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Research Article

Estimates of the norms of some cosine and sine series

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ABSTRACT. In the work, we estimate the $\mathbb{L}^1$ norms of some special cosine and sine series used in studying fractional integrals.

Keywords: Fourier series, Dirichlet kernel, cosine and sine sums.

2020 Mathematics Subject Classification: 42A10, 41A16.

1. INTRODUCTION

Let $\mathbb{L}^1$ be the (class) of all 2π-periodic, Lebesgue integrable functions f on $\mathbb{R}$ such that

$$\|f\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\, f(x) \,|\, dx < \infty.$$

For $0 < \gamma < 1$, in this work, we study properties of the series

$$\varphi_\gamma(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^\gamma} \qquad \text{and} \qquad \psi_\gamma(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^\gamma} \tag{1.1}$$

in $\mathbb{L}^1$.

Cosine series of the form

$$f(x) = \sum_{n=1}^{\infty} \mu_n \cos(nx) \tag{1.2}$$

have been studied by several authors (see [1], [14], [12] and [11]). In particular, necessary and sufficient conditions for the convergence in $\mathbb{L}^1$ of the partial sums of the series (1.2) are known (see [7], [8] and [3] and the references therein).

Here we are interested in the series given in (1.1), because of their applications in studying fractional integrals (see [5, p. 422] and [6], where the complex case was considered).

In this work, we look for estimates of the $\mathbb{L}^1$ norms of the functions in (1.1). We restrict the analysis to the case $0 < \gamma < 1$, because it follows from a result proved by Young in [13] (see also [4]) that, for $\gamma \geq 1$, $1 + \varphi_\gamma(x) \geq 0$. Moreover there exists a number α_0 such that, for $0 < \gamma < \alpha_0$, the series $\varphi_\gamma(x)$ is not uniformly bounded below (see [9] or [14, p. 191]).

Here we proof that, if $0 < \gamma < 1$, then

$$\|\varphi_\gamma\|_1 \leq 2 - \frac{1}{2^\gamma} \qquad \text{and} \qquad \|\psi_\gamma\|_1 \leq 2^{1+\gamma}\Big(1 + \frac{1}{\gamma}\Big).$$

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2. Notations and known results

Recall that, for $0 <| x |\leq \pi$ and $n \in \mathbb{N}$, the Dirichlet kernel is given by

$$D_n(x) = 1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin((2n+1)x/2)}{\sin(x/2)}, \quad D_0(x) = 1,$$

while the Fejér kernel is defined by

$$F_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(x) = \frac{1}{n+1}\Big(\frac{\sin((n+1)x/2)}{\sin(x/2)}\Big)^2$$

$$= 1 + 2\sum_{k=1}^{n}\Big(1 - \frac{k}{n+1}\Big)\cos(kx) \tag{2.3}$$

(see [5, p. 42-43]).

The associated conjugate Dirichlet kernel is defined by (see [5, p. 48] or [14, p. 49])

$$\widetilde{D}_n(x) = 2\sum_{k=1}^{n} \sin(kx) = \frac{\cos(x/2) - \cos((2n+1)x/2)}{\sin(x/2)} \tag{2.4}$$

and the conjugate Fejér kernel is given by (see [14, p. 91])

$$\widetilde{F}_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} \widetilde{D}_k(x) = \frac{1}{\tan(x/2)} - \frac{1}{2(n+1)}\frac{\sin((n+1)x)}{\sin^2(x/2)}.$$

Recall that, for $n \geq 2$ (see [10, p. 151]),

$$\|D_n\|_1 \leq 2 + \ln n. \tag{2.5}$$

3. Auxiliary results

As usually, for a given sequence $\{c_k\}$, we denote $\Delta c_k = c_k - c_{k+1}$ and $\Delta^2 c_k = c_k - 2c_{k+1} + c_{k+2}$.

The first identity in the next lemma is well known, but the second and third ones will help us to simplify some computations.

Lemma 3.1. *Let $\{c_k\}_{k=0}^{\infty}$ and $\{d_k\}_{k=0}^{\infty}$ be two numerical sequences. Set $E_k = \sum_{j=0}^{k} d_j$. For each $n \in \mathbb{N}$, $n > 1$,*

$$\sum_{k=0}^{n} c_k d_k = c_n E_n + \Delta c_{n-1}\sum_{k=0}^{n-1} E_k + \sum_{k=0}^{n-2}\Delta^2 c_k \sum_{j=0}^{k} E_j,$$

$$\sum_{k=0}^{n-2}(k+1)\Delta^2 c_k = c_0 - c_n - n\Delta c_{n-1}$$

and

$$\sum_{k=n-1}^{n+m-2}(k+1)\Delta^2 c_k = c_n - c_{n+m} + n\Delta c_{n-1} - (n+m)\Delta c_{n+m-1}.$$

Proof. The first identity is obtained by applying twice the Abel transform

$$\sum_{k=0}^{n} c_k d_k = c_n\sum_{k=0}^{n} d_k + \sum_{k=0}^{n-1}(c_k - c_{k+1})\sum_{j=0}^{k} d_j. \tag{3.6}$$

That is

$$co\sum_{k=0}^{n} c_k d_k = c_n E_n + \sum_{k=0}^{n-1}(c_k - c_{k+1})E_k$$
$$= c_n E_n + (c_{n-1} - c_n)\sum_{k=0}^{n-1} E_k + \sum_{k=0}^{n-2}(c_k - 2c_{k+1} + c_{k+2})\sum_{j=0}^{k} E_j.$$

In particular, if

$$d_k = \begin{cases} 1, & k = 0, \\ 0, & k \geq 1, \end{cases} \tag{3.7}$$

one has $E_k = 1$ ($k \geq 0$). Hence

$$c_0 = c_n + n(c_{n-1} - c_n) + \sum_{k=0}^{n-2}(k+1)(c_k - 2c_{k+1} + c_{k+2})$$

and

$$0 = \sum_{k=0}^{n+m} c_k d_k - \sum_{k=0}^{n} c_k d_k = c_{n+m} - c_n + (n+m)\Delta c_{n+m-1}$$
$$- n\Delta c_{n-1} + \sum_{k=n-1}^{n+m-2}(k+1)\Delta^2 c_k.$$

□

Lemma 3.2. *If $\gamma > 0$ and $c_k = k^{-\gamma}$ for $k \in \mathbb{N}$, then*

$$0 < c_k - c_{k+1} < \frac{\gamma}{k^{1+\gamma}} \quad \textit{and} \quad 0 < c_k - 2c_{k+1} + c_{k+2} < \frac{\gamma(1+\gamma)}{k^{2+\gamma}}.$$

Proof. Set $f_\gamma(x) = x^{-\gamma}$. If $x \geq 1$, then

$$f_\gamma(x) - f_\gamma(x+1) = -\int_x^{x+1} f'_\gamma(y)dy = \int_x^{x+1} \frac{\gamma}{y^{1+\gamma}}dy < \frac{\gamma}{x^{1+\gamma}}$$

and

$$f_\gamma(x) - 2f_\gamma(x+1) + f_\gamma(x+2) = \gamma\int_x^{x+1}\Big(f_{1+\gamma}(y) - f_{1+\gamma}(y+1)\Big)dy$$
$$= \gamma(1+\gamma)\int_x^{x+1}\int_y^{y+1}\frac{dz}{z^{2+\gamma}}dy < \frac{\gamma(1+\gamma)}{x^{2+\gamma}}.$$

□

Proposition 3.1. *If $0 < \gamma < 1$ and $n \geq 2$, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k=1}^{n}\frac{\cos(kx)}{k^\gamma}\Big|dx \leq 2 - \frac{1}{2^\gamma} + \frac{1+\ln n}{2n^\gamma}.$$

Moreover, if $m \in \mathbb{N}$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\Big|\sum_{k=n+1}^{n+m}\frac{\cos(kx)}{k^\gamma}\Big|dx \leq \frac{1+\ln(n+m)}{(n+m)^\gamma} + \frac{(3+\ln n)}{n^\gamma} + \frac{2n}{(n-1)^{1+\gamma}}. \tag{3.8}$$

Proof. Set $a_k = 1/k^\gamma$ for $k \in \mathbb{N}$, $a_0 = 2 - 1/2^\gamma$ and

$$L_n(x) = 2 - \frac{1}{2^\gamma} + 2\sum_{k=1}^{n} \frac{\cos(kx)}{k^\gamma}. \tag{3.9}$$

Notice that

$$a_0 - 2a_1 + a_2 = 0$$

and $\Delta^2 a_k \geq 0$ for $k \geq 0$ (see Lemma 3.2).

Taking into account Lemma 3.1 (with $d_0 = 1$ and $d_k = 2\cos(kx)$ for $k \geq 1$) and the definition of the Dirichlet and Fejér kernels, we obtain

$$\begin{aligned} L_n(x) &= a_n D_n(x) + \Delta a_{n-1} \sum_{k=0}^{n-1} D_k(x) + \sum_{k=0}^{n-2} \Delta^2 a_k \sum_{j=0}^{k} D_j(x) \\ &= a_n D_n(x) + n\Delta a_{n-1} F_{n-1}(x) + \sum_{k=0}^{n-2} (k+1)\Delta^2 a_k F_k(x). \end{aligned}$$

Hence

$$2\sum_{k=1}^{n} \frac{\cos(kx)}{k^\gamma} = -a_0 + a_n D_n(x) + n\Delta a_{n-1} F_{n-1}(x) + \sum_{k=0}^{n-2} (k+1)\Delta^2 a_k F_k(x).$$

Recall that (see (2.3)) $F_k(x) \geq 0$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_k(x)dx = 1.$$

Taking into account (2.5) and Lemma 3.1, for $n \geq 2$ and $0 < \gamma < 1$, one has

$$\begin{aligned} \frac{2}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{k=1}^{n} \frac{\cos(kx)}{k^\gamma}\right| dx &\leq a_0 + a_n(2 + \ln n) + n\Delta a_{n-1} + \sum_{k=0}^{n-2} (k+1)\Delta^2 a_k \\ &= a_0 + a_n(2 + \ln n) + n\Delta a_{n-1} + a_0 - a_n - n\Delta a_{n-1} \\ &= 2a_0 + a_n(1 + \ln n). \end{aligned}$$

Moreover

$$\begin{aligned} L_{n+m}(x) - L_n(x) &= a_{n+m} D_{n+m}(x) + (n+m)\Delta a_{n+m-1} F_{n+m-1}(x) \\ &\quad - a_n D_n(x) - n\Delta a_{n-1} F_{n-1}(x) + \sum_{k=n-1}^{n+m-2} (k+1)\Delta^2 a_k F_k(x). \end{aligned} \tag{3.10}$$

Therefore

$$\begin{aligned}
&\frac{2}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k=1}^{n+m}\frac{\cos(kx)}{k^\gamma}-\sum_{k=1}^{n}\frac{\cos(kx)}{k^\gamma}\Big|dx\\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|L_{n+m}(x)-L_n(x)\Big|dx\\
\le& a_{n+m}(2+\ln(n+m))+(n+m)\Delta a_{n+m-1}+a_n(2+\ln n)+n\Delta a_{n-1}\\
+&\sum_{k=n-1}^{n+m-2}(k+1)\Delta^2 a_k\\
=&a_{n+m}(2+\ln(n+m))+(n+m)\Delta a_{n+m-1}+a_n(2+\ln n)+n\Delta a_{n-1}\\
+&a_n-a_{n+m}+n\Delta a_{n-1}-(n+m)\Delta a_{n+m-1}\\
=&a_{n+m}(1+\ln(n+m))+a_n(3+\ln n)+2n\Delta a_{n-1}\\
\le&\frac{1+\ln(n+m)}{(n+m)^\gamma}+\frac{(3+\ln n)}{n^\gamma}+\frac{2n}{(n-1)^{1+\gamma}}.
\end{aligned}$$

□

Remark 3.1. *We know that (see [5, p. 50 and 43]), if $0<\delta<\pi$ and $n\in\mathbb{N}$, then*

$$\sup_{\delta\le|x|\le\pi}|\,D_n(x)\,|\le\frac{1}{\sin(\delta/2)}\quad\text{and}\quad\sup_{\delta\le|x|\le\pi}F_n(x)\le\frac{1}{(n+1)\sin^2(\delta/2)}.$$

Therefore, it follows from (3.10) that $\{L_n\}$ is a Cauchy sequence in the uniform norm in $[-\pi,-\delta)\cup(\delta,\pi]$. Hence $\{L_n\}$ converges uniformly to a continuous function in this fixed interval. Since $\delta\in(0,\pi)$ is arbitrary, it implies continuity in the open interval. In particular

$$\varphi_\gamma(x)=-\frac{a_0}{2}+\frac{1}{2}\sum_{k=0}^{\infty}(k+1)\Delta^2 a_k F_k(x).$$

We have not found good estimates for the $\mathbb{L}^1$ norm of the conjugate of the Dirichlet kernel in the existing literature, that is the reason why we include the following lemma.

