{e}_1,\tilde{e}_6),(\tilde{e}_2,\tilde{e}_4),(\tilde{e}_2,\tilde{e}_5),(\tilde{e}_2,\tilde{e}_6),(\tilde{e}_3,\tilde{e}_6),(\tilde{e}_4,\tilde{e}_7),(\tilde{e}_4,\tilde{e}_9),(\tilde{e}_5,\tilde{e}_7),(\tilde{e}_5,\tilde{e}_8),\\&(\tilde{e}_5,\tilde{e}_9),(\tilde{e}_6,\tilde{e}_9),(\tilde{e}_7,\tilde{e}_1),(\tilde{e}_7,\tilde{e}_3),(\tilde{e}_8,\tilde{e}_1),(\tilde{e}_8,\tilde{e}_2),(\tilde{e}_8,\tilde{e}_3),(\tilde{e}_9,\tilde{e}_3)\}.\end{aligned}$$

Also, the soft relations produced by the classical relations R_3 and R_4 are obtained as

$$\begin{aligned}\mathcal{R}^{(3)} = \{&(\tilde{e}_1,\tilde{e}_1),\ldots,(\tilde{e}_9,\tilde{e}_9),(\tilde{e}_1,\tilde{e}_3),(\tilde{e}_1,\tilde{e}_7),(\tilde{e}_1,\tilde{e}_9),(\tilde{e}_2,\tilde{e}_1),(\tilde{e}_2,\tilde{e}_3),(\tilde{e}_2,\tilde{e}_7),(\tilde{e}_2,\tilde{e}_8),\\&(\tilde{e}_2,\tilde{e}_9),(\tilde{e}_3,\tilde{e}_9),(\tilde{e}_4,\tilde{e}_1),(\tilde{e}_4,\tilde{e}_3),(\tilde{e}_4,\tilde{e}_6),(\tilde{e}_4,\tilde{e}_7),(\tilde{e}_4,\tilde{e}_9),(\tilde{e}_5,\tilde{e}_1),\ldots,(\tilde{e}_5,\tilde{e}_9),\\&(\tilde{e}_6,\tilde{e}_3),(\tilde{e}_6,\tilde{e}_9),(\tilde{e}_7,\tilde{e}_9),(\tilde{e}_8,\tilde{e}_7),(\tilde{e}_8,\tilde{e}_9)\},\end{aligned}$$

$$\mathcal{R}^{(4)} = \{(\tilde{e}_1,\tilde{e}_5),(\tilde{e}_2,\tilde{e}_6),(\tilde{e}_3,\tilde{e}_4),(\tilde{e}_4,\tilde{e}_8),(\tilde{e}_5,\tilde{e}_9),(\tilde{e}_6,\tilde{e}_7),(\tilde{e}_7,\tilde{e}_2),(\tilde{e}_8,\tilde{e}_3),(\tilde{e}_9,\tilde{e}_1)\}.$$

Then, the properties of these soft relations are as follows:

	Reflexive	Irreflexive	Symmetric	Antisymmetric	Asymmetric	Total	Transitive
$\mathcal{R}^{(34)}$	✗	✓	✗	✓	✓	✗	✗
$\mathcal{R}^{(43)}$	✗	✓	✗	✓	✓	✗	✗
$\mathcal{R}^{(3)}$	✓	✗	✗	✓	✗	✗	✓
$\mathcal{R}^{(4)}$	✗	✓	✗	✓	✓	✗	✗

Remark 3.2. From Example 3.1, any parametrized family of total classical relations cannot be considered as a total soft relation. Also, the parametrized family of classical relations with various properties cannot be considered as the soft relations with the same properties.

Proposition 3.3. *Every reflexive, symmetric and total soft relation can be considered as a parametrized family of reflexive, symmetric and total classical relations, respectively.*

Proof. Suppose that $\mathcal{R}$ is a reflexive soft relation on $\tilde{U}$ with parameter set P. Then, for all $\tilde{x} \in SE(\tilde{U})$, $(\tilde{x}, \tilde{x}) \in \mathcal{R}$. Hence, for all $\alpha \in P$, $(\tilde{x}(\alpha), \tilde{x}(\alpha)) \in \mathcal{R}_\alpha$ and thus all the classical relations $\mathcal{R}_\alpha$ are reflexive.

Also, for every $\tilde{x}, \tilde{y} \in SE(\tilde{X})$, $(\tilde{x}, \tilde{y}) \in \mathcal{R}$ implies $(\tilde{y}, \tilde{x}) \in \mathcal{R}$ if $\mathcal{R}$ is symmetric soft relation. Hence, for all $\alpha \in P$, $(\tilde{x}(\alpha), \tilde{y}(\alpha)) \in \mathcal{R}_\alpha$ implies $(\tilde{y}(\alpha), \tilde{x}(\alpha)) \in \mathcal{R}_\alpha$ and thus all the classical relations $\mathcal{R}_\alpha$ are symmetric.

In the case of the total soft relation, the proof obtains similarly to above. □

Example 3.2. From Example 3.1, suppose that a soft relation $\mathcal{R}_1$ is defined as

$$\mathcal{R}_1 = \{(\tilde{e}_1, \tilde{e}_1), \ldots, (\tilde{e}_9, \tilde{e}_9), (\tilde{e}_1, \tilde{e}_5), (\tilde{e}_5, \tilde{e}_1), (\tilde{e}_4, \tilde{e}_6), (\tilde{e}_6, \tilde{e}_4), (\tilde{e}_7, \tilde{e}_8), (\tilde{e}_8, \tilde{e}_7)\}.$$

So, $\mathcal{R}_1$ is reflexive, symmetric, and transitive and hence it is equivalence relation. Then, $\mathcal{R}_{1_\alpha}$ is reflexive, symmetric, and transitive and $\mathcal{R}_{1_\beta}$ is reflexive, symmetric, and non-transitive such that

$$\begin{aligned} \mathcal{R}_{1_\alpha} &= \{(u, u), (v, v), (w, w), (u, v), (v, u)\}, \\ \mathcal{R}_{1_\beta} &= \{(u, u), (v, v), (w, w), (u, v), (v, u), (u, w), (w, u)\}. \end{aligned}$$

Thus, although $\mathcal{R}_{1_\alpha}$ is an equivalence relation, $\mathcal{R}_{1_\beta}$ is not.

Suppose that a soft relation $\mathcal{R}_2$ is defined as

$$\mathcal{R}_2 = \{(\tilde{e}_1, \tilde{e}_1), \ldots, (\tilde{e}_9, \tilde{e}_9), (\tilde{e}_1, \tilde{e}_2), (\tilde{e}_1, \tilde{e}_5), (\tilde{e}_2, \tilde{e}_5), (\tilde{e}_8, \tilde{e}_7)\}.$$

So, $\mathcal{R}_2$ is reflexive, antisymmetric, and transitive and hence it is partial order. Then, $\mathcal{R}_{2_\alpha}$ is reflexive, antisymmetric, and transitive and $\mathcal{R}_{2_\beta}$ is reflexive, symmetric, and transitive such that

$$\begin{aligned} \mathcal{R}_{2_\alpha} &= \{(u, u), (v, v), (w, w), (u, v)\}, \\ \mathcal{R}_{2_\beta} &= \{(u, u), (v, v), (w, w), (u, v), (v, u)\}. \end{aligned}$$

Thus, although $\mathcal{R}_{2_\alpha}$ is a partial order, $\mathcal{R}_{2_\beta}$ is an equivalence relation.

Suppose that soft relation $\mathcal{R}_3$ is defined as

$$\mathcal{R}_3 = \{(\tilde{e}_1, \tilde{e}_7), (\tilde{e}_2, \tilde{e}_1), (\tilde{e}_2, \tilde{e}_8), (\tilde{e}_5, \tilde{e}_4), (\tilde{e}_7, \tilde{e}_8)\}.$$

So, $\mathcal{R}_3$ is irreflexive, asymmetric, and non-transitive. Then, $\mathcal{R}_{3_\alpha}$ is reflexive, antisymmetric, and transitive and $\mathcal{R}_{3_\beta}$ is non-reflexive, symmetric, and transitive such that

$$\begin{aligned} \mathcal{R}_{3_\alpha} &= \{(u, u), (v, v), (w, w), (u, w)\}, \\ \mathcal{R}_{3_\beta} &= \{(u, u), (v, v), (u, v), (v, u)\}. \end{aligned}$$

Remark 3.3. From Example 3.2, the irreflexive, transitive, asymmetric, and antisymmetric soft relation cannot be considered as a parametrized family of irreflexive, transitive, antisymmetric, and asymmetric classical relations, respectively.

Definition 3.3. Let $\tilde{U}$ be an absolute soft set with parameter set P having soft relation $\mathcal{R}$ and s be a property of $\mathcal{R}$. The closure of soft relation $\mathcal{R}$, denoted by $cl\mathcal{R}$ according to the property s is a soft relation on $\tilde{U}$ with property s which contains $\mathcal{R}$ such that $cl\mathcal{R}$ is a subset of each soft relation containing $\mathcal{R}$ with property s.

- The reflexive closure of $\mathcal{R}$ is $cl\mathcal{R}^r = \mathcal{R} \cup \Delta$, where Δ denotes the diagonal or identity soft relation on $\tilde{U}$ such that
$$\Delta = \{(\tilde{x}, \tilde{x}) : \tilde{x} \in SE(\tilde{U})\}.$$
- The symmetric closure of $\mathcal{R}$ is $cl\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$, where $\mathcal{R}^{-1}$ denotes the inverse soft relation of $\mathcal{R}$ on $\tilde{U}$ such that
$$\mathcal{R}^{-1} = \{(\tilde{y}, \tilde{x}) : (\tilde{x}, \tilde{y}) \in \mathcal{R}\}.$$
- The transitive closure of $\mathcal{R}$ is $cl\mathcal{R}^t = \bigcup\limits_{n=1}^{\infty} \mathcal{R}^n$, where $\mathcal{R}^n$ denotes the nth power of soft relation of $\mathcal{R}$ on $\tilde{U}$ such that $\mathcal{R}^1 = \mathcal{R}$ and $\mathcal{R}^n = \mathcal{R}^{n-1} \circ \mathcal{R}$ with
$$\mathcal{R} \circ \mathcal{R} = \{(\tilde{x}, \tilde{z}) : \exists \tilde{y} \in SE(\tilde{U}) \ni (\tilde{x}, \tilde{y}), (\tilde{y}, \tilde{z}) \in \mathcal{R}\}.$$

Remark 3.4. From Proposition 3.3, the reflexive and symmetric closures of the soft relation $\mathcal{R}$ can be considered as a parametrized family of reflexive and symmetric closures of the classical relations $\mathcal{R}_\alpha$ for $\alpha \in P$, respectively. But, from Remark 3.3, the transitive closures of the soft relation $\mathcal{R}$ cannot be considered as a parametrized family of transitive closures of the classical relations $\mathcal{R}_\alpha$ for $\alpha \in P$.

Definition 3.4. Let $\mathcal{R}$ be a soft equivalence relation on $\tilde{U}$ and $\tilde{x} \in SE(\tilde{U})$. The soft equivalence class of $\tilde{x}$ determined by $\mathcal{R}$ is the subset of $\tilde{U}$ defined by
$$[\tilde{x}]_{\mathcal{R}} = SS(\{\tilde{y} \in SE(\tilde{U}) : (\tilde{x}, \tilde{y}) \in \mathcal{R}\}).$$
The family of all soft equivalence classes of $\tilde{U}$ is called as soft quotient set of $\tilde{U}$ reduced by $\mathcal{R}$ and denoted by $\tilde{U}/\mathcal{R}$.

Remark 3.5. From Example 3.2, if the soft equivalence relation $\mathcal{R}_1$ is considered, then the soft equivalence classes are obtained as follows:
$$\begin{aligned}
&[\tilde{e}_1] = [\tilde{e}_5] = SS(\{\tilde{e}_1, \tilde{e}_5\}) = \{(\alpha, \{u, v\}), (\beta, \{u, v\})\},\\
&[\tilde{e}_2] = SS(\{\tilde{e}_2\}) = \{(\alpha, \{u\}), (\beta, \{v\})\},\\
&[\tilde{e}_3] = SS(\{\tilde{e}_3\}) = \{(\alpha, \{u\}), (\beta, \{w\})\},\\
&[\tilde{e}_4] = [\tilde{e}_6] = SS(\{\tilde{e}_4, \tilde{e}_6\}) = \{(\alpha, \{v\}), (\beta, \{u, w\})\},\\
&[\tilde{e}_7] = [\tilde{e}_8] = SS(\{\tilde{e}_7, \tilde{e}_8\}) = \{(\alpha, \{w\}), (\beta, \{u, v\})\},\\
&[\tilde{e}_9] = SS(\{\tilde{e}_9\}) = \{(\alpha, \{w\}), (\beta, \{w\})\}.
\end{aligned}$$
Here, $[\tilde{e}_1]$ or $[\tilde{e}_5]$ is generated by the class of soft elements $\{\tilde{e}_1, \tilde{e}_5\}$ but $\tilde{e}_2$ and $\tilde{e}_4$ are also members of these soft equivalence classes. Hence, $[\tilde{e}_1] \Cap [\tilde{e}_2] \neq \Phi$ and $[\tilde{e}_1] \Cap [\tilde{e}_4] \neq \Phi$. Thus, unlike the classical case, it is encountered that the soft equivalence classes are not disjoint.

Theorem 3.1. *Let $\mathcal{R}$ be a soft equivalence relation on $\tilde{U}$.*

1. *There exists a soft equivalence class of all soft elements of $\tilde{U}$ that is different from the null soft set.*
2. *The ϵ-union of all soft equivalence classes is equal to the soft set $\tilde{U}$, i.e.*
$$\Cup_{\tilde{x} \tilde{\in} \tilde{U}} [\tilde{x}]_{\mathcal{R}} = \tilde{U}.$$
3. *For a pair of soft equivalence classes, they are equal or one is a subset of the other or disjoint if and only if the classical relations $\mathcal{R}_\alpha$ are equivalence relations for all $\alpha \in P$.*

Proof. Since the first and second items can be proven similarly to the classical cases, only the third is proven.

Suppose that for any $\tilde{x}, \tilde{y} \in SE(\tilde{U})$, $[\tilde{x}] \Cap [\tilde{y}] \neq \Phi$. There exists $\tilde{z} \in SE(\tilde{U})$ such that $\tilde{z} \in SE([\tilde{x}])$ and $\tilde{z} \in SE([\tilde{y}])$. If $\tilde{z}$ is not a member of one of the classes of soft elements generating $[\tilde{x}]$ and $[\tilde{y}]$, then $[\tilde{x}] \tilde{\subset} [\tilde{y}]$ or $[\tilde{y}] \tilde{\subset} [\tilde{x}]$. Hence, for all $\alpha \in P$, $[\tilde{x}](\alpha) \subset [\tilde{y}](\alpha)$ or $[\tilde{y}](\alpha) \subset [\tilde{x}](\alpha)$. If $\tilde{z}$ is a member of the classes of soft elements generating $[\tilde{x}]$ and $[\tilde{y}]$, then it is clear that $[\tilde{x}] = [\tilde{y}]$. Hence, for all $\alpha \in P$, $[\tilde{x}](\alpha) = [\tilde{y}](\alpha)$. Suppose that for any $\tilde{x}, \tilde{y} \in SE(\tilde{U})$, $[\tilde{x}] \Cap [\tilde{y}] = \Phi$. Then, for at least one $\alpha \in P$, $[\tilde{x}](\alpha) \cap [\tilde{y}](\alpha) = \emptyset$. In case of $[\tilde{x}](\alpha) \cap [\tilde{y}](\alpha) \neq \emptyset$, $[\tilde{x}](\alpha) = [\tilde{y}](\alpha)$ or $[\tilde{x}](\alpha) \subset [\tilde{y}](\alpha)$ or

$[\tilde{y}](\alpha) \subset [\tilde{x}](\alpha)$. Thus, $[\tilde{x}](\alpha)$ is a partition on U for all $\tilde{x} \in SE(\tilde{U})$ and $\alpha \in P$. Since every partition of U determines an equivalence relation on U, the classical relations $\mathcal{R}_\alpha$ coincide with these relations for all $\alpha \in P$. Thus, the classical relations $\mathcal{R}_\alpha$ are equivalence relations for all $\alpha \in P$.

Conversely, suppose that the classical relations $\mathcal{R}_\alpha$ are equivalence relations for all $\alpha \in P$. From Proposition 3.2, the classical relations $\mathcal{R}_\alpha$ produce the $\mathcal{R}$ that is a soft equivalence relation. Then, the equivalence classes of $\mathcal{R}_\alpha$, which are equal or disjoint, correspond to the sets $[\tilde{x}](\alpha)$ for all $\tilde{x} \in SE(\tilde{U})$ and $\alpha \in P$. Hence, the soft equivalence classes of any $\tilde{x}, \tilde{y} \in SE(\tilde{U})$ are equal or disjoint. □

4. Soft relations applied to decision making

In this section, an application of how the soft relations can be used in decision making is presented and an algorithm for dealing with decision making problems is provided based on the weighted method in [3].

In decision making applications, where the concept of relation is used in the previously mentioned (hybrid) soft sets, the decision is made as a single element among the alternatives by determining the attributes and their weights. However, the decision may consist of certain factors and the decision-makers may want to determine each factor that will form the decision in accordance with their current criteria. While making this decision, the possible relations with other situations also occur as an issue. Here, it is proposed that using the soft elements and soft relations to deal with the situations mentioned in the decision making process.

Table 1. Comparison of the decision according to the decision making applications, where the relations are used in the soft sets

Soft set relation [28, 33]	$H : P \times P' \to P(U \times U')$	The decision is an element of U or U'.
Soft binary relation [32]	$\rho : P \to P(U \times U)$	The decision is an element of U.
Softarison [39]	$\mathcal{S} : U \to S_P(U)$	The decision is an element of U.
Soft relation	$\mathcal{R} \subset SE(\tilde{U}) \times SE(\tilde{U})$	The decision is a soft element of $\tilde{U}$.

The following notions are provided to obtain a mathematical framework for the proposed decision making method.

Definition 4.1. Let $\tilde{U}$ be an absolute soft set with parameter set P, $G \in S(\tilde{U})$ and $\mathcal{R}$ be a soft relation on G.

- The number of soft elements other than itself related to a soft element $\tilde{e}_m$ in $\mathcal{R}$ is called the degree of $\tilde{e}_m$, denoted by $deg(\tilde{e}_m) = d_m$. If there exists $\tilde{e}_m$ such that related to itself i.e. $(\tilde{e}_m, \tilde{e}_m) \in \mathcal{R}$, then two degrees are added to $deg(\tilde{e}_m)$.
- The tabular form of the parametrized classical relations $\mathcal{R}_{\alpha_i}$ reduced from $\mathcal{R}$ is defined by entries p_{ij} for each $\alpha_i \in P$, where $p_j \in U \times U$ such that if $p_j \in \mathcal{R}_{\alpha_i}$ then $p_{ij} = 1$, otherwise $p_{ij} = 0$.
- The weighted value of a pair p_j is defined by
$$s_j = \sum_i \omega_i p_{ij},$$
where $\omega_i \in (0, 1]$ are imposed on the parameters in P.

Now, a decision making method using the soft elements and soft relations can be created with the algorithm below.

Algorithm Decision making by using the soft elements and the soft relations

Step 1. Construct a feasible soft set G over U with the parameter set P based on the decision-maker,
Step 2. Construct a soft relation $\mathcal{R}$ on G as requested,
Step 3. Find $cl\mathcal{R}^t$ and find $\mathcal{L} = \{l : d_l = \max d_m\}$ in $cl\mathcal{R}^t$,
Step 4. If there is only one $l \in \mathcal{L}$, then $\tilde{e}_l$ may be chosen,
Step 5. Else find the pairs $(\tilde{e}_l, \tilde{e}_{l'}) \in cl\mathcal{R}^t$, where $l, l' \in \mathcal{L}$,
Step 6. Present $cl\mathcal{R}^t_{\alpha_i}$ in tabular form by computing the s_j for all $\alpha_i \in P$ and find k, for which $s_k = \max s_j$,
Step 7. If there is no pairs such that $(\tilde{e}_l, \tilde{e}_{l'})(\alpha_i) = p_k$ for all $l, l' \in \mathcal{L}$ and $\alpha_i \in P$, then any $\tilde{e}_l$ may be chosen for all $l \in \mathcal{L}$,
Step 8. Else $\tilde{e}_l$ or $\tilde{e}_{l'}$ may be chosen as the most related in $(\tilde{e}_l, \tilde{e}_{l'})$ pairs and having the most $(\tilde{e}_l, \tilde{e}_{l'})(\alpha_i) = p_k$.

4.1 Illustrative example

A company wants to create the most optimal system that can be integrated with other existing systems and choose the components required for the system with the specified parameters. The vendors offer various brands to the company for each system component according to the desired system and ensure the integration of systems that can be obtained with preferred brands.

Let $U = \{u, v, w, x, y\}$ be a set of the brands offered by the vendors for the components and $P = \{\alpha_1 =$ Adaptable, $\alpha_2 =$ Customizable, $\alpha_3 =$ Cheap, $\alpha_4 =$ Durable$\}$ be a set of the parameters determined by the company, where each parameter also corresponds to a component required for the system. Assume that there is a vendor and the soft set G corresponding to this vendor describes the brands of components provided by the vendor according to the parameters as follows.

$$G = \{(\alpha_1, \{u, w, y\}), (\alpha_2, \{v, x, y\}), (\alpha_3, \{u, x\}), (\alpha_4, \{y\})\}.$$

Table 2. The soft elements of G

$\tilde{e}_1 = \{(\alpha_1, u), (\alpha_2, v), (\alpha_3, u), (\alpha_4, y)\}$,	$\tilde{e}_{10} = \{(\alpha_1, w), (\alpha_2, x), (\alpha_3, x), (\alpha_4, y)\}$,
$\tilde{e}_2 = \{(\alpha_1, u), (\alpha_2, v), (\alpha_3, x), (\alpha_4, y)\}$,	$\tilde{e}_{11} = \{(\alpha_1, w), (\alpha_2, y), (\alpha_3, u), (\alpha_4, y)\}$,
$\tilde{e}_3 = \{(\alpha_1, u), (\alpha_2, x), (\alpha_3, u), (\alpha_4, y)\}$,	$\tilde{e}_{12} = \{(\alpha_1, w), (\alpha_2, y), (\alpha_3, x), (\alpha_4, y)\}$,
$\tilde{e}_4 = \{(\alpha_1, u), (\alpha_2, x), (\alpha_3, x), (\alpha_4, y)\}$,	$\tilde{e}_{13} = \{(\alpha_1, y), (\alpha_2, v), (\alpha_3, u), (\alpha_4, y)\}$,
$\tilde{e}_5 = \{(\alpha_1, u), (\alpha_2, y), (\alpha_3, u), (\alpha_4, y)\}$,	$\tilde{e}_{14} = \{(\alpha_1, y), (\alpha_2, v), (\alpha_3, x), (\alpha_4, y)\}$,
$\tilde{e}_6 = \{(\alpha_1, u), (\alpha_2, y), (\alpha_3, x), (\alpha_4, y)\}$,	$\tilde{e}_{15} = \{(\alpha_1, y), (\alpha_2, x), (\alpha_3, u), (\alpha_4, y)\}$,
$\tilde{e}_7 = \{(\alpha_1, w), (\alpha_2, v), (\alpha_3, u), (\alpha_4, y)\}$,	$\tilde{e}_{16} = \{(\alpha_1, y), (\alpha_2, x), (\alpha_3, x), (\alpha_4, y)\}$,
$\tilde{e}_8 = \{(\alpha_1, w), (\alpha_2, v), (\alpha_3, x), (\alpha_4, y)\}$,	$\tilde{e}_{17} = \{(\alpha_1, y), (\alpha_2, y), (\alpha_3, u), (\alpha_4, y)\}$,
$\tilde{e}_9 = \{(\alpha_1, w), (\alpha_2, x), (\alpha_3, u), (\alpha_4, y)\}$,	$\tilde{e}_{18} = \{(\alpha_1, y), (\alpha_2, y), (\alpha_3, x), (\alpha_4, y)\}$.

Each soft element of G given in Table 2 is considered to indicate the systems that the vendor provides. Also, each soft relation on this soft set is considered to correspond to the integrated version of the systems created with the components provided by the vendor. Assume that the following soft relation $\mathcal{R}$ on G is the system integrations that the vendor can provide

$$\mathcal{R} = \{(\tilde{e}_3, \tilde{e}_{13}), (\tilde{e}_6, \tilde{e}_3), (\tilde{e}_6, \tilde{e}_{16}), (\tilde{e}_{11}, \tilde{e}_8), (\tilde{e}_{13}, \tilde{e}_{11}), (\tilde{e}_{13}, \tilde{e}_{16})\}.$$

Then, the transitive closure of $\mathcal{R}$, obtained in below, is considered possible system integrations that can be created.

$$\begin{aligned} cl\mathcal{R}^t = \{&(\tilde{e}_3, \tilde{e}_8), (\tilde{e}_3, \tilde{e}_{11}), (\tilde{e}_3, \tilde{e}_{13}), (\tilde{e}_3, \tilde{e}_{16}), (\tilde{e}_6, \tilde{e}_3), (\tilde{e}_6, \tilde{e}_8), (\tilde{e}_6, \tilde{e}_{11}), \\ &(\tilde{e}_6, \tilde{e}_{13}), (\tilde{e}_6, \tilde{e}_{16}), (\tilde{e}_{11}, \tilde{e}_8), (\tilde{e}_{13}, \tilde{e}_8), (\tilde{e}_{13}, \tilde{e}_{11}), (\tilde{e}_{13}, \tilde{e}_{16})\}. \end{aligned}$$

Since the degree of soft elements are found as $d_3 = d_6 = d_{13} = 5$, $d_8 = d_{11} = 4$ and $d_{16} = 3$, there exist three soft elements which are $\tilde{e}_3$, $\tilde{e}_6$ and $\tilde{e}_{13}$ having the maximum degree. Hence, the members of $cl\mathcal{R}^t$, where these soft elements are related, are $(\tilde{e}_3, \tilde{e}_{13})$, $(\tilde{e}_6, \tilde{e}_3)$ and $(\tilde{e}_6, \tilde{e}_{13})$.

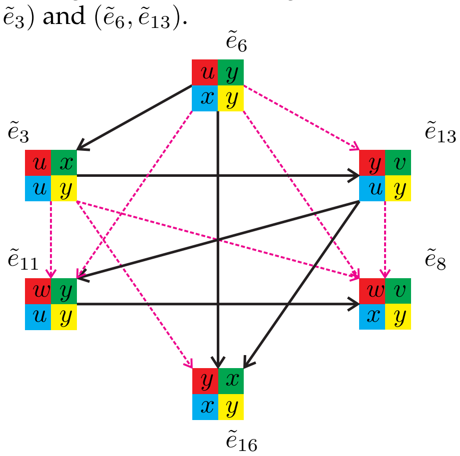

Figure 1. Visualisation of the $cl\mathcal{R}^t$ as a directed graph. Red, green, blue and yellow labels correspond to the parameters α_1, α_2, α_3 and α_4, respectively.

Table 3. The tabular form of parametrized classical relations

p_i	α_1 ($\omega_1 = 0.4$)	α_2 ($\omega_2 = 0.3$)	α_3 ($\omega_3 = 0.8$)	α_4 ($\omega_4 = 0.3$)	s_j
(u,u)	1	0	1	0	1.2
(u,v)	0	0	0	0	0
(u,w)	1	0	0	0	0.4
(u,x)	0	0	1	0	0.8
(u,y)	1	0	0	0	0.4
(v,u)	0	0	0	0	0
(v,v)	0	1	0	0	0.3
(v,w)	0	0	0	0	0
(v,x)	0	1	0	0	0.3
(v,y)	0	1	0	0	0.3
(w,u)	0	0	0	0	0
(w,v)	0	0	0	0	0
(w,w)	1	0	0	0	0.4
(w,x)	0	0	0	0	0
(w,y)	0	0	0	0	0
(x,u)	0	0	1	0	0.8
(x,v)	0	1	0	0	0.3
(x,w)	0	0	0	0	0
(x,x)	0	1	1	0	1.1
(x,y)	0	1	0	0	0.3
(y,u)	0	0	0	0	0
(y,v)	0	1	0	0	0.3
(y,w)	1	0	0	0	0.4
(y,x)	0	1	0	0	0.3
(y,y)	1	1	0	1	1.0

It can be expected that many subsystems, i.e. the components of the systems, will be integrated with each other to increase the functionality of the systems. Assume that the company assigns the weight of the parameters as $\omega_1 = 0.4$, $\omega_2 = 0.3$, $\omega_3 = 0.8$ and $\omega_4 = 0.3$ to assess the relevance between the components of the systems i.e. the pairs p_j.

