E-ISSN 2980-3020

ISTANBUL JOURNAL OF MATHEMATICS

VOLUME 1 • NUMBER 1 • JUNE 2023

İSTANBUL UNIVERSITY PRESS

İSTANBUL UNIVERSITY PRESS

OWNER
Prof. Dr. Tansel AK
İstanbul University, Faculty of Science, İstanbul, Türkiye

RESPONSIBLE MANAGER
Prof. Serap ÖZTOP KAPTANOĞLU
İstanbul University, Faculty of Science,
Department of Mathematics, İstanbul, Türkiye

CORRESPONDENCE ADDRESS
İstanbul University Faculty of Science
PK 34134 Vezneciler, Fatih, İstanbul, Türkiye
Phone : +90 (212) 455 57 00
Fax : +90 (212) 455 57 66
E-mail: ijmath@istanbul.edu.tr
https://iupress.istanbul.edu.tr/en/journal/ijmath/home
https://dergipark.org.tr/en/pub/ijmath

PUBLISHER
İstanbul University Press
İstanbul University Central Campus,
34452 Beyazıt, Fatih, İstanbul, Türkiye
Phone: +90 (212) 440 00 00

The publication language of the journal is English.

This is a scholarly, international, peer-reviewed and open-access journal published biannually in June and December.

Publication Type: Periodical

Istanbul Journal of Mathematics
JUNE 2023,
VOLUME 1, NUMBER 1

Istanbul Journal of Mathematics
JUNE 2023,
VOLUME 1, NUMBER 1

İSTANBUL UNIVERSITY PRESS

CONTENTS

RESEARCH ARTICLES

ISTANBUL UNIVERSITY PRESS

Istanbul Journal of Mathematics
ijmath 2023, 1 (1), 1–11
DOI: 10.26650/ijmath.2023.00005

RESEARCH ARTICLE

A new characterization of the Hardy space and of other Hilbert spaces of analytic functions

N. Alpay[1*]

[1]Schmid College of Science and Technology, Chapman University, One University Drive, Orange, California 92866, USA

ABSTRACT
The Fock space can be characterized (up to a positive multiplicative factor) as the only Hilbert space of entire functions in which the adjoint of derivation is multiplication by the complex variable. Similarly (and still up to a positive multiplicative factor) the Hardy space is the only space of functions analytic in the open unit disk for which the adjoint of the backward shift operator is the multiplication operator. In the present paper we characterize the Hardy space and some related reproducing kernel Hilbert spaces in terms of the adjoint of the differentiation operator. We use reproducing kernel methods, which seem to also give a new characterization of the Fock space.

Mathematics Subject Classification (2020): 46E22, 47B32, 30H20, 30H10

Keywords: Reproducing kernel, Hardy space, Fock space

1. INTRODUCTION

The Fock (or Bargmann-Fock-Segal) space consists of the entire functions f such that

$$\frac{1}{\pi}\iint_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dxdy < \infty, \tag{1}$$

and is the reproducing kernel Hilbert space with reproducing kernel

$$e^{z\overline{\omega}}. \tag{2}$$

It is (up to a positive multiplicative factor) the unique Hilbert space of entire functions in which

$$\partial_z^* = M_z, \tag{3}$$

where ∂_z denote the derivative with respect to z, and will be used throughout the work along with the notation $(\partial_z f)(z) = f'(z)$. Furthermore, in (3) M_z stands for multiplication by the variable z, e.g., $(M_z f)(z) = zf(z)$. We refer to the work of Bargmann Bargmann (1961, 1962) for this result. Formula (3) suggests to find similar characterizations for other important spaces of analytic functions. In particular, we have in mind the following spaces of functions analytic in the open unit disk $\mathbb{D}$:

(1) The Bergman space, which consists of the functions analytic in $\mathbb{D}$ and such that:

$$\frac{1}{\pi}\iint_{\mathbb{D}} |f(z)|^2 dxdy < \infty,$$

with the reproducing kernel $\dfrac{1}{(1-z\overline{\omega})^2} = \sum_{n=0}^{\infty}(n+1)z^n\overline{\omega}^n$.

(2) The Hardy space $\mathbf{H}^2$, when the condition is:

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt < \infty,$$

with the reproducing kernel $\dfrac{1}{1-z\overline{\omega}} = \sum_{n=0}^{\infty} z^n\overline{\omega}^n$.

Corresponding Author: N. Alpay **E-mail:** nalpay@chapman.edu
Submitted: 28.03.2023 • **Last Revision Received:** 15.06.2023 • **Accepted:** 16.06.2023 • **Published Online:** 19.06.2023

(3) The Dirichlet space, for which the functions vanish at the origin and satisfy

$$\frac{1}{\pi}\iint_{\mathbb{D}} |f'(z)|^2 dxdy < \infty,$$

with the reproducing kernel $-\ln(1-z\overline{\omega}) = \sum_{n=1}^{\infty} \frac{z^n\overline{\omega}^n}{n}$.

In the present work we approach this problem using reproducing kernel Hilbert spaces methods. We prove te following results.

Theorem 1.1. *The Hardy space is, up to a positive multiplicative factor, the only reproducing kernel Hilbert space of functions analytic in $\mathbb{D}$, in which the equality*

$$\partial_z^* = M_z\partial_z M_z \tag{4}$$

holds on the linear span of the kernel functions.

Note that both in this, and in the next theorem, one could assume that the functions are analytic only in a neighborhood of the origin, and then use analytic continuation. We also note that the unbounded operator $M_z\partial_z$ is diagonal, and acts on the polynomials as the number operator of quantum mechanics:

$$M_z\partial_z(z^n) = nz^n, \quad n = 0, 1, \dots,$$

see e.g. (Fayngold and Fayngold 2013, p. 548) which is the radial derivative for mathematics.

As mentioned above, the Hardy space of the open unit disk $\mathbb{D}$ has reproducing kernel $\frac{1}{1-z\overline{\omega}}$. More generally, for every $\alpha \in (0,\infty)$, the function $\frac{1}{(1-z\overline{\omega})^\alpha}$ is positive definite in $\mathbb{D}$, as can be seen from the power series expansion of the function $\frac{1}{(1-t)^\alpha}$ with center at the origin as

$$\frac{1}{(1-z\overline{\omega})^\alpha} = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} z^n\overline{\omega}^n, \quad z,\omega \in \mathbb{D}. \tag{5}$$

We will use a similar notation to Bargmann (see (Bargmann 1961, Remark 2g, page 203)), and denote $\mathfrak{H}_\alpha$ to be the associated reproducing kernel Hilbert space, characterized by the following result.

Theorem 1.2. *Let $\alpha > 0$. Then the space $\mathfrak{H}_\alpha$ is, up to a multiplicative factor, the only reproducing kernel Hilbert space of functions analytic in $\mathbb{D}$, in which the equality*

$$\partial_z^* = M_z\partial_z M_z - (1-\alpha)M_z, \quad \alpha > 0, \tag{6}$$

holds on the linear span of the kernel functions.

The case $\alpha = 1$ corresponds to the Hardy space and Theorem 1.1, and $\alpha = 2$ corresponds to the Bergman space. The case $\alpha = 0$ would "correspond" to the Dirichlet space, in the sense that

$$\lim_{\alpha\to 0} \frac{1}{\alpha}\left(\frac{1}{(1-z\overline{\omega})^\alpha} - 1\right) = -\ln(1-z\overline{\omega}).$$

Note that ∂_z is not densely defined in the Dirichlet space (since $\partial_z k_\omega$ is not in the Dirichlet space for $\omega \neq 0$), and therefore its adjoint is a relation and not an operator. We were not able to get a counterpart of Theorem 1.2 for $\alpha = 0$, but we have the following result.

Theorem 1.3. *The Dirichlet space is, up to a positive multiplicative factor, the only reproducing kernel Hilbert space of functions analytic in $\mathbb{D}$, for which the equality*

$$\partial_{z^2}^2 k = \bar{\omega}^2\partial_z\partial_{\bar{\omega}} k \tag{7}$$

holds for its kernel k, pointwise for $z,\omega \in \mathbb{D}$.

Note that (7) is not an equality in the Dirichlet space, but rather, an equality between analytic functions. We give a similar characterization of the Fock space in Proposition 2.5.

More generally, our analysis suggests a new direction in the study of the connections between reproducing kernel Hilbert spaces and operator models. In particular, the following question is of interest: For which polynomials of two variables $p(x,y)$ does the equation

$$\partial_z^* = p(M_z, \partial)$$

characterize a reproducing kernel Hilbert space?

Remark 1.4. When denoting inner products, we will sometimes mention explicitly the variable inside an inner product by writing $\langle f(z), g(z)\rangle$ rather than $\langle f, g\rangle$ to make the reading easier. See for instance equation (11).

Remark 1.5. A kernel $k(z,\omega)$ analytic in z and $\overline{\omega}$ in a neighborhood of $(0,0)$ (see Proposition 2.2) has a power series expansion at (0,0) of the form

$$k(z,\omega) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{\omega}^m. \tag{8}$$

where

$$c_{n,m} = \langle z^n, \omega^m\rangle^{-1} \tag{9}$$

To ease the presentation, we associate to (8) the infinite matrix $C(k) = (c_{m,n})_{n,m=0}^{\infty}$. Note that $C(k)$ does not necessarily need to define a bounded operator in $\ell^2(\mathbb{N}_0)$. For instance, for the Bergman kernel

$$\frac{1}{(1-z\bar{\omega})^2} = 1 + 2z\bar{\omega} + 3(z\bar{\omega})^2 + \cdots,$$

we have

$$C(k) = \begin{pmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ & 0 & & \ddots \end{pmatrix},$$

which is unbounded on $\ell^2(\mathbb{N}_0)$.

The paper consists of four sections besides the introduction. In Section 2 we review a number of definitions and results on reproducing kernel Hilbert spaces of analytic functions. Sections 3, 4, and 5 contain proofs of Theorems 1.1, 1.2, and 1.3 respectively.

2. REPRODUCING KERNEL HILBERT SPACES

In this section we will briefly review the properties of reproducing kernel Hilbert spaces needed in the following sections. We first recall a definition.

Definition 2.1. A reproducing kernel Hilbert space is a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ of functions defined in a non-empty set Ω such that there exists a complex-valued function $k(z,\omega)$ defined on $\Omega\times\Omega$ and with the following properties:

1. $\forall\omega\in\Omega, \quad k_\omega : z\mapsto k(z,\omega)\in\mathcal{H}\to\mathcal{H},$
2. $\forall f\in\mathcal{H}, \quad \langle f, k_\omega\rangle = f(\omega).$

The function $k(z,\omega)$ is uniquely defined by the Riesz representation theorem, and is called the reproducing kernel of the space. The reproducing kernel (kernel, for short) has a very important property: it is positive definite, that is, for all $N\in\mathbb{N}$, $\omega_1,\ldots\omega_N\in\Omega$, and $c_1,\ldots,c_N\in\mathbb{C}$, we have

$$\sum_{i,j=1}^{N} c_j\bar{c}_i k(\omega_i,\omega_j)\geq 0.$$

In particular, it can be shown that the equation above implies that $k(z,\omega)$ is Hermitian, i.e.,

$$k(z,\omega) = \overline{k(\omega,z)}. \tag{10}$$

We refer to the book Saitoh (1988) for more information on reproducing kernel Hilbert spaces, and we recall that there is a one-to-one correspondence between positive definite functions on a given set and reproducing kernel Hilbert spaces of functions defined on that set. In the present work we are interested in the case where Ω is an open neighborhood of the origin, and where the kernels are analytic in z and $\overline{\omega}$. The following result is a direct consequence of Hartog's theorem, and will be used in the sequel. For a different proof, see (Donoghue 1974, p. 92).

Proposition 2.2. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions analytic in $\Omega\subset\mathbb{C}$, with reproducing kernel $k(z,\omega)$. Then the reproducing kernel is jointly analytic in z and $\overline{\omega}$.*

Proof. Since the kernels belong to the space, we have that for every $\omega\in\Omega$ the function $z\mapsto k(z,\omega)$ is analytic in Ω. From (10) it follows that the kernel is also analytic in $\overline{\omega}$. Hartog's theorem (see (Chabat 1990, p. 39)) allows us to conclude that $k(z,\omega)$ is jointly analytic in z and $\overline{\omega}$.

When derivatives come into play, one then has (12) below as the counterpart of (10):

Proposition 2.3. *Under the hypotheses of the above discussion, the elements of the associated reproducing kernel Hilbert space are analytic in* Ω *and the following hold:*

$$(\partial_w f)(\omega) = \langle f(z), \partial_{\overline{\omega}} k_\omega(z)\rangle \tag{11}$$

and

$$\partial_z k(z,\omega_0)|_{z=z_0} = \overline{\partial_{\bar{\omega}} k(\omega_0,\omega)|_{\omega=z_0}}. \tag{12}$$

Proof. The proof of (11) can be found in (Saitoh 1997, Theorem 9, p. 41). We give the proof of (12), where as in Definition 2.1 and in the rest of the work, we use the notation: $k_\beta : z \mapsto k(z,\beta)$ where $\beta \in \Omega$.

Setting $f(z) = k(z,\omega_0)$ in (11) gives

$$\partial_z k(z,\omega_0)|_{z=z_0} = \langle k(z,\omega_0), \partial_{\bar{\omega}} k(z,\omega)|_{\omega=z_0}\rangle$$

and so we have

$$\overline{\partial_z k(z,\omega_0)|_{z=z_0}} = \langle \partial_{\bar{\omega}} k(z,\omega)|_{\omega=z_0}, k(z,\omega_0)\rangle = \partial_{\bar{\omega}} k(z,\omega)|_{z=\omega_0,\omega=z_0},$$

and hence the result.

For some special cases, the reader could also check (12) for $k(z,\omega) = f(z\bar{\omega})$ or for $k(z,w) = a(z)\overline{a(w)}$, where $a(z)$ is analytic in some open subset of the complex plane. In particular, for the latter example we have:

$$\partial_z k(z,\omega_0)|_{z=z_0} = a'(z_0)\overline{a(\omega_0)}$$

on the one hand, and

$$\partial_{\bar{\omega}} k(\omega_0,\omega)|_{\omega=z_0} = a(\omega_0)\overline{a'(z_0)}$$

on the other hand, and hence taking conjugates we see that (12) holds. Since every positive definite function can be represented as an infinite sum of functions of the form $a(z)\overline{a(w)}$ (this is Bergman's reproducing kernel formula, see Aronszajn (1950)), this would give another way to prove (12), after justifying interchange of sum and derivatives, but we preferred to give a direct proof.

The following is a main technical result that we will need in the proofs of the theorems.

Proposition 2.4. *Let* $k(z,\omega)$ *be positive definite and jointly analytic in* z *and* $\overline{\omega}$ *for* z,ω *in an open subset* Ω *of the complex plane. Assume that the operator* ∂_z *is densely defined in the associated reproducing kernel Hilbert space* $\mathcal{H}(k)$*. Then* ∂_z *is closed and in particular has a densely defined adjoint* ∂_z^* *which satisfies* $\partial_z^{**} = \partial_z$*.*

Proof. Let (f_n) be a sequence of elements in $\mathrm{Dom}\,\partial$ and let $f,g \in \mathcal{H}$ be such that

$$f_n \longrightarrow f$$
$$\partial f_n \longrightarrow g$$

where the convergence is in the norm. Since weak convergence follows from strong convergence, using (11), we have for every $\omega \in \Omega$ that

$$\langle f_n, \partial_{\bar{\omega}} k_\omega\rangle \longrightarrow \langle f, \partial_{\bar{\omega}} k_\omega\rangle \quad \text{and} \quad \langle \partial f_n, k_\omega\rangle \longrightarrow \langle g, k_\omega\rangle,$$

where the brackets denote the inner product in $\mathcal{H}(k)$. Hence it follows that

$$\lim_{n\to\infty} f_n'(\omega) = f'(\omega) \quad \text{and} \quad \lim_{n\to\infty} f_n'(\omega) = g(\omega).$$

Thus $g = f'$, and hence ∂ is closed. Hence, ∂ has a densely defined adjoint and $\partial^{**} = \partial$; see e.g. (Reed and Simon 1980, Theorem VIII.1, pp. 252-253).

As an application we prove the following characterization of the Fock space. In the statement, one could assume the functions analytic only in a neighborhood of the origin, and then use analytic continuation.

Proposition 2.5. *The Fock space is the unique (up to a positive multiplicative factor) reproducing kernel Hilbert space of entire functions where the equation*

$$\partial_z^* = M_z$$

holds on the linear span of the kernels (in particular the kernel functions are in the domain of ∂^* *and of* M_z*).*

Proof. Let $k(z,w)$ be the reproducing kernel of the space in the proposition. We want to show that $k(z,w) = ce^{z\overline{\omega}}$ for some $c > 0$. From Proposition 2.2 the kernel is jointly analytic in $\mathbb{D}$. Since $\partial^* = M_z$, it follows that

$$\lesssim \partial_z^* k(z,\omega), k(z,\nu)\rangle = \lesssim M_z k(z,\omega), k(z,\nu)\rangle.$$

Evaluating each side yields the following: For the right hand side we get

$$\lesssim M_z k(z,\omega), k(z,\nu)\rangle = \nu k(\nu,\omega)$$

since $M_z k(z,\omega) = zk(z,\omega)$. The left hand side yields

$$\begin{aligned}
\lesssim \partial_z^* k(z,\omega), k(z,\nu)\rangle &= \lesssim k(z,\omega), \partial_z k(z,\nu)\rangle \\
&= \overline{\lesssim \partial_z k(z,\nu), k(z,\omega)\rangle} \\
&= \overline{\partial_z k(z,\nu)|_{z=\omega}} \\
&= \overline{\partial_\omega k(\omega,\nu)} \\
&= \partial_{\bar{\omega}} k(\nu,\omega),
\end{aligned} \tag{13}$$

where we have used (12) to go from the penultimate line to the last one. Thus we obtain that $\partial_{\bar{\omega}} k(\nu,\omega) = \nu k(\nu,\omega)$, which is a differential equation with the solution

$$k(\nu,\omega) = c(\nu)e^{\nu\bar{\omega}},$$

where the function $c(\nu)$ is an entire function of ν (since $k(\nu,\omega)$ and $e^{\nu\overline{\omega}}$ are entire functions of ν). But $k(\nu,\omega) = \overline{k(\omega,\nu)}$. Hence $c(\nu) = \overline{c(\nu)}$ so that $c(\nu)$ is real valued. Using the Cauchy-Riemann equations, we see that $c(\nu)$ is a constant, which is furthermore positive since the kernel is positive.

Remark 2.6. The Fock space can be described in a geometric way by the Gaussian weight as in (1). The Gaussian weight has other characterizations. We mention in particular the one from information theory: the Gaussian distribution $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ maximizes the entropy

$$-\int_{\mathbb{R}} f(x)\ln f(x)dx$$

among all probability distributions with zero mean and second moment equal to 1; see e.g. (Petz 2008, Exercise 4, p. 50) and (Ash 1990, Theorem 8.3.3, p. 240). It can also be characterized (after normalization) as the unique continuous radial weight function $\omega(z) = \frac{1}{\pi}e^{-|z|^2}$ such that for polynomial p and q under the inner product

$$\lesssim p,q\rangle = \frac{1}{\pi}\iint_{\mathbb{C}} p(z)\bar{q}(z)\omega(z)dA(z),$$

the operator of multiplication and differentiation are adjoint to each other; see Bargmann (1961) (and J. Tung's thesis Tung (1976)).

3. PROOF OF THEOREM 1.1

We first check that the kernel $k_\omega(z) = \frac{1}{1-z\overline{\omega}}$ is a solution of (4), i.e.

$$\langle \partial_z g, k(z,\omega)\rangle = \left\langle g, \partial_z^* k(z,\omega)\right\rangle = \langle g, M_z\partial_z M_z k(z,\omega)\rangle\,,$$

with $g(z) = \frac{1}{1-z\omega^*}$. To verify the above, we compute the left side of the equation and have

$$\langle \partial_z k_\nu(z), k_\omega(z)\rangle = \left\langle \partial_z\left(\frac{1}{1-z\bar{\nu}}\right), k_\omega(z)\right\rangle = \left\langle \frac{\bar{\nu}}{(1-z\bar{\nu})^2}, k_\omega(z)\right\rangle = \frac{\bar{\nu}}{(1-\omega\bar{\nu})^2}.$$

Similarly, we independently calculate the right hand side as

$$\begin{aligned}
\langle k_{\bar{\nu}}(z), M_z\partial_z M_z k_\omega\rangle &= \left\langle k_{\bar{\nu}}(z), M_z\partial_z\left(\frac{z}{1-z\bar{\omega}}\right)\right\rangle \\
&= \left\langle k_{\bar{\nu}}(z), \frac{z}{(1-z\bar{\omega})^2}\right\rangle \\
&= \overline{\left\langle \frac{z}{(1-z\bar{\omega})^2}, k_\nu(z)\right\rangle} \\
&= \frac{\bar{\nu}}{(1-\omega\bar{\nu})^2},
\end{aligned}$$

which comes to be the same as the left hand side.

To prove the converse we apply (4) to kernels, then we use analyticity to find the kernel via its Taylor expansion at the origin. Let $\omega, \nu \in \mathbb{D}$. From (4) we get

$$\langle \partial_z k_\omega, k_\nu \rangle = \lesssim k_\omega, \partial_z^* k_\nu \rangle = \langle k_\omega, M_z \partial_z M_z k_\nu \rangle. \tag{14}$$

We rewrite (4) as

$$\partial_z^* f = z(\partial_z z f) = z(zf' + f) = z^2 f' + zf.$$

By hypothesis the kernel functions belong to the domain of ∂_z^* and we have $\partial_z^{**} = \partial_z$ by Proposition 2.4. Therefore, By by (13) we obtain

$$\lesssim \partial_z^* k_\omega, k_\nu \rangle = (\partial_{\bar{\omega}} k)(\nu, \omega). \tag{15}$$

Then, using the two end sides of (14), we get

$$\begin{aligned}
\lesssim M_z \partial_z M_z k_\nu(z), k_\omega(z) \rangle &= \overline{\lesssim k_\omega(z), M_z \partial_z M_z k_\nu(z) \rangle} \\
&= \overline{\lesssim k(z,\omega), z^2 \partial_z k(z,\nu) + z k(z,\nu) \rangle} \\
&= \overline{\lesssim k(z,\omega), z^2 \partial_z k(z,\nu) \rangle} + \overline{\lesssim k(z,\omega), z k(z,\nu) \rangle} \\
&= \bar{\omega}^2 \partial_{\bar{\omega}} k(\nu,\omega) + \bar{\omega} k(\nu,\omega)
\end{aligned}$$

where we have used (12) to go from the penultimate line to the last one. Considering $k = k(z,\omega)$ and using (14), we get the partial differential equation

$$\partial_z k = \bar{\omega}^2 \partial_{\bar{\omega}} k + \bar{\omega} k. \tag{16}$$

The kernel is analytic in z and $\overline{\omega}$ near the origin, and hence can be written as (8). So we can rewrite (16) as

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n c_{n,m} z^{n-1} \bar{\omega}^m = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m c_{n,m} z^n \bar{\omega}^{m+1} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} z^n \bar{\omega}^{m+1},$$

which can also be written as:

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+1) c_{n+1,0} z^n + \sum_{n=0}^{\infty} (n+1) c_{n+1,1} z^n \overline{\omega} + \sum_{n=0}^{\infty} \sum_{m=2}^{\infty} (n+1) c_{n+1,m} z^n \bar{\omega}^m \\
&= \sum_{n=0}^{\infty} c_{n,0} z^n \overline{\omega} + \sum_{n=0}^{\infty} \sum_{m=2}^{\infty} m c_{n,m-1} z^n \overline{\omega}^m.
\end{aligned}$$

Now we compare the terms on the two sides. First we look at the part which is constant with respect to ω and get $\sum_{n=0}^{\infty} (n+1) c_{n+1,0} z^n = 0$. Hence

$$c_{n+1,0} = 0, \tag{17}$$

for all $n \in \mathbb{N}_0$.

Consider the coefficients of $z^n \overline{\omega}$ on both sides. Then we have $\sum_{n=0}^{\infty} (n+1) c_{n+1,1} z^n \overline{\omega} = \sum_{n=0}^{\infty} c_{n,0} z^n \overline{\omega}$. Hence

$$(n+1) c_{n+1,1} = c_{n,0}, \tag{18}$$

for all $n \in \mathbb{N}_0$. Note that for $n = 0$ we get $c_{0,0} = c_{1,1}$.

Consider the terms $z^n \overline{\omega}^m$, $m \geq 2$. Then $\sum_{n=0}^{\infty} \sum_{m=2}^{\infty} m c_{n,m-1} z^n \overline{\omega}^m = \sum_{n=0}^{\infty} \sum_{m=2}^{\infty} (n+1) c_{n+1,m} z^n \bar{\omega}^m$. Hence

$$m c_{n,m-1} = (n+1) c_{n+1,m}, \tag{19}$$

for all $n \in \mathbb{N}_0$ and $m = 2, 3, \ldots$. Note if $m = n+1$, then $(n+1) c_{n+1,n+1} = (n+1) c_{n,n}$. So

$$c_{0,0} = c_{1,1} = c_{2,2} = \cdots. \tag{20}$$

We now check that $c_{n,m} = 0$ when $n \neq m$. For $0 < m < n+1$, using (18) and (19) it follows that

$$c_{n+1,m} = \alpha_{n,m} c_{n+1-m,0},$$

where $\alpha_{n,m} = \frac{m}{n+1}\frac{m-1}{n}\cdots\frac{1}{n+2-m} \neq 0$, then $c_{n+1,m} = 0$ by (17) for $n+1 > m$. The case $m > n$ is obtained by symmetry.

Hence, all off-diagonal entries of the matrix $C(k)$ (defined in Remark 1.5 will be zero, and it follows from (20) that $k(z,\omega) = \frac{c_{0,0}}{1-z\overline{\omega}}$. This ends the proof of the theorem. □

If we assume that the powers of z are in the domain of ∂^* and of M_z one has a simpler proof for the characterization given in Theorem 1.1 of the Hardy space, close in spirit to Bargmann's arguments. We note that conditions (1)-(4) in the statement of the next result are satisfied by $\mathbf{H}^2$.

Proposition 3.1. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions analytic in a neighborhood of the origin and such that*

1. *M_z bounded,*
2. *$\{z^n\}_{n=0}^{\infty} \subset \mathrm{Dom}\,\partial$,*
3. *$\mathrm{Dom}\,\partial \subset \mathrm{Dom}\,\partial^*$,*
4. *$\partial^* = M_z \partial M_z$.*

Then $\mathcal{H} = \mathbf{H}^2$.

Proof. Let the kernel K of $\mathcal{H}$ have the form in (8). From Proposition 2.2 the kernel is jointly analytic in $\mathbb{D}$. Take $f(z) = z^n$ and $g(z) = z^m$, then

$$\begin{aligned} \langle f, \partial g\rangle &= \langle z^n, mz^{m-1}\rangle \\ &= m\langle z^n, z^{m-1}\rangle, \end{aligned} \qquad \begin{aligned} \langle \partial^* f, g\rangle &= \langle z^2 f' + zf, g\rangle \\ &= \langle nz^{n+1} + z^{n+1}, z^m\rangle \\ &= (n+1)\langle z^{n+1}, z^m\rangle. \end{aligned}$$

Since $\langle f, \partial g\rangle = \langle \partial^* f, g\rangle$, we obtain

$$(n+1)\langle z^{n+1}, z^m\rangle = m\langle z^n, z^{m-1}\rangle. \tag{21}$$

For $m = n+1$, we have

$$(n+1)\langle z^n, z^n\rangle = (n+1)\langle z^{n+1}, z^{n+1}\rangle \implies \langle z^n, z^n\rangle = \langle z^{n+1}, z^{n+1}\rangle,$$

thus the diagonal entries are nonzero. Now we are left to show that if $n \neq m$, $\langle z^n, z^m\rangle = 0$. From (21) we get

$$\langle z^{n+1}, z^m\rangle = \frac{m}{n+1}\langle z^n, z^{m-1}\rangle. \tag{22}$$

Take $f(z) = z^n, n \neq 0$, and $g(z) \equiv 1$; then

$$\begin{aligned} \langle f, \partial g\rangle = \langle \partial^* f, g\rangle &= \langle z^2 f' + zf, g\rangle \\ &= \langle nz^{n+1} + z^{n+1}, 1\rangle \\ &= (n+1)\langle z^{n+1}, 1\rangle. \end{aligned}$$

However $\langle f, \partial g\rangle = 0$, hence $\langle z^{n+1}, 1\rangle = 0$, which also gives $\langle 1, z^{m+1}\rangle = 0$. Then from (9) and (22) all the off-diagonal coefficients $c_{n,m}$ are equal to 0.