Lemma 3.3. *For each $n\in\mathbb{N}$, one has*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\widetilde{D}_n(t)\Big|\,dt\;\le\;2+2\ln n.$$

Proof. It is known that

$$\frac{2x}{\pi}\le\sin x,\qquad 0<x\le\pi/2,$$

and

$$\frac{|\sin(nx)\,|}{|\sin x\,|}=\frac{\sin(nx)}{\sin x}\le n,\qquad 0<x\le\pi/(2n).$$

For instance, similar inequalities appeared in [10, p. 151]. Since the second one is less known, we include a proof. Since the function $\cos x$ decreases in the interval $(0,\pi/2]$, for $0<x\le\pi/(2n)$, $\cos(nx)\le\cos x$. If $g(x)=\sin(nx)-n\sin x$, then $g'(x)=n(\cos(nx)-\cos x)<0$. Hence $g(x)$ decreases in $[0,\pi/(2n)$. But $g(0)=0$. Therefore

$$0\le\sin(nx)\le n\sin(x),\qquad 0<x\le\pi/(2n).$$

Since $\widetilde{D}_n$ is an odd function, taking into account the trigonometric identity

$$\cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right),$$

one has

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\widetilde{D}_n(t)\right|dt &= \frac{1}{\pi}\int_{0}^{\pi}\left|\widetilde{D}_n(t)\right|dt = \frac{1}{\pi}\int_{0}^{\pi}\left|\frac{\cos(t/2)-\cos((2n+1)t/2)}{\sin(t/2)}\right|dt \\
&= \frac{2}{\pi}\int_{0}^{\pi/2}\left|\frac{\cos(s)-\cos((2n+1)s)}{\sin s}\right|ds = \frac{4}{\pi}\int_{0}^{\pi/2}\left|\frac{\sin((n+1)s)\sin(ns)}{\sin s}\right|ds \\
&\le \frac{4}{\pi}\int_{0}^{\pi/2}\left|\frac{\sin(ns)}{\sin s}\right|ds \le \frac{4}{\pi}\int_{0}^{\pi/(2n)} ndt + \frac{4}{\pi}\int_{\pi/(2n)}^{\pi/2}\frac{\pi}{2t}dt \\
&= 2 + 2\left(\ln\frac{\pi}{2} - \ln\frac{\pi}{2n}\right) = 2 + 2\ln n.
\end{aligned}$$

□

Lemma 3.4. *If* $0 < \gamma < 1$ *and* $n > 3$*, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=1}^{n}\frac{\sin(kx)}{k^\gamma}\right|dx \le 2^{1+\gamma}\left(1+\frac{1}{\gamma}\right) + \frac{(1+\ln n)}{n^\gamma}.$$

Moreover, if $m \in \mathbb{N}$*, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=n+1}^{n+m}\frac{\sin(kt)}{k^\gamma}\right|dt \le \frac{1+\ln(n+m)}{(n+m)^\gamma} + (2+2^{1+\gamma})\frac{1+\ln n}{n^\gamma} + \frac{2^{1+\gamma}}{\gamma n^\gamma}.$$

Proof. We use the notations of Lemma 3.1 by setting $c_0 = 0$, $d_0 = 1$, and $c_k = 1/k^\gamma$ and $d_k = d_k(x) = 2\sin(kx)$, for $k \ge 1$. With these notations

$$\sum_{j=1}^{k} d_j(x) = 1 + \widetilde{D}_k(x), \qquad k \ge 1.$$

If we set

$$M_n(x) = 2\sum_{k=1}^{n} c_k \sin(kx) = \sum_{k=0}^{n} c_k d_k(x),$$

it follows from (3.6) that

$$\begin{aligned}M_n(x) &= c_n\sum_{k=0}^{n} d_k(x) + \sum_{k=0}^{n-1}(c_k - c_{k+1})\sum_{j=0}^{k} d_j(x)\\ &= c_n\Big(1+\widetilde{D}_n(x)\Big) - c_1 + \sum_{k=1}^{n-1}(c_k - c_{k+1})\Big(1+\widetilde{D}_k(x)\Big)\\ &= c_n\Big(1+\widetilde{D}_n(x)\Big) - c_1 + \sum_{k=1}^{n-1}(c_k - c_{k+1}) + \sum_{k=1}^{n-1}(c_k - c_{k+1})\widetilde{D}_k(x)\\ &= c_n\widetilde{D}_n(x) + \sum_{k=1}^{n-1}(c_k - c_{k+1})\widetilde{D}_k(x).\end{aligned}$$

Taking into account Lemmas 3.2 and 3.3, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |M_n(t)|\,dt \le 2c_n(1+\ln n) + 2\gamma\sum_{k=1}^{n-1}\frac{(1+\ln k)}{k^{1+\gamma}}.$$

In order to estimate the sum in the previous inequality, we include some computations. By integration by part, we obtain

$$\begin{aligned}\gamma\sum_{k=1}^{n-1}\frac{(1+\ln k)}{k^{1+\gamma}} &\le 2^{1+\gamma}\gamma\sum_{k=1}^{n-1}\frac{(1+\ln k)}{(k+1)^{1+\gamma}} \le 2^{1+\gamma}\gamma\sum_{k=1}^{n-1}\int_k^{k+1}\frac{(1+\ln x)}{x^{1+\gamma}}dx\\ &= 2^{1+\gamma}\gamma\int_1^n\frac{(1+\ln x)}{x^{1+\gamma}}dx = 2^{1+\gamma}\Big(1-\frac{1+\ln n}{n^\gamma}+\int_1^n\frac{1}{x^{1+\gamma}}dx\Big)\\ &= 2^{1+\gamma}\Big(1-\frac{1+\ln n}{n^\gamma}+\frac{1}{\gamma}\Big(1-\frac{1}{n^\gamma}\Big)\Big) \le 2^{1+\gamma}\Big(1+\frac{1}{\gamma}\Big).\end{aligned}$$

We conclude that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=1}^{n}\frac{\sin(kt)}{k^\gamma}\right|dt = \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \mid M_n(t)\mid dt \le c_n(1+\ln n) + 2^{1+\gamma}\Big(1+\frac{1}{\gamma}\Big).$$

Moreover,

$$\begin{aligned}&\frac{2}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=1}^{n+m}\frac{\sin(kt)}{k^\gamma} - \sum_{k=1}^{n}\frac{\sin(kt)}{k^\gamma}\right|dt\\ =&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|c_{n+m}\widetilde{D}_{n+m}(x) - c_n\widetilde{D}_n(x) + \sum_{k=n}^{n+m-1}(c_k-c_{k+1})\widetilde{D}_k(x)\right|dx\\ \le& 2c_{n+m}(1+\ln(n+m)) + 2c_n(1+\ln n) + 2\gamma\sum_{k=n}^{n+m-1}\frac{(1+\ln k)}{k^{1+\gamma}}\\ \le& 2c_{n+m}(1+\ln(n+m)) + 2c_n(1+\ln n) + 2^{2+\gamma}\frac{(1+\ln n)}{n^\gamma} + \frac{2^{2+\gamma}}{\gamma n^\gamma}.\end{aligned}$$

□

Remark 3.2. *It is known that (see* [14, p. 92]*)*

$$\widetilde{F}_n(t)\,\mathrm{sign}\,t \geq 0, \qquad t \in (-\pi, \pi).$$

Hence $\widetilde{F}_n(x)$ is not a positive operator and different Cesàro means of $\widetilde{D}_n(x)$ share this properties. That is the reason why we use $\widetilde{D}_n(x)$ instead of $\widetilde{F}_n(x)$.

4. Main results

Theorem 4.1. *If $0 < \gamma < 1$, then $\varphi_\gamma, \psi_\gamma \in \mathbb{L}^1$,*

$$\|\varphi_\gamma\|_1 \leq 2 - \frac{1}{2^\gamma} \qquad \text{and} \qquad \|\psi_\gamma\|_1 \leq 2^{1+\gamma}\Big(1 + \frac{1}{\gamma}\Big).$$

Proof. If a series converges to a function $f \in \mathbb{L}^1$, then the series is the Fourier series of f (see [5, p. 51]).

If

$$H_n(x) = \sum_{k=1}^{n} \frac{\cos(kx)}{k^\gamma},$$

equation (3.8) can be rewriten as

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \Big|H_{n+m}(x) - H_n(x)\Big| dx \leq \frac{1 + \ln(n+m)}{(n+m)^\gamma} + \frac{(3 + \ln n)}{n^\gamma} + \frac{2n}{(n-1)^{1+\gamma}}.$$

Hence $\{H_n\}$ is a Cauchy sequence in $\mathbb{L}^1$. Therefore there exists a function $F \in \mathbb{L}^1$ such that $\|F - H_n\|_1 \to 0$ as $n \to \infty$. But $F(x) = \varphi_\gamma(x)$ a.e. . Since the series is continuous for $0 <| x |\leq \pi$, we have equality for $x \neq 0$.

Taking into account Proposition 3.1 (see also [2, p. 50]) and (3.9) with L_n defined as in (3.9), one has

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\, \varphi_\gamma(t) \,|\, dt = \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |\, H_n(t)\,|\, dt \leq \lim_{n\to\infty} \Big(2 - \frac{1}{2^\gamma} + \frac{1 + \ln n}{2n^\gamma}\Big)$$
$$= 2 - \frac{1}{2^\gamma}.$$

The assertions for ψ_γ follow analogously. □

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Research Article

Toward the theory of semi-linear Beltrami equations

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ABSTRACT. We study the semi-linear Beltrami equation $\omega_{\bar z} - \mu(z)\omega_z = \sigma(z)q(\omega(z))$ and show that it is closely related to the corresponding semi-linear equation of the form $\mathrm{div} A(z)\nabla U(z) = G(z)Q(U(z))$. Applying the theory of completely continuous operators by Ahlfors-Bers and Leray–Schauder, we prove existence of regular solutions both to the semi-linear Beltrami equation and to the given above semi-linear equation in the divergent form, see Theorems 1.1 and 5.2. We also derive their representation through solutions of the semi-linear Vekua type equations and generalized analytic functions with sources. Finally, we apply Theorem 5.2 for several model equations describing physical phenomena in anisotropic and inhomogeneous media.

Keywords: Semi-linear Beltrami equations, generalized analytic functions with sources, semi-linear Poisson type equations, generalized harmonic functions with sources.

2020 Mathematics Subject Classification: 30C62, 35J61, 30G30, 31A35, 35Q35.

1. INTRODUCTION

Let D be a domain in the complex plane $\mathbb{C}$. In this paper, we study semi-linear Beltrami equations of the form

$$\omega_{\bar z} - \mu(z)\omega_z = \sigma(z)q(\omega(z)), \quad z \in D, \tag{1.1}$$

where the left hand side is presented by the linear Beltrami operator $\mathcal{L}(\omega) := \omega_{\bar z} - \mu\omega_z$ with measurable coefficient $\mu : D \to \mathbb{C}$, satisfying uniform ellipticity condition $|\mu(z)| \le k < 1$ a.e., $\omega_{\bar z} := (\omega_x + i\omega_y)/2$, $\omega_z := (\omega_x - i\omega_y)/2$, $z = x + iy$, ω_x and ω_y are partial derivatives of the function ω in x and y, respectively. The non-linear part of the equation is chosen in such a way that $\sigma : D \to \mathbb{C}$ belongs to class $L_p(D)$, $p > 2$, and $q : \mathbb{C} \to \mathbb{C}$ is a continuous function, satisfying the asymptotic condition

$$\lim_{w\to\infty} \frac{q(w)}{w} = 0. \tag{1.2}$$

One of the main goals of this paper is to establish close links between semi-linear Beltrami equation (1.1) and semi-linear Poisson type equation of the form

$$\mathrm{div}\,[A(z)\,\mathrm{grad}\,U(z)] = G(z)Q(U(z)), \tag{1.3}$$

the diffusion term of which is the divergence form elliptic operator $L(u)$, whereas its reaction term $G(z)Q(U(z))$ is such that $G : D \to \mathbb{R}$ is a function of class $L_{p'}(D)$, $p' > 1$, and $Q : \mathbb{R} \to \mathbb{R}$

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stands for a continuous function such that

$$\lim_{t\to\infty} \frac{Q(t)}{t} = 0\,. \tag{1.4}$$

From now on, $A(z) = \{a_{ij}(z)\}$ is symmetric matrix function with measurable entries and $\det A(z) = 1$, satisfying the uniform ellipticity condition

$$\frac{1}{K}|\xi|^2 \le \langle A(z)\xi,\, \xi\rangle \le K|\xi|^2 \ \text{ a.e. in } D,\ \ 1\le K<\infty,\quad \forall\, \xi\in\mathbb{R}^2\,. \tag{1.5}$$

The semi-linear Poisson equation, when $A \equiv 1$ in (1.3), was studied in [15], [28] and [29]. A rather comprehensive treatment of the present state of the general theory concerning semi-linear Poisson equations can be found in the excellent books of M. Marcus and L. Véron [23] and L. Véron [34]. For the classic case $A \equiv 1$ and $Q \equiv 1$ of the Poisson equation, see e.g. the recent article [32]. The model case $G \equiv 1$ with general Q and A was first investigated in [12], see also the papers [13]–[14] and [16]-[17].

Links established by us open up new possibilities for the study both of equations (1.1) and (1.3), because one can apply a wide range of effective methods of the potential theory as well as comprehensively developed theory of quasiconformal mappings in the plane, see e.g. [1], [3], [6] and [21]. In particular, it allows us to study in detail both the regularity properties for solutions to the equations (1.1) and (1.3) and the proper representation of such solutions.

