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On The Almost $\eta-$Ricci Solitons On Pseudosymmetric Lorentz Generalized Sasakian Space Forms

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Abstract

In this paper, we consider Lorentz generalized Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz generalized Sasakian space forms admitting $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, W_1 and W_2. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz generalized Sasakian space form admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made.

1. Introduction

The notion of Ricci flow was introduced by Hamilton in 1982. With the help of this concept, Hamilton found the canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman used Ricci flow and it surgery to prove Poincare conjecture in [1,2]. The Ricci flow is a flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g(t) = -2S(g(t)).$$

A Ricci soliton emerges as the limit of the solitons of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling.
During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [3], Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Ashoka et al. in [4,5], Bagewadi et al. in [6], Ingalahalli in [7], Bejan and Crasmareanu in [8], Blaga in [9], Chandra et al. in [10], Chen and Deshmukh in [11], Deshmukh et al. in [12], He and Zhu [13], Atçeken et al. in [14], Nagaraja and Premalatta in [15], Tripathi in [16] and many others.
$\phi-$sectional curvature plays the important role for Sasakian manifold. If the $\phi-$sectional curvature of a Sasakian manifold is constant, then the manifold is a Sasakian-space-form [17]. P. Alegre and D. Blair described generalized Sasakian space forms [18]. P. Alegre and D. Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. P. Alegre and A. Carriazo later discussed generalized indefinite Sasakian space forms [19]. Generalized indefinite Sasakian space forms are also called Lorentz-Sasakian space forms, and Lorentz manifolds are of great importance for Einstein's theory of Relativity.
In this paper, we consider Lorentz generalized Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz generalized Sasakian space forms admitting $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, W_1 and W_2. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz generalized Sasakian space form admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made.

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2. Preliminaries

Let $\tilde{M}$ be a $(2n+1)-$dimensional semi-Riemannian manifold. If the $\tilde{M}$ semi-Riemannian manifold with (ϕ,ξ,η,g) structure tensors satisfies the following conditions, this manifold is called $\varepsilon-$ almost contact metric manifold and (ϕ,ξ,η) triple is called almost contact structure.

$$\begin{aligned}&\phi\xi=0,\quad \eta\circ\phi=0,\quad \eta(\xi)=1,\quad \phi^2=-Id+\eta\otimes\xi,\\ &g(Y_1,Y_2)=g(\phi Y_1,\phi Y_2)+\varepsilon\eta(Y_1)\eta(Y_2),\quad \eta(Y_1)=\varepsilon g(Y_1,\xi)\end{aligned}$$

where

$$\varepsilon=g(\xi,\xi)=\pm 1.$$

If $d\eta$ and g provide the relation

$$d\eta(Y_1,Y_2)=g(Y_1,\phi Y_2)$$

then $\tilde{M}$ is called a contact pseudometric manifold and the (ϕ,ξ,η) triple is called a contact structure.
Let be define a $\left(h\left(\frac{d}{dY_1}\right),Y_2\right)$ vector field on $\mathbb{R}\times\tilde{M}$, where Y_1 is a coordinate on $\mathbb{R}$ and h is a C^∞ function on $\mathbb{R}\times\tilde{M}$. The structure defined as

$$J\left(h\frac{d}{dY_1},Y_2\right)=\left(\eta(Y_2)\frac{d}{dY_1},\phi Y_2-h\xi\right)$$

on $\mathbb{R}\times\tilde{M}$ is called a almost complex structure and $J^2=-id$. If J is integrable, the almost contact structure (ϕ,ξ,η) is said to be normal. If Y_1 is perpendicular to ξ, the plane spanned by Y_1 and ϕY_1, is called the $\phi-$section. The curvature of the ϕ-section is called the ϕ-sectional curvature. The curvature of the indefinite Sasakian manifold defined in this way is precisely determined by the $\phi-$section curvature. If the $\phi-$section curvature of the indefinite Sasakian manifold is equal to a constant c, the curvature tensor of this manifold is defined as

$$\begin{aligned}\tilde{R}(Y_1,Y_2)Y_3=&\left(\frac{c+3\varepsilon}{4}\right)\{g(Y_2,Y_3)Y_1-g(Y_1,Y_3)Y_2\}+\left(\frac{c-\varepsilon}{4}\right)\{g(Y_1,\phi Y_3)\phi Y_2-g(Y_2,\phi Y_3)\phi Y_1+2g(Y_1,\phi Y_2)\phi Y_3\}\\ &+\left(\frac{c-\varepsilon}{4}\right)\{\eta(Y_1)\eta(Y_3)Y_2-\eta(Y_2)\eta(Y_3)Y_1+\varepsilon g(Y_1,Y_3)\eta(Y_2)\xi-\varepsilon g(Y_2,Y_3)\eta(Y_1)\xi\}.\end{aligned}$$

For an $\varepsilon-$almost contact metric manifold $\tilde{M}$, if there are $F_1,F_2,F_3\in C^\infty(\tilde{M})$ functions such that

$$\begin{aligned}\tilde{R}(Y_1,Y_2)Y_3=&F_1\{g(Y_2,Y_3)Y_1-g(Y_1,Y_3)Y_2\}+F_2\{g(Y_1,\phi Y_3)\phi Y_2-g(Y_2,\phi Y_3)\phi Y_1+2g(Y_1,\phi Y_2)\phi Y_3\}\\ &+F_3\{\eta(Y_1)\eta(Y_3)Y_2-\eta(Y_2)\eta(Y_3)Y_1+\varepsilon g(Y_1,Y_3)\eta(Y_2)\xi-\varepsilon g(Y_2,Y_3)\eta(Y_1)\xi\}\end{aligned}$$

then manifold $\tilde{M}$ is called a generalized indefinite Sasakian space form.
In this article, only the Lorentzian case, which corresponds to the $\varepsilon=-1$, where the index of the metric is 1, will be discussed. Such manifolds are called Lorentz generalized Sasakian space forms and are denoted by $M^{2n+1}(F_1,F_2,F_3)$. Thus, the curvature tensor of a $(2n+1)-$dimensional Lorentz generalized Sasakian space form is defined as

$$\begin{aligned}\tilde{R}(Y_1,Y_2)Y_3=&F_1\{g(Y_2,Y_3)Y_1-g(Y_1,Y_3)Y_2\}+F_2\{g(Y_1,\phi Y_3)\phi Y_2-g(Y_2,\phi Y_3)\phi Y_1+2g(Y_1,\phi Y_2)\phi Y_3\}\\ &+F_3\{\eta(Y_1)\eta(Y_3)Y_2-\eta(Y_2)\eta(Y_3)Y_1-g(Y_1,Y_3)\eta(Y_2)\xi+g(Y_2,Y_3)\eta(Y_1)\xi\}.\end{aligned}\tag{2.1}$$

Lemma 2.1. *Let $M^{2n+1}(F_1,F_2,F_3)$ be the $(2n+1)-$dimensional Lorentz generalized Sasakian space form. The following relations are provided for $M^{2n+1}(F_1,F_2,F_3)$.*

$$\tilde{\nabla}_{Y_1}\xi=(F_1+F_3)\phi Y_1,\tag{2.2}$$
$$\tilde{R}(Y_1,\xi)Y_3=-(F_1+F_3)[g(Y_1,Y_3)\xi+\eta(Y_3)Y_1],\tag{2.3}$$
$$\tilde{R}(\xi,Y_2)Y_3=(F_1+F_3)[g(Y_2,Y_3)\xi+\eta(Y_3)Y_2],\tag{2.4}$$
$$\tilde{R}(Y_1,Y_2)\xi=(F_1+F_3)[\eta(Y_1)Y_2-\eta(Y_2)Y_1],\tag{2.5}$$
$$\eta\left(\tilde{R}(Y_1,Y_2)Y_3\right)=(F_1+F_3)g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_3),\tag{2.6}$$
$$S(Y_1,Y_2)=(2nF_1+3F_2+F_3)g(Y_1,Y_2)+(3F_2-(2n-1)F_3)\eta(Y_1)\eta(Y_2),\tag{2.7}$$
$$S(Y_1,\xi)=-2n(F_1+F_3)\eta(Y_1),\tag{2.8}$$
$$QY_1=(2nF_1+3F_2+F_3)Y_1+((2n-1)F_3-3F_2),\tag{2.9}$$
$$Q\xi=2n(F_1+F_3)\xi,\tag{2.10}$$

where $\tilde{R},S$ and Q are the Riemann curvature tensor, Ricci curvature tensor and Ricci operator of $M^{2n+1}(F_1,F_2,F_3)$, respectively.

Let M be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1,...,X_K;Y_1,Y_2)=-T((Y_1\wedge_A Y_2)X_1,...,X_k)-...-T(X_1,...,X_{k-1},(Y_1\wedge_A Y_2)X_k),\tag{2.11}$$

where

$$(Y_1\wedge_A Y_2)Y_3=A(Y_2,Y_3)Y_1-A(Y_1,Y_3)Y_2,\tag{2.12}$$

$k \geq 1$, $X_1, X_2, ..., X_k, Y_1, Y_2 \in \Gamma(TM)$.
Precisely, a Ricci soliton on a Riemannian manifold $(\tilde{M}, g)$ is defined as a triple (g, ξ, λ) on $\tilde{M}$ satisfying

$$L_\xi g + 2S + 2\lambda g = 0 \tag{2.13}$$

where L_ξ is the Lie derivative operator along the vector field ξ and λ is a real constant. We note that if ξ is a Killing vector field, then the Ricci soliton reduces to an Einstein metric (g, λ). Futhermore, in [20], generalization is the notion of $\eta-$Ricci soliton defined by J.T. Cho and M. Kimura as a quadruple (g, ξ, λ, μ) satisfying

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \oplus \eta = 0 \tag{2.14}$$

where λ and μ are real constants and η is the dual of ξ and S denotes the Ricci tensor of $\tilde{M}$. Furthermore if λ and μ are smooth functions on $\tilde{M}$, then it called almost $\eta-$Ricci soliton on $\tilde{M}$ [20].
Suppose the quartet (g, ξ, λ, μ) is almost $\eta-$Ricci soliton on manifold $\tilde{M}$. Then,

- If $\lambda < 0$, then $\tilde{M}$ is shriking.
- If $\lambda = 0$, then $\tilde{M}$ is steady.
- If $\lambda > 0$, then $\tilde{M}$ is expanding.

3. Almost $\eta-$Ricci Solitons on Ricci Pseudosymmetric and Ricci Semisymmetric Lorentz Generalized Sasakian Space Forms

Now let (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form. Then we have

$$\begin{aligned}\left(L_\xi g\right)(Y_1, Y_2) =& L_\xi g(Y_1, Y_2) - g\left(L_\xi Y_1, Y_2\right) - g\left(Y_1, L_\xi Y_2\right)\\ =& \xi g(Y_1, Y_2) - g([\xi, Y_1], Y_2) - g(Y_1, [\xi, Y_2])\\ =& g\left(\nabla_\xi Y_1, Y_2\right) + g\left(Y_1, \nabla_\xi Y_2\right) - g\left(\nabla_\xi Y_1, Y_2\right) + g(\nabla_{Y_1}\xi, Y_2) - g\left(\nabla_\xi Y_2, Y_1\right) + g(Y_1, \nabla_{Y_2}\xi),\end{aligned}$$

for all $Y_1, Y_2 \in \Gamma(TM)$. By using ϕ is anti-symmetric, we have

$$\left(L_\xi g\right)(Y_1, Y_2) = 0. \tag{3.1}$$

Thus, in a Lorentz generalized Sasakian space form, from (2.14) and (3.1), we have

$$S(Y_1, Y_2) + \lambda g(Y_1, Y_2) + \mu\eta(Y_1)\eta(Y_2) = 0. \tag{3.2}$$

It is clear from (16) that the $(2n+1)-$dimensional Lorentz generalized Sasakian admitting almost $\eta-$Ricci soliton $\left(M^{2n+1}, g, \xi, \lambda, \mu\right)$ is an $\eta-$Einstein manifold.
For $Y_2 = \xi$ in (3.2), this implies that

$$S(\xi, Y_1) = (\lambda - \mu)\eta(Y_1). \tag{3.3}$$

Taking into account of (3.3), we conclude that

$$\mu - \lambda = 2n(F_1 + F_3).$$

Definition 3.1. *Let M^{2n+1} be an $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $\tilde{R} \cdot S$ and $Q(g, S)$ are linearly dependent, then the M^{2n+1} is said to be **Ricci pseudosymmetric**.*

In this case, there exists a function L_1 on M^{2n+1} such that

$$\tilde{R} \cdot S = L_1 Q(g, S).$$

In particular, if $L_1 = 0$, the manifold M^{2n+1} is said to be **Ricci semisymmetric**.
Let us now investigate the Ricci pseudosymmetric case of the $(2n+1)-$dimensional Lorentz generalized Sasakian space forms.

Theorem 3.2. *Let M^{2n+1} be Lorentz generalized Sasakian space forms and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a Ricci pseudosymmetric, then*

$$L_1 = \frac{(F_1 + F_3)[\lambda - 2n(F_1 + F_3)]}{\mu}$$

provided $\mu \neq 0$.

Proof. Let be assume that Lorentz generalized Sasakian space form M^{2n+1} be Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space forms M^{2n+1}. That is mean

$$\left(\tilde{R}(Y_1, Y_2) \cdot S\right)(Y_4, Y_5) = L_1 Q(g, S)(Y_4, Y_5; Y_1, Y_2),$$

for all $Y_1, Y_2, Y_4, Y_5 \in \Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S\left(\tilde{R}(Y_1, Y_2)Y_4, Y_5\right) + S\left(Y_4, \tilde{R}(Y_1, Y_2)Y_5\right) = L_1\left\{S\left(\left(Y_1 \wedge_g Y_2\right)Y_4, Y_5\right) + S\left(Y_4, \left(Y_1 \wedge_g Y_2\right)Y_5\right)\right\}. \tag{3.4}$$

If we choose $Y_5=\xi$ in (3.4), we get

$$S\left(\tilde{R}(Y_1,Y_2)Y_4,\xi\right)+S\left(Y_4,\tilde{R}(Y_1,Y_2)\xi\right)=L_1\left\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\right\}. \tag{3.5}$$

If we make use of (2.5) and (2.8) in (3.5), we have

$$\begin{aligned} &S(Y_4,(F_1+F_3)[\eta(Y_2)Y_1-\eta(Y_1)Y_2])-2n(F_1+F_3)\eta\left(\tilde{R}(Y_1,Y_2)Y_4\right)\\ &=L_1\left\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\right\}. \end{aligned} \tag{3.6}$$

If we use (2.6) in the (3.6), we get

$$\begin{aligned} &-2n(F_1+F_3)^2g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)+(F_1+F_3)S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\ &=L_1\left\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\right\}. \end{aligned}$$

If we use (3.2) in (3.5), we can write

$$[(F_1+F_3)[2n(F_1+F_3)-\lambda]+[\lambda+2n(F_1+F_3)]L_1]\times g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)=0. \tag{3.7}$$

It is clear from (3.7)

$$L_1=\frac{(F_1+F_3)[\lambda-2n(F_1+F_3)]}{\lambda+2n(F_1+F_3)}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 3.3. *Let M^{2n+1} be a Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a Ricci semisymmetric, then $\lambda=2n(F_1+F_3)$ and $\mu=4n(F_1+F_3)$.*

Corollary 3.4. *Let M^{2n+1} be a Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a Ricci semisymmetric, then M^{2n+1} is an $\eta-$Einstein manifold.*

Corollary 3.5. *Let M^{2n+1} be a Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a Ricci semisymmetric, then we observe that:*

(i) M^{2n+1} is expanding, if $F_1+F_3>0$.
(ii) M^{2n+1} is shriking, if $F_1+F_3<0$.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the concircular curvature tensor is defined as

$$C(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3-\frac{r}{2n(2n+1)}[g(Y_2,Y_3)Y_1-g(Y_1,Y_3)Y_2]. \tag{3.8}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.8), we can write

$$C(Y_1,Y_2)\xi=\left[(F_1+F_3)-\frac{r}{2n(2n+1)}\right][\eta(Y_1)Y_2-\eta(Y_2)Y_1], \tag{3.9}$$

and similarly if we take the inner product of both sides of (24) by ξ, we get

$$\eta(C(Y_1,Y_2)Y_3)=\left[(F_1+F_3)-\frac{r}{2n(2n+1)}\right]g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_3). \tag{3.10}$$

Definition 3.6. *Let M^{2n+1} be a $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $C\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be **concircular Ricci pseudosymmetric.***

In this case, there exists a function L_2 on M^{2n+1} such that

$$C\cdot S=L_2Q(g,S).$$

In particular, if $L_2=0$, the manifold M^{2n+1} is said to be **concircular Ricci semisymmetric.**
Let us now investigate the concircular Ricci pseudosymmetric case of the Lorentz generalized Sasakian space form.

Theorem 3.7. *Let M^{2n+1} be a Lorentz generalized Sasakian space forms and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a concircular Ricci pseudosymmetric, then*

$$L_2=\frac{[\lambda-2n(F_1+F_3)][2n(2n+1)(F_1+F_3)-r]}{2n(2n+1)\mu},$$

provided $\mu\neq 0$.

Proof. Let be assume that Lorentz generalized Sasakian space form M^{2n+1} be concircular Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form M^{2n+1}. That is mean

$$(C(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_2Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(C(Y_1,Y_2)Y_4,Y_5)+S(Y_4,C(Y_1,Y_2)Y_5)=L_2\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}. \tag{3.11}$$

If we choose $Y_5=\xi$ in (3.11), we get

$$S(C(Y_1,Y_2)Y_4,\xi)+S(Y_4,C(Y_1,Y_2)\xi)=L_2\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \tag{3.12}$$

By using of (2.8) and (3.9) in (3.12), we have

$$\begin{aligned}&S(Y_4,A[\eta(Y_1)Y_2-\eta(Y_2)Y_1])-2n(F_1+F_3)\eta(C(Y_1,Y_2)Y_4)\\&=L_2\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\},\end{aligned} \tag{3.13}$$

where $A=(F_1+F_3)-\frac{r}{2n(2n+1)}$. Substituting (3.10) into (3.13), we have

$$\begin{aligned}&-2n(F_1+F_3)Ag(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)+AS(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&=L_2\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\}.\end{aligned} \tag{3.14}$$

If we use (3.2) in the (3.14), we can write

$$\{A[2n(F_1+F_3)-\lambda]+[\lambda+2n(F_1+F_3)]L_2\}\times g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)=0. \tag{3.15}$$

It is clear from (3.15),

$$L_2=\frac{[\lambda-2n(F_1+F_3)][2n(2n+1)(F_1+F_3)-r]}{2n(2n+1)\mu}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 3.8. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a concircular Ricci semisymmetric, then M^{2n+1} is either manifold with scalar curvature $r=2n(2n+1)(F_1+F_3)$ or $\lambda=2n(F_1+F_3)$.*

Corollary 3.9. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a concircular Ricci semisymmetric, then we observe that:*

(i) The soliton M^{2n+1} is expanding, if $(F_1+F_3)>0$.
(ii) The soliton M^{2n+1} is shriking, if $(F_1+F_3)<0$.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the projective curvature tensor is defined as

$$P(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3-\frac{1}{2n}[S(Y_2,Y_3)Y_1-S(Y_1,Y_3)Y_2]. \tag{3.16}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.16) we can write

$$P(Y_1,Y_2)\xi=0, \tag{3.17}$$

and similarly if we take the inner product of both sides of (3.16) by ξ, we get

$$\eta(P(Y_1,Y_2)Y_3)=0. \tag{3.18}$$

Definition 3.10. *Let M^{2n+1} be an $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $P\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be **projective Ricci pseudosymmetric.***

In this case, there exists a function L_3 on M^{2n+1} such that

$$P\cdot S=L_3Q(g,S).$$

In particular, if $L_3=0$, the manifold M^{2n+1} is said to be **projective Ricci semisymmetric.**
Let us now investigate the projective Ricci pseudosymmetry case of the Lorentz generalized Sasakian space form.

Theorem 3.11. *Let M^{2n+1} be Lorentz Sasakian space forms and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a projective Ricci pseudosymmetric, then M^{2n+1} is either projective Ricci semisymmetric or almost $\eta-$Ricci soliton (g,ξ,λ,μ) reduces almost Ricci soliton (g,ξ,λ).*

Proof. Let be assume that Lorentz generalized Sasakian space form M^{2n+1} be projective Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form M^{2n+1}. Then we have

$$(P(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_3Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(P(Y_1,Y_2)Y_4,Y_5)+S(Y_4,P(Y_1,Y_2)Y_5)=L_3\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}. \tag{3.19}$$

If we choose $Y_5=\xi$ in (3.19), we get

$$S(P(Y_1,Y_2)Y_4,\xi)+S(Y_4,P(Y_1,Y_2)\xi)=L_3\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \tag{3.20}$$

If we make use of (2.8) and (3.17) in (3.20) we have

$$-2n(\digamma_1+\digamma_3)\eta(P(Y_1,Y_2)Y_4)=L_3\{-2n(\digamma_1+\digamma_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \tag{3.21}$$

If we use (3.18) in the (3.21), we get

$$L_3\{-2n(\digamma_1+\digamma_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\}=0. \tag{3.22}$$

If we use (3.2) in the (3.22), we can write

$$L_3[\lambda+2n(\digamma_1+\digamma_3)]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)=0. \tag{3.23}$$

It is clear from (3.23),

$$\mu L_3=0.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 3.12. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a projective Ricci pseudosymmetric, then M^{2n+1} is either projective Ricci semisymmetric or we observe that:*

(i) The soliton M^{2n+1} is expanding, if $\digamma_1+\digamma_3<0$.
(ii) The soliton M^{2n+1} is shriking, if $\digamma_1+\digamma_3>0$.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the $\mathscr{M}-$projective curvature tensor is defined as

$$\mathscr{M}(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3-\tfrac{1}{2n}[S(Y_2,Y_3)Y_1-S(Y_1,Y_3)Y_2\ +g(Y_2,Y_3)QY_1-g(Y_1,Y_3)QY_2] \tag{3.24}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.24), we obtain

$$\mathscr{M}(Y_1,Y_2)\xi=\frac{1}{2n}[\eta(Y_2)QY_1-\eta(Y_1)QY_2], \tag{3.25}$$

and similarly if we take the inner product of both of sides of (3.24) by ξ, we get

$$\eta(\mathscr{M}(Y_1,Y_2)Y_3)=\frac{1}{2n}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_3). \tag{3.26}$$

Definition 3.13. *Let M^{2n+1} be an $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $\mathscr{M}\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be $\mathscr{M}-$**projective Ricci pseudosymmetric.***

In this case, there exists a function L_4 on M^{2n+1} such that

$$\mathscr{M}\cdot S=L_4Q(g,S).$$

In particular, if $L_4=0$, the manifold M^{2n+1} is said to be $\mathscr{M}-$**projective Ricci semisymmetric.**
Let us now investigate the $\mathscr{M}-$projective Ricci pseudosymmetric case of the Lorentz generalized Sasakian space form.

Theorem 3.14. *Let M^{2n+1} be Lorentz generalized Sasakian space forms and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a $\mathscr{M}-$projective Ricci pseudosymmetric, then*

$$L_4=\frac{\lambda^2-2n(\digamma_1+\digamma_3)\lambda}{2n\mu},$$

provided $\mu\neq 0$.

Proof. Let be assume that Lorentz generalized Sasakian space form M^{2n+1} be projective $\mathscr{M}-$projective Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form M^{2n+1}. That is mean

$$(\mathscr{M}(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_4Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(\mathscr{M}(Y_1,Y_2)Y_4,Y_5)+S(Y_4,\mathscr{M}(Y_1,Y_2)Y_5)=L_4\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}. \tag{3.27}$$

If we choose $Y_5=\xi$ in (3.27) we get

$$S(\mathscr{M}(Y_1,Y_2)Y_4,\xi)+S(Y_4,\mathscr{M}(Y_1,Y_2)\xi)=L_4\left\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\right\}. \tag{3.28}$$

If we make use of (2.8) and (3.25) in (3.28), we have

$$\begin{aligned}&-2n(F_1+F_3)\eta(\mathscr{M}(Y_1,Y_2)Y_4)+S\left(Y_4,\frac{1}{2n}[\eta(Y_2)QY_1-\eta(Y_1)QY_2]\right)\\&=L_4\left\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\right\}.\end{aligned} \tag{3.29}$$

By using (3.26) in the (3.29), we get

$$\begin{aligned}&-(F_1+F_3)S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)+\frac{1}{2n}S(\eta(Y_2)QY_1-\eta(Y_1)QY_2,Y_4)\\&=L_4\left\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right\}.\end{aligned} \tag{3.30}$$

If we put (3.2) in (3.30), we can write

$$\begin{aligned}&\lambda(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)-\frac{\lambda}{2n}S(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)\\&=L_4[\lambda+2n(F_1+F_3)]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\end{aligned} \tag{3.31}$$

Again, if we use (3.2) in the (3.31), we obtain

$$\left\{\tfrac{\lambda^2}{2n}-(F_1+F_3)\lambda-L_4[\lambda+2n(F_1+F_3)]\right\}\times g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)=0. \tag{3.32}$$

It is clear from (3.32),

$$L_4=\frac{\lambda^2-2n(F_1+F_3)\lambda}{2n[2n(F_1+F_3)+\lambda]}.$$

This completes the proof. $\square$

We can give the following corollaries.

Corollary 3.15. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a $\mathscr{M}-$projective Ricci semisymmetric, then M^{2n+1} is either steady or $\eta-$Einstein manifold.*

Corollary 3.16. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a $\mathscr{M}-$projective Ricci semisymmetric, then M^{2n+1} is either steady or we observe that:*

(i) The soliton M^{2n+1} is shriking if λ is between 0 and $2n(F_1+F_3)$.
(ii) The soliton M^{2n+1} is steady if $\lambda=0$.
(iii) The soliton M^{2n+1} is expanding for other cases of λ.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the W_1-curvature tensor is defined as

$$W_1(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3+\frac{1}{2n}[S(Y_2,Y_3)Y_1-S(Y_1,Y_3)Y_2]. \tag{3.33}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.33), we can write

$$W_1(Y_1,Y_2)\xi=2(F_1+F_3)[\eta(Y_1)Y_2-\eta(Y_2)Y_1], \tag{3.34}$$

and similarly if we take the inner product of both of sides of (3.33) by ξ, we get

$$\eta(W_1(Y_1,Y_2)Y_3)=2(F_1+F_3)g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_3). \tag{3.35}$$

Definition 3.17. *Let M^{2n+1} be a $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $W_1\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_1-**Ricci pseudosymmetric.***

In this case, there exists a function L_5 on M^{2n+1} such that

$$W_1\cdot S=L_5Q(g,S).$$

In particular, if $L_5=0$, the manifold M^{2n+1} is said to be W_1-**Ricci semisymmetric.**
Let us now investigate the W_1-Ricci pseudosymmetric case of the Lorentz generalized Sasakian space form.