More generally, with the same hypothesis as in Proposition 3.1, one could replace $M_z\partial_z$ by a (possibly unbounded) diagonal operator defined as follows:

$$D(z^n) = \alpha_n z^n, \quad n = 0, 1, 2, \ldots,$$

with $\alpha_n > 0$ for $n \geq 1$ and α_0 arbitrary. Such D is called a radial differential operator in the literature. Then we get

$$\langle z^n, z^m\rangle = \delta_{n,m}\frac{n!}{\alpha_n \cdots \alpha_1}\langle 1, 1\rangle.$$

Taking $\beta^{-1} = \langle 1, 1\rangle$, and using (9), the reproducing kernel is given by

$$k(z,\omega) = \beta\sum_{n=0}^{\infty}\frac{\alpha_n\cdots\alpha_1}{n!}z^n\overline{\omega}^n$$

by (9), provided the radius of convergence of the above series is strictly positive.

4. PROOF OF THEOREM 1.2

To prove Theorem 1.2, we use the same strategy as in the previous section. The kernel $\frac{1}{(1-z\bar{\omega})^{\alpha}}$ is a solution of $\partial^* = M_z\partial_z M_z - (1-\alpha)M_z$. This operator applied to this kernel gives us

$$\begin{aligned}
\partial^* k(z,\omega) &= (M_z\partial_z M_z - (1-\alpha)M_z)\left(\frac{1}{(1-z\bar{\omega})^{\alpha}}\right)\\
&= \frac{z}{(1-z\bar{\omega})^{\alpha}} + \alpha\frac{z^2\bar{\omega}}{(1-z\bar{\omega})^{\alpha+1}} - (1-\alpha)\frac{z}{(1-z\bar{\omega})^{\alpha}}\\
&= \frac{\alpha z}{(1-z\bar{\omega})^{\alpha+1}}\\
&= \partial_{\bar{\omega}}\left(\frac{1}{(1-z\bar{\omega})^{\alpha}}\right)\\
&= \partial_{\bar{\omega}} k(z,\omega).
\end{aligned}$$

which implies

$$\lesssim \partial^* k_\nu(z), k_\omega(z)\rangle = \lesssim k_\nu(z), \partial_{\bar{\omega}} k_\omega(z)\rangle.$$

Additionally, we get the relation $z(1-z\bar{\omega}) + \alpha z^2\bar{\omega} - (1-\alpha)z(1-z\bar{\omega}) = \alpha z$.

As we see again, indeed for $\alpha = 1$ we have the Hardy case. To prove the converse we apply (6) to kernels, and find a partial differential equation satisfied by the reproducing kernel. Then we use analyticity to find the kernel via its Taylor expansion at the origin. Let $\omega, \nu \in \mathbb{D}$, then from (6) we get

$$\langle \partial k_\omega, k_\nu\rangle = \lesssim k_\omega, \partial^* k_\nu\rangle = \langle k_\omega, M_z\partial M_z k_\nu + (\alpha-1)M_z k_\nu\rangle. \tag{23}$$

We rewrite (6) as

$$\begin{aligned}
\partial^* f = z(\partial z f) + (\alpha-1)zf &= z^2 f' + zf + \alpha z f - zf\\
&= z^2 f' + \alpha z f.
\end{aligned}\tag{24}$$

From the calculation above similar to (13), it follows that $\lesssim \partial_z k(z,w), k(z,\nu)\rangle = \partial_z k(\nu,\omega)$, thus from (24) and the two end sides of (23). Equation (13) still holds here (it is a general computation valid for kernels analytic in z and ω) and we have

$$\begin{aligned}
\partial_z k(\nu,\omega) &= \partial_z k(z,\omega)|_{z=\nu}\\
&= \lesssim \partial_z k_\omega, k_\nu\rangle\\
&= \lesssim k_\omega, \partial_z^* k_\nu\rangle\\
&= \lesssim k_\omega, M_z\partial M_z k_\nu - (\alpha-1)M_z k_\nu\rangle\\
&= \lesssim k_\omega, \nu^2\partial_z k_\nu + \alpha\nu k_\nu\rangle\\
&= \overline{\lesssim \nu^2\partial k_\nu + \alpha\nu k_\nu, k_\omega\rangle}\\
&= \bar{\omega}^2\partial k(\nu,\omega) + \alpha\bar{\omega}k(\nu,\omega).
\end{aligned}$$

Thus we get the partial differential equation

$$\partial_z k = \bar{\omega}^2\partial_{\bar{\omega}} k + \alpha\bar{\omega}k. \tag{25}$$

The kernel is analytic in z and $\overline{\omega}$ near the origin, and hence can be written as

$$k(\nu,w) = \sum_{n,m=0}^{\infty} c_{n,m}\nu^n\bar{\omega}^m.$$

So we can rewrite (25) as

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} n c_{n,m}\nu^{n-1}\bar{\omega}^m = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} m c_{n,m}\nu^n\bar{\omega}^{m+1} + \alpha\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\nu^n\bar{\omega}^{m+1},$$

which can also be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+1)c_{n+1,0}\nu^n + \sum_{n=0}^{\infty}(n+1)c_{n+1,1}\nu^n\bar{\omega} + \sum_{n=0}^{\infty}\sum_{m=2}^{\infty}(n+1)c_{n+1,m}\nu^n\bar{\omega}^m\\
&= \sum_{n=0}^{\infty}\alpha c_{n,0}\nu^n\bar{\omega} + \sum_{n=0}^{\infty}\sum_{m=2}^{\infty}(\alpha+(m-1))c_{n,m-1}\nu^n\bar{\omega}^m.
\end{aligned}$$

Now we can consider the following cases: First we compare the coefficients for the terms with constant $\bar{\omega}$. Then we have $\sum_{n=0}^{\infty}(n+1)c_{n+1,0}\nu^n = 0$. Hence

$$c_{n+1,0} = 0$$

for all $n \in \mathbb{N}_0$.

Consider the coefficients of $\nu^n\overline{\omega}$. Then we have: $\sum_{n=0}^{\infty}(n+1)c_{n+1,1}\nu^n\bar{\omega} = \sum_{n=0}^{\infty}\alpha c_{n,0}\nu^n\overline{\omega}$. Hence

$$(n+1)c_{n+1,1} = \alpha c_{n,0},$$

for all $n \in \mathbb{N}_0$. Note that for $n = 0$ we get $c_{0,0} = \alpha c_{1,1}$.

Consider the terms $\nu^n\overline{\omega}^m$, $m \geq 2$; then we have

$$\sum_{n=0}^{\infty}\sum_{m=2}^{\infty}(n+1)c_{n+1,m}\nu^n\bar{\omega}^m = \sum_{n=0}^{\infty}\sum_{m=2}^{\infty}(\alpha+(m-1))c_{n,m-1}\nu^n\bar{\omega}^m.$$

Hence

$$(n+1)c_{n+1,m} = (m+\alpha-1)c_{n,m-1}, \tag{26}$$

for all $n \in \mathbb{N}_0$. Note that if $m = n+1$, then $(n+1)c_{n+1,n+1} = (n+\alpha)c_{n,n}$. So

$$c_{n,n} = \left(\frac{n+1}{n+\alpha}\right)c_{n+1,n+1}.$$

we see that the diagonal entries are equal (up to a constant) to the Taylor coefficients in (5).

We now check that $c_{n,m} = 0$ when $n \neq m$. For $0 \leq m \leq n+1$, it follows from (26) that

$$c_{n+1,m} = \phi_{\alpha,n,m}c_{n+1-m,0},$$

for $\phi_{\alpha,n,m} = \frac{m+\alpha-1}{n+1}\frac{m+\alpha-2}{n}\cdots\frac{\alpha}{n+2-m} \neq 0$, and hence the conclusion using (17). The case $m > n$ follows by symmetry. Hence from these cases and by symmetry, all off-diagonal entries of $C(k)$ will be zero, and this completes the proof. □

5. PROOF OF THEOREM 1.3

While with similar spirit in proof structure, unlike in proofs for Theorems 1.1 and 2, we prove (7) for the kernel pointwise for $z, \omega \in \mathbb{D}$. Let $k(\nu, \omega)$ be a solution of (7), with power series expansion

$$k(\nu,\omega) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\nu^n\bar{\omega}^m.$$

Since $k(0,0) = 0$ by hypothesis, we have $c_{0,0} = 0$ (without the condition $k(0,0) = 0$ any constant function is a solution of (7)). We have

$$\partial_\nu^2 k = \sum_{n=2}^{\infty}\sum_{m=0}^{\infty}c_{n,m}n(n-1)\nu^{n-2}\bar{\omega}^m$$

$$\bar{\omega}^2\partial_\nu\partial_{\bar{\omega}}k = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{n,m}nm\nu^{n-1}\bar{\omega}^{m+1}.$$

So we can rewrite (7) in terms of the power series expansion of kernel as:

$$\sum_{n=2}^{\infty}\sum_{m=0}^{\infty}c_{n,m}n(n-1)\nu^{n-2}\bar{\omega}^m = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{n,m}nm\nu^{n-1}\bar{\omega}^{m+1}, \tag{27}$$

which is equivalent to

$$\sum_{n=2}^{\infty}\sum_{m=0}^{\infty}c_{n,m}n(n-1)\nu^{n-2}\bar{\omega}^m = \sum_{m=1}^{\infty}c_{1,m}m\bar{\omega}^{m+1} + \sum_{n=2}^{\infty}\sum_{m=1}^{\infty}c_{n,m}nm\nu^{n-1}\bar{\omega}^{m+1}. \tag{28}$$

Comparing on both sides the part independent of ν we get

$$\sum_{m=1}^{\infty}c_{1,m}m\bar{\omega}^{m+1} = 0, \tag{29}$$

as we have no corresponding terms on the left side.

Let $n = 2$. Then

$$\sum_{m=0}^{\infty} c_{2,m} 2\bar{\omega}^m = \sum_{m=1}^{\infty} c_{2,m} 2m\nu\bar{\omega}^{m+1}. \tag{30}$$

We make the change of index $M = m + 1$ in (29), and obtain

$$\sum_{M=2}^{\infty} c_{1,m-1}(M-1)\bar{\omega}^M = 0. \tag{31}$$

From equations (31) and (30), it follows now that

$$c_{2,0} = c_{2,1} = 0 \quad \text{and} \quad 2c_{2,M} = (M-1)c_{1,M-1} \ \text{ for } \ M > 2.$$

Considering equation (27) and making the change of index $N = n - 2$, $M = m$ to the right side, and $N = n - 1$, $M = m + 1$ to the left side, we get

$$\sum_{N=0}^{\infty}\sum_{M=0}^{\infty} c_{N+2,M}(N+2)(N+1)\nu^N\bar{\omega}^M = \sum_{N=0}^{\infty}\sum_{M=2}^{\infty} c_{N+1,M-1}(N+1)(M-1)\nu^N\bar{\omega}^M. \tag{32}$$

From (32) for $N \in \mathbb{N}_0$ and $M \geq 2$, we have

$$c_{N+2,M}(N+2) = (M-1)c_{N+1,M-1}. \tag{33}$$

We now check that all off diagonal entries of $C(k)$ are indeed zero. Let $M = 0$; then from (27) with the change of variable $N = n - 2$ gives us

$$\sum_{N=0}^{\infty} c_{N+2,0}(N+2)(N+1)\nu^N = 0,$$

so we have

$$c_{N+2,0} = 0 \quad \text{for} \quad N \geq 0.$$

Let $M = 1$; then from (32) we get

$$c_{N+2,1} = 0 \quad \text{for} \quad N \geq 0.$$

Hence all off diagonal entries of $C(k)$ are zero. Since $k(0,0) = 0$ we get that $c_{0,0} = 0$. Finally we set $M = N + 2$ in (32), and get

$$c_{N+2,N+2}(N+2) = (N+1)c_{N+1,N+1}, \quad N = 0, 1, \ldots \tag{34}$$

From (34) we get $c_{N,N} = \frac{1}{N}$ for $N \geq 1$, and the proof is complete. □

Peer Review: Externally peer-reviewed.
Conflict of Interest: Author declared no conflict of interest.
Financial Disclosure: Author declared no financial support.

ACKNOWLEDGEMENTS

I would like to thank the referee for his/her time and comments that helped improve the paper.

LIST OF AUTHOR ORCIDS

N. Alpay https://orcid.org/0000-0002-1505-7531

REFERENCES

Aronszajn, N., 1950, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68, 337–404.
Ash, R.B., 1990, Information Ttheory, Dover Publications Inc., New York, Corrected reprint of the 1965 original.
Bargmann, V., 1961, On a Hilbert space of analytic functions and an associated integral transform, Comm. Pure Appl. Math., 14, 187–214.
Bargmann, V., 1962, Remarks on a Hilbert space of analytic functions, Proceedings of the National Academy of Arts, 48, 199–204.
Chabat,B., 1990, Introduction à l'analyse complexe. Tome 2, Traduit du Russe: Mathématiques. [Translations of Russian Works: Mathematics]. "Mir", Moscow, Fonctions de plusieurs variables. [Functions of several variables], Translated from the Russian by Djilali Embarek.
Donoghue, W.F., 1974, Monotone Matrix Functions and Analytic Continuation, volume 207 of Die Grundlehren der mathematischen Wissenschaften, Springer–Verlag.
Fayngold, M., Fayngold, V., 2013, Quantum mechanics and quantum information: a guide through the quantum world, John Wiley & Sons.
Petz, D., 2008, Quantum Information Theory and Quantum Statistics, Theoretical and Mathematical Physics. Springer-Verlag, Berlin.
Reed, M., Simon, B., 1980, Methods of modern mathematical physics. I, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, Functional analysis.
Saitoh, S., 1988, Theory of Reproducing Kernels and Its Applications, volume 189, Longman scientific and technical.
Saitoh, S., 1997, Integral Transforms, Reproducing Kernels and Their Applications, volume 369 of Pitman Research Notes in Mathematics Series, Longman, Harlow.
Tung, Y.C.J., 1976, Fock Spaces, A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Mathematics) in The University of Michigan, 2005.

Istanbul Journal of Mathematics
ijmath 2023, 1 (1), 12–19
DOI: 10.26650/ijmath.2023.00004

RESEARCH ARTICLE

Unitary weighted composition operators on Bergman-Besov and Hardy Hilbert spaces on the ball

H.T. Kaptanoğlu[1*]

[1]Bilkent University, Department of Mathematics, 06800 Ankara, Türkiye

ABSTRACT
On weighted Bergman and Hardy Hilbert spaces on the unit ball of the complex N-space, we consider weighted compositon operators T_ψ in which the composition is by an automorphism ψ of the unit ball and the weight is a power of the Jacobian of ψ in such a way that the operator is unitary. Assuming that the homogeneous expansion of an f in one of these spaces contains only terms with total degree even (odd, respectively) and the homogeneous expansion of $T_\psi f$ contains only terms with total degree odd (even, respectively), we prove that f is the zero function. We also find related operators on the remaining Bergman-Besov Hilbert spaces including the Drury-Arveson space and the Dirichlet space for which the same result holds. Our results generalize the results obtained in Montes-Rodríguez (2023) on three function spaces on the unit disc to a wider family of function spaces on the unit ball.

Mathematics Subject Classification (2020): Primary 47B33; Secondary 47B32, 47B38, 32A36, 32A35, 32A37, 46E20, 46E22

Keywords: Weighted composition operator, unitary operator, Bergman-Besov space, Hardy space, Dirichlet space, Drury-Arveson space

1. INTRODUCTION

In a recent paper Montes-Rodríguez (2023), the author proves that specific unitary weighted composition operators by the automorphisms of the unit disc on three (Bergman, Hardy, and Dirichlet) Hilbert spaces of holomorphic functions have the property that if a function in one of these spaces and its image under the corresponding operator have different parity, then it is the zero function.

Our objective in this paper is to extend this result to a wider classes of Hilbert spaces of holomorphic functions and to the case of the unit ball of $\mathbb{C}^N$. Moving up to arbitrary-dimensional balls where mappings of several complex variables are used complicate matters considerably. The geometry of Möbius transformations in the ball is more complicated and simple derivatives in the disc need to be replaced by complex Jacobians whose fractional powers are used in the generalizations of the operators of interest to the weighted spaces. Further, Besov Hilbert spaces such as the Dirichlet space have to be handled differently, because the derivatives used in the integral norms of such spaces simply are not compatible with the natural unitary weighted composition operators on them.

To present our result, we now introduce the necessary definitions and notation. Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^N$ with respect to the usual Hermitian inner product $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$, and the associated norm $|z| = \sqrt{\langle z, z\rangle}$. When $N = 1$, the unit ball is the unit disc $\mathbb{D}$ in the complex plane.

Definition 1.1. For $q \in \mathbb{R}$ and $z, w \in \mathbb{B}$, the *Bergman-Besov kernels* are

$$K_q(z,w) := \begin{cases} \dfrac{1}{(1-\langle z,w\rangle)^{1+N+q}} = \displaystyle\sum_{k=0}^{\infty} \frac{(1+N+q)_k}{k!} \langle z,w\rangle^k, & q > -(1+N), \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) = \displaystyle\sum_{k=0}^{\infty} \frac{k!\ \langle z,w\rangle^k}{(1-(N+q))_k}, & q \le -(1+N), \end{cases}$$

where ${}_2F_1 \in H(\mathbb{D})$ is the Gauss hypergeometric function and $(a)_b$ is the Pochhammer symbol.

Corresponding Author: H.T. Kaptanoğlu **E-mail:** kaptan@fen.bilkent.edu.tr
Submitted: 17.05.2023 • **Last Revision Received:** 07.06.2023 • **Accepted:** 01.06.2023 • **Published Online:** 12.06.2023

Definition 1.2. For $q \in \mathbb{R}$, the *Bergman-Besov Hilbert space* $\mathcal{D}_q$ is the reproducing kernel Hilbert space on $\mathbb{B}$ generated by the kernel K_q endowed with the inner product and norm induced by K_q.

The kernels K_q are sesquiholomorphic on $\mathbb{B}^2$ and hence the functions in the $\mathcal{D}_q$ are holomorphic on $\mathbb{B}$. In particular, $\mathcal{D}_q$ is the standard weighted Bergman space A^2_q for $q > -1$, the Hardy space H^2 for $q = -1$, the Drury-Arveson space $\mathcal{A}$ for $q = -N$, and the Dirichlet space $\mathcal{D}$ for $q = -(1+N)$, that is, $\mathcal{D}_{-(1+N)} = \mathcal{D}$.

Let $H(\mathbb{B})$ be the space of all holomorphic functions on $\mathbb{B}$. Every $f \in H(\mathbb{B})$ and thus every $f \in \mathcal{D}_q$ has homogeneous and Taylor expansions

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{|\alpha|=0}^{\infty} f_\alpha z^\alpha \qquad (z \in \mathbb{B}) \tag{1}$$

converging absolutely and uniformly on compact subsets of $\mathbb{B}$, where f_k is a homogeneous polynomial of degree k in $z_1, \dots, z_N$, α is a multi-index, and $k = |\alpha|$. We use the expression $f \in H(\mathbb{B})$ has *even parity* (respectively, *odd parity*) to mean that the homogeneous expansion of f as in (1) has f_k with only even k (respectively, only odd k).

Denote by $\mathcal{M}$ the group of all one-to-one onto holomorphic maps (*automorphisms*) of $\mathbb{B}$. Let $J\psi$ be the complex Jacobian of $\psi \in \mathcal{M}$. For $\psi \in \mathcal{M}$, also $\psi^{-1} \in \mathcal{M}$ and $J\psi \neq 0$ on $\mathbb{B}$.

Definition 1.3. For $q \geq -(1+N)$, $\psi \in \mathcal{M}$ and $f \in \mathcal{D}_q$, define the operator $T^q_\psi : \mathcal{D}_q \to \mathcal{D}_q$ by

$$T^q_\psi f(z) := f(\psi(z))(J\psi(z))^{1+\frac{q}{1+N}}$$

using an appropriate, say the principal, branch of the logarithm for the fractional power of $J\psi(z)$.

So T^q_ψ is the product

$$T^q_\psi = M_{\theta^q_\psi} C_\psi,$$

where C_ψ is the *composition operator* given by $C_\psi f = f \circ \psi$ and $M_{\theta^q_\psi}$ is the *multiplication operator* by

$$\theta^q_\psi(z) = (J\psi(z))^{1+\frac{q}{1+N}}. \tag{2}$$

When $q = -(1+N)$, $T^{-(1+N)}_\psi$ reduces simply to C_ψ on the Dirichlet space. When $q = 0$, $\theta^0_\psi = J\psi$ for the unweighted Bergman space. When $q = -1$, $\theta^{-1}_\psi = (J\psi)^{\frac{N}{1+N}}$ for the Hardy space. When $q = -N$, $\theta^{-N}_\psi = (J\psi)^{\frac{1}{1+N}}$ for the Drury-Arveson space.

In (Beatrous and Burbea 1989, Theorem 1.10), it is proved that T^q_ψ is a unitary operator for $q > -(1+N)$ with respect to the standard inner product that the kernel K_q induces on $\mathcal{D}_q$ given in (5) below. By (Zhu 2005, Section 6.4), $T^{-(1+N)}_\psi$ is unitary on the space $\mathcal{D}_0 = \mathcal{D}/\mathbb{C}$ with respect to the slightly different inner product (6).

Our main result is the following.

Theorem 1.4. *Let $q \geq -1$ and $\psi \in \mathcal{M}$. Suppose the homogeneous expansions (1) of an $f \in \mathcal{D}_q$ and of $T^q_\psi f$ are of different parity, that is, one contains terms only with k even and the other only with k odd. Then $f = 0$. There are also operators on $\mathcal{D}_q$ for $q < -1$ for which the same conclusion is true.*

We prove Theorem 1.4 in Section 3. In the next Section 2, we provide further details and properties on notation, the spaces, and the automorphisms.

2. PRELIMINARIES

In multi-index notation, $\alpha = (\alpha_1, \dots, \alpha_N)$ is an N-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $0^0 = 1$, and $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$. An overbar $\overline{(\ )}$ indicates complex conjugate for numbers and functions and closure for sets. The boundary of $\mathbb{B}$ is the unit sphere $\mathbb{S}$.

The *Pochhammer symbol* $(a)_b$ is defined by

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. This is a shifted rising factorial since $(a)_k = a(a+1)\cdots(a+k-1)$ for positive integer k. In particular, $(1)_k = k!$ and $(a)_0 = 1$. Stirling formula gives

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b}, \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b}, \qquad \frac{(c)_a}{(c)_b q} \sim c^{a-b} \qquad (\operatorname{Re} c \to \infty), \tag{3}$$

where $A \sim B$ means that $|A/B|$ is bounded above and below by two strictly positive constants, that is, $A = O(B)$ and $B = O(A)$ for all A, B of interest.

The *Gauss hypergeometric function* ${}_2F_1 \in H(\mathbb{D})$ is defined by

$$ {}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k. $$

2.1. Spaces

A function $K(z, w)$ is called the *reproducing kernel* of a Hilbert space H of functions defined on $\mathbb{B}$ and with inner product $\langle \cdot, \cdot \rangle_H$ if $K(\cdot, w) \in H$ for each $w \in \mathbb{B}$ and

$$ u(z) = \langle u(\cdot), K(z, \cdot) \rangle_H \qquad (u \in H,\ z \in \mathbb{B}). $$

There is a one-to-one correspondence between reproducing kernel Hilbert spaces and positive definite kernels.

Let $c_k(q)$ be the coefficient of $\langle z, w \rangle^k$ in the series for $K_q(z, w)$. Then $c_0(q) = 1$, $c_k(q) > 0$ for al k, and by (3),

$$ c_k(q) \sim k^{N+q} \qquad (k \to \infty), \tag{4} $$

for every q. This explains the choice of the parameters of the hypergeometric function in K_q for $q < -(1 + N)$. The positive definiteness of $\langle z, w \rangle$ and the positivity of the $c_k(q)$ yield that the K_q are positive definite and thus reproducing kernels. The kernels K_q for $q < -(1 + N)$ appear in the literature first in (Beatrous and Burbea 1989, p. 13). The kernels K_q for $q > -(1 + N)$ can also be written as ${}_2F_1(1, 1 + (N + q); 1; \langle z, w \rangle)$. For $q < -(1 + N)$, the functions in $\mathcal{D}_q$ are bounded on $\mathbb{B}$ while the other $\mathcal{D}_q$ contain unbounded functions.

All Bergman-Besov kernels can be written of the form

$$ K_q(z, w) = \sum_{k=0}^{\infty} c_k(q) \langle z, w \rangle^k = \sum_{k=0}^{\infty} c_k(q) \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w}^\alpha. $$

Then by the theory of reproducing kernel Hilbert spaces and (4), the space $\mathcal{D}_q$ consists of all $f \in H(\mathbb{B})$ with Taylor expansions as in (1) for which

$$ \|f\|^2_{\mathcal{D}_q} := \sum_{|\alpha|=0}^{\infty} |f_\alpha|^2 \|z^\alpha\|^2_{\mathcal{D}_q} := \sum_{|\alpha|=0}^{\infty} |f_\alpha|^2 \frac{1}{c_{|\alpha|}(q)} \frac{\alpha!}{|\alpha|!} \sim \sum_{|\alpha|=1}^{\infty} |f_\alpha|^2 \frac{1}{|\alpha|^{N+q}} \frac{\alpha!}{|\alpha|!} < \infty $$

equipped with the inner product

$$ \langle f, g \rangle_{\mathcal{D}_q} := \sum_{|\alpha|=0}^{\infty} \frac{1}{c_{|\alpha|}(q)} \frac{\alpha!}{|\alpha|!} f_\alpha \overline{g}_\alpha. \tag{5} $$

The case of the Drury-Arveson space is especially simple, because then $q = -N$ and $c_k(-N) = 1$ for all $k = 1, 2, \ldots$. For $q > -(1 + N)$, it is with respect to the inner product in (5) that the operators T^q_ψ are unitary.

Notice that the reproducing kernel of the Dirichlet space is

$$ K_{-(1+N)}(z, w) = \frac{1}{\langle z, w \rangle} \log \frac{1}{1 - \langle z, w \rangle} = \sum_{k=0}^{\infty} \frac{1}{1 + k} \langle z, w \rangle^k $$

and this gives

$$ \langle f, g \rangle_{\mathcal{D}} = \sum_{|\alpha|=0}^{\infty} (1 + |\alpha|) \frac{\alpha!}{|\alpha|!} f_\alpha \overline{g}_\alpha $$

with which $\|1\|_{\mathcal{D}} = 1$. The inner product with respect to which the operator $T^{-(1+N)}_\psi = C_\psi$ is unitary on $\mathcal{D}_0 = \mathcal{D}/\mathbb{C}$ is

$$ \langle f, g \rangle_{\mathcal{D}_0} = \sum_{|\alpha|=1}^{\infty} |\alpha| \frac{\alpha!}{|\alpha|!} f_\alpha \overline{g}_\alpha \tag{6} $$

with which $\|1\|_{\mathcal{D}_0} = 0$.

For $s, t \in \mathbb{R}$, we define the *radial fractional differential operator* D^t_s on $H(\mathbb{B})$ by

$$ D^t_s f := \sum_{k=0}^{\infty} d_k(s, t) f_k := \sum_{k=0}^{\infty} \frac{c_k(s + t)}{c_k(s)} f_k. $$

We have $d_0(s,t) = 1$ so that $D_s^t(1) = 1$, $d_k(s,t) > 0$ for any k, and by (4),

$$d_k(s,t) \sim k^t \qquad (k \to \infty),$$

for any s,t. So D_s^t is a continuous operator on $H(\mathbb{B})$ and is of order t. In particular, $D_s^t z^\alpha = d_{|\alpha|}(s,t) z^\alpha$ for any multi-index α. More importantly,

$$D_s^0 = I, \qquad D_{s+t}^u D_s^t = D_s^{t+u}, \qquad \text{and} \qquad (D_s^t)^{-1} = D_{s+t}^{-t} \tag{7}$$

for $s,t,u \in \mathbb{R}$, where the inverse is two-sided. Here and in any other context, I is the identity operator, Any D_s^t maps $H(\mathbb{B})$ onto itself continuously.

