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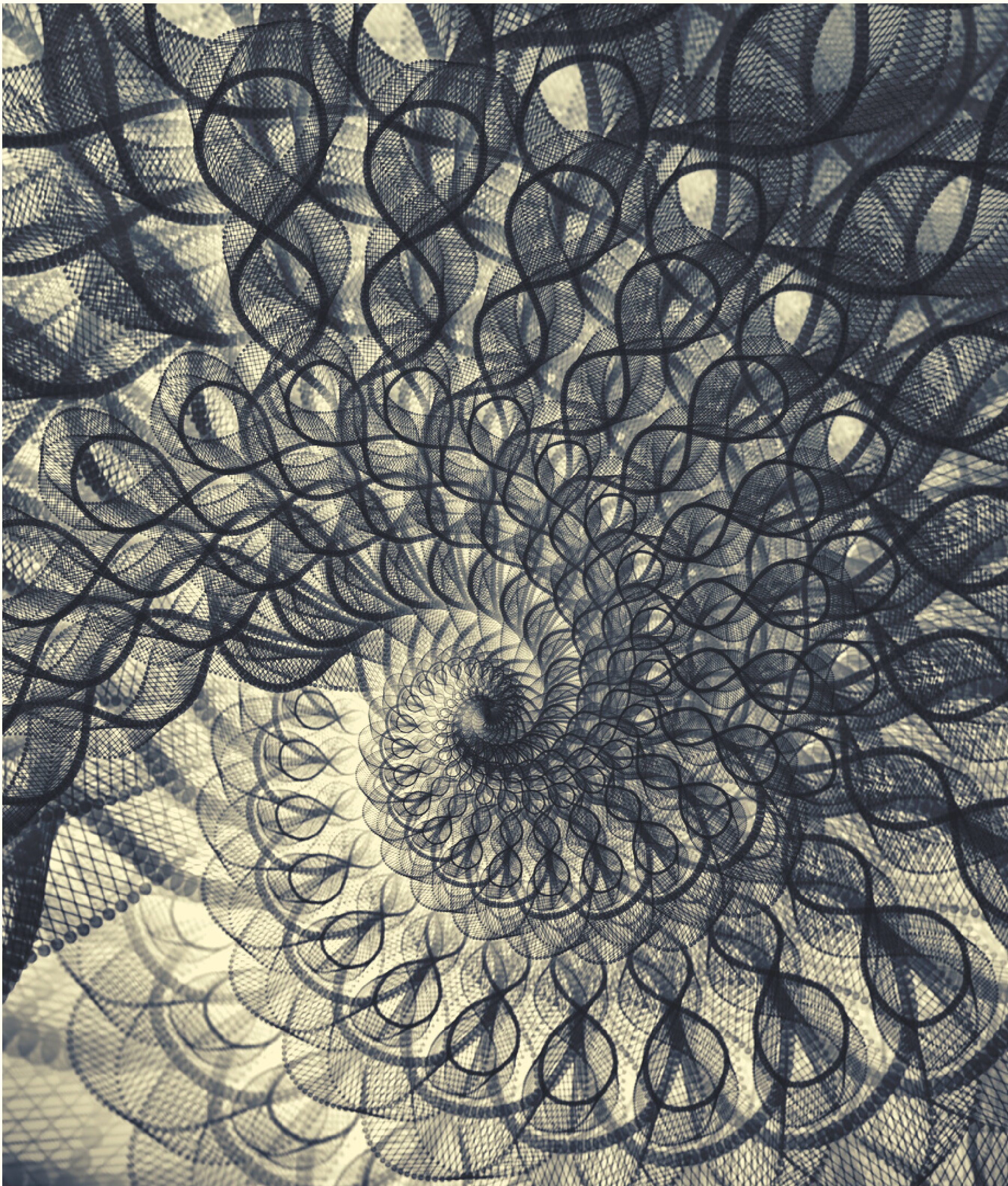

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Contents

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Global Solution and Blow-up for a Thermoelastic System of p-Laplacian Type with Logarithmic Source

Carlos Alberto Raposo da Cunha*, Adriano Pedreira Cattai, Octavio Paulo Vera Villagran, Ganesh Chandra Gorain and Ducival Carvalho Pereira

Abstract
This manuscript deals with global solution, polynomial stability and blow-up behavior at a finite time for the nonlinear system

$$\begin{cases} u'' - \Delta_p u + \theta + \alpha u' = |u|^{p-2}\, u \ln |u| \\ \theta' - \Delta\theta = u' \end{cases}$$

where Δ_p is the nonlinear p-Laplacian operator, $2 \leq p < \infty$. Taking into account that the initial data is in a suitable stability set created from the Nehari manifold, the global solution is constructed by means of the Faedo-Galerkin approximations. Polynomial decay is proven for a subcritical level of initial energy. The blow-up behavior is shown on an instability set with negative energy values.

Keywords: *Global solution; blow-up; thermoelastic system of p-Laplacian type; logarithmic source.*

AMS Subject Classification (2020): *Primary: 35A01 ; Secondary: 35B40; 74F05; 93D20.*

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1. Introduction

A thermoelastic system is the result of the coupling of a hyperbolic equation with a parabolic equation. As is well known, these systems describe the elastic and thermal behavior of elastic, heat-conducting media, especially the interactions between elastic stresses and temperature differences. The pioneering work on thermoelasticity without p -Laplacian was presented by C. M. Dafermos [1] in 1968. Since then, a great interest has been aroused in different contexts and nowadays there are many results on global and local solutions, stability, and burst behavior of solutions in thermoelasticity theory. We can cite [2–11] with references therein.

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Nonlinear hyperbolic problems have always been much studied by mathematicians and physicists. From the mathematical point of view, in [12] was investigated the initial boundary value problem of a nonlinear wave equation with weak and strong damping terms and logarithmic term, and in [13] the viscoelastic wave equation with a strong damping and nonlinearity logarithmic source was considered. In physics, the nonlinear logarithmic source $|u|^{p-2}u\ln|u|$ arises in inflation cosmology, supersymmetric led theories, quantum mechanics, nuclear physics, and fluid mechanics, [14–17].

Regarding global solution for wave equation of p-Laplacian type without an additional dissipation term

$$u'' - \Delta_p u = 0, \tag{1.1}$$

for $n = 1$, M. Derher [18] proved the local in time existence of solution and showed by a generic counter-example that the global in time solution can not be expected. Adding a strong damping $-\Delta u'$ in (1.1) the well-posedness and asymptotic behavior was studied by J. M. Greenberg [19]. In fact, the strong damping plays an important role on the existence and stability for p-Laplacian wave equation see for instance for $n \geq 2$ [20–27]. Nevertheless, if the strong damping is replaced by a weaker damping u', then global existence and uniqueness are only know for $n = 1; 2$, see [28]. For the intermediary damping given by $(-\Delta)^{\alpha}u'$, with $0 < \alpha \leq 1$, in [29] was proved the global solution depending on the growth of a forcing term. The background of these problems are in physics, especially in solid mechanics. The p-Laplacian problem for the electromagnetic effects in high-temperature Type II superconductors is considered in [30] where authors presented an extension of previous work on relaxation schemes applied to degenerate parabolic problems. Global boundedness of weak solution in an attraction–repulsion chemotaxis system with p-Laplacian diffusion was considered in [31]. In [32], the entire blow-up solutions for a quasilinear p-Laplacian Schrödinger elliptic equation with a non-square diffusion term. By using the dual approach and some new iterative techniques, the difficulty due to the non-square diffusion term and the p-Laplacian operator is overcome and the nonexistence and existence of entire blow-up solutions are established.

Thermoelastic problems involving the p-Laplacian are becoming the new object of research. The following thermoelastic system which contains corner-edge Laplacian and p-Laplacian type operators with potential function

$$u'' - \Delta_{p,\mathbb{K}}u - \varepsilon V(\tilde{x})u + \theta = |u|^{\alpha-1}u,$$
$$\theta' - \Delta_{\mathbb{K}}u = u',$$

with $\alpha > 1$ was studied in [33] where $\mathbb{K}$ is the stretched manifold with respect to the manifold K with corner-edge singularity and $\tilde{x} \in \mathbb{K}$. The operator $\Delta_{p,\mathbb{K}} + \varepsilon V(\tilde{x})$ with $p \neq 2$ arises from a diversity of physical phenomena, like in reaction-diffusion problems, in nonlinear elasticity, in non-Newtonian fluids and petroleum extraction. In [34] the relationship with non-Newtonian Mechanics was considered. Authors present a full classification of the short-time behavior of the interfaces and local solutions to the nonlinear parabolic p-Laplacian type reaction-diffusion equation of non-Newtonian elastic filtration

$$u' - \left(|u_x|^{p-2}u_x\right)_x + bu^{\beta} = 0,\ 1 < p < 2,\ \beta > 0.$$

In [35] was studied the problem for a parabolic equation involving fractional p-Laplacian with logarithmic nonlinearity. For $2 \leq p < \infty$ the existence of a global solution for the thermoelastic system of p-Laplacian type given by

$$\begin{cases} u'' - \Delta_p u + \theta &= |u|^{r-1}u, \\ \theta' - \Delta\theta &= u'. \end{cases} \tag{1.2}$$

has been proven in [36]. Later, in [37], by employing the potential well theory, authors discuss the properties of finite-time blow-up and give the lower and upper bounds of blow-up time to the solutions.

Regarding the model (1.2) in this manuscript, we analyze the competition between the weak damping $\alpha u',\ \alpha > 0$ and the logarithmic source $|u|^{p-2}u\ln|u|$. To our goal we consider the following system

$$u'' - \Delta_p u + \theta + \alpha u' = |u|^{p-2}u\ln|u|,\ \ (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.3}$$

$$\theta' - \Delta\theta = u',\ \ (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.4}$$

$$u(x,0) = u_0(x),\ \ u'(x,0) = u_1(x),\ \ \theta(x,0) = \theta_0(x),\ \ x \in \Omega, \tag{1.5}$$

$$u(x,t) = \theta(x,t) = 0 \ \text{ on } \ \partial\Omega \times [0,\infty). \tag{1.6}$$

This paper is organized as follows. In the Section 2, we introduce the notation and some technical lemmas. Section 3 deals with the potential well, we introduce some notations and the stability set for the problem. In the section 4 we introduce a suitable Galerkin basis necessary to deal with the operator p-Laplacian. In the section 5 we prove the existence of global solution by Faedo-Galerkin method. In section 6 we prove the polynomial. Finally in section 7 we prove the blow-up in finite time for initial data in the instability set.

2. Preliminaries

The duality pairing between the space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$ will be denoted using the form $\langle \cdot\,,\,\cdot \rangle_p$. According to Poincaré's inequality, the standard norm $\|\,\cdot\,\|_{W_0^{1,p}(\Omega)}$ is equivalent to the norm $\|\nabla\,\cdot\,\|_p$ on $W_0^{1,p}(\Omega)$. Henceforth, we put $\|\,\cdot\,\|_{W_0^{1,p}(\Omega)} = \|\nabla\,\cdot\,\|_p$. We denote $\|\,\cdot\,\|_{L^2(\Omega)} = |\,\cdot\,|_2$ and the usual inner product by $(\,\cdot\,,\,\cdot\,)$.

Let B be a Banach space and $u\,:\,[0,T]\to B$ a mensurable function. We denote by

$$L^p(0,T;B) = \left\{ u\;:\; \left(\int_0^T ||u(t)||_B^p\,dt\right)^{1/p} < \infty,\;\text{if}\;\,1\le p<\infty \right\},$$

$$L^\infty(0,T;B) = \left\{ u\;:\; \sup_{t\in(0,T)}\text{ess}\,||u(t)||_B < \infty,\;\text{if}\;\,p=\infty \right\}.$$

The p-Laplacian operator is given by $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$. $\Delta_p u$ can be extended to a monotone, bounded, hemicontinuos and coercive operator between the spaces $W_0^{1,p}(\Omega)$ and its dual by

$$-\Delta_p\colon W_0^{1,p}(\Omega)\to W^{-1,p'}(\Omega), \qquad \langle -\Delta_p u, v\rangle_p = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x.$$

We assume that the parameter p satisfies the following assumptions.

(H): $p\ge 2\;$ if $\;n=1,2$ and $2\le p\le \dfrac{2n-2}{n-2}\;$ if $\;n\ge 3$.

By (H) we have

$$W_0^{1,2(p-1)}(\Omega)\hookrightarrow H_0^1(\Omega)\hookrightarrow L^2(\Omega).$$

Now, we present some results that will be used in this manuscript.

Lemma 2.1 (Kim [38], Lemma 1.4). *Let u_m be a sequence of functions such that as $m\to\infty$*

$$u^m \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H^\beta(\Omega)), \quad \textit{weakly star,}$$
$$u_t^m \rightharpoonup u_t \text{ in } L^2(0,T;H^\alpha(\Omega)), \quad \textit{weakly,}$$

where $-1\le\alpha<\beta\le 1$. Then, we have

$$u^m \rightharpoonup u \text{ in } C([0,T]\,;\,H^\eta(\Omega)), \quad \textit{for any}\;\;\eta<\beta.$$

Lemma 2.2 (Lions [39], Lemma 1.3). *Let $Q=\Omega\times(0,T)$, $T>0$ a bounded open set of $\mathbb{R}^n\times\mathbb{R}$ and $g_m, g\,:\,Q\to\mathbb{R}$ functions of $L^p(0,T;L^p(\Omega))=L(Q)$, $1<p<\infty$ such that $||g_m||_{L^p(Q)}\le C$, $\;g_m\to g$ a.e. in Q. Then*

$$g_m \rightharpoonup g \;\;\text{in}\;\; L^p(0,T;L^p(\Omega)) \text{ as } m\to\infty.$$

Lemma 2.3 (Lions-Aubin [39], Theorem 5.1). *Let $T>0$, $1<p_0,p_1<\infty$. Consider $B_0\subset B\subset B_1$ Banach spaces, B_0,B_1 reflexives, B_0 with compact immersion in B. Define $W=\{u\mid u\in L^{p_0}(0,T;B_0)\,,\,u'\in L^{p_1}(0,T;B_1)\}$ equipped with the norm $||u||_W=||u||_{L^{p_0}(0,T;B_0)}+||u||_{L^{p_1}(0,T;B_1)}$. Then, W has compact immersion in $L^{p_0}(0,T;B)$.*

Lemma 2.4 (Martinez [40]). *Let $E:(0,\infty)\to(0,\infty)$ be a nonincreasing function and $\phi:[0,\infty)\to[0,\infty)$ an increasing C^1 function such that $\phi(0)=0$ and $\phi(t)\to\infty$ as $t\to\infty$.*

Assume that there exist $\sigma>-1$ and $\omega>0$ such that

$$\int_S^\infty E^{1+\sigma}(t)\phi'(t)\,dt \le \frac{1}{\omega}E^\sigma(0)E(S),\ 0\le S<\infty.$$

Then

$$\begin{aligned}
&E(t)=0\ \forall t\ge \frac{E(0)^\sigma}{\omega|\sigma|},\ \text{ if } -1<\sigma<0,\\
&E(t)\le E(0)\big(\frac{1+\sigma}{1+\omega\phi(t)}\big)^{1/\sigma}\ \forall t\ge 0,\ \text{ if } \sigma>0,\\
&E(t)\le E(0)e^{1-\omega\phi(t)}\ \forall t\ge 0,\ \text{ if } \sigma=0.
\end{aligned}$$

Lemma 2.5 (Levine [41], Qin-Rivera [42]). *Suppose that $\phi(t)\in C^2[0,\infty)$ is a positive function satisfying*

$$\phi(t)\phi''(t)-(1+\gamma)(\phi'(t))^2\ge -2C_1\phi(t)\phi'(t)-C_2(\phi(t)^2,$$

being $C_1,C_2\ge 0$ and $\gamma>0$ are constants. If

$$C_1+C_2\ge 0,\phi(0)>0,\phi'(0)+\gamma_2\frac{1}{\gamma}\phi(0)>0,$$

then

$$\lim_{t\to T_-}\phi(t)=+\infty,$$

where

$$T\le \frac{1}{2\sqrt{C_1^2+\gamma C_2}}\ln\left[\frac{\gamma_1\phi(0)+\gamma\phi'(0)}{\gamma_2\phi(0)+\gamma\phi'(0)}\right],$$

and

$$\gamma_1=-C_1+\sqrt{C_1^2+\gamma C_2},\quad \gamma_2=-C_1-\sqrt{C_1^2+\gamma C_2}.$$

3. The potential well

In this section we use the potential theory, a power full tool in the study of the global existence of solution to partial differential equation. See Payne-Sattinger [43]. It is well-known that the energy of a PDE system, in some sense, splits into the kinetic and the potential energy.

The energy of the problem (1.3)-(1.6) is given by

$$E(t)=\frac{1}{2}\int_\Omega |u'(t)|^2\,\mathrm{d}x+\frac{1}{p^2}\int_\Omega |u(t)|^p\,\mathrm{d}x+\frac{1}{2}\int_\Omega|\theta(t)|^2\,\mathrm{d}x+\frac{1}{p}\int_\Omega|\nabla u(t)|^p\,\mathrm{d}x-\frac{1}{p}\int_\Omega |u(t)|^p\ln|u(t)|\,\mathrm{d}x.$$

Mutiplying (1.3) by u', (1.4) by θ, performing integration by parts and using (1.6) we obtain

$$\frac{d}{\mathrm{d}t}E(t)=-\alpha||u'(t)||_2^2-||\nabla\theta(t)||_2^2. \tag{3.1}$$

We introduce the functional

$$J(u(t))=\frac{1}{p^2}\int_\Omega |u(t)|^p\,\mathrm{d}x+\frac{1}{p}\int_\Omega|\nabla u(t)|^p\,\mathrm{d}x-\frac{1}{p}\int_\Omega |u(t)|^p\ln|u(t)|\,\mathrm{d}x.$$

The Nehari functional associated with $J(u(t))$ is $I:W_0^{1,p}(\Omega)\cap W_0^{1,2(p-1)}(\Omega)\to\mathbb{R}$ defined by

$$I(u(t))=\int_\Omega|\nabla u(t)|^p\,\mathrm{d}x-\int_\Omega |u(t)|^p\ln|u(t)|\,\mathrm{d}x. \tag{3.2}$$

Associated with the $J(\lambda u(t))$ we have the well known *Nehari Manifold* given by

$$\mathcal{N} \overset{\text{def}}{=} \left\{ u(t) \in W_0^{1,p}(\Omega) \cap W_0^{1,2(p-1)}(\Omega)/\{0\} : \left[\frac{\mathrm{d}}{\mathrm{d}\,\lambda} I(\lambda u(t)) \right]_{\lambda=1} = 0 \right\}$$
$$= \left\{ u(t) \in W_0^{1,p}(\Omega) \cap W_0^{1,2(p-1)}(\Omega)/\{0\} : \int_\Omega |\nabla u(t)|^p \,\mathrm{d}x = \int_\Omega |u(t)|^p \ln|u(t)| \,\mathrm{d}x \right\}.$$

Now, we introduce the potential well (stable set)

$$\mathcal{W}_1 = \left\{ u(t) \in W_0^{1,p}(\Omega) \cap W_0^{1,2(p-1)}(\Omega)/\{0\} : \int_\Omega |\nabla u(t)|^p \,\mathrm{d}x > \int_\Omega |u(t)|^p \ln|u(t)| \,\mathrm{d}\,x \right\} \cup \{0\}.$$

and the unstable set

$$\mathcal{W}_2 = \left\{ u(t) \in W_0^{1,p}(\Omega) \cap W_0^{1,2(p-1)}(\Omega)/\{0\} : \int_\Omega |\nabla u(t)|^p \,\mathrm{d}x < \int_\Omega |u(t)|^p \ln|u(t)| \,\mathrm{d}\,x \right\}.$$

We define as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [44],

$$d \overset{\text{def}}{=} \inf_{u(t)\in W_0^{1,p}(\Omega)/\{0\}} \sup_{0\le\lambda} J(\lambda u(t)).$$

It is well-known that under H the depth of the well d is a strictly positive constant, see [[45], Theorem 4.2], and

$$d = \inf_{u(t)\in\mathcal{N}} J(u(t)).$$

The source term induces a potential energy in the system that act in opposed to effect of the stabilizing mechanism. In this sense, it is possible that the energy from the source term destabilize all the system and produce a blow-up a finite time. For provide a global solution, the stability set $\mathcal{W}_1$ create a valley or a well of the depth d, see Y. Ye [27], where the potential energy of the solution can never escape the potential well.

We will prove that $\mathcal{W}_1$ is invariant set for sub-critical initial energy.

Proposition 3.1. *Let* $u_0 \in \mathcal{W}_1$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$. *If* $E(0) < d$ *then* $u(t) \in \mathcal{W}_1$.

Proof. Let $T > 0$ be the maximum existence time. From (3.1) we get

$$E(t) \le E(0) < d, \text{ for all } t \in [0, T).$$

and then,

$$\frac{1}{2}\int_\Omega |u'(t)|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta(t)|^2 \,\mathrm{d}x + J(u(t)) < d, \text{ for all } t \in [0, T),$$

that is,

$$E(t) < d, \text{ for all } t \in [0, T). \tag{3.3}$$

Arguing by contradiction, we suppose that there exists a first $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$ and $I(u(t)) > 0$ for all $0 \le t < t_0$, that is,

$$\int_\Omega |\nabla u(t_0)|^p \,\mathrm{d}x = \int_\Omega |u(t_0)|^p \ln|u(t_0)| \,\mathrm{d}x.$$

From the definition of $\mathcal{N}$, we have that $u(t_0) \in \mathcal{N}$, which leads to

$$J(u(t_0)) \ge \inf_{u(t)\in\mathcal{N}} J(u(t)) = d.$$

By definition of $E(t)$,

$$\frac{1}{2}\int_\Omega |u'(t_0)|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta(t_0)|^2 \,\mathrm{d}x + J(u(t_0)) \ge d, \text{ it holds that, } E(t_0) \ge d,$$

which contradicts with (3.3). Then $u(t) \in \mathcal{W}_1$ for all $t \in [0, T)$. □

4. Galerkin basis

From Sobolev immersion, we have

$$W_0^{\nu,q}(\Omega) \hookrightarrow W_0^{\nu-k,q_k}(\Omega), \ \frac{1}{q_k} = \frac{1}{q} - \frac{k}{n}.$$

Choosing $q_k = p$, $\nu - k = 1$, and $q = 2$, we get

$$\nu = 1 + \frac{n}{2} - \frac{n}{p} = 1 + \frac{n(p-2)}{2p} > 0$$

and we obtain a Hilbert Space $H_0^{\nu}(\Omega)$ such that

$$H_0^{\nu}(\Omega) = W_0^{\nu,2}(\Omega) \hookrightarrow W_0^{1,p}(\Omega).$$

Let s an integer for which $s > \nu$. We have

$$H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,2(p-1)}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

According to the Rellich-Kondrachov theorem, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, so is also the immersion $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$. From spectral theory, there exists an operator defined by

$$\{H_0^s(\Omega),\ L^2(\Omega),\ ((\cdot,\cdot))_{H_0^s(\Omega)}\}$$

and a sequence of eigenvectors $(v_j)_{j\in\mathbb{N}}$ of this operator such that

$$((v_j, v))_{H_0^s(\Omega)} = \lambda_j (v_j, v), \ \text{for all } v \in H_0^s(\Omega)$$

with $\lambda_j > 0,\ \lambda_j \leq \lambda_{j+1},\ \text{ and }\ \lambda_j \to +\infty\ \text{ as }\ j \to +\infty$. Moreover $(v_j)_{j\in\mathbb{N}}$ is a complete orthonormal system in $L^2(\Omega)$ and $\left(w_j = \frac{v_j}{\sqrt{\lambda_j}}\right)_{j\in\mathbb{N}}$ is a complete orthonormal system in $H_0^s(\Omega)$. Then $(w_j)_{j\in\mathbb{N}}$ yields a "Galerkin basis" for both $W_0^{1,p}(\Omega)$ and $L^2(\Omega)$.