Theorem 3.18. *Let M^{2n+1} be Lorentz generalized Sasakian space forms and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1-Ricci pseudosymmetric, then*

$$L_5=\frac{2(F_1+F_3)[\lambda-2n(F_1+F_3)]}{\mu}$$

provided $\mu\neq 0$.

Proof. Let be assume that Lorentz Sasakian space form M^{2n+1} be W_1-Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form M^{2n+1}. That is mean

$$(W_1(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_5Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(W_1(Y_1,Y_2)Y_4,Y_5)+S(Y_4,W_1(Y_1,Y_2)Y_5)=L_5\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}. \tag{3.36}$$

If we choose $Y_5=\xi$ in (3.36), we get

$$S(W_1(Y_1,Y_2)Y_4,\xi)+S(Y_4,W_1(Y_1,Y_2)\xi)=L_5\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \tag{3.37}$$

If we make use of (2.8) and (3.34) in (3.37) we have

$$\begin{aligned}&2(F_1+F_3)S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)-2n(F_1+F_3)\eta(W_1(Y_1,Y_2)Y_4)\\&=L_5\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}.\end{aligned} \tag{3.38}$$

If we use (3.35) in the (3.38), we get

$$\begin{aligned}&-4n(F_1+F_3)^2g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)+2(F_1+F_3)S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&=L_5\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\}.\end{aligned} \tag{3.39}$$

If we use (3.2) in the (3.39), we can write

$$\{2(F_1+F_3)[2n(F_1+F_3-\lambda)]+L_5[\lambda+2n(F_1+F_3)]\}\times g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)=0 \tag{3.40}$$

It is clear from (3.40),

$$L_5=\frac{2(F_1+F_3)[\lambda-2n(F_1+F_3)]}{\lambda+2n(F_1+F_3)}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 3.19. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1-Ricci semisymmetric, then $\lambda=2n(F_1+F_3)$ provided $\mu\neq 0$.*

Corollary 3.20. *Let M^{2n+1} be Lorentz generalized Sasakian space form and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_1-Ricci semisymmetric, then we observe that:*

(i) The soliton M^{2n+1} is expanding, if $(F_1+F_3)>0$.
(ii) The soliton M^{2n+1} is shriking, if $(F_1+F_3)<0$.

For a $(2n+1)-$dimensional semi-Riemann manifold M, the W_2-curvature tensor is defined as

$$W_2(Y_1,Y_2)Y_3=R(Y_1,Y_2)Y_3-\frac{1}{2n}[g(Y_2,Y_3)QY_1-g(Y_1,Y_3)QY_2]. \tag{3.41}$$

For a $(2n+1)-$dimensional Lorentz generalized Sasakian space form, if we choose $Y_3=\xi$ in (3.41), we can write

$$W_2(Y_1,Y_2)\xi=(F_1+F_3)[\eta(Y_1)Y_2-\eta(Y_2)Y_1]-\tfrac{1}{2n}[\eta(Y_1)QY_2-\eta(Y_2)QY_1], \tag{3.42}$$

and similarly if we take the inner product of both sides of (3.41) by ξ, we get

$$\eta(W_2(Y_1,Y_2)Y_3)=(F_1+F_3)g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_3)+\tfrac{1}{2n}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_3). \tag{3.43}$$

Definition 3.21. *Let M^{2n+1} be an $(2n+1)-$dimensional Lorentz generalized Sasakian space form. If $W_2\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_2-**Ricci pseudosymmetric.***

In this case, there exists a function L_6 on M^{2n+1} such that

$$W_2\cdot S=L_6Q(g,S).$$

In particular, if $L_6=0$, the manifold M^{2n+1} is said to be W_2-**Ricci semisymmetric.**
Let us now investigate the W_2-Ricci pseudosymmetric case of the Lorentz generalized Sasakian space form.

Theorem 3.22. *Let M^{2n+1} be Lorentz generalized Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on M^{2n+1}. If M^{2n+1} is a W_2-Ricci pseudosymmetric, then*

$$L_6=-\frac{\lambda^2+4n^2(F_1+F_3)^2}{2n\mu}$$

provided $\mu\neq 0$.

Proof. Let be assume that Lorentz generalized Sasakian space form be W_2-Ricci pseudosymmetric and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on Lorentz generalized Sasakian space form. That is mean

$$(W_2(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_6Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma\left(TM^{2n+1}\right)$. From the last equation, we can easily write

$$S(W_2(Y_1,Y_2)Y_4,Y_5)+S(Y_4,W_2(Y_1,Y_2)Y_5)=L_6\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}. \tag{3.44}$$

If we choose $Y_5=\xi$ in (3.44), we get

$$S(W_2(Y_1,Y_2)Y_4,\xi)+S(Y_4,W_2(Y_1,Y_2)\xi)=L_6\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\ +S(Y_4,\eta(Y_2)Y_1-\eta(Y_1)Y_2)\}. \tag{3.45}$$

If we make use of (2.8) and (3.42) in (3.45), we have

$$\begin{aligned}&-2n(F_1+F_3)\eta(W_2(Y_1,Y_2)Y_4)+S\left(Y_4,(F_1+F_3)[\eta(Y_1)Y_2-\eta(Y_2)Y_1]-\frac{1}{2n}[\eta(Y_1)QY_2-\eta(Y_2)QY_1]\right)\\&=L_6\{-2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\ +S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}.\end{aligned} \tag{3.46}$$

If we use (3.43) in the (3.46), we get

$$\begin{aligned}&-2n(F_1+F_3)^2g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)-\frac{1}{2n}S(Y_4,\eta(Y_1)QY_2-\eta(Y_2)QY_1)\\&=L_6\{S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\ -2n(F_1+F_3)g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\}.\end{aligned} \tag{3.47}$$

If we use (3.2) in the (3.47), we have

$$\begin{aligned}&-2n(F_1+F_3)^2g(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4)-\frac{\lambda}{2n}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&=-L_6[\lambda+2n(F_1+F_3)]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\end{aligned} \tag{3.48}$$

Again, if we use (3.2) in (3.48), we obtain

$$\left\{\tfrac{\lambda^2}{2n}+2n(F_1+F_3)^2+L_6[\lambda+2n(F_1+F_3)]\right\}\times g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)=0. \tag{3.49}$$

It is clear from (3.49),

$$L_6=-\frac{\lambda^2+4n^2(F_1+F_3)^2}{2n[\lambda+2n(F_1+F_3)]}.$$

This completes the proof. □

4. Conclusion

In this paper, we consider Lorentz generalized Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz generalized Sasakian space forms admitting $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, W_1 and W_2. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz generalized Sasakian space form admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made.

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References

[1] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Mathematics, (2002), 1–39.
[2] G. Perelman, *Ricci flow with surgery on three manifolds*, http://arXiv.org/abs/math/0303109, (2003), 1–22.
[3] R. Sharma, *Certain results on k-contact and* $(k,\mu)-$*contact manifolds*, J. Geom., **89** (2008), 138–147.
[4] S. R. Ashoka, C. S. Bagewadi, G. Ingalahalli, *Certain results on Ricci Solitons in* $\alpha-$*Sasakian manifolds*, Hindawi Publ. Corporation, Geometry, **2013**, Article ID 573925, (2013), 4 Pages.
[5] S. R. Ashoka, C. S. Bagewadi, G. Ingalahalli, *A geometry on Ricci solitons in* $(LCS)_n$ *manifolds*, Diff. Geom.-Dynamical Systems, **16** (2014), 50–62.
[6] C. S. Bagewadi, G. Ingalahalli, *Ricci solitons in Lorentzian-Sasakian manifolds,* Acta Math. Acad. Paeda. Nyire., **28** (2012), 59–68.
[7] G. Ingalahalli, C. S. Bagewadi, *Ricci solitons in* $\alpha-$*Sasakian manifolds*, ISRN Geometry, **2012**, Article ID 421384, (2012), 13 Pages.
[8] C. L. Bejan, M. Crasmareanu, *Ricci Solitons in manifolds with quasi-contact curvature*, Publ. Math. Debrecen, **78** (2011), 235–243.
[9] A. M. Blaga, $\eta-$*Ricci solitons on para-kenmotsu manifolds*, Balkan J. Geom. Appl., **20** (2015), 1–13.
[10] S. Chandra, S. K. Hui, A. A. Shaikh, *Second order parallel tensors and Ricci solitons on* $(LCS)_n$*-manifolds*, Commun. Korean Math. Soc., **30** (2015), 123–130.
[11] B. Y. Chen, S. Deshmukh, *Geometry of compact shrinking Ricci solitons*, Balkan J. Geom. Appl., **19** (2014), 13–21.
[12] S. Deshmukh, H. Al-Sodais, H. Alodan, *A note on Ricci solitons, Balkan J. Geom. Appl.*, **16** (2011), 48–55.
[13] C. He, M. Zhu, *Ricci solitons on Sasakian manifolds*, arxiv:1109.4407V2, [Math DG], (2011).
[14] M. Atçeken, T. Mert, P. Uygun, *Ricci-Pseudosymmetric* $(LCS)_n-$*manifolds admitting almost* $\eta-$*Ricci solitons*, Asian Journal of Math. and Computer Research, **29**(2), (2022), 23–32.
[15] H. Nagaraja, C. R. Premalatta, *Ricci solitons in Kenmotsu manifolds*, J. Math. Analysis, **3(2)**, (2012), 18–24.
[16] M. M. Tripathi, *Ricci solitons in contact metric manifolds*, arxiv:0801,4221 V1, [Math DG], (2008).
[17] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Volume 203 of Progress in Mathematics, Birkhauser Boston, Inc., Boston, MA, USA, 2nd edition, 2010.
[18] P. Alegre, D.E. Blair, A. Carriazo, *Generalized Sasakian space form*, Israel Journal of Mathematics, **141** (2004), 157–183.
[19] P. Alegre, A. Carriazo, *Semi-Riemannian generalized Sasakian space forms*, Bulletin of the Malaysian Mathematical Sciences Society, **41**(1) (2001), 1–14.
[20] J. T Cho, M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J., **61** (2) (2009), 205–212.

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Analytical Rational Solitons of the Modified Lakshmanan-Porsezian-Daniel Equation

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Abstract

In this paper, the Lakshmanan-Porsezian-Daniel (LPD) equation is studied. New analytical rational solitons to the LPD equation are presented by an ansatz method. Wave solutions of three types, such as parabolic, trigonometric and hyperbolic function solutions have been retrieved. All solutions are plotted in 3D to enhance the understanding of their physical characteristics. These simulations, which represent the behavior of the resulting hyperbolic, parabolic and trigonometric solitons, are provided by choosing different appropriate values of the parameters.

1. Introduction

Many researchers in fields such as mathematics, physics, engineering and more are very interested in nonlinear partial equations, since most physical systems are not linear in nature. The Lakshmanan-Porsezian-Daniel (LPD) equation is a widely known nonlinear partial differential equation. It is a generalization of the nonlinear Schrödinger equation that includes higher order nonlinear and dispersed terms. In recent years, it has attracted great attention from mathematicians and physicists. The LPD equation has also been generalized and extended in many ways, including the addition of external forcing, the inclusion of damping effects, and the consideration of higher-dimensional versions of the equation. The LPD equation and its variants have been used to model a variety of physical systems in many areas of physics and engineering, including plasma physics, fluid dynamics, and nonlinear optics, and has been studied extensively from different perspectives such as integrability, symmetry analysis, solution methods and applications: Ricatti equation [1], $\tan(\frac{\psi(\eta)}{2})$-expansion technique [2], collective variable [3], modified simple equation method [4], method of undetermined coefficients [5], Darboux transformations [6, 7], Rogue wave equation [8], the modified auxiliary equation method [9] etc.

This paper investigates the Lakshmanan–Porsezian–Daniel equation [10, 11], a well known partial differential equation that describes the pulse propagation in an optical fiber which is in the form

$$iq_t + aq_{xx} + bq_{xt} + \zeta|q^2|q = \sigma q_{xxxx} + \beta|q_x|^2 q + \gamma|q|^2 q_{xx} + \delta|q|^4 q \tag{1.1}$$

where the complex valued function $q(x,t)$ depends on space x and time t. The term iq_t denotes the temporal evolution of pulse. The group velocity dispersion and spatio-temporal dispersion are given by a and b, respectively. The fourth-order dispersion and two-photon absorption are represented by constants σ and δ, respectively. The parameters β and γ indicate the non-linear forms of dispersion.

In this paper, our aim is finding solutions in the form of parabolic, trigonometric and hyperbolic solitons of the LPD equation. First we start by using traveling wave variables to find a solution for Eq. (1.1). After analyzing the resulting system of equations to find the condition of its compatibility, it turned out that the structure for the system of equations. At the second stage, a special logarithmic transformation is applied. At the last stage, three different methods are applied to retrieved equation. The solutions obtained by appropriate selection of some parameters affecting the shape and velocity of the solitons are observed with three-dimensional plots.

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2. System of Differential Equations Corresponding to Eq. (1.1)

In order to find soliton solutions of Eq. (1.1), we use the traveling wave reduction in the form

$$q(x,t) = y(z)e^{i\theta}, \quad z = x - ct, \quad \theta = kx - wt \tag{2.1}$$

where $y(z)$ is a complex-valued function that represents the structure of the pulse, k, w, c, θ are parameters of solution. The phase component of the soliton is θ, k is represents the frequency of the soliton, c is the velocity of the soliton, while w is the wave number.
Substituting solution (2.1) into Eq. (1.1) and equating the real and imaginary parts of expression to zero, respectively the following equations are obtained

$$\sigma y_{zzzz} + (-a + bc - 6\sigma k^2)y_{zz} + \gamma y^2 y_{zz} + \beta y {y_z}^2 + (\beta k^2 - \gamma k^2 - \zeta)y^3 + \delta y^5 + (\sigma k^4 - w + ak^2 - bwk)y = 0 \tag{2.2}$$

and

$$4\sigma k y_{zzz} + \big(-4\sigma k^3 + c - 2ak + b(ck + w)\big)y_z + 2\gamma k y^2 y_z = 0. \tag{2.3}$$

The $y(z)$ function must satify both of the above third and fourth order differential equations obtained, respectively (2.2) and (2.3). For the purpose of evaluating the solution of the Eq. (1.1), we implement the following logaritmic transformation from [12] on y,

$$y = 2(\ln f)_z. \tag{2.4}$$

If transformation (2.4) is substituted into in (2.2) and (2.3), then the resulting expressions obtained as:

$$\begin{pmatrix} -4\zeta f^2 f_z^3 - 60\sigma f f_{zz} f_z^3 - 12\gamma f f_{zz} f_z^3 - 8\beta f f_{zz} f_z^3 + 4\gamma f^2 f_z^2 f_{zzz} + 20\sigma f^2 f_z^2 f_{zzz} + 30\sigma f^2 f_{zzz}^2 f_z + 4\beta f^2 f_{zz}^2 f_z \\ -10\sigma f^3 f_{zz} f_{zzz} + 3af^3 f_{zz} f_z + bcf^4 f_{zzz} - 5\sigma f^3 f_z f_{zzzz} - 6\sigma k^2 f^4 f_{zzz} + 8\gamma f_z^5 + 4\beta f_z^5 + 16\delta f_z^5 + 24\sigma f_z^5 - wf^4 f_z \\ -2af^2 f_z^3 - bkwf^4 f_z + k^4\sigma f^4 f_z + ak^2 f^4 f_z - 4\gamma k^2 f^2 f_z^3 + 4\beta k^2 f^2 f_z^3 - 12\sigma k^2 f^2 f_z^3 + 2bcf^2 f_z^3 \\ -af^4 f_{zzz} + \sigma f_{zzzzz} f^4 - 3bcf^3 f_{zz} f_z + 18\sigma k^2 f^3 f_{zz} f_z \end{pmatrix} = 0,$$

$$\begin{pmatrix} -4\sigma k^3 f^3 f_{zz} + 4\sigma k^3 f^2 f_z^2 + bckf^3 f_{zz} - bxkf^2 f_z^2 - 2akf^3 f_{zz} + bwf^3 f_{zz} + 2akf^2 f_z^2 - bwf^2 f_z^2 + 8\gamma k f f_z^2 f_{zz} \\ -8\gamma k f_z^4 + 4\sigma f_{zzz} f^3 + cf^3 f_{zz} - 16\sigma f^2 f_{zzz} f_z - cf^2 f_z^2 - 12\sigma f^2 f_{zz}^2 + 48\sigma f f_z^2 f_{zz} - 24\sigma f_z^4 \end{pmatrix} = 0. \tag{2.5}$$

Now, we use this form to evaluate various rational solitons.

3. Hyperbolic Solitons of Differential Equations Corresponding to Eq. (1.1)

To get hyperbolic solitons of (2.5), we use the following transformation:

$$f = b_0 \cosh z + b_1 \sinh z \tag{3.1}$$

where b_0, b_1 are any constants to be determined. Substituting (3.1) into (2.5) and equating the coefficient terms that are containing independent combinations of cosh and sinh functions to zero, we obtain a system of algebraic equations:

$$-b_1(5b_0^4 + 10b_0^2 b_1^2 + b_1^4)(-k^4\sigma - ak^2 + bkw - 4\beta k^2 + 4\gamma k^2 + 4\zeta - 16\delta + w) = 0,$$

$$-b_0(b_0^4 + 10b_0^2 b_1^2 + 5b_1^4)(-k^4\sigma - ak^2 + bkw - 4\beta k^2 + 4\gamma k^2 + 4\zeta - 16\delta + w) = 0,$$

$$2b_1\begin{pmatrix} \big(-2k^4\sigma - 2ak^2 + 2bkw - 14\beta k^2 + 14\gamma k^2 + 18k^2\sigma - 3bc + 14\zeta + 3a - 80\delta - 12\gamma - 12\sigma + 2w\big)b_0^4 \\ + \big(-7k^4\sigma - 7ak^2 + 7bkw - 24\beta k^2 + 24\gamma k^2 - 12k^2\sigma + 2bc + 24\zeta - 2a - 80\delta + 8\gamma + 8\sigma + 7w\big)b_1^2 b_0^2 \\ + \big(-k^4\sigma - ak^2 + bkw - 2\beta k^2 + 2\gamma k^2 - 6k^2\sigma + bc + 2\zeta - a + 4\gamma + 4\sigma + w\big)b_1^4 \end{pmatrix} = 0,$$

$$2b_0\begin{pmatrix} \big(-2\beta k^2 + 2\gamma k^2 + 6k^2\sigma - bc + 2\zeta + a - 16\delta - 4\gamma - 4\sigma\big)b_0^4 \\ + \big(-3k^4\sigma - 3ak^2 + 3bkw - 16\beta k^2 + 16\gamma k^2 + 12k^2\sigma - 2bc + 16\zeta + 2a - 80\delta - 8\gamma - 8\sigma + 3w\big)b_0^2 b_1^2 \\ + \big(-3k^4\sigma - 3ak^2 + 3bkw - 6\beta k^2 + 6\gamma k^2 - 18k^2\sigma + 3bc + 6\zeta - 3a + 12\gamma + 12\sigma + 3w\big)b_1^4 \end{pmatrix} = 0,$$

$$-b_1\begin{pmatrix} \big(-8\beta k^2 + 8\gamma k^2 + 24k^2\sigma - 4bc + 8\zeta + 4a - 4\beta - 80\delta - 24\gamma - 40\sigma\big)b_0^4 \\ + \big(-4k^4\sigma - 4ak^2 + 4bkw - 12\beta k^2 + 12\gamma k^2 - 12k^2\sigma + 2bc + 12\zeta - 2a + 8\beta + 24\gamma + 56\sigma + 4w\big)b_0^2 b_1^2 \\ + \big(-k^4\sigma - ak^2 + bkw - 12k^2\sigma + 2bc - 2a - 4\beta - 16\sigma + w\big)b_1^4 \end{pmatrix} = 0,$$

$$b_0\begin{pmatrix} (8\gamma + 4\beta + 16\delta + 24\sigma)b_0^4 \\ + \big(4\beta k^2 - 4\gamma k^2 - 12k^2\sigma + 2bc - 4\zeta - 2a - 8\beta - 8\gamma - 40\sigma\big)b_0^2 b_1^2 \\ + \big(k^4\sigma + ak^2 - bkw + 12k^2\sigma - 2bc + 2a + 4\beta + 16\sigma - w\big)b_1^4 \end{pmatrix} = 0, \tag{3.2}$$

$$(b_0 - b_1)(b_0 + b_1)(b_0^2 + b_1^2)(-4k^3\sigma + bck - 2ak + bw + 8\gamma k + c + 16\sigma) = 0,$$

$$2b_0 b_1(b_0 - b_1)(b_0 + b_1)(-4k^3\sigma + bck - 2ak + bw + 8\gamma k + c + 16\sigma) = 0,$$

$$-(b_0 - b_1)(b_0 + b_1)(-4b_1^2 k^3\sigma + bb_1^2 ck - 2ab_1^2 k + bb_1^2 w + 8b_0^2\gamma k + 24b_0^2\sigma + b_1^2 c - 8b_1^2\sigma) = 0.$$

After solving the system (3.2) with the help of Maple software, three cases of parametric values are obtained as follows:

Case 1:

$$b_1 = b_0, \quad \sigma = \frac{-ak^2 + bkw - 4\beta k^2 + 4\gamma k^2 + 4\zeta - 16\delta + w}{k^4}. \tag{3.3}$$

Via the parametric values in (3.3), we have

$$f = b_0(\cosh z + \sinh z). \tag{3.4}$$

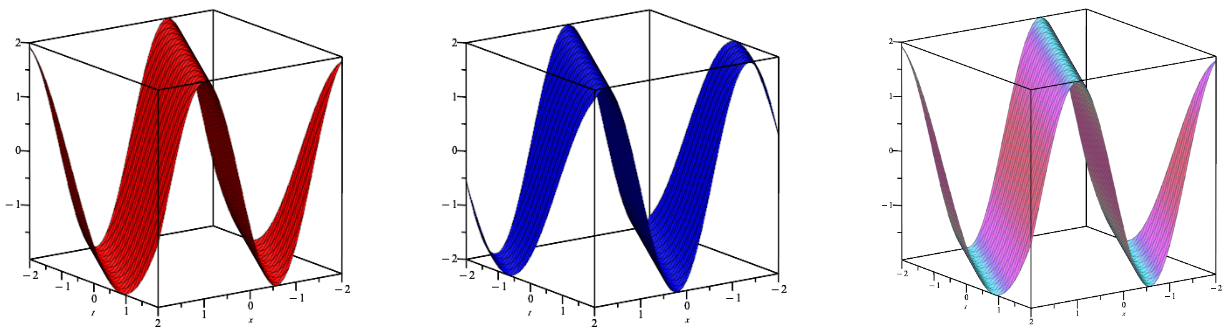
By using $y = 2(\ln f)_z$, we have

$$y = 2. \tag{3.5}$$

By using Eq. (3.5) into Eq. (2.1), we obtain a first type of rational hyperbolic solution of Eq. (1.1):

$$q(x,t) = 2e^{i(kx-wt)}. \tag{3.6}$$

(See Figure 3.1)



(a) **(b)** **(c)**

Figure 3.1: 3D plots of the rational solution (3.6) in Case 1 with the values of $k = 2$, $w = 1$, $c = 1$, (a) Real, (b) Imaginary and (c) Complex.

Case 2:

$$\begin{aligned}
\zeta &= \frac{-\left(\begin{array}{c} bck^8 + 2ak^8 - 3bk^7w + 2bck^6 + ck^7 - 20ak^6 + 18bk^5w - 4k^6w + 8ak^5 + 12bck^4 \\ -8bk^4w + 10ck^5 - 16ak^4 - 28bck^3 + 4bk^3w + 8k^4w + 24ak^3 + 8bck^2 + 4bk^2w + 4ck^3 \\ -8k^3w - 16ak^2 - 8bck + 8bkw - 12ck^2 + 32ak - 24bw + 8ck + 16kw - 24c \end{array}\right)}{16k(k^3+2)}, \\
\delta &= \frac{\begin{array}{c}(bc+2a)k^6 + (-3bw+c)k^5 + 4(bc-4a-w)k^4 + 4(3bw+2a+3c)k^3 \\ +8(-(c+w)b+2a)k^2 + 8((c-w)b-4a-c-w)k + 24(bw+c)\end{array}}{64k(k^3+2)}, \\
\gamma &= \frac{-3((bc-2a)k+bw+c)}{4k(k^3+2)}, \\
\beta &= \frac{\begin{array}{c}-(bc+2a)k^5 + (3bw-c)k^4 + 4(-bc+4a+w)k^3 - 4(3bw+2a+3c)k^2 \\ +8((-2c+w)b+4a)k + 8(2(c-w)b-2(a+c)+w)\end{array}}{16k^3+32}, \\
\sigma &= \frac{(bc-2a)k+bw+c}{4k^3+8}.
\end{aligned} \tag{3.7}$$

By using values in (3.7) into (3.1), we have

$$f = b_0 \cosh z + b_1 \sinh z. \tag{3.8}$$

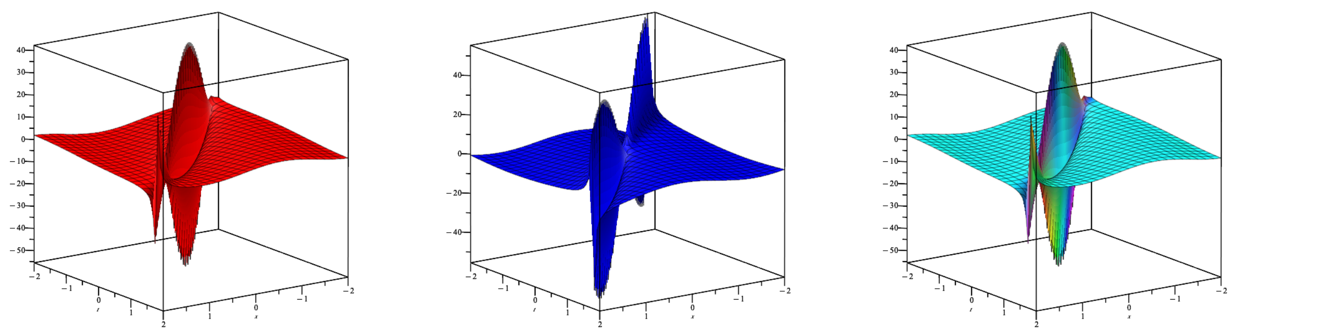
By using $y = 2(\ln f)_z$, we have

$$y = \frac{2(b_0 \sinh(z) + b_1 \cosh(z))}{b_0 \cosh(z) + b_1 \sinh(z)}. \tag{3.9}$$

By using Eq. (3.9) into Eq. (2.1), we obtain a second type of rational hyperbolic solution of Eq. (1.1):

$$q(x,t) = \frac{2(b_0 \sinh(z) + b_1 \cosh(z))e^{i(kx-wt)}}{b_0 \cosh(z) + b_1 \sinh(z)}. \tag{3.10}$$

(See Figure 3.2)



(a) (b) (c)

Figure 3.2: 3D plots of the rational solution (3.10) in Case 2 with the values of $k = 2$, $w = 1$, $c = 0.1$, $b_0 = -5$, $b_1 = 0.1$, (a) Real, (b) Imaginary and (c) Complex.