Before to formulate the main theorem on semi-linear Beltrami equation (1.1), we need to introduce some definitions. Similarly to [2], see also monograph [1], we assume that the function $\sigma:\mathbb{C}\to\mathbb{C}$ in equation (1.1) belongs to class $L_p(\mathbb{C})$ for some $p>2$ with the condition

$$k\,C_p \;<\; 1\,, \qquad k \;:=\; \|\mu\|_\infty \;<\; 1\,, \tag{1.6}$$

guaranteing the existence of suitable solutions of the equations (1.1), where C_p is the norm of the known operator $T : L_p(\mathbb{C}) \to L_p(\mathbb{C})$ defined through the Cauchy principal limit of the singular integral

$$(Tg)(\zeta) \;:=\; \lim_{\varepsilon\to 0}\left\{-\frac{1}{\pi}\int\limits_{|z-\zeta|>\varepsilon}\frac{g(z)}{(z-\zeta)^2}\,dxdy\right\},\ \ z=x+iy\,. \tag{1.7}$$

As known, $\|Tg\|_2 = \|g\|_2$, i.e., $C_2 = 1$, and by the Riesz convexity theorem $C_p \to 1$ as $p \to 2$, see e.g. Lemma 2 in [1] and Lemma 4 in [2]. Thus, there are such p, whatever the value of k in (1.6).

Let us denote by B_p the Banach space of functions $\omega : \mathbb{C} \to \mathbb{C}$, which satisfy a Hölder condition of order $1-2/p$, which vanish at the origin, and whose generalized derivatives ω_z and $\omega_{\bar z}$ exist and belong to $L_p(\mathbb{C})$. The norm in B_p is defined by

$$\|\omega\|_{B_p} \;:=\; \sup_{\substack{z_1,z_2\in\mathbb{C},\\ z_1\neq z_2}} \frac{|\omega(z_1)-\omega(z_2)|}{|z_1-z_2|^{1-2/p}} \;+\; \|\omega_z\|_p \;+\; \|\omega_{\bar z}\|_p\,. \tag{1.8}$$

Theorem 1.1. *Let $\mu:\mathbb{C}\to\mathbb{C}$ and $\sigma:\mathbb{C}\to\mathbb{C}$ have compact supports, $\mu\in L_\infty(\mathbb{C})$ with $k:=\|\mu\|_\infty<1$, $\sigma\in L_p(\mathbb{C})$ for some $p>2$ satisfying (1.6). Suppose that $q:\mathbb{C}\to\mathbb{C}$ is a continuous function satisfying condition (1.2). Then the semi-linear Beltrami equation (1.1) has a solution ω of the class $B_p(\mathbb{C})$.*

Moreover, we show that the given solution ω has the representation as a composition $H\circ f$, where f stands for a suitable quasiconformal mapping and H is a generalized analytic function, see Section 2 and Remark 4.3.

Theorem 1.1 together with the standard complexification approach allows us to prove the corresponding existence, representation and regularity result for semi-linear Poisson type equations (1.3), see Theorem 5.2.

The paper is organized as follows. Section 2 contains some definitions and preliminary results. The factoring of solutions for the semi-linear Beltrami equations (1.1) can be found in Section 3. The proof of Theorem 1.1 is given in Section 4. Section 5 includes the statement and the proof of Theorem 5.2. Finally, in Section 6 we apply Theorem 5.2 for several model equations describing some physical phenomena in anisotropic and inhomogeneous media.

2. Definitions and preliminary results

Recall that monograph [33] was devoted to **generalized analytic functions**, i.e., continuous complex valued functions $H(z)$ of one complex variable $z = x + iy$ of class $W^{1,1}_{\rm loc}$ in a domain D satisfying the equations

$$\partial_{\bar z} H \ + \ aH \ + \ b\overline{H} \ = \ S\,, \quad \partial_{\bar z} \ := (\partial_x + i\partial_y)/2\,, \tag{2.9}$$

with complex valued coefficients $a, b, S \in L_p(D)$, $p > 2$. If $a \equiv 0 \equiv b$, then H will be called **generalized analytic functions with sources** S. Later on, we also need some results on the **semi-linear Vekua type equation**

$$\partial_{\bar z} H(z) \ = \ g(z) \cdot q(H(z)) \tag{2.10}$$

that have been obtained in our preceding papers [17], [18] and [28].

According to the works [15] and [28], a continuous function $h : D \to \mathbb{R}$ of class $W^{2,p}_{\rm loc}$ is also called a **generalized harmonic function with a source** $s\colon D \to \mathbb{R}$ in $L_p(D)$, $p > 2$, if h a.e. satisfies the Poisson equation

$$\triangle h(z) \ = \ s(z)\,, \tag{2.11}$$

where, as usual, $\triangle := \partial^2/\partial x^2 + \partial^2/\partial y^2$, $z = x + iy$, is the Laplacian. Note that by the Sobolev embedding theorem, see Theorem I.10.2 in [31], such functions h belong to the class C^1.

Let H be a generalized analytic function with a complex valued source S. Then we say that a function $h : D \to \mathbb{R}$ is a **weak generalized harmonic function with the source** S, if $h = {\rm Re}\, H$.

It is well known that the homogeneous Beltrami equation

$$f_{\bar z} = \mu(z) f_z \tag{2.12}$$

is the basic equation in analytic theory of quasiconformal and quasiregular mappings in the plane with numerous applications in nonlinear elasticity, gas flow, hydrodynamics and other sections of natural sciences. For the corresponding quasilinear homogeneous Beltrami equations, when the complex coefficient μ depends not only on z but also on f, see the recent papers [10] and [30].

Recall that the equation (2.12) is said to be **nondegenerate** or uniformly elliptic if $||\mu||_{\infty} < 1$, i.e., if $K_\mu \in L_\infty$,

$$K_\mu(z) \ := \ \frac{1 \ + \ |\mu(z)|}{1 \ - \ |\mu(z)|}\,. \tag{2.13}$$

Homeomorphic solutions f of nondegenerate equation (2.12) of the class $W^{1,2}_{\rm loc}$ are called **quasiconformal mappings** or sometimes $\mu-$**conformal mappings**. Its continuous solutions in $W^{1,2}_{\rm loc}$ are called $\mu-$**conformal functions**. On the corresponding existence theorems for nondegenerate Beltrami equation (2.12), see e.g. [1], [6] and [21].

The **inhomogeneous Beltrami equations**

$$\omega_{\bar{z}} \;=\; \mu(z)\cdot\omega_z \;+\; \sigma(z) \tag{2.14}$$

have been introduced and investigated by L. Ahlfors and L. Bers in paper [2], see also the Ahlfors monograph [1].

One of the principal results in [2], Theorem 1, is the following statement:

Theorem A. *Let $\sigma \in L_p(\mathbb{C})$ for $p > 2$, satisfying condition (1.6). Then the equation (2.14) has a unique solution $\omega^{\mu,\sigma} \in B_p$. This is its only solution with $\omega(0) = 0$ and $\omega_z \in L_p(\mathbb{C})$.*

As a consequence one deduces, see Theorem 4 and Lemma 8 in [2],

Theorem B. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be in $L_\infty(\mathbb{C})$ with compact support and $\|\mu\|_\infty < 1$. Then there exists a unique $\mu-$conformal mapping f^μ in $\mathbb{C}$ which vanishes at the origin and satisfies condition $f^\mu_z - 1 \in L_p(\mathbb{C})$ for any $p > 2$, satisfying (1.6). Moreover, $f^\mu(z) = z + \omega^{\mu,\mu}(z)$.*

3. Factoring of solutions to semi-linear Beltrami equations

Let us start with the following factorization lemma for the linear inhomogeneous Beltrami equations (2.14).

Lemma 3.1. *Let D be a bounded domain in $\mathbb{C}$, $\mu : D \to \mathbb{C}$ be in class $L_\infty(D)$ with $k := \|\mu\|_\infty < 1$, $\sigma : D \to \mathbb{C}$ be in class $L_p(D)$, $p > 2$, with condition (1.6). Suppose that $f^\mu : \mathbb{C} \to \mathbb{C}$ is the $\mu-$conformal mapping from Theorem B with an arbitrary extension of μ onto $\mathbb{C}$ keeping compact support and condition (1.6).*

Then each continuous solution ω of equation (2.14) in D of class $W^{1,p}(D)$ has the representation as a composition $H \circ f^\mu|_D$, where H is a generalized analytic function in $D_* := f^\mu(D)$ with the source $g \in L_{p_*}(D_*)$, $p_* := p^2/2(p-1) \in (2,p)$,

$$g \;:=\; \left(\frac{f^\mu_z}{J^\mu}\cdot\sigma\right)\circ\left(f^\mu\right)^{-1}\ , \tag{3.15}$$

where J^μ is the Jacobian of f^μ.

Vice versa, if H is a generalized analytic function with the source $g \in L_{p_*}(D_*)$, $p_* > 2$, in (3.15), then $\omega := H \circ f^\mu$ is a solution of (2.14) of class $C^\alpha_{\rm loc} \cap W^{1,p^*}_{\rm loc}(D)$, where $\alpha = 1 - 2/p^*$ and $p^* := p_*^2/2(p_*-1) \in (2, p_*)$.

Proof. To be short, let us apply here the notation f instead of f^μ. Let us consider the function $H := \omega \circ f^{-1}$. First of all, note that by point (iii) of Theorem 5 in [2] $f^* := f^{-1}|_{D^*}$, $D^* := f(D)$, is of class $W^{1,p}(D^*)$. Then, arguing in a perfectly similar way as under the proof of Lemma 10 in [2], we obtain that $H \in W^{1,p_*}(D^*)$, where $p_* := p^2/2(p-1) \in (2,p)$. Hence it has no sense to repeat these arguments here. Since $\omega = H \circ f$, we get also, see e.g. formulas (28) in [2], see also formulas I.C(1) in [1], that

$$\omega_z \;=\; (H_\zeta \circ f)\cdot f_z \;+\; (H_{\overline{\zeta}} \circ f)\cdot \overline{f_{\bar{z}}}\ ,$$

$$\omega_{\bar{z}} \;=\; (H_\zeta \circ f)\cdot f_{\bar{z}} \;+\; (H_{\overline{\zeta}} \circ f)\cdot \overline{f_z}\ ,$$

and, thus,

$$\sigma(z) \;=\; \omega_{\bar{z}} \;-\; \mu(z)\,\omega_z \;=\; (H_{\overline{\zeta}}\circ f)\,\overline{f_z}\,(1-|\mu(z)|^2) \;=\; (H_{\overline{\zeta}}\circ f)\,J(z)/f_z\ ,$$

where $J(z) = |f_z|^2 - |f_{\bar{z}}|^2 = |f_z|^2(1-|\mu(z)|^2)$ is the Jacobian of f, i.e.,

$$H_{\overline{\zeta}} \;=\; g(\zeta) \;:=\; \left(\frac{f_z}{J}\cdot\sigma\right)\circ f^{-1}(\zeta)\ .$$

Similarly, applying Lemma 10 in [2] and the Sobolev embedding theorem, see Theorem I.10.2 in [31], we come to the inverse conclusion. □

Remark 3.1. *Note that if H is a generalized analytic function with the source g in the domain D_*, then $h = H + \mathcal{A}$ is so for any analytic function $\mathcal{A}$ in D_*, but $|\mathcal{A}'|^{p_*}$ can be integrable only locally in D_*. By Lemma 3.1, the source in (3.15) is always in class $L_{p_*}(D_*)$, $p_* := p^2/2(p-1) \in (2,p)$, in view of Theorem A with σ extended onto $\mathbb{C}$ by zero outside of D. Here we may assume that μ is extended onto $\mathbb{C}$ by zero outside of D. However, any other extension of μ keeping condition (1.6) is suitable here, too. Moreover, we may apply here as f^μ any $\mu-$conformal mappings with different normalizations, in particular, with the hydrodynamic normalization $f^\mu(z) = z + o(1)$ as $z \to \infty$.*

Next statement makes it is possible to reduce the study of the semi-linear Beltrami equations (1.1) to the study of the corresponding semi-linear Vekua type equations (2.10).

Lemma 3.2. *Let D be a bounded domain in $\mathbb{C}$, $\mu : D \to \mathbb{C}$ be measurable with $\|\mu\|_\infty < 1$, $\sigma : D \to \mathbb{C}$ be in class $L_p(D)$, $p > 2$. Suppose that $q : \mathbb{C} \to \mathbb{C}$ is continuous and $f^\mu : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B with an arbitrary extension of μ onto $\mathbb{C}$ keeping compact support and condition (1.6).*

Then each continuous solution ω of equation (1.1) in D of class $W^{1,p}(D)$ has the representation as a composition $H \circ f^\mu|_D$, where H is a continuous solution of (2.10) in class $W^{1,p_*}_{\rm loc}(D_*)$, where $D_* := f^\mu(D)$, $p_* := p^2/2(p-1) \in (2,p)$, with the multiplier g in (2.10) of class $L_{p_*}(D_*)$ defined by formula (3.15).