The $d_k(s,t)$ are chosen the way they are in order to have

$$D_q^t K_q(z,w) = K_{q+t}(z,w) \qquad (q,t \in \mathbb{R}),$$

where differentiation is performed on the holomorphic variable z. More interestingly, by (Alpay and Kaptanoğlu 2007, Proposition 3.2),

$$D_s^t(\mathcal{D}_q) = \mathcal{D}_{q+2t} \tag{8}$$

is an isomorphism of Hilbert spaces for any s,t and an isometry when the norms are chosen suitably.

The spaces $\mathcal{D}_q$ have also equivalent inner products and norms that are integrals of functions or their sufficiently high-order derivatives. For fixed $q \in \mathbb{R}$, let $s,t \in \mathbb{R}$ be such that $q + 2t > -1$. By (Alpay and Kaptanoğlu 2007, Definition 3.1c), a family of norms each of which is equivalent to $\|\cdot\|_{\mathcal{D}_q}$ is

$$|||f|||^2_{\mathcal{D}_q} := \int_{\mathbb{B}} |D_s^t f(z)|^2 (1-|z|^2)^{q+2t} \, d\nu(z), \tag{9}$$

where ν is the normalized volume measure on $\mathbb{B}$. Setting $d\nu_q(z) := (1-|z|^2)^q \, d\nu(z)$, equivalently $f \in \mathcal{D}_q$ if and only if $f \in H(\mathbb{B})$ and $D_s^t f \in L^2(\nu_{q+2t})$ for some s,t with $q+2t > -1$, where L^p denotes the Lebesgue classes. For Bergman Hilbert spaces, $q > -1$, we take $t = 0$ and obtain the usual integral norms of these spaces as

$$|||f|||^2_{A^2_q} := \int_{\mathbb{B}} |f(z)|^2 \, d\nu_q(z) \qquad (q > -1).$$

The Hardy space also has an equivalent norm which is the well-known

$$|||f|||^2_{H^2} := \int_{\mathbb{S}} |f(z)|^2 \, d\sigma(z),$$

where σ is the normalized surface measure on $\mathbb{S}$. Each integral norm on every $\mathcal{D}_q$ also has an accompanying integral inner product.

2.2. Möbius Transformations

Following (Rudin 1980, Chapter 2), the *Möbius transformation* that exchanges 0 and $0 \neq a \in \mathbb{B}$ is the map

$$\varphi_a(z) := \frac{a - P_a(z) - \sqrt{1-|a|^2}(I - P_a)(z)}{1 - \langle z, a \rangle} \qquad (z \in \overline{\mathbb{B}}),$$

where $P_a(z) := \langle z, a \rangle a / |a|^2$ is the projection on the complex line passing through 0 and a. It reduces to $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ for $a, z \in \mathbb{D}$ when $N = 1$. Each φ_a is an involution, that is, $\varphi_a^{-1} = \varphi_a$. An extremely useful identity for φ_a is

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1-|a|^2)\,(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)\,(1-\langle a,w\rangle)} \qquad (z,w \in \overline{\mathbb{B}}). \tag{10}$$

The complex Jacobian of $\varphi_a(z)$ is $\det \varphi_a'(z)$ and equals

$$J\varphi_a(z) = \gamma(z) \left(\frac{1 - |\varphi_a(z)|^2}{1-|z|^2} \right)^{(1+N)/2} = \gamma(z) \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2} \right)^{(1+N)/2} \qquad (z \in \overline{\mathbb{B}})$$

for some $\gamma(z) \in \mathbb{C}$ with $|\gamma(z)| = 1$, where in obtaining the second form, (10) is used. Its real Jacobian is

$$J_{\mathbb{R}}\varphi_a(z) = |J\varphi_a(z)|^2 > 0.$$

We need two changes of variables formulas involving $\psi \in \mathcal{M}$. Let $G \subset \mathbb{B}$ and $Q \subset \mathbb{S}$ be Borel sets, $f \in L^1(\nu_q)$, and $F \in L^1(\sigma)$. The first is the usual

$$\int_G f \, d\nu = \int_{\psi^{-1}(G)} f(\psi(w)) J_{\mathbb{R}}\psi(w) \, d\nu(w). \tag{11}$$

The less common second one is obtained by explicitly writing (Rudin 1980, p. 45, (5)) using (10) and is

$$\int_Q F\,d\sigma = \int_{\psi^{-1}(Q)} F(\psi(\eta))(J_{\mathbb{R}}\psi(\eta))^{\frac{N}{1+N}}\,d\sigma(\eta). \tag{12}$$

Let also $\mathcal{U}$ denote the group of all unitary transformations of $\mathbb{C}^N$. All $U \in \mathcal{U}$ are characterized by $\langle Uz, Uw\rangle = \langle z, w\rangle$. If $\psi \in \mathcal{M}$ and $a = \psi^{-1}(0)$, then there is a unique $U \in \mathcal{U}$ such that

$$\psi(z) = U(\varphi_a(z)) \qquad (z \in \mathbb{B}). \tag{13}$$

Since $JU \in \mathbb{C}$ with $|JU| = 1$, we see that $J\psi$ has the same form as $J\varphi_a$ with a (possibly) different $\tilde{\gamma}(z)$ in place of $\gamma(z)$.

If $U \in \mathcal{U}$, then

$$\varphi_a = U^{-1}\varphi_{Ua}U; \tag{14}$$

this is (Cowen and MacCluer 1991, Lemma 2.71). This is useful, because $\mathcal{U}$ acts on $\mathbb{S}$ transitively and we can choose U in such way that Ua has only the first component nonzero and real. The automorphism that maps such a Ua to 0 is especially simple. For example, we use $\varphi_r(z) = -\varphi_{-b}(z)$ with $b = (r, 0, \ldots, 0)$ and $0 < r < 1$ that has the explicit form

$$\varphi_r(z) := \left(\frac{r + z_1}{1 + rz_1}, \frac{\sqrt{1 - r^2}}{1 + rz_1}z'\right) \qquad (z \in \mathbb{B}), \tag{15}$$

where $z = (z_1, z')$ and z' denotes the remaining $N - 1$ components; see (Cowen and MacCluer 1991, p. 98). This φ_r has exactly 2 fixed points, $e_1 = (1, 0, \ldots, 0)$ and $-e_1$, both on $\mathbb{S}$ and none in $\mathbb{B}$.

Möbius transformations map balls onto ellipsoids. We need the ellipsoids described in (Cowen and MacCluer 1991, p. 103) given by

$$E(e_1, u) := \{\, z \in \mathbb{B} : |1 - \langle z, e_1\rangle|^2 \le u(1 - |z|^2) \,\}$$

with $u > 0$. Equivalently, $z \in E(e_1, u)$ if and only if

$$\left|z_1 - \frac{1}{1+u}\right|^2 + \frac{u}{1+u}\,|z'|^2 < \left(\frac{u}{1+u}\right)^2.$$

The ellipsoid $E(e_1, u)$ lies in $\mathbb{B}$, has center $e_1/(1 + u)$, and is tangent to $\mathbb{S}$ at e_1.

3. PROOF OF MAIN RESULT

We prove Theorem 1.4 and a corollary to it, and make some further comments.

Proof of Theorem 1.4. We follow the proof of (Montes-Rodríguez 2023, Theorem 1) with many detailed modifications to adapt it to several complex variables. For $\psi \in \mathcal{M}$, note that $C_{\psi^{-1}}(C_\psi f(z)) = C_{\psi^{-1}} f(\psi(z)) = f(\psi(\psi^{-1}(z))) = f(z)$ and hence $C_\psi^{-1} = C_{\psi^{-1}}$.

First we look at the case of weighted Bergman spaces. But the initial stages of the proof work for $q > -(1 + N)$ and that is what we assume for now. Let $f \in \mathcal{D}_q$. By (13) and (2), $T^q_\psi = M_{\theta^q_\psi} C_{\varphi_a} C_U$ for some $a \in \mathbb{B}$ and $U \in \mathcal{U}$. By the remarks following (13), also

$$\beta T^q_\psi = M_{\theta^q_{\varphi_a}} C_{\varphi_a} C_U$$

for some $\beta \in \mathbb{C}$ with $|\beta| = 1$. By a simple computation with matrices, each $U \in \mathcal{U}$ carries a monomial z^α to a homogeneous polynomial of the same degree $|\alpha|$. Consequently C_U preserves parity. Thus f and $f_1 = C_U f$ have the same parity, and also $g = T^q_\psi f$ and $g_1 = \beta T^q_\psi f$ have the same parity that is opposite to that of f_1 by hypothesis. So without loss of generality we can replace T^q_ψ by $T^q_{\varphi_a}$ and it suffices to consider

$$T^q_{\varphi_a} = M_{\theta^q_{\varphi_a}} C_{\varphi_a}.$$

The fact that $\varphi_a^{-1} = \varphi_a$ implies $C_{\varphi_a}^{-1} = C_{\varphi_a}$ and $C_{\varphi_a}^2 = I$. We have

$$\begin{aligned}(T^q_{\varphi_a})^2 f(z) &= T^q_{\varphi_a}\left((J\varphi_a(z))^{1+\frac{q}{1+N}} f(\varphi_a(z))\right)\\ &= (J\varphi_a(z))^{1+\frac{q}{1+N}}(J\varphi_a(\varphi_a(z)))^{1+\frac{q}{1+N}} f(\varphi_a(\varphi_a(z))).\end{aligned}$$

Since $\varphi_a(\varphi_a(z)) = z$, by the chain rule, $\varphi_a'(\varphi_a(z))\varphi_a'(z) = I$. Taking determinants give $J\varphi_a(\varphi_a(z))J\varphi_a(z) = 1$. This shows that $(T^q_{\varphi_a})^2 f(z) = f(z)$ and $(T^q_{\varphi_a})^2 = I$. Setting $g = T^q_{\varphi_a} f$ gives $f = T^q_{\varphi_a} g$. Thus the case $f \in \mathcal{D}_q$ having even parity and $T^q_{\varphi_a} g \in \mathcal{D}_q$ having odd parity coexists with the case $g \in \mathcal{D}_q$ having even parity and $T^q_{\varphi_a} f \in \mathcal{D}_q$ having odd parity. So it does not matter which case is investigated; let's assume the former and keep the notation $g = T^q_{\varphi_a} f$, Let $V(z) = -z$, which is unitary. Then $C_V f = f$, $C_V g = -g$, and $C_V^{-1} = C_V$. Then also $g = -C_V T^q_{\varphi_a} f$ and $f = C_V T^q_{\varphi_a} g$. Therefore

$$f = -(C_V T^q_{\varphi_a})^2 f; \tag{16}$$

that is, -1 is an eigenvalue of $(C_V T^q_{\varphi_a})^2$ with f as the eigenvector. By the spectral mapping theorem, $+i$ or $-i$ is an eigenvalue of $C_V T^q_{\varphi_a}$ with f as the eigenvector. But

$$C_V T^q_{\varphi_a} f(z) = (T^q_{\varphi_a} f)(-z) = (J\varphi_a(-z))^{1+\frac{q}{1+N}} f(\varphi_a(-z)). \tag{17}$$

Set $\mu_a(z) := \varphi_a(-z) = -\varphi_{-a}(z)$. Then $\mu_a \in \mathcal{M}$, $\mu_a'(z) = \varphi_a'(-z)(-I)$, and $J\mu_a(z) = (-1)^N J\varphi_a(-z)$. Hence

$$\begin{aligned} T^q_{\mu_a} f(z) = (J\mu_a(z))^{1+\frac{q}{1+N}} f(\mu_a(z)) &= (-1)^{N\left(1+\frac{q}{1+N}\right)} (J\varphi_a(-z))^{1+\frac{q}{1+N}} f(\varphi_a(-z)) \\ &= (-1)^{N\left(1+\frac{q}{1+N}\right)} C_V T^q_{\varphi_a} f(z) \end{aligned}$$

using (17), and

$$(T^q_{\mu_a})^2 f(z) = (-1)^{2N\left(1+\frac{q}{1+N}\right)} (C_V T^q_{\varphi_a})^2 f(z) = (-1)^{1+2N\left(1+\frac{q}{1+N}\right)} f(z) = \kappa f(z)$$

using (16), where

$$\kappa = (-1)^{1+2N\left(1+\frac{q}{1+N}\right)}$$

and $|\kappa| = 1$. Thus κ is an eigenvalue of $(T^q_{\mu_a})^2$, and $+\sqrt{\kappa}$ or $-\sqrt{\kappa}$ is an eigenvalue of $T^q_{\mu_a}$, both with eigenvector f. Clearly also $|\pm\sqrt{\kappa}| = 1$.

By (14), $\mu_a = -\varphi_{-a} = -U^{-1}\varphi_{U(-a)}U = U^{-1}(-\varphi_{-U(a)})U$. Choosing $U \in \mathcal{U}$ such that $b := U(a) = (r, 0, \ldots, 0)$ with $0 < r < 1$, we obtain $\mu_a = U^{-1}\varphi_r U$, where φ_r is as in (15). Let $\eta = (\det U)^{1+\frac{q}{1+N}}$. For $f \in \mathcal{D}_q$, we have $T^q_{U^{-1}} f(z) = \overline{\eta} f(U^{-1}z)$ and $T^q_U T^q_{U^{-1}} f(z) = \eta\overline{\eta} f(UU^{-1}z) = f(z)$; hence $(T^q_U)^{-1} = T^q_{U^{-1}}$. Further, we compute that $T^q_{\varphi_r} T^q_{U^{-1}} f(z) = \overline{\eta}(J\varphi_r(z))^{1+\frac{q}{1+N}} f(U^{-1}\varphi_r(z))$ and

$$\begin{aligned} T^q_U T^q_{\varphi_r} T^q_{U^{-1}} f(z) &= \eta\overline{\eta}(J\varphi_r(Uz))^{1+\frac{q}{1+N}} f(U^{-1}\varphi_r(Uz)) \\ &= (J\varphi_r(Uz))^{1+\frac{q}{1+N}} f(\mu_a(z)) \\ &= (J\mu_a(z))^{1+\frac{q}{1+N}} f(\mu_a(z)) = T^q_{\mu_a} f(z), \end{aligned}$$

where the equality before the last one can be seen by evaluating $J\mu_a(z)$ using the chain rule. In other words, $T^q_{\varphi_r} = (T^q_U)^{-1} T^q_{\mu_a} T^q_U$. Since a similarity transformation preserves eigenvalues and eigenvectors, we conclude that $+\sqrt{\kappa}$ or $-\sqrt{\kappa}$ is an eigenvalue of $T^q_{\varphi_r}$ with eigenvector $f \in \mathcal{D}_q$.

We have $\lim_{z\to e_1} \varphi_r(z) = e_1$ and let

$$\delta := \lim_{z\to e_1} \frac{1-|\varphi_r(z)|}{1-|z|} = \lim_{z\to e_1} \frac{1-|\varphi_r(z)|^2}{1-|z|^2},$$

where the limits are unrestricted from within $\mathbb{B}$. A quick computation shows that $\delta = 1/(1+r) < 1$. By (Cowen and MacCluer 1991, Lemma 2.77) due to Julia, $\varphi_r(E(e_1, u)) \subset E(e_1, \delta u)$, and by (Cowen and MacCluer 1991, Proposition 2.85), $E(e_1, \delta u) \subset E(e_1, u)$. Together we have the inclusion $\varphi_r(E(e_1, u)) \subset E(e_1, u)$.

For $n = 1, 2, \ldots$, denote the *forward iterates* of φ_r by $\varphi_r^n = \varphi_r \circ \varphi_r^{n-1}$, where φ_r^0 is the identity, and its *backward iterates* by $\varphi_r^{-n} = (\varphi_r^n)^{-1}$. By the properties on the ellipsoids $E(e_1, u)$ and of φ_r and the Denjoy-Wolff theorem, as $n \to \infty$, φ_r^n converges uniformly on compact subsets of $\mathbb{B}$ to e_1; see (Cowen and MacCluer 1991, Theorem 2.83 and Proposition 2.88). In other words, e_1 is the attracting fixed point of φ_r and its *Denjoy-Wolff point*. Now fix $u = 1$, call the corresponding $E(e_1, 1) =: E$, and let $G = E \setminus \varphi_r(E)$, which is nonempty by above. As a consequence of all the discussion about the ellipsoids, for any $0 < r < 1$ we have

$$\mathbb{B} = \bigcup_{n\in\mathbb{Z}} \varphi_r^n(G) \tag{18}$$

and the sets $\varphi_r^n(G)$ for different n's are disjoint.

In the remaining part of the proof we first restrict to $q > -1$ for which $\mathcal{D}_q = A^2_q$, weighted Bergman spaces. Now applying the change of variables $z = \varphi_r(w)$, using (11) and that f is an eigenvector yield

$$\begin{aligned}\int_{\varphi_r^{n+1}(G)} |f(z)|^2\,d\nu_q(z) &= \int_{\varphi_r^n(G)} |f(\varphi_r(w))|^2(1-|\varphi_r(w)|^2)^q J_{\mathbb{R}}\varphi_r(w)\,d\nu(w)\\ &= \int_{\varphi_r^n(G)} |f(\varphi_r(w))|^2\frac{(1-|\varphi_r(w)|^2)^{1+N+q}}{(1-|w|^2)^{1+N}}\,d\nu(w)\\ &= \int_{\varphi_r^n(G)} |f(\varphi_r(w))|^2\big|(J\varphi_r(w))^{1+\frac{q}{1+N}}\big|^2(1-|w|^2)^q\,d\nu(w)\\ &= \int_{\varphi_r^n(G)} |T^q_{\varphi_r}f(w)|^2\,d\nu_q(w) = \int_{\varphi_r^n(G)} |\pm\sqrt{\kappa}f(w)|^2\,d\nu_q(w)\\ &= \int_{\varphi_r^n(G)} |f(z)|^2\,d\nu_q(z).\end{aligned}$$

Thus the above integrals have the same value on all the sets $\varphi_r^n(G)$ for $n \in \mathbb{Z}$ which is equal to the value of the integral on $\varphi_r^0(G) = G$. But f is an eigenvector and hence is not the zero function, and since $f \in H(\mathbb{B})$, none of the integrals on the $\varphi_r^n(G)$ is 0. On the other hand, $f \in \mathcal{D}_q$ and hence $|||f|||_{\mathcal{D}_q} < \infty$. But by (18) we also have

$$|||f|||^2_{\mathcal{D}_q} = \int_{\mathbb{B}} |f(z)|^2\,d\nu_q(z) = \sum_{n\in\mathbb{Z}}\int_{\varphi_r^n(G)} |f(z)|^2\,d\nu_q(z) = \sum_{n\in\mathbb{Z}}\int_G |f(z)|^2\,d\nu_q(z) = \infty.$$

This contradiction shows that a nonzero f having the parity properties in the statement of the theorem cannot exist for $q > -1$.

Next we take care of the case $q = -1$, the Hardy space. Let D be the intersection of the ellipsoid $E = E(e_1, 1)$ with the complex line $[e_1]$ through 0 and e_1, which is given by $|z_1 - 1/2| < 1/4$. The set $\widetilde{G} = D \setminus \varphi_r(D)$ is nonempty just like $G \neq \emptyset$. Let also $Q = \{\,(z_1, z') : z_1 \in \widetilde{G},\ |z_1|^2 + |z'|^2 = 1\,\}$; this is that part of $\mathbb{S}$ that lies "above" $\widetilde{G}$. We have $\mathbb{B} \cap [e_1] = \bigcup_{n\in\mathbb{Z}} \varphi_r^n(\widetilde{G})$ and $\mathbb{S} = \{\,(z_1, z') : z_1 \in \mathbb{D},\ |z_1|^2 + |z'|^2 = 1\,\}$, the second modulo a set of σ-measure 0. Then just like (18), we have $\mathbb{S} = \bigcup_{n\in\mathbb{Z}} \varphi_r^n(Q)$ modulo a set of σ-measure 0, which is a disjoint union.

Now we apply the change of variables $\zeta = \varphi_r(\eta)$, use (12) and that f is an eigenvector to obtain

$$\begin{aligned}\int_{\varphi_r^{n+1}(Q)} |f(\zeta)|^2\,d\sigma(\zeta) &= \int_{\varphi_r^n(Q)} |f(\varphi_r(\eta))|^2\big|(J\varphi_r(\eta))^{1-\frac{1}{1+N}}\big|^2\,d\sigma(\eta)\\ &= \int_{\varphi_r^n(Q)} |T^{-1}_{\varphi_r}f(\eta)|^2\,d\sigma(\eta) = \int_{\varphi_r^n(Q)} |\pm\sqrt{\kappa}f(\eta)|^2\,d\sigma(\eta)\\ &= \int_{\varphi_r^n(Q)} |f(\zeta)|^2\,d\sigma(\zeta).\end{aligned}$$

As in the case $q > -1$, each integral on $\varphi_r^n(Q)$ can be replaced by one on Q. But $f \in H^2$ is an eigenvector, so is not the zero function, and by (Rudin 1980, Theorem 5.6.4 (b)), its boundary values on $\mathbb{S}$ are nonzero σ-a.e.. Then none of the integrals on the $\varphi_r^n(Q)$ is 0. On the other hand, $f \in H^2$ and hence $|||f|||_{H^2} < \infty$. Similar to the case $q > -1$, we have

$$|||f|||^2_{H^2} = \int_{\mathbb{S}} |f(\zeta)|^2\,d\sigma(\zeta) = \sum_{n\in\mathbb{Z}}\int_{\varphi_r^n(Q)} |f(\zeta)|^2\,d\sigma(\zeta) = \sum_{n\in\mathbb{Z}}\int_Q |f(\zeta)|^2\,d\sigma(\zeta) = \infty.$$

By this contradiction, the theorem is proved for the case $q = -1$ too.

Lastly, we consider the case $q < -1$. Pick $s, t \in \mathbb{R}$ such that $p = q + 2t > -1$. Here we prove the result not for T^q_ψ but for $Y^q_\psi = D^{-t}_{s+t}T^p_\psi D^t_s$ where $T^p_\psi : A^2_p \to A^2_p$. By (8) and (7), we have $Y^q_\psi : \mathcal{D}_q \to \mathcal{D}_q$. We have also $T^p_\psi = D^t_s Y^q_\psi D^{-t}_{s+t}$ and that Y^q_ψ and T^p_ψ are similar operators. Now suppose, without loss of generality, that $f \in \mathcal{D}_q$ has even parity and $Y^q_\psi f$ has odd parity. By their very definitions, the D^t_s preserve parity and $g = D^t_s f \in A^2_p$ also has even parity. On the other hand, since $f = D^{-t}_{s+t}g$, we see that $T^p_\psi g = D^t_s Y^q_\psi f$ has odd parity. By the already proved case for the Bergman space A^2_p, we conclude that $g = 0$. Then also $f = 0$. In fact, $Y^q_\psi f = \lambda f$ if and only if $D^{-t}_{s+t}T^p_\psi g = \lambda f$ if and only if $T^p_\psi g = \lambda g$ for some $\lambda \in \mathbb{C}$.

The proof of Theorem 1.4 is now complete.

Remark 3.1. The change of variables that is used in transformung the integrals in the case $q > -1$ can be expressed in the form that the measures ν_q are invariant under the transformations $Z^q_\psi f(z) := f(\psi(z))(J\psi(z))^{2\left(1+\frac{q}{1+N}\right)}$ in the sense that

$$\int_{\mathbb{B}} Z^q_\psi f\,d\nu_q = \int_{\mathbb{B}} f\,d\nu_q \qquad (f \in L^1(\nu_q),\ q \in \mathbb{R}).$$

This is already noted in (Kaptanoğlu 2005, (21)).

Remark 3.2. The last part of the proof involving integrals does not work for $q < -1$, because a positive-order derivative on f is required in the integral norms of $\mathcal{D}_q$ on $\mathbb{B}$ for all $q < -1$; see (Kaptanoğlu and Üreyen 2018, Corollary 7.2). So, for example, if we pick t so that $q + 2t = 0$ for simplicity in (9), we end up with an integral of $|T^0_{\varphi_r} D^t_s f|^2$ on $\varphi^n_r(G)$. For the proof to go through, we need $D^t_s f \in A^2_0$ to be an eigenvector of $T^0_{\varphi_r}$. This would be implied by $T^0_{\varphi_r} D^t_s f$ having odd parity when f and hence $D^t_s f$ have even parity. But what we know is that $T^q_{\varphi_r} f$ has odd parity and this need not imply that $T^0_{\varphi_r} D^t_s f$ has odd parity because of the differences between T^q_ψ and T^0_ψ.

Such differences do not prevent Montes-Rodríguez (2023) from obtaining the theorem for the Dirichlet space, because when $N = 1$, the first-order ordinary derivative and the chain rule are enough to move between that space and the unweighted Bergman space. Neither of these tools is available for $N \geq 2$. These are exactly the reasons why we resort to the other operators Y^q_ψ when $q < -1$.

Remark 3.3. The value κ depends in general on both N and q. For the unweighted Bergman space, $q = 0$, $\kappa = (-1)^{1+2N} = -1$, and the eigenvalues that are shown not to exist in the proof of Theorem 1.4 are $+\sqrt{\kappa} = +i$ and $-\sqrt{\kappa} = -i$ independently of dimension N. For the Hardy space, $q = -1$ and $\kappa = (-1)^{\frac{1+N+2N^2}{1+N}}$. If also $N = 1$, $\kappa = (-1)^2 = +1$ and the eigenvalues that are shown not to exist in the proof of Theorem 1.4 are $+\sqrt{\kappa} = +1$ and $-\sqrt{\kappa} = -1$, contrary to what is claimed in the proof of (Montes-Rodríguez 2023, Theorem 1). But as already noted in (Montes-Rodríguez 2023, Remark 1), the proof of (Montes-Rodríguez 2023, Theorem 1) as well as of Theorem 1.4 here depend only on $|\pm\sqrt{\kappa}| = 1$ and are unaffected.

Corollary 3.4. Let $q \geq -1$ and $\psi \in \mathcal{M}$. There is a nonzero function $f \in \mathcal{D}_q$ such that f and $T^q_\psi f$ have the same parity if and only if $\psi = U \in \mathcal{U}$. For $q < -1$, the same result holds for the operators Y^q_ψ.

Proof. Let $q \geq -1$ first. If $\psi = U \in \mathcal{U}$, since compositions with U and multiplication with complex numbers that are Jacobians of such composition operators preserve parity, there are f as claimed.

Conversely, let $\psi = \varphi_a$ in which $a \neq 0$ and suppose an f as claimed exists. If we repeat the proof of Theorem 1.4 carefully considering the case f even and $T^q_{\varphi_a} f$ even and the case f odd and $T^q_{\varphi_a} f$ odd, we obtain $f = (C_V T^q_{\varphi_a})^2 f$ instead of (16). This in turn yields two complex numbers of modulus 1 one of which is an eigenvalue of $T^q_{\mu_a}$ and of $T^q_{\varphi_r}$ with f as an eigenvector. In the rest of the proof, the only property of the eigenvalues used is that they are of modulus 1. Again we conclude that $f = 0$.

The case $q \leq -1$ is automatic since it depends on the conclusion of the case $q > -1$.

Peer Review: Externally peer-reviewed.
Conflict of Interest: Author declared no conflict of interest.
Financial Disclosure: Author declared no financial support.

ACKNOWLEDGEMENTS

The author thanks A. Ersin Üreyen of Eskişehir Technical University for useful discussions and an anonymous referee for careful attention to details.

LIST OF AUTHOR ORCIDS

H.T. Kaptanoğlu https://orcid.org/0000-0002-8795-4426

REFERENCES

Alpay, D., Kaptanoğlu, H.T., 2007, Toeplitz operators on Arveson and Dirichlet spaces, Integral Equations Operator Theory, 58, 1–33.
Beatrous, F., Burbea, J., 1989, Holomorphic Sobolev spaces on the ball, Dissertationes Math., 276, 57 pp.
Cowen, C.C., MacCluer, B.D., 1991, Composition Operators on Spaces of Analytic Functions, Stud. Adv. Math., CRC Press, Boca Raton.
Kaptanoğlu, H.T., 2005, Bergman projections on Besov spaces on balls, Illinois J. Math., 49, 385–403.
Kaptanoğlu , H.T., Üreyen, A.E., 2018, Precise inclusion relations among Bergman-Besov and Bloch-Lipschitz spaces and H^∞ on the unit ball of $\mathbb{C}^N$, Math. Nachr., 291, 2236–2251.
Montes-Rodríguez, A., 2023, Zeros under unitary weighted composition operators in the Hardy and Bergman spaces, Mediterr. J. Math., 20, #82, 9 pp.
Rudin, W., 1980, Function Theory in the Unit Ball of $\mathbb{C}^n$, Grundlehren Math. Wiss., vol. 241, Springer, New York.
Zhu, K., 2005, Spaces of Holomorphic Functions in the Unit Ball, Grad. Texts in Math., vol. 226, Springer, New York.

