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FUNDAMENTALS OF CONTEMPORARY MATHEMATICAL SCIENCES

E-ISSN 2717-6185

FUNDAMENTALS OF CONTEMPORARY MATHEMATICAL SCIENCES

BIANNUALLY SCIENTIFIC JOURNAL

VOLUME 4 - ISSUE 2

2023

https://dergipark.org.tr/en/pub/fcmathsci

FUNDAMENTALS OF CONTEMPORARY MATHEMATICAL SCIENCES

e-ISSN: 2717-6185

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e-ISSN: 2717-6185

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1145287
(**2023**) **4**(2) 56 – 65
Research Article

On Different Definitions of Hyper Pseudo BCC-algebras

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Received: 20 July 2022 **Accepted:** 06 June 2023

Abstract: We study hyper pseudo BCC-algebras which are a common generalization of hyper BCC-algebras and hyper BCK-algebras. In particular, we introduce different notion of hyper pseudo BCC-algebras and describe the relationship among them. Then, by choosing one of these definitions, we investigate for its related properties.

Keywords: Hyper pseudo order, hyper operation, hyper pseudo BCC-algebras.

1. Introduction

Hyper structures and pseudo structures have an important place in the field of algebra. These notions help to create new structures in algebraic system and to investigate their properties. The notions of hyper operation and hyper order were first defined by Marty in 1934 [7].

BCK-algebras were first studied by Iseki and Tanaka [4]. BCC-algebras, a generalization of BCK-algebras, were defined in 1990 by Dudek and their related properties were investigated [3]. The concept of Hyper BCK-algebra was introduced in 2000 by Jun, Zahedi, Xin and Borzooei [5]. Borzooei, Dudek and Koohestani in 2006 carried similar definitions and applications of hyper BCK-algebras to hyper BCC-algebras and defined various ideal types [1].

In this study, the notion of hyper pseudo order is defined. Then, different notions of hyper pseudo BCC-algebras are defined and their existences are proven with examples. In addition, the relationship between them is examined and some related properties are obtained. As a result, it is aimed to transfer hyper pseudo structures to BCC-algebras so that new algebraic structures can be built.

2. Preliminaries

Definition 2.1 *[3] Let X be a nonempty set, '$*$' be a operation on X and '0' be a constant*

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2020 *AMS Mathematics Subject Classification*: 06F35, 03G25

element. $(X,*,0)$ *is called to be a BCC-algebra, if it supplies the following conditions for all* $x,y,z \in X$ *:*

(BCC1) $((x*y)*(z*y))*(x*z)=0$,

(BCC2) $x*0=x$,

(BCC3) $x*x=0$,

(BCC4) $0*x=0$,

(BCC5) $x*y$ *and* $y*x=0 \Rightarrow x=y$.

Definition 2.2 *[7] Let* H *be a nonempty set*

$$\circ : H \times H \to P(H) - \{\emptyset\}$$

be a hyper operation. If "$x \ll y \Leftrightarrow 0 \in x \circ y$ *for all* $x,y \in H$ *and* $S \ll T \Leftrightarrow$ *for every* $S,T \subset H$, $\forall s \in S$, $\exists t \in T$ *such that* $s \ll t$ *", then '*$\ll$*' is named to be a hyper order in* H.

Definition 2.3 *[1] Let* H *be a nonempty set, '*$\circ$*' be a hyper operation on* H*, '*$\ll$*' be a hyper order on* H *and '*0*' be a constant element of* H. $(H,\circ,\ll,0)$ *is called to be a hyper BCC-algebra if it supplies the following conditions, for all* $x,y,z \in H$ *:*

(HBCC1) $(x\circ z)\circ(y\circ z) \ll x\circ y$,

(HBCC2) $0\circ x=0$,

(HBCC3) $x\circ 0=x$,

(HBCC4) $x \ll y$ *and* $y \ll x \Rightarrow x=y$.

Definition 2.4 *[1] Let* $(H,\circ,\ll,0)$ *be a hyper BCC-algebra and* I *be a subset of* H *such that* $0 \in I$ *is named as follows, for all* $x,y,z \in H$ *:*

(1) a hyper BCC-ideal of type1*, if*
$(x\circ y)\circ z \ll I,\ y \in I \Rightarrow x\circ z \subseteq I$,

(2) a hyper BCC-ideal of type2*, if*
$(x\circ y)\circ z \subseteq I,\ y \in I \Rightarrow x\circ z \subseteq I$,

(3) a hyper BCC-ideal of type3*, if*
$(x\circ y)\circ z \ll I,\ y \in I \Rightarrow x\circ z \ll I$,

(4) a hyper BCC-ideal of type4*, if*
$(x\circ y)\circ z \subseteq I,\ y \in I \Rightarrow x\circ z \ll I$.

Definition 2.5 *[5] Let H be a nonempty set '$\circ$' be a hyper operation on H, '$\ll$' be a hyper order in H and '0' be a constant element of H. $(H,\circ,\ll,0)$ is named to be a hyper BCK-algebra if it supplies the following conditions, for all $x,y,z\in H$:*

(HBCK1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,

(HBCK2) $(x\circ y)\circ z=(x\circ z)\circ y$,

(HBCK3) $x\circ y\ll x$,

(HBCK4) $x\ll y$ *and* $y\ll x\Rightarrow x=y$.

Definition 2.6 *[2] Let H be a nonempty set, '$*$', '$\circ$' be hyper operations on H, '$\ll$' be a hyper order in H and '0' be a constant element of H, $(H,\circ,*,\ll,0)$ is named to be a hyper pseudo BCK-algebra, if it supplies the following conditions, for all $x,y,z\in H$:*

(HPBCK1) $(x\circ z)\circ(y\circ z)\ll x\circ y$, $(x*z)*(y*z)\ll x*y$,

(HPBCK2) $(x\circ y)*z=(x*z)\circ y$,

(HPBCK3) $x\circ y\ll x$, $x*y\ll x$,

(HPBCK4) $x\ll y$ *and* $y\ll x\Rightarrow x=y$.

3. Hyper Pseudo BCC-algebras

In this section, different definitions of Hyper Pseudo BCC-algebras, these definitions relationship between them and some of their related properties are given.

Definition 3.1 *Let H be a nonempty set and*

$$\circ: H\times H\to P(H)-\{\emptyset\}$$

be a hyper operation.

If "$x\ll y\Leftrightarrow 0\in x\circ y\Leftrightarrow 0\in x*y$ for all $x,y\in H$ and $S\ll T\Leftrightarrow$ for every $S,T\subset H$, $\forall s\in S$ $\exists t\in T$ such that $s\ll t$", then '$\ll$' is called to be a hyper pseudo order in H.

Definition 3.2 *Let H be a nonempty set, '$\circ$', '$*$' be hyper operations on H, '$\ll$' be a hyper pseudo order in H, '0' be a constant element of H. $(H,\circ,*,\ll,0)$ is named to be hyper pseudo BCC_1-algebra if it supplies the following conditions, for all $x,y,z\in H$:*

$(HPBCC_1 1)$ $(x\circ z)\circ(y\circ z)\ll x\circ y$, $(x*z)*(y*z)\ll x*y$,

$(HPBCC_1 2)$ $0\circ x=\{0\}$, $0*x=\{0\}$,

$(HPBCC_1 3)$ $x\circ 0=\{x\}$, $x*0=\{x\}$,

$(HPBCC_1 4)$ $x \ll y$ *and* $y \ll x \Rightarrow x = y$,

$(HPBCC_1 5)$ $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$.

Example 3.3 *Let* $H = \{0, m, n\}$ *and '*$\circ$*', '*$*$*' be hyper operations on* H *with Cayley table give as in Table 1.*

Table 1: Hyper operations.

∘	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{0}
n	{n}	{n}	{0,n}

*	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{0}
n	{n}	{n}	{0,m,n}

Then, it is easily controlled that $(H, \circ, *, \ll, 0)$ *is a hyper pseudo* BCC_1*-algebra and hyper pseudo BCK-algebra. Also, '*$\circ$*' and '*$*$*' hyper operations with* $(H, \circ, \ll, 0)$ *and* $(H, *, \ll, 0)$ *be hyper BCC-algebras.*

Remark 3.4 *Let* H *be a nonempty set, '*$\circ$*', '*$*$*' be hyper operations on* H*, '*$\ll$*' be a hyper pseudo order in* H*, '0' be a constant element of* H*. According to both hyper operations, the* $(H, \circ, *, 0)$ *system is always a hyper pseudo* BCC_1*-algebra when the system is hyper BCC-algebra.*

Definition 3.5 *Let* H *be a nonempty set, '*$\circ$*', '*$*$*' be hyper operations on* H*, '*$\ll$*' be a hyper pseudo order in* H*, '0' be a constant element of* H*.* $(H, \circ, *, \ll, 0)$ *is named to be hyper pseudo* BCC_2*-algebra if it supplies the following conditions, for all* $x, y, z \in H$*:*

$(HPBCC_2 1)$ $(x \circ z) * (y \circ z) \ll x * y$, $(x * z) \circ (y * z) \ll x \circ y$,

$(HPBCC_2 2)$ $0 \circ x = \{0\}$, $0 * x = \{0\}$,

$(HPBCC_2 3)$ $x \circ 0 = \{x\}$, $x * 0 = \{x\}$,

$(HPBCC_2 4)$ $x \ll y$ *and* $y \ll x \Rightarrow x = y$,

$(HPBCC_2 5)$ $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$.

Example 3.6 *Let* $H = \{0, m, n\}$ *and '*$\circ$*', '*$*$*' be hyper operations on* H *with Cayley table give as in Table 2.*

Table 2: Hyper operations.

$\circ$	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{n}
n	{n}	{n}	{0,n}

$*$	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{m}
n	{n}	{n}	{0,m,n}

Then, it is easily controlled that $(H, \circ, *, \ll, 0)$ *is a hyper pseudo* BCC_2*-algebra but* $(H, \circ, \ll, 0)$ *is not hyper BCC-algebra. Moreover,* $(H, \circ, *, \ll, 0)$ *is not hyper pseudo BCK-algebra because it does not satisfy the* $(HPBCK1)$ *condition of hyper pseudo BCK-algebra. For example; it has been* $(m \circ n) \circ (0 \circ n) \ll m \circ 0$ *such that* $m, n, 0 \in H$. *Then, it can be written* $\{n\} \ll \{m\}$ *so that the condition* $(HPBCK1)$ *is satisfied because* 0 *is not an element of this equation* $\{n\} = n \circ m$.

Definition 3.7 *Let* H *be a nonempty set, '*$\circ$*', '*$*$*' be hyper operations on* H*, '*$\ll$*' be a hyper pseudo order in* H*, '*0*' be a constant element of* H. $(H, \circ, *, \ll, 0)$ *is named to be hyper pseudo* BCC_3*-algebra if it supplies the following conditions, for all* $x, y, z \in H$:

$(HPBCC_31)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$, $(x * z) * (y * z) \ll x * y$,

$(HPBCC_32)$ $0 \circ x = \{0\}$, $0 * x = \{0\}$,

$(HPBCC_33)$ $x \circ 0 = \{x\}$, $x * 0 = \{x\}$,

$(HPBCC_34)$ $0 \in x \circ y \wedge y * x \Rightarrow x = y$,

$(HPBCC_35)$ $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$.

Example 3.8 *Let* $H = \{0, m, n\}$ *and '*$\circ$*', '*$*$*' be hyper operations on* H *with Cayley table give as in Table 3.*

Table 3: Hyper operations.

$\circ$	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{0}
n	{n}	{0}	{0,n}

$*$	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{m}
n	{n}	{n}	{0,m,n}

Then, it is easily controlled that $(H, \circ, *, \ll, 0)$ *is a hyper pseudo* BCC_3*-algebra but according to operation '*$\circ$*',* $(H, \circ, \ll, 0)$ *is not hyper BCC-algebra because it does not satisfy the* $(HBCC4)$ *condition of hyper BCC-algebra. Also, this structure isn't hyper pseudo BCK-algebra because the system does not satisfy the condition* $(HPBCK4)$*.*

Definition 3.9 *Let* H *be a nonempty set, '*$\circ$*', '*$*$*' be hyper operations on* H*, '*$\ll$*' be a hyper pseudo order in* H*, '*0*' be a constant element of* H*.* $(H, \circ, *, \ll, 0)$ *is named to be hyper pseudo* BCC_4*-algebra if it supplies the following conditions, for all* $x, y, z \in H$*:*

$(HPBCC_4 1)$ $(x \circ z) * (y \circ z) \ll x * y$, $(x * z) \circ (y * z) \ll x \circ y$,

$(HPBCC_4 2)$ $0 \circ x = \{0\}$, $0 * x = \{0\}$,

$(HPBCC_4 3)$ $x \circ 0 = \{x\}$, $x * 0 = \{x\}$,

$(HPBCC_4 4)$ $0 \in x \circ y$, $0 \in y * x \Rightarrow x = y$,

$(HPBCC_4 5)$ $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$.

Example 3.10 *Let* $H = \{0, m, n, k\}$ *and '*$\circ$*', '*$*$*' be hyper operations on* H *with Cayley table give as in Table 4.*

Table 4: Hyper operations.

$\circ$	0	m	n	k
0	{0}	{0}	{0}	{0}
m	{m}	{0}	{0}	{n}
n	{n}	{0}	{0,n}	{n}
k	{k}	{0}	{0}	{0,k}

$*$	0	m	n	k
0	{0}	{0}	{0}	{0}
m	{m}	{0}	{k}	{n}
n	{n}	{n}	{0,m,n}	{m}
k	{k}	{k}	{0}	{0,k}

Then, it is easily controlled that $(H, \circ, *, \ll, 0)$ *is a hyper pseudo* BCC_4*-algebra. Also,* $(H, \circ, \ll, 0)$ *and* $(H, *, \ll, 0)$ *systems built with* H *and hyper operations '*$\circ$*', '*$*$*' are not hyper BCC-algebra as they do not satisfy* $(HBCC4)$ *and* $(HBCC1)$*, respectively. Finally, it is not hyper pseudo BCK-algebra because the system does not satisfy the conditions* $(HPBCK1)$ *and* $(HPBCK4)$*.*

Definition 3.11 *Let H be a nonempty set, '$\circ$', '$*$' be hyper operations on H, '$\ll$' be a hyper pseudo order in H, '0' be a constant element of H. $(H,\circ,*,\ll,0)$ is named to be hyper pseudo BCC_5-algebra if it supplies the following conditions, for all $x,y,z\in H$:*

$(HPBCC_51)$ $(x\circ z)*(y\circ z)\ll x*y$, $(x*z)\circ(y*z)\ll x\circ y$,

$(HPBCC_52)$ $x*(0\circ y)=\{x\}$, $x\circ(0*y)=\{x\}$,

$(HPBCC_53)$ $x\ll y$ *and* $y\ll x\Rightarrow x=y$,

$(HPBCC_54)$ $x\ll y\Leftrightarrow 0\in x\circ y\Leftrightarrow 0\in x*y$.

Example 3.12 *Let $H=\{0,m,n,k\}$ and '$\circ$', '$*$' be hyper operations on H with Cayley table give as in Table 5.*

Table 5: Hyper operations.

∘	0	m	n	k
0	{0}	{0}	{0}	{0}
m	{m}	{0}	{k}	{m}
n	{n}	{0}	{0,n}	{k}
k	{k}	{0}	{0}	{0,k}

∗	0	m	n	k
0	{0}	{0}	{0}	{0}
m	{m}	{0}	{n}	{k}
n	{n}	{0}	{0,m,n}	{n}
k	{k}	{0}	{k}	{0,k}

Then, it is easily controlled that $(H,\circ,*,\ll,0)$ is a hyper pseudo BCC_4-algebra. Also, $(H,\circ,\ll,0)$ and $(H,*,\ll,0)$ systems built with H and hyper operations '$\circ$', '$*$' are not hyper BCC-algebra as they do not satisfy $(HBCC1)$. Finally, it is not hyper pseudo BCK-algebra because the system does not satisfy the condition $(HPBCK1)$.

Theorem 3.13 *Let $(H,\circ,*,\ll,0)$ be a hyper pseudo BCC_1-algebra or hyper pseudo BCC_3-algebra. If $x*y=x\circ y$ for all $x,y\in H$, then H is a hyper BCC-algebra.*

Proof Let H be a hyper pseudo BCC_1-algebra. If $x*y=x\circ y$ for all $x,y\in H$, then proof follows from conditions of hyper pseudo BCC_1-algebra. Let H be a hyper pseudo BCC_3-algebra. If $x*y=x\circ y$ for all $x,y\in H$, then proof follows from conditions of hyper pseudo BCC_3-algebra. □

Proposition 3.14 *Let $(H,\circ,*,\ll,0)$ be any of the hyper pseudo BCC_1-algebra, hyper pseudo BCC_2-algebra, hyper pseudo BCC_3-algebra, hyper pseudo BCC_4-algebra. Then, the following conditions are satisfied for every nonempty subset S, T of H and for all $x,y,z\in H$:*

(i) $0 \circ 0 = \{0\}\,,\ 0 * 0 = \{0\}\,,$

(ii) $0 \ll x\,,$

(iii) $x \ll x\,,$

(iv) $x \circ y \ll \{x\}\,,\ x * y \ll \{x\}\,,$

(v) $S \circ 0 = S\,,\ S * 0 = T\,,$

(vi) $0 \circ S = \{0\}\,,\ 0 * S = \{0\}\,,$

(vii) $x * y = \{0\} \Rightarrow x \circ z \ll y \circ z\,,\ x \circ y = \{0\} \Rightarrow x * z \ll y * z\,,$

(viii) $S \ll S\,,$

(ix) $S \subseteq T \Rightarrow S \ll T\,,$

(x) $S \ll \{0\} \Rightarrow S = \{0\}\,,$

(xi) $x \circ 0 \ll \{y\} \Rightarrow x \ll y\,,\ x * 0 \ll \{y\} \Rightarrow x \ll y\,.$

Proof Let $(H, \circ, *, \ll, 0)$ be a hyper pseudo BCC_4-algebra.

(i) In $(HPBCC_4 2)$, let $x = 0$. Then

$$0 \circ 0 = \{0\},\ 0 * 0 = \{0\}.$$

(ii) Using $(HPBCC_4 2)$ condition,

$$0 \in 0 \circ x\,,\ 0 \in 0 * x$$

and so $0 \ll x$.

(iii) Using $(HPBCC_4 1)$ condition, let $y = z = 0$. Then, by (i) and $(HPBCC3)$ condition, we get that $x \ll x$.

(iv) By $(HPBCC_4 1)$ condition, we conclude that

$$(x \circ y) * (z \circ y) \ll (x * z),\ (x \circ y) * (z \circ y) \ll (x * z).$$

Therefore let $z = 0$. Then, by $(HPBCC_4 2)$ and $(HPBCC_4 3)$ we can write,

$$x \circ y \ll \{x\},\ x * y \ll \{x\}.$$

(v) Using $(HPBCC_4 3)$ condition,

$$S \circ 0 = S\,,\ S * 0 = S$$

is shown.

(vi) Using $(HPBCC_4 2)$ condition,

$$0 \circ S = \{0\}\,,\ 0 \circ S = \{0\}$$

is shown.

(vii) Let $x*y=\{0\}$. From the $(HPBCC_41)$ condition, since

$$(x\circ z)*(y\circ z)\ll(x*y),\ (x*z)\circ(y*z)\ll(x\circ y),$$

then for all

$$a\in(x\circ z)*(y\circ z),$$

$a\ll 0$ and then for all

$$b\in(x*z)\circ(y*z),$$

$b\ll 0$ and so, by the help of conditions $(HPBCC_43)$ and $(HPBCC_44)$, we can find $a=0$ and $b=0$. Hence

$$(x\circ z)*(y\circ z)=\{0\},\ (x*z)\circ(y*z)=\{0\}.$$

Then, we can write this,

$$x\circ z\ll y\circ z,\ x*z\ll y*z.$$

(viii) By (iii), $S\ll S$ can be proved.

(ix) Let $S\subseteq T$ and $m\in S$. For $n=m$ we can find $n\in T$. Hence, by (iii), we get $m\ll n$. Therefore we have $S\ll T$.

(x) Let $s\in S$ and $S\ll\{0\}$. Then using $s\ll 0$ and (i) we can find $s=0$. Hence $S=\{0\}$ is satisfied.

(xi) From $(HPBCC_43)$ condition,

$$0\in(x\circ 0)\circ\{y\}=0\in\{x\}\circ\{y\},$$

we can get $x\ll y$. Similarly, using $(HPBCC_43)$, since

$$0\in(x*0)*\{y\}=0\in\{x\}*\{y\},$$

then we can find $x\ll y$.

□

Theorem 3.15 *Let $(H,\circ,*,\ll,0)$ be a hyper pseudo BCK-algebra. Then, $(H,\circ,*,\ll,0)$ is a hyper pseudo BCC_1-algebra and hyper pseudo BCC_3-algebra.*

Proof Using the $(HPBCK1)$, $(HPBCK4)$ conditions hyper pseudo BCC_1-algebra and hyper pseudo BCC_3-algebra are obtained. □

Theorem 3.16 *Let $(H,\circ,*,\ll,0)$ be a hyper pseudo BCC_1-algebra. Then, H is a hyper pseudo BCK-algebra if and only if $(x\circ y)*z=(x*z)\circ y$, for all $x,y,z\in H$.*

Proof Every hyper pseudo BCC_1-algebra supplies this identity. Conversely, using $(HPBCC_1 1)$, we have $(HPBCK1)$ and using $(HPBCC_1 4)$, we get $(HPBCK4)$. Next in a hyper pseudo BCC_1-algebra satisfying this identity, for all $x, y \in H$, we get using Proposition 3.14 (iv);

$$x \circ y \ll \{x\} \Leftrightarrow x * y \ll \{x\}.$$

Then, we have the $(HPBCK_1 3)$ condition. Hence, H is a hyper pseudo BCK-algebra. □

Example 3.17 *Let* $(H, \circ, *, \ll, 0)$ *given in Example 3.3 be a hyper pseudo* BCC_1*-algebra. We can find*

$$(n \circ m) * n \neq (n * n) \circ m$$

for $m, n \in H$. *Hence,* H *is not hyper pseudo BCK-algebra.*

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Didem Sürgevil Uzay]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%60).

