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# Global Stability and Bifurcation Analysis in a Discrete-Time Two Predator-One Prey Model with Michaelis-Menten Type Prey Harvesting

Debasis Mukherjee[1]*

**Abstract**
This article studies a discrete-time Leslie-Gower two predator-one prey system with Michaelis-Menten type prey harvesting. Positivity and boundedness of the model solution are investigated. Existence and stability of fixed points are examined. Using an iteration scheme and the comparison principle of difference equations, we find out the sufficient condition for global stability of the positive fixed point. It is shown that the sufficient criterion for Neimark-Sacker bifurcation can be developed. It is observed that the system behaves in a chaotic manner when a specific set of system parameters is chosen, which are regulated by a hybrid control method. Examples are provided to illustrate our conclusions.

**Keywords:** Bifurcation, Chaos control, Leslie-Gower, Michaelis-Menten type harvesting, Predator-prey model, Stability.
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[1] *Department of Mathematics, Vivekananda College, Thakurpukur, Kolkata-700063, India, ORCID: 0000-0003-2717-3940*
***Corresponding author**: mukherjee1961@gmail.com
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## 1. Introduction

In the real world, the interaction between prey and their predator create a major interest to the researchers to explore the dynamics of the system. Most of the existing predator-prey models come from the Lotka-Volterra system. The Lotka-Volterra models cannot justify all the predator-prey interaction. For example, when the size of the prey decreases, then the predator will search for other prey. This fact motivated Leslie to form an appropriate model known as Leslie-Gower predator-prey system to investigate the behaviour of the system. Several studies have been done on modified Leslie-Gower model with various aspects [1]-[3].

In spite of the vast research over the last few years, the knowledge about the effect of non-linear Michaelis-Menten type of harvest on one prey-two predator models is insufficient. We observe that the ecological system is often perturbed by the growing human needs for more food and more energy. For example, the fish population has decreased due to the rapid progress of fishing technology and substantial growth in human populations. Therefore, the exploitation of renewable resources, which associates immediately to sustainable development. Clark [4, 5] introduced harvesting of species through mathematical models. There are three types of harvesting namely constant rate, proportionate and Michaelis-Menten type found in the literatures [6]-[9]. Out of these, non-linear harvesting is more realistic and exhibits saturation effects with respect to both the stock abundance and effort level. Das et al. [10] analysed a prey-predator model considering Michaeli-Menten type harvesting on both the populations. They discussed boundedness, local and global stability of the proposed system. Gupta and Chandra [8] followed the similar type of harvesting in prey and derived different bifurcations such as transcritical, saddle-node, Hopf and Bogdanov-Takens in the Leslie-Gower prey-predator model. Hu and Cao [11] discussed stability and bifurcation for a predator-prey system with Michaelis-Menten type predator harvesting. Ang and Safuan [12] investigated the dynamical behaviour of an intraguild prey-predator fishery model with the non-linear harvesting of prey species.

Mathematical models followed by differential equations are reasonable for the species in which populations are overlapped. In case of non-overlapping generations, discrete-time models governed by difference equations are more appropriate than the differential equations. In real ecosystem, a discrete time system can be seen, for example, fish populations reproduce at specific timed moments or for insect populations, for which non-overlapping generations are occurring. Moreover, discrete-time models also allow more efficient computational results for numerical simulations and exhibit a rich dynamics as compared to the continuous ones [13]-[16]. Even discrete time models can admit chaotic dynamics [13, 14]. More interesting and significant results on discrete prey-predator models can be seen in [17]-[21]. Ajaz et al. [22] investigated the dynamical behaviour of a modified Leslie-Gower prey-predator model with harvesting in prey population and showed the existence and directions of period doubling and Neimark-Sacker at positive fixed point and also indicated chaos control when chaos emerge through bifurcation. Khan et al. [23] discussed a discrete-time Michaelis-Menten type prey harvesting in the modified Leslie-Gower predator-prey model and obtained the conditions for the existence of flip and Neimark-Sacker bifurcations. Chen et al. [24] studied a discrete Leslie-Gower predator-prey model with Michaelis-Menten prey harvesting and observed that the system can exhibit fold, flip and Neimark-Sacker bifurcations by the application of center manifold theorem and bifurcation theory.

The above studies are mainly confined into two species models. However, it is a common fact that several predators compete for a prey in the real world. To our knowledge, there is limited works that highlight discrete-time non-linear harvesting in the modified Leslie-Gower Holling type II two-predator one-prey model.

Now we first present a model which is a modified Leslie-Gower two predator- one prey system with Michaelis-Menten type prey harvesting:

$$
\begin{aligned}
\frac{dx}{dt} &= x(r_1 - ax - \frac{c_1 y}{h_1 + x} - \frac{c_2 z}{h_2 + x} - \frac{qE}{d_1 E + d_2 x}),\\
\frac{dy}{dt} &= y(r_2 - \frac{f_1 y}{h_1 + x}),\\
\frac{dz}{dt} &= z(r_3 - \frac{f_2 z}{h_2 + x}),
\end{aligned}
\tag{1.1}
$$

where $x$, $y$ and $z$ denote the densities of prey, the first predator and the second predator respectively. $r_1, r_2, r_3$ stands for the intrinsic growth rate of the prey and two predators respectively. $a$ represents the intra-specific competition among the the prey species. $c_1$ and $c_2$ denote the per-capita reduction of prey $x$. $f_1$ and $f_2$ carry the same meaning as of $c_1$ and $c_2$. $h_1$ and $h_2$ signifies the environmental protection for predator $y$ and $z$ respectively. In the prey harvesting term $\frac{qEx}{d_1 E + d_2 x}$, $q$ is the catchability coefficient, $d_1$ and $d_2$ are the degree of competition in the harvesting business and handling time respectively. $E$ describes the harvesting effort.

For qualitative analysis, including global stability, bifurcation analysis and chaos control for a discrete analogue of system (1.1), a piecewise constant argument is introduced to describe the following exponential form of nonlinear difference equations:

$$
\begin{aligned}
x_{n+1} &= x_n \exp\{r_1 - a x_n - \frac{c_1 y_n}{h_1 + x_n} - \frac{c_2 z_n}{h_2 + x_n} - \frac{qE}{d_1 E + d_2 x_n}\},\\
y_{n+1} &= y_n \exp\{r_2 - \frac{f_1 y_n}{h_1 + x_n}\},\\
z_{n+1} &= z_n \exp\{r_3 - \frac{f_2 z_n}{h_2 + x_n}\}
\end{aligned}
\tag{1.2}
$$

where $x_n$, $y_n$ and $z_n$ represent the densities of prey and both the predator at generation $n \in \mathbb{N}$ respectively.

The rest of the paper is formatted as follows. Positivity and boundedness of solutions are presented in Section 2. The existence and stability of the interior fixed point are discussed in Section 3. Global stability criterion is derived in Section 4. Neimark-Sacker bifurcation and flip bifurcation are described in Section 5. Chaos control mechanism is presented in Section 6. Numerical examples are given in Section 7. Section 8 concludes the paper.

## 2. Positivity and Boundedness of Solutions

In this section, we discuss positivity and boundedness of solutions of system (1.2). The first lemma follows immediately from the system structure and its proof is omitted.

**Lemma 2.1.** *Solutions of system (1.2) with positive initial conditions remain positive.*

To prove the boundedness of solutions of system (1.2), we require the following lemma:

**Lemma 2.2.** *(see [25]) Suppose that $x_m$ satisfies $x_0 > 0$ and $x_{m+1} \le x_m exp[\alpha(1-\beta x_m)]$ for $m \in [m_1, \infty)$ where $\beta$ is a positive constant. Then $limsup_{n\to\infty} x_m \le \frac{1}{\alpha\beta} exp(\alpha - 1)$.*

We now state the theorem which ensures that every positive solution of system (1.2) is uniformly bounded.

**Theorem 2.3.** *Every positive solution $\{(x_n, y_n, z_n)\}$ of system (1.2) is uniformly bounded.*

*Proof.* Assume that $\{(x_n, y_n, z_n)\}$ be an arbitrary positive solution of system (1.2). From the first equation of system (1.2), we get

$$x_{n+1} \le x_n \exp(r_1 - a x_n), n = 0, 1, 2, \ldots.$$

Assume that $x_0 > 0$, then following Lemma 2.2, we get $\limsup_{n\to\infty} x_n \le \frac{1}{a}\exp(r_1 - 1) := M_1$. From the second equation of system (1.2),

$$y_{n+1} \le y_n \exp(r_2 - \frac{f_1}{h_1 + M_1} y_n), n = 0, 1, 2, \ldots.$$

It follows from Lemma 2.2 that $\limsup_{n\to\infty} y_n \le \frac{h_1+M_1}{f_1}\exp(r_2 - 1) := M_2$ whenever $y_0 > 0$. Assume that $z_0 > 0$. From the third equation of system (1.2), we get

$$z_{n+1} \le z_n \exp(r_3 - \frac{f_2}{h_2 + M_1} z_n).$$

Applying again Lemma 2.2, we get

$$\limsup_{n\to\infty} z_n \le \frac{h_2 + M_1}{f_2}\exp(r_3 - 1) := M_3.$$

Then it follows that $\limsup_{n\to\infty}(x_n, y_n, z_n) \le M$, where $M = \max\{M_1, M_2, M_3\}$.
This completes the proof. □

## 3. Existence of Fixed Points

In this section, we determine the fixed points and their dynamics. Evidently, system (1.1) has at most twelve non-negative fixed points $E_0 = (0,0,0)$. If $q < r_1 d_1$ then the fixed point $E_1 = (\bar{x}, 0, 0)$ exists uniquely where

$$\bar{x} = \frac{r_1 d_2 - a d_1 E + \sqrt{(r_1 d_2 - a d_1 E)^2 - 4 a d_2 E(q - r_1 d_1)}}{2 a d_2}.$$

If $q > r_1 d_1, r_1 d_2 > a d_1 E$ and $(r_1 d_2 - a d_1 E)^2 - 4 a d_2 E(q - r_1 d_1) > 0$ then multiple fixed points exist $E_{1\pm} = (\bar{x}_\pm, 0, 0)$ where

$$\bar{x}_\pm = \frac{r_1 d_2 - a d_1 E \pm \sqrt{(r_1 d_2 - a d_1 E)^2 - 4 a d_2 E(q - r_1 d_1)}}{2 a d_2}.$$

There always exists $E_2 = (0, \frac{r_2 h_1}{f_1}, 0)$ and $E_3 = (0, 0, \frac{r_3 h_2}{f_2})$. If $q f_1 + d_1 c_1 r_2 < d_1 r_1 f_1$ then there exists a unique fixed point $E_{12} = (\hat{x}, \hat{y}, 0)$ where

$$\hat{x} = \frac{d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E + \sqrt{(d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E)^2 - 4 a f_1 d_2 E(q f_1 + d_1 c_1 r_2 - d_1 r_1 f_1)}}{2 a f_1 d_2}$$

and

$$\hat{y} = \frac{r_2(h_1 + \hat{x})}{f_1}.$$

If $q f_1 + d_1 c_1 r_2 > d_1 r_1 f_1, r_1 f_1 d_2 > c_1 r_2 d_2 + a f_1 d_1 E$ and $\{d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E\}^2 > 4 a f_1 d_2 E(q f_1 + d_1 c_1 r_2 - d_1 r_1 f_1)$ then there exists multiple fixed points $E_{12\pm} = (\hat{x}_\pm, \hat{y}_\pm, 0)$ where

$$\hat{x}_\pm = \frac{d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E \pm \sqrt{(d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E)^2 - 4 a f_1 d_2 E(q f_1 + d_1 c_1 r_2 - r_1 f_1 d_1)}}{2 a f_1 d_2}$$

and

$$\hat{y}_{\pm} = \frac{r_2(h_1 + \hat{x}_{\pm})}{f_1}.$$

If $qf_2 + d_1c_2r_3 < d_1r_1f_2$ then there exists a unique fixed point $E_{13} = (\tilde{x}, 0, \tilde{z})$ where

$$\tilde{x} = \frac{d_2(r_1f_2 - c_2r_3) - af_2d_1E + \sqrt{(d_2(r_1f_2 - c_2r_3) - af_2d_1E)^2 - 4af_2d_2E(qf_2 + d_1c_2r_3 - d_1r_1f_2)}}{2af_2d_2}$$

and

$$\tilde{y} = \frac{r_3(h_2 + \tilde{x})}{f_2}.$$

If $qf_2 + d_1c_2r_3 > d_1r_1f_2, r_1f_2d_2 > c_2r_3d_2 + af_2d_1E$ and $\{d_2(r_1f_2 - c_2r_3) - af_2d_1E\}^2 > 4af_2d_2E(qf_2 + d_1c_2r_3 - d_1r_1f_2)$ then there exists multiple fixed points $E_{13\pm} = (\tilde{x}_{\pm}, 0, \tilde{z}_{\pm})$ where

$$\tilde{x}_{\pm} = \frac{d_2(r_1f_2 - c_2r_3) - af_2d_1E \pm \sqrt{(d_2(r_1f_2 - c_2r_3) - af_2d_1E)^2 - 4af_2d_2E(qf_2 + d_1c_2r_3 - r_1f_2d_1)}}{2af_2d_2}$$

and

$$\tilde{z}_{\pm} = \frac{r_3(h_2 + \tilde{x}_{\pm})}{f_2}.$$

There exists a unique fixed point $E_{23} = (0, \frac{r_2h_1}{f_1}, \frac{r_3h_2}{f_2})$. To determine the positive fixed point $E^* = (x^*, y^*, z^*)$ , we have to solve the following system of equations:

$$x = x(r_1 - ax - \frac{c_1y}{h_1 + x} - \frac{c_2z}{h_2 + x} - \frac{qE}{d_1E + d_2x}), \tag{3.1}$$

$$y = y(r_2 - \frac{f_1y}{h_1 + x}), \tag{3.2}$$

$$z = z(r_3 - \frac{f_2z}{h_2 + x}). \tag{3.3}$$

where $x^*, y^*$ and $z^*$ are the positive solutions of equations (3.1), (3.2) and (3.3). Solving (3.2) and (3.3) we get $y = \frac{r_2(h_1+x)}{f_1}$ and $z = \frac{r_3(h_2+x)}{f_2}$ and substituting the value of $y$ and $z$ in (3.1), we obtain the following equation:

$$Ax^2 + Bx + C = 0 \tag{3.4}$$

where

$$A = f_1f_2ad_2, B = f_1f_2ad_2E - d_2(r_1f_1f_2 - c_1r_2f_2 - c_2r_3f_1), C = E\{f_1f_2q + d_1(c_1r_2f_2 + c_2r_3f_1) - d_1r_1f_1f_2\}$$

If $C < 0$ then there exists a unique positive root $x^*$ of equation (3.4). In that case there exists a unique fixed point $E^* = (x^*, y^*, z^*)$ where

$$x^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, y^* = \frac{r_2(h_1 + x^*)}{f_1}$$

and

$$z^* = \frac{r_3(h_2 + x^*)}{f_2}.$$

If $B < 0, C > 0$ and $B^2 > 4AC$ then there exists multiple fixed points $E^*_{\pm} = (x^*_{\pm}, y^*_{\pm}, z^*_{\pm})$ where

$$x^*_{\pm} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, y^*_{\pm} = \frac{r_2(h_1 + x^*_{\pm})}{f_1}$$

and

$$z^*_{\pm} = \frac{r_3(h_2 + x^*_{\pm})}{f_2}.$$

### 3.1 Stability of fixed points

To investigate the local stability of the fixed points of system (1.2), we require the following lemma.

**Lemma 3.1.** *([26]) Consider the cubic equation*

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0 \tag{3.5}$$

*where $p_1$, $p_2$ and $p_3$ are real numbers. Then necessary and sufficient conditions that all the roots of equation (3.5) lie in an open disk $|\lambda| < 1$ are $|p_1 + p_3| < 1 + p_2, |p_1 - 3p_3| < 3 - p_2$ and $p_3^2 + p_2 - p_3p_1 < 1$.*

The Jacobian matrix $J(E_0)$ for system (1.2) is given by

$$J(E_0) = \begin{pmatrix} \exp(r_1 - \frac{q}{d_1}) & 0 & 0 \\ 0 & \exp r_2 & 0 \\ 0 & 0 & \exp r_3 \end{pmatrix}.$$

Then it follows from $J(E_0)$ that $E_0$ is an unstable fixed point for system (1.2). Again

$$J(E_1) = \begin{pmatrix} 1 - a\bar{x} + \frac{qEd_2\bar{x}}{(d_1E+d_2\bar{x})^2} & -\frac{c_1\bar{x}}{h_1+\bar{x}} & -\frac{c_2\bar{x}}{h_2+\bar{x}} \\ 0 & \exp r_2 & 0 \\ 0 & 0 & \exp r_3 \end{pmatrix}.$$

From $J(E_1)$, we conclude that that $E_1$ is an unstable fixed point for system (1.2). Similarly, it can be shown that $E_{1\pm}$ are also unstable. Now

$$J(E_2) = \begin{pmatrix} \exp(r_1 - \frac{c_1r_2}{f_1} - \frac{q}{d_1}) & 0 & 0 \\ \frac{r_1^2}{f_1} & 1 - r_2 & 0 \\ 0 & 0 & \exp r_3 \end{pmatrix}.$$

It is obvious from $J(E_2)$ that $E_2$ is an unstable fixed point for system (1.2). For $E_3$,

$$J(E_3) = \begin{pmatrix} \exp(r_1 - \frac{c_2r_3}{f_2} - \frac{q}{d_1}) & 0 & 0 \\ 0 & \exp r_2 & 0 \\ \frac{r_3^2}{f_2} & 0 & 1 - r_3 \end{pmatrix}.$$

Again we see that from $J(E_3)$ that $E_3$ is an unstable fixed point for system (1.2). For $E_{12}$,

$$J(E_{12}) = \begin{pmatrix} 1 - \hat{x}(a - \frac{c_1\hat{y}}{(h_1+\hat{x})^2} - \frac{qEd_2}{(d_1E+d_2\hat{x})^2}) & -\frac{c_1\hat{x}}{h_1+\hat{x}} & -\frac{c_2\hat{x}}{h_2+\hat{x}} \\ \frac{f_1\hat{y}^2}{(h_1+\hat{x})^2} & 1 - \frac{\hat{y}f_1}{h_1+\hat{x}} & 0 \\ 0 & 0 & \exp r_3 \end{pmatrix}.$$

Again we see that from $J(E_{12})$ that $E_{12}$ is an unstable fixed point for system (1.2). Similarly, it can be shown that $E_{12\pm}$ are also unstable. For $E_{13}$,

$$J(E_{13}) = \begin{pmatrix} 1 - \tilde{x}(a - \frac{c_2\tilde{z}}{(h_2+\tilde{x})^2} - \frac{qEd_2}{(d_1E+d_2\tilde{x})^2}) & -\frac{c_1\tilde{x}}{h_1+\tilde{x}} & -\frac{c_2\tilde{x}}{h_2+\tilde{x}} \\ 0 & \exp r_2 & 0 \\ \frac{\tilde{z}^2f_2}{(h_2+\tilde{x})^2} & 0 & 1 - \frac{f_2\tilde{z}}{h_2+\tilde{x}} \end{pmatrix}.$$

It is clear from $J(E_{13})$ that $E_{13}$ is an unstable fixed point for system (1.2). Similarly, it can be shown that $E_{13\pm}$ are also unstable. Now

$$J(E_{23}) = \begin{pmatrix} \exp(r_1 - \frac{c_1r_2}{f_1} - \frac{c_2r_3}{f_2} - \frac{q}{d_1}) & 0 & 0 \\ \frac{r_2^2}{f_1} & 1 - r_2 & 0 \\ \frac{r_3^2}{f_2} & 0 & 1 - r_3 \end{pmatrix}.$$

If $r_1 < \frac{c_1r_2f_2d_1 + c_2r_2f_1d_1 + qf_1f_2}{f_1f_2d_1}, r_2 < 2$ and $r_3 < 2$ then it follows from $J(E_{23})$ that $E_{23}$ is locally asymptotically stable fixed point for system (1.2). Let $E^* = (x^*, y^*, z^*)$ be the unique interior fixed point of system (1.2). The Jacobian matrix for (1.2) at $E^*$ is given by

$$J(x^*, y^*, z^*) = \begin{pmatrix} a_{11} & -\frac{c_1x^*}{h_1+x^*} & -\frac{c_2x^*}{h_2+x^*} \\ \frac{f_1y^{*2}}{(h_1+x^*)^2} & 1 - r_2 & 0 \\ \frac{f_2z^{*2}}{(h_2+x^*)^2} & 0 & 1 - r_3 \end{pmatrix}$$

where

$$a_{11} = 1 - ax^* + \frac{qEd_2x^*}{(d_1E + d_2x^*)^2} + \frac{c_2x^*z^*}{(h_2 + x^*)^2} + \frac{c_1x^*y^*}{(h_1 + x^*)^2}.$$

The characteristic polynomial of $J(E^*)$ is given by

$$P(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 \tag{3.6}$$

where

$$\begin{aligned} p_1 &= r_2 + r_3 - 2 - a_{11}, \\ p_2 &= a_{11}(2 - r_2 - r_3) + (1 - r_2)(1 - r_3) + \frac{c_1f_1x^*y^{*2}}{(h_1 + x^*)^3} + \frac{c_2f_2x^*z^{*2}}{(h_2 + x^*)^3}, \\ p_3 &= a_{11}(1 - r_2)(r_3 - 1) + \frac{c_1f_1x^*y^{*2}(r_3 - 1)}{(h_1 + x^*)^3} + \frac{c_2f_2x^*z^{*2}(r_2 - 1)}{(h_2 + x^*)^3}. \end{aligned} \tag{3.7}$$

We now use Lemma 3.1 to investigate stability of $E^*$.