Vice versa, if H is a continuous solution in class $W^{1,p_*}_{\rm loc}(D_*)$ of (2.10) with multiplier $g \in L_{p_*}(D_*)$, $p_* > 2$, given by (3.15), then $\omega := H \circ f^\mu$ is a solution of (1.1) in class $C^\alpha_{\rm loc} \cap W^{1,p^*}_{\rm loc}(D)$, $\alpha = 1 - 2/p^*$, where $p^* := p_*^2/2(p_* - 1) \in (2, p_*)$.

Proof. Indeed, if ω is a continuous solution of (1.1) in D of class $W^{1,p}(D)$, then ω is a solution of (2.14) in D with the source $\Sigma := \sigma \cdot q \circ \omega$ in the same class. Then by Lemma 3.1 and Remark 3.1 $\omega = H \circ f^\mu$, where H is a generalized analytic function with the source G of class $L_{p_*}(D_*)$ after replacement of σ by Σ in (3.15). Note that $H \in W^{1,p_*}_{\rm loc}(D_*)$, see e.g. Theorems 1.16 and 1.37 in [33]. The proof of the vice versa conclusion of Lemma 3.2 is similar and it is again based on its reduction to Lemma 3.1. □

4. ON SOLUTIONS OF SEMI-LINEAR BELTRAMI EQUATIONS

First of all, recall that a **completely continuous** mapping from a metric space M_1 into a metric space M_2 is defined as a continuous mapping on M_1 which takes bounded subsets of M_1 into relatively compact subsets of M_2, i.e., with compact closures in space M_2. When a continuous mapping takes M_1 into a relatively compact subset of M_1, it is nowadays said to be **compact** on M_1. Note that the notion of completely continuous (compact) operators is due essentially to Hilbert in a special space that, in reflexive spaces, is equivalent to Definition VI.5.1 for the Banach spaces in [11], which is due to F. Riesz, see also further comments of Section VI.12 in [11].

Recall some further definitions and one fundamental result of the celebrated paper [22]. Leray and Schauder extend as follows the Brouwer degree, see e.g. [7] and [9], to compact perturbations of the identity I in a Banach space B, i.e., a complete normed linear space. Namely, given an open bounded set $\Omega \subset B$, a compact mapping $F : B \to B$ and $z \notin \Phi(\partial\Omega)$, $\Phi := I - F$, the **(Leray-Schauder) topological degree** $\deg\,[\Phi, \Omega, z]$ of Φ in Ω over z is constructed from the Brouwer degree by approximating the mapping F over Ω by mappings F_ε with range in a finite-dimensional subspace B_ε (containing z) of B. It is showing that the Brouwer degrees $\deg\,[\Phi_\varepsilon, \Omega_\varepsilon, z]$ of $\Phi_\varepsilon := I_\varepsilon - F_\varepsilon$, $I_\varepsilon := I|_{B_\varepsilon}$, in $\Omega_\varepsilon := \Omega \cap B_\varepsilon$ over z stabilize for sufficiently small positive ε to a common value defining $\deg\,[\Phi, \Omega, z]$ of Φ in Ω over z.

This topological degree algebraically counts the number of fixed points of $F(\cdot) - z$ in Ω and conserves the basic properties of the Brouwer degree as additivity and homotopy invariance. Now, let a be an isolated fixed point of F. Then the **local (Leray-Schauder) index** of a is defined by $\text{ind}\,[\Phi, a] := \deg[\Phi, B(a,r), 0]$ for small enough $r > 0$. $\text{ind}\,[\Phi, 0]$ is called by **index** of F. In particular, if $F \equiv 0$, correspondingly, $\Phi \equiv I$, then the index of F is equal to 1. For our goals, we need only the latter fact from the index theory.

Now, let us formulate one of the main results in the Leray-Schauder article [22], Theorem 1, see also the survey [25].

Proposition 4.1. *Let B be a Banach space, and let $F(\cdot, \tau) : B \to B$ be a family of operators with $\tau \in [0,1]$. Suppose that the following hypotheses hold:*

- *(H1) $F(\cdot, \tau)$ is completely continuous on B for each $\tau \in [0,1]$ and uniformly continuous with respect to the parameter $\tau \in [0,1]$ on each bounded set in B;*
- *(H2) the operator $F := F(\cdot, 0)$ has finite collection of fixed points whose total index is not equal to zero;*
- *(H3) the collection of all fixed points of the operators $F(\cdot, \tau)$, $\tau \in [0,1]$, is bounded in B.*

Then the collection of all fixed points of the family of operators $F(\cdot, \tau)$ contains a continuum along which τ takes all values in $[0,1]$.

For introduction in the modern fixed point theory, see e.g. survey [20] and monograph [26].

Remark 4.2. *By Lemma 5 in [2] the mapping $\sigma \to \omega^{\mu,\sigma}$ from Theorem A is a bounded linear operator from $L_p(\mathbb{C})$ to $B_p(\mathbb{C})$ with a bound that depends only on k and p in (1.6). In particular, this is a bounded linear operator from $L_p(\mathbb{C})$ to $C(\mathbb{C})$. Namely, by (15) in [2] we have that $\omega^{\mu,\sigma}$ is Hölder continuous:*

$$|\omega^{\mu,\sigma}(z_1) \ - \ \omega^{\mu,\sigma}(z_2)| \ \leq \ c \cdot \|\sigma\|_p \cdot |z_1 - z_2|^{1-2/p} \qquad \forall\ z_1\ z_2\ \in \mathbb{C}\,, \tag{4.16}$$

where the constant c may depend only on k and p in (1.6). Moreover, $\omega^{\mu,\sigma}(z)$ is locally bounded because $\omega^{\mu,\sigma}(0) = 0$. Thus, the linear operator $\sigma \to \omega^{\mu,\sigma}|_S$ is completely continuous for each compact set S in $\mathbb{C}$ by Arzela-Ascoli theorem, see e.g. Theorem IV.6.7 in [11].

Finally, we are ready to give a proof of Theorem 1.1.

Proof for Theorem 1.1. If $\|\sigma\|_p = 0$ or $\|q\|_C = 0$, then Theorem A above gives the desired solution $\omega := \omega^{\mu,0}$ of equation (1.1). Thus, we may assume that $\|\sigma\|_p \neq 0$ and $\|q\|_C \neq 0$. Set $q_*(t) = \max\limits_{|w|\leq t} |q(w)|$, $t \in \mathbb{R}^+ := [0, \infty)$. Then the function $q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and, moreover, by (1.2)

$$\lim_{t\to\infty} \frac{q_*(t)}{t} \ = \ 0\,. \tag{4.17}$$

Let us show that the family of operators $F(g;\tau) : L_p^\sigma(\mathbb{C}) \to L_p^\sigma(\mathbb{C})$,

$$F(g;\tau) \ := \ \tau\sigma \cdot q(\omega^{\mu,g}) \qquad \forall\, \tau \in [0,1]\,, \tag{4.18}$$

where $L_p^\sigma(\mathbb{C})$ consists of functions $g \in L_p(\mathbb{C})$ with supports in the support S of σ, satisfies hypotheses H1-H3 of Theorem 1 in [22], see Proposition 4.1 above. Indeed:

H1). First of all, the function $F(g;\tau) \in L_p^\sigma(\mathbb{C})$ for all $\tau \in [0,1]$ and $g \in L_p^\sigma(\mathbb{C})$ because the function $q(\omega^{\mu,g})$ is continuous and, furthermore, the operators $F(\cdot;\tau)$ are completely continuous for each $\tau \in [0,1]$ and even uniformly continuous with respect to parameter $\tau \in [0,1]$ by Theorem A and Remark 4.2.

H2). The index of the operator $F(g;0)$ is obviously equal to 1.

H3). Let us assume that the collection of all solutions of the equations $g = F(g;\tau)$, $\tau \in [0,1]$, is not bounded in $L_p^\sigma(\mathbb{C})$, i.e., there is a sequence of functions $g_n \in L_p^\sigma(\mathbb{C})$ with $\|g_n\|_p \to \infty$ as $n \to \infty$ such that $g_n = F(g_n;\tau_n)$ for some $\tau_n \in [0,1]$, $n = 1, 2, \ldots$.

However, then by Remark 4.2, we have that

$$\|g_n\|_p \;\leq\; \|\sigma\|_p \, q_*\left(\|\omega^{\mu,g_n}|_S\|_C\right) \;\leq\; \|\sigma\|_p \, q_*\left(\, M\, \|g_n\|_p\right)$$

for some constant $M > 0$ and, consequently,

$$\frac{q_*(\, M\, \|g_n\|_p)}{M\, \|g_n\|_p} \;\geq\; \frac{1}{M\, \|\sigma\|_p} \;>\; 0\,. \tag{4.19}$$

The latter is impossible by condition (4.17). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [22], see Proposition 4.1 above, there is a function $g \in L_p^\sigma(\mathbb{C})$ with $F(g;1) = g$, and then by Theorem A the function $\omega := \omega^{\mu,g}$ gives the desired solution of (1.1). □

Remark 4.3. *By Lemma 3.2, ω has the representation as a composition $H \circ f^\mu$, where $f^\mu : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B and H is a continuous solution of (2.10) in class $W^{1,p_*}_{\rm loc}(\mathbb{C})$, $p_* := p^2/2(p-1) \in (2,p)$, with the multiplier g in (2.10) of class $L_{p_*}(\mathbb{C})$ defined by formula (3.15). Note also that H is a generalized analytic function with a source of the same class.*

Let us also give the following lemma on semi-linear Beltrami equations that may be of independent interest and will be first applied in the next section.

Lemma 4.3. *Let D be a bounded domain in $\mathbb{C}$, $\mu : D \to \mathbb{C}$ in class $L_\infty(D)$, $k := \|\mu\|_\infty < 1$, $G : D \to \mathbb{C}$ be in class $L_{p'}(D)$ for some $p' > 1$ and $\mathcal{L} : L_{p'}(D) \to L_p(D)$ be a linear bounded operator for some $p > 2$ satisfying (1.6).*

Suppose that $q : \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying condition (1.2). Then the semi-linear Beltrami equation of the form

$$\omega_{\bar{z}} \;=\; \mu(z) \cdot \omega_z \;+\; \mathcal{L}[Gq(\omega)](z)\,, \qquad z \in D, \tag{4.20}$$

has a solution ω of class $C^\alpha(D) \cap W^{1,p}(D)$ with $\alpha = 1 - 2/p$.

Proof. Indeed, arguing perfectly similar to the proof of Theorem 1.1 for

$$F(g;\tau) \;:=\; \mathcal{L}\left[\tau G q(\omega^{\mu,g})\right] \;:\; L_p(D) \to L_p(D)\,, \qquad \tau \in [0,1] \tag{4.21}$$

with μ, G and g extended by zero outside of D, we see that the family of the operators $F(g;\tau)$, $\tau \in [0,1]$, satisfies all the hypotheses of Theorem 1 in [22], see Proposition 4.1 above. Thus, there is $g \in L_p(\mathbb{C})$ with $F(g;1) = g$, and then by Theorem A the function $\omega := \omega^{\mu,g}|_D$ gives the desired solution of (4.20). □

Remark 4.4. *Moreover, arguing similarly to the proofs of Lemmas 3.1 and 3.2 one can show that $\omega = H \circ f^\mu|_D$, where $f^\mu : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B with μ extended onto $\mathbb{C}$ by zero outside of D and $H : D_* \to \mathbb{C}$ is a generalized analytic function in the domain $D_* := f^\mu(D)$ with the source*

$$S \;:=\; \left\{ \frac{f^\mu_z}{J^\mu} \cdot \mathcal{L}\left[Gq(\omega)\right] \right\} \circ (f^\mu)^{-1} \;\in\; L_{p_*}(D_*)\,, \tag{4.22}$$

where J^μ is the Jacobian of f^μ and $p_* := p^2/2(p-1) \in (2,p)$.

5. Toward semi-linear Poisson type equations

For convenience of presentation, let us denote by $\mathbb{S}^{2\times 2}$ the collection of all 2×2 matrices with real valued elements

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{5.23}$$

which are symmetric, i.e., $a_{12} = a_{21}$, with $\det A = 1$ and **ellipticity condition** $\det\,(I+A) > 0$, where I is the unit 2×2 matrix. The latter condition means in terms of elements of A that $(1+a_{11})(1+a_{22}) > a_{12}a_{21}$.

Now, let us first consider in a domain D of the complex plane $\mathbb{C}$ the linear Poisson type equation

$$\mathrm{div}\,[A(z)\,\nabla\,u(z)] \;=\; g(z)\;, \tag{5.24}$$

where $A : D \to \mathbb{S}^{2\times 2}$ is a measurable matrix valued function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded, $g : D \to \mathbb{R}$ is a scalar function in $L_{1,\mathrm{loc}}$, and here and further ∇ denotes the gradient of the corresponding functions.

Note that (5.24) is one of the main equations of hydromechanics (fluid mechanics) in anisotropic and inhomogeneous media.

