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A Modelling on the Exponential Curves as $Cubic$, 5^{th} and 7^{th} Bézier Curve in Plane

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Abstract
In this study, it has been researched the exponential curve as a 3^{rd}, 5^{th} and 7^{th} order Bézier curve in $\mathbf{E}^2$. Also, the numerical matrix representations of these curves have been calculated using the Maclaurin series in the plane via the control points.

Keywords: Bézier curves, Exponential curve, Maclaurin series, 5^{th} and 7^{th} $order$ Bézier curve.
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1. Introduction and Preliminaries

Bezier curves have special mathematical representations and are obtained with the help of polynomial functions. Since these curves are used in computer aided geometric design and modelling [1], they have an important place in applied fields. The Bezier curve has a control polygon that contains it, and only the start and end points are on the curve, so it provides an advantage in terms of use in modelling. Thus, it provides the opportunity to make the desired changes over the control polygon. Users outline the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For three dimension animation Bézier curves are often used to define 3D paths as well as two dimension curves for keyframe interpolation. Apart from the Bézier-curves' frequent use in applied sciences, the theory has been studied by many researchers in mathematical points of view. The matrix form was first coined in [2]. The derivatives of the Bezier curves in matrix notation was studied in [3]. Particularly, the 5^{th} order Bezier curve and its derivatives were studied by matrices in [4]. Besides, it has been investigated approximation methods in matrix form for Helix, sin waves and cosin curves by different order Bézier curves in [5–7]. The curve is also subjected to the differential geometry. For example: In [8], A dual unit spherical Bézier-like curve corresponds to a ruled surface by using Study's transference principle and closed ruled surfaces are determined via control points and also, integral invariants of these surfaces are investigated. In [9], Bezier-curves with curvature and torsion continuity has been examined. In [10–12], Bezier curves and surfaces has been given and Bezier curves are designed for Computer-Aided Geometric [13]. Recently equivalence conditions of control points and application to planar Bézier curves have been examined. In [14], Frenet apparatus of the cubic Bezier curves has been examined in E^3. In here, first 5^{th} order Bezier curve and its first, second and third derivatives have been examined based on the control points of 5^{th}order Bezier Curve in E^3. Subsequently, in [15, 16] involutes of cubic Bezier curves, in [17] and [18] the Bertrand and the Mannheim mate of a cubic Bézier curve by using matrix representation have been researched in E^3. In [19], it has been researched the answer of the question "How to find a n^{th} order Bezier curve if we know the first, second and third derivatives?".

Generally Bézier curves can be defined by $n+1$ control points $P_0, P_1, \ldots, P_n$ with the parametrization

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i}(t)[P_i]. \tag{1.1}$$

In this study, it will be researched the exponential curve as a 3^{rd}, 5^{th} and 7^{th} order Bézier curve in E^2. Also, the numerical matrix representations of these curves will be calculated via the control points. For more detail, see respectively [20, 21].

It is well known that Taylor series of a function $f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$ is an infinite sum of the functions derivatives at a single point a, also a Maclaurin series $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$ is a taylor series where $a = 0$.

2. The Curve e^x as a *Cubic* Bézier Curve

We will examine the curve e^x as a *cubic* or 3^{rd} *order* Bézier curve.

Theorem 2.1. *The numerical matrix representation of the curve $f(x) = e^x$ as a cubic Bézier curve is*

$$\left(t, e^t\right) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{11}{6} \\ 1 & \frac{8}{3} \end{bmatrix}$$

where the control points P_0, P_1, P_2, and P_3 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{11}{6} \\ 1 & \frac{8}{3} \end{bmatrix}.$$

Proof. For e^x function cubic Maclaurin series expansion is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

It can be written as in parametric form and a 5^{th} degree polynomial function

$$\left(t, e^t\right) = \left(t, 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right) = \left(t, a_3 t^3 + a_2 t^2 + a_1 t + a_0\right).$$

Also this can be written as a *cubic* Bézier curve in matrix representation with the coefficients

$$\begin{aligned} a_3 &= \tfrac{1}{3!}, \\ a_2 &= \tfrac{1}{2!}, \\ a_1 &= 1, \\ a_0 &= 1. \end{aligned}$$

Hence we get the following equation

$$\begin{aligned} \left(t, e^t\right) &= \left(t, 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right) \\ &= \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{3!} \\ 0 & \frac{1}{2!} \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}, \end{aligned}$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3!} \\ 0 & \frac{1}{2!} \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

where the coefficients matrix of any cubic Bézier curve and inverse matrix are respectively

$$[B^3] = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad [B^3]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

For more detail see in [18]. □

3. The Curve e^{ax+b} as a $Cubic$ Bézier Curve

Theorem 3.1. *The numerical matrix representation of the curve $f(x) = e^{ax+b}$ as a cubic Bézier curve is*

$$\left(t, e^{at+b}\right) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^3] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

where the control points $P_0, P_1, P_2,$ and P_3 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 & e^b \\ \frac{1}{3} & \frac{1}{3}e^b(a+3) \\ \frac{2}{3} & \frac{1}{6}e^b\left(a^2+4a+6\right) \\ 1 & \frac{1}{6}e^b\left(a^3+3a^2+6a+6\right) \end{bmatrix}.$$

Proof. Taylor series of a function is an infinite sum of terms of the functions derivatives at a single point a , also a Maclaurin series is a taylor series where $a = 0$. 5th degree Maclaurin series expansion for the function e^{ax+b} is

$$\begin{aligned} f(x) = e^{ax+b} &= \sum_{n=0}^{3} f^{(n)}(0)\frac{x^n}{n!} \\ &= e^b + ae^b x + a^2 e^b \frac{x^2}{2!} + a^3 e^b \frac{x^3}{3!}. \end{aligned}$$

It can be written as in parametric form and a *cubic* polynomial function

$$\begin{aligned} \left(t, e^{at+b}\right) &= \left(t, \frac{a^3 e^b}{3!}t^3 + \frac{a^2 e^b}{2!}t^2 + ae^b t + e^b\right) \\ &= \left(t, a_3 t^3 + a_2 t^2 + a_1 t + a_0\right). \end{aligned}$$

Also this can be written as a *cubic* Bézier curve in matrix representation with the coefficients

$$\begin{aligned} a_3 &= \tfrac{a^3 e^b}{3!}, \\ a_2 &= \tfrac{a^2 e^b}{2!}, \\ a_1 &= ae^b, \\ a_0 &= e^b. \end{aligned}$$

Hence we get the following equation

$$\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{a^3 e^b}{3!} \\ 0 & \frac{a^2 e^b}{2!} \\ 1 & ae^b \\ 0 & e^b \end{bmatrix} = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix},$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{a^3e^b}{3!} \\ 0 & \frac{a^2e^b}{2!} \\ 1 & ae^b \\ 0 & e^b \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^b \\ \frac{1}{3} & \frac{1}{3}e^b\left(a+3\right) \\ \frac{2}{3} & \frac{1}{6}e^b\left(a^2+4a+6\right) \\ 1 & \frac{1}{6}e^b\left(a^3+3a^2+6a+6\right) \end{bmatrix}.$$

□

4. The Curve e^x as a 5^{th} *Order* Bézier Curve

Now, we will examine the curve e^x as a 5^{th} *order* Bézier curve. We have already known that the matrix representation of $\alpha(t) = (t, a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + at_1 + a_0)$ is

$$\alpha(t) = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B^5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}$$

where the coefficient matrix and inverse matrix of 5^{th} order Bézier curve are

$$\begin{bmatrix} B^5 \end{bmatrix} = \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} B^5 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Theorem 4.1. *The numerical matrix representation of the curve $f(x) = e^x$ as a 5^{th} order Bézier curve is*

$$\left(t, e^t\right) = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{7}{12} \\ \frac{4}{5} & \frac{11}{15} \\ 1 & \frac{101}{120} \end{bmatrix}$$

where the control points $P_0, P_1, P_2, P_3, P_4,$ and P_5 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{7}{12} \\ \frac{4}{5} & \frac{11}{15} \\ 1 & \frac{101}{120} \end{bmatrix}.$$

Proof. 5th degree Maclaurin series expansion for the function e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.$$

It can be written as in parametric form and a 5^{th} degree polynomial function

$$\left(t,e^{t}\right)=\left(t,1+t+\frac{t^{2}}{2!}+\frac{t^{3}}{3!}+\frac{t^{4}}{4!}+\frac{t^{5}}{5!}\right)=\left(t,a_{5}t^{5}+a_{4}t^{4}+a_{3}t^{3}+a_{2}t^{2}+a_{1}t+a_{0}\right).$$

Also this can be written as a 5^{th} *order* Bézier curve in matrix representation with the coefficients

$$\left[\begin{array}{cccccc} a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{array}\right]=\left[\begin{array}{cccccc} \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 1 \end{array}\right].$$

Hence we get the following equation

$$\begin{aligned}\left(t,e^{t}\right)&=\left(t,1+t+\frac{t^{2}}{2!}+\frac{t^{3}}{3!}+\frac{t^{4}}{4!}+\frac{t^{5}}{5!}\right)\\ &=\begin{bmatrix} t^5\\ t^4\\ t^3\\ t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} 0 & \frac{1}{5!}\\ 0 & \frac{1}{4!}\\ 0 & \frac{1}{3!}\\ 0 & \frac{1}{2!}\\ 1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix} t^5\\ t^4\\ t^3\\ t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1\\ 5 & -20 & 30 & -20 & 5 & 0\\ -10 & 30 & -30 & 10 & 0 & 0\\ 10 & -20 & 10 & 0 & 0 & 0\\ -5 & 5 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix},\end{aligned}$$

$$\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1\\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1\\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1\\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1\\ 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix} 0 & \frac{1}{5!}\\ 0 & \frac{1}{4!}\\ 0 & \frac{1}{3!}\\ 0 & \frac{1}{2!}\\ 1 & 1\\ 0 & 1\end{bmatrix},$$

solving these equation we obtained the control numbers

$$\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix}=\begin{bmatrix} 0 & 1\\ \frac{1}{5} & \frac{6}{5}\\ \frac{2}{5} & \frac{29}{20}\\ \frac{3}{5} & \frac{53}{30}\\ \frac{4}{5} & \frac{87}{40}\\ 1 & \frac{163}{60}\end{bmatrix}.$$

□

5. The Curve e^{ax+b} as a 5^{th} $Order$ Bézier Curve

In this section we have investigated the curve e^{ax+b} as a 5^{th} *order* Bézier curve.

$$\begin{aligned} f(x)&=\sum_{n=0}^{\infty}f^{(n)}(0)\frac{x^n}{n!},\\ f(x)&=e^{ax+b},\\ f'(x)&=ae^{ax+b},\\ f''(x)&=a^2e^{ax+b},\\ f'''(x)&=a^3e^{ax+b},\\ f^{(4)}(x)&=a^4e^{ax+b},\\ f^{(5)}(x)&=a^5e^{ax+b},\\ f^{(6)}(x)&=a^6e^{ax+b},\\ f^{(7)}(x)&=a^7e^{ax+b}.\end{aligned}$$

Theorem 5.1. *The numerical matrix representation of the curve $f(x) = e^{ax+b}$ as a 5^{th} order Bézier curve is*

$$\left(t, e^{at+b}\right) = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}$$

where the control points P_0, P_1, P_2, P_3, P_4 and P_5 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & e^b \\ \frac{1}{5} & \frac{1}{5}e^b(a+5) \\ \frac{2}{5} & \frac{1}{20}e^b\left(a^2+8a+20\right) \\ \frac{3}{5} & \frac{1}{60}e^b\left(a^3+9a^2+36a+60\right) \\ \frac{4}{5} & \frac{1}{120}e^b\left(a^4+8a^3+36a^2+96a+120\right) \\ 1 & \frac{1}{120}e^b\left(a^5+5a^4+20a^3+60a^2+120a+120\right) \end{bmatrix}.$$

Proof. Taylor series of a function is an infinite sum of terms of the functions derivatives at a single point a , also a Maclaurin series is a taylor series where $a = 0$. 5th degree Maclaurin series expansion for the function e^{ax+b} is

$$\begin{aligned} f(x) = e^{ax+b} &= \sum_{n=0}^{5} f^{(n)}(0)\frac{x^n}{n!} \\ &= e^b + ae^bx + a^2e^b\frac{x^2}{2!} + a^3e^b\frac{x^3}{3!} + a^4e^b\frac{x^4}{4!} + a^5e^b\frac{x^5}{5!} \end{aligned}$$

and it can be written as in parametric form and a 5^{th} degree polynomial function

$$\begin{aligned} \left(t, e^{at+b}\right) &= \left(t, \frac{a^5e^b}{5!}t^5 + \frac{a^4e^b}{4!}t^4 + \frac{a^3e^b}{3!}t^3 + \frac{a^2e^b}{2!}t^2 + ae^bt + e^b\right) \\ &= \left(t, a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0\right). \end{aligned}$$

Also this can be written as a 5^{th} *order* Bézier curve in matrix representation with the coefficients

$$\begin{bmatrix} a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} = \begin{bmatrix} \frac{a^5e^b}{5!} & \frac{a^4e^b}{4!} & \frac{a^3e^b}{3!} & \frac{a^2e^b}{2!} & ae^b & e^b \end{bmatrix}.$$

Hence we get the following equation

$$\begin{aligned} \left(t, e^{at+b}\right) &= \left(t, \frac{a^5e^b}{5!}t^5 + \frac{a^4e^b}{4!}t^4 + \frac{a^3e^b}{3!}t^3 + \frac{a^2e^b}{2!}t^2 + ae^bt + e^b\right) \\ &= \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{a^5e^b}{5!} \\ 0 & \frac{a^4e^b}{4!} \\ 0 & \frac{a^3e^b}{3!} \\ 0 & \frac{a^2e^b}{2!} \\ 1 & ae^b \\ 0 & e^b \end{bmatrix} = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}, \end{aligned}$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{a^5e^b}{5!} \\ 0 & \frac{a^4e^b}{4!} \\ 0 & \frac{a^3e^b}{3!} \\ 0 & \frac{a^2e^b}{2!} \\ 1 & ae^b \\ 0 & e^b \end{bmatrix},$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & e^b \\ \frac{1}{5} & e^b + \frac{1}{5}ae^b \\ \frac{2}{5} & \frac{1}{20}e^b a^2 + \frac{2}{5}e^b a + e^b \\ \frac{3}{5} & \frac{1}{60}e^b a^3 + \frac{3}{20}e^b a^2 + \frac{3}{5}e^b a + e^b \\ \frac{4}{5} & \frac{1}{120}e^b a^4 + \frac{1}{15}e^b a^3 + \frac{3}{10}e^b a^2 + \frac{4}{5}e^b a + e^b \\ 1 & \frac{1}{120}e^b a^5 + \frac{1}{24}e^b a^4 + \frac{1}{6}e^b a^3 + \frac{1}{2}e^b a^2 + e^b a + e^b \end{bmatrix},$$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!},$$
$$f(x) = e^{ax+b}\frac{x^n}{0!} + ae^{ax+b}\frac{x^n}{1!} + a^2 e^{ax+b}\frac{x^n}{2!} + a^3 e^{ax+b}\frac{x^n}{3!} + a^4 e^{ax+b}\frac{x^n}{4!} + a^5 e^{ax+b}\frac{x^n}{5!} + a^6 e^{ax+b}\frac{x^n}{6!} + a^7 e^{ax+b}\frac{x^n}{7!}.$$

□

6. The Curve e^x as a 7^{th} *Order* Bézier Curve

Theorem 6.1. *The matrix of any 7^{th} order Bézier curve is*

$$[B^7] = \begin{bmatrix} -\binom{7}{0}\binom{7}{7} & \binom{7}{1}\binom{7-1}{7-1} & -\binom{7}{2}\binom{7-2}{7-2} & \binom{7}{3}\binom{7-3}{7-3} & -\binom{7}{4}\binom{7-4}{7-4} & \binom{7}{5}\binom{7-5}{7-5} & -\binom{7}{6}\binom{7-6}{7-6} & \binom{7}{7}\binom{0}{0} \\ \binom{7}{0}\binom{7}{7-1} & -\binom{7}{1}\binom{7-1}{7-2} & \binom{7}{2}\binom{7-2}{7-3} & -\binom{7}{3}\binom{7-3}{7-4} & \binom{7}{4}\binom{7-4}{7-5} & -\binom{7}{5}\binom{7-5}{7-6} & \binom{7}{6}\binom{7-6}{7-7} & 0 \\ -\binom{7}{0}\binom{7}{7-2} & \binom{7}{1}\binom{7-1}{7-3} & -\binom{7}{2}\binom{7-2}{7-4} & \binom{7}{3}\binom{7-3}{7-5} & -\binom{7}{4}\binom{7-4}{7-6} & \binom{7}{5}\binom{7-5}{7-7} & 0 & 0 \\ \binom{7}{0}\binom{7}{7-3} & -\binom{7}{1}\binom{7-1}{7-4} & \binom{7}{2}\binom{7-2}{7-5} & -\binom{7}{3}\binom{7-3}{7-6} & \binom{7}{4}\binom{7-4}{7-7} & 0 & 0 & 0 \\ -\binom{7}{0}\binom{7}{7-4} & \binom{7}{1}\binom{7-1}{7-5} & -\binom{7}{2}\binom{7-2}{7-6} & \binom{7}{3}\binom{7-3}{7-7} & 0 & 0 & 0 & 0 \\ \binom{7}{0}\binom{7}{7-5} & -\binom{7}{1}\binom{7-1}{7-6} & \binom{7}{2}\binom{7-2}{7-7} & 0 & 0 & 0 & 0 & 0 \\ -\binom{7}{0}\binom{7}{7-6} & \binom{7}{1}\binom{7-1}{7-7} & 0 & 0 & 0 & 0 & 0 & 0 \\ \binom{7}{0}\binom{7}{7-7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 7 & -21 & 35 & -35 & 21 & -7 & 1 \\ 7 & -42 & 105 & -140 & 105 & -42 & 7 & 0 \\ -21 & 105 & -210 & 210 & -105 & 21 & 0 & 0 \\ 35 & -140 & 210 & -140 & 35 & 0 & 0 & 0 \\ -35 & 105 & -105 & 35 & 0 & 0 & 0 & 0 \\ 21 & -42 & 21 & 0 & 0 & 0 & 0 & 0 \\ -7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also the inverse matrix of 7^{th} order Bézier curves in $\mathbf{E}^2$ is

$$[B^7]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{2}{7} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{35} & \frac{1}{7} & \frac{3}{7} & 1 \\ 0 & 0 & 0 & \frac{1}{35} & \frac{4}{35} & \frac{2}{7} & \frac{4}{7} & 1 \\ 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{2}{7} & \frac{10}{21} & \frac{5}{7} & 1 \\ 0 & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now, we will examine the e^x curve as a 7^{th} *order* Bézier curve.