5. Global solution

Theorem 5.1. *Consider $E(0) < d$. Given $u_0 \in \mathcal{W}_1$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$, there exist functions $u,\ \theta\colon \Omega \times (0,T) \to \mathbb{R}$ in the class*

$$\begin{aligned} u &\in L^\infty(0,T; W_0^{1,p}(\Omega)),\\ u' &\in L^\infty(0,T; L^2(\Omega)),\\ \theta &\in L^\infty(0,T; H_0^1(\Omega)), \end{aligned}$$

such that, for all $\phi \in W_0^{1,p}(\Omega)$, $\psi \in L^2(\Omega)$

$$\frac{d}{dt}(u',\phi) + \langle -\Delta_p u, \phi\rangle_p + (\theta,\phi) = (|u|^{p-2}u\ln|u|, \phi) \ \textit{in}\ D'(0,T), \tag{5.1}$$

$$\frac{d}{dt}(\theta,\psi) + (-\Delta\theta,\psi) = (u',\psi) \ \textit{in}\ D'(0,T), \tag{5.2}$$

$$u(x,0) = u_0(x),\ \ u'(x,0) = u_1(x),\ \ \theta(x,0) = \theta_0(x)\ \ \textit{a.e. in}\ \Omega. \tag{5.3}$$

Proof. Let's use the Galerkin basis obtained in the previous section. For each $m \in \mathbb{N}$, let us put

$$V_m = \mathrm{Span}\{w_1, w_2, \dots, w_m\}.$$

We search for functions

$$u_m(t) = \sum_{j=1}^{m} f_{jm}(t) w_j, \quad \theta_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j,$$

such that any $\phi, \psi \in V_m$, $u_m(t)$ and $\theta_m(t)$ satisfies the following approximate problem

$$\frac{d}{\mathrm{d}t}(u_m'(t),\phi)+\langle -\Delta_p u_m(t),\phi\rangle_p + (\theta_m(t),\phi) = (|u_m(t)|^{p-2}u_m(t)\ln|u_m(t)|,\phi), \tag{5.4}$$

$$\frac{d}{\mathrm{d}t}(\theta_m(t),\psi) + (-\Delta\theta_m(t),\psi) = (u_m'(t),\psi), \tag{5.5}$$

with the initial conditions $u_m(0)=u_{0m}$, $u_m'(0)=u_{1m}$ and $\theta_m(0)=\theta_{0m}$, where u_{0m}, u_{1m} and θ_{0m} are choose so that

$$u_{0m}\to u_0 \;\in\; W_0^{1,p}(\Omega),\;\; u_{1m}\to u_1 \; in \; L^2(\Omega) \;\;\text{ and }\;\; \theta_{0m}\to\theta_0 \; in \; H_0^1(\Omega). \tag{5.6}$$

Putting $\phi = w_i$, $\psi = w_i$, $i = 1, 2, \ldots, m$, and using

$$u_m''(t) = \sum_{j=1}^{m} f_{jm}''(t)w_j(x), \quad \Delta_p u_m(t) = \sum_{j=1}^{m} f_{jm}(t)\Delta_p w_j(x),$$
$$\theta_m'(t) = \sum_{j=1}^{m} g_{jm}'(t)w_j(x), \quad \Delta\theta_m(t) = \sum_{j=1}^{m} g_{jm}(t)\Delta w_j(x),$$

we observe that (5.4)-(5.5) leads to a system of ODEs in the variable t that has a local solution $u_m(t), \theta_m(t)$ in a interval $[0, t_m)$ by virtue of Carathéodory's theorem. In the next step we obtain a priori estimates for the solution $u_m(t), \theta_m(t)$ so that they can be extended to the whole interval $[0, T]$, $T > 0$.

5.1 A priori estimates

Replacing $\phi = u_m'(t)$, $\psi = \theta_m(t)$ in the approximate equation (5.4), (5.5) we get

$$(u_m''(t),u_m'(t)) + \langle -\Delta_p u_m(t),u_m'(t)\rangle_p + (\theta_m(t),u_m'(t)) = (|u_m(t)|^{p-2}u_m(t)\ln|u_m(t)|,u_m'(t)), \tag{5.7}$$
$$(\theta_m'(t),\theta_m(t)) + (-\Delta\theta_m(t),\theta_m(t)) = (u_m'(t),\theta_m(t)), \tag{5.8}$$

Let $z \in D(0, t_m)$. We denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the duality pairing between D' and D. So we have

$$\langle (u_m''(t),u_m'(t)),z\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |u_m'(t)|^2\,\mathrm{d}x, z\right\rangle, \tag{5.9}$$

$$\langle \langle -\Delta_p u_m(t),u_m'(t)\rangle_p,z\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p}\int_\Omega |\nabla u_m(t)|^p\,\mathrm{d}x, z\right\rangle, \tag{5.10}$$

$$\langle (u_m'(t),u_m'(t)),z\rangle = \left\langle \int_\Omega |u_m'(t)|^2\,\mathrm{d}x, z\right\rangle, \tag{5.11}$$

$$\begin{aligned}\big\langle (|u_m(t)|^{p-2}u_m(t)\ln u_m(t),u_m'(t)),z\big\rangle &= \left\langle \frac{1}{p}\frac{d}{\mathrm{d}t}\int_\Omega |u_m(t)|^p\ln u_m(t)\,\mathrm{d}x, z\right\rangle \\ &\quad - \left\langle \frac{1}{p^2}\frac{d}{\mathrm{d}t}\int_\Omega |u_m(t)|^p\,\mathrm{d}x, z\right\rangle,\end{aligned} \tag{5.12}$$

$$\langle (\theta_m'(t),\theta_m(t)),z\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\theta_m(t)|^2\,\mathrm{d}x, z\right\rangle, \tag{5.13}$$

$$\langle (-\Delta\theta_m(t),\theta_m(t))\,,z\rangle = \left\langle \int_\Omega |\nabla\theta_m(t)|^2\,\mathrm{d}x, z\right\rangle. \tag{5.14}$$

Replacing (5.9), (5.10), (5.11), (5.12), (5.13), (5.14) in (5.7) and (5.8) we obtain in $D'(0, t_m)$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m(t) = -\int_\Omega |\nabla\theta_m(t)|^2\,\mathrm{d}x - \int_\Omega |u_m'(t)|^2\,\mathrm{d}x, \tag{5.15}$$

from where follows that the approximate energy

$$\begin{aligned}E_m(t) &= \frac{1}{2}\int_\Omega |u_m'(t)|^2\,\mathrm{d}x + \frac{1}{p^2}\int_\Omega |u_m(t)|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta_m(t)|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u_m(t)|^p\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u_m(t)|^p\ln|u_m(t)|\,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |u_m'(t)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta_m(t)|^2\,\mathrm{d}x + J(u(t))\end{aligned}$$

satisfies

$$\begin{aligned} E_m(t) &\le E_m(0) \\ &= \frac{1}{2}\int_\Omega |u_m'(0)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta_m(0)|^2\,\mathrm{d}x + J(u_m(0)). \end{aligned}$$

We have that $J(u_m(0)) < d$ in $\mathcal{W}_1$. By to convergence of initial data (5.6), there exists a constant $C > 0$ independent of t and m such that

$$\frac{1}{2}\int_\Omega |u_m'(0)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta_m(0)|^2\,\mathrm{d}x \le C.$$

With the estimate $E_m(t) \le E_m(0) \le C$ we can extend the approximate solutions $u_m(t)$, $\theta_m(t)$ to the interval $[0,T], T > 0$. By using (5.15) we deduce

$$\int_0^T\!\!\int_\Omega |\nabla\theta_m(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega |u_m'(t)|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\!\!\int_\Omega |\nabla\theta_m(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega |u_m'(t)|^2\,\mathrm{d}x\,\mathrm{d}t + E_m(t) \le E_m(0) \le C. \tag{5.16}$$

To prove that (1.4)-(1.6) carrying a good energy structure in $\mathcal{W}_1$, we need show that the forcing term is $L^2(0,T;L^2(\Omega))$. Consider $\Omega = \Omega_1 \cup \Omega_2$ where

$$\Omega_1 = \{x \in \Omega \,:\, |u_m(t)(x)| \le 1\} \text{ and } \Omega_2 = \{x \in \Omega \,:\, |u_m(t)(x)| > 1\}.$$

From

$$\int_\Omega ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x = \int_{\Omega_1} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x + \int_{\Omega_2} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x.$$

We have

$$\int_{\Omega_1} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x \le |\Omega|. \tag{5.17}$$

Note that,

$$\begin{aligned} \int_{\Omega_2} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x &= \int_{\Omega_2} |u_m(t)|^{2p-4}|u_m(t)|^2|\ln|u_m(t)||^2\,\mathrm{d}x \\ &\le \int_{\Omega_2} |u_m(t)|^{2p-4}|u_m(t)|^4|\ln|u_m(t)||^2\,\mathrm{d}x \\ &= \int_{\Omega_2} |u_m(t)|^{2p}|\ln|u_m(t)|^2\,\mathrm{d}x \\ &= \int_{\Omega_2} ||u_m(t)|^{p}\ln|u_m(t)||^2\,\mathrm{d}x. \end{aligned}$$

Taking into account that $u_m(t) \in \mathcal{W}_1$ we obtain

$$\int_{\Omega_2} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x \le \int_\Omega |\nabla u|^p\,\mathrm{d}x. \tag{5.18}$$

From (5.17) and (5.18) we get

$$\int_\Omega ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x \le |\Omega| + \int_\Omega |\nabla u|^p\,\mathrm{d}x \le C. \tag{5.19}$$

Then we have

$$\begin{aligned} u_m(t) &\quad \text{is bounded in} \quad L^\infty(0,T;W_0^{1,p}(\Omega)), && (5.20)\\ u_m'(t) &\quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), && (5.21)\\ u_m'(t) &\quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)), && (5.22)\\ |u_m(t)|^{p-2}u_m(t)\ln u_m(t) &\quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)), && (5.23)\\ -\Delta_p u_m(t) &\quad \text{is bounded in} \quad L^\infty(0,T;W^{-1,p'}(\Omega)), && (5.24)\\ \theta_m(t) &\quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), && (5.25)\\ -\Delta\theta_m(t) &\quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)). && (5.26) \end{aligned}$$

Since our Galerkin basis was taken in the Hilbert space $L^2(\Omega)$ we can use the standard projection arguments as described in Lions [39], pages 75-76, to obtain an estimate for $u''_m(t)$. Let P_m be the orthogonal projection $P_m : L^2(\Omega) \to V_m$, that is

$$P_m h = \sum_{n=1}^{m} (h, w_j) w_j, \quad h \in L^2(\Omega).$$

Approximated problem (5.7) leads to

$$u''_m(t) = P_m \Delta_p u_m(t) - P_m \theta_m(t) - P_m u'_m(t) + P_m |u_m(t)|^{p-2} u_m(t) \ln |u_m(t)|.$$

As $-\Delta_p u_m(t) \in L^2(0,T;(W^{-1,p'}(\Omega))$, from estimates (5.23), (5.25) we obtain

$$u''_m(t) \quad \text{is bounded in} \quad L^\infty(0,T;W^{-1,p'}(\Omega)). \tag{5.27}$$

5.2 Passage to the limit

From (5.20)-(5.27) going to the suitable subsequence if necessary (which we continue to denote in the same way), there exist $u(t)$, $\theta(t)$ such that

$$\begin{aligned}
u_m(t) &\overset{*}{\rightharpoonup} u(t) \text{ in } L^\infty(0,T;W_0^{1,p}(\Omega)), &(5.28)\\
u'_m(t) &\overset{*}{\rightharpoonup} u'(t) \text{ in } L^\infty(0,T;L^2(\Omega)), &(5.29)\\
u'_m(t) &\rightharpoonup u'(t) \text{ in } L^2(0,T;L^2(\Omega)), &(5.30)\\
-\Delta_p u_m(t) &\overset{*}{\rightharpoonup} \mathcal{X}_1(t) \text{ in } L^\infty(0,T;W^{-1,p'}(\Omega)), &(5.31)
\end{aligned}$$

$$\begin{aligned}
|u_m(t)|^{p-2} u_m(t) \ln u_m(t) &\rightharpoonup \mathcal{X}_2(t) \text{ in } L^2(0,T;L^2(\Omega)), &(5.32)\\
\theta_m(t) &\rightharpoonup \theta(t) \text{ in } L^2(0,T;L^2(\Omega)), &(5.33)\\
-\Delta\theta_m(t) &\overset{*}{\rightharpoonup} -\Delta\theta(t) \text{ in } L^\infty(0,T;L^2(\Omega). &(5.34)
\end{aligned}$$

Applying the Lions-Aubin compactness lemma, from (5.27), (5.28) and (5.29) we get

$$\begin{aligned}
u_m(t) &\to u(t) \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ and } a.e. \text{ in } Q, &(5.35)\\
u'_m(t) &\to u'(t) \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ and } a.e. \text{ in } Q. &(5.36)
\end{aligned}$$

We need to prove that $\mathcal{X}_1(t) = -\Delta_p u(t)$. The following elementary inequality

$$\left| |x|^{p-2}x - |y|^{p-2}y \right| \le C \left(|x|^{p-2} + |y|^{p-2} \right) |x-y| \tag{5.37}$$

is a consequence of the Mean Value Theorem. Using (5.37) and Hölder generalized inequality with

$$\frac{p-2}{2(p-1)} + \frac{1}{2} + \frac{1}{2(p-1)} = 1,$$

we deduce, for $z \in \mathcal{D}(0,T)$ and $v \in V_m$, that

$$\begin{aligned}
\left| \int_0^T \langle (-\Delta_p u_m(t)) - (-\Delta_p u(t)), v \rangle z(t) dt \right| &= \left| \int_0^T\!\!\int_\Omega \left(|\nabla u_m(t)|^{p-2} \nabla u_m(t) - |\nabla u(t)|^{p-2} \nabla u(t) \right) \nabla v \; dx z(t)\, dt \right| \\
&\le C|\theta|_\infty \int_0^T\!\!\int_\Omega \left(|\nabla u_m(t)|^{p-2} + |\nabla u(t)|^{p-2} \right) |\nabla u_m(t) - \nabla u(t)| |\nabla v| \, dx dt \\
&\le C_1 \int_0^T \Big(\|\nabla u_m(t)\|_{2(p-1)}^{p-2} + \|\nabla u(t)\|_{2(p-1)}^{p-2} \Big) |\nabla u_m(t) - \nabla u(t)| \, \|\nabla v\|_{2(p-1)} dt,
\end{aligned}$$

that leads to

$$\left| \int_0^T \langle (-\Delta_p u_m(t)) - (-\Delta_p u(t)), v \rangle_p \, z(t) \, \mathrm{d}t \right| \le C \int_0^T |\nabla u_m(t) - \nabla u(t)| \, \mathrm{d}t. \tag{5.38}$$

Now, from (5.28) and (5.29), by lemma 2.1 we have

$$u_m \to u \text{ in } C([0,T]\,;\, L^2(\Omega)).$$

whence

$$\nabla u_m(t) \to \nabla u(t) \text{ a. e. in } [0,T].$$

Therefore, by (5.31) and (5.38) we have $\mathcal{X}_1(t) = -\Delta_p u$, that is

$$-\Delta_p u_m(t) \rightharpoonup -\Delta_p u(t) \text{ in } L^2(0,T;W^{-1,p'}(\Omega)), \tag{5.39}$$

Now we will prove $\mathcal{X}_2(t) = |u(t)|^{p-2}u(t)\ln u(t)$. From (5.19) we have

$$|u_m|^{p-2}u_m \ln|u_m| \text{ is bounded in } L^2(0,T;L^2(\Omega)) = L^2(Q). \tag{5.40}$$

Using continuity of function $s \to |s|^{p-2}s\ln|s|$ and (5.35) we have

$$|u_m|^{p-2}u_m \ln|u_m| \to |u|^{p-2}u\ln|u| \text{ a.e. in } Q. \tag{5.41}$$

Then, by using Lions's lemma, (5.40) and (5.41) leads to

$$|u_m|^{p-2}u_m \ln|u_m| \rightharpoonup |u|^{p-2}u\ln|u| \text{ in } L^2(0,T;L^2(\Omega)). \tag{5.42}$$

Now, with the convergences (5.29), (5.39), (5.42), (5.33) and (5.34) we can pass to the limit in the approximate system and we get (5.1),(5.2). The verification of the initial data is a routine procedure. The prove of existence is complete. □

6. Polynomial decay for $E(0) < d$

In this section, we prove the $||u||_p^p$ decay polynomially for subcritical level of initial energy.

Theorem 6.1. *Let u_0 in the stability set $\mathcal{W}_1$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$. If $E(0) < d$ then the weak solution $u(t)$ of the problem* (1.3)-(1.6) *decay polynomially. That is,*

$$\|u(t)\|_p^p \le \|u(0)\|_p^p \left[\frac{1+\sigma}{1+\omega t}\right]^{\frac{1}{\sigma}}$$

where $\sigma > \dfrac{1}{2}$, $\omega = \dfrac{[\|u(0)\|_p^p]^\sigma}{C}$, $C > 0$.

Proof. As $\ln|u| \le |u|$, we have

$$\int_\Omega |u|^p \ln|u| \,\mathrm{d}x \le \int_\Omega |u|^{p+1} \,\mathrm{d}x = \|u\|_{p+1}^{p+1}.$$

By Hölder inequality we obtain

$$\|u\|_{p+1}^{p+1} \le \|u\|_p^{\nu(p+1)} \|u\|_q^{(1-\nu)(p+1)}, \quad \nu \in (0,1).$$

Applying Young inequality

$$\|u\|_{p+1}^{p+1} \le \frac{\varepsilon}{p}\|u\|_p^{\nu(p+1)p} + \frac{C_0(\varepsilon)}{q}\|u\|_q^{q(1-\nu)(p+1)}$$

with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $q < p$, and then,

$$\|u\|_{p+1}^{p+1} \le \frac{\varepsilon}{p}\|u\|_p^{\nu(p+1)p} + C(\varepsilon)\|u\|_q^{q(1-\nu)(p+1)}.$$

For $\nu = \dfrac{1}{2}$ we have

$$\int_\Omega |u|^p \ln|u| \,\mathrm{d}x \le \|u\|_{p+1}^{p+1} \le \frac{\varepsilon}{p}\|u\|^{[\frac{p+1}{2}]p} + C(\varepsilon)\|u\|_q^{[\frac{p+1}{2}]q}. \tag{6.1}$$

We define

$$L(t) = N\|u\|_q^{[\frac{p+1}{2}]q} + \|\nabla u\|_p^p - \int_\Omega |u|^p \ln|u| \,\mathrm{d}x. \tag{6.2}$$

As $u \in W_1$ we get $L(t) > 0$. By using (6.1) and Poincaré inequality in (6.2) we obtain

$$\begin{aligned} L(t) &\geq N\|u\|_q^{[\frac{p+1}{2}]q} + C_p\|u\|_p^p - \|u\|_{p+1}^{p+1} \\ &\geq N\|u\|_q^{[\frac{p+1}{2}]q} + C_p\|u\|_p^p - \frac{\varepsilon}{p}\|u\|_p^{[\frac{p+1}{2}]p} - C(\varepsilon)\|u\|_p^{[\frac{p+1}{2}]q} \\ &\geq (N - C(\varepsilon))\,\|u\|_q^{[\frac{p+1}{2}]q} + \|u\|_p^p\left(C_p - \frac{\varepsilon}{p}\right)\|u\|_p^{\frac{p+1}{2}} \end{aligned}$$

Choosing $N, \varepsilon > 0$ such that $C_p - \dfrac{\varepsilon}{p} > C > 0$ and $N - C(\varepsilon) > 0$ we have

$$L(t) \geq C\left[\|u\|_p^p\right]^{\frac{p+1}{2}}.$$

As $p > 2$,

$$\begin{aligned} \frac{p+1}{2} &= \frac{p}{2} + \frac{1}{2} = \frac{p}{2} + \frac{1}{2} - 1 + 1 = \frac{p}{2} - \frac{1}{2} + 1 \\ &= \sigma + 1, \;\; \sigma > \frac{1}{2}. \end{aligned}$$

Then

$$\|u\|_p^{p[\frac{p+1}{2}]} = \left[\|u\|_p^p\right]^{\sigma+1}, \;\; \sigma > \frac{1}{2} \; and$$

we obtain

$$L(t) \geq C\left[\|u\|_p^p\right]^{\sigma+1}, \;\; \sigma > \frac{1}{2}. \tag{6.3}$$

By other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p \leq p^2\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq 0,$$

that is, $\|u(t)\|_p^p$ is nonincreasing function. Then $-\dfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p \geq 0$. For each $\infty > T > S \geq 0$, let $t > 0$ such that $t \in (S,T)$ and define

$$A = \left\{t \in (S,T)\,;\; -\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p > L(t)\right\}.$$

If $t \in (S,T)$ satisfy

$$-\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p \leq L(t)$$

consider $0 < \eta(t) < \infty$ such that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p\,\eta(t) \geq L(t),$$

and take

$$\overline{A} = \left\{t \in (S,T)\,;\; -\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p\,\eta(t) \geq L(t)\right\}.$$

Let

$$\eta = \sup\{\eta(t);\; t \in \overline{A},\; 0 < \eta(t) < \infty\}.$$

Then $0 < \eta < \infty$ and

$$\begin{aligned} \int_S^T L(t)\,\mathrm{d}t &= \int_A L(t)\,\mathrm{d}t + \int_{\overline{A}} L(t)\,\mathrm{d}t \\ &\leq (1+\eta)\int_S^T -\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p\,\mathrm{d}t \\ &\leq (1+\eta)\|u(S)\|_p^p, \;\; \forall\, S \geq 0. \end{aligned} \tag{6.4}$$

From (6.3) and (6.4)

$$\begin{aligned}\int_S^T \left[\|u(t)\|_p^p\right]^{\sigma+1} \mathrm{d}t &\leq C^{-1} \int_S^T L(t)\,\mathrm{d}t \\ &\leq C^{-1}(1+\eta)\|u(S)\|_p^p \\ &\leq \frac{1}{\omega}\left[\|u(0)\|_p^p\right]^{\sigma} \|u(S)\|_p^p\end{aligned}$$

where $\omega = \dfrac{[\|u(0)\|_p^p]^{\sigma}}{C^{-1}(1+\eta)}$.