**Lemma 3.2.** *Assume that $C < 0$ holds. Then, the fixed point $E^*$ is locally asymptotically stable if and only if the following conditions are satisfied:*

$$|p_1 + p_3| < 1 + p_2, |p_1 - 3p_3| < 3 - p_2$$

*and $p_3^2 + p_2 - p_3p_1 < 1$ where $p_1$, $p_2$ and $p_3$ are defined in (3.7).*

**Remark 3.3.** *In case of multiple fixed points $E^*_\pm = (x^*_\pm, y^*_\pm, z^*_\pm)$ , we can find similar type of conditions as in Lemma 3.2.*

## 4. Global Stability

In this section, we will utilize the process of iteration scheme and the comparison principle of difference equations to investigate the global stability of the positive fixed point of system (1.2). To establish global stability result, we require the following lemmas:

**Lemma 4.1.** *([27]) Let $f(u) = u exp(\delta - \eta u)$, where $\delta$ and $\eta$ are positive constants. Then $f(u)$ is nondecreasing for $u \in (0, \frac{1}{\eta}]$.*

**Lemma 4.2.** *([27]) Assume that the sequence $u_n$ satisfies*

$$u_{n+1} = u_n exp(\delta - \eta u_n), n = 1, 2, 3, ...$$

*where $\delta$ and $\eta$ are positive constants and $u_0 > 0$. Then, (i) If $\delta < 2$, then $lim_{n\to\infty} u_n = \frac{\delta}{\eta}$.*
*(ii) If $\delta \le 1$, then $u_n \le \frac{1}{\eta}, n = 2, 3, ...$.*

**Lemma 4.3.** *[28] Suppose that functions $f, g : \mathbb{Z}_+ \times [0, \infty)$ satisfy $f(n, x) \le g(n, x)$ $(f(n, x) \ge g(n, x))$ for $n \in \mathbb{Z}_+$ and $g(n, x)$ is nondecreasing with respect to x. If $u_n$ are the nonnegative solutions of the difference equations*

$$x_{n+1} = f(n, x_n), u_{n+1} = g(n, u_n)$$

*respectively, and $x_0 \le u_0$ $(x_0 \ge u_0)$ then $x_n \le u_n$ $(x_n \ge u_n)$ for all $n \ge 0$.*

**Theorem 4.4.** *Assume that $C < 0$, $\frac{c_1r_2h_2f_2d_1(ah_1+r_1)+c_2r_3h_1f_1d_1(ah_2+r_1)+qh_1h_2f_1f_2}{d_1} < r_1 < 1$, $\frac{f_1}{h_1} < r_2 < 1$ and $\frac{f_2}{h_2} < r_3 < 1$ hold. Then, the fixed point $E^*(x^*, y^*, z^*)$ of system (1.2) is globally asymptotically stable.*

*Proof.* Assume that $(x_n, y_n, z_n)$ is any solution of system (1.2) with initial values $x_0 > 0, y_0 > 0, z_0 > 0$. Let

$$U_1 = \text{limsup}_{n\to\infty} x_n,\ V_1 = \text{liminf}_{n\to\infty} x_n,$$

$$U_2 = \text{limsup}_{n\to\infty} y_n,\ V_2 = \text{liminf}_{n\to\infty} y_n,$$

$$U_3 = \text{limsup}_{n\to\infty} z_n,\ V_3 = \text{liminf}_{n\to\infty} z_n.$$

In the following, we will prove that $U_1 = V_1 = x^*, U_2 = V_2 = y^*, U_3 = V_3 = z^*$.
First we show that $U_1 \le M_1^x, U_2 \le M_1^y, U_3 \le M_1^z$. From the first equation of system (1.2), we get

$$x_{n+1} \le x_n \exp(r_1 - a x_n), n = 0, 1, 2, ...$$

Considering the auxiliary equation

$$u_{n+1} = u_n \exp(r_1 - a u_n) \tag{4.1}$$

by Lemma 4.2 (ii), because of $r_1 \le 1$, we get $u_n \le \frac{1}{a}$ for all $n \ge 2$. By Lemma 4.1, we obtain $f(u) = u\exp(r_1 - au)$ is nondecreasing for $u \in (0, \frac{1}{a}]$. Thus from Lemma 4.3, we get $x_n \le u_n$ for all $n \ge 2$, where $u_n$ is the solution of equation (4.1) with initial value $u_2 = x_2$. By Lemma 4.2 (i), we get

$$U_1 = \limsup_{n\to\infty} x_n \le \lim_{n\to\infty} u_n = \frac{r_1}{a} \triangleq M_1^x.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists a $n_1 > 2$ such that if $n \ge n_1$, then $x_n \le M_1^x + \varepsilon$. From the second equation of system (1.2), we obtain,

$$y_{n+1} \le y_n \exp(r_2 - \frac{f_1}{h_1 + M_1^x + \varepsilon} y_n), n = 0, 1, 2, ...$$

Again considering the auxiliary equation

$$u_{n+1} = u_n \exp(r_2 - \frac{f_1}{h_1 + M_1^x + \varepsilon} u_n) \tag{4.2}$$

by Lemma 4.2 (ii), because of $r_2 \le 1$, we get $u_n \le \frac{h_1 + M_1^x + \varepsilon}{f_1}$ for all $n \ge 2$. By Lemma 4.1, we obtain $f(u) = u\exp(r_2 - \frac{f_1}{h_1 + M_1^x + \varepsilon} u)$ is nondecreasing for $u \in (0, \frac{h_1 + M_1^x + \varepsilon}{f_1}]$. Thus from Lemma 4.3, we get $x_n \le u_n$ for all $n \ge 2$, where $u_n$ is the solution of Eq. (4.2) with initial value $u_2 = x_2$. By Lemma 4.2 (i), we get

$$U_2 = \limsup_{n\to\infty} x_n \le \lim_{n\to\infty} u_n = \frac{r_2(h_1 + M_1^x + \varepsilon)}{f_1} \triangleq M_1^y.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists a $n_2 > n_1$ such that if $n \ge n_2$, then $y_n \le M_1^y + \varepsilon$. Similarly, from the third equation of system (1.2) for $r_3 < 1$, we obtain

$$U_3 = \limsup_{n\to\infty} z_n \le \lim_{n\to\infty} u_n = \frac{r_3(h_2 + M_1^x + \varepsilon)}{f_2} \triangleq M_1^z.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists $n_3 > n_2$ such that for $n \ge n_3, z_n \le M_1^z + \varepsilon$. Next we show that $V_1 \ge N_1^x, V_2 \ge N_1^y, V_3 \ge N_1^z$. From the first equation of system (1.2), we have

$$x_{n+1} \ge x_n \exp[a - a x_n - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1}], n \ge n_3.$$

Consider the auxiliary equation

$$u_{n+1} = u_n \exp[r_1 - a u_n - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1}]. \tag{4.3}$$

Since we have $r_1 - \frac{c_1(M_1^y+\varepsilon)}{h_1} - \frac{c_2(M_1^z+\varepsilon)}{h_2} - \frac{q}{d_1} < 1$, by Lemma 4.2 (ii), we have, $u_n \le \frac{1}{a}$ for $n \ge n_3$. By Lemma 4.1, we obtain $f(u) = u\exp(r_1 - \frac{c_1(M_1^y+\varepsilon)}{h_1} - \frac{c_2(M_1^z+\varepsilon)}{h_2} - \frac{q}{d_1} - au)$ is nondecreasing for $u \in (0, \frac{1}{a}]$. Thus from Lemma 4.3, we get $x_n \ge u_n$ for all $n \ge n_3$. By Lemma 4.2 (i), we get

$$V_1 = \liminf_{n\to\infty} x_n \ge \lim_{n\to\infty} u_n = \frac{1}{a}[r_1 - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1}].$$

From the arbitrariness of $\varepsilon > 0$, we have

$$V_1 \ge N_1^x = \frac{1}{a}[r_1 - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1}].$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_4 > n_3$ such that for $n \geq n_4, x_n \geq N_1^x - \varepsilon$. From the second equation of system (1.2), we have

$$y_{n+1} \geq y_n \exp[r_2 - \frac{f_1}{h_1} y_n], n \geq n_4.$$

By the same way, we can get

$$V_2 = \liminf_{n\to\infty} y_n \geq \lim_{n\to\infty} u_n = \frac{r_2 h_1}{f_1}.$$

From the arbitrariness of $\varepsilon > 0$, we have,

$$V_2 \geq N_1^y = \frac{r_2 h_1}{f_1}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_5 > n_4$ such that for $n \geq n_5, y_n \geq N_1^y - \varepsilon$. Similarly, from the third equation of system (1.2), we have

$$z_{n+1} \geq z_n \exp[r_3 - \frac{f_2}{h_2} z_n], n \geq n_5.$$

with

$$V_3 = \liminf_{n\to\infty} z_n \geq \lim_{n\to\infty} u_n = \frac{r_3 h_2}{f_2}.$$

From the arbitrariness of $\varepsilon > 0$, we have,

$$V_3 \geq N_1^z = \frac{r_3 h_2}{f_2}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_6 > n_5$ such that for $n \geq n_6, z_n \geq N_1^z - \varepsilon$. Now we show that $U_1 \leq M_2^x, U_2 \leq M_2^y$ and $U_3 \leq M_2^z$, where $M_2^x \leq M_1^x, M_2^y \leq M_1^y$ and $M_2^z \leq M_1^z$ respectively. From the first equation of system (1.2) for $n > n_6$, we get

$$x_{n+1} \leq x_n \exp[r_1 - a x_n - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon} - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)}].$$

Consider the auxiliary equation

$$u_{n+1} = u_n \exp[r_1 - a u_n - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon} - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)}]. \tag{4.4}$$

Using the similar argument as in above, we can get

$$U_1 = \limsup_{n\to\infty} x_n \leq \frac{1}{a}[r_1 - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon} - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)}],$$

since

$$r_1 - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon}) - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)} \leq 1.$$

From the arbitrariness of $\varepsilon > 0$, we claim that

$$U_1 \leq M_2^x = \frac{1}{a}[r_1 - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon} - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)}].$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_7 > n_6$ such that for $n \geq n_7, x_n \leq M_2^x + \varepsilon$. Similarly, from the second equation of system (1.2) for $n > n_7$, we get

$$y_{n+1} \leq y_n \exp[r_2 - \frac{f_1}{h_1 + M_2^x + \varepsilon} y_n].$$

Similarly to the above argument, we get

$$U_2 \leq M_2^y = \frac{r_2(h_1 + M_2^x + \varepsilon)}{f_1}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_8 > n_7$ such that for $n \geq n_8, y_n \leq M_2^y + \varepsilon$. From the third equation of system (1.2) for $n > n_8$, we get

$$z_{n+1} \leq z_n \exp[r_3 - \frac{f_2}{h_2 + M_2^x + \varepsilon} y_n].$$

Similarly to the above argument, we get

$$U_3 \leq M_2^z = \frac{r_3(h_2 + M_2^x + \varepsilon)}{f_2}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_9 > n_8$ such that for $n \geq n_9, z_n \leq M_2^z + \varepsilon$. Now we show that $V_1 \geq N_2^x, V_2 \geq N_2^y$ and $V_3 \geq N_2^z$, where $N_2^x \geq N_1^x, N_2^y \geq N_1^y$ and $N_2^z \geq N_1^z$ respectively. Further, from the first equation of system (1.2) for $n > n_9$, we get

$$x_{n+1} \geq x_n \exp[r_1 - a x_n - \frac{c_1(M_2^y + \varepsilon)}{h_1 + N_1^x - \varepsilon} - \frac{c_2(M_2^z + \varepsilon)}{h_2 + N_1^x - \varepsilon} - \frac{qE}{d_1 E + d_2(N_1^x - \varepsilon)}].$$

Using a similar argument, we get

$$V_1 = \liminf_{n\to\infty} x_n \geq \frac{1}{a}[r_1 - \frac{c_1(M_2^y + \varepsilon)}{h_1 + N_1^x - \varepsilon} - \frac{c_2(M_2^z + \varepsilon)}{h_2 + N_1^x - \varepsilon} - \frac{qE}{d_1 E + d_2(N_1^x - \varepsilon)}] \leq 1.$$

From the arbitrariness of $\varepsilon > 0$, we claim that

$$V_1 \geq N_2^x = \frac{1}{a}[r_1 - \frac{c_1(M_2^y + \varepsilon)}{h_1 + N_1^x - \varepsilon} - \frac{c_2(M_2^z + \varepsilon)}{h_2 + N_1^x - \varepsilon} - \frac{qE}{d_1 E + d_2(N_1^x - \varepsilon)}].$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_{10} > n_9$ such that for $n \geq n_{10}, x_n \geq N_2^x - \varepsilon$. Similarly, from the second equation of system (1.2) for $n > n_{10}$, we have

$$y_{n+1} \geq y_n \exp[r_2 - \frac{f_1}{h_1 + N_2^x - \varepsilon} y_n]$$

with

$$V_2 = \liminf_{n\to\infty} y_n \geq \frac{r_2(h_1 + N_2^x - \varepsilon)}{f_1}.$$

From the arbitrariness of $\varepsilon > 0$, we claim that $V_2 \geq N_2^y = \frac{r_2(h_1+N_2^x-\varepsilon)}{f_1}$. Hence for any sufficiently small $\varepsilon > 0$, there exists $n_{11} > n_{10}$ such that for $n \geq n_{11}, y_n \geq N_2^y - \varepsilon$. Similarly, from the third equation of system (1.2) for $n > n_{11}$, we have

$$z_{n+1} \geq z_n \exp[r_3 - \frac{f_2}{h_2 + N_2^x - \varepsilon} z_n].$$

with

$$V_3 = \liminf_{n\to\infty} z_n \geq \frac{r_3(h_2 + N_2^x - \varepsilon)}{f_2}.$$

From the arbitrariness of $\varepsilon > 0$, we conclude that $V_3 \geq N_2^z = \frac{r_3(h_2+N_2^x-\varepsilon)}{f_2}$. Hence for any sufficiently small $\varepsilon > 0$, there exists $n_{12} > n_{11}$ such that for $n \geq n_{12}, z_n \geq N_2^z - \varepsilon$. Repeating the above process, we ultimately get six sequences $\{M_n^x\}, \{M_n^y\}, \{M_n^z\}, \{N_n^x\}$,

$\{N_n^y\}$, and $\{N_n^z\}$ such that for all $n \geq 2$,

$$\begin{aligned}
M_n^x &= \frac{1}{a}[r_1 - \frac{c_1 N_{n-1}^y}{h_1 + M_{n-1}^x} - \frac{c_2 N_{n-1}^z}{h_2 + M_{n-1}^x} - \frac{qE}{d_1 E + d_2 M_{n-1}^x}],\\
M_n^y &= \frac{r_2(h_1 + M_n^x)}{f_1},\\
M_n^z &= \frac{r_3(h_2 + M_n^x)}{f_2},\\
N_n^x &= \frac{1}{a}[r_1 - \frac{c_1 M_n^y}{h_1 + N_{n-1}^x} - \frac{c_2 M_n^z}{h_2 + N_{n-1}^x} - \frac{qE}{d_1 E + d_2 N_{n-1}^x}],\\
N_n^y &= \frac{r_2(h_1 + N_n^x)}{f_1},\\
N_n^z &= \frac{r_3(h_2 + N_n^x)}{f_2}.
\end{aligned} \tag{4.5}$$

Clearly, we have for any integer $n > 0$,

$$N_n^x \leq V_1 \leq U_1 \leq M_n^x, N_n^y \leq V_2 \leq U_2 \leq M_n^y, \text{and } N_n^z \leq V_3 \leq U_3 \leq M_n^z.$$

In the following, we will prove that $\{M_n^x\}, \{M_n^y\}$ and $\{M_n^z\}$ are monotonically decreasing and $\{N_n^x\}, \{N_n^y\}$ and $\{N_n^z\}$ are monotonically increasing, with the help of inductive method. Firstly, it is clear that

$$M_2^x \leq M_1^x, M_2^y \leq M_1^y, M_2^z \leq M_1^z, N_2^x \geq N_1^x, N_2^y \geq N_1^y, \text{and } \; N_2^z \geq N_1^z.$$

For $n = k(k \geq 2)$, we assume that

$$M_k^x \leq M_{k-1}^x, M_k^y \leq M_{k-1}^y, M_k^z \leq M_{k-1}^x, N_k^x \geq N_{k-1}^x, N_k^y \geq N_{k-1}^y, \text{and } N_k^z \geq N_{k-1}^z.$$

Now

$$\begin{aligned}
M_{k+1}^x - M_k^x \;&=\; -\frac{1}{a}[\frac{c_1\{(N_k^y M_{k-1}^x - M_k^x N_{k-1}^y) + h_1(N_k^y - N_{k-1}^y)\}}{(h_1 + M_k^x)(h_1 + M_{k-1}^x)} + \frac{c_2\{(N_k^z M_{k-1}^x - N_{k-1}^z M_k^x) + h_2(N_k^z - N_{k-1}^z)\}}{(h_2 + M_k^x)(h_2 + M_{k-1}^x)}\\
&+\; \frac{qEd_2(M_k^x - M_{k-1}^x)}{(d_1 E + d_2 M_k^x)(d_1 E + d_2 M_{k-1}^x)}] \leq 0\\
M_{k+1}^y - M_k^y \;&=\; \frac{r_2(M_{k+1}^x - M_k^x)}{f_1} \leq 0\\
M_{k+1}^z - M_k^z \;&=\; \frac{r_3(M_{k+1}^x - M_k^x)}{f_2} \leq 0\\
N_{k+1}^x - N_k^x \;&=\; -\frac{1}{a}[\frac{c_1\{(M_{k+1}^y N_{k-1}^x - M_k^y N_k^x) + h_1(M_{k+1}^y - M_k^y)\}}{(h_1 + N_k^x)(h_1 + N_{k-1}^x)} + \frac{c_2\{(M_{k+1}^z N_{k-1}^x - M_k^z N_k^x) + h_2(M_{k+1}^z - M_k^z)\}}{(h_2 + N_k^x)(h_2 + N_{k-1}^x)}\\
&+\; \frac{qEd_2(N_{k-1}^x - N_k^x)}{(d_1 E + d_2 N_k^x)(d_1 E + d_2 N_{k-1}^x)}] \geq 0\\
N_{k+1}^y - N_k^y \;&=\; \frac{r_2(N_{k+1}^x - N_k^x)}{f_1} \geq 0\\
N_{k+1}^z - N_k^z \;&=\; \frac{r_3(N_{k+1}^x - N_k^x)}{f_2} \geq 0
\end{aligned}$$

This shows that $\{M_n^x\}, \{M_n^y\}$ and $\{M_n^z\}$ are monotonically decreasing and $\{N_n^x\}, \{N_n^y\}$ and $\{N_n^z\}$ are monotonically increasing. Therefore, by the criterion of monotonic bounded, we have established that every one of this six sequences has a limit.
Let

$$\lim_{n\to\infty} M_n^x = x_1, \lim_{n\to\infty} M_n^y = x_2, \lim_{n\to\infty} M_n^z = x_3, \lim_{n\to\infty} N_n^x = y_1, \lim_{n\to\infty} N_n^y = y_2, \lim_{n\to\infty} N_n^z = y_3.$$

Passing to the limit as $n \to \infty$ in (4.5), we get

$$\begin{aligned}
x_1 &= \frac{1}{a}[r_1 - \frac{c_1 y_2}{h_1 + x_1} - \frac{c_2 y_3}{h_2 + x_1} - \frac{qE}{d_1 E + d_2 x_1}],\\
x_2 &= \frac{r_2(h_1 + x_1)}{f_1},\\
x_3 &= \frac{r_3(h_2 + x_1)}{f_2},\\
y_1 &= \frac{1}{a}[r_1 - \frac{c_1 x_2}{h_1 + y_1} - \frac{c_2 x_3}{h_2 + y_1} - \frac{qE}{d_1 E + d_2 y_1}]\\
y_2 &= \frac{r_2(h_1 + y_1)}{f_1},\\
y_3 &= \frac{r_3(h_2 + y_1)}{f_2}.
\end{aligned} \tag{4.6}$$

It is clear that $x_1 = y_1, x_2 = y_2$ and $x_3 = y_3$. Thus we obtain $x_1 = x^*, x_2 = y^*, x_3 = z^*$ as a solution of (15). Hence, the global asymptotic stability of $(x^*, y^*, z^*)$ is obtained. This completes the proof of the theorem. □

## 5. Bifurcation Study

In this section, we discuss the parametric restrictions for obtaining Neimark-Sacker bifurcation at the interior fixed point $E^*$ of system (1.2).

### 5.1 Neimark-Sacker bifurcation

To examine Neimark-Sacker bifurcation in system (1.2), we need the following result [29].

**Lemma 5.1.** *Consider an n-dimensional discrete dynamical system $U_{k+1} = f_m(U_k)$ where $m \in \mathbb{R}$ is a bifurcation parameter. Let $U^*$ be fixed point of $f_m$ and the characteristic polynomial for Jacobian matrix $J(U^*) = (b_{ij})_{n\times n}$ of n-dimensional map $f_m(U_k)$ is given by*

$$P_m(\lambda) = \lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n \tag{5.1}$$

*where $b_i = b_i(m,u), i = 1,2,3,\cdots,n$ and $u$ is a control parameter or another parameter to be deduced. Let $\Delta_0^{\pm}(m,u) = 1, \Delta_1^{\pm}(m,u), \cdots, \Delta_n^{\pm}(m,u)$ be a sequence of determinants defined by $\Delta_i^{\pm}(m,u) = det(M_1 \pm M_2), i = 1,2,3,\cdots,n$ where*

$$M_1 = \begin{pmatrix} 1 & b_1 & b_2 & \cdots & b_{i-1} \\ 0 & 1 & b_1 & \cdots & b_{i-2} \\ 0 & 0 & 1 & \cdots & b_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} b_{n-i+1} & b_{n-i+2} & \cdots & b_{n-1} & b_n \\ b_{n-i+2} & b_{n-i+3} & \cdots & b_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_n & \cdots & 0 & 0 \\ b_n & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

*Moreover, the following conditions hold:*

**A1** *Eigenvalue assignment*

$$\Delta_{n-1}^{-}(m_0,u) = 0, \Delta_{n-1}^{+}(m_0,u) > 0, P_{m_0}(1) > 0, (-)^n P_{m_0}(-1) > 0, \Delta_i^{\pm}(m_0,u) > 0, i = n-3, n-5, \cdots, 1 (or\ 2),$$

*when n is even or odd, respectively.*

**A2** *Transversality condition:* $[\frac{d(\Delta_{n-1}^{-}(m,u))}{dm}]_{m=m_0} \neq 0.$

**A3** *Non-resonance condition:*

$$cos(2\pi/j) \neq \psi, or\ resonance\ condition\ cos(2\pi/j) = \psi\ where\ j = 3,4,5,\cdots$$

*and $\psi = 1 - 0.5P_{m_0}(1)\Delta_{n-3}^{-}(m_0,u)/\Delta_{n-2}^{+}(m_0,u)$. Then Neimark-Sacker bifurcation occurs at $m_0$.*

Now we state bifurcation result by considering $a$ as a bifurcation parameter of system (1.2).

**Theorem 5.2.** *The fixed point $E^*$ of system (1.2) admits Neimark-Sacker bifurcation if the following conditions are satisfied:*

$$\begin{aligned} 1-p_2+p_3(p_1-p_3)&=0,\\ 1+p_2-p_3(p_1+p_3)&>0,\\ 1+p_1+p_2+p_3&>0,\\ 1-p_1+p_2-p_3&>0 \end{aligned} \tag{5.2}$$

*where $p_1$, $p_2$ and $p_3$ are defined in (3.7).*

*Proof.* Following Lemma 4.1, we have found the following equalities and inequalities:

$$\begin{aligned} \Delta_2^-(a^*)&=1-p_2+p_3(p_1-p_3)=0,\\ \Delta_2^+(a^*)&=1+p_2-p_3(p_1+p_3)>0,\\ P_{a^*}(1)&=1+p_1+p_2+p_3>0,\\ (-1)^3P_{a^*}(-1)&=1-p_1+p_2-p_3>0. \end{aligned} \tag{5.3}$$

□

## 6. Chaos Control

Here, we examine chaos control for system (1.2). It is more pertinent for model related with biological species. It is normally seen that discrete-time models are more chaotic and complicated than the continuous systems. Thus it is justifiable to execute control method to prevent any uncertainty. We primarily apply hybrid control process discussed in [30]. This technique takes a single control parameter which lies in the open unit interval. Various types of methods are available for regulating chaos in discrete systems, for example, state feed back method, pole-placement technique and hybrid control method [31]-[**?**] in which, hybrid control technique is most simple to apply. We use hybrid control technique to system (1.2) for controlling chaos developed through bifurcation. Assume that the system admits Neimark-Sacker bifurcation at its fixed point $(x^*,y^*,z^*)$, then the corresponding controlled system using the hybrid control method is given by:

$$\begin{aligned} x_{n+1}&=\rho x_n\exp\{r_1-ax_n-\frac{c_1y_n}{h_1+x_n}-\frac{c_2z_n}{h_2+x_n}-\frac{qE}{d_1E+d_2x_n}\}+(1-\rho)x_n,\\ y_{n+1}&=\rho y_n\exp\{r_2-\frac{f_1y_n}{h_1+x_n}\}+(1-\rho)y_n,\\ z_{n+1}&=\rho z_n\exp\{r_3-\frac{f_2y_n}{h_2+x_n}\}+(1-\rho)z_n. \end{aligned} \tag{6.1}$$

where $0<\rho<1$ is taken as a control parameter. The Jacobian matrix of controlled system (6.1) evaluated at $E^*$ is given by

$$J(x^*,y^*,z^*)=\begin{pmatrix} 1-\rho x^*(a-\frac{c_1y^*}{(h_1+x^*)^2}-\frac{c_2z^*}{(h_2+x^*)^2}-\frac{qEd_2}{(d_1E+d_2x^*)^2}) & -\frac{\rho x^*c_1}{h_1+x^*} & \frac{\rho x^*c_2}{h_2+x^*}\\ \frac{\rho {y^*}^2f_1}{(h_1+x^2)^2} & 1-\rho r_2 & 0\\ \frac{\rho {z^*}^2f_2}{(h_2+x^2)^2} & 0 & 1-\rho r_3 \end{pmatrix} \tag{6.2}$$

The fixed point $E^*$ of controlled system (6.1) is locally asymptotically stable if all the roots of the characteristic polynomial of (6.2) lie in an unit open disk.

## 7. Numerical Simulations

In this section, we present some numerical computations to justify our analytical results. We show the role of the intra-specific competition coefficient among the prey species, harvesting effort and the maximum value of per capita reduction rate of $y$ can attain on the discrete system visually through numerical simulations.