From the tabular form of parametrized classical relations reduced from $cl\mathcal{R}^t$ in Table 3, it is seen that the company will choose the system $\tilde{e}_3$ according to the parameters and the system integrations since $(\tilde{e}_3, \tilde{e}_{13})(\alpha_3) = (\tilde{e}_6, \tilde{e}_3)(\alpha_1) = (u,u)$ such that $p_k = (u,u)$ and the most related in the pairs $(\tilde{e}_3, \tilde{e}_{13})$ and $(\tilde{e}_6, \tilde{e}_3)$ is $\tilde{e}_3$.

5. Conclusion

In this study, a basis for researches is presented that will use soft relations via soft elements and ε-soft set operations. By using this basis, one can concentrate on the theoretical foundations of the concepts extended to soft set theory. In addition, while someone makes a decision, it should be noted that the decision can consist of certain factors, and it can be desirable to determine these factors according to their attributes. In such cases, which are not considered in any decision making application using the (hybrid) soft sets, it is shown that the concepts of soft element and soft relation are useful. These concepts and the mentioned decision making method can be integrated into the fuzzy sets, vague sets, rough sets, intuitionistic fuzzy sets, and neutrosophic sets and more confirmative solutions can be obtained in decision making problems.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] Molodtsov, D.: *Soft set theory–First results*. Comput. Math. Appl. **37**, 19-31 (1999).

[2] John, S.: Soft sets-theory and applications. Springer Nature. Switzerland (2021).

[3] Maji, P. K., Roy, A. R., Biswas, R.: *An application of soft sets in a decision making problem*. Comput. Math. Appl. **44**, 1077-1083 (2002).

[4] Chen, D., Tsang, E. C. C., Yeung, D. S., Wang, X.: *The parameterization reduction of soft sets and its applications*. Comput. Math. Appl. **49**, 757-763 (2005).

[5] Alcantud, J. C. R.: *Some formal relationships among soft sets, fuzzy sets, and their extensions*. International Journal of Approximate Reasoning. **68**, 45-53 (2016).

[6] Deli, I.: *Interval-valued neutrosophic soft sets and its decision making*. International Journal of Machine Learning and Cybernetics. **8**, 665-676 (2017).

[7] Çetkin, V., Güner, E., Aygün, H.: *On 2S-metric spaces*. Soft Computing. **24**, 12731-12742 (2020).

[8] Akgüller, Ö.: *Discrete Ricci curvature-based statistics for soft sets*. Soft Computing. **25**, 599-612 (2021).

[9] Zorlutuna, İ.: *Soft set-valued mappings and their application in decision making problems*. Filomat **35**, 1725-1733 (2021).

[10] Maji, P. K., Biswas, R., Roy, A. R.: *Soft set theory*. Computers & Mathematics with Applications. **45**, 555-562 (2003).

[11] Ali, M. I., Feng, F., Liu, X., Min, W. K., Shabir, M.: *On some new operations in soft set theory*. Computers & Mathematics with Applications. **57**, 1547-1553 (2009).

[12] Aktaş, H., Çağman, N.: *Soft sets and soft groups*. Information Sciences. **177**, 2726-2735 (2007).

[13] Çağman, N., Enginoğlu, S.: *Soft set theory and uni–int decision making*. European Journal of Operational Research. **207** (2), 848-855 (2010).

[14] Sezgin, A., Atagün, A. O.: *On operations of soft sets*. Computers & Mathematics with Applications. **61**, 1457-1467 (2011).

[15] Abbas, M., Ali, B., Romaguera, S.: *On generalized soft equality and soft lattice structure*. Filomat. **28**, 1191-1203 (2014).

[16] Gündüz, Ç., Sönmez, A., Çakallı, H.: *An approach to soft functions*. Journal of Mathematical Analysis. **8**, 129–138 (2017).

[17] Kandemir, M. B.: *The concept of σ-algebraic soft set*. Soft Computing. **22**, 4353-4360 (2018).

[18] Aygün, E., Kamacı, H.: *Some generalized operations in soft set theory and their role in similarity and decision making*. Journal of Intelligent & Fuzzy Systems. **36**, 6537-6547 (2019).

[19] Das, S., Samanta, S. K.: *Soft real sets, soft real numbers and their properties*. J. Fuzzy Math. **20**, 551-576 (2012).

[20] Das, S., Samanta, S. K.: *On soft metric spaces*. J. Fuzzy Math. **21**, 707-734 (2013).

[21] Güler, A. Ç., Yıldırım, E. D., Özbakır, O. B.: *A fixed point theorem on soft G-metric spaces*. J. Nonlinear Sci. Appl. **9**, 885-894 (2016).

[22] Chiney, M., Samanta, S.: *Soft topology redefined*. J. Fuzzy Math. **27**, 459-486 (2019).

[23] Altıntaş, I., Taşköprü, K.: *Compactness of soft cone metric space and fixed point theorems related to diametrically contractive mapping*. Turkish Journal of Mathematics. **44**, 2199-2216 (2020).

[24] Demir, İ.: *Some soft topological properties and fixed soft element results in soft complex valued metric spaces*. Turkish Journal of Mathematics **45**, 971-987 (2021).

[25] Taşköprü, K., Altıntaş, I.: *A new approach for soft topology and soft function via soft element*. Math. Meth. Appl. Sci. **44**, 7556-7570 (2021).

[26] Altıntaş, I., Taşköprü, K., Selvi, B.: *Countable and separable elementary soft topological space*. Math. Meth. Appl. Sci. **44**, 7811-7819 (2021).

[27] Altıntaş, I., Taşköprü, K., Esengul kyzy, P.: *Soft partial metric spaces*. Soft Computing. **26**, 8997-9010 (2022).

[28] Babitha, K. V., Sunil, J. J.: *Soft set relations and functions*. Computers & Mathematics with Applications. **60**, 1840-1849 (2010).

[29] Babitha, K. V., Sunil, J. J.: *Transitive closures and orderings on soft sets*. Computers & Mathematics with Applications. **62**, 2235-2239 (2011).

[30] Yang, H. L., Guo, Z. L.: *Kernels and closures of soft set relations, and soft set relation mappings*. Computers & Mathematics with Applications. **61**, 651-662 (2011).

[31] Park, J. H., Kim, O. H., Kwun, Y. C.: *Some properties of equivalence soft set relations*. Computers & Mathematics with Applications. **63**, 1079-1088 (2012).

[32] Feng, F., Ali, M. I., Shabir, M.: *Soft relations applied to semigroups*. Filomat. **27**, 1183-1196 (2013).

[33] Qin, K., Liu, Q., Xu, Y.: *Redefined soft relations and soft functions*. International Journal of Computational Intelligence Systems. **8**, 819-828 (2015).

[34] Polat, Ç. N., Yaylalı, G., Tanay, B.: *A method for decision making problems by using graph representation of soft set relations*. Intelligent Automation and Soft Computing. **25**, 305-311 (2019).

[35] Kanwal, R. S., Qurashi, S. M., Shabir, M.: *Generalized approximation of substructures in quantales by soft relations*. Computational and Applied Mathematics. **39**, 1-22 (2020).

[36] Yaylalı, G., Polat, N., Tanay, B.: *Soft intervals and soft ordered topology*. Celal Bayar University Journal of Science. **13**, 81-89 (2017).

[37] Al-Shami, T. M., El-Shafei, M. E., Abo-Elhamayel, M.: *On soft topological ordered spaces*. Journal of King Saud University-Science. **31**, 556-566 (2019).

[38] Al-Shami, T. M., El-Shafei, M. E.: *Two new forms of ordered soft separation axioms*. Demonstratio Mathematica. **53**, 8-26 (2020).

[39] Alcantud, J. C. R.: *Softarisons: theory and practice*. Neural Computing and Applications. **33**, 16759-16771 (2021).

[40] Alhazaymeh, K., Hassan, N.: *Vague soft set relations and functions*. Journal of Intelligent & Fuzzy Systems. **28**, 1205-1212 (2015).

[41] Karaaslan, F.: *Bipolar soft rough relations*. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics. **65**, 105-126 (2016).

[42] Qamar, M. A., Hassan, N.: *Q-Neutrosophic soft relation and its application in decision making*. Entropy. **20**, 1-14 (2018).

[43] Kanwal, R. S., Shabir, M.: *Rough approximation of a fuzzy set in semigroups based on soft relations*. Computational and Applied Mathematics. **38**, 1-23 (2019).

[44] El-Shafei, M. E., Al-shami, T. M.: *Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem*. Computational and Applied Mathematics. **39**, 1-17 (2020).

[45] Dalkılıç, O.: *Relations on neutrosophic soft set and their application in decision making*. Journal of Applied Mathematics and Computing. **67**, 257-273 (2021).

[46] Dalkılıç, O., Demirtaş, N.: *A novel perspective for Q-neutrosophic soft relations and their application in decision making*. Artificial Intelligence Review. 1-21 (2022).

Affiliations

KEMAL TAŞKÖPRÜ
ADDRESS: Bilecik Şeyh Edebali University, Department of Mathematics, Faculty of Science, Bilecik 11230, Türkiye.
E-MAIL: kemal.taskopru@bilecik.edu.tr
ORCID ID: 0000-0002-0760-3782

ELIF KARAKÖSE
ADDRESS: Bilecik Şeyh Edebali University, Department of Mathematics, Faculty of Science, Bilecik 11230, Türkiye.
E-MAIL: elf.ozdamar.1.9.9.3@gmail.com
ORCID ID: 0000-0001-6204-0890

MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753/mathenot.1194255
11 (1) 14-28 (2023) - Research Article
ISSN: 2147-6268

Two Numerical Schemes for the Solution of the Generalized Rosenau Equation with the help of Operator Splitting Techniques

Melike Karta*

Abstract
In the present manuscript, numerical solution of generalized Rosenau equation are applied quintic B-spline collocation and cubic B-spline lumped-Galerkin finite element methods (FEMs) together with both Strang splitting technique and the Ext4 and Ext6 techniques based on Strang splitting and derived from extrapolation. In the first instance, the problem is divided into two sub-equations as linear $U_t = \hat{A}(U)$ and nonlinear $U_t = \hat{B}(U)$ in the time term. Later, these sub-equations is implemented collocation and lumped-Galerkin (FEMs) using quintic and cubic B-spline functions respectively, with Strang ($S\Delta t = \hat{A} - \hat{B} - \hat{A}$), Ext4 and Ext6 splitting techniques. The numerical solutions of the system of ordinary differential equations obtained in this way are solved with help fourth order Runge-Kutta method. The aim of this study is to obtain superior results. For this, a test problem is selected to show the accuracy and efficiency of the method and the error norm results produced by these techniques have been compared among themselves and with the current study in the literature. İt can be clearly stated that it is concluded that the approximate results obtained with the proposed method are better than the study in the literature. So that one can see that the study has achieved its purpose.

Keywords: *Generalized Rosenau equation, quintic and cubic B-splines, collocation and Galerkin methods, Splitting techniques.*

AMS Subject Classification (2020): *35Q51; 74J35; 33F10.*

****Corresponding author***

1. Introduction

The Rosenau equation has been an important research and application topic in the fields of mathematics and physics by Philip Rosenau [1] since the 80s. Because the Korteweg-de Vries (KdV) equation which is one of the most

Received : *25-10-2022,* ***Accepted :*** *07-02-2023*

*(**Cite as** "M. Karta, Two Numerical Schemes for the Solution of the Generalized Rosenau Equation with the help of Operator Splitting Techniques, Math. Sci. Appl. E-Notes, 11(1) (2023), 14-28")*

important partial differential equations doesn't introduce wave to wave interaction and wave to wall interaction Chung [2], Rosenau [3] has exposed equation

$$Ut + Uxxxxt + UUx + Ux = 0, (x,t) \in \Omega x(0,T] \tag{1.1}$$

with the condition given at initial time

$$U(x,0) = g_0(x), \tag{1.2}$$

and the conditions given at the boundaries

$$\begin{aligned} U(0,t) &= U(1,t) = 0, \\ U_{xx}(0,t) &= U_{xx}(1,t) = 0 \quad t > 0. \end{aligned} \tag{1.3}$$

Existence and uniqueness of equation with initial and boundary condition given in above have been studied by Park [4] and she has shown that the solution $U \in H^4(\Omega)$ is the only one solution for $U_0 \in H^4(\Omega)$. There are many studies in the literature about the Rosenau equation. These can be summarized as follows: Chung and Ha [5] proposed finite element Galerkin approximate solutions for a KDV-like Rosenau equation that models the dynamics of dense discrete systems to show existence and uniqueness of exact solutions and discussed the error estimates of the continuous time Galerkin solutions. Manickam *et al*.[6] performed a KDV-like Rosenau equation in one space variable using a second-order splitting method. Hence they employed an orthogonal cubic spline collocation procedure to approximate the resulting system. Chung [2] indicates existence and uniqueness of numerical solutions for the KDV-like Rosenau equation describing the dynamics of dense discrete systems. Lee [7] used the discrete Galerkin type approximations for solutions of the Rosenau equation. Sportisse [8] analyzed that the evolution equations to be simulated are stiff. Chung and Pani [9] showed a continuous in time finite element Galerkin method for a KDV-like Rosenau equation in several space variables and suggested several fully discrete schemes and build up connected convergence results.The Rosenau equation in several space variables was split into two second order equations and submited a lumped mass finite element method for piece wise linear elements by Chung and Pani [10]. Barreto *et al* [11] showed the existence of solutions of the Rosenau and Benjamin-Bona-Mahony equations known as hyperbolic equation. Choo *et al*. [12] achieved a posteriori error estimates of Rosenau equation using a discontinuous Galerkin method. Omrani *et al* [13] presented a conservative difference scheme for the KDV-like Rosenau equation appeared the unique solvability of numerical solutions. Hu and Zheng [14] suggested numerical solutions of generalized Rosenau equation and considered two energy conservative finite difference schemes. Wang *et al*.[15] considered the generalized Rosenau equation with a finite difference scheme and researched existence and uniqueness of numerical solution of equation. Mittal and Jain [16] solved some Rosenau type non-linear higher order evolution equations with Dirichlet's boundary conditions with the help of quintic B-splines collocation method. Atouani and Omrani [17] applied high-order conservative difference scheme for Rosenau equation. Abazari and Abazari [18] devoted numerical solution of KDV-like Rosenau equation using the quintic B-spline collocation scheme. Cai *et al*.[19] employed the variational discretizations for the generalized Rosenau-type equations. Ramos and Garcia-Lopez [20] studied numerically a generalized viscous Rosenau equation with the help of an implicit second-order accurate method in time. Safdari-Vaighani *et al*. [21] implemented radial basis function method (RBF) approximations methods for numerical solution of Rosenau equation, where they extented the fictitious point method and the resampling method to study by means of an RBF collocation. References to [22–25] and [26–35] can also be looked at for different methods applied for this type of partial differential equations. Atouani *et al*. [36] performed mixed finite element methods for Rosenau equation via splitting technique. Also, one can have a look at the references [37–43] on the splitting technique. In this study, we will consider generalized Rosenau equation given with form

$$2Ut + Uxxxxt + 3Ux - 60U^2U_x + 120U^4U_x = 0 \tag{1.4}$$

with an initial condition

$$U(x,0) = sech(x) = g_0(x), x \in [x_L, x_R], \tag{1.5}$$

and with boundary condition

$$\begin{aligned} U(x_L,t) &= sech(x_L - t) = g_1(x), \\ U(x_R,t) &= sech(x_R - t) = g_2(x), \end{aligned} \tag{1.6}$$

$$\begin{aligned} U_x(x_L,t) &= -sech(x_L - t)tanh(x_L - t) = g_3(x), \\ U_x(x_R,t) &= sech(x_R - t)tanh(x_R - t) = g_4(x). \end{aligned} \tag{1.7}$$

The present work is summarized as follow: In section 2, about operator splitting method is informed. In section 3-4, generalized Rosenau equation is split into two sub-equations and each equation is implemented collocation and lumped-Galerkin (FEMs) with quintic and cubic B-splines respectively, and they are coverted of system order-differential equations and solved with operator splitting techniques using fourth order Runge-Kutta technque (RK-4). In Section 5, a test problem given with initial and boundary conditions is considered. Error norms L_2 and L_∞ obtained by operator splitting techniques are among themselves and with study available in the literature are compared. In Section 6, to emphasize the importance of the present method, a brief conclusion is given.

By using fourth order Runge-Kutta method via quintic B-spline collocation and cubic B-spline lumped-Galerkin finite element methods (FEMs) together with operator time splitting techniques, numerical solutions of the main problem in the study can be easily obtained. Thus, one can see that it has been produced quite good results with the method proposed in the study.

2. Splitting techniques

The splitting technique achieved with the half-time step Δt is sometimes known as Marhuk [42] and is the second-order symmetric technique proposed by Strang [44]. The mentioned technique can be defined as follows by changing the locations of the operators

$$S_{\Delta t} = e^{\frac{\Delta t}{2}\hat{A}} e^{\Delta t \hat{B}} e^{\frac{\Delta t}{2}\hat{A}} \text{ or } S^*_{\Delta t} = e^{\frac{\Delta t}{2}\hat{B}} e^{\Delta t \hat{A}} e^{\frac{\Delta t}{2}\hat{B}}. \tag{2.1}$$

This technique has the local turncation error called as splitting error which is in form

$$\begin{aligned} Te &= \frac{(e^{\Delta t(\hat{A}+\hat{B})} - e^{\frac{\Delta t}{2}\hat{A}} e^{\Delta t\hat{B}} e^{\frac{\Delta t}{2}\hat{A}})U(t_n)}{\Delta t} \\ &= \frac{\Delta t^2}{24}(2\left[\hat{B},\left[\hat{B},\hat{A}\right]\right] - \left[\hat{A},\left[\hat{A},\hat{B}\right]\right])U(t_n) + O(\Delta t^3) \end{aligned}$$

and this shows the fact that the proposed technique is of the second-order. The procedure for Strang splitting scheme can be presented as

$$\begin{aligned} \frac{dU^*(t)}{dt} &= \hat{A}U^*(t), \;\; U^*(t_n) = U_n^0, && t \in \left[t_n, t_{n+\frac{1}{2}}\right] \\ \frac{dU^{**}(t)}{dt} &= \hat{B}U^{**}(t), \;\; U^{**}(t_n) = U^*(t_{n+\frac{1}{2}}), && t \in [t_n, t_{n+1}] \\ \frac{dU^{***}(t)}{dt} &= \hat{A}U^{***}(t), \;\; U^{***}(t_{n+\frac{1}{2}}) = U^{**}(t_{n+1}), && t \in \left[t_{n+\frac{1}{2}}, t_{n+1}\right] \end{aligned} \tag{2.2}$$

in which $t_{n+\frac{1}{2}} = t_n + \frac{\Delta t}{2}$. Here the desired solutions are easily obtained through the equation of $U(t_{n+1}) = U^{***}(t_{n+1})$. When this scheme is defined as $\hat{A}-\hat{B}-\hat{A}$, a new scheme can be obtained as $\hat{B}-\hat{A}-\hat{B}$. In the present study, extrapolation techniques [45] presented below are used to further improve convergence

$$\frac{4}{3}\phi\frac{\Delta t}{2} * \phi\frac{\Delta t}{2} - \frac{1}{3}\phi\Delta t,$$

and

$$\frac{81}{40}\phi\frac{\Delta t}{3} * \phi\frac{\Delta t}{3} * \frac{\Delta t}{3} * -\frac{16}{15}\phi\frac{\Delta t}{2} * \phi\frac{\Delta t}{2} * +\frac{1}{24}\phi\Delta t.$$

In addition to the Strang splitting technique, the fourth and sixth order convergence techniques presented below are used, respectively.

$$Ext4 = \frac{4}{3}\phi^{\hat{B}}_{\frac{\Delta t}{4}} * \phi^{\hat{A}}_{\frac{\Delta t}{2}} * \phi^{\hat{B}}_{\frac{\Delta t}{2}} * \phi^{\hat{A}}_{\frac{\Delta t}{2}}\phi^{\hat{B}}_{\frac{\Delta t}{2}} - \frac{1}{3}\phi^{\hat{B}}_{\frac{\Delta t}{2}}\phi^{\hat{A}}\Delta t\phi^{\hat{B}}_{\frac{\Delta t}{2}}$$

$$\begin{aligned} Ext6 = \frac{81}{40}\phi^{\hat{B}}_{\frac{\Delta t}{6}} * \phi^{\hat{A}}_{\frac{\Delta t}{3}} * \phi^{\hat{B}}_{\frac{\Delta t}{3}}\phi^{\hat{A}}_{\frac{\Delta t}{3}} * \phi^{\hat{B}}_{\frac{\Delta t}{3}} * \phi^{\hat{A}}_{\frac{\Delta t}{3}}\phi^{\hat{B}}_{\frac{\Delta t}{6}} - \frac{16}{15}\phi^{\hat{B}}_{\frac{\Delta t}{4}} * \phi^{\hat{A}}_{\frac{\Delta t}{2}} * \phi^{\hat{B}}_{\frac{\Delta t}{2}} * \phi^{\hat{A}}_{\frac{\Delta t}{2}}\phi^{\hat{B}}_{\frac{\Delta t}{4}} \\ +\frac{1}{24}\phi^{\hat{B}}_{\frac{\Delta t}{2}}\phi^{\hat{A}}\Delta t\phi^{\hat{B}}_{\frac{\Delta t}{2}}. \end{aligned}$$

3. Scheme 1: Operator time-splitting solution by quintic B-spline collocation method of generalized Rosenau equation

Let be given as $x_L = x_0 \leq x_1 \leq ... \leq x_N = x_R$ at the knots x_m, $(m = 0(1)N-1)$ a uniform partition of closed interval $x_L \leq x \leq x_R$ about to be the solution domain $x_L \leq x \leq x_R$ in which $h = \frac{x_R - x_L}{N}$. Quintic B-spline functions $\varphi_m(x)$ are defined such as in the following on the domain $x_L \leq x \leq x_R$ with knot points x_m, $(m = -2(1)N+2)$

$$\varphi_m(x) = \frac{1}{h^5}\begin{cases} (x-x_{m-3})^5, & x \in [x_{m-3}, x_{m-2}] \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5, & x \in [x_{m-2}, x_{m-1}] \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5, & x \in [x_{m-1}, x_m] \\ (x_{m+3}-x)^5 - 6(x_{m+2}-x)^5 + 15(x_{m+1}-x)^5, & x \in [x_m, x_{m+1}] \\ (x_{m+3}-x)^5 - 6(x_{m+2}-x)^5, & x \in [x_{m+1}, x_{m+2}] \\ (x_{m+3}-x)^5, & x \in [x_{m+2}, x_{m+3}] \\ 0, & otherwise. \end{cases} \tag{3.1}$$

[46]. All the quintic B-spline functions outside of $\varphi_{m-2}(x), \varphi_{m-1}(x), \varphi_m(x), \varphi_{m+1}(x), \varphi_{m+2}(x), \varphi_{m+3}(x)$, are zero except that the elements on $[x_m, x_{m+1}]$. Global approximation $U_N(x,t)$ corresponding exact solution $U(x,t)$ of eq.(1.4) can be written with the following formula on $[x_m, x_{m+1}]$ in terms of quintic B-spline functions and quantities $\delta_j(t)$ that need to be found

$$U_N(x,t) = \sum_{j=m-2}^{m+3} \varphi_j(x)\delta_j(t). \tag{3.2}$$

Using knot values in eqs.(3.1) and (3.2), derivatives up to fourth order of U and U with respect to variable x in terms of time dependent parameters are obtained as follows:

$$\begin{aligned} U_N^e(x_m,t) = (U_N^e)_m &= (\delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2}), \\ (U_N^e)'_m = U'_m &= \frac{5}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}), \\ (U_N^e)''_m = U''_m &= \frac{20}{h^2}(\delta_{m-2} + 2\delta_{m-1} - 6\delta_m + 2\delta_{m+1} + \delta_{m+2}), \\ (U_N^e)'''_m = U'''_m &= \frac{60}{h^3}(-\delta_{m-2} + 2\delta_{m-1} - 6\delta_m - 2\delta_{m+1} + \delta_{m+2}), \\ (U_N^e)^{(4)}_m = U^{(4)}_m &= \frac{120}{h^4}(\delta_{m-2} - 4\delta_{m-1} + 6\delta_m - 4\delta_{m+1} + \delta_{m+2}). \end{aligned} \tag{3.3}$$

Generalized Rosenau equation (1.4) is split into as follows:

$$2Ut + Uxxxxt + 3Ux = 0 \tag{3.4}$$

$$2Ut + Uxxxxt - 60U^2U_x + 120U^4U_x = 0. \tag{3.5}$$

If the values of U, U', U'', U''' and $U^{(4)}$ in (3.3) are replaced in equations (3.4) and (3.5), the following system of first ordinary differential equations (3.6) and (3.7) to be consisted of $(N+1)$ equations and $(N+5)$ unknowns are found as follows:

$$\begin{aligned} 2\dot{\delta}_{m-2} + 52\dot{\delta}_{m-1} + 132\dot{\delta}_m + 52\dot{\delta}_{m+1} + 2\dot{\delta}_{m+2} + \frac{120}{h^4}(\dot{\delta}_{m-2} - 4\dot{\delta}_{m-1} + 6\dot{\delta}_m - 4\dot{\delta}_{m+1} + \dot{\delta}_{m+2}) \\ + \frac{15}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}) \end{aligned} \tag{3.6}$$

$$\begin{aligned} 2\dot{\delta}_{m-2} + 52\dot{\delta}_{m-1} + 132\dot{\delta}_m + 52\dot{\delta}_{m+1} + 2\dot{\delta}_{m+2} - \frac{120}{h^4}(\dot{\delta}_{m-2} - 4\dot{\delta}_{m-1} + 6\dot{\delta}_m - 4\dot{\delta}_{m+1} + \dot{\delta}_{m+2}) \\ \frac{-300z_m}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}) + \frac{600g_m}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}) = 0 \end{aligned} \tag{3.7}$$

where the symbol $''.''$ denotes derivation with respect to time t and $z_m = U^2, g_m = U^4$ are considered as linearization process respectively in the following form

$$z_m = (\delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2})^2,$$

$$g_m = (\delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2})^4.$$

Eliminating the parameters δ_{-2}, δ_{-1} and $\delta_{N+2}, \delta_{N+1}$ which are outside of the solution region from systems (3.6) and (3.7) using the boundary conditions $U(x_L,t) = sech(x_L - t), U(x_R,t) = sech(x_R - t)$ and $U_x(x_L,t) = sech(x_L - t)tanh(x_L - t), U_x(x_R,t) = sech(x_R - t)tanh(x_R - t)$ given by equation (1.4) , we obtain matrix systems $(N+1)$ x $(N+1)$ for systems (3.6) and (3.7) with form

$$\dot{\delta}^{n+1} = A_1^{-1} B_1 \delta^n$$

$$\dot{\delta}^{n+1} = A_1^{-1} B_2 \delta^n$$

such that $\delta^n = (\delta_0, ..., \delta_0 \delta_N)^T$ where A_1, B_1 and B_2 are matrices of dimensional $(N+1)$ x $(N+1)$ acquired as

$$A_1 = [a_{ij}] = \begin{cases} a_{11} = \frac{7650}{h^4}, a_{12} = \frac{4500}{h^4}, a_{13} = \frac{450}{h^4}, \\ a_{21} = \frac{175}{4} - \frac{975}{h^4}, a_{22} = \frac{255}{2} - \frac{990}{h^4}, a_{23} = \frac{207}{4} - \frac{495}{h^4}, a_{23} = a_{24} = 2 + \frac{120}{h^4} \\ a_{i,i-2} = a_{24}, a_{i,i-1} = 52 - \frac{480}{h^4}, a_{ii} = 132 + \frac{720}{h^4} \\ a_{i,i+1} = a_{i,i-1}, a_{i,i+2} = a_{24}; i = 3(1)N-1, \\ a_{N,N-2} = a_{24}, a_{N,N-1} = a_{23}, a_{N,N} = a_{22}, a_{N,N+1} = a_{21} \\ a_{N+1,N+1} = a_{11} \end{cases} \tag{3.8}$$

$$B_1 = [b_{ij}] = \begin{cases} b_{11} = 0, \\ b_{21} = \frac{705}{8h}, b_{22} = -\frac{135}{4h}, b_{23} = -\frac{1215}{8h}, b_{24} = -\frac{15}{h} \\ b_{i,i-2} = -b_{24}, a_i, a_{i-1} = \frac{150}{h}, b_{ii} = 0, \\ b_{i,i+1} = -\frac{150}{h}, b_{i,i+2} = b_{24}, i = 3(1)N-1, \\ b_{N,N-2} = -b_{24}, b_{N,N-1} = -b_{23}, b_{N,N} = -b_{22}, b_{N,N+1} = -b_{21} \\ a_{N+1,N+1} = 0. \end{cases} \tag{3.9}$$

$$B_2 = [b_{ij}] = \begin{cases} b_{11} = 0, \\ b_{21} = -\frac{3525}{2h} z_1 + \frac{3525}{h} g_1, b_{22} = \frac{675}{h} z_1 - \frac{1350}{h} g_1, \\ b_{23} = \frac{6075}{2h} z_1 - \frac{6075}{h} g_1, b_{24} = \frac{300}{h} z_1 - \frac{600}{h} g_1 \\ b_{i,i-2} = -\frac{300}{2h} z_m + \frac{600}{h} g_m, a_i, a_{i-1} = -\frac{3000}{2h} z_m + \frac{6000}{h} g_m, b_{ii} = 0, \\ b_{i,i+1} = -b_{i,i-1}, b_{i,i+2} = -b_{i,i-2}; i = 3(1)N-1, \\ b_{N,N-2} = -\frac{300}{h} z_N + \frac{600}{h} g_N, b_{N,N-1} = -\frac{6075}{2h} z_N + \frac{6075}{h} g_N, b_{N,N} = -\frac{675}{h} z_N + \frac{1350}{h} g_N, \\ b_{N,N+1} = \frac{3525}{2h} z_N - \frac{3525}{h} g_N, \\ b_{N+1,N+1} = 0. \end{cases} \tag{3.10}$$

