MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

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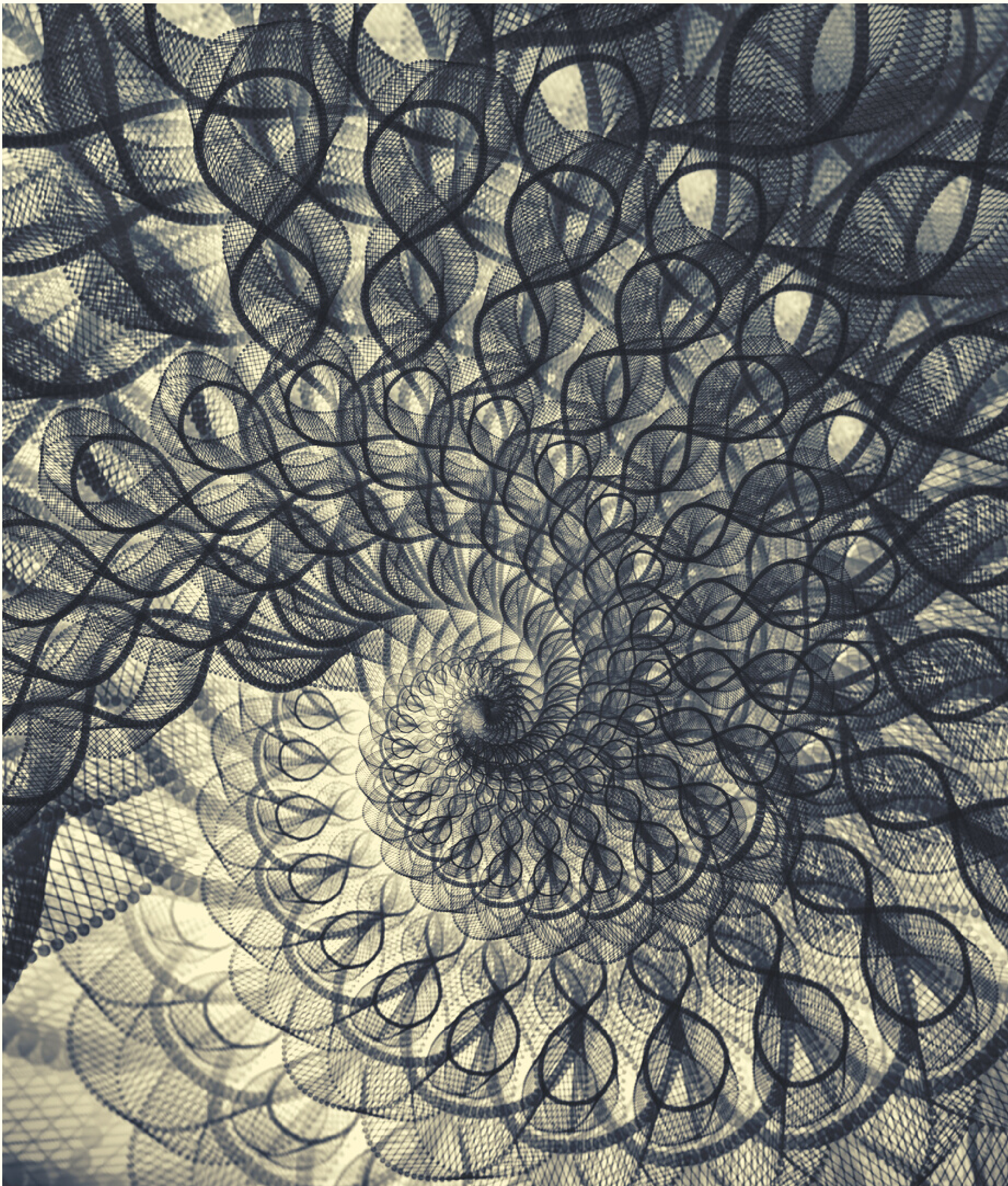
VOLUME XI - ISSUE II



2023-2

VOLUME XI ISSUE II
ISSN 2147-6268

June 2023
www.dergipark.org.tr/en/pub/mathenot

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Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753/mathenot.1117985
11 (2) 67-79 (2023) - Research Article
ISSN: 2147-6268

On the Algebra of Interval Vectors

Yılmaz Yılmaz, Halise Levent and Hacer Bozkurt*

Abstract

In this study, we examine some important subspaces by showing that the set of n-dimensional interval vectors is a quasilinear space. By defining the concept of dimensions in these spaces, we show that the set of n-dimensional interval vectors is actually a (n_r, n_s)-dimensional quasilinear space and any quasilinear space is $(n_r, 0_s)$-dimensional if and only if it is n-dimensional linear space. We also give examples of $(2_r, 0_s)$ and $(0_r, 2_s)$-dimensional subspaces. We define the concept of dimension in a quasilinear space with natural number pairs. Further, we define an inner product on some spaces and talk about them as inner product quasilinear spaces. Further, we show that some of them have Hilbert quasilinear space structure.

Keywords: *Quasilinear space; interval vectors; inner product quasilinear space; Hilbert quasilinear space.*

AMS Subject Classification (2020): *Primary: 46C50 ; Secondary: 06B99; 47H99; 46B99.*

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1. Introduction

Interval analysis is one of the main areas developed to determine the solutions of many problems in a certain compact interval. Modeling such situations sometimes emerges as a linear interval equation system and the solutions of such systems are often difficult. One of the main studies on the solutions of this type of equations is given by [1]. Another important fundamental work is [2]. Further, linear programming problems with incomplete information also appear as a system of linear interval equations, and some of the important studies on the solution of such problems were given by J.Rohn [3, 4]. The existence of the solution of linear interval equation systems or the determination of the properties of the solution set is also a difficult process, and the results obtained in [5, 6] are also important studies for this purpose. Moreover, [7] is another important work that examines the solubility of equations of this type based on some specific conditions. Since the solutions of such equations appear as interval vectors, it is important to know the properties of n-dimensional interval vectors and the algebraic structure of the set formed by these types of vectors. But, we know that the set of n-dimensional interval vectors is not a vector space. We can see this immediately for 1-dimensional interval vectors. The reverse of the shuffle between intervals may

Received : *17-05-2022,* ***Accepted :*** *10-08-2022*

*(**Cite as** "Y. Yılmaz, H. Levent, H. Bozkurt, On the Algebra of Interval Vectors, Math. Sci. Appl. E-Notes, 11(2) (2023), 67-79")*

not be available. First of all, let's specify that the interval $[0,0]=\{0\}$ is the unit element of the addition operation between intervals. But, we know that it is not possible to find an interval $[\underline{x},\overline{x}]$ such that $[1,0]+[\underline{x},\overline{x}]=[0,0]$. Although, the set of interval vectors does not have a vector space structure, it has an algebraic structure that we call quasilinear space, which is a generalization of vector spaces.

The concept of a quasilinear space on the field of real numbers was first introduced by Aseev in [8]. In this study, the normed quasilinear space and the finite quasilinear operator definitions defined between these types of spaces are also given and some properties are examined. However, in this study, the definition of subspace has not been characterized and there is no such definition as a quasilinear space or a quasi-stretch. Moreover, whether it is a generalization of a definition such as the linear dependence or independence of a subset in quasilinear space is not given in Aseev's pioneering work. In fact, the definition of these concepts is extremely vital for the establishment of a healthy quasilinear algebra. In our [9–12] referenced articles, we tried to eliminate some of these shortcomings in quasilinear algebra. Then we also introduced the concept of inner product in quasilinear spaces, and thus we were able to define the concept of Hilbert quasilinear space definition [13–16]. The introduction of these concepts also provides us with the opportunity to make many applications. For example, in [17, 18] we gave examples of how quasilinear spaces can be used in signal processing. In addition, normed and Hilbert quasilinear space examples of some fuzzy number sets are given in [19] and their properties are examined. A recent study on qasilinear spaces is the concept of quasi-algebra its details can be found in [20, 21].

In this study, we examine some important subspaces by showing that the set of n-dimensional interval vectors is a quasilinear space. By defining the concept of dimensions in these spaces, we show that the set of n-dimensional interval vectors is actually a (n_r,n_s)-dimensional quasilinear space and any quasilinear space is $(n_r,0_s)$-dimensional if and only if it is n-dimensional linear space. We also give examples of $(2_r,0_s)$ and $(0_r,2_s)$-dimensional subspaces. We define the concept of dimension in a quasilinear space with natural number pairs. Further, we define an inner product on some spaces and talk about them as inner product quasilinear spaces.

2. Preliminaries

Let us give basic facts on interval vectors from [22]. The term interval will mean closed interval $x=[\underline{x},\overline{x}]$ in this work and the left and right endpoints of x will be denoted by $\underline{x}$ and $\overline{x}$, respectively. We say that x is *degenerate* if $\underline{x}=\overline{x}$. The *width* of x is defined and denoted by $w(x)=\overline{x}-\underline{x}$ and the *absolute value* of x, denoted $|x|$, is the maximum of the absolute value of its endpoints: $|x|=|[\underline{x},\overline{x}]|=\max\{|\underline{x}|,|\overline{x}|\}$. The *midpoint* of x is given by $m(x)=\frac{1}{2}(\underline{x}+\overline{x})$. By an n-dimensional interval vectors, we mean an ordered n-tuble of intervals

$$x=(x_1,x_2,...x_n).=\left(\left[\underline{x_1},\overline{x_1}\right],...,\left[\underline{x_n},\overline{x_n}\right]\right).$$

For example, a two-dimensional interval vector

$$x=(x_1,x_2)=\left(\left[\underline{x_1},\overline{x_1}\right],\left[\underline{x_2},\overline{x_2}\right]\right)$$

can be represented as a rectangle in the plane. Addition of interval vectors is defined by coordinate-wise addition of intervals and the scalar real multiplication by an interval vectors is also similar. For example; if two-dimensional interval vectors $x=([-1,2],[3,6])$ and $y=([-1,2],[3,6])$ are given, then

$$\begin{aligned}2x-3y&=(2[-1,2],2[3,6])+((-3)[-1,2],(-3)[3,6])\\&=([-2,4],[6,12])+([-6,3],[-18,-9])\\&=([-8,7],[-12,3]).\end{aligned}$$

Note that the set of all n-dimensional interval vectors is not a vector space. $x\preceq y$ iff $x_k\subseteq y_k$ for each $k=1,2,...,n$ is a partial order relation on the set of all n-dimensional interval vectors. The set of all n-dimensional interval vectors is denoted by $\mathbb{I}^n_{\mathbb{R}}$.

The product of two intervals $x=[\underline{x},\overline{x}]$ and $y=\left[\underline{y},\overline{y}\right]$ is given by $xy=[\underline{x},\overline{x}]\left[\underline{y},\overline{y}\right]=[\min S,\max S]$ where $S=\{\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\}$.

Although we use the term n-dimensional, the algebraic meaning of this term should be questioned, since the set A is not a vector space. However, the set A has an algebraic structure, which we call quasilinear space, which is a generalization of classical vector spaces, first given by Aseev [8]. First, let's give the definition of quasilinear space.

A set X is called a quasilinear space, [8], on the field $\mathbb{K}$ of real or complex numbers, if a partial order relation "$\preceq$", an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for all elements $x,y,z,v\in X$ and all $\alpha,\beta\in\mathbb{K}$:

$$x\preceq x, \tag{2.1}$$

$$x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \tag{2.2}$$

$$x = y \text{ if } x \preceq y \text{ and } y \preceq x, \tag{2.3}$$

$$x + y = y + x, \tag{2.4}$$

$$x + (y + z) = (x + y) + z, \tag{2.5}$$

$$\text{there exists an element (zero) } \theta \in X \text{ such that } x + \theta = x, \tag{2.6}$$

$$\alpha(\beta x) = (\alpha\beta)x, \tag{2.7}$$

$$\alpha(x + y) = \alpha x + \alpha y, \tag{2.8}$$

$$1x = x, \tag{2.9}$$

$$0x = \theta, \tag{2.10}$$

$$(\alpha + \beta)x \preceq \alpha x + \beta x, \tag{2.11}$$

$$x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \tag{2.12}$$

$$\alpha x \preceq \alpha y \text{ if } x \preceq y. \tag{2.13}$$

Any linear space is a QLS with the partial order relation " $=$ ". Perhaps the most popular example of a nonlinear QLS on the field real numbers is $\mathbb{I}_{\mathbb{R}}^1$ with the inclusion relation "$\subseteq$".

Let us record some basic results from [8].

In a QLS X, the element θ is minimal, i.e., $x = \theta$ if $x \preceq \theta$. An element x' is called *inverse* of $x \in X$ if $x + x' = \theta$. The inverse is unique whenever it exists. An element x possessing inverse is called *regular*, otherwise is called *singular*.

Lemma 2.1. *[8] Suppose that each element x in QLS X has inverse element $x' \in X$. Then the partial order in X is determined by equality, the distributivity conditions hold, and consequently X is a linear space.*

In a real linear space, the equality is the only way to define a partial order such that the conditions (1)-(13) hold.

Let us give some assumption in quasilinear spaces. It will be assumed in what follows that $-x = (-1)x$. Note that the additive inverse x' may not be exists but if it exists then $x' = -x$. For example, the interval $[1, 2]$ is a singular element in $\mathbb{I}_{\mathbb{R}}^1$ since the inverse of the element $[1, 2]$ does not exists. However, $-[1, 2] = (-1)[1, 2] = [-2, -1] \in \mathbb{I}_{\mathbb{R}}^1$. Let us give an easy characterization of regular elements. An element x is regular in a QLS if and only if $x' = -x$, or equivalently, $x - x = \theta$. We should note that in a linear QLS, briefly in a linear space, each element is regular. Hence, the notions of regular and singular elements in linear spaces are redundant. Regular elements in $\mathbb{I}_{\mathbb{R}}^1$ is known as degenerate intervals and they are just the real numbers.

Definition 2.1. [10] Suppose that X is a QLS and $Y \subseteq X$. Then Y is called a *subspace of* X whenever Y is a QLS with the same partial order and the restriction to Y of the operations on X.

In [8] the concept of a subspace for a QLS was not defined. After detailed investigations we saw that the characterization of the definition is just the same as in linear subspaces.

Theorem 2.1. *[10] Y is a subspace of a QLS X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y \in Y$.*

Let Y be a subspace of a QLS X and suppose that each element x in Y has an inverse in Y. Then by Lemma 2.1 the partial order on Y is determined by the equality. In this case Y is a linear subspace of X.

An element x in X is said to be *symmetric* if $-x = x$ and X_{sym} denotes the set of all symmetric elements. In a linear QLS, equivalently, in a linear space zero is the only symmetric element. X_r and X_s stand for the set of all regular and singular elements with zero in X, respectively. Further, it can be easily shown that X_r, X_{sym} and X_s are subspaces of X. They are called *regular, symmetric* and *singular subspaces* of X, respectively. Regular subspace of X is a linear space while the singular subspace is a nonlinear QLS. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element.

3. Main results

Theorem 3.1. *$\mathbb{I}_{\mathbb{R}}^{n}$ is a quasilinear space by the partial order relation $x \preceq y$ iff $[\underline{x_k}, \overline{x_k}] \subseteq [\underline{y_k}, \overline{y_k}]$ for each $k = 1, 2, ..., n$.*

Proof. Most of the proof comes from the known result in interval analysis, see [22]. Let us only verify two axioms. The zero is $\theta = (0, 0, ...0). = ([0, 0], ..., [0, 0])$ in $\mathbb{I}_{\mathbb{R}}^{n}$ and if $x, y \in \mathbb{I}_{\mathbb{R}}^{n}$ and $\alpha, \beta \in \mathbb{R}$ then

$$\begin{aligned} (\alpha+\beta)x &= \left((\alpha+\beta)\left[\underline{x_1}, \overline{x_1}\right], ..., (\alpha+\beta)\left[\underline{x_n}, \overline{x_n}\right]\right) \\ &\preceq \left(\alpha\left[\underline{x_1}, \overline{x_1}\right], ..., \alpha\left[\underline{x_n}, \overline{x_n}\right]\right) + \left(\beta\left[\underline{x_1}, \overline{x_1}\right], ..., \beta\left[\underline{x_n}, \overline{x_n}\right]\right) \\ &= \alpha x + \beta x. \end{aligned}$$

Further, $x \preceq y$ means $[\underline{x_k}, \overline{x_k}] \subseteq [\underline{y_k}, \overline{y_k}]$ for each k and hence for every (positive or negative) $\alpha \in \mathbb{R}$, $\alpha [\underline{x_k}, \overline{x_k}] \subseteq \alpha [\underline{y_k}, \overline{y_k}]$. This implies $\alpha x \preceq \alpha y$. □

Example 3.1. The symmetric subspace of $\mathbb{I}_{\mathbb{R}}^{2}$ is $\left(\mathbb{I}_{\mathbb{R}}^{2}\right)_{sym} = \{([-a, a], [-b, b]) : a, b \in \mathbb{R}\}$. Further, the singular subspace of $\mathbb{I}_{\mathbb{R}}^{2}$ is just

$$\left(\mathbb{I}_{\mathbb{R}}^{2}\right)_{s} = \left\{\left(\left[\underline{x_1}, \overline{x_1}\right], \left[\underline{x_2}, \overline{x_2}\right]\right) : \underline{x_1} \neq \overline{x_1} \ \text{ or } \underline{x_2} \neq \overline{x_2}\right\} \cup \{([0, 0], [0, 0])\}$$

and the regular subspace is

$$\left(\mathbb{I}_{\mathbb{R}}^{2}\right)_{r} = \{([a, a], [b, b]) : a, b \in \mathbb{R}\} \equiv \{(\{a\}, \{b\}) : a, b \in \mathbb{R}\} \equiv \mathbb{R}^2.$$

Thus, we can see $\mathbb{R}^2$ as a regular subspace of $\mathbb{I}_{\mathbb{R}}^{2}$. The equivalence mentioned here means that there is a linear bijection and even an isometry when the normed space structure is introduced between these spaces. In general, $\mathbb{I}_{\mathbb{R}}^{n}$ has these special subspaces and we can see $\mathbb{R}^n \equiv (\mathbb{I}_{\mathbb{R}}^{n})_r$ is a linear part of $\mathbb{I}_{\mathbb{R}}^{n}$.