We say that a function $u : D \to \mathbb{R}$ is a **generalized A−harmonic function with the source g**, cf. [19], if u is a weak solution of (5.24), i.e., if $u \in W^{1,1}_{\mathrm{loc}}(D)$ and

$$\int\limits_D \langle A(z)\nabla u(z), \nabla\psi(z)\rangle\, d\,m(z) + \int\limits_D g(z)\,\psi(z)\,d\,m(z) = \;0 \tag{5.25}$$

for all $\psi \in C_0^\infty(D)$, where $C_0^\infty(D)$ denotes the collection of all infinitely differentiable functions $\psi : D \to \mathbb{R}$ with compact support in D, $\langle a, b\rangle$ means the scalar product of vectors a and b in $\mathbb{R}^2$, and $d\,m(z)$ corresponds to the Lebesgue measure in the plane $\mathbb{C}$.

Later on, we use the **logarithmic (Newtonian) potential of sources** $g \in L_1(\mathbb{C})$ with compact supports given by the formula:

$$\mathcal{N}^g(z) \;:=\; \frac{1}{2\pi}\int\limits_{\mathbb{C}} \ln|z-w|\,g(w)\,d\,m(w)\;. \tag{5.26}$$

By Lemmas 3 in [14] and in [15], we have its following basic properties.

Remark 5.5. *Let $g : \mathbb{C} \to \mathbb{R}$ has compact support. If $g \in L_1(\mathbb{C})$, then $\mathcal{N}^g \in L_{r,\mathrm{loc}}(\mathbb{C})$ for all $r \in [1,\infty)$, $\mathcal{N}^g \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for all $p \in [1,2)$, moreover, there exist generalized derivatives by Sobolev $\frac{\partial^2 N^g}{\partial z\partial\overline{z}}$ and $\frac{\partial^2 N^g}{\partial\overline{z}\partial z}$ satisfying the equalities, where $\triangle := \partial^2/\partial x^2 + \partial^2/\partial y^2$, $z = x+iy$, is the Laplacian,*

$$4\cdot\frac{\partial^2 N^g}{\partial z\partial\overline{z}} \;=\; \triangle N^g \;=\; 4\cdot\frac{\partial^2 N^g}{\partial\overline{z}\partial z} \;=\; g \quad a.e.\,. \tag{5.27}$$

Furthermore, if $g \in L_{p'}(\mathbb{C})$ for some $p' > 1$, then $\mathcal{N}^g \in W^{2,p'}_{\mathrm{loc}}(\mathbb{C})$, moreover, $\mathcal{N}^g \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for some $p > 2$ and, consequently, $\mathcal{N}^g \in C^{\alpha}_{\mathrm{loc}}(\mathbb{C})$ with $\alpha = 1-2/p$. Finally, if $g \in L_{p'}(\mathbb{C})$ for some $p' > 2$, then $\mathcal{N}^g \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{C})$ with $\alpha = 1-2/p'$.

Next, we say that a function $v : D \to \mathbb{R}$ is **A−conjugate of a generalized A−harmonic function u with a source** $g : D \to \mathbb{R}$ if $v \in W^{1,1}_{\mathrm{loc}}(D)$ and

$$\nabla v(z) \;=\; \mathbb{H}\,[\,A(z)\nabla u(z) \;-\; \nabla\mathcal{N}^g(z)\,] \quad \text{a.e.}\,, \qquad \mathbb{H} \;:=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{5.28}$$

Lemma 5.4. *Let D be a bounded domain in $\mathbb{C}$, $g : D \to \mathbb{R}$ be in $L_1(D)$ and let u be a weak solution of equation (5.24) with a matrix function $A : D \to \mathbb{S}^{2\times 2}$ whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

If v is $A-$conjugate of u, then $\omega := u + iv$ satisfies the nondegenerate inhomogeneous Beltrami equation (2.14) with

$$\mu(z) \ := \ \mu_A(z) \ = \frac{1}{\det\,[\,I + A(z)\,]}\ [\,a_{22}(z) - a_{11}(z) \ - \ 2ia_{21}(z)\,]\ , \tag{5.29}$$

$$\sigma(z) \ := \ \mathcal{N}^g_{\bar z}(z) \ + \ \mu(z)\,\overline{\mathcal{N}^g_{\bar z}(z)}\ . \tag{5.30}$$

Conversely, if $\omega \in W^{1,1}_{\rm loc}(D)$ is a solution of the nondegenerate inhomogeneous Beltrami equation (2.14) with σ given by (5.30), then $u := {\rm Re}\,\omega$ is a weak solution of equation (5.24) with the matrix valued function $A : D \to \mathbb{S}^{2\times 2}$,

$$A(z) \ := \ \begin{bmatrix} \frac{|1-\mu(z)|^2}{1-|\mu(z)|^2} & \frac{-2{\rm Im}\,\mu(z)}{1-|\mu(z)|^2} \\ \frac{-2{\rm Im}\,\mu(z)}{1-|\mu(z)|^2} & \frac{|1+\mu(z)|^2}{1-|\mu(z)|^2} \end{bmatrix}\ , \tag{5.31}$$

whose elements are bounded and measurable.

Remark 5.6. *Hence, in the case $A \equiv I$ and $g \in L_{p'}(D)$, $p' > 2$, we conclude that every generalized harmonic function u with the source g is a weak generalized harmonic function with the same source, see e.g. Theorem 1.16 in* [33]. *The inverse conclusion is, generally speaking, not true and has no sense at all because in the weak case the source can be complex, not real.*

Proof of Lemma 5.4. Indeed, let u be a weak solution of equation (5.24) with $g : D \to \mathbb{R}$ in $L^1(D)$ and a matrix function $A : D \to \mathbb{S}^{2\times 2}$ whose elements are bounded and measurable. Then by (5.27), because the Laplacian $\triangle = {\rm div}\,{\rm grad}$, we have that u is a weak solution of the equation

$${\rm div}\,[A(z)\,\nabla\,u(z)] \ = \ {\rm div}\,\nabla\mathcal{N}^g(z)\ . \tag{5.32}$$

If v is $A-$conjugate of u, then by Theorem 16.1.6 in [3] the function $\omega := u + iv$ satisfies the nondegenerate inhomogeneous Beltrami equation (2.14) with μ and σ given by (5.29) and (5.30).

Conversely, if $\omega \in W^{1,1}_{\rm loc}(D)$ is a solution of the nondegenerate inhomogeneous Beltrami equation (2.14) with σ given by (5.30), then, again by Theorem 16.1.6 in [3], the functions $u := {\rm Re}\,\omega$ and $v := {\rm Im}\,\omega$ satisfy the relation (5.28) with the matrix function $A : D \to \mathbb{S}^{2\times 2}$ given by (5.31) whose elements $a_{ij}(z)$ are measurable in $z \in D$ and bounded because $|a_{ij}| \le \|K_\mu\|_\infty$. Note that (5.28) is equivalent to the equation

$$A(z)\nabla\,u(z) \ - \ \nabla\mathcal{N}^g(z) \ = \ -\mathbb{H}\nabla\,v(z) \tag{5.33}$$

because $\mathbb{H}^2 = -I$. As known, the curl of any gradient field is zero in the sense of distributions and, moreover, the Hodge operator $\mathbb{H}$ transforms curl-free fields into divergence-free fields, and vice versa, see e.g. 16.1.3 in [3]. Hence u is a weak solution of equation (5.32) as well as of (5.24) in view of (5.27). □

Further we say that a function $u : D \to \mathbb{R}$ is a **weak solution** of (1.3), if $u \in W^{1,1}_{\rm loc}(D)$ and

$$\int\limits_D \langle A(z)\nabla u(z), \nabla\psi(z)\rangle\, d\,m(z) \ + \int\limits_D G(z)\,Q(u(z))\,\psi(z)\,d\,m(z) \ = \ 0 \tag{5.34}$$

for all $\psi \in C_0^\infty(D)$, where $C_0^\infty(D)$ denotes the collection of all infinitely differentiable functions $\psi : D \to \mathbb{R}$ with compact support in D, $\langle a, b\rangle$ means the scalar product of vectors a and b in $\mathbb{R}^2$, and $d\,m(z)$ corresponds to the Lebesgue measure in the plane $\mathbb{C}$.

Theorem 5.2. *Let D be a bounded domain in $\mathbb{C}$, a scalar function $G : D \to \mathbb{R}$ be in class $L_{p'}(D)$ for some $p' > 1$, a continuous function $Q : \mathbb{R} \to \mathbb{R}$ satisfy condition (1.4) and let $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

Then the semi-linear Poisson type equation (1.3) has a weak solution u of class $C^{\alpha}(D) \cap W^{1,p}(D)$ with $\alpha = 1 - 2/p$ for some $p > 2$.

Moreover, $u = h \circ f^{\mu}|_D$, where $f^{\mu} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B and h is a weak generalized harmonic function in the domain $D_* := f^{\mu}(D)$ with the source

$$S \ := \ \left\{ \frac{f^{\mu}_{z}}{J^{\mu}} \cdot \sigma \right\} \circ (f^{\mu})^{-1} \in \ L_{p_*}(D_*) \ , \tag{5.35}$$

where J^{μ} is the Jacobian of f^{μ}, $p_* := p^2/2(p-1) \in (2, p)$, μ is defined by formula (5.29) and σ is calculated by formula (5.30) with $g = G\,Q(u)$.

As it is clear from the proof below the degree $p > 2$ and the exponent $\alpha \in (0, 1)$ of the Hölder continuity, correspondingly, cannot be connected with p' in an explicit form.

Proof. With no loss of generality, we may assume here that $p' \in (1, 2]$ and that $g \equiv 0$ outside of D, and then $\mathcal{N}^g \in W^{1,p}(D)$ for all $p \in (1, p^*)$, where $p^* = 2p'/(2 - p') > 2$, see Lemma 3 in [14]. Hence later on, we may also assume that $p > 2$ satisfies condition (1.6) for μ in (5.29). Moreover, again by Lemma 3 in [14], the correspondence $g \to \mathcal{N}^g_{\bar{z}}$ generates a completely continuous linear operator L acting from real valued $L_{p'}(D)$ to complex valued $L_p(D)$. Thus, the linear operator $\mathcal{L} := L + \mu \bar{L}$ with the multiplier $\mu \in L_{\infty}(D)$ is bounded. Then by Lemma 4.3, the semi-linear Beltrami equation (4.20) with $q(\omega) := Q(\mathrm{Re}\,\omega)$ has a solution ω of class $C^{\alpha}(D) \cap W^{1,p}(D)$ with $\alpha = 1 - 2/p$. Moreover, by Lemma 5.4, the function $u := \mathrm{Re}\,\omega$ is a weak solution of equation (5.24) of the given class. Finally, by Lemma 3.1, we conclude that u has the representation as the composition $h \circ f^{\mu}|_D$, where $f^{\mu} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping from Theorem B and h is a weak generalized harmonic function in the domain $D_* := f^{\mu}(D)$ with the source (5.35). ☐

6. Some examples of applications

We apply Theorem 5.2 for several model equations describing some physical phenomena in anisotropic and inhomogeneous media.

The first group of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [8], p. 4, and, in details, in [4]. A nonlinear system is obtained for the density U and the temperature T of the reactant. Upon eliminating T the system can be reduced to equations of the form

$$\triangle U \ = \ \sigma \cdot Q(U) \tag{6.36}$$

with $\sigma > 0$ and, for isothermal reactions, $Q(U) = U^{\lambda}$, where $\lambda > 0$ that is called the order of the reaction. It turns out that the density of the reactant U may be zero in a subdomain called a **dead core**. A particularization of results in Chapter 1 of [8] shows that a dead core may exist just if and only if $\beta \in (0, 1)$, see also the corresponding examples in [13].

In the case of anisotropic and inhomogeneous media, we come to the semi-linear Poisson type equations (1.3). In this connection, the following statement may be of independent interest.

Corollary 6.1. *Let D be a bounded domain in $\mathbb{C}$, a scalar function $\sigma : D \to \mathbb{R}$ be in class $L_{p'}(D)$ for some $p' > 1$ and let $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

Then there is a weak solution $u : D \to \mathbb{R}$ of class $C^{\alpha}_{\rm loc} \cap W^{1,p}_{\rm loc}$ with $\alpha = 1 - 2/p$ for some $p > 2$ to the semi-linear Poisson type equation

$$\text{div}\,[A(z)\,\nabla\,u(z)] \;=\; \sigma(z)\cdot u^{\lambda}(z)\,, \quad 0\,<\,\lambda\,<\,1\,, \qquad \textit{a.e. in } D. \tag{6.37}$$

Note also that certain mathematical models of a thermal evolution of a heated plasma lead to nonlinear equations of the type (6.36). Indeed, it is known that some of them have the form $\triangle\psi(u) = f(u)$ with $\psi'(0) = \infty$ and $\psi'(u) > 0$ if $u \neq 0$ as, for instance, $\psi(u) = |u|^{q-1}u$ under $0 < q < 1$, see e.g. [8]. With the replacement of the function $U = \psi(u) = |u|^q \cdot \text{sign}\,u$, we have that $u = |U|^Q \cdot \text{sign}\,U$, $Q = 1/q$, and, with the choice $f(u) = |u|^{q^2} \cdot \text{sign}\,u$, we come to the equation $\triangle U = |U|^q \cdot \text{sign}\,U = \psi(U)$. For anisotropic and inhomogeneous media, we obtain the corresponding equation (6.38) below:

Corollary 6.2. *Let D be a bounded domain in $\mathbb{C}$, a scalar function $\sigma : D \to \mathbb{R}$ be in class $L_{p'}(D)$ for some $p' > 1$ and let $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

Then there is a weak solution $u : D \to \mathbb{R}$ of class $C^{\alpha}_{\rm loc} \cap W^{1,p}_{\rm loc}$ with $\alpha = 1 - 2/p$ for some $p > 2$ to the semi-linear Poisson type equation

$$\text{div}\,[A(z)\,\nabla\,u(z)] \;=\; \sigma(z)\cdot |u(z)|^{\lambda-1}u(z)\,, \quad 0\,<\,\lambda\,<\,1\,, \qquad \textit{a.e. in } D. \tag{6.38}$$

Finally, we recall that in the combustion theory, see e.g. [5] and [27] and the references therein, the following model equation

$$\frac{\partial u(z,t)}{\partial t} \;=\; \frac{1}{\delta}\cdot\triangle u \;+\; e^{u}\,, \quad \delta\;>\;0\,,\; t \geq 0,\; z \in D \tag{6.39}$$

takes a special part. Here $u \geq 0$ is the temperature of the medium. We restrict ourselves here by the stationary case, although our approach makes it possible to study the parabolic equation (6.39), see [13]. The corresponding equation of the type (1.3), see (6.40) below, appears in anisotropic and inhomogeneous media with the function $Q(u) = e^{-|u|}$ that is uniformly bounded at all.