Istanbul Journal of Mathematics
ijmath 2023, 1 (1), 20–27
DOI: 10.26650/ijmath.2023.00002

RESEARCH ARTICLE

Notes on multipliers on weighted Orlicz spaces

B. Arıs[1,2], R. Üster[1*]

[1]İstanbul University, Faculty of Science, Department of Mathematics, Vezneciler, 34134, İstanbul, Türkiye
[2]İstanbul University, Institue of Graduate Studies in Science, Süleymaniye, 34134, İstanbul, Türkiye

ABSTRACT
Let G be a locally compact abelian group with Haar measure μ, Φ be a Young function and ω be a weight function. In this paper, we consider the weighted Orlicz space $L^{\Phi}(G,\omega)$ and we investigate the relationship between the multipliers $L1(G,\omega)$-module and the multipliers on a certain Banach algebra. For this purpose, we firstly define temperate function space with respect to the weighted Orlicz space $L^{\Phi}(G,\omega)$ which we denote by $L^{\Phi}t(G,\omega)$ and give its basic properties. Later, we define a subalgebra of the space of multipliers on $L^{\Phi}(G,\omega)$ and study its basic properties. We also show that this subalgebra is isometrically isomorphic to the space of multipliers of a certain Banach algebra. Moreover, we obtain a characterization for the space of multipliers of $L1(G,\omega)\cap L^{\Phi}(G,\omega)$.

Mathematics Subject Classification (2020): 43A15, 43A22, 46E30

Keywords: multiplier, weighted Orlicz space, locally compact group

1. INTRODUCTION

An Orlicz space is a type of function space generalizing the L^p-space. Besides the L^p spaces, a variety of function spaces arises naturally in analysis in this way such as $L\log^+ L$, which is a Banach space related to Hardy-Littlewood maximal functions. Orlicz spaces could also contain certain Sobolev spaces as subspaces. Linear properties of Orlicz spaces have been studied thoroughly (see Başar E., Öztop, S., Uysal, B.H., Yaşar, Ş. (2023); Osançlıol, A., Öztop, S. (2015); Öztop, S., Samei, E. (2017); Öztop, S., Samei, E. (2019); Rao, M. M., Ren, Z. D. (1991) for example). Similar to L^p spaces, one could also consider weighted Orlicz spaces and studied their properties. Very recently the weighted Orlicz space is studied as Banach algebra with respect to convolution for which the corresponding space becomes an algebra and studied their properties such as existence of an approximate identity in compactly supported continuous function spaces of norm one (see Osançlıol, A., Öztop, S. (2015)).

On the other hand, there are a lot of results in abstract harmonic analysis on locally compact groups regarding multipliers for various function spaces. The multipliers of the group algebras of L^p were studied by many authors (see Feichtinger, H. (1976); Fisher, M. J. (1974); Griffin, J., McKennon, K. (1973); McKennon, K. (1972)). In Öztop, S. (2003), Öztop studied the space of multiplier of $L^1(G,A)\cap L^p(G,A)$ where A is a commutative Banach algebra and G is a locally compact abelian group. In Üster, R., Öztop, S. (2020), Üster and Öztop studied compact multiplier problem for $L^{\Phi}(G)$ and in Üster, R. (2021), this concept is extended to $L^{\Phi}(G,\omega)$ spaces by Üster.

Let A be a Banach algebra and E be an A-module. Then, E is essential if the linear span of the elements a, x for $a\in A$ and $x\in E$ is dense in E. A Banach algebra A is called without order, if for all $x\in A$, $xA = Ax = \{0\}$ implies $x = 0$. It is known that if A has an approximate identity, then it is without order (see (Larsen, R. 1971, p.13)). A multiplier of A is a mapping $T : A\to A$ such that

$$T(fg) = fT(g) = (Tf)g, \qquad f,g\in A. \tag{1}$$

Let us denote the collection of all multipliers of A by $M(A)$. Then, every multiplier turns out to be a bounded linear operator on A. If A is commutative Banach algebra without order, then $M(A)$ is a commutative operator algebra and $M(A)$ is called the multiplier algebra of A (see (Wang, J. K. 1961, Theorem 2.2)).

Our goal in this paper is to study the relationship between the multipliers $L^1(G,\omega)$-module and the multipliers on a certain Banach algebra. It is well known that $L^{\Phi}(G,\omega)$ is an essential Banach $L^1(G,\omega)$-module with respect to convolution product (see (Öztop, S., Samei, E. 2017, Lemma 3.2)). Moreover, we obtain a characterization for the space of multipliers of $L^1(G,\omega)\cap L^{\Phi}(G,\omega)$.

Corresponding Author: R. Üster **E-mail:** ruya.uster@istanbul.edu.tr
Submitted: 15.04.2023 • **Revision Requested:** 27.05.2023 • **Last Revision Received:** 31.05.2023 • **Accepted:** 01.06.2023 • **Published Online:** 06.06.2023

This paper is organized as follows. First we present necessary definitions and some basic results that will be used in this paper. In Section 3, we construct the space of Φ-temperate functions for $L^{\Phi}(G,w)$ and study their basic properties. In Section 4, we characterize the space of multipliers of $L^{\Phi}(G,w)$ as a certain Banach algebra and extend the results in Öztop, S. (2003) to weighted Orlicz space. In Section 5, we study the space of multipliers for $L^{1}(G,\omega)\cap L^{\Phi}(G,\omega)$.

2. PRELIMINARIES

Let us recall some facts concerning Young functions and Orlicz spaces.

An Orlicz space is determined by a Young function. A convex function $\Phi:[0,\infty)\to[0,\infty]$ is called a Young function if $\Phi(0)=0$, $\lim\limits_{x\to 0^{+}}\Phi(x)=0$ and $\lim\limits_{x\to\infty}\Phi(x)=\infty$.

For a Young function Φ, its complementary function Ψ is given by

$$\Psi(y)=\sup\{xy-\Phi(x):x\geq 0\},\qquad y\geq 0$$

and Ψ is also a Young function. Then, (Φ,Ψ) is called a complementary Young pair.

By our definition, a Young function can have the value ∞ at a point, and hence be discontinuous at such a point. However, we always consider a pair of complementary Young functions (Φ,Ψ) with both Φ and Ψ being continuous and strictly increasing. In particular, they attain positive values on $(0,\infty)$. Note that even though Φ is continuous, it may happen that Ψ is not continuous.

A Young function Φ satisfies the Δ_2 condition if there exist a constant $K>0$ and an $x_0\geq 0$ such that $\Phi(2x)\leq K\Phi(x)$ for all $x\geq x_0$. In this case, we write $\Phi\in\Delta_2$.

Let G be a locally compact abelian group with a Haar measure μ. Given a Young function Φ, the Orlicz space $L^{\Phi}(G)$ on G is defined by

$$L^{\Phi}(G)=\Big\{f:G\to\mathbb{C}:\int\limits_{G}\Phi(\alpha|f(x)|)d\mu(x)<\infty \text{ for some } \alpha>0\Big\}.$$

The Orlicz space is a Banach space under the Orlicz norm $\|\cdot\|_{\Phi}$ defined for $f\in L^{\Phi}(G)$ by

$$\|f\|_{\Phi}=\sup\left\{\int\limits_{G}|f(x)v(x)|d\mu(x):\int\limits_{G}\Psi(|v(x)|)d\mu(x)\leq 1\right\}$$

where Ψ is the complementary Young function of Φ.

Let (Φ,Ψ) be a complementary Young pair. If $\Phi\in\Delta_2$, then the dual space $L^{\Phi}(G)^{*}$ is $L^{\Psi}(G)$ (Rao, M. M., Ren, Z. D. 1991, Corollary 3.4.5). If in addition $\Psi\in\Delta_2$, then the Orlicz space $L^{\Phi}(G)$ is a reflexive Banach space. We have already mentioned that Orlicz spaces are generalizations of Lebesgue spaces. For $1\leq p<\infty$ and $\Phi(x)=\frac{x^p}{p}$, the space $L^{\Phi}(G)$ becomes the Lebesgue space $L^{p}(G)$ and the norm $\|\cdot\|_{\Phi}$ is equivalent to the classical norm $\|\cdot\|_{p}$. Particularly, if $p=1$ and $\Phi(x)=x$, then Ψ the complementary Young function of Φ is 0 when $0\leq x\leq 1$, and ∞ when $1<x<\infty$. In this case $\|f\|_{\Phi}=\|f\|_{1}$ for all $f\in L^{1}(G)$. If $p=\infty$, then for the defined function Ψ, the space $L^{\Psi}(G)$ is equal to the space $L^{\infty}(G)$ and we have $\|f\|_{\Psi}=\|f\|_{\infty}$ for all $f\in L^{\infty}(G)$.

For further information on Orlicz spaces, the reader is referred to Rao, M. M., Ren, Z. D. (1991).

On the other hand, weights and weighted function spaces play an important role in mathematical analysis and their applications. In addition to this, weights appear naturally in analysis.

Let G be a locally compact group. In this paper, we consider a weight function as a function $\omega:G\to\mathbb{R}^{+}$ with $\omega(xy)\leq\omega(x)\omega(y)$, $(x,y\in G)$ that $\omega(e)=1$ and $\frac{1}{\omega}\in L^{\infty}_{\text{loc}}(G)$, here $L^{\infty}_{\text{loc}}(G)$ denotes the space of all locally essentially bounded functions on G. There is no loss of generality in assuming that the weight ω is continuous (see (Reiter H., Stegeman J. D. 2000, Section 3.7)).

In Osançlıol, A., Öztop, S. (2015), Osançlıol and Öztop introduced the weighted Orlicz space $L^{\Phi}(G,\omega)$ on a locally compact group G as

$$L^{\Phi}(G,\omega)=\{f:f\omega\in L^{\Phi}(G)\}$$

with the norm

$$\|f\|_{\Phi,\omega}=\|f\omega\|_{\Phi}$$

for $f\in L^{\Phi}(G,\omega)$. Also, they studied them as Banach algebras with respect to the convolution product. One can observe that if $\omega=1$, then the weighted Orlicz spaces $L^{\Phi}_{\omega}(G)$ become the space $L^{\Phi}(G)$.

Now, for each $f\in L^{1}(G,\omega)$ define the mapping T_f by $T_f(g)=f*g$ whenever $g\in L^{\Phi}(G,\omega)$. T_f is an element of $B(L^{\Phi}(G,\omega))$, which is Banach algebra of all continuous linear operators from $L^{\Phi}(G,\omega)$ to $L^{\Phi}(G,\omega)$, and $\|T_f\|\leq\|f\|_{1,\omega}$. Identifying $f\mapsto T_f$, we obtain an embedding of $L^{1}(G,\omega)$ in $B(L^{\Phi}(G,\omega))$. We denote the space of all of $L^{1}(G,\omega)$-module homomorphisms of

$L^\Phi(G,\omega)$ by $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$, that is, an operator $T \in B(L^\Phi(G,\omega))$ satisfies $T(f*g) = f*T(g)$ for each $f \in L^1(G,\omega)$ and $g \in L^\Phi(G,\omega)$.

We define

$$(f \circ T)(g) = f * T(g) = T(f*g) \tag{2}$$

for all $g \in L^\Phi(G,\omega)$ and $f \in L^1(G,\omega)$. The module homomorphisms space $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$ is an essential $L^1(G,\omega)$-module with respect to product defined in (2) and is called the space of multipliers of $L^\Phi(G,\omega)$.

Throughout this paper, G is an abelian locally compact group and we are mainly interested in weighted Orlicz spaces $L^\Phi(G,\omega)$ with the weight ω and the Δ_2-condition on a Young function Φ.

The definitions, notations and results of this section are adjusted according to the corresponding content of Section 2 of Öztop, S. (2003).

3. THE Φ-TEMPERATE SPACE

In this section, we define the Φ-temperate function space $L_t^\Phi(G,\omega)$ and give a closed linear subspace of $B(L^\Phi(G,\omega))$ by using the Φ-temperate functions. Moreover, we study some basic properities of these spaces.

Definition 3.1. An element $f \in L^\Phi(G,\omega)$ is called Φ-temperate function if

$$\|f\|^t_{\Phi,\omega} = \sup\{\|g*f\|_{\Phi,\omega} : g \in L^\Phi(G,\omega), \|g\|_{\Phi,\omega} \leq 1\} < \infty$$

or equivalently

$$\|f\|^t_{\Phi,\omega} = \sup\{\|g*f\|_{\Phi,\omega} : g \in C_c(G), \|g\|_{\Phi,\omega} \leq 1\} < \infty.$$

The space of all Φ-temperate functions f is denoted by $L_t^\Phi(G,w)$. One can observe that

$$(L_t^\Phi(G,w), \|\cdot\|^t_{\Phi,\omega})$$

is a normed space. Indeed, let $f \in L_t^\Phi(G,w)$. By the definition of the norm $\|\cdot\|^t_{\Phi,\omega}$, we obtain $\|f\|^t_{\Phi,\omega} \geq 0$. On the other hand, if $f=0$, then $\|f\|^t_{\Phi,\omega} = 0$ is obvious. Conversely, let $\|f\|^t_{\Phi,\omega} = 0$. Then, we have $\sup\{\|g*f\|_{\Phi,\omega} : g \in C_c(G), \|g\|_{\Phi,\omega} \leq 1\} = 0$ and so $g*f(x) = 0$ for all $x \in G$. Since $g \in C_c(G)$ and $C_c(G)$ is dense in $L^1(G,\omega)$, we obtain $g*f \longrightarrow f$ and so $f=0$. For each $f \in L_t^\Phi(G,w)$ and $\alpha \in \mathbb{K}$, we have

$$\begin{aligned}
\|\alpha f\|^t_{\Phi,\omega} &= \sup\{\|g*(\alpha f)\|_{\Phi,\omega} : g \in C_c(G), \|g\|_{\Phi,\omega} \leq 1\} \\
&= \sup\{\|\alpha(g*f)\|_{\Phi,\omega} : g \in C_c(G), \|g\|_{\Phi,\omega} \leq 1\} \\
&= |\alpha| \sup\{\|g*f\|_{\Phi,\omega} : g \in C_c(G), \|g\|_{\Phi,\omega} \leq 1\} \\
&= |\alpha| \|f\|^t_{\Phi,\omega}.
\end{aligned}$$

Finally, for any $f_1, f_2 \in L_t^\Phi(G,w)$, we have

$$\begin{aligned}
\|f_1+f_2\|^t_{\Phi,\omega} &= \sup\{\|g*(f_1+f_2)\|_{\Phi,\omega} : g \in C_c(G), \|g\|_{\Phi,\omega} \leq 1\} \\
&\leq \sup\{\|(g*f_1)\|_{\Phi,\omega} + \|(g*f_2)\|_{\Phi,\omega} : g \in C_c(G), \|g\|_{\Phi,\omega} \leq 1\} \\
&\leq \|f_1\|^t_{\Phi,\omega} + \|f_2\|^t_{\Phi,\omega}.
\end{aligned}$$

For each $f \in L_t^\Phi(G,w)$, there exists precisely one bounded linear operator on $L^\Phi(G,\omega)$, denoted by W_f, such that $W_f : L^\Phi(G,\omega) \longrightarrow L^\Phi(G,\omega)$

$$W_f(g) = g*f \text{ and } \|W_f\| = \|f\|^t_{\Phi,\omega}. \tag{3}$$

The linearity of W_f is obvious and since we have

$$\|W_f\| = \|f\|^t_{\Phi,\omega} = \sup\{\|g*f\|_{\Phi,\omega} : g \in L^\Phi(G,\omega), \|g\|_{\Phi,\omega} \leq 1\} \leq \|f\|_{\Phi,\omega},$$

then W_f is bounded.

Also, we observe that $W_f(h*g) = (h*g)*f = h*(g*f) = h*W_f(g)$ for each $f \in L_t^\Phi(G,w)$, $g \in L^\Phi(G,\omega)$. Hence, we obtain $W_f \in \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$.

Proposition 3.2. *Let Φ be a Young function. Then $L_t^\Phi(G,w)$ is a dense subspace of $L^\Phi(G,\omega)$.*

Proof. Since each $f \in C_c(G)$ belongs to $L_t^\Phi(G,w)$ and $C_c(G)$ is dense in $L^\Phi(G,\omega)$, we have the required result. □

Lemma 3.3. *The space $L_t^\Phi(G,w)$ is a normed algebra with the convolution product.*

Proof. By (3), we have

$$\begin{aligned}\|f*g\|^t_{\Phi,\omega} &= \sup\{\|h*(f*g)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\}\\ &= \sup\{\|g*(h*f)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\}\\ &= \sup\{\|W_g(h*f)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\}\\ &\le \|W_g\| \sup\{\|h*f\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\}\\ &= \|g\|^t_{\Phi,\omega}\|f\|^t_{\Phi,\omega}\end{aligned}$$

for all $f, g \in L^\Phi_t(G, w)$. Hence, $(L^\Phi_t(G, w), \|\cdot\|^t_{\Phi,\omega})$ is a normed algebra. □

Note that

$$W_{f*g} = W_f \circ W_g = W_g \circ W_f \tag{4}$$

for all $f, g \in L^\Phi_t(G, w)$. Moreover, the closed linear subspace of $B(L^\Phi(G, \omega))$ spanned by $\{W_{f*g} : f \in L^\Phi_t(G, w), g \in C_c(G)\}$ is denoted by $\Lambda_{L^\Phi(G,\omega)}$.

Theorem 3.4. *The space $\Lambda_{L^\Phi(G,\omega)}$ is a complete subalgebra of $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$ and it has a minimal approximate identity, that is, a net $\{T_\alpha\}_\alpha$ such that $\overline{\lim}_\alpha\|T_\alpha\| \le 1$ and $\lim_\alpha \|T_\alpha \circ T - T\| = 0$ for all $T \in \Lambda_{L^\Phi(G,\omega)}$.*

Proof. If $f \in L^\Phi_t(G, w)$, then $W_f \in B(L^\Phi(G, \omega))$. Since $L^\Phi(G, \omega)$ is an $L^1(G, \omega)$-module, we have

$$W_f(g*h) = g*h*f = g*W_f(h)$$

for all $g \in L^1(G, \omega)$ and $h \in L^\Phi(G, \omega)$.

Hence W_f belongs to $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G, \omega))$. Since $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G, \omega))$ is a Banach algebra under the usual operator norm, $\Lambda_{L^\Phi(G,\omega)}$ is a complete subalgebra of $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G, \omega))$.

Now, we show the existence of minimal approximate identity of $\Lambda_{L^\Phi(G,\omega)}$. Let $\{e_{U_\alpha}\}$ be a minimal approximate identity for $L^1(G, \omega)$ Dinculeanu, N. (1974). If $\{e_\alpha\}$ denotes the product net of $\{e_{U_\alpha}\}$ with itself, then $\{e_\alpha\}$ is also minimal approximate identity for $L^1(G, \omega)$. It can be observed that the net $W_{e_\alpha} \in \Lambda_{L^\Phi(G,\omega)}$ and $\overline{\lim}_\alpha\|W_{e_\alpha}\| \le 1$.

Let $f \in L^\Phi_t(G, w)$ and $g \in C_c(G)$. Since $\{e_\alpha\}$ is a minimal approximate identity for $L^1(G, \omega)$, using (4) we obtain

$$\begin{aligned}\overline{\lim}_\alpha\|W_{e_\alpha} \circ W_{f*g} - W_{f*g}\| &= \overline{\lim}_\alpha\|(W_{e_\alpha} \circ W_g - W_g) \circ W_f\|\\ &\le \overline{\lim}_\alpha\|W_{g*e_\alpha-g}\|\|W_f\|\\ &\le \overline{\lim}_\alpha\|g*e_\alpha - g\|_{1,\omega}\|W_f\| = 0.\end{aligned}$$

Thus we have $\overline{\lim}_\alpha\|W_{e_\alpha} \circ T - T\| = 0$ for all $T \in \Lambda_{L^\Phi(G,\omega)}$. □

Let $g \in L^1(G, \omega)$, $f \in L^\Phi_t(G, \omega)$ and $W_f \in \Lambda_{L^\Phi(G,\omega)}$. We define the module action $g \circ W_f$ of $L^1(G, \omega)$ from $L^\Phi(G, \omega)$ to $L^\Phi(G, \omega)$ by

$$(g \circ W_f)(h) = W_f(h*g) = W_f(g*h)$$

for each $h \in L^\Phi(G, \omega)$.

Proposition 3.5. *The space $\Lambda_{L^\Phi(G,\omega)}$ is an essential $L^1(G, \omega)$-module.*

Proof. Let $g \in L^1(G, \omega)$, $f \in L^\Phi_t(G, \omega)$ and $W_f \in \Lambda_{L^\Phi(G,\omega)}$. We have

$$\|g \circ W_f\| = \sup\{\|W_f(g*h)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\} \le \|f\|^t_{\Phi,\omega}\|g\|_{1,\omega}.$$

Hence, $\Lambda_{L^\Phi(G,\omega)}$ is an $L^1(G, \omega)$-module. On the other hand, since $L^1(G, \omega)$ has a minimal approximate identity $\{e_\alpha\}$ with a compact support, it is also an approximate identity in $L^\Phi(G, \omega)$.

For any $W_f \in \Lambda_{L^\Phi(G,\omega)}$, we have

$$\begin{aligned}\|e_\alpha \circ W_f - W_f\| &= \sup\{\|(e_\alpha \circ W_f - W_f)(h)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\}\\ &= \sup\{\|W_f(e_\alpha * h - h)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\}\\ &\le \|f\|^t_{\Phi,\omega}\|e_\alpha * h - h\|_{\Phi,\omega} = 0\end{aligned}$$

for all $h \in L^\Phi(G, \omega)$. Thus, $\Lambda_{L^\Phi(G,\omega)}$ is an essential $L^1(G, \omega)$-module. Moreover, $\Lambda_{L^\Phi(G,\omega)}$ contains $L^1(G, \omega)$. □

4. A CHARACTERIZATION FOR THE SPACE OF MULTIPLIERS OF $\Lambda_{L^\Phi(G,\omega)}$

In this section, we give an identification for the space of $L^1(G,\omega)$-module multiplier with the space of multipliers of certain normed algebra.

The definitions, notations and proofs of this section are adjusted according to the corresponding content of Section 3 of Öztop, S. (2003).

Proposition 4.1. *Let $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$.*

(i) If $f \in L_t^\Phi(G,\omega)$, then $T(f) \in L_t^\Phi(G,\omega)$.
(ii) If $g \in L_t^\Phi(G,\omega)$, then $T(f * g) = f * T(g)$

for all $f, g \in L^\Phi(G,\omega)$.

Proof. (i) Let $f \in L_t^\Phi(G,\omega)$. Since $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$ we have

$$\begin{aligned}\|T(f)\|_{\Phi,\omega}^t &= \sup\{\|h * T(f)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\}\\ &= \sup\{\|T(h * f)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\}\\ &\le \|T\| \sup\{\|h * f\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\}\\ &= \|T\|\|f\|_{\Phi,\omega}^t < \infty.\end{aligned}$$

(ii) Let $g \in L_t^\Phi(G,\omega)$. Since $\overline{C_c(G)} = L^\Phi(G,\omega)$, for each $f \in L^\Phi(G,\omega)$ there exists $(f_n) \subseteq C_c(G)$ such that $\lim\limits_{n\to\infty} \|f_n - f\|_{\Phi,\omega} = 0$. Using (3), we obtain $\lim\limits_{n\to\infty} \|f_n * g - f * g\|_{\Phi,\omega} = 0$. By (i), we have

$$\lim_{n\to\infty} \|f_n * T(g) - f * T(g)\|_{\Phi,\omega} = 0$$

and $f * T(g) = \lim\limits_{n\to\infty} f_n * T(g) = \lim\limits_{n\to\infty} T(f_n * g) = T(f * g)$. □

Definition 4.2. For the space $\Lambda_{L^\Phi(G,\omega)}$, we define $\Lambda^\circ_{L^\Phi(G,\omega)}$ by

$$\Lambda^\circ_{L^\Phi(G,\omega)} = \{T \in \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega)) : T \circ W \in \Lambda_{L^\Phi(G,\omega)} \text{ for all } W \in \Lambda_{L^\Phi(G,\omega)}\}.$$

Lemma 4.3. *The space $\Lambda^\circ_{L^\Phi(G,\omega)}$ is equal to $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$.*

Proof. It is obvious that

$$\Lambda^\circ_{L^\Phi(G,\omega)} \subseteq \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega)). \tag{5}$$

Conversely, let $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$. For any $S \in \Lambda_{L^\Phi(G,\omega)}$ which is $S = W_{f*g}$ for some $f \in L_t^\Phi(G,\omega)$ and $g \in C_c(G)$, we have

$$\begin{aligned}(T \circ W_{f*g})(h) &= T(h * f * g)\\ &= h * T(f * g)\\ &= W_{T(f*g)}(h)\\ &= W_{f*T(g)}(h),\end{aligned}$$

for all $h \in L^\Phi(G,\omega)$. So, $T \circ S \in \Lambda_{L^\Phi(G,\omega)}$ implies that $T \in \Lambda^\circ_{L^\Phi(G,\omega)}$. Hence, we have $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega)) \subseteq \Lambda^\circ_{L^\Phi(G,\omega)}$ by the continuity of T. □

Let us note that we have the inclusion $M(\Lambda_{L^\Phi(G,\omega)}) \subset \mathrm{Hom}_{L^1(G,\omega)}(\Lambda_{L^\Phi(G,\omega)})$.

Theorem 4.4. *The space of multipliers $M(\Lambda_{L^\Phi(G,\omega)})$ is isometrically isomorphic to the space $\Lambda^\circ_{L^\Phi(G,\omega)}$.*

Proof. Define the mapping $F : \Lambda^\circ_{L^\Phi(G,\omega)} \to M(\Lambda_{L^\Phi(G,\omega)})$ by letting $F(T) = \rho_T$ for each $T \in \Lambda^\circ_{L^\Phi(G,\omega)}$, where $\rho_T(S) = T \circ S$ for all $S \in \Lambda_{L^\Phi(G,\omega)}$. Thus F is well-defined and moreover $\rho_T(S \circ K) = T \circ S \circ K = \rho_T(S) \circ K$ for all $S, K \in \Lambda_{L^\Phi(G,\omega)}$, so $\rho_T \in M(\Lambda_{L^\Phi(G,\omega)})$, since G is an abelian group and so the convolution multiplication is commutative.

It is clear that the mapping $T \mapsto \rho_T$ is linear and that $\|\rho_T\| \le \|T\|$. Since W_{e_α} is minimal approximate identity for $\Lambda_{L^\Phi(G,\omega)}$ by Theorem 3.4, we have

$$\begin{aligned}\|\rho_T\| = \sup_{S \in \Lambda_{L^\Phi(G,\omega)}} \frac{\|\rho_T(S)\|}{\|S\|} &= \sup_{S \in \Lambda_{L^\Phi(G,\omega)}} \frac{\|T \circ S\|}{\|S\|}\\ &\ge \sup_{S \in \Lambda_{L^\Phi(G,\omega)}} \frac{\|T \circ W_{e_\alpha}\|}{\|W_{e_\alpha}\|} \ge \|T\|.\end{aligned}$$

Thus $\|\rho_T\| = \|T\|$.