Theorem 6.2. *The numerical matrix representation of the curve $f(x) = e^x$ as a 7^{th} order Bézier curve is*

$$\left(t, e^t\right) = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{a^7 e^b}{7!} \\ 0 & \frac{a^6 e^b}{6!} \\ 0 & \frac{a^5 e^b}{5!} \\ 0 & \frac{a^4 e^b}{4!} \\ 0 & \frac{a^3 e^b}{3!} \\ 0 & \frac{a^2 e^b}{2!} \\ 1 & a e^b \\ 0 & e^b \end{bmatrix}^T = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \left[B^7\right] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix}$$

where the control points $P_0, P_1, P_2, \ldots, P_7$ are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{7} & \frac{8}{7} \\ \frac{2}{7} & \frac{55}{42} \\ \frac{3}{7} & \frac{158}{105} \\ \frac{4}{7} & \frac{1457}{840} \\ \frac{5}{7} & \frac{632}{315} \\ \frac{6}{7} & \frac{11743}{5040} \\ 1 & \frac{685}{252} \end{bmatrix}.$$

Proof. 7^{th} degree Maclaurin series expansion for the function e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$$

and it can be written as in parametric form and a 7^{th} degree polynomial function

$$\left(t, e^t\right) = \left(t, 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!}\right) = \left(t, a_7 t^7 + a_6 t^6 + a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0\right).$$

Also this can be written as a 7^{th} *order* Bézier curve in matrix representation with the coefficients. Hence we get the following equation

$$\left(t, e^t\right) = \left(t, 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!}\right)$$

$$= \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{7!} \\ 0 & \frac{1}{6!} \\ 0 & \frac{1}{5!} \\ 0 & \frac{1}{4!} \\ 0 & \frac{1}{3!} \\ 0 & \frac{1}{2!} \\ 1 & 1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \left[B^7\right] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix},$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{2}{7} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{35} & \frac{1}{7} & \frac{3}{7} & 1 \\ 0 & 0 & 0 & \frac{1}{35} & \frac{4}{35} & \frac{2}{7} & \frac{4}{7} & 1 \\ 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{2}{7} & \frac{10}{21} & \frac{5}{7} & 1 \\ 0 & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{7!} \\ 0 & \frac{1}{6!} \\ 0 & \frac{1}{5!} \\ 0 & \frac{1}{4!} \\ 0 & \frac{1}{3!} \\ 0 & \frac{1}{2!} \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

□

7. The Curve e^{ax+b} as a 7^{th} *Order* Bézier Curve

In this section, we will research the curve e^{ax+b} as a 7^{th} order Bézier curve.

$$\begin{aligned} f(x) = e^{ax+b} &= \sum_{n=0}^{7} f^{(n)}(0)\frac{x^n}{n!} \\ &= e^{ax+b} + ae^{ax+b}x + a^2e^{ax+b}\frac{x^2}{2!} + a^3e^{ax+b}\frac{x^3}{3!} + a^4e^{ax+b}\frac{x^4}{4!} + a^5e^{ax+b}\frac{x^5}{5!} + a^6e^b\frac{x^6}{6!} + a^7e^b\frac{x^7}{7!}. \end{aligned}$$

Theorem 7.1. *The numerical matrix representation of the curve $f(x) = e^{ax+b}$ as a 7^{th} order Bézier curve is*

$$\left(t, e^{at+b}\right) = \alpha(t) = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 7 & -21 & 35 & -35 & 21 & -7 & 1 \\ 7 & -42 & 105 & -140 & 105 & -42 & 7 & 0 \\ -21 & 105 & -210 & 210 & -105 & 21 & 0 & 0 \\ 35 & -140 & 210 & -140 & 35 & 0 & 0 & 0 \\ -35 & 105 & -105 & 35 & 0 & 0 & 0 & 0 \\ 21 & -42 & 21 & 0 & 0 & 0 & 0 & 0 \\ -7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix}$$

where the control points $P_0, P_1, P_2, \ldots, P_7$ are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & e^b \\ \frac{1}{7} & e^b + \frac{1}{7}ae^b \\ \frac{2}{7} & \frac{1}{42}e^ba^2 + \frac{2}{7}e^ba + e^b \\ \frac{3}{7} & \frac{1}{210}e^ba^3 + \frac{1}{14}e^ba^2 + \frac{3}{7}e^ba + e^b \\ \frac{4}{7} & \frac{1}{840}e^ba^4 + \frac{2}{105}e^ba^3 + \frac{1}{7}e^ba^2 + \frac{4}{7}e^ba + e^b \\ \frac{5}{7} & \frac{1}{2520}e^ba^5 + \frac{1}{168}e^ba^4 + \frac{1}{21}e^ba^3 + \frac{5}{21}e^ba^2 + \frac{5}{7}e^ba + e^b \\ \frac{6}{7} & \frac{1}{5040}e^ba^6 + \frac{1}{420}e^ba^5 + \frac{1}{56}e^ba^4 + \frac{2}{21}e^ba^3 + \frac{5}{14}e^ba^2 + \frac{6}{7}e^ba + e^b \\ 1 & \frac{1}{5040}e^ba^7 + \frac{1}{720}e^ba^6 + \frac{1}{120}e^ba^5 + \frac{1}{24}e^ba^4 + \frac{1}{6}e^ba^3 + \frac{1}{2}e^ba^2 + e^ba + e^b \end{bmatrix}.$$

Proof. 7^{th} degree Maclaurin series expansion for the function e^{ax+b} is

$$f(x) = e^{ax+b} = \sum_{n=0}^{7} f^{(n)}(0)\frac{x^n}{n!},$$

$$f(x) = e^{ax+b} + ae^{ax+b}x + a^2e^{ax+b}\frac{x^2}{2!} + a^3e^{ax+b}\frac{x^3}{3!} + a^4e^{ax+b}\frac{x^4}{4!} + a^5e^{ax+b}\frac{x^5}{5!} + a^6e^b\frac{x^6}{6!} + a^7e^b\frac{x^7}{7!},$$

and it can be written as in parametric form and a 5^{th} degree polynomial function

$$\begin{aligned} \left(t, e^{at+b}\right) &= \left(t, e^{at+b} + ae^{at+b}t + \frac{a^2e^{ax+b}}{2!}t^2 + \frac{a^3e^{ax+b}}{3!}t^3 + \frac{a^4e^{ax+b}}{4!}t^4 + \frac{a^5e^{ax+b}}{5!}t^5\right) \\ &= \left(t, a^7e^b\frac{t^7}{7!} + a^6e^b\frac{t^6}{6!} + \frac{a^5e^{ax+b}}{5!}t^5 + \frac{a^4e^{ax+b}}{4!}t^4 + \frac{a^3e^{ax+b}}{3!}t^3 + \frac{a^2e^{ax+b}}{2!}t^2 + ae^{ax+b}t + e^{ax+b}\right) \\ &= \left(t, a_7t^7 + a_6t^6 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0\right). \end{aligned}$$

Also, this can be written as a 7^{th} *order* Bézier curve in matrix representation with the coefficients. Hence we get the following equation

$$\left(t, e^{at+b}\right) = \left(t, \frac{a^5 e^{ax+b}}{5!} t^5 + \frac{a^4 e^{ax+b}}{4!} t^4 + \frac{a^3 e^{ax+b}}{3!} t^3 + \frac{a^2 e^{ax+b}}{2!} t^2 + a e^{ax+b} t + e^{ax+b}\right),$$

$$\left(t, e^{at+b}\right) = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \frac{a^7 e^b}{7!} \\ 0 & \frac{a^6 e^b}{6!} \\ 0 & \frac{a^5 e^b}{5!} \\ 0 & \frac{a^4 e^b}{4!} \\ 0 & \frac{a^3 e^b}{3!} \\ 0 & \frac{a^2 e^b}{2!} \\ 1 & a e^b \\ 0 & e^b \end{bmatrix}^T = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \left[B^7\right] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_7 \\ P_7 \end{bmatrix},$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{2}{7} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{35} & \frac{1}{7} & \frac{3}{7} & 1 \\ 0 & 0 & 0 & \frac{1}{35} & \frac{4}{35} & \frac{2}{7} & \frac{4}{7} & 1 \\ 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{2}{7} & \frac{10}{21} & \frac{5}{7} & 1 \\ 0 & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{a^7 e^b}{7!} \\ 0 & \frac{a^6 e^b}{6!} \\ 0 & \frac{a^5 e^b}{5!} \\ 0 & \frac{a^4 e^b}{4!} \\ 0 & \frac{a^3 e^b}{3!} \\ 0 & \frac{a^2 e^b}{2!} \\ 1 & a e^b \\ 0 & e^b \end{bmatrix}^T,$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = \begin{bmatrix} 0 & e^b \\ \frac{1}{7} & \frac{1}{7} e^b (a+7) \\ \frac{2}{7} & \frac{1}{42} e^b \left(a^2 + 12a + 42\right) \\ \frac{3}{7} & \frac{1}{210} e^b \left(a^3 + 15a^2 + 90a + 210\right) \\ \frac{4}{7} & \frac{1}{840} e^b \left(a^4 + 16a^3 + 120a^2 + 480a + 840\right) \\ \frac{5}{7} & \frac{1}{2520} e^b \left(a^5 + 15a^4 + 120a^3 + 600a^2 + 1800a + 2520\right) \\ \frac{6}{7} & \frac{1}{5040} e^b \left(a^6 + 12a^5 + 90a^4 + 480a^3 + 1800a^2 + 4320a + 5040\right) \\ 1 & \frac{1}{5040} e^b \left(a^7 + 7a^6 + 42a^5 + 210a^4 + 840a^3 + 2520a^2 + 5040a + 5040\right) \end{bmatrix},$$

and so, the result give us the proof. □

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A Qualitative Investigation of the Solution of the Difference Equation $\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(\pm 1 \pm \Psi_{m-3}\Psi_{m-5})}$

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Abstract
We explore the dynamics of adhering to rational difference formula

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(\pm 1 \pm \Psi_{m-3}\Psi_{m-5}\right)} \quad m \in \mathbb{N}_0$$

where the initials Ψ_{-5}, Ψ_{-4}, Ψ_{-3},Ψ_{-2}, Ψ_{-1}, Ψ_0 are arbitrary nonzero real numbers. Specifically, we examine global asymptotically stability. We also give examples and solution diagrams for certain particular instances.

Keywords: Boundedness, Equilibrium point, Global asymptotic stability, Solution of difference equation, Stability.
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1. Introduction

Because of its employment in discrete-time systems with microprocessors, difference equations are becoming increasingly important in engineering. The study of rational difference equations and their qualitative features has recently sparked a surge of interest. We refer the reader to [1–3] for some literature in this field.

Important rational difference equations were investigated by several authors. As examples:
Aloqeili, [4] has actually gotten the solutions to the difference equation

$$\Psi_{m+1} = \frac{\Psi_{m-1}}{a - \Psi_m \Psi_{m-1}}.$$

Çınar [5], researched adhering to problems with positive first values:

$$\Psi_{m+1} = \frac{Q_{m-1}}{-1 + a\Psi_m \Psi_{m-1}}$$

for $m=0,1,2,\ldots$

Gelişken [6] investigated behaviors of

$$\Psi_{m+1}=\frac{A_1M_{m-(3k-1)}}{B_1+C_1M_{m-(3k-1)}\Psi_{m-(2k-1)}M_{m-(k-1)}},$$
$$M_{m+1}=\frac{A_2\Psi_{m-(3k-1)}}{B_2+C_2\Psi_{m-(3k-1)}M_{m-(2k-1)}\Psi_{m-(k-1)}}.$$

Karataş et al. [7] deal with

$$\Psi_{m+1}=\frac{\Psi_{m-5}}{1+\Psi_{m-2}\Psi_{m-5}}.$$

Oğul et al. [8] deal with

$$\Psi_{m+1}=\frac{\Psi_{m-17}}{\pm1\pm\Psi_{m-2}\Psi_{m-5}\Psi_{m-8}\Psi_{m-11}\Psi_{m-14}\Psi_{m-17}}.$$

Şimşek et al. [9] examine the equation

$$\Psi_{m+1}=\frac{\Psi_{m-13}}{1+\Psi_{m-1}\Psi_{m-3}\Psi_{m-5}\Psi_{m-7}\Psi_{m-9}\Psi_{m-11}}.$$

Yalçınkaya et al. [10] have studied

$$\Psi_{m+1}=\frac{a\Psi_{m-k}}{b+c_m^p}.$$

For more related works we refer to [11–18].

Our objective in this study is to check out actions of the solution of adhering to nonlinear difference formula

$$\Psi_{m+1}=\frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(\pm1\pm\Psi_{m-3}\Psi_{m-5}\right)},\quad m\in\mathbb{N}_0$$

where the initials are arbitrary real numbers. Additionally, we obtain these types of solutions.

2. Solution of $\Psi_{m+1}=\frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(1+\Psi_{m-3}\Psi_{m-5})}$

In this part we give the solutions of

$$\Psi_{m+1}=\frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(1+\Psi_{m-3}\Psi_{m-5}\right)},\quad m\in\mathbb{N}_0 \tag{2.1}$$

where the initials are real numbers.

Theorem 2.1. *Let* $\{\Psi_m\}_{m=-5}^{\infty}$ *be a solution of* (2.1)*. Then for* $m\in\mathbb{N}_0$

$$\Psi_{4m+1}=\frac{DF^{m+1}}{B^{m+1}}\prod_{i=0}^{m}\left(\frac{1+(i)BD}{1+(i+1)DF}\right),\qquad \Psi_{4m+2}=\frac{CE^{m+1}}{A^{m+1}}\prod_{i=0}^{m}\left(\frac{1+(i)CA}{1+(i+1)CE}\right),$$
$$\Psi_{4m+3}=\frac{B^{m+2}}{F^{m+1}}\prod_{i=0}^{m}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right),\qquad \Psi_{4m+4}=\frac{A^{m+2}}{E^{m+1}}\prod_{i=0}^{m}\left(\frac{1+(i+1)CE}{1+(i+1)CA}\right),$$

where, $\Psi_{-5}=F,\quad \Psi_{-4}=E,\ \Psi_{-3}=D,\quad \Psi_{-2}=C,\quad \Psi_{-1}=B,\quad \Psi_0=A.$

Proof. Assume $m>0$ and this our supposition remains true for $m-1$.
That is,

$$\Psi_{4m-3}=\frac{DF^m}{B^m}\prod_{i=0}^{m-1}\left(\frac{1+(i)BD}{1+(i+1)DF}\right),\quad \Psi_{4m-2}=\frac{CE^m}{A^m}\prod_{i=0}^{m-1}\left(\frac{1+(i)CA}{1+(i+1)CE}\right),$$
$$\Psi_{4m-1}=\frac{B^{m+1}}{F^m}\prod_{i=0}^{m-1}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right),\quad \Psi_{4m}=\frac{A^{m+1}}{E^m}\prod_{i=0}^{m-1}\left(\frac{1+(i+1)CE}{1+(i+1)CA}\right),\quad \Psi_{4m-5}=\frac{B^m}{F^{m-1}}\prod_{i=0}^{m-2}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right).$$

At the present time, using the main (2.1), one has

$$\Psi_{4m+1} = \frac{\Psi_{4m-3}\Psi_{4m-5}}{\Psi_{4m-1}\left(1+\Psi_{4m-3}\Psi_{4m-5}\right)}$$
$$= \frac{\frac{DF^m}{B^m}\prod_{i=0}^{m-1}\left(\frac{1+(i)BD}{1+(i+1)DF}\right)\frac{B^m}{F^{m-1}}\prod_{i=0}^{m-2}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right)}{\frac{B^{m+1}}{F^m}\prod_{i=0}^{m-1}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right)+\frac{B^{m+1}}{F^m}\prod_{i=0}^{m-1}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right)\frac{DF^m}{B^m}\prod_{i=0}^{m-1}\left(\frac{1+(i)BD}{1+(i+1)DF}\right)\frac{B^m}{F^{m-1}}\prod_{i=0}^{m-2}\left(\frac{1+(i+1)DF}{1+(i+1)BD}\right)}.$$

Hence, we have

$$\Psi_{4m+1} = \frac{DF^{m+1}}{B^{m+1}}\prod_{i=0}^{m}\left(\frac{1+(i)BD}{1+(i+1)DF}\right).$$

Similarly, it is easily obtained in other relationships. □

Theorem 2.2. (2.1) *has one equilibrium* $\overline{\Psi} = 0$ *and this equilibrium isn't locally asymptotically stable.*

Proof. We may express the equilibrium points of (2.1) as

$$\overline{\Psi} = \frac{\overline{\Psi}^2}{\overline{\Psi}(1+\overline{\Psi}^2)},$$

$$\overline{\Psi}^2\left(1+\overline{\Psi}^2\right) = \overline{\Psi}^2.$$

After that

$$\overline{\Psi}^4 = 0.$$

As a result, the equilibrium of (2.1) is $\overline{\Psi} = 0$.
Assume that $f : (0,\infty)^4 \to (0,\infty)$ is the function defined by

$$f(\tau,\kappa,\rho) = \frac{\tau\rho}{\kappa(1+\tau\rho)}.$$

As a result, it follows that

$$f_\tau(\tau,\kappa,\rho) = \frac{\rho}{\kappa(1+\tau\rho)^2}, \qquad f_\kappa(\tau,\kappa,\rho) = -\frac{\tau\rho}{\kappa^2(1+\tau\rho)}, \qquad f_\rho(\tau,\kappa,\rho) = \frac{\tau}{\kappa(1+\tau\rho)^2}.$$

We see that

$$f_\tau(\overline{\Psi},\overline{\Psi},\overline{\Psi}) = 1, \qquad f_\kappa(\overline{\Psi},\overline{\Psi},\overline{\Psi}) = 1, \qquad f_\rho(\overline{\Psi},\overline{\Psi},\overline{\Psi}) = 1.$$

□

We confirm our results with the following numerical examples.

Example 2.3. *Assume that*

$$\Psi_{-5} = 0.3, \quad \Psi_{-4} = 0.32, \quad \Psi_{-3} = 0.34, \quad \Psi_{-2} = 0.36, \quad \Psi_{-1} = 0.38, \quad \Psi_0 = 0.4.$$

See Figure 2.1.

Example 2.4. *Assume that*

$$\Psi_{-5} = 0.35, \quad \Psi_{-4} = 0.32, \quad \Psi_{-3} = 0.34, \quad \Psi_{-2} = 0.38, \quad \Psi_{-1} = 0.42, \quad \Psi_0 = 0.43.$$

See Figure 2.2.