Corollary 6.3. *Let D be a bounded domain in $\mathbb{C}$, a scalar function $\sigma : D \to \mathbb{R}$ be in class $L_{p'}(D)$ for some $p' > 1$ and let $A : D \to \mathbb{S}^{2\times 2}$ be a matrix function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are bounded and measurable.*

Then there is a weak solution $u : D \to \mathbb{R}$ of class $C^{\alpha}_{\rm loc} \cap W^{1,p}_{\rm loc}$ with $\alpha = 1 - 2/p$ for some $p > 2$ to the semi-linear Poisson type equation

$$\text{div}\,[A(z)\,\nabla\,u(z)] \;=\; \sigma(z)\cdot e^{-|u(z)|} \qquad \textit{a.e. in } D. \tag{6.40}$$

Remark 6.7. *Such solutions u in Corollaries 6.1, 6.2, 6.3 have the representation as the composition $h \circ f^{\mu}|_D$, where $f^{\mu} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping in Theorem B with μ extended onto $\mathbb{C}$ by zero outside of D, and all h are weak generalized harmonic functions with sources of class $L_{p_*}(D_*)$, $D_* := f^{\mu}(D)$ and $p_* := p^2/2(p-1) \in (2, p)$.*

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Research Article

Generalizations of the drift Laplace equation over the quaternions in a class of Grushin-type spaces

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ABSTRACT. Beals, Gaveau, and Greiner established a formula for the fundamental solution to the Laplace equation with drift term in Grushin-type planes. The first author and Childers expanded these results by invoking a p-Laplace-type generalization that encompasses these formulas while the authors explored a different natural generalization of the p-Laplace equation with drift term that also encompasses these formulas. In both, the drift term lies in the complex domain. We extend these results by considering a drift term in the quaternion realm and show our solutions are stable under limits as p tends to infinity.

Keywords: p-Laplace equation, Grushin-type plane.

2020 Mathematics Subject Classification: 53C17, 35H20, 35A09, 35R03, 17B70.

1. MOTIVATION AND BACKGROUND

1.1. **Motivation.** In [2], Beals, Gaveau, and Greiner established a formula for the fundamental solution to the Laplace equation with drift term in a large class of sub-Riemannian spaces, which includes the so-called Grushin-type planes. In [4], the first author and Childers expanded these results by invoking a p-Laplace-type generalization that encompasses the formulas of [2] while in [3], the authors explored a different natural generalization of the p-Laplace equation with drift term that also encompasses the formulas of [2]. In both cases, the drift term lies in the complex domain. In this paper, we will consider both approaches, but with a drift term in the quaternion realm and create an extension of both cases. We will then show our solutions are stable under limits when $\mathrm{p} \to \infty$.

1.2. **Grushin-type planes.** We begin with a brief discussion of our environment. The Grushin-type planes are a class of sub-Riemannian spaces lacking an algebraic group law. We begin with $\mathbb{R}^2$ possessing coordinates (y_1, y_2), $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$. We use them to construct the vector fields

$$Y_1 = \frac{\partial}{\partial y_1} \text{ and } Y_2 = c(y_1 - a)^n \frac{\partial}{\partial y_2}.$$

For these vector fields, the only (possibly) nonzero Lie bracket is

$$[Y_1, Y_2] = cn(y_1 - a)^{n-1} \frac{\partial}{\partial y_2}.$$

Because $n \in \mathbb{N}$, it follows that Hörmander's condition (see, for example, [1]) is satisfied by these vector fields.

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We endow $\mathbb{R}^2$ with a (singular) inner product, denoted $\langle\cdot,\cdot\rangle$, with related norm $\|\cdot\|$, so that the collection $\{Y_1, Y_2\}$ forms an orthonormal basis. We then have a sub-Riemannian space that we will call g_n, which is also the tangent space to a generalized Grushin-type plane $\mathbb{G}_n$. Points in $\mathbb{G}_n$ will also be denoted by $p = (y_1, y_2)$. The Carnot-Carathéodory distance on $\mathbb{G}_n$ is defined for points p and q as follows:

$$d_{\mathbb{G}}(p,q) = \inf_{\Gamma} \int \|\gamma'(t)\|\, dt$$

with Γ the set of curves γ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma'(t) \in \operatorname{span}\{Y_1(\gamma(t)), Y_2(\gamma(t))\}$. By Chow's theorem, this is an honest metric.

We shall now discuss calculus on the Grushin-type planes. Given a smooth function f on $\mathbb{G}_n$, we define the horizontal gradient of f as

$$\nabla_0 f(p) = \big(Y_1 f(p), Y_2 f(p)\big).$$

Using these derivatives, we consider a key operator on $C^2_{\mathbb{G}}$ functions, namely the p-Laplacian for $1 < \mathrm{p} < \infty$, given by

$$\begin{aligned}\Delta_{\mathrm{p}} f &= \operatorname{div}(\|\nabla_0 f\|^{\mathrm{p}-2}\nabla_0 f) = Y_1\big(\|\nabla_0 f\|^{\mathrm{p}-2} Y_1 f\big) + Y_2\big(\|\nabla_0 f\|^{\mathrm{p}-2} Y_2 f\big)\\ &= \frac{1}{2}(\mathrm{p}-2)\|\nabla_0 f\|^{\mathrm{p}-4}\big(Y_1\|\nabla_0 f\|^2 Y_1 f + Y_2\|\nabla_0 f\|^2 Y_2 f\big)\\ &+ \|\nabla_0 f\|^{\mathrm{p}-2}\big(Y_1 Y_1 f + Y_2 Y_2 f\big).\end{aligned} \tag{1.1}$$

For more recent results concerning Grushin-type spaces, see [6] and references therein.

2. Motivating Results

2.1. Grushin-type Planes. The first author and Gong [5] proved the following in the Grushin-type planes.

Theorem 2.1 ([5]). *Let $1 < p < \infty$ and define*

$$f(y_1, y_2) = c^2(y_1 - a)^{(2n+2)} + (n+1)^2(y_2 - b)^2.$$

For $p \neq n+2$, consider

$$\tau_p = \frac{n+2-p}{(2n+2)(1-p)}$$

so that in $\mathbb{G}_n \setminus \{(a,b)\}$ we have the well-defined function

$$\psi_p = \begin{cases} f(y_1,y_2)^{\tau_p}, & p \neq n+2 \\ \log f(y_1,y_2), & p = n+2 \end{cases}.$$

Then, $\Delta_p \psi_p = 0$ in $\mathbb{G}_n \setminus \{(a,b)\}$.

In the Grushin-type planes, Beals, Gaveau and Greiner [2] extended this equation as shown in the following theorem.

Theorem 2.2 ([2]). *Let $L \in \mathbb{R}$. Consider the following quantities*

$$\alpha = \frac{-n}{(2n+2)}(1+L) \ \text{ and } \ \beta = \frac{-n}{(2n+2)}(1-L).$$

We use these constants with the functions

$$\begin{aligned} g(y_1,y_2) &= c(y_1-a)^{n+1} + i(n+1)(y_2-b)\\ h(y_1,y_2) &= c(y_1-a)^{n+1} - i(n+1)(y_2-b)\end{aligned}$$

to define our main function $f(y_1, y_2)$, given by

$$f(y_1, y_2) = g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}.$$

Then, $\mathcal{D}(f) := \Delta_2 f + iL[Y_1, Y_2]f = 0$ *in* $\mathbb{G}_n \setminus \{(a, b)\}$.

Non-linear generalizations of Theorem 2.2 have been explored by the first author and Childers in [4] and by the authors in [3]. The following theorem extends Theorem 2.2 through a p-Laplace type divergence form.

Theorem 2.3 ([4]). *For $L \in \mathbb{R}$ with $L \neq \pm 1$, consider the following parameters for $p \neq n+2$:*

$$\alpha = \frac{n+2-p}{(1-p)(2n+2)}(1+L) \quad \text{and} \quad \beta = \frac{n+2-p}{(1-p)(2n+2)}(1-L)$$

with the functions:

$$g(y_1, y_2) = c(y_1 - a)^{n+1} + i(n+1)(y_2 - b)$$
$$h(y_1, y_2) = c(y_1 - a)^{n+1} - i(n+1)(y_2 - b)$$

to define the main function:

$$f_{p,L} = \begin{cases} g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}, & p \neq n+2 \\ \log\left(g(y_1, y_2)^{1+L} h(y_1, y_2)^{1-L}\right), & p = n+2 \end{cases}.$$

Then

$$\overline{\Delta_p} f_{p,L} := \operatorname{div}\left(\left\| \begin{matrix} Y_1 f_{p,L} + iLY_2 f_{p,L} \\ Y_2 f_{p,L} - iLY_1 f_{p,L} \end{matrix} \right\|^{p-2} \begin{pmatrix} Y_1 f_{p,L} + iLY_2 f_{p,L} \\ Y_2 f_{p,L} - iLY_1 f_{p,L} \end{pmatrix} \right) = 0.$$

The following theorem of the authors takes an alternative approach to extending Theorem 2.2 through a generalization of the drift term.

Theorem 2.4 ([3]). *For $L \in \mathbb{R}$ with:*

$$L \neq \pm \frac{n+2-p}{n(1-p)}$$

consider the parameters:

$$\alpha = \frac{n+2-p-Ln(1-p)}{2(n+1)(1-p)} \text{ and } \beta = \frac{n+2-p+Ln(1-p)}{2(n+1)(1-p)}$$

with the functions

$$g(y_1, y_2) = c(y_1 - a)^{n+1} + i(n+1)(y_2 - b)$$
$$h(y_1, y_2) = c(y_1 - a)^{n+1} - i(n+1)(y_2 - b)$$

to define the main function:

$$f_{p,L}(y_1, y_2) = g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}. \tag{2.2}$$

Then on $\mathbb{G}_n \setminus \{(a, b)\}$, we have:

$$\mathcal{G}_{p,L}(f_{p,L}) := \Delta_p f_{p,L} + iL[Y_1, Y_2]\left(\|\nabla_0 f_{p,L}\|^{p-2} f_{p,L}\right) = 0.$$

Main Question. *We wish to extend the preceding generalizations of Theorem 2.2 over the quaternions, denoted $\mathbb{H}$. Recall that the solved partial differential equation of Theorem 2.2, namely*

$$\Delta_2 f + iL[Y_1, Y_2]f = 0,$$

features a drift term bearing the purely complex-imaginary coefficient $iL \in \mathbb{C}$. We ask if this coefficient can be generalized to a purely quaternion-imaginary coefficient of the form

$$Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R},$$

where the case of $Q = 0$ reduces to the result of Theorem 2.1. With respect to Theorem 2.3, we explore smooth solutions to the generalization

$$\overline{\Delta_p} f := \operatorname{div}\left(\left\| \begin{matrix} Y_1 f + QY_2 f \\ Y_2 f - QY_1 f \end{matrix} \right\|^{p-2} \begin{pmatrix} Y_1 f + QY_2 f \\ Y_2 f - QY_1 f \end{pmatrix} \right) = 0.$$

With respect to Theorem 2.4, we explore smooth solutions to the generalization

$$\mathcal{G}_{p,Q}(f) := \Delta_p f + Q\,[Y_1, Y_2]\left(\|\nabla_0 f\|^{p-2} f \right) = 0.$$

3. A p-Laplacian Type Generalization over $\mathbb{H}$

3.1. Case I: $\mathbf{L} + \mathbf{M} + \mathbf{N} \neq \mathbf{0}$.

Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N \neq 0$. We consider the following parameters:

$$\begin{aligned} \mu &= \frac{\sqrt{|Q^2|}}{|L + M + N|} \\ \omega &= \frac{Q}{L + M + N} \\ \xi &= \sqrt{|Q^2|}(L + M + N) \\ \alpha &= \frac{n + 2 - \mathrm{p}}{(1 - \mathrm{p})(2n + 2)}(1 + \xi) \\ \text{and } \beta &= \frac{n + 2 - \mathrm{p}}{(1 - \mathrm{p})(2n + 2)}(1 - \xi), \end{aligned}$$

where $\xi \neq \pm 1$. We use these constants with the functions:

$$\begin{aligned} g(y_1, y_2) &= \mu c(y_1 - a)^{n+1} + \omega(n + 1)(y_2 - b) \\ h(y_1, y_2) &= \mu c(y_1 - a)^{n+1} - \omega(n + 1)(y_2 - b) \end{aligned}$$

to define our main function:

$$f_{\mathrm{p},Q}(y_1, y_2) = \begin{cases} g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}, & \mathrm{p} \neq n + 2 \\ \log\left(g(y_1, y_2)^{1+\xi} h(y_1, y_2)^{1-\xi} \right), & \mathrm{p} = n + 2 \end{cases}. \tag{3.3}$$

Using equation 3.3, we have the following theorem.