Finally, we show that F is onto. Let $\rho \in M(\Lambda_{L^\Phi(G,\omega)})$ and $\{e_\alpha\} \subseteq L^1(G,\omega)$ be a minimal approximate identity of $L^1(G,\omega)$. The limit of ρW_{e_α} exists for strong operator topology. Let $T = \lim_\alpha \rho W_{e_\alpha}$. We prove $\rho_T = \rho$. By (1), we have $(\rho W_{e_\alpha})(f * g) = (\rho W_{e_\alpha})(W_f g) = (\rho W_{e_\alpha * f})(g)$ for every $f \in L^1(G,\omega)$, $g \in L^\Phi(G,\omega)$. So we have

$$T(f * g) = \lim_\alpha (\rho W_{e_\alpha})(f * g) = (\rho W_f)g. \tag{6}$$

Since $L^\Phi(G,\omega)$ is an essential $L^1(G,\omega)$-module, the limit of $(\rho W_{e_\alpha})(f * g)$ exists in $L^\Phi(G,\omega)$. Let this limit be denoted by $T_g \in \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$. From (6) we obtain for all $f \in L^1(G,\omega)$,

$$f \circ T = \rho f. \tag{7}$$

Thus we have

$$T \circ W_{e_\alpha} \circ W = (\rho W_{e_\alpha}) \circ W = \rho(W_{e_\alpha} \circ W) \tag{8}$$

for all $W \in \Lambda_{L^\Phi(G,\omega)}$. Since $L^\Phi(G,\omega)$ is essential $L^1(G,\omega)$-module, we have $T \circ W = \rho(W)$ and so $\rho_T(W) = \rho(W)$ for all $W \in \Lambda_{L^\Phi(G,\omega)}$, which gives $\rho_T = \rho$. □

Corollary 4.5. $M(\Lambda_{L^\Phi(G,\omega)}) \cong \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$.

Proof. From Lemma 4.3 and Theorem 4.4, the result is obtained. □

5. THE IDENTIFICATION FOR THE SPACE $L^1(G,\omega) \cap L^\Phi(G,\omega)$

In this section, adapted from Chapter of Öztop, S. (2003), we study some basic properties of the space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ and we characterize the space of multipliers $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$.

Given a Young function Φ, the space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Banach space with the norm

$$|||f||| = \|f\|_{1,\omega} + \|f\|_{\Phi,\omega} \tag{9}$$

for $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$.

Lemma 5.1. *For $L^1(G,\omega) \cap L^\Phi(G,\omega)$ the following is true.*

(i) $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is dense in $L^1(G,\omega)$ with respect to the norm $\|\cdot\|_{1,\omega}$.
(ii) For every $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$ and $x \in G$ the mapping $x \mapsto L_x f$ is continuous where $L_x f(y) = f(x^{-1}y)$ for all $y \in G$.

Proof. (i) Since $C_c(G)$ is dense in $L^1(G,\omega)$ with respect to the norm $\|\cdot\|_{1,\omega}$ and $C_c(G) \subseteq L^1(G,\omega) \cap L^\Phi(G,\omega) \subseteq L^1(G,\omega)$ we have the required result.

(ii) Let $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$. Observe that by (Osançlıol, A., Öztop, S. 2015, Lemma 2.3) $|||L_x f||| \le w(x)|||f|||$ for all $x \in G$ and the function $x \mapsto L_x f$ is continuous from G into $L^\Phi(G,\omega)$ and $L^1(G,\omega)$. Thus for any $x_0 \in G$ and $\varepsilon > 0$, there exists $U_1 \in V_{(x_0)}$ and $U_2 \in V_{(x_0)}$ such that for every $x \in U_1$

$$\|L_x f - L_{x_0} f\|_{\Phi,\omega} < \frac{\varepsilon}{2}$$

and for every $x \in U_2$

$$\|L_x f - L_{x_0} f\|_{1,\omega} < \frac{\varepsilon}{2},$$

where $V_{(x_0)}$ denotes the neighborhood of x_0. Set $V = U_1 \cap U_2$, then for all $x \in V$ we have $|||L_x f - L_{x_0} f||| < \varepsilon$. □

Since $L^1(G,\omega)$ has a minimal approximate identity and $L^\Phi(G,\omega)$ is an essential $L^1(G,\omega)$-module, the following proposition and lemma are hold trivially.

Proposition 5.2. *The space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ has a minimal approximate identity in $L^1(G,\omega)$.*

Lemma 5.3. *The space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is an essential $L^1(G,\omega)$-module.*

Corollary 5.4. $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Banach ideal in $L^1(G,\omega)$.

Proposition 5.5. *$L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Banach algebra with the norm $|||\cdot|||$.*

Proof. For any $f, g \in L^1(G,\omega) \cap L^\Phi(G,\omega)$ we have

$$\begin{aligned} |||f * g||| &= \|f * g\|_{1,\omega} + \|f * g\|_{\Phi,\omega} \\ &\le \|f\|_{1,\omega}\|g\|_{\Phi,\omega} + \|f\|_{1,\omega}\|g\|_{\Phi,\omega} \le |||f|||\ |||g|||. \end{aligned}$$

□

Note that let G be a locally compact abelian group. A subalgebra $S^1(G)$ of $L^1(G)$ is called a Segal algebra if it satisfies the following conditions (see Reiter H., Stegeman J. D. (2000)).

(i) The space $S^1(G)$ is dense in $L^1(G)$.

(ii) The subalgebra $S^1(G)$ is a Banach algebra which is invariant under translations and for each $f \in L^1(G)$ there is a neighborhood $U = U_\varepsilon$ of the identity element e such that

$$\|L_y f - f\| < \varepsilon, \quad y \in U.$$

Corollary 5.6. The space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Segal algebra.

Proof. By Lemma 5.1 and Proposition 5.5 we obtain that $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Segal algebra. □

Remark 5.7. Since $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is an $L^1(G,\omega)$-module and a Banach algebra, using the similar methods in Section 3 we obtain $M(L^1(G,\omega) \cap L^\Phi(G,\omega)) \cong \mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$.

Proposition 5.8. $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ *is an essential Banach module over* $L^1(G,\omega)$.

Proof. Let $f \in L^1(G,\omega)$ and $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$. Define the operator fT on $L^1(G,\omega) \cap L^\Phi(G,\omega)$ by

$$(fT)(g) = T(f * g) \tag{10}$$

for all $g \in L^1(G,\omega) \cap L^\Phi(G,\omega)$ and $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$. Since $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Banach algebra, the mapping (10) is well defined. On the other hand, we have

$$\begin{aligned}\|fT\| &= \sup_{|||g|||\le 1} |||(fT)(g)||| \\ &\le \sup_{|||g|||\le 1} |||T(f * g)||| \\ &\le \sup_{|||g|||\le 1} \|T\|\ |||f * g||| \\ &\le \sup_{|||g|||\le 1} \|T\|\ \|f\|_{1,\omega}|||g||| \\ &\le \|T\|\ \|f\|_{1,\omega}.\end{aligned}$$

Hence, $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ is an $L^1(G,\omega)$ module.

Let $\{e_\alpha\}$ be a minimal approximate identity for $L^1(G,\omega)$ and T be in $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$. We have

$$\lim_\alpha \|e_\alpha \circ T - T\| = 0.$$

Then, we obtain that $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ is an essential Banach module over $L^1(G,\omega)$. □

Define $\mathcal{P}$ to be the closure of $L^1(G,\omega)$ in $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ for the operator norm. Clearly we have

$$\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega)) = (\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega)))_e = \mathcal{P} = (\mathcal{P})_e, \tag{11}$$

where $(.)_e$ denotes the essential part and we have

$$\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega)) = (\mathcal{P}).$$

Here $(\mathcal{P})$ is defined as the space of the elements $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ such that $T \circ \mathcal{P} \subset \mathcal{P}$.

Using the same method as in Theorem 4.4, we obtain the following lemma.

Lemma 5.9. *The space of multipliers of Banach algebra* $\mathcal{P}$ *is isometrically isometric to the space* $(\mathcal{P})$.

Corollary 5.10. $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega)) \cong M(\mathcal{P})$.

Proof. The proof is obtained by using Lemma 4.3 and Theorem 4.4. □

Remark 5.11. It is evident that every measure $\mu \in M(G)$ defines multiplier for $L^1(G,\omega) \cap L^\Phi(G,\omega)$. This is obvious from the fact that $\|\mu * f\| \le \|\mu\|\|f\|$, $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$. On the other hand, for $\mu \in M(G)$, we have $\mu \circ L^1(G,\omega) \subset L^1(G,\omega)$, the inclusion in the space $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$. Thus, $\mu \circ \mathcal{P} \subset \mathcal{P}$ and $M(G)$ can be embeded into $(\mathcal{P})$.

Moreover, if G is noncompact locally compact abelian, we have the more general results than Corollary in (Larsen, R. 1971, Corollary 3.5.1).

Peer Review: Externally peer-reviewed.
Author Contribution: All authors have contributed equally.
Conflict of Interest: Authors declared no conflict of interest.
Financial Disclosure: Authors declared no financial support.

ACKNOWLEDGEMENTS

We are grateful to anonymous referees for careful reading of the manuscript and for helpful comments.

LIST OF AUTHOR ORCIDS

B. Arıs https://orcid.org/0000-0002-4699-4122
R. Üster https://orcid.org/0000-0003-4063-5118

REFERENCES

Başar E., Öztop, S., Uysal, B.H., Yaşar, Ş. 2023, Extreme points in Orlicz spaces equipped with s-norms and its closedness, Math. Nachr. 00, 1–21.
Dinculeanu, N., 1974, Integration on locally compact spaces, Springer.
Feichtinger, H., 1976, Multipliers of Banach spaces of functions on groups, Math. Z. 152, 47-58.
Fisher, M. J., 1974, Properties of three algebras related to L^p-multipliers, Bull. Amer. Math. Soc. 80, 262-265.
Griffin, J., McKennon, K., 1973, Multipliers and the group L^p-algebras, Pacific J. Math. 49, 365-370.
Larsen, R., 1971, An introduction to the theory of multipliers, Die Grundlehren der mathematischen Wissenschaften, 175, Springer-Verlag, Berlin, Heidelberg and New York.
McKennon, K., 1972, Multipliers of type $(p, p)'$, Pacific J. Math. 43, 429-436.
Osançlıol, A., Öztop, S., 2015, Weighted Orlicz algebras on locally compact groups, Journal of Australian Mathematical Society, 99, 399-414.
Öztop, S., 2003, A note on multipliers of $L^p(G, A)$, J. Aust. Math. Soc., 74, 25–34.
Öztop, S., Samei, E., 2017, Twisted Orlicz algebras I, Studia Mathematica, 236, 271-296.
Öztop, S., Samei, E., 2019, Twisted Orlicz algebras II, Mathematische Nachrichten, 292, 1122-1136.
Rao, M. M., Ren, Z. D., 1991, Theory of Orlicz spaces, Marcel Dekker, New York.
Reiter H., Stegeman J. D., 2000, Classical harmonic analysis and locally compact groups, Clarendon Press, Oxford.
Üster, R., Öztop, S., 2020, Invariant subsets and homological properties of Orlicz modules over group algebras, Taiwanese Journal of Mathematics, 24, 959-973.
Üster, R., 2021, Multipliers for the weighted Orlicz spaces of a locally compact abelian group, Results in Mathematics, 76 (4), Paper No. 183.
Wang, J. K., 1961, Multipliers of commutative Banach algebras, Pacific J. Math. 11, 1131-114.

İSTANBUL UNIVERSITY PRESS

Istanbul Journal of Mathematics
ijmath 2023, 1 (1), 28–39
DOI: 10.26650/ijmath.2023.00003

RESEARCH ARTICLE

Clairaut and Einstein conditions for locally conformal Kaehler submersions

B. Pirinççi[1*], Ç. Çimen[2], D. Ulusoy[2]

[1]İstanbul University, Faculty of Science, Department of Mathematics, Vezneciler, 34134, İstanbul, Türkiye
[2]İstanbul University, Institue of Graduate Studies in Science, Süleymaniye, 34134, İstanbul, Türkiye

ABSTRACT
In the present paper, we study Clairaut submersions and Einstein conditions whose total manifolds are locally conformal Kaehler manifolds. We first give a necessary and sufficient condition for a curve to be geodesic on total manifold of a locally conformal Kaehler submersion. Then, we investigate conditions for a locally conformal Kaehler submersion to be a Clairaut submersion.We find the Ricci and scalar curvature formulas between any fiber of the total manifold and the base manifold of a locally conformal Kaehler submersion and give necessary and sufficient conditions for the total manifold of a locally conformal Kaehler submersion to be Einstein. Finally, we obtain some formulas for sectional and holomorphic sectional curvatures for a locally conformal Kaehler submersion.

Mathematics Subject Classification (2020): 53B35, 53C18, 53C25

Keywords: Riemannian submersion, almost Hermitian submersion, locally conformal Kaehler submersion, Clairaut submersion, Einstein manifold.

1. INTRODUCTION

The notion of Riemannian submersion was introduced by O'Neill (1966) and Gray (1967), independently. Watson (1976) introduced almost Hermitian submersions by adding the condition to be almost complex mappings for Riemannian submersions, and proved that the vertical and the horizontal distributions are invariant with respect to the almost complex structure of the total space of the submersion. Then various kinds of Riemannian and almost Hermitian submersions Falcitelli et al. (2004), Şahin (2017) have been introduced and studied widely such as anti-invariant submersions, Lagrangian submersions, slant submersions, semi-slant submersions, hemi-slant submersions, etc. Moreover, these submersions have been studied for different kinds of manifolds like Kaehler, almost Kaehler, Sasakian and examined under some particular conditions, for example Einstein and Clairaut Lee et al. (2015). Especially, Clairaut submersions were studied in Lorentizian manifolds Allison (1996), Sasakian and Kenmotsu manifolds Taştan and Gerdan (2016), cosymplectic manifolds Taştan and Gerdan Aydın (2019), and locally product Riemannian manifolds Gündüzalp (2020). An important class of these manifolds is locally conformal Kaehler manifold, whose metric is conformal to a Kaehler metric locally. Vaisman studied locally conformal Kaehler manifolds and obtained some curvature properties of these manifolds Vaisman (1980). A comprehensive review for locally conformal Kaehler manifolds was made by Dragomir and Ornea (1998). An almost Hermitian submersion whose total manifold is a locally conformal Kaehler is called a locally conformal Kaehler submersion. Marrero and Rocha (1994) gave some conditions for the fibers of a locally conformal Kaehler submersion to be minimal and studied some relations between the Betti numbers of the total space and the base space of this submersion. In a recent paper Çimen et al. (2023) obtained Gauss and Weingarten equations for a locally conformal Kaehler submersion.

In this paper, we study Clairaut submersions and Einstein conditions whose total manifolds are locally conformal Kaehler manifolds. In section 2, we give basic informations about Riemannian submersions, almost Hermitian submersions and locally conformal Kaehler manifolds. In section 3, we derive conditions for a curve to be geodesic, with respect to two connections which are determined by the Riemannian metric and its conformally related Kaehler metric, on total manifold of a locally conformal Kaehler submersion. After that, we give a necessary and sufficient condition for a locally conformal Kaehler submersion to be Clairaut. In section 4, we derive the formulas for Ricci and scalar curvatures between any fiber of the total manifold and the base manifold of a locally conformal Kaehler submersion. Afterwards, we give necessary and sufficient conditions for the total manifold of a

Corresponding Author: B. Pirinççi **E-mail:** beranp@istanbul.edu.tr

Submitted: 01.05.2023 • **Revision Requested:** 24.05.2023 • **Last Revision Received:** 31.05.2023 • **Accepted:** 01.06.2023 • **Published Online:** 06.06.2023

locally conformal Kaehler submersion to be Einstein. At the end of this section, we obtain the sectional and holomorphic sectional curvatures for a locally conformal Kaehler submersion.

2. PRELIMINAIRES

In this section, we will give some informations about locally conformal Kaehler manifolds.

Let (M, g) and (N, g') be Riemannian manifolds. A mapping π of (M, g) onto (N, g') is called a *Riemannian submersion* if it satisfies the following conditions:

(i) For every $p \in M$, the derivative map π_* of π is surjective;

hence for each $q \in N$, $\pi^{-1}(q)$ is a submanifold of dimension $dim(M) - dim(N)$. These submanifolds are called *fibers* of the submersion and a vector field on M which is tangent (resp. orthogonal) to fibers is called *vertical* (resp. *horizontal*). Thus, we can write a vector field E on M uniquely as $E = E^v + E^h$, where E^v and E^h are vertical and horizontal parts of E, respectively.

(ii) For every horizontal vector fields X, Y we have $g(X, Y) = g'(\pi_*X, \pi_*Y)$;

that is, π_* is a linear isometry of horizontal distribution.

To find the Gauss and Weingarten formulas of a Riemannian submersion, O'Neill introduced two new tensors of types $(1, 2)$ as follows;

$$\mathcal{T}_E F = (\nabla_{E^v} F^h)^v + (\nabla_{E^v} F^v)^h,$$

$$\mathcal{A}_E F = (\nabla_{E^h} F^h)^v + (\nabla_{E^h} F^v)^h,$$

where E and F are vector fields on M and ∇ is the Levi-Civita connection of g (see for the properties $\mathcal{T}$ and $\mathcal{A}$ in O'Neill (1966)). It is easy to see that,

$$\nabla_U V = (\nabla_U V)^v + \mathcal{T}_U V, \tag{1}$$
$$\nabla_U X = \mathcal{T}_U X + (\nabla_U X)^h, \tag{2}$$
$$\nabla_X U = (\nabla_X U)^v + \mathcal{A}_X U, \tag{3}$$
$$\nabla_X Y = \mathcal{A}_X Y + (\nabla_X Y)^h, \tag{4}$$

where U and V are vertical, and X and Y are horizontal vector fields on M.

Let (M, J, g) and (N, J', g') be almost Hermitian manifolds and $\pi : (M, J, g) \longrightarrow (N, J', g')$ be a Riemannian submersion. π is called an *almost Hermitian submersion* if $\pi_* \circ J = J' \circ \pi_*$, i.e., π is an almost complex mapping. The vertical and horizontal distributions are invariant under the almost complex structure J (see Proposition 2.1 in Watson (1976)).

Let a Hermitian manifold (M, J, g) is called a *locally conformal Kaehler* manifold (briefly l.c.K.), if M has an open cover $\{\mathcal{U}_i\}_{i\in I}$ and for every $i \in I$ with family of positive differentiable functions $\sigma_i : \mathcal{U}_i \longrightarrow \mathbb{R}$ such that

$$g_i = e^{-\sigma_i} g|_{\mathcal{U}_i}$$

are Kaehler metrics on $\mathcal{U}_i$.

Let (M, J, g) be a Hermitian manifold and let Ω be a 2-form defined by $\Omega(E, F) = g(E, JF)$ where E and F are vector fields on M. Dragomir and Ornea (1998) showed that (M, J, g) is a l.c.K. manifold if and only if there exists a globally defined closed 1-form ω such that

$$d\Omega = \omega \wedge \Omega.$$

The 1-form ω is called the *Lee form* and the vector field B defined by

$$\omega(E) = g(B, E), \tag{5}$$

is called *Lee vector field* of M, where E is a vector field of M.

Let ∇^i be the Levi-Civita connection of the locally conformal Kaehler metrics g_i, for every $i \in I$. Then the Levi-Civita connections ∇^i glue up to a globally defined linear connection $\tilde{\nabla}$ on M (see Theorem 2.1 (Dragomir and Ornea (1998))) is given by

$$\tilde{\nabla}_E F = \nabla_E F - \frac{1}{2}\Big\{\omega(E)F + \omega(F)E - g(E, F)B\Big\}, \tag{6}$$

for any vector fields E and F on M. One can see that $\tilde{\nabla}$ is torsion-free and satisfies

$$\tilde{\nabla} g = \omega \otimes g, \tag{7}$$

and

$$\tilde{\nabla} J = 0. \tag{8}$$

$\tilde{\nabla}$ is called *Weyl connection* of the l.c.K manifold M. From (6) and (8), it can be obtained

$$(\nabla_E J)F = \frac{1}{2}\Big\{\omega(JF)E - \omega(F)JE - g(E, JF)B + g(E, F)JB\Big\}. \tag{9}$$

Çimen et al. (2023) showed that the following equations hold for a l.c.K. submersion:

$$\mathcal{T}_U JV = J\mathcal{T}_U V + \frac{1}{2}\Big\{g(U, V)(JB)^h - g(U, JV)B^h\Big\}, \tag{10}$$

$$\mathcal{T}_V JX = J\mathcal{T}_V X + \frac{1}{2}\Big\{\omega(JX)V - \omega(X)JV\Big\}, \tag{11}$$

$$\mathcal{A}_X JV = J\mathcal{A}_X V + \frac{1}{2}\Big\{\omega(JV)X - \omega(V)JX\Big\}, \tag{12}$$

$$\mathcal{A}_X JY = J\mathcal{A}_X Y + \frac{1}{2}\Big\{g(X, Y)(JB)^v - g(X, JY)B^v\Big\}, \tag{13}$$

where U and V are vertical, X and Y are horizontal, and B is the Lee vector field of the total manifold of the submersion.

3. CLAIRAUT LOCALLY CONFORMAL KAEHLER SUBMERSIONS

In this section we shall give a necessary and sufficient condition for a locally conformal Kaehler submersion to be Clairaut. First, we recall the definition of a Clairaut submersion.

Let $\rho(p)$ be the distance from a point p on a surface of revolution in $\mathbb{R}^3$ to the rotation axis of this surface and α be a geodesic in this surface. Clairaut's theorem says that for the angle $\theta(s)$ between the velocity vector $\dot{\alpha}(s)$ and the meridian through $\alpha(s)$, $(\rho \sin\theta)(s)$ is constant. Motivated by this idea, Bishop (1972) introduced the notion of Clairaut submersion in the following way:

Definition 3.1. A Riemannian submersion $\pi : (M, g) \to (N, g')$ is called a *Clairaut submersion* if there exists a positive function ρ on M such that for any geodesic α on M, the function $\rho sin\theta$ is constant, where θ is the angle between $\dot{\alpha}$ and the horizontal distribution at every point of M.

Bishop (1972) gave the following characterization for Clairaut submersions.

Theorem 3.2. *Let $\pi : (M, g) \to (N, g')$ be a Riemannian submersion with connected fibers. Then π is a Clairaut submersion with $\rho = e^f$ if and only if each fiber is totally umbilical and has the mean curvature vector field $H = -\mathrm{grad} f$.*

We shall obtain a necessary and sufficient condition for a curve on the total space of a l.c.K submersion to be geodesic.

Lemma 3.3. *Let π be a l.c.K. submersion from (M, J, g) onto (N, J', g'), and let α be a curve on M whose tangent vector field has horizontal and vertical components X and V, respectively. Then α is a geodesic with respect to the Weyl connection $\tilde{\nabla}$ if and only if*

$$(\nabla_{\dot{\alpha}} JX)^h + \mathcal{T}_V JV + \mathcal{A}_X JV - \frac{1}{2}\Big\{\omega(\dot{\alpha})JX + \omega(J\dot{\alpha})X\Big\} = 0, \tag{14}$$

$$(\nabla_{\dot{\alpha}} JV)^v + \mathcal{T}_V JX + \mathcal{A}_X JX - \frac{1}{2}\Big\{\omega(\dot{\alpha})JV + \omega(J\dot{\alpha})V\Big\} = 0. \tag{15}$$

Proof. From (6) and (8), we have

$$\begin{aligned}
\tilde{\nabla}_{\dot{\alpha}}\dot{\alpha} &= -J\tilde{\nabla}_{\dot{\alpha}} J\dot{\alpha} \\
&= -J\Big(\nabla_{\dot{\alpha}} J\dot{\alpha} - \frac{1}{2}\Big\{\omega(\dot{\alpha})J\dot{\alpha} + \omega(J\dot{\alpha})\dot{\alpha} - g(\dot{\alpha}, J\dot{\alpha})B\Big\}\Big) \\
&= -J\Big(\nabla_V JV + \nabla_V JX + \nabla_X JV + \nabla_V JV \\
&\qquad - \frac{1}{2}\Big\{\omega(\dot{\alpha})JX + \omega(\dot{\alpha})JV + \omega(J\dot{\alpha})X + \omega(J\dot{\alpha})V\Big\}\Big).
\end{aligned}$$

Then nonsingular J implies that α is geodesic if and only if

$$\nabla_V JV + \nabla_V JX + \nabla_X JV + \nabla_V JV - \frac{1}{2}\Big\{\omega(\dot\alpha)JX + \omega(\dot\alpha)JV + \omega(J\dot\alpha)X + \omega(J\dot\alpha)V\Big\} = 0.$$

Taking the horizontal and vertical parts of this equation, we get (14) and (15), respectively. □

Lemma 3.4. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion with connected fibers. If α is a geodesic on M with respect to both ∇ and $\tilde\nabla$, then we have*

$$\omega(\dot\alpha)\dot\alpha = \frac{1}{2}B. \tag{16}$$

Proof. *Suppose that α is a geodesic curve with respect to both ∇ and $\tilde\nabla$, that is $\nabla_{\dot\alpha}\dot\alpha = 0$ and $\tilde\nabla_{\dot\alpha}\dot\alpha = 0$. Then we get (16) immediately from (6).* □

A geodesic curve whose vertical component of its velocity vector is zero is called a *horizontal geodesic* by O'Neill (1967). For a horizontal geodesic of a l.c.K. submersion, we have the following result.

Theorem 3.5. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion with connected fibers. If a curve α is a horizontal geodesic on M with respect to both ∇ and $\tilde\nabla$, then the dimension of horizontal distribution is equal to 2 or the submersion π is a Kaehler submersion, i.e., its total manifold is Kaehler.*

Proof. *Let $\{X_1, ..., X_m\}$ be an orthonormal basis of the horizontal distribution of the submersion π at $p \in \pi^{-1}(q)$, where $q \in N$. Then there exist horizontal geodesic curves $\alpha_1, ..., \alpha_m$ such that $\dot\alpha_i = X_i$, $i = 1, ..., m$. Thus, for every $i = 1, ..., m$, we have*

$$g(B, X_i)X_i = \frac{1}{2}B^h \tag{17}$$

from (5) and (16). Taking summation of the equation (17) over i, we obtain

$$\Big(1 - \frac{m}{2}\Big)B^h = 0.$$

Hence, it follows that $m = 2$ or $B^h = 0$. In the case of $B^h = 0$, B is a zero vector field since B cannot be vertical by Theorem 2 of Çimen et al. (2023). It means that M is Kaehler. □

Now, we shall give the condition for a l.c.K. submersion to be Clairaut.

Theorem 3.6. *Let π be a l.c.K. submersion from (M, J, g) onto (N, J', g'). Then π is a Clairaut submersion with $\rho = e^f$ if and only if*

$$g(\dot\alpha, \mathrm{grad} f)g(V, V) + \frac{1}{2}\omega(\dot\alpha)g(V, V) + \frac{1}{2}||V||^2\omega(X) - \frac{1}{2}||X||^2\omega(V) - g(\mathcal{T}_V X, V) = 0, \tag{18}$$

where X and V denote the horizontal and vertical components of $\dot\alpha$ of the geodesic α on M with respect to $\tilde\nabla$, respectively.

Proof. *Let α be a geodesic on M. Then we have*

$$g(X, X) = cos^2\theta \quad and \quad g(V, V) = sin^2\theta.$$

From (6) and (7) , we have

$$\begin{aligned}\omega(\dot\alpha)g(V, V) &= (\tilde\nabla_{\dot\alpha} g)(V, V)\\ &= \tilde\nabla_{\dot\alpha} g(V, V) - 2g(\tilde\nabla_{\dot\alpha} JV, JV)\\ &= \tilde\nabla_{\dot\alpha} g(V, V) - 2g(\nabla_{\dot\alpha} JV, JV) + g(\omega(\dot\alpha)JV + \omega(JV)\dot\alpha - g(JV, \dot\alpha)B, JV).\end{aligned}$$

Then we obtain,

$$\tilde\nabla_{\dot\alpha} g(V, V) = 2g(\nabla_{\dot\alpha} JV, JV). \tag{19}$$

On the other hand, we have

$$\tilde\nabla_{\dot\alpha} g(V, V) = \nabla_{\dot\alpha} g(V, V) = sin\theta cos\theta\frac{d\theta}{dt}. \tag{20}$$

Then, π is Clairaut if and only if $\frac{d}{ds}(e^f sin\theta) = 0$. Hence from (20), we get

$$\tilde\nabla_{\dot\alpha} g(V, V) = -2\frac{df}{ds}sin^2\theta. \tag{21}$$

By using (15) and (21), in (19) we obtain

$$g(\dot{\alpha}, gradf)g(V,V) + \frac{1}{2}\omega(\dot{\alpha})||V||^2 - g(\mathcal{T}_V JX, JV) - g(\mathcal{A}_X JX, JV) = 0. \tag{22}$$

With the help of (11) and (13), the equation (18) follows from (22). □

Let $\pi : (M, J, g) \longrightarrow (N, J', g')$ be a l.c.K. submersion with totally umbilical fibers. Then, for any vertical vector fields U and V, and horizontal vector field X, we have

$$g(\mathcal{T}_U V, X) = -g(U,V)g(H,X),$$

from Theorem (3.2). Hence, using (6) and (1), we obtain

$$g(\tilde{\nabla}_U V, X) = -g(U,V)g(H,X) + \frac{1}{2}g(U,V)g(B^h, X).$$

Here, we know that $H = -\frac{1}{2}B^h$ from Proposition 3.34 of Falcitelli et al. (2004). Thus, we get

$$g(\tilde{\nabla}_U V, X) = 0,$$

and so we have

$$g(\tilde{\nabla}_U X, V) = 0.$$

Hence, we obtain the following result.