Author [Alev Fırat]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%40).

Conflicts of Interest

The authors declare no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1146523
(**2023**) **4**(2) 66 – 76
Research Article

Certain Weighted Fractional Integral Inequalities for Convex Functions

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Received: 21 July 2022 **Accepted:** 02 May 2023

Abstract: In this study, by using the monotonicity properties of functions, several inequalities for convex functions are obtained with the help of a weighted fractional integral operator which provides a function f to be integrated in fractional order with respect to another function. It is also seen that the results obtained were generalizations of the previous results presented.

Keywords: Convex functions, weighted fractional operators, fractional integral inequality.

1. Introduction

Fractional calculus plays an important role in the field of inequality theory with its rich content and new fractional operators have been added day by day, especially in recent years. Some of these operators have certain algebraic properties such as semigroup property while some do not. Also, some of them have a singularity problem at some points while some of them do not. Therefore, the application areas of the operators can also differ. Convex analysis has become one of the important application areas of fractional analysis [1–3].

In addition, severel mathematicians have studied certain inequalities for convex functions using different type (for example; R-L fractional integral operator, tempered fractional integral operators, generalized proportional integral operators, generalized proportional Hadamard integral operators) of integral operators. These studies have helped to develop different aspects of operator analysis [9–12].

At first, we recall the elementary notation in convex analysis:

Definition 1.1 *A set $\digamma \subset \mathbb{R}$ is said to be convex if*

$$\varphi a + (1-\varphi)b \in \digamma$$

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2020 *AMS Mathematics Subject Classification*: 26A15, 26A51, 26D10

for each $a,b \in \digamma$ *and* $\varphi \in [0,1]$.

Definition 1.2 *The mapping* $f_1 : \digamma \to \mathbb{R}$, *is said to be convex if the following inequality holds:*

$$f_1(\varphi a + (1-\varphi)b) \leq \varphi f_1(a) + (1-\varphi) f_1(b)$$

for all $a,b \in \digamma$ *and* $\varphi \in [0,1]$. *We say that* f_1 *is concave if* $(-f_1)$ *is convex.*

The properties and definitions of the convex functions have recently ascribed a significant role to its theory and practice in the field of fractional integral operators.

In [7], Ngo et al. established the following inequalities:

$$\int_0^1 g_1^{\zeta+1}(\rho)d\rho \geq \int_0^1 \rho^{\zeta} g_1^{\zeta}(\rho)d\rho$$

and

$$\int_0^1 g_1^{\zeta+1}(\rho)d\rho \geq \int_0^1 \rho g_1^{\zeta}(\rho)d\rho,$$

where $\zeta > 0$ and the positive continuous function g_1 on $[0,1]$ such that

$$\int_x^1 g_1(\rho)d\rho \geq \int_x^1 \rho d\rho, \quad x \in [0,1].$$

Then, in [8], Liu et al. established the following inequalities:

$$\int_a^b g_1^{\zeta+\vartheta}(\rho)d\rho \geq \int_a^b (\rho-a)^{\zeta} g_1^{\vartheta}(\rho)d\rho,$$

where $\zeta > 0$, $\vartheta > 0$, and the positive continuous g_1 on $[a,b]$ is such that

$$\int_a^b g_1^{\xi}(\rho)d\rho \geq \int_0^1 (\rho-a)^{\xi} d\rho, \quad \xi = \min(1,\vartheta), \ \rho \in [0,1].$$

The following two theorems are obtained by Liu in [1]:

Theorem 1.3 *Let* $\hbar_1$ *and* $\hbar_2$ *be continuous and positive functions with* $\hbar_1 \leq \hbar_2$ *on* $[a,b]$ *such that* $\hbar_1$ *is increasing and* $\frac{\hbar_1}{\hbar_2}$ *(*$\hbar_2 \neq 0$*) is decreasing. If* ϕ *is a convex function, then the inequality*

$$\frac{\int_a^b \hbar_1(t)dt}{\int_a^b \hbar_2(t)dt} \geq \frac{\int_a^b \phi(\hbar_1(t))\,dt}{\int_a^b \phi(\hbar_2(t))\,dt}$$

holds, where $\phi(0) = 0$.

Theorem 1.4 *Let $\hbar_1$, $\hbar_2$ and $\hbar_3$ be continuous and positive functions with $\hbar_1 \leq \hbar_2$ on $[a,b]$ such that $\hbar_1$ and $\hbar_3$ are increasing and $\frac{\hbar_1}{\hbar_2}$ ($\hbar_2 \neq 0$) is decreasing. If ϕ is a convex function, then the inequality*

$$\frac{\int_a^b \hbar_1(t)dt}{\int_a^b \hbar_2(t)dt} \geq \frac{\int_a^b \phi(\hbar_1(t))\hbar_3(t)dt}{\int_a^b \phi(\hbar_2(t))\hbar_3(t)dt}$$

holds, where $\phi(0) = 0$.

Now some fractional integral operators used to obtain integral inequalities will be given. First of them is Riemann-Liouville fractional integral operators (see [6]) which is widely used in fractional calculus.

Definition 1.5 *Let $\hbar \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}\hbar$ and $J_{b-}^{\alpha}\hbar$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}\hbar(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\hbar(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}\hbar(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\hbar(t)dt, \quad x < b$$

where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$, respectively. Here is $J_{a+}^{0}\hbar(x) = J_{b-}^{0}\hbar(x) = \hbar(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Definition 1.6 *Let $(a,b) \subseteq \mathbb{R}$ and $\sigma(x)$ be an increasing positive and monotonic function on the interval $(a,b]$ with a continuous derivative $\sigma'(x)$ on the interval (a,b) with $\sigma(0) = 0$, $0 \in [a,b]$. Then, the left-side and right-side of the weighted fractional integrals of a function $\hbar$ with respect to another function $\sigma(x)$ on $[a,b]$ are defined by [3]*

$$\begin{aligned} \left({}_{a+}\Im_w^{\ell:\sigma}\hbar\right)(x) &= \frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar(t)w(t)dt, \quad (1)\\ \left({}_{w}\Im_{b-}^{\ell:\sigma}\hbar\right)(x) &= \frac{w^{-1}(x)}{\Gamma(\ell)}\int_x^b \sigma'(t)\left[\sigma(t)-\sigma(x)\right]^{\ell-1}\hbar(t)w(t)dt, \quad \ell > 0 \end{aligned}$$

where $w^{-1}(x) = \frac{1}{w(x)}$, $w(x) \neq 0$ $(w(x) > 0)$.

Remark 1.7 *In Definition 1.6,*

- *To obtain Riemann-Liouville fractional integral operator, one can choose $w(x) = 1$ and $\sigma(x) = x$ in definition of the weighted fractional integral operators (1).*

- *To obtain the following version of fractional integral operator which is defined in [4, 5], one can choose* $w(x) = 1$ *in (1):*

$$\left({}_{a+}\Im^{\ell:\sigma}\hbar\right)(x) = \frac{1}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar(t)dt,$$

$$\left(\Im_{b-}^{\ell:\sigma}\hbar\right)(x) = \frac{1}{\Gamma(\ell)}\int_x^b \sigma'(t)\left[\sigma(t)-\sigma(x)\right]^{\ell-1}\hbar(t)dt, \quad \ell>0.$$

2. Main Results

In this section, inequalities for convex functions by utilizing weighted fractional operators presented.

Theorem 2.1 *Let* $\hbar_1$ *and* $\hbar_2$ *be two positive continuous functions on the interval* $[a,b]$ *and* $\hbar_1 \leq \hbar_2$ *on* $[a,b]$*. If* $\frac{\hbar_1}{\hbar_2}$ *is decreasing and* $\hbar_1$ *is increasing on* $[a,b]$*, then for a convex function* ϕ *with* $\phi(0)=0$*, the weighted fractional operator given by (1) satisfies the following inequality*

$$\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)} \geq \frac{\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_2\right)(x)}, \tag{2}$$

where $x>a>0,\ \ell\in\mathbb{C},\ Re(\ell)>0.$

Proof $\frac{\phi(x)}{x}$ is increasing since ϕ is defined as convex function satisfying $\phi(0)=0$. Besides the function $\frac{\phi(\hbar_1(x))}{\hbar_1(x)}$ is also increasing as $\hbar_1$ is increasing. Obviously, the function $\frac{\hbar_1(x)}{\hbar_2(x)}$ is decreasing.

Thus, for all $[a,x],\ a<x\leq b$, it can be written $\varphi \leq t$

$$\left(\frac{\phi(\hbar_1(t))}{\hbar_1(t)} - \frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\right)\left(\frac{\hbar_1(\varphi)}{\hbar_2(\varphi)} - \frac{\hbar_1(t)}{\hbar_2(t)}\right) \geq 0.$$

It follows that

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\frac{\hbar_1(\varphi)}{\hbar_2(\varphi)} + \frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\frac{\hbar_1(t)}{\hbar_2(t)} - \frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\frac{\hbar_1(\varphi)}{\hbar_2(\varphi)} - \frac{\phi(\hbar_1(t))}{\hbar_1(t)}\frac{\hbar_1(t)}{\hbar_2(t)} \geq 0. \tag{3}$$

Multiplying (3) by $\hbar_2(t)\hbar_2(\varphi)$, we have

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_1(\varphi)\hbar_2(t) + \frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(t)\hbar_2(\varphi) - \frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(\varphi)\hbar_2(t) - \frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_1(t)\hbar_2(\varphi) \geq 0. \tag{4}$$

Now, multiplying both sides of (4) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}w(t)$ and then integrating with respect to the variable t from a to x, we have

$$\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_1(\varphi)\hbar_2(t)w(t)dt$$

$$+\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(t)\hbar_2(\varphi)w(t)dt$$

$$-\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(\varphi)\hbar_2(t)w(t)dt$$

$$-\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_1(t)\hbar_2(\varphi)w(t)dt\geq 0.$$

Then, it follows that

$$\hbar_1(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)+\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_2(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)$$

$$-\frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_1(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)-\hbar_2(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_1\right)(x)\geq 0. \tag{5}$$

Again, multiplying both sides of (5) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(\varphi)\left[\sigma(x)-\sigma(\varphi)\right]^{\ell-1}w(\varphi)$ and then integrating with respect to φ from a to x, we obtain

$$\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x) \tag{6}$$

$$\geq\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x).$$

It follows that

$$\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)}\geq\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)}. \tag{7}$$

Now, since $\frac{\phi(x)}{x}$ is an increasing function and $\hbar_1\leq\hbar_2$ on $[a,b]$, we get

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\leq\frac{\phi(\hbar_2(t))}{\hbar_2(t)} \tag{8}$$

for $t\in[a,x]$.

Multiplying both sides of (8) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar_2(t)w(t)$ and then integrating with respect to the variable t from a to x, we have

$$\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_2(t)w(t)dt$$

$$\leq\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_2(t))}{\hbar_2(t)}\hbar_2(t)w(t)dt,$$

which yields

$$\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)\le\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_2\right)(x).\tag{9}$$

Hence from (7) and (9), we have (2). □

Remark 2.2 *In Theorem 2.1, if we choose $w(x)=1$ and $\sigma(x)=x$, then we obtain Theorem 3.1 in [9].*

Remark 2.3 *In Theorem 2.1, if we choose $w(x)=1=\ell$, $\sigma(x)=x$ and $x=b$, then we obtain Theorem 1.3.*

Theorem 2.4 *Let $\hbar_1$ and $\hbar_2$ be two positive continuous functions and $\hbar_1\le\hbar_2$ on $[a,b]$. If $\frac{\hbar_1}{\hbar_2}$ is decreasing and $\hbar_1$ is increasing on $[a,b]$, then for a convex function ϕ with $\phi(0)=0$, the weighted fractional operator given by (1) satisfies the following inequality*

$$\frac{\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\phi\circ\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\rho:\sigma}\phi\circ\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x)}\ge 1,$$

where $x>a>0$, $\ell,\rho\in\mathbb{C}$, $Re(\ell)>0$ and $Re(\rho)>0$.

Proof $\frac{\phi(x)}{x}$ is increasing since ϕ is defined as convex function satisfying $\phi(0)=0$. Besides the function $\frac{\phi(\hbar_1(x))}{\hbar_1(x)}$ is also increasing as $\hbar_1$ is increasing. Obviously, the function $\frac{\hbar_1(x)}{\hbar_2(x)}$ is decreasing for all $[a,x]$, $a<x\le b$. Multiplying both sides of (5) by $\frac{w^{-1}(x)}{\Gamma(\rho)}\sigma'(\varphi)\left[\sigma(x)-\sigma(\varphi)\right]^{\rho-1}w(\varphi)$ and then integrating the resulting identity from a to x, we obtain

$$\begin{aligned}&\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)\\ \ge\;&\left({}_{a+}\Im_w^{\rho:\sigma}\phi\circ\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\phi\circ\hbar_1\right)(x).\end{aligned}\tag{10}$$

Similar to the (9) inequality, multiplying both sides of (8) by

$$\frac{w^{-1}(x)}{\Gamma(\rho)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\rho-1}\hbar_2(t)w(t)$$

and then integrating with respect to the variable t from a to x, we have

$$\left({}_{a+}\Im_w^{\rho:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\right)(x)\le\left({}_{a+}\Im_w^{\rho:\sigma}\phi\circ\hbar_2\right)(x).\tag{11}$$

Hence, from (9), (11) and (10), we have the needful result. □

Remark 2.5 *If we choose $\ell = \rho$, then Theorem 2.4 will lead to Theorem 2.1.*

Remark 2.6 *In Theorem 2.4, if we choose $w(x) = 1$ and $\sigma(x) = x$, then we obtain Theorem 3.3 in [9].*

Remark 2.7 *In Theorem 2.4, if we choose $w(x) = 1 = \ell = \rho$, $\sigma(x) = x$ and $x = b$, then we obtain Theorem 1.3.*

Theorem 2.8 *Let $\hbar_1$, $\hbar_2$ and $\hbar_3$ be positive continuous functions and $\hbar_1 \leq \hbar_2$ on $[a,b]$. If $\frac{\hbar_1}{\hbar_2}$ is decreasing and $\hbar_1$ and $\hbar_3$ are increasing on $[a,b]$, then for a convex function ϕ with $\phi(0) = 0$, then the following inequality holds for the weighted fractional operator (1)*

$$\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)} \geq \frac{\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)},$$

where $x > a > 0$, $\ell \in \mathbb{C}$, $Re(\ell) > 0$.

Proof Since $\hbar_1 \leq \hbar_2$ on $[a,b]$ and $\frac{\phi(x)}{x}$ is increasing for $t,\varphi \in [a,x]$, $a < x \leq b$, we get

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)} \leq \frac{\phi(\hbar_2(t))}{\hbar_2(t)}. \tag{12}$$

Multiplying both sides of (12) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar_2(t)\hbar_3(t)w(t)$ and then integrating with respect to the variable t from a to x, we have

$$\begin{aligned} &\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_2(t)\hbar_3(t)w(t)dt \\ \leq\; &\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_2(t))}{\hbar_2(t)}\hbar_2(t)\hbar_3(t)w(t)dt \end{aligned}$$

which, in view of (1), can be written as

$$\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x) \leq \left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x). \tag{13}$$

Also, since the function ϕ is convex and such that $\phi(0) = 0$, $\frac{\phi(t)}{t}$ is increasing. Since $\hbar_1$ is increasing, so is $\frac{\phi(\hbar_1(t))}{\hbar_1(t)}$. Clearly, the function $\frac{\hbar_1(t)}{\hbar_2(t)}$ is decreasing for $t,\varphi \in [a,x]$, $a < x \leq b$. Thus

$$\left(\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\hbar_3(t) - \frac{\phi(\hbar_1(\varphi))}{\hbar_1(\varphi)}\hbar_3(\varphi)\right)\left(\hbar_1(\varphi)\hbar_2(t) - \hbar_1(t)\hbar_2(\varphi)\right) \geq 0.$$

It becomes

$$\frac{\phi(\hbar_1(t))\hbar_3(t)}{\hbar_1(t)}\hbar_1(\varphi)\hbar_2(t) + \frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(t)\hbar_2(\varphi)$$

$$-\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(\varphi)\hbar_2(t)-\frac{\phi(\hbar_1(t))\hbar_3(t)}{\hbar_1(t)}\hbar_1(t)\hbar_2(\varphi)\geq 0. \tag{14}$$

Multiplying both sides of (14) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}w(t)$ and then integrating with respect to the variable t from a to x, we obtain

$$\begin{aligned}
&\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))\hbar_3(t)}{\hbar_1(t)}\hbar_1(\varphi)\hbar_2(t)w(t)dt\\
&+\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(t)\hbar_2(\varphi)w(t)dt\\
&-\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(\varphi)\hbar_2(t)w(t)dt\\
&-\frac{w^{-1}(x)}{\Gamma(\ell)}\int_a^x \sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\frac{\phi(\hbar_1(t))\hbar_3(t)}{\hbar_1(t)}\hbar_1(t)\hbar_2(\varphi)w(t)dt\geq 0.
\end{aligned}$$

This follows that

$$\begin{aligned}
&\hbar_1(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)+\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_2(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)\\
&-\frac{\phi(\hbar_1(\varphi))\hbar_3(\varphi)}{\hbar_1(\varphi)}\hbar_1(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)-\hbar_2(\varphi)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)\geq 0. \quad (15)
\end{aligned}$$

Again, multiplying both sides of (15) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(\varphi)\left[\sigma(x)-\sigma(\varphi)\right]^{\ell-1}w(\varphi)$ and then integrating with respect to the variable φ from a to x, we have

$$\begin{aligned}
&\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)\\
\geq\ &\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x).
\end{aligned}$$

Therefore, we can write

$$\frac{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)}\geq\frac{\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)}. \tag{16}$$

Hence, from (13) and (16), we obtain the required result. □

Remark 2.9 *In Theorem 2.8, if we choose $w(x)=1$ and $\sigma(x)=x$, then we obtain Theorem 3.5 in [9].*

Remark 2.10 *In Theorem 2.8, if we choose $w(x)=1=\ell$, $\sigma(x)=x$ and $x=b$, then we obtain Theorem 1.4.*

Theorem 2.11 *Let $\hbar_1$, $\hbar_2$ and $\hbar_3$ be positive continuous functions and $\hbar_1 \leq \hbar_2$ on $[a,b]$. If $\frac{\hbar_1}{\hbar_2}$ is decreasing and $\hbar_1$ and $\hbar_3$ are increasing on $[a,b]$, then for a convex function ϕ with $\phi(0)=0$ then the following inequality holds for the weighted fractional operator (1)*

$$\frac{\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)}\geq 1,$$

where $x>a>0$, $\ell,\rho\in\mathbb{C}$, $Re(\ell)>0$ and $Re(\rho)>0$.

Proof By the assumption of Theorem 2.11, multiplying both sides of (15) by

$$\tfrac{w^{-1}(x)}{\Gamma(\rho)}\sigma'(\varphi)\left[\sigma(x)-\sigma(\varphi)\right]^{\rho-1}w(\varphi)$$

and then integrating with respect to the variable φ from a to x, we have

$$\begin{aligned}&\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x) \qquad (17)\\ \geq\ &\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x).\end{aligned}$$

Since $\hbar_1\leq\hbar_2$ on $[a,b]$ and $\frac{\phi(x)}{x}$ is increasing for $t,\varphi\in[a,x]$, $a<x\leq b$, we get

$$\frac{\phi(\hbar_1(t))}{\hbar_1(t)}\leq\frac{\phi(\hbar_2(t))}{\hbar_2(t)}. \qquad (18)$$

Multiplying both sides of (18) by $\frac{w^{-1}(x)}{\Gamma(\ell)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\ell-1}\hbar_2(t)\hbar_3(t)w(t)$ and then integrating with respect to the variable t from a to x, we have

$$\left({}_{a+}\Im_w^{\ell:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)\leq\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x). \qquad (19)$$

Similarly, multiplying both sides of (18) by $\frac{w^{-1}(x)}{\Gamma(\rho)}\sigma'(t)\left[\sigma(x)-\sigma(t)\right]^{\rho-1}\hbar_2(t)\hbar_3(t)w(t)$ and then integrating with respect to the variable t from a to x, we can write

$$\left({}_{a+}\Im_w^{\rho:\sigma}\frac{\phi\circ\hbar_1}{\hbar_1}\hbar_2\hbar_3\right)(x)\leq\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x). \qquad (20)$$

So, from (17), (19) and (20) we have

$$\frac{\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_1\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_2)\hbar_3\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_1\right)(x)}{\left({}_{a+}\Im_w^{\ell:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\rho:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)+\left({}_{a+}\Im_w^{\rho:\sigma}\hbar_2\right)(x)\left({}_{a+}\Im_w^{\ell:\sigma}(\phi\circ\hbar_1)\hbar_3\right)(x)}\geq 1.$$

□

Remark 2.12 *If we choose $\ell = \rho$, then Theorem 2.11 will lead to Theorem 2.8.*

Remark 2.13 *In Theorem 2.11, if we choose $w(x) = 1$ and $\sigma(x) = x$, then we obtain Theorem 3.7 in [9].*

3. Conclusion

In this paper, first we gave different definitions of fractional integral operators and then we introduced some inequalities using the monotonicity properties of functions for weighted fractional operators. The obtained results are an extension of some known results in the literature. Especially, we would like to emphasize that different types of integral inequalities can be obtained using this operators.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Çetin Yıldız]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).