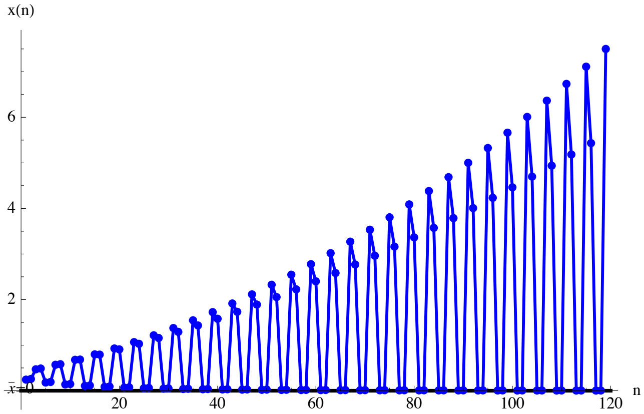

Figure 2.1

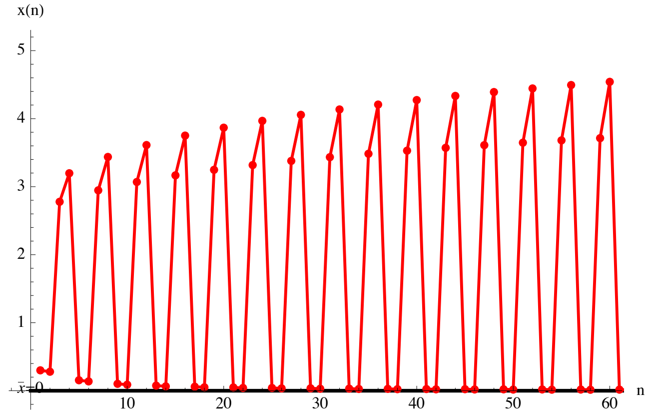

Figure 2.2

3. Solution of $\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(1-\Psi_{m-3}\Psi_{m-5})}$

We deal with the difference equation

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(1-\Psi_{m-3}\Psi_{m-5}\right)}, \quad m \in \mathbb{N}_0. \tag{3.1}$$

Theorem 3.1. *Let $\{\Psi_m\}_{m=-7}^{\infty}$ represent a solution of (3.1). In that case for $m \in \mathbb{N}_0$*

$$\Psi_{4m+1} = \frac{DF^{m+1}}{B^{m+1}} \prod_{i=0}^{m} \left(\frac{-1+(i)BD}{-1+(i+1)DF} \right), \qquad \Psi_{4m+2} = \frac{CE^{m+1}}{A^{m+1}} \prod_{i=0}^{m} \left(\frac{-1+(i)CA}{-1+(i+1)CE} \right),$$

$$\Psi_{4m+3} = \frac{B^{m+2}}{F^{m+1}} \prod_{i=0}^{m} \left(\frac{-1+(i+1)DF}{-1+(i+1)BD} \right), \qquad \Psi_{4m+4} = \frac{A^{m+2}}{E^{m+1}} \prod_{i=0}^{m} \left(\frac{-1+(i+1)CE}{-1+(i+1)CA} \right),$$

where, $\Psi_{-5} = F, \quad \Psi_{-4} = E, \quad \Psi_{-3} = D, \quad \Psi_{-2} = C, \quad \Psi_{-1} = B, \quad \Psi_0 = A.$

Proof. The proof is similar to the proof of Theorem 2.1 and therefore it will be omitted. □

Theorem 3.2. *The unique equilibrium $\overline{\Psi} = 0$ in (3.1) isn't locally asymptotically stable.*

Proof. For confirming outcomes of this section, we take into consideration mathematical instances which stand for various kind of solutions to (3.1). □

Example 3.3. *Figure 3.1 depicts the actions taken when*

$$\Psi_{-5} = 3, \quad \Psi_{-4} = 3.9, \quad \Psi_{-3} = 3.1, \quad \Psi_{-2} = 2.8, \quad \Psi_{-1} = 2.5, \quad \Psi_0 = 3.5.$$

Example 3.4. *Figure 3.2 depicts the actions taken when*

$$\Psi_{-5} = 5.1, \quad \Psi_{-4} = 4.9, \quad \Psi_{-3} = 4.3, \quad \Psi_{-2} = 5.3, \quad \Psi_{-1} = 4.5, \quad \Psi_0 = 4.6.$$

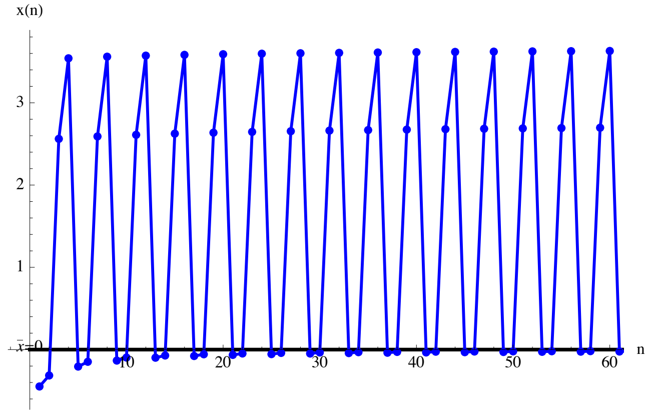

Figure 3.1

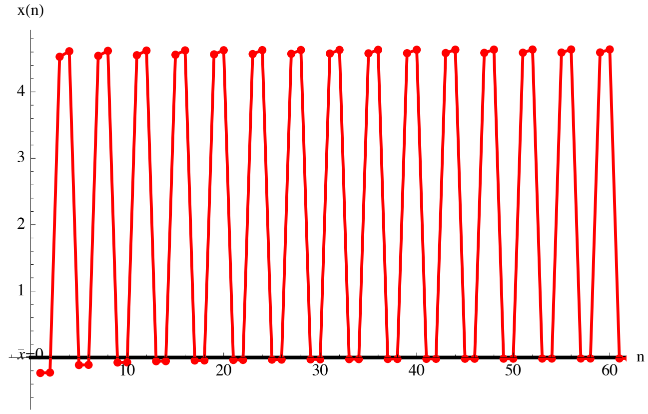

Figure 3.2

4. Solution of $\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(-1+\Psi_{m-3}\Psi_{m-5})}$

In this part, we study

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(-1+\Psi_{m-3}\Psi_{m-5}\right)}, \quad m \in \mathbb{N}_0. \tag{4.1}$$

Theorem 4.1. *Let $\{\Psi_m\}_{m=-5}^{\infty}$ represent a solution of (4.1). In that case for, $m = 0, 1, 2, \ldots$*

$$\Psi_{8m+1} = \frac{-DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^{m+1}}, \quad \Psi_{8m+2} = \frac{-CE^{2m+1}(1+AC)^m}{A^{2m+1}(1+CE)^{m+1}}, \quad \Psi_{8m+3} = \frac{B^{2m+2}(1+DF)^{m+1}}{F^{2m+1}(1+BD)^{m+1}},$$

$$\Psi_{8m+4} = \frac{A^{2m+2}(1+CE)^{m+1}}{E^{2m+1}(1+AC)^{m+1}}, \quad \Psi_{8m+5} = \frac{DF^{2m+2}(1+BD)^{m+1}}{B^{2m+2}(1+DF)^{m+1}}, \quad \Psi_{8m+6} = \frac{CE^{2m+2}(1+AC)^{m+1}}{A^{2m+2}(1+CE)^{m+1}},$$

$$\Psi_{8m+7} = \frac{B^{2m+3}(1+DF)^{m+1}}{F^{2m+2}(1+BD)^{m+1}}, \quad \Psi_{8m+8} = \frac{A^{2m+3}(1+CE)^{m+1}}{E^{2m+2}(1+AC)^{m+1}}.$$

Proof. Assume that $m > 0$ and our supposition hold for $m-1$.

$$\Psi_{8m-7} = \frac{-DF^{2m}(1+BD)^{m-1}}{B^{2m}(1+DF)^m}, \quad \Psi_{8m-6} = \frac{-CE^{2m}(1+AC)^{m-1}}{A^{2m}(1+CE)^m}, \quad \Psi_{8m-5} = \frac{B^{2m+1}(1+DF)^m}{F^{2m}(1+BD)^m},$$

$$\Psi_{8m-4} = \frac{A^{2m+1}(1+CE)^m}{E^{2m}(1+AC)^m}, \quad \Psi_{8m-3} = \frac{DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^m}, \quad \Psi_{8m-2} = \frac{CE^{2m+1}(1+AC)^m}{A^{2m+1}(1+CE)^m},$$

$$\Psi_{8m-1} = \frac{B^{2m+2}(1+DF)^m}{F^{2m+1}(1+BD)^m}, \quad \Psi_{8m} = \frac{A^{2m+2}(1+CE)^m}{E^{2m+1}(1+AC)^m}.$$

Now, it follows from (4.1) that

$$\begin{aligned}\Psi_{8m+1} &= \frac{\Psi_{8m-3}\Psi_{8m-5}}{\Psi_{8m-1}(-1+\Psi_{8m-3}\Psi_{8m-5})} \\ &= \frac{\frac{DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^m}\frac{B^{2m+1}(1+DF)^m}{F^{2m}(1+BD)^m}}{-\frac{B^{2m+2}(1+DF)^m}{F^{2m+1}(1+BD)^m}+\frac{B^{2m+2}(1+DF)^m}{F^{2m+1}(1+BD)^m}\frac{DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^m}\frac{B^{2m+1}(1+DF)^m}{F^{2m}(1+BD)^m}}.\end{aligned}$$

Then, we have

$$\Psi_{8m+1} = \frac{-DF^{2m+1}(1+BD)^m}{B^{2m+1}(1+DF)^{m+1}}.$$

The other relations can be provided in the same way. □

Theorem 4.2. (4.1) *contains three equilibriums, 0, $\pm\sqrt{2}$ and they aren't locally asymptotically stable.*

Proof. The proof is similar to the proof of Theorem 2.2 and therefore it will be omitted. □

Example 4.3. *Figure 4.1 depicts the actions taken when*

$$\Psi_{-5} = 4.3, \quad \Psi_{-4} = 4.7, \quad \Psi_{-3} = 4.9, \quad \Psi_{-2} = 3.8, \quad \Psi_{-1} = 3.6, \quad \Psi_0 = 3.3.$$

Example 4.4. *Figure 4.2 depicts the actions taken when*

$$\Psi_{-5} = 4, \quad \Psi_{-4} = 4.5, \quad \Psi_{-3} = 5.3, \quad \Psi_{-2} = 4.7, \quad \Psi_{-1} = 5.1, \quad \Psi_0 = 5.5.$$

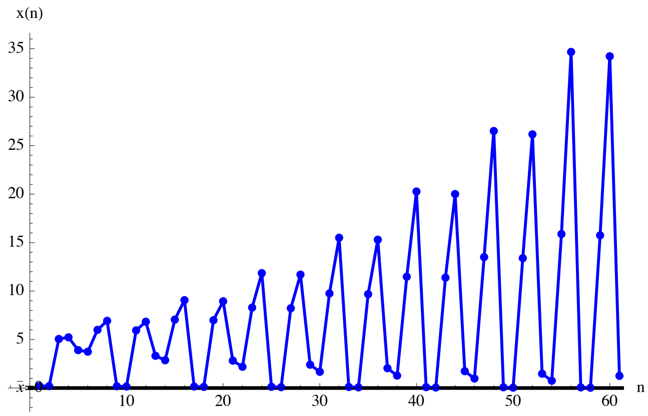

Figure 4.1

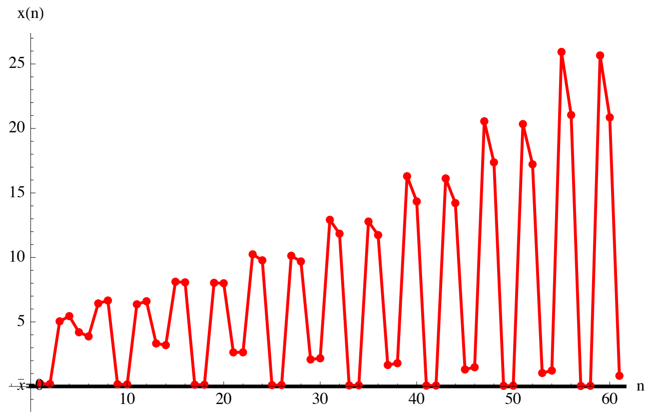

Figure 4.2

5. Solution of $\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(-1-\Psi_{m-3}\Psi_{m-5})}$

In this section, we find the solutions of

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(-1-\Psi_{m-3}\Psi_{m-5}\right)}, \quad m \in \mathbb{N}_0. \tag{5.1}$$

Theorem 5.1. *Assume that, $\{\Psi_m\}_{m=-5}^{\infty}$ represent a solution of (5.1).*

$$\Psi_{8m+1} = \frac{DF^{2m+1}(-1+BD)^m}{B^{2m+1}(-1+DF)^{m+1}}, \quad \Psi_{8m+2} = \frac{CE^{2m+1}(-1+AC)^m}{A^{2m+1}(-1+CE)^{m+1}}, \quad \Psi_{8m+3} = \frac{B^{2m+2}(-1+DF)^{m+1}}{F^{2m+1}(-1+BD)^{m+1}},$$

$$\Psi_{8m+4} = \frac{A^{2m+2}(-1+CE)^{m+1}}{E^{2m+1}(-1+AC)^{m+1}}, \quad \Psi_{8m+5} = \frac{DF^{2m+2}(-1+BD)^{m+1}}{B^{2m+2}(-1+DF)^{m+1}}, \quad \Psi_{8m+6} = \frac{CE^{2m+2}(-1+AC)^{m+1}}{A^{2m+2}(-1+CE)^{m+1}},$$

$$\Psi_{8m+7} = \frac{B^{2m+3}(-1+DF)^{m+1}}{F^{2m+2}(-1+BD)^{m+1}}, \quad \Psi_{8m+8} = \frac{A^{2m+3}(-1+CE)^{m+1}}{E^{2m+2}(-1+AC)^{m+1}}.$$

Proof. The proof is similar to the proof of Theorem 4.1 and therefore it will be omitted. □

Theorem 5.2. (5.1) *contains three equilibriums, 0, $\pm\sqrt{-2}$ and these aren't locally asymptotically stable.*

Proof. The proof is similar to the proof of Theorem 2.2 and therefore it will be omitted. □

Example 5.3. *See Figure 5.1 for the initials*

$$\Psi_{-5} = 2.85, \quad \Psi_{-4} = 2.8, \quad \Psi_{-3} = 2.75, \quad \Psi_{-2} = 2.7, \quad \Psi_{-1} = 2.6, \quad \Psi_0 = 2.55.$$

Example 5.4. *We consider*

$$\Psi_{-5} = 2, \quad \Psi_{-4} = 2.8, \quad \Psi_{-3} = 2.4, \quad \Psi_{-2} = 2.7, \quad \Psi_{-1} = 2.3, \quad \Psi_0 = 2.5.$$

See Figure 5.2.

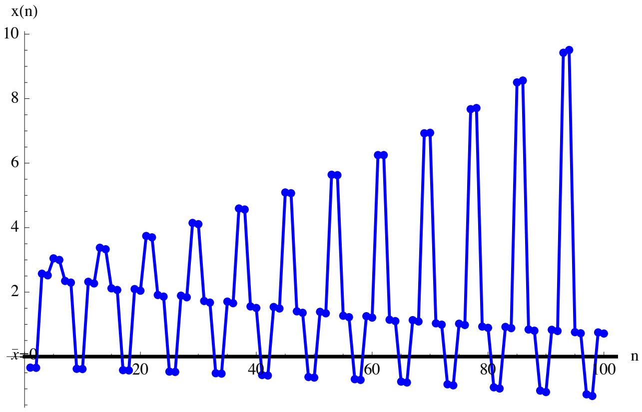

Figure 5.1

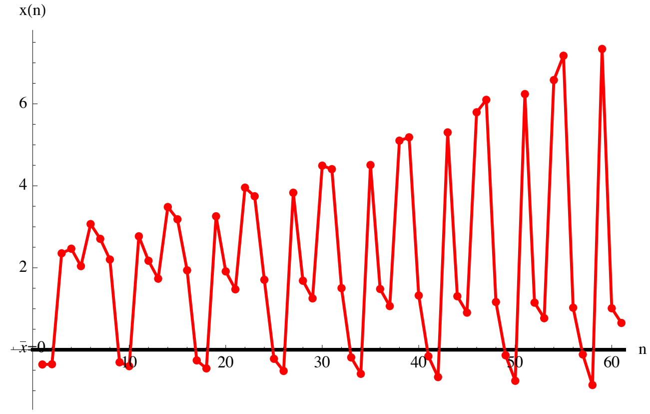

Figure 5.2

6. Conclusion

We explore the behavior of the following difference equation

$$\Psi_{m+1} = \frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}\left(\pm 1 \pm \Psi_{m-3}\Psi_{m-5}\right)}, \quad m \in \mathbb{N}_0$$

with positive real integers as initials. Local stability is discussed. Furthermore, we obtain the solution to several exceptional circumstances. Finally, a few numerical examples are shown.

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On Quasi Hemi-Slant Submersions

Pramod Kumar Rawat[1] and Sushil Kumar[2]*

Abstract
The paper deals with the notion of quasi hemi-slant submersions from Lorentzian para Sasakian manifolds onto Riemannian manifolds. These submersions are generalization of hemi-slant submersions and semi-slant submersions. In this paper, we also study the geometry of leaves of distributions which are involved in the definition of the submersion. Further, we obtain the conditions for such distributions to be integrable and totally geodesic. Moreover, we also give the characterization theorems for proper quasi hemi-slant submersions and provide some examples of it.

Keywords: Hemi-slant submersions, Lorentzian para Sasakian manifolds, Quasi hemi-slant submersions, Slant submersions.
2010 AMS:53C12, 53C15, 53C25, 53C50, 55D15.

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1. Introduction

In differential geometry the theory of Riemannian submersions was firstly defined and studied by O'Neill [1] and Gray [2], in 1966 and 1967, respectively. In 1976, Watson [3] studied almost complex type of Riemannian submersions and introduced almost Hermitian submersions between almost Hermitian manifolds. Latar on, Chinea [4] extended the idea of almost Hermitian submersion to different sub-classes of almost contact manifolds. There are so many important and interesting results about Riemannian and almost Hermitian submersion which are studied in ([5]- [7]). As a natural generalization of holomorphic submersions and totally real submersions, B. Sahin introduced the notion of slant submersions [8], semi-invariant submersions [9] and hemi-slant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds in 2011, 2013 and 2015 respectively. There are many research articles on Riemannian submersions between Riemannian manifolds equipped with different structures have been published by several geometers ([10]- [27]).

Magid and Falcitelli et. al. stablished the theory of Lorentzian submersions in [28] and [29], respectively. In 1989, Matsumoto [30] introduced the notion of Lorentzian para Sasakian manifolds. Later, Mihai and Rosca studied the same notion independently in [31]. Recently, Gunduzalp and Sahin studied paracontact and Lorentzian almost paracontact structures in [32] and [33]. Kumar et. al. in [34] defined and studied conformal semi-slant submersions from Lorentzian para Sasakian manifolds onto Riemannian manifolds. As a natural generalization of hemi-slant submersions, semi-slant submersions and bi-slant submersions, Prasad, Shukla and Kumar in [35] introduced the notion of quasi bi-slant submersions from Kaehler manifold onto a Riemannian manifold.

Beside the introduction this paper contains three sections. In the second section, we present some basic informations related to quasi hemi-slant Riemannian submersion needed throughout this paper. In the third section, we obtain some results on quasi hemi-slant Riemannian submersions from Lorentzian para Sasakian manifold onto Riemannian manifold. We also study the geometry of leaves of distribution involved in above submersion. Finally, we obtain certain conditions for such submersions to be totally geodesic. In the last section, we provide some examples for such submersions.

2. Preliminaries

In this section, we recall main definitions and properties of Lorentzian para Sasakian manifolds.

An $(2n+1)$-dimensional differentiable manifold M_1 which admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η is called Lorentzian para Sasakian manifold with Lorentzian metric g_{M_1} ([31], [36]) which satisfy:

$$\phi^2 = I+\eta\otimes\xi, \quad \phi\circ\xi=0, \quad \eta\circ\phi=0, \tag{2.1}$$

$$\eta(\xi) = -1, \quad g_{M_1}(Z_1,\xi)=\eta(Z_1), \tag{2.2}$$

$$g_{M_1}(\phi Z_1,\phi Z_2) = g_{M_1}(Z_1,Z_2)+\eta(Z_1)\eta(Z_2), \quad g_{M_1}(\phi Z_1,Z_2)=g_{M_1}(Z_1,\phi Z_2), \tag{2.3}$$

$$\nabla_{Z_1}\xi = \phi Z_1, \tag{2.4}$$

$$(\nabla_{Z_1}\phi)Z_2 = g_{M_1}(Z_1,Z_2)\xi+\eta(Z_2)X+2\eta(Z_1)\eta(Z_2)\xi, \tag{2.5}$$

where ∇ represents the operator of covariant differentiation with respect to the Lorentzian metric g_{M_1} and Z_1, Z_2 vector fields on M_1.

In a Lorentzian para Sasakian manifold, it is clear that

$$rank(\phi)=2n. \tag{2.6}$$

Now, if we put

$$\Phi(Z_1,Z_2)=\Phi(Z_2,Z_1)=g_{M_1}(Z_1,\phi Z_2)=g_{M_1}(\phi Z_1,Z_2) \tag{2.7}$$

then the tensor field Φ is symmetric $(0,2)$ tensor field, for any vector fields Z_1 and Z_2 on M_1.