From Lemma 2.4, with $E(t) = \|u(t)\|_p^p$ and $\phi(t) = t$ we obtain

$$\|u(t)\|_p^p \leq \|u(0)\|_p^p \left[\frac{1+\sigma}{1+\omega t}\right]^{\frac{1}{\sigma}}$$

where $\sigma > \dfrac{1}{2}$, $\omega > 0$, $C > 0$. □

7. Blow-up in finite time

As in section 3 we can prove that $\mathcal{W}_2$ is invariant for sub-critical initial energy, that is,

Proposition 7.1. *Let $u_0 \in \mathcal{W}_2$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$. If $E(0) < d$ then $u(t) \in \mathcal{W}_2$.*

Theorem 7.1. *Let u_0 in the instability set $\mathcal{W}_2$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$ and $r > 1$ a fixed real number. If $||u_0||_2^2 < \sqrt{r-1}(u_0, u_1)$ and $E(0) < d$ then the weak solution $u(t)$ of the problem (1.3)-(1.6) will blow up at finite time. Namely, the maximum existence time $T < \infty$ and*

$$\lim_{t \to T_-} ||u(t)||_p^p = +\infty,$$

where

$$T < \frac{1}{\sqrt{r-1}} \ln\left[\frac{(r-1)(u_0, u_1) + \sqrt{r-1}||u_0||_2^2}{(r-1)(u_0, u_1) - \sqrt{r-1}||u_0||_2^2}\right].$$

Proof. By contradiction, suppose that the solution $u(t) \in \mathcal{W}_2$ is global. That is, we let $T = \infty$. Let $\phi(t) = |u(t)|^2$. We have $\phi'(t) = 2(u(t), u'(t))$. Applying Hölder inequality we get

$$2(u(t), u'(t)) \leq 2|u(t)|\,|u'(t)|$$

and

$$[\phi'(t)]^2 \leq 4|u(t)|^2\,|u'(t)|^2$$

that leads to

$$[\phi'(t)]^2 \leq 4\phi(t)|u'(t)|^2. \tag{7.1}$$

We have

$$(u''(t), u(t)) = -\|\nabla u(t)\|_p^p - \int_\Omega u(t)\theta(t)\,\mathrm{d}x - \frac{\alpha}{2}\frac{d}{\mathrm{d}t}|u(t)|^2 + \int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x.$$

Note that,

$$\begin{aligned}\phi''(t) &= 2|u'(t)|^2 + 2(u''(t), u(t)) \\ &= 2|u'(t)|^2 - 2\|\nabla u(t)\|_p^p - 2\int_\Omega u(t)\theta(t)\,\mathrm{d}x - \alpha\frac{d}{\mathrm{d}t}|u(t)|^2 + 2\int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x.\end{aligned}$$

By using

$$I(u(t)) = \|\nabla u(t)\|_p^p - \int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x$$

we get

$$\phi''(t) = 2|u'(t)|^2 - 2I(u(t)) - 2\int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x - 2\int_\Omega u(t)\theta(t)\,\mathrm{d}x - \alpha\frac{d}{\mathrm{d}t}|u(t)|^2 + 2\int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x.$$

that is

$$\phi''(t) = 2|u'(t)|^2 - 2I(u(t)) - 2\int_\Omega u(t)\theta(t)\,\mathrm{d}x - \alpha\frac{d}{\mathrm{d}t}|u(t)|^2.$$

Let $r > 0$ be a real number. By using (7.1) we obtain

$$\phi(t)\phi''(t) - \frac{r+3}{4}(\phi'(t))^2 \geq \phi(t)\left(2|u'(t)|^2 - 2I(u(t)) - 2\int_\Omega u(t)\theta(t)\,\mathrm{d}x\right) - \alpha\phi(t)\frac{d}{\mathrm{d}t}|u(t)|^2 - (r+3)\phi(t)|u'(t)|^2.$$

Applying Young inequality we get

$$\phi(t)\phi''(t) - \frac{r+3}{4}(\phi'(t))^2 \geq \phi(t)\left[-(r+1)|u'(t)|^2 - 2I(u(t)) - |u(t)|^2 - |\theta(t)|^2\right] - \alpha\phi(t)\frac{d}{\mathrm{d}t}|u(t)|^2. \tag{7.2}$$

From,

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|\theta(t)|^2 + J(u(t)).$$

we get

$$\begin{aligned}\frac{1}{2}|u'(t)|^2 &= -\frac{1}{2}|\theta(t)|^2 + E(t) - J(u(t)),\\ &\leq -\frac{1}{2}|\theta(t)|^2 + E(0) - J(u(t)),\\ &\leq -\frac{1}{2}|\theta(t)|^2 + d - J(u(t)).\end{aligned}$$

Then,

$$-(r+1)|u'(t)|^2 \geq (r+1)|\theta(t)|^2 + 2(r+1)(J(u(t)) - d). \tag{7.3}$$

By using (7.3) in (7.2) we obtain

$$\begin{aligned}\phi(t)\phi''(t) - \frac{r+3}{4}(\phi'(t))^2 &\geq \phi(t)\big[(r+1)|\theta(t)|^2 - |\theta(t)|^2\big]\\ &\quad + \phi(t)\big[2(r+1)(J(u(t)) - d)\big] + \phi(t)\big[-2I(u(t))\big]\\ &\quad - \alpha\phi(t)\frac{d}{\mathrm{d}t}|u(t)|^2 - \phi(t)|u(t)|^2.\end{aligned}$$

Now, observe that $\big[(r+1)|\theta(t)|^2 - |\theta(t)|^2\big] > 0$, $-2I(u(t)) > 0$ in $\mathcal{W}_2$, and $J(u(t)) - d > 0$ because

$$d = \inf_{u\in\mathcal{N}} J(u).$$

Namely, we have

$$\phi(t)\phi''(t) - (1+\gamma)(\phi(t))^2 \geq -2c_1\phi(t)\phi'(t) - c_2(\phi(t))^2,$$

where $c_1 = \dfrac{\alpha}{2}$, $c_2 = 1$, $\gamma = \dfrac{r-1}{4}$. By $\sqrt{r-1}(u_0, u_1) > |u_0|^2$, $c_1 + c_2 > 0$, $\phi(0) > 0$ we get $\phi'(0) + \gamma_2\gamma^{-1}\phi(0) > 0$, for $\gamma_1 = \dfrac{\sqrt{r-1}}{2}$ and $\gamma_2 = -\dfrac{\sqrt{r-1}}{2}$.

Finally, from Lemma 2.5 we concludes that

$$\lim_{t\to T_-} \|u(t)\|_p^p \geq c\lim_{t\to T_-} |u(t)|^2 = +\infty,$$

where

$$T < \frac{1}{\sqrt{r-1}}\ln\left[\frac{(r-1)(u_0,u_1) + \sqrt{r-1}|u_0|^2}{(r-1)(u_0,u_1) - \sqrt{r-1}|u_0|^2}\right],$$

which contradicts $T = \infty$. Then $u(t)$ blows up in finite time. □

8. Final comment

In recent years, results on global well-posedness, local well-posedness, blow-up, and asymptotic behavior of thermoelastic system have been studied. However, when considering the p-Laplacian operator, few results are known. We analyze the competition between the logarithmic source and the stabilization power given by the temperature difference. We show the existence of a global solution and the polynomial decay in a suitable stability set created from the Nehari Manifold. On the other hand, we prove the blow-up in finite time out of the stability set. We hope that the results presented here will be a font of inspiration for future research related to the topic.

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Interpolative KMK-Type Fixed-Figure Results

Nihal Taş*

Abstract
Fixed-figure problem has been introduced as a generalization of fixed circle problem and investigated a geometric generalization of fixed point theory. In this sense, we prove new fixed-figure results with some illustrative examples on metric spaces. For this purpose, we use KMK-type contractions, that is, Kannan type and Meir-Keeler type contractions.

Keywords: *Fixed figure; fixed point; KMK-type contraction; metric space.*

AMS Subject Classification (2020): *Primary: 54H25 ; Secondary: 47H09; 47H10.*

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1. Introduction

In recent years, fixed-point theory has been generalized using the geometric approaches. For this purpose, fixed-circle problem has been occurred as a geometric generalization to the fixed-point theory when the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ has more than one fixed point [1]. In many studies, there are different solutions to this problem with applications on metric and some generalized metric spaces (for example, see [2], [3], [4], [5], [6], [7], [8] and [9]). After than, this problem has been extended to fixed-figure problem [10]. For this problem, the following notions were defined (see [11], [12], [1] and [10]).

Let $(\mathfrak{X}, \mathfrak{d})$ be a metric space, $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ a self-mapping and $\mathfrak{x}_0, \mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$, $\mathfrak{r} \in [0, \infty)$. Then,

(a) the circle $\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}}$ is defined by

$$\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_0) = \mathfrak{r}\}.$$

(b) the disc $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}}$ is defined by

$$\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_0) \leq \mathfrak{r}\}.$$

(c) the ellipse $\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ is defined by

$$\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2) = \mathfrak{r}\}.$$

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(d) the hyperbola $\mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ is defined by

$$\mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x} \in \mathfrak{X} : |\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)| = \mathfrak{r}\}.$$

(e) the Cassini curve $\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ is defined by

$$\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)\, \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2) = \mathfrak{r}\}.$$

(f) the Apollonius circle $\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ is defined by

$$\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \left\{\mathfrak{x} \in \mathfrak{X} - \{\mathfrak{x}_2\} : \frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)} = \mathfrak{r}\right\}.$$

(g) the k-ellipse $\mathfrak{E}[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}]$ is defined by

$$\mathfrak{E}[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}] = \left\{\mathfrak{x} \in \mathfrak{X} : \sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_i) = \mathfrak{r}\right\}.$$

A geometric figure $\mathcal{F}$ contained in the fixed point set $Fix(\mathfrak{T}) = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{x} = \mathfrak{T}\mathfrak{x}\}$ is called a *fixed figure* (a fixed circle, a fixed disc, a fixed ellipse, a fixed hyperbola, a fixed Cassini curve, etc.) of the self-mapping $\mathfrak{T}$ (see [10]). Some fixed-figure results were obtained using different aspects (see [13], [11], [12], [3], [10], [14] and [15] for more details).

In this paper, we investigate some solutions to the fixed-figure problem on metric spaces. To do this, we modify the Kannan type and Meir-Keeler type contractions used in the fixed-point theorems. We give some illustrative examples related to the proved fixed-figure results.

2. Main results

In this section, we present some solutions to the fixed-figure problem using Kannan type (see [16] and [17]) and Meir-Keeler type (see [18]) contractions on metric spaces. To do this, we inspire the used approaches in [19] and [20].

In the sequel, let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a self-mapping of a metric space $(\mathfrak{X}, \mathfrak{d})$ and the number $\mathfrak{r}$ defined as

$$\mathfrak{r} = \inf\{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} \notin Fix(\mathfrak{T})\}. \tag{2.1}$$

Also, in the examples of this section, we use the usual metric $\mathfrak{d}$.

The following theorem can be considered as a new fixed-disc or fixed-circle theorem.

Theorem 2.1. *If there exist $\mathfrak{x}_0 \in \mathfrak{X}$ and $\gamma \in (0,1)$ such that*

(a) There exists a $\delta(\mathfrak{r}) > 0$ so that

$$\frac{\mathfrak{r}}{2} < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_0)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_0) \le \mathfrak{r},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(b)

$$1 \le \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_0)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x})]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$, then we have

(i) $\mathfrak{x}_0 \in Fix(\mathfrak{T})$,

(ii) $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$,

(iii) $\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$.

Proof. (i) Let $\mathfrak{x}_0 \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we have

$$1 \le \mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x}_0) < [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x}_0)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x}_0)]^{1-\gamma} = \mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x}_0),$$

a contradiction. So it should be $\mathfrak{x}_0 \in Fix(\mathfrak{T})$.

(ii) If $\mathfrak{r} = 0$, then we have $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x}_0\}$ and from the condition (i), we get $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$.

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}}$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_0)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x})]^{1-\gamma} \tag{2.2}$$

and by the condition (a), we have

$$\frac{\mathfrak{r}}{2} < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_0)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_0) \leq \mathfrak{r}. \tag{2.3}$$

If we combine the inequalities (2.2) and (2.3), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_0)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x})]^{1-\gamma} \leq \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$.

(iii) It can be easily seen that $\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$ since $\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}}$ is a boundary of $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}}$. □

Example 2.1. Let $\mathfrak{X} = \{-1, 0, 1, 2\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}x = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 1 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.1 for $\mathfrak{x}_0 = 0$, $\gamma = \frac{1}{2}$ and $\delta(\mathfrak{r}) = 2$. Also, we have

$$\mathfrak{r} = \inf \{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = 2\} = 1$$

and

$$Fix(\mathfrak{T}) = \{-1, 0, 1\}$$

Consequently, $0 \in Fix(\mathfrak{T})$, $\mathfrak{D}_{0,1} = \{-1, 0, 1\} \subseteq Fix(\mathfrak{T})$ and $\mathfrak{C}_{0,1} = \{-1, 1\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.2. *If there exist $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$ and $\gamma \in (0, 1)$ such that*

(a) There exists a $\delta(\mathfrak{r}) > 0$ so that

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, T\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2) \leq \mathfrak{r}, \end{aligned}$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x}) + \mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(c) $\mathfrak{x}_1, \mathfrak{x}_2 \in Fix(\mathfrak{T})$,

then we have

$$\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Proof. Let $\mathfrak{r} = 0$. Then we have $\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x}_1\} = \{\mathfrak{x}_2\}$. From the condition (c), we get

$$\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x}) + \mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})]^{1-\gamma} \tag{2.4}$$

and by the condition (a), we have

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2) \leq r. \end{aligned} \tag{2.5}$$

If we combine the inequalities (2.4) and (2.5), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

□

Example 2.2. Let $\mathfrak{X} = \{-1, 1, 2, 3\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}\mathfrak{x} = \begin{pmatrix} -1 & 1 & 2 & 3 \\ -1 & 1 & 2 & 1 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.2 for $\mathfrak{x}_1 = -1$, $\mathfrak{x}_2 = 1$, $\gamma = \frac{1}{2}$ and $\delta(\mathfrak{r}) = 2$. Also, we have

$$\mathfrak{r} = \inf \{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = 3\} = 2$$

and

$$Fix(\mathfrak{T}) = \{-1, 1, 2\}$$

Consequently, $-1, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{E}_2(-1, 1) = \{-1, 1\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.3. *If there exist* $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$, $\gamma \in (0, 1)$ *and* $\mathfrak{r} > 0$ *such that*

(a) *There exists a* $\delta(\mathfrak{r}) > 0$ *so that*

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left|\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)\right|^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \left|\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2)\right| \leq \mathfrak{r}, \end{aligned}$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left|\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)\right|^{\gamma} \left|\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x}) - \mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})\right|^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(c) $\mathfrak{x}_1, \mathfrak{x}_2 \in Fix(\mathfrak{T})$,

then we have

$$\mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Proof. Let $\mathfrak{x} \in \mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left|\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)\right|^{\gamma} \left|\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x}) - \mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})\right|^{1-\gamma} \tag{2.6}$$

and by the condition (a), we have

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left|\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)\right|^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \left|\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2)\right| \leq \mathfrak{r}. \end{aligned} \tag{2.7}$$

If we combine the inequalities (2.6) and (2.7), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

□

Example 2.3. Let $\mathfrak{X} = \left\{-1, \frac{1}{2}1, 2, \frac{5}{2}, 3, 4\right\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}\mathfrak{x} = \begin{pmatrix} -1 & \frac{1}{2} & 1 & 2 & \frac{5}{2} & 3 & 4 \\ -1 & \frac{5}{2} & 1 & 2 & \frac{5}{2} & 3 & 4 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.3 for $\mathfrak{x}_1 = -1$, $\mathfrak{x}_2 = 1$, $\gamma = \frac{1}{3}$ and $\delta(\mathfrak{r}) = 2$. Also, we have

$$\mathfrak{r} = \inf \left\{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = \frac{1}{2}\right\} = 2$$

and

$$Fix(\mathfrak{T}) = \left\{-1, 1, 2, \frac{5}{2}, 3, 4\right\}$$

Consequently, $-1, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{H}_2(-1, 1) = \left\{-1, 1, 2, \frac{5}{2}, 3, 4\right\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.4. *If there exist* $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$ *and* $\gamma \in (0,1)$ *such that*

(a) *There exists a* $\delta(\mathfrak{r}) > 0$ *so that*

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1)\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2) \leq \mathfrak{r}, \end{aligned}$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1)\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x})\mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(c) $\mathfrak{x}_1, \mathfrak{x}_2 \in Fix(\mathfrak{T})$,

then we have

$$\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Proof. Let $\mathfrak{r} = 0$. Then we have $\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x}_1\} = \{\mathfrak{x}_2\}$. From the condition (c), we get

$$\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1)\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x})\mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})]^{1-\gamma} \tag{2.8}$$

and by the condition (a), we have

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1)\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2) \leq \mathfrak{r}. \end{aligned} \tag{2.9}$$

If we combine the inequalities (2.8) and (2.9), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

□

Example 2.4. Let $\mathfrak{X} = \{-\sqrt{3}, -1, 0, 1, \sqrt{3}, 2\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}\mathfrak{x} = \begin{pmatrix} -\sqrt{3} & -1 & 0 & 1 & \sqrt{3} & 2 \\ -\sqrt{3} & 1 & 0 & 1 & \sqrt{3} & 0 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.4 for $\mathfrak{x}_1 = -1$, $\mathfrak{x}_2 = 1$, $\gamma = \frac{8}{9}$ and $\delta(\mathfrak{r}) = 4$. Also, we have

$$\mathfrak{r} = \inf \left\{ \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = \frac{1}{2} \right\} = 2$$

and

$$Fix(\mathfrak{T}) = \left\{ -\sqrt{3}, -1, 0, 1, \sqrt{3} \right\}$$

Consequently, $-1, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{C}_2(-1, 1) = \{-\sqrt{3}, \sqrt{3}\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.5. *If there exist* $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$ *and* $\gamma \in (0,1)$ *such that*

(a) *There exists a* $\delta(\mathfrak{r}) > 0$ *so that*

$$\frac{\mathfrak{r}}{2} < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left[\frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)} \right]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \Longrightarrow \frac{\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2)} \leq \mathfrak{r},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,
(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left[\frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)}\right]^{\gamma} \left[\frac{\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x})}{\mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})}\right]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,
(c) $\mathfrak{x}_1, \mathfrak{x}_2 \in Fix(\mathfrak{T})$,
then we have

$$\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Proof. Let $\mathfrak{r} = 0$. Then we have $\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x}_1\} = \{\mathfrak{x}_2\}$. From the condition (c), we get

$$\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left[\frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)}\right]^{\gamma} \left[\frac{\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x})}{\mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})}\right]^{1-\gamma} \tag{2.10}$$

and by the condition (a), we have

$$\frac{\mathfrak{r}}{2} < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left[\frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)}\right]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \Longrightarrow \frac{\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2)} \leq \mathfrak{r}. \tag{2.11}$$

If we combine the inequalities (2.10) and (2.11), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

□

Example 2.5. Let $\mathfrak{X} = \left\{-1, 0, \frac{1}{3}, 1, 2, 3\right\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}x = \begin{pmatrix} -1 & 0 & \frac{1}{3} & 1 & 2 & 3 \\ -1 & 0 & \frac{1}{3} & 1 & 0 & 3 \end{pmatrix},$$

for all $x \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.5 for $\mathfrak{x}_1 = -1, \mathfrak{x}_2 = 1, \gamma = \frac{8}{9}$ and $\delta(\mathfrak{r}) = 4$. Also, we have

$$\mathfrak{r} = \inf \left\{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = \frac{1}{2}\right\} = 2$$

and

$$Fix(\mathfrak{T}) = \left\{-1, 0, \frac{1}{3}, 1, 3\right\}$$

Consequently, $-1, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{A}_2(-1, 1) = \left\{\frac{1}{3}, 3\right\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.6. *If there exist* $\mathfrak{x}_1, \mathfrak{x}_2, \ldots \mathfrak{x}_k \in \mathfrak{X}$ *and* $\gamma \in (0, 1)$ *such that*
(a) *There exists a* $\delta(\mathfrak{r}) > 0$ *so that*

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_i)\right]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \sum_{i=1}^{k} \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_i) \leq \mathfrak{r}, \end{aligned}$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_i)\right]^{\gamma} \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_i)\right]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(c) $\mathfrak{x}_1, \mathfrak{x}_2, \ldots \mathfrak{x}_k \in Fix(\mathfrak{T})$,

then we have

$$\mathfrak{C}\left[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}\right] \subseteq Fix(T).$$

Proof. Let $\mathfrak{r} = 0$. Then we have $\mathfrak{C}\left[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}\right] = \{\mathfrak{x}_1\} = \ldots = \{\mathfrak{x}_k\}$. From the condition (c), we get

$$\mathfrak{C}\left[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}\right] \subseteq Fix(\mathfrak{T}).$$

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{C}\left[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}\right]$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_i)\right]^{\gamma} \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_i)\right]^{1-\gamma} \tag{2.12}$$

and by the condition (a), we have

$$\begin{aligned} \frac{\mathfrak{r}}{2} \quad &< \quad [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_i)\right]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \sum_{i=1}^{k} \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_i) \leq \mathfrak{r}. \end{aligned} \tag{2.13}$$

If we combine the inequalities (2.12) and (2.13), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{C}\left[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}\right] \subseteq Fix(\mathfrak{T}).$$

□

Example 2.6. Let $\mathfrak{X} = \{-1, 0, 1, 2\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}\mathfrak{x} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 0 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.6 for $\mathfrak{x}_1 = -1, \mathfrak{x}_2 = 0, \mathfrak{x}_3 = 1, \gamma = \frac{1}{2}$ and $\delta(\mathfrak{r}) = 4$. Also, we have

$$\mathfrak{r} = \inf\left\{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = \frac{1}{2}\right\} = 2$$

and

$$Fix(\mathfrak{T}) = \{-1, 0, 1\}$$

Consequently, $-1, 0, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{C}[-1, 0, 1; 2] = \{0\} \subseteq Fix(\mathfrak{T})$.

3. Conclusion and future works

This paper is an example of the geometric approaches to fixed-point theory. The aim of this paper is to gain new solutions to the fixed-figure problem. For this paper, we use KMK-type contractions, that is, Kannan type and Meir-Keeler type contractions on metric spaces. This problem can be studied with different approaches on both metric spaces and some generalized metric spaces (for example, see [21], [22], [23] and the references therein).

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Family of Analytic Functions with Negative Coefficients Involving $q-$Analogue of Multiplier Transformation Operator

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Abstract
We introduce a new class of analytic functions with negative coefficients by using the $q-$analogue of multiplier transformation operator. Coefficient inequalities, distortion theorems, closure theorems, and some properties involving the modified Hadamard products, radii of close-to-convexity, starlikeness, and convexity, and integral operators associated with functions belonging to this class are obtained.