Theorem 3.7. *If $\pi : (M, J, g) \to (N, J', g')$ is a Clairaut l.c.K. submersion, then we have*

$$(\tilde{\nabla}_U V)^h = 0 \quad and \quad (\tilde{\nabla}_U X)^v = 0,$$

where U and V are vertical, and X is horizontal vector fields on M.

4. EINSTEIN LOCALLY CONFORMAL KAEHLER SUBMERSIONS

In this section, we shall give the conditions for the fibers and the base manifold of a l.c.K. submersion to be Einstein.

Definition 4.1. A Riemannian manifold (M, g) with $\dim(M) = m > 2$ is said to be an *Einstein manifold* if its Ricci tensor $S = \frac{r}{m}g$, where r denotes the scalar curvature of M.

Lemma 4.2. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion. If the Lee vector field B is horizontal, then the Ricci tensor S is given by*

$$\begin{aligned} S(U,V) = \hat{S}(U,V) &- \sum_{i=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_U V) + \sum_{i=1}^{k} g(\mathcal{T}_U U_i, \mathcal{T}_V U_i) \\ &+ \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{T})(U,V), E_j) - \sum_{j=1}^{l} g(\mathcal{T}_U E_j, \mathcal{T}_V E_j), \end{aligned} \tag{23}$$

$$S(X,Y) = S^*(X,Y) + \sum_{i=1}^{k} g((\nabla_X T)(U_i,U_i), Y) - \sum_{i=1}^{k} g(T_{U_i}X, T_{U_i}Y), \tag{24}$$

$$S(U,X) = \sum_{i=1}^{k} g((\nabla_U \mathcal{T})(U_i,U_i), X) - \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{T})(U,U_i), X), \tag{25}$$

and the scalar curvature r is given by

$$\begin{aligned} r = \hat{r} + r^* &- \sum_{i,j=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_{U_j}U_j) - 2\sum_{j=1}^{l}\sum_{i=1}^{k} g(\mathcal{T}_{U_i}E_j, \mathcal{T}_{U_i}E_j) \\ &+ \sum_{i,j=1}^{k} g(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_i}U_j) + \sum_{j=1}^{l}\sum_{i=1}^{k} g((\nabla_{E_j}\mathcal{T})(U_i,U_i), E_j), \end{aligned} \tag{26}$$

where $\{U_1, ..., U_k\}$ and $\{E_1, ..., E_l\}$ are orthonormal frames of vertical and horizontal distributions, respectively, S^* is the horizontal lift of Ricci tensor of N, $\hat{S}$ is Ricci tensor of any fiber, r^* is the lift of scalar curvature of N and $\hat{r}$ is scalar curvature of any fiber.

Proof. From Proposition 2 of Lee et al. (2015) we have,

$$\begin{aligned} S(U,V) = \hat{S}(U,V) &- \sum_{i=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_U V) + \sum_{i=1}^{k} g(\mathcal{T}_U U_i, \mathcal{T}_V U_i) \\ &+ \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{T})(U,V), E_j) - \sum_{j=1}^{l} g(\mathcal{T}_U E_j, \mathcal{T}_V E_j) \\ &+ \sum_{j=1}^{l} g(\mathcal{A}_{E_j}U, \mathcal{A}_{E_j}V), \end{aligned}$$

$$\begin{aligned} S(X,Y) = S^*(X,Y) &+ \sum_{i=1}^{k} g((\nabla_X T)(U_i,U_i), Y) - \sum_{i=1}^{k} g(T_{U_i}X, T_{U_i}Y) \\ &+ \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{A})(X,Y), U_i) + \sum_{i=1}^{k} g(\mathcal{A}_X U_i, \mathcal{A}_Y U_i) \\ &-3 \sum_{j=1}^{l} g(\mathcal{A}_{E_j}X, \mathcal{A}_{E_j}Y), \end{aligned}$$

$$\begin{aligned} S(U,X) = &\sum_{i=1}^{k} g((\nabla_U \mathcal{T})(U_i,U_i), X) - \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{T})(U,U_i), X) \\ &+ \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{A})(X,E_j), U) - 2\sum_{j=1}^{l} g(\mathcal{A}_X E_j, \mathcal{T}_U E_j), \end{aligned}$$

and

$$\begin{aligned} r = \hat{r} + r^* &- \sum_{i,j=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_{U_j}U_j) - 2\sum_{j=1}^{l}\sum_{i=1}^{k} g(\mathcal{T}_{U_i}E_j, \mathcal{T}_{U_i}E_j) \\ &+ \sum_{i,j=1}^{k} g(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_i}U_j) + \sum_{j=1}^{l}\sum_{i=1}^{k} g((\nabla_{E_j}\mathcal{T})(U_i,U_i), E_j) \\ &+2\sum_{j=1}^{l}\sum_{i=1}^{k} g(\mathcal{A}_{E_j}U_i, \mathcal{A}_{E_j}U_i) - 3\sum_{i,j=1}^{l} g(\mathcal{A}_{E_i}E_j, \mathcal{A}_{E_i}E_j). \end{aligned}$$

Since the Lee vector field B is horizontal, then we have $\mathcal{A} \equiv 0$, see Proposition 4.3 of Marrero and Rocha (1994). Thus, (23) ~ (26) can be obtained from the above equations, respectively. □

Theorem 4.3. *$\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion with horizontal Lee vector field B. Then (M, J, g) is an Einstein manifold if and only if the following relations hold:*

$$\begin{aligned} \hat{S}(U,V) = &\frac{r}{m}g(U,V) + \sum_{i=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_U V) - \sum_{i=1}^{k} g(\mathcal{T}_U U_i, \mathcal{T}_V U_i) \\ &- \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{T})(U,V), E_j) + \sum_{j=1}^{l} g(\mathcal{T}_U E_j, \mathcal{T}_V E_j), \end{aligned}$$

$$S^*(X,Y) = \frac{r}{m}g(X,Y) - \sum_{i=1}^{k} g((\nabla_X T)(U_i,U_i), Y) + \sum_{i=1}^{k} g(T_{U_i}X, T_{U_i}Y),$$

and

$$\sum_{i=1}^{k} g((\nabla_U \mathcal{T})(U_i,U_i), X) - \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{T})(U,U_i), X) = 0.$$

Proof. If π is a l.c.K. submersion with horizontal Lee vector field B, then $\mathcal{A}$ vanishes. So, from (23), (24) and (25), we have the result. □

For a l.c.K. manifold (M, J, g), the relation between the curvature tensors R and $\tilde{R}$ of ∇ and $\tilde{\nabla}$ respectively, is given by Vaisman (1980)

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z\\ &-\frac{1}{2}\Big\{L(X,Z)Y - L(Y,Z)X - g(Y,Z)\Big[\nabla_X B + \frac{1}{2}\omega(X)B\Big]\\ &+g(X,Z)\Big[\nabla_Y B + \frac{1}{2}\omega(Y)B\Big]\Big\}\\ &-\frac{||\omega||^2}{4}\Big\{g(Y,Z)X - g(X,Z)Y\Big\},\end{aligned} \tag{27}$$

where

$$L(X,Y) = (\nabla_X\omega)(Y) + \frac{1}{2}\omega(X)\omega(Y) = g(\nabla_X B, Y) + \frac{1}{2}\omega(X)\omega(Y), \tag{28}$$

and X, Y and Z are vector fields on M.

As ω is closed and L is a symmetric 2-tensor, we have from (27) that

$$\begin{aligned}e^{\sigma_i}R^i(X,Y,Z,W) = R(X,Y,Z,W)\\ &-\frac{1}{2}\Big\{L(X,Z)g(Y,W) - L(Y,Z)g(X,W)\\ &+L(Y,W)g(X,Z) - L(X,Z)g(Y,W)\Big\}\\ &-\frac{||\omega||^2}{4}\Big\{g(Y,Z)g(X,W) - L(X,Z)g(Y,W)\Big\},\end{aligned} \tag{29}$$

where R^i is the curvature tensor of the locally conformal Kaehler metric g_i.
If $\pi : (M, J, g) \to (N, J', g')$ is a l.c.K. submersion, then (28) takes the form

$$\begin{aligned}L(U,X) = &\ g(\nabla_U B, X) + \tfrac{1}{2}\omega(U)\omega(X)\\ = &\ Ug(B,X) - g(B, \nabla_U X) + \tfrac{1}{2}\omega(U)\omega(X)\\ = &\ U\omega(X) - g(B, \mathcal{T}_U X) - g(B, (\nabla_U X)^h) + \tfrac{1}{2}\omega(U)\omega(X)\\ = &\ U\omega(X) - \omega(\mathcal{T}_U X) - \omega((\nabla_U X)^h) + \tfrac{1}{2}\omega(U)\omega(X),\end{aligned} \tag{30}$$

where U is a vertical and X is a horizontal vector field of M. Similarly, we obtain

$$L(X,Y) = X\omega(Y) - \omega(\mathcal{A}_X Y) - \omega((\nabla_X Y)^h) + \frac{1}{2}\omega(X)\omega(Y), \tag{31}$$

$$L(U,V) = U\omega(V) - \omega(\mathcal{T}_U V) - \omega((\nabla_U V)^v) + \frac{1}{2}\omega(U)\omega(V), \tag{32}$$

where U and V are vertical, and X and Y are horizontal vector fields on M.

Theorem 4.4. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion. Then the Riemannian curvature tensor R^i is given by*

$$\begin{aligned}e^{\sigma_i}R^i(U,V,W,W') = &\ \hat{R}(U,V,W,W') + g(\mathcal{T}_U W, \mathcal{T}_V W') - g(\mathcal{T}_V W, \mathcal{T}_U W')\\ &-\frac{1}{2}\Big\{\big(U\omega(W) - \omega(\mathcal{T}_U W) - \omega((\nabla_U W)^v)\big) + \frac{1}{2}\omega(U)\omega(W)\big)g(V,W')\\ &-\big(V\omega(W) - \omega(\mathcal{T}_V W) - \omega((\nabla_V W)^v) + \frac{1}{2}\omega(V)\omega(W)\big)g(U,W')\\ &-\big(U\omega(W') - \omega(\mathcal{T}_U W') - \omega((\nabla_U W')^v) + \frac{1}{2}\omega(U)\omega(W')\big)g(V,W)\\ &+\big(V\omega(W') - \omega(\mathcal{T}_V W') - \omega((\nabla_V W')^v) + \frac{1}{2}\omega(V)\omega(W')\big)g(U,W)\Big\}\\ &-\frac{||\omega||^2}{4}\Big\{g(V,W)g(U,W') - g(U,W)g(V,W')\Big\},\end{aligned} \tag{33}$$

$$\begin{aligned}e^{\sigma_i}R^i(U,V,W,X) = &\ g((\nabla_U\mathcal{T})(V,W),X) - g((\nabla_V\mathcal{T})(U,W),X)\\ &-\frac{1}{2}\Big\{\big(V\omega(X) - \omega(\mathcal{T}_V X) - \omega((\nabla_V X)^h) + \frac{1}{2}\omega(V)\omega(X)\big)g(U,W)\\ &-\big(U\omega(X) - \omega(\mathcal{T}_U X) - \omega((\nabla_U X)^h) + \frac{1}{2}\omega(U)\omega(X)\big)g(V,W)\Big\},\end{aligned} \tag{34}$$

$$\begin{aligned} e^{\sigma_i} R^i(X,Y,Z,V) = \;& g(\mathcal{A}_Y Z, \mathcal{T}_V X) + g(\mathcal{A}_Z X, \mathcal{T}_V Y) \\ & -g((\nabla_Z \mathcal{A})(X,Y),V) - g(\mathcal{A}_X Y, \mathcal{T}_V Z) \\ & -\frac{1}{2}\Big\{\big(Y\omega(V) - \omega(\mathcal{A}_Y V) - \omega((\nabla_Y V)^v) + \frac{1}{2}\omega(Y)\omega(V)\big)g(X,Z) \\ & -\big(V\omega(X) - \omega(\mathcal{T}_V X) - \omega((\nabla_V X)^h) + \frac{1}{2}\omega(V)\omega(X)\big)g(Y,Z)\Big\}, \end{aligned} \tag{35}$$

$$\begin{aligned} e^{\sigma_i} R^i(X,Y,Z,H) = \;& R^*(X,Y,Z,H) + 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ & -g(\mathcal{A}_Y Z, \mathcal{A}_X H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H) \\ & -\frac{1}{2}\Big\{\big(X\omega(Z) - \omega(\mathcal{A}_X Z) - \omega((\nabla_X Z)^h) + \frac{1}{2}\omega(X)\omega(Z)\big)g(Y,H) \\ & -\big(Y\omega(Z) - \omega(\mathcal{A}_Y Z) - \omega((\nabla_Y Z)^h) + \frac{1}{2}\omega(Y)\omega(Z)\big)g(X,H) \\ & -\big(X\omega(H) - \omega(\mathcal{A}_X H) - \omega((\nabla_X H)^h) + \frac{1}{2}\omega(X)\omega(H)\big)g(Y,Z) \\ & +\big(Y\omega(H) - \omega(\mathcal{A}_Y H) - \omega((\nabla_Y H)^h) + \frac{1}{2}\omega(Y)\omega(H)\big)g(X,Z)\Big\} \\ & -\frac{||\omega||^2}{4}\Big\{g(Y,Z)g(X,H) - g(X,Z)g(Y,H)\Big\}, \end{aligned} \tag{36}$$

$$\begin{aligned} e^{\sigma_i} R^i(X,Y,V,W) = \;& g((\nabla_W \mathcal{A})(X,Y),V) - g((\nabla_V \mathcal{A})(X,Y),W) - g(\mathcal{A}_X V, \mathcal{A}_Y W) \\ & +g(\mathcal{A}_X W, \mathcal{A}_Y V) + g(\mathcal{T}_V X, \mathcal{T}_W Y) - g(\mathcal{T}_W X, \mathcal{T}_V Y) \end{aligned} \tag{37}$$

$$\begin{aligned} e^{\sigma_i} R^i(X,V,Y,W) = \;& g(\mathcal{T}_V X, \mathcal{T}_W Y) - g((\nabla_V \mathcal{A})(X,Y),W) \\ & -g((\nabla_X \mathcal{T})(V,W),Y) - g(\mathcal{A}_X V, \mathcal{A}_Y W) \\ & -\frac{1}{2}\Big\{\big(X\omega(Y) - \omega(\mathcal{A}_X Y) - \omega((\nabla_X Y)^h) + \frac{1}{2}\omega(X)\omega(Y)\big)g(V,W) \\ & +\big(V\omega(W) - \omega(\mathcal{T}_V W) - \omega((\nabla_V W)^v) + \frac{1}{2}\omega(V)\omega(W)\big)g(X,Y)\Big\} \\ & +\frac{||\omega||^2}{4}g(X,Y)g(V,W), \end{aligned} \tag{38}$$

where U, V, W and W' are vertical, and X, Y, Z and H are horizontal vector fields on M, $\hat{R}$ is Riemannian curvature tensor of any fiber, and R^* is the horizontal lift of Riemannian curvature tensor of N.

Proof. *(33) ∼ (38) can be obtained from (29) ∼ (32) by direct computation.* □

Using (33) ∼ (38), we have the following proposition.

Corollary 4.5. Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K submersion. Then the Ricci tensor S^i is given by

$$\begin{aligned} S^i(U,V) = \;& e^{-\sigma_i}\Big\{\hat{S}(U,V) - \sum_{i=1}^{k} g(\mathcal{T}_{U_i} U_i, \mathcal{T}_U V) + \sum_{i=1}^{k} g(\mathcal{T}_U U_i, \mathcal{T}_V U_i) \\ & + \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{T})(U,V),E_j) - \sum_{j=1}^{l} g(\mathcal{T}_U E_j, \mathcal{T}_V E_j) + \sum_{j=1}^{l} g(\mathcal{A}_{E_j} U, \mathcal{A}_{E_j} V) \\ & +\Big(\frac{k+l-2}{2}\Big)(U\omega(V) - \omega(\mathcal{T}_U V) - \omega((\nabla_U V)^v) + \frac{1}{2}\omega(U)\omega(V)) \\ & + g(U,V)\Big[\frac{1}{2}\sum_{i=1}^{k+l} g(\nabla_{E_i} B, E_i) - \frac{||\omega||^2}{4}(k+l-2)\Big]\Big\}, \end{aligned}$$

$$\begin{aligned}
S^i(X,Y) = e^{-\sigma_i}\Bigg\{ & S^*(X,Y) + \sum_{i=1}^{k} g((\nabla_X T)(U_i,U_i),Y) - \sum_{i=1}^{k} g(T_{U_i}X, T_{U_i}Y) \\
& + \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{A})(X,Y),U_i) + \sum_{i=1}^{k} g(\mathcal{A}_X U_i, \mathcal{A}_Y U_i) - 3\sum_{j=1}^{l} g(\mathcal{A}_{E_j}X, \mathcal{A}_{E_j}Y) \\
& + \Big(\frac{k+l-2}{2}\Big)(X\omega(Y) - \omega(\mathcal{A}_X Y) - \omega((\nabla_X Y)^h) + \frac{1}{2}\omega(X)\omega(Y)) \\
& + g(X,Y)\Big[\frac{1}{2}\sum_{i=1}^{k+l} g(\nabla_{E_i}B, E_i) - \frac{||\omega||^2}{4}(k+l-2)\Big]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
S^i(U,X) = \quad e^{-\sigma_i}\Bigg\{ & \sum_{i=1}^{k} g((\nabla_U \mathcal{T})(U_i,U_i),X) - \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{T})(U,U_i),X) \\
& + \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{A})(X,E_j),U) - 2\sum_{j=1}^{l} g(\mathcal{A}_X E_j, \mathcal{T}_U E_j) \\
& + \Big(\frac{k+l-2}{2}\Big)(U\omega(X) - \omega(\mathcal{T}_U X) - \omega((\nabla_U X)^h) + \frac{1}{2}\omega(U)\omega(X))\Bigg\},
\end{aligned}$$

and the scalar curvature r^i is given by

$$r^i = e^{-\sigma_i}\Bigg\{ r + (k+l-1)\Big[\sum_{i=1}^{k+l} g(\nabla_{E_i}B, E_i) - \frac{||\omega||^2}{4}(k+l-2)\Big]\Bigg\},$$

where $\{E_1, ..., E_{k+l}\}$ is an orthonormal frame field of tangent bundle of M.

Theorem 4.6. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion. Then the curvature tensor R^i has the relation*

$$\begin{aligned}
R^i(X,Y,Z,W) = & \; R^i(JX,JY,JZ,JW) \\
& + \frac{1}{2}\Big\{\delta(X,Z)g(Y,W) - \delta(Y,Z)g(X,W) \\
& - \delta(X,W)g(Y,Z) + \delta(Y,W)g(X,Z)\Big\},
\end{aligned} \tag{39}$$

where

$$\delta(X,Y) = L(X,Y) - L(JX,JY).$$

and X, Y, Z and W are vector fields on M.

Proof. *Since g^i is a Kaehler metric then $R^i(X,Y,Z,W) = R^i(JX,JY,JZ,JW)$. If we write the last equation in (29), we get (39).* □

Using (33), (38) and (36), we get the following equations, respectively.

Theorem 4.7. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion. Then the sectional curvature tensor K^i is given by*

$$\begin{aligned}
K^i(U,V) = e^{-\sigma_i} \quad \Bigg\{ & \hat{K}(U,V) + \frac{g(\mathcal{T}_U U, \mathcal{T}_V V) - ||\mathcal{T}_U V||^2}{||U \wedge V||^2} \\
& - \frac{1}{2||U \wedge V||^2}\Big[\Big(U\omega(U) - \omega(\mathcal{T}_U U) - \omega((\nabla_U U)^v) + \frac{1}{2}(\omega(U))^2\Big)g(V,V) \\
& - 2\Big(U\omega(V) - \omega(\mathcal{T}_U V) - \omega((\nabla_U V)^v) + \frac{1}{2}\omega(U)\omega(V)\Big)g(U,V) \\
& + \Big(V\omega(V) - \omega(\mathcal{T}_V V) - \omega((\nabla_V V)^v) + \frac{1}{2}(\omega(V))^2\Big)g(U,U)\Big] + \frac{||\omega||^2}{4}\Bigg\},
\end{aligned}$$

$$\begin{aligned}
K^i(X,U) = \quad & e^{-\sigma_i}\Bigg\{\frac{-g((\nabla_X\mathcal{T})(U,U),X) - ||\mathcal{A}_XU||^2 + ||\mathcal{T}_UX||^2}{||X||^2||U||^2} \\
& -\frac{1}{2||X||^2||U||^2}\Big[\Big(X\omega(X) - \omega((\nabla_XX)^h) + \frac{1}{2}(\omega(X))^2\Big)g(U,U) \\
& +\Big(U\omega(U) - \omega(\mathcal{T}_UU) - \omega((\nabla_UU)^v) + \frac{1}{2}(\omega(U))^2\Big)g(X,X)\Big] + \frac{||\omega||^2}{4}\Bigg\},
\end{aligned}$$

$$\begin{aligned}
K^i(X,Y) = \quad & e^{-\sigma_i}\Bigg\{K^*(X,Y) + \frac{3||\mathcal{A}_XY||^2}{||X\wedge Y||^2} \\
& -\frac{1}{2||X\wedge Y||^2}\Big[\Big(X\omega(X) - \omega((\nabla_XX)^h) + \frac{1}{2}(\omega(X))^2\Big)g(Y,Y) \\
& -2\Big(X\omega(Y) - \omega(\mathcal{A}_XY) - \omega((\nabla_XY)^h) + \frac{1}{2}\omega(X)\omega(Y)\Big)g(X,Y) \\
& +\Big(Y\omega(Y) - \omega((\nabla_YY)^h) + \frac{1}{2}(\omega(Y))^2\Big)g(X,X)\Big] + \frac{||\omega||^2}{4}\Bigg\},
\end{aligned}$$

where U,V are vertical and X,Y are horizontal vector fields on M.

Definition 4.8. Let $\pi : (M,J,g) \longrightarrow (N,J',g')$ be a l.c.K. submersion. The holomorphic bisectional curvature is defined for any pair of nonzero vector fields E and F by

$$\mathcal{B}(E,F) = \frac{R(E,JE,F,JF)}{||E||^2||F||^2}, \tag{40}$$

and the holomorphic sectional curvature of the 2-plane spanned by E and JE is

$$\mathcal{H}(E) = \mathcal{B}(E,E). \tag{41}$$

Using (40) and (41) we get the following two propositions.

Proposition 4.9. *Let $\pi : (M,J,g) \longrightarrow (N,J',g')$ be a l.c.K. submersion. Then the holomorphic bisectional curvature $\mathcal{B}^i$ is given by*

$$\begin{aligned}
\mathcal{B}^i(U,V) = \frac{e^{-\sigma_i}}{||U||^2||V||^2} \quad & \Bigg\{\hat{R}(U,JU,V,JV) + g(\mathcal{T}_UV,\mathcal{T}_{JU}JV) - g(\mathcal{T}_{JU}V,\mathcal{T}_UJV) \\
& -\frac{1}{2}\Big[\Big(U\omega(V) - \omega(\mathcal{T}_UV) - \omega((\nabla_UV)^v) + \frac{1}{2}\omega(U)\omega(V) \\
& +JU\omega(JV) - \omega(\mathcal{T}_{JU}JV) - \omega((\nabla_{JU}JV)^v) + \frac{1}{2}\omega(JU)\omega(JV)\Big)g(U,V) \\
& +\Big(U\omega(JV) - \omega(\mathcal{T}_UJV) - \omega((\nabla_UJV)^v) + \frac{1}{2}\omega(U)\omega(JV) \\
& -JU\omega(V) + \omega(\mathcal{T}_{JU}V) + \omega((\nabla_{JU}V)^v) - \frac{1}{2}\omega(JU)\omega(V)\Big)g(U,JV)\Big] \\
& +\frac{||\omega||^2}{4}\Big[(g(U,JV))^2 + (g(U,V))^2\Big]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}^i(X,U) = \frac{e^{-\sigma_i}}{||X||^2||U||^2}\Bigg\{ & -g((\nabla_U\mathcal{A})(X,JX),JU) + g((\nabla_{JU}\mathcal{A})(X,JX),U) \\
& - g(\mathcal{A}_XU,\mathcal{A}_{JX}JU) + g(\mathcal{A}_XJU,\mathcal{A}_{JX}U) \\
& + g(\mathcal{T}_UX,\mathcal{T}_{JU}JX) - g(\mathcal{T}_{JU}X,\mathcal{T}_UJX)\Bigg\}
\end{aligned}$$

$$\mathcal{B}^i(X,Y) = \frac{e^{-\sigma_i}}{||X||^2||Y||^2} \Big\{ R^*(X,JX,Y,JY) + 2g(\mathcal{A}_X JX, \mathcal{A}_Y JY)$$
$$-g(\mathcal{A}_{JX}Y, \mathcal{A}_X JY) + g(\mathcal{A}_X Y, \mathcal{A}_{JX}JY)$$
$$-\frac{1}{2}\Big[\Big(X\omega(Y) - \omega(\mathcal{A}_X Y) - \omega((\nabla_X Y)^h) + \frac{1}{2}\omega(X)\omega(Y)$$
$$+JX\omega(JY) - \omega(\mathcal{A}_{JX}JY) - \omega((\nabla_{JX}JY)^h) + \frac{1}{2}\omega(JX)\omega(JY)\Big)g(X,Y)$$
$$+\Big(X\omega(JY) - \omega(\mathcal{A}_X JY) - \omega((\nabla_X JY)^h) + \frac{1}{2}\omega(X)\omega(JY)$$
$$-JX\omega(Y) + \omega(\mathcal{A}_{JX}Y) + \omega((\nabla_{JX}Y)^h) - \frac{1}{2}\omega(JX)\omega(Y)\Big)g(X,JY)\Big]$$
$$+\frac{||\omega||^2}{4}\Big[(g(X,JY))^2 + (g(X,Y))^2\Big]\Big\},$$

where U, V are vertical and X, Y are horizontal vector fields.

Proposition 4.10. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion. Then the holomorphic sectional curvature $\mathcal{H}^i$ is given by*

$$\mathcal{H}^i(U) = \frac{e^{-\sigma_i}}{||U||^4} \Big\{ \hat{R}(U,JU,U,JU) + g(\mathcal{T}_U U, \mathcal{T}_{JU}JU) - ||\mathcal{T}_U JU||^2$$
$$-\frac{1}{2}\Big(U\omega(U) - \omega(\mathcal{T}_U U) - \omega((\nabla_U U)^v) + \frac{1}{2}(\omega(U))^2$$
$$+JU\omega(JU) - \omega(\mathcal{T}_{JU}JU) - \omega((\nabla_{JU}JU)^v) + \frac{1}{2}(\omega(JU))^2\Big)||U||^2 + \frac{||\omega||^2||U||^4}{4}\Big\},$$

$$\mathcal{H}^i(X) = \frac{e^{-\sigma_i}}{||X||^4} \Big\{ R^*(X,JX,X,JX) + 3||\mathcal{A}_X JX||^2$$
$$-\frac{1}{2}\Big(X\omega(X) - \omega(\nabla_X X) + \frac{1}{2}(\omega(X))^2$$
$$+JX\omega(JX) - \omega(\nabla_{JX}JX) + \frac{1}{2}(\omega(JX))^2\Big)||X||^2 + \frac{||\omega||^2||X||^4}{4}\Big\},$$

where U is vertical and X is horizontal vector fields on M.

Peer Review: Externally peer-reviewed.
Author Contribution: All authors have contributed equally.
Conflict of Interest: Authors declared no conflict of interest.
Financial Disclosure: Authors declared no financial support.