Author [Mustafa Gürbüz]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1176710
(**2023**) **4**(2) 77 – 86
Research Article

A New Characterization of Tzitzeica Curves in Euclidean 4-Space

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Received: 17 September 2022 **Accepted:** 10 July 2023

Abstract: In this study, we are interested in Tzitzeica curves (Tz-curves) in Euclidean 4-space $\mathbb{E}^4$. Tz-curve condition for Euclidean 4-space are determined as three types for three hyperplanes and some examples are given.

Keywords: Tzitzeica condition, Tzitzeica curve, hyperplane, Frenet frame.

1. Introduction

Gheorgha Tzitzeica, Romanian mathematician (1872-1939), introduced a class of surfaces [11], nowadays called Tzitzeica surfaces in 1907 and a class of curves [12], called Tzitzeica curves in 1911. A Tzitzeica curve in $\mathbb{E}^3$ is a spatial curve $x = x(s)$ with the Frenet frame $\{T, N_1, N_2\}$ and curvatures $\{k_1, k_2\}$, for which the ratio of its torsion k_2 and the square of the distance d_{osc} from the origin to the osculating plane at an arbitrary point $x(s)$ of the curve is constant, i.e., a Tzitzeica curve in $\mathbb{E}^3$ is a curve satisfying the condition (Tzitzeica condition)

$$\frac{k_2}{{d_{osc}}^2} = a, \tag{1}$$

where $d_{osc} = \langle N_2, x \rangle$ and $a \neq 0$ is a real constant, N_2 is the binormal vector field of x.

A Tzitzeica surface in $\mathbb{E}^3$ is a spatial surface M given with the parametrization $X(u,v)$, for which the ratio of its Gaussian curvature K and the distance d_{tan} from the origin to the tangent plane at any arbitrary point of the surface is constant, i.e., a Tzitzeica surface in $\mathbb{E}^3$ is a surface satisfying the condition (Tzitzeica condition)

$$\frac{K}{{d_{tan}}^4} = a_1 \tag{2}$$

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2020 *AMS Mathematics Subject Classification*: 53A04

for a constant $a_1 \neq 0$. The orthogonal distance from the origin to the tangent plane is defined by

$$d_{tan} = \langle X, N \rangle, \tag{3}$$

where X is the position vector of surface and N is unit normal vector field of the surface.

In [1] the authors gave the connections between Tzitzeica curve and Tzitzeica surface in Minkowski 3-space and the original ones from the Euclidean 3-space. Besides, the asymptotic lines of a Tzitzeica surface with the negative Gaussian curvature are Tzitzeica curves [3]. In [3], the authors determined the elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition in Euclidean space. In [**? ?**], hyperbolic and elliptic cylindrical curves verifying Tzitzeica condition were adapted to Minkowski 3-space, respectively.

Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. Let us denote $T(s) = x^{'}(s)$ and call $T(s)$ a unit tangent vector of x at s. We denote the first Serret-Frenet curvature of x by $k_1(s) = \|x^{''}(s)\|$. If $k_1(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve x at s is given by $T^{'}(s) = k_1(s)N_1(s)$. If $k_2(s) \neq 0$, then the unit second principal normal vector $N_2(s)$ of the curve x at s is given by $N_1{}^{'}(s) + k_1(s)T(s) = k_2(s)N_2(s)$, where k_2 is the second Serret-Frenet curvature of x. $N_2{}^{'}(s) + k_2(s)N_1(s) = k_3(s)N_3(s)$, where k_3 is the third Serret-Frenet curvature of x. Then, we have the Serret-Frenet formulae [5]:

$$\begin{aligned}
T^{'}(s) &= k_1(s)N_1(s), \\
N_1{}^{'}(s) &= -k_1(s)T(s) + k_2(s)N_2(s), \\
N_2{}^{'}(s) &= -k_2(s)N_1(s) + k_3(s)N_3(s), \\
N_3{}^{'}(s) &= -k_3(s)N_2(s).
\end{aligned} \tag{4}$$

If the Serret-Frenet curvatures $k_1(s), k_2(s)$ and $k_3(s)$ of x are constant functions then x is called a screw line or a helix [4]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations, Klein and Lie called them W-curves [8]. If the tangent vector T of the curve x makes a constant angle with a unit vector U of $\mathbb{E}^4$ then this curve is called a general helix (or inclined curve) in $\mathbb{E}^4$ [9].

Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. Position vector of $x = x(s)$ satisfies parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s) + m_3(s)N_3(s), \tag{5}$$

where

$$m_0(s) = \langle x(s), T(s), \rangle \qquad m_1(s) = \langle x(s), N_1(s), \rangle$$
$$m_2(s) = \langle x(s), N_2(s), \rangle \qquad m_3(s) = \langle x(s), N_3(s). \rangle \tag{6}$$

By taking the derivative of (5) with respect to arclength parameter s and using Serret-Frenet equations (4) , we obtain

$$\begin{aligned} T(s) = x'(s) &= {m_0}'(s)T(s) + m_0(s)T'(s) + {m_1}'(s)N_1(s) + m_1(s){N_1}'(s) + {m_2}'(s)N_2(s) \\ &+ m_2(s){N_2}'(s) + {m_3}'(s)N_3(s) + m_3(s){N_3}'(s) \\ &= \left({m_0}'(s) - m_1(s)k_1(s)\right)T(s) + \left(m_0(s)k_1(s) + {m_1}'(s) - m_2(s)k_2(s)\right)N_1(s) \\ &+ \left(m_1(s)k_2(s) + {m_2}'(s) - m_3(s)k_3(s)\right)N_2(s) + (m_2(s)k_3(s) + m_3'(s))N_3(s). \end{aligned}$$

It follows that

$$\begin{aligned} m_0' - k_1 m_1 &= 1, \\ m_1' + k_1 m_0 - k_2 m_2 &= 0, \\ m_2' + k_2 m_1 - k_3 m_3 &= 0, \\ m_3' + k_3 m_2 &= 0. \end{aligned} \tag{7}$$

We consider Tzitzeica curves in Euclidean 4-space $\mathbb{E}^4$ whose position vector $x = x(s)$ satisfies the parametric equation (5). We determine Tz-curve condition for Euclidean 4-space $\mathbb{E}^4$ as three types for three hyperplanes and give some examples. Besides, we express Tzitzeica curve conditions in terms of their curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$.

2. A Characterization of Tzitzeica Curves in Euclidean 4-Space

Definition 2.1 *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. A first type Tzitzeica curve $x = x(s)$, for which the ratio of its second Frenet curvature k_2 and the square of the distance $d_{\{T,N_1,\ N_3\}}$ from the origin to the hyperplane spanned by $\{T, N_1,\ N_3\}$ at an arbitrary point $x(s)$ of the curve is constant, i.e.,*

$$\frac{k_2}{d^2_{\{T,N_1,\ N_3\}}} = a_1, \tag{8}$$

where

$$d_{\{T,N_1,\ N_3\}} = \langle x, N_2 \rangle \tag{9}$$

and $a_1 \neq 0$ is a real constant.

Definition 2.2 *Let $x : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. A second type Tzitzeica curve $x = x(s)$, for which the ratio of its first Frenet curvature k_1 and the square of the distance $d_{\{T,N_2,\ N_3\}}$ from the origin to the hyperplane spanned by $\{T, N_2,\ N_3\}$ at an arbitrary point $x(s)$ of the curve is constant, i.e.,*

$$\frac{k_1}{d^2_{\{T,N_2,\ N_3\}}} = a_2, \tag{10}$$

where

$$d_{\{T,N_2,\ N_3\}} = \langle x, N_1 \rangle \tag{11}$$

and $a_2 \neq 0$ is a real constant.

Definition 2.3 *Let $x : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. A third type Tzitzeica curve $x = x(s)$, for which the ratio of its second Frenet curvature k_3 and the square of the distance $d_{\{T,N_1,\ N_2\}}$ from the origin to the hyperplane spanned by $\{T, N_1,\ N_2\}$ at an arbitrary point $x(s)$ of the curve is constant, i.e.,*

$$\frac{k_3}{d^2_{\{T,N_1,\ N_2\}}} = a_3, \tag{12}$$

where

$$d_{\{T,N_1,\ N_2\}} = \langle x, N_3 \rangle \tag{13}$$

and $a_3 \neq 0$ is a real constant.

Theorem 2.4 *Let $x : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5). x is first type Tzitzeica curve if and only if the equation*

$$k_2' m_2 + 2k_2^2 m_1 - 2k_2 k_3 m_3 = 0 \tag{14}$$

holds.

Proof Let x be the first type Tzitzeica curve. By taking the derivative of (8) with respect to arc length parameter s and using (4) and (6), we get (14). The opposite of the proof is clear. □

Proposition 2.5 *Let $x : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be a unit speed spherical curve in $\mathbb{E}^4$ given with the* *parametrization* (5)*. Then*

$$
\begin{aligned}
m_0 &= 0,\\
m_1 &= \frac{-1}{k_1},\\
m_2 &= \frac{k_1'}{k_2 k_1^2},\\
m_3 &= \frac{k_1''}{k_1^2 k_2 k_3} - \frac{2{k_1'}^2}{k_1^3 k_2 k_3} - \frac{k_1' k_2'}{k_1^2 k_2^2 k_3} - \frac{k_2}{k_1 k_3}
\end{aligned}
\tag{15}
$$

hold.

Proof Let x be a unit speed spherical curve. Then, $\langle x, x\rangle = r^2$. By taking the derivative of this expression, we get

$$\langle x, T\rangle = 0 = m_0. \tag{16}$$

By taking the derivative of (16) and using (4) and (6), we get

$$\langle x, N_1\rangle = \frac{-1}{k_1} = m_1. \tag{17}$$

Again, by taking the derivative of (17) and using (4), (16) and (6), we get

$$\langle x, N_2\rangle = \frac{{k_1}'}{k_2 {k_1}^2} = m_2. \tag{18}$$

Similarly, by taking the derivative of (18) and using (4), (17) and (6), we get

$$\langle x, N_3\rangle = \frac{{k_1}''}{{k_1}^2 k_2 k_3} - \frac{2{k_1}'^2}{{k_1}^3 k_2 k_3} - \frac{{k_1}'{k_2}'}{{k_1}^2 {k_2}^2 k_3} - \frac{k_2}{k_1 k_3} = m_3.$$

□

Theorem 2.6 *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed spherical curve in $\mathbb{E}^4$ given with the parametrization* (5)*. x is first type Tzitzeica curve if and only if the equations*

$$3k_1 k_1' k_2' - 2k_1 k_1'' k_2 + 4{k_1'}^2 k_2 = 0 \tag{19}$$

and $k_2 = c.\left[\left(\frac{-1}{k_1}\right)'\right]^{\frac{2}{3}}$ hold, where c is integral constant.

Proof Let x be a first type Tzitzeica curve. Then, substituing (15) into (14) and arranging the expression, we get (19). From the solution of (19), we get $k_2 = c.\left[\left(\frac{-1}{k_1}\right)'\right]^{\frac{2}{3}}$. The opposite of the proof is clear. □

Corollary 2.7 *Let x be a first type spherical Tzitzeica curve. If k_2 is constant, then we get $k_1 = \frac{c_2}{c_1+s}$.*

Proof If k_2 is constant, equation $k_1 k_1'' - 2{k_1'}^2 = 0$ is obtained from (19). If this equation is solved, then we get $k_1 = \frac{c_2}{c_1+s}$. □

Theorem 2.8 *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5). x is second type Tzitzeica curve if and only if the equation*

$$k_1' m_1 + 2k_1^2 m_0 - 2k_1 k_2 m_2 = 0 \tag{20}$$

holds.

Proof Let x be the second type Tzitzeica curve. By taking the derivative of (10) with respect to arc length parameter s and using (4) and (6), we get (20). The opposite of the proof is clear. □

Proposition 2.9 *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed spherical curve in $\mathbb{E}^4$ given with the parametrization (5). x is second type Tzitzeica curve if and only if $k_1 = c$, where c is a constant.*

Proof Let x be the second type spherical Tzitzeica curve. Substituing (15) into (20), we get $3\frac{k_1'}{k_1} = 0$. Which means that, $k_1 = c$ (constant). □

Theorem 2.10 *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5). x is third type Tzitzeica curve if and only if the equation*

$$k_3' m_3 + 2k_3^2 m_2 = 0 \tag{21}$$

holds.

Proof Let x be the third type Tzitzeica curve. By taking the derivative of (12) with respect to arc length parameter s and using (4) and (6), we get (21). The opposite of the proof is clear. □

Proposition 2.11 *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed spherical curve in $\mathbb{E}^4$ given with the parametrization (5). x is third type Tzitzeica curve if and only if the equation*

$$k_3' \left(k_1'' - 2\frac{{k_1'}^2}{k_1} - \frac{k_1 k_2'}{k_2} - k_1 k_2^2 \right) + 2k_1' k_3^3 = 0 \tag{22}$$

holds.

Proof Let x be third type spherical Tzitzeica curve. Then, substituing (15) into (21) and arranging the expression, we get (22). The opposite of the proof is clear. □

Corollary 2.12 *Let x be third type spherical Tzitzeica curve. If k_1 and k_2 are non-zero constants, then x is a W-curve.*

Example 2.13 *Let $x = x(s)$ be regular W-curve in $\mathbb{E}^4$ given with the parametrization*

$$x(s) = (a\cos(cs), a\sin(cs), b\cos(ds), b\sin(ds)) \tag{23}$$

is a second type and third type Tzitzeica curve, where $0 \leq s \leq 2\pi$, a, b, c, d real constants and $c > 0$, $d > 0$.

Then, x without loss of generality, let x be unit speed curve, i.e., $a^2c^2 + b^2d^2 = 1$. If $c = d$, then x is a circle, otherwise $(c \neq d)$ x is a curve in $\mathbb{E}^4$.

The Frenet curvatures k_1, k_3 and the Frenet vector fields N_1, N_3 of the curve x can be given by

$$k_1 = \sqrt{a^2c^4 + b^2d^4}, \tag{24}$$

$$k_3 = \frac{cd}{\sqrt{a^2c^4 + b^2d^4}}, \tag{25}$$

$$N_1 = \frac{1}{k_1}\left[-ac^2\cos(cs), -ac^2\sin(cs), -bd^2\cos(ds), -bd^2\sin(ds)\right], \tag{26}$$

$$N_3 = \frac{1}{k_1}\left[bd^2\cos(cs), bd^2\sin(cs), -ac^2\cos(ds), -ac^2\sin(ds)\right] \tag{27}$$

[2]. By the use of (23) and (26) at (11), we get

$$d_{\{T, N_2,\ N_3\}} = \frac{-1}{\sqrt{a^2c^4 + b^2d^4}}. \tag{28}$$

Substituting (24) and (28) into (10), we get $a_2 = \left(a^2c^4 + b^2d^4\right)^{\frac{3}{2}}$, which means that a_2 is constant and x is a second type Tzitzeica curve.

Further by the use of (23) and (27) at (13), we obtain

$$d_{\{T, N_1,\ N_2\}} = \frac{ab\left(d^2 - c^2\right)}{\sqrt{a^2c^4 + b^2d^4}}. \tag{29}$$

Substituing (25) and (29) into (12), we get $a_3 = \frac{cd\sqrt{a^2c^4+b^2d^4}}{a^2b^2(d^2-c^2)^2}$, which means that a_3 is constant and x is a third type Tzitzeica curve.

Then, the projection of W-curve with the parametrization (23) on $x_4 = 0$ coordinate hyperplane in $\mathbb{E}^4$ is $x(s) = (\cos(s\sqrt{10}), \sin(s\sqrt{10}), \cos(3s\sqrt{10}))$ if we take $a = 1,\ b = 1,\ c = 1\sqrt{10}, d = 3\sqrt{10}$.

We can plot this W-curve with maple command with (plots):

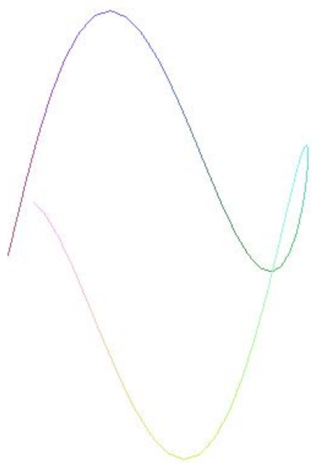

Figure 1: Second type and third type Tzitzeica curves, m=0, n=5*pi

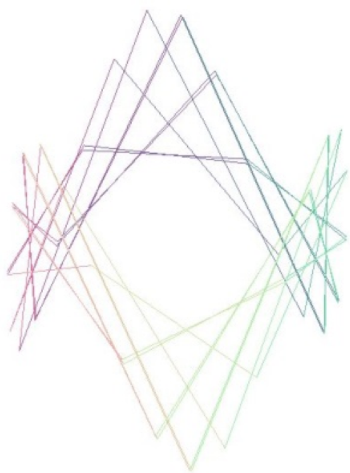

Figure 2: Second type and third type Tzitzeica curves, m=0, n=50*pi

*spacecurve([cos(t/sqrt(10)),sin(t/sqrt(10)),cos(3*t/sqrt(10))], t=m.n, grid=[30,30]*

Example 2.14 *Let $x = x(s)$ be a helix on the unit 3-sphere $S^3(1)$ embedded in $\mathbb{E}^4$ given with the parametrization*

$$x(s) = (\cos\theta\cos(as), \cos\theta\sin(as), \sin\theta\cos(bs), \sin\theta\sin(bs)), \tag{30}$$

where $a^2\cos^2\theta + b^2\sin^2\theta = 1$ and ${x_1}^2 + {x_2}^2 = \cos^2\theta, {x_3}^2 + {x_4}^2 = \sin^2\theta$. Then, x is a second type and third type Tzitzeica curve.
The Frenet curvatures k_1, k_3 and the Frenet vector fields N_1, N_3 of the curve x can be given by

$$k_1 = \sqrt{a^4\cos^2\theta + b^4\sin^2\theta}, \tag{31}$$

$$k_3 = \frac{ab}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}}, \tag{32}$$

$$N_1 = \frac{\left(-a^2\cos\theta\cos(as), -a^2\cos\theta\sin(as), -b^2\sin\theta\cos(bs), -b^2\sin\theta\sin(bs)\right)}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}} \tag{33}$$

$$N_3 = \frac{(b^2\sin\theta\cos(as), b^2\sin\theta\sin(as), -a^2\cos\theta\cos(bs), -a^2\cos\theta\sin(bs))}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}}, \tag{34}$$

[10]. By the use of (30) and (33) at (11), we get

$$d_{\{T,N_2,\ N_3\}} = \frac{-1}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}}. \tag{35}$$

Substituting (31) and (35) into (10), we get $a_2 = \left(\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}\right)^3$, *which means that* a_2 *is constant and* x *is a second type Tzitzeica curve.*

Further, by the use of (30) and (34) at (13), we obtain

$$d_{\{T,N_1,\ N_2\}} = \frac{\cos\theta\sin\theta\left(b^2 - a^2\right)}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}}. \tag{36}$$

Substituting (35) and (36) into (12), we get $a_3 = \frac{ab\left(a^4\cos^2\theta+b^4\sin^2\theta\right)^{\frac{1}{2}}}{\cos^2\theta\sin^2\theta\left(b^2-a^2\right)^2}$, *which means that* a_3 *is constant and* x *is a third type Tzitzeica curve.*

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Emrah Tunç]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).

Author [Bengü Bayram]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1203544
(**2023**) **4**(2) 87 – 95
Research Article

The Source of Semi-Primeness of Γ-Rings

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Received: 13 November 2022 **Accepted:** 22 May 2023

Abstract: The notion of source of semi-primeness is firstly given by Aydın, Demir and Camcı in 2018 as the set of all elements a of R that satisfy $aRa = (0)$ for any associative ring R. They investigated some basic properties of this set and defined three types of rings which have not appeared in literature before. The theory of gamma ring has been introduced by Nobusawa in 1964 as a generalization of rings. In this work, we generalized the notion of source of semi-primeness for gamma rings and investigated its basic algebraic properties. We also defined $|S_M|$-strongly reduced Γ-ring, $|S_M|$-domain, $|S_M|$-division ring and examined the relationship between these structures. We determined all possible characteristic values of a $|S_M|$-domain and proved every finite $|S_M|$-domain Γ-ring M is a $|S_M|$-division Γ-ring.

Keywords: Γ-ring, source of semi-primeness, strong unity.

1. Introduction

The theory of gamma rings has been introduced by Nobusawa as a generalization of rings by defining triple products on two abelian groups [11]. His model was a pair (Γ, M), where M is a subgroup of $\mathrm{Hom}(A, B)$ and Γ is a subgroup of $\mathrm{Hom}(B, A)$ for additive abelian groups A and B and products $M \times \Gamma \times M$ and $\Gamma \times M \times \Gamma$, which are defined as ordinary composition of mappings. W. Barnes dropped the closedness of multiplications in Γ and then defined slightly generalized gamma rings [2]. After Barnes' definition a number of authors have done a lot of works and have obtained various generalizations analogous to the corresponding results in ring theory [3–6, 8, 9].