Definition 3.1. [8] In a QLS X, a real function $\|.\|_X : X \longrightarrow \mathbb{R}$ is called a *norm* if the following conditions hold:

$$\|x\|_X > 0 \text{ if } x \neq 0, \tag{3.1}$$

$$\|x + y\|_X \leq \|x\|_X + \|y\|_X, \tag{3.2}$$

$$\|\alpha x\|_X = |\alpha| \, \|x\|_X, \tag{3.3}$$

$$\text{if } x \preceq y, \text{ then } \|x\|_X \leq \|y\|_X, \tag{3.4}$$

$$\begin{gathered}\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \\ x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\|_X \leq \varepsilon \text{ then } x \preceq y.\end{gathered} \tag{3.5}$$

A quasilinear space X with a norm defined on it, is called *normed quasilinear* space (briefly, *normed QLS*). It follows from Lemma 2 that if any $x \in X$ has an inverse element $x' \in X$ then the concept of normed QLS coincides with the concept of real normed linear space. Hausdorff metric or norm metric on X is defined by the equality

$$h_X(x, y) = \inf\left\{r \geq 0 : x \preceq y + a_1^r,\, y \preceq x + a_2^r \text{ and } \|a_i^r\| \leq r, i = 1, 2\right\}.$$

Since $x \preceq y + (x - y)$ and $y \preceq x + (y - x)$, the quantity $h_X(x, y)$ is well-defined for any elements $x, y \in X$, and the function h_X satisfies all axioms of the metric. Further, $h_X(x, y)$ may not equal to $\|x - y\|_X$ if X is not a linear space, but always $h_X(x, y) \leq \|x - y\|_X$ for every $x, y \in X$ [8].

Example 3.2. A norm on $\mathbb{I}_{\mathbb{R}}^{n}$ is defined by

$$\|x\|_\infty = \max_{1 \leq k \leq n} |x_k| = \max_k \left\{\max\left\{\left|\underline{x_k}\right|, \left|\overline{x_k}\right|\right\}\right\}$$

where $k \in \{1, 2, ..., n\}$ and $|x_k|$ is the absolute value of the interval x_k. Another important norm on $\mathbb{I}_{\mathbb{R}}^{n}$ is

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} = \left(\sum_{k=1}^{n} \left\{\max\left\{\left|\underline{x_k}\right|, \left|\overline{x_k}\right|\right\}\right\}^2\right)^{1/2}$$

which is perhaps the most important one. This norm is the classical norm of $\mathbb{I}_{\mathbb{R}}^{n}$. To prove $\|.\|_2$ is a norm on $\mathbb{I}_{\mathbb{R}}^{n}$ let us only verify the last condition. Let $\varepsilon > 0$ be given and let $x = (x_1, x_2, ...x_n). = \left(\left[\underline{x_1}, \overline{x_1}\right], ..., \left[\underline{x_n}, \overline{x_n}\right]\right)$ and $y = (y_1, y_2, ...y_n). = \left(\left[\underline{y_1}, \overline{y_1}\right], ..., \left[\underline{y_n}, \overline{y_n}\right]\right) \in \mathbb{I}_{\mathbb{R}}^{n}$. Assume that there exists an element $x_\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, ..., x_n^\varepsilon). = \left(\left[\underline{x_1^\varepsilon}, \overline{x_1^\varepsilon}\right], ..., \left[\underline{x_n^\varepsilon}, \overline{x_n^\varepsilon}\right]\right) \in \mathbb{I}_{\mathbb{R}}^{n}$ such that $x \preceq y + x_\varepsilon$ and $\|x_\varepsilon\|_2 = \left(\sum_{k=1}^{n} |x_k^\varepsilon|^2\right)^{1/2} \leq \varepsilon$. This implies, for each $k \in \{1, 2, ..., n\}$, $\left[\underline{x_k}, \overline{x_k}\right] \subseteq \left[\underline{y_k}, \overline{y_k}\right] + \left[\underline{x_k^\varepsilon}, \overline{x_k^\varepsilon}\right]$ and $|x_k^\varepsilon| = \max\left\{\left|\underline{x_k^\varepsilon}\right|, \left|\overline{x_k^\varepsilon}\right|\right\} \leq \varepsilon$. Now for $\varepsilon \to 0$ we get $|x_k^\varepsilon| \to 0$ for each $k \in \{1, 2, ..., n\}$ and this means $\|x_\varepsilon\|_2 \to 0$ and hence $x_\varepsilon \to 0$ in $\mathbb{I}_{\mathbb{R}}^{n}$. Eventually, we get $x \preceq y$.

4. Quasilinear independence and basis

In this section, we will give some algebraic definitions [9, 11]. Let X be a QLS and $\{x_k\}_{k=1}^{n}$ be a subset of X where n is a positive integer. *A (linear) combination of the set* $\{x_k\}_{k=1}^{n}$ is an element z of X in the form

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = z$$

where the coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ are real scalars. On the other hand, *a* quasilinear *combination of the set* $\{x_k\}_{k=1}^{n}$ is an element $z \in X$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \preceq z$$

for some real scalars $\alpha_1, \alpha_2, ..., \alpha_n$. Hence, the quasilinear combination, briefly ql-combination, is defined by the partial order relation on X. In fact, the definition of linear combination in a QLS is also depend on the partial order relation and it can be defined as in the following form; a linear combination of the set $\{x_k\}_{k=1}^{n}$ is an element z of X such that

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \preceq z \text{ and } z \preceq \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n,$$

where the coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ are real scalars. In a linear QLS, this is the definition of the classical linear combination since the relation "$\preceq$" turns to the relation "$=$". Clearly, a linear combination of $\{x_k\}_{k=1}^{n}$, is a quasilinear combination of $\{x_k\}_{k=1}^{n}$, but not conversely. For any nonempty subset A of a QLS X, we know that the *span* of A is written by SpA and

$$SpA = \{\sum_{k=1}^{n} \alpha_k x_k : x_1, x_2, ..., x_n \in A,\ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R},\ n \in \mathbb{N}\}.$$

However, $QspA$, the *quasispan (q-span,* for short) *of* A, is defined by the set of all possible quasilinear combinations of A, that is,

$$\begin{aligned} QspA &= \{x \in X : \sum_{k=1}^{n} \alpha_k x_k \preceq x, \\ \text{for some } x_1, x_2, ..., x_n &\in A \text{ and for some scalars } \alpha_1, \alpha_2, ..., \alpha_n\}. \end{aligned}$$

Obviously, $SpA \subseteq QspA$. Further, $SpA = QspA$ for some linear QLS (linear space), hence, the notion of $QspA$ is redundant in linear spaces. Moreover, we say A *quasi spans* X whenever $QspA = X$.

Let us give an example from the quasilinear space of compact intervals.

Example 4.1. Let $X = \mathbb{I}_{\mathbb{R}}^{1}$ and take $A = \{[1, 3]\}$, a singleton in X. The q-span of A is

$$QspA = \{x \in \mathbb{I}_{\mathbb{R}}^{1} : \lambda [1, 3] \subseteq x,\ \lambda \in \mathbb{R}\}.$$

For example, $[2, 7] \in QspA$ since $2 [1, 3] \subseteq [2, 7]$ whereas $[2, 7] \notin SpA$ since there is no $\lambda \in \mathbb{R}$ satisfying $\lambda [1, 3] = [2, 7]$. Further, $[2, 3] \notin QspA$ since we cannot find any $\lambda \in \mathbb{R}$ satisfying the condition $\lambda [1, 3] \subseteq [2, 3]$. Clearly, $QspA \neq \mathbb{I}_{\mathbb{R}}^{1}$ Let $B = \{\{1\}\}$, another singleton in X. It consist of a regular element or degenerate interval. For any $x \in X$, clearly, we can write $\lambda.\{1\} \subseteq x$ for some $\lambda \in \mathbb{R}$. This means $QspB = X$. It can be easily shown that a singleton arising from nonzero regular element can quasispans X. A singular element cannot quasi spans X.

Theorem 4.1. *Let $A = \{x_1, x_2, ..., x_n\}$ be a subset of the QLS X. Then $QspA$ is a subspace of X.*

Definition 4.1. (*Quasilinear independence* and *dependence)* A set $A = \{x_1, x_2, ..., x_n\}$ in a QLS X is called *quasilinear independent (briefly ql-independent*) whenever the inequality

$$\theta \preceq \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n \tag{4.1}$$

holds if and only if $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$. Otherwise, A is called *quasilinear dependent (briefly ql-dependent).*

If we recall again that every linear space is a QLS with the relation "=", it can be seen that the notions of quasilinear independence and dependence coincide with linear independence and dependence.

Example 4.2. Consider the singleton $A = \{[1, 2]\}$ in $\mathbb{I}^1_{\mathbb{R}}$. It is obvious that $\{0\} = [0, 0] \subseteq \alpha[1, 2]$ if and only if $\alpha = 0$ where $\{0\}$ is the zero element of $\mathbb{I}^1_{\mathbb{R}}$. Therefore, A is ql-independent. However, the singleton $B = \{[-1, 2]\}$ is ql-dependent since $[0, 0] \subseteq \beta[-1, 2]$ for $\beta = 2 \neq 0$. This is a unusual case since a non-zero singleton is obviously linear independent in linear spaces. On the other hand, the set $\{[1, 2], [-1, 2]\}$ is ql-dependent. In general, we can see from the definition that any subset including an element related to zero must be ql-dependent in a QLS. This is a generalization of the well-known fact that a subset including zero must be linear independent in linear spaces.

Example 4.3. In $\mathbb{I}^2_{\mathbb{R}}$, let $v_1 = ([-2, 1], [0, 0])$ and $v_2 = ([0, 0], [-2, 3])$. Then the set $\{v_1, v_2\}$ is ql-dependent since

$$([0, 0], [0, 0]) \subseteq \lambda_1 v_1 + \lambda_2 v_2 = ([-2, 1], [-2, 3])$$

for $\lambda_1 = \lambda_2 = 1$ where $([0, 0], [0, 0])$ is the zeros of $\mathbb{I}^2_{\mathbb{R}}$. However, $\{u_1, u_2\}$ is ql-independent where $u_1 = ([-2, -1], [0, 0])$ and $u_2 = ([0, 0], [2, 3])$. On the other hand, let $u = ([-2, 2], [-3, 3])$ then the singleton $\{u\}$ is ql-dependent in $\mathbb{I}^2_{\mathbb{R}}$ since $([0, 0], [0, 0]) \subseteq u$.

Definition 4.2. A ql-independent subset A of a QLS X which quasi spans X is called *a basis (or Hamel basis)* for X.

Remark 4.1. For any $a \in \mathbb{R}$, the singleton $\{\{a\}\}$ is a basis for $\mathbb{I}^1_{\mathbb{R}}$. Further, $B = \{([1, 1], [0, 0]), ([0, 0], [1, 1])\}$ is a basis for $\mathbb{I}^2_{\mathbb{R}}$. In general, $B = \{([1, 1], [0, 0], ..., [0, 0]), ..., ([0, 0], [0, 0], ..., [1, 1])\}$ is a basis for $\mathbb{I}^n_{\mathbb{R}}$. As can be seen, a basis of $\mathbb{I}^n_{\mathbb{R}}$ is a set of degenerate intervals of $\mathbb{I}^n_{\mathbb{R}}$.

Following example is extraordinary since it presents an example of QLS which has no basis. This is an unusual case since all linear spaces have a (Hamel) basis.

Example 4.4. Let us consider singular subspace

$$\{\{0\}\} \cup \{[a, b] : a < b \text{ and } a, b \in \mathbb{R}\} = (\mathbb{I}^1_{\mathbb{R}})_s$$

of $\mathbb{I}^1_{\mathbb{R}}$. This quasilinear space has no basis. Any singleton $\{[a, b]\}$ in $(\mathbb{I}^1_{\mathbb{R}}))_s$ cannot quasi spans $(\mathbb{I}^1_{\mathbb{R}})_s$ where $a < b$.

Now let us introduce the notion of dimension of a QLS. Our investigation shows that it is necessary to split it into two different notion as regular and singular dimension. Previously, let us give analog of a classical definition.

Definition 4.3. Let S be a ql-independent subset of the QLS X. S is called *maximal ql-independent subset* of X whenever S is ql-independent, but any superset of S is ql-dependent.

Definition 4.4. *Regular (Singular) dimension* of any QLS X is the cardinality of any maximal ql-independent subsets of $X_r(X_s)$. If this number is finite then X is said to be finite regular (singular)-dimensional, otherwise; is said to be infinite regular (singular)-dimensional. Regular dimension is denoted by r-$\dim X$ and singular dimension is denoted by s-$\dim X$. If r-$\dim X = a$ and s-$\dim X = b$ then we say that X *is an* (a_r, b_s)*-dimensional QLS* where a and b are natural numbers or ∞.

Remark 4.2. The above definition means that r-$\dim X$ is classical definition of dimension of the linear space X_r. So, r-$\dim X = \dim X_r$. Notice that a non-trivial singular subspace of a QLS cannot be a linear space. Further, we can easily see that any QLS is $(n_r, 0_s)$-dimensional if and only if it is n-dimensional linear space. In this respect, the trivial linear space $\{0\}$ is a $(0_r, 0_s)$-dimensional QLS. Later, we will give an example of a $(0_r, 0_s)$-dimensional QLS other than $\{0\}$.

Let us determine dimensions of some nonlinear QLSs.

Example 4.5. It isn't hard to prove that $\mathbb{I}^n_{\mathbb{R}}$ is (n_r, n_s)-dimensional QLSs, that is, n-dimensional nonlinear QLS. Consider again the singular subspace $(\mathbb{I}^1_{\mathbb{R}})_s$ of $\mathbb{I}^1_{\mathbb{R}}$. $r - \dim (\mathbb{I}^1_{\mathbb{R}})_s = 0$ since $((\mathbb{I}^1_{\mathbb{R}})_s)_r = \{0\}$. Further, $\{[1, 2]\}$ is ql-independent in $((\mathbb{I}^1_{\mathbb{R}})_s)_s$ and so $s - \dim (\mathbb{I}^1_{\mathbb{R}})_s = 1$. Hence, $(\mathbb{I}^1_{\mathbb{R}})_s$ is $(0_r, 1_s)$ $-$dimensional. Obviously, $\left(\mathbb{I}^1_{\mathbb{R}}\right)_r$ is $(1_r, 0_s)$ $-$dimensional. In this respect, $\mathbb{R}$ is also $(1_r, 0_s)$ $-$dimensional

If $X = (\mathbb{I}^2_{\mathbb{R}})_s \cup \{([t, t], [0, 0]) : t \in \mathbb{R}\}$ then X is a subspace of $\mathbb{I}^2_{\mathbb{R}}$ and $r - \dim X = 1$ since $X_r = \{([t, t], [0, 0]) : t \in \mathbb{R}\}$. Further, the set $\{u_1, u_2\}$ in Example 4.3 is ql-independent. This proves $s - \dim X = 2$. Hence X is a $(1_r, 2_s)$ $-$dimensional QLS.

Consider the QLS $X = \Omega_C(c_0)$, the set of all closed bounded subsets of the Banach space c_0. Regular subspace X_r is equivalent to c_0, the linear space of all sequences convergent to zero, and so $r - \dim X = \infty$. Let us define the set

$$\begin{aligned}\{\{(t,0,0,...) &: 1 \leq t \leq 4\}, \{(0,t,0,...) : 1 \leq t \leq 4\}, ...\} \\ &= \{[1,4] \odot e_1, [1,4] \odot e_2, ...\}\end{aligned}$$

where

$$[1,4] \odot e_k = \{(0,...,0, \overset{k.\ term}{s}, 0...) : s \in [1,4]\}$$

is ql-independent in X_s, where e_k's are coordinate vectors of c_0, $k = 1, 2, ...$.Therefore, $s - \dim X = \infty$ and so $X = \Omega_C(c_0)$ is an $(\infty_r : \infty_s)$-dimensional QLS. In general, an infinite-dimensional linear space E is a $(\infty_r, 0_s)$-dimensional QLS while $\Omega_C(E)$ is (∞_r, ∞_s)-dimensional QLS.

In a finite dimensional linear space X let us recall that each $x \in X$ has a unique representation

$$x = \sum_{k=1}^{n} a_k b_k$$

where n is the dimension of X, $B = \{b_1, b_2, ..., b_n\}$ is a basis of X and $a_1, a_2, ..., a_n$ are corresponding scalars. Since a consolidate QLS has a basis we can give a similar representation. Let X be a $(n_r : n_s)$ $-$dimensional (finite-dimensional) QLS where n_r and n_s are positive integers, and $n_r = n_s$. Let us try to give a representation in X. If y is any element of X then the *floor* $F_y = \{x \in X_r : x \preceq y\}$ of y have many regular elements. From linear algebra any $x \in F_y$ has a unique representation

$$x = \sum_{k=1}^{n} \alpha_k^x b_k$$

where each α_k^x, $k = 1, 2, ..., n$, is a real scalar depending on x. Now let us consider the supremum with respect to the partial order relation "$\preceq$" on the QLS X. Thus, by the definition of consolidate space, we get the representation

$$y = \sup\{x \in X_r : x \preceq y\} = \sup\{\sum_{k=1}^{n} \alpha_k^x b_k : x \preceq y,\ x \in X_r\}$$

of each element y in X. That is, any element of a (nonlinear) consolidate QLS can be represented by the basis elements and by the supremum with respect to "$\preceq$". More practically, we can write

$$y = \sup_{\substack{x \preceq y \\ x \in X_r}} \sum_{k=1}^{n} \alpha_k^x b_k. \tag{4.2}$$

Theorem 4.2. *Any $y \in \mathbb{I}_{\mathbb{R}}^n$ has a unique representation*

$$y = \sup_{\substack{x \preceq y \\ x \in X_r}} \sum_{k=1}^{n} \alpha_k^x b_k$$

where $B = \{b_k\}_{k=1}^n = \{([1,1], [0,0], ..., [0,0]), ..., ([0,0], [0,0], ..., [1,1])\}$ is the standard basis of $\mathbb{I}_{\mathbb{R}}^n$ and the supremum is calculated by the partial order "$\preceq$" on $\mathbb{I}_{\mathbb{R}}^n$.