Keywords: *Univalent function; convolution; $q-$convex, $q-$starlike, $q-$analogue of multiplier transformation operator.*

AMS Subject Classification (2020): *30C45*

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1. Introduction

Let $\mathcal{A}(j)$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}(1) = \mathcal{A}$. For functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=j+1}^{\infty} b_k z^k \quad (j \in \mathbb{N}), \tag{1.2}$$

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the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by

$$(f*g)(z) = z + \sum_{k=j+1}^{\infty} a_k b_k z^k = (g*f)(z). \tag{1.3}$$

Quantum calculus or $q-$calculus is an ordinary calculus without limit. In recent years, the study of $q-$theory attracted the researches due to its applications in various branches of mathematics and physics, for example, in the areas of special functions, ordinary fractional calculus, $q-$difference, $q-$integral equations and in $q-$transform analysis (see, for instance, [1], [2], [3], [4], [5], [6], [7], [8], [9] and [10]).

For $f \in \mathcal{A}(j)$ given by (1.1) and $0 < q < 1$, the $q-$derivative of f is defined by (see [11], [12], [13], [14], [15] and [16])

$$D_{q,j}f(z) = \begin{cases} f'(0) & \text{if } z = 0, \\ \dfrac{f(z) - f(qz)}{(1-q)z} & \text{if } z \neq 0, \end{cases} \tag{1.4}$$

and $D^2_{q,j}f(z) = D_{q,j}\left(D_{q,j}f(z)\right)$. From (1.1) and (1.4), we deduce that

$$D_{q,j}f(z) = 1 + \sum_{k=j+1}^{\infty} [k]_q \, a_k z^{k-1} \quad (j \in \mathbb{N}; z \neq 0), \tag{1.5}$$

where $[k]_q$ is $q-$integer number k defined by

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + ... + q^{k-1} \quad (0 < q < 1). \tag{1.6}$$

We note that $D_{q,1}f(z) = D_q f(z)$ and

$$\lim_{q \to 1^-} D_{q,j}f(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. As a right inverse, the $q-$integral of f is introduced by

$$\int_0^z f(t)\, d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f\left(zq^k\right),$$

provided that the series converges (see [17] and [18]). For a function f given by (1.1), we observe that

$$\int_0^z f(t)\, d_q t = \frac{z^2}{[2]_q} + \sum_{k=j+1}^{\infty} \frac{a_k z^{k+1}}{[k+1]_q}$$

and

$$\lim_{q \to 1^-} \int_0^z f(t)\, d_q t = \frac{z^2}{2} + \sum_{k=j+1}^{\infty} \frac{a_k z^{k+1}}{k+1} = \int_0^z f(t)\, dt,$$

where $\int_0^z f(t)\, dt$ is the ordinary integral.

Making use of the $q-$derivative $D_{q,j}f(z)$, we introduce the subclasses $\mathcal{S}_{q,j}(\alpha)$ and $\mathcal{C}_{q,j}(\alpha)$ of the class $\mathcal{A}(j)$ for $0 < q < 1, j \in \mathbb{N}$ and $0 \leq \alpha < 1$ as follows:

$$\mathcal{S}_{q,j}(\alpha) = \left\{ f \in \mathcal{A}(j) : \Re \frac{zD_{q,j}f(z)}{f(z)} > \alpha, z \in \mathbb{U} \right\}, \tag{1.7}$$

$$\mathcal{C}_{q,j}(\alpha) = \left\{ f \in \mathcal{A}(j) : \Re \frac{D_{q,j}(zD_{q,j}f(z))}{D_{q,j}f(z)} > \alpha, z \in \mathbb{U} \right\}, \tag{1.8}$$

From (1.7) and (1.8), we have

$$f \in \mathcal{C}_{q,j}(\alpha) \Leftrightarrow zD_{q,j}f \in \mathcal{S}_{q,j}(\alpha).$$

We note that $\mathcal{S}_{q,1}(\alpha) = \mathcal{S}_q(\alpha)$ and $\mathcal{C}_{q,1}(\alpha) = \mathcal{C}_q(\alpha)$ (see [16]) and

$$\lim_{q\to 1^-} \mathcal{S}_{q,1}(\alpha) = \mathcal{S}(\alpha) \quad and \quad \lim_{q\to 1^-} \mathcal{C}_{q,1}(\alpha) = \mathcal{C}(\alpha),$$

where $\mathcal{S}(\alpha)$ and $\mathcal{C}(\alpha)$ are, respectively, the classes of starlike of order α and convex of order α in $\mathbb{U}$.

Now, we define the $q-$analogue of multiplier transformation operator

$$\mathcal{J}_{q,j}^{m}(l) : \mathcal{A}(j) \to \mathcal{A}(j) \quad (l > -1; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; j \in \mathbb{N}),$$

as follows:

$$\begin{aligned}
\mathcal{J}_{q,j}^{-m}(l) f(z) &= \frac{[l+1]_q}{z^l} \int_0^z t^{l-1} \mathcal{J}_{q,j}^{-(m-1)}(l) f(t)\, d_q t \quad (z \in \mathbb{U}),\\
&\;\;\vdots\\
\mathcal{J}_{q,j}^{-2}(l) f(z) &= \frac{[l+1]_q}{z^l} \int_0^z t^{l-1} \mathcal{J}_{q,j}^{-1}(l) f(t)\, d_q t \quad (z \in \mathbb{U}),\\
\mathcal{J}_{q,j}^{-1}(l) f(z) &= \frac{[l+1]_q}{z^l} \int_0^z t^{l-1} f(t)\, d_q t \quad (z \in \mathbb{U}),\\
\mathcal{J}_{q,j}^{0}(l) f(z) &= f(z) \quad (z \in \mathbb{U}),\\
\mathcal{J}_{q,j}^{1}(l) f(z) &= \frac{z^{1-l}}{[l+1]_q} D_{q,j}\left(z^l f(z)\right) \quad (z \in \mathbb{U}),\\
\mathcal{J}_{q,j}^{2}(l) f(z) &= \frac{z^{1-l}}{[l+1]_q} D_{q,j}\left(z^l \mathcal{J}_{q,j}^{1}(l) f(z)\right) \quad (z \in \mathbb{U}).\\
&\;\;\vdots\\
\mathcal{J}_{q,j}^{m}(l) f(z) &= \frac{z^{1-l}}{[l+1]_q} D_{q,j}\left(z^l \mathcal{J}_{q,j}^{m-1}(l) f(z)\right) \quad (z \in \mathbb{U}).
\end{aligned}$$

We see that for $f \in \mathcal{A}(j)$, we have

$$\mathcal{J}_{q,j}^{m}(l) f(z) = z + \sum_{k=j+1}^{\infty} \left(\frac{[l+k]_q}{[l+1]_q}\right)^m a_k z^k \tag{1.9}$$

$$(0 < q < 1; l > -1; m \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}; j \in \mathbb{N}).$$

It is readily verified from (1.9) that

$$q^l\, z D_{q,j}\left(\mathcal{J}_{q,j}^{m}(l) f(z)\right) = [l+1]_q\; \mathcal{J}_{q,j}^{m+1}(l) f(z) - [l]_q\, \mathcal{J}_{q,j}^{m}(l) f(z) \quad (m \in \mathbb{Z}). \tag{1.10}$$

We observe that the operator $\mathcal{J}_{q,j}^{m}(l)$ generalize several previously familiar operators, and we will show some of the interesting particular cases as follows:

(i) $\mathcal{J}_{q,j}^{m}(0) f(z) = \mathcal{S}_{q,j}^{m} f(z)$ and $\mathcal{J}_{q,1}^{m}(0) f(z) = \mathcal{S}_q^m f(z)\,(m \in \mathbb{N}_0)$ (see [19]);

(ii) $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{m}(0) f(z) = \mathcal{D}^m f(z)\,(m \in \mathbb{N}_0)$ (see [20], [21], [22] and [23]);

(iii) $\lim_{q\to 1^-} \mathcal{J}_{q,j}^{m}(l) f(z) = \mathcal{I}_{l,j}^{m} f(z)$ and $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{m}(l) f(z) = \mathcal{I}_l^m f(z)\,(l \geq 0; m \in \mathbb{N}_0)$ (see [24] and [25]);

(iv) $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{m}(1) f(z) = D^m f(z)\,(m \in \mathbb{N}_0)$ (see [26]);

(v) $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{-m}(1) f(z) = I^m f(z)\,(m \in \mathbb{N}_0)$ (see [27]);

(vi) $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{-1}(c) f(z) = F_c f(z) = \frac{1+c}{z^c}\int\limits_0^z t^{c-1} f(t)\, dt$ $(c > -1)$ is the well-known Bernardi integral operator [28].

With the help of the operator $\mathcal{J}_{q,j}^m(l)$, we say that a function f belonging to the class $\mathcal{A}(j)$ is in the class $\mathcal{L}_q^m(l,\lambda,\alpha;j)$ if and only if

$$\Re\left\{\frac{zD_{q,j}\left(\mathcal{J}_{q,j}^m(l) f(z)\right) + \lambda q z^2 D_{q,j}^2\left(\mathcal{J}_{q,j}^m(l) f(z)\right)}{(1-\lambda)\mathcal{J}_{q,j}^m(l) f(z) + \lambda z D_{q,j}\left(\mathcal{J}_{q,j}^m(l) f(z)\right)}\right\} > \alpha \tag{1.11}$$

$$(z \in \mathbb{U}; m \in \mathbb{Z}; 0 < q < 1; l > -1; 0 \leq \lambda \leq 1; 0 \leq \alpha < 1).$$

Let $\mathcal{T}(j)$ denote the subclass of $\mathcal{A}(j)$ consisting of functions of the form:

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k > 0; j \in \mathbb{N}) \tag{1.12}$$

Further, we define the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$ by

$$\mathcal{H}_q^m(l,\lambda,\alpha;j) = \mathcal{L}_q^m(l,\lambda,\alpha;j) \cap \mathcal{T}(j).$$

We note that

(i) $\lim_{q\to 1^-} \mathcal{H}_q^m(0,\lambda,\alpha;j) = \mathcal{P}(j;\lambda,\alpha,m)$ $(m \in \mathbb{N})$ (Aouf and Srivastava [29]);

(ii) $\lim_{q\to 1^-} \mathcal{H}_q^0(0,0,\alpha;1) = \mathcal{S}(\alpha)$ and $\lim_{q\to 1^-} \mathcal{H}_q^0(0,1,\alpha;1) = \mathcal{C}(\alpha)$ (Silverman [30]);

(iii) $\lim_{q\to 1^-} \mathcal{H}_q^0(0,0,\alpha;j) = \mathcal{S}(\alpha;j)$ and $\lim_{q\to 1^-} \mathcal{H}_q^0(0,1,\alpha;j) = \mathcal{C}(\alpha;j)$ (Chatterjea [31] and Srivastava et al. [32]);

(iv) $\mathcal{H}_q^m(0,\lambda,\alpha;j) = \mathcal{H}_q^m(\lambda,\alpha;j)$

$$= \left\{f \in \mathcal{T}(j) : \Re\left\{\frac{zD_{q,j}\left(\mathcal{S}_{q,j}^m f(z)\right) + \lambda q z^2 D_{q,j}^2\left(\mathcal{S}_{q,j}^m f(z)\right)}{(1-\lambda)\mathcal{S}_{q,j}^m f(z) + \lambda z D_{q,j}\left(\mathcal{S}_{q,j}^m f(z)\right)}\right\} > \alpha\right\};$$

(v) $\lim_{q\to 1^-} \mathcal{H}_q^m(l,\lambda,\alpha;j) = \mathcal{H}^m(l,\lambda,\alpha;j)$

$$= \left\{f \in \mathcal{T}(j) : \Re\left\{\frac{z\left(\mathcal{I}_{l,j}^m f(z)\right)' + \lambda z^2\left(\mathcal{I}_{l,j}^m f(z)\right)''}{(1-\lambda)\mathcal{I}_{l,j}^m f(z) + \lambda z\left(\mathcal{I}_{l,j}^m f(z)\right)'}\right\} > \alpha\right\}.$$

The present paper aims at providing a systematic investigation of the various interesting properties and characteristics of the general class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$.

2. Coefficient estimates

Unless otherwise mentioned, we assume throughout this section that $m \in \mathbb{Z}$, $j \in \mathbb{N}$, $0 < q < 1$, $l > -1$, $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $z \in \mathbb{U}$ and $[k]_q$ is given by (1.6).

Theorem 2.1. *Let the function f be defined by (1.12). Then $f \in \mathcal{H}_q^m(l,\lambda,\alpha;j)$ if and only if*

$$\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right) a_k \leq 1 - \alpha. \tag{2.1}$$

Proof. Assume that the inequality (2.1) holds true. Then we find that

$$\left|\frac{zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)+\lambda qz^{2}D_{q,j}^{2}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}{(1-\lambda)\mathcal{J}_{q,j}^{m}(l)f(z)+\lambda zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}-1\right|$$

$$\leq \frac{\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left([k]_q-1\right)\left\{1+\left([k]_q-1\right)\lambda\right\}a_k|z|^{k-1}}{1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k|z|^{k-1}}$$

$$\leq \frac{\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left([k]_q-1\right)\left\{1+\left([k]_q-1\right)\lambda\right\}a_k}{1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k}\leq 1-\alpha.$$

This shows that the values of the function

$$\phi(z)=\frac{zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)+\lambda qz^{2}D_{q,j}^{2}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}{(1-\lambda)\mathcal{J}_{q,j}^{m}(l)f(z)+\lambda zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)} \tag{2.2}$$

lie in a circle which is centered at $w=1$ and whose radius is $1-\alpha$. Hence f satisfies the condition (1.11).

Conversely, assume that the function f is in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then we have

$$\Re\left\{\frac{zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)+\lambda qz^{2}D_{q,j}^{2}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}{(1-\lambda)\mathcal{J}_{q,j}^{m}(l)f(z)+\lambda zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}\right\}$$

$$=\Re\left\{\frac{1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k|z|^{k-1}}{1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k|z|^{k-1}}\right\}>\alpha, \tag{2.3}$$

for some $\alpha\ (0\leq\alpha<1)$, $m\in\mathbb{Z}$, $0<q<1$, $l>-1$, $0\leq\lambda\leq1$ and $z\in\mathbb{U}$. Choose values of z on the real axis so that ϕ given by (2.2) is real. Upon clearing the denominator in (2.3) and letting $z\to1^-$ through real values, we can see that

$$1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k$$

$$\geq\alpha\left(1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k\right). \tag{2.4}$$

Thus we have the inequality (2.1). This completes the proof of Theorem 2.1. □

Corollary 2.1. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then*

$$a_k\leq\frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}\ (k\geq j+1;j\in\mathbb{N}) \tag{2.5}$$

The equality in (2.5) is attained for the function f given by

$$f(z)=z-\frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}z^k\quad(k\geq j+1;j\in\mathbb{N}). \tag{2.6}$$

Theorem 2.2. *If $0\leq\alpha_1<\alpha_2<1$, then*

$$\mathcal{H}_q^m(l,\lambda,\alpha_2;j)\subseteq\mathcal{H}_q^m(l,\lambda,\alpha_1;j). \tag{2.7}$$

Proof. Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha_2;j)$. Then, by Theorem 2.1, we have

$$\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_2\right)a_k\le 1-\alpha_2 \tag{2.8}$$

and

$$\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k\le \frac{1-\alpha_2}{[j+1]_q-\alpha_2}<1. \tag{2.9}$$

Consequently,

$$\begin{aligned}&\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_1\right)a_k\\ =&\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_2\right)a_k\\ &+(\alpha_2-\alpha_1)\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k\\ \le&\ 1-\alpha_1.\end{aligned} \tag{2.10}$$

This completes the proof of Theorem 2.2 with the aid of Theorem 2.1. □

Theorem 2.3. *If* $0\le\lambda_1\le\lambda_2\le 1$, *then*

$$\mathcal{H}_q^m(l,\lambda_2,\alpha;j)\subseteq\mathcal{H}_q^m(l,\lambda_1,\alpha;j). \tag{2.11}$$

Proof. It follows from Theorem 2.1 that

$$\begin{aligned}&\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda_1\right\}\left([k]_q-\alpha\right)a_k\\ \le&\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda_2\right\}\left([k]_q-\alpha\right)a_k\\ \le&\ 1-\alpha.\end{aligned}$$

for $f\in\mathcal{H}_q^m(l,\lambda_2,\alpha;j)$. This completes the proof of Theorem 2.3 □

Similarly we can prove

Theorem 2.4. *If* $m\in\mathbb{Z}$, *then*

$$\mathcal{H}_q^{m+1}(l,\lambda,\alpha;j)\subseteq\mathcal{H}_q^m(l,\lambda,\alpha;j).$$

3. Distortion theorems and convex linear combinations

Theorem 3.1. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then, for $|z|<r<1$,*

$$\begin{aligned}r-\frac{1-\alpha}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)}r^{j+1}&\le|f(z)|\\ \le r+\frac{1-\alpha}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)}r^{j+1}.&\end{aligned} \tag{3.1}$$

The equality in (3.1) is attained for the function f given by

$$f(z)=z-\frac{1-\alpha}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)}z^{j+1}. \tag{3.2}$$

Proof. It is easy to see from Theorem 2.1 that

$$\begin{aligned}
&\left(\tfrac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)\sum_{k=j+1}^{\infty} a_k \\
\leq\ &\sum_{k=j+1}^{\infty}\left(\tfrac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right) a_k \leq 1-\alpha,
\end{aligned}$$

so that

$$\sum_{k=j+1}^{\infty} a_k \leq \frac{1-\alpha}{\left(\tfrac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)}. \tag{3.3}$$

Making use of (3.3), we have

$$\begin{aligned}
|f(z)| &\geq r-\sum_{k=j+1}^{\infty} a_k r^k \leq r-r^{j+1}\sum_{k=j+1}^{\infty} a_k \\
&\geq r-\frac{(1-\alpha)}{\left(\tfrac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)} r^{j+1}
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\leq r+\sum_{k=j+1}^{\infty} a_k r^k \leq r+r^{j+1}\sum_{k=j+1}^{\infty} a_k \\
&\leq r+\frac{(1-\alpha)}{\left(\tfrac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)} r^{j+1}
\end{aligned}$$

which prove the assertion (3.1). Finally, we note that the equality in (3.1) is attained for the function f defined by (3.2). This completes the proof of Theorem 3.1. □

Now, we shall prove that the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$ is closed under convex linear combinations.

Theorem 3.2. *$\mathcal{H}_q^m(l,\lambda,\alpha;j)$ is a convex set.*

Proof. Let the functions

$$f_v(z)=z-\sum_{k=j+1}^{\infty} a_{v.k} z^k \; (a_{v,k}>0; v=1,2; j\in\mathbb{N}) \tag{3.4}$$

be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z)=(1-\gamma) f_1(z)+\gamma f_2(z) \; (0\leq\gamma\leq 1) \tag{3.5}$$

is also in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Since, for $0\leq\gamma\leq 1$,

$$h(z)=z-\sum_{k=j+1}^{\infty}\left\{(1-\gamma) a_{1,k}+\gamma a_{2,k}\right\} z^k, \tag{3.6}$$

with the aid of Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\left(\tfrac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)\left\{(1-\gamma) a_{1,k}+\gamma a_{2,k}\right\} \leq 1-\alpha, \tag{3.7}$$

which implies that $h\in\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Hence $\mathcal{H}_q^m(l,\lambda,\alpha;j)$ is a convex set. □

Theorem 3.3. *Let* $f_j(z) = z$ *and*

$$f_k(z) = z - \frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)} z^k \,(k \geq j+1; j \in \mathbb{N}). \tag{3.8}$$

Then f *is in the class* $\mathcal{H}_q^m(l, \lambda, \alpha; j)$ *if and only if it can be expressed in the form:*

$$f(z) = \sum_{k=j}^{\infty} \mu_k z^k \left(\mu_k \geq 0, k \geq j; \sum_{k=j}^{\infty} \mu_k = 1\right). \tag{3.9}$$

Proof. Assume that

$$f(z) = \sum_{k=j}^{\infty} \mu_k z^k = z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)} \mu_k z^k. \tag{3.10}$$

Then it follows that

$$\sum_{k=j+1}^{\infty} \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha} \frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)} \mu_k$$

$$= \sum_{k=j+1}^{\infty} \mu_k = 1 - \mu_j \leq 1$$

So, by Theorem 2.1, $f \in \mathcal{H}_q^m(l, \lambda, \alpha; j)$.

Conversely, assume that the function f defined by (1.12) belongs to the class $\mathcal{H}_q^m(l, \lambda, \alpha; j)$. Then

$$a_k \leq \frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)} \,(k \geq j+1; j \in \mathbb{N}) \tag{3.11}$$

Setting

$$\mu_k = \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)}{1-\alpha} \,(k \geq j+1; j \in \mathbb{N}) \tag{3.12}$$

and

$$\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k,$$

we can see that f can be expressed in the form (3.9). This completes the proof of Theorem 3.3. □

4. Radii of close-to-convexity, starlikeness and convexity

Theorem 4.1. *Let the function* f *defined by (1.12) be in the class* $\mathcal{H}_q^m(l, \lambda, \alpha; j)$. *Then* f *is close-to-convex of order* $\rho\,(0 \leq \rho < 1)$ *in* $|z| < r_1$, *where*

$$r_1 = \inf_k \left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{k(1-\alpha)}\right]^{\frac{1}{k-1}} (k \geq j+1). \tag{4.1}$$

The result is sharp, the extremal function f *being given by (2.6).*

Proof. We must show that

$$\left|f'(z) - 1\right| \leq 1 - \rho \quad \text{for} \quad |z| < r_1,$$

where r_1 is given by (4.1). Indeed we find from the definition (1.12) that

$$\left|f'(z) - 1\right| = \sum_{k=j+1}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$\left|f'(z)-1\right| \le 1-\rho,$$

if

$$\sum_{k=j+1}^{\infty} \frac{k}{1-\rho} a_k |z|^{k-1} \le 1. \tag{4.2}$$

But, by Theorem 2.1, (4.2) will be true if

$$\frac{k}{1-\rho} a_k |z|^{k-1} \le \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha},$$

that is, if

$$|z| \le \left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{k(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{4.3}$$

Theorem 4.1 follows easily from (4.3). □

Theorem 4.2. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then f is starlike of order $\rho\,(0 \le \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = \inf_k \left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{4.4}$$

The result is sharp, the extremal function f being given by (2.6).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho \quad \text{for} \quad |z| < r_2,$$

where r_2 is given by (4.4). Indeed we find, again from the definition (1.12), that

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \frac{\sum\limits_{k=j+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}a_k|z|^{k-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho,$$

if

$$\sum_{k=j+1}^{\infty} \frac{k-\rho}{1-\rho} a_k |z|^{k-1} \le 1. \tag{4.5}$$

But, by Theorem 2.1, (4.5) will be true if

$$\frac{k-\rho}{1-\rho} a_k |z|^{k-1} \le \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha},$$

that is, if

$$|z| \le \left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{4.6}$$

Theorem 4.2 follows easily from (4.6). □

Similarly, we can prove the following theorem.

Theorem 4.3. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then f is convex of order $\rho\,(0\le\rho<1)$ in $|z|<r_3$, where*

$$r_3=\inf_k\left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{k(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}}\quad(k\ge j+1). \tag{4.7}$$

The result is sharp, the extremal function f being given by (2.6).