As a solution method, after splitting the main equation, the FEMs presented in the article are applied to each equation. Then, the obtained system of ordinary differential equations is solved using the Runge Kutta method (RK4) with the help of Strang, Ext4 and Ext6 splitting algorithms. Furthermore, in order to obtain better results at each time step, systems (3.6) and (3.7) are applied three-five times an inner iteration given by

$$(\delta^*)^n = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1}).$$

Now, for solution of the systems (3.6) and (3.7), we need to obtain the initial vector δ_m^0 with the help of initial condition $U(x,0) = g_0(x)$ and boundary conditions

$$\begin{aligned} U_x(x_L,t) = g_3(x), U_x(x_R,t) = g_4(x), \\ U_{xx}(x_L,t) = g_5(x), U_{xx}(x_R,t) = g_6(x). \end{aligned} \tag{3.11}$$

Finally, the matrix equation for the initial vector δ_m^0 is acquired as

$$\begin{bmatrix} 54 & 60 & 6 & & & & & \\ 25.25 & 67.5 & 26.25 & 1 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & \\ & & & \ddots & & & & \\ & & 1 & 26 & 66 & 26 & 1 & \\ & & & 1 & 26.25 & 67.5 & 25.25 & \\ & & & & 6 & 60 & 54 & \end{bmatrix} \begin{bmatrix} {\delta^0}_0 \\ {\delta^0}_1 \\ {\delta^0}_2 \\ . \\ . \\ . \\ {\delta^0}_{N-2} \\ {\delta^0}_{N-1} \\ {\delta^0}_N \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ . \\ . \\ . \\ U_{N-2} \\ U_{N-1} \\ U_N \end{bmatrix}.$$

These matrices are easy to obtain with a symbolic programming language. In this study, Matlab 2019b with a memory 20GB and 64 bit has been used.

4. Scheme 2: Operator time-splitting solution by cubic B-spline Lumped Galerkin method of generalized Rosenau equation

In this section, we will handle with cubic B-spline lumped Galerkin method to numerical solution of (1.4) equation with the initial-boundary conditions (1.5) and (1.6)-(1.7). Here, the solution region of the problem is taken as in section 3. Cubic B-spline functions $\varphi_m(x)$, $(m = -1(1)N + 1)$ at knots point x_m on the solution domain $[x_L, x_R]$ are described by Prenter [46] as follows:

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}) \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & x \in [x_{m-1}, x_m) \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & x \in [x_m, x_{m+1}) \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}] \\ 0, & otherwise. \end{cases} \tag{4.1}$$

Approximate solution $U_N(x,t)$ corresponding exact solution $U(x,t)$ of eq.(1.4) can be given in the following form in terms of cubic B-splines on $[x_m, x_{m+1}]$

$$U_N(x,t) = \sum_{j=-1}^{N+1} \varphi_j(x)\delta_j(t) \tag{4.2}$$

where $\varphi_j(x)$ are element shape functions and $\delta_j(t)$ are unknown time-dependent element parameters obtained with boundary conditions and weighted residual conditions. Using the local coordinate transformation given by $\xi = x - x_m$ such that $0 \le \xi \le h$ on the finite element $[x_m, x_{m+1}]$, cubic B-spline shape functions in terms of ξ on the region [0,h] can be submitted as follows:

$$\begin{aligned} \varphi_{m-1} &= \frac{1}{h^3}(h - \xi)^3, \\ \varphi_m &= \frac{1}{h^3}(4h^3 - 6h\xi^2 + 3\xi^3), \\ \varphi_{m+1} &= \frac{1}{h^3}(h^3 + 3h^2\xi + 3h\xi^2 - 3\xi^3), \\ \varphi_{m+2} &= \frac{1}{h^3}(\xi^3). \end{aligned} \tag{4.3}$$

Thus, the approximate solution $U_N(x,t)$ can be given as

$$U_N(\xi,t) = \sum_{j=m-1}^{m+2} \delta_j\varphi_j. \tag{4.4}$$

whole splines except $\varphi_{m-1}(x), \varphi_m(x), \varphi_{m+1}(x), \varphi_{m+2}(x)$ are zero over the domain $[x_m, x_{m+1}]$. Using cubic B-spline functions (4.1) and trial functions (4.2), u_N and its first and second derivatives at knots x_m according to x in terms of the element parameters δ_j are presented by

$$\begin{aligned} U_m &= Ux_m = \delta_{m+1} + 4\delta_m + \delta_{m-1} \\ U_m^{'} &= U^{'}(x_m) = \frac{3}{h}(\delta_{m+1} - \delta_{m-1}) \\ U_m^{''} &= U^{''}(x_m) = \frac{6}{h^2}(\delta_{m+1} - 2\delta_m + \delta_{m-1}). \end{aligned} \tag{4.5}$$

Generalized Rosenau equation (1.4) is split into two sub-equations as follows:

$$2U_t + U_{xxxxt} + 3U_x = 0, \tag{4.6}$$

$$2U_t + U_{xxxxt} - 60U^2U_x + 120U^4U_x = 0. \tag{4.7}$$

When applying the Galerkin method to (4.6) and (4.7) equations, respectively, the weak form of (4.6) and (4.7) equations is obtained as follows:

$$\int_{x_L}^{x_R} W\Big[2U_t + U_{xxxxt} + 3U_x\Big]dx = 0 \tag{4.8}$$

$$\int_{x_L}^{x_R} W\Big[2U_t + U_{xxxxt} - 60U^2U_x + 120U^4U_x\Big]dx = 0. \tag{4.9}$$

Here, due to the use of the Galerkin method, the weight function is chosen the same as the approximate functions and the approximate functions are B-splines and at the same time the smoothness of the weight function is guaranteed. If is used transformation $\xi = x - x_m$, we can get the following equations

$$\int_0^h W\Big[2U_t + U_{\xi\xi\xi\xi t} + 3U_\xi\Big]d\xi = 0 \tag{4.10}$$

$$\int_0^h W\Big[2U_t + U_{\xi\xi\xi\xi t} - 60U^2U_\xi + 120U^4U_\xi\Big]d\xi = 0. \tag{4.11}$$

In which U^2 and U^4 are considerd to be a constant such that z_m and g_m respectively. Applying partial integration to (4.10) and (4.11) equations lead to

$$\int_0^h \Big[2WU_t + W_{\xi\xi}U_{\xi\xi t} + 3WU_\xi\Big]d\xi = [-WU_{\xi\xi\xi t} + W_\xi U_{\xi\xi t}]\ |_0^h \tag{4.12}$$

$$\int_0^h \Big[2WU_t + W_{\xi\xi}U_{\xi\xi t} - 60z_mWU_\xi + 120Wg_mU_\xi\Big]d\xi = [-WU_{\xi\xi\xi t} + W_\xi U_{\xi\xi t}]\ |_0^h\ . \tag{4.13}$$

If it is taken the weight function as cubic B-spline base functions presented by equation (4.3) and replacing approximation (4.4) in integral equations (4.12) and (4.13) with some manipulation, the element contributions are given in the form

$$\sum_{j=m-1}^{m+2}\Big[2\Big(\int_0^h(\varphi_i\varphi_j + \varphi_i^{''}\varphi_j^{''})d\xi\Big)\Big]\dot{\delta}_j + 3\Big(\int_0^h \varphi_i\varphi_j^{'}d\xi\Big)\delta_j = -\Big[\Big((\varphi_i\varphi_j^{'''}) + (\varphi_i^{'}\varphi_j^{''})\Big)\ |_0^h\ \Big]\dot{\delta}_j \tag{4.14}$$

$$\sum_{j=m-1}^{m+2}\Big[2\Big(\int_0^h(\varphi_i\varphi_j + \varphi_i^{''}\varphi_j^{''})d\xi\Big)\Big]\dot{\delta}_j + \Big(-60z_m(\int_0^h \varphi_i\varphi_j^{'}d\xi) + 120g_m(\int_0^h \varphi_i\varphi_j^{'}d\xi)\Big)\delta_j = -\Big[\Big((\varphi_i\varphi_j^{'''}) + (\varphi_i^{'}\varphi_j^{''})\Big)\ |_0^h\ \Big]\dot{\delta}_j \tag{4.15}$$

In matrix form, (4.14) and (4.15) equations can be written as follows:

$$(2A^e + 3B^e - C^e + D^e)\dot{\delta}^e + 3C^e\delta^e = 0 \tag{4.16}$$

$$(2A^e + 3B^e - C^e + D^e)\dot{\delta}^e - (60C_1^e + 120C_2^e)\delta^e = 0 \tag{4.17}$$

respectively, where

$$A^e = \int_0^h \varphi_i\varphi_j d\xi = \frac{h}{140}\begin{bmatrix} 20 & 129 & 60 & 1 \\ 129 & 1188 & 933 & 60 \\ 160 & 933 & 1188 & 129 \\ 1 & 60 & 1129 & 20 \end{bmatrix}$$

$$B^e = \int_0^h \varphi_i^{''} \varphi_j^{''} d\xi = \frac{6}{h^3} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 1 & 0 & -3 & 2 \end{bmatrix}$$

$$C^e = \int_0^h \varphi_i \varphi_j^{'} d\xi = \frac{1}{20} \begin{bmatrix} -10 & -9 & 18 & 1 \\ -71 & -150 & 183 & 38 \\ -38 & -183 & 150 & 71 \\ -1 & -18 & 9 & 10 \end{bmatrix}$$

$$D^e = \varphi_i \varphi_j^{'''} \mid_h^0 = \frac{6}{h^3} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 3 & -9 & 9 & -3 \\ -3 & 9 & -9 & 3 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$E^e = \varphi_i^{'} \varphi_j^{''} \mid_h^0 = \frac{18}{h^3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

with sub-indexes $i, j = m-1, m, m+1, m+2$. Since U^2 and U^4 are considered constants like z_m and g_m, respectively, their lumped values are taken as $z_m = (\dfrac{\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})}{2})^2$ and $g_m = (\dfrac{\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2}}{2})^4$ respectively. Combining together contributions from all elements, we have the following matrix equations

$$\dot{\delta} = (2A + 3B - C + D)^{-1}(3C)\delta \tag{4.18}$$

$$\dot{\delta} = (2A + 3B - C + D)^{-1}(60C_1 + 120C_2)\delta^e \tag{4.19}$$

respectively, where C_1 and C_2 are matrices Z_mC and g_mC, $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$ is a vector and "$\cdot$" shows the derivative according to time. Here $\delta = (\delta_{-1}, \delta_0, ..., \delta_N, \delta_{N+1})^T$ is global element parameters. The A,B,C,D and z_mC and g_mC, are septa-diagonal matrices and their m.th rows are

$$A = \frac{1}{140}(1, 120, 1191, 2416, 1191, 120, 1),$$

$$B = \frac{6}{h^3}(1, 0, -9, 16, -9, 0, 1),$$

$$C = \frac{1}{20}(-1, -56, -245, 0, 245, 56, 1),$$

$$D = \frac{6}{h^3}(-1, 0, 9, -16, 9, 0, -1),$$

$$E = (0, 0, 0, 0, 0, 0, 0),$$

$$Z_mD = \frac{1}{20}(-Z_1, -18Z_1 - 38Z_2, 9Z_1 - 183Z_2 - 71Z_3, 10Z_1 + 150Z_2 - 150Z_3 - 10Z_4, 71Z_2 + 183Z_3 - 9Z_4, 38Z_3 + 18Z_4, Z_4)$$

in which

$$Z_1 = \frac{1}{4}(\delta_{m-2} + 5\delta_{m-1} + 5\delta_m + \delta_{m+1})^2$$

$$Z_2 = \frac{1}{4}(\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^2$$

$$Z_3 = \frac{1}{4}(\delta_m + 5\delta_{m+1} + 5\delta_{m+2} + \delta_{m+3})^2$$

$$Z_4 = \frac{1}{4}(\delta_{m+1} + 5\delta_{m+2} + 5\delta_{m+3} + \delta_{m+4})^2.$$

Similar operations are written for g_m to the fourth power. Equation systems (4.18) and (4.19) comprise $(N+3)$ unknowns and $(N+3)$ equations. Using boundary conditions (1.6) and (1.7) given in (1.4) and the values $U_N(x,t)$ at knot points for $m=0$ and $m=N$, we can get the following equations

$$\delta_{-1}(t) + 4\delta_0(t) + 4\delta_1(t)$$

$$\delta_{N-1}(t) + 4\delta_N(t) + 4\delta_{N+1}(t).$$

If the δ_{-1} and $\delta_{N+1}(t)$ parameters from equations system (4.18) and (4.19) are eliminated using the above equations, we obtain system of matrices $(N+1)$ x $(N+1)$ for systems (4.18) and (4.19) This system is solved by means of the Thomas algorithm. In order to mimimize the nonlinearity , we need to two - five time inner iterations $(\delta^*)^n = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ via fourth-order Runge-Kutta technique. In order to start the Runge-Kutta technique, the initial values of the parameters are needed. These values are obtained from $U(x,0) = f(x)$ initial conditions and approximate solutions $U_N(x_m,0) = \sum_{j=-1}^{N+1} \varphi_j(x_m)\delta_j^0(t)$ at $t=0$. Thus, equation systems consisted from $(N+1)$ equation and $(N+3)$ unknown for equations (4.18) and (4.19) are obtained in the following form

$$U(x_0,0) = \delta_{-1}(t) + 4\delta_0(t) + 4\delta_1(t)$$

$$U(x_1,0) = \delta_0(t) + 4\delta_1(t) + 4\delta_2(t)$$

...

$$U(x_{N-1},0) = \delta_{N-2}(t) + 4\delta_{N-1}(t) + 4\delta_N(t).$$

$$U(x_N,0) = \delta_{N-1}(t) + 4\delta_N(t) + 4\delta_{N+1}(t).$$

To solve this systems, we need to two auxiliary equations. These assistant equations are obtained utilizing the second derivative boundary conditions submited by (1.7) at $t=0$.

$$U_m^{''}(x_0,0) = \frac{6}{h^2}(\delta_{-1} - 2\delta_0 + \delta_1)$$

$$U_m^{''}(x_N,0) = \frac{6}{h^2}(\delta_{N-1} - 2\delta_N + \delta_{N+1}).$$

As a result, the systems (4.18) and (4.19) is $(N+3)$ x $(N+3)$-dimensional and we can be easily calculated the initial vector δ^0 from the following matrix equations

$$\begin{bmatrix} 1 & -2 & 1 & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & & & . & & & & & \\ & & & & . & & & & \\ & & & & & . & & & \\ & & & & & & 1 & 4 & 1 \\ & & & & & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \delta^0{}_{-1} \\ \delta^0{}_0 \\ . \\ . \\ . \\ \delta^0{}_N \\ \delta^0{}_{N+1} \end{bmatrix} = \begin{bmatrix} U_0'' \\ U_0 \\ . \\ . \\ U_N \\ U_N'' \end{bmatrix}.$$

5. Numerical examples and results

In this section, we have calculated with one example existing in the literature the difference between numerical solution with exact solution to demonstrate the accuracy and performance of the presented method. For this purpose, we have utilized error norms L_2 and L_∞ presented in the following form with the Matlab 2019b computer program which has a memory 20GB and 64 bit

$$L_2 = ||U - U_N||_2 = \sqrt{h\sum_{j=0}^{N}(U - U_N)^2}$$

$$L_\infty = ||U - U_N||_\infty = \max_j |U - U_N|.$$

Table 1. The error norm values for different values of h at $t = 0.2$ of Scheme I.

	$S\Delta t$		Ext 4		Ext 6	
N	L_2	L_∞	L_2	L_∞	L_2	L_∞
80	$1.57E-3$	$1.48E-3$	$1.51E-3$	$1.45E-3$	$1.48E-3$	$1.36E-3$
100	$1.06E-3$	$9.91E-4$	$0.98E-3$	$9.33E-4$	$0.95E-3$	$8.34E-4$
120	$8.02E-4$	$6.91E-4$	$7.09E-4$	$6.43E-4$	$6.92E-4$	$5.44E-4$
140	$6.72E-4$	$5.06E-4$	$5.69E-4$	$4.53E-4$	$5.62E-4$	$4.17E-4$

Table 2. The error norm values for different values of h at $t = 0.2$ of Scheme II.

	$S\Delta t$		Ext 4		Ext 6	
N	L_2	L_∞	L_2	L_∞	L_2	L_∞
80	$2.69E-3$	$2.23E-3$	$1.40E-3$	$1.11E-3$	$0.81E-3$	$0.74E-3$
100	$2.65E-3$	$22.07E-4$	$0.98E-3$	$7.35E-4$	$0.54E-3$	$5.30E-4$
120	$26.93eE-4$	$22.09E-4$	$7.88E-4$	$6.22E-4$	$5.23E-4$	$4.22E-4$
140	$27.37eE-4$	$22.16E-4$	$6.92E-4$	$5.60E-4$	$5.61E-4$	$3.96E-4$

Example 5.1. In this study, the solution region of the problem given with (1.4)-(1.7) is taken as $[-10, 10]$. Initial condition and the exact solution of the problem are submitted as follows:

$$U(x, 0) = sech(x), \quad x \in [-10, 10],$$

and

$$U(x, t) = sech(x - t), \quad x \in [-10, 10].$$

For different space values $h = 1/4, 1/5, 1/6, 1/7$ at $t = 0.2$, firstly, Tables 1 and 2 present comparison of the error norm values L_2 and L_∞ produced for both scheme I and scheme II with the help of operator splitting techniques (SΔt, Ext4,Ext6) and fourth order Runge-Kutta technque (RK-4) using quintic B-spline collocation and cubic B-spline lumped Galerkin methods respectively, for generalized Rosenau equation. As can be seen from these tables, the techniques Ext4 and Ext6 have lower error norm results produced for decreasing h values. In addition, the Strang splitting technique with the quintic B-spline collocation method applied in scheme I produces better results than the Strang splitting technique with the cubic B-spline lumped Galerkin method applied in scheme II. However, the results obtained with Ext4 and Ext6 techniques in scheme II are better than those in scheme I. After, Table 3-4 presents comparison with those in study [16] of the error norm values obtained for schemes I-II. It is clear from this table that the error norms L_2 and L_∞ acquired with the techniques Ext4 and Ext6 for both scheme I and scheme II are better from those of the values given in [16]. In the Table 5, for descending values of Δt and different space step length at time $t = 0.2$, we have calculated the error norms L_2 and L_∞ to show effectiveness of SΔt technique in Scheme II. We have seen that the error norms are significantly reduced, when the time step Δt become smaller and also we have observed that scheme I produces the same results as scheme II for the same parameter values.

Table 3. A comparison of the error norms with those given in [16] for different values of h at $t = 0.2$ of Scheme I.

	$S\Delta t$		Ext 4		Ext 6		[16]	
N	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
80	$1.57E-3$	$1.48E-3$	$1.51E-3$	$1.45E-3$	$1.48E-3$	$1.36E-3$	$1.53eE-3$	$1.49eE-3$
100	$1.06E-3$	$9.91E-4$	$0.98E-3$	$9.33E-4$	$0.95E-3$	$8.34E-4$	$1.01E-3$	$9.84eE-4$
120	$8.02E-4$	$6.91E-4$	$7.09E-4$	$6.43E-4$	$6.92E-4$	$5.44E-4$	$7.38eE-4$	$6.91eE-4$
140	$6.72E-4$	$5.06E-4$	$5.69E-4$	$4.53E-4$	$5.62E-4$	$4.17E-4$	$6.04E-4$	$5.00eE-4$

Table 4. A comparison of the error norms with those given in [16] for different values of h at $t = 0.2$ of Scheme II.

	SΔt		Ext 4		Ext 6		[16]	
N	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
80	$2.69E-3$	$2.23E-3$	$1.40E-3$	$1.11E-3$	$0.81E-3$	$0.74E-3$	$1.53E-3$	$1.49eE-3$
100	$2.65E-3$	$22.07E-4$	$0.98E-3$	$7.35E-4$	$0.54E-3$	$5.30E-4$	$1.01E-3$	$9.84E-4$
120	$26.93eE-4$	$22.09E-4$	$7.88E-4$	$6.22E-4$	$5.23E-4$	$4.22E-4$	$7.38E-4$	$6.91E-4$
140	$27.37eE-4$	$22.16E-4$	$6.92E-4$	$5.60E-4$	$5.61E-4$	$3.96E-4$	$6.04E-4$	$5.00E-4$

Table 5. The computed of the error norms for different values of h and Δt at $t = 0.2$ of Scheme II.

		L_2	L_∞			L_2	L_∞
h=1/4	$\Delta t = 0.02$	$1.66E-3$	$1.35E-3$	$h = 1/5$	$\Delta t = 0.02$	$1.33E-3$	$11.46E-4$
	$\Delta t = 0.01$	$1.40E-3$	$1.08E-3$		$\Delta t = 0.01$	$0.98E-3$	$8.13E-4$
	$\Delta t = 0.005$	$1.30E-3$	$1.03E-3$		$\Delta t = 0.005$	$0.83E-3$	$6.43E-4$
	$\Delta t = 0.0025$	$1.26E-3$	$1.04E-3$		$\Delta t = 0.0025$	$0.78E-3$	$6.43E-4$
h=1/6	$\Delta t = 0.02$	$12.17E-4$	$10.49E-4$	$h = 1/7$	$\Delta t = 0.02$	$11.70E-4$	$9.96E-4$
	$\Delta t = 0.01$	$7.94E-4$	$6.83E-4$		$\Delta t = 0.01$	$7.03E-4$	$6.17E-4$
	$\Delta t = 0.005$	$6.15E-4$	$5.08E-4$		$\Delta t = 0.005$	$4.97E-4$	$4.29E-4.$
	$\Delta t = 0.0025$	$5.42E-4$	$4.20E-4$		$\Delta t = 0.0025$	$4.12E-4$	$3.36E-4$

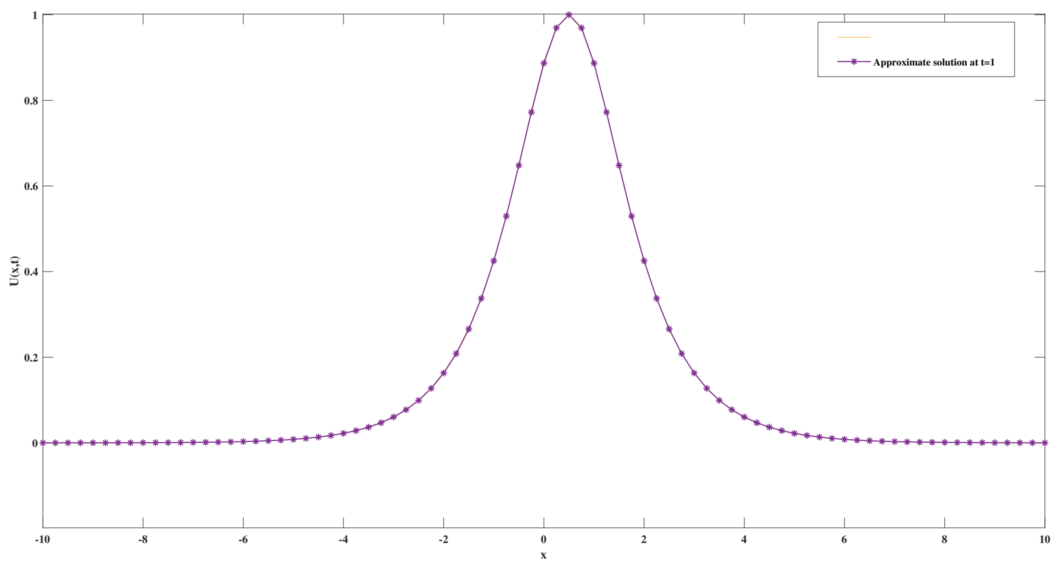

Figure 1. The overlapping of the approximate and the exact solution at $t = 1$ for Scheme I with $h = 0.25, \Delta t = 0.1$.

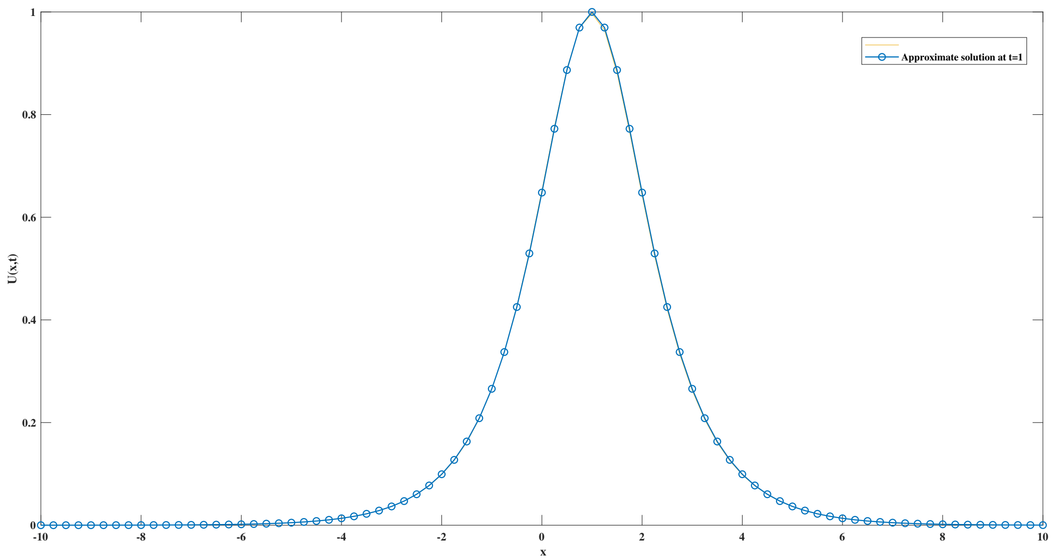

Figure 2. The overlapping of the approximate and the exact solution at $t = 1$ for Scheme II with $h = 0.25, \Delta t = 0.1$.