Theorem 3.5. *Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N \neq 0$. On $G_n \setminus \{(a, b)\}$, we have:*

$$\overline{\Delta_p} f_{p,Q} := \operatorname{div}_{\mathbb{G}}\left(\left\| \begin{matrix} Y_1 f_{p,Q} + QY_2 f_{p,Q} \\ Y_2 f_{p,Q} - QY_1 f_{p,Q} \end{matrix} \right\|^{p-2} \begin{pmatrix} Y_1 f_{p,Q} + QY_2 f_{p,Q} \\ Y_2 f_{p,Q} - QY_1 f_{p,Q} \end{pmatrix} \right) = 0.$$

Proof. Suppressing arguments and subscripts, we let:

$$\Upsilon := \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \begin{pmatrix} Y_1 f + QY_2 f \\ Y_2 f - QY_1 f \end{pmatrix}.$$

Observing that:

$$\begin{aligned}\overline{\Delta_{\mathrm{p}}} f &= \operatorname{div}\left(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon\right)\\ &= \|\Upsilon\|^{\mathrm{p}-4}\left(\frac{\mathrm{p}-2}{2}\sum_{s=1}^{2} Y_s\|\Upsilon\|^2\Upsilon_s + \|\Upsilon\|^2(Y_1\Upsilon_1+Y_2\Upsilon_2)\right)\end{aligned}$$

it suffices to show:

$$\Lambda := \frac{\mathrm{p}-2}{2}\sum_{s=1}^{2} Y_s\|\Upsilon\|^2\Upsilon_s + \|\Upsilon\|^2(Y_1\Upsilon_1+Y_2\Upsilon_2) = 0.$$

For $\mathrm{p} \neq n+2$, we compute the following:

$$\begin{aligned} Y_1 f &= \mu c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h+\beta g)\\ Y_2 f &= \omega c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h-\beta g)\\ Y_1 f + QY_2 f &= \mu c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h(1-\xi)+\beta g(1+\xi))\\ Y_2 f - QY_1 f &= \omega c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h(1-\xi)-\beta g(1+\xi))\\ \text{and } \|\Upsilon\|^2 &= 2\mu^2c^2(n+1)^2(y_1-a)^{2n}g^{\alpha+\beta-1}h^{\alpha+\beta-1}\left(\alpha^2(1-\xi)^2+\beta^2(1+\xi)^2\right).\end{aligned}$$

We then calculate:

$$\begin{aligned} Y_1\Upsilon_1 + Y_2\Upsilon_2 &= \frac{1}{(-1+\mathrm{p})^2 gh}\mu^2c^2(-1+\xi^2)(1+n)(2+n-\mathrm{p})(-2+\mathrm{p})(y_1-a)^{2n}g^{\alpha}h^{\beta}\\ Y_1\|\Upsilon\|^2 &= -\frac{1}{(-1+\mathrm{p})^3 gh}\Big(2\mu^2c^2(1-\xi^2)^2(n+1)(n+2-\mathrm{p})^2(y_1-a)^{2n-1}\\ &\quad\times g^{\alpha+\beta-1}h^{\alpha+\beta-1}\left(\mu^2c^2(y_1-a)^{2n+2}-\mu^2 n(n+1)(-1+\mathrm{p})(y_2-b)^2\right)\Big)\\ \text{and } Y_2\|\Upsilon\|^2 &= \frac{1}{(-1+\mathrm{p})^3 gh}2\mu^4c^3(1-\xi^2)^2(n+1)(n+2-\mathrm{p})^2(1+n\mathrm{p})\\ &\quad\times(y_1-a)^{3n}(b-y_2)g^{\alpha+\beta-1}h^{\alpha+\beta-1}.\end{aligned}$$

Using the above quantities, we compute:

$$\begin{aligned} \frac{\mathrm{p}-2}{2}\sum_{s=1}^{2} Y_s\|\Upsilon\|^2\Upsilon_s &= -\frac{1}{(-1+\mathrm{p})^4}\mu^4c^4(-1+\xi^2)^3(n+1)(n+2-\mathrm{p})^3\\ &\quad\times(y_1-a)^{4n}g^{2\alpha+\beta-2}h^{\alpha+2\beta-2}(\mathrm{p}-2)\\ \text{and } \|\Upsilon\|^2(Y_1\Upsilon_1+Y_2\Upsilon_2) &= \frac{1}{(-1+\mathrm{p})^4}\mu^4c^4(n+1)(y_1-a)^{4n}g^{2\alpha+\beta-2}h^{\alpha+2\beta-2}\\ &\quad\times(n+2-\mathrm{p})^3(-1+\xi^2)^3(\mathrm{p}-2)\end{aligned} \tag{3.4}$$

whereby it follows that $\Lambda = 0$, as desired. The case $\mathrm{p} = n+2$ is similar and omitted. □

3.2. **Case II: L + M + N = 0.**

Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N = 0$. We consider the following parameters:

$$\begin{aligned}
\xi &= \sqrt{2|LM + LN + MN|} \\
\alpha &= \frac{n+2-\mathrm{p}}{(1-\mathrm{p})(2n+2)}(1+\xi) \\
\text{and } \beta &= \frac{n+2-\mathrm{p}}{(1-\mathrm{p})(2n+2)}(1-\xi),
\end{aligned}$$

where $\xi \neq \pm 1$. We use these constants with the functions:

$$\begin{aligned}
g(y_1, y_2) &= \xi c (y_1 - a)^{n+1} + Q(n+1)(y_2 - b) \\
h(y_1, y_2) &= \xi c (y_1 - a)^{n+1} - Q(n+1)(y_2 - b)
\end{aligned}$$

to define our main function:

$$f_{\mathrm{p},Q}(y_1, y_2) = \begin{cases} g(y_1,y_2)^{\alpha} h(y_1,y_2)^{\beta}, & \mathrm{p} \neq n+2 \\ \log\left(g(y_1,y_2)^{1+\xi} h(y_1,y_2)^{1-\xi}\right), & \mathrm{p} = n+2 \end{cases}. \tag{3.5}$$

Using equation 3.5, we have the following theorem.

Theorem 3.6. *Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N = 0$. On $G_n \setminus \{(a,b)\}$, we have:*

$$\overline{\Delta_p} f_{p,Q} := \operatorname{div}_{\mathbb{G}} \left(\left\| \begin{matrix} Y_1 f_{p,Q} + Q Y_2 f_{p,Q} \\ Y_2 f_{p,Q} - Q Y_1 f_{p,Q} \end{matrix} \right\|^{p-2} \begin{pmatrix} Y_1 f_{p,Q} + Q Y_2 f_{p,Q} \\ Y_2 f_{p,Q} - Q Y_1 f_{p,Q} \end{pmatrix} \right) = 0.$$

Proof. The proof of Theorem 3.6 is similar to that of Theorem 3.5 and left to the reader. □

We then conclude the following corollary.

Corollary 3.1. *Let $p > n+2$. The function $f_{p,Q}$, as above, is a nontrivial smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_p} f_{p,Q}(\mathbf{y}) = 0, & \mathbf{y} \in \mathbb{G}_n \setminus \{(a,b)\} \\ 0, & \mathbf{y} = (a,b) \end{cases}.$$

4. A Generalization of the Drift Term over $\mathbb{H}$

4.1. Case I: $\mathbf{L + M + N \neq 0}$.

Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N \neq 0$. We consider the following parameters:

$$\begin{aligned}
\mu &= \frac{\sqrt{|Q^2|}}{|L+M+N|} \\
\omega &= \frac{Q}{L+M+N} \\
\xi &= \sqrt{|Q^2|}(L+M+N) \\
\alpha &= \frac{n+2-\mathrm{p}-\xi n(1-\mathrm{p})}{2(n+1)(1-\mathrm{p})} \\
\text{and } \beta &= \frac{n+2-\mathrm{p}+\xi n(1-\mathrm{p})}{2(n+1)(1-\mathrm{p})},
\end{aligned}$$

where:

$$\xi \neq \pm \frac{n+2-\mathrm{p}}{n(\mathrm{p}-1)}.$$

We use these constants with the functions:

$$g(y_1,y_2) = \mu c(y_1-a)^{n+1} + \omega(n+1)(y_2-b)$$
$$h(y_1,y_2) = \mu c(y_1-a)^{n+1} - \omega(n+1)(y_2-b)$$

to define our main function:

$$f_{\mathrm{p},Q}(y_1,y_2) = g(y_1,y_2)^{\alpha} h(y_1,y_2)^{\beta}. \tag{4.6}$$

Using equation 4.6, we have the following theorem.

Theorem 4.7. *Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L+M+N \neq 0$. On $G_n \setminus \{(a,b)\}$, we have:*

$$\mathcal{G}_{p,Q}\left(f_{p,Q}\right) := \Delta_p f_{p,Q} + Q\left[Y_1,Y_2\right]\left(\|\nabla_0 f_{p,Q}\|^{p-2} f_{p,Q}\right) = 0.$$

Proof. Suppressing arguments and subscripts, we compute the following:

$$Y_1 f = \mu c(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}(\alpha h + \beta g) \tag{4.7}$$
$$\overline{Y_1 f} = \mu c(n+1)(y_1-a)^n g^{\beta-1} h^{\alpha-1}(\alpha g + \beta h)$$
$$Y_2 f = \omega c(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}(\alpha h - \beta g) \tag{4.8}$$
$$\overline{Y_2 f} = -\omega c(n+1)(y_1-a)^n g^{\beta-1} h^{\alpha-1}(\alpha g - \beta h)$$
$$\text{and } \|\nabla_0 f\|^2 = 2\mu^2 c^2 (n+1)^2 (y_1-a)^{2n} g^{\alpha+\beta-1} h^{\alpha+\beta-1}\left(\alpha^2+\beta^2\right).$$

Using the above, we compute:

$$\begin{aligned}
Y_1Y_1 f &= \mu c(n+1)(y_1-a)^{n-1} g^{\alpha-2} h^{\beta-2} \\
&\quad \times \Big(ngh(\alpha h+\beta g) + \mu c(n+1)(y_1-a)^{n+1} \\
&\quad \times \big((\alpha h+\beta g)\big((\alpha-1)h + (\beta-1)g\big) + gh(\alpha+\beta)\big)\Big) \\
Y_2Y_2 f &= -\mu^2 c^2 (n+1)^2 (y_1-a)^{2n} g^{\alpha-2} h^{\beta-2} \\
&\quad \times \big((\alpha h-\beta g)\big((\alpha-1)h - (\beta-1)g\big) - gh(\alpha+\beta)\big)
\end{aligned}$$

$$\begin{aligned}
Y_1\|\nabla_0 f\|^2 &= 4\mu^2 c^2 (n+1)^2 (y_1-a)^{2n-1} g^{\alpha+\beta-2} h^{\alpha+\beta-2}(\alpha^2+\beta^2)x \\
&\quad \times \big(ngh + \mu^2 c^2 (n+1)(y_1-a)^{2n+2}(\alpha+\beta-1)\big)
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
Y_2\|\nabla_0 f\|^2 &= -4\omega^2 \mu^2 c^3 (n+1)^4 (y_1-a)^{3n}(y_2-b) g^{\alpha+\beta-2} h^{\alpha+\beta-2} \\
&\quad \times \big(\alpha^2+\beta^2\big)(\alpha+\beta-1)
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
\sum_{s=1}^{2} Y_s \|\nabla_0 f\|^2 (Y_s f) &= 4\mu^3 c^3 (n+1)^3 (y_1-a)^{3n-1} g^{2\alpha+\beta-3} h^{\alpha+2\beta-3}(\alpha^2+\beta^2) \\
&\quad \times \big((\alpha h+\beta g)\big(ngh + \mu^2 c^2 (n+1)(y_1-a)^{2n+2}(\alpha+\beta-1)\big) \\
&\quad + \omega\mu c(n+1)^2 (y_1-a)^{n+1}(y_2-b)(\alpha+\beta-1)(\alpha h-\beta g)\big) \\
\|\nabla_0 f\|^2 (Y_1Y_1 + Y_2Y_2 f) &= 2\mu^3 c^3 (n+1)^3 (y_1-a)^{3n-1} g^{2\alpha+\beta-3} h^{\alpha+2\beta-3} \\
&\quad \times \big(\alpha^2+\beta^2\big)\big(ngh(\alpha h+\beta g) + 4\mu c(n+1)(y_1-a)^{n+1} gh\alpha\beta\big)
\end{aligned}$$

so that

$$\Delta_{\rm p} f = \|\nabla_0 f\|^{\rm p-4}\left(\frac{({\rm p}-2)}{2}\sum_{s=1}^{2} Y_s\|\nabla_0 f\|^2(Y_s f) + \|\nabla_0 f\|^2(Y_1Y_1 f + Y_2Y_2 f)\right)$$
$$= -\xi 2^{\frac{\rm p-2}{2}}\mu^{\rm p-1}c^{\rm p-1}n^2(n+1)^{\rm p-2}(y_1-a)^{n({\rm p}-1)-1}g^{\frac{\alpha{\rm p}+\beta({\rm p}-2)-{\rm p}}{2}}h^{\frac{\alpha({\rm p}-2)+\beta{\rm p}-{\rm p}}{2}}\left(\alpha^2+\beta^2\right)^{\frac{\rm p-2}{2}}$$
$$\times\left(\xi\mu c(y_1-a)^{n+1} + \omega(1-{\rm p})(n+1)(y_2-b)\right).$$