5. Modified Hadamard products and integral operator

Let the functions $f_v\,(v=1,2)$ be defined by (3.4). The modified Hadamard product of f_1 and f_2 is defined by

$$\left(f_1*f_2\right)(z)=z-\sum_{k=j+1}^{\infty}a_{1,k}a_{2,k}z^k. \tag{5.1}$$

Theorem 5.1. *Let each of the functions $f_v(z)\,(v=1,2)$ defined by (3.4) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then*

$$\left(f_1*f_2\right)(z)\in\mathcal{H}_q^m(l,\lambda,\beta;j),$$

where

$$\beta=\frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-[j+1]_q(1-\alpha)^2}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-(1-\alpha)^2}. \tag{5.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [33], we need to find the largest β such that

$$\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\beta\right)}{1-\beta}a_{1,k}a_{2,k}\le1. \tag{5.3}$$

Since

$$\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}a_{1,k}\le1 \tag{5.4}$$

and

$$\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}a_{2,k}\le1, \tag{5.5}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\sqrt{a_{1,k}a_{2,k}}\le1. \tag{5.6}$$

Thus it is sufficient to show that

$$\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\beta\right)}{1-\beta}a_{1,k}a_{2,k}\le\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\sqrt{a_{1,k}a_{2,k}} \tag{5.7}$$

that is, that

$$\sqrt{a_{1,k}a_{2,k}}\le\frac{(1-\beta)\left([k]_q-\alpha\right)}{(1-\alpha)\left([k]_q-\beta\right)}\quad(k>j+1). \tag{5.8}$$

Note that

$$\sqrt{a_{1,k}a_{2,k}}\le\frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}\quad(k\ge j+1). \tag{5.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)} \leq \frac{(1-\beta)\left([k]_q-\alpha\right)}{(1-\alpha)\left([k]_q-\beta\right)} \qquad (k \geq j+1)\,, \tag{5.10}$$

or, equivalently, that

$$\beta \leq \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-[k]_q(1-\alpha)^2}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-(1-\alpha)^2} \qquad (k \geq j+1)\,. \tag{5.11}$$

Since

$$\Psi_q\left(k\right) = \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-[k]_q(1-\alpha)^2}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-(1-\alpha)^2} \qquad (k \geq j+1) \tag{5.12}$$

is an increasing function of k $(k \geq j+1)$, letting $k = j+1$ in (5.12). we obtain

$$\beta \leq \Psi_q\left(j+1\right) = \frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-[j+1]_q(1-\alpha)^2}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-(1-\alpha)^2} \tag{5.13}$$

which proves the main assertion of Theorem 5.1. Finally, by taking the functions

$$f_i\left(z\right) = z - \frac{1-\alpha}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)} z^{j+1} \qquad (i=1,2)\,, \tag{5.14}$$

we can see that the result is sharp. □

Theorem 5.2. *Let* $f_i \in \mathcal{H}_q^m\left(l,\lambda,\alpha_i;j\right)$ $(i=1,2)$*. Then* $(f_1 * f_2) \in \mathcal{H}_q^m\left(l,\lambda,\delta;j\right)$*, where*

$$\delta = \frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha_1\right)\left([j+1]_q-\alpha_2\right)-[j+1]_q(1-\alpha_1)(1-\alpha_2)}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha_1\right)\left([j+1]_q-\alpha_2\right)-(1-\alpha_1)(1-\alpha_2)}. \tag{5.15}$$

The result is the best possible for the functions

$$f_i\left(z\right) = z - \frac{1-\alpha_i}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha_i\right)} z^{j+1} \,(i=1,2)\,. \tag{5.16}$$

Proof. Proceeding as in the proof of Theorem 5.1, we get

$$\delta \leq \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_1\right)\left([k]_q-\alpha_2\right)-[k]_q(1-\alpha_1)(1-\alpha_2)}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_1\right)\left([k]_q-\alpha_2\right)-(1-\alpha_1)(1-\alpha_2)} \,(k \geq j+1)\,. \tag{5.17}$$

Since the right-hand side of (5.17) is an increasing function of k, setting $k = j+1$ in (5.17), we obtain (5.15). This completes the proof of Theorem 5.2. □

Theorem 5.3. *Let each of the functions* $f_i\,(i=1,2)$ *defined by (3.4) be in the class* $\mathcal{H}_q^m\left(l,\lambda,\alpha;j\right)$*. Then the function*

$$h\left(z\right) = z - \sum_{k=j+1}^{\infty}\left(a_{1,k}^2 + a_{2,k}^2\right) z^k \tag{5.18}$$

belongs to the class $\mathcal{H}_q^m\left(l,\lambda,\zeta;j\right)$*, where*

$$\zeta = \frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-2[j+1]_q(1-\alpha)^2}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-2(1-\alpha)^2}. \tag{5.19}$$

The result is sharp for the functions $f_i\,(i=1,2)$ *defined by (5.14).*

Proof. By virtue of Theorem 2.1, we obtain

$$\sum_{k=j+1}^{\infty}\left[\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\right]^2 a_{1,k}^2$$

$$\leq\left[\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}a_{1,k}\right]^2\leq 1 \tag{5.20}$$

and

$$\sum_{k=j+1}^{\infty}\left[\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\right]^2 a_{2,k}^2$$

$$\leq\left[\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}a_{2,k}\right]^2\leq 1. \tag{5.21}$$

It follows from (5.20) and (5.21) that

$$\sum_{k=j+1}^{\infty}\frac{1}{2}\left[\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\right]^2\left(a_{1,k}^2+a_{2,k}^2\right)\leq 1 \tag{5.22}$$

Therefore, we need to find the largest ζ such that

$$\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\zeta\right)}{1-\zeta}\leq\frac{1}{2}\left[\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\right]^2 \tag{5.23}$$

that is,

$$\zeta\leq\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-2[k]_q(1-\alpha)^2}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-2(1-\alpha)^2}\qquad(k\geq j+1). \tag{5.24}$$

Since

$$\chi_q(k)=\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-2[k]_q(1-\alpha)^2}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-2(1-\alpha)^2} \tag{5.25}$$

is an increasing function of k $(k\geq j+1)$, we readily have

$$\zeta\leq\chi_q(j+1)=\frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-2[j+1]_q(1-\alpha)^2}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-2(1-\alpha)^2} \tag{5.26}$$

and Theorem 5.3 follows at once. □

Theorem 5.4. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$, and let c be a real number such that $c>-1$. Then the function*

$$\mathcal{J}_{q,j}^{-1}(c)\,f(z)=F_{c,q,j}(z)=\frac{[c+1]_q}{z^c}\int_0^z t^{c-1}f(t)\,d_qt\quad(c>-1) \tag{5.27}$$

also belongs to the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$.

Proof. From the representation (5.27) of $F_{c,q,j}(z)$, it follows that

$$F_{c,q,j}(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,$$

where $b_k = \frac{[c+1]_q}{[c+k]_q} a_k$ (see [34] and [35]). Therefore, we have

$$\begin{aligned}
&\sum_{k=j+1}^{\infty} \left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right) b_k z^k \\
= &\sum_{k=j+1}^{\infty} \left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right) \frac{[c+1]_q}{[c+k]_q} a_k z^k \\
\leq &\sum_{k=j+1}^{\infty} \left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right) a_k z^k \\
\leq & \; 1-\alpha,
\end{aligned}$$

since $f \in \mathcal{H}_q^m(l,\lambda,\alpha;j)$. Hence, by Theorem 2.1, $F_{c,q,j} \in \mathcal{H}_q^m(l,\lambda,\alpha;j)$. □

Remark 5.1. Taking $l=0, m \in \mathbb{N}_0$ and $q \to 1^-$ in the above results, we obtain the results of Aouf and Srivastava [29] for the class $\mathcal{P}(j;\lambda,\alpha,m)$.

Remark 5.2. Putting $l=0$ in the above results, we obtain the the corresponding results for the class $\mathcal{H}_q^m(\lambda,\alpha;j)$ involving an operator $\mathcal{S}_{q,j}^m$.

Remark 5.3. Putting $q \to 1^-$ in the above results, we obtain the corresponding results for the class $\mathcal{H}^m(l,\lambda,\alpha;j)$ involving multiplier transformation operator $\mathcal{I}_{l,j}^m$.

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Blow up and Exponential Growth to a Kirchhoff-Type Viscoelastic Equation with Degenerate Damping Term

Fatma Ekinci* and Erhan Pişkin

Abstract
In this paper, we consider a Kirchhoff-type viscoelastic equation with degenerate damping term have initial and Dirichlet boundary conditions. We obtain the blow up and exponential growth of solutions with negative initial energy.

Keywords: *Blow up; degenerate damping; exponential growth; Kirchhoff-type equation; viscoelastic equation.*

AMS Subject Classification (2020): *35B40; 35B44; 35G61; 35L75.*

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1. Introduction

We deal with the following nonlinear Kirchhoff-type viscoelastic problem:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int\limits_0^t g(t-s)\,\Delta u(s)\,ds + |u|^{\upsilon}\,\partial j(u_t) = |u|^{r-1}u \text{ in } \Omega\times(0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{on } x\in\Omega, \\ u(x,t) = 0 \quad \text{on } x\in\partial\Omega, \end{cases} \tag{1.1}$$

here $\partial j(s)$ denotes the sub-differential $j(s)$ [1], Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$. $M(\alpha)$ is a nonnegative C^1 function for $\alpha \geq 0$ satisfying

$$M(\alpha) = 1 + \alpha^{\kappa}, \ \kappa > 0.$$

The Kirchhoff type equations orginated from the nonlinear vibration of an elastic string and was firstly considered by Kirchhoff for $f = g = \delta = 0$:

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} + g\left(\frac{\partial u}{\partial t}\right) = \left\{\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.2}$$

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where $0 < x < L, t \geq 0.$

Now, we focus on a chronological literature overview. Eq. (1.2) with $f = g = 0$ was investigated by Nishihara and Yamada [2]. The author studied the global solvability of solution for non-analytic initial data. In [3], Ikehata and Matsuyama investigated Eq.(1.2) with $\delta = 0,\ g = \delta \left|u_t\right|^{p-1} u_t$ and $f = \left|u\right|^{r-1} u,$ and employed the global solvability and the energy decay of solution. Moreover, Ono [4] studied Eq. (1.2) with $g = 0,$ the author employed the local and the global existence, decay properties of solutions for degenerate and non-degenerate equations with a dissipative term. Also, the author studied the blow up of solution with nonpositive and positive initial energy. The other work related to Kirchhoff type equations is Taniguchis's work. Taniguchi [5] considered the existence of local solution, also discuss the global existence and exponential asymptotic behaviour of solutions for weakly damped Eq. (1.2).

In case of $M \equiv 1$, the problem (1.1) discussed by Han and Wang [6] and the authors proved the global existence of generalized solutions, weak solutions. Moreover, they studied finite time blow-up of solutions with negative initial energy.

Furthermore, in case of $M \equiv 1$ and $g = 0,$ the problem (1.1) becomes the following form

$$u_{tt} - \Delta u + |u|^{v}\, \partial j\left(u_t\right) = |u|^{r-1} u,$$

has been studied by some authors see [7–11].

In [12], Ekinci and Pişkin studied following equation:

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right) \Delta u + |u|^{v}\, \partial j\left(u_t\right) = |u|^{r-1} u, \tag{1.3}$$

with initial and boundary conditions. They studied blow up of solutions with arbitrary positive initial energy by constructing a energy perturbation function.

In the work [13], Piskin investigated the following equation:

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right) \Delta u + \left|u_t\right|^{p-1} u_t = |u|^{r-1} u \tag{1.4}$$

and proved the existence, decay and blow up of the solution. Then, Pişkin and Irkıl [14] investigated the same problem treated in [13] and studied blow up results for positive initial energy. In 2018, Pişkin and Yüksekkaya [15] considered problem (1.4) in case $p = 1$ and proved the blow up of solutions with positive and negative initial energy. Furthermore, Periera et al. [16] discussed problem (1.4) in case $p = 1$ and studied existence of the global solutions via the Faedo-Galerkin method. The authors also obtained the asymptotic behavior via the Nakao method. Then, in 2021, Periera et al. [17] investigated the existence and the energy decay estimate of global solutions for problem (1.4) in case $p \geq 1.$

The hyperbolic models with degenerate damping also are of much interest in material science and physics. It particularly appears in physics when the friction is modulated by the strains. There are a lot of studies have Kirchhoff-type viscoelastic problem with degenerate damping term. But, most of these studies are system problem. For instance, Pişkin and Ekinci [18] investigated the following system

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2)\Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + \left(|u|^k + |v|^l\right)\left|u_t\right|^{p-1} u_t = f_1\left(u, v\right), \\ v_{tt} - M(\|\nabla v\|^2)\Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + \left(|v|^\theta + |u|^\varrho\right)\left|v_t\right|^{q-1} v_t = f_2\left(u, v\right), \end{cases} \tag{1.5}$$

in $\Omega \times (0, T)$. The authors discussed global existence, general decay and blow up results of solutions. Recently, Piskin and Ekinci [19] considered same problem and proved local existence result. In [20], the author studied blow up of solutions with positive initial energy for problem (1.5) without viscoelastic term. In addition, they gave some estimates for lower bound of the blow up time. On the other hand, the other some studies with degenerate damping terms are see (see [21–29]).

The equation (1.5) in case $M \equiv 1$, Pişkin et al. [29] studied local existence and uniqueness of the solution by using the Faedo-Galerkin method. Furthermore, they proved the blow up of weak solutions.

To the best of our knowledge too many system problems with Kirchhoff type and degenerate damping terms. But there are a few studies as single equation with degenerate damping and Kirchhoff type. Motivated by previous works, we prove several results concerning the blow up and exponential growth of solution for the problem (1.1).

To analyze the blow up and growth of solution for problem (1.1), we are interested in effect caused by the source term $|u|^{r-1} u$, memory $\int_0^t g(t-s)\Delta u(s)\, ds$ and degenerate damping $|u|^{v} \partial j(u_t)$. In our problem is that the source term of type $|u|^{r-1}u$ overcomes the stabilizing mechanisms, memory $\int_0^t g(t-s)\Delta u(s)\,ds$ and degenerate damping $|u|^{\upsilon}\partial j(u_t)$, thus causing a destabilization of the model with the blow up of the solution at a finite time [30].

The remaining part of this paper is organized as follows: In the next section, we introduce some assumptions, notations and present a lemma needed in the proof of our results. In Section 3, we prove the blow up of solution with negative initial energy. In Section 4, we prove the exponential growth of solution with negative initial energy.

2. Preliminaries

Now, we present some preliminary material which will be helpful in the proof of our results. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^q(\Omega)$ norm $\|.\|_q = \|.\|_{L^q(\Omega)}$.

$(\mathbf{A1})$ $\upsilon, p \geq 0$, $r > 1$; $\upsilon \leq \frac{n}{n-2}$, $r+1 \leq \frac{2n}{n-2}$ if $n \geq 3$. There exists positive constants c, c_0, c_1 such that for all $s, k \in R$ $j(s) : R \to R$ be a C^1 convex real function satisfies

- $j(s) \geq c|s|^{p+1}$,
- $j'(s)$ is single valued and $|j'(s)| \leq c_0 |s|^p$,
- $(j'(s) - j'(k))(s-k) \geq c_1 |s-k|^{p+1}$.

$(\mathbf{A2})$ $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^2(\Omega)$.

$(\mathbf{A3})$ Assume $g(\tau) : R^+ \to R^+$ satisfies

$$g(\tau) \geq 0,\ g'(\tau) \leq 0,$$

for all $\tau \in R^+$ and

$$\int_0^t g(\tau)\,d\tau < 1.$$

$(\mathbf{A4})$ $\int_0^t g(s)\,ds < \frac{r-1}{r+1}$.

We use the following notations

$$\begin{aligned} l &= 1 - \int_0^t g(\tau)\,d\tau, \\ (g \diamond \theta)(t) &= \int_0^t g(t-\tau) \int_\Omega |\theta(t) - \theta(\tau)|\,dx d\tau. \end{aligned}$$

Lemma 2.1. *Suppose that $(A1)$, $(A2)$ and $(A3)$ hold. Let u be a solution of (1.1). Then, $E(t)$ is nonincreasing, namely,*

$$E'(t) \leq 0.$$

Proof. A multiplication of Eq.(1.1) by u_t and integration over Ω give

$$E'(t) = -\frac{1}{2} g(t) \|\nabla u\|^2 + \frac{1}{2}(g' \diamond \nabla u)(t) - \int_0^t \int_\Omega |u(\tau)|^{\upsilon} j(u_t)(\tau)\,dx d\tau \leq 0, \tag{2.1}$$

where

$$\begin{aligned} E(t) &= \frac{1}{2}\left[\|u_t\|^2 + \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2\right] \\ &\quad + \frac{1}{2}\left[\frac{1}{\kappa+1}\|\nabla u\|^{2(\kappa+1)} + (g \diamond \nabla u)(t)\right] - \frac{1}{r+1}\|u\|_{r+1}^{r+1} \end{aligned} \tag{2.2}$$

Thus, we have

$$E(t) \leq E(0). \tag{2.3}$$

□

3. Blow up

In this section, we shall prove the blow up of solutions for problem (1.1).

Theorem 3.1. *Let* $(A1)$-$(A4)$ *hold.* u *be a any solution to (1.1) on the interval* $[0,T]$. *Assume further that* $r > \upsilon + p$, $E(0) < 0$ *and*

$$\int_0^t g(s)\,ds \geq \frac{\kappa}{\kappa+1}.$$

Then T *is necessarily finite, i.e.* u *can't be continued for all* $t > 0$.

Proof. Set

$$H(t) = -E(t). \tag{3.1}$$

By using (2.1), we have

$$\begin{aligned} H'(t) &= -E'(t) \\ &= \frac{1}{2}g(t)\|\nabla u\|^2 - \frac{1}{2}(g' \diamond \nabla u)(t) + \int_\Omega |u(t)|^\upsilon j(u_t)\,u_t dx \\ &\geq \int_\Omega |u(t)|^\upsilon j(u_t)\,u_t dx \\ &\geq c_0 \int_\Omega |u(t)|^\upsilon |u_t|^{p+1} dx. \end{aligned} \tag{3.2}$$

Thus, we arrive at

$$0 < H(0) \leq H(t) \leq \frac{1}{r+1}\|u\|_{r+1}^{r+1}, \quad t \geq 0. \tag{3.3}$$

Now, we define

$$L(t) = H^{1-\rho}(t) + \varepsilon \int_\Omega u u_t dx,$$

where $\rho = \min\left\{\frac{r-p-\upsilon}{p(r+1)}, \frac{r-1}{2(r+1)}\right\}$ and ε is a positive constant.

By derivating $L(t)$ and using Eq.(1.1), we obtain

$$\begin{aligned} L'(t) &= (1-\rho)H^{-\rho}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla u\|^{2(\kappa+1)} \\ &\quad +\varepsilon \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\,dxds \\ &\quad -\varepsilon \int_\Omega |u(t)|^\upsilon u(t)\,\partial j(u_t)(t)\,dx + \varepsilon\|u\|_{r+1}^{r+1} \\ &= (1-\rho)H^{-\rho}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 \\ &\quad -\varepsilon\|\nabla u\|^{2(\kappa+1)} - \varepsilon\int_0^t g(s)\,ds\,\|\nabla u\|^2 \\ &\quad +\varepsilon \int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(s) - \nabla u(t))\,dxds \\ &\quad -\varepsilon \int_\Omega |u(t)|^\upsilon u(t)\,\partial j(u_t)(t)\,dx + \varepsilon\|u\|_{r+1}^{r+1}. \end{aligned} \tag{3.4}$$

By applying Young's inequality to estimate the fifth term of (3.4) as follows

$$\begin{aligned} &\left|\int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(s) - \nabla u(t))\,dxds\right| \\ \leq\ &\int_0^t g(s)\,ds\,\|\nabla u\|^2 + \frac{1}{4}(g \diamond \nabla u)(t). \end{aligned}$$

From $(A3)$, since $0 < l \leq 1$. Then it follows from the definition of $H(t)$ that

$$\begin{aligned} -\|\nabla u\|^2 &= \frac{2}{l}H(t) + \frac{1}{l}\|u_t\|^2 + \frac{1}{l}(g \diamond \nabla u)(t) \\ &\quad + \frac{1}{l(\kappa+1)}\|\nabla u\|^{2(\kappa+1)} - \frac{2}{l(r+1)}\|u\|_{r+1}^{r+1}. \end{aligned} \tag{3.5}$$

Combining (3.4)-(3.5), we obtain

$$\begin{aligned} L'(t) &\geq (1-\rho)H^{-\rho}(t)H'(t) + \varepsilon\left(1+\frac{1}{l}\right)\|u_t\|^2 \\ &\quad + \varepsilon\frac{2}{l}H(t) + \varepsilon\left(\frac{1}{l}-\frac{1}{4}\right)(g \diamond \nabla u)(t) + \varepsilon\left(\frac{1}{l(\kappa+1)}-1\right)\|\nabla u\|^{2(\kappa+1)} \\ &\quad - \varepsilon\int_\Omega |u(t)|^{\upsilon} u(t)\,\partial j(u_t)(t)\,dx + \varepsilon\left(1-\frac{2}{l(r+1)}\right)\|u\|_{r+1}^{r+1}. \end{aligned} \tag{3.6}$$

By condition $\int_0^t g(s)\,ds < \frac{r-1}{r+1}$, we have $1 - \frac{2}{l(r+1)} > 0$.

In order to estimate fifth term in (3.6), since $r > \upsilon + p$, from assumption $(A1)$ and thanks to Holder's inequality and Young's inequality, we get

$$\begin{aligned} &\left|\int_\Omega |u(t)|^{\upsilon} u(t)\,\partial j(u_t)(t)\,dx\right| \\ &\leq \int_\Omega |u(t)|^{\upsilon+1-\frac{\upsilon+p+1}{p+1}} |u(t)|^{\frac{\upsilon+p+1}{p+1}} |u_t(t)|^p\,dx \\ &\leq C_0\left(\int_\Omega |u(t)|^{\upsilon}|u_t(t)|^{p+1}\,dx\right)^{\frac{p}{p+1}}\left(\int_\Omega |u(t)|^{\upsilon+p+1}\,dx\right)^{\frac{1}{p+1}} \\ &\leq C_0|\Omega|^{\frac{r-\upsilon-p}{r+1}}\left(\int_\Omega |u(t)|^{\upsilon}|u_t(t)|^{p+1}\,dx\right)^{\frac{p}{p+1}}\|u(t)\|_{r+1}^{\frac{\upsilon+p+1}{p+1}} \\ &\leq \beta\left(H'(t)\right)^{\frac{p}{p+1}}\|u(t)\|_{r+1}^{\frac{\upsilon+p+1}{p+1}} \\ &\leq \beta\left(\delta^{-\frac{1}{p}}H'(t) + \delta\|u(t)\|_{r+1}^{\upsilon+p+1}\right), \end{aligned} \tag{3.7}$$

where constant $\delta > 0$ is specified later and $\beta = C_0 C_1^{-\frac{p}{p+1}}|\Omega|^{\frac{r-\upsilon-p}{r+1}}$.