Proof. Let us first write y explicitly;

$$y = (y_1, y_2, ..., y_n) = \left(\left[\underline{y_1}, \overline{y_1}\right], ..., \left[\underline{y_n}, \overline{y_n}\right]\right).$$

Now take an arbitrary $t_k \in y_k$ and constitute the degenerate interval $[t_k, t_k]$. Obviously, $[t_k, t_k] \subseteq y_k$ for each k and hence, $([t_k, t_k]) \preceq (y_k) = y$. Since $t = (t_k) \in \mathbb{R}^n$ has a unique representation $t = \sum_{k=1}^{n} t_k e_k$, we can say

$([t_k, t_k]) := x \in (\mathbb{I}_{\mathbb{R}}^n)_r$ has the unique representation $x = ([t_k, t_k]) = \sum_{k=1}^{n} t_k b_k$. On the other hand, for any y_k there may be a lot of $t_k \in y_k$. In other words, there may be a lot of $[t_k, t_k] \subseteq y_k$. You can easily see that, for each k,

$$\begin{aligned} y_k &= \sup_{\subseteq} \{[t_k, t_k] : [t_k, t_k] \subseteq y_k\} \\ &= \sup_{\subseteq} \left\{ \sum_{k=1}^{n} t_k b_k : [t_k, t_k] \subseteq y_k \right\}. \end{aligned}$$

This representation is still unique by the properties of the suprema by the partial order relation $\subseteq$. Thus, we get

$$\begin{aligned} y &= (y_k)_{k=1}^n = \left(\sup_{\subseteq} \left\{ \sum_{k=1}^{n} t_k b_k : [t_k, t_k] \subseteq y_k \right\} \right)_{k=1}^n \\ &= \sup_{\preceq} \left\{ \sum_{k=1}^{n} t_k^x b_k : ([t_k, t_k])_{k=1}^n = x \preceq y = (y_k)_{k=1}^n \right\}. \end{aligned}$$

The last supremum, of course, is taken over "$\preceq$" relation on $\mathbb{I}_{\mathbb{R}}^n$ and the representation is obviously unique. □

The representation is also known as the *super position* of y in $\mathbb{I}_{\mathbb{R}}^n$.

Example 4.6. Let us give the super position of $y = ([-1, 3], [2, 2])$ in $\mathbb{I}_{\mathbb{R}}^2$ where $y_1 = [-1, 3]$ and $y_2 = [2, 2]$. By the discussion in the proof

$$y_1 = [-1, 3] = \sup_{\subseteq} \{[t, t] : [t, t] \subseteq y_1\} = \sup_{\subseteq} \{[t, t] : t \in [-1, 3]\}$$

and

$$\begin{aligned} y_2 &= [2, 2] = \sup_{\subseteq} \{[t, t] : [t, t] \subseteq y_2\} = \sup_{\subseteq} \{[t, t] : t \in [2, 2]\} \\ &= \sup_{\subseteq} \{[2, 2]\} = [2, 2]. \end{aligned}$$

Hence,

$$\begin{aligned} y &= ([-1, 3], [2, 2]) = \left(\sup_{\subseteq} \left\{ \sum_{k=1}^{2} t_k b_k : [t_k, t_k] \subseteq y_k \right\} \right)_{k=1}^2 \\ &= \sup_{\subseteq} \{t_1 ([1, 1], [0, 0]) + t_2 ([0, 0], [1, 1]) : [t_k, t_k] \subseteq y_k, k = 1, 2\} \\ &= \sup_{\preceq} \left\{ \begin{array}{c} [t_1, t_1] ([1, 1], [0, 0]) + [t_2, t_2] ([0, 0], [1, 1]) \\ : x = ([t_1, t_1], [t_2, t_2]) \preceq y \end{array} \right\}. \end{aligned}$$

Definition 4.5. A quasilinear space X is called *consolidate (solid-floored) QLS* whenever $y = \sup\{x \in X_r : x \preceq y\}$ for each $y \in X$. Otherwise, X is called *a non-consolidate QLS*, briefly, *nc-QLS*.

The supremum in this definition is taken on the order relation "$\preceq$" in the definition of a QLS. Above definition assumes $\sup\{x \in X_r : x \preceq y\}$ exists for each $y \in X$. Implicitly, we say that X is consolidate if and only if $y = \sup F_y$, for each $y \in X$.

We signify that any linear space is a consolidate QLS: Indeed, $X_r = X$ for any linear space X and so

$$y = \sup\{x \in X_r : x \preceq y\} = \sup\{x \in X_r : x = y\} = \sup\{y\} = y$$

for any element y in X.

Example 4.7. $\mathbb{I}_{\mathbb{R}}^n$ is a consolidate QLS. Singular subspace of $\mathbb{I}_{\mathbb{R}}^1$ is a nc-QLS since $F_y = \emptyset$ for the element $y = [1, 2]$ in $(\mathbb{I}_{\mathbb{R}}^1)_s$. Further,

$$\mathcal{B} = \{[a, b] : a \leq 0 \leq b, \ \ a, b, 0 \in \mathbb{R}\}$$

is another nc-subspace of $\mathbb{I}_{\mathbb{R}}^1$. $(\mathbb{I}_{\mathbb{R}}^1)_{sym}$ is also a nc-QLS subspace of $\mathcal{B}$ and of $\mathbb{I}_{\mathbb{R}}^1$.

Definition 4.6. For two quasilinear spaces $(X, \leq)$ and $(Y, \preceq)$, Y is called compatible contains X whenever $X \subseteq Y$ and the partial order relation $\leq$ on X is the restriction of the partial order relation $\preceq$ on Y. We briefly use the symbol $X \overline{\subseteq} Y$ in this case. We write $X \equiv Y$ whenever $X \overline{\subseteq} Y$ and $Y \overline{\subseteq} X$.

Remark 4.3. Hence $X \equiv Y$ means X and Y are the same sets with the same partial order relations which make one each quasilinear space. However, we may write $X = Y$ for $X \equiv Y$ whenever the relations are clear from context.

Definition 4.7. Let X be a QLS. *Consolidation of* X is the smallest consolidate QLS $\widehat{X}$ which compatible contains X, that is, if there exists another consolidate QLS Y which compatible contains X then $\widehat{X} \subseteq Y$.

Clearly, $\widehat{X} = X$ for some consolidate QLS X. Whether each QLS has a consolidation is not know yet. This notion is unnecessary for consolidate QLSs, hence is in linear spaces.

Theorem 4.3. *Consolidation of* $(\mathbb{I}^1_{\mathbb{R}})_s$ *is* $\mathbb{I}^1_{\mathbb{R}}$.

Proof. Obviously, $\mathbb{I}^1_{\mathbb{R}}$ compatible contains $(\mathbb{I}^1_{\mathbb{R}})_s$. Suppose that Z is another consolidate QLS containing $(\mathbb{I}^1_{\mathbb{R}})_s$. For an arbitrary element x of $\mathbb{I}^1_{\mathbb{R}}$ we will show that $x \in Z$. If $x \in (\mathbb{I}^1_{\mathbb{R}})_s$ then the proof is clear. If $x \notin (\mathbb{I}^1_{\mathbb{R}})_s$ then x have to be a degenerate interval that is an element of $(\mathbb{I}^1_{\mathbb{R}})_r$. Hence, $x = [a, a]$ for an $a \in \mathbb{R}$. Assume that $[a, a] \notin Z$. For any $\varepsilon > 0$ we have that $[a - \varepsilon, a + \varepsilon] \in (\mathbb{I}^1_{\mathbb{R}})_s$ and so $[a - \varepsilon, a + \varepsilon] \in Z$. Since Z is consolidate,

$$[a - \varepsilon, a + \varepsilon] = \sup\{y \subseteq [a - \varepsilon, a + \varepsilon] : y \in Z_r\}$$

for any $\varepsilon > 0$. This means there exists an element $u_\varepsilon \in Z_r$ such that $u_\varepsilon \subseteq [a - \varepsilon, a + \varepsilon]$ in Z. Therefore, we have $[a, a] \in Z_r$, otherwise; the set $[a - \varepsilon, a + \varepsilon]$ cannot be a closed set in $\mathbb{R}$ and so this conflicts with the fact that $[a - \varepsilon, a + \varepsilon] \in (\mathbb{I}^1_{\mathbb{R}})_s$. Thus, the assumption $[a, a] \notin Z$ is incorrect. □

For any element y of a QLS X, the set

$$F_y^{\widehat{X}} = \left\{z \in \left(\widehat{X}\right)_r : z \preceq y\right\}$$

denotes the floor of y in $\widehat{X}$ and sometimes $F_y^{\widehat{X}}$ is said to be the *floor of* y *in the consolidation.* For a consolidate QLS, this notion is unnecessary. But the concept is important in a nc-QLS, especially, in producing of an inner-product on a QLS.

Definition 4.8. Let X be a quasilinear space having a consolidation $\widehat{X}$. A mapping $\langle\, ,\, \rangle : X \times X \to \Omega(\mathbb{K})$ is called an inner product on X if for any $x, y, z \in X$ and $\alpha \in \mathbb{K}$ the following conditions are satisfied :

$$\text{If } x, y \in X_r \text{ then } \langle x, y\rangle \in \Omega(\mathbb{K})_r \equiv \mathbb{K}, \tag{4.3}$$

$$\langle x + y, z\rangle \subseteq \langle x, z\rangle + \langle y, z\rangle\,, \tag{4.4}$$

$$\langle \alpha x, y\rangle = \alpha \langle x, y\rangle\,, \tag{4.5}$$

$$\langle x, y\rangle = \langle y, x\rangle\,, \tag{4.6}$$

$$\langle x, x\rangle \geq 0 \text{ for } x \in X_r \text{ and } \langle x, x\rangle = 0 \Leftrightarrow x = 0, \tag{4.7}$$

$$\|\langle x, y\rangle\|_{\Omega(\mathbb{R})} = \sup\left\{\|\langle a, b\rangle\|_{\Omega(\mathbb{R})} : a \in F_x^{\widehat{X}}, b \in F_y^{\widehat{X}}\right\}, \tag{4.8}$$

$$\text{if } x \preceq y \text{ and } u \preceq v \text{ then } \langle x, u\rangle \subseteq \langle y, v\rangle\,, \tag{4.9}$$

$$\begin{gathered}\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that}\\ x \preceq y + x_\varepsilon \text{ and } \langle x_\varepsilon, x_\varepsilon\rangle \subseteq S_\varepsilon(\theta) \text{ then } x \preceq y,\end{gathered} \tag{4.10}$$

where $\mathbb{K}$ is real or complex field and $\Omega(\mathbb{K})$ denotes the quasilinear space of the family of all compact subsets of $\mathbb{K}$. Further $S_\varepsilon(\theta)$ is the zero-centered ε-radius closed circle in $\mathbb{K}$. A quasilinear space with an inner product is called an inner product quasilinear space, briefly, IPQLS.

Theorem 4.4. *For* $x = \left(\left[\underline{x_1}, \overline{x_1}\right], ..., \left[\underline{x_n}, \overline{x_n}\right]\right)$ *and* $y = \left(\left[\underline{y_1}, \overline{y_1}\right], ..., \left[\underline{y_n}, \overline{y_n}\right]\right) \in \mathbb{I}_{\mathbb{R}}^n$*, the equality*

$$\langle x, y\rangle = \sum_{k=1}^{n} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{y_k}, \overline{y_k}\right]$$

defines an inner-product and hence $\mathbb{I}_{\mathbb{R}}^n$ *is an IPQLS on the field* $\mathbb{R}$ *by this inner product.*

Proof. Let $x, y \in \left(\mathbb{I}_{\mathbb{R}}^n\right)_r$ then $x = ([x_1, x_1], ..., [x_n, x_n])$ and $y = ([y_1, y_1], ..., [y_n, y_n])$. So,

$$\begin{aligned} \langle x, y\rangle &= \sum_{k=1}^{n} [x_k, x_k]\, [y_k, y_k] \\ &= \sum_{k=1}^{n} \{x_k y_k\} \in \Omega(\mathbb{R})_r \equiv \mathbb{R}. \end{aligned}$$

Later three condition can be easily verified. Now for $x \in \left(\mathbb{I}_{\mathbb{R}}^n\right)_r$, $\langle x, x\rangle = \sum_{k=1}^{n} \{x_k x_k\} = \sum_{k=1}^{n} \left\{|x_k|^2\right\} \in \Omega(\mathbb{R})_r \equiv \mathbb{R}$ and so we can write $\langle x, x\rangle \geq 0$. Easily we can see that $\langle x, x\rangle = 0 \Leftrightarrow x = 0$. Let us now verify the equality

$$\|\langle x, y\rangle\|_{\Omega(\mathbb{R})} = \sup\left\{\|\langle a, b\rangle\|_{\Omega(\mathbb{R})} : a \in F_x^{\widehat{X}}, b \in F_y^{\widehat{X}}\right\}$$

where $X = \mathbb{I}_{\mathbb{R}}^n$. Since X is consolidate $\widehat{X} = X$ and

$$\begin{aligned} \|\langle x, y\rangle\|_{\Omega(\mathbb{R})} &= \left\|\sum_{k=1}^{n} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{y_k}, \overline{y_k}\right]\right\|_{\Omega(\mathbb{R})} \\ &= \left\|\sum_{k=1}^{n} \sup_{\subset} \left\{\langle [t_k, t_k], [s_k, s_k]\rangle : [t_k, t_k] \subset \left[\underline{x_k}, \overline{x_k}\right], [s_k, s_k] \subset \left[\underline{y_k}, \overline{y_k}\right]\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \left\|\sum_{k=1}^{n} \sup_{\subset} \left\{\langle [t_k, t_k], [s_k, s_k]\rangle : [t_k, t_k] \in F_{\left[\underline{x_k}, \overline{x_k}\right]}^{\mathbb{I}_{\mathbb{R}}^1}, [s_k, s_k] \in F_{\left[\underline{y_k}, \overline{y_k}\right]}^{\mathbb{I}_{\mathbb{R}}^1}\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \left\|\sup\left\{\sum_{k=1}^{n} \langle [t_k, t_k], [s_k, s_k]\rangle : [t_k, t_k] \in F_{\left[\underline{x_k}, \overline{x_k}\right]}^{\mathbb{I}_{\mathbb{R}}^1}, [s_k, s_k] \in F_{\left[\underline{y_k}, \overline{y_k}\right]}^{\mathbb{I}_{\mathbb{R}}^1}\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \left\|\sup\left\{\langle a, b\rangle : a \in F_x^{\mathbb{I}_{\mathbb{R}}^n}, b \in F_y^{\mathbb{I}_{\mathbb{R}}^n}\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \sup\left\|\left\{\langle a, b\rangle : a \in F_x^{\mathbb{I}_{\mathbb{R}}^n}, b \in F_y^{\mathbb{I}_{\mathbb{R}}^n}\right\}\right\|_{\Omega(\mathbb{R})} \\ &= \sup\left\{\|\langle a, b\rangle\| : a \in F_x^{\mathbb{I}_{\mathbb{R}}^n}, b \in F_y^{\mathbb{I}_{\mathbb{R}}^n}\right\} \end{aligned}$$

were $a = ([t_1, t_1], [t_2, t_2], ..., [t_n, t_n])$ and $b = ([s_1, s_1], [s_2, s_2], ..., [s_n, s_n])$ are degenerate interval vectors obeying the above equality chain. Now let us only verify the last axiom of the inner product. Let us assume that for any $\varepsilon > 0$ there exists an element $x_\varepsilon = \left(\left[\underline{x_{1_\varepsilon}}, \overline{x_{1_\varepsilon}}\right], ..., \left[\underline{x_{n_\varepsilon}}, \overline{x_{n_\varepsilon}}\right]\right) \in \mathbb{I}_{\mathbb{R}}^n$ such that

$$x = \left(\left[\underline{x_1}, \overline{x_1}\right], ..., \left[\underline{x_n}, \overline{x_n}\right]\right) \preceq y = \left(\left[\underline{y_1}, \overline{y_1}\right], ..., \left[\underline{y_n}, \overline{y_n}\right]\right) + x_\varepsilon$$

and $\langle x_\varepsilon, x_\varepsilon\rangle \subseteq S_\varepsilon(\theta)$. This implies, for each $k \in \{1, 2, ..., n\}$, $[x_k, x_k] \subseteq [y_k, y_k] + \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right]$. Since

$$\langle x_\varepsilon, x_\varepsilon\rangle = \sum_{k=1}^{n} \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] \subseteq S_\varepsilon(\theta),$$

we get

$$\left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] = \left\langle \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right], \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right]\right\rangle \subseteq S_\varepsilon(\theta)$$

for each k. Since $\varepsilon \to 0$ implies $\|S_\varepsilon(\theta)\|_{\Omega(\mathbb{R})} \to 0$, we obtain

$$\left\langle \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right], \left[\underline{x_{k_\varepsilon}}, \overline{x_{k_\varepsilon}}\right] \right\rangle \to \{0\}$$

in $\Omega(\mathbb{R})$. This brings us $[x_k, x_k] \subseteq [y_k, y_k]$ for each k. Eventually we can say $x \preceq y$.
Verification of remaining axioms are easy. $\square$

Let us see verification of the condition (26) in the proof by an easy example in $\mathbb{I}^n_{\mathbb{R}}$ in order to well-understanding of the condition.