4. Parabolic Solutions of Differential Equations Corresponding to Eq. (1.1)

To get parabolic solution of (2.5), we choose f as following:

$$f = b_2 z^2 + b_1 z + b_0 \tag{4.1}$$

where b_0, b_1 and b_2 represent any constant parameters. By substituting (4.1) into (2.5) and equating the various coefficient terms of z, we then solve the following system of algebraic equations to find the values of the parameters:

$$2b_2^4(4k^3\sigma - bck + 2ak - bw - c) = 0$$

$$6b_1 b_2^3(4k^3\sigma - bck + 2ak - bw - c) = 0,$$

$$b_2^2\left((8k^3\sigma - 2bck + 4ak - 2bw - 2c)b_2 b0 + (28k^3\sigma - 7bck + 14ak - 7bw - 7c)b_1^2 + (-64\gamma k - 48\sigma)b_2^2\right) = 0,$$

$$4b_1 b_2\left((4k^3\sigma - bck + 2ak - bw - c)b_2 b0 + (4k^3\sigma - bck + 2ak - bw - c)b_1^2 + (-32\gamma k - 24\sigma)b_2^2\right) = 0,$$

$$\begin{bmatrix} \left((16k^3\sigma - 4bck + 8ak - 4bw - 4c)b_2 b_1^2 + (64\gamma k + 288\sigma)b_2^3\right) b_0, \\ +(-8k^3\sigma + 2bck - 4ak + 2bw + 2c)b_2^2 b0^2 + (4k^3\sigma - bck + 2ak - bw - c)b_1^4 + (-112\gamma k - 144\sigma)b_2^2 b_1^2 \end{bmatrix} = 0,$$

$$-2b_1\left((4k^3\sigma - bck + 2ak - bw - c)b_2 b0^2 + ((-4k^3\sigma + bck - 2ak + bw + c)b_1^2 + (-32\gamma k - 144\sigma)b_2^2)b0 + (24\gamma k + 48\sigma)b_2 b_1^2\right) = 0,$$

$$\begin{bmatrix} (4k^3\sigma - bck + 2ak - bw - c)b_1^2 - 48b_2^2\sigma \\ (-8k^3\sigma + 2bck - 4ak + 2bw + 2c)b_2 b_0^3 + (16\gamma k + 96\sigma)b_2 b_1^2 b0 + (-8\gamma k - 24\sigma)b_1^4 \end{bmatrix} = 0, \tag{4.2}$$

$$2b_2^5(k^4\sigma+ak^2-bkw-w)=0,$$

$$9b_1b_2^4(k^4\sigma+ak^2-bkw-w)=0,$$

$$-4b_2^3\left((-2k^4\sigma-2ak^2+2bkw+2w)b_2b_0+(-4k^4\sigma-4ak^2+4bkw+4w)b_1^2+(-8\beta k^2+8\gamma k^2+6k^2\sigma-bc+8\zeta+a)b_2^2\right)=0,$$

$$-14b_1b_2^2\left((-2k^4\sigma-2ak^2+2bkw+2w)b_2b_0+(-k^4\sigma-ak^2+bkw+w)b_1^2+(-8\beta k^2+8\gamma k^2+6k^2\sigma-bc+8\zeta+a)b_2^2\right)=0,$$

$$-2b_2\left[\begin{array}{c}(-18k^4\sigma-18ak^2+18bkw+18w)b_2b_1^2+(-32\beta k^2+32\gamma k^2-12k^2\sigma+2bc+32\zeta-2a)b_2^3\\(-6k^4\sigma-6ak^2+6bkw+6w)b_2^2b_0^2+(-3k^4\sigma-3ak^2+3bkw+3w)b_1^4\\+(-76\beta k^2+76\gamma k^2+66k^2\sigma-11bc+76\zeta+11a)b_2^2b_1^2+(-16\beta-256\delta-32\gamma-24\sigma)b_2^4\end{array}\right]=0,$$

$$-b_1\left[\begin{array}{c}((-20k^4\sigma-20ak^2+20bkw+20w)b_2b_1^2+(-160\beta k^2+160\gamma k^2-60k^2\sigma+10bc+160\zeta-10a)b_2^3)b_0\\+(-30k^4\sigma-30ak^2+30bkw+30w)b_2^2b_0^2+(-k^4\sigma-ak^2+bkw+w)b_1^4\\(-100\beta k^2+100\gamma k^2+120k^2\sigma-20bc+100\zeta+20a)b_2^2b_1^2+(-80\beta-1280\delta-160\gamma-120\sigma)b_2^4\end{array}\right]=0,$$

$$\left[\begin{array}{c}((24k^4\sigma+24ak^2-24bkw-24w)b_2b_1^2+(32\beta k^2-32\gamma k^2+120k^2\sigma-20bc-32\zeta+20a)b_2^3)b_0^2\\+(8k^4\sigma+8ak^2-8bkw-8w)b_2^2b_0^3+(32\beta k^2-32\gamma k^2-60k^2\sigma+10bc-32\zeta-10a)b_2b_1^4\\+(96\beta+1280\delta+208\gamma+240\sigma)b_2^3b_1^2\\+((4k^4\sigma+4ak^2-4bkw-4w)b_1^4+(144\beta k^2-144\gamma k^2-144\zeta)b_2^2b_1^2+(-64\beta-192\gamma-480\sigma)b_2^4)b_0\end{array}\right]=0,$$

$$-2b_1\left[\begin{array}{c}((-3k^4\sigma-3ak^2+3bkw+3w)b_1^2+(-24\beta k^2+24\gamma k^2-90k^2\sigma+15bc+24\zeta-15a)b_2^2)b_0^2\\+(-6k^4\sigma-6ak^2+6bkw+6w)b_2b_0^3+(-2\beta k^2+2\gamma k^2+6k^2\sigma-bc+2\zeta+a)b_1^4\\+(-32\beta-320\delta-76g-120\sigma)b_2^2b_1^2\\+((-28\beta k^2+28\gamma k^2+30k^2\sigma-5bc+28\zeta+5a)b_2b_1^2+(48\beta+144\gamma+360\sigma)b_2^3)b_0\end{array}\right]=0,$$

$$\left[\begin{array}{c}(2k^4\sigma+2ak^2-2bkw-2w)b_2b_0^4+((4k^4\sigma+4ak^2-4bkw-4w)b_1^2+(72k^2\sigma-12bc+12a)b_2^2)b_0^3\\+((24\beta k^2-24\gamma k^2+36k^2\sigma-6bc-24\zeta+6a)b_2b_1^2+(32\beta+240\sigma)b_2^3)b_0^2\\+((8\beta k^2-8\gamma k^2-24k^2\sigma+4bc-8\zeta-4a)b_1^4+(-64\beta-144\gamma-480\sigma)b_2^2b_1^2)b_0+(24\beta+160\delta+56\gamma+120\sigma)b_2b_1^4\end{array}\right]=0,$$

$$-b_1\left[\begin{array}{c}(-k^4\sigma-ak^2+bkw+w)b_0^4+(-36k^2\sigma+6bc-6a)b_2b_0^3+(-4\beta-16\delta-8\gamma-24\sigma)b_1^4\\+((-4\beta k^2+4\gamma k^2+12k^2\sigma-2bc+4\zeta+2a)b_1^2+(-16\beta-120\sigma)b_2^2)b_0^2+(16\beta+24\gamma+120\sigma)b_2b_1^2b_0\end{array}\right]=0.$$

Resolution of the system (4.2) with the help of Maple gives five cases of parametric values as follows:

Case 1:

$$a=\frac{4\gamma k^5+3bkw+3w}{3k^2},\quad b_0=b_1=0,\quad \beta=-\frac{12b\gamma k^6-24b\gamma k^5+20\gamma k^5-24\zeta bk^3-24\gamma k^4-24\zeta k^2-3w}{24k^4(bk+1)},$$
$$c=-\frac{8\gamma k^5-3bkw-6w}{3k(bk+1)},\quad \delta=\frac{60b\gamma k^6-72b\gamma k^5+68\gamma k^5-24\zeta bk^3-72\gamma k^4-24\zeta k^2-3w}{384k^4(bk+1)},\quad \sigma=-\frac{4\gamma k}{3}. \tag{4.3}$$

By using values in (4.3) into (4.1), we have

$$f=b_2z^2. \tag{4.4}$$

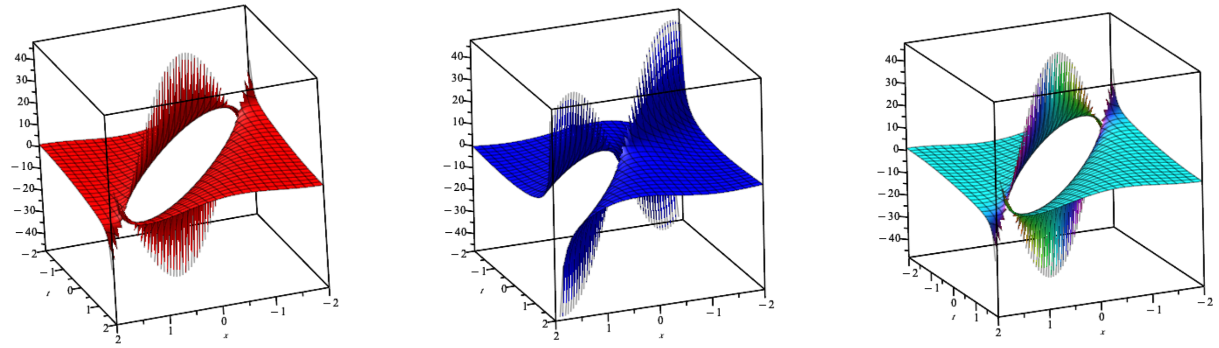
By using $y=2(\ln f)_z$, we have

$$y=\frac{4}{z}. \tag{4.5}$$

By replacing Eq. (2.1) with Eq. (4.5), we obtain the first type of rational parabolic solution of Eq. (1.1):

$$q(x,t)=\frac{4e^{i(kx-wt)}}{-ct+x}. \tag{4.6}$$

(See Figure 4.1)



(a) (b) (c)

Figure 4.1: 3D plots of the rational solution (4.6) in Case 1 with the values of $k=2$, $w=1$, $b=128$, $\gamma=-3$, $b_1=0.1$, (a) Real, (b) Imaginary and (c) Complex.

Case 2:

$$a=\frac{4\gamma k^3}{3},\quad b=-\frac{1}{k},\quad b_0=b_1=0,\quad \beta=\frac{-20\gamma k^4+24\gamma k^3+24\zeta k+3c}{24k^3},$$
$$\delta=-\frac{-68\gamma k^4+72\gamma k^3+24\zeta k+3c}{384k^3},\quad \sigma=-\frac{4\gamma k}{3},\quad w=\frac{8\gamma k^5}{3}. \tag{4.7}$$

By using values in (4.7) into (4.1), we have

$$f=b_2z^2. \tag{4.8}$$

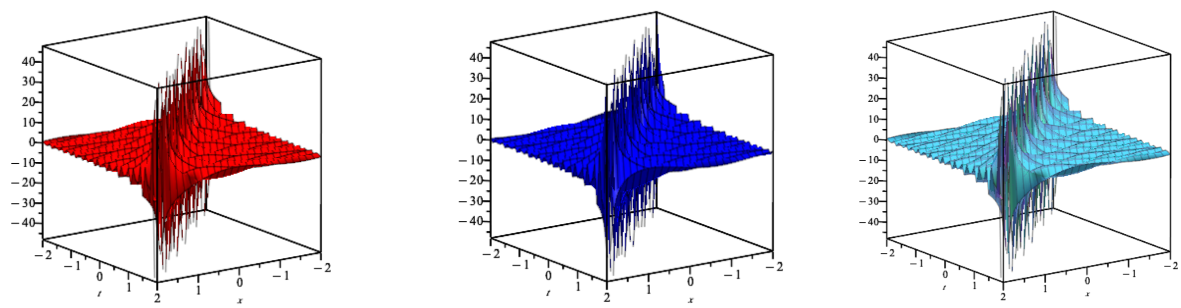
By using $y=2(\ln f)_z$, we have

$$y=\frac{4}{z}. \tag{4.9}$$

By using Eq. (4.9) into Eq. (2.1), we obtain a second type of rational parabolic solution of Eq. (1.1):

$$q(x,t)=\frac{4e^{i\left(-\frac{8}{3}\gamma k^5t+kx\right)}}{-ct+x}. \tag{4.10}$$

(See Figure 4.2)



(a) (b) (c)

Figure 4.2: 3D plots of the rational solution (4.10) in Case 2 with the values of $k=2$, $c=1$, $\gamma=3$, (a) Real, (b) Imaginary and (c) Complex.

Case 3:

$$a=\frac{\gamma k^5+3bkw+3w}{3k^2},\quad b_2=0,\quad \beta=\frac{-3b\gamma k^6+6b\gamma k^5-5\gamma k^5+6\zeta bk^3+6\gamma k^4+6\zeta k^2+3w}{6k^4(bk+1)},$$
$$c=\frac{-2\gamma k^5+3bkw+6w}{3k(bk+1)},\quad \delta=-\frac{-15b\gamma k^6+18b\gamma k^5-17\gamma k^5+6\zeta bk^3+18\gamma k^4+6\zeta k^2+3w}{24k^4(bk+1)},\quad \sigma=-\frac{\gamma k}{3}. \tag{4.11}$$

By using values in (4.11) into (4.1), we have

$$f = b_1 z + b_0. \tag{4.12}$$

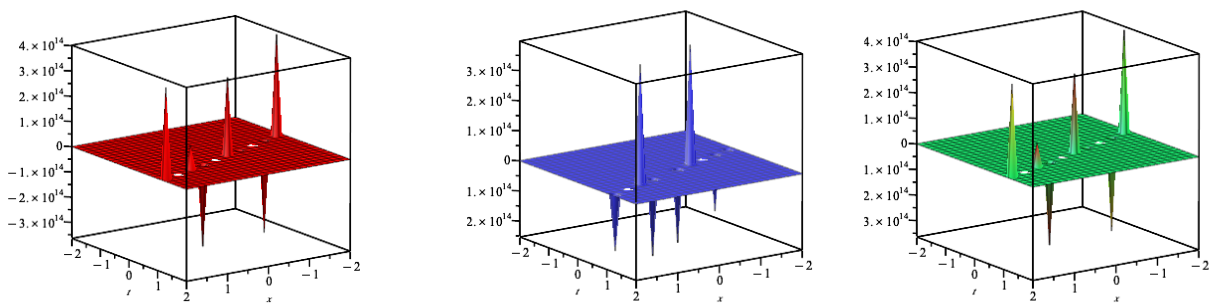
By using $y = 2(\ln f)_z$, we have

$$y = \frac{2b_1}{b_1 z + b_0}. \tag{4.13}$$

By replacing Eq. (2.1) with Eq. (4.13) , we obtain a third type of rational parabolic solution of Eq. (1.1):

$$q(x,t) = \frac{2b_1 e^{i(kx-wt)}}{b_1\left(-\dfrac{(-2\gamma k^5 + 3bkw + 6w)t}{3k(bk+1)} + x\right) + b_0}. \tag{4.14}$$

(See Figure 4.3)



(a) (b) (c)

Figure 4.3: 3D plots of the rational solution (4.14) in Case 3 with the values of $b_1 = 1$, $b_0 = 1$, $k = 2$, $w = 1$, $b = \frac{-5}{12}$, $\gamma = \frac{1}{64}$, (a) Real, (b) Imaginary and (c) Complex.

Case 4:

$$a = \frac{k^3\gamma}{3}, \quad b = -\frac{1}{k}, \quad b_2 = 0, \quad \beta = \frac{-5\gamma k^4 + 6\gamma k^3 + 6\zeta k + 3c}{6k^3},$$
$$\delta = -\frac{-17\gamma k^4 + 18\gamma k^3 + 6\zeta k + 3c}{24k^3}, \quad \sigma = -\frac{k\gamma}{3}, \quad w = \frac{2k^5\gamma}{3}. \tag{4.15}$$

By using values in (4.15) into (4.1), we have

$$f = b_1 z + b_0. \tag{4.16}$$

By using $y = 2(\ln f)_z$, we have

$$y = \frac{2b_1}{b_1 z + b_0}. \tag{4.17}$$

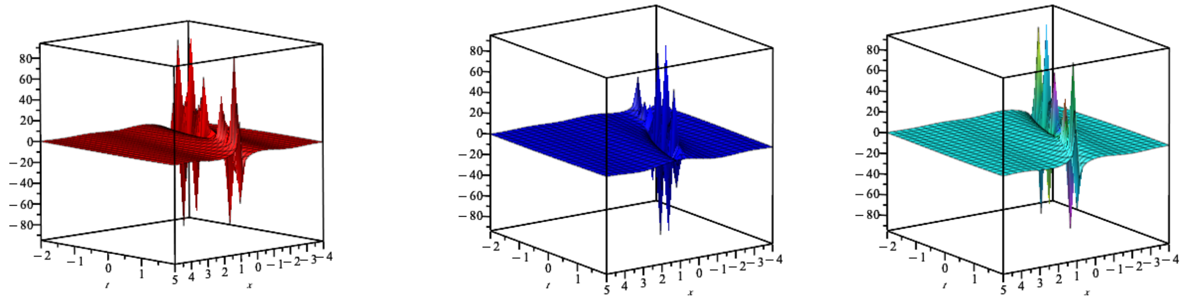
By using Eq. (4.17) into Eq. (2.1), we obtain a fourth type of rational parabolic solution of Eq. (1.1):

$$q(x,t) = \frac{2b_1 e^{i\left(\frac{-2}{3}\gamma k^5 t + kx\right)}}{b_1(-ct + x) + b_0}. \tag{4.18}$$

(See Figure 4.4)

Case 5:

$$a = \frac{4\gamma k^5 + 3bkw + 3w}{3k^2}, \quad b_2 = \frac{b_1^2}{4b_0}, \quad \beta = \frac{-12b\gamma k^6 + 24b\gamma k^5 - 20\gamma k^5 + 24\zeta bk^3 + 24\gamma k^4 + 24\zeta k^2 + 3w}{24k^4(bk+1)},$$
$$c = \frac{-8\gamma k^5 + 3bkw + 6w}{3k(bk+1)}, \quad \delta = -\frac{-60b\gamma k^6 + 72b\gamma k^5 - 68\gamma k^5 + 24\zeta bk^3 + 72\gamma k^4 + 24\zeta k^2 + 3w}{384k^4(bk+1)}, \quad \sigma = -\frac{4\gamma k}{3}. \tag{4.19}$$



(a) **(b)** **(c)**

Figure 4.4: 3D plots of the rational solution (4.18) in Case 4 with the values of $b_1 = 1$, $b_0 = 1$, $k = 2$, $w = 1$, $b = \frac{-5}{12}$, $\gamma = \frac{1}{64}$, $c = 1$, (a) Real, (b) Imaginary and (c) Complex.

By using values in (4.19) into (4.1), we have

$$f = \frac{b_1^2}{4b_0}z^2 + b_1 z + b_0. \tag{4.20}$$

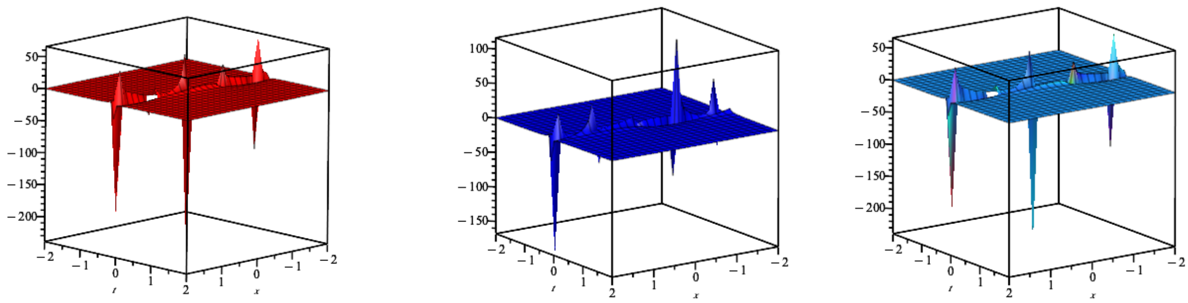
By using $y = 2(\ln f)_z$, we have

$$y = \frac{4b_1}{b_1 z + 2b_0}. \tag{4.21}$$

By using Eq. (4.21) into Eq. (2.1), we obtain a fifth type of rational parabolic solution of Eq. (1.1):

$$q(x,t) = \frac{4b_1 e^{i(kx-wt)}}{b_1\left(-\frac{(-8\gamma k^5 + 3bkw + 6w)t}{3k(bk+1)} + x\right) + 2b_0}. \tag{4.22}$$

(See Figure 4.5)



(a) **(b)** **(c)**

Figure 4.5: 3D plots of the rational solution (4.22) in Case 5 with the values of $b_1 = 2$, $b_0 = -1$, $k = 2$, $w = 1$, $b = \frac{-1}{3}$, $\gamma = 0.1$, (a) Real, (b) Imaginary and (c) Complex.

Case 6:

$$\begin{aligned} &a = \frac{4k^3\gamma}{3}, \quad b = \frac{-1}{k}, \quad b_2 = \frac{b_1^2}{4b_0}, \quad \beta = \frac{-20\gamma k^4 + 24\gamma k^3 + 24\zeta k + 3c}{24k^3}, \\ &\delta = -\frac{-68\gamma k^4 + 72\gamma k^3 + 24\zeta k + 3c}{384k^3}, \quad \sigma = -\frac{4\gamma k}{3}, \quad w = \frac{8k^5\gamma}{3}. \end{aligned} \tag{4.23}$$

By using values in (4.23) into (4.1), we have

$$f = \frac{b_1^2}{4b_0}z^2 + b_1 z + b_0. \tag{4.24}$$

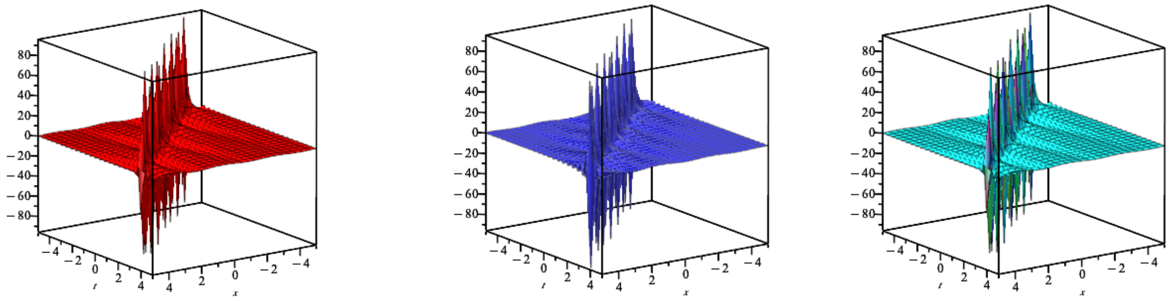
By using $y=2(\ln f)_z$, we have

$$y=\frac{4b_1}{b_1z+2b_0}. \tag{4.25}$$

By using Eq. (4.25) into Eq. (2.1), we obtain a sixth type of rational parabolic solution of Eq. (1.1):

$$q(x,t)=\frac{4b_1e^{i\left(kx-\frac{8k^5\gamma}{3}t\right)}}{b_1(x-ct)+2b_0}. \tag{4.26}$$

(See Figure 4.6)



(a) **(b)** **(c)**

Figure 4.6: 3D plots of the rational solution (4.26) in Case 5 with the values of $b_1=2$, $b_0=-1$, $k=2$, $c=1$, $\gamma=3$, (a) Real, (b) Imaginary and (c) Complex.