**Example 7.1.** *Suppose $r_1=0.8, r_2=0.5, r_3=0.4, c_1=0.01, c_2=0.02, h_1=1, h_2=1, d_1=1, d_2=1, f_1=0.2, f_2=0.1, a=0.1, q=0.1, E=1$ for system (1.2). Then all the conditions of Theorem 4.4 are satisfied. Thus the fixed point $E^*=(6.878, 19.94, 30.72)$ is globally asymptotically stable (see Fig. 7.1). The Fig. 7.1) shows that initially all the population increases and eventually all the interacting populations get their steady states and finally become globally asymptotically stable.*

**Example 7.2.** *Suppose* $r_1 = 3.5, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1.5, f_2 = 1, a = 0.3, q = 0.2, E = 1$ *initial points (0.5, 0.5, 0.) for system (2). Then the conditions of Lemma 3.2 are violated. Thus the fixed point* $E^* = (3.894, 7.196, 9.813)$ *is unstable. Moreover, system (1.2) admits chaotic behaviour (see 7.2(a)). In order to show the effectiveness of hybrid control method implemented in system (6.1), we choose* $\rho = 0.5$ *and other parameters are same as in Example 7.2. The 7.2(b) shows that the solutions initiating from (0.5,0.5,0.5) approaches to the fixed point* $E^* = (3.894, 7.196, 9.813)$. *i.e., the steady state for controlled system (6.1) is a sink.*

**Example 7.3.** *Suppose* $r_1 = 3, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, q = 0.2, E = 1$ *and initial points (0.5, 0.5, 0.5) and* $a \in (0.1, 1.5)$ *in system (1.2) with the initial condition* $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$. *When a is considered as a bifurcation parameter, then at* $a = a^* = 0.326$, *the interior fixed point* $E^* = (1.46935, 5.43257, 4.9387)$ *becomes unstable and system (1.2) undergoes Neimark-Sacker bifurcation by Theorem 5.2. Bifurcation diagrams and maximum Lyapunov exponents (MLE) respect to the parameter a of system (1.2) are depicted in Fig. 7.3. As a increases, we observe that a transition from unstable to stable.*

**Example 7.4.** *Suppose* $r_1 = 2.98, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, q = 0.2, a = 0.3$ *and initial points (0.5, 0.5, 0.5) and* $a \in (0.5, 1.5)$ *in system (1.2) with the initial condition* $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$. *When E is considered as a bifurcation parameter, then at* $E = E_* = 0.978$, *the interior fixed point* $E^* = (1.435, 5.373, 4.884)$ *becomes unstable and system (1.2) undergoes Neimark-Sacker bifurcation by Theorem 5.2. Bifurcation diagrams and MLE respect to the parameter E of system (1.2) are depicted in Fig. 7.4. As E increases, we observe that a transition from unstable to stable.*

**Example 7.5.** *Suppose* $r_1 = 2.98, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, E = 1, f_2 = 1, q = 0.2, a = 0.3$ *and initial points (0.5, 0.5, 0.5) and* $f_1 \in (0.6, 2)$ *in system (2) with the initial condition* $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$. *When* $f_1$ *is considered as a bifurcation parameter, then at* $f_1 = f_1^* = 0.998$, *the interior fixed point* $E^* = (1.534, 5.584, 5.066)$ *becomes unstable and system (1.2) undergoes Neimark-Sacker bifurcation by Theorem 5.2. Bifurcation diagrams and MLE respect to the parameter* $f_1$ *of system (1.2) are depicted in Fig. 7.5. As* $f_1$ *increases, we observe that a transition from stable to unstable and then bifurcation within a limit cycle to a periodic window and finally to chaos.*

**Example 7.6.** *Suppose* $r_1 = 5.8, r_2 = 2, r_3 = 3, c_1 = 1, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, E = 0.2, f_1 = 1, f_2 = 1, q = 1, a = 1$ *and initial points (0.5, 3, 4) ,we obtained two interior fixed points* $E_+^* = (0.523607, 3.047214, 4.570821)$ *and* $E_-^* = (0.0763932, 2.1527864, 3.2291796)$ *both are unstable (see Fig. 7.6). Fig. 7.6(b) represents the time series plot of system (2) when* $E = 0.28$

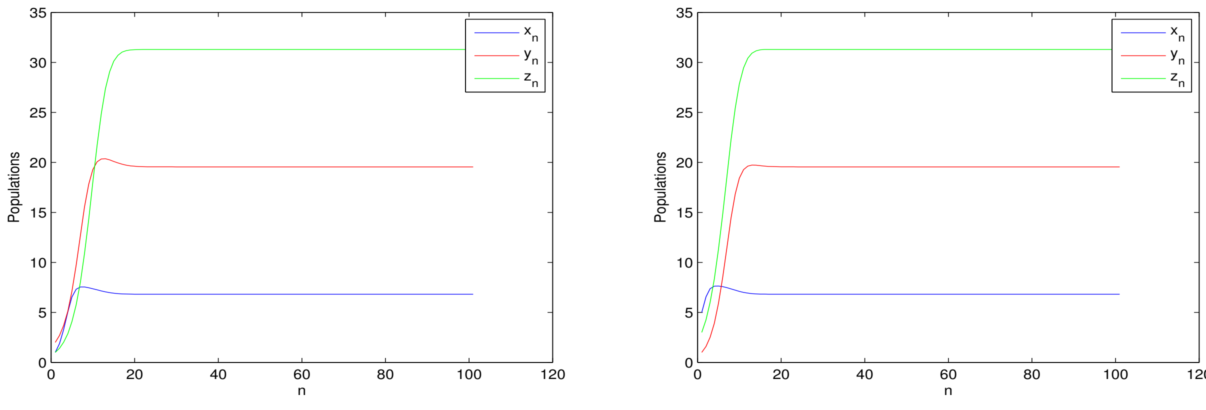


**Figure 7.1.** Time series plots of system (1.2) with parameter values $r_1 = 0.8, r_2 = 0.5, r_3 = 0.4, c_1 = 0.01, c_2 = 0.02, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 0.2, f_2 = 0.1, a = 0.1, q = 0.1, E = 1$ and initial points (1, 2, 1) and (5, 1, 3).

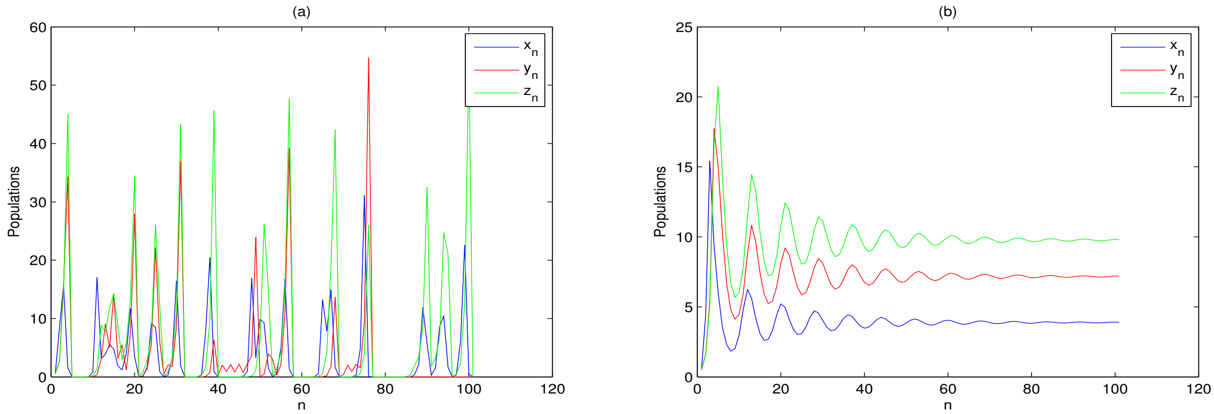


**Figure 7.2.** (a) Time series plots of system (1.2) with parameter values $r_1 = 3.5, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1.5, f_2 = 1, a = 0.3, q = 0.2, E = 1$ with initial points (0.5, 0.5, 0.5) and (b) phase portrait of controlled system (6.1) for $\rho = 0.5$

.

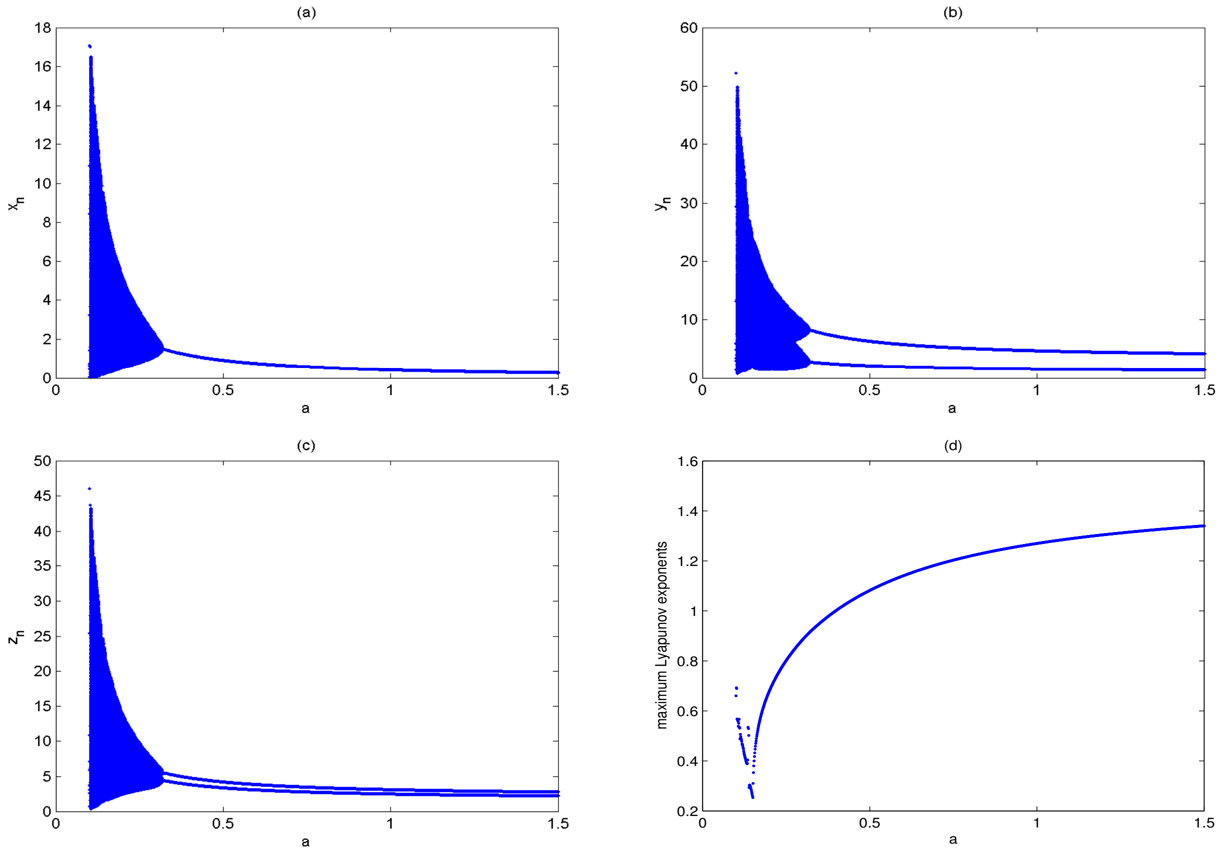


**Figure 7.3.** Bifurcation diagrams and MLE for system (1.2) with parameter values $r_1 = 3, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, q = 0.2, E = 1, a \in (0.1, 1.5)$ and initial point (0.5, 0.5, 0.5).

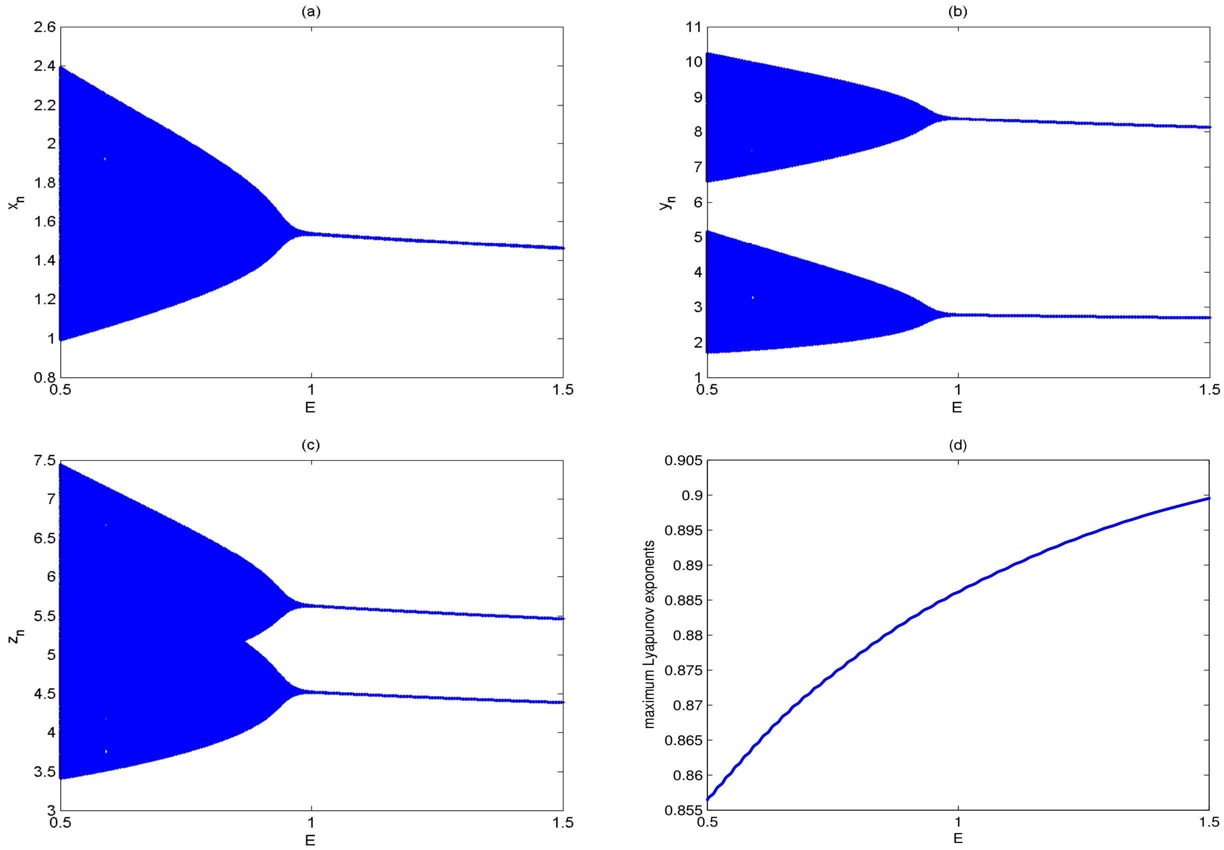


**Figure 7.4.** Bifurcation diagrams and MLE for system (1.2) with parameter values $r_1 = 2.98, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, q = 0.2, a = 0.3, E \in (0.5, 1.5)$ and initial point (0.5, 0.5, 0.5)

.

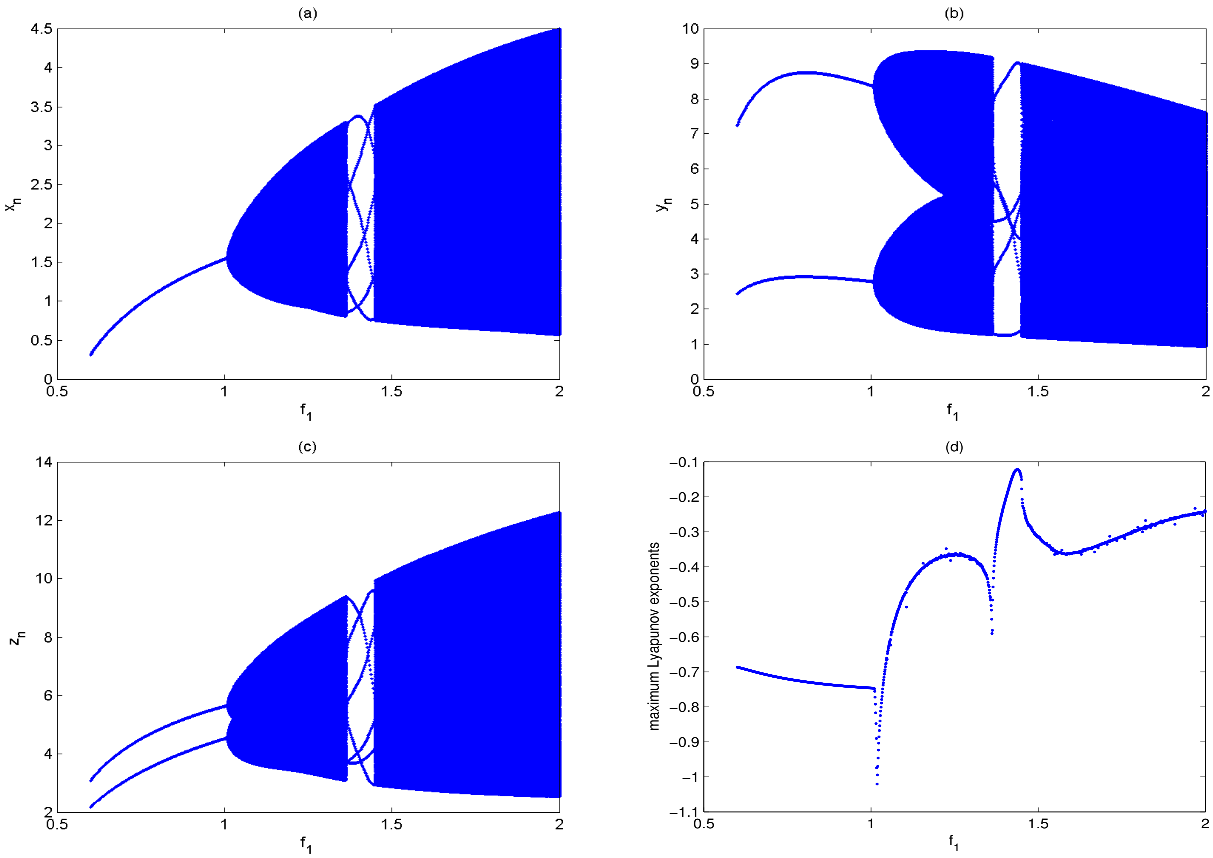


**Figure 7.5.** Bifurcation diagrams and MLE for system (1.2) with parameter values $r_1 = 2.98, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, E = 1, f_2 = 1, q = 0.2, a = 0.3, f_1 \in (0.6, 2)$ and initial point (0.5, 0.5, 0.5)

.

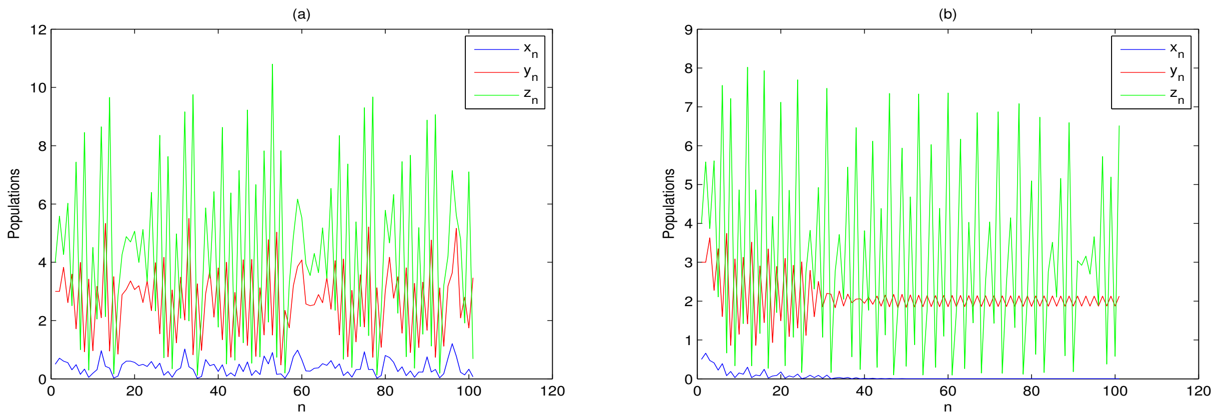


**Figure 7.6.** Time series plots of system (1.2) with parameter values $r_1 = 5.8, r_2 = 2, r_3 = 3, c_1 = 1, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, a = 1, q = 1$ for $E = 0.2$ and 0.28 respectively. initial point (0.5, 3, 4).

## 8. Discussion

In this article, a discrete-time Leslie-Gower two predator-one prey system with Michaelis-Menten type prey harvesting is investigated. To our knowledge, there are a few works that address the impact of non-linear harvesting on System (1.2). It is

shown that the system has at most twelve fixed points. Qualitative analysis shows that all the boundary fixed points, excepting $E_{23}$ are unstable. Under certain restrictions on the system parameters, $E_{23}$ may be stable, which in turn implies that that the prey population goes into extinction. As the trivial fixed point always exists and unstable, the three species cannot go to extinction together. It is established that multiple fixed points exist due to the presence of non-linear harvesting term. It is shown that Neimark-Sacker bifurcation occurs at the unique positive fixed point when the parameters $a, E, f_1$ are varied. The choice of these parameters is arbitrary, one may find similar type of bifurcations for other parameters also. Numerical simulations show that when the parameters $a$ and $E$ exceed a certain critical value, the system becomes stable (see Figs. 7.3 and 7.4) whereas the opposite holds $f_1$ is increased. In case of multiple fixed points, chaotic behaviour is observed. In particular, we observe when the predator population is chaotic, the prey population ultimately tends to extinct. This fact is clear when we increase the harvest rate from 0.2 to 0.28 (see Fig. 7.6 ). The proposed model admits more rich characteristics and more complicated dynamics than that exist in the continuous case. We have derived the condition for global stability of the positive fixed point by applying the iteration scheme and comparison principle of difference equations. Conditions of Theorem 4.4 indicate that when the intrinsic growth rate of the three species remains below one, the positive fixed point is globally asymptotically stable.
Sometimes bifurcation and chaotic behaviour are in fact unwanted situations in discrete dynamical systems, because there may be an extinction of the population due to chaos. So chaos control becomes a crucial issue. To prevent chaos, we have used the hybrid control method so that the stability of the system can be regained.
To our understanding, the dynamical study of discrete time model considering a Leslie-Gower two predator-one prey system with Michaelis-Menten type prey harvesting has not investigated yet.

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# Miscellaneous Properties of Generalized Fubini Polynomials

Muhammet Ağca[1], Nejla Özmen[2]*

**Abstract**
This article attempts to present the generalized Fubini polynomials $F_n(x,y,z,q)$. The results obtained here include various families of multilinear and multilateral generating functions, various properties, as well as some special cases for these generalized Fubini polynomials $F_n(x,y,z,q)$. Finally, we get several interesting results of this generalized Fubini polynomials and obtain an integral representation.

**Keywords:** Generalized Fubini polynomials, Generating function, Multilinear and multilateral generating function, Recurrence relations.
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[1] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, ORCID: 0000-0003-1818-3098*
[2] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, ORCID: 0000-0001-7555-1964*
***Corresponding author**: nejlaozmen06@gmail.com
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## 1. Introduction

Numerous studies on families of special polynomials, including the Bernoulli, Euler, Genocchi, and Fubini polynomials, as well as their generalizations and unifications (see, for example, the most recent works in [1]- [6], have gained significant popularity due to the wide range of their applications in various branches of mathematics, including p-adic analytic number theory, umbral calculus, special functions, and mathematical analysis. The special functions of mathematical physics have undergone a major evolution in recent years, especially in their generalized and multivariable forms. Thus, research on the multivariate Fubini polynomials was done for this work. Now let's go through the fundamental terms and theories that we will be using for the duration of the entire study.

For $n \geq 0$, let

$$F_n = \sum_{k=0}^{n} k!S(n,k),$$

where $S(n,k)$ denotes the Stirling numbers of the second kind [11]. In [12], the Fubini numbers $F_n$ were connected with preference arrangements and the recursion for $F_n$ was derived. In [12], [13], the exponential generating function

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}$$

and an asymptotic estimate for $F_n$ were established. In [14], the Fubini polynomials $F_n(y)$ were defined by

$$F_n(y) = \sum_{k=0}^{n} k!S(n,k)y^k$$

and generated by

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}.$$

It is clear that $F_n(1) = F_n$. Due to the relation

$$\left(y\frac{d}{dy}\right)^m \frac{1}{1-y} = \sum_{k=0}^{\infty} k^m y^k = \frac{1}{1-y}F_m\left(\frac{y}{1-y}\right), \quad |y| < 1$$

in [15], one also calls $F_n(y)$ the geometric polynomials. In [16], the Fubini polynomials $F_n(x,y)$ of two variables $x$, $y$ are defined by means of the generating function

$$\frac{e^{xt}}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!}.$$

It is apparent that $F_n(0,y) = F_n(y)$. In Particular, the special polynomials of two variables provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization has been suggested by physical problems (see, e.g., [7]-[10] and the references therein). In [17], the bivariate Fubini polynomials $F_n^{(r)}(x,y)$ of order $r$, generated by

$$\frac{e^{xt}}{[1-y(e^t-1)]^r} = \sum_{n=0}^{\infty} F_n^{(r)}(x,y)\frac{t^n}{n!}, \; r \in \mathbb{N}$$

were studied. It is obvious that $F_n^{(1)}(x,y) = F_n(x,y)$. The generating functions of $F_n$, $F_n(y)$, $F_n(x,y)$ and $F_n^{(r)}(x,y)$ remind us to consider the generating function

$$\frac{e^{xt}}{[z-y(e^t-1)]^q} = \sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!}, \; x,q \in \mathbb{R} \tag{1.1}$$

and the generalized Fubini polynomials $F_n(x,y,z,q)$ of four variables $x$, $y$, $z$, $q$ [18]. It is clear that, since

$$\frac{e^{xt}}{[z-y(e^t-1)]^q} = \frac{1}{z^q}\frac{e^{xt}}{[1-(y/z)(e^t-1)]^q},$$

we have

$$F_n(x,y,z,q) = \frac{F_n^{(r)}(x,y/z)}{z^r}.$$

The aim of this paper is to derive various families of multilinear and multilateral generating functions for the polynomials $F_n(x,y,z,q)$ given by (1.1). We present some special cases of our results and also obtain some other properties for these special cases.