6. Conclusion

In this study, the numerical solution of generalized Rosenau equation with the initial and boundary conditions are computed by applying the fourth order Runge -Kutta method to systems obtained using collocation and lumped Galerkin methods (FEMs) with quintic and cubic B-splines with help operator time splitting techniques (Strang($S\Delta t$), Ext4 and ext6). It is selected a test problem available in the literature to measure the effectiveness of the method. Results obtained by the application of the method have been compared among themselves and with the present study in the literature. As result of comparisons, it is understood that the Ex6 technique are better than the Ext4 and the Ext4 technique than Strang splitting technique. Here, it is clear that the best among operator splitting techniques is the Ext6 technique. As a conclusion, one can observe that performence of the present method applied for generalized Rosenau equation is very well. Furthermore, operator time splitting techniques can be easily applied to partial differential equations used in different types of science.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] Rosenau, P.: *Dynamics of dense discrete systems.* Progress of Theoretical Physics. **79** (5), 1028-1042 (1988).

[2] Chung, S. K.: *Finite difference approximate solutions for the Rosenau equation.* Applicable Analysis. **69** (1-2), 149-156 (1998).

[3] Rosenau, P.: *A quasi-continuous description of a non -linear transmissionline.*Physica Scripta. **34**, 827-829 (1986).

[4] Park, M. A.: *On the Rosenau equation.* Math. Appl. Comput. **9**, 145-152 (1990).

[5] Chung,S. K., Ha, S. N.: *Finite element Galerkin solutions for the Rosenau equation.* Applicable Analysis.**54**, 30-56 (1994).

[6] Manickam, S. A., Pani, A. K., Chung, S. K.: *A second-order splitting combined with orthogonal cubik spline collocation method for the Rosenau equation.* Numerical Methods for Partial Differential Equations: An International Journal. **14**, 695-716 (1998).

[7] Lee, H. Y., Ohm, M. R., Shin, J. Y.: *The convergence of fully discrete Galerkin approximations of the Rosenau equation.* The Korean Journal of Computational & Applied Mathematics. **6** (1), 1-13 (1999).

[8] Sportisse, B.: *An analysis of operator splitting techniques in the Stiff case.* Journal of Computational Physics. **161**, 140-168 (2000). https://doi:10.1006/jcph.2000.6495.

[9] Chung, S. K., Pani, A. K.: *Numerical methods for the Rosenau equation.* Applicable Analysis. **77**, 351-369 (2001).

[10] Chung, S. K., Pani, A. K.: *A second order splitting lumped mass finite element method for the Rosenau equation.* Differ. Equ. Dyn. Syst. **15**, 331-351 (2004).

[11] Barreto, R. K., De Caldas, C. S . Q., Gamboa, P., Limaco, J.: *Existence of solutions to the Rosenau and Benjamin-Bano-mahony equation in domains with moving boundary.* Electronic Journal of Differential Equations. **2004**, 35 (2004).

[12] Choo, S. M., Chung, S. K., Kim, K. I.: *A discontinuous Galerkin method for the Rosenau equation.* Applied Numerical Mathematics. **58**, 783-799 (2008) .

[13] Omrani, K., Abidi, F., Achouri, T., Khiari, N.: *A new conservative finite difference scheme for the Rosenau equation.* Applied Mathematics and Computation. **201**, 35-43 (2008).

[14] Hu, J., Zheng, K.: *Two conservative difference schemes for the generalized Rosenau equation.* Boundary Value Problems. **2010**, 543503 (2010). https://doi:10.1155/2010/543503.

[15] Wang, M., Li, D., Cui, P.: *A Conservative finite difference scheme for the generalized Rosenau equation.* International Journal of Pure and Applied Mathematics. **71**(4), 539-549 (2011).

[16] Mittal,R. C., Jain,R.: *Application of quintic B-splines collocation method on some Rosenau type nonlinear higher order evolution equations.* International Journal of Nonlinear Science. **13**(2), 142-152 (2012).

[17] Atouani, N., Omrani, K.: *A new conservative high-order accurate difference scheme for the Rosenau equation.* Applicable Analysis. **94**(12), 2435-2455 (2015). http://dx.doi.org/10.1080/00036811.2014.987134.

[18] Abazari, R., Abazari, R.: *Numerical solution of the Rosenau equation using quintic collocation B-spline method.* Iranian Journal of Science and Technology. **39**(3), 281-288 (2015).

[19] Cai, W., Sun, Y., Wang, Y.: *Variational discretizations for the generalized Rosenau-type equations.* Applied Mathematics and Computation. **271**, 860-873 (2015).

[20] Ramos J. I., García-López, C. M.:*Solitary Wave Formation from a Generalized Rosenau Equation.* Hindawi Publishing Corporation Mathematical Problems in Engineering. **2016**, 4618364 (2016). http://dx.doi.org/10.1155/2016/4618364.

[21] Safdari-Vaighani, A., Larsson, E., Heryudono, A.:*Radial basis function methods for the Rosenau equation and other higher order PDEs.* Journal of Scientific Computing. **75**, 1555-1580 (2018). https://doi.org/10.1007/s10915-017-0598-1.

[22] Kutluay, S., Karta, M., Yağmurlu, N. M.:*Operator time-splitting techniques combined with quintic B-spline collocation method for the generalized Rosenau–KdV equation.* Numerical Methods for Partial Differential Equations. **35**(6), 2221-2235 (2019).

[23] Hundsdorfer, W.: Numerical solution of advection-diffusion-reaction equations. Lecture notes for PH.D. course, Thomas Stieltjes Institute, Amsterdam (2000).

[24] McLachlan, R. I.:*On the numerical integration of ordinary differential equations by symmetric composition methods.* SIAM Journal on Scientific Computing. **16**(1), 151-168 (1995).

[25] Smith, G. D.:Numerical solution of partial differential equations: Finite difference methods. Clarendon Press. Oxford (1985).

[26] Özer, S., Yağmurlu, N. M.:*Numerical solutions of nonhomogeneous Rosenau type equations by quintic B-spline collocation method.* Math. Meth. Appl. Sci. **45**(9), 5545-5558 (2022). https://doi.org/10.1002/mma.8125.

[27] Yağmurlu, N. M., Karakaş, A. S.:*Numerical solutions of the equal width equation by trigonometric cubic B-spline collocation method based on Rubin–Graves type linearization.* Numerical Methods for Partial Differential Equations. **36**(5), 1170-1183 (2020). https://doi.org/10.1002/num.22470.

[28] Yağmurlu, N. M., Karaagac, B., Kutluay, S.: *Numerical solutions of Rosenau-RLW equation using Galerkin cubic B-spline finite element method.* American Journal of Computational and Applied Mathematics. **7**(1), 1-10 (2017).

[29] Yağmurlu, N. M., Uçar, Y., Çelikkaya, İ.: *Operator splitting for numerical solutions of the RLW equation.* J. Appl. Anal. Comput. **8**(5), 1494-1510 (2018).

[30] Başhan, A.: *Solitary wave, undular-bore and wave-maker solutions of the cubic, quartic and quintic nonlinear generalized equal width (GEW) wave equation.* The European Physical Journal Plus. **138**, 53 (2023). https://doi.org/10.1140/epjp/s13360-023-03648-4.

[31] Başhan, A., Esen, A.: *Single soliton and double soliton solutions of thequadratic-nonlinear Korteweg-de Vries equation for small and long-times.* Numer Methods Partial Differential Equation. **37**, 1561–1582 (2021).

[32] Başhan, A., Uçar, Y., Yağmurlu, N. M.: *Numerical solution of the complex modified Korteweg-de Vries equation by DQM.* Journal of Physics: Conference Series. **766**, 012028 (2016).

[33] Başhan, A., Uçar, Y., Yağmurlu, N. M., Esen, A.: *Numerical approximation to the MEW equation for the single solitary wave and different types of interactions of the solitary waves.* Journal of Difference Equations and applications. **28**(9), 1193–1213 (2022). https://doi.org/10.1080/10236198.2022.2132154.

[34] Yağmurlu, N. M., Uçar, Y., Başhan, A.: *Numerical approximation of the combined KdV-mKdV equation via the quintic B-spline differential quadrature method.* Adıyaman University Journal of Science. **9**(2), 386-403 (2019). https://dergipark.org.tr/en/pub/adyujsci https://doi.org/10.37094/adyujsci.526264.

[35] Başhan, A.: *Nonlinear dynamics of the Burgers' equation and numerical experiments.* Mathematical Sciences **16**, 183–205 (2022). https://doi.org/10.1007/s40096-021-00410-8.

[36] Atouani, N., Ouali, Y., Omrani, K.:*Mixed finite element methods for the Rosenau equation.* Journal of Applied Mathematics and Computing. **57**, 393-420 (2018). https://doi.org/10.1007/s12190-017-1112-5.

[37] Holden, H., Karlsen, K. H., Lie, K. A., Risebro, N. H.: In: Splitting methods for partial differential equations with Rough solutions. European Mathematical Society (2010).

[38] Geiser, J.: Iterative splitting methods for differential equations. CRC Press. Boca Raton (2011).

[39] Pazy, A.:Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences. **44**, Springer, New York (1983).

[40] Trotter, H. F.: *On the product of semi-groups of operators.* Proceedings of the American Mathematical Society. **10**, 545-551 (1959).

[41] Geiser, J.:*Recent advances in splitting methods for multiphysics and multiscale: Theory and applications.* Journal of Algorithms and Computational Technology. **9**(1), 65-93 (2013).

[42] Marchuk, G. I.:*Some application of splitting-up methods to the solution of mathematical physics problems.* Aplikace Matematiky. **13**, 103-132 (1968).

[43] Varadara jan, S.:Lie groups, Lie algebras, and their representations (First Edition). Springer-Verlag. New York (1985).

[44] Strang, G.: *On the construction and comparison of difference schemes.* SIAM Journal on Numerical Analysis. **5**, 506–517 (1968).

[45] Seydaoglu, M., Erdogan, U., Özis, T.: *Numerical solution of Burgers' equation with high order splitting methods.* Journal of Computational and Applied Mathematics. **291**, 410–421 (2016). https://doi.org/10.1016/j.cam.2015.04.02.

[46] Prenter, P. M.: Splines and variational methods. Wiley. New York (1975).

Affiliations

MELIKE KARTA
ADDRESS: Department of Mathematics, Faculty of Science and Arts, Ağrı İbrahim Çeçen University,Ağrı,Turkey.
E-MAIL: mkarta@agri.edu.tr
ORCID ID:0000-0003-3412-4370

Mathematical Sciences and Applications E-Notes
https://doi.org/10.36753/mathenot.1110497
11 (1) 29-42 (2023) - Research Article
ISSN: 2147-6268

Computing Eigenvalues of Sturm–Liouville Operators with a Family of Trigonometric Polynomial Potentials

Cemile Nur*

Abstract
We provide estimates for the periodic and antiperiodic eigenvalues of non-self-adjoint Sturm–Liouville operators with a family of complex-valued trigonometric polynomial potentials. We even approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Moreover, we give a numerical example with error analysis.

Keywords: *Eigenvalue estimations; periodic and antiperiodic boundary conditions; trigonometric polynomial potentials*

AMS Subject Classification (2020): *34L05; 34L15; 65L15.*

***Corresponding author**

1. Introduction

In the present paper, we consider the operators $T_s(v)$, for $s = 0, 1$, generated in $L_2[0, \pi]$ by the differential expression

$$-y''(x) + v(x)y(x) \tag{1.1}$$

and the boundary conditions

$$y(\pi) = e^{i\pi s}y(0), \qquad y'(\pi) = e^{i\pi s}y'(0), \tag{1.2}$$

which are periodic and antiperiodic boundary conditions, where v is the complex-valued trigonometric polynomial potential of the form

$$v(x) = v_{-1}e^{-i2x} + v_2e^{i4x}, \qquad v_{-1}, v_2 \in \mathbb{C}. \tag{1.3}$$

Note that, the trigonometric polynomial potential (1.3) is a PT-symmetric potential if $v_{-1}, v_2 \in \mathbb{R}$. For the properties of the general PT-symmetric potentials, see [1–6] and references therein. Here, we only note that, the investigations of PT-symmetric periodic potentials were begun by Bender et al. [7].

Received : *28-04-2022,* ***Accepted :*** *02-08-2022*

*(**Cite as** "C. Nur, Computing Eigenvalues of Sturm–Liouville Operators with a Family of Trigonometric Polynomial Potentials, Math. Sci. Appl. E-Notes, 11(1) (2023), 29-42")*

It is well known that the spectra of the operators $T_0(v)$ and $T_1(v)$ are discrete and for large enough n, there are two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues (counting with multiplicities) in the neighborhood of n^2. See the basic and detailed classical results in [8–11] and references therein.

The eigenvalues of the operators $T_0(0)$ and $T_1(0)$ are $(2n)^2$ and $(2n+1)^2$, for $n \in \mathbb{Z}$, respectively and all eigenvalues of $T_0(0)$ and $T_1(0)$, except 0, are double. The eigenvalues of $T_0(v)$ and $T_1(v)$ are called the periodic and antiperiodic eigenvalues and they are denoted by $\mu_n(v)$, for $n \in \mathbb{Z}$ and $\lambda_n(v)$, for $n \in \mathbb{Z} - \{0\}$, respectively.

It is well known that (see [10–12]), if v is real-valued, then all eigenvalues of the operator $T_s(v)$ are real, for all $s \in (-1, 1]$, and the spectrum $\sigma(T(v))$ of the Hill operator $T(v)$, generated in $L_2(-\infty, \infty)$ by expression (1.1) with the real-valued potential (1.3), consists of the real intervals

$$\Gamma_1 := [\mu_0(v), \lambda_{-1}(v)], \quad \Gamma_2 := [\lambda_{+1}(v), \mu_{-1}(v)], \quad \Gamma_3 := [\mu_{+1}(v), \lambda_{-2}(v)], \quad \Gamma_4 := [\lambda_{+2}(v), \mu_{-2}(v)], \ldots,$$

where $\mu_0(v)$, $\mu_{-n}(v)$, $\mu_{+n}(v)$, for $n = 1, 2, \ldots$ are the eigenvalues of $T_0(v)$ and $\lambda_{-n}(v)$, $\lambda_{+n}(v)$, for $n = 1, 2 \ldots$ are the eigenvalues of $T_1(v)$ and the following inequalities hold:

$$\mu_0(v) < \lambda_{-1}(v) \leq \lambda_{+1}(v) < \mu_{-1}(v) \leq \mu_{+1}(v) < \lambda_{-2}(v) \leq \lambda_{+2}(v) < \mu_{-2}(v) \leq \mu_{+2}(v) < \cdots.$$

The bands $\Gamma_1, \Gamma_2, \ldots$ of the spectrum $\sigma(T(v))$ of $T(v)$ are separated by the gaps

$$\Delta_1 := (\lambda_{-1}(v), \lambda_{+1}(v)), \quad \Delta_2 := (\mu_{-1}(v), \mu_{+1}(v)), \quad \Delta_3 := (\lambda_{-2}(v), \lambda_{+2}(v)), \ldots$$

if and only if the eigenvalues at the endpoints of the intervals are simple. In other notation, $\Gamma_n = \{\gamma_n(s) : s \in [0, 1]\}$, where $\gamma_1(s), \gamma_2(s), \ldots$ are the eigenvalues of $T_s(v)$, called as Bloch eigenvalues corresponding to the quasimomentum s. The Bloch eigenvalue $\gamma_n(s)$, continuously depends on s and $\gamma_n(-s) = \gamma_n(s)$.

Obviously, $\mu_{-n}(v)$ and $\mu_{+n}(v)$, for $n = 1, 2, \ldots$ are the $(2n)$th and $(2n+1)$th periodic eigenvalues; $\lambda_{-n}(v)$ and $\lambda_{+n}(v)$, for $n = 1, 2, \ldots$ are the $(2n-1)$th and $(2n)$th antiperiodic eigenvalues, respectively.

If one of the numbers v_{-1} and v_2 is zero and the other is real in (1.3), then all eigenvalues of the operator $T_0(v)$, except 0, are double and they are equal to $(2n)^2$. This fact was proved for the first time in [13]. This case was investigated also in [14–16].

In this paper, we provide estimates for the eigenvalues of $T_0(v)$, when $v_{-1}, v_2 \in \mathbb{C}$. We even approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Finally, we give a numerical example with error analysis using Rouche's theorem.

It is well known that [17]

$$|\mu_{\pm n}(v) - \mu_{\pm n}(0)| \leq \sup_{x \in [0,\pi]} |v(x)| = M,$$

$$|\lambda_{\pm n}(v) - \lambda_{\pm n}(0)| \leq \sup_{x \in [0,\pi]} |v(x)| = M,$$

for $n = 1, , 2 \ldots$, where

$$\mu_{\pm n}(0) = (2n)^2, \qquad \lambda_{\pm n}(0) = (2n-1)^2$$

and $M \leq |v_{-1}| + |v_2| \leq 2c$, $c = \max\{|v_{-1}|, |v_2|\}$. Moreover, for $n = 0$, $|\mu_0(v)| \leq M$ holds. Therefore, we have

$$(2n)^2 - M \leq |\mu_n| \leq (2n)^2 + M \tag{1.4}$$

and

$$|\mu_n - (2k)^2| \geq |(2n)^2 - (2k)^2| - M = 4|n-k||n+k| - M \geq 4|2n-1| - M,$$

for $n \in \mathbb{Z}$ and $k \neq \pm n$. In particular, if $n = 1$, we have $|\mu_{\pm 1}| \leq 4 + M$ and

$$|\mu_{\pm 1} - (2k)^2| \geq ||\mu_{\pm 1}| - (2k)^2| \geq 16 - |\mu_{\pm 1}| \geq 12 - M, \tag{1.5}$$

for $k \geq 2$. Besides, if $|n| \geq 2$, we have $|\mu_n| \geq |\mu_{-2}| \geq 16 - M$ and

$$|\mu_n - (2k)^2| \geq ||\mu_{-2}| - (2k)^2| \geq |\mu_{-2}| - 4 \geq 12 - M, \tag{1.6}$$

for $k \neq \pm n$. The analogous inequalities can be written for the antiperiodic eigenvalues from

$$(2n-1)^2 - M \leq |\lambda_{\pm n}| \leq (2n-1)^2 + M,$$

for $n = 1, 2, \ldots$.

2. Main results

We shall focus only on the operator $T_0(v)$ which is associated with the periodic boundary conditions. The investigation of $T_1(v)$ is similar. From now on, when we use the notation μ_n, we mean the $(2n)$th and $(2n+1)$th periodic eigenvalues $\mu_{-n}(v)$ and $\mu_{+n}(v)$, for $n = 1, 2, \ldots$. In order to obtain iteration formulas, we use the equations

$$(\mu_N - (2n)^2)(\Psi_N, e^{i2nx}) = (v\Psi_N, e^{i2nx}), \tag{2.1}$$
$$(\mu_N - (2n)^2)(\Psi_N, e^{-i2nx}) = (v\Psi_N, e^{-i2nx}), \tag{2.2}$$

which are obtained from

$$-\Psi_N''(x) + v(x)\Psi_N(x) = \mu_N \Psi_N(x),$$

by multiplying both sides of the equality by e^{i2nx} and e^{-i2nx}, respectively, where $\Psi_N(x)$ is the eigenfunction corresponding to the eigenvalue μ_N. Iterating equation (2.1) k times, the way it was done in the paper [18], we obtain

$$\big(\mu_n - (2n)^2 - \sum_{j=1}^{k} a_j(\mu_n)\big)(\Psi_n, e^{i2nx}) - \big(v_{2n} + \sum_{j=1}^{k} b_j(\mu_n)\big)(\Psi_n, e^{-i2nx}) = r_k(\mu_n), \tag{2.3}$$

where

$$a_j(\mu_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{v_{n_1} v_{n_2} \cdots v_{n_j} v_{-n_1-n_2-\cdots-n_j}}{[\mu_n - (2(n-n_1))^2] \cdots [\mu_n - (2(n-n_1-n_2-\cdots-n_j))^2]},$$
$$b_j(\mu_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{v_{n_1} v_{n_2} \cdots v_{n_j} v_{2n-n_1-n_2-\cdots-n_j}}{[\mu_n - (2(n-n_1))^2] \cdots [\mu_n - (2(n-n_1-n_2-\cdots-n_j))^2]},$$
$$r_k(\mu_n) = \sum_{n_1, n_2, \ldots, n_{k+1}} \frac{v_{n_1} v_{n_2} \cdots v_{n_k} v_{n_{k+1}} (v\Psi_n, e^{i2(n-n_1-\cdots-n_{k+1})x})}{[\mu_n - (2(n-n_1))^2] \cdots [\mu_n - (2(n-n_1-\cdots-n_{k+1}))^2]}.$$

Here, the sums are taken under the conditions $n_l = -1, 2$, $\sum_{i=1}^{l} n_i \neq 0, 2n$ for $l = 1, 2, ..., k+1$. Note that, for the trigonometric polynomial potential of the form (1.3), we have $v_i = 0$ for $i \neq -1, 2$.

Similarly, iterating equation (2.2) k times, we obtain

$$\big(\mu_n - (2n)^2 - \sum_{j=1}^{k} a_j^*(\mu_n)\big)(\Psi_n, e^{-i2nx}) - \big(v_{-2n} + \sum_{j=1}^{k} b_j^*(\mu_n)\big)(\Psi_n, e^{i2nx}) = r_k^*(\mu_n), \tag{2.4}$$

where

$$a_j^*(\mu_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{v_{n_1} v_{n_2} \cdots v_{n_j} v_{-n_1-n_2-\cdots-n_j}}{[\mu_n - (2(n+n_1))^2] \cdots [\mu_n - (2(n+n_1+\cdots+n_j))^2]},$$
$$b_j^*(\mu_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{v_{n_1} v_{n_2} \cdots v_{n_j} v_{-2n-n_1-n_2-\cdots-n_j}}{[\mu_n - (2(n+n_1))^2] \cdots [\mu_n - (2(n+n_1+\cdots+n_j))^2]},$$
$$r_k^*(\mu_n) = \sum_{n_1, n_2, \ldots, n_{k+1}} \frac{v_{n_1} v_{n_2} \cdots v_{n_k} v_{n_{k+1}} (v\Psi_n, e^{-i2(n+n_1+\cdots+n_{k+1})x})}{[\mu_n - (2(n+n_1))^2] \cdots [\mu_n - (2(n+n_1+\cdots+n_{k+1}))^2]}.$$

Here, the sums are taken under the conditions $n_l = -1, 2$, $\sum_{i=1}^{l} n_i \neq 0, -2n$ for $l = 1, 2, ..., k+1$. Since the potential v is the trigonometric polynomial potential of the form (1.3), we have the followings, after some calculations:

$$a_{3j-1}^*(\mu_n) = a_{3j-1}(\mu_n), \qquad a_{3j-2}^*(\mu_n) = a_{3j-2}(\mu_n) = a_{3j}^*(\mu_n) = a_{3j}(\mu_n) = 0, \tag{2.5}$$

for $j = 1, 2, \ldots$. Now, in order to give the main results, we prove the following lemma. Without loss of generality, we assume that $\Psi_n(x)$ is the normalized eigenfunction corresponding to the eigenvalue μ_n.

Lemma 2.1. *The statements*

(a) $|p_n|^2 + |q_n|^2 > 0$, *where* $p_n = (\Psi_n, e^{i2nx})$ *and* $q_n = (\Psi_n, e^{-i2nx})$,

(b) $\lim_{k\to\infty} r_k(\mu_n) = 0, \qquad \lim_{k\to\infty} r_k^*(\mu_n) = 0$,

hold in the following cases:

(i) *if* $\max\{|v_{-1}|, |v_2|\} = c \le 97/50$, *for* $n = 1$,

(ii) *if* $c < 2t - 1$, *for* $n \ge t$, $t = 2, 3, \ldots$.

Proof. **(a)** Assume the contrary, $p_n = 0$ and $q_n = 0$. Since the system of the root functions $\{e^{2ikx}/\sqrt{\pi} : k \in \mathbb{Z}\}$ of $T_0(0)$ forms an orthonormal basis for $L_2[0, \pi]$, we have the decomposition

$$\pi\Psi_n = p_n e^{i2nx} + q_n e^{-i2nx} + \sum_{k\in\mathbb{Z}, k\neq\pm n} \left(\Psi_n, e^{i2kx}\right) e^{i2kx}$$

for the normalized eigenfunction Ψ_n corresponding to the eigenvalue μ_n of $T_0(v)$. By Parseval's equality, we obtain

$$\sum_{k\in\mathbb{Z}, k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 = \pi.$$

On the other hand, by (1.4)-(1.6), we have

$$|\mu_1 - 16| \ge 12 - M, \qquad |\mu_1 - 36| \ge 32 - M, \qquad |\mu_1 - 64| \ge 60 - M, \tag{2.6}$$

$$16 - M \le |\mu_2| \le 16 + M, \qquad |\mu_2 - 4| \ge 12 - M, \qquad |\mu_2 - 36| \ge 20 - M. \tag{2.7}$$

First, we consider the case $n = 1$, namely, the case (i). Using the relations (2.1) and (2.6), the Bessel inequality, and taking

$$(v\Psi_1, 1) = v_{-1}(\Psi_1, e^{i2x}) + v_2(\Psi_1, e^{-i4x}) = v_2(\Psi_1, e^{-i4x}),$$
$$(v\Psi_1, e^{-i4x}) = v_{-1}(\Psi_1, e^{-i2x}) + v_2(\Psi_1, e^{-i8x}) = v_2(\Psi_1, e^{-i8x})$$

into account, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}, k\neq\pm 1} |(\Psi_1, e^{i2kx})|^2 &= \frac{|(v\Psi_1, 1)|^2}{|\mu_1|^2} + \sum_{k\neq 0,\pm 1} \frac{|(v\Psi_1, e^{i2kx})|^2}{|\mu_1 - (2k)^2|^2} \\ &< \frac{|v_2|^4 |(v\Psi_1, e^{-i8x})|^2}{|\mu_1|^2 |\mu_1 - 16|^2 |\mu_1 - 64|^2} + \sum_{k\neq 0,\pm 1} \frac{|(v\Psi_1, e^{i2kx})|^2}{|\mu_1 - 16|^2} \\ &\le \frac{c^4 \pi (2c)^2}{(4 - 2c)^2 (12 - 2c)^2 (60 - 2c)^2} + \frac{1}{(12 - 2c)^2} \sum_{k\neq 0,\pm 1} |(v\Psi_1, e^{i2kx})|^2 \\ &\le \frac{4\pi (97/50)^6}{(3/25)^2 (203/25)^2 (1403/25)^2} + \frac{\pi (97/50)^2}{(203/25)^2} < \frac{13\pi}{100} < \pi,\end{aligned}$$

which contradicts $\sum_{k\in\mathbb{Z}, k\neq\pm 1} |(\Psi_1, e^{i2kx})|^2 = \pi$.

Now, we consider the case (ii), namely the case $c < 2t - 1$, for $n \ge t$, $t = 2, 3, \ldots$. Using

$$(2n)^2 - 2c \le |\mu_n| \le (2n)^2 + 2c,$$

we obtain

$$\begin{aligned}|\mu_n - (2k)^2| &\ge |\mu_n - (2(n-1))^2| \ge (2n)^2 - 2c - (2(n-1))^2 \\ &= 4(2n - 1) - 2c > 4(2t - 1) - 2(2t - 1) = 4t - 2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k\in\mathbb{Z}, k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 &= \sum_{k\in\mathbb{Z}, k\neq\pm n} \frac{|(v\Psi_n, e^{i2kx})|^2}{|\mu_n - (2k)^2|^2} \\ &< \frac{1}{(4t - 2)^2} \sum_{k\in\mathbb{Z}, k\neq\pm n} |(v\Psi_n, e^{i2kx})|^2 \le \frac{\pi (2c)^2}{(4t - 2)^2} < \pi,\end{aligned}$$

which contradicts $\sum_{k\in\mathbb{Z},k\neq\pm n}|(\Psi_n,e^{i2kx})|^2=\pi$ and completes the proof of (a).