5. Trigonometric Solutions of Differential Equations Corresponding to Eq. (1.1)

To get trigonometric solution, we suppose a solution of (2.5) in the following form:

$$f=b_0\cos z+b_1\sin z \tag{5.1}$$

where b_0, b_1 are all constants. Inserting (5.1) into (2.5) and setting the coefficient terms that are containing independent combinations of cos and sin functions to zero, we get a system of algebraic equations:

$$\begin{aligned}
&4b_1^4(4k^3\sigma-bck+2ak-bw+8\gamma k-c+16\sigma)=0,\\
&2b_1^4(4k^3\sigma-bck+2ak-bw-8\gamma k-c-32\sigma)=0,\\
&4b_1^5(k^4\sigma+ak^2-bkw-4\beta k^2+4\gamma k^2+4\zeta+16\delta-w)=0,\\
&8b_1^5(k^4\sigma+ak^2-bkw-2\beta k^2+2\gamma k^2-6k^2\sigma+bc+2\zeta-a-4\gamma-4\sigma-w)=0,\\
&-8b_1^5(-2\beta k^2+2\gamma k^2+6k^2\sigma-bc+2\zeta+a+16\delta+4\gamma+4\sigma)=0,\\
&-b_1^5(3k^4\sigma+3ak^2-3bkw-4\beta k^2+4\gamma k^2-24k^2\sigma+4bc+4\zeta-4a-16\beta-80\delta-48\gamma-112\sigma-3w)=0,\\
&b_1^5(-k^4\sigma-ak^2+bkw-4\beta k^2+4\gamma k^2+24k^2\sigma-4bc+4\zeta+4a-16\beta-16\delta-16\gamma-80\sigma+w)=0.
\end{aligned} \tag{5.2}$$

After solving the system (5.2) with the help of Maple, we obtain three cases of parametric values as follows:

Case 1:

$$
\begin{aligned}
a &= \frac{\begin{pmatrix} 3b\gamma k^7+3b^2ck^4+18b\gamma k^5+4\gamma k^6-8b\gamma k^4+12b^2ck^2 \\ +6bck^3-4b\gamma k^3+8\gamma k^4-4b\gamma k^2-8\gamma k^3+6b^2c \\ +12bck+8b\gamma k+3ck^2+12\zeta b-24b\gamma-16\gamma k+6c \end{pmatrix}}{3(bk^4+4bk^2+2k^3+2b+4k)}, \\
\beta &= -\frac{\begin{pmatrix} 3b\gamma k^5-6b\gamma k^4-20b\gamma k^3+8\gamma k^4-6\zeta bk^2 \\ +8b\gamma k^2-12\gamma k^3-24b\gamma k-40\gamma k^2-12\zeta b \\ -12\zeta k+24b\gamma+32\gamma k-3c \end{pmatrix}}{6(bk^4+4bk^2+2k^3+2b+4k)}, \\
\delta &= \frac{\begin{pmatrix} 15b\gamma k^5-18b\gamma k^4+28b\gamma k^3+32\gamma k^4 \\ -6\zeta bk^2-40\gamma k^2-36\gamma k^3+8b\gamma k^2 \\ -12\zeta b-12\zeta k-16\gamma k-3c \end{pmatrix}}{24(bk^4+4bk^2+2k^3+2b+4k)}, \\
w &= \frac{\begin{pmatrix} 2\gamma k^8+3bck^5+20\gamma k^6-8\gamma k^5+12bck^3 \\ +3ck^4-16\gamma k^4+24\gamma k^3+6bck+16\gamma k^2 \\ +24\zeta k-32\gamma k-6c \end{pmatrix}}{3(bk^4+4bk^2+2k^3+2b+4k)}, \\
\sigma &= -\frac{1}{3}\gamma k.
\end{aligned}
\tag{5.3}
$$

By utilizing values in (5.3) into (5.1), we have

$$f = b_1\cos z + b_1 \sin z. \tag{5.4}$$

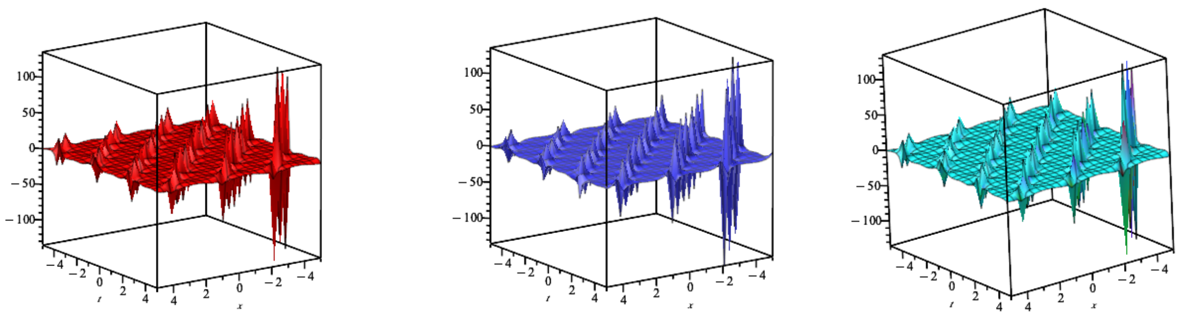
Inserting Eq. (5.4) into (2.4) yields

$$y = -\frac{2(\sin(z)-\cos(z))}{\sin(z)+\cos(z)}. \tag{5.5}$$

By using Eq. (5.5) into Eq. (2.1), we obtain a first type of trigonometric solution of Eq. (1.1):

$$q(x,t) = -\frac{2(\sin(x-ct)-\cos(x-ct))e^{i(kx-wt)}}{\sin(x-ct)+\cos(x-ct)}. \tag{5.6}$$

(See Figure 5.1)



(a) **(b)** **(c)**

Figure 5.1: 3D plots of the rational solution (5.6) in Case 1 with the values of $k=2$, $c=1$, $\gamma=3$, $b=1$, $\zeta=-1$ (a) Real, (b) Imaginary and (c) Complex.

Case 2:

$$a=-b^2w+2\zeta-4\gamma-w, \quad \beta=\zeta-2\gamma-\frac{1}{4}w, \quad c=-bw, \quad \delta=-\frac{1}{4}\zeta+\frac{1}{16}w, \quad k=0, \quad \sigma=0, \tag{5.7}$$

Using the obtained values in (5.1) gives

$$f = b_1\cos z + b_1 \sin z. \tag{5.8}$$

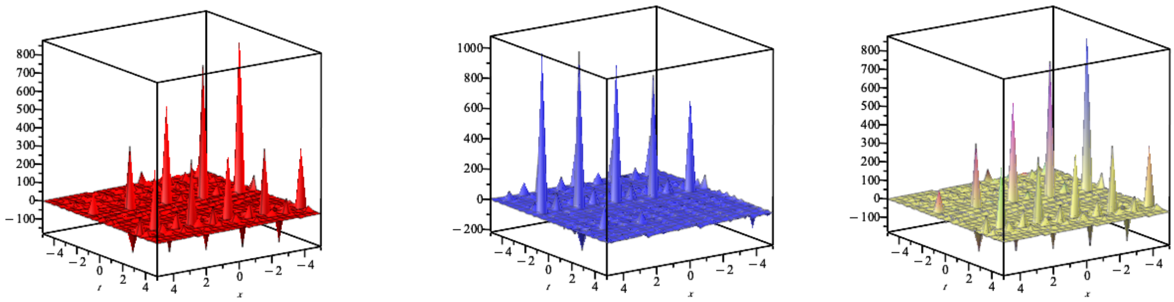
Inserting Eq. (5.8) into (2.4) yields

$$y = -\frac{2(\sin(z)-\cos(z))}{\sin(z)+\cos(z)}. \tag{5.9}$$

By using Eq. (5.9) into Eq. (2.1), we obtain a second type of trigonometric solution of Eq. (1.1):

$$q(x,t) = -\frac{2(\sin(x-ct)-\cos(x-ct))e^{i(kx-wt)}}{\sin(x-ct)+\cos(x-ct)}. \tag{5.10}$$

(See Figure 5.2)



(a) **(b)** **(c)**

Figure 5.2: 3D plots of the rational solution (5.10) in Case 2 with the values of $b=10$, $w=1$ (a) Real, (b) Imaginary and (c) Complex.

Case 3:

$$\begin{aligned}
a &= -\frac{\begin{pmatrix} -\gamma k^{11}-6\gamma k^9-50\gamma k^7+44\gamma k^6+3k^6w-44\gamma k^5+12\zeta k^4+72\gamma k^4 \\ +18k^4w+8\gamma k^3+40\gamma k^2+30k^2w-16\gamma k+48\gamma+12w-24\zeta \end{pmatrix}}{3(k^4+4k^2+2)^2},\\
\beta &= \frac{\begin{pmatrix} -\gamma k^9+6\gamma k^8+28\gamma k^7+8\gamma k^6+6k^6\zeta+82\gamma k^5-20\gamma k^4-3k^4w+36k^4\zeta \\ +152\gamma k^3-144\gamma k^2-12k^2w+84k^2\zeta+48\gamma k-48\gamma-6w+24\zeta \end{pmatrix}}{6(k^4+4k^2+2)^2},\\
\delta &= -\frac{\begin{pmatrix} -13\gamma k^9+18\gamma k^8-68\gamma k^7+104\gamma k^6+6k^6\zeta-158\gamma k^5+220\gamma k^4 \\ -3k^4w+36k^4\zeta-40\gamma k^3+48\gamma k^2-12k^2w+84k^2\zeta+24\zeta-6w \end{pmatrix}}{24(k^4+4k^2+2)^2},\\
c &= \frac{2k\left(\gamma k^7+10\gamma k^5-4\gamma k^4-8\gamma k^3+12\gamma k^2+8\gamma k-16\gamma+12\zeta\right)}{3(k^4+4k^2+2)}, \quad b=-\frac{2(k^2+2)k}{k^4+4k^2+2}, \quad \sigma=-\frac{1}{3}\gamma k.
\end{aligned} \tag{5.11}$$

By utilizing values in (5.11) into (5.1), we have

$$f = b_1\cos z + b_1\sin z. \tag{5.12}$$

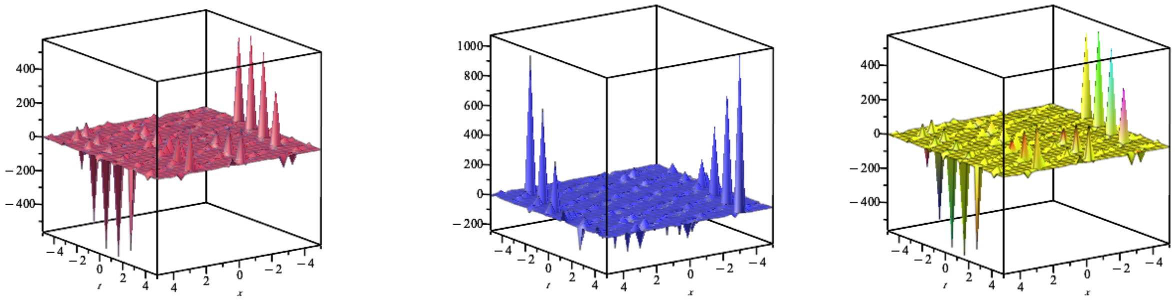
Inserting Eq. (5.12) into (2.4) yields

$$y = -\frac{2(\sin(z)-\cos(z))}{\sin(z)+\cos(z)}. \tag{5.13}$$

By using Eq. (5.13) into Eq. (2.1), we obtain a first type of trigonometric solution of Eq. (1.1):

$$q(x,t) = -\frac{2(\sin(x-ct)-\cos(x-ct))e^{i(kx-wt)}}{\sin(x-ct)+\cos(x-ct)}. \tag{5.14}$$

(See Figure 5.3)



(a) (b) (c)

Figure 5.3: 3D plots of the rational solution (5.14) in Case 3 with the values of $k=5$, $b=10$, $\gamma=0.5$, $w=1$, $\zeta=-1$ (a) Real, (b) Imaginary and (c) Complex.

6. Conclusion

New soliton solutions of the LPD equation were obtained with three different current, systematic and powerful methods. In order to understand how the obtained solutions change under different conditions, the solutions obtained by appropriate selection of some parameters affecting the shape and velocity of the solitons are observed. In this context, to understand the mechanism of the original equation (1.1) real, imaginary and complex three-dimensional plots have been drawn for each case of solitons. This paper presents novel solutions of LPD equation that have not been reported in the literature before. Also, comparing with the existing literature, our result is complete and our method is simple and direct. By providing novel solutions, this study contributes to the knowledge of the dynamical aspects of various physical phenomena that are modeled by the LPD equation.

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References

[1] G. Akram, M. Sadaf, S. Arshed, F. Sameen, *Bright, dark, kink, singular and periodic soliton solutions of Lakshmanan-Porsezian-Daniel model by generalized projective Riccati equations method*, Optik, (2021), Article ID 167051, 241.
[2] G. Akram, M. Sadaf, M. Dawood, D. Baleanu, *Optical solitons for Lakshmanan-Porsezian-Daniel equation with Kerr law non-linearity using improved* $\tan\left(\frac{\psi(\eta)}{2}\right)$*-expansion technique*, Results in Physics, (2021), Article ID 104758, 29.
[3] A. Al Qarni, A. Alshaery, H. Bakodah, *Optical solitons for the Lakshmanan-Porsezian-Daniel model by collective variable method*, Results in Optics, (2020), Article ID 100017, 1.
[4] A. Biswas, Y. Yildirim, E. Yasar, Q. Zhou, S.P. Moshokoa, M. Belic, *Optical solitons for Lakshmanan–Porsezian–Daniel model by modified simple equation method*, Optik, **160** (2018), 24–32.
[5] J. Vega-Guzman, R. T. Alqahtani, Q. Zhou, M. F. Mahmood, S. P. Moshokoa, M. Z. Ullah, A. Biswas, M. Belic, *Optical solitons for Lakshmanan–Porsezian–Daniel model with spatio-temporal dispersion using the method of undetermined coefficients*, Optik, **144** (2017), 115–123.
[6] Y. Yang, T. Suzuki, X. Cheng, *Darboux transformations and exact solutions for the integrable nonlocal Lakshmanan-Porsezian-Daniel equation*, Appl. Math. Lett., (2020), Article ID 105998, 99.
[7] X. H. Wu, Y. T. Gao, X. Yu, C. C. Ding, L. Q. Li, *Modified generalized Darboux transformation and solitons for a Lakshmanan-Porsezian-Daniel equation*, M. Chaos, Solitons & Fractals, **162** (2022), Article ID 112399.
[8] Y. Ye, C. Hou, D. Cheng, S. Chen, *Rogue wave solutions of the vector Lakshmanan–Porsezian–Daniel equation*, Phys. Lett. A, **384** (11) (2020), Article ID 126226.
[9] G. Akram, M. Sadaf, M. A. U. Khan, *Abundant optical solitons for Lakshmanan–Porsezian–Daniel model by the modified auxiliary equation method*, Optik, **251** (2022), Article ID 168163.
[10] M. Lakshmanan, K. Porsezian, M. Daniel, *Effect of discreteness on the continuum limit of the Heisenberg spin chain*, Phys. Lett. A, **133** (9) (1988), 483-488.
[11] N. A. Kudryashov, *The Lakshmanan–Porsezian–Daniel model with arbitrary refractive index and its solution*, Optik, (2021), Article ID 167043, 241.
[12] J. Y. Yang, W. X. Ma, Z. Qin, *Lump and lump-soliton solutions to the* $(2+1)$*-dimensional Ito equation*, Anal. Math. Phys., **8** (2018), 427–436.

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Exact Traveling Wave Solutions of the Schamel-KdV Equation with Two Different Methods

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Abstract

The Schamel-Korteweg-de Vries (S-KdV) equation including a square root nonlinearity is very important pattern for the research of ion-acoustic waves in plasma and dusty plasma. As known, it is significant to discover the traveling wave solutions of such equations. Therefore, in this paper, some new traveling wave solutions of the S-KdV equation, which arises in plasma physics in the study of ion acoustic solitons when electron trapping is present and also it governs the electrostatic potential for a certain electron distribution in velocity space, are constructed. For this purpose, the Bernoulli Sub-ODE and modified auxiliary equation methods are used. It has been shown that the suggested methods are effective and give different types of function solutions as: hyperbolic, trigonometric, power, exponential, and rational functions. The applied computational strategies are direct, efficient, concise and can be implemented in more complex phenomena with the assistant of symbolic computations. The results found in the paper are of great interest and may also be used to discover the wave sorts and specialities in several plasma systems.

1. Introduction

Nonlinear partial differential equations (NPDEs) are used to describe complex problems with numerous phenomena in different fields, including engineering, chemical kinematics, biology, wave theory, optics, physics, fluid mechanics, biomedical science, and others [1]- [4]. S-KdV equation, based upon both usual KdV equation (when $\alpha = 0$) [5]- [10],

$$u_t + \beta u u_x + \delta u_{xxx} = 0,$$

and Schamel equation (when $\beta = 0$)

$$u_t + \alpha u^{1/2} u_x + \delta u_{xxx} = 0,$$

which was derived a German scientist Hans Schamel in 1973 has the form [11]- [13]

$$u_t + (\alpha u^{1/2} + \beta u)_x + \delta u_{xxx} = 0, \quad \alpha\beta \neq 0 \tag{1.1}$$

where α, β and δ are constants which they are refer to the activation trapping, the convection and the dispersion coefficients, respectively. The advantage of implementing the nonlinear S-KdV equation to analyze dynamics of modulated waves in dispersive media lies in the diversity of its solutions [14]- [17]. Here we should point out that the S-KdV equation has a stronger nonlinearity than the usual KdV

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equation in that the single soliton solution possesses a smaller width and higher velocity [18]. This equation is contained in many physical phenomena involving electromagnetic theory, physical chemistry, geophysics and other fields are examples [19, 20]. The square root in the nonlinear term then translates to lowest order some of the kinetic effects, associated with electron trapping [21]. Schamel [22] stated that when uu_x is replaced by $(\mid u\mid^{3/2})_x$, compared to the classical KdV equation, the Schamel equation possesses a stronger nonlinearity, which reveals that the wave has a smaller width and higher velocity and exact traveling wave solutions for the regularized Schamel equation [23].To create different exact solutions and to notice their properties, various significant methods have been developed [19]- [21], [24]- [29].
The implementations of the Sub-ODE and modified auxiliary equation methods in this paper highlight our main motivation and indicates its capacity to handle nonlinear equations, permitting it to be utilized to solve many types of nonlinearity models. The body of our paper is structured as follows: Methodologies of the Sub-ODE and modified auxiliary equation methods and their detailed structures are given in Section 2. In Section 3, we apply the different methods, introduced in Section 2, to the Schamel-KdV equation to find the exact solutions. Different forms of the exact solutions are derived from these methods. Section 4 is devoted to graphical illustrations of the methods. A discussion section is presented in Section 5. Finally, Section 6 provides conclusions stemming from the results of our work.

2. The Bernoulli Sub-ODE Method [30]- [33]

Herein, we introduce the steps of the Bernoulli Sub-ODE method. Suppose, nonlinear partial differential equation is given by

$$F(v, v_t, v_x, v_{xx}, v_{xxx}, ...) = 0, \tag{2.1}$$

where $v = v(x,t)$ is wave function to be calculated.

Step 1: Apply the traveling wave transformation,

$$v(x,t) = v(\eta),\ \ \eta = kx - ct \tag{2.2}$$

where k is constant and c is speed of the traveling wave. Substituting (2.2) into (2.1), then (2.1) converted to an ordinary differential equation:

$$F(v', v'', v''', ...) = 0 \tag{2.3}$$

where F is a polynomial in $v(\eta)$ and its derivatives.

Step 2: Presume solutions of (2.3) presented by a series in G:

$$v(\eta) = \sum_{i=0}^{N} b_i (G(\eta))^i \tag{2.4}$$

where $b_i (0 \leq i \leq N)$ are constants to be calculated, $b_N \neq 0$, and $G(\eta)$ satisfies the next ODE,

$$G'(\eta) + \lambda G(\eta) = \mu G(\eta)^2 \tag{2.5}$$

which has the following solution:

$$G(\eta) = \frac{1}{\frac{\mu}{\lambda} + de^{\lambda \eta}}$$

where $\lambda, \mu \neq 0$ are arbitrary constants.

Step 3: The positive integer N determined by balancing the highest order derivative term with the highest power nonlinear term in (2.3).

Step 4: Replacing (2.4) into (2.3), we acquire a polynomial in $G(\eta)$. Gathering all terms with the same power and equating each one to zero. We get a system of equations which can be solved by using Mathematica program.

2.1. The modified auxiliary equation method (MAE) [34]- [37]

Main steps of the modified auxiliary equation method are explained as follows:

Step 1: Solution of (2.3) is given by:

$$v(\eta) = \sum_{i=0}^{N} (b_i a^{if(\eta)} \tag{2.6}$$

where $f(\eta)$ satisfies the following ODE:

$$f' = \frac{1}{\ln a} \left(\mu a^{-f(\eta)} + \sigma + \lambda a^{f(\eta)} \right) \tag{2.7}$$

where $b_i\, (i = 0, 1, 2, \ldots, N)$, $b_N \neq 0$, λ, σ and μ, are constants to be calculated.

Step 2: In (2.3), N is a positive integer determined via the homogeneous balance principle as illustrated before.

Step 3: Substituting (2.6) and (2.7) in (2.3), and gathering the terms which had like powers of $(a^{f(\eta)})$ and putting their coffecients equal to zero, we obtain a set of algebraic equations, which can be solved by the aid of Mathematica program.

Step 4: There is various sets of solutions of (2.7):

Set 1: $\sigma^2 - 4\lambda\mu < 0$ and $\lambda \neq 0$,

$$a^{f(\eta)} = \frac{-\sigma}{2\lambda} + \frac{\sqrt{4\mu\lambda - \sigma^2}}{2\lambda} \tan\left(\frac{\sqrt{4\mu\lambda - \sigma^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{-\sigma}{2\lambda} + \frac{\sqrt{4\mu\lambda - \sigma^2}}{2\lambda} \cot\left(\frac{\sqrt{4\mu\lambda - \sigma^2}\eta}{2}\right).$$

Set 2: $\sigma^2 - 4\lambda\mu > 0$ and $\lambda \neq 0$,

$$a^{f(\eta)} = \frac{-\sigma}{2\lambda} - \frac{\sqrt{\sigma^2 - 4\mu\lambda}}{2\lambda} \tanh\left(\frac{\sqrt{\sigma^2 - 4\mu\lambda}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{-\sigma}{2\lambda} - \frac{\sqrt{\sigma^2 - 4\mu\lambda}}{2\lambda} \coth\left(\frac{\sqrt{\sigma^2 - 4\mu\lambda}\eta}{2}\right).$$

Set 3: $\sigma^2 + 4\mu^2 < 0, \lambda \neq 0 and \lambda = -\mu$,

$$a^{f(\eta)} = \frac{\sigma}{2\mu} - \frac{\sqrt{-\sigma^2 - 4\mu^2}}{2\mu} \tan\left(\frac{\sqrt{-\sigma^2 - 4\mu^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{\sigma}{2\mu} - \frac{\sqrt{-\sigma^2 - 4\mu^2}}{2\mu} \cot\left(\frac{\sqrt{-\sigma^2 - 4\mu^2}\eta}{2}\right).$$

Set 4: $\sigma^2 + 4\mu^2 > 0, \lambda \neq 0$ and $\lambda = -\mu$,

$$a^{f(\eta)} = \frac{\sigma}{2\mu} + \frac{\sqrt{\sigma^2 + 4\mu^2}}{2\mu} \tanh\left(\frac{\sqrt{\sigma^2 + 4\mu^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{\sigma}{2\mu} + \frac{\sqrt{\sigma^2 + 4\mu^2}}{2\mu} \coth\left(\frac{\sqrt{\sigma^2 + 4\mu^2}\eta}{2}\right).$$

Set 5:$\sigma^2 - 4\mu^2 < 0$ and $\lambda = \mu$,

$$a^{f(\eta)} = \frac{-\sigma}{2\mu} + \frac{\sqrt{-\sigma^2 + 4\mu^2}}{2\mu} \tan\left(\frac{\sqrt{-\sigma^2 + 4\mu^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{-\sigma}{2\mu} + \frac{\sqrt{-\sigma^2 + 4\mu^2}}{2\mu} \cot\left(\frac{\sqrt{-\sigma^2 + 4\mu^2}\eta}{2}\right).$$

Set 6: $\sigma^2 - 4\mu^2 > 0$ and $\lambda = \mu$,

$$a^{f(\eta)} = \frac{-\sigma}{2\mu} - \frac{\sqrt{\sigma^2 - 4\mu^2}}{2\mu} \tanh\left(\frac{\sqrt{\sigma^2 - 4\mu^2}\eta}{2}\right),$$

or

$$a^{f(\eta)} = \frac{-\sigma}{2\mu} - \frac{\sqrt{\sigma^2 - 4\mu^2}}{2\mu} \coth\left(\frac{\sqrt{\sigma^2 - 4\mu^2}\eta}{2}\right).$$

Set 7: $\sigma^2 = 4\lambda\mu$ and $\lambda = \mu$,

$$a^{f(\eta)} = -\frac{2 + \sigma\eta}{2\lambda\eta}.$$

Set 8: $\lambda\mu < 0, \sigma = 0$ and $\lambda \neq 0$,

$$a^{f(\eta)} = -\sqrt{\frac{-\mu}{\lambda}}\tanh\left(\sqrt{-\mu\lambda}\eta\right),$$

or

$$a^{f(\eta)} = -\sqrt{\frac{-\mu}{\lambda}}\coth\left(\sqrt{-\mu\lambda}\eta\right).$$

Set 9:$\sigma = 0$ and $\mu = -\lambda$,

$$a^{f(\eta)} = \frac{1+e^{-2\lambda\eta}}{-1+e^{-2\lambda\eta}}.$$

Set 10: $\mu = \lambda = 0$,

$$a^{f(\eta)} = \cosh(\sigma\eta)+\sinh(\sigma\eta).$$

Set 11: $\mu = \sigma =h$ and $\lambda = 0$,

$$a^{f(\eta)} = e^{h\eta}-1.$$

Set 12: $\lambda = \sigma =h$ and $\mu = 0$,

$$a^{f(\eta)} = \frac{e^{h\eta}}{1-e^{h\eta}}.$$

Set 13: $\sigma = \lambda+\mu$,

$$a^{f(\eta)} = -\frac{1-\mu e^{(\mu-\lambda)\eta}}{1-\lambda e^{(\mu-\lambda)\eta}}.$$

Set 14: $\sigma = -(\lambda+\mu)$,

$$a^{f(\eta)} = \frac{\mu-e^{(\mu-\lambda)\eta}}{\lambda-e^{(\mu-\lambda)\eta}}.$$

Set 15: $\mu = 0$,

$$a^{f(\eta)} = \frac{\sigma e^{\sigma\eta}}{1-\lambda e^{\sigma\eta}}.$$

Set 16: $\lambda = \mu = \sigma \neq 0$,

$$a^{f(\eta)} = \sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\mu\eta\right)-1.$$

Set 17: $\lambda = \sigma = 0$,

$$a^{f(\eta)} = \mu\eta.$$

Set 18: $\mu = \sigma = 0$,

$$a^{f(\eta)} = \frac{-1}{\lambda\eta}.$$

Set 19: $\lambda = \mu$ and $\sigma = 0$,

$$a^{f(\eta)} = \tan(\mu\eta).$$

Set 20: $\lambda = 0$,

$$a^{f(\eta)} = e^{\sigma\eta}-\frac{\mu}{\sigma}.$$

3. Applications of the Methods

Begin with the following transformation:

$$v(x,t)=u(x,t)^2,$$

with wave transformation (2.2) into (1.1), we obtain following ordinary differential equation:

$$2k\beta u^3u'+6k^3\delta u'u''+2u((-c+k\alpha u)u'+k^3\delta u''')=0.$$

Integrating once with respect to η, we get

$$-cu^2+\frac{2}{3}k\alpha u^3+\frac{1}{2}k\beta u^4+2k^3\delta(u')^2+2k^3\delta uu''=0. \tag{3.1}$$

Balancing uu'' with u^4 in (3.1), we get $4N=2N+2$, then $N=1$.