## 2. Multilinear and Multilateral Generating Functions

The goal of this section is to derive several families of multilinear and multilateral generating functions for a class of polynomials in four variables given by equation (1.1) with the help of the method considered in refs. [20], [21].

**Lemma 2.1.** *The following addition formula holds for the generalized Fubini polynomials $F_n(x,y,z,q)$ :*

$$F_n(x_1+x_2,y,q_1+q_2) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(x_1,y,z,q_1)F_m(x_2,y,z,q_2). \tag{2.1}$$

*Proof.* Replacing $x$ by $x = x_1 + x_2$ and $q$ by $q = q_1 + q_2$ in (1.1), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n(x_1+x_2, y, q_1+q_2)\frac{t^n}{n!} &= \frac{e^{x_1 t + x_2 t}}{[z-y(e^t-1)]^{q_1+q_2}} \\
&= \frac{e^{x_1 t}}{[z-y(e^t-1)]^{q_1}} \frac{e^{x_2 t}}{[z-y(e^t-1)]^{q_2}} \\
&= \sum_{n=0}^{\infty} F_n(x_1, y, z, q_1)\frac{t^n}{n!} \sum_{m=0}^{\infty} F_m(x_2, y, z, q_2)\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_n(x_1, y, z, q_1)F_m(x_2, y, z, q_2)\frac{t^{n+m}}{n!.m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \tbinom{n}{m} F_{n-m}(x_1, y, z, q_1)F_m(x_2, y, z, q_2)\frac{t^n}{n!}.
\end{aligned}$$

From the coefficients of $t^n$ on the both sides of the last equality, one can get the desired result. □

**Theorem 2.2.** *Corresponding to an identically non-vanishing function $\Omega_\mu(s_1,\ldots,s_r)$ of $r$ complex variables $s_1,\ldots,s_r$ $(r \in \mathbb{N})$ and of complex order $\mu$, $\psi$, let*

$$\Lambda_{\mu,\psi}(s_1,\ldots,s_r;\zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(s_1,\ldots,s_r)\zeta^k,$$

$$\theta_{n,p}^{\mu;\psi}(x,y,z,q;s_1,\ldots,s_r;\xi) := \sum_{k=0}^{[n/p]} a_k F_{n-pk}(x,y,z,q)\Omega_{\mu+\psi k}(s_1,\ldots,s_r)\frac{\xi^k}{(n-pk)!}.$$

*where $a_k \neq 0$, $n$, $p \in \mathbb{N}$ and the notation $[n/p]$ means the greatest integer less than or equal $p \in \mathbb{N}$. Then, for $p \in \mathbb{N}$ we have*

$$\sum_{n=0}^{\infty} \theta_{n,p}^{\mu,\psi}(x,y,z,q;s_1,\ldots,s_r;\frac{\eta}{t^p})t^n = \frac{e^{xt}}{[z-y(e^t-1)]^q}\Lambda_{\mu,\psi}(s_1,\ldots,s_r;\eta), \tag{2.2}$$

*provided that each member of (2.2) exists.*

*Proof.* For convenience, let $S$ denote the first member of the assertion of Theorem 2.2. Then,

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k F_{n-pk}(x,y,z,q)\Omega_{\mu+\psi k}(s_1,\ldots,s_r)\eta^k \frac{t^{n-pk}}{(n-pk)!}.$$

Replacing $n$ by $n+pk$; we may write that

$$\begin{aligned}
S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k F_n(x,y,z,q)\Omega_{\mu+\psi k}(s_1,\ldots,s_r)\eta^k \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!}\sum_{k=0}^{\infty} a_k\, \Omega_{\mu+\psi k}(s_1,\ldots,s_r)\eta^k \\
&= \frac{e^{xt}}{[z-y(e^t-1)]^q}\Lambda_{\mu,\psi}(s_1,\ldots,s_r;\eta),
\end{aligned}$$

which completes the proof. □

Using Lemma 1, we have the following theorem.

**Theorem 2.3.** *Corresponding to an identically non-vanishing function $\Omega_\mu$ $(s_1,\ldots,s_r)$ of $r$ complex variables $s_1,\ldots,s_r$ $(r \in \mathbb{N})$ and of complex order $\mu$, $\psi$, let*

$$\Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y,z,q_1+q_2;s_1,\ldots,s_r;t) := \sum_{k=0}^{[n/p]} a_k F_{n-pk}(x_1+x_2,y,q_1+q_2)\Omega_{\mu+\psi k}(s_1,\ldots,s_r)t^k$$

*where* $a_k \neq 0$, $n$, $p \in \mathbb{N}$. *Then, for* $p \in \mathbb{N}$, *we have*

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l \binom{n-pl}{k-pl} F_{n-k}(x_1,y,z,q_1)F_{k-pl}(x_2,y,z,q_2)\Omega_{\mu+\psi l}(s_1,\ldots,s_r)t^l\\ =\;& \Lambda^{n,p}_{\mu,\psi}(x_1+x_2,y,z,q_1+q_2;s_1,\ldots,s_r;t),\end{aligned} \tag{2.3}$$

*provided that each member of (2.3) exists.*

*Proof.* For convenience, let T denote the first member of the assertion of Theorem 2.3. Then, upon substituting for the polynomials $F_n(x_1+x_2,y,z,q_1+q_2)$ from the (2.3) into the left-hand side of (2.1), we obtain

$$\begin{aligned}T &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} a_l \binom{n-pl}{k} F_{n-k-pl}(x_1,y,z,q_1)F_k(x_2,y,z,q_2)\Omega_{\mu+\psi l}(s_1,\ldots,s_r)t^l\\ &= \sum_{l=0}^{[n/p]} a_l \left(\sum_{k=0}^{n-pl}\binom{n-pl}{k} F_{n-k-pl}(x_1,y,z,q_1)F_k(x_2,y,z,q_2)\right)\Omega_{\mu+\psi l}(s_1,\ldots,s_r)t^l\\ &= \sum_{l=0}^{[n/p]} a_l F_{n-pl}(x_1+x_2,y,q_1+q_2)\Omega_{\mu+\psi l}(s_1,\ldots,s_r)t^l\\ &= \Lambda^{n,p}_{\mu,\psi}(x_1+x_2;s_1,\ldots,s_r;t),\end{aligned}$$

which completes the proof. □

## 3. Special Cases

When the multivariable function $\Omega_{\mu+\psi k}(s_1,\ldots,s_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}_0$ is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$\Omega_{\mu+\psi k}(s_1,\ldots,s_r) = T^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}_{\mu+\psi k,\lambda,l}(s_1,\ldots,s_r;s)$$

in Theorem 2.2, where the Lagrange-based Apostol- type polynomials $T^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}_{n,\lambda,k}(x_1,\ldots,x_r;x)$ [19], generated by

$$\sum_{n=0}^{\infty} T^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}_{n,\lambda,l}(x_1,\ldots,x_r;x)t^n = \left(\prod_{j=1}^{r}(1-x_jt)^{-\alpha_j}\right)\left(\frac{2^l t}{\lambda e^t+(-1)^{l+1}}\right)^{\alpha} e^{xt} \quad (\lambda;\,\alpha_j\in\mathbb{C}) \tag{3.1}$$

We are thus led to the following result which provides a class of bilateral generating functions for the Lagrange-based Apostol- type polynomials $T^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}_{n,\lambda,l}(x_1,\ldots,x_r;x)$ and the generalized Fubini polynomials $F_n(x,y,z,q)$.

**Corollary 3.1.** *If*

$$\Lambda_{\mu,\psi}(s_1,\ldots,s_r;s;\zeta) := \sum_{k=0}^{\infty} a_k T^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}_{\mu+\psi k,\lambda,l}(s_1,\ldots,s_r;s)\zeta^k \quad (a_k\neq 0,\ \mu,\psi\in C)$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k F_{n-pk}(x,y,z,q) T^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}_{\mu+\psi k,\lambda,l}(s_1,\ldots,s_r;s)\frac{\eta^k}{t^{pk}}\frac{t^n}{(n-pk)!} = \frac{e^{xt}}{[z-y(e^t-1)]^q}\Lambda_{\mu,\psi}(s_1,\ldots,s_r;s;\eta), \tag{3.2}$$

*provided that each member of (3.2) exists.*

**Remark 3.2.** *Using the generating relation (3.1) for the Lagrange-based Apostol-type polynomials* $T^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}_{n,\lambda,l}(s_1,\ldots,s_r;s)$ *and getting* $a_k=1$, $\mu=0$, $\psi=1$ *in Corollary 1, we find that*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} F_{n-pk}(x,y,z,q)\, T^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}_{k,\lambda,l}(s_1,\ldots,s_r;s)\eta^k \frac{t^{n-pk}}{(n-pk)!}\\ =\;& \frac{e^{xt}}{[z-y(e^t-1)]^q}\left(\prod_{j=1}^{r}(1-s_j\eta)^{-\alpha_j}\right)\left(\frac{2^l\eta}{\lambda e^{\eta}+(-1)^{l+1}}\right)^{\alpha} e^{s\eta},\ (\lambda\in\mathbb{C};\,\alpha_j\in\mathbb{C}).\end{aligned}$$

In the particular cases when $l=0$, $l=1$ in the Corollary 1 and Remak 1, we have bilateral generating functions the Lagrange-based Apostol-Bernoulli polynomials $B_{k,\lambda}^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}(s_1,\ldots,s_r;s)$, the Lagrange-based Apostol-Genocchi polynomials $G_{k,\lambda}^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}(s_1,\ldots,s_r;s)$ and the generalized Fubini polynomials [28].

If we set $r=4$ and

$$\Omega_{\mu+\psi k}(s_1,s_2,s_3,s_4)=F_{\mu+\psi k}(s_1,s_2,s_3,s_4)$$

in Theorem 2.2, we have the following bilinear generating functions for the generalized Fubini polynomils.

**Corollary 3.3.** *If*

$$\Lambda_{\mu,\psi}(s_1,s_2,s_3,s_4;\zeta):=\sum_{k=0}^{\infty}a_kF_{\mu+\psi k}(s_1,s_2,s_3,s_4)\zeta^k,\ \ (a_k\neq 0\ \mu,\psi\in\mathbb{C})$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kF_{n-pk}(x,y,z,q)F_{\mu+\psi k}(s_1,s_2,s_3,s_4)\frac{\eta^k}{t^{pk}}\frac{t^n}{(n-pk)!}=\frac{e^{xt}}{[z-y(e^t-1)]^q}\Lambda_{\mu,\psi}(s_1,s_2,s_3,s_4;\eta),\tag{3.3}$$

*provided that each member of (3.3) exists.*

**Remark 3.4.** *Using the generating relation (1.1) for the generalized Fubini polynomials $F_n(x,y,z,q)$ and getting*

$$a_k=\frac{1}{k!},\ \mu=0,\ \psi=1$$

*in Corollary 2, we find that*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\frac{1}{k!}F_{n-pk}\left(x,y,z,q\right)F_k(s_1,s_2,s_3,s_4)\eta^k\frac{t^{n-pk}}{(n-pk)!}\\ =&\sum_{n=0}^{\infty}F_n\left(x,y,z,q\right)\frac{t^n}{n!}\sum_{k=0}^{\infty}F_k(s_1,s_2,s_3,s_4)\frac{\eta^k}{k!}\\ =&\frac{e^{xt}}{[z-y\left(e^t-1\right)]^q}\frac{e^{s_1t}}{[s_3-s_2\left(e^t-1\right)]^{s_4}}.\end{aligned}$$

If we set $r=1$ and

$$\Omega_{\mu+\psi k}(s_1)=F_{\mu+\psi k}(x_3,y,z,q_3)$$

in Theorem 2.3, we have the following summation formula for the generalized Fubini polynomials.

**Corollary 3.5.** *If*

$$\begin{aligned}&\Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y,z,q_1+q_2;x_3,y,z,q_3;\eta):=\sum_{k=0}^{[n/p]}a_kF_{n-pk}(x_1+x_2,y,z,q_1+q_2)F_{\mu+\psi k}(x_3,y,z,q_3)\eta^k\,,\\ &(a_k\neq 0,\mu,\psi\in\mathbb{C})\,,\end{aligned}$$

*then, we have*

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l\binom{n}{k}F_{n-k}(x_1,y,z,q_1)F_{k-pl}(x_2,y,z,q_2)F_{\mu+\psi l}(x_3,y,z,q_3)\eta^l\\ =&\Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y,z,q_1+q_2;x_3,y,z,q_3;\eta),\end{aligned}\tag{3.4}$$

*provided that each member of (3.4) exists.*

**Remark 3.6.** *Using (2.1) and taking*

$$a_l=1,\ \mu=0,\ \psi=1,\ p=1,\ \eta^l=\binom{k}{l}$$

*in Corollary 3, we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}F_{n-k}(x_1;y,z,q_1)F_{k-l}(x_2,y,z,q_2)F_l(x_3,y,z,q_3)=F_n(x_1+x_2+x_3,y,z,q_1+q_2+q_3).$$

Furthermore, for every suitable choice of the coefficients $a_k$ $(k \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(s_1,\ldots,s_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 2.2, Theorem 2.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the generalized Fubini polynomials given explicitly by (1.1).

## 4. Miscellaneous Properties

In this section, we give some properties for the generalized Fubini polynomials $F_n(x,y,z,q)$ given by (1.1).

Firstly, recall that the classical Frobenius-Euler polynomials $H_n^{(r)}(u;x)$ of order $r$ are generated by (see, e.g., [22]-[26])

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(u;x)\frac{t^n}{n!}, \tag{4.1}$$

where $u \neq 1$.

We note that, for $r=1$ in (4.1), the $H_n^{(1)}(u;x) = H_n(u;x)$, which denotes the Frobenius-Euler polynomials and for $u=0$ in (4.1), the $H_n^{(r)}(0;x) = H_n^{(r)}(x)$, which denotes the Frobenius-Euler numbers of order $r$. For $x=-1$ in (4.1), the $H_n^{(r)}(u;-1) = E_n(u)$, which denotes the Euler polynomials (cf. [27]).

**Theorem 4.1.** *For $n \geq 0$, $y,\ z \neq 0$; we have*

$$F_n(x,y,z,q) = \frac{H_n^{(q)}(\frac{z+y}{y};x)}{z^q}. \tag{4.2}$$

*Proof.* Using (1.1) and (4.1), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!} &= \frac{e^{xt}}{[z-y(e^t-1)]^q} \\
&= \left[\frac{1-\frac{z+y}{y}}{e^t-\frac{z+y}{y}}\right]^q e^{xt} \\
&= z^{-q}\sum_{n=0}^{\infty} H_n^{(q)}(\frac{z+y}{y};x)\frac{t^n}{n!}.
\end{aligned}$$

Hence, we have

$$F_n(x,y,z,q) = \frac{H_n^{(q)}(\frac{z+y}{y};x)}{z^q},\ \ (y,\ z \neq 0),$$

or

$$H_n^{(q)}(\frac{z+y}{y};x) = z^q F_n(x,y,z,q).$$

□

Some special cases of Theorem 4.1 are examined below.

**Corollary 4.2.** *For $n \geq 0$, $q=1$, $z,\ y \neq 0$; we have*

$$H_n^{(1)}(\frac{z+y}{y};x) = H_n(\frac{z+y}{y};x) = zF_n(x,y,z,1).$$

**Corollary 4.3.** *For $n \geq 0$, $z=-y \neq 0$; we have*

$$H_n^{(q)}(0;x) = H_n^{(r)}(x) = (-y)^q F_n(x,y,-y,q).$$

**Corollary 4.4.** *For $n \geq 0$, $z,\ y \neq 0$, $x=-1$; we have*

$$H_n^{(q)}(\frac{z+y}{y};-1) = E_n(\frac{z+y}{y}) = z^q F_n(-1,y,z,q).$$

We now discuss some miscellaneous recurrence relations of the generalized Fubini polynomials.

**Theorem 4.5.** *The following (differential) recurrence relation for the generalized Fubini polynomials holds:*

$$\frac{\partial}{\partial x}F_n(x,y,z,q) = n.F_{n-1}(x,y,z,q) \tag{4.3}$$

*and* $\deg F_n(x,y,z,q) = n$.

*Proof.* If we take the derivative of (1.1) with respect to $x$ both sides of the expression, we have

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!}\right) &= \frac{\partial}{\partial x}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right],\\
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}F_n(x,y,z,q)\frac{t^n}{n!} &= \frac{t e^{xt}}{[z-y(e^t-1)]^q},\\
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}F_n(x,y,z,q)\frac{t^n}{n!} &= t\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},\\
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}F_n(x,y,z,q)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^{n+1}}{n!}\\
\sum_{n=1}^{\infty}\frac{\partial}{\partial x}F_n(x,y,z,q)\frac{t^n}{n!} &= \sum_{n=1}^{\infty}F_{n-1}(x,y,z,q)\frac{t^n}{(n-1)!}.
\end{aligned}$$

On equating like powers of $t^n$ in the above expression, which completes the proof. ☐

**Theorem 4.6.** *The following (differential) recurrence relation for the generalized Fubini polynomials holds:*

$$(z+y)\frac{\partial}{\partial y}F_n(x,y,z,q) + q\sum_{n=0}^{\infty}F_n(x,y,z,q) = y\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q) + q\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q) \tag{4.4}$$

*and* $\deg F_n(x,y,z,q) = n$.

*Proof.* If we take the derivative of (1.1) with respect to $y$ both sides of the expression, we have

$$\frac{\partial}{\partial y}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} = \frac{\partial}{\partial y}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} = e^{xt}\left[-q\left(z-y\left(e^t-1\right)\right)\right]^{-q-1}(-1)(e^t-1),$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} = \frac{e^{xt}}{[z-y(e^t-1)]^q}\frac{q(e^t-1)}{z-y(e^t-1)},$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} - y\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!}\sum_{p=0}^{\infty}\frac{t^p}{p!} = q\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}F_n(x,y,z,q)\frac{t^{n+p}}{n!p!} - q\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} - y\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^{n+p}}{n!p!} = q\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^{n+p}}{n!p!} - q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$\begin{aligned}
&(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} - y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!}\\
= \quad & q\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!} - q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},
\end{aligned}$$

$$
\begin{aligned}
&(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!}+q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}\\
=\;& y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!}+q\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!}
\end{aligned}
$$

$$
\begin{aligned}
&(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!}+q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}\\
=\;& y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{n!}+q\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{n!}
\end{aligned}
$$

which upon comparison of the coefficients of $\frac{t^n}{n!}$ yields our stated result (4.4). □

**Theorem 4.7.** *The following (differential) recurrence relation for the generalized Fubini polynomials holds:*

$$(z+y)\frac{\partial}{\partial z}F_n(x,y,z,q)=y\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial z}F_{n-p}(x,y,z,q)-qF_n(x,y,z,q)$$

*and* $\deg F_n(x,y,z,q)=n$.

*Proof.* If we take the derivative of (1.1) with respect to $z$ both sides of the expression, we have

$$\frac{\partial}{\partial z}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}=\frac{\partial}{\partial z}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=\left[e^{xt}\left(-q\left[z-y(e^t-1)\right]^{-q-1}\right)\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=-q\frac{e^{xt}}{[z-y(e^t-1)]^q\,(z-y(e^t-1))},$$

$$(z-y(e^t-1))\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}=z\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}-y\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^{n+p}}{n!p!}+y\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}=z\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}-y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial z}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!}+y\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}.$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial z}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!}-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial z}F_{n-p}(x,y,z,q)\frac{t^n}{n!}-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}.$$

From the coefficients of $\frac{t^n}{n!}$ on the both sides of the last equality, one can get the desired result. □

**Theorem 4.8.** *The following (differential) recurrence relation for the generalized Fubini polynomials holds:*

$$\frac{\partial}{\partial q}F_n(x,y,z,q)=\sum_{m=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\left(\frac{y}{z+y}\right)^{m+1}(m+1)^{p-1}F_{n-p}(x,y,z,q)-\ln(z+y)F_n(x,y,z,q)$$

*and* $\deg F_n(x,y,z,q)=n$.

*Proof.* If we take the derivative of (1.1) with respect to $q$ both sides of the expression, we have

$$\frac{\partial}{\partial q}\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!} = \frac{\partial}{\partial q}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = e^{xt}((-1)\left[z-y(e^t-1)\right]^{-q}\ln(z-y(e^t-1)),$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = \frac{-e^{xt}}{[z-y(e^t-1)]^q}\ln(z+y)(1-\frac{ye^t}{z+y}),$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = -\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}\left[\ln(z+y)+\ln(1-\frac{ye^t}{z+y})\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = -\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} - \ln(1-\frac{ye^t}{z+y})\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = -\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} - \left[-\frac{ye^t}{z+y}F(1,1;2;\frac{ye^t}{z+y})\right]\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = -\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} + \frac{y}{z+y}e^t\sum_{m=0}^{\infty}\frac{(1)_m(1)_m}{(2)_m}\frac{(\frac{ye^t}{z+y})^m}{m!}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = -\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F_n(x,y,z,q)(\frac{y}{z+y})^{m+1}\frac{(e^t)^{m+1}}{m+1}\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = -\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{F_n(x,y,z,q)}{m+1}(\frac{y}{z+y})^{m+1}\frac{t^p(m+1)^p}{p!}\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = -\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}F_n(x,y,z,q)(\frac{y}{z+y})^{m+1}\frac{(m+1)^{p-1}}{p!}\frac{t^{n+p}}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = -\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{n}F_{n-p}(x,y,z,q)(\frac{y}{z+y})^{m+1}\frac{(m+1)^{p-1}}{p!}\frac{t^n}{(n-p)!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!} = -\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}F_{n-p}(x,y,z,q)(\frac{y}{z+y})^{m+1}(m+1)^{p-1}\frac{t^n}{n!}.$$

On equating like powers of $\frac{t^n}{n!}$ on both sides in the above expression and after some simplification, we arrive at our desired result. □

**Theorem 4.9.** *The following recurrence relation for the generalized Fubini polynomials holds:*

$$(z+y)F_{n+1}(x,y,z,q) - x(z+y)F_n(x,y,z,q) = y\sum_{m=0}^{n+1}F_{n-m+1}(x,y,z,q) + (q-x)y\sum_{m=0}^{n}\binom{n}{m}F_{n-m}(x,y,z,q).$$

*Proof.* If we take the derivative of (1.1) with respect to $t$ both sides of the expression, we have

$$\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!}\right] = \frac{\partial}{\partial t}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right],$$

$$\left[\sum_{n=1}^{\infty} nF_n(x,y,z,q)\frac{t^{n-1}}{n!}\right] = xe^{xt}\left[\frac{1}{[z-y(e^t-1)]^q}\right] - q\left[z-y\left(e^t-1\right)\right]^{q-1}\left[-ye^t\right]e^{xt},$$

$$\left[\sum_{n=1}^{\infty} nF_n(x,y,z,q)\frac{t^{n-1}}{n!}\right] = x\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!} + \frac{qy\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}}{z-y(e^t-1)},$$

$$\left[z-y\left(e^t-1\right)\right]\sum_{n=1}^{\infty} F_n(x,y,z,q)\frac{t^{n-1}}{n!} = x\left[z-y\left(e^t-1\right)\right]\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!} + qy\sum_{n=0}^{\infty}\sum_{m=0}^{n} F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!},$$

$$\begin{aligned}
&\left(z-y\left(e^t-1\right)\right)\sum_{n=1}^{\infty} F_n(x,y,z,q)\frac{t^{n-1}}{n!}\\
= \quad & x(z+y)\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!} - xy\sum_{n=0}^{\infty}\sum_{m=0}^{n} F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!} + qy\sum_{n=0}^{\infty}\sum_{m=0}^{n} F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!},
\end{aligned}$$

$$\begin{aligned}
&(z+y)\sum_{n=o}^{\infty} F_{n+1}(x,y,z,q)\frac{t^n}{n!} - y\sum_{n=0}^{\infty}\sum_{m=0}^{n+1} F_{n+1-m}(x,y,z,q)\frac{t^n}{n!}\\
= \quad & x(z+y)\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!} - xy\sum_{n=0}^{\infty}\sum_{m=0}^{n} F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!} + qy\sum_{n=0}^{\infty}\sum_{m=0}^{n} F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!},
\end{aligned}$$

which yields our stated result. □

**Theorem 4.10.** *The following integral representation*

$$\int_{\alpha}^{\beta} F_n(x,y,z,q)dx = \frac{F_{n+1}(\beta,y,z,q) - F_{n+1}(\alpha,y,z,q)}{n+1} \tag{4.5}$$

*holds for $n \geq 0$.*

*Proof.* From (4.3), we derive that

$$\begin{aligned}
\int_{\alpha}^{\beta} F_n(x,y,z,q)dx &= \frac{1}{n+1}\int_{\alpha}^{\beta}\frac{\partial}{\partial x}F_{n+1}(x,y,z,q)dx\\
&= \frac{F_{n+1}(\beta,y,z,q) - F_{n+1}(\alpha,y,z,q)}{n+1},
\end{aligned}$$

which means the asserted result (4.5). □

## 5. Conclusion

In this paper, we have established some generating functions for the generalized Fubini polynomials by using series rearrangement techniques. Also, some summation formulae for that polynomials are derived by using certain operational techniques and by using different analytical means on its generating function. Further, many generating functions and summation formulae for the polynomials related to generalized Fubini polynomials are obtained as applications of main results. The approach presented in this paper is general and can be extended to establish other properties of special polynomials.