Example 2.1 ([36]). *Let $R^{2k+1}=\{(x^1,x^2,\dots,x^k,y^1,y^2,\dots,y^k,z) : x^i,y^i,z\in R, \quad i=1,2,\dots,k\}$. Consider R^{2k+1} with the following structure:*

$$\phi\left(\sum_{i=1}^{k}\left(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i}\right)+Z\frac{\partial}{\partial z}\right)=-\sum_{i=1}^{k}Y_i\frac{\partial}{\partial x_i}-\sum_{i=1}^{k}X_i\frac{\partial}{\partial y_i}+\sum_{i=1}^{k}Y_iy^i\frac{\partial}{\partial z},$$

$$g_{R^{2k+1}}=-(\eta\otimes\eta)+\frac{1}{4}\sum_{i=1}^{k}\left(dx^i\otimes dx^i+dy^i\otimes dy^i\right), \quad \eta=-\frac{1}{2}\left(dz-\sum_{i=1}^{k}y^idx^i\right), \quad \xi=2\frac{\partial}{\partial z}.$$

Then, $(R^{2k+1},\phi,\xi,\eta,g_{R^{2k+1}})$ is a Lorentzian para-Sasakian manifold. The vector fields $E_i=2\frac{\partial}{\partial y^i}, E_{k+i}=2\left(\frac{\partial}{\partial x^i}+y^i\frac{\partial}{\partial z}\right)$ and ξ form a ϕ-basis for the contact metric structure.

Let $\Pi:(M_1,g_{M_1})\to(M_2,g_{M_2})$ be Riemannian submersions between Riemannian manifolds [7]. Define O'Neill's tensors $\mathscr{T}$ and $\mathscr{A}$ [1] by

$$\mathscr{A}_E L = \mathscr{H}\nabla_{\mathscr{H}E}\mathscr{V}L+\mathscr{V}\nabla_{\mathscr{H}E}\mathscr{H}L, \tag{2.8}$$

$$\mathscr{T}_E L = \mathscr{H}\nabla_{\mathscr{V}E}\mathscr{V}L+\mathscr{V}\nabla_{\mathscr{V}E}\mathscr{H}L, \tag{2.9}$$

for any vector fields E, L on M_1, where ∇ is the Levi-Civita connection of g_{M_1}. It is easy to see that $\mathscr{T}_E$ and $\mathscr{A}_E$ are skew-symmetric operators on the tangent bundle of M_1 reversing the vertical and the horizontal distributions.

From equations (2.8) and (2.9), we have

$$\nabla_{Y_1}Y_2 = \mathscr{T}_{Y_1}Y_2+\mathscr{V}\nabla_{Y_1}Y_2, \tag{2.10}$$

$$\nabla_{Y_1}Z_1 = \mathscr{T}_{Y_1}Z_1+\mathscr{H}\nabla_{Y_1}Z_1, \tag{2.11}$$

$$\nabla_{Z_1}Y_1 = \mathscr{A}_{Z_1}Y_1+\mathscr{V}\nabla_{Z_1}Y_1, \tag{2.12}$$

$$\nabla_{Z_1}Z_2 = \mathscr{H}\nabla_{Z_1}Z_2+\mathscr{A}_{Z_1}Z_2 \tag{2.13}$$

for $Y_1,Y_2\in\Gamma(\ker\Pi_*)$ and $Z_1,Z_2\in\Gamma(\ker\Pi_*)^\perp$, where $\mathscr{H}\nabla_{Y_1}Z_1=\mathscr{A}_{Z_1}Y_1$, if Z_1 is basic. It is not difficult to observe that $\mathscr{T}$ acts on the fibers as the second fundamental form, while $\mathscr{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

Since $\mathscr{T}_{Z_1}$ is skew-symmetric, we observe that Π has totally geodesic fibres if and only if $\mathscr{T} \equiv 0$.

Let $(M_1, \phi, \xi, \eta, g_{M_1})$ be a Lorentzian para Sasakian manifold and (M_2, g_{M_2}) be a Riemannian manifold and $\Pi : M_1 \to M_2$ is smooth map. Then the second fundamental form of Π is given by

$$(\nabla \Pi_*)(U_1, U_2) = \nabla^{\Pi}_{U_1} \Pi_* U_2 - \Pi_*(\nabla_{U_1} U_2) \quad \text{for } U_1, U_2 \in \Gamma(T_p M_1), \tag{2.14}$$

where we denote conveniently by ∇ the Levi-Civita connections of the matrices g_{M_1} and g_{M_2} and ∇^{Π} is the pullback connection.

We recall that a differentiable map Π between two Riemannian manifolds is totally geodesic if

$$(\nabla \Pi_*)(U_1, U_2) = 0 \text{ for all } U_1, U_2 \in \Gamma(TM_1). \tag{2.15}$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Now, we can easily prove the following lemma as in [12].

Lemma 2.2. *Let Π be a Riemannian submersion from a Lorentzian para Sasakian manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto Riemannian manifold (M_2, g_{M_2}), then we have*

(i) $(\nabla \Pi_*)(W_1, W_2) = 0$,

(ii) $(\nabla \Pi_*)(Z_1, Z_2) = -\Pi_*(\mathscr{T}_{Z_1} Z_2) = -\Pi_*(\nabla_{Z_1} Z_2)$,

(iii) $(\nabla \Pi_*)(W_1, Z_1) = -\Pi_*(\nabla_{W_1} Z_1) = -\Pi_*(\mathscr{A}_{W_1} Z_1)$,

where W_1, W_2 are horizontal vector fields and Z_1, Z_2 are vertical vector fields.

3. Quasi Hemi-Slant Submersions

In this section, quasi hemi-slant submersions Π from a Lorentzian para Sasakian manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto a Riemannian manifold (M_2, g_{M_2}) is defined and studied.

Definition 3.1 ([37]). *Let $(M_1, \phi, \xi, \eta, g_{M_1})$ be a Lorentzian para Sasakian manifold and (M_2, g_{M_2}) a Riemannian manifold. A Riemannian submersion $\Pi : (M_1, \phi, \xi, \eta, g_{M_1}) \to (M_2, g_{M_2})$ is called a quasi hemi-slant submersion if there exist four mutually orthogonal distribution $D, D^\theta, D^\perp$ and $< \xi >$ such that*

(i) $\ker \Pi_* = D \oplus_{orth} D^\theta \oplus_{orth} D^\perp \oplus_{orth} < \xi >$,

(ii) $\phi(D) = D$ *i.e., D is invariant,*

(iii) for any non-zero vector field $Z_1 \in (D^\theta)_p$, $p \in M_1$, the angle θ between ϕZ_1 and $(D^\theta)_p$ is constant and independent of the choice of point p and Z_1 in $(D^\theta)_p$.

The angle θ is called slant angle of the submersion, where D, D^θ and $D^\perp$ are space like subspaces.

Let Π be quasi hemi-slant submersion from an almost contact metric manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto a Riemannian manifold (M_2, g_{M_2}). Then, we have

$$TM_1 = \ker \Pi_* \oplus (\ker \Pi_*)^\perp. \tag{3.1}$$

Now, for any vector field $V_1 \in \Gamma(\ker \Pi_*)$, we put

$$V_1 = PV_1 + QV_1 + RV_1 - \eta(V_1)\xi, \tag{3.2}$$

where P, Q and R are projection morphisms of $\ker \Pi_*$ onto D, D^θ and $D^\perp$, respectively.

For $Y_1 \in \Gamma(\ker \Pi_*)$, we set

$$\phi Y_1 = \psi Y_1 + \omega Y_1, \tag{3.3}$$

where $\psi Y_1 \in \Gamma(\ker \Pi_*)$ and $\omega Y_1 \in \Gamma(\omega D^\theta \oplus \omega D^\perp)$.

Using equations (3.2) and (3.3), we have

$$\begin{aligned} \phi V_1 &= \phi(PV_1) + \phi(QV_1) + \phi(RV_1), \\ &= \psi(PV_1) + \omega(PV_1) + \psi(QV_1) + \omega(QV_1) + \psi(RV_1) + \omega(RV_1). \end{aligned}$$

Since $\phi(D) = D$ and $\phi(D^{\perp}) \subset (\ker \Pi_*)^{\perp}$, we get $\omega(PV_1) = 0$ and $\psi(RV_1) = 0$.

Hence above equation reduces to

$$\phi V_1 = \psi(PV_1) + \psi QV_1 + \omega QV_1 + \omega RV_1. \tag{3.4}$$

Thus we have the following decomposition

$$\phi(\ker \Pi_*) = D \oplus \psi D^{\theta} \oplus (\omega D^{\theta} \oplus \omega D^{\perp}), \tag{3.5}$$

where $\oplus$ denotes orthogonal direct sum. Since $\omega D^{\theta} \subseteq (\ker \Pi_*)^{\perp}$, $\omega D^{\perp} \subseteq (\ker \Pi_*)^{\perp}$. So, we can write

$$(\ker \Pi_*)^{\perp} = \omega D^{\theta} \oplus \omega D^{\perp} \oplus \mu,$$

where μ is orthogonal complement of $(\omega D^{\theta} \oplus \omega D^{\perp})$ in $(\ker \Pi_*)^{\perp}$.

Also for any non-zero vector field $W_1 \in \Gamma(\ker \Pi_*)^{\perp}$, we have

$$\phi W_1 = BW_1 + CW_1, \tag{3.6}$$

where $BW_1 \in \Gamma(\ker \Pi_*)$ and $CW_1 \in \Gamma(\mu)$.
Span$\{\xi\} = \langle \xi \rangle$ defines time like vector field distribution. If Z_1 is a space-like vector field and is orthogonal to ξ, then

$$g_{M_1}(\phi Z_1, \phi Z_2) = g_{M_1}(Z_1, Z_2) > 0,$$

so ϕZ_1 is also space like. Also ψZ_1 is space-like.

For space-like vector fields the Cauchy-Schwartz inequality, $g_{M_1}(Z_1, Z_2) \leq \mid Z_1 \mid\mid Z_2 \mid$ is verified.

Therefore the Wirtinger angle θ is given by

$$cos\theta = \frac{g_{M_1}(\phi Z_1, \psi Z_2)}{\mid \phi Z_1 \mid\mid \psi Z_2 \mid}.$$

$g_{M_1}\big|_{\ker \mathrm{F}_*}$ is non degenerate metric of index 1 at any point of M_1. So $(\ker \Pi_*)_q$ is time like subspace of T_qM_1 at any point of M_1,

so $(\ker \Pi_*)_q^{\perp}$ is space like subspace of T_qM_1 at any point $q \in M_1$.

We will denote a quasi hemi-slant submersion from a Lorentzian para Sasakian manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto a Riemannian manifold (M_2, g_{M_2}) by Π.

Lemma 3.2. *If Π be a quasi hemi-slant submersion then we have*

$$\psi^2 U_1 + B\omega U_1 = U_1 + \eta(U_1)\xi, \quad \omega\psi U_1 + C\omega U_1 = 0, \quad \omega BX_1 + C^2 X_1 = X_1, \quad \psi BX_1 + BCX_1 = 0,$$

for all $U_1 \in \Gamma(\ker \Pi_*)$ and $X_1 \in \Gamma(\ker \Pi_*)^{\perp}$.

Proof. Using equations (2.1), (3.3) and (3.5), we have Lemma 3.2. □

Lemma 3.3. *If Π be a quasi hemi-slant submersion then we have*

(i) $\psi^2 U_1 = (\cos^2\theta) U_1$,

(ii) $g_{M_1}(\psi U_1, \psi U_2) = \cos^2\theta\, g_{M_1}(U_1, U_2)$,

(iii) $g_{M_1}(\omega U_1, \omega U_2) = \sin^2\theta\, g_{M_1}(U_1, U_2)$,

for all $U_1, U_2 \in \Gamma(D^{\theta})$.

Proof. (i) Let Π be a quasi hemi-slant submersion from a Lorentzian para Sasakian manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto a Riemannian manifold (M_2, g_{M_2}) with the quasi hemi-slant angle θ.

Then for a non-vanishing vector field $U_1 \in \Gamma(D^{\theta})$, we have

$$\cos\theta = \frac{\mid \psi U_1 \mid}{\mid \phi U_1 \mid}, \tag{3.7}$$

and

$$\cos\theta = \frac{g_{M_1}(U_1, \psi U_1)}{\mid U_1 \mid\mid \psi U_1 \mid}. \tag{3.8}$$

By using equations (2.1), (3.3) and (3.8) we have

$$\cos\theta = \frac{g_{M_1}(\psi U_1, \psi U_1)}{\mid \phi U_1 \mid\mid \psi U_1 \mid},$$

$$\cos\theta = \frac{g_{M_1}(U_1, \psi^2 U_1)}{|\phi U_1||\psi U_1|}. \tag{3.9}$$

From equations (3.8) and (3.9), we get $\psi^2 U_1 = (\cos^2\theta)U_1$, for $U_1 \in \Gamma(D^\theta)$.

(ii) For all $U_1, U_2 \in \Gamma(D^\theta)$, using equations (2.3), (3.3) and Lemma 3.3 (i), we have

$$\begin{aligned} g_{M_1}(\psi U_1, \psi U_2) &= g_{M_1}(\phi U_1 - \omega U_1, \psi U_2) \\ &= g_{M_1}(U_1, \psi^2 U_2) \\ &= \cos^2\theta g_{M_1}(U_1, U_2). \end{aligned}$$

(iii) Using equation (2.3), (3.3) and Lemma 3.3 (i), (ii) we have Lemma 3.3 (iii).

□

Lemma 3.4. *If Π be a quasi hemi-slant submersion then we have*

$$\mathscr{V}\nabla_{Y_1}\psi Y_2 + \mathscr{T}_{Y_1}\omega Y_2 - g_{M_1}(Y_1,Y_2)\xi - 2\eta(Y_1)\eta(Y_2)\xi - \eta(Y_2)Y_1 = \psi\mathscr{V}\nabla_{Y_1}Y_2 + B\mathscr{T}_{Y_1}Y_2, \tag{3.10}$$
$$\mathscr{T}_{Y_1}\psi Y_2 + \mathscr{H}\nabla_{Y_1}\omega Y_2 = \omega\mathscr{V}\nabla_{Y_1}Y_2 + C\mathscr{T}_{Y_1}Y_2, \tag{3.11}$$
$$\mathscr{V}\nabla_{U_1}BU_2 + \mathscr{A}_{U_1}CU_2 - g_{M_1}(CU_1,U_2)\xi = \psi\mathscr{A}_{U_1}U_2 + B\mathscr{H}\nabla_{U_1}U_2, \tag{3.12}$$
$$\mathscr{A}_{U_1}BU_2 + \mathscr{H}\nabla_{U_1}CU_2 = \omega\mathscr{A}_{U_1}U_2 + C\mathscr{H}\nabla_{U_1}U_2, \tag{3.13}$$
$$\mathscr{V}\nabla_{Y_1}BU_1 + \mathscr{T}_{Y_1}CU_1 = \psi\mathscr{T}_{Y_1}U_1 + B\mathscr{H}\nabla_{Y_1}U_1, \tag{3.14}$$
$$\mathscr{T}_{Y_1}BU_1 + \mathscr{H}\nabla_{Y_1}CU_1 = \omega\mathscr{T}_{Y_1}U_1 + C\mathscr{H}\nabla_{Y_1}U_1, \tag{3.15}$$
$$\mathscr{V}\nabla_{U_1}\psi Y_1 + \mathscr{A}_{U_1}\omega Y_1 = B\mathscr{A}_{U_1}Y_1 + \psi\mathscr{V}\nabla_{U_1}Y_1, \tag{3.16}$$
$$\mathscr{A}_{U_1}\psi Y_1 + \mathscr{H}\nabla_{U_1}\omega Y_1 - \eta(Y_1)U_1 = C\mathscr{A}_{U_1}Y_1 + \omega\mathscr{V}\nabla_{U_1}Y_1, \tag{3.17}$$

for any $Y_1, Y_2 \in \Gamma(\ker\Pi_*)$ and $U_1, U_2 \in \Gamma(\ker\Pi_*)^\perp$.

Proof. Using equations (2.5), (2.10)-(2.13), (3.3) and (3.5), we get equations (3.10)-(3.17). □

Now, we define

$$(\nabla_{Y_1}\psi)Y_2 = \mathscr{V}\nabla_{Y_1}\psi Y_2 - \psi\mathscr{V}\nabla_{Y_1}Y_2, \tag{3.18}$$
$$(\nabla_{Y_1}\omega)Y_2 = \mathscr{H}\nabla_{Y_1}\omega Y_2 - \omega\mathscr{V}\nabla_{Y_1}Y_2, \tag{3.19}$$
$$(\nabla_{X_1}C)X_2 = \mathscr{H}\nabla_{X_1}CX_2 - C\mathscr{H}\nabla_{X_1}X_2, \tag{3.20}$$
$$(\nabla_{X_1}B)X_2 = \mathscr{V}\nabla_{X_1}BX_2 - B\mathscr{H}\nabla_{X_1}X_2 \tag{3.21}$$

for any $Y_1, Y_2 \in \Gamma(\ker\Pi_*)$ and $X_1, X_2 \in \Gamma(\ker\Pi_*)^\perp$.

Lemma 3.5. *If Π be a quasi hemi-slant submersion then we have*

$$(\nabla_{Y_1}\phi)Y_2 = B\mathscr{T}_{Y_1}Y_2 - \mathscr{T}_{Y_1}\omega Y_2 + g_{M_1}(Y_1,Y_2)\xi + 2\eta(Y_1)\eta(Y_2)\xi + \eta(Y_2)Y_1,$$
$$(\nabla_{Y_1}\omega)Y_2 = C\mathscr{T}_{Y_1}Y_2 - \mathscr{T}_{Y_1}\psi Y_2,$$
$$(\nabla_{U_1}C)U_2 = \omega\mathscr{A}_{U_1}U_2 - \mathscr{A}_{U_1}BU_2,$$
$$(\nabla_{U_1}B)U_2 = \psi\mathscr{A}_{U_1}U_2 - \mathscr{A}_{U_1}CU_2 + g_{M_1}(U_1,U_2)\xi,$$

for any vectors $Y_1, Y_2 \in \Gamma(\ker\Pi_*)$ and $U_1, U_2 \in \Gamma(\ker\Pi_*)^\perp$.

Proof. Using equations (3.10), (3.11), (3.12), (3.13) and (3.18)-(3.21), we get all equations of Lemma 3.5. □

If the tensors ϕ and ω are parallel with respect to the linear connection ∇ on M_1 respectively, then

$$B\mathscr{T}_{Y_1}Y_2 = \mathscr{T}_{Y_1}\omega Y_2 - g_{M_1}(Y_1,Y_2)\xi - 2\eta(Y_1)\eta(Y_2)\xi - \eta(Y_2)Y_1, C\mathscr{T}_{Y_1}Y_2 = \mathscr{T}_{Y_1}\psi Y_2$$

for any $Y_1, Y_2 \in \Gamma(TM_1)$.

Theorem 3.6. *Let Π be a quasi hemi-slant submersion. Then, the invariant distribution D is integrable if and only if*

$$g_{M_1}(\mathscr{T}_{X_1}\phi X_2 - \mathscr{T}_{X_2}\phi X_1, \omega QY_1 + \omega RY_1) = g_{M_1}(\mathscr{V}\nabla_{X_2}\phi X_1 - \mathscr{V}\nabla_{X_1}\phi X_2, \psi QY_1),$$

for $X_1, X_2 \in \Gamma(D)$ and $Y_1 \in \Gamma(D^\theta \oplus D^\perp)$.

Proof. For $X_1, X_2 \in \Gamma(D)$, and $Y_1 \in \Gamma(D^\theta \oplus D^\perp)$, using equations (2.3), (2.5), (2.10), (3.2) and (3.3), we have

$$\begin{aligned} g_{M_1}([X_1,X_2],Y_1) &= g_{M_1}(\nabla_{X_1}\phi X_2, \phi Y_1) - g_{M_1}(\nabla_{X_2}\phi X_1, \phi Y_1), \\ &= g_{M_1}(\mathscr{T}_{X_1}\phi X_2 - \mathscr{T}_{X_2}\phi X_1, \omega QY_1 + \omega RY_1) + g_{M_1}(\mathscr{V}\nabla_{X_1}\phi X_2 - \mathscr{V}\nabla_{X_2}\phi X_1, \psi QY_1), \end{aligned}$$

which completes the proof. □

Theorem 3.7. *Let Π be a quasi hemi-slant submersion. Then, the slant distribution D^θ is integrable if and only if*

$$g_{M_1}(\mathscr{H}\nabla_{Z_2}\omega Z_1 - \mathscr{H}\nabla_{Z_1}\omega Z_2, \phi RX_1) = g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2 - \mathscr{T}_{Z_2}\omega Z_1, \phi PX_1) + g_{M_1}(\mathscr{T}_{Z_1}\omega\psi Z_2 - \mathscr{T}_{Z_2}\omega\psi Z_1, X_1)$$

for all $Z_1, Z_2 \in \Gamma(D^\theta)$ and $X_1 \in \Gamma(D \oplus D^\perp)$.