Prime and semiprime ideals of a Γ-ring M are beneficial to obtain the algebraic structure of M. The notion of a prime ideal was firstly defined by W. Barnes as an ideal P that satisfies $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any ideals A and B of M [2]. Barnes also defined prime ideal and prime radical in this work. He obtained some equivalent conditions that of an ideal to be a prime ideal and introduced prime radical of a Γ-ring M by defining m-system in a manner

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2020 *AMS Mathematics Subject Classification*: 16N60, 16U10, 16Y80

analogous to that of McCoy [10]. Kyuno is also obtained some results on prime ideal, semiprime ideal and prime radical of a Γ-ring M [6].

The source of semi-primeness of a ring R which is denoted by S_R was firstly defined by Aydın et al. in 2018 as the set of all elements a of R satisfying $aRa = (0)$ [1]. They proved some of basic properties of the set S_R. Aydın et al. also defined other new notions which are $|S_R|$-strongly reduced ring, $|S_R|$-domain and $|S_R|$-field and obtained their relations with each other.

Our main interest is to define the source of semi-primeness $S_M(A)$ for any subset A of a Γ-ring M and to introduce some new notions such as $|S_M|$-strongly reduced ring, $|S_M|$-integral domain and $|S_M|$-field to understand the algebraic structure of the Γ-ring M.

2. Preliminaries

Let M and Γ be two additive Abelian groups. M is said to be a Γ-ring (in the sense of Barnes) if there exists ternary multiplication $M \times \Gamma \times M \to M$ satisfying below conditions for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

(1) $(a + b)\,\alpha c = a\alpha c + b\alpha c,$
$a(\alpha + \beta)c = a\alpha c + a\beta c,$
$a\alpha\,(b + c) = a\alpha b + a\alpha c,$

(2) $(a\alpha b)\,\beta c = a\alpha\,(b\beta c).$

Let M be a Γ-ring. If there exist $\delta \in \Gamma$ and $e \in M$ such that $a\delta e = e\delta a = a$ for any $a \in M$, then a pair (δ, e) is called strong unity of the Γ-ring M [9]. A subset N of the Γ-ring M is said to be a subring if N is a subgroup of M and $n\alpha n' \in N$ for all $n, n' \in N$ and $\alpha \in \Gamma$. A subgroup U of M is called left ideal (resp. right ideal) if $M\Gamma U \subseteq U$ (resp. $U\Gamma M \subseteq U$). If U is both left and right ideal, then U is called an ideal of M. An ideal P of the Γ-ring M is said to be prime if $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any ideals A and B of M [2]. An ideal Q of M is said to be semi-prime if $A\Gamma A \subseteq P$ implies $A \subseteq P$ for any ideal A of M [6]. A Γ-ring M is said to be prime (resp. semi-prime) if the zero ideal is prime (resp. semi-prime) [9].

A nonzero element a in M is called zero divisor if there are nonzero elements $b, c \in M$ and $\beta, \gamma \in \Gamma$ such that $a\beta b = 0 = c\gamma a$. An element x of a Γ-ring M is called strongly nilpotent if there exists a positive integer n such that $(x\Gamma)^n x = (x\Gamma x\Gamma \ldots x\Gamma)x = (0)$ [8]. The smallest such n is called the index of x. A Γ-ring M with no nonzero strongly nilpotent elements is called a strongly reduced Γ-ring. A Γ-ring M is said to be a division Γ-ring if it has a strong unity (δ, e) and for each nonzero element a of M there exists b of M such that $a\delta b = b\delta a = e$. The prime radical of a Γ-ring M is the intersection of all prime ideals of M [9]. If there exists a positive integer n such that $nx = 0$ for all $x \in M$, then the smallest such positive integer is called the characteristic of M and denoted by char M. If there is no such positive integer, then M is said to be characteristic zero. Let M_1 be a Γ_1-ring and M_2 be a Γ_2-ring. An ordered pair (θ, φ) is called homomorphism if $\varphi : M_1 \longrightarrow M_2$ is a group homomorphism, $\theta : \Gamma_1 \longrightarrow \Gamma_2$ is a group homomorphism and $\varphi(a\alpha b) = \varphi(a)\theta(\alpha)\varphi(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$ [9]. A subset A of a Γ-ring M is called semi-group ideal if $a\alpha m, m\alpha a \in A$ for all $a \in A$, $\alpha \in \Gamma$ and $m \in M$.

In this study, we introduced the notion of source of semi-primeness $S_M(A)$ as the set of all elements m of M that satisfy $m\Gamma A\Gamma m = (0)$ for any subset A of a Γ-ring M and prove some of its set theoretical properties. For instance, we show that $S_M(A)$ is a semi-group ideal of M and a condition is obtained for $S_M(A)$ to be an ideal of M. Also, the definitions of $|S_M|$-strongly reduced Γ-ring, $|S_M|$-domain and $|S_M|$-division Γ-ring are given and obtained some results about their relations. We determine all possible characteristic values of a $|S_M|$-domain and prove every finite $|S_M|$-domain Γ-ring M is a $|S_M|$-division Γ-ring.

3. Main Results

Definition 3.1 *Let A be a subset of a Γ-ring M. We define the source of semi-primeness of A as the set $S_M(A) = \{m \in M \mid m\Gamma A\Gamma m = (0)\}$. We write S_M instead of $S_M(M)$, when $A = M$.*

From the definition of source of semi-primeness it is clear that $S_A = S_M(A) \cap A$ and $S_M(B) \subseteq S_M(A)$ for any $A \subseteq B$. One can easily show that the source of semiprimeness of a Γ-ring M is equal to zero if and only if M is a semi-prime Γ-ring. Another observation about the source of semiprimeness of a Γ-ring M is that if $S_M = M$, then the Jordan product $(m,n)_{\alpha m'\beta} := m\alpha m'\beta n + n\alpha m'\beta m$ for any elements $m, m', n \in M$ with $\alpha, \beta \in \Gamma$ is equal to zero. Conversely, if the Jordan product for any elements $m, m', n \in M$ with $\alpha, \beta \in \Gamma$ is equal to zero, then S_M may not be equal to M. Indeed, if $M = \left\{\begin{bmatrix} 2\overline{a} & \overline{b} \end{bmatrix} \middle| \overline{a}, \overline{b} \in \mathbb{Z}_{18}\right\}$ and $\Gamma = \left\{\begin{bmatrix} 0 \\ 3\overline{x} \end{bmatrix} \middle| \overline{x} \in \mathbb{Z}_{18}\right\}$, then the equation $(m,n)_{\alpha m'\beta} = 0$ holds for all $m, m', n \in M$ and $\alpha, \beta \in \Gamma$. But, it can be shown that S_M is not equal to M. However, if one assume that the Γ-ring M being 2-torsion free, then converse of the proposition is true. It is also clear that every element in S_M is nilpotent of index at most 3.

We now give the other set-theoretical properties of the source of semi-primeness of a subset for a Γ-ring M.

Proposition 3.2 *Let M_1 and M_2 be two Γ-rings. If A and B are nonempty subsets of M_1 and M_2, respectively, then $S_{M_1 \times M_2}(A \times B) = S_{M_1}(A) \times S_{M_2}(B)$.*

Proof If M_1 and M_2 are two Γ-rings, then $M_1 \times M_2$ is a $\Gamma \times \Gamma$-ring with the ternary multiplication

$$(a,b)(\alpha,\beta)(c,d) = (a\alpha c, b\beta d).$$

Let $(a,b) \in S_{M_1 \times M_2}(A \times B)$. Then, $(a,b)(\alpha,\beta)(x,y)(\gamma,\theta)(a,b) = (0,0)$ for all $(x,y) \in A \times B$ and $(\alpha,\beta),(\gamma,\theta) \in \Gamma \times \Gamma$. Therefore, we get $a\alpha x\gamma a = 0$ and $b\beta y\theta b = 0$ for all $x \in A$, $y \in B$, $\alpha,\beta,\gamma,\theta \in \Gamma$, $a \in M_1$ and $b \in M_2$. Hence, $(a,b) \in S_{M_1}(A) \times S_{M_2}(B)$. Similarly, one can show that $S_{M_1}(A) \times S_{M_2}(B) \subseteq S_{M_1 \times M_2}(A \times B)$. Thus, the equality is obtained. □

Proposition 3.3 *Let M be a Γ-ring and A be an ideal of M. Then, the followings hold:*

(i) The source of semi-primeness of A is a semi-group ideal of M. In particular, it is a multiplicatively closed subset of M.

(ii) If $S_M(A)\Gamma S_M(A) = (0)$, then $S_M(A)$ is an ideal of M.

Proof (i) Let $m \in S_M(A)$, $\alpha \in \Gamma$ and $x \in M$. Then, $(x\alpha m)\Gamma A\Gamma(x\alpha m) = (0)$ since $m\Gamma A\Gamma m = (0)$. It follows that $x\alpha m \in S_M(A)$. Similarly, we have $m\alpha x \in S_M(A)$. Therefore, $S_M(A)$ is a semi-group ideal of M. The last part of the proposition is obvious.

(ii) Let $S_M(A)\Gamma S_M(A) = (0)$. It is enough to show that $S_M(A)$ is additively closed. Let $x,y \in S_M(A)$. Then,

$$(x+y)\Gamma A\Gamma(x+y) = x\Gamma A\Gamma x + x\Gamma A\Gamma y + y\Gamma A\Gamma x + y\Gamma A\Gamma y \subseteq x\Gamma A\Gamma y + y\Gamma A\Gamma x.$$

Since $S_M(A)$ is a semi-group ideal, we have $A\Gamma x \subseteq S_M(A)$ and $x\Gamma A \subseteq S_M(A)$. Therefore, $x\Gamma A\Gamma y + y\Gamma A\Gamma x = (0)$. Thus, $x+y \in S_M(A)$, that is, $S_M(A)$ is an ideal of M. □

Proposition 3.4 *If Q is a semi-prime ideal of a Γ-ring M, then $S_M \subseteq Q$. Moreover, S_M is contained in the prime radical of M.*

Proof Let $a \in S_M$. Since Q is semi-prime and $a\Gamma M\Gamma a = (0) \subseteq Q$, we have $a \in Q$. Therefore, $S_M \subseteq Q$. This also shows that S_M is contained in the prime radical of M. □

Theorem 3.5 *Let M_1 be a Γ_1-ring and M_2 be a Γ_2-ring. If the ordered pair (θ,φ) is a gamma ring homomorphism, then $\varphi(S_{M_1})$ is contained in $S_{\varphi(M_1)}$. Moreover, if φ is injective, then $\varphi(S_{M_1}) = S_{\varphi(M_1)}$.*

Proof Since (θ,φ) is a gamma ring homomorphism, we have $\varphi(M_1)$ is a $\theta(\Gamma_1)$-ring with ternary multiplication

$$\varphi(a)\theta(\alpha)\varphi(b) = \varphi(a\alpha b).$$

Therefore, the source of semi-primeness of $\varphi(M_1)$ is

$$\{\varphi(a) \in \varphi(M_1) \mid \varphi(a)\theta(\Gamma_1)\varphi(M_1)\theta(\Gamma_1)\varphi(a) = (0)\}.$$

Now, it is obvious that the set $\varphi(S_{M_1})$ is contained in $S_{\varphi(M_1)}$. Conversely, let φ be injective and $\varphi(a) \in S_{\varphi(M_1)}$. Then, we have $\varphi(a\Gamma_1 M_1 \Gamma_1 a) = \varphi(0)$. Hence, $a \in S_{M_1}$ since φ is injective. This shows that $S_{\varphi(M_1)} \subseteq \varphi(S_{M_1})$. □

Theorem 3.6 *Let M be a Γ-ring and $a \in S_M$. If $M\Gamma a \neq (0)$ and $a\Gamma M \neq (0)$, then a is a zero divisor. Consequently, an element of M which is a not a zero divisor is contained in $M - S_M$.*

Proof By hypothesis, there exist $b, c \in M$ and $\alpha, \gamma \in \Gamma$ such that $a\alpha b \neq 0 \neq c\gamma a$. Therefore, we get a is a zero divisor since $a\alpha b\delta a = 0 = a\varepsilon c\gamma a$, $a\alpha b \neq 0$ and $c\gamma a \neq 0$. Now assume that b is not a zero divisor of M. Hence, $b \in M - S_M$ since $b\Gamma M \neq (0) \neq M\Gamma b$. Otherwise, b would be a zero divisor. □

4. $|S_M|$-strongly Reduced Γ-ring, $|S_M|$-domain Γ-ring, $|S_M|$-division Γ-ring

Definition 4.1 *Let M be a Γ-ring and $M \neq S_M$.*

(1) *M is said to be a $|S_M|$-strongly reduced ring if there are no strongly nilpotent elements of $M - S_M$.*

(2) *M is said to be a $|S_M|$-domain if there are no left or right zero divisors of $M - S_M$. A $|S_M|$-domain M is called $|S_M|$-integral domain if M is commutative with strong unity.*

(3) *M is said to be a $|S_M|$-division ring if M has a strong unity and every element of $M - S_M$ is unit. A $|S_M|$-division ring M is called $|S_M|$-field if M is commutative.*

It is necessary to assume $M \neq S_M$ in the above definition. For instance, if M is the set of all 2×3 matrices of the form $\begin{bmatrix} \overline{a} & 0 & \overline{a} \\ 0 & \overline{b} & 0 \end{bmatrix}$ with $\overline{a}, \overline{b} \in 4\mathbb{Z}_{16}$ and Γ is the set of all 3×2 matrices of the form $\begin{bmatrix} \overline{x} & 0 \\ 0 & \overline{x} \\ \overline{x} & 0 \end{bmatrix}$ with $\overline{x} \in 4\mathbb{Z}_{16}$, then M is a Γ-ring with $S_M = M$.

From the Definition 4.1, it is clear that if M is a strongly reduced Γ-ring (Γ-domain or Γ-division ring), then M is a $|S_M|$-strongly reduced ring ($|S_M|$-domain or $|S_M|$-division ring). Also, one can show that every $|S_M|$-domain is a $|S_M|$-strongly reduced ring. Conversely, $|S_M|$-strongly reduced rings are not a $|S_M|$-domain in general. For example, if $M = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \end{bmatrix} | a, b, c \in \mathbb{Z} \right\}$ and $\Gamma = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & x \\ x & 0 \end{bmatrix} | x \in \mathbb{Z} \right\}$, then M is a $|S_M|$-strongly reduced Γ-ring but not a $|S_M|$-domain. Similarly, a $|S_M|$-division ring M may not be a $|S_M|$-domain. Let $M = \left\{ \begin{bmatrix} \overline{a} & \overline{a} \end{bmatrix} | \overline{a} \in \mathbb{Z}_p \right\}$ for any prime p

and $\Gamma=\left\{\begin{bmatrix}x\\0\end{bmatrix}|x\in\mathbb{Z}\right\}$. Then, one can show that M is a $|S_M|$-division Γ-ring, but not a $|S_M|$-domain. Another observation on the Definition 4.1 is that if M_1 is a $|S_{M_1}|$-domain and M_2 is a $|S_{M_2}|$-domain, then the direct product $M_1\times M_2$ is $|S_{M_1}\times S_{M_2}|$-strongly reduced ring. It is easy to show that the prime radical of a $|S_M|$-strongly reduced Γ-ring M contains every strongly nilpotent element. By the very nature of the gamma ring, every division gamma ring is not a gamma domain. Similarly, every $|S_M|$-division Γ-ring is not a $|S_M|$-domain. For example, the $\Gamma=\left\{\begin{bmatrix}x\\0\end{bmatrix}|x\in\mathbb{Z}\right\}$-ring $M=\left\{\begin{bmatrix}\overline{a} & \overline{a}\end{bmatrix}|\overline{a}\in\mathbb{Z}_p\right\}$ is a $|S_M|$-division Γ-ring that is not a $|S_M|$-domain for any prime p.

Proposition 4.2 *Let M be a Γ-ring with $M\neq S_M$ and $a\in M$. Then the followings are equivalent:*

(i) M is a $|S_M|$-strongly reduced ring.

(ii) If $a\Gamma a\subseteq S_M$, then $a\in S_M$.

(iii) If $(a\Gamma)^n a\subseteq S_M$ for any positive integer n, then $a\in S_M$.

Proof (i)$\Rightarrow$(ii) Let M be a $|S_M|$-strongly reduced ring and $a\Gamma a\subseteq S_M$. Therefore, we have $(a\Gamma)^4a=(0)$ that is a is a strongly nilpotent element. Hence, $a\in S_M$ since M is a $|S_M|$-strongly reduced ring.

(ii)$\Rightarrow$(iii) Let $a\in M$ and n be the smallest positive integer such that $(a\Gamma)^n a\subseteq S_M$. There exists a positive integer k such that $n\leq 2k\leq n+1$. By Proposition 3.3, we have $(a\Gamma)^{2k+1}a\subseteq S_M$, that is, $(a\Gamma)^k a\subseteq S_M$. If $k=1$, then $a\in S_M$ by (ii). Assume that $k>1$. But, this contradicts with n to be the smallest positive integer since $k\leq n-k+1<n$. Hence, n cannot exceed 2.

(iii)$\Rightarrow$(i) Assume that $a\in M$ is a strongly nilpotent element. Then, there exists a positive integer n such that $(a\Gamma)^n a=(0)$. By hypothesis, we get $a\in S_M$ since $(a\Gamma)^n a\subseteq S_M$. Therefore, there is no strongly nilpotent element in $M-S_M$. So, M is a $|S_M|$-strongly reduced ring. □

Corollary 4.3 *If M is a $|S_M|$-strongly reduced Γ-ring, then $S_M=\left\{a\in M\,\middle|\,(a\Gamma)^2a=(0)\right\}$.*

Proof Let $T=\left\{a\in M\,\middle|\,(a\Gamma)^2a=(0)\right\}$ and $a\in S_M$. Then, clearly $a\in T$. Conversely, assume that $a\in T$. Then, we have $(a\Gamma)^2a=(0)$, that is, a is a strongly nilpotent element. It follows that $a\in S_M$ since M is a $|S_M|$-strongly reduced Γ-ring. Consequently, we get $S_M=T$. □

Proposition 4.4 *Let M be a Γ-ring. If M is a $|S_M|$-domain, then $S_M(A)=S_M$ for any nonzero Γ-subring A of M. Besides, A is a $|S_A|$-domain.*

Proof From the definition of source of semi-primeness, it is clear that $S_M \subseteq S_M(A)$. Assume that there exists an element $m \in S_M(A)$ such that $m \notin S_M$. Then, we get $m\Gamma A = (0) = A\Gamma m$ since $m\Gamma A\Gamma m = (0)$ and M is a $|S_M|$-domain. This implies $A = (0)$, which is a contradiction. Hence, $S_M(A) = S_M$. Now, let $a \in A$ be a zero-divisor. Therefore, $a \in S_M$ since M is a $|S_M|$-domain. This implies $a \in S_M(A) \cap A = S_A$. It follows that A is a $|S_A|$-domain. □

We should note that $S_M(A) = S_A$ may not be provided even if M is a $|S_M|$-domain Γ-ring. For the $\Gamma = \left\{\begin{bmatrix} x & 0 \\ 0 & x \\ x & 0 \end{bmatrix} | x \in \mathbb{Z}\right\}$-ring $M = \left\{\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \end{bmatrix} | a, b, c \in \mathbb{Z}\right\}$, one can show that the M is a $|S_M|$-domain and $S_M(A) \neq S_A$ for the subset $A = \left\{\begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \end{bmatrix} | a, b \in \mathbb{Z}\right\}$ of M.

Proposition 4.4 is not true for a $|S_M|$-strongly reduced Γ-ring M in general. For example, let $M = \left\{\begin{bmatrix} a & 0 & c \\ 0 & b & 0 \end{bmatrix} | a, b, c \in \mathbb{Z}\right\}$ and $\Gamma = \left\{\begin{bmatrix} x & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix} | x \in \mathbb{Z}\right\}$. Then, M is $|S_M|$-strongly reduced Γ-ring since there is no strongly nilpotent element in the set

$$M - S_M = \left\{\begin{bmatrix} a & 0 & c \\ 0 & b & 0 \end{bmatrix} | a, b, c \in \mathbb{Z},\ a \neq 0 \text{ or } b \neq 0\right\}.$$

For the Γ-subring $A = \left\{\begin{bmatrix} a & 0 & c \\ 0 & 0 & 0 \end{bmatrix} | a, c \in \mathbb{Z}\right\}$ of M, we have $S_M(A) = \left\{\begin{bmatrix} 0 & 0 & c \\ 0 & b & 0 \end{bmatrix} | b, c \in \mathbb{Z}\right\}$. Therefore, it is clear that $S_M(A) \neq S_M$.