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# Global Asymptotic Stability of a System of Difference Equations with Quadratic Terms

Mohamed Abd El-Moneam[1]*

**Abstract**
In this article, we discuss the global asymptotic stability of following system of difference equations with quadratic terms: $x_{i+1} = \alpha + \beta \frac{y_{i-1}}{y_i^2}, \quad y_{i+1} = \alpha + \beta \frac{x_{i-1}}{x_i^2}$ where $\alpha$, $\beta$ are positive numbers and the initial values are positive numbers. We also study the rate of convergence and oscillation behaviour of the solutions of related system. We will give also, some numerical examples to illustrate our results.

**Keywords:** Difference equations, Equilibrium, Globally asymptotically stable, Oscillates, Prime period two solution, Qualitative properties of solutions of difference equations, Rational difference equations.
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[1] *Department of Mathematics, Faculty of Science, Jazan University, Kingdom of Saudi Arabia, ORCID: 0000-0002-1676-2662*
***Corresponding author**: maliahmedibrahim@jazanu.edu.sa
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## 1. Introduction

The difference equations or systems have too many applications among many branches of science. over the last two decades, difference equations or their systems have been huge interest between scholars which are mathematicians . For example, in [22] discussed global dynamics of an one-dimensional discrete-time laser model. Further in [8] Din et al. discussed stability of a discrete ecological model. Studies of difference equations are increasing day by day and will continue to increase. Therefore, there are many papers related to applications of difference equations or systems. More specifically, some scientists studied the dynamics of solutions of difference equations or systems (for example, see [1]-[5],[7, 9, 12], [14]-[21], [23], [25]-[30]). Additionally, there are many results related to our study as follows:
In [31], Yang et al. studied the solutions, stability and asymptotic behaviour of the system of the two nonlinear difference equations

$$x_{n+1} = \frac{Ax_n}{1+y_n^p}, \quad y_{n+1} = \frac{By_n}{1+x_n^p}.$$

In [11], Elabbasy et al. investigated the global behaviour of following system of difference equations

$$x_{n+1} = \frac{a_1 x_n}{a_2 + a_3 y_n^r}, \quad y_{n+1} = \frac{b_1 y_n}{b_2 + b_3 x_n^r}.$$

In [6], Bacani et al. discussed solutions of the following two nonlinear difference equations

$$x_{n+1} = \frac{q}{p + x_n^v}, \quad y_{n+1} = \frac{q}{-p + y_n^v}.$$

In [24], Hadziabdic et al. examined the global behaviours of following system of difference equations

$$x_{n+1} = \frac{b_1 x_n^2}{A_1 + y_n^2}, \quad y_{n+1} = \frac{a_2 + c_2 y_n^2}{x_n^2}$$

In [8], Burgic et al. investigated the global stability properties and asymptotic behaviour of solutions for the system of difference equations

$$x_{n+1} = \frac{x_n}{a + y_n^2}, \quad y_{n+1} = \frac{y_n}{b + x_n^2}.$$

In [10], Beso et al. concentrates on discussing boundedness of solutions of following difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2}.$$

In [13], Tasdemir et al. discussed the global asymptotic stability of a system of difference equations with quadratic terms

$$x_{n+1} = A + B\frac{y_n}{y_{n-m}^2}, \quad y_{n+1} = A + B\frac{x_n}{x_{n-m}^2}$$

They also studied global asymptotic stability of related difference equation. Motivated by difference equations and their systems, we consider the following system of difference equations

$$x_{i+1} = \alpha + \beta \frac{y_{i-1}}{y_i^2}, \quad y_{i+1} = \alpha + \beta \frac{x_{i-1}}{x_i^2} \tag{1.1}$$

where $\alpha$ and $\beta$ are positive numbers and the initial values are positive numbers. In this paper we study the stability, global behaviour and rate of convergence of solutions of system (1.1). We also discussed the oscillation behaviour of solutions of related system. In this here, we obtain two theorems which are used during this study.

**Theorem 1.1.** *(Linearized Stability Theorem [25]) Assume that*

$$X_{i+1} = F(X_i), i = 0, 1, \ldots$$

*is a system of difference equations such that $\bar{X}$ is a fixed point of $F$.*
*(i) If all eigenvalues of the Jacobian matrix $\beta$ about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, that is, if all of them have absolute value less than one, then $\bar{X}$ is locally asymptotically stable.*
*(ii) If at least one of them has a modulus greater than one, then $\bar{X}$ is unstable.*

**Theorem 1.2.** *[5] Let $i \in N_{i_0}^{+}$ and $g(i,u,v)$ be a decreasing function in $u$ and $v$ for any fixed $n$. Suppose that for $i \leq i_0$,the inqualities*

$$y_{i+1} \leq g(i, y_i, y_{i-1})$$

$$u_{i+1} \geq g(i, y_i, y_{i-1})$$

*hold. Then*

$$y_{i_0-1} \leq u_{i_0-1}, y_{i_0} \leq u_{i_0}$$

*implies that*

$$y_i \leq u_i, i \geq i_0.$$

## 2. Linearized Stability of System (1.1)

First of all, we consider the change of the variables for system (1.1) as follows:

$$\zeta_i = \frac{x_i}{\alpha}, \ \eta_i = \frac{y_i}{\alpha}.$$

From this, system (1.1) transform into following system:

$$\zeta_{i+1} = 1 + \mu \frac{\eta_{i-1}}{\eta_i^2}, \eta_{i+1} = 1 + \mu \frac{\zeta_{i-1}}{\zeta_i^2} \tag{2.1}$$

where $\mu = \frac{\beta}{\alpha^2} > 0$. From now on, we study the system (2.1).

**Lemma 2.1.** *Let $\mu > 0$. Unique positive equilibrium point of system (2.1) is*

$$(\bar{\zeta},\bar{\eta}) = \left(\frac{1+\sqrt{1+4\mu}}{2}, \frac{1+\sqrt{1+4\mu}}{2}\right).$$

*Now, we consider a transformation as follows:*

$$(\zeta_i, \zeta_{i-1}, \eta_i, \eta_{i-1}) \to (t, t_1, z, z_1)$$

*where $t = 1+\mu\ \frac{\eta_{i-1}}{\eta_i^2}, t_1 = \zeta_i, z = 1+\mu\ \frac{\zeta_{i-1}}{\zeta_i^2}, z_1 = \eta_i$ . Thus we get the jacobian matrix about equilibrium point $(\bar{\zeta},\bar{\eta})$:*

$$\beta(\bar{\zeta},\bar{\eta}) = \begin{pmatrix} 0 & 0 & \frac{\mu}{\bar{\eta}^2} & \frac{-2\mu}{\bar{\eta}^2} \\ 1 & 0 & 0 & 0 \\ \frac{\mu}{\bar{\zeta}^2} & \frac{-2\mu}{\bar{\zeta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

*Thus, the linearized system of system (2.1) about the unique positive equilibrium point is given by $X_{I+1} = \beta(\zeta,\eta)X_I$ , where*

$$X_I = \begin{pmatrix} \zeta_i \\ \zeta_{i-1} \\ \eta_i \\ \eta_{i-1} \end{pmatrix},$$

$$\beta(\bar{\zeta},\bar{\eta}) = \begin{pmatrix} 0 & 0 & \frac{\mu}{\bar{\eta}^2} & \frac{-2\mu}{\bar{\eta}^2} \\ 1 & 0 & 0 & 0 \\ \frac{\mu}{\bar{\zeta}^2} & \frac{-2\mu}{\bar{\zeta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

*Hence, the characteristic equation of $\beta(\zeta,\eta)$ about the unique positive equilibrium point $(\bar{\zeta},\bar{\eta})$is*

$$\lambda^4 - \frac{\mu^2}{\bar{\zeta}^2\bar{\eta}^2}\lambda^2 + \frac{4\mu^2}{\bar{\zeta}^2\bar{\eta}^2}\lambda - \frac{4\mu^2}{\bar{\zeta}^2\bar{\eta}^2} = 0.$$

*Due to $\bar{\zeta} = \bar{\eta}$, we can rearrange the characteristic equation such that*

$$\lambda^4 - \frac{\mu^2}{\bar{\zeta}^4}\lambda^2 + \frac{4\mu^2}{\bar{\zeta}^4}\lambda - \frac{4\mu^2}{\bar{\zeta}^4} = 0.$$

*Therefore, we obtain the four roots of characteristic equation as follows:*

$$\lambda_1 = \frac{\mu + \sqrt{\mu^2 - 8\mu\ \bar{\zeta}^2}}{2\ \bar{\zeta}^2},$$

$$\lambda_2 = \frac{\mu - \sqrt{\mu^2 - 8\mu\ \bar{\zeta}^2}}{2\ \bar{\zeta}^2},$$

$$\lambda_3 = \frac{-\mu + \sqrt{\mu^2 + 8\mu\ \bar{\zeta}^2}}{2\ \bar{\zeta}^2},$$

$$\lambda_4 = \frac{-\mu - \sqrt{\mu^2 + 8\mu\ \bar{\zeta}^2}}{2\ \bar{\zeta}^2}.$$

*Now, we calculate $\bar{\zeta}^2$ and write in $\lambda_1$. Then we have*

$$\begin{aligned} \lambda_1 &= \frac{\mu + \sqrt{\mu^2 - 4\mu(1+2\mu+\sqrt{4\mu+1})}}{1+2\mu+\sqrt{4\mu+1}} \\ &= \frac{\mu + \sqrt{-7\mu^2 - 4\mu - 4\mu\sqrt{1+4\mu}}}{1+2\mu+\sqrt{4\mu+1}} \\ &= \frac{\mu + \sqrt{7\mu^2 + 4\mu + 4\mu\sqrt{1+4\mu}}i}{1+2\mu+\sqrt{4\mu+1}}. \end{aligned}$$

*Thus straightforward calculations show that*

$$|\lambda_1| = \frac{2\sqrt{2\mu}}{1+\sqrt{1+4\mu}}.$$

*Additionally, we obtain similarly calculations that*

$$|\lambda_2| = \frac{2\sqrt{2\mu}}{1+\sqrt{1+4\mu}}.$$

*On the other hand, we consider $\lambda_3$ as follows:*

$$\begin{aligned}\lambda_3 &= \frac{-\mu+\sqrt{9\mu^2+4\mu+4\mu\sqrt{4\mu+1}}}{1+2\mu+\sqrt{4\mu+1}} \\ &= \frac{-\mu+\sqrt{(3\mu+\sqrt{1+4\mu})^2-1-2\mu\sqrt{4\mu+1}}}{1+2\mu+\sqrt{4\mu+1}} \\ &< \frac{-\mu+\sqrt{(3\mu+\sqrt{1+4\mu})^2}}{1+2\mu+\sqrt{4\rho+1}} \\ &= \frac{2\mu+\sqrt{1+4\mu}}{1+2\mu+\sqrt{4\mu+1}} < 1.\end{aligned}$$

*Moreover, we clearly see that $\lambda_3 > 0$. So $0 < \lambda_3 < 1$ for all $\mu > 0$ . Similar calculations we have that $-1 < \lambda_4 < 0$ for all $\mu > 0$.*

**Theorem 2.2.** *Suppose that $\mu > 0$.Then the following cases hold for system (2.1):*
*(i) If $\mu < 2$ then the equilibrium point of system (2.1) is locally asymptotically stable.*
*(ii) If $\mu = 2$ then the equilibrium point of system (2.1) is a non-hyperbolic equilibrium .*
*(iii) If $\mu > 2$ then the equilibrium point of system (2.1) is a repeller.*

*Proof.* Firstly we know that $|\lambda_3|, |\lambda_4| < 1$ for all $\mu > 0$ . Now we consider

$$|\lambda_1| = |\lambda_2| = \frac{2\sqrt{2\mu}}{1+\sqrt{1+4\mu}}.$$

If the equilibrium point of system (2.1) is locally asymptotically stable, then all roots of characteristic equation must lie the unit disk. Therefore,we must show that $|\lambda_1|, |\lambda_2| < 1$. Hence

$$|\lambda_1| = |\lambda_2| = \frac{2\sqrt{2\mu}}{1+\sqrt{1+4\mu}} < 1.$$

Thus, we have $2\sqrt{2\mu} < 1+\sqrt{1+4\mu}$. From this, we obtain that $\mu < 2$. The proofs of other cases can be obtained in a similar way. □

## 3. An Oscillation Result of Solutions of System (2.1)

In this here, we investigate the oscillation behaviour of solutions of system (2.1).

**Theorem 3.1.** *Assume $\{(\zeta_i, \eta_i)\}$ be a positive solution of system (2.1) $\mu > 0$. Then for any $i \geq 0$ the following cases are true.*

*(i) if $\zeta_{i+1}, \eta_i < \bar{\zeta} = \bar{\eta} < \zeta_i, \eta_{i+1}$ then*

$$\begin{aligned}&(\zeta_{i+2k-1})_{k=1}^{\infty} < \bar{\zeta} < (\zeta_{i+2k})_{k=1}^{\infty}, \\ &(\eta_{i+2k})_{k=1}^{\infty} < \bar{\eta} < (\eta_{i+2k-1})_{k=1}^{\infty}.\end{aligned} \tag{3.1}$$

*(ii) if $\zeta_i, \eta_{i+1} < \bar{\zeta} = \bar{\eta} < \zeta_{i+1}, \eta_i$ then*

$$\begin{aligned}&(\zeta_{i+2k})_{k=1}^{\infty} < \bar{\zeta} < (\zeta_{i+2k-1})_{k=1}^{\infty}, \\ &(\eta_{i+2k-1})_{k=1}^{\infty} < \bar{\eta} < (\eta_{i+2k})_{k=1}^{\infty}.\end{aligned} \tag{3.2}$$

*Proof.* Firstly we consider case (3.1). Assume that $\zeta_{i+1},\eta_i<\bar{\zeta}=\bar{\eta}<\zeta_i,\eta_{i+1}$. Then we obtain that

$$\zeta_{i+2}=1+\mu\ \frac{\eta_i}{\eta_{i+1}^2}>1+\mu\ \frac{\bar{\eta}}{\bar{\eta}^2}=\bar{\eta}=\bar{\zeta},$$
$$\eta_{i+2}=1+\mu\ \frac{\zeta_i}{\zeta_{i+1}^2}<1+\mu\ \frac{\bar{\zeta}}{\bar{\zeta}^2}=\bar{\zeta}=\bar{\eta},$$
$$\zeta_{i+3}<\bar{\zeta},\eta_{i+3}>\bar{\eta},\zeta_{i+4}>\bar{\zeta},\eta_{i+4}<\bar{\eta}.$$

Therefore we have by using induction

$$\zeta_i,\zeta_{i+2},\ldots,\zeta_{i+2k},\ldots>\bar{\zeta}>\zeta_{i+1},\zeta_{i+3},\ldots,\zeta_{i+2k-1},\ldots$$
$$\eta_{i+1},\eta_{i+3},\ldots,\eta_{i+2k-1},\ldots>\bar{\eta}>\eta_i,\eta_{i+2},\ldots,\eta_{i+2k},\ldots$$

Thus the proof of (3.1) is completed as desired. The proof of (3.2) is similar to proof of (3.1). □

## 4. Boundedness of System (2.1)

**Lemma 4.1.** *Let $\{(\zeta_i,\eta_i)\}$ be a positive solution of system (2.1) and $\mu>0$ . Then $\zeta_i>1$ and $\eta_i>1$ for $i\geq 1$.*

*Proof.* Assume $\{(\zeta_i,\eta_i)\}$ be a positive solution of system (2.1). Then we have from system (2.1):

$$\zeta_1=1+\mu\ \frac{\eta_{-1}}{\eta_0^2}>1,$$
$$\eta_1=1+\mu\ \frac{\zeta_{-1}}{\zeta_0^2}>1.$$

Therefore, we obtain by induction

$$\zeta_{i+1}=1+\mu\ \frac{\eta_{i-1}}{\eta_i^2}>1,$$
$$\eta_{i+1}=1+\mu\ \frac{\zeta_{i-1}}{\zeta_i^2}>1.$$

So, the proof of lemma is completed. □

**Theorem 4.2.** *If $0<\mu<1$ then every solution of system (2.1) is bounded.*

*Proof.* Firstly we have from system (2.1) $\zeta_i>1$ and $\eta_i>1$ for $i\geq 1$ and $\mu>0$. Moreover, every solution of system (2.1) satisfies

$$\zeta_{i+1}\leq 1+\mu+\mu^2\ \zeta_{i-1},\ i\geq 1, \tag{4.1}$$

which due to Theorem 1.2, means that $\zeta_i\leq q_i$, $i=0,1,\ldots$, where $\{u_i\}$ satisfy

$$u_{i+1}=1+\mu+\mu^2\ u_{i-1}, i\geq 1, \tag{4.2}$$

such that

$$u_s=\zeta_s,u_{s+1}=\zeta_{s+1},s\in\{-1,0,1,\ldots\},i\geq s.$$

Hence the solution $u_i$ of the difference equation (4.2) is

$$u_i=\frac{1}{1-\mu}+\mu^iC_1+(-\mu)^iC_2. \tag{4.3}$$

Actually, we have from (4.2)

$$u_{i+1}=1+\mu+\mu^2u_{i-1}\Rightarrow\lambda^2-\mu^2=0\Rightarrow\lambda_{1,2}=\pm\mu.$$

From this, the homogeneous solution of difference equation (4.2) is

$$u_n=\mu^iC_1+(-\mu)^iC_2.$$

In additon, from (4.2), the equilibrium solution of difference equation (4.2) is

$$\bar{u} = 1 + \mu + \mu^2 \bar{u} \Rightarrow \bar{u} = \frac{1}{1-\mu}.$$

Additionally, relations (4.1) and (4.2) imply that

$$\zeta_{i+1} - u_{i+1} \leq \mu^2 \left(\zeta_{i-1} - u_{s-1}\right), \; i > s, \mu \in (0,1).$$

Therefore we have

$$\zeta_i \leq u_i, i > s \tag{4.4}$$

Hence, we obtain from (4.3), (4.4) and Lemma 4.1,

$$1 < \zeta_i \leq \frac{1}{1-\mu} + \mu^i C_1 + (-\mu)^i C_2 = N_1,$$

where

$$C_1 = \frac{1}{2\mu}\left(\mu\zeta_0 + \zeta_1 - \frac{1+\mu}{1-\mu}\right),$$
$$C_2 = \frac{1}{2\mu}\left(\mu\zeta_0 - \zeta_1 + 1\right).$$

Similarly we can write that

$$1 < \eta_i \leq \frac{1}{1-\mu} + \mu^i C_3 + (-\mu)^i C_4 = N_2$$

where

$$C_3 = \frac{1}{2\mu}\left(\mu\zeta_0 + \zeta_1 - \frac{1+\mu}{1-\mu}\right),$$
$$C_4 = \frac{1}{2\mu}\left(\mu\zeta_0 - \zeta_1 + 1\right).$$

□

## 5. Convergence Results of Solutions of System (2.1)

**Theorem 5.1.** *If $\zeta_i \geq \bar{\zeta}$ and $\eta_i \geq \bar{\eta}$ (resp., $\zeta_i \geq \bar{\zeta}$ and $\eta_i \geq \bar{\eta}$ ) for $i \geq s$ and $s \in \{-1, 0, \ldots\}$ then the solution $\{(\zeta_i, \eta_i)\}$ of system (2.1) tends to equilibrium point $\left\{\left(\bar{\zeta}, \bar{\eta}\right)\right\}$ as $i \to \infty$.*

*Proof.* Let $\{(\zeta_i, \eta_i)\}$ be a positive solution of system (2.1) such that

$$\zeta_i \geq \bar{\zeta}, \; \eta_i \geq \bar{\eta}, \; i \geq s, \tag{5.1}$$

where $s \in \{-1, 0, \ldots\}$. Hence, we obtain from (5.1), system (2.1) and Lemma 4.1:

$$\zeta_{i+1} \leq 1 + \mu + \mu^2 \zeta_{i-1}. \tag{5.2}$$

$$u_{i+1} = 1 + \mu + \mu^2 u_{i-1}, \tag{5.3}$$

$$u_s = \zeta_s, u_{s+1} = \zeta_{s+1}, s \in \{-1, 0, \ldots\}, \; i \geq s. \tag{5.4}$$

Therefore, we get from the solution of the difference equation (5.3):

$$u_i = \frac{1}{1-\mu} + \mu^i C_1 + (-\mu)^i C_2 \tag{5.5}$$

where $C_1, C_2$ depent on $\zeta_s$ ,$\zeta_{s+1}$. Moreover, we have from (5.2) and (5.3):

$$\zeta_{i+1} - u_{s+1} \leq \mu^2 (\zeta_{i-1} - u_{s-1}), \ i > s \tag{5.6}$$

Thus we obtain from (5.4), (5.6) and by induction

$$\zeta_i \leq u_i, \ i \geq s. \tag{5.7}$$

From (5.1), (5.5) and (5.7), we obtain that

$$\lim_{i\to\infty} \zeta_i = \bar{\zeta}.$$

Then we similarly obtain that $\lim_{i\to\infty} \eta_i = \bar{\eta}$. The proof of the other case of this theorem is similar to this case, so we leave it to readers. □

**Theorem 5.2.** *Suppose that* $0 < \mu < \frac{1}{2}$. *Then the positive equilibrium point of system* (2.1) *is globally asymptotically stable.*

*Proof.* We have from Theorem 4.2,

$$1 < m_1 = \liminf_{i\to\infty} \zeta_i \leq N_1,$$
$$1 < m_2 = \liminf_{i\to\infty} \eta_i \leq N_2,$$
$$1 < U_1 = \limsup_{i\to\infty} \zeta_i \leq N_1,$$
$$1 < U_2 = \limsup_{i\to\infty} \eta_i \leq N_2.$$

By system (2.1), we can write

$$U_1 \leq 1 + \mu \frac{U_2}{m_2^2}, m_1 \geq 1 + \mu \frac{m_2}{U_2^2},$$
$$U_2 \leq 1 + \mu \frac{U_1}{m_1^2}, m_2 \geq 1 + \mu \frac{m_1}{U_1^2}.$$

Hence we have

$$U_1 + \mu \frac{m_1}{U_1} \leq U_1 m_2 \leq m_2 + \mu \frac{U_2}{m_2},$$
$$U_2 + \mu \frac{m_2}{U_2} \leq U_2 m_1 \leq m_1 + \mu \frac{U_1}{m_1}.$$

Therefore we obtain that

$$U_1 + \mu \frac{m_1}{U_1} + U_2 + \mu \frac{m_2}{U_2} \leq m_2 + \mu \frac{U_2}{m_2} + m_1 + \mu \frac{U_1}{m_1},$$
$$U_1 + \mu \frac{m_1}{U_1} + U_2 + \mu \frac{m_2}{U_2} - m_2 - \mu \frac{U_2}{m_2} - m_1 - \mu \frac{U_1}{m_1} \leq 0,$$
$$(U_1 - m_1)\left(1 - \mu\left(\frac{1}{m_1} + \frac{1}{U_1}\right)\right) + (U_2 - m_2)\left(1 - \mu\left(\frac{1}{m_2} + \frac{1}{U_2}\right)\right) \leq 0.$$

In this here if $\mu \in (0, \frac{1}{2})$ than

$$1 - \mu\left(\frac{1}{m_1} + \frac{1}{U_1}\right) > 0,$$
$$1 - \mu\left(\frac{1}{m_2} + \frac{1}{U_2}\right) > 0.$$