3.1. The Bernoulli sub-ODE method

Using (2.4), solution of (3.1) is given by

$$u(\eta)=b_0+b_1G(\eta). \tag{3.2}$$

Substituting (3.2) in (3.1), then collecting terms of the same powers and putting their coefficients equal to zero, next system of equations are acquired :

$$\begin{aligned}
&-cb_0^2+\frac{2}{3}k\alpha b_0^3+\frac{1}{2}k\beta b_0^4=0,\\
&-2cb_0b_1+2k^3\delta\lambda^2b_0b_1+2k\alpha b_0^2b_1+2k\beta b_0^3b_1=0,\\
&-6k^3\delta\lambda\mu b_0b_1-cb_1^2+4k^3\delta\lambda^2b_1^2+2k\alpha b_0b_1^2+3k\beta b_0^2b_1^2=0,\\
&4k^3\delta\mu^2b_0b_1-10k^3\delta\lambda\mu b_1^2+\frac{2}{3}k\alpha b_1^3+2k\beta b_0b_1^3=0,\\
&6k^3\delta\mu^2b_1^2+\frac{1}{2}k\beta b_1^4=0.
\end{aligned}$$

In what follows, we present the two sets of solution:
Set 1:

$$\delta=-\frac{4\alpha^2}{75k^2\beta\lambda^2},\quad b_0=-\frac{4\alpha}{5\beta},\quad b_1=\frac{4\alpha\mu}{5\beta\lambda},\quad c=-\frac{16k\alpha^2}{75\beta},\quad v(x,t)=\left(-\frac{4\alpha}{5\beta}+\frac{4\alpha\mu}{5\beta\lambda}\left(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\left(kx+\left(\frac{16k\alpha^2}{75\beta}\right)t\right)}}\right)\right)^2. \tag{3.3}$$

Set 2:

$$\delta=-\frac{4\alpha^2}{75k^2\beta\lambda^2},\quad b_0=0,\quad b_1=-\frac{4\alpha\mu}{5\beta\lambda},\quad c=-\frac{16k\alpha^2}{75\beta},\quad v(x,t)=\left(-\frac{4\alpha\mu}{5\beta\lambda}\left(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\left(kx+\left(\frac{16k\alpha^2}{75\beta}\right)t\right)}}\right)\right)^2. \tag{3.4}$$

3.2. The modified auxiliary equation method (MAE)

(2.6) presents the solution in the form:

$$u(\eta)=b_0+b_1a^{f(\eta)}. \tag{3.5}$$

Substituting (3.5)in (3.1), then summing terms of like powers and setting their coefficients equal to zero, the next system of equations are obtained:

$$\begin{aligned}
&-cb_0^2+\frac{2}{3}k\alpha b_0^3+\frac{1}{2}k\beta b_0^4+2k^3\delta\mu\sigma b_0b_1+2k^3\delta\mu^2b_1^2=0,\\
&-2cb_0b_1+4k^3\delta\lambda\mu b_0b_1+2k^3\delta\sigma^2b_0b_1+2k\alpha b_0^2b_1+2k\beta b_0^3b_1+6k^3\delta\mu\sigma b_1^2=0,\\
&6k^3\delta\lambda\sigma b_0b_1-cb_1^2+8k^3\delta\lambda\mu b_1^2+4k^3\delta\sigma^2b_1^2+2k\alpha b_0b_1^2+3k\beta b_0^2b_1^2=0,\\
&4k^3\delta\lambda^2b_0b_1+10k^3\delta\lambda\sigma b_1^2+\frac{2}{3}k\alpha b_1^3+2k\beta b_0b_1^3=0,\\
&6k^3\delta\lambda^2b_1^2+\frac{1}{2}k\beta b_1^4=0.
\end{aligned}$$

Solving the previous system yields two sets of solutions:

$$c=4k^3\delta(-4\lambda\mu+\sigma^2),\quad \alpha=\pm\frac{15k^2\delta\lambda\sqrt{(-4\lambda\mu+\sigma^2)b_1^2}}{b_1^2},\quad \beta=-\frac{12k^2\delta\lambda^2}{b_1^2},\quad b_0=\frac{\sigma b_1\pm\sqrt{(-4\lambda\mu+\sigma^2)b_1^2}}{2\lambda}.$$

Therefore, using the above sets gives the solitary wave solutions to (2.5) in the following formulas:

$(kx-(4k^3\delta(-4\lambda\mu+\sigma^2))t)$

Set 1: $\sigma^2-4\lambda\mu<0$ and $\lambda\neq 0$,

$$v_{1,2}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\lambda\mu+\sigma^2\right)b_1^2}}{2\lambda}+b_1\left(\frac{-\sigma}{2\lambda}+\frac{\sqrt{4\mu\lambda-\sigma^2}}{2\lambda}\tan\left(\frac{\sqrt{4\mu\lambda-\sigma^2}\left(kx-\left(4k^3\delta\left(-4\lambda\mu+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{3,4}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\lambda\mu+\sigma^2\right)b_1^2}}{2\lambda}+b_1\left(\frac{-\sigma}{2\lambda}+\frac{\sqrt{4\mu\lambda-\sigma^2}}{2\lambda}\cot\left(\frac{\sqrt{4\mu\lambda-\sigma^2}\left(kx-\left(4k^3\delta\left(-4\lambda\mu+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2.$$

Set 2: $\sigma^2-4\lambda\mu>0$ and $\lambda\neq 0$,

$$v_{5,6}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\lambda\mu+\sigma^2\right)b_1^2}}{2\lambda}+b_1\left(\frac{-\sigma}{2\lambda}-\frac{\sqrt{\sigma^2-4\mu\lambda}}{2\lambda}\tanh\left(\frac{\sqrt{\sigma^2-4\mu\lambda}\left(kx-\left(4k^3\delta\left(-4\lambda\mu+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{7,8}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\lambda\mu+\sigma^2\right)b_1^2}}{2\lambda}+b_1\left(\frac{-\sigma}{2\lambda}-\frac{\sqrt{\sigma^2-4\mu\lambda}}{2\lambda}\coth\left(\frac{\sqrt{\sigma^2-4\mu\lambda}\left(kx-\left(4k^3\delta\left(-4\lambda\mu+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2. \tag{3.6}$$

Set 3: $\sigma^2+4\mu^2<0, \lambda\neq 0$ and $\lambda=-\mu$,

$$v_{9,10}(x,t)=\left(-\frac{\sigma b_1\pm\sqrt{\left(4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(\frac{\sigma}{2\mu}-\frac{\sqrt{-\sigma^2-4\mu^2}}{2\mu}\tan\left(\frac{\sqrt{-\sigma^2-4\mu^2}\left(kx-\left(4k^3\delta\left(4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{11,12}(x,t)=\left(-\frac{\sigma b_1\pm\sqrt{\left(4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(\frac{\sigma}{2\mu}-\frac{\sqrt{-\sigma^2-4\mu^2}}{2\mu}\cot\left(\frac{\sqrt{-\sigma^2-4\mu^2}\left(kx-\left(4k^3\delta\left(4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2.$$

Set 4: $\sigma^2+4\mu^2>0$, $\lambda\neq 0$ and $\lambda=-\mu$,

$$v_{13,14}(x,t)=\left(-\frac{\sigma b_1\pm\sqrt{\left(4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(\frac{\sigma}{2\mu}+\frac{\sqrt{\sigma^2+4\mu^2}}{2\mu}\tanh\left(\frac{\sqrt{\sigma^2+4\mu^2}\left(kx-\left(4k^3\delta\left(4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{15,16}(x,t)=\left(-\frac{\sigma b_1\pm\sqrt{\left(4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(\frac{\sigma}{2\mu}+\frac{\sqrt{\sigma^2+4\mu^2}}{2\mu}\coth\left(\frac{\sqrt{\sigma^2+4\mu^2}\left(kx-\left(4k^3\delta\left(4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2.$$

Set 5: $\sigma^2-4\mu^2<0$ and $\lambda=\mu$,

$$v_{17,18}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(-\frac{\sigma}{2\mu}+\frac{\sqrt{-\sigma^2+4\mu^2}}{2\mu}\tan\left(\frac{\sqrt{-\sigma^2+4\mu^2}\left(kx-\left(4k^3\delta\left(-4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{19,20}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(-\frac{\sigma}{2\mu}+\frac{\sqrt{-\sigma^2+4\mu^2}}{2\mu}\cot\left(\frac{\sqrt{-\sigma^2+4\mu^2}\left(kx-\left(4k^3\delta\left(-4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2. \tag{3.7}$$

Set 6: $\sigma^2-4\mu^2>0$ and $\lambda=\mu$,

$$v_{21,22}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(-\frac{\sigma}{2\mu}-\frac{\sqrt{\sigma^2-4\mu^2}}{2\mu}\tanh\left(\frac{\sqrt{\sigma^2-4\mu^2}\left(kx-\left(4k^3\delta\left(-4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2,$$

or

$$v_{23,24}(x,t)=\left(\frac{\sigma b_1\pm\sqrt{\left(-4\mu^2+\sigma^2\right)b_1^2}}{2\mu}+b_1\left(-\frac{\sigma}{2\mu}-\frac{\sqrt{\sigma^2-4\mu^2}}{2\mu}\coth\left(\frac{\sqrt{\sigma^2-4\mu^2}\left(kx-\left(4k^3\delta\left(-4\mu^2+\sigma^2\right)\right)t\right)}{2}\right)\right)\right)^2.$$

Set 7: $\sigma^2=4\lambda\mu$ and $\lambda=\mu$,

$$v_{25,26}(x,t)=\left(\frac{\sqrt{\lambda\mu}b_1}{\lambda}-\frac{\left(2+2kx\sqrt{\lambda\mu}\right)b_1}{2kx\lambda}\right)^2.$$

Set 8: $\lambda\mu<0$, $\sigma=0$ and $\lambda\neq 0$,

$$v_{27,28}(x,t)=\left(\pm\frac{\sqrt{-\lambda\mu b_1^2}}{\lambda}-\sqrt{-\frac{\mu}{\lambda}}b_1\tanh\left(\sqrt{-\lambda\mu}\left(kx+16k^3t\delta\lambda\mu\right)\right)\right)^2,$$

or (3.8)

$$v_{29,30}(x,t)=\left(\pm\frac{\sqrt{-\lambda\mu b_1^2}}{\lambda}-\sqrt{-\frac{\mu}{\lambda}}b_1\coth\left(\sqrt{-\lambda\mu}\left(kx+16k^3t\delta\lambda\mu\right)\right)\right)^2.$$

Set 9: $\sigma=0$ and $\mu=-\lambda$,

$$v_{31,32}(x,t)=\left(\pm\frac{\sqrt{\lambda^2b_1^2}}{\lambda}+b_1\left(\frac{1+e^{-2\lambda\left(kx-16k^3\delta\lambda^2t\right)}}{-1+e^{-2\lambda\left(kx-16k^3\delta\lambda^2t\right)}}\right)\right)^2.$$

Set 10: $\lambda=\sigma=h$ and $\mu=0$,

$$v_{33,34}(x,t)=\left(\frac{hb_1\pm\sqrt{h^2b_1^2}}{2h}+b_1\left(\frac{e^{h\left(kx-4h^2k^3\delta t\right)}}{1-e^{h\left(kx-4h^2k^3\delta t\right)}}\right)\right)^2.$$

Set 11: $\sigma=\lambda+\mu$,

$$v_{35,36}(x,t)=\left(\frac{b_1\left(\lambda+\mu\right)\pm b_1\left(\lambda-\mu\right)}{2\lambda}-b_1\left(\frac{1-\mu e^{(\mu-\lambda)\left(kx-4k^3\delta(\lambda-\mu)^2t\right)}}{1-\lambda e^{(\mu-\lambda)\left(kx-4k^3\delta(\lambda-\mu)^2t\right)}}\right)\right)^2.$$

Set 12: $\sigma=-(\lambda+\mu)$,

$$v_{37,38}(x,t)=\left(\frac{b_1\left(-\lambda-\mu\right)\pm b_1\left(\lambda-\mu\right)}{2\lambda}+b_1\left(\frac{\mu-e^{(\mu-\lambda)\left(kx-4k^3\delta(\lambda-\mu)^2t\right)}}{\lambda-e^{(\mu-\lambda)\left(kx-4k^3\delta(\lambda-\mu)^2t\right)}}\right)\right)^2$$

Set 13: $\mu=0$,

$$v_{39,40}(x,t)=\left(\frac{\sigma b_1\pm\sigma b_1}{2\lambda}+b_1\left(\frac{\sigma e^{\sigma\left(kx-4k^3\delta\sigma^2t\right)}}{1-\lambda e^{\sigma\left(kx-4k^3\delta\sigma^2t\right)}}\right)\right)^2.$$

Set 14: $\lambda=\mu=\sigma\neq 0$,

$$v_{41,42}(x,t)=\left(\frac{b_1\pm\sqrt{-3b_1^2}}{2}+\frac{1}{2}b_1\left(-1+\sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\sigma\left(kx+12k^{3\delta}\sigma^2t\right)\right)\right)\right)^2.$$

Set 15: $\mu=\sigma=0$,

$$v_{43,44}(x,t)=\frac{b_1^2}{k^2\lambda^2x^2}.$$

Set 16: $\lambda=\mu$ and $\sigma=0$,

$$v_{45,46}(x,t)=\left(\pm\sqrt{-b_1^2}+b_1\tan\left(\mu\left(kx+16k^3\delta\mu^2t\right)\right)\right)^2.$$

4. Graphical Illustrations

The majority of our solutions are presented in the following graphs to illustrate solutions.
In Figure 4.1, we present graph of (3.3) using the Bernoulli Sub-ODE method at $k=2,\alpha=0.5,\beta=0.3,\mu=0.3,\lambda=0.3,d=2$. Figure 4.2 shows graph of (3.4) using the Bernoulli Sub-ODE method at $k=2,\alpha=0.5,\beta=0.3,\mu=0.3,\lambda=0.3,d=2$. Graph of (3.6) using the modified auxiliary equation method at $k=2,b_1=0.3,\mu=0.02,\lambda=0.1,\delta=0.1,\sigma=0.3$ is presented in Figure 4.3. Graph of (3.7) using the modified auxiliary equation method at $k=0.6,b_1=0.1,\mu=0.03,\delta=1.6,\sigma=0.04$ is given in Figure 4.4. Lastly, Figure 4.5 presents graph of (3.8) using the modified auxiliary equation method at $k=0.7,b_1=0.1,\mu=0.3,\delta=0.3,\sigma=0,\lambda=-0.5$.

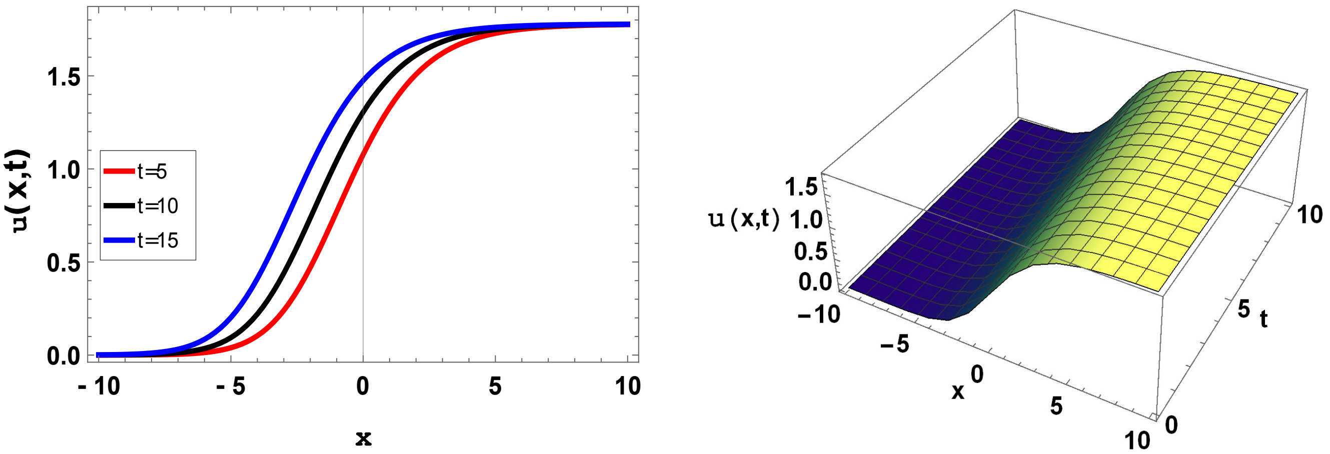

Figure 4.1: Graph of Eq. (3.3) using the Bernoulli Sub-ODE method at $k=2,\alpha=0.5,\beta=0.3,\mu=0.3,\lambda=0.3,d=2$.

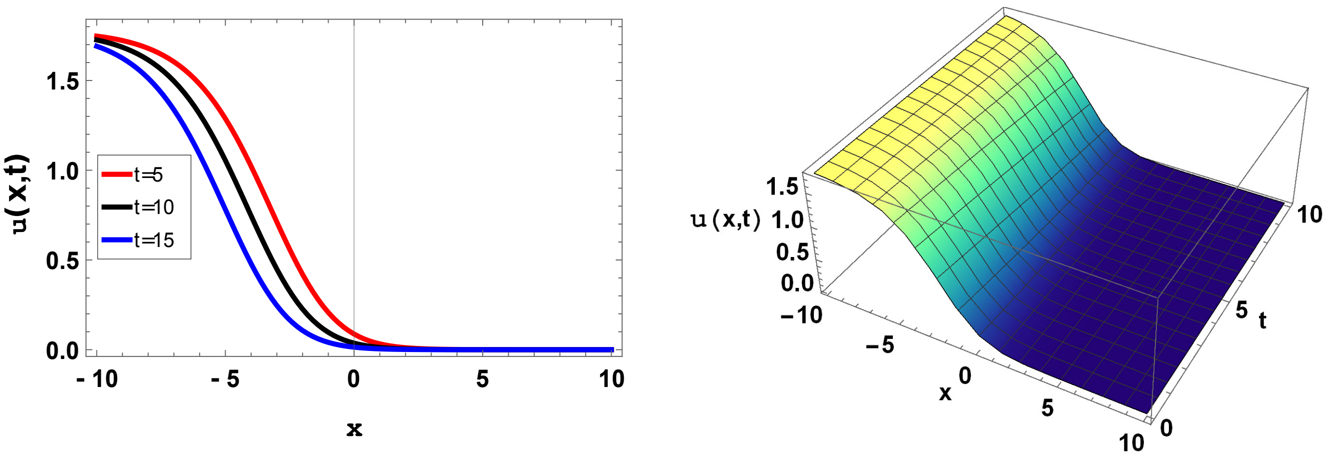

Figure 4.2: Graph of Eq. (3.4) using the Bernoulli Sub-ODE method at $k=2,\alpha=0.5,\beta=0.3,\mu=0.3,\lambda=0.3,d=2$.

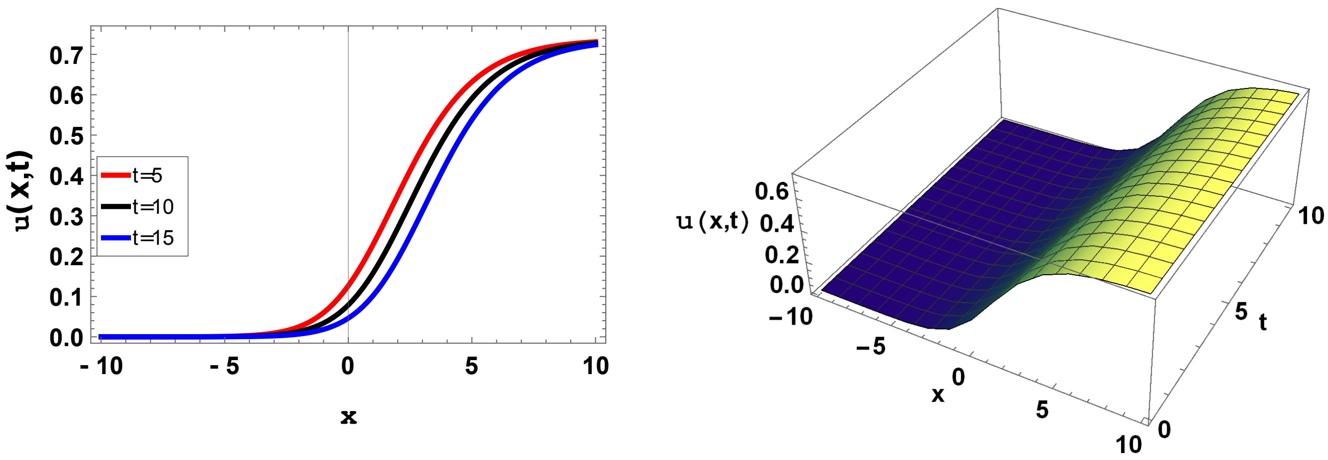

Figure 4.3: Graph of Eq. (3.6) using the modified auxiliary equation method at $k = 2, b_1 = 0.3, \mu = 0.02, \lambda = 0.1, \delta = 0.1, \sigma = 0.3$.

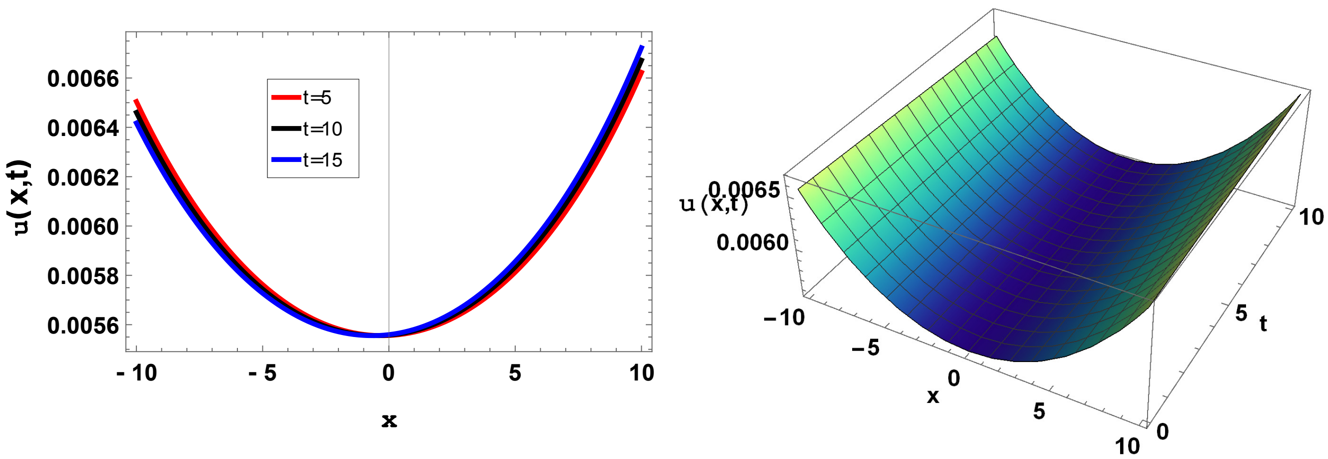

Figure 4.4: Graph of Eq. (3.7) using the modified auxiliary equation method at $k = 0.6, b_1 = 0.1, \mu = 0.03, \delta = 1.6, \sigma = 0.04$.

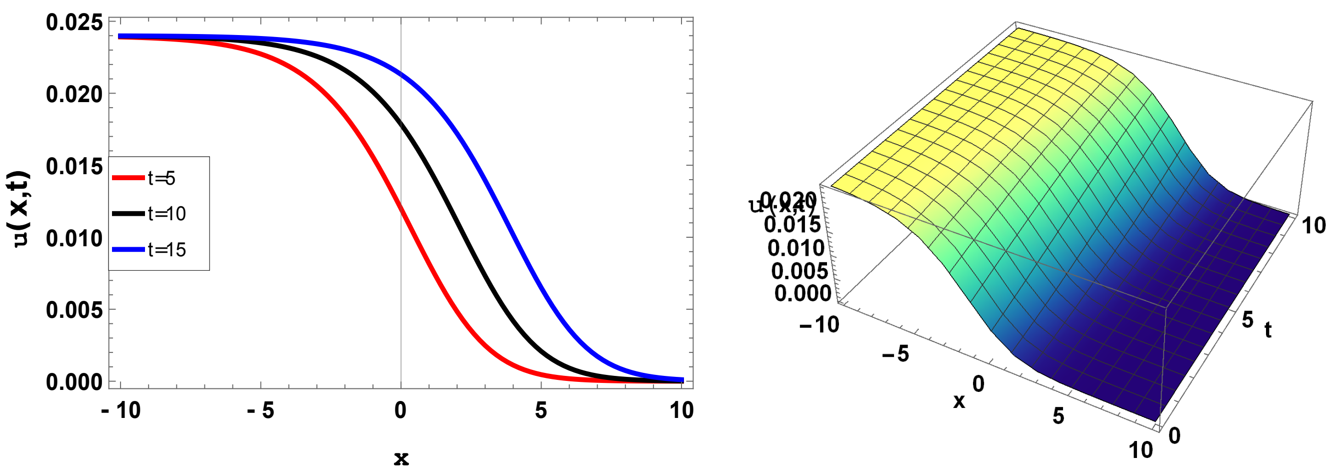

Figure 4.5: Graph of Eq. (3.8) using the modified auxiliary equation method at $k = 0.7, b_1 = 0.1, \mu = 0.3, \delta = 0.3, \sigma = 0, \lambda = -0.5$.

5. Discussion

The graph is one of the best tools for describing and presenting solutions. In the following, we review the behavior of the wave in the solutions presented: In Figures 4.1-4.2 the wave travels to the left with increasing time$t = 0,5,10$. Contrarily, in Figures 4.3-4.5 the wave moves towards left as time passes $t = 0,5,10$. The flipped wave is presented in Figure 4.4 as time goes on.