Example 4.8. Let us consider $x = ([-3,3], [2,5])$, $y = ([-1,3], [2,2])$ in $\mathbb{I}^2_{\mathbb{R}}$.

$$\begin{aligned}
\|\langle x, y\rangle\|_{\Omega(\mathbb{R})} &= \left\| \sum_{k=1}^{2} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{y_k}, \overline{y_k}\right] \right\|_{\Omega(\mathbb{R})} = \|[-3,3][-1,3] + [2,5][2,2]\|_{\Omega(\mathbb{R})} \\
&= \left\| \begin{array}{c} \sup\{[t,t][s,s] : [t,t] \subset [-3,3], [s,s] \subset [-1,3]\} \\ + \sup\{[t,t][s,s] : [t,t] \subset [2,5], [s,s] \subset [2,2]\} \end{array} \right\|_{\Omega(\mathbb{R})} \\
&= \left\| \begin{array}{c} \sup\left\{[t,t][s,s] : [t,t] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[-3,3]}, [s,s] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[-1,3]}\right\} \\ + \sup\left\{[t,t][s,s] : [t,t] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[2,5]}, [s,s] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[2,2]}\right\} \end{array} \right\|_{\Omega(\mathbb{R})} \\
&= \left\| \begin{array}{c} \sup\{\left\{[t,t][s,s] : [t,t] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[-3,3]}, [s,s] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[-1,3]}\right\} \\ + \left\{[t,t][s,s] : [t,t] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[2,5]}, [s,s] \in F^{\mathbb{I}^1_{\mathbb{R}}}_{[2,2]}\right\}\} \end{array} \right\|_{\Omega(\mathbb{R})} \\
&= \left\| \sup\left\{\langle a, b\rangle : a \in F^{\mathbb{I}^2_{\mathbb{R}}}_x, b \in F^{\mathbb{I}^2_{\mathbb{R}}}_y\right\} \right\|_{\Omega(\mathbb{R})} \\
&= \sup \left\| \left\{\langle a, b\rangle : a \in F^{\mathbb{I}^2_{\mathbb{R}}}_x, b \in F^{\mathbb{I}^2_{\mathbb{R}}}_y\right\} \right\|_{\Omega(\mathbb{R})} \\
&= \sup\left\{\|\langle a, b\rangle\| : a \in F^{\mathbb{I}^2_{\mathbb{R}}}_x, b \in F^{\mathbb{I}^2_{\mathbb{R}}}_y\right\}.
\end{aligned}$$

Remark 4.4. The norm derived from this inner product is obtained in a usual way for any

$$x = (x_1, x_2, ...x_n). = \left(\left[\underline{x_1}, \overline{x_1}\right], ..., \left[\underline{x_n}, \overline{x_n}\right]\right) \in \mathbb{I}^n_{\mathbb{R}} :$$

$$\begin{aligned}
\|x\| &= \sqrt{\|\langle x, x\rangle\|_{\Omega(\mathbb{R})}} = \left(\left\| \sum_{k=1}^{n} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{x_k}, \overline{x_k}\right] \right\|_{\Omega(\mathbb{R})} \right)^{1/2} \\
&= \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2} = \left(\sum_{k=1}^{n} \left\{\max\left\{\left|\underline{x_k}\right|, \left|\overline{x_k}\right|\right\}\right\}^2 \right)^{1/2} = \|x\|_2 .
\end{aligned}$$

This shows that the inner-product norm is just the 2-norm on $\mathbb{I}^n_{\mathbb{R}}$. For $n = 1$ if $x = [\underline{x}, \overline{x}] \in \mathbb{I}^1_{\mathbb{R}}$ then

$$\begin{aligned}
\|x\| &= \sqrt{\|\langle x, x\rangle\|_{\Omega(\mathbb{R})}} = \left(\left\| \sum_{k=1}^{1} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{x_k}, \overline{x_k}\right] \right\|_{\Omega(\mathbb{R})} \right)^{1/2} = \left(\left|[\underline{x}, \overline{x}][\underline{x}, \overline{x}]\right|\right)^{1/2} \\
&= \left(\left|[\min S, \max S]\right|\right)^{1/2}, \, where \; S = \{\underline{x}^2, \underline{x}\overline{x}, \overline{x}\underline{x}, \overline{x}^2\} \\
&= \left(\max\{\min S, \max S\}\right)^{1/2}, \, where \; S = \{\underline{x}^2, \underline{x}\overline{x}, \overline{x}\underline{x}, \overline{x}^2\} \\
&= \left(\max\left\{\left|a^2\right| : a \in [\min S, \max S]\right\}\right)^{1/2} \\
&= \left(\left|[\underline{x}, \overline{x}]\right|^2\right)^{1/2} = \left|[\underline{x}, \overline{x}]\right| .
\end{aligned}$$

Note in general that $[\underline{x}, \overline{x}][\underline{x}, \overline{x}] \neq [\underline{x}, \overline{x}]^2$ where $[\underline{x}, \overline{x}]^2$ is defined as $[\underline{x}, \overline{x}]^2 = \left\{t^2 : t \in [\underline{x}, \overline{x}]\right\}$ in $\mathbb{I}^1_{\mathbb{R}}$. However, $\|x\|^2 = \left|[\underline{x}, \overline{x}][\underline{x}, \overline{x}]\right| = \left|[\underline{x}, \overline{x}]\right|^2$.

Definition 4.9. Let $x = ([x_1, x_1], ..., [x_n, x_n])$ and $y = ([y_1, y_1], ..., [y_n, y_n])$ be two elements in $\mathbb{I}_{\mathbb{R}}^n$. x and y are called *orthogonal* if

$$\langle x, y\rangle = \sum_{k=1}^{n} \left[\underline{x_k}, \overline{x_k}\right] \left[\underline{y_k}, \overline{y_k}\right] = [0,0] = \{0\}\,.$$

Any set A in $\mathbb{I}_{\mathbb{R}}^n$ is called *orthogonal* if each two elements in A are orthogonal. Moreover, if we know each elements of A has norm 1 then A is called *orthonormal*.

Example 4.9. Let us consider $x = ([-3,3], [0,0])$, $y = ([0,0], [2,5])$ in $\mathbb{I}_{\mathbb{R}}^2$. Obviously x and y are orthogonal. These two elements are singular elements which are orthogonal. $([-3,-3], [0,0])$ and $y = ([0,0], [2,2])$ are regular (degenerate) orthogonal elements. The set

$$A = \{([1,1], [0,0], ..., [0,0]), ([0,0], [1,2], ..., [0,0]), ..., ([0,0], [0,0], ..., [1,n])$$

is an orthogonal set in $\mathbb{I}_{\mathbb{R}}^n$ which is not a basis. However,

$$B = \{([1,1], [0,0], ..., [0,0]), ([0,0], [1,1], ..., [0,0]), ..., ([0,0], [0,0], ..., [1,1])$$

is an orthonormal basis in $\mathbb{I}_{\mathbb{R}}^n$.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753/mathenot.1008750
11 (2) 80-88 (2023) - Research Article
ISSN: 2147-6268

Degree of Approximation of Functions by Nörlund Summability of Double Fourier Series

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Abstract
In this research paper, the author studies some problems which are relating to harmonic summability of double Fourier series on Nörlund summability. These results constitute substantial extension and generalization of related works of F. Moricz and B.E Rhodes [1] and H.K. Nigam and K. Sharma [2].

Keywords: *Nörlund sumambility, double Fourier series, double matrix summability.*

AMS Subject Classification (2020): *40G05, 42B05,42B08.*

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1. Introduction

Let $f(\alpha,\beta)$ be Lebesgue integral in the square $R(-\pi,\pi;-\pi,\pi)$ and be of period 2π in each of the variables α and β. Then the series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\gamma_{mn}\Big\{r_{mn} cos\ m\alpha\ cos\ n\beta + s_{mn}\ sin\ m\alpha\ cos\ n\beta + t_{mn}\ cos\ m\alpha\ sin\ n\beta + q_{mn} sin\ m\alpha\ sin\ n\beta\Big\} \tag{1.1}$$

is called the double Fourier series associated with the function $f(\alpha,\beta)$([2],[3]) where

$$\gamma_{mn} = \begin{cases} \frac{1}{4} & \text{for } m=0, n=0 \\ \frac{1}{2} & \text{for } m=0, n>0 \text{ or } m>0, n=0 \\ 1 & \text{for } m,n>0 \end{cases}$$

$$r_{mn} = \frac{1}{\pi^2}\int\int_R f(\alpha,\beta)\ cos\ m\alpha\ cos\ n\beta\ d\alpha\ d\beta \tag{1.2}$$

Received : *12-10-2021,* ***Accepted :*** *29-07-2022*

*(**Cite as** "S.K. Sahani, V.N. Mishra, Degree of Approximation of Functions by Nörlund Summability of Double Fourier Series, Math. Sci. Appl. E-Notes, 11(2) (2023), 80-88")*

$$s_{mn} = \frac{1}{\pi^2}\int\int_R f(\alpha,\beta)\ sin\ m\alpha\ cos\ n\beta\ d\alpha\ d\beta \tag{1.3}$$

$$t_{mn} = \frac{1}{\pi^2}\int\int_R f(\alpha,\beta)\ cos\ m\alpha\ \ sin\ n\beta\ d\alpha\ d\beta \tag{1.4}$$

$$q_{mn} = \frac{1}{\pi^2} = \int\int_R f(\alpha,\beta)\ sin\ m\alpha\ sin\ n\beta\ d\alpha\ d\beta. \tag{1.5}$$

We have

$$\chi(\alpha,\beta) = \chi_{x,y}(\alpha,\beta) = \frac{1}{4}\Big\{f(x+\alpha,y+\beta)+f(x-\alpha,y+\beta)+f(x+\alpha,y-\beta)+f(x-\alpha,y-\beta)-4f(\alpha,\beta)\Big\}. \tag{1.6}$$

1.Definition([4],[5])

Let$\{p_m^{(1)}\}$and $\{p_n^{(2)}\}$ are two sequence of constants, real or complex.
Let

$$\begin{aligned} P_m^{(1)} &= p_0^{(1)} + p_1^{(1)} + p_2^{(1)} + ... + p_m^{(1)} \\ P_n^{(2)} &= p_0^{(2)} + p_1^{(2)} + p_2^{(2)} + ... + p_n^{(2)}. \end{aligned}$$

We shall also consider a double Nörlund transform of $\{a_{mn}\}$. Then the double Nörlund transform is

$$V_{mn} = \frac{1}{P_m^{(1)}P_n^{(2)}}\sum_{l=0}^{m}\sum_{g=0}^{n} p_{m-l}^{(1)}p_{g-n}^{(2)}a_{lg}. \tag{1.7}$$

2. Definition([4],[5])

The double sequence $\{\alpha_{lg}\}$ is said to be Nörlund summable to a limit V if

$$V_{mn} \to V, (m,n) \to (\infty,\infty). \tag{1.8}$$

It is also known as summable$(N, p_m^{(1)}, p_n^{(2)})$.

3. Definition([1], [2], [4], [5], [6])

If

$$\left.\begin{aligned} p_m^{(1)} &= 1\ for\ m = 0,1,2,... \\ p_n^{(2)} &= 1\ for\ m = 0,1,2,... \end{aligned}\right\} \tag{1.9}$$

then the double Nörlund transform reduces to double Cesàro transform of order one. This summability method is known as Cesàro summability (C,1,1).

4.Definition([1], [2], [5], [6])

$$p_m^{(1)} = \frac{1}{m+1}, m = 0,1,2,...\ and\ p_n^{(2)} = \frac{1}{n+1},\ n = 0,1,2,...$$

then the double Nörlund summability $(N, p_m^{(1)}, p_n^{(2)}$ becomes Harmonic summability and is denoted by $(H,1,1)$.

5. Definition([5])

If, for any $\gamma \geq 1, Vmn \to V, (m,n) \to (\infty,\infty)$in such a manner that $\gamma \geq \frac{m}{n}, \gamma \geq \frac{n}{m}$ then the sequence $\{\alpha_l g\}$ is said to be restrictedly summable N_p at(x,y) to the same limit.

There are several results on Nörlund summability of Fourier series. Nörlund summability of Fourier series has been studied by the authors[1–16]. This motivates us to study on the Nörlund summability of Fourier series in more generalized as particular cases. Therefore, an attempt to make an advance in this research work, we study on the double Fourier series and its conjugate series by Nörlund method. T. Sing [7] proved the following theorem:

Theorem 1.1. *If*

$$\int_0^v \Big|\chi(y)\Big|dy = O\bigg(\frac{v}{\log\ v^{-1}}\bigg),$$

where $\chi(y) = f(v+y) + f(v-y) - 2f(y)$ *a s* $v \to 0$, *then the Fourier series of* $f(u)$ *at* $v = y$ *is summable* (N, p_n) *to* $f(y)$ *where* $\{p_n\}$ *is real non-increasing sequence such that*

$$\sum_{a=2}^{n} \left(\frac{p_a}{a\log a}\right) = o(P_n).$$

In this present research paper, we established the following theorem which is the extended forms of Singh [7] and also the generalized results of [2].

2. Main Results

Theorem 2.1. *If* $(\alpha, \beta) \to (0, 0)$,

$$\int_0^{\alpha}\int_0^{\beta} \Big|\chi(s,t)\Big|dsdt = o\left(\frac{\alpha}{\log\alpha^{-1}}\frac{\beta}{\log\beta^{-1}}\right) \tag{2.1}$$

$$\int_{\delta}^{\pi} ds \int_0^{\beta} \Big|\chi(s,t)\Big|dt = o\left(\frac{\beta}{\log\beta^{-1}}\right), (0 < \delta < \pi) \tag{2.2}$$

and

$$\int_{\delta}^{\pi} dt \int_0^{\alpha} \Big|\chi(s,t)\Big|ds = o\left(\frac{\alpha}{\log\alpha^{-1}}\right), (0 < \delta < \pi) \tag{2.3}$$

then the double Fourier series of $f(\alpha, \beta)$ *at* $\alpha = x$, $\beta = y$ *is summable* $\left(N, p_m^{(1)} p_n^{(2)}\right)$ *to* $f(x, y)$ *where* $\{p_n^{(v)}\}$ *are real non-negative, non-increasing sequence of constants such that*

$$\sum_{k=2}^{n} \left(\frac{p_k^{(v)}}{k\log k}\right) = o\Big(P_n^{(v)}\Big), (v = 1, 2). \tag{2.4}$$

The following lemmas are required in the proof of our theorem.

Lemma 2.1. *If* $\{p_n\}$ *is non-negative and non-increasing, then for* $0 \le a \le b \le \infty, 0 \le t \le \pi$ *and for any n, we have*

$$\left|\sum_{k=a}^{b} p_k e^{i(n-k)t}\right| \le AP_{[t^{-1}]}. \tag{2.5}$$

Lemma 2.2. *Under the condition of lemma 2.1,*

$$\left|\sum_{k=a}^{b} \frac{P_k sin\left(n-k+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right| = o\left(nP_n\right), 0 \le t \le \frac{1}{n}. \tag{2.6}$$

Lemma 2.3. *Under the condition of lemma 2.1,*

$$\left|\sum_{k=a}^{b} p_k \frac{sin\left(n-k+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right| = o\left[\frac{1}{t}P_{[t^{-1}]}\right] \qquad for\ \frac{1}{n} \le t \le \delta. \tag{2.7}$$

Lemma 2.4. *Under the condition of lemma 2.1,*

$$\left|\sum_{k=0}^{n} p_k \frac{sin\left(n-k+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right| = o(1) \quad for \;\; 0 \le \delta < t \le \pi. \tag{2.8}$$

They are uniformly in each of the intervals.

Proof. Let $U_{mn}(x, y; f) = U_{mn}$ denotes the rectangular $(m, n)^{th}$ partial sum of the series (1.1), then we must have

$$U_{mn}(x, y; f) - f(x, y) = \frac{1}{\pi^2}\int_0^T \int_0^\pi \chi(\alpha, \beta) D_m^1(\alpha) D_n^2(\beta) d\alpha d\beta \tag{2.9}$$

where

$$D_m^1(\alpha) = \frac{sin\left(m+\frac{1}{2}\right)\alpha}{2sin\frac{\alpha}{2}} \tag{2.10}$$

and

$$D_n^2(\beta) = \frac{sin\left(n+\frac{1}{2}\right)\beta}{2sin\frac{\beta}{2}} \tag{2.11}$$

where $D_m^1(\alpha)$ and $D_n^2(\beta)$ are respectively denote the Dirichlet kernels.
Let $\left\{V_{mn}(x, y)\right\}$ denote the double Nörlund transform of the sequence $\left\{V_{mn} - f(x, y)\right\}$ then

$$V_{mn}(x, y) = \frac{1}{P_m^{(1)} P_n^{(2)}} \sum_{l=0}^{m}\sum_{g=0}^{n} p_{m-l}^{(1)}\, p_{n-g}^{(2)} \left\{U_{lg} - f(x, y)\right\},$$

$$= \frac{1}{P_m^{(1)} P_n^{(2)}} \sum_{l=0}^{m}\sum_{g=0}^{n} \left\{p_{m-l}^{(1)} p_{n-g}^{(2)} \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \chi(\alpha, \beta) D_l^1(\alpha) D_g^2(\beta) d\alpha d\beta\right\}$$

$$= \int_0^\pi \int_0^\pi \chi(\alpha, \beta)\left\{\frac{1}{2\pi P_m^{(1)}} \sum_{l=0}^{m} p_{m-l}^{(1)} \frac{sin\left(l+\frac{1}{2}\right)l}{sin\frac{\alpha}{2}}\right\}\left\{\frac{1}{2\pi P_n^{(2)}} \sum_{g=0}^{n} p_{g-n}^{(2)} \frac{sin\left(g+\frac{1}{2}\right)\beta}{sin\frac{\beta}{2}}\right\} d\alpha d\beta \tag{2.12}$$

$$= \int_0^\pi \int_0^\pi \chi(\alpha, \beta) N_m^{(1)}(\alpha) N_n^{(2)}(\beta) d\alpha d\beta \tag{2.13}$$

where

$$N_m^{(1)}(\alpha) = \frac{1}{2\pi P_m^{(1)}} \sum_{l=0}^{m} p_{m-l}^{(1)} \frac{sin\left(m-l+\frac{1}{2}\right)\alpha}{sin\frac{\alpha}{2}} \tag{2.14}$$

and

$$N_n^{(2)}(\beta) = \frac{1}{2\pi P_n^{(2)}} \sum_{g=0}^{n} p_{g-n}^{(2)} \frac{sin\left(n-g+\frac{1}{2}\right)\beta}{sin\frac{\beta}{2}}. \tag{2.15}$$