(b) By the definition of $r_k(\mu_n)$ and the conditions imposed on the summations, the number of each of the greatest summands

$$\frac{(v_{-1})^{2k-1}(v_2)^{k+1}(v\Psi_1,e^{-i4x})}{\mu_1(\mu_1-16)^{k+1}(\mu_1-36)^{k-1}(\mu_1-64)^{k-1}}$$

and

$$\frac{(v_{-1})^{2k-1}(v_2)^{k+1}(v\Psi_2,e^{-i2x})}{\mu_2^k(\mu_2-4)^{2k}}$$

of $r_{3k-1}(\mu_1)$ and $r_{3k-1}(\mu_2)$ in absolute value, is not greater than 4^k. Therefore, using (2.6), (2.7) and $M\leq|v_{-1}|+|v_2|\leq 2c$ and considering the greatest summands of $r_{3k-1}(\mu_n)$ in absolute value, we obtain for case (i)

$$\begin{aligned}|r_{3k-1}(\mu_1)|&<\frac{4^k|v_{-1}|^{2k-1}|v_2|^{k+1}M\sqrt{\pi}}{|\mu_1||\mu_1-16|^{k+1}|\mu_1-36|^{k-1}|\mu_1-64|^{k-1}}\leq\frac{4^kc^{2k-1}c^{k+1}2c\sqrt{\pi}}{(4-2c)(12-2c)^{k+1}(32-2c)^{k-1}(60-2c)^{k-1}}\\&\leq\frac{2\sqrt{\pi}4^k(97/50)^{3k+1}}{(3/25)(203/25)^{k+1}(703/25)^{k-1}(1403/25)^{k-1}}<6284\sqrt{\pi}(\frac{1}{438})^k,\qquad k\geq 2\end{aligned}$$

and for case (ii)

$$\begin{aligned}|r_{3k-1}(\mu_n)|\leq|r_{3k-1}(\mu_2)|&<\frac{4^k|v_{-1}|^{2k-1}|v_2|^{k+1}M\sqrt{\pi}}{|\mu_2|^k|\mu_2-4|^{2k}}\leq\frac{4^kc^{3k+1}2\sqrt{\pi}}{(16-2c)^k(12-2c)^{2k}}\\&<\frac{4^k3^{3k+1}2\sqrt{\pi}}{10^k6^{2k}}=6\sqrt{\pi}\frac{4^k27^k}{10^k36^k}=6\sqrt{\pi}(\frac{3}{10})^k.\end{aligned}$$

Thus, in any case $|r_{3k-1}(\mu_n)|<\alpha a^k$, for some constant $\alpha>0$ and $0<a<1$, which implies $\lim_{k\to\infty}r_k(\mu_n)=0$. Similarly, we prove that $\lim_{k\to\infty}r_k^*(\mu_n)=0$. ☐

Now, we consider the statements of Lemma 2.1 for the case $n=0$:

Lemma 2.2. *If $\max\{|v_{-1}|,|v_2|\}=c\leq 36/25$, for $n=0$, then the statements **(a)** $|(\Psi_0,1)|>0$ and **(b)** $\lim_{k\to\infty}r_k(\mu_0)=0$ are valid.*

Proof. **(a)** Assume the contrary $(\Psi_0,1)=0$. Isolating the terms $|(\Psi_0,e^{-i2x})|^2$ and $|(\Psi_0,e^{i2x})|^2$ in Parseval's equality, we can write

$$|(\Psi_0,e^{-i2x})|^2+|(\Psi_0,e^{i2x})|^2+\sum_{k\neq0,\pm1}|(\Psi_0,e^{i2kx})|^2=\pi.$$

First, we estimate $|(v\Psi_0,e^{-i2x})|^2+|(v\Psi_0,e^{i2x})|^2$. Using (2.1), the relations

$$|\mu_0-4|\geq 4-M,\qquad|\mu_0-16|\geq 16-M,\qquad|\mu_0-36|\geq 36-M,\tag{2.8}$$

and

$$\begin{aligned}(v\Psi_0,e^{-i2x})&=v_{-1}(\Psi_0,1)+v_2(\Psi_0,e^{-i6x})=v_2(\Psi_0,e^{-i6x}),\\(v\Psi_0,e^{i2x})&=v_{-1}(\Psi_0,e^{i4x})+v_2(\Psi_0,e^{-i2x}),\\(v\Psi_0,e^{i4x})&=v_{-1}(\Psi_0,e^{i6x})+v_2(\Psi_0,1)=v_{-1}(\Psi_0,e^{i6x}),\end{aligned}$$

we obtain

$$|(v\Psi_0,e^{-i2x})|\leq\frac{|v_2(v\Psi_0,e^{-i6x})|}{|\mu_0-36|}\leq\frac{2c^2\sqrt{\pi}}{(36-2c)}\leq\frac{2(36/25)^2\sqrt{\pi}}{(828/25)}<\frac{13\sqrt{\pi}}{100}$$

and

$$\begin{aligned}|(v\Psi_0,e^{i2x})|&\leq\frac{|(v_{-1})^2(q\Psi_0,e^{i6x})|}{|\mu_0-16||\mu_0-36|}+\frac{|(v_2)^2(q\Psi_0,e^{-i6x})|}{|\mu_0-4||\mu_0-36|}\\&\leq\frac{2c^3\sqrt{\pi}}{(16-2c)(36-2c)}+\frac{2c^3\sqrt{\pi}}{(4-2c)(36-2c)}\\&\leq\frac{2(36/25)^3\sqrt{\pi}}{(328/25)(828/25)}+\frac{2(36/25)^3\sqrt{\pi}}{(28/25)(828/25)}<\frac{3\sqrt{\pi}}{10},\end{aligned}$$

and hence,

$$|(v\Psi_0, e^{-i2x})|^2 + |(v\Psi_0, e^{i2x})|^2 < \pi/9. \tag{2.9}$$

Using (2.1), (2.8), (2.9) and the Bessel inequality, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z},k\neq 0} |(\Psi_0, e^{i2kx})|^2 &= \frac{|(v\Psi_0, e^{-i2x})|^2}{|\mu_0-4|^2} + \frac{|(v\Psi_0, e^{i2x})|^2}{|\mu_0-4|^2} + \sum_{k\neq 0,\pm1} \frac{|(v\Psi_0, e^{i2kx})|^2}{|\mu_0-(2k)^2|^2} \\ &< \frac{\pi}{9(4-2c)^2} + \frac{1}{(16-2c)^2} \sum_{k\neq 0,\pm1} |(v\Psi_0, e^{i2kx})|^2 \\ &\leq \frac{\pi}{9(28/25)^2} + \frac{4(36/25)^2\pi}{(328/25)^2} < \frac{\pi}{11} + \frac{\pi}{20} = \frac{31\pi}{220} < \pi,\end{aligned}$$

which contradicts $\sum_{k\in\mathbb{Z},k\neq 0} |(\Psi_0, e^{i2kx})|^2 = \pi$ and completes the proof of (a).

(b) The number of the greatest summand

$$\frac{2(v_{-1})^{2k-1}(v_2)^{k+1}(v\Psi_0, e^{-i6x})}{(\mu_0-4)^{k+1}(\mu_0-16)^{k-1}(\mu_0-36)^k}$$

of $r_{3k-1}(\mu_0)$ in absolute value, is not greater than 4^k. Hence, using (2.8) and $M \leq |v_{-1}| + |v_2| \leq 2c$, we obtain

$$\begin{aligned}|r_{3k-1}(\mu_0)| &< \frac{4^k 2|v_{-1}|^{2k-1}|v_2|^{k+1} M\sqrt{\pi}}{|\mu_0-4|^{k+1}|\mu_0-16|^{k-1}|\mu_0-36|^k} \leq \frac{4^{k+1}c^{3k+1}\sqrt{\pi}}{(4-2c)^{k+1}(16-2c)^{k-1}(36-2c)^k} \\ &\leq \frac{\sqrt{\pi}4^{k+1}(36/25)^{3k+1}}{(28/25)^{k+1}(328/25)^{k-1}(828/25)^k} < 68\sqrt{\pi}(\frac{1}{40})^k, \qquad k\geq 2,\end{aligned}$$

which implies $\lim_{k\to\infty} r_k(\mu_0) = 0$. □

Now, letting k tend to infinity in the equations (2.3) and (2.4), we obtain the following results. First, we consider the case $n \geq 2$.

Theorem 2.1. *If* $\max\{|v_{-1}|, |v_2|\} = c < 2t-1$, *for* $n \geq t$, $t = 2, 3, \ldots$, *then* μ *is an eigenvalue of* $T_0(v)$ *if and only if it is a root of the equation*

$$\big(\mu-(2n)^2 - \sum_{j=1}^{\infty} a_{3j-1}(\mu)\big)^2 - \sum_{j=1}^{\infty} b_j(\mu) \sum_{j=1}^{\infty} b_j^*(\mu) = 0 \tag{2.10}$$

lying inside the circle $C_n := \{\mu \in \mathbb{C} : |\mu-(2n)^2| = 2c\}$ *and each of the series in equation* (2.10) *converges uniformly to an analytic function on the disk* $D_n := \{\mu \in \mathbb{C} : |\mu-(2n)^2| \leq 2c\}$.

Proof. **(a)** By Lemma 2.1, letting k tend to infinity in the equations (2.3) and (2.4), we obtain

$$\big(\mu_n-(2n)^2 - \sum_{j=1}^{\infty} a_{3j-1}(\mu_n)\big)p_n = \big(v_{2n} + \sum_{j=1}^{\infty} b_j(\mu_n)\big)q_n, \tag{2.11}$$

$$\big(\mu_n-(2n)^2 - \sum_{j=1}^{\infty} a^*_{3j-1}(\mu_n)\big)q_n = \big(v_{-2n} \sum_{j=1}^{\infty} b^*_j(\mu_n)\big)p_n, \tag{2.12}$$

where $p_n = (\Psi_n, e^{i2nx})$ and $q_n = (\Psi_n, e^{-i2nx})$. If one of the numbers p_n and q_n is zero, then the proof is obvious. If they are both different from zero, multiplying these equations side by side and then cancelling the term p_nq_n, by (2.5), we have

$$\big(\mu_n-(2n)^2 - \sum_{j=1}^{\infty} a_{3j-1}(\mu_n)\big)^2 - \big(v_{2n} + \sum_{j=1}^{\infty} b_j(\mu_n)\big)\big(v_{-2n} \sum_{j=1}^{\infty} b^*_j(\mu_n)\big) = 0. \tag{2.13}$$

Since $v_{2n} = v_{-2n} = 0$ for $n \geq 2$, the eigenvalue μ of $T_0(v)$ is a root of (2.10).

Now, we prove that the roots of (2.10) lying in the disk D_n are the eigenvalues of T_0. The equation $f(\mu) := (\mu-(2n)^2)^2 = 0$, has two roots in the disk D_n and

$$|f(\mu_n)| = |\mu_n-(2n)^2|^2 = 4c^2,$$

for all $\mu_n \in C_n$. Define the function

$$g(\mu) := \big(\mu-(2n)^2 - \sum_{j=1}^{\infty} a_{3j-1}(\mu)\big)^2 - \sum_{j=1}^{\infty} b_j(\mu)\sum_{j=1}^{\infty} b_j^*(\mu) = 0.$$

Estimating the summands of $|a_{3j-1}(\mu_n)|$, $|b_j(\mu_n)|$ and $|b_j^*(\mu_n)|$ for $n=2$, we obtain

$$|a_{3j-1}(\mu_2)| < \frac{2^j|v_{-1}|^{2j}|v_2|^j}{|\mu_2|^j|\mu_2-4|^{2j-1}}, \quad |b_{3j-2}(\mu_2)| < \frac{2^{j-1}|v_{-1}|^{2j-2}|v_2|^{j+1}}{|\mu_2|^j|\mu_2-4|^{2j-2}}, \quad |b_{3j}^*(\mu_2)| < \frac{|v_{-1}|^{2j+2}|v_2|^{j-1}}{|\mu_2|^j|\mu_2-4|^{2j}}, \tag{2.14}$$

for $j \geq 1$. Using the relations $|\mu_2| \geq 16-2c$ and $|\mu_2-4| \geq 12-2c$, it follows by the geometric series formula that

$$\begin{aligned}\sum_{j=1}^{\infty}|a_{3j-1}(\mu_n)| &\leq \sum_{j=1}^{\infty}|a_{3j-1}(\mu_2)| < \frac{2c^3}{(16-2c)(12-2c)} + \frac{2^2c^6}{(16-2c)^2(12-2c)^3} + \frac{2^3c^9}{(16-2c)^3(12-2c)^5} + \cdots \\ &= \frac{2c^3}{(16-2c)(12-2c)}\Big(1 + \frac{2c^3}{(16-2c)(12-2c)^2} + \frac{2^2c^6}{(16-2c)^2(12-2c)^4} + \cdots\Big) \\ &= \frac{2c^3}{(16-2c)(12-2c)}\frac{1}{1-\frac{2c^3}{(16-2c)(12-2c)^2}} = \frac{2c^3(12-2c)}{(16-2c)(12-2c)^2-2c^3} < \frac{12.3^3}{360-54} = \frac{18}{17},\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{\infty}|b_j(\mu_n)| &\leq \sum_{j=1}^{\infty}|b_{3j-2}(\mu_2)| < \frac{c^2}{(16-2c)} + \frac{2c^5}{(16-2c)^2(12-2c)^2} + \frac{2^2c^8}{(16-2c)^3(12-2c)^4} + \cdots \\ &= \frac{c^2}{(16-2c)}\Big(1 + \frac{2c^3}{(16-2c)(12-2c)^2} + \frac{2^2c^6}{(16-2c)^2(12-2c)^4} + \cdots\Big) \\ &= \frac{c^2}{(16-2c)}\frac{1}{1-\frac{2c^3}{(16-2c)(12-2c)^2}} = \frac{c^2(12-2c)^2}{(16-2c)(12-2c)^2-2c^3} < \frac{18}{17},\end{aligned}$$

and that

$$\begin{aligned}\sum_{j=1}^{\infty}|b_j^*(\mu_n)| &\leq \sum_{j=1}^{\infty}|b_{3j}^*(\mu_2)| < \frac{c^4}{(16-2c)(12-2c)^2} + \frac{c^7}{(16-2c)^2(12-2c)^4} + \frac{c^{10}}{(16-2c)^3(12-2c)^6} + \cdots \\ &= \frac{c^4}{(16-2c)(12-2c)^2}\Big(1 + \frac{c^3}{(16-2c)(12-2c)^2} + \frac{c^6}{(16-2c)^2(12-2c)^4} + \cdots\Big) \\ &= \frac{c^4}{(16-2c)(12-2c)^2}\frac{1}{1-\frac{c^3}{(16-2c)(12-2c)^2}} = \frac{c^4}{(16-2c)(12-2c)^2-c^3} < \frac{3^4}{360-27} = \frac{9}{37}.\end{aligned}$$

Hence,

$$\begin{aligned}|g(\mu_n)-f(\mu_n)| &\leq 2|\mu_n-(2n)^2|\sum_{j=1}^{\infty}|a_{3j-1}(\mu_n)| + \Big(\sum_{j=1}^{\infty}|a_{3j-1}(\mu_n)|\Big)^2 + \sum_{j=1}^{\infty}|b_j(\mu_n)|\sum_{j=1}^{\infty}|b_j^*(\mu_n)| \\ &< \frac{8c^4(12-2c)}{(16-2c)(12-2c)^2-2c^3} + \Big(\frac{2c^3(12-2c)}{(16-2c)(12-2c)^2-2c^3}\Big)^2 \\ &+ \frac{c^2(12-2c)^2}{(16-2c)(12-2c)^2-2c^3}\frac{c^4}{(16-2c)(12-2c)^2-c^3} < 4c^2.\end{aligned}$$

Therefore, $|g(\mu)-f(\mu)| < |f(\mu)|$ holds for all $\mu \in C_n$. By Rouche's theorem, $g(\mu)$ has two roots inside the circle C_n. Hence, T_0 has two eigenvalues (counting with multiplicities) lying inside C_n, which are the roots of (2.10). On the other hand, equation (2.10) has exactly two roots (counting with multiplicities) inside C_n. Thus, $\mu \in C_n$ is an eigenvalue of T_0 if and only if, it is a root of (2.10) and the roots of (2.10) coincide with the eigenvalues μ_{-n} and μ_{+n} of T_0.

Now, in order to estimate $\sum_{j=1}^{\infty}|a'_{3j-1}(\mu_n)|$, $\sum_{j=1}^{\infty}|b'_j(\mu_n)|$ and $\sum_{j=1}^{\infty}|b^{*\prime}_j(\mu_n)|$, first we estimate the summands $|a'_{3j-1}(\mu_2)|$, $|b'_j(\mu_2)|$ and $|b^{*\prime}_j(\mu_2)|$, by differentiating $a_{3j-1}(\mu_2)$, $b_j(\mu_2)$ and $b^*_j(\mu_2)$ with respect to μ_2:

$$|\frac{d(a_{3j-1}(\mu_2))}{d\mu_2}|< \frac{2^{j+1}|v_{-1}|^{2j}|v_2|^j}{|\mu_2|^j|\mu_2-4|^{2j}}, \qquad |\frac{d(b^*_{3j}(\mu_2))}{d\mu_2}|< \frac{3^{j+1}|v_{-1}|^{2j+2}|v_2|^{j-1}}{2^j|\mu_2|^j|\mu_2-4|^{2j+1}},, \qquad j\geq 1$$
$$|\frac{d(b_1(\mu_2))}{d\mu_2}|< \frac{|v_2|^2}{|\mu_2|^2}, \qquad |\frac{d(b_{3j-2}(\mu_2))}{d\mu_2}|< \frac{2^{j+1}|v_{-1}|^{2j-2}|v_2|^{j+1}}{|\mu_2|^j|\mu_2-4|^{2j-1}}, \qquad j\geq 2,$$

and hence, we have

$$\begin{aligned}\sum_{j=1}^{\infty}|a'_{3j-1}(\mu_n)| &\leq \sum_{j=1}^{\infty}|a'_{3j-1}(\mu_2)| < \frac{2^2c^3}{(16-2c)(12-2c)^2}+\frac{2^3c^6}{(16-2c)^2(12-2c)^4}+\frac{2^4c^9}{(16-2c)^3(12-2c)^6}+\cdots\\ &= \frac{2^2c^3}{(16-2c)(12-2c)^2}\left(1+\frac{2c^3}{(16-2c)(12-2c)^2}+\frac{2^2c^6}{(16-2c)^2(12-2c)^4}+\cdots\right)\\ &= \frac{4c^3}{(16-2c)(12-2c)^2}\frac{1}{1-\frac{2c^3}{(16-2c)(12-2c)^2}} = \frac{4c^3}{(16-2c)(12-2c)^2-2c^3} < \frac{4.3^3}{360-54} = \frac{6}{17},\end{aligned}$$

and

$$\sum_{j=1}^{\infty}|b'_j(\mu_n)| \leq \sum_{j=1}^{\infty}|b'_{3j-2}(\mu_2)| < \frac{c^2}{(16-2c)^2}+\frac{8c^5}{(16-2c)(12-2c)[(16-2c)(12-2c)^2-2c^3]} < \frac{7}{37},$$
$$\sum_{j=1}^{\infty}|b^{*\prime}_j(\mu_n)| \leq \sum_{j=1}^{\infty}|b^{*\prime}_{3j}(\mu_2)| < \frac{9c^4}{(12-2c)[2(16-2c)(12-2c)^2-3c^3]} < \frac{27}{142}.$$

Therefore, each of the series $\sum_{j=1}^{\infty}a_{3j-1}(\mu_n)$, $\sum_{j=1}^{\infty}b_j(\mu_n)$ and $\sum_{j=1}^{\infty}b^*_j(\mu_n)$, converges uniformly to an analytic function on the disk D_n. $\square$

Now, to estimate the periodic eigenvalues μ_{-1} and μ_1, we consider the case $n=1$. In this case, substituting $b_{3j-1}(\mu_1)=0$, $b_{3j-2}(\mu_1)=0$, for $j\geq 1$, and $\sum_{j=1}^{\infty}b^*_j(\mu_1)=b^*_1(\mu_1)=(v_{-1})^2/\mu_1$, in (2.3) and (2.4) as $k\to\infty$, by Lemma 2.1, we obtain

$$\left(\mu_1-4-\sum_{j=1}^{\infty}a_{3j-1}(\mu_1)\right)^2-\frac{(v_{-1})^2}{\mu_1}\left(v_2-\sum_{j=1}^{\infty}b_{3j}(\mu_1)\right)=0. \tag{2.15}$$

Therefore, we have the following results.

Theorem 2.2. *If $\max\{|v_{-1}|,|v_2|\}=c\leq 97/50$, for $n=1$, then μ is an eigenvalue of $T_0(v)$ if and only if it is a root of the equation*

$$\left(\mu-4-\sum_{j=1}^{\infty}a_{3j-1}(\mu)\right)^2-\frac{(v_{-1})^2v_2}{\mu}-\frac{(v_{-1})^2}{\mu}\sum_{j=1}^{\infty}b_{3j}(\mu)=0 \tag{2.16}$$

lying inside the circle $C_1:=\{\mu\in\mathbb{C}:|\mu|=4+2c\}$ and each of the series in equation (2.16) converges uniformly to an analytic function on the disk $D_1:=\{\mu\in\mathbb{C}:|\mu|\leq 4+2c\}$.

Proof. **(a)** Equation (2.16) follows from (2.15). Let $F(\mu):=(\mu-4)^2=0$ and

$$G(\mu):=\left(\mu-4-\sum_{j=1}^{\infty}a_{3j-1}(\mu)\right)^2-\frac{(v_{-1})^2v_2}{\mu}-\frac{(v_{-1})^2}{\mu}\sum_{j=1}^{\infty}b_{3j}(\mu)=0.$$

Then, $|F(\mu_1)| = |\mu_1 - 4|^2 \geq (|\mu_1| - 4)^2 = 4c^2$, for all $\mu_1 \in C_1$. Using the estimations

$$\sum_{j=1}^{\infty} |a_{3j-1}(\mu_1)| < \sum_{j=1}^{\infty} \frac{(3/2)^j |v_{-1}|^{2j} |v_2|^j}{|\mu_1||\mu_1 - 16|^j |\mu_1 - 36|^{j-1} |\mu_1 - 64|^{j-1}} \tag{2.17}$$
$$< \frac{3c^3(32-2c)(60-2c)}{(4+2c)[2(12-2c)(32-2c)(60-2c) - 3c^3]} < \frac{9}{50},$$
$$\sum_{j=1}^{\infty} |b_{3j}(\mu_1)| < \sum_{j=1}^{\infty} \frac{2^{j-1} |v_{-1}|^{2j} |v_2|^{j+1}}{|\mu_1||\mu_1 - 16|^{j+1} |\mu_1 - 36|^{j-1} |\mu_1 - 64|^{j-1}} \tag{2.18}$$
$$< \frac{c^4(32-2c)(60-2c)}{(4+2c)(12-2c)[(12-2c)(32-2c)(60-2c) - 2c^3]} < \frac{7}{250},$$

and

$$\sum_{j=1}^{\infty} |a'_{3j-1}(\mu_1)| < \sum_{j=1}^{\infty} \frac{2^{j+1} |v_{-1}|^{2j} |v_2|^j}{|\mu_1|^2|\mu_1 - 16|^j |\mu_1 - 36|^{j-1} |\mu_1 - 64|^{j-1}}$$
$$< \frac{4c^3(32-2c)(60-2c)}{(4+2c)^2[(12-2c)(32-2c)(60-2c) - 2c^3]} < \frac{3}{50},$$
$$\sum_{j=1}^{\infty} |b'_{3j}(\mu_1)| < \sum_{j=1}^{\infty} \frac{2^{j+1} |v_{-1}|^{2j} |v_2|^{j+1}}{|\mu_1|^2|\mu_1 - 16|^{j+1} |\mu_1 - 36|^{j-1} |\mu_1 - 64|^{j-1}}$$
$$< \frac{4c^4(32-2c)(60-2c)}{(4+2c)^2(12-2c)[(12-2c)(32-2c)(60-2c) - 2c^3]} < \frac{7}{500},$$

for all $\mu_1 \in C_1$, and arguing as in the proof of Theorem 2.1, by Rouche's theorem, we complete the proof. □

Finally, in order to estimate the first periodic eigenvalue μ_0, we consider the case $n = 0$. By Lemma 2.2, we have:

Theorem 2.3. *If* $\max\{|v_{-1}|, |v_2|\} = c \leq 36/25$, *for* $n = 0$, *then* μ *is an eigenvalue of* $T_0(v)$ *if and only if it is the root of the equation*

$$\mu - \frac{(v_{-1})^2 v_2}{(\mu - 4)^2} - \frac{2(v_{-1})^2 v_2}{(\mu - 4)(\mu - 16)} - \sum_{j=2}^{\infty} a_{3j-1}(\mu) = 0 \tag{2.19}$$

lying inside the circle $C_0 := \{\mu \in \mathbb{C} : |\mu| = 2c\}$ *and the series in equation* (2.19) *converges uniformly to an analytic function on the disk* $D_0 := \{\mu \in \mathbb{C} : |\mu| \leq 2c\}$.