LIST OF AUTHOR ORCIDS

B. Pirinççi https://orcid.org/0000-0002-4692-9590
Ç. Çimen https://orcid.org/0009-0000-6331-9615
D. Ulusoy https://orcid.org/0000-0002-0742-4047

REFERENCES

Allison, D., 1996, Lorentzian Clairaut submersions, Geom. Dedicata, 63(3), 309-319.
Bishop, R.L., 1972, Clairaut submersions, differential geometry (in Honor of Kentaro Yano), Tokyo, Kinokuniya, 21-31.
Çimen Ç., Pirinççi B., Taştan H.M., Ulusoy D., 2023, On locally conformal Kaehler submersions (Submitted).
Dragomir, S., Ornea, L., 1998, Locally conformal Kähler geometry, Birkhauser: Boston, Basel, Berlin.
Falcitelli, M., Lanus, S., Pastore, A.M., 2004, Riemannian Submersion and Related Topics, World Scientific Publishing Co. Pte. Ltd.: Singapore
Gray, A., 1967, Pseudo-Riemannian Almost Product Manifolds and Submersions, Journal of Mathematics and Mechanics, (16)7, 715-737.
Gündüzalp, Y., 2020, Clairaut anti-invariant submersions from locally product Riemannian manifolds, Beitr. Algebra Geom., 61, 605-614.
Lee, J., Park, J.H., Şahin, B., Song, D.Y., 2015, Einstein conditions for the base space of anti-invariant Riemannian submersions and Clairaut submersions, Taiwanese J. Math., 19(4), 1145-1160.
Marrero, J.C., Rocha, J., 1994, Locally conformal Kaehler submersions, Geom. Dedicata, 52(3), 271-289.

O'Neill, B., 1966, The fundamental equations of a submersion, Mich. Math. J., 13, 458–469,
O'Neill, B. 1967 Submersions and Geodesics, Duke Math. J., 34(2), 363-373.
Şahin, B., 2017, Riemannian submersions, Riemannian maps in Hermitian geometry, and their application, Elsiever.
Taştan, H.M., Gerdan S., 2016, Clairaut anti-invariant submersions from Sasakian and Kenmotsu manifolds, Mediterr. J. Math., 14(6), 234-251.
Taştan, H.M., Gerdan Aydın, S., 2019, Clairaut anti-invariant submersions from cosymplectic manifolds, Honam Math. J., 41(4), 707-724.
Vaisman, I., 1980, Some curvature properties of locally conformal Kaehler manifolds, Trans. Amer. Math. Soc., 259, 439-447.
Watson, B., 1976, Almost Hermitian submersions, J. Differ. Geom., 11(1), 147–165.

Istanbul Journal of Mathematics
ijmath 2023, 1 (1), 40–47
DOI: 10.26650/ijmath.2023.00001

RESEARCH ARTICLE

Complex extreme points and complex rotundity in Orlicz spaces equipped with the s-norm

E. Başar [1,4], B.H. Uysal [2,4*], Ş. Yaşar [3,4]

[1] Yeditepe University, Faculty of Arts and Sciences, Department of Mathematics, Ataşehir, 34755, İstanbul, Türkiye
[2] İstanbul University, Faculty of Science, Department of Mathematics, Vezneciler, 34134, İstanbul, Türkiye
[3] Gebze Technical University, Faculty of Basic Sciences, Department of Mathematics, Gebze, 41400, Kocaeli, Türkiye
[4] İstanbul University, Institue of Graduate Studies in Science, Süleymaniye, 34134, İstanbul, Türkiye

ABSTRACT
Let Φ be an Orlicz function and $L^{\Phi}(X, \Sigma, \mu)$ be the corresponding Orlicz space on a non-atomic, σ-finite, complete measure space (X, Σ, μ). It is known that extreme points which are connected with rotundity of the whole spaces are the most essential and important geometric notion in the geometric theory of Banach spaces. On the other hand, geometric theory of complex Banach spaces has significant applications that differ from the geometric theory of real Banach spaces. In this paper, we first describe the complex extreme points of unit ball of Orlicz spaces equipped with the s-norm where s is a strictly increasing outer function. We also give criteria for complex rotundity. Our study generalizes and unifies the results that have been obtained for the Orlicz norm and the p-Amemiya norm ($1 < p < \infty$) separately.

Mathematics Subject Classification (2020): 46E30, 46B20

Keywords: Orlicz space, complex extreme points, complex strictly rotund, s-norm

1. INTRODUCTION

The notion of extreme points plays a crucial role for geometric theory of Banach spaces. Also, rotundity properties are very important in geometry of Banach spaces and its applications. Since the early 1980's, the investigations concerning the geometric theory of complex Banach spaces have been developed because it has significant applications that differ from the geometric theory of real Banach spaces. For instance, the notion of complex rotundity, which was introduced by Thorp, E., Whitley, R. (1967), has an important application in the theory of analytic functions. It is known that if f is a function from the unit disc of $\mathbb{C}$ into a complex Banach space X, f is analytic, i.e. $x^* \circ f$ is analytic in the classical sense for any $x^* \in X^*$ (the dual space of X) and the maximum of the function $F(z) = \|f(z)\|$ is attained in an interior point of unit disc, then F is a constant function. However, in the case when X is complex rotund, more can be deduced, namely that f is a constant function.

On the other hand, Orlicz spaces comprise an important class of Banach spaces that are a kind of generalization of Lebesgue spaces. The theory of Orlicz spaces has been greatly developed because of its important theoretical properties and value in applications. Some examples for applications of Orlicz spaces can be found in Arıs B., Öztop S., (2023) and Üster R. (2021). Structure of complex extreme points and complex rotundity in the class of Musielak–Orlicz spaces have been first studied by Wu, C.X., Sun, H. (1987) and Wu, C.X., Sun, H. (1987). Then Chen, L., Cui, Y. (2010) gave criteria for complex extreme points and complex rotundity in Orlicz function spaces equipped with the p-Amemiya norm.

Wisła, M. (2020), using the concept of an outer function, presented a general and universal method of introducing norms in Orlicz spaces that covered the classical Orlicz and Luxemburg norms, and p-Amemiya norms ($1 \leqslant p \leqslant \infty$). After then, Başar E., Öztop, S., Uysal, B.H., Yaşar, Ş. (2023), classified s-norms with respect to the constant σ_s and described real extreme points as well.

Our first aim in this work is to describe the complex extreme points in Orlicz spaces equipped with s-norms where s is strictly increasing. Then we give criteria for complex rotundity by using description of extreme points.

The structure of this paper as follows. In Section 2, we provide necessary definitions. In Section 3, we recall some technical results for Orlicz spaces equipped with s-norms that will be used and we make some observations from these known results. In

Corresponding Author: B.H. Uysal **E-mail:** huseyinuysal@istanbul.edu.tr
Submitted: 17.04.2023 • **Last Revision Received:** 31.05.2023 • **Accepted:** 01.06.2023 • **Published Online:** 06.06.2023

Section 4, we first describe complex extreme points of unit ball in Orlicz spaces equipped with s-norms for a strictly increasing outer function s. Then we obtain a necessary and sufficient condition for complex rotundity.

2. PRELIMINARIES

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if $\Phi(0) = 0$, Φ is not identically equal to zero, Φ is even and convex on the interval $(-b_\Phi, b_\Phi)$, and Φ is left continuous at b_Φ, where $b_\Phi = \sup\{u > 0 : \Phi(u) < \infty\}$. From these properties it follows that an Orlicz function Φ is continuous on $(-b_\Phi, b_\Phi)$, increasing on $[0, b_\Phi)$, and satisfies $\lim\limits_{u\to\infty} \Phi(u) = \infty$. If Φ is an Orlicz function, letting also $a_\Phi = \sup\{u \geqslant 0 : \Phi(u) = 0\}$, then $a_\Phi = 0$ means that Φ vanishes only at 0 while $b_\Phi = \infty$ means that Φ takes only finite values. In this work, we assume that Orlicz function satisfies $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = \infty$.

For an Orlicz function Φ, we define its complementary function Ψ by the formula

$$\Psi(v) = \sup_{u\geq 0}\{u|v| - \Phi(u)\}.$$

It is well-known that the complementary function is an Orlicz function as well. Let p_+ denote the right derivative of an Orlicz function Φ and q_+ denote the right derivative of its complementary function Ψ with the conventions that $\lim\limits_{u\to\infty} p_+(u) = p_+(\infty)$ and $p_+(u) = \infty$ for all $u \geqslant b_\Phi$. If there exists a constant $K > 0$ such that $\Phi(2u) \leqslant K\Phi(u)$ for all $u \in \mathbb{R}$, we say that Orlicz function Φ satisfies the Δ_2 condition and we denote this by $\Phi \in \Delta_2$. We know that the pair (Φ, Ψ) satisfies Young's inequality, that is,

$$xy \leqslant \Phi(x) + \Psi(y) \qquad (x, y \in \mathbb{R}),$$

where equality holds when $y = p_+(x)$ or $x = q_+(y)$ for $x, y \in \mathbb{R}$ (Rao, M. M. and Ren, Z. D. (1991)).

Throughout the paper, we will assume that (X, Σ, μ) is a measure space with a σ-finite, non-atomic and complete measure μ and denote by $L^c(X, \Sigma, \mu)$ (for short, $L^c(X)$) the space of all μ-equivalence classes of complex-valued and Σ-measurable functions defined on X. In addition, we use the conventions $0 \cdot \infty = 0$, $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

For a given Orlicz function Φ we define on $L^c(X, \Sigma, \mu)$ a convex functional I_Φ by

$$I_\Phi(f) = \int\limits_X \Phi(|f(t)|)\, d\mu \ \text{ for any } f \in L^c(\mu).$$

The Orlicz space $L^\Phi(X, \Sigma, \mu)$ generated by an Orlicz function Φ is a linear space of measurable functions defined by Orlicz, W. (1932)

$$L^\Phi(X, \Sigma, \mu) = \{f \in L^c(X, \Sigma, \mu) : I_\Phi(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

We denote the Orlicz space $L^\Phi(X, \Sigma, \mu)$ shortly by L^Φ.

The Orlicz space L^Φ is usually equipped with the Orlicz norm (Orlicz, W. (1932))

$$\|f\|_\Phi^o = \sup\left\{\int_X |f(t)g(t)|\, d\mu : g \in L^\Psi,\ I_\Psi(g) \leq 1\right\},$$

where Ψ is the complementary function to Φ, or with the equivalent Luxemburg norm

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : I_\Phi\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

Further, for all $1 \leq p \leq \infty$ the p-Amemiya norm is defined on L^Φ by

$$\|f\|_{\Phi,p} = \begin{cases} \inf\limits_{k>0} k^{-1}(1 + I_\Phi(kf)^p)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \inf\limits_{k>0} k^{-1}\max\{1, I_\Phi(kf)\}, & \text{if } p = \infty. \end{cases}$$

The family of p-Amemiya norms includes the Orlicz and Luxemburg norms (see Cui, Y., Duan, L., Hudzik, H. and Wisła, M. (2008)).

In 2020, the notion of the s-norm was introduced by M. Wisła and all of the following definitions can be found in Wisła, M. (2020).

Definition 2.1. A function $s : [0, \infty] \to [1, \infty]$ is called an outer function if it is convex and satisfies the inequality

$$\max\{u, 1\} \leqslant s(u) \leqslant u + 1$$

for all $u \geqslant 0$.

Let us note that an outer function s is continuous and increasing on $[0,\infty)$. Evidently $s(0)=1$ and set $s(\infty)=\infty$.

Since it is convex, an outer function s has both right and left derivatives. Let s'_+ be the right derivative of s so that $s'_+:[0,\infty)\to[0,1]$ is an increasing function. Let ${s'_+}^{-1}:[0,1]\to[0,\infty]$ be a general inverse of s'_+ as defined in (Wisła, M. 2020, p. 11). Then ${s'_+}^{-1}$ is an increasing function as well.

Let us give some examples of families of outer functions (see Wisła, M. (2020)).

Example 2.1. (i) For $1\leqslant p\leqslant\infty$,

$$s_p(u)=\begin{cases}(1+u^p)^{1/p}, & \text{if } 1\leqslant p<\infty,\\ \max\{1,u\}, & \text{if } p=\infty.\end{cases} \tag{1}$$

(ii) For $0\leqslant c\leqslant 1$,

$$s_c(u)=\max\{1,u+c\}. \tag{2}$$

(iii) For $1\leqslant m\leqslant 2$,

$$s_m(u)=\begin{cases}(m-1)u+1, & \text{if } 0\leqslant u\leqslant 1,\\ u+m-1, & \text{if } u>1.\end{cases} \tag{3}$$

Definition 2.2. Let s be an outer function and Φ be an Orlicz function. Then the s-norm of $f\in L^\Phi$ is defined by

$$\|f\|_{\Phi,s}=\inf_{k>0}\frac{1}{k}s(I_\Phi(kf)).$$

The Orlicz space equipped with the s-norm will be denoted by L^Φ_s.

Observe that each of the families given in Example 2.1 generates both the Orlicz norm and the Luxemburg norm. In (1), if we take $s=s_1$ then $\|f\|_{\Phi,s}=\|f\|^o_\Phi$; if $s=s_\infty$, then $\|f\|_{\Phi,s}=\|f\|_\Phi$; if $s=s_p$ for $1<p<\infty$ then $\|f\|_{\Phi,s}=\|f\|_{\Phi,p}$ (see Cui, Y., Duan, L., Hudzik, H. and Wisła, M. (2008)). Similarly, in (2), $c=0$ gives the Luxemburg norm and $c=1$ the Orlicz norm. Further, in (3), $m=1$ yields the Luxemburg norm and $m=2$ the Orlicz norm.

It is known that the s-norm $\|\cdot\|_{\Phi,s}$ is equivalent to the Luxemburg norm $\|\cdot\|_\Phi$ with $\|f\|_\Phi\leqslant\|f\|_{\Phi,s}\leqslant 2\|f\|_\Phi$ for any $f\in L^\Phi_s$ (see Wisła, M. (2020)). Note that the Orlicz space L^Φ_s is a Banach space with the s-norm.

Definition 2.3. Let s be an outer function. For all $0\leqslant v\leqslant 1$, define

$$w(v)=\int_0^v {s'_+}^{-1}(t)\,dt. \tag{4}$$

It is clear that w is a non-negative, increasing and continuous function on $[0,1]$.

Definition 2.4. Let s be an outer function. For all $0\leqslant u<\infty$ and $0\leqslant v\leqslant\infty$,

$$\beta_s(u,v)=1-w\left(s'_+(u)\right)-vs'_+(u).$$

Denote also $\beta_s(kf)=\beta_s\left(I_\Phi(kf),I_\Psi(p_+(k|f|))\right)$ for all $f\in L^\Phi_s$.

Note that the function $k\mapsto\beta_s(kf)$ is decreasing on $[0,\infty)$.

Definition 2.5. Let s be an outer function and Φ be an Orlicz function. For $f\in L^\Phi\setminus\{0\}$ and $0<k<\infty$, we define the following functions.

$$D:L^\Phi_s\to\mathcal{P}([0,\infty)),\qquad D(f)=\{0<k<\infty: I_\Phi(kf)<\infty\}$$
$$k^*:L^\Phi_s\to(0,\infty],\qquad k^*(f)=\inf\{k\in D(f):\beta_s(kf)\leqslant 0\}$$
$$k^{**}:L^\Phi_s\to[0,\infty),\qquad k^{**}(f)=\sup\{k\in D(f):\beta_s(kf)\geqslant 0\}$$

It is easy to see that $0<k^*(f)\leqslant k^{**}(f)\leqslant\infty$. Let us also define

$$K(f):=\{0<k<\infty: k^*(f)\leqslant k\leqslant k^{**}(f)\}.$$

Obviously, $K(f)\neq\emptyset\Leftrightarrow k^*(f)<\infty$. If $k^*(f)<\infty$ for any $f\in L^\Phi_s\setminus\{0\}$, then the s-norm is called k^*-finite; if $k^{**}(f)<\infty$ for any $f\in L^\Phi_s\setminus\{0\}$, then the s-norm is called k^{**}-finite. Further, if $k^*(f)=k^{**}(f)<\infty$ for any $f\in L^\Phi_s\setminus\{0\}$, then the s-norm is called k-unique.

Definition 2.6. Let s be an outer function. Define the constant σ_s by

$$\sigma_s=\sup\{u\geqslant 0: s(u)=1\}.$$

Note that $0 \leqslant \sigma_s \leqslant 1$ and it is obvious that s is strictly increasing on $[\sigma_s, \infty)$. We focus on the cases of $\sigma_s > 0$ and $\sigma_s = 0$ in the rest of this paper. The key point in defining this constant is that the equality $\sigma_s = 0$ provides an inverse function for the outer function s since this function is strictly increasing on the entire interval $[0, \infty)$ whenever $\sigma_s = 0$.

Let $\mathcal{S}$ denote the set of outer functions and define the sets

$$\mathcal{S}_0 = \{s \in \mathcal{S} : \sigma_s = 0\} \qquad \text{and} \qquad \mathcal{S}_+ = \{s \in \mathcal{S} : \sigma_s > 0\}.$$

The constants σ_s of the outer functions in Example 2.1 are obtained as follows.

(i) For s_p of (1),

$$\sigma_{s_p} = \begin{cases} 0, & 1 \leqslant p < \infty, \\ 1, & p = \infty. \end{cases}$$

(ii) For s_c of (2),

$$\sigma_{s_c} = \sup\{u \geqslant 0 : u + c \leqslant 1\} = 1 - c.$$

Note that $0 \leqslant c \leqslant 1$.

(iii) For s_m of (3),

$$\sigma_{s_m} = \sup\{u \geqslant 0 : (m-1)u + 1 = 1\} = \begin{cases} 1, & m = 1, \\ 0, & 1 < m \leqslant 2. \end{cases}$$

As a consequence, we can classify the given outer functions as follows. The outer functions $s_p, s_c, s_m \in \mathcal{S}_0$ for $1 \leqslant p < \infty$, $c = 1$, $1 < m \leqslant 2$ and $s_p, s_c, s_m \in \mathcal{S}_+$ for $p = \infty$, $0 \leqslant c < 1$, $m = 1$.

3. AUXILIARY RESULTS

We recall some technical results that will be used in the rest of paper.

Lemma 3.1. (Chen, S. (1996), Proposition 5.17) *For any $\varepsilon > 0$, there exists $\delta \in (0, \frac{1}{2})$ such that if $u, v \in \mathbb{C}$ and*

$$|v| \geqslant \frac{\delta}{8} \max_j |u + jv|,$$

then

$$|u| \leqslant \frac{1 - 2\delta}{4} \sum_j |u + jv|,$$

where

$$\max_j |u + jv| = \max\{|u + v|, |u - v|, |u + iv|, |u - iv|\},$$

$$\sum_j |u + jv| = |u + v| + |u - v| + |u + iv| + |u - iv|.$$

Lemma 3.2. (Wisła, M. (2020), Lemma 3.2) *For every outer function s and Orlicz function Φ,*

$$\|f\|_{\Phi,\infty} \leq \|f\|_{\Phi,s} \leq \|f\|_{\Phi,1} \leq 2\|f\|_{\Phi,\infty}$$

for all $f \in L_s^\Phi$.

Lemma 3.3. (Cui, Y., Zhan, Y. (2019), Lemma 7) *If $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$ then $K(f) \neq \emptyset$ for any $f \in L_s^\Phi \setminus \{0\}$.*

Theorem 3.4. (Wisła, M. (2020), Theorem 7.3) *Let s be an outer function and Φ be an Orlicz function.*

(i) The s-norm is k^*-finite if and only if one of the following conditions is satisfied.

(a) Φ takes infinite values, i.e., $b_\Phi < \infty$,

(b) $w\left(s'_+(u)\right) = 1$ for some $0 < u < \infty$,

(c) $w(1) = 1$ and Φ is not linear on $[0, \infty)$,

(d) Φ does not admit an oblique asymptote.

(ii) The s-norm is k^{**}-finite if and only if one of the conditions (a), (c) or (d) is satisfied.

(iii) If Φ does not admit an oblique asymptote, then the s-norm is k^{**}-finite if and only if it is k^*-finite.

Theorem 3.5. (Wisła, M. (2020), Theorem 6.1) *Let s be an outer function and Φ be an Orlicz function. For all $f \in L_s^\Phi \setminus \{0\}$, we have*

$$k^*(f) = \inf\left\{k > 0 : \|f\|_{\Phi,s} = \frac{1}{k}s(I_\Phi(kf))\right\},$$
$$k^{**}(f) = \sup\left\{k > 0 : \|f\|_{\Phi,s} = \frac{1}{k}s(I_\Phi(kf))\right\}.$$

Corollary 3.1 (Wisła, M. (2020), **Corollary 6.2**). Let s and Φ be an outer and an Orlicz function, respectively. The followings hold for any $f \in L_s^\Phi \setminus \{0\}$.
(i) For every $k \in (0,\infty) \cap [k^*(f), k^{**}(f)]$, we have $\|f\|_{\Phi,s} = \frac{1}{k}s(I_\Phi(kf))$.
(ii) If $k^{**}(f) = \infty$, then $\|f\|_{\Phi,s} = \lim\limits_{k\to\infty} \frac{1}{k}s(I_\Phi(kf))$.

4. MAIN RESULTS

In this section, we will give some results for s-norms that generalize the results obtained for the Orlicz and the p-Amemiya norms $(1 < p < \infty)$. Then, we will give our main results on complex extreme points of unit ball and complex rotundity of Orlicz space (Theorems 4.3 and Theorem 4.4).

Definition 4.1. (see Chen, S. (1996)) Let $B(L_s^\Phi)$ (resp. $S(L_s^\Phi)$) be the closed unit ball (resp. the unit sphere) of a Orlicz space L_s^Φ. A function $f \in S(L_s^\Phi)$ is called an complex extreme point of $B(L_s^\Phi)$ if for any non-zero $g \in L_s^\Phi$ implies $\max_{|\lambda|=1} \|f + \lambda g\|_{\Phi,s} > 1$. The set of all complex extreme points of $B(L_s^\Phi)$ is denoted by $\operatorname{Ext} B(L_s^\Phi)$. Orlicz space is called complex strictly rotund if every element of $S(L_s^\Phi)$ is a complex extreme point of $B(L_s^\Phi)$.

Lemma 4.2. *If $f \in B(L_s^\Phi)$, then $|f(t)| \leqslant b_\Phi$ μ-a.e. on X.*

Proof. Assume that $f \in B(L_s^\Phi)$. By Lemma 3.2, we have $\|f\|_{\Phi,\infty} \le 1$. Therefore, we obtain $I_\Phi(f) \leqslant 1$ (see Chen, S. (1996)). Hence, $\Phi(|f(t)|) < \infty$ for μ-a.e. $t \in X$. By definition of b_Φ, we have $|f(t)| \leqslant b_\Phi$ μ- a.e. on X.

Theorem 4.3. *Let $s \in \mathcal{S}_0$. Then $f \in S(L_s^\Phi)$ is a complex extreme point of the unit ball $B(L_s^\Phi)$ if and only if $\mu(\{t \in X \ : \ k|f(t)| < a_\Phi\}) = 0$ for any $k \in K(f)$.*

Proof. Necessity. Suppose that $f \in S(L_s^\Phi)$ with $\sigma_s = 0$ is a complex extreme point of the unit ball $B(L_s^\Phi)$. Let us prove for any $k \in K(f)$, $\mu(\{t \in X \ : \ k|f(t)| < a_\Phi\}) = 0$. Assume that there exists $k_0 \in K(f)$ such that $\mu(\{t \in X \ : \ k_0|f(t)| < a_\Phi\}) > 0$. Then we can find $d > 0$ and measurable subset A of X such that $\mu(A) > 0$ and

$$k_0|f(t)| + d \le a_\Phi$$

for any $t \in A$. Letting $g = \frac{d}{k_0}\chi_A$, we obtain $g \neq 0$ and for any $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$,

$$\begin{aligned}\|f + \lambda g\|_{\Phi,s} \le \frac{1}{k_0}s(I_\Phi(k_0(f+\lambda g))) &= \frac{1}{k_0}s(I_\Phi(k_0 f\chi_{X\setminus A}) + I_\Phi(k_0 f\chi_A + \lambda d\chi_A)) \\ &\le \frac{1}{k_0}s(I_\Phi(k_0 f\chi_{X\setminus A}) + I_\Phi((k_0 f + d)\chi_A)) \\ &\le \frac{1}{k_0}s(I_\Phi(k_0 f\chi_{X\setminus A})) \\ &\le \frac{1}{k_0}s(I_\Phi(k_0 f)) = \|f\|_{\Phi,s} = 1.\end{aligned}$$

This gives that $f \notin \operatorname{Ext} B(L_s^\Phi)$.

Sufficiency. Suppose that for any $k \in K(f)$, $\mu(\{t \in X \ : \ k|f(t)| < a_\Phi\}) = 0$. Let us prove $f \in S(L_s^\Phi)$ with $\sigma_s = 0$ is a complex extreme point of the unit ball $B(L_s^\Phi)$. Assume that $f \in S(L_s^\Phi)$ with $\sigma_s = 0$ is not a complex extreme point of the unit ball $B(L_s^\Phi)$. Therefore, there exist $\varepsilon_0 > 0$ and $g_0 \in L_s^\Phi$ with $\|g_0\|_{\Phi,s} > \varepsilon_0$ such that

$$\max_{|\lambda|\le 1} \|f + \lambda g_0\|_{\Phi,s} \le 1. \tag{5}$$

By Lemma 3.1, there exists $\delta_0 \in (0, \frac{1}{2})$ such that if $u, v \in \mathbb{C}$ and

$$|v| \ge \frac{\varepsilon_0}{8}\max_j |u + jv|,$$

then we have

$$|u| \leq \frac{1-2\delta_0}{4} \sum_j |u+jv|.$$

Define $A = \{t \in X \ : \ |g_0(t)| \geq \frac{\varepsilon_0}{8} \max_j |f(t)+jg_0(t)|\}$. We obtain by using (5)

$$\|g_0\chi_{X\setminus A}\|_{\Phi,s} < \frac{\varepsilon_0}{8}\|\max_j |f+jg_0|\|_{\Phi,s} \leq \frac{\varepsilon_0}{8}\sum_j \|f+jg_0\|_{\Phi,s} \leqslant \frac{\varepsilon_0}{2}.$$

Consequently, we have $\|g_0\chi_A\|_{\Phi,s} > \frac{\varepsilon_0}{2}$ which shows that $\mu(A) > 0$. For any $t \in A$, we obtain

$$|f(t)| \leq \frac{1-2\delta_0}{4}\sum_j |f(t)+jg_0(t)|.$$

By Lemma 3.3, we can take any $k \in K(\frac{1}{4}\sum_j |f+jg_0|)$ and we have by (5)

$$\begin{aligned} 1 \geq \left\|\tfrac{1}{4}\textstyle\sum_j |f+jg_0|\right\|_{\Phi,s} &= \tfrac{1}{k}s\left(I_\Phi\left(\tfrac{k}{4}\textstyle\sum_j |f+jg_0|\right)\right) \geq \tfrac{1}{k}s\left(I_\Phi\left(k\tfrac{1-2\delta_0}{4}\textstyle\sum_j |f+jg_0|\right)\right) \\ &\geq \tfrac{1}{k}s(I_\Phi(kf)) \geq \|f\|_{\Phi,s} = 1 \end{aligned}$$

which implies that $\|f\|_{\Phi,s} = \frac{1}{k}s(I_\Phi(kf)) = \|\frac{1}{4}\sum_j |f+jg_0|\|_{\Phi,s} = 1$ and $k \geq 1$. Since $k|f(t)| \geq a_\Phi$ for μ-a.e. $t \in X$, we obtain that

$$\frac{1}{1-2\delta_0}k|f(t)| \geq \frac{a_\Phi}{1-2\delta_0}, \quad \mu - \text{a.e.} t \in A,$$

we conclude that $I_\Phi(k\frac{|f|}{1-2\delta_0}\chi_A) \geq \Phi(\frac{a_\Phi}{1-2\delta_0})\mu(A) > 0$. Let us define $b = \Phi(\frac{a_\Phi}{1-2\delta_0})\mu(A)$. To complete the proof, we consider the following two cases.