Hence, (3.6) becomes

$$\begin{aligned} L'(t) &\geq \left[(1-\rho)H^{-\rho}(t) - \varepsilon\beta\delta^{-\frac{1}{p}}\right]H'(t) \\ &\quad + \varepsilon\left(1+\frac{1}{l}\right)\|u_t\|^2 + \varepsilon\left(\frac{1}{l(\kappa+1)}-1\right)\|\nabla u\|^{2(\kappa+1)} \\ &\quad + \varepsilon\frac{2}{l}H(t) + \varepsilon\left(\frac{1}{l}-\frac{1}{4}\right)(g \diamond \nabla u)(t) \\ &\quad + \varepsilon\left(1-\frac{2}{l(r+1)}\right)\|u\|_{r+1}^{r+1} - \varepsilon\beta\delta\|u(t)\|_{r+1}^{\upsilon+p+1}. \end{aligned} \tag{3.8}$$

The choice of δ $\left(\text{i.e. } \delta = \frac{1}{\beta}\left(\frac{1}{2} - \frac{1}{l(r+1)}\right)\|u\|_{r+1}^{r-\upsilon-p}\right)$, then

$$\varepsilon\beta\delta\|u(t)\|_{r+1}^{r+p+1} = \varepsilon\left(\frac{1}{2}-\frac{1}{l(r+1)}\right)\|u\|_{r+1}^{r+1}.$$

Furthermore, since $\|u\|_{r+1} \geq [(r+1)H(0)]^{\frac{1}{r+1}}$ by (3.3) and $\upsilon + p - r + p(r+1)\rho \leq 0$, then

$$\begin{aligned} & (1-\rho) H^{-\rho}(t) - \varepsilon\beta\delta^{-\frac{1}{p}} \\ = \;& H^{-\rho}(t)\left[1-\rho-\varepsilon\beta\delta^{-\frac{1}{p}}H^{\rho}(t)\right] \\ \geq \;& H^{-\rho}(t)\left[1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\left(\frac{1}{2}-\frac{1}{l(r+1)}\right)^{-\frac{1}{p}}(r+1)^{-\rho}\|u\|_{r+1}^{\frac{p+\upsilon-r+p(r+1)\rho}{p}}\right] \\ \geq \;& H^{-\rho}(t)\left[1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\left(\frac{1}{2}-\frac{1}{l(r+1)}\right)^{-\frac{1}{p}}(r+1)^{-\rho-\frac{r-p-\upsilon}{p(r+1)}}H(0)_{r+1}^{\rho-\frac{r-\upsilon-p}{p(r+1)}}\right] \\ \geq \;& H^{-\rho}(t)\left[1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\chi\right], \end{aligned} \tag{3.9}$$

where $\chi = \left(\frac{1}{2}-\frac{1}{l(r+1)}\right)^{-\frac{1}{p}}(r+1)^{\rho-\frac{r-p-\varrho}{p(r+1)}}H(0)_{r+1}^{\rho-\frac{r-\upsilon-p}{p(r+1)}}$. Now, we choose ε to be sufficiently small such that

$$1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\chi > 0.$$

Then (3.9) and (3.8) yield

$$L'(t) \geq \varepsilon C\left[H(t)+\|u_t(t)\|^2+\|u\|_{r+1}^{r+1}+(g\diamond\nabla u)(t)\right], \tag{3.10}$$

where $C>0$ is a constant that does not depended on ε. Especially, (3.10) means that $L(t)$ is increasing on $[0,T)$, with

$$L(t) = H^{1-\rho}(t)+\varepsilon\int_{\Omega}uu_t dx \geq H^{1-\rho}(0)+\varepsilon\int_{\Omega}u_0u_1 dx.$$

We also select ε to be sufficiently small such that $L(0)>0$, thus $L(t)\geq L(0)>0$ for $t\geq 0$.

Let $\eta=\frac{1}{1-\rho}$. Since $0<\rho<\frac{1}{2}$, it is evident that $2>\eta>1$. By using the following inequality

$$|x+y|^{\eta} \leq 2^{\eta-1}\left(|x|^{\eta}+|y|^{\eta}\right) \text{ for } \eta\geq 1,$$

applying Young's inequality, we get

$$\begin{aligned} L^{\eta}(t) &\leq 2^{\eta-1}\left(H(t)+\varepsilon\|u(t)\|^{\eta}\|u_t(t)\|^{\eta}\right) \\ &\leq C\left(H(t)+\|u_t(t)\|^2+\|u(t)\|_{r+1}^{\frac{1}{\frac{1}{2}-\rho}}\right). \end{aligned} \tag{3.11}$$

By the choice of ρ, we have $\frac{1}{2}-\rho>\frac{1}{r+1}$. Now applying the inequality

$$a^{\sigma} \leq \left(1+\frac{1}{b}\right)(b+a), \quad a\geq 0, \quad 0\leq\sigma\leq 1, \quad b>0,$$

and taking $a=\|u(t)\|_{r+1}^{r+1}$, $\eta=\frac{1}{\left(\frac{1}{2}-\rho\right)(r+1)}<1$, and $b=H(0)$, we obtain

$$\begin{aligned} \|u(t)\|_{r+1}^{\frac{1}{\frac{1}{2}-\rho}} &\leq \left(1+\frac{1}{H(0)}\right)\left(H(0)+\|u(t)\|_{r+1}^{r+1}\right) \\ &\leq C\left(H(t)+\|u(t)\|_{r+1}^{r+1}\right). \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) gives the inequality

$$\begin{aligned} L^{\eta}(t) &\leq C\left(H(t)+\|u_t(t)\|^2+\|u(t)\|_{r+1}^{r+1}\right) \\ &\leq C\left(H(t)+\|u_t(t)\|^2+\|u(t)\|_{r+1}^{r+1}+(g\diamond\nabla u)(t)\right). \end{aligned} \tag{3.13}$$

Thus, (3.10) and (3.13) arrive at

$$L'(t) \geq CL^{\eta}(t), \quad t\in[0,T]. \tag{3.14}$$

In the end, from (3.14) and $\eta=\frac{1}{1-\rho}>1$, we see that $L(t)=H^{1-\rho}(t)+\varepsilon\int_{\Omega}uu_t dx$ blow up in finite time. This completes the proof. □

4. Exponential growth

In this section, we aim to indicate that the energy grow up as an exponential function as time as goes to infinity.

Theorem 4.1. *Let* $(A1)$-$(A3)$ *hold.* u *be a any solution to (1.1). Suppose further that* $r > \upsilon + p$ *and* $E(0) < 0$ *and*

$$\int_0^t g(s)\,ds \geq \frac{\kappa}{\kappa + 1/2}$$

Then, the solution to (1.1) grows exponentially.

Proof. We define

$$F(t) = H(t) + \varepsilon \int_\Omega u u_t dx, \tag{4.1}$$

where $H(t) = -E(t)$ and choose $0 < \varepsilon \leq 1$ in this interval to obtain small perturbation of $E(t)$ and we will indicate that $F(t)$ grows exponentialy, namely $F(t)$ satisfies a differential inequality of the form

$$\frac{dF(t)}{dt} \geq \Gamma F(t).$$

By derivating (4.1) and using Eq.(1.1), we have

$$\begin{aligned}
F'(t) &= H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\kappa+1)} \\
&\quad + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(s) \nabla u(t)\,dxds \\
&\quad - \varepsilon \int_\Omega |u(t)|^\upsilon u(t)\, \partial j(u_t)(t)\,dx + \varepsilon \|u\|_{r+1}^{r+1} \\
&= H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 \\
&\quad + \varepsilon \int_0^t g(s)\,ds \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\kappa+1)} \\
&\quad + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) (\nabla u(s) - \nabla u(t))\,dxds \\
&\quad - \varepsilon \int_\Omega |u(t)|^\upsilon u(t)\, \partial j(u_t)(t)\,dx + \varepsilon \|u\|_{r+1}^{r+1}.
\end{aligned} \tag{4.2}$$

Terms in (4.2) is estimated as follows:

$$\begin{aligned}
&\left| \int_0^t g(t-s) \int_\Omega \nabla u(t) (\nabla u(s) - \nabla u(t))\,dxds \right| \\
\leq\ & \frac{1}{2} \int_0^t g(s)\,ds \|\nabla u\|^2 + \frac{1}{2} (g \diamond \nabla u)(t).
\end{aligned}$$

$$\begin{aligned}
F'(t) &\geq H'(t)+\varepsilon\|u_t\|^2-\varepsilon\left(1-\frac{1}{2}\int_0^t g(s)\,ds\right)\|\nabla u\|^2\\
&\quad -\varepsilon\|\nabla u\|^{2(\kappa+1)}-\varepsilon\frac{1}{2}(g\diamond\nabla u)(t)\\
&\quad -\varepsilon\int_\Omega |u(t)|^{\upsilon}u(t)\,\partial j(u_t)(t)\,dx+\varepsilon\|u\|_{r+1}^{r+1}\\
&\geq H'(t)+\varepsilon\|u_t\|^2-\varepsilon\left(\frac{1-\frac{1}{2}\int_0^t g(s)\,ds}{1-\int_0^t g(s)\,ds}\right)l\|\nabla u\|^2\\
&\quad -\varepsilon\|\nabla u\|^{2(\kappa+1)}-\varepsilon\frac{1}{2}(g\diamond\nabla u)(t)\\
&\quad -\varepsilon\int_\Omega |u(t)|^{\upsilon}u(t)\,\partial j(u_t)(t)\,dx+\varepsilon\|u\|_{r+1}^{r+1}.\\
&\geq H'(t)+\varepsilon\|u_t\|^2-\varepsilon\zeta l\|\nabla u\|^2\\
&\quad -\varepsilon\|\nabla u\|^{2(\kappa+1)}-\varepsilon\frac{1}{2}(g\diamond\nabla u)(t)\\
&\quad -\varepsilon\int_\Omega |u(t)|^{\upsilon}u(t)\,\partial j(u_t)(t)\,dx+\varepsilon\|u\|_{r+1}^{r+1},
\end{aligned}\tag{4.3}$$

where $\zeta=\frac{1-\frac{1}{2}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}$.

By using the assumption $(A3)$ and the definition $H(t)$, we have $0<l\leq 1$ and

$$\begin{aligned}
F'(t) &\geq H'(t)+\varepsilon(1+\zeta)\|u_t\|^2+\varepsilon\left(\frac{\zeta}{\kappa+1}-1\right)\|\nabla u\|^{2(\kappa+1)}\\
&\quad +\varepsilon\left(\zeta-\frac{1}{2}\right)(g\diamond\nabla u)(t)+\varepsilon\left(1-\frac{2\zeta}{\gamma+1}\right)\|u\|_{r+1}^{r+1}\\
&\quad +2\varepsilon\zeta H(t)-\varepsilon\int_\Omega |u(t)|^{\upsilon}u(t)\,\partial j(u_t)(t)\,dx.
\end{aligned}$$

By using (3.7), we get

$$\begin{aligned}
F'(t) &\geq \left[1-\varepsilon\beta\delta^{-\frac{1}{p}}\right]H'(t)+\varepsilon(1+\zeta)\|u_t\|^2\\
&\quad \varepsilon\left(\frac{\zeta}{\kappa+1}-1\right)\|\nabla u\|^{2(\kappa+1)}\\
&\quad +2\varepsilon\zeta H(t)+\varepsilon\left(\zeta-\frac{1}{2}\right)(g\diamond\nabla u)(t)\\
&\quad +\varepsilon\left(1-\frac{2\zeta}{r+1}\right)\|u\|_{r+1}^{r+1}-\varepsilon\beta\delta\|u(t)\|_{r+1}^{\upsilon+p+1}.
\end{aligned}\tag{4.4}$$

The choice of δ $\left(\text{i.e. }\ \delta=\frac{1}{\beta}\left(\frac{1}{2}-\frac{\zeta}{r+1}\right)\|u\|_{r+1}^{r-\upsilon-p}\right)$, then

$$\varepsilon\beta\delta\|u(t)\|_{r+1}^{\upsilon+p+1}=\varepsilon\left(\frac{1}{2}-\frac{\zeta}{r+1}\right)\|u\|_{r+1}^{r+1}.$$

Furthermore, since $\|u\|_{r+1}\geq[(r+1)H(0)]^{\frac{1}{r+1}}$ by (3.3) and assumption $\upsilon+p-r\leq 0$, then

$$\begin{aligned}
&1-\varepsilon\beta\delta^{-\frac{1}{p}}\\
\geq\ &1-\varepsilon\beta^{1+\frac{1}{p}}\left(\frac{1}{2}-\frac{\zeta}{r+1}\right)^{-\frac{1}{p}}(r+1)^{-\frac{r-p-\upsilon}{p(r+1)}}H(0)_{r+1}^{-\frac{r-\upsilon-p}{p(r+1)}}\\
\geq\ &1-\varepsilon\beta^{1+\frac{1}{p}}K,
\end{aligned}$$

where $K=\left(\frac{1}{2}-\frac{\zeta}{r+1}\right)^{-\frac{1}{p}}(r+1)^{-\frac{r-p-\upsilon}{p(r+1)}}H(0)_{r+1}^{-\frac{r-\upsilon-p}{p(r+1)}}$. Now, we choose ε to be sufficiently small such that

$$1-\varepsilon\beta^{1+\frac{1}{p}}K>0.$$

Thus,

$$F'(t) \geq \varepsilon C \left[H(t) + \|u_t(t)\|^2 + \|u\|_{r+1}^{r+1} + (g \diamond \nabla u)(t) \right] \tag{4.5}$$

where $C > 0$ is a constant that does not depended on ε.

Now, applying Young's inequality, and Sobolev Poincare inequality we have

$$\begin{aligned} F(t) &\leq H(t) + \varepsilon \|u\| \|u_t\| \\ &\leq C\left(H(t) + \|u_t\|^2 + \|u\|^2 \right). \end{aligned}$$

Now, in order the estimate $\|u\|^2$ term we apply the inequality $a^l \leq (a+1) \leq (1+\frac{1}{b})(a+b)$ for $a = \|u\|_{r+1}^{r+1}$, $l = 2/r+1 < 1$, $b = H(0)$, we have

$$\begin{aligned} \|u\|^2 &\leq C\|u\|_{r+1}^2 \\ &= C\left(\|u\|_{r+1}^{r+1} \right)^{\frac{2}{r+1}} \\ &\leq \left(1 + \frac{1}{H(0)} \right) \left(\|u\|_{r+1}^{r+1} + H(0) \right) \\ &\leq C\left(\|u\|_{r+1}^{r+1} + H(t) \right). \end{aligned} \tag{4.6}$$

Thus,

$$F(t) \leq C\left[H(t) + \|u_t(t)\|^2 + \|u\|_{r+1}^{r+1} + (g \diamond \nabla u)(t) \right]. \tag{4.7}$$

Therefore, (4.5) and (4.7) arrive at

$$F'(t) \geq \xi F(t), \quad t \geq 0$$

This completes the proof. □

5. Conclusion

As far as we know, there is not any blow up and exponential growth results in the literature known for viscoelastic Kirchhoff type equation with degenerate damping term. Our work extends the works for some viscoelastic Kirchhoff type equations treated in the literature to the viscoelastic Kirchhoff equations with degenerate damping terms.

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On the Codes over a Family of Rings and Their Applications to DNA Codes

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Abstract
In this paper, the structures of the linear codes over a family of the rings $A_t = Z_4[u_1,\ldots,u_t]/\langle u_i^2-u_i, u_iu_j-u_ju_i\rangle$ are given, where $i,j=1,2,\ldots,t$, $i\neq j$, $Z_4=\{0,1,2,3\}$. A map between the elements of the A_t and the alphabet $\{A,T,C,G\}^{2^t}$ is constructed. The DNA codes are obtained with three different methods, by using the cyclic, skew cyclic codes over a family of the rings A_t and θ_i-set, where θ_i is a non trivial automorphism on A_i, for $i=1,2,\ldots,t$.

Keywords: *DNA codes; cyclic codes; skew cyclic codes; reversibility.*
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1. Introduction

There are many methods in order to obtain DNA codes. In [1], it was used the cyclic codes over the finite ring $F_2[u]/\langle u^4-1\rangle$ in order to obtain DNA codes. The sufficient and necessary conditions of cyclic codes over the finite ring satisfying the reverse complement constraints was given. By introducing a map, the DNA codes were obtained from these types codes. In different method, it was used the skew cyclic codes over $Z_4[u,v]/\langle u^2-u, v^2-v, uv-vu\rangle$ in order to obtain reversible DNA codes, in [2]. Thanks to this, reversibility problem was solved for DNA 4-bases. This problem arises from the fact that the pairing of nucleotides in two different strands of a DNA sequence is done in opposite direction and reverse order. For example, take $t=1$. Let $(\alpha_1,\alpha_2)\in A_1^2$ be a codeword corresponding to $CTCG$, where $A_1=Z_4+u_1Z_4, u_1^2=u_1$. The reverse of (α_1,α_2) is (α_2,α_1). The vector (α_2,α_1) corresponding to $CGCT$. It is not reverse of $CTCG$. The reverse of $CTCG$ is $GCTC$. In order to solve reversibility problem, there is a different approach. In [3], it was used θ-set, where θ is a non trivial automorphism on $F_2[u,v]/\langle u^2, v^2-v, uv-vu\rangle$ in order to obtain reversible and reversible complement DNA codes.

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Moreover, there are similar papers in the literature, [4–6]. Motivated from all these works in which were considered the codes over one ring and were used one method in order to DNA codes, we decide to consider the codes over a family of rings and use three methods in order to obtain DNA codes.

In this paper, we use the cyclic, skew cyclic codes over a family of the rings $A_t = Z_4[u_1, \ldots, u_t]/\langle u_i^2 - u_i, u_i u_j - u_j u_i\rangle$, where $i, j = 1, 2, \ldots, t, i \neq j$ and $Z_4 = \{0, 1, 2, 3\}$ and θ_i-set, where θ_i is a non trivial automorphism on A_i, for $i = 1, 2, \ldots, t$ in order to obtain DNA codes. Section 2 includes some knowledge about a family of the rings A_t. A map ϕ_i is defined from A_i to A_{i-1}^2, for $i = 1, 2, \ldots, t$. A map ξ_i is defined from A_i to $\{A, T, C, G\}^{2^i}$, for $i = 1, 2, \ldots, t$. A Gray map is defined on A_i, for $i = 1, \ldots, t$. In the section 3 and 4, the structures of linear and cyclic codes over A_t are given, respectively. In the section 5.1 and 5.2 the sufficient and necessary conditions of cyclic codes over A_t satisfying the reverse and reverse complement constraints are given, respectively. The DNA codes are obtained with first method. In the section 6, by defining a non trivial automorphism on A_i for $i = 1, \ldots, t$, the skew cyclic codes over a family of the finite rings are introduced. By using the skew cyclic codes over A_t, the DNA codes are obtained with second method. In the section 7, by using the θ_i-set, where θ_i is a non trivial automorphism on A_i, for $i = 1, 2, \ldots, t$, the DNA codes are obtained with third method.

2. Preliminaries

A family of the finite rings $A_t = Z_4[u_1, \ldots, u_t]/\langle u_i^2 - u_i, u_i u_j - u_j u_i\rangle$, where $i, j = 1, 2, \ldots, t, i \neq j$ contains the commutative the finite rings with characteristic 4 and cardinality 4^{2^t}. The finite rings of the family are written as recursively

$$A_r = A_{r-1} + u_r A_{r-1}$$

where $r = 1, 2, \ldots, t$ and $A_1 = Z_4 + u_1 Z_4, u_1^2 = u_1$, where $A_0 = Z_4 = \{0, 1, 2, 3\}$.

We define a map as follows for every $a_i = x_{i-1} + u_i y_{i-1} \in A_i$,

$$\begin{array}{ccl} \phi_i & : & A_i \longrightarrow A_{i-1}^2 \\ a_i & \longmapsto & \phi_i(a_i) = (x_{i-1}, x_{i-1} + y_{i-1}) \end{array}$$

where $i = 1, 2, \ldots, t$ and

$$\begin{array}{ccl} \phi_1 & : & A_1 \longrightarrow A_0^2 \\ a_1 & = & x_0 + u_1 y_0 \longmapsto \phi_1(a_1) = (x_0, x_0 + y_0) \end{array}$$

where $A_0 = Z_4$.

The map ϕ_i can be extended to A_i^n naturally, for $i = 1, \ldots, t$.

Let $S_{D_4} = \{A, T, C, G\}$ represent the DNA alphabet. The Watson Crick Complement is given $A^c = T, T^c = A, G^c = C, C^c = G$. We use the same notation for the set $S_{D_{16}} = \{AA, TT, \ldots, CG\}$ which was presented in [7]. It is extended the notation to the elements of $S_{D_{16}}$ such that $AA^c = TT, AT^c = TA, \ldots, GG^c = CC$. By using the matching the elements of A_0 and $S_{D_4} = \{A, T, C, G\}$ which is given as $\xi_0(0) = A, \xi_0(1) = T, \xi_0(3) = C, \xi_0(2) = G$ and by using the map ϕ_1 from $A_1 = Z_4 + u_1 Z_4$ to Z_4^2, we defined a ξ_1 correspondence between the elements of the finite ring $A_1 = Z_4 + u_1 Z_4$ and DNA double pairs by $a_1 = x_0 + u_1 y_0 \mapsto (\xi_0(x_0), \xi_0(x_0 + y_0))$ in [7],

Elements a_1	DNA double pairs $\xi_1(a_1)$
0	AA
1	TT
2	GG
3	CC
u_1	AT
$1+u_1$	TG
u_1+2	GC
u_1+3	CA
$2u_1$	AG
$1+2u_1$	TC
$2+2u_1$	GA
$3+2u_1$	CT
$3u_1$	AC
$1+3u_1$	TA
$2+3u_1$	GT
$3+3u_1$	CG

Table 1. Identifying codons with the elements of the ring A_1.

By using the map ϕ_2 and ξ_1, we established ξ_2 correspondence between the elements of A_2 and DNA 4-bases by $a_2 = x_1 + u_1 y_1 \mapsto (\xi_1(x_1), \xi_1(x_1 + y_1))$ as follows in [2],

Elements a_2	DNA 4-bases $\xi_2(a_2)$
0	$AAAA$
1	$TTTT$
2	$GGGG$
3	$CCCC$
u_1	$ATAT$
u_2	$AATT$
$\vdots$	$\vdots$

Table 2. Identifying codons with the elements of the ring A_2.

By using the map ϕ_i and ξ_{i-1}, we can establish ξ_i correspondence between the element of A_i and DNA 2^i-bases for $i = 1, .., t$ as follows.

$$\xi_i : A_i \longrightarrow A_{i-1}^2 \longrightarrow \{A, T, C, G\}^{2^i}$$

$$a_i = x_{i-1} + u_i y_{i-1} \longmapsto \phi_i(a_i) = (x_{i-1}, x_{i-1} + y_{i-1}) \longmapsto \gamma_i(\phi_i(a_i)) = (\xi_{i-1}(x_{i-1}), \xi_{i-1}(x_{i-1} + y_{i-1}))$$

where $\xi_i = \gamma_i \phi_i$ and the map γ_i is defined from A_{i-1}^2 to DNA 2^i-bases as follows,

$$\gamma_i(s_{i-1}, t_{i-1}) = (\xi_{i-1}(s_{i-1}), \xi_{i-1}(t_{i-1}))$$

where $s_{i-1}, t_{i-1} \in A_{i-1}$, for $i = 1, ..., t$.