Proof. **(a)**Iterating $\mu_N(\Psi_N, 1) = (v\Psi_N, 1)$, for $N = 0$, k times, by isolating the terms containing $(\Psi_0, 1)$ gives

$$\left(\mu_0 - \sum_{j=1}^{k} a_j(\mu_0)\right)(\Psi_0, 1) = r_k(\mu_0). \tag{2.20}$$

Letting k tend to infinity in (2.20), by Lemma 2.2 and (2.5), we obtain (2.19). Let

$$H(\mu) := \mu - \frac{(v_{-1})^2 v_2}{(\mu_0 - 4)^2} = 0$$

and

$$K(\mu) := \mu - \sum_{j=1}^{\infty} a_{3j-1}(\mu) = \mu - \frac{(v_{-1})^2 v_2}{(\mu - 4)^2} - \frac{2(v_{-1})^2 v_2}{(\mu - 4)(\mu - 16)} - \sum_{j=2}^{\infty} a_{3j-1}(\mu) = 0.$$

Then,

$$|H(\mu_0)| \geq \left||\mu_0| - \frac{|v_{-1}|^2 |v_2|}{|\mu_0 - 4|^2}\right| \geq 2c - \frac{c^3}{(4-2c)^2},$$

for all $\mu_0 \in C_0$. Using the estimations

$$
\begin{aligned}
|K(\mu_0) - H(\mu_0)| &\leq \frac{2|v_{-1}|^2|v_2|}{|\mu_0-4||\mu_0-16|} + \sum_{j=2}^{\infty} |a_{3j-1}(\mu_0)| \\
&< \frac{2|v_{-1}|^2|v_2|}{|\mu_0-4||\mu_0-16|} + \sum_{j=2}^{\infty} \frac{2^j|v_{-1}|^{2j}|v_2|^j}{|\mu_0-4|^{j+1}|\mu_0-16|^{j-1}|\mu_0-36|^{j-1}} \\
&< \frac{2|v_{-1}|^2|v_2|}{(4-2c)(16-2c)} + \sum_{j=2}^{\infty} \frac{2^j|v_{-1}|^{2j}|v_2|^j}{(4-2c)^{j+1}(16-2c)^{j-1}(36-2c)^{j-1}} \\
&< \frac{2c^3}{(4-2c)(16-2c)} + \frac{4c^6}{(4-2c)^2[(4-2c)(16-2c)(36-2c)-2c^3]} < \frac{47}{100}
\end{aligned}
\tag{2.21}
$$

and

$$
\begin{aligned}
\sum_{j=2}^{\infty} |a'_{3j-1}(\mu_0)| &< \sum_{j=2}^{\infty} \frac{2^{j+1}|v_{-1}|^{2j}|v_2|^j}{|\mu_0-4|^{j+2}|\mu_0-16|^{j-1}|\mu_0-36|^{j-1}} \\
&< \sum_{j=2}^{\infty} \frac{2^{j+1}|v_{-1}|^{2j}|v_2|^j}{(4-2c)^{j+2}(16-2c)^{j-1}(36-2c)^{j-1}} \\
&< \frac{8c^6}{(4-2c)^3[(4-2c)(16-2c)(36-2c)-2c^3]} < \frac{11}{100}
\end{aligned}
$$

and arguing as in the proof of Theorem 2.1, by Rouche's theorem, we complete the proof. □

In order to estimate eigenvalues numerically, we take finite summations instead of the infinite series in the equations (2.10), (2.16) and (2.19). When we say the $(3k)$th approximations, we mean the equations containing $\sum_{j=1}^{k} a_{3j-1}(\mu)$, $\sum_{j=1}^{3k} b_j(\mu)$ and $\sum_{j=1}^{3k} b_j^*(\mu)$ instead of $\sum_{j=1}^{\infty} a_{3j-1}(\mu)$, $\sum_{j=1}^{\infty} b_j(\mu)$ and $\sum_{j=1}^{\infty} b_j^*(\mu)$. For instance, in the cases $n=0$, $n=1$ and $n=2$, the $(3k)$th approximations of (2.19), (2.16) and (2.10) are

$$
\mu - \frac{(v_{-1})^2 v_2}{(\mu-4)^2} - \frac{2(v_{-1})^2 v_2}{(\mu-4)(\mu-16)} - \sum_{j=2}^{k} a_{3j-1}(\mu) = 0, \tag{2.22}
$$

$$
\Big(\mu - 4 - \sum_{j=1}^{k} a_{3j-1}(\mu)\Big)^2 - \frac{(v_{-1})^2 v_2}{\mu} - \frac{(v_{-1})^2}{\mu} \sum_{j=1}^{k} b_{3j}(\mu) = 0, \tag{2.23}
$$

and

$$
\Big(\mu - 16 - \sum_{j=1}^{k} a_{3j-1}(\mu)\Big)^2 - \sum_{j=1}^{3k} b_j(\mu) \sum_{j=1}^{3k} b_j^*(\mu) = 0,
$$

respectively. Then, by (2.14), (2.17), (2.18) and (2.21), we have the following estimations for the remaining terms of the series in these equations:

$$
\begin{aligned}
\Big|\sum_{j=k+1}^{\infty} a_{3j-1}(\mu_0)\Big| \leq \sum_{j=k+1}^{\infty} |a_{3j-1}(\mu_0)| &< \sum_{j=k+1}^{\infty} \frac{2^j|v_{-1}|^{2j}|v_2|^j}{(4-2c)^{j+1}(16-2c)^{j-1}(36-2c)^{j-1}} \\
&< \frac{2^{k+1}c^{3k+3}}{(4-2c)^{k+1}(16-2c)^{k-1}(36-2c)^{k-1}[(4-2c)(16-2c)(36-2c)-2c^3]} < 2.41\Big(\frac{1}{162}\Big)^k,
\end{aligned}
$$

for $n=0$;

$$
\begin{aligned}
|\sum_{j=k+1}^{\infty} a_{3j-1}(\mu_1)| &\leq \sum_{j=k+1}^{\infty} |a_{3j-1}(\mu_1)| < \sum_{j=k+1}^{\infty} \frac{(3/2)^j|v_{-1}|^{2j}|v_2|^j}{(4-2c)(12-2c)^j(32-2c)^{j-1}(60-2c)^{j-1}} \\
&< \frac{2(3/2)^{k+1}c^{3k+3}}{(4-2c)(12-2c)^k(32-2c)^{k-1}(60-2c)^{k-1}[2(12-2c)(32-2c)(60-2c)-3c^3]} < 11.25(\frac{1}{1170})^k, \\
|\sum_{j=k+1}^{\infty} b_{3j}(\mu_1)| &\leq \sum_{j=k+1}^{\infty} |b_{3j}(\mu_1)| < \sum_{j=k+1}^{\infty} \frac{2^{j-1}|v_{-1}|^{2j}|v_2|^{j+1}}{(4-2c)(12-2c)^{j+1}(32-2c)^{j-1}(60-2c)^{j-1}} \\
&< \frac{2^k c^{3k+4}}{(4-2c)(12-2c)^{k+1}(32-2c)^{k-1}(60-2c)^{k-1}[(12-2c)(32-2c)(60-2c)-2c^3]} < 1.8(\frac{1}{877})^k,
\end{aligned}
$$

for $n=1$; and

$$
\begin{aligned}
|\sum_{j=k+1}^{\infty} a_{3j-1}(\mu_2)| &\leq \sum_{j=k+1}^{\infty} |a_{3j-1}(\mu_2)| < \sum_{j=k+1}^{\infty} \frac{2^j|v_{-1}|^{2j}|v_2|^j}{(16-2c)^j(12-2c)^{2j-1}} \\
&< \frac{2^{k+1}c^{3k+3}}{(16-2c)^k(12-2c)^{2k-1}[(16-2c)(12-2c)^2-2c^3]} < \frac{18}{17}(\frac{3}{20})^k, \\
|\sum_{j=k+1}^{\infty} b_{3j-2}(\mu_2)| &\leq \sum_{j=k+1}^{\infty} |b_{3j-2}(\mu_2)| < \sum_{j=k+1}^{\infty} \frac{2^{j-1}|v_{-1}|^{2j-2}|v_2|^{j+1}}{(16-2c)^j(12-2c)^{2j-2}} \\
&< \frac{2^k c^{3k+2}}{(16-2c)^k(12-2c)^{2k-2}[(16-2c)(12-2c)^2-2c^3]} < \frac{18}{17}(\frac{3}{20})^k, \\
|\sum_{j=k+1}^{\infty} b^*_{3j}(\mu_2)| &\leq \sum_{j=k+1}^{\infty} |b^*_{3j}(\mu_2)| < \sum_{j=k+1}^{\infty} \frac{|v_{-1}|^{2j+2}|v_2|^{j-1}}{(16-2c)^j(12-2c)^{2j}} \\
&< \frac{c^{3k+4}}{(16-2c)^k(12-2c)^{2k}[(16-2c)(12-2c)^2-c^3]} < \frac{9}{37}(\frac{3}{40})^k,
\end{aligned}
$$

for $n=2$. Obviously, we have better approximations as k grows.

Now, we approach the periodic eigenvalues by the roots of the polynomials derived from the $(3k)$th approximations (2.22) and (2.23), the way it was done in [19]. For example, for $n=0$ and $n=1$, the sixth approximations are

$$
\begin{aligned}
Q_0(\mu) := \mu &- \frac{(v_{-1})^2 v_2}{(\mu-4)^2} - \frac{2(v_{-1})^2 v_2}{(\mu-4)(\mu-16)} - \frac{2(v_{-1})^4(v_2)^2}{(\mu-4)^3(\mu-16)(\mu-36)} \\
&- \frac{2(v_{-1})^4(v_2)^2}{(\mu-4)^2(\mu-16)^2(\mu-36)} - \frac{2(v_{-1})^4(v_2)^2}{(\mu-4)(\mu-16)^2(\mu-36)(\mu-64)} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
Q_1(\mu) := \Big(\mu - 4 &- \frac{(v_{-1})^2 v_2}{\mu(\mu-16)} - \frac{(v_{-1})^2 v_2}{(\mu-16)(\mu-36)} - \frac{(v_{-1})^4(v_2)^2}{\mu(\mu-16)^2(\mu-36)(\mu-64)} \\
&- \frac{(v_{-1})^4(v_2)^2}{(\mu-16)^2(\mu-36)^2(\mu-64)} - \frac{(v_{-1})^4(v_2)^2}{(\mu-16)(\mu-36)^2(\mu-64)(\mu-100)}\Big)^2 \\
&- \frac{(v_{-1})^2 v_2}{\mu} - \frac{(v_{-1})^4(v_2)^2}{\mu^2(\mu-16)^2} - \frac{2(v_{-1})^6(v_2)^3}{\mu^2(\mu-16)^3(\mu-36)(\mu-64)} = 0,
\end{aligned}
$$

respectively. Then, the corresponding polynomials are

$$P_0(\mu) := (\mu-4)^3(\mu-16)^2(\mu-36)(\mu-64)Q_0(\mu), \tag{2.24}$$

and

$$P_1(\mu) := \mu^2(\mu-16)^4(\mu-36)^4(\mu-64)^2(\mu-100)^2 Q_1(\mu), \tag{2.25}$$

respectively. By the same token, we can derive polynomials to approximate the other periodic eigenvalues, as well. Now, we present a numerical example.

Example 2.1. Consider the potential $v(x) = e^{i4x} + e^{-i2x}$. In this case, $v_{-1} = v_2 = 1$, and we have the following approximations for the first periodic eigenvalues μ_0, μ_{-1} and μ_{-1}:

First, we show that μ_0 is the eigenvalue lying inside the circle

$$c_0 := \{\mu \in \mathbb{C} : |\mu - 0.0978293068037| = 8.8 \times 10^{-8}\}.$$

The root of the polynomial $P_0(\mu)$ defined by (2.24), lying in the disk $D_0 = \{\mu \in \mathbb{C} : |\mu| \leq 2\}$, is

$$z_1 = 0.0978293068037.$$

The other roots of $P_0(\mu)$ are

$$\begin{aligned} &z_2 = 3.43962569257, \quad z_3 = 3.99479646224, \quad z_4 = 4.45733974252, \\ &\quad z_5 = (16.0052043612 - 0.00233054592651i), \quad z_6 = (16.0052043612 + 0.00233054592651i), \\ &\quad z_7 = 36.0000000654 and z_8 = 64.0000000081. \end{aligned}$$

Using the decomposition

$$Q_0(\mu) = \frac{(\mu - z_1)(\mu - z_2) \cdots (\mu - z_8)}{(\mu - 4)^3(\mu - 16)^2(\mu - 36)(\mu - 64)},$$

we obtain by direct calculation $|Q_0(\mu)| > 7.0297 \times 10^{-8}$, for all $\mu \in c_0$. On the other hand, again by direct calculations, we have

$$|K(\mu) - Q_0(\mu)| \leq \sum_{j=3}^{\infty} |a_{3j-1}(\mu)| < 6.8948 \times 10^{-8},$$

for all $\mu \in c_0$. Therefore, by Rouche's theorem, equation (2.19) has only one root inside the circle c_0. Thus, using Theorem 2.3, we conclude that μ_0 is the eigenvalue lying inside the circle c_0.

Now, we show that μ_{-1} and μ_1 are the complex eigenvalues lying inside the circles

$$c_{-1} := \{\mu \in \mathbb{C} : |\mu - (3.9817022865 - 0.00000193582494331i)| = 2.4 \times 10^{-11}\}.$$

and

$$c_1 := \{\mu \in \mathbb{C} : |\mu - (3.9817022865 + 0.00000193582494331i)| = 2.4 \times 10^{-11}\}.$$

respectively. The roots of the polynomial $P_1(\mu)$ defined by (2.25), lying in the disk $D_1 = \{\mu \in \mathbb{C} : |\mu| \leq 6\}$ are $x_1 = (3.9817022865 - 0.00000193582494331i)$, $x_2 = (3.9817022865 + 0.00000193582494331i)$ and $x_{3,4} = (0.0156946762466 \pm 0.000000001038902733399i)$. The other roots of $P_1(\mu)$ are

$$\begin{aligned} &x_{5,6} = (15.6169913038 \pm 0.443784473665i), \quad x_{7,8} = (16.1045835081 \pm 0.60324291928i), \\ &\quad x_9 = 15.4238418891, \quad x_{10,11} = (16.5675433687 \pm 0.291363414949i), \quad x_{12} = 35.6520094578, \\ &\quad x_{13,14} = (35.8640598326 \pm 0.342968435657i), \quad x_{15,16} = (36.3114986519 \pm 0.241842187319i), \\ &\quad x_{17,18} = (63.9746729986 \pm 0.044045369207i), \quad x_{19} = 64.0506554041, \\ &\quad x_{20} = 99.9999172995, \quad x_{21} = 100.000082697. \end{aligned}$$

Using the decomposition

$$Q_1(\mu) = \frac{(\mu - x_1)(\mu - x_2) \cdots (\mu - x_{21})}{\mu^2(\mu - 16)^4(\mu - 36)^4(\mu - 64)^2(\mu - 100)^2},$$

by direct calculations, we obtain $|Q_1(\mu)| > 1.0992 \times 10^{-11}$, for all $\mu \in c_{-1}$ and $|Q_1(\mu)| > 1.0992 \times 10^{-11}$, for all $\mu \in c_1$. On the other hand, one can easily calculate that

$$|G(\mu) - Q_1(\mu)| \leq 2(|\mu - 4| + |a_2(\mu)| + |a_5(\mu)|) \sum_{j=3}^{\infty} |a_{3j-1}(\mu)| + \big(\sum_{j=3}^{\infty} |a_{3j-1}(\mu)|\big)^2 + \sum_{j=3}^{\infty} \frac{|b_{3j}(\mu)|}{|\mu|} < 4.7184 \times 10^{-12},$$

for all $\mu \in c_{-1} \cup c_1$. The proof follows from Rouche's theorem and Theorem 2.2; equation (2.16) has one root inside each of the circles c_{-1} and c_1 and μ_{-1} and μ_{+1} are the complex eigenvalues lying inside c_{-1} and c_1, respectively.

3. Conclusion

In this paper, we have given estimates for the periodic eigenvalues, when $v_{-1}, v_2 \in \mathbb{C}$. We have even approximated complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Finally, we have given a numerical example with error analysis using Rouche's theorem. In this paper, we have given a practical way to calculate the eigenvalues of the operator $T_0(v)$. The method used in this paper can be extended to compute the periodic eigenvalues of the Hill operator for different classes of potentials.

Article Information

Acknowledgements: The author is thankful for the Research Fund of Yalova University (Project no. 2019/AP/0010). The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] Bagarello, F., Gazeau, J.-P., Szafraniec, F., Znojil, M. (Eds.): Non-selfadjoint operators in quantum physics: Mathematical aspects. John Wiley & Sons (2015).

[2] Bender, C. M.: *PT-symmetric potentials having continuous spectra.* Journal of Physics A-Mathematical and Theoretical. **53** (37), 375302 (2020).

[3] Mostafazadeh, A.: *Psevdo-hermitian representation of quantum mechanics.* International Journal of Geometric Methods in Modern Physics. **11**, 1191-1306 (2010).

[4] Veliev, O. A.: *On the spectral properties of the Schrodinger operator with a periodic PT-symmetric potential.* International Journal of Geometric Methods in Modern Physics. **14**, 1750065 (2017).

[5] Veliev, O. A.: *On the finite-zone periodic PT-symmetric potentials.* Moscow Mathematical Journal. **19** (4), 807-816 (2019).

[6] Veliev, O. A.: Non-self-adjoint Schrödinger operator with a periodic potential. Springer, Cham (2021).

[7] Bender, C. M., Dunne, G. V., Meisinger, P. N.: *Complex periodic potentials with real band spectra.* Physics Letters A. **252**, 272-276 (1999).

[8] Brown, B.M., Eastham, M. S. P., Schmidt, K. M.: Periodic differential operators, Operator Theory: Advances and Applications, 230, Birkhuser/Springer: Basel AG, Basel (2013).

[9] Levy, M., Keller, B.: *Instability intervals of Hill's equation.* Communications on Pure and Applied Mathematics. **16**, 469-476 (1963).

[10] Magnus, W., Winkler, S.: Hill's equation. Interscience Publishers, New York (1966).

[11] Marchenko, V.: Sturm-Liouville operators and applications. Birkhauser Verlag, Basel (1986).

[12] Eastham, M. S. P.: The spectral theory of periodic differential operators. Hafner. New York (1974).

[13] Gasymov, M. G.: *Spectral analysis of a class of second-order nonself-adjoint differential operators.* Fankts. Anal. Prilozhen. **14**, 14-19 (1980).

[14] Kerimov, N. B.: *On a boundary value problem of N. I. Ionkin type.* Differential Equations. **49**, 1233-1245 (2013).

[15] Nur, C.: *On the estimates of periodic eigenvalues of Sturm-Liouville operators with trigonometric polynomial potentials.* Mathematical Notes. **109** (5), 794-807 (2021).

[16] Veliev, O. A.: *Spectral problems of a class of non-self-adjoint one-dimensional Schrodinger operators.* Journal of Mathematical Analysis and Applications. **422**, 1390-1401 (2015).

[17] Pöschel, J., Trubowitz, E.: Inverse Spectral Theory. Academic Press, Boston, Mass, USA (1987).

[18] Dernek, N., Veliev, O. A.: *On the Riesz basisness of the root functions of the nonself-adjoint Sturm-Liouville operators.* Israel Journal of Mathematics. **145**, 113-123 (2005).

[19] Veliev, O. A.: *The spectrum of the Hamiltonian with a PT-symmetric periodic optical potential.* International Journal of Geometric Methods in Modern Physics. **15**, 1850008 (2018).

Affiliations

CEMILE NUR
ADDRESS: Department of Computer Engineering, Yalova University, Yalova, Turkey.
E-MAIL: cnur@yalova.edu.tr
ORCID ID:0000-0001-7375-3474

Mathematical Sciences and Applications E-Notes

https://doi.org/ 10.36753/mathenot.1056913
11 (1) 43-55 (2023) - Research Article
ISSN: 2147-6268

The Effect of Corona-Virus Disease (COVID-19) Outbreak Quarantine on the Belief and Behavior of Children in Early Childhood with a Fuzzy Conjoint Method

Murat Kirişci*, Musa Bardak and Nihat Topaç

Abstract
The purpose of this paper is to measure the effect of Corona-virus quarantine on the belief and behavior of children in the early childhood period using the Fuzzy Set Theory. In this study, after the Coronavirus quarantine, the thoughts of the children and their parents' observations and thoughts about the belief and behavior of their children were questioned. This investigation was used to measure the change in children's beliefs and behavior during the Coronavirus quarantine, both in questions asked to children and in questions asked to parents. The fuzzy Conjoint Method was used to analyze the data obtained. The measurements of the effect of the Corona-virus quarantine have been recorded in the form of degrees of similarity and levels of agreement.

Keywords: *COVID-19; quarantine; fuzzy conjoint model; early childhood.*

AMS Subject Classification (2020): *Primary: 62P25 ; Secondary: 91B06; 62A01; 92D30.*

****Corresponding author***

1. Introduction

The world is faced with disasters caused by natural or human effects from time to time. Various political, economic, health, and social consequences of these disasters affect people. Sawada and Takasaki [1] state that the most important damage caused by the disasters is loss of life and inflicting severe blows on the states' economies. Regarding a similar situation today, Banerjee and Rai [2] stated that the Covid-19 virus has almost taken the world by storm. In this study, the effects of disasters on mental health are discussed. These disasters affect humans the

Received : *12-01-2022,* ***Accepted :*** *06-01-2023*

(***Cite as*** *"M. Kirişci, M. Bardak, N. Topaç, The Effect of Corona-Virus Disease (COVID-19) Outbreak Quarantine on the Belief and Behavior of Children in Early Childhood with a Fuzzy Conjoint Method, Math. Sci. Appl. E-Notes, 11(1) (2023), 43-55"*)

most among the living groups. Among humans, children are most affected.

Countries or unions generally primarily try to calculate the economic effects of natural disasters. For example, the European Parliament (EP) published a briefing on the economic effects of the Covid-19 global pandemic in February 2020 [3]. However, people and countries can be affected by disasters in different ways. As an example, the Organization for Economic Cooperation and Development (OECD) findings regarding the effects of natural disasters can be given. According to these determinations, the effects are as follows [4]:

- Individual impact: Death, injury, and other harm
- Physical impact: Damages to buildings, infrastructure, agriculture, and vital systems
- Economic impact: Changes in financial and economic relations
- Sociological impact: deterioration of family integrity, deterioration in the education system, public health, and mental health problems

In some natural disasters and especially in cases of epidemics, measures such as quarantine are taken to protect people from the harmful effects of the situation. In this context, Cliff and Smallman-Raynor [5] stated that quarantine is used to indicate restrictions on the activities of people or animals exposed to infectious disease during the contagious period. The Corona-virus quarantine, which started on January 27, 2020, in Wuhan, China, has been shown as the most extensive quarantine in human history. Schools, workplaces, meetings, social events, entrances, and exits to the city have been stopped [6]. Similar situations in other cities and countries in the following days have caused this quarantine to be implemented in many parts of the world. Within the scope of the measures announced on March 12 in Turkey, it was decided that the people would voluntarily remain in quarantine at home, and the schools would be closed. While this process continues, curfews were imposed on citizens over the age of 65 and individuals under the age of 20. According to the research conducted by Bozdag [7] in Turkey, in this process, restrictions can cause many psychological effects on people, including depression. According to Houston, First, Houston et al [8], children are the most vulnerable and vulnerable group exposed to disasters and their psychological and behavioral effects. For this reason, it can be said that the negative consequences of quarantine primarily affect children.

Children's reactions to disasters can be examined in three categories: emotion, thought, and behavior. Pfefferbaum, Houston, North, and Regens [9] stated that the behavioral responses of children and adolescents to natural disasters differ from the disaster behaviors of adults; however, traces of adult disaster reactions can be seen in children's behaviors. Children aged 5-6, at the end of early childhood, which is the target group of this study, are gaining skills in expressing their emotions. While expressing these feelings and behaviors, they are influenced by their parents. In a recent study, it has been determined that parents' attitudes significantly structure the beliefs and behaviors of children. Hammer, Scheiter, and Sturmer [10] found that parents' beliefs affect children's self-efficacy. Dekovic and Janssens [11] state that parental behavior can change a child's social behavior positively and negatively. Danseco [12], on the other hand, states that parents' beliefs shape children's development, and this is a reflection of culture. Murphy [13] states that parents' relevant beliefs and perspectives influence the context of children's development. In addition, McGillicuddy-DeLisi [14] states that parents' beliefs affect the developmental processes of children, although at different rates. Therefore, it can be said that parents' beliefs and behaviors about quarantine and pandemic affect children's beliefs and behaviors.

One of the stimuli around children is the media. The role of the media in influencing children's beliefs and behaviors is quite significant. It can be said that the primary tool for obtaining information and news about the pandemic during the quarantine period is television. Van den Bulck, Custers, and Nelissen [15] state that the media affects the development of children. Guru, Nabi, and Raslana [16] state that the content broadcast on television significantly affects the behavior and development of children. Huston and Wright [17] state that television is more effective in children's development than other media organs. Kozma [18] states that books, television, computers, and multimedia environments are important factors affecting their children's learning. For these reasons, it is thought that the media shape the beliefs and behaviors of children during the quarantine process.

Karabulut and Bekler [19] stated that most of the studies on the effects of natural disasters on adolescents and children in the world and Turkey focus on the symptoms of Post Traumatic Stress Disorder (PTSD), which is one of the psychological effects of disasters. However, studies examining the psychological effects of the quarantine process applied during natural disasters are limited. Three weeks were expected for a study on the effect of the quarantine process implemented in Turkey on children's beliefs and behaviors. At the same time, it assumed that children's awareness and sensitivity to the situation were formed during this period. In this process, it is thought that the information acquired by the children's parents, other individuals around them, and media tools affects and changes their beliefs, so behavioral changes occur in children. This research was carried out to determine how children's beliefs and behaviors, who received information about the pandemic directly from their parents, develop regarding quarantine.

1.1 Fuzzy Environment

Since 1965, the fuzzy theory as a new theory introduced by Zadeh in his celebrated paper [20], has generated a growing interest both in the theoretical mathematical framework as well as in its practical applications. Thanks to the fuzzy sets(FSs), imprecision, new mathematical construction has been obtained related to inexactness, ambiguity, and uncertainty. The applications of fuzziness are uncountable and varied. A vast literature has been created since 1965. The FS has led to major technological advances since its origin.

Since its inception in 1965, the theory of fuzzy sets has advanced in a variety of ways and many disciplines. Applications of this theory can be found, for example, in artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics. Mathematical developments have advanced to a very high standard and are still forthcoming today.

The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework that parallels in many respects the framework used in the case of ordinary sets but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables. Imprecision here is meant in the sense of vagueness rather than the lack of knowledge about the value of a parameter. The fuzzy set theory provides a strict mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied. It can also be considered as a modeling language, well suited for situations in which fuzzy relations, criteria, and phenomena exist.

Since its inception in 1965 as a generalization of dual logic and/or classical set theory, fuzzy set theory has been advanced to a powerful mathematical theory. In more than several publications, it has been applied to many mathematical areas, such as algebra, analysis, clustering, control theory, graph theory, measure theory, optimization, operations research, topology, and so on. In addition, alone or in combination with classical approaches it has been applied in practice in various disciplines, such as control, data processing, decision support, engineering, management, logistics, medicine, and others. It is particularly well suited as a 'bridge' between natural language and formal models and for the modeling of non-stochastic uncertainties [21–24].

1.2 Fuzzy Conjoint Model

Conjoint analysis is a research method that helps investigate how people make complex decisions. It is also called trade-off analysis to this scrutinize. The basic principle of this method is that difficult and complex decision-making processes such as purchasing and determining a diagnosis can be made with more than one criterion, regardless of a single criterion. Smith [25] has defined conjoint analysis as a methodology for the computation of psychological decisions, such as consumer choices. The conjoint analysis arose out of the mathematical psychology study of conjoint computation. Green and Wind [26] have emphasized that conjoint measurement is about measuring the joint impact of two or more independent variables on the ordering of a dependent variable. The conjoint analysis ventures to jointly recognize the combination model for decision options and at the same time forecast the benefit value of the attributes that are important in the option decision.

The FS conjoint model is solely based on the FSs decision model. The estimation of each attribute was performed in the condition of the degree of membership which is assumed to be able to indicate the degree of perceptions. The FS conjoint model can be modified into the metrics to ease the users' ability to estimate the usability by using natural language, which is full of vague and subjective expressions.

The fuzzy conjoint analysis has been utilized in many works that include the use of a Likert scale to symbolize linguistic terms. It is considered that the fuzzification of this Likert-scale is more demonstrates the vagueness theme of human subjectivity in giving decisions. For assessing job satisfaction among academic staff in a higher education institution implemented the fuzzy analysis by Rasmani and Shahari [27]. The study of Rasmani and Shahari has confirmed that the analysis using the fuzzy approach gave very good results and ensures more information than the statistical analysis based on percentage. Fuzzy conjoint analysis has also been used in research in the education area [28, 29].

Similarity means that the concept is defined as a generalization of the concepts of equivalence[20]. Although the idea of similarity is a basic notion in human belief, there does not exist an available, general-purpose definition of similarity. There do exist many special-purpose definitions that have been used with achievement in diagnostics, recognition, classification, search, and cluster analysis. Several Similarity Measures are proposed and used for varied purposes. The similarity is a fundamental factor in obtaining an understanding of variables that motivate behavior and intervene with the effect. Thus, the concept of similarity is of fundamental importance in applied areas. Since the notion of similarity has an extensive range of applications, there are several approaches in literature as axioms for the degree or compute of similarity. These axioms have distinctions and similarities depending upon the contexts in which they are established. Therefore, the concept of similarity measure is used in a diversity of scientific areas such as market prediction, machine learning, pattern recognition, and decision making. FS theory has improved its own computes of similarity, which applications in various scientific fields.

1.3 The problem and purpose of the study

The target audience of this study is children aged 5-6 at the end of early childhood. Children of this age group are in the process of gaining skills in expressing their feelings during this period. In addition, the emotional responses of these children can be noticed by a careful observer or even an expert. In addition to those mentioned in the literature, most of the studies related to the effects on the adolescents and children of natural disasters in the World and Turkey focused on the symptoms of "Post Traumatic Stress Disorder" which is one of the psychological effects of disasters [19]. This study aims to investigate the effects of quarantine status due to the COVID-19 pandemic on the beliefs and behaviors of children who stay at home. The precise results of the current situation can only be determined by analyzing the data to be made and collected at the end of the process.