Case 1. Let assume that $I_\Phi(k(\frac{1}{4}\sum_j |f+jg_0|)) \geq 2\delta_0 b$. In this case, we obtain the following contradiction

$$\begin{aligned} 1 = \|f\|_{\Phi,s} &= \frac{1}{k}s(I_\Phi(kf\chi_A) + I_\Phi(kf\chi_{X\setminus A})) \\ &\leq \frac{1}{k}s\left(I_\Phi\left(k\left(\frac{1-2\delta_0}{4}\sum_j |f+jg_0|\right)\chi_A\right) + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f+jg_0|\right)\chi_{X\setminus A}\right)\right) \\ &\leq \frac{1}{k}s\left((1-2\delta_0)I_\Phi\left(k\left(\frac{1}{4}\sum_j |f+jg_0|\right)\chi_A\right) + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f+jg_0|\right)\chi_{X\setminus A}\right)\right) \\ &\leq \frac{1}{k}s\left(I_\Phi\left(k\left(\frac{1}{4}\sum_j |f+jg_0|\right)\right) - 2\delta_0 I_\Phi\left(k\left(\frac{1}{4}\sum_j |f+jg_0|\right)\chi_A\right)\right) \\ &\leq \frac{1}{k}s\left(I_\Phi\left(k\left(\frac{1}{4}\sum_j |f+jg_0|\right)\right) - 2\delta_0 I_\Phi\left(k\frac{|f|}{1-2\delta_0}\sum_j |f+jg_0|\chi_A\right)\right) \\ &\leq \frac{1}{k}s\left(I_\Phi\left(k\left(\frac{1}{4}\sum_j |f+jg_0|\right)\right) - 2\delta_0 b\right) \\ &< \frac{1}{k}s(I_\Phi(k(\frac{1}{4}\sum_j |f+jg_0|))) = \|\frac{1}{4}\sum_j |f+jg_0|\|_{\Phi,s} = 1. \end{aligned}$$

Therefore, we obtain a contradiction.

Case 2. Let assume that $I_\Phi(k(\frac{1}{4}\sum_j |f+jg_0|)) < 2\delta_0 b$. By using the fact that for all outer functions $s(u) \leq 1+u$ for any $u \in \mathbb{R}$.

$$
\begin{aligned}
1 = \|f\|_{\Phi,s} &= \frac{1}{k}s(I_\Phi(kf\chi_A) + I_\Phi(kf\chi_{X\setminus A})) \le \frac{1}{k}(1 + I_\Phi(kf\chi_A) + I_\Phi(kf\chi_{X\setminus A})) \\
&\le \frac{1}{k}\left(1 + I_\Phi\left(k\left(\frac{1-2\delta_0}{4}\sum_j |f + jg_0|\right)\chi_A\right) + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_{X\setminus A}\right)\right) \\
&\le \frac{1}{k}\left(1 + (1-2\delta_0)I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_A\right) + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_{X\setminus A}\right)\right) \\
&\le \frac{1}{k}\left(1 + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\right) - 2\delta_0 I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_A\right)\right) \\
&\le \frac{1}{k}\left(1 + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\right) - 2\delta_0 I_\Phi\left(k\frac{|f|}{1-2\delta_0}\sum_j |f + jg_0|\chi_A\right)\right) \\
&\le \frac{1}{k}\left(1 + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\right) - 2\delta_0 b\right) < \frac{1}{k} \le 1
\end{aligned}
$$

Therefore, we obtain a contradiction.

The following theorem gives us necessary and sufficient condition for being complex rotundity of Orlicz spaces when $s \in \mathcal{S}_0$.

Theorem 4.4. *Let $s \in \mathcal{S}_0$. Then L_s^Φ is complex rotund if and only if $a_\Phi = 0$.*

Proof. Necessity. Suppose that L_s^Φ with $\sigma_s = 0$ is complex strictly rotund. Let us prove $a_\Phi = 0$. Assume that $a_\Phi > 0$. Then take $c \in (0, a_\Phi)$. Choose measurable subset A of X and $f \in S(L_s^\Phi)$ such that $\mu(A) > 0$ and $\operatorname{supp} f = X \setminus A$. Take $k \in K(f)$, and define

$$
g(t) = \begin{cases} \frac{c}{k}, & t \in A, \\ f(t), & t \in X \setminus A. \end{cases}
$$

Since $\operatorname{supp} f = X \setminus A$, we obtain $\|g\|_{\Phi,s} \ge \|f\|_{\Phi,s} = 1$. On the other hand,

$$
\begin{aligned}
\|g\|_{\Phi,s} \le \frac{1}{k}s(I_\Phi(kg)) &= \frac{1}{k}s(I_\Phi(c\chi_A) + I_\Phi(kf\chi_{X\setminus A})) \\
&= \frac{1}{k}s(I_\Phi(kf\chi_{X\setminus A})) = \|f\|_{\Phi,s} = 1.
\end{aligned}
$$

Thus, $\|g\|_{\Phi,s} = 1$. However, for $t \in A$, we have $k|g(t)| = c < a_\Phi$, which implies that $g \notin \operatorname{Ext} B(L_s^\Phi)$ by Theorem 4.3.

Sufficiency. Suppose that $a_\Phi = 0$. Let us prove L_s^Φ with $\sigma_s = 0$ is complex strictly rotund. Assume that $f \in S(L_s^\Phi)$ is not a complex extreme point of the unit ball $B(L_s^\Phi)$. It follows from Theorem 4.3 that $\mu(\{t \in X : k|f(t)| < a_\Phi\}) > 0$ for some $k \in K(f)$. Then there exists $t_0 \in X$ such that $a_\Phi > k|f(t_0)| \ge 0$, which contradicts with $a_\Phi = 0$.

5. CONCLUSION

In this work, we characterize complex extreme points and complex rotundity of Orlicz Spaces equipped with the s-norms for $\sigma_s = 0$.

Peer Review: Externally peer-reviewed.
Author Contribution: All authors have contributed equally.
Conflict of Interest: Authors declared no conflict of interest.
Financial Disclosure: Authors declared no financial support.

ACKNOWLEDGEMENTS

We are grateful to anonymous referees for careful reading of the manuscript and for helpful comments.

LIST OF AUTHOR ORCIDS

E. Başar	https://orcid.org/0000-0002-7300-6177
B.H. Uysal	https://orcid.org/0000-0002-2522-3417
Ş. Yaşar	https://orcid.org/0000-0001-9708-1107

REFERENCES

Arıs, B., Öztop, S., 2023, Wiener amalgam spaces with respect to Orlicz spaces on the affine group. J. Pseudo-Differ. Oper. Appl., 14(23).

Başar E., Öztop, S., Uysal, B.H., Yaşar, Ş. 2023, Extreme points in Orlicz spaces equipped with s-norms and its closedness, Math. Nachr. 00, 1–21.

Chen, L., Cui, Y., 2010, Complex extreme points and complex rotundity in Orlicz function spaces equipped with the p-Amemiya norm, Nonlinear Anal., 73(5), 1389-1393.

Chen, S., 1996, Geometry of Orlicz space. Dissertationes Math., Harbin University, China.

Cui, Y., Duan, L., Hudzik, H. and Wisła, M., 2008, Basic theory of p-Amemiya norm in Orlicz spaces ($1 \leq p \leq \infty$): Extreme points and rotundity in Orlicz spaces endowed with these norms, Nonlinear Anal., 69(5), 1796-1816.

Cui, Y., Zhan, Y., 2019, Strongly extreme points and middle point locally uniformly convex in Orlicz spaces equipped with s-norm, Journal of Funct. Spac., 2019, 1-7.

Orlicz, W., 1932, Über eine gewisse klasse von Räumen vom typus B, Bull. Int. Acad. Polon. Sér. A, 207-220.

Rao, M. M. and Ren, Z. D., 1991, Theory of Orlicz Spaces, Marcel Dekker, New York.

Üster, R., 2021, Multipliers for the weighted Orlicz spaces of a locally compact abelian group, Results in Mathematics, 76 (4), Paper No. 183.

Thorp, E., Whitley, R., 1967, The strong maximum modulus theorem for analytic functions into a Banach space, Proc. Amer. Math. Soc., 18, 640-646.

Wisła, M., 2020, Orlicz spaces equipped with s-norms, J. Math. Anal. Appl., 483(2), 123659-123689.

Wu, C.X., Sun, H., 1987, On complex extreme points and complex strict convexities of Musielak–Orlicz spaces, J. Sys. Sci. Math. Scis., 7, 7-13.

Wu, C.X., Sun, H., 1988 On complex uniform convexity of Musielak–Orlicz spaces, J. Northeast. Math., 4, 389-396.

AIMS AND SCOPE

İstanbul Journal of Mathematics aims to contribute to the scientific knowledge in all areas of mathematics. It is an open-acces, peer-reviewed and international journal published semiannually in June and December.

The journal is devoted to publish original research and review articles written in English. Original and high quality research articles, review articles (by invitation only), and high-quality conference proceedings (in special issues by the approval of the editorial board) are welcome. The journal targets international and national communities.

Processing and publication are free of charge with the journal. There is no article processing charges or submission fees for any submitted or accepted articles.

POLICIES

Publication Policy

The journal is committed to upholding the highest standards of publication ethics and pays regard to Principles of Transparency and Best Practice in Scholarly Publishing published by the Committee on Publication Ethics (COPE), the Directory of Open Access Journals (DOAJ), the Open Access Scholarly Publishers Association (OASPA), and the World Association of Medical Editors (WAME) on https://publicationethics.org/resources/guidelines-new/principles-transparency-and-best-practice-scholarly-publishing

The subjects covered in the manuscripts submitted to the journal for publication must be in accordance with the aim and scope of the journal. Only those manuscripts approved by every individual author and that were not published before or sent to another journal, are accepted for evaluation.

Changing the name of an author (omission, addition or order) in papers submitted to the Journal requires written permission from all the declared authors.

Plagiarism, duplication, fraudulent authorship/denied authorship, research/data fabrication, salami slicing/salami publication, breaching copyrights, and prevailing conflicts of interest are unethical behaviors. All manuscripts that are not in accordance with accepted ethical standards will be removed from publication. This also includes any possible malpractice discovered after publication.

Plagiarism:

Submitted manuscripts that pass the preliminary control are scanned for plagiarism using iThenticate software. If plagiarism/self-plagiarism is found, the authors will be informed of it. Editors may resubmit the manuscript for a similarity check at any peer-review or production stage if required. High similarity scores may lead to rejection of the manuscript before and even after acceptance. Depending on the type of article and the percentage of the similarity score taken from each article, the overall similarity score is generally expected to be less than 15 or 20%.

Double Blind Peer-Review:

After the plagiarism check, the eligible ones are evaluated by the editors-in-chief for their originality, methodology, importance of the subject covered and compliance with the journal scope. The editor provides a fair double-blind peer review of the submitted articles and hands over the papers matching the formal rules to at least two national/international referees for evaluation and gives green light for publication upon modification by the authors in accordance with the referees' claims.

Open Access Statement

The journal is open access and all content is freely available without charge to the user or his/her institution. Except for commercial purposes, users are allowed to read, download, copy, print, search, or link to the full text of the articles in this journal without asking prior permission from the publisher or author. This is in accordance with the BOAI definition of open access.

The open access articles in the journal are licensed under the terms of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0) license.

Article Processing Charge

All expenses of the journal are covered by İstanbul University. The processing and publication are free of charge. There are no article processing charges or submission fees for any submitted or accepted articles.

Copyright Notice

Authors publishing with the journal retain the copyright to their work licensed under the Creative Commons Attribution-NonCommercial 4.0 International license (CC BY-NC 4.0) (https://creativecommons.org/licenses/by-nc/4.0/) and grant the Publisher non-exclusive commercial right to publish the work. CC BY-NC 4.0 license permits unrestricted, non-commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Correction, Retraction, Expression of Concern

The editor should consider publishing corrections if minor errors that do not affect the results, interpretations and conclusions of the published paper are detected. The editor should consider retraction if major errors and/or misconduction invalidate results and conclusions are detected.

The editors should consider issuing an expression of concern if there is evidence of research or publication misconduct by the authors; there is evidence that the findings are not reliable and institutions of the authors do not investigate the case or the possible investigation seems to be unfair or nonconclusive.

The COPE and ICJME guidelines are considered for correction, retractions or expression of concern.

Archiving Policy

To guarantee that all papers published in the journal are maintained and permanently accessible, articles are stored in Dergipark, which serves as a national archival website and simultaneously permits LOCKSS to collect, preserve, and serve content.

Additionally, the authors are encouraged to self-archive the final PDF version of their articles in open electronic archives, which conform to the standards of the Open Archives Initiative (https:// www.openarchives.org/). Authors should provide a link from the deposited version to the URL of IUPress journal website.

PEER REVIEW

Peer Review Policies

Only those manuscripts approved by every individual author and that were not published before or submitted to another journal, are accepted for evaluation.

Submitted manuscripts that pass the preliminary control are scanned for plagiarism using iThenticate software. After the plagiarism check, the eligible ones are evaluated by the editor-in-chief for their originality, methodology, importance of the subject covered, and compliance with the journal scope.

The editor hands over the papers matching the formal rules to at least two national/international referees for double-blind peer review evaluation and gives green light for publication upon modification by the authors in accordance with the referees' claims.

Responsibility for the Editor and Reviewers

The editor-in-chief evaluates manuscripts for their scientific content without regard to ethnic origin, gender, sexual orientation, citizenship, religious beliefs or the authors'political philosophy. He/She provides a fair double-blind peer review of the submitted articles for publication and ensures that all the information related to submitted manuscripts is kept as confidential before publishing.

The editor-in-chief is responsible for the content and overall quality of publications. He/She must publish errata pages or make corrections when needed. The editor-in-chief does not allow any conflicts of interest between the authors, editors and reviewers. Only he has the full authority to assign a reviewer and is responsible for the final decision to publish the manuscripts in the journal.

The reviewers must have no conflict of interest with respect to the research, authors and/or research funders. Their judgment must be objective.

Reviewers must ensure that all information related to submitted manuscripts is kept confidential and must report to the editor if they are aware of copyright infringement and plagiarism on the author's side.

A reviewer who feels unqualified to review the topic of a manuscript or knows that its prompt review will be impossible should notify the editor and excuse him/herself from the review process.

The editor informs the reviewers that the manuscripts are confidential information and that this is a privileged interaction. The reviewers and editorial board cannot discuss the manuscripts with other people. The anonymity of referees must be ensured. In particular situations, the editor may share a review by one reviewer with other reviewers to clarify a particular point.

Peer Review Process

Only those manuscripts approved by every individual author and that were not published before or submitted to another journal, are accepted for evaluation.

Submitted manuscripts that pass the preliminary control are scanned for plagiarism using iThenticate software. After the plagiarism check, the eligible ones are evaluated by the editor-in-chief for their originality, methodology, importance of the subject covered and compliance with the journal scope. Editor-in-chief evaluates anuscripts for their scientific content without regard to ethnic origin, gender, sexual orientation, citizenship, religious belief or political philosophy of the authors and ensures a fair double-blind peer review of the selected manuscripts.

The selected manuscripts are sent to at least two national/international referees for evaluation, and publication decision is made by the editor-in-chief upon modification by the authors in accordance with the referees' claims.

Editor-in-chief does not allow any conflicts of interest between the authors, editors, and reviewers and is responsible for the final decision for publication of the manuscript in the journal.

The reviewers' judgments must be objective. Reviewers' comments on the following aspects are expected during the review.

- Does the manuscript contain new and significant information?
- Does the abstract clearly and accurately describe the content of the manuscript?
- Is this problem significant and concisely stated?
- Are the methods comprehensively described?
- Are the interpretations and conclusions justified by the results?
- Is adequate reference made to other works in the field?
- Is the language acceptable?

Reviewers must ensure that all information related to submitted manuscripts is kept confidential and must report to the editor if they are aware of copyright infringement and plagiarism on the author's side.

A reviewer who feels unqualified to review the topic of a manuscript or knows that its prompt review will be impossible should notify the editor and excuse him/herself from the review process.

The editor informs the reviewers that the manuscripts are confidential information, and that this is a privileged interaction. The reviewers and editorial board cannot discuss the manuscripts with other people. The anonymity of the referees is important.

ETHICS

PUBLICATION ETHICS AND PUBLICATION MALPRACTICE STATEMENT

İstanbul Journal of Mathematics is committed to upholding the highest standards of publication ethics and pays regard to Principles of Transparency and Best Practice in Scholarly Publishing published by the Committee on Publication

Ethics (COPE), the Directory of Open Access Journals (DOAJ), the Open Access Scholarly Publishers Association (OASPA), and the World Association of Medical Editors (WAME) on https://publicationethics.org/resources/guidelines-new/principles-transparency-and-best-practice-scholarly-publishing

All submissions must be original, unpublished (including full text in conference proceedings), and not under the review of any other publication. The authors must ensure that the submitted work is original in nature. They must certify that the manuscript has not already been published or submitted elsewhere, in any language. Applicable copyright laws and conventions must be followed. Copyright materials (e.g., tables, figures or extensive quotations) must be reproduced only with appropriate permission and acknowledgement. Any work or words by other authors, contributors, or sources must be appropriately credited and referenced.

Each manuscript is reviewed by at least two referees using a double-blind peer review process. Plagiarism, duplication, fraudulant authorship/ denied authorship, research/data fabrication, salami slicing/salami publication, breaching of copyrights, and prevailing conflicts of interest are unethical behaviors. All manuscripts that are not in accordance with accepted ethical standards will be removed from publication. This also includes any possible malpractice discovered after publication.

Research Ethics

İstanbul Journal of Mathematics adheres to the highest standards in research ethics and follows the principles of international research ethics as defined below. The authors are responsible for the compliance of the manuscripts with ethical rules.

- Principles of integrity, quality and transparency should be sustained in designing the research, reviewing the design and conducting the research.
- The research team and participants should be fully informed about the aims, methods, possible uses, requirements of the research, and risks of participation in research.
- The confidentiality of the information provided by the research participants and the confidentility of the respondents should be ensured. The research should be designed to protect the autonomy and dignity of the participants.
- Research participants should voluntarily participate in the research, not under any coercion.
- Any possible harm to the participants must be avoided. Research should be planned in such a way that the participants are not at risk.
- The independence of research must be clear, and any conflicts of interest must be disclosed.
- In experimental studies with human subjects, written informed consent from the participants who decided to participate in the research must be obtained. In the case of children and those under wardship or with confirmed insanity, legal custodian assent must be obtained.
- If the study is to be carried out in any institution or organization, approval must be obtained from that institution or organization.
- In studies with human subjects, it must be noted in the method section of the manuscript that the informed consent of the participants and ethics committee approval from the institution where the study was conducted have been obtained.

Author's Responsibilities

The authors are responsible for ensuring that the article is in accordance with scientific and ethical standards and rules. The authors must ensure that the submitted work is original in nature. They must certify that the manuscript has not already been published or submitted elsewhere, in any language. Applicable copyright laws and conventions must be followed. Copyright materials (e.g., tables, figures or extensive quotations) must be reproduced only with appropriate permission and acknowledgement. Any work or words by other authors, contributors, or sources must be appropriately credited and referenced.

All the authors of the submitted manuscript must have direct scientific and academic contributions to the manuscript.

The author(s) of the original research articles is defined as a person who is significantly involved in "conceptualization and design of the study", "collecting the data", "analyzing the data", "writing the manuscript", "reviewing the manuscript with a critical perspective" and "planning/conducting the study of the manuscript and/or revising it". Fund raising, data collection and supervision of the research group are not sufficient roles to be accepted as authors. The author(s) must meet all these criteria described above. The order of names in the author list of an article must be a co-decision and must be indicated in the Copyright Agreement Form.

Individuals who do not meet the authorship criteria but have contributed to the study must take place in the acknowledgement section. Individuals providing technical support, general support, material, financial support and assisting in writing are examples to be indicated in the acknowledgement section.

All authors must disclose any issues concerning financial relationships, conflicts of interest, and competing interests that may potentially influence the results of the research or scientific judgment. When an author discovers a significant error or inaccuracy in his/her own published paper, it is the author's obligation to promptly cooperate with the editor-in-chief to provide retractions or corrections of mistakes.

Responsibility for the Editor and Reviewers

The editor-in-chief evaluates manuscripts for their scientific content without regard to ethnic origin, gender, sexual orientation, citizenship, religious beliefs or the authors'political philosophy. He/She provides a fair double-blind peer review of the submitted articles for publication and ensures that all the information related to submitted manuscripts is kept as confidential before publishing.

The editor-in-chief is responsible for the content and overall quality of publications. He/She must publish errata pages or make corrections when needed.

The editor-in-chief does not allow any conflicts of interest between the authors, editors and reviewers. Only he has the full authority to assign a reviewer and is responsible for the final decision to publish the manuscripts in the journal.

The reviewers must have no conflict of interest with respect to the research, authors and/or research funders. Their judgment must be objective.

Reviewers must ensure that all information related to submitted manuscripts is kept confidential and must report to the editor if they are aware of copyright infringement and plagiarism on the author's side.

A reviewer who feels unqualified to review the topic of a manuscript or knows that its prompt review will be impossible should notify the editor and excuse him/herself from the review process.

The editor informs the reviewers that the manuscripts are confidential information and that this is a privileged interaction. The reviewers and editorial board cannot discuss the manuscripts with other people. The anonymity of referees must be ensured. In particular situations,the editor may share a review by one reviewer with other reviewers to clarify a particular point.

AUTHOR GUIDELINES

Manuscript Organization and Submission

1. The publication language of the journal is English
2. Authors are required to submit Copyright Agreement Form, Author Form and Title Page together with the main manuscript document
3. Manuscripts should be prepared using the article template in Latex format
4. Due to double-blind peer review, the main manuscript document must not include any author information.
5. Title page should be submitted together with the main manuscript document and should include the information below:
    - Category of the manuscript
    - The title of the manuscript.

- All authors' names and affiliations (institution, faculty/department, city, country), e-mail addresses, and ORCIDs.
- Information of the corresponding author (in addition to the author's information e-mail address, open correspondence address, and mobile phone number).
- Financial support
- Conflict of interest.
- Acknowledgment.

6. Submitted manuscripts should have an abstract of 100-150 words before the introduction, summarizing the scope, the purpose, the results of the study, and the methodology used. Under the abstract, a minimum of 3 and a maximum of 6 keywords that inform the reader about the content of the study should be specified.
7. Keywords must be in accordance with the Mathematics Subject Classification standard available at www.ams.org/msc
8. The manuscripts should contain mainly these components: title, abstract and keywords; sections, references, tables and figures. The main text of research articles should include introduction, methods, results, discussion, conclusion and references subheadings.
9. Tables and figures should be given with a number and a caption. Every Figure or Table should be referred within the text of the article in numerical order with no abbreviations (ie: Table 1, Figure 1)
10. References should be prepared in line with Harvard reference system. For information: https://www.easybib.com/guides/citation-guides/harvard-referencing/
11. Authors are responsible for all statements made in their work submitted to the journal for publication.

References

İstanbul Journal of Mathematics complies with Harvard system for referencing and in-text citations. For information: https:/ / www.easybib.com/ guides/ citation-guides/ harvard-referencing/ . Accuracy of citations is the author's responsibility. All references should be cited in the text. It is strongly recommended that authors use Reference Management Software such as Zotero, Mendeley, etc.

Submission Checklist

Please ensure the following:

- The title page was prepared according to the journal rules.
- This manuscript has not been submitted to any other journal.
- The manuscript has been checked for English language.
- The manuscript was written in accordance with the full-text writing rules determined by the journal.
- The manuscript has an abstract of 100-150 words and the number of keywords should be 3-6.
- The references are in line with the Harvard reference system.
- The Copyright Agreement Form has been filled in and is ready for submission together with the manuscript.
- The Author Contribution Form has been filled in and is ready for submission together with the manuscript.
- Permission for previously published copyrighted material (text,figure,table) has been obtained if used in the present manuscript.
- The Ethics Committee Report (if necessary) has been obtained and ready for submission together with the manuscript, and the ethics committee report date and number have been given in the manuscript text.
- Review of the journal policies.
- All authors have read and approved the final version of the manuscript.

EK 2-B

Istanbul University
İstanbul Üniversitesi

Journal name: Istanbul Journal of Mathematics
Dergi Adı: Istanbul Journal of Mathematics

Copyright Agreement Form
Telif Hakkı Anlaşması Formu

Responsible/Corresponding Author *Sorumlu Yazar*	
Title of Manuscript *Makalenin Başlığı*	
Acceptance Date *Kabul Tarihi*	
List of Authors *Yazarların Listesi*	

Sıra No	**Name - Surname** *Adı-Soyadı*	**E-mail** *E-Posta*	**Signature** *İmza*	**Date** *Tarih*
1				
2				
3				
4				
5				

Manuscript Type (Research Article, Review, etc.) *Makalenin türü (Araştırma makalesi, Derleme, v.b.)*	

Responsible/Corresponding Author:
Sorumlu Yazar:

University/company/institution	*Çalıştığı kurum*	
Address	*Posta adresi*	
E-mail	*E-posta*	
Phone; mobile phone	*Telefon no; GSM no*	

The author(s) agrees that:
The manuscript submitted is his/her/their own original work, and has not been plagiarized from any prior work,
all authors participated in the work in a substantive way, and are prepared to take public responsibility for the work,
all authors have seen and approved the manuscript as submitted,
the manuscript has not been published and is not being submitted or considered for publication elsewhere,
the text, illustrations, and any other materials included in the manuscript do not infringe upon any existing copyright or other rights of anyone.
ISTANBUL UNIVERSITY will publish the content under Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0) license that gives permission to copy and redistribute the material in any medium or format other than commercial purposes as well as remix, transform and build upon the material by providing appropriate credit to the original work.
The Contributor(s) or, if applicable the Contributor's Employer, retain(s) all proprietary rights in addition to copyright, patent rights.
I/We indemnify ISTANBUL UNIVERSITY and the Editors of the Journals, and hold them harmless from any loss, expense or damage occasioned by a claim or suit by a third party for copyright infringement, or any suit arising out of any breach of the foregoing warranties as a result of publication of my/our article. I/We also warrant that the article contains no libelous or unlawful statements, and does not contain material or instructions that might cause harm or injury.
This Copyright Agreement Form must be signed/ratified by all authors. Separate copies of the form (completed in full) may be submitted by authors located at different institutions; however, all signatures must be original and authenticated.

Yazar(lar) aşağıdaki hususları kabul eder:
Sunulan makalenin yazar(lar)ın orijinal çalışması olduğunu ve intihal yapmadıklarını,
Tüm yazarların bu çalışmaya asli olarak katılmış olduklarını ve bu çalışma için her türlü sorumluluğu aldıklarını,
Tüm yazarların sunulan makalenin son halini gördüklerini ve onayladıklarını,
Makalenin başka bir yerde basılmadığını veya basılmak için sunulmadığını,
Makalede bulunan metnin, şekillerin ve dokümanların diğer şahıslara ait olan Telif Haklarını ihlal etmediğini kabul ve taahhüt ederler.
İSTANBUL ÜNİVERSİTESİ'nin bu fikri eseri, Creative Commons Atıf-GayrıTicari 4.0 Uluslararası (CC BY-NC 4.0) lisansı ile yayınlamasına izin verirler. Creative Commons Atıf-GayrıTicari 4.0 Uluslararası (CC BY-NC 4.0) lisansı, eserin ticari kullanım dışında her boyut ve formatta paylaşılmasına, kopyalanmasına, çoğaltılmasına ve orijinal esere uygun şekilde atıfta bulunmak kaydıyla yeniden düzenleme, dönüştürme ve eserin üzerine inşa etme dâhil adapte edilmesine izin verir.
Yazar(lar)ın veya varsa yazar(lar)ın işvereninin telif dâhil patent hakları, fikri mülkiyet hakları saklıdır.
Ben/Biz, telif hakkı ihlali nedeniyle üçüncü şahıslarca vuku bulacak hak talebi veya açılacak davalarda İSTANBUL ÜNİVERSİTESİ ve Dergi Editörlerinin hiçbir sorumluluğunun olmadığını, tüm sorumluluğun yazarlara ait olduğunu taahhüt ederim/ederiz.
Ayrıca Ben/Biz makalede hiçbir suç unsuru veya kanuna aykırı ifade bulunmadığını, araştırma yapılırken kanuna aykırı herhangi bir malzeme ve yöntem kullanılmadığını taahhüt ederim/ederiz.
Bu Telif Hakkı Anlaşması Formu tüm yazarlar tarafından imzalanmalıdır/onaylanmalıdır. Form farklı kurumlarda bulunan yazarlar tarafından ayrı kopyalar halinde doldurularak sunulabilir. Ancak, tüm imzaların orijinal veya kanıtlanabilir şekilde onaylı olması gerekir.

	Signature / *İmza*	**Date** / *Tarih*
Responsible/Corresponding Author; *Sorumlu Yazar;*		......../......../...............