Also equation (2.13) can be written as

$$\begin{aligned} U_{mn}(x, y) - f(x, y) &= \int_0^\pi \int_0^\pi \chi(\alpha, \beta) N_m^{(1)}(\alpha) N_n^{(2)}(\beta) d\alpha d\beta \\ &= \left(\int_0^\pi \int_0^\tau + \int_0^\delta \int_\tau^\pi + \int_\delta^\pi \int_0^\tau + \int_\delta^\pi \int_\tau^\pi\right) \chi(\alpha, \beta) N_m^{(1)}(\alpha) N_n^{(2)}(\beta) d\alpha d\beta \\ &= I_1 + I_2 + I_3 + I_4 \;\; say \end{aligned} \tag{2.16}$$

by hypothesis and using the results of equations (2.6) and equations (2.7), we easily obtain

$$\begin{aligned}
|I_4| &= \left|\int_\delta^\pi \int_\tau^\pi \chi(\alpha,\beta) N_m^{(1)}(\alpha) N_n^{(2)}(\beta) d\alpha d\beta\right| \\
&= o\left(\frac{1}{P_m^{(1)} P_n^{(2)}} \int_\delta^\pi \int_\tau^\pi \left|\chi(\alpha,\beta) N_m^{(1)}(\alpha) N_n^{(2)}(\beta)\right| d\alpha d\beta\right) \\
&= \left(\frac{1}{P_m^{(1)} P_n^{(2)}} \int_0^\pi \int_0^\pi \left|\chi(\alpha,\beta)\right| d\alpha . d\beta\right) \left(as\ N_n^{(2)}(\beta), N_m^{(1)}(\alpha) \text{ are even function}\right) \\
&= o(1). \qquad (2.17)
\end{aligned}$$

Also, for I_3,

$$\begin{aligned}
I_3 &= \int_\delta^\pi N_m^{(1)}(\alpha) d\alpha \int_0^\tau \chi(\alpha,\beta) N_n^{(2)}(\beta) d\beta \\
&= \int_\delta^\pi N_m^{(1)}(\alpha) d\alpha \left\{ \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^\delta \right\} \chi(\alpha,\beta) N_n^{(2)}(\beta) d\beta \\
&= I_{3,1} + I_{3,2} \quad say. \qquad (2.18)
\end{aligned}$$

Thus

$$\begin{aligned}
|I_{3,1}| &= o\left(\frac{n}{P_m^{(1)}} \int_0^\pi \int_0^{\frac{1}{n}} \left|\chi(\alpha,\beta)\right| d\beta\right) \\
&= o\left(\frac{n}{P_m}\right) o\left(\frac{\frac{1}{n}}{\log n}\right) \\
&= o(1). \qquad (2.19)
\end{aligned}$$

Again by equation (2.6) and equation (2.7) and hypothesis,

$$\begin{aligned}
\left|I_{3,2}\right| &= o\left(\frac{1}{P_m^{(1)}} \int_0^\pi d\alpha \int_{\frac{1}{n}}^\tau \left|\chi(\alpha,\beta)\right| \frac{1}{P_n^{(2)}} \frac{P_{[\beta^{-1}]}^{(2)}}{\beta} d\beta\right) \\
&= o\left(\frac{1}{P_m^{(1)} P_n^{(2)}} \int_\delta^\pi d\alpha \left\{ \frac{P_{[\beta^{-1}]}}{\beta} \chi_1(\alpha,\beta) \right\}_{\frac{1}{n}}^\tau - \int_{\frac{1}{n}}^\tau \chi_1(\alpha,\beta) d \left[\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\right]\right) \\
&= (|I_{3,2,1}|) + o\left(|I_{3,2,2}|\right) \quad say \qquad (2.20)
\end{aligned}$$

where

$$\chi_1(\alpha,\beta) = \int_0^\beta |\chi(\alpha\beta)| dw$$

and $I_{3,2,1}$ and $I_{3,2,2}$ stands for two inner integrals.

$$\begin{aligned}
|I_{3,2,1}| &= o\left(\frac{1}{P_m^{(1)} P_n^{(2)}} \int_\delta^\pi d\alpha \left\{ \frac{P_{[\tau^{-1}]}^{(2)}}{\tau} \phi_1(\alpha,\tau) - n\ P_n^{(2)} \phi_1(\alpha, \frac{1}{n}) \right\}\right) \\
&= o\left(\frac{1}{P_m^{(1)} P_n^{(2)}} \frac{p_{[\tau^{-1}]}^{(2)}}{\tau} \int_\delta^\pi d\alpha \int_0^\tau |\chi(\alpha,\beta)| d\beta\right) + o\left(\frac{n}{P_m^{(1)}} \int_\delta^\pi d\alpha \int_0^{\frac{1}{n}} |\chi(\alpha,\beta)| d\beta\right) \\
&= o(1) + o\left(\frac{n}{P_m^{(1)}} \frac{\frac{1}{n}}{\log\ n}\right) \\
&= o(1) \qquad (2.21)
\end{aligned}$$

and

$$\begin{aligned}
|I_{3,2,2}| &= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi d\alpha\left[\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\right]\chi(\alpha,\beta)\right)\\
&= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\frac{1}{n}}^\tau d\left[\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\right]\int_\delta^\pi d\alpha\int_0^\beta|\chi_1(\alpha,w)|dw\right)\\
&= o\left(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_{\frac{1}{n}}^\tau d\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\frac{\beta}{\log(\frac{1}{\beta})}\right). \qquad (2.22)
\end{aligned}$$

Also,

$$\int_{\frac{1}{n}}^\tau \frac{\beta}{\log(\frac{1}{\beta})}d\left[\frac{P_{[\beta^{-1}]}^{(2)}}{\beta}\right]=\int_{\frac{1}{\tau}}^n\frac{1}{y\log y}d\left[y\,P_{[y]}^{(2)}\right]$$

for

$$\int_j^{j+1}\frac{1}{y\log y}d\left[yP_{[y]}^{(2)}\right]<\frac{1}{j\log j}\int_j^{j+1}d\left[yP_{[y]}^{(2)}\right]=\frac{1}{j\log j}\left[yP_{[y]}^{(2)}\right]_j^{j+1}$$

$$=\frac{1}{j\log j}\left\{(j+1)P_{j+1}^{(2)}-k\,P_j^{(2)}\right\}<\frac{1}{j\log j}\left\{P_j^{(2)}+P_j^{(2)}+P_j^{(2)}\right\}$$

for

$$p_{k+1}^{(2)}\le p_k^{(2)}\ \text{ and }\ kp_k^{(2)}\le p_k^2\le\frac{2p_j^{(2)}}{j\log j}+\frac{p_j^{(2)}}{j\log j}$$

thus,

$$\begin{aligned}
\int_{\frac{1}{\tau}}^n\frac{1}{y\log y}d\left[xP_{[x]}^{(2)}\right] &< A+\sum_{j=c}^n\left(\frac{2P_k^{(2)}}{j\log j}+\frac{P_j^{(2)}}{j\log j}\right)\\
&= o\left(P_n^{(2)}\right). \qquad (2.23)
\end{aligned}$$

Now, by hypothesis equation (2.4) and using the equation (2.23) and equation (2.13), we get

$$|I_{3,2,1}|=o(1). \qquad (2.24)$$

Combining equations (2.18), (2.19),(2.20), (2.21), (2.22), (2.23) and (2.24), we get

$$|I_3|=o(1). \qquad (2.25)$$

Similarly, we can show that

$$|I_2|=o(1). \qquad (2.26)$$

Now, for I_1,

$$\begin{aligned}
I_1 &= \int_0^\delta\int_0^\tau\chi(\alpha,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)d\alpha d\beta\\
&= \left(\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}+\int_0^{\frac{1}{n}}\int_{\frac{1}{n}}^\delta+\int_{\frac{1}{m}}^\delta\int_0^{\frac{1}{n}}+\int_{\frac{1}{m}}^\delta\int_{\frac{1}{n}}^\tau\right)\chi(\alpha\,\beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta)d\alpha d\beta\\
&= I_{1,1}+I_{1,2}+I_{1,3}+I_{1,4}\ \text{ say.} \qquad (2.27)
\end{aligned}$$

Then by (2.6) and (2.7)

$$\begin{aligned} |I_{1.1}| &= o\left(\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} |\chi(\alpha,\beta)| mn\, d\alpha d\beta\right) \\ &= o(mn)o\left(\frac{\frac{1}{m}}{\log m}\frac{\frac{1}{n}}{\log n}\right) \\ &= o(1). \end{aligned} \tag{2.28}$$

Similarly,

$$\left.\begin{aligned} |I_{1,2}| = o(1) \\ |I_{1,3}| = o(1) \end{aligned}\right\} \tag{2.29}$$

and

$$\begin{aligned} \int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau}\left|\chi(\alpha,\beta)\right|\frac{P_{[\alpha-1]}}{\alpha}\frac{P_{[\beta-1]}}{\beta}d\alpha d\beta &= \chi(\delta,\tau)\frac{1}{\delta}P^{(1)}_{[\delta-1]}\frac{1}{\tau}\,P^{(2)}_{[\tau-1]} - \frac{1}{\tau}\,P^{(2)}_{[\tau-1]} \\ &\quad -\frac{1}{\tau}\,P^{(2)}_{[\tau-1]}\int_{\frac{1}{m}}^{\delta}\phi(\alpha,\tau)d\frac{P^{(1)}_{[\alpha^{-1}]}}{\alpha} - \frac{1}{\delta}P^{(1)}_{[\delta^{-1}]}\int_{\frac{1}{n}}^{\tau}\chi(\alpha,\beta)d\left[\frac{P^{(2)}_{[\beta^{-1}]}}{\beta}\right]. \end{aligned}$$

Thus,

$$\begin{aligned} |I_{1,4}| &= o\left(\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\tau}\left|\chi(\alpha,\beta)\right|\frac{1}{P^{(1)}_m P^{(2)}_n}\frac{P^{(1)}_{[\alpha^{-1}]}}{\alpha}\frac{P^{(2)}_{[\beta^{-1}]}}{\beta}d\alpha d\beta\right) \\ &= o(1) + o\left(\frac{1}{P^{(1)}_m P^{(2)}_n}\left(C_1 + C_2 + C_3\right)\right) \end{aligned} \tag{2.30}$$

where $o(1)$ corresponds to the integrated part in (2.29) and C_1, C_2 and C_3 are repetitively denote the remaining there integrals

$$\left.\begin{aligned} C_2 = o(1) \\ C_3 = o(1) \end{aligned}\right\}. \tag{2.31}$$

Again for C_4

$$\begin{aligned} C_4 &= o\left(\int_{\frac{1}{m}}^{\delta}\frac{\alpha}{\log(\frac{1}{\alpha})}d\left(\frac{P^{(1)}_{[\alpha^{-1}]}}{\alpha}\right)\int_{\frac{1}{n}}^{\tau}\frac{\beta}{\log(\frac{1}{\beta})}\,d\left[\frac{P^{(2)}_{[\beta^{-1}]}}{\beta}\right]\right) \\ &= o\left(P^{(1)}_m P^{(2)}_n\right) \end{aligned} \tag{2.32}$$

as in (2.23), using the estimate (2.31), we get from (2.30) that

$$|I_{1,4}| = o(1) \tag{2.33}$$

thus

$$|I_1| = O(1). \tag{2.34}$$

Combining equations (2.17),(2.23),(2.24),(2.34), we get equation (2.16). Which competes the proof of the theorem. □

Conclusion

Mathematical analysis is primarily concerned with the notion of limit of a sequences of real or complex number which forms the basis for study of infinite series. The general theory of the convergence and Summability of a double Fourier series has also been discussed by [1–16]. In 1913, in connection with the study of summation by arithmetic means of double Fourier series corresponding to function having discontinuities along a curve Moore [16] was led to the introduction of the notion of restricted summability of a double series. This differs from summability in the general sense in that the indices of the sequences whose limit is involved, become infinite in such a manner that there ratios remain bounded by two ordinary positive constants.
Corresponding to the classical tests for convergence of ordinary Fourier series, tests for pringsheim convergence of the double fourier series have been given by a number of writers. A main point of difference in which double, or multiple, Fourier series differ from ordinary series is the fact that the behavior of the former, as regards convergence, divergence, or oscillation, at a point, does not, as in the later case, depend only on the nature of the function in a neighborhood of the point, but upon its nature in cross-neighborhood of the point. The purpose of this research paper is to formulate the least conditions for Nörlund summability of double Fourier series. The main theorem for Nö"rlund summability of double Fourier series provide more stability to the system. Summability methods are used to decrease error. In this research we may find that our main theorem is a extended version by which many well known results on summabilities, can be obtained that is shown in above part. A function of two variables may be associated with a double fourier series.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753/mathenot.1136328
11 (2) 89-96 (2023) - Research Article
ISSN: 2147-6268

Lacunary Strongly Invariant Convergence in Fuzzy Normed Spaces

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Abstract
In this study, firstly, we defined the notions of lacunary invariant convergence and lacunary invariant Cauchy sequence in fuzzy normed spaces. Then, we introduced lacunary strongly invariant convergence in fuzzy normed spaces and we investigated some properties of these new concepts.

Keywords: *Fuzzy normed space; invariant convergence; lacunary convergence.*

AMS Subject Classification (2020): *Primary: 03E72 ; Secondary: 46S40; 40A05; 40A30.*

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1. Introduction

The idea of fuzzy sets initially introduced by Zadeh [1] to deal with imprecise phenomena as an alternative to classical set theory. After that, several classical concepts were reconstructed. Fuzzy topological spaces [2, 3], fuzzy metric [4–6], fuzzy norm [7–10] are just some of the examples. Felbin's fuzzy norm [9], which is associated with Kaleva and Seikkala [5] type metric space by assigning a non-negative fuzzy real number to each element of a linear space, forms the basis of this study. Das and Das [11] studied fuzzy topology generated by fuzzy norm. Diamond and Kloeden [12] investigated the metric spaces of fuzzy sets-theory and applications. Fang and Huang [13] studied on the level convergence of a sequence of fuzzy numbers. Recently Yalvaç and Dündar [14] defined the notions of invariant convergence and invariant Cauchy sequences with some properties and inclusions in fuzzy normed spaces. Also, some other authors [15–18] studied the concepts related to fuzzy numbers and fuzzy normed space.

Banach [19] defined the generalized limit, an application of Hahn-Banach theorem on the set of all bounded real valued sequences. It is also known as Banach limit. Later, Lorentz [20] offered that if all Banach limits of the given bounded sequence are equal, it is called almost convergent. In further studies [21, 22], invariant mean and invariant convergence are given as more general cases of Banach limit and almost convergence. Also, several authors including Schaefer [23], Mursaleen and Edely [24], Mursaleen [25, 26], Savaş [27, 28] had significant studies on invariant convergence.

Received : *27-06-2022,* ***Accepted :*** *10-11-2022*

*(**Cite as** "Ş. Yalvaç, E. Dündar, Lacunary Strongly Invariant Convergence in Fuzzy Normed Spaces, Math. Sci. Appl. E-Notes, 11(2) (2023), 89-96")*

Freedman et al.[29] gave the relation between strong Cesaro convergent space and the sequence of integers (2^r) and offered lacunary convergence by taking lacunary sequences instead of (2^r). Further studies on this convergence were done by several authors [30, 31].

Now, we recall the basic notions and some essential definitions used in our paper (See [1, 7–10, 15, 16, 18, 20–26, 28–36]).

A fuzzy number is a fuzzy set provided that

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$ for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$;

(iii) u is upper semi-continuous;

(iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is a compact set.

The set of all fuzzy numbers is denoted by $L(\mathbb{R})$. $\mathbb{R}$ can be embedded in $L(\mathbb{R})$ since each $r \in \mathbb{R}$ considered a fuzzy real number $\tilde{r}$ defined by $\widetilde{r}(t) = 1$ if $t = r$ and $\widetilde{r}(t) = 0$ if $t \neq r$.

For $u \in L(\mathbb{R})$, the α-level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1], \\ cl\,\{x \in \mathbb{R} : u(x) > \alpha\}, & \text{if } \alpha = 0. \end{cases}$$

The α-level set of a fuzzy number, denoted by $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, is a non-empty, bounded and closed interval for each $\alpha \in [0,1]$ where $u_\alpha^- = -\infty$ and $u_\alpha^+ = \infty$ are also admissible.

If $u \in L(\mathbb{R})$ and $u(x) = 0$ for $x < 0$, then u is called a non-negative fuzzy number. The set of all non-negative fuzzy numbers is denoted by $L^*(\mathbb{R})$. It is easy to see $\tilde{0} \in L^*(\mathbb{R})$.

A partial ordering $\preceq$ in $L(\mathbb{R})$ is defined by for $u, v \in L(\mathbb{R})$,

$$u \preceq v \quad \text{iff} \quad u_\alpha^- \leq v_\alpha^- \quad \text{and} \quad u_\alpha^+ \leq v_\alpha^+ \quad \text{for all} \quad \alpha \in [0,1].$$

Arithmetic equations addition, multiplication and multiplication with a scaler on $L(\mathbb{R})$ are defined by

(i) $(u \oplus v)(t) = \sup_{s\in\mathbb{R}} \{u(s) \wedge v(t-s)\}, \quad t \in \mathbb{R}$,

(ii) $(u \odot v)(t) = \sup_{s\in\mathbb{R}, s\neq 0} \{u(s) \wedge v(t/s)\}, \quad t \in \mathbb{R}$,

(iii) For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \tilde{0}$, $t \in \mathbb{R}$.

Let $u, v \in L(\mathbb{R})$. Arithmetic equations in terms of α-level sets are defined by

(i) $[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$,

(ii) $[u \odot v]_\alpha = [u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+]$, $u, v \in L^*(\mathbb{R})$,

(iii) $[ku]_\alpha = k[u]_\alpha = \begin{cases} [ku_\alpha^-, ku_\alpha^+], & k \geq 0, \\ [ku_\alpha^+, ku_\alpha^-], & k < 0. \end{cases}$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined by

$$D(u,v) = \sup_{0\leq\alpha\leq 1} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\}.$$

One can see that

$$D\left(u, \tilde{0}\right) = \sup_{0\leq\alpha\leq 1} \max\left\{\left|u_\alpha^-\right|, \left|u_\alpha^+\right|\right\} = \max\left\{\left|u_0^-\right|, \left|u_0^+\right|\right\}.$$

Obviously, $D\left(u, \tilde{0}\right) = u_0^+$ when $u \in L^*(\mathbb{R})$.