We established ξ_i correspondence between the elements of A_i and DNA 2^i-bases as follows

Elements a_i	DNA 2^i-bases $\xi_i(a_i)$
0	$\underbrace{AA\ldots A}_{2^i \text{ times}}$
1	$\underbrace{TT\ldots T}_{2^i \text{ times}}$
2	$\underbrace{GG\ldots G}_{2^i \text{ times}}$
3	$\underbrace{CC\ldots C}_{2^i \text{ times}}$
u_1	$\underbrace{ATAT\ldots AT}_{2^i \text{ times}}$
$\vdots$	$\vdots$

Table 3. Identifying codons with the elements of the ring A_i.

for $i = 1, \ldots, t$.

We can also express an element of A_t as follows uniquely.

Let $B \subseteq \{1, 2, \ldots, t\}$ and $u_B = \prod_{i \in B} u_i$. In particular $u_\emptyset = 1$. Each element of A_t is of the form $\sum_{B \in P_t} \alpha_B u_B$, where $\alpha_B \in Z_4$, P_t is the power set of the set $\{1, 2, \ldots, t\}$. For $A, B \subseteq \{1, 2, \ldots, t\}$, we have that $u_A u_B = u_{A \cup B}$ which gives that $\sum_{B \in P_t} \alpha_B u_B . \sum_{C \in P_t} \beta_C u_C = \sum_{D \in P_t} \left(\sum_{B \cup C = D} \alpha_B \beta_C \right) u_D$. Moreover,

$$e_{u_\emptyset} = 1 + (-1)^{|B|} \sum_{B \in P_t} u_B$$

and the number of $e_{u_\emptyset}$ is $\binom{t}{0}$.

$$e_{u_i} = u_i + (-1)^{|B|+1} \sum_{\substack{i \in B \in P_t, \\ |B| \geq 2}} u_B$$

for $i = 1, 2, \ldots, t$ and the number of e_{u_i} is $\binom{t}{1}$.

$$e_{u_i u_j} = \underset{i<j}{u_i u_j} + (-1)^{|B|+2} \sum_{\substack{i,j \in B \in P_t, \\ |B| \geq 3}} u_B$$

for $i, j = 1, 2, \ldots, t$ and the number of $e_{u_i u_j}$ is $\binom{t}{2}$.

$$e_{u_i u_j u_s} = \underset{i<j<s}{u_i u_j u_s} + (-1)^{|B|+3} \sum_{\substack{i,j,s \in B \in P_t, \\ |B| \geq 4}} u_B$$

for $i, j, s = 1, 2, \ldots, t$ and the number of $e_{u_i u_j u_s}$ is $\binom{t}{3}$

$$\vdots$$

$$e_{u_1 u_2 \ldots u_t} = u_1 u_2 \ldots u_t$$

and the number of $e_{u_1 u_2 \ldots u_t}$ is $\binom{t}{t}$.

Then we have $\sum_{B \in P_t} e_{u_B} = 1, (e_{u_B})^2 = e_{u_B}$ and $e_{u_B} e_{u_A} = 0$ if $A \neq B$ for any $A, B \subseteq \{1, 2, \ldots, t\}$. Hence $A_t = \bigoplus_{B \in P_t} A_t e_{u_B} \cong \bigoplus_{B \in P_t} Z_4 e_{u_B}$. So every element z of A_t can be uniquely expressed as $z = \sum_{B \in P_t} a_{u_B} e_{u_B}$, where $a_{u_B} \in Z_4$.

Example 2.1. Let t be 3. Then $A_3 = Z_4 + u_1Z_4 + u_2Z_4 + u_3Z_4 + u_1u_2Z_4 + u_1u_3Z_4 + u_2u_3Z_4 + u_1u_2u_3Z_4$. Consider the elements of A_3 below

$$e_{u_\emptyset} = e_1 = 1 - u_1 - u_2 - u_3 + u_1u_2 + u_1u_3 + u_2u_3 - u_1u_2u_3$$
$$e_{u_1} = u_1 - u_1u_2 - u_1u_3 + u_1u_2u_3$$
$$e_{u_2} = u_2 - u_1u_2 - u_2u_3 + u_1u_2u_3$$
$$e_{u_3} = u_3 - u_1u_3 - u_2u_3 + u_1u_2u_3$$
$$e_{u_1u_2} = u_1u_2 - u_1u_2u_3$$
$$e_{u_1u_3} = u_1u_3 - u_1u_2u_3$$
$$e_{u_2u_3} = u_2u_3 - u_1u_2u_3$$
$$e_{u_1u_2u_3} = u_1u_2u_3$$

We can also define Gray map as follows,

$$\begin{array}{rcl} \Psi_t & : & A_t \longrightarrow Z_4^{2^t} \\ z = \sum\limits_{B\in P_t} a_{u_B}e_{u_B} & \longmapsto & \Psi_t(z) = \gamma \end{array}$$

where $\gamma = \begin{pmatrix} \sum\limits_{B=\emptyset} a_{u_B}, \sum\limits_{B\subseteq\{1\}} a_{u_B}, \ldots, \sum\limits_{B\subseteq\{t\}} a_{u_B}, \sum\limits_{B\subseteq\{1,2\}} a_{u_B}, \sum\limits_{B\subseteq\{1,3\}} a_{u_B}, \ldots, \\ \sum\limits_{\substack{B\subseteq\{i,j\},\\ i<j}} a_{u_B}, \sum\limits_{B\subseteq\{1,2,3\}} a_{u_B}, \ldots, \sum\limits_{\substack{B\subseteq\{i,j,s\},\\ i<j<s}} a_{u_B}, \ldots, \sum\limits_{B\subseteq\{1,2,\ldots,t\}} a_{u_B} \end{pmatrix}$ and $a_{u_B} \in Z_4$, for $i, j, s, \ldots \in \{1, 2, \ldots, t\}$.

The map Ψ_t can be extended from A_t^n, naturally.

Example 2.2. Let $t = 3$. Then

$$\begin{array}{rcl} \Psi_3 & : & A_3 \longrightarrow Z_4^{8} \\ z = \sum\limits_{B\in P_3} a_{u_B}e_{u_B} & \longmapsto & \Psi_3(z) = \gamma \end{array}$$

where $\gamma = (a_1, a_1 + a_{u_1}, a_1 + a_{u_2}, a_1 + a_{u_3}, a_1 + a_{u_1} + a_{u_2} + a_{u_1u_2}, a_1 + a_{u_1} + a_{u_3} + a_{u_1u_3}, a_1 + a_{u_2} + a_{u_3} + a_{u_2u_3}, a_1 + a_{u_1} + a_{u_2} + a_{u_3} + a_{u_1u_2} + a_{u_2u_3} + a_{u_1u_3} + a_{u_1u_2u_3})$.

The Lee weight on Z_4, denoted w_L, is defined as $w_L(p) = 0$ if $p = 0$, $w_L(p) = 1$ if $p = 1$ or $p = 3$, $w_L(p) = 2$ if $p = 2$. For any $x = \sum\limits_{B\in P_t} a_{u_B}e_{u_B} \in A_t$, the Gray weight of x is defined as

$$w_G(x) = w_L(\Psi_t(x)) = \sum_{i=1}^{2^t} w_L(x_i)$$

where $\Psi_t(x) = (x_1, \ldots, x_{2^t})$ and $x_i \in Z_4$ for $i = 1, 2, \ldots, 2^t$. The Gray weight of a vector $\mathbf{a} = (a_1, \ldots, a_n) \in A_t^n$ is defined to be a rational sum of the Gray weight of its components. Moreover, for any $\mathbf{c}, \mathbf{d} \in A_t^n$, the Gray distance between $\mathbf{c}$ and $\mathbf{d}$ is defined as $d_G(\mathbf{c}, \mathbf{d}) = w_G(\mathbf{c} - \mathbf{d})$.

Theorem 2.1. *The map Ψ_i is a linear and distance preserving map, for $i = 1, \ldots, t$.*

3. Linear codes over A_t

A non empty subset $C \subseteq A_t^n$ is called linear code over A_t if C is a submodule of A_t.

Let $\mathbf{x} = (x_0, x_1, \dots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \dots, y_{n-1})$ be two vectors in A_t^n. The Euclidean inner product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=0}^{n-1} x_j y_j$$

where the operations are performed in the ring A_t.

Dual of the code $C \subseteq A_t^n$ is the code

$$C^{\perp} = \{\mathbf{x} \in A_t^n : \langle \mathbf{x}, \mathbf{y} \rangle = 0, \forall \mathbf{y} \in C\}.$$

Clearly, $C^{\perp}$ is also linear.

Denote $\mathbf{r} = (r^{(0)}, \dots, r^{(n-1)}) \in A_t^n$, where $r^{(i)} = \sum_{B \in P_t} a_{iu_B} e_{u_B}$ for $i = 0, 1, 2, \dots, n-1$. Then $\mathbf{r}$ can be uniquely expressed as $\mathbf{r} = \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B}$, where $\mathbf{a}_{u_B} = (a_{0u_B}, a_{1u_B}, \dots, a_{n-1u_B})$, each $B \in P_t$.

Let

$$R_1 \oplus \dots \oplus R_{2^t} = \{r_1 + \dots + r_{2^t} | r_i \in R_i, i = 1, \dots, 2^t\},$$
$$R_1 \oplus \dots \oplus R_{2^t} = \{(r_1, \dots, r_{2^t}) | r_i \in R_i, i = 1, \dots, 2^t\}.$$

Define the codes C_{u_B} as follows

$$C_{u_\emptyset} = C_1 = \{\mathbf{a}_{u_\emptyset} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \emptyset} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$C_{u_1} = \{\mathbf{a}_{u_1} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{1\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$C_{u_2} = \{\mathbf{a}_{u_2} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{2\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$\vdots$$

$$C_{u_t} = \{\mathbf{a}_{u_t} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{t\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$C_{u_1 u_2} = \{\mathbf{a}_{u_1 u_2} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{1,2\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$\vdots$$

$$C_{u_1 u_2 \dots u_t} = \{\mathbf{a}_{u_1 u_2 \dots u_t} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{1, \dots, t\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}.$$

The number of C_{u_B} is 2^t. Clearly C_{u_B} is a linear code of length n over Z_4. C can be uniquely decomposed into

$$C = \bigoplus_{B \in P_t} C_{u_B} e_{u_B}$$

and hence we have $|C| = \prod_{B \in P_t} |C_{u_B}|$.

The following theorems can be proved as in [8].

Theorem 3.1. *Let* $C = \bigoplus_{B \in P_t} C_{u_B} e_{u_B}$ *be a linear code of length* n *over* A_t. *Then the dual* $C^{\perp} = \bigoplus_{B \in P_t} {C_{u_B}}^{\perp} e_{u_B}$ *is also a linear code of length* n *over* A_t.

Theorem 3.2. *If C is a (n, M, d_G) linear code over A_i, then $\Psi_i(C)$ is a $(2^i n, M, d_L)$ linear code over Z_4 for $i = 1, \ldots, t$, where $d_G = d_L$.*

Theorem 3.3. *Let C be a linear code of length n over A_i. Then $\Psi_i(C) = \bigotimes_{B \in P_i} C_{u_B}$, for $i = 1, \ldots, t$.*

4. Cyclic codes over A_t

In [9], the structures of cyclic codes of length n over Z_4 were determined as follows. By using this, we will obtain the structures of cyclic codes over A_i for $i = 1, \ldots, t$.

Theorem 4.1. *[9] Let C be a cyclic code of length n over $R_n = Z_4[x]/\langle x^n - 1\rangle$.*
1. If n is odd, then R_n is a principal ideal ring and $C = \langle g(x), 2a(x)\rangle = \langle g(x) + 2a(x)\rangle$, where $g(x)$ and $a(x)$ are polynomials with $a(x)|g(x)|x^n - 1 \pmod 4$.
2. If n is not odd, then

i. If $g(x) = a(x)$, then $C = \langle g(x) + 2a(x)\rangle$, where $g(x)|x^n - 1 \pmod 2$, $g(x) + 2a(x)|x^n - 1 \pmod 4$,

ii. $C = \langle g(x) + 2p(x), 2a(x)\rangle$, where $g(x), a(x)$ and $p(x)$ are polynomials with $g(x)|x^n - 1 \pmod 2$ and $a(x)|p(x)\,(x^n - 1/g(x)) \pmod 2$, deg $a(x) >$ deg $p(x)$.

Theorem 4.2. *Let $C = \bigoplus_{B \in P_t} C_{u_B} e_{u_B}$ be a linear code over A_t. Then C is a cyclic code over A_t if and only if C_{u_B} are cyclic codes over Z_4 for all $B \in P_t$. Moreover, if C is a cyclic code over A_t, then*

$$C = \langle f_1(x)e_1, f_{u_1}(x)e_{u_1}, \ldots, f_{u_t}(x)e_{u_t}, f_{u_1u_2}(x)e_{u_1u_2}, \ldots, f_{u_1u_2\ldots u_t}(x)e_{u_1u_2\ldots u_t}\rangle$$

where $f_{u_B}(x)$ are generator polynomials of C_{u_B}, for all $B \in P_t$, respectively.

Proof. This can be proven similarly to [7]. □

5. The reversible codes and reversible complement codes

In [7], the sufficient and necessary conditions of cyclic codes over A_1 satisfying the reverse constraint and reverse complement constraint were given. In this section, the sufficient and necessary conditions of cyclic codes over A_i satisfying the reverse constraint and reverse complement constraint are given for $i = 2, \ldots, t$.

Definition 5.1. A cyclic code C of length n over A_t is said to be reversible if $\mathbf{x}^r = (x_{n-1}, \ldots, x_0) \in C$, for all $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in C$.

Definition 5.2. For each polynomial $c(x) = c_0 + c_1x + \ldots + c_mx^m$ with $c_m \neq 0$, the reciprocal polynomial of $c(x)$ is defined to be the polynomial $c^*(x) = x^m c(x^{-1})$. The polynomial $c(x)$ and $c^*(x)$ always have the same degree. The polynomial $c(x)$ is called reciprocal if and only if $c(x) = c^*(x)$.

Lemma 5.1. *Let $f(x)$ and $g(x)$ be polynomials in $A_t[x]$. Suppose that deg$f(x)$-deg $g(x) = m$, then*

$$(f(x).g(x))^* = f^*(x)g^*(x)$$

and

$$(f(x) + g(x))^* = f^*(x) + x^m g^*(x).$$

5.1 The reversible codes

In [9], the author studied the reversible codes over Z_4 as follows, by using this, the sufficient and necessary conditions of cyclic codes over A_i satisfying the reverse constraint are given for $i = 2, \ldots, t$.

Lemma 5.2. *[9] Let $C = \langle g(x), 2a(x)\rangle = \langle g(x) + 2a(x)\rangle$ be a cyclic code of odd length n over Z_4. Then C is reversible if and only if both $g(x)$ and $a(x)$ are self reciprocal.*

Theorem 5.1. *[9] Let $C = \langle g(x) + 2p(x)\rangle$ be a cyclic code of even length n over Z_4. Then C is reversible if and only if*

i. $g(x)$ is self reciprocal,

ii. $a(x)|\left(x^i p^*(x)+p(x)\right)$, *where* i =*deg* $g(x)$- *deg* $p(x)$.

Theorem 5.2. *[9] Let* $C=\langle g(x)+2p(x),2a(x)\rangle$ *with* $g(x)|x^n-1 \pmod 2$, $a(x)|g(x) \pmod 2$, $a(x)|p(x)|\left(x^n-1/g(x)\right) \pmod 2$ *and deg* $a(x)>$ *deg* $p(x)$ *be a cyclic code of even length* n *over* Z_4. *Then* C *is reversible if and only if*

i. $g(x)$ *and* $a(x)$ *are self reciprocal,*

ii. $a(x)|(x^i p^*(x)+p(x))$, *where* i =*deg* $g(x)$-*deg* $p(x)$.

Theorem 5.3. *Let* $C=\bigoplus_{B\in P_t} C_{u_B}e_{u_B}$ *be a cyclic code of length* n *over* A_t. *Then* C *is reversible if and only if* C_{u_B} *are reversible, where* C_{u_B} *are cyclic codes over* Z_4*, for all* $B\in P_t$.

Proof. This can be proven similarly to [7]. □

5.2 The reversible complement codes

In this section, the sufficient and necessary conditions of cyclic codes over A_i satisfying the reverse complement constraint are given for $i=2,\dots,t$ and DNA codes are obtained by using cyclic DNA codes over A_t.

Definition 5.3. A cyclic code C of length n over A_t is said to be complement if $\mathbf{x}^c=(x_0^c,\dots,x_{n-1}^c)\in C$, for all $\mathbf{x}=(x_0,\dots,x_{n-1})\in C$.

A cyclic code C of length n over A_t is said to be reversible complement if $\mathbf{x}^{rc}=(x_{n-1}^c,\dots,x_0^c)\in C$, for all $\mathbf{x}=(x_0,\dots,x_{n-1})\in C$.

A cyclic code C of length n over A_t that has reversible complement property is said to be cyclic DNA code.

Lemma 5.3. *The following conditions hold,*

i. For any element $a_i\in A_i$, $a_i^c=(x_{i-1}+u_iy_{i-1})^c=x_{i-1}^c+3u_iy_{i-1}$, *where* $x_{i-1},y_{i-1}\in A_{i-1}$, $i=1,2,\dots,t$.

ii. For all $a\in A_t$, *we have* $a+a^c=1$.

iii. For all $a,b\in A_t$, *we have* $(a+b)^c=a^c+b^c+3$.

Proof. i., ii. According the tables, the computations are easy.
iii. Let $a,b\in A_t$. From ii., $(a+b)^c=1-(a+b)=(1-a)+(1-b)-1=a^c+b^c+3$. □

Theorem 5.4. *Let* $C=\bigoplus_{B\in P_t} C_{u_B}e_{u_B}$ *be a cyclic code of length* n *over* A_t. *Then* C *is reversible complement if and only if* C *is reversible and* $(0^c,\dots,0^c)\in C$, *where* C_{u_B} *are cyclic codes over* Z_4*, for all* $B\in P_t$.

Proof. This can be proven similarly to [7]. □

Corollary 5.1. *Let* C *be a cyclic DNA code of length* n *over* A_t *and minimum Hamming distance* d. *Then* $\xi_t(C)$ *is a DNA code of length* $2^t n$ *over the alphabet* $\{A,C,G,T\}$ *with minimum Hamming distance at least* d.

6. Skew cyclic codes over A_t

For $i=2$, the reversibility problem was solved in [2]. In this section, by using the skew cyclic codes over A_i, the reversibility problem for DNA 2^i-mers is solved for $i=1,3,\dots,t$.

Definition 6.1. Let B be a finite ring and θ be a non trivial automorphism over B. A subset C of B^n is called a skew cyclic code of length n if C satisfies the following conditions,

i. C is a submodule of B^n

ii. If $c=(c_0,\dots,c_{n-1})\in C$, then $\sigma_\theta(c)=(\theta(c_{n-1}),\theta(c_0),\dots,\theta(c_{n-2}))\in C$,

where σ_θ is the skew cyclic shift operator.

By defining a non trivial automorphism on A_t as follows, we can define the skew cyclic codes over A_t.

$$\begin{array}{rcl}\theta_i & : & A_i \longrightarrow A_i \\ x_{i-1}+u_i y_{i-1} & \longmapsto & \theta_{i-1}(x_{i-1}+y_{i-1})-u_i\theta_{i-1}(y_{i-1})\end{array}$$

and

$$\begin{array}{rcl}\theta_1 & : & A_1 \longrightarrow A_1 \\ x_0+u_1 y_0 & \longmapsto & (x_0+y_0)-u_1 y_0\end{array}$$

where $i=2,3,\ldots,t$. The order of θ_i is 2, where $i=1,2,\ldots,t$.

The rings

$$A_i[x,\theta_i]=\{b_0^i+b_1^i x+\ldots+b_{n-1}^i x^{n-1}: b_j^i\in A_i, n\in N, i=1,\ldots,t, j=0,\ldots,n-1\}$$

are called skew polynomial rings with the usual polynomial addition and the multiplication as follows

$$(\varrho x^s)(\eta x^v)=\varrho\theta_i^s(\eta)x^{s+v}$$

where $i=1,\ldots,t$. They are non commutative rings.

The set $A_{\theta_i,n}=A_i[x,\theta_i]/\langle x^n-1\rangle=\{f_i(x)+\langle x^n-1\rangle : f_i(x)\in A_i[x,\theta_i]\}$ is a left $A_i[x,\theta_i]$-module with the multiplication from left as follows,

$$r_i(x)(f_i(x)+\langle x^n-1\rangle)=r_i(x)f_i(x)+\langle x^n-1\rangle$$

where for any $r_i(x)\in A_i[x,\theta_i]$, for $i=1,\ldots,t$.

A code C_i over A_i of length n is a skew cyclic code if and only if C_i is a left $A_i[x,\theta_i]$-submodule of $A_{\theta_i,n}$, where $i=1,\ldots,t$. Let $f_i(x)$ be a polynomial in C_i of minimal degree. If the leading cofficient of $f_i(x)$ is a unit in A_i, then $C_i=\langle f_i(x)\rangle$, where $f_i(x)$ is a right divisor of x^n-1.

We can express the matching the elements A_1 and $S_{D_{16}}=\{AA,TT,\ldots,GG\}$ by means of the automorphism θ_1 as follows.

Each element $\alpha_1=x_0+u_1y_0\in A_1$ and $\theta_1(\alpha_1)$ are mapped to DNA 2-bases which are reverse of each other. Let ξ_1 be a correspondence the elements of the finite ring A_1 and DNA 2-bases. For example

$$\xi_1(u_1)=AT,\ \text{while}\ \xi_1(\theta_1(u_1))=TA$$

By using a map $\xi_i=\gamma_i\circ\phi_i$, where the map γ_i is defined from A_{i-1}^2 to DNA 2^i-bases as foolows

$$\gamma_i(s_{i-1},t_{i-1})=(\xi_{i-1}(s_{i-1}),\xi_{i-1}(t_{i-1}))$$

where $s_{i-1},t_{i-1}\in A_{i-1}$, for $i=1,\ldots,t$, we can explain a relationship between skew cyclic codes and DNA codes. Actually, $\xi_i(r_i)$ and $\xi_i(\theta_i(r_i))$ are DNA reverse of each other, where $r_i=a_{i-1}+u_ib_{i-1}$, $a_{i-1},b_{i-1}\in A_{i-1}$ for $i=1,\ldots,t$.

For $r_i=a_{i-1}+u_ib_{i-1}\in A_i$, we have

$$\begin{array}{rcl}\xi_i(r_i) & = & \gamma_i(\phi_i(a_{i-1}+u_ib_{i-1}))=\gamma_i(a_{i-1},a_{i-1}+b_{i-1}) \\ & = & (\xi_{i-1}(a_{i-1}),\xi_{i-1}(a_{i-1}+b_{i-1}))\end{array}$$

On the other hand,

$$\begin{array}{rcl}\xi_i(\theta_i(r_i)) & = & \xi_i(\theta_{i-1}(a_{i-1}+b_{i-1})-u_i\theta_{i-1}(b_{i-1})) \\ & = & \gamma_i(\phi_i(\theta_{i-1}(a_{i-1}+b_{i-1})-u_i\theta_{i-1}(b_{i-1}))) \\ & = & \gamma_i(\theta_{i-1}(a_{i-1}+b_{i-1}),\theta_{i-1}(a_{i-1})) \\ & = & (\xi_{i-1}(\theta_{i-1}(a_{i-1}+b_{i-1})),\xi_{i-1}(\theta_{i-1}(a_{i-1})))\end{array}$$

where $i=1,\ldots,t$.