6. Conclusion

In this work, a class of some new travelling wave solutions of the Schamel–Korteweg-de Vries equation are successfully found out by using the Bernoulli Sub-ODE and modified auxiliary equation methods. The Bernoulli Sub-ODE is a simple and straightforward method and is applicable to a wide range of problems in science and engineering, but it can be time-consuming to apply the method if the equation involves complex functions. The modified auxiliary equation method can be used to solve a wide range of differential equations, also it can provide closed-form solutions, but it can be difficult to determine the appropriate auxiliary equation to use for a given differential equation. The presented exact solutions provided here may describe various new characteristics of waves and then may be useful in the theoretical and numerical studies of the considered equation. A graphical representation of newly discovered solutions are also shown to explain the dynamics of soliton profiles. The found new soliton solutions of the S-KdV equation are of significant importance and can be used in other areas of physics such as plasma physics.

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References

[1] B. Karaagac, A. Esen, Y. Ucar, N. M. Yagmurlu, *A new outlook for analysis of Noyes-Field model for the nonlinear Belousov-Zhabotinsky reaction using operator splitting method*, Computers Math. Appl. **136** (2023), 127-135.
[2] S. Kutluay, S. Ozer, N. M. Yagmurlu, *A new highly accurate numerical scheme for Benjamin-Bona-Mahony-Burgers equation describing small amplitude long wave propagation*, Mediterr. J. Math. **20**(3) (2023), 1-24.
[3] S. Kutluay, N. M. Yagmurlu, A. S. Karakas, *Operator time-splitting techniques combined with quintic B-spline collocation method for the generalized Rosenau–KdV equation*, Numerical Meth. Partial Differential Equ., **35** (2019), 2221-2235.
[4] S. Kutluay, M. Karta, N. M. Yagmurlu, *An Effective Numerical approach based on cubic Hermite B-spline collocation method for solving the 1D Heat conduction equation*, New Trends Math. Sci., **10**(4) (2022), 20-31.
[5] D. J. Korteweg, G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and a new type of long stationary wave*, Philosophical Magazine, **39** (1895) 422-443.
[6] N. J. Zabusky, *A synergetic approach to problem of nonlinear dispersive wave propagation and interaction, in: W. Ames (Ed.).* Proc. Symp. Nonlinear Partial Diff. Equ., Academic Press (1967) 223-258.
[7] S. B. G. Karakoc, *A quartic subdomain finite element method for the modified KdV equation*, Statistics, Optimization Information Comp. **6** (2018), 609-618.
[8] A. Bashan, A. Esen, *Single soliton and double soliton solutions of the quadratic-nonlinear Korteweg-de Vries equation for small and long-times*, Numerical Meth. Partial Differential Equ., **37**(2) (2021), 1561-1582.
[9] S. B. G. Karakoc, *Numerical solutions of the modified KdV equation with collocation method* Malaya J. Mat., **6**(4) (2018), 835-842.
[10] S. B. G. Karakoc, A. Saha, D. Sucu, *A novel implementation of Petrov-Galerkin method to shallow water solitary wave pattern and superperiodic traveling wave and its multistability generalized Korteweg-de Vries equation*, Chinese J. Phys., **68** (2020), 605-617.
[11] A. I. Zemlyanukhin, I. V. Andrianov, A. V. Bochkarev, L. I. Mogilevich, *The generalized Schamel equation in nonlinear wave dynamics of cylindrical shells*, Nonlinear Dyn. **98** (2019), 185-194.
[12] M. G.Kuzyk, *Polymer Fiber Optics: Materials, Materials, Physics, and Applications*, CRC Press, 2018.
[13] K. Shimoda, *Introduction to Laser Physics*. Springer-Verlag, Heidelberg, 1986.
[14] M. W. Coffey, *On the integrability of schamel's modified Korteweg-de Vries equation*, J. Phys. A: Mathematical and General **24**(23) (1991).
[15] Y. Wu, Z. Liu, *New types of nonlinear waves and bifurcation phenomena in Schamel-Korteweg-de Vries equation*, Abstr. Appl. Anal., **2013** (2013), 1-18.
[16] J. Yang, S. Q. Tand, *Exact traveling wave solutions of the Schamel-Korteweg-de Vries equation*, J. Math. Sci. Adv. Appl., **31**(25) (2015).
[17] E. Kengne, A. Lakhssassi, W. M. Liu, *Nonlinear Schamel–Korteweg deVries equation for a modified Noguchi nonlinear electric transmission network: Analytical circuit modeling*, Chaos, Solitons, Fractals **140** (2020), 110229.
[18] H. Schamel, *Stationary solitary, snoidal and sinusoidal ion acoustic waves*, Plasma Phys. **14**(10) (1972), 905-924.
[19] K. U. Tariq, H. Rezazadeh, M. Zubair, M. S. Osman, L. Akinyemi, *New exact and solitary wave solutions of nonlinear Schamel–KdV equation*, Int. J. Appl. Comput. Math., **8**(114) (2022), 1-16.
[20] Z. Pinar, A. Yildirim, S. T. Mohyud-Din, K. F. Oğuz, S. Djabrailov, A. Biswas, *New exact solutions for Schamel-Korteweg-de-Vries equation*, Studies in Nonlinear Sci. **3**(3) (2012), 102-106.
[21] F. Kangalgil, *Travelling wave solutions of the Schamel–Korteweg–de Vries and the Schamel equations*, J. Egyptian Math. Soc., **24** (2016), 526-531.
[22] H. Schamel, *A modified Korteweg-de Vries equation for ion acoustic waves due to resonant electrons*, J. Plasma Phys. **9**(3) (1973), 377-387.
[23] Q. Cai, K. Tan, J. Li, *Bifurcations and exact traveling wave solutions for the regularized Schamel equation*,Open Math., **19** (2021), 1699-1712.
[24] O. Dönmez, D. Dağhan, *Analytic solutions of the Schamel-KdV equation by using different methods: Application to a dusty space plasma*, Süleyman Demirel Univ. J. Natural and Appl. Sci. **21**(1) (2017), 208-215.

[25] I. B. Giresunlu, Y. S. Özkan, E. Yaşar, *On the exact solutions, lie symmetry analysis, and conservation laws of Schamel–Korteweg–de Vries equation*, Math. Methods Appl. Sci. **40**(11) (2017), 3927-3936.

[26] K. U. Tariq, M. Inc, H. Y. Martinez, M. M. A. Khater, *Explicit, periodic and dispersive soliton solutions to the Schamel-KdV equation with constant coefficients*, Journal of Ocean Engineering and Science, https://doi.org/10.1016/j.joes.2022.05.025 (In press).

[27] S. K. Mohanty, A. N. Dev, *Recent trends in applied mathematics*, Lect. Notes Mech. Engrg., **174** (2021), 109-136.

[28] J. Lee, R. Sakthivel, *Exact travelling wave solutions of the Schamel-Korteweg- de Vries equation*, Rep. Mathematical Phys., **68**(2) (2011), 153-161.

[29] H. I. Abdel-Gawad, M. Tantawy, *Exact solutions of the Shamel-Korteweg-de Vries equation with time dependent coefficients*, Inf. Sci. Lett. **3** (2014), 103-109.

[30] N. Taghizadeh, M. Akbari, P. Esmaeelnejhad, *Parirokh Esmaeelnejhad, application of Bernoulli Sub-ODE method for finding travelling wave solutions of Schrödinger Equation Power Law Nonlinearity*, Appl. Appl. Math., **12**(1) (2017), 596-603.

[31] Q. Feng, *Traveling wave solution of (3+1) dimensional potential-YTSF equation by Bernoulli Sub-ODE method*, Adv. Mater. Res., **403** (2012), 212-216.

[32] B. Zheng, *Application of a Generalized Bernoulli Sub-ODE Method For Finding Traveling Solutions Of Some Nonlinear Equations*, Wseas Trans. Math., **7**(11) (2012), 618-626.

[33] B. Zheng, *New analytical solutions for two equations by a proposed Sub-ODE method*, International Conference on Computer Technology and Science (ICCTS 2012) **47** (2012), 360-364.

[34] M. A. Akbar, N. H. Mohd Ali, T. Tanjim, *Outset of multiple soliton solutions to the nonlinear Schrödinger equation and the coupled Burgers equation*, J. Phys. Commun., **3** (2019), 1-17.

[35] S. M. R.Islam, S. Khan, S. M. Y. Arafat, M. A. Akbar, *Diverse analytical wave solutions of plasma physics and water wave equations*, Results in Physics, **40** (2022), 1-8.

[36] M. A. Khani, M. A. Akbar, N. HJ. M. Ali, M. Abbas, *The new auxiliary method in the solution of the generalized Burgers-Huxley equation*, J. Prime Res. Math., **16**(2) (2020), 16-26.

[37] M. M. A. Khater, A. R. Seadawy, D. Lu, *Dispersive optical soliton solutions for higher order nonlinear Sasa-Satsuma equation in mono mode fibers via new auxiliary equation method*, Superlattices and Microstructures **113** (2018), 346-358.

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Characterizations of Matrix and Compact Operators on BK Spaces

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Abstract

In the present paper, by estimating operator norms, we give some characterizations of infinite matrix classes $\left(\left|E_{\mu}^{r}\right|_{q},\Lambda\right)$ and $\left(\left|E_{\mu}^{r}\right|_{\infty},\Lambda\right)$, where the absolute spaces $\left|E_{\mu}^{r}\right|_{q}$, $\left|E_{\mu}^{r}\right|_{\infty}$ have been recently studied by Gökçe and Sarıgöl [1] and Λ is one of the well-known spaces $c_0, c, l_\infty, l_q (q \geq 1)$. Also, we obtain necessary and sufficient conditions for each matrix in these classes to be compact establishing their identities or estimates for the Hausdorff measures of noncompactness.

1. Introduction

The summability theory is one of the most important field in mathematics specially analysis, applied mathematics, engineering sciences, quantum mechanics and probability theory, therefore, it has been chosen as the subject of study by many authors. The theory of sequence space, which is one of the main topics of the summability theory, is mainly about generalizing the concepts of convergence-divergence for sequences and series. In this context, the primary aim is to assign a limit value for non-convergent sequences or series by using a transformation given by the most general linear mappings of infinite matrices. So, several studies can be traced in the literature dealing with characterization of matrix transformation between special sequence spaces. To mention few of them are [2–6]. On the one hand, from a different perspective, using the notion of absolute summability, a lot of new spaces of series summable by the absolute summability methods have studied and introduced by authors (see [1, 7–14]). In recent paper [1], the infinite matrix classes $\left(\left|E_{\mu}^{r}\right|,\left|E_{\mu}^{r}\right|_{q}\right)$ and $\left(\left|E_{\mu}^{r}\right|_{q},\left|E_{\mu}^{r}\right|\right)$ have been investigated. In the present paper, the matrix classes $\left(\left|E_{\mu}^{r}\right|_{q},\Lambda\right)$ and $\left(\left|E_{\mu}^{r}\right|_{\infty},\Lambda\right)$ have been characterized with operator norms, where $1 \leq q < \infty$ and $\Lambda \in \{c, c_0, l_\infty, l_q\}$. Besides, establishing their identities or estimates for the Hausdorff measures of noncompactness, the necessary and sufficient conditions for each matrix in these classes to be compact have been investigated .

A linear subspaces of ω, the set of all sequences of complex numbers, is called a *sequence space*. Let Δ, Γ be any subspaces of ω and $U = (u_{nv})$ be any infinite matrix of complex components. The transform of a sequence $\delta = (\delta_v) \in \omega$ is the sequence $U(\delta)$ deduced by the usual matrix product and its terms are written as

$$U_n(\delta) = \sum_{j=0}^{\infty} u_{nj}\delta_j,$$

provided that the series converges for all $n \geq 0$. Then, U is called a matrix mapping from the space Δ into the another spaces Γ, if the sequence $U(\delta)$ exists and $U(\delta) \in \Gamma$ for all $\delta \in \Delta$. The collection, containing all such infinite matrices, is denoted by (Δ, Γ).
A triangle matrix U is given as $u_{nn} \neq 0$ for all n and $u_{nj} = 0$ for $n > j$.
The concept of domain of an infinite matrix U in the Δ is described by

$$\Lambda_U = \{\delta = (\delta_n) \in \omega : U(\delta) \in \Delta\} \tag{1.1}$$

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and also the β-dual of the sequence space Λ is given by the set

$$\Delta^{\beta}=\left\{y:\sum_{v=0}^{\infty}y_v\delta_v \text{ converges for all } \delta\in\Delta\right\}.$$

If $\Delta\subset\omega$ is a Frechet space that is a complete locally convex linear metric space, on which all coordinate functionals $r_n(\delta)=\delta_n$ are continuous for all n, then it is said to be an FK space; an FK space whose metric is given by a norm is called a BK space.
BK-spaces have a significant role in summability theory. For instance, the matrix operators between BK-spaces are continuous and when Δ is a BK-space, the matrix domain Δ_U is also a BK-space, and also its norm is given by

$$\|\delta\|_{\Delta_U}=\|U(\delta)\|_{\Delta}.$$

A BK-space $\Lambda\supset\phi$ is said to have AK property if, for all sequence $\delta=(\delta_v)\in\Delta$, there is a unique representation $\delta=\sum\limits_{v=0}^{\infty}\delta_v e^{(v)}$ where $(e^{(v)})$ is the sequence whose only nonzero term is 1 in v-th place for $v\geq 0$ and ϕ is the set of all finite sequences. For example, while the space l_∞ does not have AK property, the sequence space l_q has AK property in respect to its natural norm where $q\geq 1$.

Let Δ and Γ be two Banach spaces. The set of all continuous linear operators from Δ into Γ is represented by $\mathscr{B}(\Delta,\Gamma)$ and, for $U\in\mathscr{B}(\Delta,\Gamma)$, the norm of U is stated by

$$\|U\|=\sup_{\delta\in S_\Delta}\|U(\delta)\|_{\Gamma}.$$

If $y\in\omega$ and $\Delta\supset\phi$ is a BK-space, then

$$\|y\|^{*}_{\Delta}=\sup_{\delta\in S_\Delta}\left|\sum_{k=0}^{\infty}y_k\delta_k\right|,$$

and it is finite for $y\in\Delta^{\beta}$. Here, S'_Δ is the unit sphere in Δ.
Throughout this study, $\mu=(\mu_n)$ is any sequence of positive real numbers, $U=(u_{nj})$ be an infinite matrix of complex components for all $n,j\geq 0$ and q^* is the conjugate of q, that is $1/q+1/q^*=1$ for $q>1$, and $1/q^*=0$ for $q=1$.
Let $\sum\delta_k$ be an infinite series with partial sums s_n, and (μ_n) be a sequence of positive terms. The series $\sum\delta_v$ is said to be summable $|U,\mu_n|_q$, $1\leq q<\infty$, if (see [15])

$$\sum_{n=0}^{\infty}\mu_n^{q-1}|U_n(s)-U_{n-1}(s)|^q<\infty, \tag{1.2}$$

where $U_{-1}(s)=0$.
Point out that the method includes certain well known methods. For instance, for Cesàro matrix with $\mu_n=n$ and the weighted mean matrix, it reduces to the absolute Cesàro summability due to Flett [7] and the absolute weighted summability given by Sulaiman [6], respectively. For more applications, we refer readers to ([1, 8–10, 12]).
Also, if we choose the Euler matrix $E^r=(e^r_{ni})$ instead of U, the summability $|U,\mu_n|_q$ is reduced to the absolute Euler summability $|E^r,\mu_n|_q$ of order r. Here the terms of the matrix $E^r=(e^r_{ni})$ is given by

$$e^r_{ni}=\begin{cases}\binom{n}{i}(1-r)^{n-i}r^i, & 0\leq i\leq n\\ 0, & i>n\end{cases}$$

for all $n,i\geq 0$ and $0<r\leq 1$, [1].
The spaces of all series summable by the methods $|E^r,\mu_n|_q$, $1\leq q<\infty$, and $|E^r,\mu_n|_\infty$ have recently been introduced by Gökçe and Sarıgöl [1] as follows:

$$\left|E^r_\mu\right|_q=\left\{\delta=(\delta_v):\sum_{n=1}^{\infty}|T^r_n(q)(\delta)|^q<\infty\right\}$$

$$\left|E^r_{\mu,q}\right|_\infty=\left\{\delta=(\delta_v):\sup_n|T^r_n(q)(\delta)|<\infty\right\}$$

where $T^r_0(q)(\delta)=\delta_0$ and

$$T^r_n(q)(\delta) = \mu_n^{1/q^*}\sum_{i=1}^{n}\binom{n-1}{i-1}(1-r)^{n-i}r^i\delta_i. \tag{1.3}$$

Also, with the notation of domain, we can state $\left|E^r_\mu\right|_q=(l_q)_{T^r(q)}$ and $\left|E^r_{\mu,q}\right|_\infty=(l_\infty)_{T^r(q)}$, if we define the matrix $T^r(q)=(t^r_{nj}(q))$ by

$$t^r_{ni}(q)=\begin{cases}\mu_n^{1/q^*}\binom{n-1}{i-1}(1-r)^{n-i}r^i, & 1\leq i\leq n\\ 0, & i>n.\end{cases}$$

The inverse transformation of $T^r_n(q)$ can be written as

$$\delta_n=\sum_{i=1}^{n}\mu_i^{-1/q^*}\binom{n-1}{i-1}(r-1)^{n-i}r^{-n}T^r_i(q)(\delta), \tag{1.4}$$

[1].
Now, we list some known lemmas:

Lemma 1.1 ([1]). *Let $1 \leq q < \infty$. The spaces $\left|E^r_\mu\right|_q$ and $\left|E^r_{\mu,q}\right|_\infty$ are BK-spaces with the norms $\|\delta\|_{\left|E^r_\mu\right|_q} = \|T^r(q)(\delta)\|_{l_q}$ and $\|\delta\|_{\left|E^r_{\mu,q}\right|_\infty} = \|T^r(q)(\delta)\|_\infty$. Also, these are linearly isomorphic to the space l_q and l_∞, respectively.*

Lemma 1.2 ([16]). *The following statements hold:*

1. $U \in (l,c)$ *iff (i)* $\lim_n u_{nj}$ *exists for all* $j \geq 0$, *(ii)* $\sup_{n,j} |u_{nj}| < \infty$,
   $U \in (l,l_\infty)$ *iff (ii) holds.*
2. *If* $1 < q < \infty$, *then,* $U \in (l_q, c)$ *if and only if (i)holds, (iii)* $\sup_n \sum_{j=0}^{\infty} |u_{nj}|^{q^*} < \infty$,
   $U \in (l_q, l_\infty)$ *iff (iii) holds.*
3. $U \in (l, c_0)$ *iff (iv)* $\lim_n u_{nj} = 0$ *for all* $j \geq 0$, *(ii) hold.*
4. *If* $1 < q < \infty$, *then,* $U \in (l_q, c_0)$ *iff (iii) and (iv) hold.*
5. $U \in (l_\infty, c)$ *iff (i) holds, (v)* $\sum_{j=0}^{\infty} |u_{nj}| < \infty$ *uniformly in n,*
   $U \in (l_\infty, l_\infty)$ *iff (vi)* $\sup_n \sum_{j=0}^{\infty} |u_{nj}| < \infty$.
6. $U \in (l_\infty, c_0)$ *iff (vii)* $\lim_n \sum_{j=0}^{\infty} |u_{nj}| = 0$.
7. *If* $1 \leq p < \infty$, *then* $U \in (l_\infty, l_q)$ *iff (viii)* $\sup_K \sum_{n=0}^{\infty} \left|\sum_{k\in K} u_{nj}\right|^q < \infty$.

Lemma 1.3 ([17]). *Let $1 \leq q < \infty$. Then, $U \in (l, l_q)$ iff*

$$\|U\|_{(l,l_q)} = \sup_j \left\{ \sum_{n=0}^{\infty} |u_{nj}|^q \right\}^{1/p} < \infty.$$

Lemma 1.4 ([16]). *Let $1 < q < \infty$. Then, $U \in (l_q, l)$ iff*

$$\|U\|_{(l_q,l)} = \sup_{N\in\mathfrak{T}} \left\{ \sum_{j=0}^{\infty} \left| \sum_{n\in N} u_{nj} \right|^{q^*} \right\}^{1/q^*} < \infty$$

where $\mathfrak{T}$ stands for the collection of all finite subsets of $\mathbb{N}$.

It is difficult to apply Lemma 1.4 in applications. The following lemma presents to us an equivalent applicable norm.

Lemma 1.5 ([18]). *Let $1 < q < \infty$. Then, $U \in (l_q, l)$ iff*

$$\|U\|^{'}_{(l_q,l)} = \left\{ \sum_{j=0}^{\infty} \left(\sum_{n=0}^{\infty} |u_{nj}| \right)^{q^*} \right\}^{1/q^*} < \infty.$$

Since $\|U\|_{(l_q,l)} \leq \|U\|^{'}_{(l_q,l)} \leq 4\|U\|_{(l_q,l)}$, there exists $\zeta \in [1,4]$ such that $\|U\|^{'}_{(l_q,l)} = \zeta \|U\|_{(l_q,l)}$.

Using the Hausdorff measure of noncompactness χ introduced in [19], characterizations of compact operators on great number of sequence spaces are investigated by many researchers. For instance, to characterize the class of compact operators on several spaces, the Hausdorff measure of noncompactness method have been used by Malkowsky and Rakocevic in [20], Mursaleen and Noman in [21, 22], (see also [1, 23–26]).
Let (Δ, d) be a metric space and Q be a bounded subset of Δ. Then, χ and the number

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon - \text{net in } \Delta\}$$

are called the Hausdorff measure of noncompactness and the Hausdorff measure of noncompactness of Q, respectively.
Suppose that S is a linear operator between the Banach spaces Δ and Γ such that $S : \Delta \to \Gamma$. Then, it is said that S is compact if its domain is all of Δ and, for every bounded sequence (δ_n) in Δ, the sequence $(S(\delta_n))$ has a convergent subsequence in Γ.

Lemma 1.6 ([27]). *Let $Q \subset \Delta$ be a bounded set where Δ is one of the normed spaces c_0 or l_q for $1 \leq q < \infty$. If $R_r : \Delta \to \Delta$ is the operator defined by $R_r(y) = (y_0, y_1, ...y_r, 0, 0, ...)$ for all $y \in \Delta$, then*

$$\chi(Q) = \lim_{r\to\infty} \left(\sup_{\delta\in Q} \|(I - R_r)(y)\| \right).$$

Let χ_1, χ_2 be Hausdorff measures on Δ and Γ. If $S(Q)$ is a bounded subset of Γ and there exists $M > 0$ such that $\chi_2(S(Q)) \leq M\chi_1(Q)$ for each bounded subset Q of Δ, then the linear operator $S : \Delta \to \Gamma$ is called (χ_1, χ_2)- bounded. If an operator S is (χ_1, χ_2)- bounded, then the number

$$\|S\|_{(\chi_1,\chi_2)} = \inf\{M > 0 : \chi_2(S(Q)) \leq M\chi_1(Q) \text{ for all bounded sets } Q \subset \Delta\}$$

is called the (χ_1, χ_2)-measure noncompactness of L. Also, in case of $\chi_1 = \chi_2 = \chi$, it is written by $\|S\|_{(\chi,\chi)} = \|S\|_\chi$.

Lemma 1.7 ([28]). *$L \in \mathscr{B}(\Delta,\Gamma)$ and $S'_\delta = \{\delta \in \Delta : \|\delta\| \leq 1\}$ be the unit ball in Δ. Then,*

$$\|S\|_\chi = \chi\left(S\left(S'_\delta\right)\right)$$

and

$$S \text{ is compact } \Leftrightarrow \|S\|_\chi = 0.$$

Lemma 1.8 ([29]). *Let $T = (t_{nv})$ be an infinite triangle matrix, Δ be a normed sequence space and χ_T and χ stand for the Hausdorff measures of noncompactness on M_{Δ_T} and M_Δ, the collections of all bounded sets in Δ_T and Δ, respectively. Then, $\chi_T(Q) = \chi(T(Q))$ for each $Q \in M_{\Delta_T}$.*

Lemma 1.9 ([22]). *Let $\Delta = l_\infty$ or $\Delta \supset \phi$ be any BK-space with AK property. If $U \in (\Delta, c)$, then*

$$\lim_{n\to\infty} u_{nk} = \lambda_k \text{ exists for all } k,$$

$$\lambda = (\lambda_k) \in \Delta^\beta,$$

$$\sup_n \|U_n - \lambda\|^*_X < \infty,$$

$$\lim_{n\to\infty} U_n(\delta) = \sum_{k=0}^{\infty} \lambda_k \delta_k \text{ for each } \delta = (\delta_k) \in \Delta.$$

Lemma 1.10 ([22]). *Let $\Delta \supset \phi$ be a BK-space. Then,*
(a) If $U \in (\Delta, c_0)$, then

$$\|L_U\|_\chi = \lim_{r\to\infty}\left(\sup_{n>i}\|U_n\|^*_\Delta\right).$$

(b) If the space Δ has AK or $\Delta = l_\infty$ and $U \in (\Delta, c)$, then

$$\frac{1}{2}\lim_{i\to\infty}\left(\sup_{n\geq i}\|U_n - \lambda\|^*_\Delta\right) \leq \|S_U\|_\chi \leq \lim_{i\to\infty}\left(\sup_{n\geq i}\|U_n - \lambda\|^*_\Delta\right)$$

where $\lambda = (\lambda_k)$ defined by $\lambda_k = \lim_{n\to\infty} u_{nk}$, for all $n \in \mathbb{N}$.
(c) If $U \in (\Delta, l_\infty)$, then

$$0 \leq \|S_U\|_\chi \leq \lim_{i\to\infty}\left(\sup_{n>i}\|U_n\|^*_\Delta\right).$$

2. Matrix and Compact Operators on the Spaces $\left|E^r_\mu\right|_q$ and $\left|E^r_{\mu,q}\right|_\infty$

In this part of the study, firstly, by computing operator norms we obtain some characterizations of infinite matrix classes $\left(\left|E^r_\mu\right|_q, \Lambda\right)$ and $\left(\left|E^r_\mu\right|_\infty, \Lambda\right)$, where Λ is one of the spaces c, c_0, l_∞, l_q and $1 \leq q < \infty$. Moreover, we search the necessary and sufficient conditions for each matrix in these classes to be compact establishing their estimates or identities for the Hausdorff measures of noncompactness.