A sequence (u_n) in $L(\mathbb{R})$ is convergent to $u \in L(\mathbb{R})$, denoted by $D - \lim\limits_{n\to\infty} u_n = u$, if $\lim_{n\to\infty} D(u_n, u) = 0$, i.e., for all given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $D(u_n, u) < \varepsilon$, for all $n > n_0$.

Let X be a vector space over $\mathbb{R}$, $\|.\| : X \to L^*(\mathbb{R})$ and $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$, $R(1,1) = 1$.

The quadruple $(X, \|.\|, L, R)$ is called fuzzy normed linear space (FNS) and $\|.\|$ is a fuzzy norm if the following axioms are satisfied

(i) $\|x\| = \widetilde{0}$ iff $x = \theta$,

(ii) $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,

(iii) For all $x, y \in X$,

(a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x+y\|_1^-$,

(b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x+y\|_1^-$.

When $L = \min$ and $R = \max$ are taken in above (iii), triangle inequalities become

$$\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^- \text{ and } \|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$$

for all $\alpha \in (0,1]$. Since they fulfil the other conditions of norm, $\|x\|_\alpha^-$ and $\|x\|_\alpha^+$ can be seen as ordinary norms on X.

Example 1.1. Let $(X, \|.\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|.\|$ on X can be obtained

$$\|x\|(t) = \begin{cases} 0, & \text{if } 0 \le t \le a\|x\|_C \text{ or } t \ge b\|x\|_C, \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a}, & \text{if } a\|x\|_C \le t \le \|x\|_C, \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1}, & \text{if } \|x\|_C \le t \le b\|x\|_C, \end{cases}$$

where $\|x\|_C$ is the ordinary norm of $x\ (\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \widetilde{0}$. Hence $(X, \|.\|)$ is a fuzzy normed linear space.

Throughout paper let $(X, \|.\|)$ be an fuzzy normed linear space (FNS).

Let us consider the topological structure of the space X. For any $\varepsilon > 0$, $\alpha \in [0,1]$ and $x \in X$, the (ε, α)-neighborhood of x is the set $\mathcal{N}_x(\varepsilon, \alpha) := \{y \in X : \|x - y\|_\alpha^+ < \varepsilon\}$.

A sequence (x_n) in X is convergent to x with respect to the fuzzy norm, denoted by $x_n \overset{FN}{\to} x$, if it is provided that $(D) - \lim_{n\to\infty} \|x_n - x\| = \widetilde{0}$, i.e., for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$D\left(\|x_n - x\|, \widetilde{0}\right) = \sup_{\alpha\in[0,1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon,$$

for all $n > n_0$. In terms of neighborhoods, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in \mathcal{N}_x(\varepsilon, 0)$, for all $n > n_0$.

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if and only if

(i) $\phi(x) \ge 0$, when the sequence $x = (x_n)$ has $x_n \ge 0$ for all n,

(ii) $\phi(e) = 1$, where $e = (1, 1, 1...)$,

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and satisfied the condition $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m-th iterate of the mapping σ at n. Invariant mean ϕ is a extension of the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. The sequence is called invariant convergent when its invariant means are equal. In case $\sigma(n) = n + 1$, the σ-mean become Banach limit and invariant convergence become almost convergence.

A bounded sequence $x = (x_n)$ is σ-convergent to the number L if $\lim\limits_{m\to\infty} t_{mn} = L$ uniformly in n, where

$$t_{mn} = \frac{x_{\sigma(n)} + x_{\sigma^2(n)} + \cdots + x_{\sigma^m(n)}}{m}.$$

A sequence $x = (x_n)$ in X is invariant convergent to L with respect to fuzzy norm if $(D) - \lim_{m\to\infty} \|t_{mn} - L\| = \widetilde{0}$, uniformly in n. Namely, for given $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$,

$$D(\|t_{mn} - L\|, \widetilde{0}) = \sup_{\alpha\in[0,1]} \|t_{mn} - L\|_\alpha^+ = \|t_{mn} - L\|_0^+ < \varepsilon, \text{ for every } n \in \mathbb{N}.$$

Let $0 < q < \infty$. The sequence $x = (x_n)$ in X is q-strongly invariant convergent to L with respect to fuzzy norm if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \left[D\left(\|x_{\sigma^i(n)} - L\|, \widetilde{0}\right)\right]^q = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \left[\|x_{\sigma^i(n)} - L\|_0^+\right]^q = 0,$$

uniformly in n.

An increasing sequence of non-negative integers $\theta = (k_r)$ with $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ is called lacunary sequence. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is given by q_r.

For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary convergent to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} (x_i - L) = 0.$$

For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary strongly convergent to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} |x_i - L| = 0.$$

For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary strongly convergent to L with respect to fuzzy norm if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} D\left(\|x_i - L\|, \widetilde{0}\right) = 0.$$

2. Main results

Definition 2.1. For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary invariant convergent to L with respect to fuzzy norm and it is denoted by $x_n \overset{\sigma-FN_\theta}{\longrightarrow} L$ if

$$\lim_{r\to\infty} D\left(\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|, \widetilde{0}\right) = \lim_{r\to\infty}\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ = 0,$$

unifomly in n; i.e., for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$D\left(\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|, \widetilde{0}\right) = \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$.

Definition 2.2. For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary invariant Cauchy sequence with respect to fuzzy norm if for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r, s > r_0$,

$$\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - \frac{1}{h_s}\sum_{j\in I_s} x_{\sigma^j(n)}\right\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$.

Theorem 2.1. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$ be a sequence in X. If x is lacunary invariant convergent to L with respect to fuzzy norm, then x is lacunary invariant Cauchy sequence with respect to fuzzy norm.*

Proof. Assume that the sequence $x = (x_n)$ is lacunary invariant convergent to L with respect to fuzzy norm in X. Then, for every $\varepsilon > 0$ there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$. Therefore for all $r, s > r_0$,

$$\begin{aligned}\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - \frac{1}{h_s}\sum_{j\in I_s} x_{\sigma^j(n)}\right\|_0^+ &= \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ + \left\|\frac{1}{h_s}\sum_{j\in I_s} x_{\sigma^j(n)} - L\right\|_0^+ \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, x is lacunary invariant Cauchy sequence with respect to fuzzy norm. □

Definition 2.3. For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary strongly invariant convergent to L with respect to fuzzy norm and it is denoted by $x_n \overset{[\sigma-FN]_\theta}{\longrightarrow} L$ if

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} D(\|x_{\sigma^i(n)} - L\|, \widetilde{0}) = \lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L\|_0^+ = 0,$$

unifomly in n; i.e., for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r} D(\|x_{\sigma^i(n)} - L\|, \widetilde{0}) = \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$.

Theorem 2.2. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$ be a sequence in X. If x is lacunary strongly invariant convergent to L, then L is unique.*

Proof. Assume that $x_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} L_1$, $x_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} L_2$ and $L_1 \neq L_2$. Then for every $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that for all $r > r_1$,

$$\frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_1\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$ and for given $\varepsilon > 0$, there exists r_2 such that for all $r > r_2$,

$$\frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_2\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$. Take $r_0 = \max\{r_1, r_2\}$. Then for all $r > r_0$,

$$\begin{aligned} \|L_1 - L_2\|_0^+ &= \frac{1}{h_r}\sum_{i\in I_r} \|L_1 - L_2\|_0^+ \\ &\leq \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_1\|_0^+ + \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_2\|_0^+ \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

for all $n \in \mathbb{N}$. Since for all $\varepsilon > 0$,

$$\|L_1 - L_2\|_0^+ < \varepsilon,$$

we have $L_1 = L_2$. □

Theorem 2.3. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$, $y = (y_n)$ be sequences in X. If x and y are lacunary strongly invariant convergent to L_1 and L_2, respectively, then the sequence $x + y$ is lacunary strongly invariant convergent to $L_1 + L_2$.*

Proof. Assume that $x_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} L_1$ and $y_n \stackrel{[\sigma-FN]_\theta}{\longrightarrow} L_2$. Then for every $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that for all $r > r_1$,

$$\frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_1\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$ and for given $\varepsilon > 0$, there exists r_2 such that for all $r > r_2$,

$$\frac{1}{h_r}\sum_{i\in I_r} \|y_{\sigma^i(n)} - L_2\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$. Take $r_0 = \max\{r_1, r_2\}$ then for all $r > r_0$,

$$\begin{aligned} \frac{1}{h_r}\sum_{i\in I_r} \|(x_{\sigma^i(n)} + y_{\sigma^i(n)}) - (L_1 + L_2)\|_0^+ &\leq \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_1\|_0^+ + \frac{1}{h_r}\sum_{i\in I_r} \|y_{\sigma^i(n)} - L_2\|_0^+ \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, we have

$$(x_n + y_n) \stackrel{[\sigma-FN]_\theta}{\longrightarrow} (L_1 + L_2).$$

□

Theorem 2.4. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$ be sequence in X. If x is strongly lacunary invariant convergent to L and c is a scaler, then the sequence cx is strongly lacunary invariant convergent to cL.*

Proof. Assume that $x_n \overset{[\sigma-FN]_\theta}{\longrightarrow} L$ and c is a scaler. Then for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r}\|x_{\sigma^i(n)} - L_1\|_0^+ < \frac{\varepsilon}{|c|},$$

for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \frac{1}{h_r}\sum_{i\in I_r}\|cx_{\sigma^i(n)} - cL_1\|_0^+ &= |c|\frac{1}{h_r}\sum_{i\in I_r}\|x_{\sigma^i(n)} - L_1\|_0^+ \\ &< |c|\frac{\varepsilon}{|c|} = \varepsilon, \end{aligned}$$

for all $n \in \mathbb{N}$. So, we conclude

$$cx_n \overset{[\sigma-FN]_\theta}{\longrightarrow} cL.$$

□

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$ be sequence in X. If the sequence x is strongly lacunary invariant convergent to L then x is lacunary invariant convergent to L.*

Proof. Assume that $x = (x_n)$ is strongly lacunary invariant convergent to L with respect to fuzzy norm. Then for all $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r}\|x_{\sigma^i(n)} - L\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$. Since

$$\begin{aligned} \left\|\frac{1}{h_r}\sum_{i\in I_r}x_{\sigma^i(n)} - L\right\|_0^+ &\leq \frac{1}{h_r}\sum_{i\in I_r}\|x_{\sigma^i(n)} - L\|_0^+ \\ &< \varepsilon, \end{aligned}$$

for all $n \in \mathbb{N}$, then we obtain that x is lacunary invariant convergent to L with respect to fuzzy norm.

□

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753/mathenot.1057344
11 (2) 97-103 (2023) - Research Article
ISSN: 2147-6268

On the Study of Pantograph Differential Equations with Proportional Fractional Derivative

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Abstract
This manuscript is devoted to investigate the existence, uniqueness and stability of pantograph equations with Hilfer generalized proportional fractional derivative. The concerned results are obtained using standard theorems.

Keywords: *Pantograph differential equation; generalized proportional fractional derivative; existence; stability.*

AMS Subject Classification (2020): *34A08; 46E35; 58C30.*

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1. Introduction

In the present paper, we will derive some sufficient conditions on existence and stability results for pantograph equation involving fractional order of the form

$$\begin{cases} \mathscr{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t),\mathfrak{h}(\lambda t)), \quad t \in J := [a,b], \\ \mathscr{I}^{1-\nu,\vartheta;\psi}\mathfrak{h}(t)| = \mathfrak{h}_0 \end{cases} \tag{1.1}$$

where, $\mathscr{D}^{\alpha,\beta,\vartheta;\psi}$ is ψ-Hilfer proportional fractional derivative of orders $\alpha \in (0,1)$, $\beta \in [0,1]$ and $\vartheta \in (0,1]$, $\mathscr{I}^{1-\nu,\vartheta;\psi}$ is ψ-fractional integral of orders $1-\nu(\nu = \alpha+\beta-\alpha\beta)$. Let $\mathfrak{g}$ be the continuous function from J into $R \times R$ and $\mathfrak{h}$ is the given function.

Fractional calculus is extension of ordinary differentiation and integration to arbitrary order (non-integer). In recent years, fractional differential equations (FDE) arise naturally in various fields such as science and engineering. Theory of FDE has been extensively studied by many authors, see [1–9].

It is renowned that, within the settled scenario, there's an awfully special delay equation called the pantograph equations. In the following years, the pantograph equation became a prime example for a delay differential equation. The pantograph equations have been well studied over the last several decades, refer to [10–12].

Received : *13-01-2022,* ***Accepted :*** *31-12-2022*

*(**Cite as** "S. Harikrishnan, D.Vivek, E.M. Elsayed, On the Study of Pantograph Differential Equations with Proportional Fractional Derivative, Math. Sci. Appl. E-Notes, 11(2) (2023), 97-103")*

Most recently a fractional derivative with kernel of function is introduced by Vanterler Da C. Sousa and the classical properties with transformation of existing fractional derivative is discussed in [8, 13]. Motivated by the above mentioned work, we introduce a new generalized fractional calculus based on a special case of the proportional derivatives discussed in [14]. There are three features for our new generalized proportional fractional (GPF) derivative that make it different and distinctive: the kernel of the fractional operator contains exponential function, the generated fractional integrals possess a semi-group property and the obtained operators provide undeviating generalization to the existing Riemann-Liouville and Caputo fractional derivatives and integrals when the order 0 tends to 1.

The paper is organized as follows. In section 2, we declare the weighted spaces, basic definitions and results for proportional derivatives and their corresponding integral equation. In Section 3, we analyze the existence, uniqueness and stability results for proposed problem.

2. Preliminaries

Let $J(0 \leq a \leq b)$ be a finite interval. The space of continuous function $\mathfrak{h}$, defined by C associated with the norm

$$\|\mathfrak{h}\|_{C_{\nu,\psi}} = \sup\{|\mathfrak{h}(t)| : t \in J\}.$$

We denote the weighted spaces of all continuous functions defined by

$$C_{\nu,\psi} = \{\mathfrak{g} : J \to R : (\psi(t) - \psi(a))^{\nu}\, \mathfrak{g}(t) \in C\}, 0 \leq \nu < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{\nu,\psi}} = \sup_{t\in J} |(\psi(t) - \psi(a))^{\nu}\, \mathfrak{g}(t)|.$$

The weighted space $C^n_{\nu,\psi}$ of functions $\mathfrak{g}$ on J is defined by

$$C^n_{\nu,\psi} = \left\{\mathfrak{g} : J \to R : \mathfrak{g}(t) \in C^{n-1}; \mathfrak{g}(t) \in C_{\nu,\psi}\right\}, 0 \leq \nu < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C^n_{\nu,\psi}} = \sum_{k=0}^{n-1} \left\|\mathfrak{g}^k\right\|_C + \|\mathfrak{g}^n\|_{C_{\nu,\psi}}.$$

For $n = 0$, we have, $C^0_\nu = C_\nu$.
Here, we present the following weighted space for our problem as follows

$$C^{\alpha,\beta}_{1-\nu;\psi} = \left\{\mathfrak{g} \in C_{1-\nu;\psi}, \mathscr{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{g} \in C_{\nu;\psi}\right\},$$

and

$$C^{\nu}_{1-\nu;\psi} = \left\{\mathfrak{g} \in C_{1-\nu;\psi}, \mathscr{D}^{\nu,\vartheta;\psi}\mathfrak{g} \in C_{1-\nu;\psi}\right\}.$$

It is obvious that

$$C^{\nu}_{1-\nu;\psi} \subset C^{\alpha,\beta}_{1-\nu;\psi}.$$

Definition 2.1. [14] If $\vartheta \in (0, 1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$. then the fractional integral

$$\left(\mathscr{I}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \int_0^t \psi^{'}(s) e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \frac{(\psi(t) - \psi(s))^{\alpha-1}}{\vartheta^\alpha \Gamma(\alpha)} \mathfrak{h}(s) ds. \tag{2.1}$$

Definition 2.2. [14] If $\vartheta \in (0, 1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a, b]$, where $\psi^{'}(s) > 0$, the GPF derivative of order α of the function $\mathfrak{h}$ with respect to another function isdefined by with $\psi^{'}(t) \neq 0$ is describe as

$$\left(\mathscr{D}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_0^t \psi^{'}(s) e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \frac{(\psi(t) - \psi(s))^{n-\alpha-1}}{\Gamma(n - \alpha)} \mathfrak{h}(s) ds. \tag{2.2}$$

Definition 2.3. [14] If $\vartheta \in (0, 1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a, b]$, where $\psi^{'}(s) > 0$, the GPF derivative in Caputo sence of order α of the function $\mathfrak{h}$ with respect to another function isdefined by with $\psi^{'}(t) \neq 0$ is describe as

$$\left(\mathscr{D}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \mathscr{I}^{n-\alpha,\vartheta;\psi} \left(\mathscr{D}^{n,\vartheta;\psi}\mathfrak{h}\right)(t). \tag{2.3}$$

Definition 2.4. The ψ-Hilfer GPF derivative of order α and type β over $\mathfrak{h}$ with respect to another function is defined by

$$\left(\mathscr{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}\right)(t) = \mathscr{I}^{\beta(1-\alpha),\vartheta;\psi}\left(\mathscr{D}^{1,\vartheta;\psi}\right)\mathscr{I}^{(1-\beta)(1-\alpha),\vartheta;\psi}\mathfrak{h}(t). \tag{2.4}$$

Next, we shall give the definitions and the criteria of generalized Ulam-Hyers-Rassias(UHR) stability. Let $\epsilon > 0$ be a positive real number and $\varphi : J \to R^+$ be a continuous function. We consider the following inequalities:

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t) - \mathfrak{g}\left(t, \mathfrak{v}(t), \mathfrak{v}(t)\right)\right| \leq \varphi(t). \tag{2.5}$$