2. Preliminaries

Definition 2.1. Consider the G as an initial universe. A FS on G is a set defined by a function $m_S : G \to [0,1]$. m_S is called the membership function of S, and the value $m_S(x)$ is called the grade of membership of $a \in G$. The value represents the degree of a belonging to the FS S. Thus, an FS S on G can be represented as follows:

$$S = \{(m_S(a)/a) : m_S(a) \in [0,1], \forall a \in G\}.$$

In this Definition, the membership function of the fuzzy set is a crisp(real-valued) function. Zadeh also defined fuzzy sets in which the membership functions themselves are fuzzy sets:

Definition 2.2. A type m fuzzy set is a fuzzy set whose membership values are type $m-1$, $m>1$, fuzzy sets on $[0,1]$.

Example 2.1. Let $D = \{d_1, d_2, d_3\}$ represent the set of linguistic values such as "yes", "maybe", "no". Then,

$$d_1 = \left\{\frac{1}{1}, \frac{0.2}{2}, \frac{0.5}{3}\right\}, \quad d_2 = \left\{\frac{0.7}{1}, \frac{1}{2}, \frac{0.3}{3}\right\}, \quad d_3 = \left\{\frac{0.6}{1}, \frac{0,5}{2}, \frac{1}{3}\right\}.$$

These values can be shown in Table 1. The first element of d_1 is $\frac{1}{1}$, 1 is the value of the function at first option. The second element is $\frac{0.2}{2}$, which indicates the function value is 0.2 at the second option and so on.

Definition 2.3. A linguistic variable is characterized by a quintuple $(x, T(x), U, G, \widehat{M})$ in which x is the name of the variable, $T(x)$ denotes the term set of x, that is, the set of names of linguistic values of x. Each of these values is

Table 1. The fuzzy set D

	1	2	3
d_1	1	0.2	0.5
d_2	0.7	1	0.3
d_3	0.6	0.5	1

a fuzzy variable, denoted generically by X and ranging over a universe of discourse U, which is associated with the base variable u; G is a syntactic rule (which usually has the form of a grammar) for generating the name, X, of values of x. M is a semantic rule for associating with each X its meaning. $\widehat{M}(X)$ is a fuzzy subset of U. A particular X, that is, a name generated by G, is called a term.

Example 2.2. Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ represent the set of linguistic values such as "too much", "much", "some", "too little" and "none" 2. Then,

$$e_1 = \left\{\frac{1}{1}, \frac{0.7}{2}, \frac{0.4}{3}, \frac{0.2}{4}, \frac{0.6}{5}\right\}, \quad e_2 = \left\{\frac{0.3}{1}, \frac{1}{2}, \frac{0.8}{3}, \frac{0.6}{4}, \frac{0.5}{5}\right\}, \quad e_3 = \left\{\frac{0.5}{1}, \frac{0,2}{2}, \frac{1}{3}, \frac{0.9}{4}, \frac{0.1}{5}\right\}$$

$$e_4 = \left\{\frac{0.2}{1}, \frac{0,6}{2}, \frac{0.5}{3}, \frac{1}{4}, \frac{0.3}{5}\right\}, \quad e_5 = \left\{\frac{0.7}{1}, \frac{0,8}{2}, \frac{0.4}{3}, \frac{0.5}{4}, \frac{1}{5}\right\}.$$

Table 2. The fuzzy set E

	1	2	3	4	5
e_1	1	0.7	0.4	0.2	0.6
e_2	0.3	1	0.8	0.6	0.5
e_3	0.5	0.2	1	0.9	0.1
e_4	0.2	0.6	0.5	1	0.3
e_5	0.7	0.8	0.4	0.5	1

3. Methods

In this study, the survey model was used. The survey was prepared in the Likert type and the opinions of the participants were determined. Many scientific researchers use a wide variety of psychometric measurement tools. The most commonly used measurement tool is the Likert scale. An attitude or opinion determined in the problem of the study is called the Likert-type question, the choices that show the participant's optional level of participation. In Likert-type questions, more than one option is presented for the level of participation in the study. The options to be given for the questions are ranked from best to worst or from largest to smallest. In this study, there are two different participant groups and two different Likert questionnaires were prepared to be suitable for their development levels. In this research, the Likert scale is used with 3 linguistic terms. The linguistic terms and their numeric labels are:

For Questions to be asked to the child: Yes(1), maybe/some (2), no(3). For Questions to be asked to parents: too much(1), much (2), some (3), too little(4), none (5).

The survey was prepared to be answered on the internet. Survey questions were asked to children aged 5-6 and their families. The survey included the following questions:

Questions to be asked to the child:

- Do you know Corona-virus?
- Are you afraid of the Corona virus?
- Does Corona-virus harm people?

- Does Corona-virus harm animals?
- Can Corona-virus be prevented?
- Do you think it's nice not to go to school?
- Are you upset that you can't go to school?
- Is the obligation to stay home boring?
- Can we be protected from Corona virus by staying at home?
- Do you think you can go to school from now on?

Questions to be asked to parents:

- Does your child behave anxiously after the Corona virus?
- Is your child afraid when a conversation about the Corona virus has passed?
- Does your child ask about the Corona virus?
- Does your child pay more attention to cleaning after Coronavirus?
- Has your child's sleep pattern been impaired after the Corona virus?
- Have there been changes in your child's nutritional habits after Coronavirus?
- Did your child develop undesirable behaviour after Coronavirus?
- Is your child happy because she/he can't go to school?
- Has the time your child spent on the Internet after Coronavirus increased?
- Has the time your child spent in front of the TV increased after Corona-virus?

The belief and behavioural distributions of questions are as follows:

For children's belief;
Do children know about the current situation? (4 questions)
Does the current situation affect children's emotions? (4 questions)
Does the current situation affect children's thoughts? (2 questions)

For children's behavioural;
Has Corona-virus changed the basic habits of children? (3 questions)
Did behaviour change occur in children after quarantine? (5 questions)
Did children's behaviour regarding information technologies increase after quarantine? (2 questions)

In this study, from Turkey, 201 children ages $5-6$ units and 201 parents were the participants. Opinions of each child and each parent about the questions asked were got. The effect of quarantine on their belief in line with the answers given by the children and the effect of the behaviour of their children in line with the observations of the parents have been revealed.

3.1 Algorithm

Let us consider the set D containing linguistic values (from Example 2.1). The FS represented for each linguistic value $d_n, (n = 1, 2, 3)$ were defined the Table 1. Let F be a FS, D be a factor attributes. For ith respondent $(i = 1, 2, \cdots, 201)$ against attribute D,

$$F_i \in \{d_1, d_2, d_3\}.$$

The degree of membership of domain element a_k in the subject's linguistic rating F of the ith attribute of D for each element in FS F_i is denoted by $m_{F_i}(a_j, D) = \Lambda$ where $a_k = 1, 2, \cdots, r$. r is a linguistic values used.

Let's n show the number of participants. If v_i is taken as the score of the linguistic values in the ith,respondent, the weight for ith respondent $\omega_i = \frac{v_i}{\sum_{t=1}^{n} v_t}$. The approximate degree of membership for each element, $b_k, (k = 1, 2, \cdots, 201)$ in FS F is defined as

$$\Omega = \sum_{i=1}^{n} \omega_i \Lambda, \tag{3.1}$$

where $\Omega = m_F(b_j, D)$. Let $m_{d_n}(j)$ shows the FS defined for linguistic rating. Consider $m_F(j, F)$ is the calculated set for product m from (3.1). The formula for the similarity of two sets is

$$B(F, d_n) = \frac{1}{1 + \sqrt{\sum_{k=1}^{3} \left[m_F(j, F) - m_{d_n}(j)\right]^2}} \tag{3.2}$$

The above information and formulas also apply to the E set in Example 2.2. In this case, the FS represented for each linguistic value $e_n, (n = 1, 2, 3, 4, 5)$ were defined the Table 2.

Calculations for the measurement procedure will be carried out with the following algorithm.

Algorithm:
Step 1: Record children's answers.
Step 2: Record parent's answers.
Step 3: Define linguistic variables according to the answers in Step 1.
Step 4: Define linguistic variables according to the answers in step 2.
Step 5: Data handling: According to the answers of children and parents.
Step 6: Calculate Fuzzy weight vectors by children's response.
Step 7: Calculate Fuzzy weight vectors by parent's response.
Step 8: Compute of similarity degree values between respondents and experts.
Step 9: Select the maximum value of the degree of similarity.

4. Results

According to the results given in Table 3, the following evaluations can be made:

It was observed that the vast majority of children knew about the Corona-virus, while a small number of them did not know about the virus. Although the proportion of children who are afraid and not afraid of Corona-virus among the children participating study is close to each other, it has been observed is a considerable ”maybe/some” answer. Almost all the children think that Corona-virus is harming people. The proportion of children who think that Corona-virus does not harm animals is relatively high compared to those who think it will harm, but the two values are close. In addition, the ratio of those who answered ”maybe/some” is close to half of those who answered ”yes”. The proportion of children who think Corona-virus can be prevented is considerably higher than children who think it cannot be prevented or partially prevented. The rate of children who think it is not pleasant to be unable to go to school due to quarantine is higher than those who think otherwise. For the question "Are you upset that you cannot go to school" most children answered that they are sad or a little sad. The number of children who express that they are not upset is few. The majority of the participating children stated that the obligation to stay at home was tedious. However, a considerable part of the children stated that they were not dull or a little boring. It is seen that almost all of the children have the idea that they can be protected or partially protected by staying at home from Corona-virus. While most of the children answer yes to the question "Do you think you can go to school from now on", the proportion of those who state that they cannot go or maybe go is close to each other.

Table 3. Questions to be asked to the child

	d_1	d_2	d_3	maximum value
Do you know Corona-virus?	0.80132	0.21451	0.00138	yes
Are you afraid of Corona-virus?	0.58517	0.25741	0.50638	yes
Does Corona-virus harm people?	0.97336	0.00491	0.12541	yes
Does Corona-virus harm animals?	0.52037	0.24663	0.59904	no
Can Corona-virus be prevented?	0.72167	0.24471	0.49055	yes
Do you think it's nice not to go to school?	0.51007	0.18211	0.76814	no
Are you upset that you can't go to school?	0.78564	0.12847	0.27346	yes
Is the obligation to stay home boring?	0.79881	0.22089	0.19918	yes
Can we be protected from Corona-virus by staying at home?	0.95647	0.00973	0.15042	yes
Do you think you can go to school from now on?	0.63429	0.24977	0.31076	yes

According to the results given in Table 4, the following evaluations can be made:

According to parental observation, while more than half of the children showed a little anxious behavior, it was seen that a significant part of them did not have any anxiety. This result is the same as the observation about the child's fear. A large part of the parents stated that their children were a little afraid. Many parents stated that their children behave in a questioning manner regarding quarantine and Coronavirus. It has been stated that most children pay more attention to the cleaning after the Coronavirus appears. It is mentioned that very few of them did not change their cleaning behavior. Parents have declared that the sleep pattern of children, in general, has not changed. However, in part, it is present in children whose sleep patterns have changed. While it was stated that there was no change in the nutrition of the children of the participating parents, very few of them had a change in nutrition. The question about "whether unwanted behavior develops in their children during the Corona-virus quarantine period or not has given" a reply that a significant part of the parents did not develop. Parents generally answered "not happy" to the question, "Is your child happy because she/he can't go to school". The answer "some" was given to the question, "Has the time your child spent on the Internet after Coronavirus increased". Similar findings were obtained for the level of watching TV. However, the increase in the rate of watching TV is higher than on the Internet.

5. Discussion and Conclusion

As in all continents and countries of the world, natural disasters, especially different epidemics, are among the situations encountered in Anatolian lands and Turkish history. For example, in the study of cholera epidemics in the Ottoman Empire at some times, Yucel [30] emphasized that the people's unconscious behaviors caused the epidemic to be effective for a long time despite all the state's efforts. Regardless of physical or mental health, awareness and belief about disaster and emergency planning should be developed at all parts and levels of the health system [31]. As can be understood from here, both adults and children can exhibit wrong behaviors by having wrong beliefs during epidemic times. Even in quarantine, people can develop false beliefs and act accordingly.

Quarantine, one of the most important ways to prevent epidemics, requires conscious participation. However, it is also important to direct the belief and behavior of more vulnerable and disadvantaged groups such as children correctly in this process. Although the World Health Organization (WHO) states that quarantine increases people's capacity to control the spread of infectious diseases [32], this may have negative repercussions on people. In addition to the restrictions experienced during the quarantine process, such as basic needs and habits, fear and anxiety may threaten the individual's mental well-being.

In times of disaster, different tools have been developed to protect the mental health of various strata of the population. Public child and family disaster communication has been defined as a public health tool that can be used to cope with the post-disaster situation/promote resilience and improve maladaptive child responses. It has

Table 4. Questions to be asked to parents

	e_1	e_2	e_3	e_4	e_5	maximum value
Does your child behave anxiously after Corona-virus?	0.00182	0.00215	0.30128	0.39407	0.51833	none
Is your child afraid when a conversation about Corona-virus has passed?	0.0	0.00899	0.39004	0.36725	0.48913	none
Does your child ask about Corona-virus?	0.00209	0.02487	0.51004	0.31417	0.002138	some
Does your child pay more attention to cleaning after Corona-virus?	0.28433	0.40314	0.41126	0.10465	0.00679	some
Has your child's sleep pattern been impaired after Corona-virus?	0.12743	0.00977	0.36148	0.18751	0.64039	none
Have there been changes in your child's eating habits after Corona-virus?	0.00179	0.00813	0.27638	0.02097	0.71465	none
Did your child develop undesirable behaviour after Corona-virus?	0.00164	0.00197	0.2009	0.02103	0.81027	none
Is your child happy because she/he can't go to school?	0.00001	0.00193	0.22067	0.269813	0.72505	none
Has the time your child spent on the Internet after Corona-virus increased?	0.12069	0.25144	0.49217	0.30024	0.27244	some
Has the time your child spent in front of the TV increased after Corona-virus?	0.14727	0.25162	0.52403	0.29904	0.24138	some

been stated that schools are an important (promising) system for child and family disaster communication [8]. Epidemics may occur after some disasters, and epidemics may arise as a natural or artificial disaster. For example, Pascapurnama et al [33], in their study investigating the outbreaks after eight major natural disasters, stated that infectious diseases are preventable, and for this, knowledge and awareness about health risks stated should be be be continuously supported with pre-disaster and post-disaster training opportunities.

Every new experience means new knowledge, and it is a situation that needs to be learned. Significantly children should acquire information with quality experiences in natural disasters such as epidemics. Information needs to be encoded correctly and translated into behavior. For this reason, the administrators need to inform the public with the correct information and consider their psychology. As Gostin [34] stated, the principles, technical standards, and recommendations published by international organizations such as WHO to guide member countries should also be considered. It is essential to take precautions and inform the public on time so that they do not panic. People might cause the spread of the disease with panic and false information, and they may engage in behaviors that may harm themselves and the society they live. In this context, it should be kept in mind that children's beliefs and behaviors may also be affected by the negative aspects of the situation.

This study focuses on the fact that children aged 5-6, who are the target group, have been living under voluntary quarantine due to the global epidemic of corona virus, the effect of which has been felt around the world since December 2019. During quarantine's first 14 days, preliminary effects of the situation on belief and behavior of children were revealed. While striking findings were obtained in some parts of the study, some findings were not following the researchers' expectations. This situation can be shown by the short-term nature of the quarantine process.

Children have a great deal of knowledge about corona-virus shows that the stimuli in the environment are concentrated in the direction of the virus. However, considering the answers given to the other questions and the observations of the parents, it is not possible to make a definite determination about the level and accuracy of their knowledge. It can be said that the reason for this is that the data collection process is carried out on the 14th day of the corona-virus quarantine in Turkey. This result is informative about the early effects of the corona-virus quarantine. Fears are one of the emotions that direct the cognitive elements of the individual. Looking at the answers about whether children are afraid of the corona virus, it can be said that there is a situation in which their knowledge is not yet explicit. It may be related to the fact that the rate of children who are afraid and those not afraid are close to each other, and the rate of children who are not sure is not very low. However, the corona virus's strong tendency to harm humans and the opposite answers about the harm to animals reinforces the suspicion that children do not have enough knowledge on the subject. Beliefs that they harm people direct their other beliefs and behaviors in this regard.

Children's thoughts that the corona virus can be prevented and parental observations about cleanliness parallel each other. It was concluded that their belief turned into behavior as a precaution in this direction. In addition to the child's fear, the thought that the virus can be prevented by cleaning can be explained because belief affects behavior in this direction. However, one of the situations that should not be forgotten here is the possibility of one or more family members exhibiting an adverse profile. According to the National Academies of Sciences, Engineering, and Medicine (2019), having a parent with an untreated mental illness or addiction who does not seek counseling can lead to genetic risks, epigenetic changes, negative behavioral patterns, and negative social learning. It can lead to a problematic negative childhood experience that includes many mechanisms that can also cause relational skills. Therefore, the underlying cause of, some of these beliefs of children may be fed from different sources, but the effect of the current period as a trigger should also be taken into account.

The new environment formed during quarantine times, when environmental effects are reduced to a narrower framework, can leave deep traces on the child's development, mainly if it affects the child for an extended period. In this narrow framework, social, economic, cultural, and environmental factors significantly affect the child's health ecosystem and capacity to prepare for adulthood[35]. The fact that they cannot stay at home and go to school due to quarantine is expressed as an unpleasant situation by most children. It can be said that the fact that the children answered the question asked to reveal whether they were upset about this situation as a supporter at the same rate as the quarantine question shows consistency. The fact that most children expressed in this direction are in parallel with the answers given by the parents. Those who stated that staying at home was boring and that it was sad not to go to school were close to each other support the conclusion that children's beliefs on this issue are consistent. However, most children believe that they can protect themselves from the virus by staying at home, significantly higher than the rate of those who say that staying at home is boring. The belief that one can be protected by staying at home is belief, and the boredom of staying at home is an expression of emotion. From this, it can be concluded that children's belief takes precedence over their emotions. The fact that a substantial part of the children gave a negative answer to whether they can go to school, which aims to reveal their expectations from the future, can be associated with their fears and anxieties. At this point, as experts [36] stated, it is necessary to recommend that adults try to establish a balance between having sufficient knowledge about the new corona virus and answering their children's questions well enough without increasing the severity of anxiety. It is understood from the answers given by the parents regarding the observations that more than half of the children exhibit slightly anxious behaviors. In addition, as a result, that supports this situation, it can be shown that the ratio of those who answered "I am afraid" to the question of "are you afraid of corona virus" and those who exhibited anxious behavior were very close to each other. From this point of view, it can be concluded that more than half of the children have beliefs related to fear, and this is observed as anxious behaviors, while a significant part of them develop a belief that they cannot go to school. It should be kept in mind that the coronavirus pandemic may aggravate existing mental health problems or cause new cases in children who are thought to have a more significant impact than parents to citegolb. The same situation is observed for the fear situation specifically related to corona-virus within the scope of environmental effects.

The rate of children who say they know the coronavirus and the rate of children who ask their parents a few questions about this issue are very close. This situation also has consistency in itself. When the findings obtained from the behavior of paying attention to cleanliness are combined with other findings, it is seen that results that support each other emerge. More clearly, the answers are given to questions about topics such as fear, anxiety, protection, and the thought that it will harm, and the rate of increase in cleaning behavior overlap with each other. In this context, the data on depression, anxiety, and behavioral disorders in statistics about children made by the Centers for Disease Control and Prevention (2020) support this situation.

A short period of 14 days is not sufficient to describe the changes in sleep behavior, which is one of the fundamental habits, as low or to express the extent of the change. This observation of the parents, who stated that the child's sleep pattern has changed, can be explained by reasons such as not being able to go out, not having to wake up early, and therefore going to bed late. Considering these situations, the change in sleep and nutrition, which are fundamental habits, strongly relate to parental attitudes in the home environment. However, it is seen that the change in sleep pattern is slightly more than the change in the diet. It can be said that the reason for this situation is the family members being together all day long and the close attention of the children at meals. At the end of the second week of the quarantine period, the fact that radical behavioral changes, especially undesirable behaviors, are not seen intensely can be explained by the shortness of the process. However, these pre-effects may lead to more incredible behavioral changes with the boredom in the child if the quarantine period is prolonged. However, it was also observed that a small part of the families stated a severe behavioral change in the child. The fact that some have stated partial changes suggests that these effects may increase in the future, as in the snowball effect. The reason can be shown as the individual differences between children and the level of interaction within the family. There is a proportional similarity between the children's answers and the parents' observations about not being able to go to school. The lack of a significant change in the time children spend on both the Internet and TV can be explained by the possibility of the parent making extra efforts to spend quality time with the child. In addition, it can be said that there are attempts to keep children away from the internet and TV as much as possible in order not to increase their anxiety and fear.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] Sawada Y., Takasaki Y.: *Natural disaster, poverty, and development: An introduction*. World Development. **94**, 2–15 (2017).

[2] Banerjee, D., Rai, M.: *Social isolation in Covid-19: The impact of loneliness*. International Journal of Social Psychiatry. **66**(6), 525-527 (2020).

[3] *European Parliament resolution of 17 April 2020 on EU coordinated action to combat the COVID-19 pandemic and its consequences* (2020/2616(RSP)).

[4] *OECD (1994) Guidelines on Disaster Mitigation*. [online], `http://www.oecd.org/dac/environment-development/1887740.pdf`

[5] Cliff A. D., Smallman-Raynor M. R.: Quarantine: Spatial Strategies. In A. D. Cliff & M. R. Smallman-Raynor: Eds, Oxford Textbook of Infectious Disease Control: A Geographical Analysis from Medieval Quarantine to Global Eradication. Oxford: Oxford University Press (2013).

[6] Markel, H.: Will the Largest Quarantine in History Just Make Things Worse?. The New York Times, 27.01.2020. `https://www.nytimes.com/2020/01/27/opinion/china-wuhan-virus-quarantine.html?auth=link-dismiss-google1tap`

[7] Bozdağ, F.: *The psychological effects of staying home due to the COVID-19 pandemic*. The Journal of General Psychology. **148**(3), 226–248 (2021).

[8] Houston, J. B., First, J., Spialek, M. L., Sorenson, M. E., Koch, M.: *Public disaster communication and child and family disaster mental health: A review of theoretical frameworks and empirical evidence*. Current Psychiatry Reports. **18**(6), 1–9 (2016).

[9] Pfefferbaum, B., Houston, J. B., North, C. S., Regens, J. L.: *Child reactions to disasters and the factors that influence their response*. The Prevention Researcher. **15**(3), 3–6 (2008).

[10] Hammer, M., Scheiter, K., Stürmer, K.: *New technology, the new role of parents: How parents' beliefs and behavior affect students' digital media self-efficac*y. Computers in Human Behavior. **116**, 106642 (2021).

[11] Deković, M., Janssens, J. M.: *Parents' child-rearing style and child's sociometric status*. Developmental psychology. **28**(5), 925-932 (1992).

[12] Danseco, E. R.: *Parental beliefs on childhood disability: Insights on culture, child development, and intervention*. International Journal of Disability, Development, and Education. **44**(1), 41–52 (1997).

[13] Murphey, D. A.: *Constructing the child: Relations between parents' beliefs and child outcome*s. Developmental Review. **12**(2), 199–232 (1992).

[14] McGillicuddy-DeLisi, A. V.: *Parental beliefs about developmental processes*. Human development. **25**(3), 192–200 (1982).

[15] Van den Bulck, J., Custers, K., Nelissen, S.: *The child-effect in the new media environment: Challenges and opportunities for communication research*. Journal of Children and Media. **10**(1), 30–38 (2016).

[16] Guru, B. P., Nabi, A., Raslana, R.: *Role of television in child development*. Research Journal of Humanities and Social Sciences. **4**(2), 264–270 (2013).

[17] Huston, A. C., Wright, J. C.: Mass media and children's development. In W. Damon, I. E. Sigel, & K. A. Renninger (Eds.), Handbook of child psychology: Child psychology in practice. John Wiley & Sons Inc. (1998).

[18] Kozma, R. B.: *Learning with media*. Review of educational research. **61**(2), 179–211 (1991).

[19] Karabulut, D., Bekler, T.: *Effects of Natural Disasters on Children and Adolescents*. Artvin Çoruh University Journal of Natural Hazards and Environment. **5**(2), 368–376 (2019).

[20] Zadeh, L. A.: *Fuzzy sets*. Inf. Comp. **8**, 338–353 (1965).

[21] Mithchell H.B.: *Pattern recognition using type-II fuzzy sets*. Information Sciences. **170**(2–4), 409–418 (2005).

[22] Kirişci, M.: *Integrated and differentiated spaces of Triangular Fuzzy Numbers*. Fasciculi Mathematici. **59**, 75–89 (2017).

[23] Kirişci, M.: *A case study for medical decision making with the fuzzy soft sets*. Afrika Matematika. **31**, 557–564 (2020).

[24] Kirişci, M.: *Medical decision making with respect to the fuzzy soft sets*. Journal of Interdisciplinary Mathematics. **23**(4), 767–776 (2020).

[25] Smith, S.: Conjoint analysis tutorial: East, west and in between. Brigham Young University (2005).

[26] Green, P. E., Wind, Y.: Multiatribute decisions in marketing. Dryden Press (1973).

[27] Rasmani, K. A., Shahari, N. A.: *Job satisfaction evaluation using fuzzy approach*. In: Proceedings of the third International Conference on Natural Computation (ICNC 2007), IEEE. **4**, 544–548 (2007).

[28] Lazim, M. A., Abu Osman, M. T.: *Measuring teacher's beliefs about mathematics: a fuzzy set approach*. World Acad. Sci. Eng. Tech. **33**, 1098–1102 (2009).

[29] Kirişci, M.: *Fuzzy conjoint model for preservice mathematics teachers' beliefs*. Far East Journal of Mathematical Education. **19**(2), 87–105 (2019). http://dx.doi.org/10.17654/ME019020087.

[30] Yucel, G.: *Cholera epidemics in Malatya and state intervention efforts (1892-1894)*. İnönü Üniversitesi Sağlık Hizmetleri Meslek Yüksekokulu Dergisi. **7**(1), 108–121 (2019).

[31] Yesil, S. T.: *General approaches in health disaster and emergency planning and existing health plans in Turkey*. Turkish Journal of Public Health. **15**(3), 233–244 (2017).

[32] Bonita, R., Beaglehole, R., Kjellström, T.: Basic Epidemiology (2nd edition). Geneva: World Health Organization (2006).

[33] Pascapurnama D. N., Murakami, A., Chagan-Yasutan, H., Hattori, T., Sasaki, H., Egawa, S.: *Integrated health education in disaster risk reduction: Lesson learned from disease outbreak following natural disasters in Indonesia*, International Journal of Disaster Risk Reduction. **29**, 94–102 (2018).

[34] Gostin, L. O.: Global health law. Cambridge: Harvard University Press (2014).

[35] National Academies of Sciences, Engineering, and Medicine. Vibrant and Healthy Kids: Aligning Science, Practice, and Policy to Advance Health Equity. The National Academies Press, Washington DC (2019).

[36] Sperling, J.: How to talk to children about the coronavirus. Harvard Health Blog. Accessed April 15, (2020). `https://www.health.harvard.edu/blog/how-to-talk-to-children-about-the-coronavirus-2020030719111`

Affiliations

MURAT KIRIŞCI
ADDRESS: Istanbul University-Cerrahpaşa, Dept. of Biostatistics and Medical Informatics, Fatih, Istanbul-Türkiye.
E-MAIL: mkirisci@hotmail.com
ORCID ID:0000-0003-4938-5207

MUSA BARDAK
ADDRESS: İstanbul S. Zaim University, Dept. of Early Childhood Education, Kücükçekmece, Istanbul-Türkiye.
E-MAIL: musa.bardak@izu.edu.tr
ORCID ID:0000-0001-5585-8002

NIHAT TOPAÇ
ADDRESS: Istanbul University-Cerrahpaşa, Dept. of Early Childhood Education, Büyükçekmece, Istanbul-Türkiye.
E-MAIL: ntopac@istanbul.edu.tr
ORCID ID:0000-0001-9364-4072

MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753/mathenot.1243583

11 (1) 56-66 (2023) - Research Article

ISSN: 2147-6268

On the Qualitative Behavior of the Difference Equation $\delta_{m+1} = \omega + \zeta \frac{f(\delta_m, \delta_{m-1})}{\delta_{m-1}^{\beta}}$

Mehmet Gümüş* and Şeyma Irmak Eğilmez

Abstract

In this paper, we aim to investigate the qualitative behavior of a general class of non-linear difference equations. That is, the prime period two solutions, the prime period three solutions and the stability character are examined. We also use a new technique introduced in [1] by E. M. Elsayed and later developed by O. Moaaz in [2] to examine the existence of periodic solutions of these general equations. Moreover, we use homogeneous functions for the investigation of the dynamics of the aforementioned equations.