Lemma 2.1. *Let $1 \leq q < \infty$. Then,*
(i) If $u = (u_v) \in \left\{\left|E^r_\mu\right|_q\right\}^\beta$, then, $\tilde{u}^{(q)} = (\tilde{u}^{(q)}_v) \in l_{q^*}$ for all $\delta \in \left|E^r_\mu\right|_q$
(ii) If $u = (u_v) \in \left\{\left|E^r_\mu\right|\right\}^\beta$, then, $\tilde{u}^{(1)} = (\tilde{u}^{(1)}_v) \in l_\infty$ for all $\delta \in \left|E^r_\mu\right|$
(iii) If $u = (u_v) \in \left\{\left|E^r_{\mu,q}\right|_\infty\right\}^\beta$, then, $\tilde{u}^{(q)} = (\tilde{u}^{(q)}_v) \in l$ for all $\delta \in \left|E^r_{\mu,q}\right|_\infty$
and the equality

$$\sum_{v=0}^{\infty} u_v \delta v = \sum_{v=0}^{\infty} \tilde{u}^{(q)}_v y_v \tag{2.1}$$

holds, where $y = T^r(q)(\delta)$ is $T^r(q)$-transformation sequence of the sequence $\delta = (\delta_v)$ and

$$\tilde{u}^{(q)}_v = \mu_v^{-1/q^*}\sum_{n=v}^{\infty}\binom{n-1}{v-1}(r-1)^{n-v}r^{-n}u_n, \tilde{u}^{(q)}_0 = u_0.$$

Proof. (i) Let $u = (u_v) \in \left\{\left|E^r_\mu\right|_q\right\}^\beta$. Considering (1.4) the equation (2.1) is obtained immediately. Also, it follows from Theorem 1.29 in [30] that $\tilde{u}^{(q)} \in l_{q^*}$ whenever $u \in \left\{\left|E^r_\mu\right|_q\right\}^\beta$.
As (ii) and (iii) can be proved with similar lines, these parts are left to reader. □

Lemma 2.2. *Let $1<q<\infty$. Then, we have $\|u\|^*_{|E^r_\mu|_q}=\left\|\tilde{u}^{(q)}\right\|_{l_{q^*}}$ for all $u\in\left\{\left|E^r_\mu\right|_q\right\}^\beta$, $\|u\|^*_{|E^r_\mu|}=\left\|\tilde{u}^{(1)}\right\|_{l_\infty}$ for all $u\in\left\{\left|E^r_\mu\right|\right\}^\beta$ and $\|u\|^*_{|E^r_{\mu,q}|_\infty}=\left\|\tilde{u}^{(q)}\right\|_{l}$ for all $u\in\left\{\left|E^r_{\mu,q}\right|_\infty\right\}^\beta$.*

Proof. Take $u\in\left\{\left|E^r_\mu\right|_q\right\}^\beta$. Since $l_q^\beta=l_{q^*}$, we get $\tilde{u}^{(q)}\in l_{q^*}$. Also, it follows from Theorem 1.29 in [30] and Lemma 2.1 that

$$\|u\|^*_{|E^r_\mu|_q}=\sup_{\delta\in S_{|E^r_\mu|_q}}\left|\sum_{v=0}^{\infty}u_v\delta_v\right|=\sup_{y\in S_{l_q}}\left|\sum_{v=0}^{\infty}\tilde{u}^{(q)}_v y_v\right|=\left\|\tilde{u}^{(q)}\right\|^*_{l_q}=\left\|\tilde{u}^{(q)}\right\|_{l_{q^*}}.$$

For $u\in\left\{\left|E^r_\mu\right|\right\}^\beta$ and $u\in\left\{\left|E^r_{\mu,q}\right|_\infty\right\}^\beta$, the proofs are similar, so the proofs are omitted. □

Theorem 2.3. *Let $1\le q<\infty$. Further, let $W=(w_{nj})$ be a matrix satisfying*

$$w_{nj}=\mu_n^{1/q^*}\sum_{i=1}^{n}\binom{n-1}{i-1}(1-r)^{n-i}r^i u_{ij}. \tag{2.2}$$

Then, $U\in\left(\Delta,\left|E^r_\mu\right|_q\right)$ equals to $W\in\left(\Delta,l_q\right)$, and $U\in\left(\Delta,\left|E^r_{\mu,q}\right|_\infty\right)$ if and only if $W\in(\Delta,l_\infty)$.

Proof. Let take $\lambda\in\Delta$. Then, considering (2.2) it can be written that

$$\sum_{j=0}^{\infty}w_{nj}\delta_j=\mu_n^{1/q^*}\sum_{v=1}^{n}\binom{n-1}{v-1}(1-r)^{n-v}r^v\sum_{j=0}^{\infty}u_{jv}\delta_j,$$

which implies that $W_n(\delta)=T^r_n(q)(U(\delta))$. This shows that $U_n(\delta)\in\left|E^r_\mu\right|_q$ when $\delta\in\Delta$ if and only if $W(\delta)\in l_q$ when $\delta\in\Delta$, which completes the first part of the proof of the theorem.
The remaining part of the proof is omitted, as it is similar. □

Theorem 2.4. *Assume that $1\le q<\infty$ and Δ is arbitrary sequence space. Then, $U\in\left(\left|E^r_\mu\right|_q,\Delta\right)$ if and only if for all $n\ge0$*

$$V^{(n)}\in\left(l_q,c\right)\ \text{and}\ \tilde{U}^{(q)}\in\left(l_q,\Delta\right),$$

$U\in\left(\left|E^r_{\mu,q}\right|_\infty,\Delta\right)$ if and only if for all $n\ge0$

$$V^{(n)}\in\left(l_\infty,c\right)\ \text{and}\ \tilde{U}^{(q)}\in\left(l_\infty,\Delta\right).$$

Here the matrices $\tilde{U}$ and $V^{(n)}$ are described as

$$\tilde{u}^{(q)}_{nk}=\mu_k^{-1/p^*}\sum_{v=k}^{\infty}\binom{v-1}{k-1}(r-1)^{v-k}r^{-v}u_{nv}$$

and

$$v^{(n)}_{mk}=\begin{cases}u_{n0}, & k=0\\ \mu_k^{-1/q^*}\sum_{v=k}^{m}\binom{v-1}{k-1}(r-1)^{v-k}r^{-v}u_{nv}, & 1\le k\le m\\ 0, & k>m.\end{cases}$$

Proof. We only demonstrate for $U\in\left(\left|E^r_\mu\right|_q,\Delta\right)$ to avoid repetition. Assume that $U\in\left(\left|E^r_\mu\right|_q,\Delta\right)$. Given $\delta\in\left|E^r_\mu\right|_q$. Since $\left|E^r_\mu\right|_q=\left(l_q\right)_{T^{(r)}(q)}$, it follows from (1.4) that, for $n,m\ge0$,

$$\sum_{k=0}^{m}u_{nk}\delta_k=\sum_{k=0}^{m}v^{(n)}_{mk}y_k. \tag{2.3}$$

So, we get that, for all $\delta\in\left|E^r_\mu\right|_q$, $U\delta$ is well defined iff $V^{(n)}\in(l_q,c)$. Also, letting $m\to\infty$, gives (2.3) that $U\delta=\tilde{U}^{(q)}y$. Since $U\delta\in\Delta$, $\tilde{U}^{(q)}y$ is also in Δ, and so $\tilde{U}\in(l_q,\Delta)$.

On the contrary, let $V^{(n)}\in\left(l_q,c\right)$ and $\tilde{U}^{(q)}\in\left(l_q,\Delta\right)$. Then, by (2.3), we have $U_n\in\left\{\left|E^r_\mu\right|_q\right\}^\beta$ for all n, which gives that $U\delta$ exists. Also, by $\tilde{U}^{(q)}\in\left(l_q,\Delta\right)$ and (2.3), by letting $m\to\infty$, we get $U\in\left(\left|E^r_\mu\right|_q,\Delta\right)$. □

We present the following tables and conditions:

From To	c	c_0	l_∞	l	$l_p(p>1)$
$\left\|E_\mu^r\right\|_q$	1,3,12,14	2,3,12,14	3,12,14	4,12,14	–
$\left\|E_\mu^r\right\|$	1,6,11,14	2,6,11,14	6,11,14	5,11,14	5,11,14
$\left\|E_{\mu,q}^r\right\|_\infty$	1,7,13,14	8,13,14	10,13,14	9,13,14	9,13,14

Table 1: From Absolute Euler spaces to $\{l_\infty, c_0, c, l, l_p\}$

From To	c_s	b_s
$\left\|E_\mu^r\right\|_q$	1, 3,12,14	3,12,14
$\left\|E_\mu^r\right\|$	1,6,11,14	6,11,14
$\left\|E_{\mu,q}^r\right\|_\infty$	1,7,13,14	10,13,14

Table 2: From Absolute Euler spaces to $\{c_s, b_s\}$

1. $\lim_{n\to\infty} \tilde{u}_{nj}^{(q)}$ exists for all $j \in \mathbb{N}$
2. $\lim_{n\to\infty} \tilde{u}_{nj}^{(q)} = 0$ for all $j \in \mathbb{N}$
3. $\sup_n \sum_{j=0}^{\infty} \left|\tilde{u}_{nv}^{(q)}\right|^{q^*} < \infty$
4. $\sup_N \sum_v \left|\sum_{n\in N} \tilde{u}_{nj}^{(q)}\right|^{q^*} < \infty$
5. $\sup_j \sum_n \left|\tilde{u}_{nj}^{(q)}\right|^p < \infty, (1 \le p < \infty)$
6. $\sup_{n,j} \left|\tilde{u}_{nj}^{(q)}\right| < \infty$
7. $\sum_{j=0}^{\infty} |\tilde{u}_{nj}^q| < \infty$ uniformly in n
8. $\lim_n \sum_{j=0}^{\infty} |\tilde{u}_{nj}^q| = 0$
9. $\sup_K \sum_{n=0}^{\infty} \left|\sum_{k\in K}^{\infty} \tilde{u}_{nj}^q\right|^p < \infty, (1 \le p < \infty)$
10. $\sup_n \sum_{j=0}^{\infty} |\tilde{u}_{nj}^q| < \infty$
11. $\sup_{m,j} \left|v_{mj}^{(n)}\right| < \infty$
12. $\sup_m \sum_{j=0}^{\infty} \left|v_{mj}^{(n)}\right|^{q^*} < \infty$
13. $\sum_{j=0}^{\infty} |v_{mj}^{(n)}| < \infty$ uniformly in m
14. $\lim_{m\to\infty} v_{mj}^{(n)}$ exists for all $j, n \in \mathbb{N}$

We obtain following by Theorem 2.4.

Theorem 2.5. *Let $1 < p, q < \infty$. Then, Table 1 presents us the necessary and sufficient conditions for $U \in (\eta, \Lambda)$, where η is one of absolute Euler spaces and $\Lambda \in \{c, c_0, l_\infty, l, l_p\}$.*

Take the matrices $T_1 = (t_{nj}^1)$ and $T_2 = (t_{nj}^2)$ as

$$t_{nj}^1 = \begin{cases} 1, & 0 \le j \le n \\ 0, & j > n \end{cases}$$

and

$$t_{nj}^2 = \begin{cases} 1, & n = j \\ -1, & n = j+1 \\ 0, & \textit{otherwise}. \end{cases}$$

Then, since $b_s = \{l_\infty\}_{T_1}$, $c_s = \{c\}_{T_1}$ and $bv_q = \{l_q\}_{T_2}$, characterization of the matrix classes (η,Θ) can be obtained immediately as follows, where $\Theta \in \{c_s, b_s, bv_q\}$ and η is one of the any absolute Euler spaces.

Corollary 2.6. *Let's take $u(n,j) = \sum_{i=0}^{n} u_{ij}$ instead of u_{nj} in the matrices $V^{(n)} = (v_{mv}^{(n)})$ and $\tilde{U}^{(p)} = (\tilde{u}_{nv}^{(p)})$ for all $n, j \geq 0$. Then, Table 2 presents us the necessary and sufficient conditions for $U \in (\eta,\Theta)$, where $\Theta \in \{c_s, b_s\}$ and η is one of the absolute Euler spaces.*

Corollary 2.7. *Put $b_{nj} = u_{nj} - u_{n+1,j}$ instead of u_{nj} in the matrices $V^{(n)}$ and $\tilde{U}^{(q)}$ for all $n, j \geq 0$. Then,*

$$U \in \left(\left|E_\mu^r\right|, bv_p\right) \text{ iff the conditions } 5, 11, 14 \text{ hold,}$$

$$U \in \left(\left|E_{\mu,q}^r\right|_\infty, bv_p\right) \text{ iff the conditions } 9, 13, 14 \text{ hold.}$$

Theorem 2.8. *(i) Let $1 < q < \infty$ and $\Lambda \in \{c_0, c, l_\infty\}$. Then,*

$$U \in \left(\left|E_\mu^r\right|_q, \Lambda\right) \Rightarrow \|S_U\| = \sup_n \left\|\tilde{U}_n^{(q)}\right\|_{l_{q^*}} = \sup_n \left(\sum_{v=0}^{\infty} \left|\tilde{u}_{nv}^{(q)}\right|^{q^*}\right)^{1/q^*}$$

$$U \in \left(\left|E_\mu^r\right|, \Lambda\right) \Rightarrow \|S_U\| = \sup_n \left\|\tilde{U}_n^{(1)}\right\|_{l_\infty} = \sup_{n,v} \left|\tilde{u}_{nv}^{(1)}\right|$$

$$U \in \left(\left|E_{\mu,}^r\right|_\infty, \Lambda\right) \Rightarrow \|S_U\| = \sup_n \left\|\tilde{U}_n^{(q)}\right\|_l = \sup_n \sum_{v=0}^{\infty} \left|\tilde{u}_{nv}^{(q)}\right|.$$

(ii) Let $1 < q < \infty$. Then, there exists $\zeta \in [1,4]$ such that

$$U \in \left(\left|E_\mu^r\right|_q, l\right) \Rightarrow \|S_U\| = \frac{1}{\zeta}\left\|\tilde{U}^{(q)}\right\|'_{(l_q,l)} = \frac{1}{\zeta}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right|\right)^{q^*}\right\}^{1/q^*},$$

$$U \in \left(\left|E_\mu^r\right|, l_q\right) \Rightarrow \|S_U\| = \left\|\tilde{U}^{(1)}\right\|_{(l,l_q)} = \sup_v \left\{\sum_{n=0}^{\infty}\left|\tilde{u}_{nv}^{(1)}\right|^q\right\}^{\frac{1}{q}},$$

$$U \in \left(\left|E_\mu^r\right|, l\right) \Rightarrow \|S_U\| = \left\|\tilde{U}_n^{(1)}\right\|_{(l,l)} = \sup_v \sum_{n=0}^{\infty}\left|\tilde{u}_{nv}^{(1)}\right|,$$

$$U \in \left(\left|E_{\mu,q}^r\right|_\infty, l_q\right) \Rightarrow \|S_U\| = \left\|\tilde{U}^{(q)}\right\|_{(l_\infty,l_q)},$$

$$U \in \left(\left|E_{\mu,q}^r\right|_\infty, l\right) \Rightarrow \|S_U\| = \left\|\tilde{U}^{(q)}\right\|_{(l,l)}.$$

Proof. The theorem can be easily proved by using Lemma 1.3, Lemma 1.5, Lemma 2.2 and Theorem 1.23 in [30], so it have left to reader. □

Theorem 2.9. *Let $1 < q < \infty$.*

(a) If $U \in \left(\left|E_\mu^r\right|_q, c_0\right)$, then

$$\|S_U\|_\chi = \lim_{r\to\infty}\sup_{n>r}\left\|\tilde{U}_n^{(q)}\right\|_{l_{q^*}} = \lim_{r\to\infty}\sup_{n>r}\left(\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right|^{q^*}\right)^{1/q^*},$$

and

L_U is compact iff $\lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right|^{q^*} = 0$.

(b) If $U \in \left(\left|E_\mu^r\right|_q, c\right)$, then

$$\frac{1}{2}\lim_{r\to\infty}\sup_{n>r}\left(\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)} - \tilde{u}_v\right|^{q^*}\right)^{1/q^*} \leq \|S_U\|_\chi \leq \lim_{r\to\infty}\sup_{n>r}\left(\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)} - \tilde{u}_v\right|^{q^*}\right)^{1/q^*}$$

and

S_U is compact iff $\lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)} - \tilde{u}_v\right|^{q^*} = 0$, where $\tilde{u}_v = \lim_{n\to\infty}\tilde{u}_{nv}$, for all $n \in \mathbb{N}$.

(c) If $U \in \left(\left|E_\mu^r\right|_q, l_\infty\right)$, then

$$0 \leq \|S_U\|_\chi \leq \lim_{r\to\infty}\sup_{n>r}\left(\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right|^{q^*}\right)^{1/q^*},$$

and

if $\lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right|^{q^*} = 0$, S_U is compact.

Proof. (a) Let $U \in \left(\left|E_\mu^r\right|_q, c_0\right)$. Then, the series $\sum_{n=0}^{\infty} u_{nv}\lambda_v$ converges for all $\lambda \in \left|E_\mu^r\right|_q$, or, equivalently $U_n = \{u_{nv}\}_{v=0}^{\infty} \in \left\{\left|E_\mu^r\right|_q\right\}^\beta$. So, it follows from Lemma 2.2 that $\|U_n\|^*_{|E_\mu^r|_q} = \left\|\tilde{U}_n\right\|_{l_{q^*}}$. Also, by Lemma 1.10 (a), we have

$$\|S_U\|_\chi = \lim_{r\to\infty} \sup_{n>r} \left\|\tilde{U}_n\right\|_{l_{q^*}}.$$

Hence, the compactness of S_U is immediate by Lemma 1.7, which completes the proof of (a).
(b) Let take the unit sphere $S'_{|E_\mu^r|_q}$ in $\left|E_\mu^r\right|_q$. From Lemma 1.7 it follows that

$$\|S_U\|_\chi = \chi(U(S'_{|E_\mu^r|_q})).$$

Further, since $\left|E_\mu^r\right|_q \cong l_q$, $U \in \left(\left|E_\mu^r\right|_q, c\right)$ equals to $\tilde{U} \in (l_q, c)$, and

$$\|S_U\|_\chi = \chi(U(S'_{|E_\mu^r|_q})) = \chi(\tilde{U}(T(S'_{|E_\mu^r|_q}))) = \|S_{\tilde{U}}\|_\chi.$$

which implies, by Lemma 1.10 (b),

$$\frac{1}{2}\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\tilde{U}_n - \tilde{u}\right\|^*_{l_q}\right) \leq \|L_U\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\tilde{U}_n - \tilde{u}\right\|^*_{l_q}\right), \tag{2.4}$$

where $\tilde{u}_k = \lim_{n\to\infty} \tilde{u}_{nk}$, for all $k \geq 0$.
Considering Theorem 1.29 in [30], it can be easily written that $\left\|\tilde{U}_n - \tilde{u}\right\|^*_{l_q} = \left\|\tilde{U}_n - \tilde{u}\right\|_{l_{p^*}}$. The last equality and (2.4) complete the first part of the proof of (b). Also, the compactness of S_U is concluded by Lemma 1.7.
(c) can be proved by similar way, so it is omitted. □

By following the above lines,the proof of the following theorems also can be obtained immediately. Therefore, we just give the statement of the theorems.

Theorem 2.10. *(a) If $U \in \left(\left|E_\mu^r\right|, c_0\right)$. Then*

$$\|S_U\|_\chi = \lim_{r\to\infty}\sup_{n>r}\left\|\tilde{U}_n^{(1)}\right\|_{l_\infty} = \lim_{r\to\infty}\sup_{n>r}\sup_{v}\left|\tilde{u}_{nv}^{(1)}\right|,$$

and
S_U is compact iff $\lim_{r\to\infty}\sup_{n>r}\sup_{v}\left|\tilde{u}_{nv}^{(1)}\right| = 0.$
(b) If $U \in \left(\left|E_\mu^r\right|, c\right)$, then

$$\frac{1}{2}\lim_{r\to\infty}\sup_{n>r}\sup_{v}\left|\tilde{u}_{nv}^{(1)} - \tilde{u}_v\right| \leq \|S_U\|_\chi \leq \lim_{r\to\infty}\sup_{n>r}\sup_{v}\left|\tilde{u}_{nv}^{(1)} - \tilde{u}_v\right|$$

and
S_U is compact iff $\lim_{r\to\infty}\sup_{n>r}\sup_{v}\left|\tilde{u}_{nv}^{(1)} - \tilde{u}_v\right| = 0$
where $\tilde{u}_v = \lim_{n\to\infty}\tilde{u}_{nv}$, for all $v \in \mathbb{N}$.
(c) If $U \in \left(\left|E_\mu^r\right|, l_\infty\right)$, then

$$0 \leq \|S_U\|_\chi \leq \lim_{r\to\infty}\sup_{n>r}\sup_{v}\left|\tilde{u}_{nv}^{(1)}\right|,$$

and
S_U is compact if $\lim_{r\to\infty}\sup_{n>r}\sup_{v}\left|\tilde{u}_{nv}^{(1)}\right| = 0.$

Theorem 2.11. *Let $1 < q < \infty$.*
(a) If $U \in \left(\left|E_{\mu,q}^r\right|_\infty, c_0\right)$, then

$$\|S_U\|_\chi = \lim_{r\to\infty}\sup_{n>r}\left\|\tilde{U}_n^{(q)}\right\|_{l} = \lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right|,$$

and
S_U is compact iff $\lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right| = 0.$
(b) If $U \in \left(\left|E_{\mu,q}^r\right|_\infty, c\right)$, then

$$\frac{1}{2}\lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)} - \tilde{u}_v\right| \leq \|S_U\|_\chi \leq \lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)} - \tilde{u}_v\right|$$

and
S_U *is compact iff* $\lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}-\tilde{u}_v\right|=0$
where $\tilde{u}_v=\lim_{n\to\infty}\tilde{u}_{nv}$, *for all* $v\in\mathbb{N}$.
(c) *If* $U\in\left(\left|E_{\mu,q}^{r}\right|_{\infty},l_\infty\right)$, *then*

$$0\leq\|S_U\|_\chi\leq\lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right|,$$

and
S_U *is compact if* $\lim_{r\to\infty}\sup_{n>r}\sum_{v=0}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right|=0.$

Theorem 2.12. (a) *If* $U\in\left(\left|E_{\mu}^{r}\right|,l_q\right)$, $1\leq q<\infty$, *then*

$$\|S_U\|_\chi=\lim_{r\to\infty}\left\{\sup_v\left(\sum_{n=r+1}^{\infty}\left|\tilde{u}_{nv}^{(1)}\right|^q\right)^{1/q}\right\},$$

and
S_U *is compact iff* $\lim_{r\to\infty}\sup_v\sum_{n=r+1}^{\infty}\left|\tilde{u}_{nv}^{(1)}\right|^q=0.$
(b) *If* $U\in\left(\left|E_{\mu}^{r}\right|_q,l\right)$, $1<q<\infty$, *then there exists* $\zeta\in[1,4]$ *such that*

$$\|S_U\|_\chi=\frac{1}{\zeta}\lim_{r\to\infty}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=r+1}^{\infty}\left|\tilde{u}_{nv}^{(q)}\right|\right)^{q^*}\right\}^{1/q},$$

and
S_U *is compact iff* $\lim_{r\to\infty}\sum_{v=0}^{\infty}\left(\sum_{n=r+1}\left|\tilde{u}_{nv}^{(q)}\right|\right)^{q^*}=0.$

3. Conclusion

One of the most important subjects in summability theory is the theory of sequence spaces which concerns with the generalization of the concept of convergence for series and sequences. In this sense, the primary aim is to assign a limit value for divergent sequences or series by using transformation which is given by the most general linear mappings of infinite special matrices. So, there has been a large literature, concerned with characterizing completely all matrices which transform one given sequence space into another. Besides this, the literature has been also grown up in terms of the studies of many sequence spaces defined as domain of special matrices and related matrix operators (see, for instance, [1–4, 6–12]). For a recent paper [1], the infinite matrix classes $\left(\left|E_{\mu}^{r}\right|,\left|E_{\mu}^{r}\right|_q\right)$ and $\left(\left|E_{\mu}^{r}\right|_q,\left|E_{\mu}^{r}\right|\right)$ have been introduced. In this study, estimating the operator norms, the classes $\left(\left|E_{\mu}^{r}\right|_q,\Lambda\right)$ and $\left(\left|E_{\mu}^{r}\right|_\infty,\Lambda\right)$ have been characterized where $1\leq q<\infty$. Also, in case Λ is one of the spaces c_0,c,l_∞,l_q, the necessary and sufficient conditions for each matrix in these classes to be compact have been obtained and certain identities or estimates for the Hausdorff measures of noncompactness have been established.