Definition 2.5. Eq. (1.1) is generalized UHR stable with respect to $\varphi \in C_{1-\nu,\psi}$ if there exists a real number $C_{\mathfrak{g},\varphi} > 0$ such that for each solution $\mathfrak{v} \in C_{1-\nu,\psi}$ of the inequality (2.5) there exists a solution $\mathfrak{h} \in C_{1-\nu,\psi}$ of Eq. (1.1) with

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \leq C_{\mathfrak{g},\varphi}\varphi(t).$$

Lemma 2.1. *Let $\alpha, \beta > 0$, then we have the following semigroup property*

$$(\mathscr{I}^{\alpha,\vartheta;\psi}\mathscr{I}^{\beta,\vartheta;\psi}\mathfrak{g})(t) = (\mathscr{I}^{\alpha+\beta,\vartheta;\psi}\mathfrak{g})(t),$$

and

$$(\mathscr{D}^{\alpha,\vartheta;\psi}\mathscr{I}^{\alpha,\vartheta;\psi}\mathfrak{g})(t) = \mathfrak{g}(t).$$

Lemma 2.2. *Let $n - 1 < \alpha < n$ where $n \in N, \vartheta \in (0, 1]$, $0 \leq \beta \leq 1$, with $\nu = \alpha + \beta(n - \alpha)$, such that $n - 1 < \nu < n$. If $\mathfrak{g} \in C_\nu$ and $\mathfrak{I}^{n-\nu,\vartheta;\psi}\mathfrak{g} \in C^n_\nu$, then*

$$(\mathscr{I}^{\alpha,\vartheta;\psi}\mathscr{I}^{\alpha,\beta,\vartheta;\psi}\mathfrak{g})(t) = \mathfrak{g}(t) - \sum_{k=1}^{n}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\nu-k}}{\vartheta^{\nu-k}\Gamma\nu-k+1}\mathscr{I}^{k-\nu,\vartheta;\psi}(a),$$

Lemma 2.3. *(Grönwall's Lemma [13]) Let $\alpha > 0$, $a(t) > 0$ is locally integrable function on J and if $\mathfrak{g}(t)$ be a increasing and nonnegative continuous function on J, such that $|\mathfrak{g}(t)| \leq K$ for some constant K. Moreover if $\mathfrak{h}(t)$ be a nonnegative locally integrable function on J with*

$$\mathfrak{h}(t) \leq a(t) + g(t)\int_0^t \psi^{'}(s)\left(\psi(t) - \psi(s)\right)^{\alpha-1}\mathfrak{h}(s)ds, \quad (t) \in J,$$

with some $\alpha > 0$. Then

$$\mathfrak{h}(t) \leq a(t) + \int_0^t\left[\sum_{n=1}^{\infty}\frac{(\mathfrak{g}(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi^{'}(s)\left(\psi(t) - \psi(s)\right)^{n\alpha-1}\right]a(s)ds, \quad (t) \in J.$$

Theorem 2.1. *(Schauder fixed point theorem, [15]) Let B be closed, convex and nonempty subset of a Banach space C. Let $\mathscr{T} : B \to B$ be a continuous mapping such that $\mathscr{T}(B)$ is a relatively compact subset of C. Then $\mathscr{T}$ has at least one fixed point in B.*

Lemma 2.4. *A function $\mathfrak{h}$ is the solution of (1.1), if and only if $\mathfrak{h}$ satisfies the random integral equation*

$$\begin{aligned}\mathfrak{h}(t) =& \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1}\\ &+ \frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds.\end{aligned} \tag{2.6}$$

3. Main results

Utilizing the concept of Theorem 2.1, we obtain the following results for the proposed problem (1.1). First, we declare the hypotheses used to obtain the result:

(H1) There exists a constant ℓ, such that

$$|\mathfrak{g}(\cdot, \mathfrak{h}_1(\cdot), \mathfrak{h}_2(\cdot)) - \mathfrak{g}(\cdot, \mathfrak{y}_1(\cdot), \mathfrak{y}_2(\cdot))| \leq \ell\left(|\mathfrak{h}_1 - \mathfrak{y}_1| + |\mathfrak{h}_2 - \mathfrak{y}_2|\right).$$

(H2) There exists an increasing function $\varphi \in C_{1-\nu,\psi}$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$\mathfrak{I}^{\alpha;\psi}\varphi(t) \leq \lambda_\varphi \varphi(t).$$

Theorem 3.1. *Assume that hypothesis $(H1)$ is satisfied. Then, Eq.(1.1) has at least one solution.*

Proof. Consider the operator $\mathscr{T} : C_{1-\nu,\psi} \to C_{1-\nu,\psi}$. Hence $\mathfrak{h}$ is a solution for the problem (1.1) if and only if $\mathfrak{h}(t) = (\mathscr{T}\mathfrak{h})(t)$, where the equivalent integral Eq. (2.6) which can be written in the operator form

$$\begin{aligned}(\mathscr{T}\mathfrak{h})(t) =& \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} \\ &+ \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds. \end{aligned} \tag{3.1}$$

Clearly, the fixed points of the operator $\mathscr{T}$ is solution of the problem (1.1). Set $\tilde{\mathfrak{g}} = \mathfrak{g}(s,0,0)$. For any $\mathfrak{h}$, we have

$$\begin{aligned} &\left|(\mathscr{T}\mathfrak{h})(t)(\psi(t)-\psi(a))^{1-\nu}\right| \\ &\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} + \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right|ds \\ &\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} \\ &\quad + \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))-\mathfrak{g}(s,0,0)+\mathfrak{g}(s,0,0)\right|ds \\ &\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} + \frac{2\ell(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha+\nu-1}\left\|\mathfrak{h}\right\|_{C_{1-\nu,\psi}} \\ &\quad + \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha+\nu-1}\left\|\tilde{\mathfrak{g}}\right\|_{C_{1-\nu,\psi}} \\ &\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} + \frac{\ell}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^{\alpha}\left\|\mathfrak{h}\right\|_{C_{1-\nu,\psi}} \\ &\quad + \frac{B(\nu,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}(\psi(b)-\psi(a))^{\alpha}\left\|\tilde{\mathfrak{g}}\right\|_{C_{1-\nu,\psi}} \end{aligned}$$

This proves that $\mathscr{T}$ transforms the ball $B_r = \left\{\mathfrak{h} \in C_{1-\nu,\psi} : \left\|\mathfrak{h}\right\|_{C_{1-\nu,\psi}} \leq r\right\}$, into itself. We shall show that the operator $\mathscr{T} : B_r \to B_r$ satisfies all the conditions of Theorem 2.1. The proof will be given in the following steps.
Step 1: $\mathscr{T}$ is continuous.

Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n \to \mathfrak{h}$ in $C_{1-\nu,\psi}$. Then, for each $t \in J$,

$$\begin{aligned} &\left|((\mathscr{T}\mathfrak{h}_n)(t)-(\mathscr{T}\mathfrak{h})(t))(\psi(t)-\psi(a))^{1-\nu}\right| \\ &\leq \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}_n(s),\mathfrak{h}_n(\lambda s))-\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right|ds \\ &\leq \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(s))^{\alpha+\nu-1}\left\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot),\mathfrak{h}_n(\cdot))-\mathfrak{g}(\cdot,\mathfrak{h}(\cdot),\mathfrak{h}(\cdot))\right\|_{C_{1-\nu,\psi}} \\ &\leq \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^{\alpha}\left\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot),\mathfrak{h}_n(\cdot))-\mathfrak{g}(\cdot,\mathfrak{h}(\cdot),\mathfrak{h}(\cdot))\right\|_{C_{1-\nu,\psi}}. \end{aligned}$$

Due to continuity of $\mathfrak{g}$, we have

$$\left\|\mathscr{T}\mathfrak{h}_n - \mathscr{T}\mathfrak{h}\right\|_{C_{1-\nu,\psi}} \to 0 \text{ as } n \to \infty.$$

Step 2: $\mathscr{T}(B_r)$ is uniformly bounded.
This is clear since $\mathscr{T}(B_r) \subset B_r$ is bounded.

Step 3: We show that $\mathscr{T}(B_r)$ is equi-continuous.
Let $t_1 > t_2 \in J$ with B_r be a bounded set of $C_{1-\nu,\psi}$ as in Step 2, and $\mathfrak{h} \in B_r$. Then

$$\begin{aligned}
&\left|(\psi(t_1)-\psi(a))^{1-\nu}\left(\mathscr{T}\mathfrak{h}\right)(t_1)-(\psi(t_2)-\psi(a))^{1-\nu}\left(\mathscr{T}\mathfrak{h}\right)(t_2)\right| \\
&\leq \left|\frac{(\psi(t_1)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}\int_a^{t_1} e^{\frac{\vartheta-1}{\vartheta}(\psi(t_1)-\psi(a))}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds\right. \\
&\quad \left.-\frac{(\psi(t_2)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}\int_a^{t_2} e^{\frac{\vartheta-1}{\vartheta}(\psi(t_2)-\psi(a))}\psi^{'}(s)(\psi(t_2)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds\right| \\
&\leq \frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_a^{\tau_1}\left[(\psi(\tau_1)-\psi(a))^{1-\nu}(\psi(\tau_1)-\psi(s))^{\alpha-1}(\psi(\tau_2)-\psi(a))^{1-\nu}(\psi(\tau_2)-\psi(s))^{\alpha-1}\right] \\
&\quad \times\psi^{'}(s)\left|\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right|ds \\
&\quad +\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_{\tau_2}^{\tau_1}(\psi(\tau_2)-\psi(a))^{1-\nu}(\psi(\tau_2)-\psi(s))^{\alpha-1}\psi^{'}(s)\left|\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right|ds
\end{aligned}$$

right hand side of the inequality approaches to zero, as $t_1 \to t_2$. Therefore by Steps 1-3 together with the Arzela-Ascoli theorem, we say that $\mathscr{T}$ is continuous and compact. Hence by Theorem 2.1, the operator $\mathscr{T}$ has a fixed point which is a solution of the problem (1.1). □

Lemma 3.1. *Assume that the hypothesis $(H1)$ is satisfied. If*

$$\frac{2\ell}{\vartheta^\alpha\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^\alpha < 1.$$

Then, problem (1.1) *has a unique fixed point.*

Proof. Consider the operator $\mathscr{T}: C_{1-\nu,\psi} \to C_{1-\nu,\psi}$ defined by

$$\begin{aligned}
\left(\mathscr{T}\mathfrak{h}\right)(t) =& \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} \\
&+\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds.
\end{aligned}$$

Clearly the operator $\mathscr{T}$ is well defined. Now for any $\mathfrak{h}_1, \mathfrak{h}_2 \in C_{1-\nu}$, we attain

$$\begin{aligned}
&\left|\left(\left(\mathscr{T}\mathfrak{h}_1\right)(t)-\left(\mathscr{T}\mathfrak{h}_2\right)(t)\right)(\psi(t)-\psi(a))^{1-\nu}\right| \\
&\leq \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}_1(s),\mathfrak{h}_1(\lambda s))-\mathfrak{g}(s,\mathfrak{h}_2(s),\mathfrak{h}_2(\lambda s))\right|ds \\
&\leq \frac{2\ell\,(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha+\nu-1}\left\|\mathfrak{h}_1-\mathfrak{h}_2\right\|_{C_{1-\nu,\psi}} \\
&\leq \frac{2\ell}{\vartheta^\alpha\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^\alpha\left\|\mathfrak{h}_1-\mathfrak{h}_2\right\|_{C_{1-\nu,\psi}}.
\end{aligned}$$

It follows that $\mathscr{T}$ has a contraction map, there exists a unique solution of problem (1.1). □

Theorem 3.2. *The hypotheses $(H1)$ and $(H2)$ are satisfied. Then Eq.* (1.1) *is g-UHR stable.*

Proof. Let $\mathfrak{v}$ be solution of inequality (2.5) and by Theorem 3.1, $\mathfrak{h}$ is a unique solution of Eq. (1.1) is as follows

$$\begin{aligned}
\mathfrak{h}(t) =& \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} \\
&+\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds.
\end{aligned}$$

By inequality (2.5), we obtain

$$\left|\mathfrak{v}(t)-\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1}\right.$$
$$\left.-\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{v}(s),\mathfrak{v}(\lambda s))ds\right|\leq\lambda_{\varphi}\varphi(t).$$

Hence for every $t\in J$, we have

$$\begin{aligned}
&|\mathfrak{v}(t)-\mathfrak{h}(t)|\\
&\leq\left|\mathfrak{v}(t)-\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1}\right.\\
&\qquad\left.-\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds\right|\\
&\leq\left|\mathfrak{v}(t)-\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1}\right.\\
&\qquad\left.-\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{v}(s),\mathfrak{v}(\lambda s))ds\right|\\
&\qquad+\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{v}(s),\mathfrak{v}(\lambda s))-\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right|ds\\
&\leq\lambda_{\varphi}\varphi(t)+\frac{2\ell}{\Gamma(\alpha)}\int_a^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{v}(s)-\mathfrak{h}(s)\right|ds.
\end{aligned}$$

By Lemma 2.3, there exists a constant $c>0$ such that

$$|\mathfrak{v}(t)-\mathfrak{h}(t)|\leq C_{\mathfrak{g},\varphi}\lambda_{\varphi}\varphi(t).$$

Hence, Eq. (1.1) is g-UHR stable. □

4. Conclusion

We have studied a nonlinear fractional differential equation with unknown function together with its lower-order fractional derivative. Several existence and uniqueness results have been derived by applying different tools of the fixed point theory. Our results are quite general and give rise to many new cases by assigning different values to the parameters involved in the problem.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753/mathenot.1212331
11 (2) 104-111 (2023) - Research Article
ISSN: 2147-6268

Statistical Convergence of Spliced Sequences in Terms of Power Series on Topological Spaces

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Abstract
In the present paper, $P-$distributional convergence which is defined by power series method has been introduced. We give equivalent expressions for $P-$distributional convergence of spliced sequences. Moreover, convergence of a bounded ∞-spliced sequence via power series method is represented in terms of Bochner integral in Banach spaces.

Keywords: *P-distributional convergence; P-density; power series; P-statistical convergence; spliced sequences.*

AMS Subject Classification (2020): *Primary: 54A20 ; Secondary: 40J05; 40A05; 40A35; 40G10; 40C15.*

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1. Preliminaries

Results of the ordinary summability theory in topological setting can not be obtained in the lack of addition operator. Hence summability theory has been studied in topological spaces under some assumptions. In this context some convergences such as Abel-statistical convergence and Abel distributional convergence have been introduced in topological spaces. From another perspective Osikiewicz [1] has introduced the concept of spliced sequences. And then this concept has been studied by Ünver [2] and Ünver and et al. [3] in topological spaces. Also Yurdakadim and et al. [4] have generalized this concept by using bounded sequences instead of convergent sequences. On the other hand spliced sequences have been studied from a different perspective in [5]. In the present paper, we introduce P-distributional convergence which generalizes Abel distributional convergence in the Hausdorff topological space. We also give equivalent expressions for P-distributional convergence of spliced sequences. Furthermore, we show that convergence of a bounded ∞-spliced sequence via power series method can be represented in terms of Bochner integral in Banach spaces.

Received : *30-11-2022,* ***Accepted :*** *19-01-2023*

*(**Cite as** "S. Demirkale, E. Taş, Statistical Convergence of Spliced Sequences in Terms of Power Series on Topological Spaces, Math. Sci. Appl. E-Notes, 11(2) (2023), 104-111")*

Definition 1.1. If the limit

$$\delta(G) := \lim_{n\to\infty} \frac{1}{n+1} |\{j \le n : j \in G\}|$$

exists then it is said to be the natural density of the subset $G \subset \mathbb{N}_0 = \{0\} \cap \mathbb{N}$. Here by $|.|$, we denote the cardinality of elements of enclosed set. If for every $\epsilon > 0$, $\delta(G_\epsilon) = 0$ where $G_\epsilon = \{j \in \mathbb{N}_0 : |x_j - L| \ge \epsilon\}$ then it is said that $x = (x_j)$ converges statistically to L [6–8].

Definition 1.2. Let (p_j) be a real sequence such that $p_0 > 0$, $p_1, p_2, ... \ge 0$ and $p(t) := \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \le \infty$,

$$C_P := \left\{ x = (x_j) \mid P_x(t) := \sum_{j=0}^{\infty} p_j t^j x_j \text{ has radius of convergence } \ge R \text{ and } P_x \in C_p \right\}.$$

The functional $P - \lim : C_P \to \mathbb{R}$ defined by

$$P - \lim x = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j x_j$$

is called a power series method and x is said to be P-convergent [9], where

$$C_p := \left\{ f : (-R, R) \to \mathbb{R} \mid \lim_{0<t\to R^-} \frac{1}{p(t)} f(t) \text{ exists} \right\}.$$

Now consider $x = (1, -1, 1, -1, ...)$, $R = \infty$, $p(t) = e^t$ and for $j \ge 0$, $p_j = \frac{1}{j!}$. Then we immediately see that

$$\lim_{t\to\infty} \frac{1}{e^t} \sum_{j=0}^{\infty} \frac{x_j t^j}{j!} = \lim_{t\to\infty} \frac{1}{e^t} \sum_{j=0}^{\infty} \frac{(-1)^j t^j}{j!} = \lim_{t\to\infty} \frac{1}{e^t} e^{-t} = 0.$$

Hence while the sequence $x = (x_j)$ is P-convergent to 0, it does not converge in the ordinary sense. This illustrates us that ordinary convergence is not as useful as power series method.

If $\lim x = l$ implies $P - \lim x = l$, then it is said that the method P is regular. A power series method P is regular if and only if for any $j \in \mathbb{N}_0$

$$\lim_{0<t\to R^-} \frac{p_j t^j}{p(t)} = 0$$

holds [9].

Definition 1.3. Let P be regular and $G \subset \mathbb{N}_0$. If the limit

$$\delta_P(G) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j\in G} p_j t^j$$

exists then it is said to be the P-density of G [10].

Definition 1.4. The sequence $x = (x_j)$ of real numbers P-statistically converges to L if for every $\epsilon > 0$, $\delta_P(G_\epsilon) = 0$ that is for any $\epsilon > 0$,

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j\in G_\epsilon} p_j t^j = 0$$

is satisfied [10].