Proposition 4.5 *If M is a $|S_M|$-strongly reduced Γ-ring and A is a non-zero Γ-subring of M, then A is a $|S_A|$-strongly reduced Γ-ring.*

Proof Let M be a $|S_M|$-strongly reduced Γ-ring and A be a nonzero Γ-subring of M. If $a \in A$ is a strongly nilpotent element, then $a \in S_M$ by hypothesis. This implies that $a \in S_A$ since $S_M \subseteq S_M(A)$. Hence, A is a $|S_A|$-strongly reduced Γ-ring. □

Lemma 4.6 *If M is a $|S_M|$-domain Γ-ring, then $M - S_M$ is a multiplicative set.*

Proof Let M be a $|S_M|$-domain Γ-ring. Assume that $a\alpha b$ is a zero-divisor for $a, b \in M - S_M$ and $\alpha \in \Gamma$. Then, there exist nonzero elements $c \in M - S_M$ and $\gamma \in \Gamma$ such that $(a\alpha b)\gamma c = 0$. Hence, a or b must be zero-divisors which contradicts with our hypothesis. This implies $a\alpha b$ is not a zero divisor, that is, $a\alpha b \in M - S_M$ by Theorem 3.6. Therefore, $M - S_M$ is a multiplicative set. □

Theorem 4.7 *Every finite $|S_M|$-domain Γ-ring M is a $|S_M|$-division ring.*

Proof Assume that M is a $|S_M|$-domain Γ-ring. Let $T = M - S_M = \{a_1, \ldots, a_n\}$ and a be any element of T. Since T is a multiplicative set by Lemma 4.6 and a is not a left (or right) zero divisor, we define injective maps on T such that $f(x) = a\gamma x$ and $g(x) = x\gamma a$ for all $x \in T$. Then, finite cardinality requires the maps to be surjective. Therefore, there exist $1 \leq i \leq n$ and $1 \leq j \leq n$ such that $a\gamma a_i = a = a_j\gamma a$. Since $a\gamma a_i\gamma a = a\gamma a = a\gamma a_j\gamma a$, we get $a_i = a_j$ and so $a\gamma a_i = a = a_i\gamma a$. By the same argument, we have an element $a_i' \in T$ such that $b\gamma a_i' = b = a_i'\gamma b$ for $b \in T$. Accordingly, one has

$$(a\gamma b)\,\gamma a_i' = a\gamma b = a_i\gamma\,(a\gamma b)$$

and since $a\gamma b \in T$, it follows that $a_i' = a_i$. Set $e = a_i$ and $\delta = \gamma$. Then, (δ, e) is a strong unity of the semigroup T and clearly $e\delta e = e$.

For an arbitrary element x of M, we either have $x \in S_M$ or $x \in T$. If $x \in T$, then we already have that $x\delta e = e\delta x = x$. Let $x \in S_M$. Assuming $e - e\delta x \in S_M$ implies that $e = 0$. But, it is a contradiction because $e \in T$. Thus, $e - e\delta x \in T$ and similarly we have $e - x\delta e \in T$. Then,

$$(e - e\delta x)\,\delta e = e - e\delta x \qquad \text{and} \qquad e\delta\,(e - x\delta e) = e - x\delta e$$

yields us that $e\delta x = x\delta e$. Therefore, we have $x\delta e = x = e\delta x$ since e is not a zero-divisor.

Consequently, (δ, e) is a strong unity of Γ-ring M. Moreover, considering the maps f and g, there exist $x, y \in T$ such that $a\delta x = e = y\delta a$. This shows that a is a unit in M. Hence, M is a $|S_M|$-division ring. □

Corollary 4.8 *If M is a finite $|S_M|$-integral domain, then it is $|S_M|$-field.*

Theorem 4.9 *Let M be a Γ-ring with strong unity (δ, e). If M is a $|S_M|$-domain, then the characteristic of M is either 0, or p for a prime p, or p^2 for a prime p.*

Proof Assume that $\text{char}M = n > 1$ and p is a prime dividing n. Then, there exists an integer k such that $n = pk$. Hence, $0 = ne = (pe)\,\delta\,(ke)$. This implies that pe is a zero-divisor, that is, $pe \in S_M$. Therefore, we have $(pe)\,\delta m\delta\,(pe) = 0$ for all $m \in M$. It follows that $p^2 m = 0$ for all $m \in M$. Accordingly, we get $n = p$ or $n = p^2$ since $\text{char}M = n$. □

Theorem 4.10 *Let M be a Γ-ring with strong unity (δ, e). If M is a $|S_M|$-strongly reduced ring, then the characteristic of M is a cube-free integer, that is, there is no prime p such that p^3 divides $\text{char}M$.*

Proof Assume that $\text{char}M = n > 1$ and p is a prime dividing n, say $n = p^t k$ for some $t \geq 1$ and

$1 \le k < n$ with $gcd(p,k) = 1$. Since

$$(pke)^t = p^t k^t e = k^{t-1}(ne) = 0 \Rightarrow pke \in S_M$$

$$\Rightarrow (pke)\delta m \delta (pke) = 0, \forall m \in M \;\Rightarrow p^2 k^2 m = 0, \forall m \in M$$

and char$M = n$, there exits $s \in \mathbb{Z}$ such that $p^t ks = p^2 k^2$. If t were greater than or equal to 3, then we get $p \mid k$. But, this contradicts with $gcd(p,k) = 1$. Hence, n must be a cube-free integer. □

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Okan Arslan]: Thought and designed the research/problem, contributed to research method or evaluation of data, collected the data, wrote the manuscript (%70).

Author [Nurcan Düzkaya]: Collected the data, contributed to completing the research and solving the problem (%30).

Conflicts of Interest

The authors declare no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1231857
(**2023**) **4**(2) 96 – 106
Research Article

Convergence of a Four-Step Iteration Process for $G-$nonexpansive Mappings in Banach Spaces with a Digraph

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Received: 10 January 2023 **Accepted:** 16 June 2023

Abstract: This review reckons with iterative scheme of Thianwan to approximate a common fixed point for four $G-$nonexpansive mappings (tersely $G-nm$). We verify several convergence results for in this way mappings in Banach space by dint of a digraph.

Keywords: Fixed point, digraph, $G-$nonexpansive mappings.

1. Introduction and Preliminaries

Let X be a Banach space, $K \neq \varnothing, K \subseteq X$. *Directed graph* mostly enrolled qua *digraph* is a double: $G = (V(G), E(G))$, that here $V(G)$ is the set of vertices of graph and $E(G)$ is the set of its edges that involves overall the loops, scilicet $(x,x) \in E(G)$ for all $x \in V(G)$. Given that G enjoys no parallel edges. If x, y occur vertices of G, here a *path* in G ranging x from y of length N is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x = x_0$, $y = x_N$ and $(x_{i-1}, x_i) \in E(G)$ for all $i = \overline{1,N}$. Digraph G is alleged to become *transitive* if, for all $x, y, z \in V(G)$ such that (x,y) and (y,z) are in $E(G)$, we acquire $(x,z) \in E(G)$ [2]. A mapping $f : K \to K$ is asserted to become

- *$G-$nonexpansive* (tersely $G-nm$) [3] if it yields (i) $(x,y) \in E(G) \Rightarrow (fx, fy) \in E(G)$ (f preserves edges of G), (ii) $(x,y) \in E(G) \Rightarrow \|fx - fy\| \leq \|x - y\|$;
- *semi-compact* [9] if for $\{x_n\}$ in K with $\|x_n - fx_n\| \to 0$ as $n \to \infty$, there appears a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to f^* \in K$.

The mappings $f_i : K \to K$ are supply condition (A'') [1] if there is a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, $0 < g(t)$ for all $t \in (0,\infty)$ such that $\|x - f_i x\| \geq g(d(x, F_f))$ for all $i = \overline{1,k}$, $x \in K$, where $d(x, F_f) = \inf\left\{\|x - f^*\| : f^* \in F_f = \cap_{c=1}^{k} F(f_c) \neq \varnothing\right\}$.

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2020 *AMS Mathematics Subject Classification*: 47H09, 47H10

Let $x_0 \in V(G)$ and $\Upsilon \subseteq V(G)$. We state that [5], (i) Υ is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in \Upsilon$, (ii) Υ dominates x_0 if for each $x \in \Upsilon$, $(x_0, x) \in E(G)$.

Let G be a digraph such that $V(G) = K$. Then, K is alleged to get property P [8] if for each sequence $\{x_n\}$ in $K \rightharpoonup x \in K$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $(x_{n_l}, x) \in E(G)$ for all $l \in N$.

Remark 1.1 *[6] If G is transitive, then Property P is equal to the speciality: if $\{x_n\} \subseteq K$ with $(x_n, x_{n+1}) \in E(G)$ such that for any subsequence $\{x_{n_l}\}$ of $\{x_n\} \rightharpoonup x \subseteq X$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Phuengrattana and Suantai [15] gave on the rate of convergence of Mann, Ishikawa, Noor and $SP-$iterations for continuous functions on an arbitrary interval. Şahin and Başarır [16] presented on the strong and $\Delta-$convergence of $SP-$iteration on $CAT(0)$ space.

Motivated by [11–13] and above results, the iterative scheme is defined as follows:

$$\begin{aligned} t_n &= (1-\beta_n) x_n + \beta_n f_1 x_n, \\ y_n &= (1-\xi_n) x_n + \xi_n f_2 t_n, \\ s_n &= (1-\varrho_n) y_n + \varrho_n f_3 y_n, \\ x_{n+1} &= (1-\theta_n) x_n + \theta_n f_4 s_n, \ n \geq 1, \end{aligned} \tag{1}$$

where $\{\xi_n\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\varrho_n\} \subseteq [0,1]$, for all $i = \overline{1,4}$, $f_i : K \to K$ are $G-nm$. We verify several convergence results for in this way mappings in Banach space by dint of a digraph.

Lemma 1.2 *[10] Let X be a uniformly convex Banach space. Suggesting that $0 < b \leq \nu_n \leq c < 1$, $n \geq 1$. Let $\{x_n\}, \{y_n\} \subseteq X$ be such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|\nu_n x_n + (1-\nu_n) y_n\| = a$, where $a \geq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

2. Main Results

$F_f = \cap_{c=1}^{4} F(f_c) \neq \varnothing$. For $x_0 \in K$, let $\{x_n\}$ be the sequence created by (1).

Proposition 2.1 *Let $u_0 \in F_f$ be such that (x_0, u_0) and (u_0, x_0) are in $E(G)$. Then, (x_n, u_0), (u_0, x_n), (x_n, s_n), (s_n, x_n), (x_n, y_n), (y_n, x_n), (x_n, t_n), (t_n, x_n), (u_0, s_n), (s_n, u_0), (u_0, y_n), (y_n, u_0), (u_0, t_n), (t_n, u_0), (x_n, x_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$.*

Proof We shall demonstrate our deductions by induction. Let $(x_0, u_0) \in E(G)$. By virtue of edge-preserving of f_1, we have $(f_1 x_0, u_0) \in E(G)$, and thus $(t_0, u_0) \in E(G)$ from the convexity of $E(G)$. Due to edge-preserving of f_2, we get $(f_2 t_0, u_0) \in E(G)$. By using the convexity of $E(G)$

and (x_0, u_0), $(f_2 t_0, u_0) \in E(G)$, we own $(y_0, u_0) \in E(G)$. As f_3 is edge-preserving, we possess $(f_3 y_0, u_0) \in E(G)$ and $(s_0, u_0) \in E(G)$ from the convexity of $E(G)$. Owing to edge-preserving of f_4, $(f_4 s_0, u_0) \in E(G)$. Again the convexity of $E(G)$ and (x_0, u_0), $(f_4 s_0, u_0) \in E(G)$, we acquire $(x_1, u_0) \in E(G)$. Continuing in this fashion for (x_1, u_0) instead of (x_0, u_0), we get (t_1, u_0), (y_1, u_0), (s_1, u_0), $(x_2, u_0) \in E(G)$.

Suppose that $(x_v, u_0) \in E(G)$ for $v \geq 1$. Because of edge-preserving of f_1, we attain $(f_1 x_v, u_0) \in E(G)$, and thus $(t_v, u_0) \in E(G)$ from the convexity of $E(G)$. On account of edge-preserving of f_2, we achieve $(f_2 t_v, u_0) \in E(G)$. Using the convexity of $E(G)$ and (x_v, u_0), $(f_2 t_v, u_0) \in E(G)$, we obtain $(y_v, u_0) \in E(G)$. Because f_3 is edge-preserving, we own $(f_3 y_v, u_0) \in E(G)$ and so $(s_v, u_0) \in E(G)$ from the convexity of $E(G)$. In view of edge-preserving of f_4, $(f_4 s_v, u_0) \in E(G)$. Repetition the convexity of $E(G)$ and (x_v, u_0), $(f_4 s_v, u_0) \in E(G)$, we belong $(x_{v+1}, u_0) \in E(G)$. Repeating the procedure on one occasion for $(x_{v+1}, u_0) \in E(G)$, we get (t_{v+1}, u_0), (y_{v+1}, u_0), (s_{v+1}, u_0), $(x_{v+2}, u_0) \in E(G)$.

Hence, (x_n, u_0), (t_n, u_0), (y_n, u_0), $(s_n, u_0) \in E(G)$ for $n \geq 1$. Utilizing an analog argumentum, we infer that (u_0, x_n), (u_0, t_n), (u_0, y_n), $(u_0, s_n) \in E(G)$ from $(u_0, x_0) \in E(G)$. As the graph G is transitivity, we acquire for $n \geq 1$ (x_n, s_n), (s_n, x_n), (y_n, x_n), (x_n, y_n), (t_n, x_n), (x_n, t_n) and $(x_n, x_{n+1}) \in E(G)$. □

Lemma 2.2 *If K is a nonempty closed convex subset of a real uniformly convex Banach space X, $\{\xi_n\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\varrho_n\} \subseteq [a, b]$, where $0 < a < b < 1$ and (x_0, u_0), $(u_0, x_0) \in E(G)$ for $x_0 \in K$ and $u_0 \in F_f$, then*

(i) $\|x_{n+1} - u_0\| \leq \|x_n - u_0\|$ for $n \geq 1$, and hence $\|x_n - u_0\| \to 0$ as $n \to \infty$;

(ii) $\lim_{n\to\infty} \|x_n - f_i x_n\| = 0$ for all $i = \overline{1, 4}$.

Proof (i) By Proposition 2.1, (x_n, u_0), (u_0, x_n), (s_n, x_n), (x_n, s_n), (y_n, x_n), (x_n, y_n), (x_n, t_n), (t_n, x_n), (u_0, s_n), (s_n, u_0), (u_0, y_n), (y_n, u_0), (u_0, t_n), (t_n, u_0), (x_n, x_{n+1}) are in $E(G)$. It follows from (1) that

$$\begin{aligned} \|t_n - u_0\| &= \|-u_0 + (-\beta_n + 1) x_n + \beta_n f_1 x_n\| \\ &\leq (-\beta_n + 1) \|-u_0 + x_n\| + \beta_n \|f_1 x_n - u_0\| \\ &\leq (-\beta_n + 1) \|-u_0 + x_n\| + \beta_n \|-u_0 + x_n\| \\ &= \|-u_0 + x_n\|. \end{aligned} \tag{2}$$

Using (1) & (2), we have

$$\begin{aligned}\|y_n-u_0\| &\leq (1-\xi_n)\|x_n-u_0\|+\xi_n\|f_2t_n-u_0\| \\ &\leq (1-\xi_n)\|x_n-u_0\|+\xi_n\|t_n-u_0\| \\ &\leq \|x_n-u_0\|. \qquad (3)\end{aligned}$$

Similarly, along with (3), we get

$$\begin{aligned}\|s_n-u_0\| &\leq (1-\varrho_n)\|y_n-u_0\|+\varrho_n\|f_3y_n-u_0\| \\ &\leq (1-\varrho_n)\|y_n-u_0\|+\varrho_n\|y_n-u_0\| \\ &\leq \|y_n-u_0\| \\ &\leq \|x_n-u_0\|. \qquad (4)\end{aligned}$$

By (4), we possess

$$\begin{aligned}\|-u_0+x_{n+1}\| &\leq (-\theta_n+1)\|-u_0+x_n\|+\theta_n\|-u_0+f_4s_n\| \\ &\leq (-\theta_n+1)\|-u_0+x_n\|+\theta_n\|s_n-u_0\| \\ &\leq \|x_n-u_0\|. \qquad (5)\end{aligned}$$

Hence, $\lim_{n\to\infty}\|x_n-u_0\|$ exists.

(ii) By assumption (i), $\{x_n\}$ is bounded. Let

$$\lim_{n\to\infty}\|x_n-u_0\|=M. \qquad (6)$$

If $M=0$, then, by $G-nm$ of $\{f_1,f_2,f_3,f_4\}$, it is obvious. Next, suppose $M>0$. We shall show that, for all $i=\overline{1,4}$, $\|x_n-f_ix_n\|\to 0$ as $n\to\infty$.

Getting lim sup on both parts of (2), (3) & (4), we have

$$\limsup_{n\to\infty}\|t_n-u_0\| \leq M, \qquad (7)$$

$$\limsup_{n\to\infty}\|y_n-u_0\| \leq M, \qquad (8)$$

$$\limsup_{n\to\infty}\|s_n-u_0\| \leq M. \qquad (9)$$

It implies by (7), (8) & (9) and the $G-nm$ of $\{f_1,f_2,f_3,f_4\}$ that

$$\begin{aligned}\|f_1x_n-u_0\| &\leq \|x_n-u_0\| \\ \limsup_{n\to\infty}\|f_1x_n-u_0\| &\leq M, \qquad (10)\end{aligned}$$

$$\begin{aligned} \|f_2t_n - u_0\| &\leq \|t_n - u_0\| \\ \limsup_{n\to\infty} \|f_2t_n - u_0\| &\leq M, \end{aligned} \tag{11}$$

$$\begin{aligned} \|f_3y_n - u_0\| &\leq \|y_n - u_0\| \\ \limsup_{n\to\infty} \|f_3y_n - u_0\| &\leq M, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \|f_4s_n - u_0\| &\leq \|s_n - u_0\| \\ \limsup_{n\to\infty} \|f_4s_n - u_0\| &\leq M. \end{aligned} \tag{13}$$

Since $\lim_{n\to\infty} \|x_{n+1} - u_0\| = M$, we get

$$\lim_{n\to\infty} \|(1-\theta_n)(x_n - u_0) + \theta_n (f_4s_n - u_0)\| = M. \tag{14}$$

By Lemma 1.2, we obtain

$$\|x_n - f_4s_n\| \to 0 \text{ as } n \to \infty. \tag{15}$$

Now, using the $G-nm$ of $\{f_1, f_2, f_3, f_4\}$, we have

$$\begin{aligned} \|-u_0 + x_n\| &\leq \|f_4s_n - u_0\| + \|-f_4s_n + x_n\| \\ &\leq \|x_n - f_4s_n\| + \|s_n - u_0\| && (16) \\ &\leq \|x_n - f_4s_n\| + \|(1-\varrho_n)(y_n - u_0) + \varrho_n (f_3y_n - u_0)\| \\ &\leq \|x_n - f_4s_n\| + (1-\varrho_n)\|y_n - u_0\| + \varrho_n \|f_3y_n - u_0\| \\ &\leq \|x_n - f_4s_n\| + \|y_n - u_0\| && (17) \\ &\leq \|x_n - f_4s_n\| + \|(1-\xi_n)(x_n - u_0) + \xi_n (f_2t_n - u_0)\| \\ &\leq \|x_n - f_4s_n\| + (1-\xi_n)\|x_n - u_0\| + \xi_n \|f_2t_n - u_0\| \\ &\leq \frac{1}{\xi_n}\|x_n - f_4s_n\| + \|t_n - u_0\| \\ &\leq \frac{1}{a}\|x_n - f_4s_n\| + \|t_n - u_0\|. && (18) \end{aligned}$$

Taking $\liminf$ on both sides of (16), (17), (18) and using (15), we obtain

$$M \leq \liminf_{n\to\infty} \|s_n - u_0\|, \tag{19}$$

$$M \leq \liminf_{n\to\infty} \|y_n - u_0\|, \tag{20}$$

$$M \leq \liminf_{n\to\infty} \|t_n - u_0\|, \tag{21}$$

respectively.