Proof. For all $Z_1, Z_2 \in \Gamma(D^\theta)$ and $X_1 \in \Gamma(D \oplus D^\perp)$, we have

$$g_{M_1}([Z_1,Z_2],X_1) = g_{M_1}(\nabla_{Z_1}Z_2, X_1) - g_{M_1}(\nabla_{Z_2}Z_1, X_1).$$

Using equations (2.3), (2.5), (3.2), (3.3) and Lemma 3.3, we have

$$\begin{aligned} g_{M_1}([Z_1,Z_2],X_1) =& g_{M_1}(\phi\nabla_{Z_1}Z_2, \phi X_1) - g_{M_1}(\phi\nabla_{Z_2}Z_1, \phi X_1) \\ =& g_{M_1}(\nabla_{Z_1}\phi Z_2, \phi X_1) - g_{M_1}(\nabla_{Z_2}\phi Z_1, \phi X_1) \\ =& g_{M_1}(\nabla_{Z_1}\psi Z_2, \phi X_1) + g_{M_1}(\nabla_{Z_1}\omega Z_2, \phi X_1) - g_{M_1}(\nabla_{Z_2}\psi Z_1, \phi X_1) - g_{M_1}(\nabla_{Z_1}\omega Z_2, \phi X_1) \\ =& \cos^2\theta g_{M_1}(\nabla_{Z_1}Z_2, X_1) - \cos^2\theta g_{M_1}(\nabla_{Z_2}Z_1, X_1) + g_{M_1}(\mathscr{T}_{Z_1}\omega\psi Z_2 - \mathscr{T}_{Z_2}\omega\psi Z_1, X_1) \\ &+ g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2 + \mathscr{T}_{Z_1}\omega Z_2, \phi PX_1 + \phi RX_1) - g_{M_1}(\mathscr{H}\nabla_{Z_2}\omega Z_1 + \mathscr{T}_{Z_2}\omega Z_1, \phi PX_1 + \phi RX_1). \end{aligned}$$

Now, we have

$$\begin{aligned} \sin^2\theta g_{M_1}([Z_1,Z_2],X_1) =& g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2 - \mathscr{T}_{Z_2}\omega Z_1, \phi PX_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2 - \mathscr{H}\nabla_{Z_2}\omega Z_1, \phi RX_1) \\ &+ g_{M_1}(\mathscr{T}_{Z_1}\omega\psi Z_2 - \mathscr{T}_{Z_2}\omega\psi Z_1, X_1), \end{aligned}$$

which completes the proof. □

Theorem 3.8. *Let Π be a quasi hemi-slant submersion. Then the anti-invariant distribution $D^\perp$ is always integrable.*

Proof. The proof of the above theorem is exactly the same as that one for hemi-slant submersions, see Theorems 3.13 of [38]. So we omit it. □

Proposition 3.9. *Let Π be a quasi hemi-slant submersion. Then the vertical distribution* $(\ker\Pi_*)$ *does not defines a totally geodesic foliation on M_1.*

Proof. Let $Z_1 \in \Gamma(\ker\Pi_*)$ and $Z_2 \in \Gamma(\ker\Pi_*)^\perp$, using equation (2.4), we have

$$g_{M_1}(\nabla_{Z_1}\xi, Z_2) = g_{M_1}(\phi Z_1, Z_2),$$

since $g_{M_1}(\phi Z_1, Z_2) \neq 0$, so $g_{M_1}(\nabla_{Z_1}\xi, Z_2) \neq 0$. Hence, $(\ker\Pi_*)$ does not defines a totally geodesic foliation on M_1. □

Theorem 3.10. *Let Π be a proper quasi hemi-slant submersion. Then the distribution $(\ker\Pi_*)- <\xi>$ defines a totally geodesic foliation on M_1 if and only if*

$$g_{M_1}(\mathscr{T}_{Z_1}PZ_2+\cos^2\theta\mathscr{T}_{Z_1}QZ_2,V_1)=-g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega\psi QZ_2,V_1)-g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2,BV_1)-g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2,CV_1)$$

for all $Z_1,Z_2\in\Gamma(\ker\Pi_*)- <\xi>$ and $V_1\in\Gamma(\ker\Pi_*)^\perp$.

Proof. For all $Z_1,Z_2\in\Gamma(\ker\Pi_*)- <\xi>$ and $V_1\in\Gamma(\ker\Pi_*)^\perp$, using equations (2.3), (2.5) and (3.2), we have

$$g_{M_1}(\nabla_{Z_1}Z_2,V_1)=g_{M_1}(\nabla_{Z_1}\phi PZ_2,\phi V_1)+g_{M_1}(\nabla_{Z_1}\phi QZ_2,\phi V_1)+g_{M_1}(\nabla_{Z_1}\phi RZ_2,\phi V_1).$$

Now, using equations (2.10), (2.11), (3.3), (3.5) and Lemma 3.3, we have

$$\begin{aligned}g_{M_1}(\nabla_{Z_1}Z_2,V_1)=&g_{M_1}(\mathscr{T}_{Z_1}PZ_2,V_1)+\cos^2\theta g_{M_1}(\mathscr{T}_{Z_1}QZ_2,V_1)+g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega\psi QZ_2,V_1)\\&+g_{M_1}(\nabla_{Z_1}(\omega PZ_2+\omega QZ_2+\omega RZ_2),\phi V_1).\end{aligned}$$

Now, since $\omega PZ_2+\omega QZ_2+\omega RZ_2=\omega Z_2$ and $\omega PZ_2=0$, we have

$$\begin{aligned}g_{M_1}(\nabla_{Z_1}Z_2,V_1)=&g_{M_1}(\mathscr{T}_{Z_1}PZ_2+\cos^2\theta\mathscr{T}_{Z_1}QZ_2,V_1)+g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega\psi QZ_2,V_1)+g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2,BV_1)\\&+g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2,CV_1),\end{aligned}$$

which completes the proof. □

Theorem 3.11. *Let Π be a quasi hemi-slant submersion. Then, the horizontal distribution $(\ker\Pi_*)^\perp$ does not defines a totally geodesic foliation on M_1.*

Proof. Let $X_1,X_2\in\Gamma(\ker\Pi_*)^\perp$, using equation (2.4), we have

$$g_{M_1}(\nabla_{X_1}X_2,\xi)=-g_{M_1}(X_2,\nabla_{X_1}\xi)=-g_{M_1}(X_2,\phi X_1),$$

since $g_{M_1}(X_2,\phi X_1)\neq 0$, so $g_{M_1}(\nabla_{X_1}X_2,\xi)\neq 0$. Hence, $(\ker\Pi_*)^\perp$ does not defines a totally geodesic foliation on M_1. □

Proposition 3.12. *Let Π be a quasi hemi-slant submersion. Then the distribution D does not defines a totally geodesic foliation on M_1.*

Proof. For all $Y_1,Y_2\in\Gamma(D)$, using equation (2.4), we have

$$g_{M_1}(\nabla_{Y_1}Y_2,\xi)=-g_{M_1}(Y_2,\phi Y_1),$$

since $g_{M_1}(Y_2,\phi Y_1)\neq 0$, so $g_{M_1}(\nabla_{Y_1}Y_2,\xi)\neq 0$. Hence D does not defines a totally geodesic foliation on M_1. □

Theorem 3.13. *Let Π be a quasi hemi-slant submersion. Then the distribution $D\oplus<\xi>$ defines a totally geodesic foliation if and only if*

$$g_{M_1}(\mathscr{T}_{X_1}\phi PX_2,\omega QY_1+\phi RY_1)=-g_{M_1}(\mathscr{V}\nabla_{X_1}\phi PX_2,\psi QY_1),$$

$$g_{M_1}(\mathscr{V}\nabla_{X_1}\phi PX_2,BY_2)=-g_{M_1}(\mathscr{T}_{X_1}\phi PX_2,CY_2),$$

for all $X_1,X_2\in\Gamma(D\oplus<\xi>),Y_1=QY_1+RY_1\in\Gamma(D^\theta\oplus D^\perp)$ and $Y_2\in\Gamma(\ker\Pi_*)^\perp$.

Proof. For all $X_1,X_2\in\Gamma(D\oplus<\xi>),Y_1=QY_1+RY_1\in\Gamma(D^\theta\oplus D^\perp)$ and $Y_2\in\Gamma(\ker\Pi_*)^\perp$, using equations (2.3), (2.5), (2.10), (3.2) and (3.3), we have

$$\begin{aligned}g_{M_1}(\nabla_{X_1}X_2,Y_1)=&g_{M_1}(\nabla_{X_1}\phi X_2,\phi Y_1)\\=&g_{M_1}(\nabla_{X_1}\phi PX_2,\phi QY_1+\phi RY_1)\\=&g_{M_1}(\mathscr{T}_{X_1}\phi PX_2,\omega QY_1+\phi RY_1)+g_{M_1}(\mathscr{V}\nabla_{X_1}\phi PX_2,\psi QY_1).\end{aligned}$$

Now, again using equations (2.3), (2.5), (2.10), (3.2) and (3.5), we have

$$\begin{aligned}g_{M_1}(\nabla_{X_1}X_2,Y_2)=&g_{M_1}(\nabla_{X_1}\phi X_2,\phi Y_2)\\=&g_{M_1}(\nabla_{X_1}\phi PX_2,BY_2+CY_2)\\=&g_{M_1}(\mathscr{V}\nabla_{X_1}\phi PX_2,BY_2)+g_{M_1}(\mathscr{T}_{X_1}\phi PX_2,CY_2),\end{aligned}$$

which completes the proof. □

Proposition 3.14. *Let Π be a quasi hemi-slant submersion. Then the distribution D^θ does not defines a totally geodesic foliation on M_1.*

Proof. For all $Z_1, Z_2 \in \Gamma(D^\theta)$, using equation (2.4), we have

$$g_{M_1}(\nabla_{Z_1} Z_2, \xi) = -g_{M_1}(Z_2, \phi Z_1),$$

since $g_{M_1}(Z_2, \phi Z_1) \neq 0$, so $g_{M_1}(\nabla_{Z_1} Z_2, \xi) \neq 0$. Hence D^θ does not defines a totally geodesic foliation on M_1. □

Theorem 3.15. *Let Π be a quasi hemi-slant submersion. Then the distribution $D^\theta \oplus < \xi >$ defines a totally geodesic foliation on M_1 if and only if*

$$g_{M_1}(\mathscr{T}_{Z_1} \omega\psi Z_2, X_1) + g_{M_1}(\mathscr{T}_{Z_1} \omega Z_2, \phi P X_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1} \omega Z_2, \phi R X_1) = \eta(Z_2) g_{M_1}(Z_1, \phi P X_1),$$
$$g_{M_1}(\mathscr{H}\nabla_{Z_1} \omega\psi Z_2, X_2) + g_{M_1}(\mathscr{H}\nabla_{Z_1} \omega Z_2, C X_2) + g_{M_1}(\mathscr{T}_{Z_1} \omega Z_2, B X_2) = \eta(Z_2) g_{M_1}(Z_1, B X_2),$$

for all $Z_1, Z_2 \in \Gamma(D^\theta \oplus < \xi >), X_1 \in \Gamma(D \oplus D^\perp)$ and $X_2 \in \Gamma(\ker \Pi_*)^\perp$.

Proof. For all $Z_1, Z_2 \in \Gamma(D^\theta \oplus < \xi >)$, $X_1 \in \Gamma(D \oplus D^\perp)$ and $X_2 \in \Gamma(\ker \Pi_*)^\perp$, using equations (2.3), (2.5), (2.11), (3.2), (3.3) and Lemma 3.3, we have

$$\begin{aligned} g_{M_1}(\nabla_{Z_1} Z_2, X_1) =& g_{M_1}(\nabla_{Z_1} \phi Z_2, \phi X_1) - \eta(Z_2) g_{M_1}(Z_1, \phi X_1) \\ =& g_{M_1}(\nabla_{Z_1} \psi Z_2, \phi X_1) + g_{M_1}(\nabla_{Z_1} \omega Z_2, \phi X_1) - \eta(Z_2) g_{M_1}(Z_1, \phi P X_1) \\ =& \cos^2\theta_1 g_{M_1}(\nabla_{Z_1} Z_2, X_1) + g_{M_1}(\mathscr{T}_{Z_1} \omega\psi Z_2, X_1) + g_{M_1}(\mathscr{T}_Z \omega Z_2, \phi P X_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1} \omega Z_2, \phi R X_1) \\ & - \eta(Z_2) g_{M_1}(Z_1, \phi P X_1). \end{aligned}$$

Now, we have

$$\sin^2\theta_1 g_{M_1}(\nabla_{Z_1} Z_2, X_1) = g_{M_1}(\mathscr{T}_{Z_1} \omega\psi Z_2, X_1) + g_{M_1}(\mathscr{T}_{Z_1} \omega Z_2, \phi P X_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1} \omega Z_2, \phi R X_1) - \eta(Z_2) g_{M_1}(Z_1, \phi P X_1)$$

Next, from equations (2.3), (2.5), (2.11), (3.2), (3.3), (3.5) and Lemma 3.3, we have

$$\begin{aligned} g_{M_1}(\nabla_{Z_1} Z_2, X_2) =& g_{M_1}(\nabla_{Z_1} \phi Z_2, \phi X_2) - \eta(Z_2) g_{M_1}(Z_1, \phi X_2), \\ =& g_{M_1}(\nabla_{Z_1} \psi Z_2, \phi X_2) + g_{M_1}(\nabla_{Z_1} \omega Z_2, \phi X_2) - \eta(Z_2) g_{M_1}(Z_1, \phi X_2), \\ =& \cos^2\theta_1 g_{M_1}(\nabla_{Z_1} Z_2, X_2) + g_{M_1}(\mathscr{H}\nabla_{Z_1} \omega\psi Z_2, X_2) + g_{M_1}(\mathscr{H}\nabla_{Z_1} \omega Z_2, C X_2) + g_{M_1}(\mathscr{T}_{Z_1} \omega Z_2, B X_2) \\ & - \eta(Z_2) g_{M_1}(Z_1, B X_2). \end{aligned}$$

Now, we have

$$\sin^2\theta_1 g_{M_1}(\nabla_{Z_1} Z_2, X_2) = g_{M_1}(\mathscr{H}\nabla_{Z_1} \omega\psi Z_2, X_2) + g_{M_1}(\mathscr{H}\nabla_{Z_1} \omega Z_2, C X_2) + g_{M_1}(\mathscr{T}_{Z_1} \omega Z_2, B X_2) - \eta(Z_2) g_{M_1}(Z_1, B X_2),$$

which completes the proof. □

Theorem 3.16. *Let Π be a quasi hemi-slant submersion. Then the distribution $D^\perp$ defines a totally geodesic foliation on M_1 if and only if*

$$g_{M_1}(\mathscr{T}_{X_1} X_2, \omega\psi Q Y_1) = -g_{M_1}(\mathscr{H}\nabla_{X_1} \omega R X_2, \omega Y_1),$$
$$g_{M_1}(\mathscr{T}_{X_1} \omega R X_2, B Y_2) = g_{M_1}(\nabla_{\omega R X_2} \phi C Y_2, \omega R X_1),$$

for all $X_1, X_2 \in \Gamma(D^\perp), Y_1 \in \Gamma(D \oplus D^\theta)$, and $Y_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. For all $X_1, X_2 \in \Gamma(D^\perp), Y_1 \in \Gamma(D \oplus D^\theta)$, and $Y_2 \in \Gamma(\ker \pi_*)^\perp$. Using equation (2.4), we have

$$g_{M_1}(\nabla_{X_1} X_2, \xi) = 0.$$

Next, using equations (2.3), (2.5), (3.2), (3.3) and Lemma 3.3, we have

$$g_{M_1}(\nabla_{X_1} X_2, Y_1) = g_{M_1}(\phi \nabla_{X_1} X_2, \phi P Y_1 + \psi Q Y_1) + g_{M_1}(\nabla_{X_1} \phi X_2, \omega Q Y_1),$$
$$g_{M_1}(\nabla_{X_1} X_2, P Y_1 + Q Y_1) = g_{M_1}(\nabla_{X_1} X_2, P Y_1) + \cos^2\theta g_{M_1}(\nabla_{X_1} X_2, Q Y_1) + g_{M_1}(\nabla_{X_1} X_2, \omega\psi Q Y_1) + g_{M_1}(\nabla_{X_1} \phi X_2, \omega Q Y_1).$$

Now, using equations (2.10) and (2.11), we have

$$\sin^2\theta g_{M_1}(\nabla_{X_1}X_2, QY_1) = g_{M_1}(\mathscr{T}_{X_1}X_2, \omega\psi QY_1) + g_{M_1}(\mathscr{H}\nabla_{X_1}\omega RX_2, \omega Y_1).$$

Next, using equations (2.3), (2.5), (2.11), (2.13), (3.3) and (3.5), we have

$$\begin{aligned} g_{M_1}(\nabla_{X_1}X_2, Y_2) =& g_{M_1}(\nabla_{X_1}\omega RX_2, BY_2) + g_{M_1}(\nabla_{X_1}\omega RX_2, CY_2), \\ =& g_{M_1}(\mathscr{T}_{X_1}\omega RX_2, BY_2) - g_{M_1}(\mathscr{H}\nabla_{\omega RX_2}\phi CY_2, \omega RX_1), \end{aligned}$$

which is complete proof. □

Using Proposition 3.9 and Theorem 3.11, one can give the following theorem:

Theorem 3.17. *Let Π be a quasi hemi-slant submersion. Then the map Π is not a totally geodesic map.*

4. Examples

Example 4.1. *Consider the Euclidean space R^{11} with coordinates $(x_1,...,x_5,,y_1.....,y_5,z)$ and base field $\{E_i,\ E_{5+i},\xi\}$ where $E_i = 2\frac{\partial}{\partial y^i}$, $E_{5+i} = 2\left(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}\right)$, $i = 1,\dots,5$ and contravariant vector field $\xi = 2\frac{\partial}{\partial z}$. Define Lorentzian almost para contact structure on R^{11} as follows:*

$$\phi\left(\sum_{i=1}^{5}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial z}\right) = -\sum_{i=1}^{5}Y_i\frac{\partial}{\partial x^i} - \sum_{i=1}^{5}X_i\frac{\partial}{\partial y^i} + \sum_{i=1}^{5}Y_iy^i\frac{\partial}{\partial z},$$

$$\xi = 2\frac{\partial}{\partial z}, \quad \eta = -\frac{1}{2}\left(dz - \sum_{i=1}^{5}y^idx^i\right), \quad g_{R^{11}} = -(\eta\otimes\eta) + \frac{1}{4}\left(\sum_{i=1}^{5}dx^i\otimes dx^i + \sum_{i=1}^{5}dy^i\otimes dy^i\right).$$

Then $(R^{11},\phi,\xi,\eta,g_{R^{11}})$ is Lorentzian para Sasakian manifold. Let the Riemannian metric tensor field g_{R^4} is defined by

$$g_{R^4} = \frac{1}{4}\sum_{i=1}^{4}(dv_i\otimes dv_i).$$

on R^4, where $\{v_1,v_2,v_3,v_4\}$ is local coordinate system on R^4.