Thus, we get that

$$U_1 - m_1 = 0, \quad U_2 - m_2 = 0.$$

So, $U_1 = m_1$ and $U_2 = m_2$. The proof is completed as desired. □

## 6. Rate of Convergence of System (2.1)

Now we study the rate of convergence of system (2.1). Hence, we consider the following system:

$$E_{i+1} = (\alpha + \beta(i))E_i, \tag{6.1}$$

where $E_i$ is a k-dimensional vector, $\alpha \in C^{k\times k}$ is a constant matrix, and $\beta : \mathbf{Z}^+ \to C^{k\times k}$ is a matrix function satisfying

$$\|\beta(i)\| \to 0, \tag{6.2}$$

as $i \to \infty$, where $\|\cdot\|$ denotes any matrix norm that is associated with the vector norm

$$\|(x,y)\| = \sqrt{x^2+y^2}.$$

**Theorem 6.1.** *(Perronas Theorem, [24]) Assume that condition (6.2) holds. If $E_i$ is a solution of (6.1), then either $E_i = 0$ for all as $i \to \infty$,or*

$$\lim_{i\to\infty} \sqrt[i]{\|E_i\|},$$

*or*

$$\lim_{i\to\infty} \frac{\|E_{i+1}\|}{\|E_i\|},$$

*exists and is equal to modulus of one of the eigenvalues of matrix $\alpha$.*

**Theorem 6.2.** *Suppose that $0 < \mu < \frac{1}{2}$ and $\{(\zeta_i, \eta_i)\}$ be a solution of the system (2.1) such that $\lim\limits_{i\to\infty} \zeta_i = \bar{\zeta}$ and $\lim\limits_{i\to\infty} \eta_i = \bar{\eta}$. Then the error vector*

$$E_i = \begin{pmatrix} e_i^1 \\ e_{i-1}^1 \\ e_i^2 \\ e_{i-1}^2 \end{pmatrix} = \begin{pmatrix} \zeta_i - \bar{\zeta} \\ \zeta_{i-1} - \bar{\zeta} \\ \eta_i - \bar{\eta} \\ \eta_{i-1} - \bar{\eta} \end{pmatrix}$$

*of every solution of system (2.1) satisfies both of the following asymptotic relations:*

$$\lim_{i\to\infty} \sqrt[i]{\|E_i\|} = \left|\lambda_{1,2,3,4}\ F_J(\bar{\zeta},\bar{\eta})\right|,$$
$$\lim_{i\to\infty} \frac{\|E_{i+1}\|}{\|E_i\|} = \left|\lambda_{1,2,3,4}\ F_J(\bar{\zeta},\bar{\eta})\right|.$$

*where $\lambda_{1,2,3,4}\ F_J(\bar{\zeta},\bar{\eta})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{\zeta},\bar{\eta})$.*

*Proof.* To find the error terms, we set

$$\zeta_{i+1} - \bar{\zeta} = \sum_{n=0}^{1} A_n \left(t_{i-n} - \bar{\zeta}\right) + \sum_{n=0}^{1} B_n \left(z_{i-n} - \bar{\eta}\right),$$
$$\eta_{i+1} - \bar{\eta} = \sum_{n=0}^{1} D_n \left(\zeta_{i-n} - \bar{\zeta}\right) + \sum_{n=0}^{1} G_n \left(\eta_{i-n} - \bar{\eta}\right),$$

and $e_i^1 = \zeta_i - \bar{\zeta}, e_i^2 = \eta_i - \bar{\eta}$. Thus we have

$$e_{i+1}^1 = \sum_{n=0}^{1} A_n e_{i-n}^1 + \sum_{n=0}^{1} B_n e_{i-n}^2,$$

$$e_{i+1}^1 = \sum_{n=0}^{1} D_n e_{i-n}^1 + \sum_{n=0}^{1} G_n e_{i-n}^2,$$

where

$$\begin{aligned}
&A_0 = A_1 = 0,\\
&B_0 = \frac{\mu}{\eta_i^2}, B_1 = \frac{-\mu\left(\bar{\eta}+\eta_i\right)}{\bar{\eta}\eta_i^2},\\
&D_0 = \frac{\mu}{\zeta_i^2}, D_1 = \frac{-\mu\left(\bar{\zeta}+\zeta_i\right)}{\bar{\zeta}\zeta_i^2},\\
&G_0 = G_1 = 0.
\end{aligned}$$

Now we take the limits

$$\begin{aligned}
&\lim_{i\to\infty} A_0 = \lim_{i\to\infty} A_1 = 0,\\
&\lim_{i\to\infty} B_0 = \frac{\mu}{\bar{\eta}^2}, \quad \lim_{i\to\infty} B_1 = \frac{-2\mu}{\bar{\eta}^2},\\
&\lim_{i\to\infty} D_0 = \frac{\mu}{\bar{\zeta}^2}, \quad \lim_{i\to\infty} D_1 = \frac{-2\mu}{\bar{\zeta}^2},\\
&\lim_{i\to\infty} G_0 = \lim_{i\to\infty} G_1 = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&B_0 = \frac{\mu}{\bar{\eta}^2} + b_i, \quad B_1 = \frac{-2\mu}{\bar{\eta}^2} + r_i,\\
&D_0 = \frac{\mu}{\bar{\zeta}^2} + d_i, \quad D_1 = \frac{-2\mu}{\bar{\zeta}^2} + t_i,
\end{aligned}$$

where $b_i \to 0,\ r_i \to 0,\ d_i \to 0,\ t_i \to 0$ as $i \to \infty$. Therefore, we obtain the system of the form (6.1)

$$E_{i+1} = (\alpha + \beta(i))E_i$$

where

$$\begin{aligned}
&\alpha = \begin{pmatrix} 0 & 0 & \frac{\mu}{\bar{\eta}^2} & \frac{-2\mu}{\bar{\eta}^2} \\ 1 & 0 & 0 & 0 \\ \frac{\mu}{\bar{\zeta}^2} & \frac{-2\mu}{\bar{\zeta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\\
&\beta(i) = \begin{pmatrix} 0 & 0 & b_i & r_i \\ 1 & 0 & 0 & 0 \\ d_i & t_i & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\end{aligned} \tag{6.3}$$

and $\|\beta(i)\| \to 0$ as $i \to \infty$. So, the limiting system of error terms about the equilibrium point can be written as follows:

$$\begin{pmatrix} e_i^1 \\ e_i^1 \\ e_{i+1}^2 \\ e_i^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\mu}{\bar{\eta}^2} & \frac{-2\mu}{\bar{\eta}^2} \\ 1 & 0 & 0 & 0 \\ \frac{p}{\bar{\zeta}^2} & \frac{-2\mu}{\bar{\zeta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_i^1 \\ e_{i-1}^1 \\ e_i^2 \\ e_{i-1}^2 \end{pmatrix}$$

which is same as linearized system of system (2.1) about equilibrium point$(\bar{\zeta},\, \bar{\eta})$. □

## 7. Numerical Examples

In this section, we give two examples which include three figures to verify our theoretical results.

**Example 7.1.** *We consider system* (2.1) *for* $\mu = 0.43$. *With the initial values* $\zeta_{-1} = 1$, $\zeta_0 = 1.2$, $\eta_{-1} = 3$ *and* $\eta_0 = 0.95$ *positive equilibrium point of system* (2.1) *is globally asymptotically stable. Figures 7.1, 7.2 verify our theoretical results.*

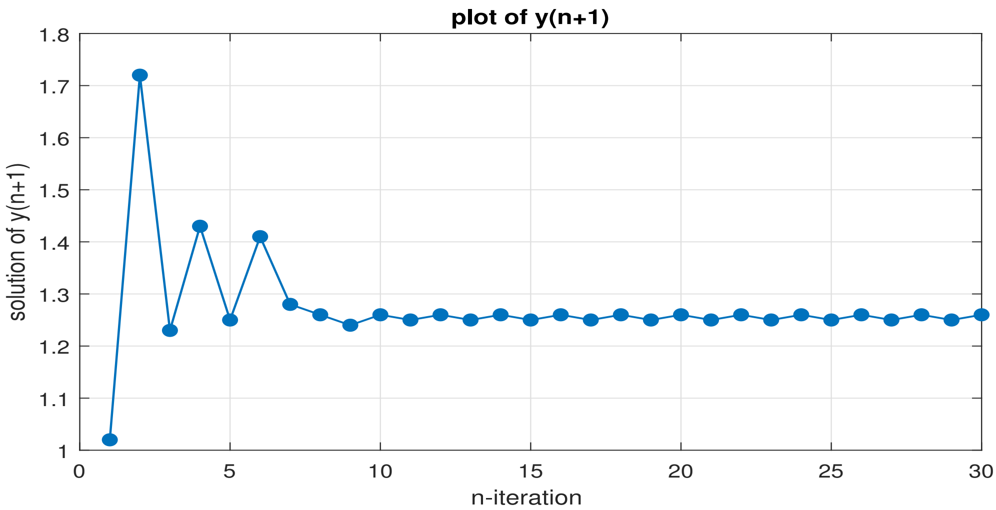


**Figure 7.1**

**Example 7.2.** *We consider system* (2.1) *for* $\mu = 2.2$. *With the initial values* $\zeta_{-1} = 2.08$, $\zeta_0 = 2.02$, $\eta_{-1} = 2.03$ *and* $\eta_0 = 2.08$, *solutions of system* (2.1) *oscillate about positive equilibrium point* $(\bar{\zeta},\ \bar{\eta} = (0.0652, 0.0652)$. *Figure 7.3 verifies our theoretical results.*

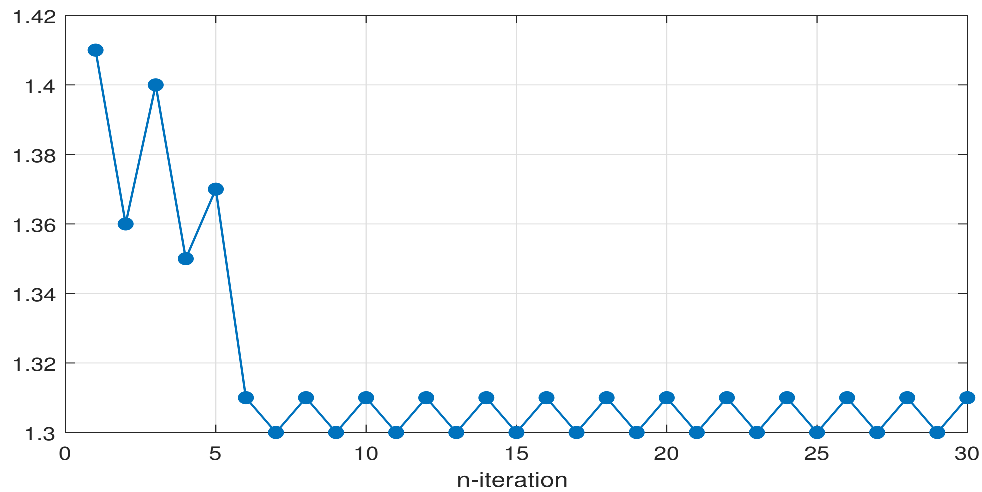


**Figure 7.2**

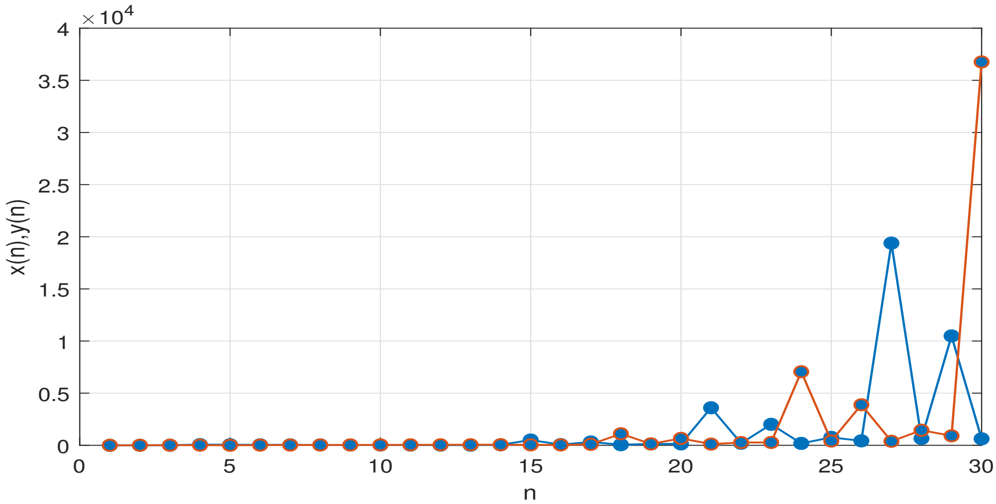


**Figure 7.3**

## 8. Conclusions

In this paper we studied convergence results of a system of second order difference equations . Firstly we deal with the unique positive equilibrium point of system(2.1). Then we analyse the bounded solutions of system (2.1). We also investigate the oscillation of solutions of system. More specifically, we focus on the convergence results of solutions of system. According to our results, if $0 < \mu < \frac{1}{2}$ then the positive equilibrium point of system (2.1) is globally asymptotically stable. After this we concentrates on discussing the rate of convergence of solutions of system(2.1). Moreover to this we give two numerical examples to verify our theoretical results.

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# Almost $\eta-$Ricci Solitons on Pseudosymmetric Lorentz Sasakian Space Forms

Tuğba Mert[1]*, Mehmet Atçeken[2]

**Abstract**
In this paper, we consider pseudosymmetric Lorentz Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz Sasakian space forms admits $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, $W_1$ and $W_2$. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz Sasakian space form admits $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made under the some conditions.

**Keywords:** $\eta-$Ricci Soliton, Lorentz Sasakian Space Form, Ricci-pseudosymmetric Manifold.
**2010 AMS:** 53C15, 53C25, 53D25

[1] *Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, Sivas, Turkey, ORCID: 0000-0001-8258-8298*
[2] *Department of Mathematics, Faculty of Science and Arts, Aksaray University, Aksaray, Turkey, ORCID: 0000-0002-1242-4359*
***Corresponding author**: tmert@cumhuriyet.edu.tr
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## 1. Introduction

The notion of Ricci flow was introduced by Hamilton in 1982. With the help of this concept, Hamilton found the canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman used Ricci flow and it surgery to prove Poincare conjecture in [1, 2]. The Ricci flow is an flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g(t) = -2S(g(t)).$$

A Ricci soliton emerges as the limit of the solitons of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [3], Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Ashoka et al. in [4, 5], Bagewadi et al. in [6], Ingalahalli in [7], Bejan and Crasmareanu in [8], Blaga in [9], Chandra et al. in [10], Chen and Deshmukh in [11], Deshmukh et al. in [12], He and Zhu [13], Atçeken et al. in [14], Nagaraja and Premalatta in [15], Tripathi in [16] and many others.

$\phi-$sectional curvature plays an important role for Sasakian manifold. If the $\phi-$sectional curvature of a Sasakian manifold is constant, then the manifold is a Sasakian-space-form [17]. P. Alegre and D. Blair described generalized Sasakian space forms [18]. P. Alegre and D. Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. P. Alegre and A. Carriazo later discussed generalized indefinite Sasakian space forms [19]. Generalized indefinite Sasakian space forms are also called Lorentz-Sasakian space forms, and Lorentz manifolds are of great importance for Einstein's theory of Relativity.

In this paper, we consider Lorentz Sasakian space form admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz Sasakian space form admits $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemannian, concircular, projective, $\mathscr{M}-$projective, $W_1$ and $W_2$. Then, again according to the choice of the curvature tensor, necessary conditions for Lorentz Sasakian space form admits $\eta-$Ricci soliton to be Ricci semisymmetric are given. Then some characterizations are obtained and some classifications have been made

## 2. Preliminaries

Let $\tilde{N}$ be a $(2m+1)-$dimensional Lorentz manifold. If the $\tilde{N}$ Lorentz manifold with $(\phi,\xi,\eta,g)$ structure tensors satisfies the following conditions, it is called a Lorentz-Sasakian manifold

$$\phi^2 Y_1 = -Y_1+\eta(Y_1)\xi, \eta(\xi)=1, \eta(\phi Y_1)=0,$$

$$g(\phi Y_1,\phi Y_2)=g(Y_1,Y_2)+\eta(Y_1)\eta(Y_2), \eta(Y_1)=-g(Y_1,\xi),$$

$$\left(\tilde{\nabla}_{Y_1}\phi\right)Y_2=-g(Y_1,Y_2)\xi-\eta(Y_2)Y_1, \tilde{\nabla}_{Y_1}\xi=-\phi Y_1,$$

where, $\tilde{\nabla}$ is the Levi-Civita connection according to the Riemannian metric $g$.

The plane section $\Pi$ in $T_{Y_1}\tilde{N}$. If the $\Pi$ plane is spanned by $Y_1$ and $\phi Y_1$, this plane is called the $\phi$-section. The curvature of the $\phi$-section is called the $\phi$-sectional curvature. If the Lorentz-Sasakian manifold has a constant $\phi$-sectional curvature, this manifold is called the Lorentz-Sasakian space form and is denoted by $\tilde{N}(c)$. The curvature tensor of the Lorentz-Sasakian space form $\tilde{N}(c)$ is defined as

$$\begin{aligned}
&\tilde{R}(Y_1,Y_2)Y_3=\left(\tfrac{c-3}{4}\right)\{g(Y_2,Y_3)Y_1-g(Y_1,Y_3)Y_2\}\\
&+\left(\tfrac{c+1}{4}\right)\{g(Y_1,\phi Y_3)\phi Y_2-g(Y_2,\phi Y_3)\phi Y_1\\
&+2g(Y_1,\phi Y_2)\phi Y_3+\eta(Y_2)\eta(Y_3)Y_1-\eta(Y_1)\eta(Y_3)Y_2\\
&+g(Y_1,Y_3)\eta(Y_2)\xi-g(Y_2,Y_3)\eta(Y_1)\xi\},
\end{aligned}\tag{2.1}$$

for all $Y_1,Y_2,Y_3\in\chi\left(\tilde{N}\right)$.

**Lemma 2.1.** *Let $\tilde{N}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. The following relations are hold for the Lorentz-Sasakian space forms.*

$$\tilde{\nabla}_{Y_1}\xi=-\phi Y_1,\tag{2.2}$$

$$\left(\tilde{\nabla}_{Y_1}\phi\right)Y_2=-g(Y_1,Y_2)\xi-\eta(Y_2)Y_1,$$

$$\left(\tilde{\nabla}_{Y_1}\eta\right)Y_2=g(\phi Y_1,Y_2),$$

$$\tilde{R}(Y_1,Y_2)\xi=\eta(Y_2)Y_1-\eta(Y_1)Y_2,\tag{2.3}$$

$$\eta\left(\tilde{R}(Y_1,Y_2)Y_3\right)=g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_3),\tag{2.4}$$

$$S(Y_1,Y_2)=\left[\frac{(m+2)c-(3m-2)}{2}\right]g(Y_1,Y_2)$$

$$+\frac{(c+1)(m+1)}{2}\eta(Y_1)\eta(Y_2),$$

$$S(Y_1,\xi)=-\left[\frac{(c+1)-4m}{2}\right]\eta(Y_1), \tag{2.5}$$

$$QY_1=\left[\frac{(m+2)c-(3m-2)}{2}\right]Y_1-\frac{(c+1)(m+1)}{2}\eta(Y_1)\xi$$

$$Q\xi=\frac{(c+1)-4m}{2}\xi$$

*where $\tilde{R}$,$S$ are the Riemannian curvature tensor, Ricci curvature tensor of $\tilde{N}(c)$, respectively.*

Precisely, Ricci soliton on a Riemannian manifold $(\tilde{N},g)$ is defined as a triple $(g,\xi,\kappa_1)$ on $\tilde{N}$ satisfying

$$L_\xi g+2S+2\kappa_1 g=0,$$

where $L_\xi$ is the Lie derivative operator along the vector field $\xi$ and $\kappa_1$ is a real constant. We note that if $\xi$ is a Killing vector field, then the Ricci soliton reduces to an Einstein metric $(g,\kappa_1)$. Futhermore, in [20], generalization is the notion of $\eta-$Ricci soliton defined by J.T. Cho and M. Kimura as a quadruple $(g,\xi,\kappa_1,\kappa_2)$ satisfying

$$L_\xi g+2S+2\kappa_1 g+2\kappa_2\mu\eta\oplus\eta=0, \tag{2.6}$$

where $\kappa_1$ and $\kappa_2$ are real constants and $\eta$ is the dual of $\xi$ and $S$ denotes the Ricci tensor of $\tilde{N}$. Furthermore if $\kappa_1$ and $\kappa_2$ are smooth functions on $\tilde{N}$, then it called almost $\eta-$Ricci soliton on $\tilde{N}$ [20].

Suppose the quartet $(g,\xi,\kappa_1,\kappa_2)$ is almost $\eta-$Ricci soliton on manifold $\tilde{N}$. Then,

· If $\kappa_1<0$, then $\tilde{N}$ is shrinking.
· If $\kappa_1=0$, then $\tilde{N}$ is steady.
· If $\kappa_1>0$, then $\tilde{N}$ is expanding.

## 3. Almost $\eta-$Ricci Solitons on Ricci Pseudosymmetric and Ricci Semisymmetric Lorentz Sasakian Space Form

Now let $(g,\xi,\kappa_1,\kappa_2)$ be an almost $\eta-$Ricci soliton on Lorentz Sasakian space form. Then we have

$$(L_\xi g)(Y_1,Y_2)=L_\xi g(Y_1,Y_2)-g(L_\xi Y_1,Y_2)-g(Y_1,L_\xi Y_2)$$

$$=\xi g(Y_1,Y_2)-g([\xi,Y_1],Y_2)-g(Y_1,[\xi,Y_2])$$

$$=g(\nabla_\xi Y_1,Y_2)+g(Y_1,\nabla_\xi Y_2)-g(\nabla_\xi Y_1,Y_2)$$

$$+g(\nabla_{Y_1}\xi,Y_2)-g(\nabla_\xi Y_2,Y_1)+g(Y_1,\nabla_{Y_2}\xi),$$

for all $Y_1,Y_2\in\Gamma(TM)$. By using $\phi$ is anti-symmetric and taking into account (2.2) we have

$$(L_\xi g)(Y_1,Y_2)=0. \tag{3.1}$$

Thus, in a Lorentz Sasakian space form, from (2.6) and (3.1) we have

$$S(Y_1,Y_2)+\kappa_1 g(Y_1,Y_2)+\kappa_2\eta(Y_1)\eta(Y_2)=0. \tag{3.2}$$

It is clear from (3.2) that the $(2m+1)-$dimensional Lorentz Sasakian $\eta-$Ricci soliton $\left(\tilde{N}^{2m+1}, g, \xi, \kappa_1, \kappa_2\right)$ is an $\eta-$Einstein manifold.

For $Y_2 = \xi$ in (3.2) this implies that

$$S(\xi, Y_1) = (\kappa_1 - \kappa_2)\,\eta(Y_1). \tag{3.3}$$

Taking into account of (3.3) we conclude that

$$\kappa_1 - \kappa_2 = \frac{4m-(c+1)}{2}.$$

**Definition 3.1.** *Let $\tilde{N}(c)$ be an $(2m+1)-$dimensional Lorentz Sasakian space form. If $\tilde{R}\cdot S$ and $Q(g,S)$ are linearly dependent, then the $\tilde{N}(c)$ is said to be **Ricci pseudosymmetric**.*

In this case, there exists a function $L_1$ on $\tilde{N}(c)$ such that

$$\tilde{R}\cdot S = L_1 Q(g,S).$$

In particular, if $L_1 = 0$, the manifold $\tilde{N}(c)$ is said to be **Ricci semisymmetric**.

Let us now investigate the Ricci pseudosymmetry case of the $(2m+1)-$dimensional Lorentz Sasakian space form.

**Theorem 3.2.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a Ricci pseudosymmetric, then*

$$L_1 = \frac{2\kappa_1-(c+1)+4m}{4m-2\kappa_1-(c+1)},$$

*provided $2\kappa_1 \neq 4m-(c+1)$.*

*Proof.* Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. Then we have

$$\left(\tilde{R}(Y_1,Y_2)\cdot S\right)(Y_4,Y_5) = L_1 Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5 \in \Gamma\left(T\tilde{N}\right)$. From the last equation, we can easily write

$$\begin{aligned} & S\left(\tilde{R}(Y_1,Y_2)Y_4,Y_5\right) + S\left(Y_4,\tilde{R}(Y_1,Y_2)Y_5\right) \\ & = L_1\left\{S((Y_1\wedge_g Y_2)Y_4,Y_5) + S(Y_4,(Y_1\wedge_g Y_2)Y_5)\right\}. \end{aligned} \tag{3.4}$$

If we choose $Y_5 = \xi$ in (3.4) we get

$$\begin{aligned} & S\left(\tilde{R}(Y_1,Y_2)Y_4,\xi\right) + S\left(Y_4,\tilde{R}(Y_1,Y_2)\xi\right) \\ & = L_1\{S(g(Y_2,Y_4)Y_1 - g(Y_1,Y_4)Y_2,\xi) \\ & + S(Y_4,\eta(Y_1)Y_2 - \eta(Y_2)Y_1)\}. \end{aligned} \tag{3.5}$$

If we make use of (2.3) and (2.5) in (3.5) we have

$$\begin{aligned} & -\left[\tfrac{(c+1)-4m}{2}\right]\eta\left(\tilde{R}(Y_1,Y_2)Y_4\right) + S(Y_4,\eta(Y_2)Y_1 - \eta(Y_1)Y_2) \\ & = L_1\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1,Y_4)\right. \\ & + S(Y_4,\eta(Y_1)Y_2 - \eta(Y_2)Y_1)\}. \end{aligned} \tag{3.6}$$

If we use (2.4) in the (3.6), we get

$$\begin{aligned} & -\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1,Y_4) \\ & + S(\eta(Y_2)Y_1 - \eta(Y_1)Y_2,Y_4) \\ & = L_1\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1,Y_4)\right. \\ & + S(Y_4,\eta(Y_1)Y_2 - \eta(Y_2)Y_1)\}. \end{aligned} \tag{3.7}$$

If we use (3.2) in the (3.7), we can write

$$\left[\left(\kappa_1-\tfrac{(c+1)-4m}{2}\right)+\left(\kappa_1+\tfrac{(c+1)-4m}{2}\right)L_1\right]\times \\ g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)=0. \tag{3.8}$$

It is clear from (3.8)

$$L_1=\frac{2\kappa_1-(c+1)+4m}{4m-2\kappa_1-(c+1)}.$$

This completes the proof. □

Thus we have the following corollaries.