By combining (7) & (21), (8) & (20), (9) & (19), we get

$$\lim_{n\to\infty}\|t_n - u_0\| = \lim_{n\to\infty}\|y_n - u_0\| = \lim_{n\to\infty}\|s_n - u_0\| = M, \tag{22}$$

respectively. Namely,

$$\begin{aligned}
\lim_{n\to\infty}\|(1-\beta_n)(x_n - u_0) + \beta_n(f_1 x_n - u_0)\| &= M,\\
\lim_{n\to\infty}\|(1-\xi_n)(x_n - u_0) + \xi_n(f_2 t_n - u_0)\| &= M,\\
\lim_{n\to\infty}\|(1-\varrho_n)(y_n - u_0) + \varrho_n(f_3 y_n - u_0)\| &= M,
\end{aligned}$$

respectively. It follows from (6), (8), (10), (11) & (12) and Lemma 1.2 that

$$\lim_{n\to\infty}\|x_n - f_1 x_n\| = 0, \tag{23}$$

$$\lim_{n\to\infty}\|x_n - f_2 t_n\| = 0, \tag{24}$$

$$\lim_{n\to\infty}\|y_n - f_3 y_n\| = 0, \text{ resp.} \tag{25}$$

It implies by (23) & (24) that

$$\begin{aligned}
\|x_n - f_2 x_n\| &\leq \|x_n - f_2 t_n\| + \|f_2 t_n - f_2 x_n\|\\
&\leq \|x_n - f_2 t_n\| + \|t_n - x_n\|\\
&\leq \|x_n - f_2 t_n\| + \beta_n\|f_1 x_n - x_n\|\\
&\leq \|x_n - f_2 t_n\| + b\|f_1 x_n - x_n\|\\
&\to 0 \text{ as } n\to\infty.
\end{aligned} \tag{26}$$

By (1) & (24), we have

$$\begin{aligned}
\|x_n - y_n\| &= \|x_n - [(1-\xi_n)x_n + \xi_n f_2 t_n]\|\\
&\leq \xi_n\|x_n - f_2 t_n\|\\
&\leq b\|x_n - f_2 t_n\|\\
&\to 0 \text{ as } n\to\infty.
\end{aligned} \tag{27}$$

It follows from (25) & (27), we get

$$\begin{aligned}
\|x_n - f_3 x_n\| &\leq \|-y_n + x_n\| + \|y_n - f_3 y_n\| + \|f_3 y_n - f_3 x_n\|\\
&\leq \|-y_n + x_n\| + \|y_n - f_3 y_n\|\\
&\quad + \|-x_n + y_n\| \to 0 \text{ as } n\to\infty.
\end{aligned} \tag{28}$$

By (1), (25) & (27), we have

$$\begin{aligned} \|s_n - x_n\| &\leq \|-y_n + s_n\| + \|y_n - x_n\| \\ &= \|[(1-\varrho_n) y_n + \varrho_n f_3 y_n] - y_n\| + \|-x_n + y_n\| \\ &\leq \varrho_n \|y_n - f_3 y_n\| + \|-x_n + y_n\| \\ &\leq b \|y_n - f_3 y_n\| + \|-x_n + y_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{29}$$

Using (15) & (29), we obtain

$$\begin{aligned} \|x_n - f_4 x_n\| &\leq \|x_n - f_4 s_n\| + \|f_4 s_n - f_4 x_n\| \\ &\leq \|x_n - f_4 s_n\| \\ &\quad + \|s_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{30}$$

From (23), (26), (28) & (30), we get

$$\|x_n - f_i x_n\| \to 0 \text{ as } n \to \infty \text{ for all } i = \overline{1,4}. \tag{31}$$

□

Theorem 2.3 *Let K is a nonempty closed convex subset of a real uniformly convex Banach space X and $\{\xi_n\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\varrho_n\} \subseteq [a,b]$, where $0 < a < b < 1$. Let $u_0 \in F_f$ such that (x_0, u_0), (u_0, x_0) are in $E(G)$ for $x_0 \in K$. Supposing that K hold the property P, $\{f_1, f_2, f_3, f_4\}$ satisfy the condition(A''), F_f is dominated by x_0 and F_f dominates x_0, then $\{x_n\} \longrightarrow u_0 \in F_f$.*

Proof Let $u_0 \in F_f$ be such that (x_n, u_0), (u_0, x_n), (s_n, x_n), (x_n, s_n), (x_n, y_n), (y_n, x_n), (x_n, t_n), (t_n, x_n), (u_0, s_n), (s_n, u_0), (u_0, y_n), (y_n, u_0), (u_0, t_n), (t_n, u_0), (x_n, x_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$. Due to Lemma 2.2 (ii) and *condition* (A''), we attain that $\lim_{n\to\infty} g(d(x_n, F_f)) = 0$. As g is nondecreasing with $g(0) = 0$, we hold $d(x_n, F_f) \to 0$ as $n \to \infty$. Thus, we can receive a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ and $\{u_l^*\} \subset F_f$ such that $\|x_{n_l} - u_l^*\| < 2^{-l}$. Due to the fact that strong convergence implies weak convergence and by Remark 1.1, we hold $(x_{n_l}, u_l^*) \in E(G)$. Using the proof method of [11], we own

$$\|x_{n_{l+1}} - u_l^*\| \leq \|x_{n_l} - u_l^*\| < \frac{1}{2^l},$$

and so

$$\|-u_{l+1}^* + u_l^*\| \leq \|-x_{n_{l+1}} + u_l^*\| + \|-u_{l+1}^* + x_{n_{l+1}}\| \leq 3.2^{-(1+l)}.$$

We deduce that $\{u_{l+1}^*\}$ is a Cauchy sequence. Therefore, we have $u_l^* \to r$. By closed of F_f, $r \in F_f$ in that case $x_{n_l} \to r$. Because of Lemma 2.2 (i), $x_n \to r \in F_f$. □

Theorem 2.4 *Let K is a nonempty closed convex subset of a real uniformly convex Banach space X and $\{\xi_n\}$, $\{\theta_n\}$, $\{\beta_n\}$, $\{\varrho_n\} \subseteq [a,b]$, where $0 < a < b < 1$. Let $u_0 \in F_f$ such that (x_0, u_0), (u_0, x_0) are in $E(G)$ for $x_0 \in K$. Supposing that K has the property P and one of $\{f_1, f_2, f_3, f_4\}$ is semi-compact, F_f is dominated by x_0 and F_f dominates x_0, then $\{x_n\} \longrightarrow u_0 \in F_f$.*

Proof Let $u_0 \in F_f$ be such that (x_n, u_0), (u_0, x_n), (x_n, s_n), (s_n, x_n), (x_n, y_n), (y_n, x_n), (x_n, t_n), (t_n, x_n), (u_0, s_n), (s_n, u_0), (u_0, y_n), (y_n, u_0), (u_0, t_n), (t_n, u_0), (x_n, x_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$. We have $\lim_{n\to\infty} \|x_n - f_j x_n\| = 0$ from Lemma 2.2 (ii). Assume that f_j is semi-compact for all $j = \overline{1,4}$. Then, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\lim_{l\to\infty} \|x_{n_l} - \upsilon\| = 0$ for some $\upsilon \in K$. This together with Remark 1.1 implies that $(x_{n_l}, \upsilon) \in E(G)$. It follows from the $G-nm$ of $\{f_1, f_2, f_3, f_4\}$ and Lemma 2.2 (ii) that

$$\begin{aligned} \|\upsilon - f_j\upsilon\| &\leq \|\upsilon - x_{n_l}\| + \|x_{n_l} - f_j x_{n_l}\| + \|f_j x_{n_l} - f_j \upsilon\| \\ &\to 0 \text{ as } l \to \infty, \end{aligned}$$

for all $j = \overline{1,4}$. Hereat, $\upsilon \in F_f$ so that $\lim_{n\to\infty} \|x_n - \upsilon\|$ exists. Thus, $x_n \to \upsilon$ as $n \to \infty$. □

We indicate an instance which is inspired by Example 4.5 in [7].

Example 2.5 $K = [0,2] \subseteq X = \mathbb{R}$. *Let G be a digraph described by $V(G) = K$ and $(x,y) \in E(G)$ iff $1.20 \geq y \geq x \geq 0.50$. Denote $\{f_1, f_2, f_3, f_4\} : K \to K$ by $f_1 x = 1 + \frac{23}{49}\tan(-1+x)$, $f_2 x = 1 + \frac{29}{45}\tan(-1+x)$, $f_3 x = 1 + \frac{23}{49}\arcsin(-1+x)$, $f_4 x = 1 + \frac{29}{45}\arcsin(-1+x)$ for any $x \in K$ and $i = 1,2,3,4$. It is easy to see that f_1, f_2, f_3, f_4 are $G-nm$, but f_1, f_2, f_3, f_4 are not nonexpansive. Let $\beta_n = \frac{6n+5}{8n+15}$, $\xi_n = \frac{3n+1}{9n+20}$, $\varrho_n = \frac{10n+3}{11n+4}$, $\theta_n = \frac{7n+11}{13n+47}$ for $n \geq 1$. $F_f = \cap_{c=1}^4 F(f_c) = \{1\}$ as in Figure 1.*

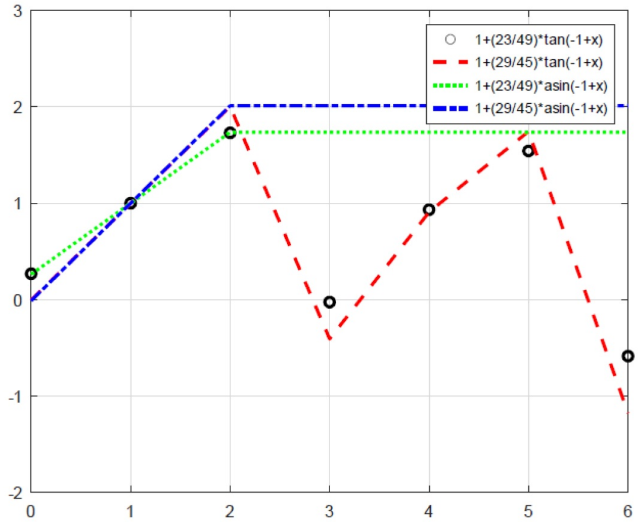

Figure 1: Plot showing $F_f = \cap_{c=1}^{4} F(f_c) = \{1\}$

Table 1 The value of the sequence $\{x_n\}$ *with initial value* $x_0 = 1.20000$, $x_0 = 0.80000$ *and* $n = 20$, *respectively.*

n	x_n	x_n
1	1.20000	0.80000
2	1.15950	0.84047
3	1.12180	0.87822
4	1.09010	0.90994
5	1.06500	0.93499
6	1.04600	0.95395
7	1.03210	0.96788
8	1.02210	0.97787
9	1.01510	0.98492
10	1.01020	0.98981
11	1.00680	0.99317
12	1.00450	0.99545
13	1.00300	0.99699
14	1.00200	0.99802
15	1.00130	0.99870
16	1.00090	0.99915
17	1.00060	0.99945
18	1.00040	0.99964
19	1.00030	0.99977
20	1.00020	0.99985

Remark 2.6 *(i) If* $\xi_n \equiv 0$ *and* $f_1 = f_2 = f_3 = f_4 = f$ *in (1), then Theorem 2.3 generalize the results of Theorem 3.6 in [14] for self-map.*

(ii) If $\xi_n = \varrho_n \equiv 0$ *and* $f_1 = f_2 = f_3 = f_4 = f$ *in (1), we attain convergence of the Mann iteration to some fixed points of* f *on Banach space involving a digraph.*

(iii) If $f_1 = f_2 = f_3 = f_4 = f$ *in (1), then Theorem 2.3 extends the results of [12] without errors for self-map.*

(iv) If $f_1 = f_2$, $f_3 = f_4$ in (1), then Theorem 2.3 improves the results of [13] without errors for self-map.

(v) If $\xi_n \equiv 0$ in (1), then Theorem 2.4 reduces to the results of [4].

3. Conclusion

In this writting, we reckons with four step iteration scheme to common fixed points of four $G-nm$ described on Banach space involving a digraph. Our findings evolve the equal results of Shahzad (2005) [14], Thianwan (2008) [12], Kızıltunç et al. (2010) [13] and Tripak (2016) [4]. Within the future scope of the idea, reader can show that (1) compare convergence rate Picard, Mann, Ishikawa and $SP-$iteration process for contractions.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1246015
(**2023**) **4**(2) 107 – 127
Research Article

Non-lightlike Helices Associated with Helical Curves, Relatively Normal-Slant Helices and Isophote Curves in Minkowski 3-space

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Received: 01 February 2023 **Accepted:** 01 July 2023

Abstract: In this paper, we introduce a new type of non-lightlike general helix that we name non-lightlike associated helix which is associated with a non-lightlike special surface curve. By using the Darboux frame of a surface curve, we generate the position vector of a non-lightlike associated helix in parametric form. We investigate special cases when the non-lightlike surface curve is a helical curve, a relatively normal-slant helix or an isophote curve. In every case, we obtain the position vector of the non-lightlike associated helix by solving differential equations and examples are given for the achieved results.

Keywords: Non-lightlike associated helix, non-lightlike isophote curve, non-lightlike relatively normal-slant helix.

1. Introduction

Geometrical structures of special type such as special surfaces or curves have always been a focus of interest for different disciplines. Without a doubt, the helix curve is the most fascinating of such special geometric structures. A general helix is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix) and a necessary and sufficient condition that a curve to be a general helix is that the ratio of curvature κ to torsion τ be constant [3]. Helices arise in carbon nano-tubes, nano-springs, DNA double and collagen triple helix, α-helices, bacterial flagella in salmonella and escherichia coli, lipid bilayers, bacterial shape in spirochetes, aerial hyphae in actinomycetes, tendrils, horns, screws, springs, vines, helical staircases and sea shells [4, 14, 17]. Helical structures such as hyper-helices are used in fractal geometry [22]. In the realm of computer-aided design and computer graphics, helix shapes can be utilized for describing tool paths, simulating movement, and creating designs for roads, etc. [25].

Instead of tangent, by considering principal normal vector, a new type of special curve called slant helix has been defined by Izumiya and Takeuchi [10]. Later, further studies have been

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2020 *AMS Mathematics Subject Classification*: 53A04, 53A05, 53A35

done. For instance, Ali investigated the position vector of spacelike slant helices, Ali and Turgut investigated the position vector of timelike slant helices in Minkowski 3-space [1, 2].

A surface curve is a curve that lies on a surface. While properties of any arbitrary curve are examined by Frenet frame, properties of surface curves can also be examined by Darboux frame $\{T, g, n\}$ (see Section 2 for details). On a surface, helical curves, relatively normal-slant helices and isophote curves have been defined considering the vectors of Darboux frame, by the property that the vector T, g and n makes a constant angle with a fixed straight line, respectively. Puig-Pey, Gálvez and Iglesias have studied helical surface curves and for the parametric and the implicit forms of a surface, they introduced a new method of generating helical tool paths [20]. In 2017, Macit and Düldül introduced relatively normal-slant helices and studied their axis in Euclidean 3-space [15]. El Haimi and Chahdi investigated the parametric equations of relatively normal-slant helices also in Euclidean 3-space [8]. Further studies have been done by Yadav and Pal, Yadav and Yadav in Minkowski 3-space [23, 24]. On the other hand, isophote curves have been studied in both Euclidean and Lorentzian spaces [5–7]. An isophote curve on a surface is also a result of Lambert's cosine law in optics. Lambert's cosine law indicates that the intensity of illumination on a diffuse surface is proportional to the cosine of the angle between the surface normal and the light vector. According to this law, the intensity is irrespective of the actual viewpoint; hence the illumination is the same when viewed from any direction [12]. By considering Lambert's law Doğan and Yaylı introduced the geometric description of isophote curves in [7]. Isophote curves have many applications in different areas such as car body construction, local shading of a surface or geometry of surfaces of rotation and canal surfaces [11, 19, 21]. Öztürk, Nešović and Koç Öztürk have presented a method for numerical computing of general helices, relatively normal-slant helices, and isophote curves lying on a non-degenerate surface in Minkowski space $\mathbb{E}_1^3$ [18].

In [16], Önder defined new types of associated helices that are associated with special surface curves such as helical curves, relatively normal-slant helices and isophote curves in Euclidean 3-space. He introduced parametric forms of some special associated helices with respect to Darboux frame of special surface curves.

In this paper, we define new types of non-lightlike associated helices in Minkowski 3-space. We name these new helices as non-lightlike (spacelike or timelike) surface curve-connected (SCC) associated helices and we obtain parametrizations for such helices by considering helical curves, relatively normal-slant helices and isophote curves on a non-lightlike surface in Minkowski 3-space.

2. Preliminaries

Minkowski 3-space which is denoted by $\mathbb{E}_1^3$ is a real vector space endowed with the metric $\langle , \rangle = -dx^2 + dy^2 + dz^2$, where (x, y, z) is a rectangular coordinate system. This metric is also called Lorentzian metric. In $\mathbb{E}_1^3$, a vector u is called spacelike (resp. timelike or lightlike) if $\langle u, u\rangle > 0$ or $u = 0$ (resp. $\langle u, u\rangle < 0$ or $\langle u, u\rangle = 0$). Similarly, a curve is called spacelike (resp. timelike or lightlike) if its velocity vector is spacelike (resp. timelike or lightlike). In the case of surfaces, a surface is called spacelike (timelike or lightlike) if the induced metric on the surface is Riemannian (Lorentzian or degenerate), i.e., the normal vector on the surface is timelike (spacelike or lightlike, respectively) [13]. Throughout this paper, we only consider non-lightlike curves and surfaces. Therefore, whenever we talk about a surface or a curve, we assume that they are either spacelike or timelike.

The Lorentzian cross product for any vectors $u, v \in \mathbb{E}_1^3$ is defined by

$$u \times v = (u_2v_3 - u_3v_2, u_1v_3 - u_3v_1, u_2v_1 - u_1v_2),$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ [13]. The Frenet formulae $\{T, N, B\}$ for a unit speed non-lightlike curve α with arc-length parameter s is given by

$$T' = \kappa N, \;\; N' = \varepsilon_B \kappa T + \tau B, \;\; B' = \varepsilon_T \tau N, \tag{1}$$

where T, N, B are the tangent (velocity) vector, principal normal vector, binormal vector, respectively, $\varepsilon_T = \langle T, T\rangle$, $\varepsilon_B = \langle B, B\rangle$, $'$ denotes derivative with respect to s, κ is curvature and τ is torsion of the curve α. Here, ε_T and ε_B determines the Lorentzian character of the vectors T and B, respectively. If $\varepsilon_T = \varepsilon_B = 1$, then α is a spacelike curve with timelike principal normal vector. If $\varepsilon_T = 1$ and $\varepsilon_B = -1$, then α is a spacelike curve with spacelike principal normal vector. If $\varepsilon_T = -1$, then α is a timelike curve [13].

Let φ be a regular surface in $\mathbb{E}_1^3$ and $\alpha : I \subset \mathbb{R} \to \varphi$ be a non-lightlike smooth curve on φ. Then, the Darboux frame $\{T, g, n\}$ along the surface curve α is well defined and its formulae is given by

$$T' = \kappa_g g + \varepsilon_g k_n n, \;\; g' = \varepsilon_n \kappa_g T + \varepsilon_T \tau_g n, \;\; n' = k_n T + \tau_g g, \tag{2}$$

where T, $g = \varepsilon_g T \times n$, n are tangent vector of α, intrinsic normal, surface normal along α, respectively, k_n is normal curvature, κ_g is geodesic curvature, τ_g is geodesic torsion, $\varepsilon_T = \langle T, T\rangle$, $\varepsilon_g = \langle g, g\rangle$ and $\varepsilon_n = \langle n, n\rangle$. If $\varepsilon_T = \varepsilon_g = 1$, then both φ and α are spacelike. If $\varepsilon_T = 1$ and $\varepsilon_g = -1$, then φ is timelike and α is spacelike. Finally, if $\varepsilon_T = -1$ and $\varepsilon_n = 1$, then both φ and α are timelike [5, 6].

Considering Darboux vector fields defined in [9], we define following vector fields for non-lightlike surface curves on non-lightlike surfaces.

Definition 2.1 *Let α be a unit speed non-lightlike curve on a regular non-lightlike surface φ with Darboux frame $\{T, g, n\}$. Then, the vector fields D_n, D_r and D_o along α defined by*

$$D_n = -k_n g + \varepsilon_n \kappa_g n, \;\; D_r = -\tau_g T - \kappa_g n, \;\; D_o = \varepsilon_T \tau_g T + \varepsilon_g k_n g$$

are called normal Darboux vector field, rectifying Darboux vector field and osculating Darboux vector field, respectively.

Lemma 2.2 *[16] Let φ be a regular non-lightlike surface and α be a smooth non-lightlike curve on φ with Darboux frame $\{T, g, n\}$, normal curvature k_n, geodesic curvature κ_g and geodesic torsion τ_g. We have the followings:*

(i) α *is a geodesic curve* $\Leftrightarrow$ $\kappa_g = 0$.

(ii) α *is an asymptotic curve* $\Leftrightarrow$ $k_n = 0$.

(iii) α *is a line of curvature* $\Leftrightarrow$ $\tau_g = 0$.

Definition 2.3 *[24] Let α be a unit speed non-lightlike curve on a regular non-lightlike surface φ with Darboux frame $\{T, g, n\}$. Then, α is called a relatively normal-slant helix if the vector g makes a constant angle with a fixed unit direction.*

Definition 2.4 *[5, 6] Let α be a unit speed non-lightlike curve on a regular non-lightlike surface φ with Darboux frame $\{T, g, n\}$. Then, α is called an isophote curve if the vector n makes a constant angle with a fixed unit direction.*

Similar to the definition given by Önder in [16], we give the following definition for non-lightlike surface curves in Minkowski 3-space.

Definition 2.5 *Let α be a unit speed non-lightlike curve on a regular non-lightlike surface φ with Darboux vector fields D_n, D_r and D_o. Then, α is called a D_i-Darboux slant helix if the Darboux vector field D_i makes a constant angle with a fixed unit direction, where $i \in \{n, r, o\}$.*

By using the above definitions, we introduce helices associated with special surface curves in the following section.

3. Helices Associated with Surface Curves in $\mathbb{E}_1^3$

Let φ be a regular non-lightlike surface and $\alpha : I \subset \mathbb{R} \to \varphi$ be a smooth, unit speed non-lightlike curve with arc-length parameter s, Frenet frame $\{T, N, B\}$ and Darboux frame $\{T, g, n\}$. We consider another non-lightlike curve $\beta : J \subset \mathbb{R} \to \mathbb{E}_1^3$ which is given by the parametrization

$$\beta(s) = \alpha(s) + x(s)T(s) + y(s)g(s) + z(s)n(s), \tag{3}$$

where $x = x(s)$, $y = y(s)$ and $z = z(s)$ are smooth functions of s. The non-lightlike curve β is called non-lightlike associated curve of surface curve α" or SCC-associated curve", where SCC stands for surface curve connected. As well as the associated curve β might be on φ, it might be totally apart from φ. The position that β is on φ or not relies on the values which the functions x, y, z take. We investigate special cases for the functions x, y, z in the following subsections.

Moreover to the definition of the curve β, considering that β is a general helix it would be called SCC-associated helix. Now, let us differentiate the equation (3) with respect to s by using (1) and (2). As the result of this differentiation, we get

$$\beta'(s) = R_1(s)T(s) + R_2(s)g(s) + R_3(s)n(s), \tag{4}$$

where $R_1 = R_1(s)$, $R_2 = R_2(s)$ and $R_3 = R_3(s)$ are smooth functions of s which are defined by

$$R_1 = x' + \varepsilon_n \kappa_g y + k_n z + 1, \quad R_2 = \kappa_g x + y' + \tau_g z, \quad R_3 = \varepsilon_g k_n x + \varepsilon_T \tau_g y + z'. \tag{5}$$

In the following subsections, we investigate special cases when β is a helix and it is associated with a special surface curve.