Let $\Pi : R^{11} \to R^4$ be a map defined by

$$\Pi(x_1,...,x_5,y_1....,y_5,z) = (x_2, \sin\alpha x_3 - \cos\alpha x_4, y_1, y_4).$$

which is quasi hemi-slant submersion map such that

$$\begin{aligned} X_1 &= 2\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}\right), \quad X_2 = 2\cos\alpha\left(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}\right) + 2\sin\alpha\left(\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial z}\right), \quad X_3 = 2\left(\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial z}\right), \\ X_4 &= 2\frac{\partial}{\partial y_2}, \quad X_5 = 2\frac{\partial}{\partial y_3}, \quad X_6 = 2\frac{\partial}{\partial y_5}, \quad X_7 = \xi = 2\frac{\partial}{\partial z}, \end{aligned}$$

$$(\ker\Pi_*) = (D\oplus D^\theta\oplus D^\perp\oplus <\xi>),$$

$$D = \left\langle X_3 = 2\left(\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial z}\right), X_6 = 2\frac{\partial}{\partial y_5}\right\rangle,$$

$$D^\theta = \left\langle X_2 = 2\cos\alpha\left(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}\right) + 2\sin\alpha\left(\frac{\partial}{\partial x_4} + y_1\frac{\partial}{\partial z}\right), X_5 = 2\frac{\partial}{\partial y_3}\right\rangle,$$

$$D^\perp = \left\langle X_1 = 2\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}\right), X_4 = 2\frac{\partial}{\partial y_2}\right\rangle, \quad \langle\xi\rangle = \left\langle X_7 = 2\frac{\partial}{\partial z}\right\rangle,$$

$$(\ker\Pi_*)^\perp = \left\langle V_1 = 2\left(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z}\right), V_2 = 2\sin\alpha\left(\frac{\partial}{\partial x_3} + y_2\frac{\partial}{\partial z}\right) - 2\cos\alpha\left(\frac{\partial}{\partial x_4} + y_1\frac{\partial}{\partial z}\right), V_3 = 2\frac{\partial}{\partial y_1}, V_4 = 2\frac{\partial}{\partial y_4}\right\rangle,$$

with quasi hemi-slant angle α. Also by direct computations, we obtain

$$\Pi_*V_1 = 2\frac{\partial}{\partial v_1}, \quad \Pi_*V_2 = 2\frac{\partial}{\partial v_2}, \quad \Pi_*V_3 = 2\frac{\partial}{\partial v_3}, \quad \Pi_*V_4 = 2\frac{\partial}{\partial v_4}.$$

Example 4.2. *Consider R^{11} and R^4 has same structure as in Example 4.1. Let $\Pi : R^{11} \to R^4$ be a map defined by*

$$\Pi(x_1,\dots,x_5,y_1,\dots,y_5,z) = \left(\frac{\sqrt{3}x_1+x_2}{2}, x_4, y_1, y_3\right).$$

which is quasi hemi-slant submersion map such that

$$X_1 = 2\left(\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial z}\right) - 2\sqrt{3}\left(\frac{\partial}{\partial x_2}+y_2\frac{\partial}{\partial z}\right), \quad X_2 = 2\left(\frac{\partial}{\partial x_3}+y_3\frac{\partial}{\partial z}\right), \quad X_3 = 2\left(\frac{\partial}{\partial x_5}+y_5\frac{\partial}{\partial z}\right),$$
$$X_4 = 2\frac{\partial}{\partial y_2}, \quad X_5 = 2\frac{\partial}{\partial y_4}, \quad X_6 = 2\frac{\partial}{\partial y_5}, \quad X_7 = 2\frac{\partial}{\partial z},$$
$$(\ker\Pi_*) = (D \oplus D^\theta \oplus D^\perp \oplus <\xi>),$$

$$D = \left\langle X_3 = 2\left(\frac{\partial}{\partial x_5}+y_5\frac{\partial}{\partial z}\right), X_6 = 2\frac{\partial}{\partial y_5}\right\rangle,$$
$$D^\theta = \left\langle X_1 = 2\left(\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial z}\right) - 2\sqrt{3}\left(\frac{\partial}{\partial x_2}+y_1\frac{\partial}{\partial z}\right), X_4 = 2\frac{\partial}{\partial y_2}\right\rangle,$$
$$D^\perp = \left\langle X_5 = 2\left(\frac{\partial}{\partial x_3}+y_3\frac{\partial}{\partial z}\right), X_2 = 2\frac{\partial}{\partial y_4}\right\rangle, \langle\xi\rangle = < X_7 = 2\frac{\partial}{\partial z} >,$$

$$(\ker\Pi_*)^\perp = \left\langle V_1 = 2\sqrt{3}\left(\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial z}\right) + 2\left(\frac{\partial}{\partial x_2}+y_2\frac{\partial}{\partial z}\right), V_2 = 2\left(\frac{\partial}{\partial x_4}+y_4\frac{\partial}{\partial z}\right), V_3 = 2\frac{\partial}{\partial y_1}, V_4 = 2\frac{\partial}{\partial y_3}\right\rangle,$$

with quasi hemi-slant angle $\theta = \frac{\pi}{6}$. Also by direct computations, we obtain

$$\Pi_* V_1 = 2\frac{\partial}{\partial v_1}, \quad \Pi_* V_2 = 2\frac{\partial}{\partial v_2}, \quad \Pi_* V_3 = 2\frac{\partial}{\partial v_3}, \quad \Pi_* V_4 = 2\frac{\partial}{\partial v_4}.$$

5. Conclusion

In this paper, integrability conditions and conditions for defining a totally geodesic foliation by certain distributions were found. Then, by applying the notion of quasi hemi-slant submersions from Lorentzian para Sasakian manifolds onto Riemannian manifolds.

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Multistability in a Circulant Dynamical System

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Abstract
In this paper we report on a two parameter four-dimensional dynamical system with cyclic symmetry, namely a circulant dynamical system. This system is a twelve-term polynomial system with four cubic nonlinearities. Reported are some parameter-space diagrams for this system, all of them considering the same range of parameters, but generated from different initial conditions. We show that such diagrams display the occurrence of multistability in this system. Properly generated bifurcation diagrams confirm this finding. Basins of attraction of coexisting attractors in the related phase-space are presented, as well as an example showing phase portraits for periodic and chaotic coexisting attractors.

Keywords: Basin of attraction, Circulant dynamical system, Multistability, Parameter-space
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1. Introduction

In this paper we report numerical results referring to a four-dimensional dynamical system with cyclic symmetry, the so-called circulant dynamical system [1], which is modeled by an autonomous nonlinear set of four first-order ordinary differential equations. Such a system was recently proposed by Rajagopal and co-workers [2], being given by

$$\begin{aligned}\dot{x} &= ax + by - y^3,\\ \dot{y} &= ay + bz - z^3,\\ \dot{z} &= az + bw - w^3,\\ \dot{w} &= aw + bx - x^3,\end{aligned} \tag{1.1}$$

where x, y, z, w are the dynamical variables, and a, b are the parameters responsible for the type of behavior presented by the system. We draw attention to the fact that the only nonlinearity present in system (1.1) is of the cubic type, and that reports on nonlinear models dominated by such terms are not abundant in the literature. Also, it is important to note that the parameter a must always be negative to guarantee the existence of attractors in the respective phase-space. It is easy to see that negative values of parameter a make system (1.1) dissipative, since is straightforward to show that the flow divergence is equal to $4a$.

System (1.1) was investigated numerically in Ref. [2], both the integer and the fractional order versions. Also, system (1.1) was investigated in Ref. [2] through a circuit design. Bifurcation diagrams with parameter a kept fixed, and parameter b being considered as the bifurcation parameter, were used to detect the presence of the multistability phenomenon. Our contribution to advancing knowledge of this system considers the simultaneous variation of both parameters a and b in the investigation of multistability. In Sect. 2 we report (a,b) parameter-space diagrams which consider the same ranges for the parameters, but generated from different initial conditions. Such procedure will allow, as we will see in detail in the next section, the detection of multistability areas, instead of the multistability lines obtained in the procedure that uses bifurcation diagrams for this purpose. Finally, concluding remarks are given in Sect. 3.

2. The Dynamics in Parameter-Space

Here we report on a numerical experiment related to the investigation of the long-term dynamical behavior of system (1.1). More specifically, five (a,b) parameter-space diagrams are presented, for $-3.5 \leq a \leq -3.0$ and $8.0 \leq b \leq 10.0$. Each of these diagrams was generated in a different way which we will detail in the following, but they all use the largest Lyapunov exponent (LLE), computed by using the algorithm in Wolf and collaborators [3], to characterize the dynamical behavior for each choice of a and b in the respective parameter-space diagram. For each of them the parameter interval was discretized in a grid of 800×800 points, being system (1.1) numerically integrated by using the fourth-order Runge-Kutta algorithm with a time step equal to 10^{-3}. The average that must be considered in the computation of each of the 6.4×10^5 LLEs takes into account 4×10^6 integration steps, after discarding an appropriate transient. As is well known, system (1.1) has four Lyapunov exponents for each choice of parameters a and b, and its dynamical behavior is characterized by the LLE: *(i)* equilibrium point if LLE< 0, *(ii)* periodic or quasi-periodic motion if LLE$= 0$, and *(iii)* chaotic or hyperchaotic motion if LLE> 0. The main purpose of presenting these five diagrams is to detect differences between the parameter-spaces which, if any, will be a numerical proof of the occurrence of multistability in system (1.1).

Figure 2.1 shows five versions of a same global view of the (a,b) parameter-space of system (1.1), for $-3.5 \leq a \leq -3.0$ and $8.0 \leq b \leq 10.0$. Color in each diagram is related to the magnitude of the LLE. Parameter regions with a positive LLE, painted in a color that ranges from yellow to red, relate to chaotic behavior, while parameter regions in black color stand for periodic solutions and have LLE$= 0$. The small gray region at the bottom left in each diagram, for which the LLE< 0, concerns to parameters that lead the system to equilibrium points.

The diagram in Fig. 2.1(a) was generated always from a same arbitrary initial condition, regardless of the values of the parameters a and b. Once the set of parameters is defined, system (1.1) is numerically integrated, the respective time series obtained, and the related Lyapunov exponents spectrum is computed. In order to generate the diagram in Fig. 2.1(b) we fix $(a,b) = (-3.5, 8.0)$, and initialize system (1.1) with an arbitrary initial condition. Then system (1.1) is numerically integrated, the respective time series obtained, and the related Lyapunov exponents spectrum is computed. Parameter a is increased, and system (1.1) is initialized with the variables related to the final point obtained for the prior value of a. The numerical integration is performed, and a new Lyapunov exponents spectrum is computed from the new time series obtained. Such procedure is repeated until the highest value of a, namely $a = -3.0$, is reached. Then parameter b is increased, and the entire procedure is repeated until the parameter set $(a,b) = (-3.0, 10.0)$ is considered in computing. The diagram in Fig. 2.1(c) is constructed in a manner analogous to that in Fig. 2.1(b), but starting from $(a,b) = (-3.0, 8.0)$. Parameter a is decreased until $a = -3.5$. For each increased b until $(a,b) = (-3.5, 10.0)$, this last procedure is repeated. In short, the diagram in Fig. 2.1(b) [Fig. 2.1(c)] was generated by using the method *following the attractor* along lines of constant b, increasing (decreasing) a from -3.5 (-3.0).

Diagrams in Figs. 2.1(d) and 2.1(e) also were generated by using the method *following the attractor*, but in a different way from the one used in the generation of Figs. 2.1(b) and 2.1(c), where each time parameter b is changed the system (1.1) is initialized from a same arbitrary initial condition. This time, however, this initialization happens only once for each of the diagrams. In the case of Fig. 2.1(d), the parameters are fixed at the lowest values $(a,b) = (-3.5, 8.0)$, system (1.1) is initialized from an arbitrary initial condition, and the attractor is followed until the highest values $(a,b) = (-3.0, 10.0)$ are reached. A similar procedure allows generating Fig. 2.1(e), only now going from the highest values $(a,b) = (-3.0, 10.0)$ to the lowest values $(a,b) = (-3.5, 8.0)$.

A cursory glance at the diagrams in Fig. 2.1 misleadingly concludes that they are all identical. However, a closer look reveals that significant differences exist between them. One of these differences appears, for example, in the chaotic stripe in yellow crossed by the small line segment in magenta, which is in the same *geographical position* in each of the diagrams. In two of them, namely in Figs. 2.1(c) and 2.1(e), there is only one periodic stripe in black embedded in this chaotic stripe, while in the other three diagrams there are two periodic stripes in black embedded. Thus, we have just identified a region in the parameter-space of system (1.1), near the magenta line, whose long-term dynamical behavior can be different depending on the initial condition adopted for the numerical integration of system (1.1). In other words, we can say that system (1.1) presents at least more than one coexisting attractors in the phase-space, for a kept fixed set of parameters (a,b) in this region, and this is a signature of the multistability phenomenon [4]. What makes multistable systems worth studying is the fact that this phenomenon has been observed, for a long time, in mathematical models of nonlinear dynamical systems, in the most varied fields of knowledge [5–9].

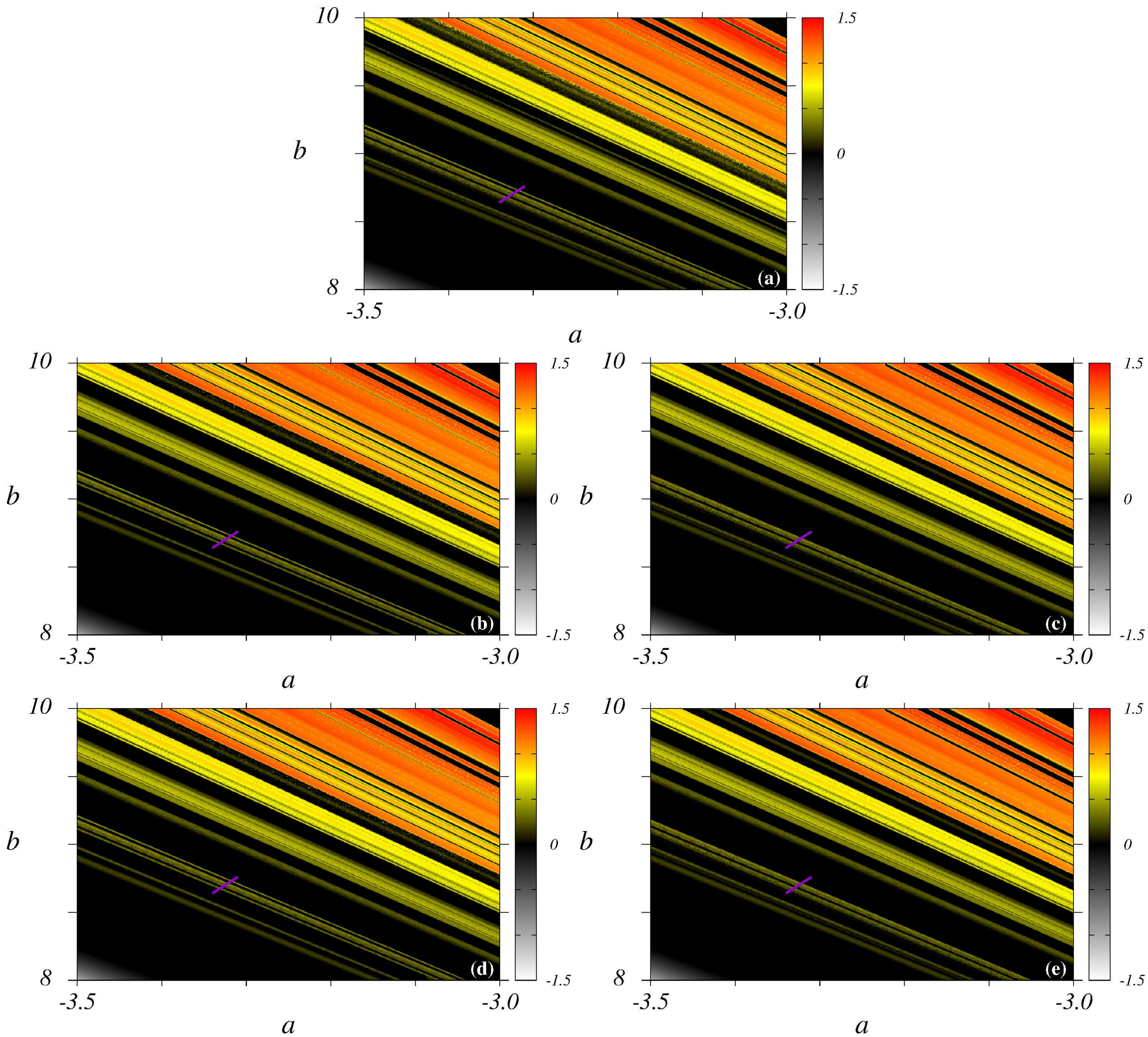

Figure 2.1. Regions of different dynamical behaviors in the (a,b) parameter-space of system (1.1). Color in each diagram is related to the magnitude of the largest Lyapunov exponent. (a) Same initial condition, regardless of the values of a and b. (b) Following the attractor along lines of constant parameter b, from $(a,b) = (-3.5, 8.0)$ to $(a,b) = (-3.0, 10.0)$. (c) Following the attractor along lines of constant parameter b, from $(a,b) = (-3.0, 8.0)$ to $(a,b) = (-3.5, 10.0)$. (d) Following the attractor from $(a,b) = (-3.5, 8.0)$ to $(a,b) = (-3.0, 10.0)$. (e) Following the attractor from $(a,b) = (-3.0, 10.0)$ to $(a,b) = (-3.5, 8.0)$.

Figure 2.2 shows two bifurcation diagrams for system (1.1), both generated by *following the attractor*, for points along the line segment $b = 4a + 22$ in magenta connecting the points $(a,b) = (-3.34, 8.64)$ and $(a,b) = (-3.31, 8.76)$ in any of the diagrams in Fig. 2.1. In each of them are shown the local maxima (the peaks) of the variable x, commonly called period and denoted by x_m, for one thousand values of the parameter a. The diagram in blue was generated considering the increase of the parameter a from -3.34 to -3.31, while that in red considers the decrease of a from -3.31 to -3.34. There are clear differences between the two bifurcation diagrams in Fig. 2.2 and, as a consequence, a clear evidence of the occurrence of multistability. For example, in the right region, inside the green box for $-3.318 < a < -3.316$, we can observe the coexistence of a chaotic attractor, in red, and a period-5 attractor, in blue.

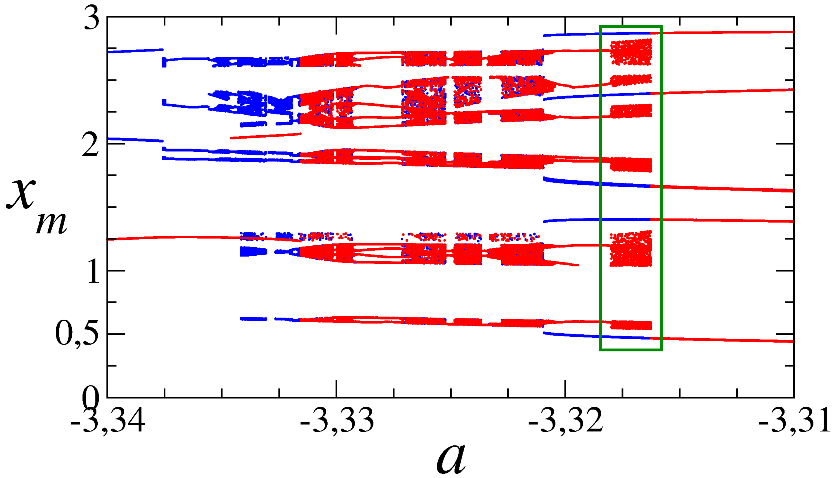

Figure 2.2. Two bifurcation diagrams for points along the line segment $b = 4a + 22$ in any of the diagrams in Fig. 2.1. Diagram in blue (red) considers the increase (decrease) of the parameter a.