Keywords: *Homogeneous function; difference equation; periodicity; qualitative behavior; stability.*

AMS Subject Classification (2020): *Primary: 39A05 ; Secondary: 39A21; 39A23; 39A30.*

****Corresponding author***

1. Introduction

Since the emergence of the difference equation theory, many pioneering studies have been carried out that will benefit both the development of the theory and other applied sciences (see [3], [4], [5], [6], [7]). The use of difference equations is not only in theory but also in many applied sciences outside the field of mathematics in terms of applying mathematical models of physical phenomena to daily life. Especially in mathematical biology, ecology, and economics, different mathematical models are needed to study populations, population growth and the spread of epidemics (see [8], [9], [10]). Therefore, the difference equations create mathematical models that can be applied to the basic living conditions of physical phenomena. At the same time, it can be said that the solution of even the simplest problem encountered in differential equations, which is another branch of applied mathematics, is more complex and more difficult than the difference equations. For this reason, difference equations have been used as an approach in the mathematical modelling of many physical, chemical and biological phenomena that can

Received : *27-01-2023,* ***Accepted :*** *11-03-2023*

*(**Cite as** "M. Gümüş, Ş. I. Eğilmez, On the Qualitative Behavior of the Difference Equation $\delta_{m+1} = \omega + \zeta \frac{f(\delta_m, \delta_{m-1})}{\delta_{m-1}^{\beta}}$, Math. Sci. Appl. E-Notes, 11(1) (2023), 56-66")*

express with ordinary and partial differential equations (ODE and PDE) and in equations that are difficult to solve analytically [11].

In recent years, difference equation research has attracted great interest from researchers. In particular, applications of higher-order non-linear difference equations have influenced many researchers (see [12], [13], [14], [15], [16], [17], [18], [19]). It is very important to examine especially the oscillation, the asymptotic behavior and the stability character of the solutions of the general classes of the higher-order non-linear difference equations, which have a complex structure and contain different states. However, there are not many articles and books that handle with the qualitative studies of non-linear difference equations. Therefore, there are many aspects of non-linear difference equations that need to be investigated and developed.

In [1], Elsayed introduced a new method for the prime period two solutions and the prime period three solutions of the rational difference equation

$$\delta_{m+1} = \mu + \phi \frac{\delta_m}{\delta_{m-1}} + \gamma \frac{\delta_{m-1}}{\delta_m},\ m = 0, 1, \ldots$$

where the parameters $\mu, \phi, \gamma \in \mathbb{R}^+$ and initial values $\delta_{-1}, \delta_0 \in \mathbb{R}^+$. Besides, the global convergence and the boundedness nature have been investigated.

In [20], Moaaz *et al.* examined the dynamical behaviors of solutions of a general class difference equation

$$z_{m+1} = g(z_m, z_{m-1}),\ m = 0, 1, \ldots$$

where the initial conditions $z_{-1}, z_0 \in \mathbb{R}$ and g is a continuous homogeneous function with degree zero. Namely, the stability, the oscillation and the periodicity character have been investigated.

In [21], Moaaz studied the global dynamics of solutions of the following general class of difference equations

$$\delta_{m+1} = g(\delta_{m-l}, \delta_{m-k}),\ m = 0, 1, \ldots$$

where l, k are positive integers, the initial conditions $\delta_{-\rho}, \delta_{-\rho+1}, \ldots, \delta_0 \in \mathbb{R}$ for $\rho = \max\{l, k\}$ and g is a continuous homogeneous real function of degree γ. That is, the global attractiveness, the periodic character, and the stability nature have been investigated. Furthermore, the author investigated the periodic solutions with used the new method [1].

In [22], Moaaz *et al.* studied the existence and the non-existence of periodic solutions of some non-linear difference equations. Especially, they investigated the existence of periodic solutions of the difference equation

$$\omega_{m+1} = \gamma \omega_{m-1} F(\omega_m, \omega_{m-1}),\ m = 0, 1, \ldots$$

where the parameter $\gamma \in \mathbb{R}^+$, the initial values $\omega_{-1}, \omega_0 \in \mathbb{R}^+$ and F is a homothetic function, namely there exists a strictly increasing function $F_1 : \mathbb{R} \to \mathbb{R}$ and $F_2 : \mathbb{R}^2 \to \mathbb{R}$ are homogenous function with degree ρ, such that $F = F_1(F_2)$ and also studied the following second-order difference equation

$$\omega_{m+1} = \mu + \eta \frac{\omega_{m-1}^{\rho}}{h(\omega_m, \omega_{m-1})},\ m = 0, 1, \ldots$$

where $\rho \in \mathbb{R}^+$, the parameters $\mu, \eta \in \mathbb{R}$, the initial values $\omega_{-1}, \omega_0 \in \mathbb{R}$ and h is a continuous homogeneous function with degree ρ.

In [23], Abdelrahman investigated the dynamical behavior of solutions of the general class of difference equations

$$\omega_{m+1} = h(\omega_m, \omega_{m-1}, \ldots, \omega_{m-k}),\ m = 0, 1, \ldots$$

where $h : (0, \infty)^{k+1} \to (0, \infty)$ is a continuously homogeneous function of degree zero and k is positive integer. That is, the stability, the periodicity and the oscillation nature have been examined.

The aim of this paper is to investigate the global behavior of solutions, that is, the prime period two solutions, the prime period three solutions and the stability character of a new general class of the second-order difference equation

$$\delta_{m+1} = \omega + \zeta \frac{f(\delta_m, \delta_{m-1})}{\delta_{m-1}^{\beta}},\ m = 0, 1, \ldots \tag{1.1}$$

where the parameters $\omega, \zeta \in \mathbb{R}$, the initial conditions $\delta_{-1}, \delta_0 \in \mathbb{R}$ and $f : (0, \infty)^2 \to (0, \infty)$ is a continuous homogeneous function with degree β. Also, in particular, the two periodic solutions and the three periodic solutions are examined by using the new method [1, 2]. In addition, we specify the new sufficient conditions for the stability character of the positive equilibrium point.

2. Preliminaries

In the following, we give some basic definitions and theorems that we will benefit from in this paper.

Assume that J be an interval of real numbers and let the initial conditions for every $z_{-1}, z_0 \in J$. If $h : J \times J \to J$ be a continuously differentiable function, then the difference equation

$$z_{n+1} = h(z_n, z_{n-1}),\ n \in \mathbb{N} \tag{2.1}$$

has a unique positive solution $\{z_n\}_{n=-1}^{\infty}$.

Definition 2.1. [24] (Periodicity) Let t be a positive integer. Then, the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(2.1) is said to be periodic with period t if

$$x_{n+t} = x_n,\ n = 0, 1, \ldots$$

where t is the smallest integer.

Theorem 2.1. *[24] (The Linearized Stability Theorem)*
(i) If both roots of the quadratic equation

$$\lambda^2 - p\lambda - q = 0 \tag{2.2}$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{x}$ of Eq.(2.1) is locally asymptotic stable.

(ii) If at least one of the roots of Eq.(2.2) has absolute value greater than one, then the equilibrium point $\overline{x}$ of Eq.(2.1) is unstable.

Theorem 2.2. *[5] (Clark Theorem) Assume that $\rho_0, \rho_1 \in \mathbb{R}$ and $k \in \{0, 1, \ldots\}$. Then, the difference equation*

$$\delta_{m+1} + \rho_0\delta_m + \rho_1\delta_{m-k} = 0,\ m = 0, 1, \ldots$$

is the asymptotic stability if

$$|\rho_0| + |\rho_1| < 1.$$

Definition 2.2. [25] (Homogeneous Function) Assume that $f : \mathbb{R}_+^n \to \mathbb{R}$ is called a homogeneous function with degree k if for every $x \in \mathbb{R}_+^n$ and every $\lambda > 0$

$$f(\lambda x) = \lambda^k f(x).$$

Theorem 2.3. *[25] (Euler's Homogeneous Function Theorem) Assume*

$$f : \mathbb{R}_+^n \to \mathbb{R}$$

is a continuous function and also differentiable on $\mathbb{R}_+^n$. Then, f is homogeneous function with degree k if only if for every $x \in \mathbb{R}_+^n$

$$kf(x) = \sum_{i=1}^{n} D_i f(x) x_i.$$

Corollary 2.1. *[25] Let $f : \mathbb{R}_+^n \to \mathbb{R}$ be continuous function, and also differentiable on $\mathbb{R}_+^n$. If f is homogeneous function with degree k, then $D_j f(x)$ is homogeneous with degree $k-1$.*

3. Asymptotic behavior of solutions of the non-linear difference equation (1.1)

In this section, we will examine the two periodic solutions, the three periodic solutions and the stability character of the second-order non-linear difference equation (1.1).

Here, we investigate the stability character of the positive equilibrium point of Eq.(1.1). From the definition equilibrium point, we obtain that

$$\begin{aligned} \bar{\delta} &= \omega + \zeta \frac{f(\bar{\delta}, \bar{\delta})}{\bar{\delta}^{\beta}} \\ &= \omega + \zeta \frac{\bar{\delta}^{\beta} f(1,1)}{\bar{\delta}^{\beta}}. \end{aligned}$$

Thus, the positive equilibrium point is

$$\bar{\delta} = \omega + \zeta f(1,1).$$

Now, let's define the function $f : (0,\infty)^2 \to (0,\infty)$ by

$$f(u,v) = \omega + \zeta \frac{f(u,v)}{v^{\beta}}.$$

Thus, we obtain that

$$\frac{\partial f}{\partial u}(u,v) = \zeta \frac{f_u(u,v)v^{\beta}}{(v^{\beta})^2},$$

and

$$\frac{\partial f}{\partial v}(u,v) = \zeta \frac{f_v(u,v)v^{\beta} - \beta v^{\beta-1} f(u,v)}{(v^{\beta})^2}.$$

In the next theorem, the locally asymptotic stability for Eq.(1.1) will be investigated.

Theorem 3.1. *The equilibrium point of Eq.(1.1) $\bar{\delta} = \omega + \zeta f(1,1)$ is locally asymptotically stable if*

$$|f_u(1,1)| + |f_v(1,1) - \beta f(1,1)| < \left| \frac{\omega + \zeta f(1,1)}{\zeta} \right|.$$

Proof. Using Euler's Homogeneous Function Theorem, and from Corollary (2.1), we can easily obtain that

$$\begin{aligned} f_u(\bar{\delta},\bar{\delta}) &= \zeta \frac{f_u(\bar{\delta},\bar{\delta})\bar{\delta}^{\beta}}{\bar{\delta}^{2\beta}} \\ &= \zeta \frac{\bar{\delta}^{2\beta-1} f_u(1,1)}{\bar{\delta}^{2\beta}} \\ &= \zeta \frac{f_u(1,1)}{\bar{\delta}}, \end{aligned}$$

and

$$\begin{aligned} f_v(\bar{\delta},\bar{\delta}) &= \zeta \frac{f_v(\bar{\delta},\bar{\delta})\bar{\delta}^{\beta} - \beta v^{\beta-1} f(\bar{\delta},\bar{\delta})}{\bar{\delta}^{2\beta}} \\ &= \zeta \frac{\bar{\delta}^{2\beta-1} f_v(1,1) - \beta \bar{\delta}^{2\beta-1} f(1,1)}{\bar{\delta}^{2\beta}} \\ &= \zeta \frac{f_v(1,1) - \beta f(1,1)}{\bar{\delta}}. \end{aligned}$$

Hence, by using Clark Theorem, we obtain that

$$\left| \zeta \frac{f_u(1,1)}{\bar{\delta}} \right| + \left| \zeta \frac{f_v(1,1) - \beta f(1,1)}{\bar{\delta}} \right| < 1.$$

Since $\bar{\delta} = \omega + \zeta f(1,1)$, we find

$$\left| \zeta \frac{f_u(1,1)}{\omega + \zeta f(1,1)} \right| + \left| \zeta \frac{f_v(1,1) - \beta f(1,1)}{\omega + \zeta f(1,1)} \right| < 1,$$

and so

$$|f_u(1,1)| + |f_v(1,1) - \beta f(1,1)| < \left| \frac{\omega + \zeta f(1,1)}{\zeta} \right|.$$

The proof is completed. □

In the following theorem, the two periodic solutions of Eq.(1.1) will be examined.

Theorem 3.2. *Eq.(1.1) has the prime period two solution*

$$\ldots, \sigma, \mu, \sigma, \mu, \ldots$$

if and only if

$$\omega = \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} \tag{3.1}$$

where $\phi = \frac{\sigma}{\mu}$, $\phi \in \mathbb{R} - \{0, \pm 1\}$.

Proof. Suppose that Eq.(1.1) has a prime period two solution in the following form

$$\ldots, \sigma, \mu, \sigma, \mu, \ldots.$$

Let's define $\delta_{m-(2s+1)} = \sigma$ and $\delta_{m-2s} = \mu$ for $s = 0, 1, 2, \ldots$. From the definition of the periodicity, we can rewrite the following equalities

$$\sigma = \omega + \zeta \frac{f(\mu, \sigma)}{\sigma^{\beta}},$$

and

$$\mu = \omega + \zeta \frac{f(\sigma, \mu)}{\mu^{\beta}}.$$

Therefore, we obtain that

$$\sigma = \omega + \zeta \frac{\sigma^{\beta} f\left(\frac{\mu}{\sigma}, 1\right)}{\sigma^{\beta}} \Rightarrow \sigma = \omega + \zeta f\left(\frac{1}{\phi}, 1\right), \tag{3.2}$$

and

$$\mu = \omega + \zeta \frac{\sigma^{\beta} f\left(1, \frac{\mu}{\sigma}\right)}{\mu^{\beta}} \Rightarrow \mu = \omega + \zeta \phi^{\beta} f\left(1, \frac{1}{\phi}\right). \tag{3.3}$$

Now, by using the fact $\sigma - \phi\mu = 0$, we get

$$0 = \sigma - \phi\mu = \omega + \zeta f\left(\frac{1}{\phi}, 1\right) - \phi\left(\omega + \zeta \phi^{\beta} f\left(1, \frac{1}{\phi}\right)\right),$$

and so

$$\omega(1-\phi) = \zeta \phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - \zeta f\left(\frac{1}{\phi}, 1\right).$$

Therefore, we find

$$\omega = \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)}.$$

Thus, from (3.2) and (3.3), we obtain that

$$\begin{aligned} \sigma &= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta f\left(\frac{1}{\phi}, 1\right) \\ &= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - \phi f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \mu &= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \phi^{\beta} f\left(1, \frac{1}{\phi}\right) \\ &= \zeta \frac{\phi^{\beta} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)}. \end{aligned} \tag{3.5}$$

Secondly, suppose (3.1) holds. Let's choose the initial conditions

$$\delta_{-1} = \sigma \text{ and } \delta_0 = \mu,$$

where σ, μ defined as (3.2) and (3.3), respectively. Therefore, we see that

$$\begin{aligned}
\delta_1 &= \omega + \zeta \frac{f(\delta_0, \delta_{-1})}{\delta_{-1}^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \frac{f(\mu, \sigma)}{\sigma^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \frac{\sigma^{\beta} f\left(\frac{\mu}{\sigma}, 1\right)}{\sigma^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta f\left(\frac{1}{\phi}, 1\right) \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - \phi f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} = \sigma
\end{aligned}$$

and

$$\begin{aligned}
\delta_2 &= \omega + \zeta \frac{f(\delta_1, \delta_0)}{\delta_0^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \frac{f(\sigma, \mu)}{\mu^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \frac{\sigma^{\beta} f\left(1, \frac{\mu}{\sigma}\right)}{\mu^{\beta}} \\
&= \zeta \frac{\phi^{\beta+1} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} + \zeta \phi^{\beta} f\left(1, \frac{1}{\phi}\right) \\
&= \zeta \frac{\phi^{\beta} f\left(1, \frac{1}{\phi}\right) - f\left(\frac{1}{\phi}, 1\right)}{(1-\phi)} = \mu.
\end{aligned}$$

Then, by induction, we can obtain for all $m \geq 0$

$$\delta_{2m-1} = \sigma \text{ and } \delta_{2m} = \mu.$$

Hence, Eq.(1.1) has a prime period two solution. The proof is completed. □

In the following theorem, the three periodic solutions of Eq.(1.1) will be investigated.

Theorem 3.3. *Eq.(1.1) has the prime period three solution $\{\delta_m\}_{m=-1}^{\infty}$ where*

$$\delta_m = \begin{cases} \sigma, & \text{for } m = 3z - 1 \\ \mu, & \text{for } m = 3z \\ \rho, & \text{for } m = 3z + 1 \end{cases}, \ z = 0, 1, \ldots$$

if and only if

$$\begin{aligned}
\eta \left(\omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}}\right) &= \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}} \\
\psi \left(\omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}}\right) &= \omega + \zeta f(\eta, 1)
\end{aligned} \tag{3.6}$$

where $\eta = \frac{\mu}{\sigma}$ and $\psi = \frac{\rho}{\sigma}$, $\eta, \psi \in \mathbb{R} - \{0, \pm 1\}$.

Proof. Suppose that Eq.(1.1) has a prime period three solution in the following form

$$\ldots, \sigma, \mu, \rho, \sigma, \mu, \rho, \ldots.$$

From Eq.(1.1), we find

$$\sigma = \omega + \zeta \frac{f(\rho, \mu)}{\mu^{\beta}},$$

$$\mu = \omega + \zeta \frac{f(\sigma, \rho)}{\rho^{\beta}},$$

and

$$\rho = \omega + \zeta \frac{f(\mu, \sigma)}{\sigma^{\beta}}.$$

Since f is a homogeneous function with degree β, we find

$$\sigma = \omega + \zeta \frac{\sigma^{\beta} f(\psi, \eta)}{\mu^{\beta}} \Rightarrow \sigma = \omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}},$$

$$\mu = \omega + \zeta \frac{\sigma^{\beta} f(1, \psi)}{\rho^{\beta}} \Rightarrow \mu = \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}},$$

and

$$\rho = \omega + \zeta \frac{\sigma^{\beta} f(\eta, 1)}{\sigma^{\beta}} \Rightarrow \rho = \omega + \zeta f(\eta, 1).$$

Therefore, we can obtain that

$$\eta = \frac{\mu}{\sigma} = \frac{\omega + \zeta \frac{f(1,\psi)}{\psi^{\beta}}}{\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}},$$

and

$$\psi = \frac{\rho}{\sigma} = \frac{\omega + \zeta f(\eta, 1)}{\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}}.$$

Hence, we find

$$\eta \left(\omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}} \right) = \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}},$$

$$\psi \left(\omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}} \right) = \omega + \zeta f(\eta, 1).$$

Secondly, assume that (3.6) holds. Let's choose the initial values for all $\eta, \psi \in \mathbb{R} - \{0, 1\}$

$$\delta_{-1} = \omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}}$$

and

$$\delta_0 = \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}}.$$

Thus, we obtain that

$$\begin{aligned}
\delta_1 &= \omega + \zeta \frac{f(\delta_0, \delta_{-1})}{\delta_{-1}^{\beta}} \\
&= \omega + \zeta \frac{f\left(\omega + \zeta \frac{f(1,\psi)}{\psi^{\beta}}, \omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)}{\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)^{\beta}} \\
&= \omega + \zeta \frac{f\left(\eta\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right), \omega + \zeta \frac{f(\psi,\eta)}{\psi^{\beta}}\right)}{\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)^{\beta}} \\
&= \omega + \zeta \frac{\left(\omega + \zeta \frac{f(\psi,\eta)}{\psi^{\beta}}\right)^{\beta} f(\eta, 1)}{\left(\omega + \zeta \frac{f(\psi,\eta)}{\psi^{\beta}}\right)^{\beta}} \\
&= \omega + \zeta f(\eta, 1) = \rho,
\end{aligned}$$

$$\begin{aligned}
\delta_2 &= \omega + \zeta \frac{f(\delta_1, \delta_0)}{\delta_0^{\beta}} \\
&= \omega + \zeta \frac{f\left(\omega + \zeta f(\eta, 1), \omega + \zeta \frac{f(1,\psi)}{\psi^{\beta}}\right)}{\left(\omega + \zeta \frac{f(1,\psi)}{\psi^{\beta}}\right)^{\beta}} \\
&= \omega + \zeta \frac{f\left(\psi\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right), \eta\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)}{\left(\eta\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)^{\beta}} \\
&= \omega + \zeta \frac{\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)^{\beta} f(\psi, \eta)}{\left(\eta\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)^{\beta}} \\
&= \omega + \zeta \frac{f(\psi, \eta)}{\eta^{\beta}} = \sigma
\end{aligned}$$

and

$$\begin{aligned}
\delta_3 &= \omega + \zeta \frac{f(\delta_2, \delta_1)}{\delta_1^{\beta}} \\
&= \omega + \zeta \frac{f\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}, \omega + \zeta f(\eta, 1)\right)}{(\omega + \zeta f(\eta, 1))^{\beta}} \\
&= \omega + \zeta \frac{f\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}, \psi\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)}{\left(\psi\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)^{\beta}} \\
&= \omega + \zeta \frac{\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)^{\beta} f(1, \psi)}{\left(\psi\left(\omega + \zeta \frac{f(\psi,\eta)}{\eta^{\beta}}\right)\right)^{\beta}} \\
&= \omega + \zeta \frac{f(1, \psi)}{\psi^{\beta}} = \mu.
\end{aligned}$$

Then, by induction, we can obtain that for all $m \geq 0$

$$\delta_{3m+1} = \rho, \ \delta_{3m+2} = \sigma \text{ and } \delta_{3m+3} = \mu.$$

Hence, Eq.(1.1) has a prime period three solution. The proof is completed. □

4. Conclusions and suggestions

Mathematical models are of great importance in the natural sciences, including biology, ecology, engineering sciences and genetics (see [8], [9], [10]). Mathematical models are developed to explain a system, study the effects of its various components, and make predictions about their behavior. Discrete models treat time or system states as discrete. Mathematical models can be created with the help of difference equations. In this respect, each study in the field of difference equation theory is very valuable both in terms of its own importance and has applications in other disciplines.

In this work, we introduced a new general class of non-linear difference equations. we investigated the qualitative behavior of solutions of the introduced second-order non-linear difference equations. In other words, we dealt with the two periodic solutions, the three periodic solutions and the stability character of given difference equations. In particular, we obtained the periodic solutions using the new method. Finally, we obtained new sufficient conditions for local asymptotic stability for the given difference equations. We can say that the results we have obtained here have gathered and developed many previous studies under one roof. Contributing to the theory of difference equations introduced with the help of homogeneous functions [20–23, 26] in this article has been one of the main aims.

It can be suggested to those who do research in this field that research can be done in the equations established with the help of homogeneous functions. Difference equations created with these functions are very convenient and useful for researching general classes of difference equations.

In our future studies, we will aim to investigate some general classes of difference equations formed by homogeneous functions of different degrees.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] Elsayed, E. M.: *New method to obtain periodic solutions of period two and three of a rational difference equation*. Nonlinear Dynamics. **79** (1), 241-250 (2014).

[2] Moaaz, O.: *Comment on "new method to obtain periodic solutions of period two and three of a rational difference equation" [Nonlinear Dyn 79:241–250]*. Nonlinear Dyn. **88**, 1043-1049 (2017).

[3] Elaydi, S.: An introduction to difference equations. 3rd ed. Springer-Verlag. New York (2005).

[4] Kelley, W. G., Peterson, A. C.: Difference equations: An introduction with applications. Academic Press. New York (1991).

[5] Kocić, V., Ladas, G.: Global behavior of non-linear difference equations of higher-order with applications. Kluwer Academic Publishers. Dordrecht (1993).

[6] Levin, S. A., May, R. M.: *A note on difference-delay equations.* Theoretical Population Biology. **9** (2), 178-187 (1976).

[7] Mickens, R. E.: Difference equations, theory and applications. Van Nostrand Rheinhold. (1990).

[8] Allen, L. J. S.: An introduction to mathematical biology. Pearson/Prentice Hall. New Jersey (2007).

[9] Murray, J. D.: Mathematical biology I: An introduction. 3rd ed. Springer. (2002).

[10] Pielou, E. C.: An introduction to mathematical ecology. Wiley Interscience. New York (1969).

[11] Oztepe, G. S.: *An investigation on the Lasota-Wazewska model with a piecewise constant argument.* Hacettepe Journal of Mathematics and Statistics. **50** (5), 1500-1508 (2021).

[12] Abo-Zeid, R.: *Global attractivity of a higher-order difference equation.* Discrete Dynamics in Nature and Society. **2012**, 930410 (2012).

[13] Abo-Zeid, R.: *Global behavior of a higher-order difference equation.* Mathematica Slovaca. **64** (4), 931-940 (2014).

[14] Belhannache, F., Touafek, N., Abo-Zeid, R.: *Dynamics of a third-order rational difference equation.* Bulletin Mathematique de La Societe Des Sciences Mathematiques de Roumanie. **59(107)** (1), 13-22 (2016).

[15] Gumus, M.: *Global dynamics of solutions of a new class of rational difference equations.* Konuralp Journal of Mathematics. **7** (2), 380-387 (2019).

[16] Gumus, M.: *Analysis of periodicity for a new class of non-linear difference equations by using a new method.* Electron. J. Math. Anal. Appl. **8**, 109-116 (2020).

[17] Halim, Y., Touafek, N.,Yazlik, Y.: *Dynamic behavior of a second-order non-linear rational difference equation.* Turkish Journal of Mathematics. **39** (6), 1004-1018 (2015).

[18] Touafek, N., Halim, Y.: *Global attractivity of a rational difference equation.* Mathematical Sciences Letters. **2** (3), 161-165 (2013).

[19] Yalçınkaya, I.: *On the difference equation* $x_{n+1} = \alpha + x_{n-m}/x_n^k$. Discrete Dynamics in Nature and Society. **2008**, 805460 (2008).

[20] Moaaz, O., Chalishajar, D., Bazighifan, O.: *Some qualitative behavior of solutions of general class of difference equation.* Mathematics. **7**, 585 (2019).

[21] Moaaz, O.: *Dynamics of difference equation* $x_{n+1} = f(x_{n-l}, x_{n-k})$. Advances in Difference Equations. **2018**, 447 (2018).

[22] Moaaz, O., Mahjoub, H., Muhib, A.: *On the periodicity of general class of difference equations.* Axioms. 9, 75 (2020).

[23] Abdelrahman, M. A. E.: *On the difference equation* $z_{m+1} = f(z_m, z_{m-1}, \ldots, z_{m-k})$. Journal of Taibah University for Science. **13** (1), 1014-1021 (2019).

[24] Kulenović, M. R. S., Ladas, G.: Dynamics of second-order rational difference equations. Chapman & Hall/CRC. (2001).

[25] Border, K. C.: *Euler's theorem for homogeneous function.* Caltech Division of The Humanities and Social Sciences. **27**, 16-34 (2017).

[26] Boulouh, M., Touafek, N., Tollu, D. T.: *On the behavior of the solutions of an abstract system of difference equations*. Journal of Applied Mathematics and Computing. **68**, 2937-2969 (2022).

Affiliations

Mehmet Gümüş
Address: Zonguldak Bülent Ecevit University, Faculty of Science, Department of Mathematics, Farabi Campus, 67100, Zonguldak, TURKEY.
e-mail: m.gumus@beun.edu.tr
ORCID ID:0000-0002-7447-479X

Şeyma Irmak Eğilmez
Address: Zonguldak Bülent Ecevit University, Faculty of Science, Department of Mathematics, Farabi Campus, 67100, Zonguldak, TURKEY.
e-mail: e.seymairmak@gmail.com
ORCID ID:0000-0003-1781-5399