3.1. Non-lightlike Helices Associated with Helical Curves on a Surface in $\mathbb{E}_1^3$

In this first subsection, we assume that the tangent vector β' of the non-lightlike associated curve β of any arbitrary non-lightlike surface curve α is linearly dependent with the tangent vector of α. For this special case, from (4), we get $R_1 \neq 0$, $R_2 = 0$, $R_3 = 0$ and thus $\beta'(s) = R_1(s)T(s)$. Let s_β be the arc-length parameter of the associated curve β. Then, from $\beta'(s) = R_1(s)T(s)$, we obtain $ds_\beta = \pm R_1 ds$ and the Frenet vectors of β are computed as

$$\begin{cases} T_\beta = \pm T, \quad N_\beta = \pm \dfrac{1}{\sqrt{|\varepsilon_g \kappa_g^2 + \varepsilon_n k_n^2|}} (\kappa_g g + \varepsilon_g k_n n), \\ B_\beta = \dfrac{\varepsilon_{B_\beta}}{\sqrt{|\varepsilon_g \kappa_g^2 + \varepsilon_n k_n^2|}} (\varepsilon_n \kappa_g n - k_n g) = \varepsilon_{B_\beta} \dfrac{D_n}{\|D_n\|}, \end{cases} \tag{6}$$

where $\varepsilon_{B_\beta} = \langle B_\beta, B_\beta \rangle$ and T_β, N_β, B_β are tangent vector, principal normal vector, binormal vector of β, respectively. By using Definition 2.1 and (6), we obtain the following Theorem 3.1:

Theorem 3.1 *Let β be a non-lightlike associated curve of an arbitrary non-lightlike surface curve α with $(k_n, \kappa_g) \neq (0,0)$ which lies on a regular surface φ with the condition that β' and $\alpha' = T$ are linearly dependent. Then, followings are equivalent:*

(i) β is a helix.

(ii) α is a helical curve on φ.

(iii) α is a D_n-Darboux slant helix on φ.

Remark 3.2 *The non-lightlike helix curve β which is associated with a non-lightlike helical surface curve α can be referred to as: Non-lightlike helical curve-connected associated helix or non-lightlike HCC-associated helix.*

Let us now, investigate special cases when x, y or z vanishes, respectively. Such special cases allow us to determine the position vector of β in parametric form. From (5), we have the following system

$$x' + \varepsilon_n \kappa_g y + k_n z + 1 \neq 0, \quad \kappa_g x + y' + \tau_g z = 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y + z' = 0. \tag{7}$$

Case 1: $x = 0$. Then, from (7) we have the system

$$\varepsilon_n \kappa_g y + k_n z + 1 \neq 0, \quad y' + \tau_g z = 0, \quad \varepsilon_T \tau_g y + z' = 0. \tag{8}$$

If $\tau_g \neq 0$, then the solution of system (8) depends on the sign of ε_T. Let $\varepsilon_T = 1$. By using a variable change $t = \int \tau_g(s)ds$, for constants $c_1, c_2 \in \mathbb{R}$ the solution of the system (8) is calculated as

$$y = -c_1 \sinh\left(\int \tau_g(s)ds\right) - c_2 \cosh\left(\int \tau_g(s)ds\right),$$

$$z = c_1 \cosh\left(\int \tau_g(s)ds\right) + c_2 \sinh\left(\int \tau_g(s)ds\right),$$

which we substitute in (3) and obtain the parametric form of the position vector of β as follows

$$\begin{aligned} \beta(s) = \alpha(s) &- \left[c_1 \sinh\left(\int \tau_g(s)ds\right) + c_2 \cosh\left(\int \tau_g(s)ds\right)\right] g(s) \\ &+ \left[c_1 \cosh\left(\int \tau_g(s)ds\right) + c_2 \sinh\left(\int \tau_g(s)ds\right)\right] n(s). \end{aligned} \tag{9}$$

In this case, α, β are spacelike curves and φ is a non-lightlike, i.e., spacelike or timelike, surface.

Let $\varepsilon_T = -1$. Then, for constants $c_3, c_4 \in \mathbb{R}$ the solution of system (8) is given by

$$y = c_3 \cos\left(\int \tau_g(s)ds\right), \quad z = c_4 \sin\left(\int \tau_g(s)ds\right),$$

which similarly leads to the parametric form of the position vector of β as follows

$$\beta(s) = \alpha(s) + c_3 \cos\left(\int \tau_g(s)ds\right) g(s) + c_4 \sin\left(\int \tau_g(s)ds\right) n(s). \tag{10}$$

In this case, α, β are timelike curves and φ is a timelike surface.

If $\tau_g = 0$, then, from second and third equations of system (8), we get $y = c_5$ and $z = c_6$, respectively, where $c_5, c_6 \in \mathbb{R}$ are constants. Therefore, position vector of β curve is given by $\beta(s) = \alpha(s) + c_5 g(s) + c_6 n(s)$.

We can give the following theorem and corollary as results of the above investigation.

Theorem 3.3 *The spacelike (resp. timelike) associated curve β given in (9) (resp. (10)) is a general helix if and only if α is a spacelike (resp. timelike) helical curve on a non-lightlike (resp. timelike) surface φ.*

Remark 3.4 *The spacelike (resp. timelike) associated curve (9) (resp. (10)) can be referred to as: Spacelike (resp. timelike) helical curve-connected associated helix of type 1 or spacelike (resp. timelike) HCC-associated helix of type 1.*

Corollary 3.5 *The helical curve α is a line of curvature if and only if non-lightlike HCC-associated helix has the parametrization $\beta(s) = \alpha(s) + c_5 g(s) + c_6 n(s)$, where $c_5, c_6 \in \mathbb{R}$ are constants.*

Case 2: $y = 0$. From (7), it follows

$$x' + k_n z + 1 \neq 0, \quad \kappa_g x + \tau_g z = 0, \quad \varepsilon_g k_n x + z' = 0, \tag{11}$$

with the condition $(\kappa_g, \tau_g) \neq (0,0)$. If $k_g \neq 0$, then we get $x = -\frac{\tau_g}{\kappa_g} z$ from second equation of system (11). We substitute this equality in the third equation of system (11) and get the differential equation

$$z' - \frac{\varepsilon_g k_n \tau_g}{\kappa_g} z = 0$$

whose solution is $z = c_7 \exp\left(\int \frac{\varepsilon_g k_n \tau_g}{\kappa_g} ds\right)$, where $c_7 \in \mathbb{R}$ is constant. Hence, the position vector of β is given by

$$\beta(s) = \alpha + c_7 \exp\left(\int \frac{\varepsilon_g k_n \tau_g}{\kappa_g} ds\right)\left(-\frac{\tau_g}{\kappa_g} T + n\right). \tag{12}$$

If $\kappa_g = 0$ and $k_n \neq 0$, then we obtain $x = z = 0$ and therefore $\beta(s) = \alpha(s)$.

By the investigation above, the followings can be given.

Theorem 3.6 *The non-lightlike associated curve β given by (12) is a general helix if and only if α is a non-lightlike helical curve on φ.*

Remark 3.7 *The associated curve (12) can be referred to as: spacelike (timelike) helical curve-connected associated helix of type 2 or spacelike (timelike) HCC-associated helix of type 2.*

Corollary 3.8 *(i) The non-lightlike helical curve α is an asymptotic curve with $\kappa_g \neq 0$ if and only if non-lightlike HCC-associated helix of type 2 has the parametrization $\beta(s) = \alpha(s) - \frac{c_5 \tau_g}{\kappa_g} T + c_7 n$, where $c_7 \in \mathbb{R}$ is constant.*

(ii) *The non-lightlike helical curve* α *is a line of curvature if and only if non-lightlike HCC-associated helix of type 2 has the parametrization* $\beta(s) = c_7 n$, *where* $c_7 \in \mathbb{R}$ *is constant.*

Case 3: $z = 0$. In this case, from (7), we have the following system

$$x' + \varepsilon_n \kappa_g y \neq 0, \quad \kappa_g x + y' = 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y = 0, \tag{13}$$

with $(k_n, \tau_g) \neq (0,0)$. If $k_n \neq 0$, then from third equation of system (13), we have $x = -\frac{\varepsilon_T \tau_g}{\varepsilon_g k_n} y$. By substituting x in second equation of system (13), we get the following differential equation

$$y' - \frac{\varepsilon_T \tau_g \kappa_g}{\varepsilon_g k_n} y = 0,$$

whose solution is $y = c_8 \exp\left(\int \frac{\varepsilon_T \tau_g \kappa_g}{\varepsilon_g k_n} ds\right)$, where $c_8 \in \mathbb{R}$ is constant. Hence, the position vector of β is given by

$$\beta(s) = \alpha(s) + c_8 \exp\left(\int \frac{\varepsilon_T \tau_g \kappa_g}{\varepsilon_g k_n} ds\right)\left(-\frac{\varepsilon_T \tau_g}{\varepsilon_g k_n} T + g\right). \tag{14}$$

If $k_n = 0$, then it follows $x = y = 0$ and $\beta(s) = \alpha(s)$.

By the investigation above, we can give the followings.

Theorem 3.9 *The non-lightlike associated curve* β *given by (14) is a general helix if and only if* α *is a non-lightlike helical curve on* φ.

Remark 3.10 *The non-lightlike associated curve (14) can be referred to as: Non-lightlike helical curve-connected associated helix of type 3 or non-lightlike HCC-associated helix of type 3.*

Corollary 3.11 (i) *The non-lightlike helical curve* α *is a geodesic curve if and only if non-lightlike HCC-associated helix of type 3 has the parametrization* $\beta(s) = \alpha(s) - \frac{c_8 \varepsilon_T \tau_g}{\varepsilon_g k_n} T + c_6 g$, *where* $c_8 \in \mathbb{R}$ *is constant.*

(ii) *The non-lightlike helical curve* α *is a line of curvature if and only if non-lightlike HCC-associated helix of type 3 has the parametrization* $\beta(s) = \alpha(s) + c_8 g$, *where* $c_8 \in \mathbb{R}$ *is constant.*

3.2. Non-lightlike Helices Associated with Relatively Normal-slant Helices in $\mathbb{E}_1^3$

This subsection is to investigate non-lightlike associated helices of relatively normal-slant helices. In order to do the mentioned investigation, we assume that tangent vector β' of the associated curve β is linearly dependent with intrinsic normal vector field g of a surface curve α. Then, from

(4), it follows $\beta'(s) = R_2(s)g(s)$ and thus the Frenet vectors T_β, N_β, B_β of β are calculated as

$$\begin{cases} T_\beta = \pm g, \quad N_\beta = \pm \dfrac{1}{\sqrt{\left|\varepsilon_T \kappa_g^2 + \varepsilon_n \tau_g^2\right|}} \left(\varepsilon_n \kappa_g T + \varepsilon_T \tau_g n\right), \\ B_\beta = -\dfrac{\varepsilon_{B_\beta}}{\sqrt{\left|\varepsilon_T \kappa_g^2 + \varepsilon_n \tau_g^2\right|}} \left(\kappa_g n + \tau_g T\right) = \varepsilon_{B_\beta} \dfrac{D_r}{\|D_r\|}, \end{cases} \tag{15}$$

where $\varepsilon_{B_\beta} = \langle B_\beta, B_\beta \rangle$. We can give the following theorem by using (15) and Definition 2.1.

Theorem 3.12 *Let β be a non-lightlike associated curve of an arbitrary non-lightlike surface curve α with $(\kappa_g, \tau_g) \neq (0,0)$ who lies on a regular surface φ with the condition that β' and intrinsic normal g are linearly dependent. Then, followings are equivalent:*
(i) β is a helix.
(ii) α is a relatively normal-slant helix on φ.
(iii) α is a D_r-Darboux slant helix on φ.

Remark 3.13 *The non-lightlike helix β which is associated with relatively normal-slant helix α can be referred to as: Non-lightlike relatively normal-slant helix-connected associated helix or non-lightlike RNS-HC-associated helix.*

Investigating when x, y, z functions have special values leads us to the following cases. From (5), we have

$$x' + \varepsilon_n \kappa_g y + k_n z + 1 = 0, \quad \kappa_g x + y' + \tau_g z \neq 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y + z' = 0. \tag{16}$$

Case 1: $x = 0$. Then, the system (16) is reduced to

$$\varepsilon_n \kappa_g y + k_n z + 1 = 0, \quad y' + \tau_g z \neq 0, \quad \varepsilon_T \tau_g y + z' = 0 \tag{17}$$

with $(k_n, \kappa_g) \neq (0,0)$. If $\kappa_g \neq 0$, then first and third equations of system (16) yields the following linear differential equation

$$z' - \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} z = \frac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g},$$

whose solution can be calculated as

$$z = \exp\left(\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right)\left[\int \exp\left(-\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right) \frac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g} ds + c_9\right],$$

where $c_9 \in \mathbb{R}$ is constant. Then, position vector of associated curve beta is given by

$$\begin{aligned} \beta(s) = \alpha(s) - &\frac{1 + k_n \exp\left(\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right)\left[\int \exp\left(-\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right) \frac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g} ds + c_9\right]}{\varepsilon_n \kappa_g} g \\ &+ \exp\left(\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right)\left[\int \exp\left(-\int \frac{\varepsilon_T k_n \tau_g}{\varepsilon_n \kappa_g} ds\right) \frac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g} ds + c_9\right] n. \end{aligned} \tag{18}$$

If $\kappa_g = 0$ and $\tau_g \neq 0$, then from the first equation of system (16), we get $z = -\dfrac{1}{k_n}$. Since $z' = \dfrac{k_n'}{k_n^2}$, from the third equation of system (16), it follows $y = -\dfrac{k_n'}{\varepsilon_T k_n^2 \tau_g}$. Thus, associated curve beta is given with the position vector

$$\beta(s) = \alpha(s) - \frac{k_n'}{\varepsilon_T k_n^2 \tau_g} g - \frac{1}{k_n} n. \tag{19}$$

Theorem 3.14 *The non-lightlike associated curve β given in (18) (resp. (19)) is a general helix if and only if α is a relatively normal-slant helix on φ.*

Remark 3.15 *The non-lightlike associated curve (18) (resp. (19)) can be referred to as: Non-lightlike relatively normal-slant helix-connected associated helix of type 1 or non-lightlike RNS-HC-associated helix of type 1.*

Corollary 3.16 *(i) The non-lightlike relatively normal-slant helix α is an asymptotic curve on φ with $(k_n, \kappa_g) \neq (0,0)$ if and only if RNS-HC-associated helix has the parametrization $\beta(s) = \alpha - \dfrac{1}{\varepsilon_n \kappa_g} g + \left(\displaystyle\int \frac{\varepsilon_T \tau_g}{\varepsilon_n \kappa_g} ds + c_7 \right) n$.*

(ii) The non-lightlike relatively normal-slant helix α is a geodesic curve on φ with $(k_n, \kappa_g) \neq (0,0)$ if and only if RNS-HC-associated helix has the parametrization in (19).

(iii) The non-lightlike relatively normal-slant helix α is a line of curvature on φ with $(k_n, \kappa_g) \neq (0,0)$ if and only if RNS-HC-associated helix has the parametrization $\beta(s) = \alpha(s) - \dfrac{c_7 k_n + 1}{\varepsilon_n \kappa_g} g + c_7 n$.

Case 2: $y = 0$. The system (16) becomes

$$x' + k_n z = 0, \quad \kappa_g x + \tau_g z \neq 0, \quad \varepsilon_g k_n x + z' = 0. \tag{20}$$

If $k_n \neq 0$, then, from system (20), the following differential equation is derived

$$z'' - \frac{k_n'}{k_n} z' - \varepsilon_g k_n^2 z = \varepsilon_g k_n, \tag{21}$$

whose homogeneous part can be obtained with the aid of a variable change $t = \int k_n ds$ as follows

$$\frac{d^2 z}{dt^2} - \varepsilon_g z = 0. \tag{22}$$

The differential equation (22) has two different types of solutions with respect to the value of ε_g.

Let $\varepsilon_g = 1$. In this case, β is a spacelike curve. Then, the general solution of (21) is obtained as follows

$$\begin{aligned} z = {} & c_{10}\cosh\left(\int k_n ds\right) + c_{11}\sinh\left(\int k_n ds\right) \\ & - \cosh\left(\int k_n ds\right)\int \sinh\left(\int k_n ds\right)ds + \sinh\left(\int k_n ds\right)\int \cosh\left(\int k_n ds\right)ds, \end{aligned} \tag{23}$$

where $c_{10}, c_{11} \in \mathbb{R}$ are constants. This leads us to

$$\begin{aligned} x = {} & -c_{10}\sinh\left(\int k_n ds\right) - c_{11}\cosh\left(\int k_n ds\right) \\ & + \sinh\left(\int k_n ds\right)\int \sinh\left(\int k_n ds\right)ds - \cosh\left(\int k_n ds\right)\int \cosh\left(\int k_n ds\right)ds \end{aligned} \tag{24}$$

since $x = -\dfrac{z'}{k_n}$ from the third equation of system (20). In this case, β is a spacelike curve and α is a spacelike (resp. timelike) curve on a spacelike (resp. timelike) surface. Thus, by using (23) and (24), the position vector of spacelike associated curve β is given as follows

$$\begin{aligned} \beta(s) = \alpha(s) + \Big[& -c_{10}\sinh\left(\int k_n ds\right) - c_{11}\cosh\left(\int k_n ds\right) \\ & + \sinh\left(\int k_n ds\right)\int \sinh\left(\int k_n ds\right)ds - \cosh\left(\int k_n ds\right)\int \cosh\left(\int k_n ds\right)ds\Big] T \\ & + \Big[c_{10}\cosh\left(\int k_n ds\right) + c_{11}\sinh\left(\int k_n ds\right) \\ & + \sinh\left(\int k_n ds\right)\int \sinh\left(\int k_n ds\right)ds - \cosh\left(\int k_n ds\right)\int \cosh\left(\int k_n ds\right)ds\Big] n. \end{aligned} \tag{25}$$

Let $\varepsilon_g = -1$. In this case, T and n become spacelike vectors. Then, we get φ is a timelike surface, α is a spacelike curve and β is a timelike curve. Similar to the previous case, the general solution of (21) is obtained as follows

$$\begin{aligned} z = {} & c_{12}\cos\left(\int k_n ds\right) + c_{13}\sin\left(\int k_n ds\right) \\ & + \cos\left(\int k_n ds\right)\int \sin\left(\int k_n ds\right)ds - \sin\left(\int k_n ds\right)\int \cos\left(\int k_n ds\right)ds, \end{aligned}$$

where $c_{12}, c_{13} \in \mathbb{R}$ are constants and thus

$$\begin{aligned} x = {} & -c_{12}\sin\left(\int k_n ds\right) + c_{13}\cos\left(\int k_n ds\right) \\ & - \sin\left(\int k_n ds\right)\int \sin\left(\int k_n ds\right)ds - \cos\left(\int k_n ds\right)\int \cos\left(\int k_n ds\right)ds. \end{aligned}$$

Hence, the position vector of timelike associated curve β is stated as

$$\begin{aligned}\beta(s) = \alpha(s) + \Big[&-c_{12}\sin\Big(\int k_n ds\Big) + c_{13}\cos\Big(\int k_n ds\Big)\\ &-\sin\Big(\int k_n ds\Big)\int \sin\Big(\int k_n ds\Big) ds - \cos\Big(\int k_n ds\Big)\int \cos\Big(\int k_n ds\Big) ds\Big]T\\ &+\Big[c_{12}\cos\Big(\int k_n ds\Big) + c_{13}\sin\Big(\int k_n ds\Big)\\ &+\cos\Big(\int k_n ds\Big)\int \sin\Big(\int k_n ds\Big) ds - \sin\Big(\int k_n ds\Big)\int \cos\Big(\int k_n ds\Big) ds\Big]n.\end{aligned} \tag{26}$$

If $k_n = 0$, then from first and third equations of system (20), we get $x = -s + c_{19}$, $z = c_{20}$, respectively, and therefore the position vector of β is given by

$$\beta(s) = \alpha(s) + (-s + c_{14})T + c_{15}n, \tag{27}$$

where $c_{14}, c_{15} \in \mathbb{R}$ are constants. Now, we can give the followings:

Theorem 3.17 *The spacelike (resp. timelike and non-lightlike) associated curve β given by (25) (resp. (26) and (27)) is a general helix if and only if α is a relatively normal-slant helix on φ.*

Remark 3.18 *The associated curves (25) and (26) can be referred to as: Spacelike and timelike relatively normal-slant helix-connected associated helix of type 2 or spacelike and timelike RNS-HC-associated helix of type 2, respectively.*

Corollary 3.19 *The non-lightlike relatively normal-slant helix α is an asymptotic curve on φ if and only if non-lightlike RNS-HC-associated helix has the parametrization in (27).*

Case 3: $z = 0$. In this case, from system (16), we obtain

$$x' + \varepsilon_n \kappa_g y + 1 = 0, \quad \kappa_g x + y' \neq 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y = 0. \tag{28}$$

with $(k_n, \tau_g) \neq (0,0)$. If $\tau_g \neq 0$, then from the third equation of system (28), we have $y = -\dfrac{\varepsilon_g k_n}{\varepsilon_T \tau_g}$. Substituting y in first equation of (28), it follows $x' - \dfrac{\varepsilon_g \varepsilon_n k_n \kappa_g}{\varepsilon_T \tau_g} x + 1 = 0$, where $\dfrac{\varepsilon_g \varepsilon_n}{\varepsilon_T} = -1$. Then, following differential equation is obtained

$$x' + \frac{k_n \kappa_g}{\tau_g} x = -1,$$

whose general solution is

$$x = \exp\Big(-\int \frac{k_n \kappa_g}{\tau_g} ds\Big)\Big[-\int \exp\Big(\int \frac{k_n \kappa_g}{\tau_g} ds\Big) ds + c_{16}\Big],$$

where $c_{16} \in \mathbb{R}$ is constant. Hence, we obtain y as follows

$$y = -\frac{\varepsilon_g k_n}{\varepsilon_T \tau_g} \exp\left(-\int \frac{k_n \kappa_g}{\tau_g} ds\right)\left[-\int \exp\left(\int \frac{k_n \kappa_g}{\tau_g} ds\right) ds + c_{16}\right],$$

and the position vector of associated curve β is given by

$$\beta(s) = \alpha(s) + \exp\left(-\int \frac{k_n \kappa_g}{\tau_g} ds\right)\left[-\int \exp\left(\int \frac{k_n \kappa_g}{\tau_g} ds\right) ds + c_{16}\right]\left(T - \frac{\varepsilon_g k_n}{\varepsilon_T \tau_g} g\right). \tag{29}$$

If $\kappa_g \neq 0$ and $\tau_g = 0$, then from the system (28), we get $x = 0$ and $y = -\dfrac{1}{\varepsilon_n \kappa_g}$. Thus, the position vector of associated curve β is given by

$$\beta(s) = \alpha(s) - \frac{1}{\varepsilon_n \kappa_g} g. \tag{30}$$

Theorem 3.20 *The non-lightlike associated curve β given by (29) (resp. (30)) is a general helix if and only if α is a relatively normal-slant helix on φ.*

Remark 3.21 *The non-lightlike associated curve (29) (resp. (30)) can be referred to as: Non-lightlike relatively normal-slant helix-connected associated helix of type 3 or non-lightlike RNS-HC-associated helix of type 3.*

Corollary 3.22 *(i) The non-lightlike relatively normal-slant helix α is an asymptotic curve on φ if and only if non-lightlike RNS-HC-associated helix has the parametrization $\beta(s) = \alpha(s) + (-s + c_{16})T$, where $c_{16} \in \mathbb{R}$ is constant.*

(ii) The non-lightlike relatively normal-slant helix α is a geodesic curve on φ if and only if non-lightlike RNS-HC-associated helix has the parametrization $\beta(s) = \alpha(s) + (-s + c_{16})T + \frac{(-s+c_{16})\varepsilon_g k_n}{\varepsilon_T \tau_g} g$, where $c_{16} \in \mathbb{R}$ is constant.