A criteria has been given for P-statistical convergence in [11].

Now, we introduce P-statistical convergence in a Hausdorff topological space (X, τ).

Definition 1.5. Consider a Hausdorff topological space (X,τ). The sequence $x=(x_j)$ in X is P-statistically convergent to $\alpha\in X$ if for any open set H that contains α

$$\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{x_j\notin H}p_jt^j=0$$

holds.

Definition 1.6. Consider a Hausdorff topological space (X,τ) and the Borel sigma field $\sigma(\tau)$ on (X,τ). Let $F:\sigma(\tau)\to[0,1]$ be a set function such that $F(X)=1$ and if $H_0,H_1,...$ are pairwise disjoint sets in $\sigma(\tau)$ then

$$F(\bigcup_{j=0}^{\infty}H_j)=\sum_{j=0}^{\infty}F(H_j)$$

holds. Then F is said to be a distribution on $\sigma(\tau)$ [2].

Definition 1.7. Consider a distribution F on $\sigma(\tau)$ and a nonnegative summability matrix $A=(a_{nj})$ such that whose each row adds up to one. Let $x=(x_j)$ be a sequence in X and ∂H be the boundary of H. Then the sequence x is said to be A-distributionally convergent to F if for all $H\in\sigma(\tau)$ with $F(\partial H)=0$ we have ([2])

$$\lim_{n\to\infty}\sum_{x_j\in H}a_{nj}=F(H).$$

The next definition is power series version of the above one and Abel distributional convergence [2].

Definition 1.8. Consider a distribution F on $\sigma(\tau)$ and a sequence $x=(x_j)$ in X. Let ∂H be the boundary of H. Then the sequence x is said to be P-*distributionally convergent* to F if for any $H\in\sigma(\tau)$ with $F(\partial H)=0$, we have

$$\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{x_j\in H}p_jt^j=F(H)$$

where ∂H is the boundary of H.

Definition 1.9. Let T be a fixed positive integer. A $T-$partition of $\mathbb{N}_0$ consists of infinite sets $E_i=\{v_i(j)\}$ for $i=0,1,...,T-1$ such that $\bigcup_{i=0}^{T-1}E_i=\mathbb{N}_0$ and $E_i\cap E_j=\varnothing$ for all $i\neq j$. An $\infty-$partition of $\mathbb{N}_0$ consists of a countably infinite number of infinite sets $E_i=\{v_i(j)\}$ for $i\in\mathbb{N}_0$ such that $\bigcup_{i=0}^{\infty}E_i=\mathbb{N}_0$ and $E_i\cap E_j=\varnothing$ for all $i\neq j$.

Definition 1.10. Let $\{E_i:i=0,1,...,T-1\}$ be a fixed $T-$partition of $\mathbb{N}_0$ and $x^{(i)}=(x_j^{(i)})$ be a sequence in X with for $i=0,1,...,T-1$, $\lim_{j\to\infty}x_j^{(i)}=\alpha_i$. If $k\in E_i$, then $k=v_i(j)$ for some j. Define $x=(x_k)$ by $x_k=x_{v_i(j)}=x_j^{(i)}$. Then x is called a $T-splice$ on $\{E_i:i=0,1,...,T-1\}$ with limit points $\alpha_0,\alpha_1,...,\alpha_{T-1}$ [1].

Definition 1.11. Let $\{E_i:i\in\mathbb{N}_0\}$ be a fixed ∞ $-$partition of $\mathbb{N}_0$, consider a sequence $x^{(i)}=(x_j^{(i)})$ in X with $\lim_{j\to\infty}x_j^{(i)}=\alpha_i$, $i\in\mathbb{N}_0$. If $k\in E_i$, then $k=v_i(j)$ for some j. Define $x=(x_k)$ by $x_k=x_{v_i(j)}=x_j^{(i)}$. Then it is said that x is an $\infty-splice$ on $\{E_i:i\in\mathbb{N}_0\}$ with limit points $\alpha_0,\alpha_1,...,\alpha_T,...$ [1].

From [3], it is useful to recall the following

Proposition 1.1. *Consider a Banach space $(X,\|.\|)$ and a nonnegative regular summability matrix $A=(a_{nj})$ such that each row adds up to one and an $\infty-$partition of $\mathbb{N}$, $\{E_i=\{v_i(j)\}:i\in\mathbb{N}\}$. If $\delta_A(E_i)$ exists for $i\in\mathbb{N}$ and $\sum_{i=1}^{\infty}\delta_A(E_i)=1$ then for every bounded $\infty-spliced$ sequence $x=(x_j)$ on $\{E_i:i\in\mathbb{N}\}$*

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}a_{nj}x_j=\int_X tdF, \tag{1.1}$$

where the integral in (1.1) is the Bochner integral and

$$F(H) = \sum_{\alpha_i \in H} \delta_A(E_i),\, H \in \sigma(\tau)$$

is a distribution.

2. Statistical convergence of spliced sequences in terms of power series

In this section, we aim to characterize P-statistical convergence in topological spaces and also obtain equivalent expressions of P-distributional convergence of spliced sequences. The convergence of a bounded ∞-spliced sequence via power series method is also represented in terms of Bochner integral in Banach spaces.

The next theorem characterizes P-statistical convergence in topological spaces.

Theorem 2.1. *Consider X being a Hausdorff topological space and a sequence $x = (x_j)$ in X. The following assessments are equivalent:*

- *(i) x is P-statistically convergent to $\alpha \in X$.*
- *(ii) x is P-distributionally convergent to $F : \sigma(\tau) \to [0, 1]$ defined by*

$$F(H) = \begin{cases} 0 & , \quad \alpha \notin H \\ 1 & , \quad \alpha \in H \end{cases}.$$

Proof. Assume that x is P-statistically convergent to α. So for every open set H which contains α we get

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{x_j \notin H} p_j t^j = 0.$$

First, let us give the following notations that we will need to complete the proof and use in the remainder of the paper.

Consider $\mathbf{D} = \{(z_n) \subset \mathbb{R} : \forall n \in \mathbb{N}_0, 0 < z_n < R \text{ and } \lim_{n\to\infty} z_n = R\}$ and let

$$\mathbf{C} = \left\{ C(z) = (c_{nk}) : c_{nk} = \frac{1}{p(z_n)} p_j z_n^j \right\},$$

where

$$p(t) := \sum_{j=0}^{\infty} p_j t^j.$$

Hence for any $z \in \mathbf{D}$ we get

$$\lim_{n\to\infty} \frac{1}{p(z_n)} \sum_{x_j \notin H} p_j z_n^j = 0.$$

This limit shows that for any $C(z) \in \mathbf{C}$, the sequence x is $C(z)$-statistically convergent to α. According to Proposition 1 of [3], for any $C(z) \in \mathbf{C}$, x is $C(z)$-distributionally convergent to F. This completes the proof.

Conversely assume that x is P-distributionally convergent to F. So for any $C(z) \in \mathbf{C}$, x is $C(z)$-distributionally convergent to F. Again according to Proposition 1 of [3], x is $C(z)$-statistically convergent to α for any $C(z) \in \mathbf{C}$. This implies x is P-statistically convergent to α. □

The next theorem deals with P-distributional convergence of finite spliced sequences.

Theorem 2.2. *Consider X being a Hausdorff topological space and a T−partition of $\mathbb{N}_0$, $\{E_i : i = 0, 1, ..., T-1\}$. Then the following assessments are equivalent:*

- *(a) For each $i = 0, 1, ..., T-1$, $\delta_P(E_i)$ exists.*

- *(b) There exist* $s_0, s_1, ..., s_{T-1} \in [0,1]$ *such that* $\sum_{i=0}^{T-1} s_i = 1$ *and any* $T-$*spliced sequence on* $\{E_i : i = 0,1,...,T-1\}$ *with limit points* $\alpha_0, \alpha_1, ..., \alpha_{T-1}$ *is* P*-distributionally convergent to the distribution* $F : \sigma(\tau) \to [0,1]$ *where*
$$F(H) = \sum_{\substack{0 \le i \le T-1 \\ \alpha_i \in H}} s_i, \text{ for all } H \in \sigma(\tau).$$

- *(c) There exist* $s_0, s_1, ..., s_{T-1} \in [0,1]$ *such that* $\sum_{i=0}^{T-1} s_i = 1$ *and the* $T-$*splice of* $x^{(0)}, x^{(1)}, ..., x^{(T-1)}$ *on* $\{E_i : i = 0,1,...,T-1\}$ *where* $x^{(i)} = (\alpha_i, \alpha_i, ...)$ *is a constant sequence and is* P*-distributionally convergent to the distribution* $F : \sigma(\tau) \to [0,1]$ *where*
$$F(H) = \sum_{\substack{0 \le i \le T-1 \\ \alpha_i \in H}} s_i, \text{ for all } H \in \sigma(\tau).$$

Proof. $(a) \implies (b)$: Assume that $\delta_P(E_i)$ exists for any $i = 0,1,...,T-1$ and set $s_i = \delta_P(E_i)$ for $i = 0,1,...,T-1$. Since $\{E_i : i = 0,1,...,T-1\}$ is a $T-$partition of $\mathbb{N}_0$, it is obvious that
$$1 = \sum_{i=0}^{T-1} \delta_P(E_i) = \sum_{i=0}^{T-1} p_i.$$
Let $F : \sigma(\tau) \to [0,1]$ be defined as follows
$$F(H) = \sum_{\substack{0 \le i \le T-1 \\ \alpha_i \in H}} p_i, \text{ for all } H \in \sigma(\tau).$$
Obviously one can see that F is a distribution on $\sigma(\tau)$.

For each $i = 0,1,...,T-1$ by the existence of $\delta_P(E_i)$, we observe that for any $C(z) \in \mathbf{C}$ and any $i = 0,1,...,T-1$, $\delta_{C(z)}(E_i)$ exists and equals to $\delta_P(E_i)$. Since for each $i = 0,1,...,T-1$, $s_i = \delta_P(E_i) = \delta_{C(z)}(E_i)$ and for any $C(z) \in \mathbf{C}$ from Theorem 1 of [3] , x is obviously $C(z)$-distributionally convergent to F i.e. for any $z \in \mathbf{D}$ and for all $H \in \sigma(\tau)$ with $F(\partial H) = 0$
$$\lim_{n \to \infty} \frac{1}{p(z_n)} \sum_{x_j \in H} p_j z_n^j = F(H)$$
holds and implies
$$\lim_{0 < t \to R^-} \frac{1}{p(t)} \sum_{x_j \in H} p_j t^j = F(H).$$
Then the sequence x is P-distributionally convergent to F.

$(b) \implies (c)$: Since for each $i = 0,1,...,T-1$, $x^{(i)} = (\alpha_i, \alpha_i, ...)$ is convergent, the proof can be obtained immediately.

$(c) \implies (a)$: Consider the sequence x which is the T-spliced of $x^{(0)}, x^{(1)}, ..., x^{(T-1)}$ on $\{E_i : i = 0,1,...,T-1\}$ where $x^{(i)} = (\alpha_i, \alpha_i, ...)$ is a constant sequence and is P-distributionally convergent to the distribution F and let $s_0, s_1, ..., s_{T-1} \in [0,1]$ such that $\sum_{i=0}^{T-1} s_i = 1$. Then for any $C(z) \in \mathbf{C}$, x is $C(z)$-distributionally convergent to F. According to Theorem 1 of [3] for any $i = 0,1,...,T-1$ and for every $C(z) \in \mathbf{C}$, $\delta_{C(z)}(E_i)$ exists and equals to s_i which implies
$$\lim_{n \to \infty} \frac{1}{p(z_n)} \sum_{x_j \in E_i} p_j z_n^j = s_i.$$
Thus for each $i = 0,1,...,T-1$, we have
$$\delta_P(E_i) = \lim_{0 < t \to R^-} \frac{1}{p(t)} \sum_{x_j \in E_i} p_j t^j = s_i.$$
This completes the proof. $\square$

The following result deals with P-distributional convergence of $\infty-$spliced sequences.

Theorem 2.3. *Consider a Hausdorff topological space X and an $\infty-$partition of $\mathbb{N}_0$, $\{E_i = \{v_i(j)\} : i \in \mathbb{N}_0\}$. Then $\delta_P(E_i)$ exists for all $i \in \mathbb{N}_0$ and $\sum_{i=0}^{\infty}\delta_P(E_i) = 1$ if and only if there exist $s_i \in [0,1]$ for $i \in \mathbb{N}_0$ such that $\sum_{i=0}^{\infty}s_i = 1$ and any $\infty-$splice sequence on $\{E_i : i \in \mathbb{N}_0\}$ with limit points $\alpha_0, \alpha_1, \alpha_2, ...$ is P-distributionally convergent to the distribution $F : \sigma(\tau) \to [0,1]$ where for all $H \in \sigma(\tau)$*

$$F(H) = \sum_{\alpha_i \in H} s_i.$$

Proof. Assume that $\delta_P(E_i)$ exists for any $i \in \mathbb{N}_0$. Hence for any $C(z) \in \mathbf{C}$ and for all $i \in \mathbb{N}_0$, $\delta_{C(z)}(E_i)$ exists and equals to $\delta_P(E_i)$. Hence for any $C(z) \in \mathbf{C}$ and for each $i \in \mathbb{N}_0$, we get

$$\sum_{i=0}^{\infty}\delta_{C(z)}(E_i) = 1.$$

According to Theorem 2 of [3], for any $C(z) \in \mathbf{C}$ we get that every $\infty - spliced$ sequence $x = (x_k)$ on $\{E_i : i \in \mathbb{N}_0\}$ with limit points $\alpha_0, \alpha_1, \alpha_2, ...$ is $C(z)$-distributionally convergent to the distribution $F : \sigma(\tau) \to [0,1]$ where for all $H \in \sigma(\tau)$

$$F(H) = \sum_{\alpha_i \in H} \delta_{C(z)}(E_i) = \sum_{\alpha_i \in H} \delta_P(E_i),$$

i.e.

$$\lim_{n\to\infty} \frac{1}{p(z_n)} \sum_{x_j \in H} p_j z_n^j = F(H)$$

holds for every $H \in \sigma(\tau)$ with $F(\partial H) = 0$ and for any $z \in \mathbf{D}$. We have

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{x_j \in H} p_j t^j = F(H)$$

which means that x is P-distributionally convergent to F.

For sufficiency, let $s_i \in [0,1]$ for $i \in \mathbb{N}_0$ such that $\sum_{i=0}^{\infty}s_i = 1$ and consider every $\infty - spliced$ sequence on $\{E_i : i \in \mathbb{N}_0\}$ with limit points $\alpha_0, \alpha_1, \alpha_2, ...$ is P-distributionally convergent to F. Then

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{x_j \in H} p_j t^j = F(H)$$

holds for all $H \in \sigma(\tau)$ with $F(\partial H) = 0$. So for any $H \in \sigma(\tau)$ with $F(\partial H) = 0$ and for each $z \in \mathbf{D}$, we have

$$\lim_{n\to\infty} \frac{1}{p(z_n)} \sum_{x_j \in H} p_j z_n^j = F(H). \tag{2.1}$$

Then from Theorem 2 of [3] and by (2.1) for each $C(z) \in \mathbf{C}$ and for all $i \in \mathbb{N}_0$, $\delta_{C(z)}(E_i)$ exists and equals to s_i with $\sum_{i=0}^{\infty}s_i = 1$. Therefore for any $z \in \mathbf{D}$ and for each $i \in \mathbb{N}_0$

$$\lim_{n\to\infty} \frac{1}{p(z_n)} \sum_{j \in E_i} p_j z_n^j = s_i$$

which implies

$$\delta_P(E_i) = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j \in E_i} p_j t^j = s_i$$

for all $i \in \mathbb{N}_0$. Hence $\delta_P(E_i)$ exists for all $i \in \mathbb{N}_0$ and $\sum_{i=0}^{\infty}\delta_P(E_i) = 1$. □

In the next theorem the convergence of a bounded ∞-spliced sequence via power series method is represented by Bochner integral in Banach spaces.

Theorem 2.4. *Consider a Banach space $(X, \|.\|)$ and an $\infty-$partition of $\mathbb{N}_0$, $\{E_i = \{v_i(j)\} : i \in \mathbb{N}_0\}$. If $\delta_P(E_i)$ exists for each $i \in \mathbb{N}_0$ and $\sum_{i=0}^{\infty}\delta_P(E_i) = 1$ then for every bounded $\infty-$spliced sequence $x = (x_j)$ on $\{E_i = \{v_i(j)\} : i \in \mathbb{N}_0\}$*

$$\lim_{0<t\to R^-} \frac{1}{p\,(t)}\sum_{j=0}^{\infty} p_j t^j x_j = \int_X t dF \tag{2.2}$$

where the integral in (2.2) is the Bochner integral and

$$F(H) = \sum_{\alpha_i \in H} \delta_P(E_i), \text{ for every } H \in \sigma(\tau)$$

is a distribution.

Proof. Let $\delta_P(E_i)$ exists for all $i \in \mathbb{N}_0$ and $\sum_{i=0}^{\infty}\delta_P(E_i) = 1$. Hence for any $C(z) \in \mathbf{C}$ and for each $i \in \mathbb{N}_0$, $\delta_{C(z)}(E_i)$ exists and equals to $\delta_P(E_i)$ with $\sum_{i=0}^{\infty}\delta_{C(z)}(E_i) = 1$. Then from Proposition 1.1, we obtain for every bounded $\infty - spliced$ sequence $x = (x_j)$ on $\{E_i = \{v_i(j)\} : i \in \mathbb{N}_0\}$, for each $C(z) \in \mathbf{C}$

$$\lim_{n\to\infty} \frac{1}{p(z_n)}\sum_{j=0}^{\infty} p_j z_n^j x_j = \int_X t dF \tag{2.3}$$

where F is defined by

$$F(H) = \sum_{\alpha_i \in H} \delta_{C(z)}(E_i) = \sum_{\alpha_i \in H} \delta_P(E_i). \tag{2.4}$$

Hence from (2.3) and (2.4), we obtain

$$\lim_{0<t\to R^-} \frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j x_j = \int_X t dF.$$

This completes the proof. $\square$

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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