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References

[1] F. Gökçe, MA. Sarıgöl, *On absolute Euler spaces and related matrix operators*, Proc. Natl. Acad. Sci., India. Sec. A. Phy. Sci., **90** (2020), 769-775.
[2] B. Altay, F. Başar, M. Mursaleen, *On the Euler sequence spaces which include the spaces l_p and l_∞ I*, Information Sci., **176** (10) (2005), 1450-1462.
[3] F. Başar, NL. Braha, *Euler-Cesàro difference spaces of bounded, convergent and null sequences*, Tamkang J. Math., **47** (2016), 405-420.
[4] E E. Kara, M. Ilkhan, *Some properties of generalized Fibonacci sequence spaces*, Linear Multilinear Algebra, **64** (2016), 2208-2223.
[5] M. Mursaleen, B. Altay, F. Başar, *On the Euler sequence spaces which include the spaces l_p and l_∞ II*, Nonlinear Anal., **65** (2006), 707–717.
[6] WT. Sulaiman, *On summability factors of infinite series*, Proc. Amer. Math. Soc., **115** (1992), 313- 317.
[7] TM. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. Lond. Math. Soc., **7** (1957), 113-141.
[8] F. Gökçe, *Compact and Matrix Operators on the Space* $\left|\overline{N}_p^\phi\right|_k$, Fundam. J. Math. Appl., **4** (2021), 124-133.
[9] F. Gökçe, MA. Sarıgöl, *Some matrix and compact operators of the absolute Fibonacci series spaces*, Kragujevac J. Math., **44** (2020) 273–286.
[10] F. Gökçe, MA. Sarıgöl, *Generalization of the absolute Cesàro space and some matrix transformations*, Num. Func. Anal. Opt., **40** (2019), 1039-1052.
[11] F. Gökçe, MA. Sarıgöl, *Extension of Maddox's space $l(\mu)$ with Nörlund means*, AEJM, **12** (2019), 2040005.
[12] F. Gökçe, MA. Sarıgöl, *Generalization of the space $l(p)$ derived by absolute Euler summability and matrix operators*, J. Ineq. Appl., **2018**, (2018).
[13] RN. Mohapatra, MA. Sarıgöl, *On Matrix Operators on the Series Space* $\left|\overline{N}_p^\theta\right|_k$, Ukrain. Math. J., **69** (2018), 1772-1783.
[14] M.A. Sarıgöl, *Matrix transformations on fields of absolute weighted mean summability*, Studia Sci. Math. Hungar., **48** (2011), 331-341.
[15] M.A. Sarıgöl,*On the local properties of factored Fourier series*, Appl. Math. Comput., **216** (2010), 3386-3390.
[16] M. Stieglitz, H. Tietz, *Matrix transformationen von Folgenraumen*, Eine Ergebnisübersicht. Math. Z., **154** (1977), 1-16.
[17] IJ. Maddox, Elements of functinal analysis. Cambridge University Press, London, New York, (1970).
[18] M.A. Sarıgöl, *Extension of Mazhar's theorem on summability factors*, Kuwait J. Sci., **42** (2015), 28-35.
[19] LS. Goldenstein, IT. Gohberg, AS. Markus, *Investigations of some properties of bounded linear operators with their q-norms*, Uchen. Zap. Kishinev. Gos. Univ., **29** (1957), 29-36.
[20] E. Malkowsky, V. Rakocevic *Measure of noncompactness of Linear operators between spaces of sequences that are $(\bar{N},q)$ summable or bounded*, Czechoslovak Math. J., **51** (2001), 505-522.
[21] M. Mursaleen, AK. Noman, *Compactness of matrix operators on some new difference sequence spaces*, Linear Algebra Appl., **436** (2012), 41-52.
[22] M. Mursaleen, AK. Noman *Compactness by the Hausdorff measure of noncompactness*, Nonlinear Anal., **73** (2010), 2541-2557.
[23] M.A. Sarıgöl, *Norms and compactness of operators on absolute weighted mean summable series*, Kuwait J. Sci., **43** (2016), 68-74.
[24] M. C. Daglı, *Matrix mappings and compact operators for Schröder sequence spaces*, Turk. J. Math., **46** (2022) 2304-2320.
[25] M. C. Daglı, T. Yaying, *Some results on matrix transformation and compactness for fibonomial sequence spaces*, Acta Sci. Math. (Szeged) (2023), https://doi.org/10.1007/s44146-023-00087-6.
[26] I. Djolovic, E. Malkowsky, *Matrix transformations and compact operators on some new mth-order difference sequences*, Appl. Math. Comput., **198** (2008), 700-714.
[27] V. Rakocevic, *Measures of noncompactness and some applications*, Filomat, **12** (1998), 87-120.
[28] AM. Jarrah, E. Malkowsky, *Ordinary absolute and strong summability and matrix transformations*, Filomat, **17** (2003), 59-78.
[29] E. Malkowsky, V. Rakocevic *On matrix domains of triangles*, Appl. Math. Comput., **1890** (2007), 1146-1163.
[30] E. Malkowsky, V. Rakocevic *An introduction into the theory of sequence space and measures of noncompactness*, Zb. Rad.(Beogr)., **17** (2000), 143-234.

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On Rough $\mathscr{I}$-Convergence and $\mathscr{I}$-Cauchy Sequence for Functions Defined on Amenable Semigroups

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Abstract

In this paper, firstly we introduced the concepts of rough $\mathscr{I}$-convergence, rough $\mathscr{I}^*$-convergence, rough $\mathscr{I}$-Cauchy sequence, and rough $\mathscr{I}^*$-Cauchy sequence of a function defined on discrete countable amenable semigroups. Then, we investigated the relations between them.

1. Introduction

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [1] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of $\mathbb{N}$.
Phu [2] introduced, firstly, the notion of rough convergence in finite-dimensional normed spaces. In [2], he investigated some properties of $\mathrm{LIM}^r x$ such as boundedness, closedness and convexity, and also he defined the notion of rough Cauchy sequence. Then, Phu [3] studied on rough convergence and some important properties of this concept. Furthermore, recently some authors [4–8] investigated the rough convergence types in some normed spaces.
In [9], Day studied on the concept of amenable semigroups (or briefly ASG). Then, some authors [10–12] studied the notions of summability in ASG. Douglas [13] extended the notion of arithmetic mean to ASG and obtained a characterization for almost convergence in ASG. In [14], Nuray and Rhoades presented the concepts of convergence and statistical convergence in ASG. Dündar et al. [15] and Dündar, Ulusu [16] introduced rough convergence and investigated some properties of rough convergence in ASG. Dündar, Ulusu [17] studied rough statistical convergence in ASG. Also, Dündar et al. [18] defined rough ideal convergence and some properties in ASG. Recently, some authors studied on the new concepts in ASG (see [19–22]).
First of all, we remember the basic definitions and concepts that we will use in our study such as amenable semigroups, rough convergence, rough ideal convergence, etc. (see [2, 3, 8–16, 18–24, 26, 27]).
Let a real number $r \geq 0$ and $\mathbb{R}^n$ (the real n-dimensional space) with the norm $\|.\|$, and a sequence $x = (x_k)_{k=0}^n \subset \mathbb{R}^n$.
A sequence (x_k) is said to be r-convergent to L, denoted by $x_k \xrightarrow{r} L$, provided that

$$\forall \varepsilon > 0\ \exists k_\varepsilon \in \mathbb{N} :\ k \geq k_\varepsilon \Rightarrow \|x_k - L\| < r + \varepsilon.$$

The rough limit set of the sequence $x = (x_k)$ is showed by $\mathrm{LIM}^r x = \{L \in \mathbb{R}^n : x_k \xrightarrow{r} L\}$.
A sequence $x = (x_k)$ is said to be r-convergent if $\mathrm{LIM}^r x \neq \emptyset$ and r is called the convergence degree of the sequence (x_k). For $r = 0$, we get the ordinary convergence.
Let G be a discrete countable amenable semigroups (or briefly DCASG) with identity in which both left and right cancelation laws hold, and $w(G)$ denotes the space of all real valued functions on G.
If G is a countable amenable group, there exists a sequence $\{S_n\}$ of finite subsets of G such that

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(i) $G=\bigcup_{n=1}^{\infty} S_n$,
(ii) $S_n \subset S_{n+1}$ $(n=1,2,...)$,
(iii) $\lim\limits_{n\to\infty} \frac{|S_n g\cap S_n|}{|S_n|}=1,\ \lim\limits_{n\to\infty} \frac{|gS_n\cap S_n|}{|S_n|}=1$, for all $g\in G$.

If a sequence of finite subsets of G satisfy (i)-(iii), then it is called a Folner sequence (or briefly FS) of G.
Throughout the paper, we take G be a DCASG with identity in which both left and right cancelation laws hold.
For any FS $\{S_n\}$ of G, a function $f\in w(G)$ is said to be convergent to t if for every $\varepsilon>0$ there exists a $k_0=k_0(\varepsilon)\in\mathbb{N}$ such that $|f(g)-t|<\varepsilon$, for all $m>k_0$ and $g\in G\setminus S_m$.
Let $X\neq\emptyset$. A class $\mathscr{I}$ of subsets of X is said to be an ideal in X provided:

i) $\emptyset\in\mathscr{I}$,
ii) $A,B\in\mathscr{I}$ implies $A\cup B\in\mathscr{I}$,
iii) $A\in\mathscr{I}$, $B\subset A$ implies $B\in\mathscr{I}$.

$\mathscr{I}$ is called a nontrivial ideal if $X\notin\mathscr{I}$. A nontrivial ideal $\mathscr{I}$ in X is called admissible if $\{x\}\in\mathscr{I}$, for each $x\in X$.
Throughout the paper, we take $\mathscr{I}$ as an admissible ideal in $\mathbb{N}$.
Let $X\neq\emptyset$. A class $\emptyset\neq\mathscr{F}$ of subsets of X is said to be a filter in X provided:

i) $\emptyset\notin\mathscr{F}$,
ii) $A,B\in\mathscr{F}$ implies $A\cap B\in\mathscr{F}$,
iii) $A\in\mathscr{F}$, $A\subset B$ implies $B\in\mathscr{F}$.

If $\mathscr{I}$ is a nontrivial ideal in X, $X\neq\emptyset$, then the class

$$\mathscr{F}(\mathscr{I})=\{M\subset X:(\exists A\in\mathscr{I})(M=X\backslash A)\}$$

is a filter on X, called the filter associated with $\mathscr{I}$.
An admissible ideal $\mathscr{I}\subset 2^{\mathbb{N}}$ satisfies the property (AP), if for every countable family of mutually disjoint sets $\{A_1,A_2,\ldots\}$ belonging to $\mathscr{I}$, there exists a countable family of sets $\{B_1,B_2,\ldots\}$ such that $A_j\Delta B_j$ is a finite set for $j\in\mathbb{N}$ and $B=\bigcup_{j=1}^{\infty}B_j\in\mathscr{I}$ (hence $B_j\in\mathscr{I}$ for each $j\in\mathbb{N}$).
After then, we let $\mathscr{I}\subseteq 2^G$ be an admissible ideal for amenable semigroup G.
A function $f\in w(G)$ is said to be $\mathscr{I}$-convergent to s for any FS $\{S_n\}$ for G, if for every $\varepsilon>0$

$$\left\{g\in G:|f(g)-s|\geq\varepsilon\right\}\in\mathscr{I}.$$

In this case, we write $\mathscr{I}-\lim f(g)=s$.
A function $f\in w(G)$ is said to be $\mathscr{I}^*$-convergent to s, for any FS $\{S_n\}$ for G if there exists $M\subset G$, $M\in\mathscr{F}(\mathscr{I})$ (i.e., $G\setminus M\in\mathscr{I}$) and a $k_0=k_0(\varepsilon)\in\mathbb{N}$ such that for every $\varepsilon>0$ $|f(g)-s|<\varepsilon$, for all $n>k_0$ and all $g\in M\setminus S_n$. In this case, we write $\mathscr{I}^*-\lim f(g)=s$.
A function $f\in w(G)$ is said to be $\mathscr{I}$-Cauchy sequence, for any FS $\{S_n\}$ for G if for every $\varepsilon>0$, there exists an $h=h(\varepsilon)\in G$ such that

$$\left\{g\in G:|f(g)-f(h)|\geq\varepsilon\right\}\in\mathscr{I}.$$

A function $f\in w(G)$ is said to be $\mathscr{I}^*$-Cauchy sequence, for any FS $\{S_n\}$ for G if there exists $M\subset G$, $M\in\mathscr{F}(\mathscr{I})$ (i.e., $G\setminus M\in\mathscr{I}$) and a $k_0=k_0(\varepsilon)\in\mathbb{N}$ such that for every $\varepsilon>0$ $|f(g)-f(h)|<\varepsilon$, for all $n>k_0$ and $g,h\in M\setminus S_n$.
For any FS $\{S_n\}$ of G, a function $f\in w(G)$ is said to be rough convergent (r-convergent) to t if

$$\forall\varepsilon>0\ \exists k_\varepsilon\in\mathbb{N}:\ m\geq k_\varepsilon\Rightarrow|f(g)-t|<r+\varepsilon, \tag{1.1}$$

for all $g\in G\setminus S_m$ or equivalently if $\limsup|f(g)-t|\leq r$, for all $g\in G\setminus S_m$. In this instance, we write $r-\lim f(g)=t$ or $f(g)\xrightarrow{r}t$.
If (1.1) holds, then t is an r-limit point of the function $f\in w(G)$, which is usually no longer unique (for $r>0$). Hence, we have to think the so-called rough limit set (r-limit set) of the function $f\in w(G)$ defined by $\mathrm{LIM}^r f:=\{t:f(g)\xrightarrow{r}t\}$.
For any FS $\{S_n\}$ for G, the function $f\in w(G)$ is said to be r-convergent if $\mathrm{LIM}^r f\neq\emptyset$. In this instance, r is called the convergence degree of the $f\in w(G)$.
For any FS $\{S_n\}$ of G, a function $f\in w(G)$ is said to be a rough Cauchy sequence with roughness degree $\wp$, if $\forall\varepsilon>0\ \exists k_\varepsilon:\ m\geq k_\varepsilon\Rightarrow |f(g)-f(h)|\leq\wp+\varepsilon$ is hold for $\wp>0$ and all $g,h\in G\setminus S_m$. $\wp$ is also said to be Cauchy degree of $f\in w(G)$.

2. Main Results

In this section, we introduced the concepts of rough $\mathscr{I}$-convergence, rough $\mathscr{I}^*$-convergence, rough $\mathscr{I}$-Cauchy sequence and rough $\mathscr{I}^*$-Cauchy sequence of a function defined on discrete countable amenable semigroups. Then, we investigated relations between them.

Definition 2.1. *For any FS $\{S_n\}$ of G, a function $f\in w(G)$ is said to be rough $\mathscr{I}$-convergent (r-$\mathscr{I}$-convergent) to s if for every $\varepsilon>0$*

$$\{g\in G:|f(g)-s|\geq r+\varepsilon\}\in\mathscr{I} \tag{2.1}$$

or equivalently if

$$\mathscr{I}-\limsup|f(g)-s|\leq r$$

is satisfied. In this instance, we write

$$r-\mathscr{I}-\lim f(g)=s \text{ or } f(g)\xrightarrow{r-\mathscr{I}}s.$$

On the other hand, we say that $f(g)\xrightarrow{r-\mathscr{I}}s$ if and only if the condition

$$|f(g)-s|\leq r+\varepsilon$$

holds for every $\varepsilon>0$ and almost $g\in G$.

In this convergence r is named the roughness degree. For $r=0$, we get the $\mathscr{I}$-convergence.
If (2.1) holds, then s is an r-$\mathscr{I}$-limit point of the function $f\in w(G)$, which is usually no longer unique (for $r>0$). Hence, we have to think the so-called rough $\mathscr{I}$-limit set of the function $f\in w(G)$ defined by

$$\mathscr{I}-\mathrm{LIM}^r f := \{s : f(g) \xrightarrow{r-\mathscr{I}} s\}.$$

For any FS $\{S_n\}$ for G, the function $f\in w(G)$ is said to be r-$\mathscr{I}$-convergent if

$$\mathscr{I}-\mathrm{LIM}^r f \neq \emptyset.$$

If $\mathscr{I}-\mathrm{LIM}^r f \neq \emptyset$ for a function $f\in w(G)$, then we have

$$\mathscr{I}-\mathrm{LIM}^r f = [\mathscr{I}-\limsup f - r,\ \mathscr{I}-\liminf f + r].$$

Remark 2.2. *If $\mathscr{I}$ is an admissible ideal, then for a function $f\in w(G)$, usual rough convergence implies rough $\mathscr{I}$-convergence for any FS $\{S_n\}$ of G.*

Definition 2.3. *A function $f\in w(G)$ is said to be rough $\mathscr{I}$-Cauchy sequence, for any FS $\{S_n\}$ for G if for every $\varepsilon>0$, there exists an $h=h(\varepsilon)\in G$ such that*

$$\left\{g\in G : |f(g)-f(h)| \geq r+\varepsilon\right\} \in \mathscr{I}.$$

Theorem 2.4. *If $f\in w(G)$ is rough $\mathscr{I}$-convergent for any FS $\{S_n\}$ for G, then it is rough $\mathscr{I}$-Cauchy for same sequence.*

Proof. For any Folner sequence $\{S_n\}$ for G, let

$$r-\mathscr{I}-\lim f(g) = s.$$

Then, for every $\varepsilon>0$, we have

$$A_\varepsilon = \left\{g\in G : |f(g)-s| \geq r+\varepsilon\right\} \in \mathscr{I}.$$

Since $\mathscr{I}$ is an admissible ideal there exists an $h\in G$ such that $h\notin A_\varepsilon$. Now, let

$$B_\varepsilon = \left\{g\in G : |f(g)-f(h)| \geq 2(r+\varepsilon)\right\}.$$

Taking into account the inequality

$$|f(g)-f(h)| \leq |f(g)-s| + |f(h)-s|,$$

we observe that if $g\in B_\varepsilon$, then

$$|f(g)-s| + |f(h)-s| \geq 2(r+\varepsilon).$$

On the other hand, since $h\notin A_\varepsilon$ we have

$$|f(h)-s| < r+\varepsilon$$

and so

$$|f(g)-s| > r+\varepsilon.$$

Hence, $g\in A_\varepsilon$ and so we have

$$B_\varepsilon \subset A_\varepsilon \in \mathscr{I}.$$

Thus, $B_\varepsilon\in\mathscr{I}$ that is, f is rough $\mathscr{I}$-Cauchy sequence. □

Definition 2.5. *A function $f\in w(G)$ is said to be rough $\mathscr{I}^*$-convergent to s, for any FS $\{S_n\}$ for G if there exists $M\subset G$, $M\in\mathscr{F}(\mathscr{I})$ (i.e., $G\setminus M\in\mathscr{I}$) and a $k_0=k_0(\varepsilon)\in\mathbb{N}$ such that for every $\varepsilon>0$*

$$|f(g)-s| < r+\varepsilon, \tag{2.2}$$

for all $n>k_0$ and all $g\in M\setminus S_n$. In this case, we write

$$r-\mathscr{I}^*-\lim f(g) = s.$$

In this convergence r is named the roughness degree. For $r=0$, we get the $\mathscr{I}^*$-convergence.
If (2.2) holds, then s is an r-$\mathscr{I}^*$-limit point of the function $f\in w(G)$, which is usually no longer unique (for $r>0$).
Hence, we have to think the so-called rough $\mathscr{I}^*$-limit set of the function $f\in w(G)$ defined by

$$\mathscr{I}^*-\mathrm{LIM}^r f := \{s : f(g) \xrightarrow{r-\mathscr{I}^*} s\}.$$

For any FS $\{S_n\}$ for G, the function $f\in w(G)$ is said to be r-$\mathscr{I}^*$-convergent if

$$\mathscr{I}^*-\mathrm{LIM}^r f \neq \emptyset.$$

Theorem 2.6. *If $f \in w(G)$ is rough $\mathscr{I}^*$-convergent to s, then f is rough $\mathscr{I}$-convergent to s for any FS $\{S_n\}$ for G.*

Proof. For any FS $\{S_n\}$ for G, let

$$r - \mathscr{I}^* - \lim f(g) = s.$$

Then, there exists $M \subset G$, $M \in \mathscr{F}(\mathscr{I})$ (i.e., $H = G \setminus M \in \mathscr{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$|f(g) - s| < r + \varepsilon,$$

for all $n > k_0$ and all $g \in M \setminus S_n$. Therefore obviously,

$$A(\varepsilon) = \left\{ g \in G : |f(g) - s| \geq r + \varepsilon \right\} \subset H \cup S_{k_0}.$$

Since $\mathscr{I}$ is admissible,

$$H \cup S_{k_0} \in \mathscr{I}$$

and so

$$A(\varepsilon) \in \mathscr{I}.$$

Hence,

$$r - \mathscr{I} - \lim f(g) = s.$$

□

Definition 2.7. *A function $f \in w(G)$ is said to be rough $\mathscr{I}^*$-Cauchy sequence, for any FS $\{S_n\}$ for G if there exists $M \subset G$, $M \in \mathscr{F}(\mathscr{I})$ (i.e., $G \setminus M \in \mathscr{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$*

$$|f(g) - f(h)| < r + \varepsilon,$$

for all $n > k_0$ and $g, h \in M \setminus S_n$.

Theorem 2.8. *If $f \in w(G)$ is rough $\mathscr{I}^*$-Cauchy for any FS $\{S_n\}$ for G, then it is rough $\mathscr{I}$-Cauchy for same sequence.*

Proof. Let $f \in w(G)$ be an rough $\mathscr{I}^*$-Cauchy for any FS $\{S_n\}$ for G. Then by definition, there exists $M \subset G$, $M \in \mathscr{F}(\mathscr{I})$ (i.e., $G \setminus M \in \mathscr{I}$) and a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$

$$|f(g) - f(h)| < r + \varepsilon,$$

for all $n > k_0$ and $g, h \in M \setminus S_n$. Let $H = G \setminus M$. It is clearly $H \in \mathscr{I}$ and

$$A(\varepsilon) = \left\{ g \in G : |f(g) - f(h)| \geq r + \varepsilon \right\} \subset H \cup S_{k_0}.$$

Since $\mathscr{I}$ is admissible,

$$H \cup S_{k_0} \in \mathscr{I}$$

and so

$$A(\varepsilon) \in \mathscr{I}.$$

Consequently, f is rough $\mathscr{I}$-Cauchy for same sequence. □

Following theorems show relationships between $\mathscr{I}$-convergence and $\mathscr{I}^*$-convergence, between $\mathscr{I}$-Cauchy sequence and $\mathscr{I}^*$-Cauchy sequence. These theorems can be proved like in [19,25], these theorems are given without the proof.

Theorem 2.9. *Let $\mathscr{I} \subset 2^G$ be an admissible ideal with the property (AP). If $f(g) \in w(G)$ is rough $\mathscr{I}$-convergent to s, then f is rough $\mathscr{I}^*$-convergent to s for any FS $\{S_n\}$ for G.*

Theorem 2.10. *Let $\mathscr{I} \subset 2^G$ be an admissible ideal with the property (AP). If $f \in w(G)$ is rough $\mathscr{I}$-Cauchy for any FS $\{S_n\}$ for G, then it is rough $\mathscr{I}^*$-Cauchy for same sequence.*

3. Conclusion

In this paper, we introduced the concepts of rough $\mathscr{I}$-convergence, rough $\mathscr{I}^*$-convergence, rough $\mathscr{I}$-Cauchy sequence and rough $\mathscr{I}^*$-Cauchy sequence of a function defined on discrete countable amenable semigroups. Also, we investigated relations between them. Then after, The concepts given here can also be studied for double sequences.

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References

[1] P. Kostyrko, T. Salat, W. Wilczyński, *$\mathscr{I}$-convergence*, Real Anal. Exchange, **26**(2) (2000), 669–686.
[2] H. X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Opt., **22** (2001), 199–222.
[3] H. X. Phu, *Rough convergence in infinite dimensional normed spaces*, Numer. Funct. Anal. Opt., **24** (2003), 285–301.
[4] M. Arslan, E. Dündar, *Rough convergence in 2-normed spaces*, Bull. Math. Anal. Appl., **10**(3) (2018), 1–9.
[5] M. Arslan, E. Dündar, *On rough convergence in 2-normed spaces and some properties*, Filomat, **33**(16) (2019), 5077–5086.
[6] M. Arslan, E. Dündar, *Rough statistical convergence in 2-normed spaces*, Honam Mathematical J., **43**(3) (2021), 417–431.
[7] S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. and Optimiz., **29**(3-4) (2008), 291–303.
[8] E. Dündar, C. Çakan, *Rough $\mathscr{I}$-convergence*, Gulf J. Math., **2**(1) (2014), 45–51.
[9] M. Day, *Amenable semigroups*, Illinois J. Math., **1** (1957), 509–544.
[10] S. A. Douglass, *Summing sequences for amenable semigroups*, Michigan Math. J., **20** (1973), 169–179.
[11] P. F. Mah, *Summability in amenable semigroups*, Trans. Amer. Math. Soc., **156** (1971), 391–403.
[12] P. F. Mah, *Matrix summability in amenable semigroups*, Proc. Amer. Math. Soc., **36** (1972), 414–420.
[13] S. A. Douglass, *On a concept of summability in amenable semigroups*, Math. Scand., **28** (1968), 96–102.
[14] F. Nuray, B. E. Rhoades, *Some kinds of convergence defined by Folner sequences*, Analysis, **31**(4) (2011), 381–390.
[15] E. Dündar, U. Ulusu, F. Nuray, *Rough convergent functions defined on amenable semigroups*, Sigma J. Eng. Nat. Sci. (accepted - in press).
[16] E. Dündar, U. Ulusu, *On rough convergence in amenable semigroups and some properties*, J. Intell. Fuzzy Syst., **41** (2021), 2319–2324.
[17] E. Dündar, U. Ulusu, *Rough statistical convergent functions defined on amenable semigroups*, (under review).
[18] E. Dündar, U. Ulusu, Ş. Yalvaç, N. Akın, *Rough ideal convergent functions defined on amenable semigroups*, (under review).
[19] E. Dündar, F. Nuray, U. Ulusu, *$\mathscr{I}$-convergent functions defined on amenable semigroups*, TWMS J. Pure Appl. Math. (accepted - in press).
[20] U. Ulusu, F. Nuray, E. Dündar, *$\mathscr{I}$-limit and $\mathscr{I}$-cluster points for functions defined on amenable semigroups*, Fundam. J. Math. Appl., **4**(1) (2021), 45–48.
[21] *U. Ulusu, E. Dündar, F. Nuray, Some generalized convergence types using ideals in amenable semigroups*, Bull. Math. Anal. Appl., **11**(1) (2019), 28–35.
[22] U. Ulusu, E. Dündar, B. Aydın, *Asymptotically $\mathscr{I}$-statistical equivalent functions defined on amenable semigroups*, Sigma J. Eng. Nat. Sci., **37**(4) (2019), 1367–1373.
[23] E. Dündar, C. Çakan, *Rough convergence of double sequences*, Demonstr. Math., **47**(3) (2014), 638–651.
[24] E. Dündar, *On Rough $\mathscr{I}_2$-convergence*, Numer. Funct. Anal. Optim., **37**(4) (2016), 480–491.
[25] A. Nabiev, S. Pehlivan, M. Gürdal, *On I-Cauchy Sequences*, Taiwanese J. Math., **11**(2) (2007), 569–576.
[26] I. Namioka, *Følner's conditions for amenable semigroups*, Math. Scand., **15** (1964), 18–28.
[27] H. X. Phu, *Rough continuity of linear operators*, Numer. Funct. Anal. Optim., **23** (2002), 139–146.