(iii) The non-lightlike relatively normal-slant helix α is a line of curvature on φ if and only if non-lightlike RNS-HC-associated helix has the parametrization in (30).

3.3. Non-lightlike helices associated with isophote curves in $\mathbb{E}_1^3$

In this final subsection of Section 3, we investigate non-lightlike helices associated with isophote curves. Let the tangent vector β' of associated curve β be linearly dependent with the unit surface normal along an arbitrary non-lightlike curve α on an oriented surface φ. Then, from (4), we have $R_1 = R_2 = 0$ and $\beta'(s) = R_3(s) n(s)$. Arc-length parameter and Frenet vectors T_β, N_β, B_β of β

are calculated as $ds_\beta = \pm R_3 ds$ and

$$\begin{cases} T_\beta = \pm n, \quad N_\beta = \pm \dfrac{1}{\sqrt{\left|\varepsilon_T k_n^2 + \varepsilon_g \tau_g^2\right|}} \left(k_n T + \tau_g g\right), \\ B_\beta = \dfrac{\varepsilon_{B_\beta}}{\sqrt{\left|\varepsilon_T k_n^2 + \varepsilon_g \tau_g^2\right|}} \left(\varepsilon_g k_n g + \varepsilon_T \tau_g T\right) = \varepsilon_{B_\beta} \dfrac{D_o}{\|D_o\|}, \end{cases} \tag{31}$$

respectively, where $\varepsilon_{B_\beta} = \langle B_\beta, B_\beta. \rangle$. From (31) and Definition 2.1, we can give the following theorem.

Theorem 3.23 *Let β be a non-lightlike associated curve of an arbitrary non-lightlike surface curve α with $(k_n, \tau_g) \neq (0,0)$ who lies on a regular surface φ with the condition that β' and unit surface normal n along α are linearly dependent. Then, followings are equivalent:*

(i) β is a helix.

(ii) α is an isophote curve on φ.

(iii) α is a D_o-Darboux slant helix on φ.

Remark 3.24 *The non-lightlike helix β associated with isophote curve α can be referred to as: Non-lightlike isophote curve-connected associated helix or non-lightlike ICC-associated helix.*

We now investigate special cases when x, y, z functions have special values. From (5), we get

$$x' + \varepsilon_n \kappa_g y + k_n z + 1 = 0, \quad \kappa_g x + y' + \tau_g z = 0, \quad \varepsilon_g k_n x + \varepsilon_T \tau_g y + z' \neq 0. \tag{32}$$

Case 1: $x = 0$. Then, from (32), we have

$$\varepsilon_n \kappa_g y + k_n z + 1 = 0, \quad y' + \tau_g z = 0, \quad \varepsilon_T \tau_g y + z' \neq 0, \tag{33}$$

with $(k_n, \kappa_g) \neq (0,0)$. If $\tau_g \neq 0$, then from second equation of system (33), we have $z = -\frac{y'}{\tau_g}$ and by substituting this equality in the third equation of system (33), we obtain the following differential equation

$$y' - \frac{\varepsilon_n \kappa_g \tau_g}{k_n} y = \frac{\tau_g}{k_n},$$

whose general solution is

$$y = \exp\left(\int \frac{\varepsilon_n \kappa_g \tau_g}{k_n} ds\right)\left(\int \exp\left(-\int \frac{\varepsilon_n \kappa_g \tau_g}{k_n} ds\right) \frac{\tau_g}{k_n} ds + c_{17}\right), \tag{34}$$

where c_{17} is a real constant. Since $z = -\frac{y'}{\tau_g}$, it follows

$$z = -\frac{1}{k_n} - \frac{\varepsilon_n \kappa_g}{k_n} \exp\left(\int \frac{\varepsilon_n \kappa_g \tau_g}{k_n} ds\right)\left(\int \exp\left(-\int \frac{\varepsilon_n \kappa_g \tau_g}{k_n} ds\right) \frac{\tau_g}{k_n} ds + c_{17}\right). \tag{35}$$

Therefore, for the position vector of associated curve β, we obtain

$$\begin{aligned}\beta(s) = \alpha(s) + \exp\left(\int \frac{\varepsilon_n\kappa_g\tau_g}{k_n}ds\right)\left(\int \exp\left(-\int \frac{\varepsilon_n\kappa_g\tau_g}{k_n}ds\right)\frac{\tau_g}{k_n}ds + c_{17}\right)g \\ -\left[\frac{1}{k_n} + \frac{\varepsilon_n\kappa_g}{k_n}\exp\left(\int \frac{\varepsilon_n\kappa_g\tau_g}{k_n}ds\right)\left(\int \exp\left(-\int \frac{\varepsilon_n\kappa_g\tau_g}{k_n}ds\right)\frac{\tau_g}{k_n}ds + c_{17}\right)\right]n.\end{aligned} \tag{36}$$

If $k_n \neq 0$ and $\tau_g = 0$, then from the second equation of system (33), we get $y = c_{18}$ for a real constant c_{18}. Substituting this result in first equation of system (33) yields $z = -\dfrac{c_{18}\varepsilon_n\kappa_g + 1}{k_n}$. Therefore, the position vector of associated curve β is obtained as

$$\beta(s) = \alpha(s) + c_{18}g - \frac{c_{18}\varepsilon_n\kappa_g + 1}{k_n}n. \tag{37}$$

We state our findings with the following theorem and corollaries.

Theorem 3.25 *The non-lightlike associated curve β given by (36) (resp. (37)) is a general helix if and only if α is an isophote curve on φ.*

Remark 3.26 *The non-lightlike associated curve (36) (resp. (37)) can be referred to as: Non-lightlike isophote curve-connected associated helix of type 1 or non-lightlike ICC-associated helix of type 1.*

Corollary 3.27 *(i) The non-lightlike isophote curve α with $(k_n, \kappa_g) \neq (0,0)$ is an asymptotic curve if and only if non-lightlike ICC-associated helix has the parametrization $\beta(s) = \alpha(s) - \frac{1}{\varepsilon_n\kappa_g}g - \frac{k_g'}{\varepsilon_n\kappa_g^2\tau_g}n$.*

(ii) The non-lightlike isophote curve α with $(k_n, \kappa_g) \neq (0,0)$ is a geodesic curve if and only if non-lightlike ICC-associated helix has the parametrization $\beta(s) = \alpha(s) + \int \frac{\tau_g}{k_n}dsg - \frac{1}{k_n}n$.

(iii) The non-lightlike isophote curve α with $(k_n, \kappa_g) \neq (0,0)$ is a line of curvature if and only if non-lightlike ICC-associated helix has the parametrization (37).

Case 2: $y = 0$. From system (32), we have

$$x' + k_n z + 1 = 0, \quad \kappa_g x + \tau_g z = 0, \quad \varepsilon_g k_n x + z' \neq 0, \tag{38}$$

with $(\kappa_g, \tau_g) \neq (0,0)$. If $\tau_g \neq 0$, then, from the second equation of system (38), we get $z = -\dfrac{\kappa_g}{\tau_g}x$ which we substitute in the first equation of system (38) and obtain the following differential equation

$$x' - \frac{k_n\kappa_g}{\tau_g}x = -1,$$

whose general solution is

$$x = \exp\left(\int \frac{k_n\kappa_g}{\tau_g}ds\right)\left[-\int \exp\left(-\int \frac{k_n\kappa_g}{\tau_g}ds\right)ds + c_{19}\right], \tag{39}$$

where c_{19} is a real constant. Since $z = -\frac{\kappa_g}{\tau_g}x$, the position vector of the associated curve β is obtained as

$$\beta(s) = \alpha(s) + \exp\left(\int \frac{k_n\kappa_g}{\tau_g}ds\right)\left[-\int \exp\left(-\int \frac{k_n\kappa_g}{\tau_g}ds\right)ds + c_{19}\right]\left(T - \frac{\kappa_g}{\tau_g}n\right). \tag{40}$$

If $k_n \neq 0$ and $\tau_g = 0$, then, second and first equations of system (38) yield $x = 0$ and $z = -\frac{1}{k_n}$, respectively. Thus, the position vector of associated curve β is given by

$$\beta(s) = \alpha - \frac{1}{k_n}n. \tag{41}$$

Now, we give the following theorem and corollaries.

Theorem 3.28 *The non-lightlike associated curve β given by (40) (resp. (41)) is a general helix if and only if α is an isophote curve on φ.*

Remark 3.29 *The non-lightlike associated curve (40) (resp. (41)) can be referred to as: Non-lightlike isophote curve-connected associated helix of type 2 or non-lightlike ICC-associated helix of type 2.*

Corollary 3.30 *(i) The non-lightlike isophote curve α with $(\kappa_g, \tau_g) \neq (0,0)$ is an asymptotic curve if and only if non-lightlike ICC-associated helix has the parametrization $\beta(s) = \alpha(s) + (-s + c_{19})T + \frac{\kappa_g(s-c_{19})}{\tau_g}n$.*

(ii) The non-lightlike isophote curve α with $(\kappa_g, \tau_g) \neq (0,0)$ is a geodesic curve if and only if non-lightlike ICC-associated helix has the parametrization $\beta(s) = \alpha(s) + (-s + c_{19})T$.

(iii) The non-lightlike isophote curve α with $(\kappa_g, \tau_g) \neq (0,0)$ is a line of curvature if and only if non-lightlike ICC-associated helix has the parametrization in (41).

Case 3: $z = 0$. In this case, from (32) we obtain

$$x' + \varepsilon_n\kappa_g y + 1 = 0, \quad \kappa_g x + y' = 0, \quad \varepsilon_g k_n x + \varepsilon_T\tau_g y \neq 0. \tag{42}$$

If $\kappa_g = 0$, then, from system (42), we get $x = -s + c_{29}$ and $y = c_{30}$, where c_{20}, c_{21} are real constants. Then, the position vector of the associated curve β is given by

$$\beta(s) = \alpha(s) + (-s + c_{20})T + c_{21}g. \tag{43}$$

If $\kappa_g \neq 0$, then from second equation of system (42), we have $x = -\frac{y'}{\kappa_g}$. We take the derivative of x and substitute it in the first equation of system (42) and obtain the following differential equation

$$y'' - \frac{k_g'}{\kappa_g} y' - \varepsilon_n \kappa_g^2 y = \kappa_g,$$

whose homogeneous part can be achieved by a parameter change $t = \int \kappa_g ds$ as

$$\frac{d^2 y}{dt^2} - \varepsilon_n y = 0. \tag{44}$$

The solution of (44) depends on the value of ε_n which could be either 1 or -1. If $\varepsilon_n = 1$, then we get

$$\begin{aligned} y =\ & c_{22} \cosh\left(\int \kappa_g ds\right) + c_{23} \sinh\left(\int \kappa_g ds\right) \\ & - \cosh\left(\int \kappa_g ds\right) \int \sinh\left(\int \kappa_g ds\right) ds + \sinh\left(\int \kappa_g ds\right) \int \cosh\left(\int \kappa_g ds\right) ds, \\ x =\ & - c_{22} \sinh\left(\int \kappa_g ds\right) - c_{23} \cosh\left(\int \kappa_g ds\right) \\ & + \sinh\left(\int \kappa_g ds\right) \int \sinh\left(\int \kappa_g ds\right) ds - \cosh\left(\int \kappa_g ds\right) \int \cosh\left(\int \kappa_g ds\right) ds, \end{aligned} \tag{45}$$

where c_{22}, c_{23} are real constants.

If $\varepsilon_n = -1$, then we get

$$\begin{aligned} y =\ & c_{24} \cos\left(\int \kappa_g ds\right) + c_{25} \sin\left(\int \kappa_g ds\right) \\ & - \cos\left(\int \kappa_g ds\right) \int \sin\left(\int \kappa_g ds\right) ds + \sin\left(\int \kappa_g ds\right) \int \cos\left(\int \kappa_g ds\right) ds, \\ x =\ & c_{24} \sin\left(\int \kappa_g ds\right) - c_{25} \cos\left(\int \kappa_g ds\right) \\ & - \sin\left(\int \kappa_g ds\right) \int \sin\left(\int \kappa_g ds\right) ds - \cos\left(\int \kappa_g ds\right) \int \cos\left(\int \kappa_g ds\right) ds, \end{aligned} \tag{46}$$

where c_{24}, c_{25} are real constants. In either cases,

$$\beta(s) = \alpha(s) + xT + yg, \tag{47}$$

where x, y are as defined in (45) or (46).

Theorem 3.31 *The non-lightlike associated curve β given by (47) is a general helix if and only if α is an isophote curve on φ.*

Remark 3.32 *The non-lightlike associated curve (47) can be referred to as: Non-lightlike isophote curve-connected associated helix of type 3 or non-lightlike ICC-associated helix of type 3.*

Corollary 3.33 *The non-lightlike isophote curve α is a geodesic curve if and only if non-lightlike ICC-associated helix has the parametrization (43).*

4. Examples

Example 4.1 *Let the spacelike surface φ be given by the parametrization* $\varphi(u,v) = (\cosh u, \sinh u, v)$ *and*

$$\alpha(u) = \left(\cosh\left(\frac{u}{\sqrt{2}}\right), \sinh\left(\frac{u}{\sqrt{2}}\right), \frac{u}{\sqrt{2}}\right)$$

be a spacelike helix on φ. Then, elements of Darboux frame of α are calculated as

$$T(s) = \left(\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right),$$

$$g(s) = \left(\sinh\left(\frac{s}{\sqrt{2}}\right), \cosh\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\right), \quad n(s) = \left(\cosh\left(\frac{s}{\sqrt{2}}\right), \sinh\left(\frac{s}{\sqrt{2}}\right), 0\right),$$

$k_n = \frac{1}{2}$, $\kappa_g = 0$ *and* $\tau_g = \frac{1}{2}$. *Since $\kappa_g = 0$, α is a geodesic curve on φ. On the other hand, since g and n are Lorenztian circles or arc of a Lorenztian circle, then we have that α is also a relatively normal-slant helix and an isophote curve on φ. Figure 1 shows some β curves associated with α considering the obtained results in Section 3.*

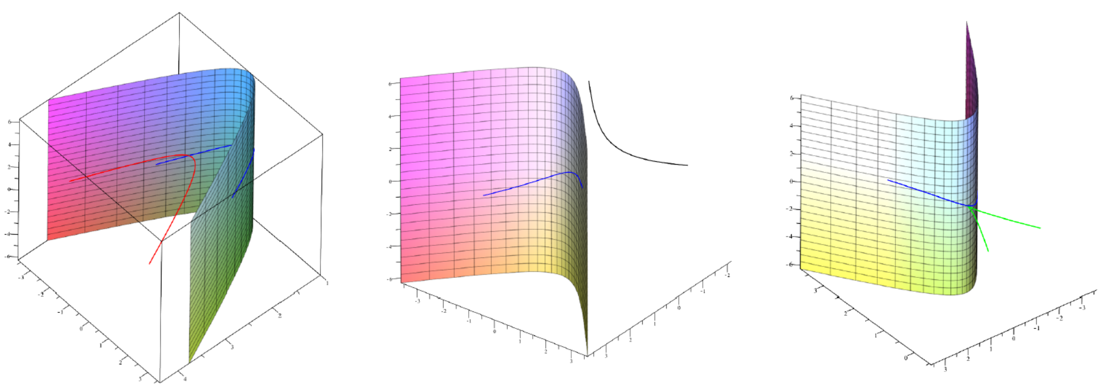

Figure 1: Spacelike surface curve α (blue), spacelike HCC-associated helix of type 1 (red), spacelike RNS-HC-associated helix of type 1 (black) and spacelike ICC-associated helix of type 2 (green), respectively

Example 4.2 *Let the timelike surface φ be given by the parametrization* $\varphi(u,v) = \left(\sqrt{3}u, v\cos(u), v sin(u)\right)$,

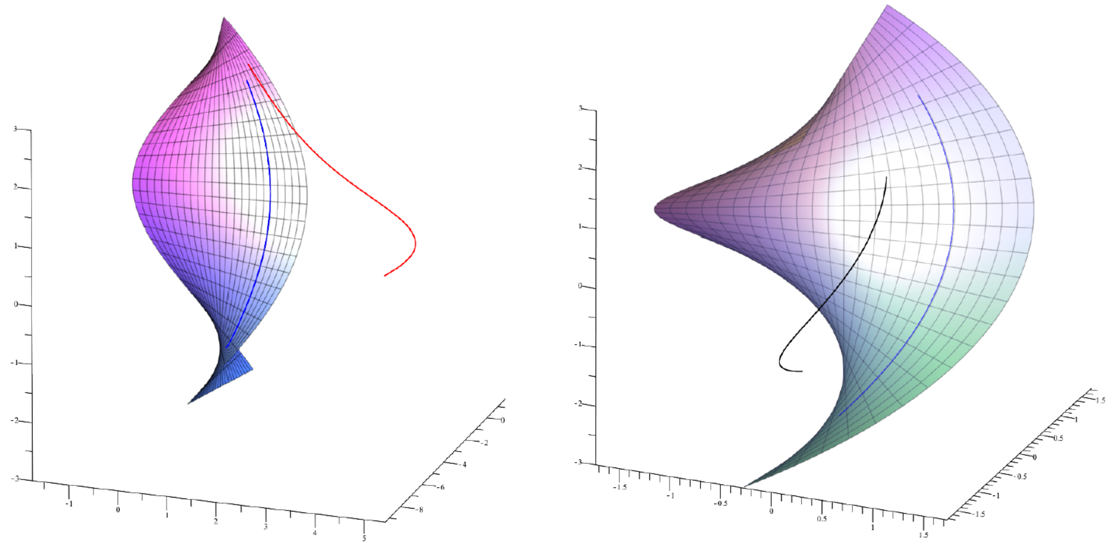

Figure 2: Timelike surface curve α (blue), timelike HCC-associated helix of type 3 (red), timelike RNS-HC-associated helix of type 3 (black), respectively

$v \in (-\sqrt{3}, \sqrt{3})$ *and*

$$\alpha(s) = \left(\sqrt{\frac{3}{2}}s, \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right)\right)$$

be a timelike helix on φ*. The elements of Darboux frame of* α *are calculated as*

$$n(s) = \left(-\frac{\sqrt{2}s}{2\sqrt{3-\frac{s^2}{2}}}, \frac{\sqrt{3}\sin\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{3-\frac{s^2}{2}}}, \frac{\sqrt{3}\cos\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{3-\frac{s^2}{2}}}\right), \quad k_n = \frac{1}{2}\cosh\left(\frac{\pi}{2}\right), \ k_n = \frac{1}{2}\sinh\left(\frac{\pi}{2}\right) \text{ and}$$

$\tau_g = \frac{\sqrt{3}}{2}$. Since g is a Lorenztian circle or an arc of a Lorenztian circle, then we have that α is also a relatively normal-slant helix. Figure 2 shows some β curves associated with α considering the obtained results in Section 3.

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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