The basins of attraction related to the chaotic and the period-5 attractors, in their respective colors, are shown in Fig. 2.3. In fact, Fig. 2.3 shows a (x_0, y_0) initial condition cross-section of a four-dimensional (x_0, y_0, z_0, w_0) basin of attraction for system (1.1), namely the one for which $z_0 = w_0 = 3.0$, and $(a, b) = (-3.317, 8.732)$, a point belonging to the line segment $b = 4a + 22$ drawn in diagrams of Fig. 2.1. We can see that the basins of the chaotic (in red) and the period-5 (in blue) attractors are not intermingled, that is, the points belonging to one basin are perfectly distinguishable from the points belonging to the other basin. Therefore, the basins of attraction in Fig. 2.3 clearly indicate initial conditions leading to either of the two attractors. Accordingly, since the parameters are kept fixed at $(a, b) = (-3.317, 8.732)$, and for $z_0 = w_0 = 3.0$, any initial condition point (x_0, y_0) chosen in the red region takes the system to a chaotic attractor in the phase-space, whereas any initial condition point (x_0, y_0) chosen in the blue region takes the system to a period-5 attractor.

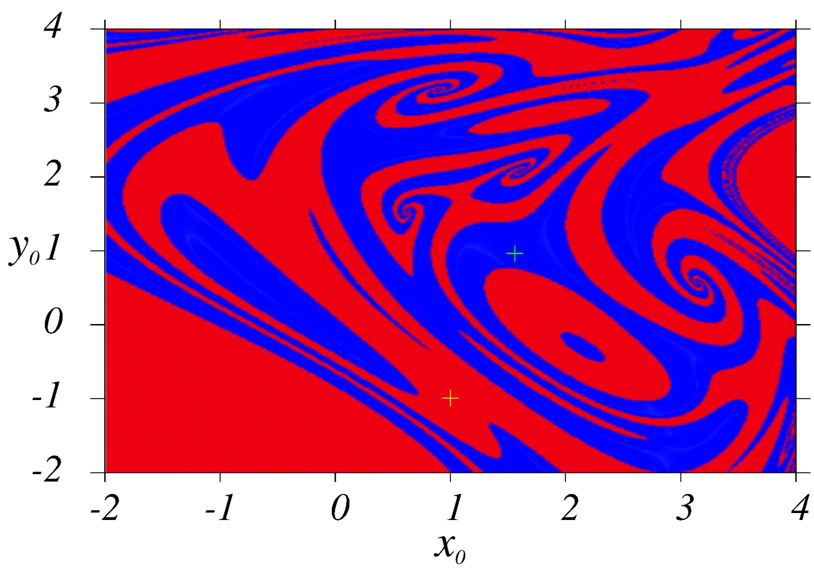

Figure 2.3. Projection of basins of attraction for system (1.1) on the (x_0, y_0) initial condition plane, for $z_0 = w_0 = 3.0$. Blue (Red) is related to the period-5 (chaotic) attractor basin.

Figure 2.4 shows two-dimensional projections of the two coexisting attractors, a period-5 and a chaotic, all of them considering the variable x in the horizontal axis, and generated for $(a, b) = (-3.317, 8.732)$. For the period-5 attractor shown in Figs. 2.4(a), 2.4(b), and 2.4(c), the initial condition is $(x_0, y_0, z_0, w_0) = (1.5, 1.0, 3.0, 3.0)$, corresponding to the point marked with a plus sign in the blue region of Fig. 2.3, while for the chaotic attractor shown in Figs. 2.4(e), 2.4(f), and 2.4(g), the initial condition is $(x_0, y_0, z_0, w_0) = (1.0, -1.0, 3.0, 3.0)$, corresponding to the point also marked with a plus sign, but this time in the red region of the same Fig. 2.3. Figures 2.4(d) and 2.4(h) show the evolution over time of the variable x, respectively for the periodic and the chaotic attractors.

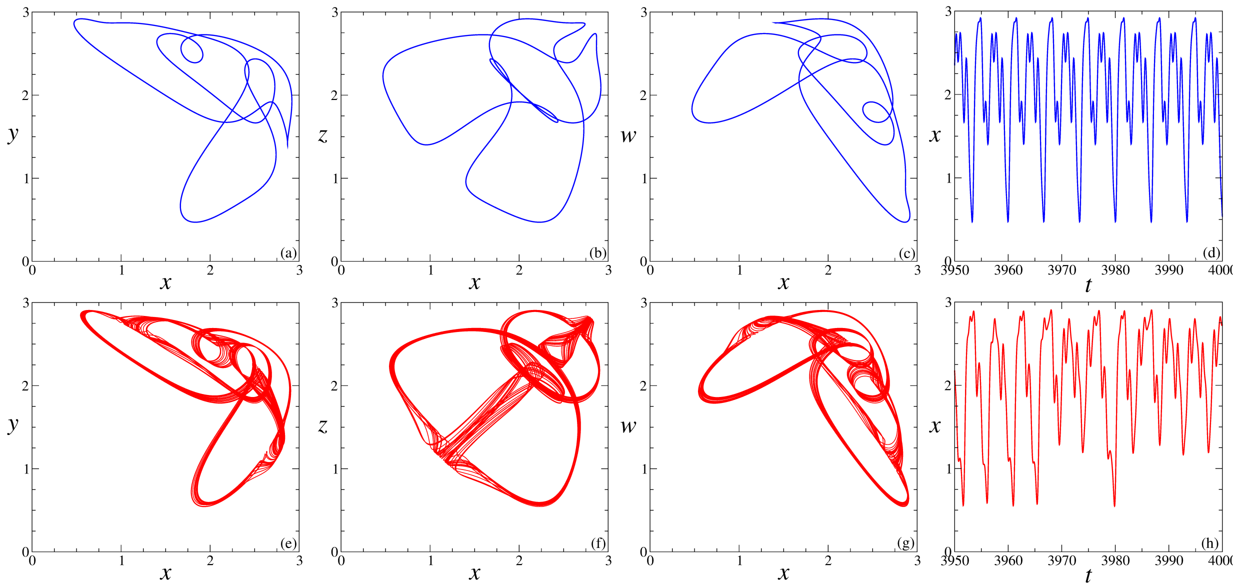

Figure 2.4. Two coexisting attractors for system (1.1). In (a) and (b) and (c) are shown projections of the period-5 attractor. In (e) (f) and (g) are shown projections of a chaotic attractor. Diagrams in (d) and (h) show the time series for the variable x, respectively for the period-5 and the chaotic attractors.

3. Summary and Outlook

We have investigated a two parameter four-dimensional dynamical system, namely a circulant dynamical system modeled by an autonomous set of four first-order ordinary differential equations which presents cubic nonlinearities in all variables, but no crossed nonlinearities. We have reported some versions of a same parameter-space plot of this system, obtained from different initial conditions. Such diagrams present sensitive differences that allow us concluding that multistability is a possible phenomenon in this system for some parameter values. Bifurcation diagrams confirm this finding. As a consequence of the multistability phenomenon, we also have reported on basins of attraction for coexisting periodic and chaotic attractors.

Therefore, we locate and investigate a region in the parameter-space of the circulant dynamical system, in which the model displays coexisting periodic and chaotic attractors, for a same set of parameters. It means the presence of an area in the parameter-space where at least two attractors coexist, depending on the choice of the initial conditions in the numerical integration of the system. As far as I know, such a result has never been reported in the literature of this field of study, for this system. Therefore, this work represents an interesting contribution to advancing knowledge of the system under study, deserving to be read. A possible future work consists of continuing to explore the parameter-space of the circulant dynamical system, in search of other regions that present multistability, including other sets of coexisting attractors, namely periodic-periodic and chaotic-chaotic. We understand, therefore, that the circulant dynamical system deserves further investigation.

With regard to the relevance of the occurrence of multistability in nonlinear dynamical systems, it is important to mention that the phenomenon has been recently reported in several other systems, among them neuron models [10, 11], electronic circuits [12, 13], memristor oscillators [14, 15], biological systems [16, 17], couplings of Duffing and van der Pol Oscillators [18], and snap and jerk systems [19, 20], just to name a few among many examples.

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Nonlinear Approximation by q-Favard-Szász-Mirakjan Operators of Max-Product Kind

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Abstract
In this study, nonlinear q-Favard-Szász-Mirakjan operators of max-product kind are defined and approximation properties of these operators are investigated. Classical approximation and A-statistical approximation theorems are given.

Keywords: Favard-Szász-Mirakjan operators, Modulus of continuity, Nonlinear max-product operators, q-integers
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1. Introduction

The approximation of functions by using linear positive operators introduced via q-Calculus and (p,q)-Calculus is currently under intensive research. Firstly, generalizations of Bernstein polynomials based on the q-integers has been investigated by Lupas [1] and Phillips [2]. Later, generalized q-Bernstein operators and the q-generalization of other operators were studied in [3]-[8]. Also, in recent years, a nonlinear modification of the classical Bernstein polynomial has been introduced by Bede and Gal [9]. All the max-product operators are nonlinear and piecewise rational, and they present, for many subclasses of functions, essentially better approximation properties than the classical linear operators. In [10]-[13], Favard-Szász-Mirakjan operator of max-product kind and Bernstein operator of max-product kind were studied. Duman constructed a nonlinear approximation operator by modifying the q-Bernstein polynomial in [14].

In this study, we define nonlinear q-Favard-Szász-Mirakjan operators of max-product kind. But, before that the classical Favard-Szász-Mirakjan operators (see [15]) and its q-generalization (see [16]) are given respectively by

$$S_n(f,x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{1.1}$$

and

$$S_{n,q}(f,x) = E_q(-[n]_q x) \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!} f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.2}$$

where $n \in \mathbb{N}$, f is bounded, $f \in C[0,+\infty)$, $x \in [0,+\infty)$, $q \in (0,1)$ and $E_q(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}$.

The aim of this paper is to study the nonlinear approximation properties of q-Favard-Szász-Mirakjan operators of max-product kind.

We first recall some basic definitions in q-calculus. Let parameter q be a positive real number and n a non-negative integer. $[n]_q$ denotes a q integer, defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

Let $q > 0$ be given. We define a q-factorial, $[n]_q!$ of $k \in \mathbb{N}$, as

$$[n]_q! = \begin{cases} [1]_q[2]_q...[n]_q, & n = 1,2,... \\ 1, & n = 0. \end{cases}$$

The q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}_q$ by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[n-r]_q![r]_q!}.$$

2. Construction of the Operators

The approximation properties of the classical Favard-Szasz-Mirakjan operators of max-product kind were investigated in [9]. In this section, we construct nonlinear q-Favard-Szász-Mirakjan operators of max-product kind. We consider the operations "$\vee$" (maximum) and "." (product) over the interval $[0,+\infty)$. Then $([0,+\infty), \vee, .)$ has a semiring structure and is called "max-product algebra" (see, for instance [13]).

Let $C_+[0,+\infty) := \{f : [0,+\infty) \to [0,+\infty) : f \quad \textit{is} \quad \textit{continuous} \quad \textit{on} \quad [0,+\infty)\}$. We define nonlinear q-Favard-Szász-Mirakjan operators of max-product kind as follows:

$$F_{n,q}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} s_{n,k}(x,q) f\left(\frac{[k]_q}{[n]_q}\right)}{\bigvee_{k=0}^{\infty} s_{n,k}(x,q)}, \tag{2.1}$$

where $n \in \mathbb{N}$, $f \in C_+[0,+\infty)$, $x \in [0,+\infty)$, $q \in (0,1)$ and $s_{n,k}(x,q)$ is given by

$$s_{n,k}(x,q) = \frac{([n]_q x)^k}{[k]_q!}. \tag{2.2}$$

Since it easy to check that $F_{n,q}^{(M)}(f)(0) - f(0) = 0$ for all n, notice that in the notations, proofs and statements of all approximation results in fact we always may suppose that $x > 0$.

Since $f \in C_+[0,+\infty)$ and $s_{n,k}(x,q)$ is positive for all $x \in [0,+\infty)$, $F_{n,q}^{(M)}(f)(x)$ is a positive operator. Now, we show that $F_{n,q}^{(M)}(f)(x)$ is not linear operator on $C_+[0,+\infty)$.

Let $f, g \in C_+[0,+\infty)$. Then, by definition we see that

$$f \leq g \implies F_{n,q}^{(M)}(f)(x) \leq F_{n,q}^{(M)}(g)(x). \tag{2.3}$$

Thus, $F_{n,q}^{(M)}(f)(x)$ is increasing with respect to $f \in C_+[0,+\infty)$. Besides, for any $f, g \in C_+[0,+\infty)$ we have

$$F_{n,q}^{(M)}(f+g)(x) \leq F_{n,q}^{(M)}(f)(x) + F_{n,q}^{(M)}(g)(x). \tag{2.4}$$

In general, $\omega_1(f,\delta)$, $\delta > 0$ denote the modulus of continuity of $f \in C_+[0,+\infty)$ defined by

$$\omega_1(f,\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,+\infty), |x-y| \leq \delta\}.$$

Now, using (2.3), (2.4) and also applying Corollary 2.3 in [11] or Corollary 3 in [13], we have the following inequality:

$$|F_{n,q}^{(M)}(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta_n} F_{n,q}^{(M)}(\varphi_x)(x)\right) \omega_1(f,\delta_n), \tag{2.5}$$

where $n \in \mathbb{N}$, $f \in C_+[0,+\infty)$, $x \in [0,+\infty)$, $q \in (0,1)$ and $\varphi_x(t) = |x-t|$.

3. Auxiliary Results

For each $k, j \in \{0,1,2,...\}$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, let us denote

$$M_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left|\frac{[k]_q}{[n]_q} - x\right|}{s_{n,j}(x,q)}, \tag{3.1}$$

$$m_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)}{s_{n,j}(x,q)}. \tag{3.2}$$

It can easily see that if $k \geq j+1$ then

$$M_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left(\frac{[k]_q}{[n]_q} - x\right)}{s_{n,j}(x,q)},$$

and if $k \leq j-1$ then

$$M_{k,n,j}(x,q) = \frac{s_{n,k}(x,q)\left(x - \frac{[k]_q}{[n]_q}\right)}{s_{n,j}(x,q)}.$$

Lemma 3.1. *Let $q \in (0,1)$. For all $k, j \in \{0,1,2,...\}$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, we get*

$$m_{k,n,j}(x,) \leq 1. \tag{3.3}$$

Proof. We consider two cases: (i) $k \geq j$ and (ii) $k < j$.

Case (i). From (3.2), we have

$$\frac{m_{k,n,j}(x,q)}{m_{k+1,n,j}(x,q)} = \frac{[k+1]_q}{[n]_q}\frac{1}{x}.$$

Since the function $h(x) = \frac{1}{x}$ is non-increasing on $\left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, from here we get

$$\begin{aligned}\frac{m_{k,n,j}(x,q)}{m_{k+1,n,j}(x,q)} &= \frac{[k+1]_q}{[n]_q}\frac{[n]_q}{[j+1]_q} \\ &= \frac{[k+1]_q}{[j+1]_q} \geq 1\end{aligned}$$

which immediately implies

$$m_{j,n,j}(x,q) \geq m_{j+1,n,j}(x,q) \geq m_{j+2,n,j}(x,q) \geq ... \quad .$$

Case (ii) We get

$$\frac{m_{k,n,j}(x,q)}{m_{k-1,n,j}(x,q)} = \frac{[n]_q}{[k]_q}x \geq \frac{[n]_q}{[k]_q}\frac{[j]_q}{[n]_q} = \frac{[j]_q}{[k]_q} \geq 1,$$

which immediately implies

$$m_{j,n,j}(x,q) \geq m_{j-1,n,j}(x,q) \geq m_{j-2,n,j}(x,q) \geq ... \geq m_{0,n,j}(x,q).$$

Since $m_{j,n,j}(x,q) = 1$ the proof of the lemma is finished. □

Lemma 3.2. *Let $q \in (0,1)$, $j \in \{1,2,...\}$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$.*

(i) If $k \in \{j+1, j+2, ...\}$ is such that $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$, then $M_{k,n,j}(x,q) \geq M_{k+1,n,j}(x,q)$.

(ii) If $k \in \{1,2,...,j-1\}$ is such that $[k]_q - \sqrt{q^{k-1}[k]_q} \leq [j]_q$, then $M_{k-1,n,j}(x,q) \leq M_{k,n,j}(x,q)$.

Proof. *(i)* Let $k \in \{j+1, j+2, ...\}$ and $[k+1]_q - \sqrt{q^k [k+1]_q} \geq [j+1]_q$. Then, we can write that

$$\frac{M_{k,n,j}(x,q)}{M_{k+1,n,j}(x,q)} = \frac{[k+1]_q}{[n]_q} \frac{1}{x} \frac{\frac{[k]_q}{[n]_q} - x}{\frac{[k+1]_q}{[n]_q} - x}.$$

Since the $g(x) = \frac{1}{x} \frac{\frac{[k]_q}{[n]_q} - x}{\frac{[k+1]_q}{[n]_q} - x}$ clearly is decreasing on the interval $\left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, we have

$$\begin{aligned} g(x) \geq g\left(\frac{[j+1]_q}{[n]_q}\right) &= \frac{[n]_q}{[j+1]_q} \frac{\frac{[k]_q}{[n]_q} - \frac{[j+1]_q}{[n]_q}}{\frac{[k+1]_q}{[n]_q} - \frac{[j+1]_q}{[n]_q}} \\ &= \frac{[n]_q}{[j+1]_q} \frac{[k]_q - [j+1]_q}{[k+1]_q - [j+1]_q}. \end{aligned}$$

Since the condition $[k+1]_q - \sqrt{q^k [k+1]_q} \geq [j+1]_q$ is equivalent to $[k+1]_q - \sqrt{[k+1]_q^2 - [k]_q [k+1]_q} \geq [j+1]_q$ which implies that $[k+1]_q ([k]_q - [j+1]_q) \geq [j+1]_q ([k+1]_q - [j+1]_q)$.

So, we achieve that

$$\frac{M_{k,n,j}(x,q)}{M_{k+1,n,j}(x,q)} \geq 1,$$

which proves Lemma 3.2 *(i)*.

(ii) Let $k \in \{1, 2, ..., j-1\}$ and $[k]_q - \sqrt{q^{k-1} [k]_q} \leq [j]_q$. Then, we can write that

$$\frac{M_{k,n,j}(x,q)}{M_{k-1,n,j}(x,q)} = \frac{[n]_q}{[k]_q} x \frac{x - \frac{[k]_q}{[n]_q}}{x - \frac{[k+1]_q}{[n]_q}}.$$

Since the $h(x) = x \frac{x - \frac{[k]_q}{[n]_q}}{x - \frac{[k+1]_q}{[n]_q}}$ clearly is increasing on the interval $\left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, we have

$$\begin{aligned} h(x) \geq h\left(\frac{[j]_q}{[n]_q}\right) &= \frac{[j]_q}{[n]_q} \frac{\frac{[j]_q}{[n]_q} - \frac{[k]_q}{[n]_q}}{\frac{[j]_q}{[n]_q} - \frac{[k-1]_q}{[n]_q}} \\ &= \frac{[j]_q}{[n]_q} \frac{[j]_q - [k]_q}{[j]_q - [k-1]_q}. \end{aligned}$$

Since the condition $[k]_q + \sqrt{q^{k-1} [k+1]_q} \leq [j]_q$ is equivalent to $[k]_q - \sqrt{[k]_q^2 - [k]_q [k1]_q} \leq [j]_q$ which implies that $[j]_q ([j]_q - [k]_q) \geq [k]_q ([j]_q - [k-1]_q)$.