**Corollary 3.3.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a Ricci semisymmetric, then $\tilde{N}(c)$ is an $\eta-$Einstein manifold with $\kappa_1=\frac{(c+1)-4m}{2}$ and $\kappa_2=(c+1)-4m$.*

**Corollary 3.4.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a Ricci semisymmetric, then we observe that:*
*i) $\tilde{N}(c)$ is expanding, if $(c+1)>4m$.*
*ii) $\tilde{N}(c)$ is shrinking, if $(c+1)<4m$.*

For a $(2m+1)-$dimensional semi-Riemannian manifold $N$, the concircular curvature tensor is defined as

$$C\left(Y_1,Y_2\right)Y_3=R\left(Y_1,Y_2\right)Y_3-\frac{r}{2m\left(2m+1\right)}\left[g\left(Y_2,Y_3\right)Y_1-g\left(Y_1,Y_3\right)Y_2\right]. \tag{3.9}$$

For a $(2m+1)-$dimensional Lorentz Sasakian space form, if we choose $Y_3=\xi$ in (3.9) we can write

$$C\left(Y_1,Y_2\right)\xi=\left[1+\frac{r}{2m\left(2m+1\right)}\right]\left[\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2\right], \tag{3.10}$$

and similarly if we take the inner product of both sides of (3.9) by $\xi$, we get

$$\eta\left(C\left(Y_1,Y_2\right)Y_3\right)=\left[1+\frac{r}{2m\left(2m+1\right)}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_3\right). \tag{3.11}$$

**Definition 3.5.** *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $C\cdot S$ and $Q(g,S)$ are linearly dependent, then it is said to be **concircular Ricci pseudosymmetric.***

In this case, there exists a function $L_2$ on $\tilde{N}(c)$ such that

$$C\cdot S=L_2Q\left(g,S\right).$$

In particular, if $L_2=0$, the manifold $\tilde{N}(c)$ is said to be **concircular Ricci semisymmetric.**

Let us now investigate the concircular Ricci pseudosymmetry case of the Lorentz Sasakian space form.

**Theorem 3.6.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a concircular Ricci pseudosymmetric, then*

$$L_2=\frac{\left[2\kappa_1-(c+1)+4m\right]\left[2m\left(2m+1\right)+r\right]}{2m\left(2m+1\right)\left[4m-(c+1)-2\kappa_1\right]},$$

*provided $4m\neq 2\kappa_1+(c+1)$.*

*Proof.* Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be concircular Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. That is mean

$$\left(C\left(Y_1,Y_2\right)\cdot S\right)\left(Y_4,Y_5\right)=L_2Q\left(g,S\right)\left(Y_4,Y_5;Y_1,Y_2\right),$$

for all $Y_1, Y_2, Y_4, Y_5 \in \Gamma\left(T\tilde{N}\right)$. From the last equation, we can easily write

$$\begin{aligned}&S\left(C\left(Y_1,Y_2\right)Y_4,Y_5\right)+S\left(Y_4,C\left(Y_1,Y_2\right)Y_5\right)\\&=L_2\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}.\end{aligned} \tag{3.12}$$

If we choose $Y_5=\xi$ in (3.12) we get

$$\begin{aligned}&S\left(C\left(Y_1,Y_2\right)Y_4,\xi\right)+S\left(Y_4,C\left(Y_1,Y_2\right)\xi\right)\\&=L_2\left\{S\left(g\left(Y_2,Y_4\right)Y_1-g\left(Y_1,Y_4\right)Y_2,\xi\right)\right.\\&\left.+S\left(Y_4,\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1\right)\right\}.\end{aligned} \tag{3.13}$$

If by using (2.5) and (3.10) in (3.13) we have

$$\begin{aligned}&S\left(Y_4,\left[1+\tfrac{r}{2m(2m+1)}\right]\left[\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2\right]\right)\\&-\left[\tfrac{(c+1)-4m}{2}\right]\eta\left(C\left(Y_1,Y_2\right)Y_4\right)\\&=L_2\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right.\\&\left.+S\left(Y_4,\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1\right)\right\}.\end{aligned} \tag{3.14}$$

Substituting (3.11) in (3.14), we get

$$\begin{aligned}&-\left[\tfrac{(c+1)-4m}{2}\right]\left[1+\tfrac{r}{2m(2m+1)}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\\&+\left[1+\tfrac{r}{2m(2m+1)}\right]S\left(\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2,Y_4\right)\\&=L_2\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right.\\&\left.+S\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right\}.\end{aligned} \tag{3.15}$$

If we use (3.2) in the (3.15), we can write

$$\begin{aligned}&\left[\left(\kappa_1-\tfrac{(c+1)-4m}{2}\right)\left(1+\tfrac{r}{2m(2m+1)}\right)+\left(\kappa_1+\tfrac{(c+1)-4m}{2}\right)L_2\right]\times\\&g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)=0.\end{aligned}$$

This implies that

$$L_2=\frac{\left[2\kappa_1-(c+1)+4m\right]\left[2m(2m+1)+r\right]}{2m(2m+1)\left[4m-(c+1)-2\kappa_1\right]}.$$

This completes the proof. □

We can give the following corollaries.

**Corollary 3.7.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a concircular Ricci semisymmetric, then $\tilde{N}(c)$ is either manifold with scalar curvature $r=-2m(2m+1)$ or $\kappa_1=\frac{(c+1)-4m}{2}$.*

**Corollary 3.8.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a concircular Ricci semisymmetric, then we conclude that:*
*i) Let $r<2m(2m+1)$.*
*a) $\tilde{N}(c)$ is expanding, if $(c+1)>4m$.*
*b) $\tilde{N}(c)$ is shrinking, if $(c+1)<4m$.*
*ii) Let $r>2m(2m+1)$.*
*c) $\tilde{N}(c)$ is shrinking, if $(c+1)>4m$.*
*d)$\tilde{N}(c)$ is expanding, if $(c+1)<4m$.*

For a $(2m+1)-$dimensional semi-Riemannian manifold $N$, the projective curvature tensor is defined as

$$P(Y_1,Y_2)Y_3 = R(Y_1,Y_2)Y_3 - \frac{1}{2m}\left[S(Y_2,Y_3)Y_1 - S(Y_1,Y_3)Y_2\right]. \tag{3.16}$$

For a $(2m+1)-$dimensional Lorentz Sasakian space form, if we choose $Y_3=\xi$ in (3.16) we can write

$$P(Y_1,Y_2)\xi = \frac{c+1}{4m}\left[\eta(Y_2)Y_1 - \eta(Y_1)Y_2\right], \tag{3.17}$$

and in the same way if we take the inner product of both sides of (3.16) by $\xi$, we get

$$\eta(P(Y_1,Y_2)Y_3) = \frac{c+1}{4m}g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1, Y_3). \tag{3.18}$$

**Definition 3.9.** *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $P\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be* ***projective Ricci pseudosymmetric.***

In this case, there exists a function $L_3$ on $\tilde{N}(c)$ such that

$$P\cdot S = L_3 Q(g,S).$$

In particular, if $L_3=0$, the manifold $\tilde{N}(c)$ is said to be **projective Ricci semisymmetric.**

Let us now investigate the projective Ricci pseudosymmetry case of the Lorentz Sasakian space form.

**Theorem 3.10.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a projective Ricci pseudosymmetric, then*

$$L_3 = \frac{(c+1)\left[2\kappa_1-(c+1)+4m\right]}{2m\left[4m-(c+1)-2\kappa_1\right]},$$

*provided* $2\kappa_1 \neq 4m-(c+1)$.

*Proof.* Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be projective Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. That is mean

$$(P(Y_1,Y_2)\cdot S)(Y_4,Y_5) = L_3 Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5 \in \Gamma\left(T\tilde{N}\right)$. From the last equation, we can easily see

$$\begin{aligned} &S(P(Y_1,Y_2)Y_4,Y_5) + S(Y_4,P(Y_1,Y_2)Y_5) \\ &= L_3\left\{S((Y_1\wedge_g Y_2)Y_4,Y_5) + S(Y_4,(Y_1\wedge_g Y_2)Y_5)\right\}. \end{aligned} \tag{3.19}$$

If we choose $Y_5=\xi$ in (3.19) we get

$$\begin{aligned} &S(P(Y_1,Y_2)Y_4,\xi) + S(Y_4,P(Y_1,Y_2)\xi) \\ &= L_3\{S(g(Y_2,Y_4)Y_1 - g(Y_1,Y_4)Y_2,\xi) \\ &+S(Y_4,\eta(Y_1)Y_2 - \eta(Y_2)Y_1)\}. \end{aligned} \tag{3.20}$$

If we taking into account (2.5) and (3.17) in (3.20), then we have

$$\begin{aligned} &S\left(Y_4,\tfrac{c+1}{4m}\left[\eta(Y_2)Y_1 - \eta(Y_1)Y_2\right]\right) \\ &-\left[\tfrac{(c+1)-4m}{2}\right]\eta(P(Y_1,Y_2)Y_4) \\ &= L_3\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1,Y_4)\right. \\ &+S(Y_4,\eta(Y_1)Y_2 - \eta(Y_2)Y_1)\}. \end{aligned} \tag{3.21}$$

If we use (3.18) in the (3.21), we get

$$\begin{aligned} &-\left[\tfrac{(c+1)-4m}{2}\right]\left(\tfrac{c+1}{4m}\right)g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\\ &+\left(\tfrac{c+1}{4m}\right)S\left(\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2,Y_4\right)\\ &=L_3\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right.\\ &\left.+S\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right\}. \end{aligned} \tag{3.22}$$

If we use (3.2) in the (3.22), we taking into account

$$\begin{aligned} &\left[\left(\kappa_1-\tfrac{(c+1)-4m}{2}\right)\left(\tfrac{c+1}{4m}\right)+\left(\kappa_1+\tfrac{(c+1)-4m}{2}\right)L_3\right]\times\\ &g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)=0. \end{aligned} \tag{3.23}$$

It is clear from (3.23)

$$L_3=\frac{(c+1)\left[2\kappa_1-(c+1)+4m\right]}{2m\left[4m-(c+1)-2\kappa_1\right]}.$$

This completes the proof. □

We have the following corollaries.

**Corollary 3.11.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a projective Ricci semisymmetric, then $\tilde{N}(c)$ is either real space form with constant section curvature $c=-1$ or $\kappa_1=\frac{(c+1)-4m}{2}$.*

**Corollary 3.12.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a projective Ricci semisymmetric, then we conclude provided that $c+1\neq 0$:*
*i) The soliton $\tilde{N}(c)$ is expanding, if $(c+1)>4m$.*
*ii) The soliton $\tilde{N}(c)$ is shrinking, if $(c+1)<4m$.*

For a $(2m+1)-$dimensional semi-Riemannian manifold $N$, the $\mathscr{M}-$projective curvature tensor is defined as

$$\begin{aligned} &\mathscr{M}\left(Y_1,Y_2\right)Y_3=R\left(Y_1,Y_2\right)Y_3-\tfrac{1}{2m}\left[S\left(Y_2,Y_3\right)Y_1-S\left(Y_1,Y_3\right)Y_2\right.\\ &\left.+g\left(Y_2,Y_3\right)QY_1-g\left(Y_1,Y_3\right)QY_2\right] \end{aligned} \tag{3.24}$$

For a $(2m+1)-$dimensional Lorentz Sasakian space form, if we choose $Y_3=\xi$ in (3.24) we can write

$$\begin{aligned} &\mathscr{M}\left(Y_1,Y_2\right)\xi=\tfrac{c+1}{4m}\left[\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2\right]\\ &+\tfrac{1}{2m}\left[\eta\left(Y_2\right)QY_1-\eta\left(Y_1\right)QY_2\right]. \end{aligned} \tag{3.25}$$

On the other hand, if we take the inner product of both sides of (3.24) by $\xi$, we get

$$\begin{aligned} &\eta\left(\mathscr{M}\left(Y_1,Y_2\right)Y_3\right)=\tfrac{c+1}{4m}g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_3\right)\\ &-\tfrac{1}{2m}S\left(\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2,Y_3\right). \end{aligned} \tag{3.26}$$

**Definition 3.13.** *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $\mathscr{M}\cdot S$ and $Q(g,S)$ are linearly dependent, then it is said to be $\mathscr{M}-$**projective Ricci pseudosymmetric.***

In this case, there exists a function $L_4$ on $\tilde{N}(c)$ such that

$$\mathscr{M}\cdot S=L_4Q(g,S).$$

In particular, if $L_4=0$, the manifold $\tilde{N}(c)$ is said to be $\mathscr{M}-$**projective Ricci semisymmetric.**

Let us now investigate the $\mathscr{M}-$projective Ricci pseudosymmetric case of the Lorentz Sasakian space form admitting almost $\eta-$Ricci soliton.

**Theorem 3.14.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $\mathscr{M}-$projective Ricci pseudosymmetric, then*

$$L_4=\frac{4\kappa_1\left[(c+1)-2m\right]-(c+1)\left[(c+1)-4m\right]-4\kappa_1^2}{4m\left[2\kappa_1-(c+1)+4m\right]},$$

*provided $2\kappa_1\neq(c+1)-4m$.*

*Proof.* Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be $\mathscr{M}-$projective Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. That is mean

$$\left(\mathscr{M}(Y_1,Y_2)\cdot S\right)(Y_4,Y_5)=L_4Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(T\tilde{N}\right)$. From the last equation, we have

$$\begin{aligned}&S\left(\mathscr{M}(Y_1,Y_2)Y_4,Y_5\right)+S\left(Y_4,\mathscr{M}(Y_1,Y_2)Y_5\right)\\&=L_4\left\{S\left((Y_1\wedge_g Y_2)Y_4,Y_5\right)+S\left(Y_4,(Y_1\wedge_g Y_2)Y_5\right)\right\}.\end{aligned}\tag{3.27}$$

If we choose $Y_5=\xi$ in (3.27) we get

$$\begin{aligned}&S\left(\mathscr{M}(Y_1,Y_2)Y_4,\xi\right)+S\left(Y_4,\mathscr{M}(Y_1,Y_2)\xi\right)\\&=L_4\left\{S\left(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi\right)\right.\\&\left.+S\left(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1\right)\right\}.\end{aligned}\tag{3.28}$$

If we make use of (2.5) and (3.25) in (3.28), we have

$$\begin{aligned}&-\left[\tfrac{(c+1)-4m}{2}\right]\eta\left(\mathscr{M}(Y_1,Y_2)Y_4\right)\\&+S\left(Y_4,\tfrac{c+1}{4m}\left[\eta(Y_2)Y_1-\eta(Y_1)Y_2\right]\right.\\&\left.+\tfrac{1}{2m}\left[\eta(Y_2)QY_1-\eta(Y_1)QY_2\right]\right)\\&=L_4\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4\right)\right.\\&\left.+S\left(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1\right)\right\}.\end{aligned}\tag{3.29}$$

If we by using (3.26) in the (3.29), we get

$$\begin{aligned}&-\tfrac{(c+1)[(c+1)-4m]}{8m}g\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4\right)\\&+\tfrac{(c+1)-4m}{4m}S\left(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4\right)\\&+S\left(Y_4,\tfrac{c+1}{4m}\left[\eta(Y_2)Y_1-\eta(Y_1)Y_2\right]\right.\\&\left.+\tfrac{1}{2m}\left[\eta(Y_2)QY_1-\eta(Y_1)QY_2\right]\right)\\&=L_4\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4\right)\right.\\&\left.+S\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4\right)\right\}.\end{aligned}\tag{3.30}$$

If we use (3.2) in the (3.30), we can write

$$\begin{aligned}
&-\tfrac{(c+1)[(c+1)-4m]}{8m}g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\\
&-\tfrac{\kappa_1[(c+1)-4m]}{4m}g\left(\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2,Y_4\right)\\
&-\tfrac{\kappa_1(c+1)}{4m}g\left(Y_4,\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2\right)\\
&-\tfrac{\kappa_1}{2m}S\left(\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2,Y_4\right)\\
&=L_4\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right.\\
&\left.-\kappa_1 g\left(Y_4,\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right\}.
\end{aligned} \tag{3.31}$$

Again, if we use (3.2) in the (3.31), we obtain

$$\begin{aligned}
&\left[\tfrac{\kappa_1[(c+1)-4m]}{4m}+\tfrac{\kappa_1(c+1)}{4m}-\tfrac{(c+1)[(c+1)-4m]}{8m}\right.\\
&\left.-\tfrac{\kappa_1^2}{2m}+L_4\left(\tfrac{(c+1)-4m}{2}-\kappa_1\right)\right]\times\\
&g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)=0.
\end{aligned} \tag{3.32}$$

It is clear from (3.32)

$$L_4=\frac{4\kappa_1\left[(c+1)-2m\right]-(c+1)\left[(c+1)-4m\right]-4\kappa_1^2}{4m\left[2\kappa_1-(c+1)+4m\right]},$$

which proves our assertion □

We have the following corollaries.

**Corollary 3.15.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $\mathscr{M}-$projective Ricci semisymmetric, then*

$$\kappa_1=\frac{(c+1)-4m}{2},$$

*or*

$$\kappa_1=\frac{c+1}{2}.$$

**Corollary 3.16.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $\mathscr{M}-$projective Ricci semisymmetric, then we observe that:*
*i) $\tilde{N}(c)$ is shrinking, if $\kappa_1$ is between $\frac{(c+1)-4m}{2}$ and $\frac{c+1}{2}$,*
*ii) $\tilde{N}(c)$ is steady for $\kappa_1=\frac{(c+1)-4m}{2}$ and $\kappa_1=\frac{c+1}{2}$,*
*iii) $\tilde{N}(c)$ is expanding for other cases of $\kappa_1$.*

For a $(2m+1)-$dimensional semi-Riemannian manifold $N$, the $W_1-$curvature tensor is defined as

$$W_1\left(Y_1,Y_2\right)Y_3=R\left(Y_1,Y_2\right)Y_3+\frac{1}{2m}\left[S\left(Y_2,Y_3\right)Y_1-S\left(Y_1,Y_3\right)Y_2\right]. \tag{3.33}$$

For a $(2m+1)-$dimensional Lorentz Sasakian space form, if we choose $Y_3=\xi$ in (3.33), we can write

$$W_1\left(Y_1,Y_2\right)\xi=\frac{8m-(c+1)}{4m}\left[\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2\right], \tag{3.34}$$

and similarly if we take the inner product of both sides of (3.33) by $\xi$, we get

$$\eta\left(W_1\left(Y_1,Y_2\right)Y_3\right)=\frac{8m-(c+1)}{4m}g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_3\right). \tag{3.35}$$

**Definition 3.17.** *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $W_1 \cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be $W_1-$**Ricci pseudosymmetric.***

In this case, there exists a function $L_5$ on $\tilde{N}(c)$ such that

$$W_1 \cdot S = L_5 Q(g,S).$$

In particular, if $L_5 = 0$, the manifold $\tilde{N}(c)$ is said to be $W_1-$**Ricci semisymmetric.**

Let us now investigate the $W_1-$Ricci pseudosymmetric case of the Lorentz Sasakian space form.

**Theorem 3.18.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $W_1-$Ricci pseudosymmetric, then*

$$L_5 = \frac{[8m-(c+1)][2\kappa_1-(c+1)+4m]}{4m[4m-(c+1)-2\kappa_1]},$$

*provided* $2\kappa_1 \neq 4m-(c+1)$.

*Proof.* Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be $W_1-$Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. That is mean

$$(W_1(Y_1,Y_2)\cdot S)(Y_4,Y_5) = L_5 Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5 \in \Gamma\left(T\tilde{N}\right)$. From the last equation, we have

$$\begin{aligned} &S(W_1(Y_1,Y_2)Y_4,Y_5) + S(Y_4,W_1(Y_1,Y_2)Y_5) \\ &= L_5\left\{S((Y_1\wedge_g Y_2)Y_4,Y_5) + S(Y_4,(Y_1\wedge_g Y_2)Y_5)\right\}. \end{aligned} \tag{3.36}$$

If we choose $Y_5 = \xi$ in (3.36) we get

$$\begin{aligned} &S(W_1(Y_1,Y_2)Y_4,\xi) + S(Y_4,W_1(Y_1,Y_2)\xi) \\ &= L_5\{S(g(Y_2,Y_4)Y_1 - g(Y_1,Y_4)Y_2,\xi) \\ &+S(Y_4,\eta(Y_1)Y_2 - \eta(Y_2)Y_1)\}. \end{aligned} \tag{3.37}$$

If we make use of (2.5) and (3.34) in (3.37), we have

$$\begin{aligned} &S\left(Y_4,\tfrac{8m-(c+1)}{4m}[\eta(Y_2)Y_1 - \eta(Y_1)Y_2]\right) \\ &-\left[\tfrac{(c+1)-4m}{2}\right]\eta(W_1(Y_1,Y_2)Y_4) \\ &= L_5\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1,Y_4)\right. \\ &+S(Y_4,\eta(Y_1)Y_2 - \eta(Y_2)Y_1)\}. \end{aligned} \tag{3.38}$$

If we use (3.35) in the (3.38), we get

$$\begin{aligned} &\tfrac{[4m-(c+1)][8m-(c+1)]}{8m}g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1,Y_4) \\ &+\tfrac{8m-(c+1)}{4m}S(\eta(Y_2)Y_1 - \eta(Y_1)Y_2,Y_4) \\ &= L_5\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1,Y_4)\right. \\ &+S(\eta(Y_1)Y_2 - \eta(Y_2)Y_1,Y_4)\}. \end{aligned} \tag{3.39}$$

If we use (3.2) in the (3.39), we can write

$$\begin{aligned} &\left\{\tfrac{8m-(c+1)}{4m}\left[\kappa_1+\tfrac{4m-(c+1)}{2}\right]+L_5\left[\tfrac{(c+1)-4m}{2}+\kappa_1\right]\right\}\times \\ &g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)=0 \end{aligned} \tag{3.40}$$

It is clear from (3.40)

$$L_5=\frac{[8m-(c+1)]\left[2\kappa_1-(c+1)+4m\right]}{4m\left[4m-(c+1)-2\kappa_1\right]}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

**Corollary 3.19.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $W_1-$Ricci semisymmetric, then $\tilde{N}(c)$ is either real space form with $c=8m-1$ constant section curvature or $\kappa_1=\frac{(c+1)-4m}{2}$.*

**Corollary 3.20.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $W_1-$Ricci semisymmetric, then we conclude that:*
*i) Let $8m>c+1$.*
*a) $\tilde{N}(c)$ is expanding, if $(c+1)>4m$.*
*b) $\tilde{N}(c)$ is shrinking, if $(c+1)<4m$.*
*ii) Let $8m<c+1$.*
*c) $\tilde{N}(c)$ is shrinking, if $(c+1)>4m$.*
*d) $\tilde{N}(c)$ is expanding, if $(c+1)<4m$.*

For a $(2m+1)-$dimensional semi-Riemannian manifold $N$, the $W_2-$curvature tensor is defined as

$$W_2(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3-\frac{1}{2m}\left[g(Y_2,Y_3)QY_1-g(Y_1,Y_3)QY_2\right]. \tag{3.41}$$

For a $(2m+1)-$dimensional Lorentz Sasakian spacew form $\tilde{N}(c)$, if we choose $Y_3=\xi$ in (3.41), we can write

$$\begin{aligned} &W_2(Y_1,Y_2)\xi=\left[\eta(Y_2)Y_1-\eta(Y_1)Y_2\right] \\ &-\tfrac{1}{2m}\left[\eta(Y_1)QY_2-\eta(Y_2)QY_1\right]. \end{aligned} \tag{3.42}$$

Furthermore, if we take the inner product of both sides of (3.41) by $\xi$, we get

$$\begin{aligned} &\eta\left(W_2(Y_1,Y_2)Y_3\right)=g\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_3\right) \\ &+\tfrac{1}{2m}S\left(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_3\right). \end{aligned} \tag{3.43}$$

**Definition 3.21.** *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $W_2\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be $W_2-$**Ricci pseudosymmetric.***

In this case, there exists a function $L_6$ on $\tilde{N}(c)$ such that

$$W_2\cdot S=L_6Q(g,S).$$

In particular, if $L_6=0$, the manifold $\tilde{N}(c)$ is said to be $W_2-$**Ricci semisymmetric.**
Let us now investigate the $W_2-$Ricci pseudosymmetric of the Lorentz Sasakian space form.