This map can be extended as follows. For any $\mathrm{s}_i=(s_0^i,\ldots,s_{n-1}^i)\in A_i^n$,

$$\left(\xi_i(s_0^i),\xi_i(s_1^i),\ldots,\xi_i(s_{n-1}^i)\right)^r=\left(\xi_i(\theta_i(s_{n-1}^i)),\ldots,\xi_i(\theta_i(s_1^i)),\xi_i(\theta_i(s_0^i))\right)$$

where $i=1,2,\ldots,t$.

Example 6.1. If $r_2 = 1 + u_1 + u_2\,(2 + 3u_1) \in A_2$, then we have

$$\begin{aligned} \xi_2\,(r_2) &= \gamma_2\,(\phi_2\,(r_3)) = \gamma_2\,(1 + u_1, 3) \\ &= (\xi_1\,(1 + u_1), \xi_1\,(3))) = (TG, CC) \end{aligned}$$

On the other hand,

$$\begin{aligned} \xi_2\,(\theta_2(r_2)) &= \xi_2\,(\theta_1(3) - u_2\theta_1(2 + 3u_1)) \\ &= \gamma_2(\theta_1(3), \theta_1(1 + u_1)) \\ &= (\xi_1(\theta_1(3)), \xi_1(\theta_1(1 + u_1))) \\ &= (CC, GT) \end{aligned}$$

Definition 6.2. Let C_i be a code of length n over A_i, for $i = 1, \ldots, t$. If $\xi_i(\mathbf{c})^r \in \xi_i(C_i)$ for all $\mathbf{c} \in C_i$, then C_i or equivalently $\xi_i(C_i)$ is called a reversible DNA code, for $i = 1, \ldots, t$.

The skew cyclic code of odd length over A_i with respect to θ_i is a cyclic code, as the order of θ_i is 2 for $i = 1, \ldots, t$. So we will take the length n to be even.

Definition 6.3. Let $g_i(x) = b_0^i + b_1^i x + b_2^i x^2 + \ldots + b_s^i x^s$ be a polynomial of degree s over A_i, for $i = 1, \ldots, t$. $g_i(x)$ is called a palindromic polynomial if $b_j^i = b_{s-j}^i$ for all $j \in \{0, 1, \ldots, s\}$. $g_i(x)$ is called a θ_i-palindromic polynomial if $b_j^i = \theta_i(b_{s-j}^i)$ for all $j \in \{0, 1, \ldots, s\}$, for $i = 1, \ldots, t$.

Theorem 6.1. *Let $C_i = \langle f_i\,(x)\rangle$ be a skew cyclic code of length n over A_i, for $i = 1, 3, \ldots, t$, where $f_i(x)$ is a right divisor of $x^n - 1$ and $\deg(f_i(x))$ is odd. If $f_i(x)$ is a θ_i-palindromic polynomial then $\xi_i(C_i)$ is a reversible DNA code.*

Proof. Let $f_i(x)$ be a θ_i-palindromic polynomial and $f_i(x) = a_0^i + a_1^i x + \ldots + a_{2s-1}^i x^{2s-1}$. So $a_j^i = \theta_i(a_{2s-1-j}^i)$, for all $j = 0, 1, \ldots, s-1$, $i = 1, 3, \ldots, t$. Let $h_i(x) = h_0^i + h_1^i x + \ldots + h_{2k-1}^i x^{2k-1}$. Let b_j^i be the coefficient of x^j in $h_i(x)f_i(x)$. For any $\kappa < n/2$, the coefficient of x^κ in $h_i(x)f_i(x)$ is

$$b_\kappa^i = \sum_{j=0}^{\kappa} h_j^i \theta_i^j (a_{\kappa-j}^i)$$

and the coefficient of $x^{(n-1)-\kappa}$ is $b_{(n-1)-\kappa}^i = \sum\limits_{j=0}^{\kappa} h_{2k-1-j}^i \theta_i^{2k-1-j}(a_{2s-1-(\kappa-j)}^i)$, for $i = 1, 3, \ldots, t$.

The polynomial $h_i(x)f_i(x) = \sum\limits_{p=0}^{2k-1} h_p^i x^p f_i(x)$ corresponds a vector $\mathbf{b} = (b_0^i, b_1^i, \ldots, b_{n-1}^i) \in C_i$, for $i = 1, 3, \ldots, t$. The vector $\xi_i\,(\mathbf{b})^{\,r} = \left(\left(\xi_i\left(b_0^i\right), \xi_i\left(b_1^i\right), \ldots, \xi_i\left(b_{n-1}^i\right)\right)\right)^r$ is equal to the vector $\xi_i\,(\mathbf{z})$, where the vector $\mathbf{z}$ corresponds the polynomial $\sum\limits_{p=0}^{2k-1} \theta_i(h_p^i) x^{2k-1-p} f_i(x)$,for $i = 1, 3, \ldots, t$. So $\xi_i(C_i)$ is a reversible DNA code. □

Theorem 6.2. *Let $C_i = \langle f_i\,(x)\rangle$ be a skew cyclic code of length n over A_i, for $i = 1, 3, \ldots, t$, where $f_i(x)$ is a right divisor of $x^n - 1$ and $\deg(f_i(x))$ is even. If $f_i(x)$ is a palindromic polynomial then $\xi_i(C_i)$ is a reversible DNA code.*

Proof. Let $f_i(x)$ be a palindromic polynomial with even degree. $f_i(x) = a_0^i + a_1^i x + \ldots + a_{2s}^i x^{2s}$ and $a_p^i = a_{2s-p}^i$, for all $p = 0, 1, \ldots, s$, for $i = 1, 3, \ldots, t$. Let $h_i(x) = h_0^i + h_1^i x + \ldots + h_{2k}^i x^{2k}$. Let b_p^i be the coefficient of x^p in $h_i(x)f_i(x)$. For any $\kappa < n/2$, the coefficient of x^κ in $h_i(x)f_i(x)$ is

$$b_\kappa^i = \sum_{j=0}^{\kappa} h_j^i \theta_i^j (a_{\kappa-j}^i)$$

and the coefficient of $x^{(n-1)-\kappa}$ is $b_{(n-1)-\kappa}^i = \sum\limits_{j=0}^{\kappa} h_{(2k)-j}^i \theta_i^{(2k)-j}(a_{2s-(\kappa-j)}^i)$, for $i = 1, 3, \ldots, t$.

The polynomial $h_i(x)f_i(x) = \sum\limits_{p=0}^{2k} h_p^i x^p f_i(x)$ corresponds a vector $\mathbf{b} = (b_0^i, b_1^i, \ldots, b_{n-1}^i) \in C_i$, for $i = 1, 3, \ldots, t$. The vector $\xi_i\,(\mathbf{b})^{\,r} = \left(\left(\xi_i\left(b_0^i\right), \xi_i\left(b_1^i\right), \ldots, \xi_i\left(b_{n-1}^i\right)\right)\right)^r$ is equal to the vector $\xi_i\,(\mathbf{z})$, where the vector $\mathbf{z}$ corresponds the polynomial $\sum\limits_{p=0}^{2k} \theta_i(h_p^i) x^{2k-p} f_i(x)$. So $\xi_i(C_i)$ is a reversible DNA code. □

7. θ_i-set

In this section, we will obtain DNA codes by using θ_i-set, where θ_i is a non trivial automorphism on A_i for $i = 1, ..., t$.

Definition 7.1. Let $f_{0,1}, \ldots, f_{0,2^i}$ be polynomials dividing $x^n - 1$ over Z_4 and let $f_{i-1,1}, f_{i-1,2}$ be polynomials with $\deg f_{i-1,1} = d_{i-1,1}, \deg f_{i-1,2} = d_{i-1,2}$ and both are over A_{i-1} for $i = 1, 2, \ldots, t$. Let

$$f_i = u_i f_{i-1,1} + (1 + u_i) f_{i-1,2} \in A_i[x]$$

and

$$\begin{aligned}
f_{i-1,1} &= u_{i-1} f_{i-2,1} + (1 + u_{i-1}) f_{i-2,2} \\
f_{i-1,2} &= u_{i-1} f_{i-2,3} + (1 + u_{i-1}) f_{i-2,4} \\
\\
f_{i-2,1} &= u_{i-2} f_{i-3,1} + (1 + u_{i-2}) f_{i-3,2} \\
f_{i-2,2} &= u_{i-2} f_{i-3,3} + (1 + u_{i-2}) f_{i-3,4} \\
f_{i-2,3} &= u_{i-2} f_{i-3,5} + (1 + u_{i-2}) f_{i-3,6} \\
f_{i-2,4} &= u_{i-2} f_{i-3,7} + (1 + u_{i-2}) f_{i-3,8} \\
&\vdots \\
f_{1,1} &= u_1 f_{0,1} + (1 + u_1) f_{0,2} \\
f_{1,2} &= u_1 f_{0,3} + (1 + u_1) f_{0,4} \\
&\vdots
\end{aligned}$$

$$f_{1,2^{i-1}} = u_1 f_{0,2^i-1} + (1 + u_1) f_{0,2^i}$$

Let $m_i = \min\{n - d_{i-1,1}, n - d_{i-1,2}\}$. The set $L(f_i)$ is called a θ_i-set and is defined as

$$L(f_i) = \{E_0, E_1, \ldots, E_{m_i-1}, F_0, F_1, \ldots, F_{m_i-1}\}$$

where $E_j = x^j f_i, F_j = x^j \theta_i(h_i), 0 \leq j \leq m_i - 1, i = 1, 2, \ldots, t$.

If $\deg f_{0,2s} \geq \deg f_{0,2s-1}$,

$$h_{i,1,s} = u_1 x^{\deg f_{0,2s} - \deg f_{0,2s-1}} f_{0,2s-1} + (1 + u_1)\, f_{0,2s}$$

otherwise

$$h_{i,1,s} = u_1 f_{0,2s-1} + (1 + u_1)\, x^{\deg f_{0,2s-1} - \deg f_{0,2s}} f_{0,2s}$$

where $s = 1, 2, \ldots, 2^{i-1}$ and

If $\deg h_{i,1,2t} \geq \deg h_{i,1,2t-1}$,

$$h_{i,2,t} = u_2 x^{\deg f_{ii,1,2t} - \deg f_{i,1,2t-1}} h_{i,1,2t-1} + (1 + u_2)\, h_{i,1,2t}$$

otherwise

$$h_{i,2,t} = u_2 h_{i,1,2t-1} + (1 + u_2)\, x^{\deg f_{i,1,2t-1} - \deg f_{ii,1,2t}} h_{i,1,2t}$$

where $t = 1, 2, \ldots, 2^{i-2}$ and

$$\vdots$$

If $\deg h_{i,i-2,2v} \geq \deg h_{i,i-2,2v-1}$,

$$h_{i,i-1,v} = u_{i-1} x^{\deg h_{i,i-2,2v} - \deg h_{i,i-2,2v-1}} h_{i,i-2,2v-1} + (1 + u_{i-1})\, h_{i,i-2,2v}$$

otherwise

$$h_{i,i-1,v} = u_{i-1} h_{i,i-2,2v-1} + (1 + u_{i-1})\, x^{\deg h_{i,i-2,2v-1} - \deg h_{i,i-2,2v}} h_{i,i-2,2v}$$

where $v = 1, 2$ and

If $\deg h_{i,i-1,2} \geq \deg h_{i,i-1,1}$,

$$h_i = u_i x^{\deg h_{i,i-1,2} - \deg h_{i,i-1,1}} h_{i,i-1,1} + (1 + u_i)\, h_{i,i-1,2}$$

otherwise

$$h_i = u_i h_{i,i-1,1} + (1 + u_i)\, x^{\deg h_{i,i-1,1} - \deg h_{i,i-1,2}} h_{i,i-1,2}.$$

$L(f_i)$ generates a linear code C_i over A_i, where $i = 1, 2, \ldots, t$. It will be denoted by $C_i = \langle f_i \rangle_{\theta_i}$ or $C_i = \langle L(f_i) \rangle$. It means that it is A_i-submodule generated by the set $L(f_i)$, where $i = 1, 2, \ldots, t$. Let $f_i = a_0^i + a_1^i x + \ldots + a_p^i x^p \in A_i[x], \theta_i(h_i) = b_0^i + b_1^i x + \ldots + b_s^i x^s$, where $i = 1, 2, \ldots, t$. The A_i-submodule can be considered to be generated by the rows of the following matrix

$$L(f_i) = \begin{bmatrix} E_0 \\ F_0 \\ E_1 \\ F_1 \\ E_2 \\ F_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_0^i & a_1^i & a_2^i & \cdots & a_p^i & 0 & \cdots & \cdots & & \cdots & 0 \\ b_0^i & b_1^i & \cdots & \cdots & b_p^i & b_{p+1}^i & \cdots & b_s^i & 0 & \cdots & 0 \\ 0 & a_0^i & a_1^i & a_2^i & \cdots & a_p^i & 0 & 0 & & \cdots & 0 \\ 0 & b_0^i & b_1^i & \cdots & \cdots & \cdots & \cdots & \cdots & b_s^i & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & & \cdots & \vdots \end{bmatrix}$$

Theorem 7.1. *Let $f_{0,1}, \ldots, f_{0,2^i}$ be self reciprocal polynomials dividing $x^n - 1$ over Z_4. Then $C_i = \langle L(f_i) \rangle$ is a linear code over A_i and $\xi_i(C_i)$ is a reversible DNA code, where the map ξ_i is from C_i to $S_{D_4}^{2^i n}$, for $i = 1, 2, \ldots, t$.*

Proof. It is proved as in the proof of the Theorem 4.3 in [3]. □

Corollary 7.1. *Let $f_{0,1}, \ldots, f_{0,2^i}$ be self reciprocal polynomials dividing $x^n - 1$ over Z_4 and $C_i = \langle L(f_i) \rangle$ be a cyclic code over A_i for $i = 1, \ldots, t$. If $\frac{x^n - 1}{x - 1} \in C_i$, then $\xi_i(C_i)$ is a reversible complement DNA code.*

Example 7.1.

$$\begin{aligned} f_{0,1}(x) &= 2\,(x+1) \\ f_{0,2}(x) &= x^4 - x^3 + x^2 - x + 1 \end{aligned}$$

where all of them divide $x^{10} - 1$ over Z_4. Hence,

$$f_1 = u_1 f_{0,1} + (1 + u_1)\, f_{0,2}$$

over A_1. That is

$$f_2 = (1 + u_1)\, x^4 - (1 + u_1)\, x^3 + (1 + u_1)\, x^2 - (1 - u_1)\, x + 1 + 3u_1.$$

We get $h_1 = u_1 x^3 h_{1,0,1} + (1 + u_1) h_{1,0,2} = (1 + 3u_1)\, x^4 - (1 - u_1)\, x^3 + (1 + u_1)\, x^2 - (1 + u_1)\, x + 1 + u_1$. So, $\theta_1(h_1) = u_1 x^4 - u_1 x^3 + (2 + 3u_1)\, x^2 - (2 + 3u_1)\, x + 2 + 3u_1$. Since $m_1 = 6$, we consider the generator matrix of C $\begin{bmatrix} E_0 \\ F_0 \\ E_1 \\ F_1 \\ \vdots \\ E_5 \\ F_5 \end{bmatrix}$, where $E_0 = f_1, E_1 = x f_1, E_2 = x^2 f_1, E_3 = x^3 f_1, E_4 = x^4 f_1, E_5 = x^5 f_1, F_0 = \theta_1(h_1), F_1 = x\theta_1(h_1), F_2 = x^2\theta_1(h_1), F_3 = x^3\theta_1(h_1), F_4 = x^4\theta_1(h_1), F_5 = x^5\theta_1(h_1)$. If we take $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = u_1, \alpha_3 = 0, \alpha_4 = 0, \alpha_5 = 0, \beta_0 = 1, \beta_1 = 0, \beta_2 = 1, \beta_3 = 0, \beta_4 = 0, \beta_5 = 3$, then $\alpha_0 E_0 + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 + \alpha_5 E_5 + \beta_0 F_0 + \beta_1 F_1 + \beta_2 F_2 + \beta_3 F_3 + \beta_4 F_4 + \beta_4 F_4 = 3u_1 x^9 + u_1 x^8 + (2 + u_1) x^7 + (2 + 2u_1)\, x^6 + (3 + 3u_1)\, x^5 + (1 + u_1)\, x^4 + (3 + u_1) x^3 + (3 + 3u_1) x^2 + 3x + 2 + 3u_1$. It corresponds to the codeword

$$\mathbf{d}_1 = (2 + 3u_1, 3, 3 + 3u_1, 3 + u_1, 1 + u_1, 3 + 3u_1, 2 + 2u_1, 2 + u_1, u_1, 3u_1)$$

Hence, $\xi_1(\mathbf{d}_1) = GTCCCGCATGCGGAGCATAC$. Moreover, $\theta_1(\alpha_0) F_5 + \theta_1(\alpha_1) F_4 + \theta_1(\alpha_2) F_3 + \theta_1(\alpha_3) F_2 + \theta_1(\alpha_4) F_1 + \theta_1(\alpha_5) F_0 + \theta_1(\beta_0) E_5 + \theta_1(\beta_1) E_4 + \theta_1(\beta_2) E_3 + \theta_1(\beta_3) E_2 + \theta_1(\beta_4) E_1 + \theta_1(\beta_5) E_0 = (1 + u_1)\, x^9 +$

$3x^8+(2+u_1)x^7+3u_1x^6+(2+3u_1)x^5+(2+u_1)x^4+2u_1x^3+(3+3u_1)x^2+(1+3u_1)x+3+u_1$ corresponds to the codeword

$$\mathbf{d}_2=(3+u_1,1+3u_1,3+3u_1,2u_1,2+u_1,2+3u_1,3u_1,2+u_1,3,1+u_1)$$

Hence, $\xi_1(\mathbf{d}_2)=CATACGAGGCGTACGCCCTG$. So, $(\xi_1(\mathbf{d}_2))^r=\xi_1(\mathbf{d}_1)$.

Example 7.2.

$$\begin{aligned} f_{0,1}(x) &= x+1\\ f_{0,2}(x) &= x^2+x+1\\ f_{0,3}(x) &= x^6+x^3+1\\ f_{0,4}(x) &= x+1 \end{aligned}$$

where all of them divide x^9-1 over Z_4. Hence,

$$f_2=u_2\left(u_1f_{0,1}+(1+u_1)f_{0,2}\right)+(1+u_2)$$

over A_2. That is

$$f_2=u_1(1+u_2)x^6+u_1(1+u_2)x^3+u_2(1+u_1)x^2+(1+u_1+2u_2+3u_1u_2)x+1+2u_1+2u_2.$$

Since $h_{2,1,1}=u_1xf_{0,1}+(1+u_1)f_{0,2}$ and $h_{2,1,2}=u_1f_{0,3}+x^5(1+u_1)f_{0,4}$, we get $h_2=u_2x^4h_{2,1,1}+(1+u_2)h_{2,1,2}=(1+2u_1+2u_2)x^6+(1+u_1+2u_2+3u_1u_2)x^5+(1+u_1)u_2x^4+(1+u_2)u_1x^3+u_1(1+u_2)$. So, $\theta_2(h_2)=(1+2u_1+2u_2)x^6+(3+3u_2+3u_1u_2)x^5+(2+3u_1+2u_2+u_1u_2)x^4+(2+2u_1+3u_2+u_1u_2)x^3+(2+2u_1+3u_2+u_1u_2)$.

Since $m_2=3$, we consider the generator matrix of C $\begin{bmatrix}E_0\\F_0\\E_1\\F_1\\E_2\\F_2\end{bmatrix}$, where $E_0=f_2, E_1=xf_2, E_2=x^2f_2, F_0=\theta_2(h_2), F_1=x\theta_2(h_2), F_2=x^2\theta_2(h_2)$. If we take $\alpha_0=0,\alpha_1=0,\alpha_2=3,\beta_0=0,\beta_1=2,\beta_2=0$, then $\alpha_0E_0+\alpha_1E_1+\alpha_2E_2+\beta_0F_0+\beta_1F_1+\beta_2F_2=3u_1(1+u_2)x^8+2x^7+(2+2u_2+2u_1u_2)x^6+(u_1+u_1u_2)x^5+(u_2+u_1u_2)x^4+(3+3u_1+2u_2+u_1u_2)x^3+(3+2u_1+2u_2)x^2+(2u_2+2u_1u_2)x$. It corresponds to the codeword

$$\mathbf{d}_1=\begin{pmatrix}0,2u_2+2u_1u_2,3+2u_1+2u_2,3+3u_1+2u_2+u_1u_2,\\ u_2+u_1u_2,u_1+u_1u_2,2+2u_2+2u_1u_2,2,3u_1+3u_1u_2\end{pmatrix}$$

Hence, $\xi_2(\mathbf{d}_1)=AAAAAAGACTTCCGTTAATGATAGGGAGGGGGACAG$. Moreover, $\theta_2(\alpha_0)F_2+\theta_2(\alpha_1)F_1+\theta_2(\alpha_2)F_0+\theta_2(\beta_0)E_2+\theta_2(\beta_1)E_1+\theta_2(\beta_2)E_0=2u_1(1+u_2)x^7+(3+2u_1+2u_2)x^6+(1+u_2+u_1u_2)x^5+(2+3u_1+2u_2+u_1u_2)x^4+(2+2u_1+3u_2+u_1u_2)x^3+(2+2u_1+2u_1u_2)x^2+2x+2+2u_1+u_2+3u_1u_2$ corresponds to the codeword

$$\mathbf{d}_2=\begin{pmatrix}2+2u_1+u_2+3u_1u_2,2,2+2u_1+2u_1u_2,2+2u_1+3u_2+u_1u_2,\\ 2+3u_1+2u_2+u_1u_2,1+u_2+u_1u_2,3+2u_1+2u_2,2u_1+2u_1u_2,0\end{pmatrix}$$

Hence, $\xi_2(\mathbf{d}_2)=GACAGGGGGAGGGATAGTAATTGCCTTCAGAAAAAA$. So, $(\xi_2(\mathbf{d}_2))^r=\xi_2(\mathbf{d}_1)$.

8. Conclusion

The DNA codes are obtained with three different methods by using cyclic, skew cyclic codes and θ_i-set over a family of the rings A_t. A one to one correspondence between A_t and $\{A,T,C,G\}^{2^t}$ is constructed by using a map.The sufficient and necessary conditions of cyclic codes over A_t satisfying the reverse and reverse complement constraints are given, respectively. By defining a non trivial automorphism θ_i on A_t, the skew cyclic codes are introduced. By using the skew cyclic codes over A_t and the θ_i-set, the DNA codes are obtained. In a future work, it can be identified the new ring family and its associated Gray map reversible and reversible complement codes to search for optimal DNA codes that meet all or some of the constraints.

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