So, we achieve that

$$\frac{M_{k,n,j}(x,q)}{M_{k-1,n,j}(x,q)} \geq 1$$

which proves Lemma 3.2 *(ii)*. □

Lemma 3.3. *Let $q \in (0,1)$, $j \in \{0, 1, 2, ...\}$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$. We get*

$$\bigvee_{k=0}^{\infty} s_{n,k}(x,q) = s_{n,j}(x,q).$$

Proof. Firstly, we show that for fixed $n \in \mathbb{N}$ and $0 \leq k$ we get

$$0 \leq s_{n,k+1}(x,q) \leq s_{n,k}(x,q) \iff x \in \left[0, \frac{[k+1]_q}{[n]_q}\right].$$

Indeed, from $s_{n,k}(x,q)=\frac{([n]_q x)^k}{[k]_q!}$ we have

$$0\le s_{n,k+1}(x,q)\le s_{n,k}(x,q)$$

$$0\le \frac{([n]_q x)^{k+1}}{[k+1]_q!}\le \frac{([n]_q x)^{k}}{[k]_q!},$$

which after simplifications is obviously equivalent to

$$0\le x\le \frac{[k+1]_q}{[n]_q}.$$

So, if we take $k=0,1,2,...$, then we achieve that

$$s_{n,1}(x,q)\le s_{n,0}(x,q) \iff x\in\left[0,\frac{[1]_q}{[n]_q}\right],$$

$$s_{n,2}(x,q)\le s_{n,1}(x,q) \iff x\in\left[0,\frac{[2]_q}{[n]_q}\right],$$

$$s_{n,3}(x,q)\le s_{n,2}(x,q) \iff x\in\left[0,\frac{[3]_q}{[n]_q}\right],$$

so on,

$$s_{n,k+1}(x,q)\le s_{n,k}(x,q) \iff x\in\left[0,\frac{[k+1]_q}{[n]_q}\right],$$

and so on.

From above inequalities, we can easily write:

$$if \quad x\in\left[0,\frac{[1]_q}{[n]_q}\right] \quad then \quad s_{n,k}(x,q)\le s_{n,0}(x,q), \quad for \quad all \quad k=0,1,2,...,$$

$$if \quad x\in\left[\frac{[1]_q}{[n]_q},\frac{[2]_q}{[n]_q}\right] \quad then \quad s_{n,k}(x,q)\le s_{n,1}(x,q), \quad for \quad all \quad k=0,1,2,...,$$

$$if \quad x\in\left[\frac{[2]_q}{[n]_q},\frac{[3]_q}{[n]_q}\right] \quad then \quad s_{n,k}(x,q)\le s_{n,2}(x,q), \quad for \quad all \quad k=0,1,2,...,$$

and so on, as a result, we obtain

$$if \quad x\in\left[\frac{[j]_q}{[n]_q},\frac{[j+1]_q}{[n]_q}\right] \quad then \quad s_{n,k}(x,q)\le s_{n,j}(x,q), \quad for \quad all \quad k=0,1,2,...,$$

which completes the proof of Lemma 3.3. □

4. Approximation Results

Theorem 4.1. *Let $f:[0,+\infty)\to[0,+\infty)$ be bounded and continuous on $[0,+\infty)$ and $q\in(0,1)$. Then we get the following estimation*

$$\left|F_{n,q}^{(M)}(f)(x)-f(x)\right|\le 8\omega_1\left(f;\frac{\sqrt{x}}{\sqrt{[n]_q}}\right), \tag{4.1}$$

where $n\in\mathbb{N}$, $x\in[0,+\infty)$ and

$$\omega_1(f,\delta)=\sup\{|f(x)-f(y)|:x,y\in[0,+\infty),|x-y|\le\delta\}.$$

Proof. Taking $q = q_n \in (0,1)$ such that $\lim_n q_n = 1$, we deduce $\lim_n [n]_{q_n} = \infty$. From (2.5), we have

$$|F_{n,q}^{(M)}(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta_n} F_{n,q}^{(M)}(\varphi_x)(x)\right) \omega_1(f, \delta_n), \tag{4.2}$$

where $\varphi_x(t) = |x - t|$. Thus, it is enough to estimate

$$A_{n,q}(x) := F_{n,q}^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{\infty} s_{n,k}(x,q) \left|\frac{[k]_q}{[n]_q} - x\right|}{\bigvee_{k=0}^{\infty} s_{n,k}(x,q)},$$

where $x \in [0, +\infty)$. Let $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$, where $j \in \{0, 1, 2, ...\}$ is fixed,
arbitrary. By Lemma 3.3 we can easily achieve

$$A_{n,q}(x) = max\{M_{k,n,j}(x,q) : x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right], k = 0, 1, ...\}.$$

Firstly, we show that for $j = 0$ and $k = 0, 1, 2, ...$ we obtain $A_{n,q}(x) \leq \frac{\sqrt{x}}{\sqrt{[n]_q}}$ for all $x \in \left[0, \frac{1}{[n]_q}\right]$.

Indeed, for $j = 0$ we get $M_{k,n,0}(x,q) = \frac{([n]_q x)^k}{[k]_q!} \left|\frac{[k]_q}{[n]_q} - x\right|$ which for $k = 0$ gives $M_{k,n,0}(x,q) = x = \sqrt{x}\sqrt{x} \leq \sqrt{x}\frac{1}{\sqrt{[n]_q}}$. Furthermore, for any $k = 1, 2, ...$ we have $\frac{1}{[n]_q} \leq \frac{[k]_q}{[n]_q}$ and we obtain

$$M_{k,n,0}(x,q) \leq \frac{([n]_q x)^k}{[k]_q!} \frac{[k]_q}{[n]_q} = \sqrt{x} \frac{[n]_q^{k-1} x^{k-\frac{1}{2}}}{[k-1]_q} \leq \sqrt{x} \frac{[n]_q^{k-1}}{[k-1]_q [n]^{k-\frac{1}{2}}{}_q} \leq \frac{\sqrt{x}}{\sqrt{[n]_q}}.$$

Now we claim that for each $M_{k,n,j}(x,q)$ when $j = 1, 2, ...$ and $k = 0, 1, 2, ...$ the following inequality

$$M_{k,n,j}(x,q) \leq \frac{4\sqrt{x}}{\sqrt{[n]_q}}, \quad \forall x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right], \tag{4.3}$$

which immediately will imply that

$$A_{n,q}(x) \leq \frac{4\sqrt{x}}{\sqrt{[n]_q}}, \quad \forall x \in [0, \infty), n \in \mathbb{N},$$

and taking $\delta_n = \frac{4\sqrt{x}}{\sqrt{[n]_q}}$ in (4.2) we complete the proof of Theorem 4.1.

In order to prove (4.3) we consider the following three cases: 1) $k = j$, 2) $k \geq j + 1$, 3) $k \leq j - 1$.

Case 1) If $k = j$ then from (3.1) $M_{j,n,j}(x,q) = \left|\frac{[j]_q}{[n]_q} - x\right|$. Since $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$ we can easily see that $M_{j,n,j}(x,q) \leq \frac{1}{[n]_q}$. Since $j \geq 1$ we have $x \geq \frac{1}{[n]_q}$ which implies

$$M_{j,n,j}(x,q) \leq \frac{1}{[n]_q} = \frac{1}{\sqrt{[n]_q}} \frac{1}{\sqrt{[n]_q}} \leq \sqrt{x} \frac{1}{\sqrt{[n]_q}}.$$

Case 2) Subcase a) We suppose that $k \geq j + 1$ and $[k+1]_q - \sqrt{q^k [k+1]_q} < [j+1]_q$. We have from Lemma 3.1 that

$$M_{k,n,j}(x,q) = m_{k,n,j}(x,q) \left(\frac{[k]_q}{[n]_q} - x\right) \leq \frac{[k]_q}{[n]_q} - \frac{[j]_q}{[n]_q}.$$

By hypothesis, since

$$q[k]_q - \sqrt{q^k [k+1]_q} < q[j]_q,$$

we have

$$M_{k,n,j}(x,q) \leq \frac{[k]_q}{[n]_q} - \frac{[k]_q - \sqrt{q^{k-2}[k+1]_q}}{[n]_q} = \frac{\sqrt{q^{k-2}[k+1]_q}}{[n]_q}.$$

Since $k \geq 2$ and $q \in (0,1)$, we obtain

$$M_{k,n,j}(x,q) \leq \frac{\sqrt{[k+1]_q}}{[n]_q}.$$

But we necessarily have $k \leq 3j$. Indeed, if we suppose that $k > 3j$, then because $g(k) = [k+1]_q - \sqrt{q^k[k+1]_q}$ is increasing with respect to k. Indeed, we can write that

$$\begin{aligned} g(k+1)-g(k) &= [k+2]_q - [k+1]_q + \sqrt{q^k[k+1]_q} - \sqrt{q^{k+1}[k+2]_q} \\ &\geq [k+2]_q - [k+1]_q + \sqrt{q^k[k+1]_q} - \sqrt{q^k[k+2]_q} \\ &= q^{k+1} - q^{\frac{k}{2}}\left(\sqrt{[k+1]_q} - \sqrt{[k+2]_q}\right) \\ &= q^{k+1} - \frac{q^{k+1}q^{\frac{k}{2}}}{\sqrt{[k+1]_q} - \sqrt{[k+2]_q}} \\ &= q^{k+1}\left(1 - \frac{q^{\frac{k}{2}}}{\sqrt{[k+1]_q} - \sqrt{[k+2]_q}}\right) \\ &\geq q^{k+1}\left(1 - \frac{1}{\sqrt{[k+1]_q} - \sqrt{[k+2]_q}}\right) \\ &> 0. \end{aligned}$$

Hence, we get that $[j+1]_q > [k+1]_q - \sqrt{q^k[k+1]_q} > [3j+1]_q - \sqrt{q^{3j}[3j+1]_q}$ which implies the obvious contradiction $[3j+1]_q - [j+1]_q < \sqrt{q^{3j}[3j+1]_q}$ is to equivalent $q^{j+1}[2j]_q < \sqrt{q^{3j}[3j+1]_q}$.

As a result, we achieve

$$\begin{aligned} M_{k,n,j}(x,q) &\leq \frac{\sqrt{[k+1]_q}}{[n]_q} \leq \frac{\sqrt{[3j+1]_q}}{[n]_q} \\ &\leq \frac{\sqrt{[4j]_q}}{[n]_q} = \sqrt{(1+q^j)(1+q^{2j})}\frac{\sqrt{[j]_q}}{[n]_q} \\ &\leq \sqrt{(1+q^j)(1+q^{2j})}\frac{\sqrt{x}}{\sqrt{[n]_q}} \leq 2\frac{\sqrt{x}}{\sqrt{[n]_q}}, \end{aligned}$$

taking into account that $\sqrt{x} \geq \frac{\sqrt{[j]_q}}{[n]_q}$.

Subcase b) We suppose that $k \geq j+1$ and $[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$. Since, the function $g(k) = [k+1]_q - \sqrt{q^k[k+1]_q}$ is increasing with respect to k, it follows that there exits $\overline{k} \in \{0,1,2,...\}$, of maximum value, such that

$$[\overline{k}+1]_q - \sqrt{q^{\overline{k}}[\overline{k}+1]_q} < [j+1]_q.$$

Let $\tilde{k} = \overline{k}+1$. Then for all $k \geq \tilde{k}$ we have

$$[k+1]_q - \sqrt{q^k[k+1]_q} \geq [j+1]_q$$

and

$$M_{\tilde{k},n,j}(x,q) = m_{\tilde{k},n,j}(x,q)\left(\frac{[\tilde{k}]_q}{[n]_q} - x\right) \leq \frac{[\overline{k}+1]_q}{[n]_q} - \frac{[j]_q}{[n]_q}.$$

Since

$$[j]_q \geq [\overline{k}+1]_q - q^j - \sqrt{q^{\overline{k}}[\overline{k}+1]_q},$$

we can see that

$$M_{\tilde{k},n,j}(x,q) \leq \frac{[\overline{k}+1]_q}{[n]_q} - \frac{[\overline{k}+1]_q - q^j - \sqrt{q^{\overline{k}}[\overline{k}+1]_q}}{[n]_q}$$
$$= \frac{q^j + \sqrt{q^{\overline{k}}[\overline{k}+1]_q}}{[n]_q}$$
$$\leq \frac{1+\sqrt{[\overline{k}+1]_q}}{[n]_q} \leq \frac{2\sqrt{[\overline{k}+1]_q}}{[n]_q}$$
$$\leq 4\frac{\sqrt{x}}{\sqrt{[n]_q}}.$$

The last above inequality follows from the fact that

$$[\overline{k}+1]_q - \sqrt{q^{\overline{k}}[\overline{k}+1]_q} < [j+1]_q,$$

necessarily implies $\overline{k} \leq 3j$ (see the similar reasoning in the above Subcase a)). Also, we get $\tilde{k} \geq j+1$. Indeed, this is a consequence of the fact that g is increasing function and because it is easy to see that $g(j) \leq [j+1]_q$.

By Lemma 3.2, (i) it follows that

$$M_{\overline{k}+1,n,j}(x,q) \geq M_{\overline{k}+2,n,j}(x,q) \geq ...$$

So, we achieve $M_{k,n,j}(x,q) \leq 4\frac{\sqrt{x}}{\sqrt{[n]_q}}$ for any $k \in \{\overline{k}+1, \overline{k}+2, ...\}$.

Case 3) Subcase a) We suppose that $k \leq j-1$ and $[k]_q + \sqrt{q^{k-1}[k]_q} \geq [j]_q$. We have from Lemma 3.1 that

$$M_{k,n,j}(x,q) = m_{k,n,j}(x,q)\left(x - \frac{[k]_q}{[n]_q}\right) \leq \frac{[j+1]_q}{[n]_q} - \frac{[k]_q}{[n]_q} = \frac{[j]_q + q^j}{[n]_q} - \frac{[k]_q}{[n]_q}$$

By hypothesis, we get

$$M_{k,n,j}(x,q) \leq \frac{[k]_q + \sqrt{q^{k-1}[k]_q} + q^j}{[n]_q} - \frac{[k]_q}{[n]_q}$$
$$= \frac{\sqrt{q^{k-1}[k]_q} + q^j}{[n]_q} \leq \frac{\sqrt{[k]_q}+1}{[n]_q}$$
$$\leq \frac{\sqrt{[j-1]_q}+1}{[n]_q} = \frac{1}{\sqrt{[n]_q}} \frac{\sqrt{[j-1]_q}+1}{\sqrt{[n]_q}}$$
$$\leq \frac{1}{\sqrt{[n]_q}} \frac{2\sqrt{[j]_q}}{\sqrt{[n]_q}} \leq 2\frac{\sqrt{x}}{\sqrt{[n]_q}}.$$

Subcase b) We suppose that $k \leq j-1$ and $[k]_q + \sqrt{q^{k-1}[k]_q} < [j]_q$. Let $\overline{k} \in \{0,1,2,...\}$ be the minimum value such that $[\overline{k}]_q + \sqrt{q^{\overline{k}-1}[\overline{k}]_q} \geq [j]_q$. Then $\tilde{k} = \overline{k}-1$ satisfies $[\overline{k}-1]_q + \sqrt{q^{\overline{k}-2}[\overline{k}-1]_q} < [j]_q$. Also we have

$$M_{\overline{k}-1,n,j}(x,q) = m_{\overline{k}-1,n,j}(x,q)\left(x - \frac{[\overline{k}-1]_q}{[n]_q}\right) \leq \frac{[j+1]_q}{[n]_q} - \frac{[\overline{k}-1]_q}{[n]_q}$$
$$= \frac{[j]_q + q^j}{[n]_q} - \frac{[\overline{k}-1]_q}{[n]_q}.$$

Since $[\bar{k}]_q + \sqrt{q^{\bar{k}-1}[\bar{k}]_q} \geq [j]_q$, we obtain

$$M_{\bar{k}-1,n,j}(x,q) \leq \frac{[\bar{k}]_q + \sqrt{q^{\bar{k}-1}[\bar{k}]_q} + q^j}{[n]_q} - \frac{[\bar{k}-1]_q}{[n]_q}$$
$$= \frac{q^{\bar{k}-1} + \sqrt{q^{\bar{k}-1}[\bar{k}]_q} + q^j}{[n]_q} \leq \frac{2 + \sqrt{[\bar{k}]_q}}{[n]_q}$$
$$\leq 3\frac{\sqrt{[j]_q}}{[n]_q} \leq 3\frac{\sqrt{x}}{\sqrt{[n]_q}}.$$

By Lemma 3.2, (ii) it follows that

$$M_{\bar{k}-1,n,j}(x,q) \geq M_{\bar{k}-2,n,j}(x,q) \geq \ldots \geq M_{0,n,j}(x,q).$$

So, we achieve $M_{k,n,j}(x,q) \leq \frac{\sqrt{x}}{\sqrt{[n]_q}}$ for any $k \leq j-1$ and $x \in \left[\frac{[j]_q}{[n]_q}, \frac{[j+1]_q}{[n]_q}\right]$.

Collecting all the above estimates we have (4.3), which completes the proof of Theorem 4.1. □

5. A-Statistical Approximation

In this section, we will give an A-statistical approximation theorem for the (2.1) operators. Firstly, we have to replace a fixed $q \in (0,1)$ consider in the previous sections with an appropriate sequence (q_n) whose terms are in the interval $(0,1)$. This idea was first used by Philips [2] for the q-Bernstein polynomials.

Let $(q_n)$is a real sequence satisfying the following conditions,

$$0 < q_n < 1 \quad for \quad every \quad n \in \mathbb{N}, \tag{5.1}$$

$$st_A - \lim_n q_n = 1 \tag{5.2}$$

and

$$st_A - \lim_n q_n^n = 1. \tag{5.3}$$

Note that the notations in (5.2) and (5.3) denote the A-statistical limit of (q_n) where $A = [a_{jn}]$, $(j,n \in \mathbb{N})$ is an infinite non-negative regular summability matrix, i.e., $a_{jn} \geq 0$ for every $j,n \in \mathbb{N}$ and $\lim_j \sum_{n=1}^{\infty} a_{jn}x_n = L$ provided that, for a given sequence (x_n), we say that (x_n) is A-statistically convergent to a number L if, for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n-L|\geq\varepsilon}^{\infty} a_{jn}x_n = 0$ (see [17]). We should remark that this method of convergence generalizes both the classical convergence and the concept of statistical convergence which first introduced by Fast [18]. We give the following A-statistical approximation theorem.

Theorem 5.1. *Let $A = [a_{nj}]$ be a non-negative regular summability matrix and (q_n) be a sequence satisfying (5.1)-(5.3). Then for every $f \in C_+[0,\infty)$ we have*

$$st_A - \lim_n \left(\sup_{x\in[0,\infty)} \left| F_{n,q}^{(M)}(f)(x) - f(x) \right| \right) = 0. \tag{5.4}$$

Proof. Let $f \in C_+[0,\infty)$. Replacing q with (q_n), taking supremum over $x \in [0,\infty)$ and using the monotonicity of the modulus of continuity, we achieve from Theorem 4.1 that

$$E_n := \sup_{x\in[0,\infty)} \left| F_{n,q}^{(M)}(f)(x) - f(x) \right| \leq 8\omega_1\left(f; \frac{\sqrt{x}}{\sqrt{[n]_q}}\right), \tag{5.5}$$

holds for every $n \in \mathbb{N}$. Then, let we prove

$$st_A - \lim_n E_n = 0.$$

From (5.1)-(5.3), we get

$$st_A - \lim_n \frac{1}{\sqrt{[n]_{q_n}}} = 0.$$

So we can write

$$st_A - \lim_n \omega_1 \left(f; \frac{\sqrt{x}}{\sqrt{[n]_q}} \right) = 0. \tag{5.6}$$

So, the proof of Theorem 5.1 follows from (5.1)-(5.6) immediately. □

We should note that the A-statistical approximation result in Theorem 5.1 includes the classical approximation by choosing $A = I$ the identity matrix.

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