**Theorem 3.22.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $W_2-$Ricci pseudosymmetric, then*

$$L_6=\frac{\kappa_1(1-2m)+m\left[(c+1)-4m\right]+\kappa_1^2}{m\left[2\kappa_1+(c+1)-4m\right]},$$

*provided $2\kappa_1\neq 4m-(c+1)$.*

*Proof.* Let be assume that Lorentz Sasakian space form be $W_2-$Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form. That is mean

$$(W_2(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_6Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned}&S(W_2(Y_1,Y_2)Y_4,Y_5)+S(Y_4,W_2(Y_1,Y_2)Y_5)\\&=L_6\left\{S((Y_1\wedge_g Y_2)Y_4,Y_5)+S(Y_4,(Y_1\wedge_g Y_2)Y_5)\right\}.\end{aligned}\tag{3.44}$$

If putting $Y_5=\xi$ in (3.44), we get

$$\begin{aligned}&S(W_2(Y_1,Y_2)Y_4,\xi)+S(Y_4,W_2(Y_1,Y_2)\xi)\\&=L_6\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\\&+S(Y_4,\eta(Y_2)Y_1-\eta(Y_1)Y_2)\}.\end{aligned}\tag{3.45}$$

If we make use of (2.5) and (3.42) in (3.45), we have

$$\begin{aligned}&-\left[\tfrac{(c+1)-4m}{2}\right]\eta(W_2(Y_1,Y_2)Y_4)\\&+S(Y_4,[\eta(Y_2)Y_1-\eta(Y_1)Y_2]\\&-\tfrac{1}{2m}[\eta(Y_1)QY_2-\eta(Y_2)QY_1])\\&=L_6\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right.\\&+S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}.\end{aligned}\tag{3.46}$$

If we use (3.43) in the (3.46), we get

$$\begin{aligned}&-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&+\tfrac{1}{2m}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&+S(Y_4,[\eta(Y_2)Y_1-\eta(Y_1)Y_2]\\&-\tfrac{1}{2m}[\eta(Y_1)QY_2-\eta(Y_2)QY_1]\\&=L_6\{S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\\&\left.-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right\}.\end{aligned}\tag{3.47}$$

If we use (3.2) in the (3.47), we have

$$\begin{aligned}&\left[\kappa_1-\tfrac{\kappa_1}{2m}-\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&+\tfrac{\kappa_1}{2m}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&=-L_6\left[\kappa_1+\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\end{aligned}\tag{3.48}$$

Again, if we use (3.2) in (3.48), we obtain

$$\begin{aligned}&\left[\kappa_1-\tfrac{\kappa_1}{2m}-\tfrac{(c+1)-4m}{2}-\tfrac{\kappa_1^2}{2m}\right.\\&\left.+L_6\left(\kappa_1+\tfrac{(c+1)-4m}{2}\right)\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\end{aligned}\tag{3.49}$$

It is clear from (3.49)

$$L_6 = \frac{\kappa_1(1-2m)+m[(c+1)-4m]+\kappa_1^2}{m[2\kappa_1+(c+1)-4m]}.$$

This completes the proof. □

We can give a result of this theorem as follows.

**Corollary 3.23.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $W_2-$ Ricci semisymmetric, then*

$$\kappa_1 = -\frac{1}{2}\left[-(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right],$$

*or*

$$\kappa_1 = \frac{1}{2}\left[(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right].$$

**Corollary 3.24.** *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $W_2-$Ricci semisymmetric, then we observe that*
*i) $\tilde{N}(c)$ is shrinking, if $\kappa_1$ is between $-\frac{1}{2}\left[-(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right]$ and $\frac{1}{2}\left[(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right]$,*
*ii) $\tilde{N}(c)$ is steady for $-\frac{1}{2}\left[-(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right]$*
*and $\frac{1}{2}\left[(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right]$,*
*iii) $\tilde{N}(c)$ is expanding for other cases of $\kappa_1$.*

## 4. Conclusion

In this paper, we consider pseudosymmetric Lorentz Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz Sasakian space forms admits $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, $W_1$ and $W_2$. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz Sasakian space form admits $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made under the some conditions.

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# Some Relations between Stieltjes Transform and Hankel Transform with Applications

Virendra Kumar[1]*

**Abstract**
In the present paper four theorems connecting Stieltjes transform and Hankel transform are established. The theorems are general in nature. Four integral formulae involving special functions are obtained with the help of these theorems. Otherwise it is very difficult to evaluate such type of integrals. Other several integrals may be evaluated with the help of these theorems.

**Keywords:** Bessel functions, Hankel transform, Stieltjes transform, Struve's functions
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*Department of Mathematics, Formerly Scientist-B, Defence Research and Development Organization, India, ORCID: 0000-0003-3597-1571*
***Corresponding author**: vkumar10147@yahoo.com
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## 1. Introduction

Several authors have made significant contributions for the development of integral transforms through a series of papers. Among other eminent authors, Bhonsle [1, 2], Sharma [5] Gupta and Agrawal [6], Goyal and Vasishta [7], Goyal and Jain [8], Saxena [14], Srivastava [15, 16, 18], Srivastava and Vyas [17], Srivastava and Tuan [19], Srivastava and Yürekli [20] and Yakubovich and Martins [21] have studied and explored Laplace, Meijer, Stieltjes, $H-$ function, Kontorovitch-Lebdev and Hankel transforms at large in the form of generalizations, convolution and interconnecting theorems.
Bhonsle [1, 2], Sharma [5], Saxena [14], Srivastava [15, 16], Srivastava and Vyas [17] have obtained integral formulae involving Legendre functions of the first kind, Bessel functions of the first kind and modified Bessel functions of the second kind.
In the present paper we have obtained four integral formulae involving Bessel functions of the first kind and second kind, modified Bessel functions of the first kind and second kind, Struve's functions and Anger functions.
Now, we define the Stieltjes transform and Hankel transform.

**Definition 1.1.** *The Stieltjes transform [4, 8, 19] of a function $f(x) \in L(0,\infty)$ is defined in the following manner.*

$$G(f;\, y) = \int_0^\infty (x+y)^{-1} f(x)dx,$$

*where $y$ is a complex variable.*

**Definition 1.2.** *The Hankel transform [4, 5, 16] of order $v$ of a function $f(x) \in L(0,\infty)$ is defined in the following manner.*

$$h_v(f;\, \zeta) = \int_0^\infty (\zeta x)^{1/2} J_v(\zeta x)\, f(x)dx, \quad \zeta > 0,$$

*where $J_v(z)$ stands for the Bessel function of the first kind ([3], Page 4, Equation (2)).*

## 2. Main Theorems

In this section we establish four theorems connecting Stieltjes transform and Hankel transform.

**Theorem 2.1.** *If $\zeta > 0$, $-1 <Re(v) < 1/2$ and $|arg\ y| < \pi$, then*

$$G\{x^{v+1/2}\ f(x);\ y\} = \int_0^\infty K(y,\ \zeta)\ h_v(f;\ \zeta)d\zeta, \tag{2.1}$$

*where*

$$K(y,\ \zeta) = 2^v \pi^{-1/2}\ \zeta^{-v-1/2}\ \Gamma(v+1/2) + \zeta^{1/2}\ 2^{-1}\pi\ y^{v+1} sec(v\pi)\ [Y_{-v}(\zeta y) - H_{-v}(\zeta y)],$$

*where $Y_{-v}(z)$ and $H_{-v}(z)$ stand for the Bessel function of the second kind ([3], Page 4, Equation (4)) and Struve's function ([3], Page 38, Equation (55)) respectively.*

*Proof.* We have by the Hankel inversion theorem [13] that

$$f(x) = \int_0^\infty (\zeta x)^{1/2}\ h_v(f;\ \zeta) J_v(\zeta x) d\zeta. \tag{2.2}$$

Hence

$$G\{x^{v+1/2}\ f(x);\ y\} = \int_0^\infty \zeta^{1/2}\ h_v(f;\ \zeta)\ G\{x^{v+1}\ J_v(\zeta x);\ y\} d\zeta. \tag{2.3}$$

The change of order of integration is justified because $\zeta > 0$, $-1 <\mathrm{Re}(v) < 1/2$ and $J_v(\zeta x)$ is a bounded function for both the variables for Landau's bounds [9] (see also [10]) i.e

$$|J_v(x)| \le b_L v^{-1/3}, \quad b_L := 2^{1/3} \sup_{x \in R_+} (\mathbf{Ai}(x)) \tag{2.4}$$

and

$$|J_v(x)| \le c_L |x|^{-1/3}, \quad c_L := \sup_{x \in R_+} (J_0(x)), \tag{2.5}$$

where $\mathbf{Ai}(x)$ stands for the familiar Airy function.
Now, using the following result ([4], Page 224, Equation (4)) in (2.3)

$$G\{x^{v+1}\ J_v(ax);\ y\} = 2^v \pi^{-1/2} a^{-v-1}\ \Gamma(v+1/2) + 2^{-1}\pi\ y^{v+1} sec(v\pi)[Y_{-v}(ay) - H_{-v}(ay)], \tag{2.6}$$

provided that $a > 0$, $-1 <\mathrm{Re}(v) < 1/2$ and $|arg\ y| < \pi$ we arrive at the desired result (2.1), where $\zeta > 0$, $-1 <\mathrm{Re}(v) < 1/2$ and $|arg\ y| < \pi$. □

**Theorem 2.2.** *If $\zeta > 0$, $Re(v) > -1$ and $|arg\ y| < \pi$, then*

$$G\{x^{-1/2}\ f(x);\ y\} = \int_0^\infty K(y,\ \zeta)\ h_v(f;\ \zeta)d\zeta, \tag{2.7}$$

*where*

$$K(y,\ \zeta) = \zeta^{1/2}\ \pi\ cosec(v\pi)\ [\mathbf{J}_v(\zeta y) - J_v(\zeta y)],$$

*where $\mathbf{J}_v(z)$ and $J_v(z)$ stand for the Anger's function ([3], Page 35, Equation (33)) and Bessel function of the first kind ([3], Page 4, Equation (2)) respectively.*

*Proof.* Again, by (2.2) we have that

$$G\{x^{-1/2}\ f(x);\ y\} = \int_0^\infty \zeta^{1/2}\ h_v(f;\ \zeta)\ G\{J_v(\zeta x);\ y\} d\zeta. \tag{2.8}$$

The change of order of integration is justified because $\zeta > 0$, $\mathrm{Re}(v) > -1$ and $J_v(\zeta x)$ is a bounded function for both the variables for Landau's bounds [9, 10] (see (2.4) and (2.5)).
Now, using the following result ([4], Page 224, Eq. (2)) in (2.8)

$$G\{J_v(ax);\ y\} = \pi\ cosec(v\pi)\ [\mathbf{J}_v(ay) - J_v(ay)],$$

provided that $a > 0$, $\mathrm{Re}(v) > -1$ and $|arg\ y| < \pi$ we arrive at the desired result (2.7), where $\zeta > 0$, $\mathrm{Re}(v) > -1$ and $|arg\ y| < \pi$. □

**Theorem 2.3.** *If* $0 < a < \zeta$, $-1 < Re(v) < 3/2$ *and* $|arg\ y| < \pi$, *then*

$$G\{x^{v/2-3/4}\ sin(ax^{1/2})\ f(x^{1/2});\ y\} = \int_0^\infty K(y,\ \zeta)\ h_v(f;\ \zeta)d\zeta, \tag{2.9}$$

*where*

$$K(y,\ \zeta) = 2\ \zeta^{1/2}\ y^{v/2-1/2}\ sinh(ay^{1/2})\ K_v(\zeta y^{1/2}),$$

*where* $K_v(z)$ *stands for the modified Bessel function of the second kind or Basset's function ([3], Page 5, Equation (13)).*

*Proof.* Again, by (2.2) we have that

$$G\{x^{v/2-3/4}\ sin(ax^{1/2})\ f(x^{1/2});\ y\} = \int_0^\infty \zeta^{1/2}\ h_v(f;\ \zeta)\ G\{x^{v/2-1/2}\ sin(ax^{1/2})\ J_v(\zeta x^{1/2});\ y\}d\zeta. \tag{2.10}$$

The change of order of integration is justified because $0 < a < \zeta$, $-1 < \mathrm{Re}(v) < 3/2$ and $J_v(\zeta x)$ is a bounded function for both the variables for Landau's bounds [9, 10] (see (2.4) and (2.5)).
Now, using the following result ([4], Page 226, Equation (18)) in (2.10)

$$G\{x^{v/2-1/2}\ sin(ax^{1/2})\ J_v(bx^{1/2});\ y\} = 2\ y^{v/2-1/2}\ sinh(ay^{1/2})K_v(by^{1/2}), \tag{2.11}$$

provided that $0 < a < b$, $-1 < \mathrm{Re}(v) < 3/2$ and $|arg\ y| < \pi$ we arrive at the desired result (2.9), where $0 < a < \zeta$, $-1 < \mathrm{Re}(v) < 3/2$ and $|arg\ y| < \pi$. □

**Theorem 2.4.** *If* $0 < \zeta < a$, $Re(v) > -1/2$ *and* $|arg\ y| < \pi$, *then*

$$G\{x^{-v/2-1/4}\ sin(ax^{1/2})\ f(x^{1/2});\ y\} = \int_0^\infty K(y,\ \zeta)\ h_v(f;\ \zeta)d\zeta, \tag{2.12}$$

*where*

$$K(y,\ \zeta) = \zeta^{1/2}\ \pi y^{-v/2}\ exp(-ay^{1/2})\ I_v(\zeta y^{1/2}),$$

*where* $I_v(z)$ *stands for the modified Bessel function of the first kind ([3], Page 5, Equation (12)).*

*Proof.* Again, by (2.2) we have that

$$G\{x^{-v/2-1/4}\ sin(ax^{1/2})\ f(x^{1/2});\ y\} = \int_0^\infty \zeta^{1/2}\ h_v(f;\ \zeta)\ G\{x^{-v/2}\ sin(ax^{1/2})\ J_v(\zeta x^{1/2});\ y\}d\zeta. \tag{2.13}$$

The change of order of integration is justified because $0 < \zeta < a$, $\mathrm{Re}(v) > -1/2$ and $J_v(\zeta x)$ is a bounded function for both the variables for Landau's bounds [9, 10] (see (2.4) and (2.5)).
Now, using the following result ([4], Page 226, Equation (19)) in (2.13)

$$G\{x^{-v/2}\ sin(ax^{1/2})\ J_v(bx^{1/2});\ y\} = \pi\ y^{-v/2}\ \exp(-ay^{1/2})\ I_v(by^{1/2}), \tag{2.14}$$

provided that $0 < b < a$, $\mathrm{Re}(v) > -1/2$ and $|arg\ y| < \pi$ we arrive at the desired result (2.12), where $0 < \zeta < a$, $\mathrm{Re}(v) > -1/2$ and $|arg\ y| < \pi$. □

## 3. Applications

In this section we make applications of our theorems to obtain integral formulae.

**Example 3.1.** *Let* $f(x) = x^{\mu-v+1/2}J_\mu(ax)$, $[a > 0,\ Re(v) > Re(\mu) > -1]$. *Then*

$$G\{x^{v+1/2}\ f(x);\ y\} = G\{x^{\mu+1}\ J_\mu(ax);\ y\}. \tag{3.1}$$

*Using the result (2.6) in (3.1), we get*

$$G\{x^{v+1/2}\ f(x);\ y\} = 2^\mu\ \pi^{-1/2}\ a^{-\mu-1}\ \Gamma(\mu+1/2) + 2^{-1}\pi\ y^{\mu+1} sec(\mu\pi)\ [Y_{-\mu}(ay) - H_{-\mu}(ay)], \tag{3.2}$$

*where* $a > 0$, $-1 < Re(\mu) < 1/2$ *and* $|arg\ y| < \pi$.
*Now, we have*

$$h_v(f;\ \zeta) = h_v\{x^{\mu-v+1/2}\ J_\mu(ax);\ \zeta\}. \tag{3.3}$$

*Using the following result ([4], Page 48, Equation (8)) in (3.3)*

$$h_v\{x^{\mu-v+1/2}\ J_\mu(ax);\ y\} = \frac{2^{\mu-v+1}a^\mu}{\Gamma(v-\mu)y^{v-1/2}}(y^2-a^2)^{v-\mu-1}, \tag{3.4}$$

*provided that* $Re(v) > Re(\mu) > -1$ *and* $0 < a < y < \infty$ *we get*

$$h_v(f;\ \zeta) = \frac{2^{\mu-v+1}a^\mu}{\Gamma(v-\mu)\zeta^{v-1/2}}(\zeta^2-a^2)^{v-\mu-1}, \tag{3.5}$$

*where* $Re(v) > Re(\mu) > -1$ *and* $0 < a < \zeta < \infty$.
*Now, using the results (3.2) and (3.5) in (2.1), we get*

$$\begin{aligned}\int_a^\infty [2^v\ \pi^{-1/2}\ \zeta^{-v-1/2}\ \Gamma(v+1/2) + \zeta^{1/2}\ 2^{-1}\pi\ \boldsymbol{sec}(v\pi)\ y^{v+1}\{Y_{-v}(\zeta y) - H_{-v}(\zeta y)\}]\ \zeta^{1/2-v}(\zeta^2-a^2)^{v-\mu-1}d\zeta \\ = 2^{v-1}\ \pi^{-1/2}\ a^{-2\mu-1}\ \Gamma(v-\mu) + \pi\ y^{\mu+1}\ 2^{v-\mu-2}a^{-\mu}\Gamma(v-\mu)\ sec(\mu\pi)\left[Y_{-\mu}(ay) - H_{-\mu}(ay)\right],\end{aligned} \tag{3.6}$$

*where* $a > 0$, $Re(v) > Re(\mu) > -1$, $Re(v-\mu) > 0$ *and* $|arg\ y| < \pi$.

**Example 3.2.** *Let* $f(x) = x^{v+1/2}$, $[0 < x < 1,\ Re(v) > -1]$. *Then*

$$G\{x^{-1/2}\ f(x);\ y\} = G\{x^v;\ y\}. \tag{3.7}$$

*Using the following result ([4], Page 216, Equation (5)) in (3.7)*

$$G\{x^v;\ y\} = -\pi\ y^v\ cosec(\pi v),$$

*where* $-1 < Re(v) < 0$ *and* $|arg\ y| < \pi$, *we get*

$$G\{x^{-1/2}\ f(x);\ y\} = -\pi\ y^v\ cosec(\pi v), \tag{3.8}$$

*where* $-1 < Re(v) < 0$ *and* $|arg\ y| < \pi$.
*Now, we have*

$$h_v(f;\ \zeta) = h_v\{x^{v+1/2};\ \zeta\}. \tag{3.9}$$

*Using the following result ([4], Page 22, Equation (6)) in (3.9)*

$$h_v\{x^{v+1/2};\ y\} = y^{-1/2}\ J_{v+1}(y),$$

*where* $0 < x < 1$, $Re(v) > -1$ *and* $y > 0$, *we get*

$$h_v(f;\ \zeta) = \zeta^{-1/2}\ J_{v+1}(\zeta), \tag{3.10}$$

*where* $0 < x < 1$, $Re(v) > -1$ *and* $\zeta > 0$.
*Now, using the results (3.8) and (3.10) in (2.7), we get*

$$\int_0^\infty [\boldsymbol{J}_v(\zeta y) - J_v(\zeta y)]J_{v+1}(\zeta)d\zeta = -y^v, \tag{3.11}$$

*where* $-1 < Re(v)$ *and* $|arg\ y| < \pi$.

**Example 3.3.** *Let* $f(x) = x^{\mu-v+1/2}J_\mu(bx)$, $[b > 0,\ Re(v) > Re(\mu) > -1]$. *Then*

$$f(x^{1/2}) = x^{\mu/2-v/2+1/4}J_\mu(bx^{1/2})$$

*and*

$$G\{x^{\nu/2-3/4}\ sin(ax^{1/2}) f(x^{1/2});\ y\} = G\{x^{\mu/2-1/2}\ sin(ax^{1/2})\ J_\mu(bx^{1/2});\ y\}. \tag{3.12}$$

*Using the result (2.11) in (3.12), we get*

$$G\{x^{\nu/2-3/4}\ sin(ax^{1/2}) f(x^{1/2});\ y\} = 2\ y^{\mu/2-1/2}\ sinh(ay^{1/2})\ K_\mu(by^{1/2}), \tag{3.13}$$

*where* $0 < a < b$, $-1 < Re(\mu) < 3/2$ *and* $|arg\ y| < \pi$.
*Now, we have*

$$h_\nu(f;\ \zeta) = h_\nu\{x^{\mu-\nu+1/2}\ J_\mu(bx);\ \zeta\}. \tag{3.14}$$

*Using the result (3.4) in (3.14), we get*

$$h_\nu(f;\ \zeta) = \frac{2^{\mu-\nu+1} b^\mu}{\Gamma(\nu-\mu)\zeta^{\nu-1/2}} (\zeta^2 - b^2)^{\nu-\mu-1}, \tag{3.15}$$

*where* $Re(\nu) > Re(\mu) > -1$ *and* $0 < b < \zeta < \infty$.
*Now, using the results (3.13) and (3.15) in (2.9), we get*

$$\int_b^\infty \zeta^{1-\nu}(\zeta^2-b^2)^{\nu-\mu-1} K_\nu(\zeta y^{1/2}) d\zeta = 2^{\nu-\mu-1} b^{-\mu} y^{\mu/2-\nu/2}\ \Gamma(\nu-\mu) K_\mu(by^{1/2}), \tag{3.16}$$

*where* $Re(\nu) > Re(\mu) > -1$, $Re(\nu-\mu) > 0$ *and* $|arg\ y| < \pi$.

**Example 3.4.** *Let* $f(x) = x^{\nu-\mu+1/2} J_\mu(bx)$, $[b > 0, -1 < Re(\nu) < Re(\mu)]$. *Then*

$$f(x^{1/2}) = x^{\nu/2-\mu/2+1/4} J_\mu(bx^{1/2})$$

*and*

$$G\{x^{-\nu/2-1/4}\ sin(ax^{1/2}) f(x^{1/2});\ y\} = G\{x^{-\mu/2}\ sin(ax^{1/2})\ J_\mu(bx^{1/2});\ y\}. \tag{3.17}$$

*Using the result (2.14) in (3.17), we get*

$$G\{x^{-\nu/2-1/4}\ sin(ax^{1/2}) f(x^{1/2});\ y\} = \pi\ y^{-\mu/2}\ \exp(-ay^{1/2})\ I_\mu(by^{1/2}), \tag{3.18}$$

*where* $0 < b < a$, $Re(\mu) > -1/2$ *and* $|arg\ y| < \pi$.
*Now, we have*

$$h_\nu(f;\ \zeta) = h_\nu\{x^{\nu-\mu+1/2}\ J_\mu(bx);\ \zeta\}. \tag{3.19}$$

*Using the following result ([4], Page 48, Equation (7)) in (3.19)*

$$h_\nu\{x^{\nu-\mu+1/2}\ J_\mu(ax);\ y\} = \frac{2^{\nu-\mu+1}\ y^{\nu+1/2}}{\Gamma(\mu-\nu)\ a^\mu}\ (a^2-y^2)^{\mu-\nu-1},$$

*provided that* $a > 0$, $-1 < Re(\nu) < Re(\mu)$ *and* $0 < y < a$ *we get*

$$h_\nu(f;\ \zeta) = \frac{2^{\nu-\mu+1} \zeta^{\nu+1/2}}{\Gamma(\mu-\nu)\ b^\mu} (b^2-\zeta^2)^{\mu-\nu-1}, \tag{3.20}$$

*where* $b > 0$, $-1 < Re(\nu) < Re(\mu)$ *and* $0 < \zeta < b$.
*Now, using the results (3.18) and (3.20) in (2.12), we get*

$$\int_0^b \zeta^{\nu+1}(b^2-\zeta^2)^{\mu-\nu-1} I_\nu(\zeta y^{1/2}) d\zeta = 2^{\mu-\nu-1} b^\mu y^{-\mu/2+\nu/2} \Gamma(\mu-\nu) I_\mu(by^{1/2}), \tag{3.21}$$

*where* $b > 0, -1 < Re(\nu) < Re(\mu)$, $Re(\mu-\nu) > 0$ *and* $|arg\ y| < \pi$.

## 4. Conclusion

Four integral formulae (3.6), (3.11), (3.16) and (3.21) involving special functions have been obtained with the help of the theorems established in this paper. Several other integral formulae extending the results given in [11, 12] may be obtained with the help of the theorems established in this paper and Stieltjes transforms available in [4].

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