We then compute:

$$Q[Y_1,Y_2]\left(\|\nabla_0 f\|^{\rm p-2} f\right) =$$
$$Q2^{\frac{p-2}{2}}\mu^{p-2}c^{p-1}n(n+1)^{p-2}(y_1-a)^{n(p-1)-1}\left(\alpha^2+\beta^2\right)^{\frac{p-2}{2}}$$
$$\times\frac{\partial}{\partial y_2}\left(g^{\frac{\alpha{\rm p}+\beta({\rm p}-2)-({\rm p}-2)}{2}}h^{\frac{\alpha({\rm p}-2)+\beta{\rm p}-({\rm p}-2)}{2}}\right)$$
$$= \xi 2^{\frac{\rm p-2}{2}}\mu^{\rm p-1}c^{\rm p-1}n^2(n+1)^{\rm p-2}(y_1-a)^{n({\rm p}-1)-1}g^{\frac{\alpha{\rm p}+\beta({\rm p}-2)-{\rm p}}{2}}h^{\frac{\alpha({\rm p}-2)+\beta{\rm p}-{\rm p}}{2}}$$
$$\times\left(\alpha^2+\beta^2\right)^{\frac{\rm p-2}{2}}\left(\xi\mu c(y_1-a)^{n+1}+\omega(1-{\rm p})(n+1)(y_2-b)\right)$$
$$= -\Delta_{\rm p} f.$$

□

4.2. Case II: $\mathbf{L+M+N=0}$.

Let $Q = Li + Mj + Nk \in \mathbb{H}\setminus\mathbb{R}$ with $L+M+N=0$. We consider the following parameters:

$$\xi = \sqrt{2|LM+LN+MN|}$$
$$\alpha = \frac{n+2-{\rm p}-\xi n(1-{\rm p})}{2(n+1)(1-{\rm p})}$$
$$\text{and } \beta = \frac{n+2-{\rm p}+\xi n(1-{\rm p})}{2(n+1)(1-{\rm p})},$$

where:

$$\xi \neq \pm\frac{n+2-{\rm p}}{n({\rm p}-1)}.$$

We use these constants with the functions:

$$g(y_1,y_2) = \xi c(y_1-a)^{n+1} + Q(n+1)(y_2-b)$$
$$h(y_1,y_2) = \xi c(y_1-a)^{n+1} - Q(n+1)(y_2-b)$$

to define our main function:

$$f_{{\rm p},Q}(y_1,y_2) = g(y_1,y_2)^{\alpha}h(y_1,y_2)^{\beta}. \tag{4.11}$$

Using equation 4.11, we have the following theorem.

Theorem 4.8. *Let $Q = Li+Mj+Nk \in \mathbb{H}\setminus\mathbb{R}$ with $L+M+N=0$. On $G_n\setminus\{(a,b)\}$, we have:*

$$\mathcal{G}_{p,Q}\left(f_{p,Q}\right) := \Delta_p f_{p,Q} + Q\left[Y_1,Y_2\right]\left(\|\nabla_0 f_{p,Q}\|^{p-2} f_{p,Q}\right) = 0.$$

Proof. The computations proving Theorem 4.8 are similar to those of the proof of Theorem 4.7 and are left to the reader. □

Observing that

$$\xi \neq \pm\frac{n(\mathrm{p}-1)}{n+2-\mathrm{p}} \quad \text{implies} \quad \mathrm{p} \neq \left|\frac{\xi(n+2)+n}{n+\xi}\right|, \left|\frac{\xi(n+2)-n}{n-\xi}\right|$$

we have immediately the following corollary.

Corollary 4.2. *Let* $p > \max\left\{\left|\frac{\xi(n+2)+n}{n+\xi}\right|, \left|\frac{\xi(n+2)-n}{n-\xi}\right|\right\}$. *Then the function* $f_{p,Q}$ *of equation 4.6 is a nontrivial smooth solution to the Dirichlet problem*

$$\begin{cases} \mathcal{G}_{p,Q}\left(f_{p,Q}(\mathbf{y})\right) = 0, & \mathbf{y} \in \mathbb{G}_n \setminus \{(a,b)\} \\ 0, & \mathbf{y} = (a,b) \end{cases}.$$

5. The Limit as $\mathrm{p} \to \infty$

5.1. **p-Laplacian Type Generalization over $\mathbb{H}$.** Recall that on $\mathbb{G}_n \setminus \{(a,b)\}$, we have

$$\begin{aligned} \overline{\Delta_{\mathrm{p}}} f &= \operatorname{div}_G(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon) \\ &= \|\Upsilon\|^{\mathrm{p}-4}\left(\frac{1}{2}(\mathrm{p}-2)\left(Y_1\|\Upsilon\|^2\Upsilon_1 + Y_2\|\Upsilon\|^2\Upsilon_2\right) + \|\Upsilon\|^2\left(Y_1\Upsilon_1 + Y_2\Upsilon_2\right)\right), \end{aligned}$$

where Υ defined by

$$\Upsilon := \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \begin{pmatrix} Y_1 f + QY_2 f \\ Y_2 f - QY_1 f \end{pmatrix}.$$

Formally letting $\mathrm{p} \to \infty$, we obtain:

$$\overline{\Delta_\infty} f = (Y_1\|\Upsilon\|^2)\Upsilon_1 + (Y_2\|\Upsilon\|^2)\Upsilon_2.$$

5.1.1. *Case I: $L + M + N \neq 0$.*

Formally letting $\mathrm{p} \to \infty$ in equation 3.3, we obtain:

$$f_{\infty,Q}(y_1,y_2) = g(y_1,y_2)^{\frac{1+\xi}{2n+2}} h(y_1,y_2)^{\frac{1-\xi}{2n+2}},$$

where we recall the functions $g(y_1,y_2)$ and $h(y_1,y_2)$ are given by:

$$\begin{aligned} g(y_1,y_2) &= \mu c(y_1-a)^{n+1} + \omega(n+1)(y_2-b) \\ h(y_1,y_2) &= \mu c(y_1-a)^{n+1} - \omega(n+1)(y_2-b). \end{aligned}$$

We then have the following theorem.

Theorem 5.9. *The function* $f_{\infty,Q}$, *as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_\infty} f_{\infty,Q}(\mathbf{y}) = 0, & \mathbf{y} \in \mathbb{G}_n \setminus \{(a,b)\} \\ 0, & \mathbf{y} = (a,b) \end{cases}.$$

Proof. We may prove this theorem by letting $\mathrm{p} \to \infty$ in a prudent multiple of Equation (3.4) and invoking continuity (cf. Corollary 3.1). For completeness, though, we compute formally. We let:

$$A = \frac{1+\xi}{2n+2} \quad \text{and} \quad B = \frac{1-\xi}{2n+2}$$

and compute:

$$\begin{aligned}
Y_1 f &= \mu c(n+1)(y_1-a)^n g^{A-1}h^{B-1}(Ah+Bg)\\
Y_2 f &= \omega c(n+1)(y_1-a)^n g^{A-1}h^{B-1}\left(Ah-Bg\right)\\
Y_1 f + QY_2 f &= \mu c(n+1)(y_1-a)^n g^{A-1}h^{B-1}\left(Ah(1-\xi)+Bg(1+\xi)\right)\\
Y_2 f - QY_1 f &= \omega c(n+1)(y_1-a)^n g^{A-1}h^{B-1}\left(Ah(1-\xi)-Bg(1+\xi)\right)\\
\|\Upsilon\|^2 &= 2\mu^2c^2(n+1)^2(y_1-a)^{2n}g^{A+B-1}h^{A+B-1}\left(A^2(1-\xi)^2+B^2(1+\xi)^2\right).
\end{aligned}$$

We then have:

$$\begin{aligned}
Y_1\|\Upsilon\|^2 &= 2\mu^2c^2(1-\xi^2)^2n(n+1)^2(y_1-a)^{2n-1}(y_2-b)^2(gh)^{\frac{-1-2n}{n+1}}\\
Y_2\|\Upsilon\|^2 &= 2\omega\mu c^3(1-\xi^2)^2n(n+1)(y_1-a)^{3n}(y_2-b)(gh)^{\frac{-1-2n}{n+1}}
\end{aligned}$$

so that:

$$\begin{aligned}
Y_1\|\xi\|^2\xi_1 &= 2\mu^3c^4(1-\xi^2)^3n(n+1)^2(y_1-a)^{4n}(y_2-b)^2(gh)^{\frac{-1-2n}{n+1}}g^{A-1}h^{B-1}\\
Y_2\|\xi\|^2\xi_2 &= -2\mu^3c^4(1-\xi^2)^3n(n+1)^2(y_1-a)^{4n}(y_2-b)^2(gh)^{\frac{-1-2n}{n+1}}g^{A-1}h^{B-1}.
\end{aligned}$$

The theorem follows. □

5.1.2. *Case II: $L+M+N=0$.*

Formally letting $\mathrm{p}\to\infty$ in equation 3.5, we obtain:

$$f_{\infty,Q}(y_1,y_2) = g(y_1,y_2)^{\frac{1+\xi}{2n+2}}h(y_1,y_2)^{\frac{1-\xi}{2n+2}},$$

where we recall the functions $g(y_1,y_2)$ and $h(y_1,y_2)$ are given by:

$$\begin{aligned}
g(y_1,y_2) &= \xi c(y_1-a)^{n+1}+Q(n+1)(y_2-b)\\
h(y_1,y_2) &= \xi c(y_1-a)^{n+1}-Q(n+1)(y_2-b).
\end{aligned}$$

We then have the following theorem.

Theorem 5.10. *The function $f_{\infty,Q}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_\infty} f_{\infty,Q}(\mathbf{y}) = 0, & \mathbf{y}\in\mathbb{G}_n\setminus\{(a,b)\}\\ 0, & \mathbf{y}=(a,b)\end{cases}.$$

Proof. The proof of Theorem 5.10 is similar to that of Theorem 5.9 and omitted. □

5.2. **Generalization of the Drift Term over $\mathbb{H}$.** Recall that the drift p-Laplace equation in the Grushin-type planes $\mathbb{G}_n$ is given by:

$$\mathcal{G}_{\mathrm{p},Q}(f) := \Delta_{\mathrm{p}} f + Q[Y_1,Y_2]\left(\|\nabla_0 f\|^{\mathrm{p}-2}f\right) = 0\,.$$

A routine expansion of the drift term yields the observation

$$\begin{aligned}
\mathcal{G}_{\mathrm{p},Q}(f) &= \Delta_{\mathrm{p}} f + Qcn(y_1-a)^{n-1}\\
&\quad\times\left(\frac{\mathrm{p}-2}{2}\|\nabla_0 f\|^{\mathrm{p}-4}\left(\frac{\partial}{\partial y_2}\|\nabla_0 f\|^2\right)f+\|\nabla_0 f\|^{\mathrm{p}-2}\frac{\partial}{\partial y_2}f\right)\\
&= 0.
\end{aligned}$$

Dividing through by $\frac{\mathrm{p}-2}{2}\|\nabla_0 f\|^{\mathrm{p}-4}$ and formally taking the limit $\mathrm{p}\to\infty$, we obtain:

$$\mathcal{G}_{\infty,Q}(f) = \Delta_\infty f + Q[Y_1,Y_2]\left(\|\nabla_0 f\|^2\right)f.$$

5.2.1. *Case I: $L+M+N\neq 0$.* Considering equation 4.6 and formally letting $\mathrm{p}\to\infty$ yields:

$$f_{\infty,Q}(y_1,y_2)=g(y_1,y_2)^{\frac{1}{2(n+1)}(1-n\xi)}h(y_1,y_2)^{\frac{1}{2(n+1)}(1+n\xi)},$$

where we recall the functions $g(y_1,y_2)$ and $h(y_1,y_2)$ are given by:

$$\begin{aligned} g(y_1,y_2)&=\mu c(y_1-a)^{n+1}+\omega(n+1)(y_2-b)\\ h(y_1,y_2)&=\mu c(y_1-a)^{n+1}-\omega(n+1)(y_2-b). \end{aligned}$$

We have the following theorem.

Theorem 5.11. *The function $f_{\infty,Q}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \mathcal{G}_{\infty,Q}f_{\infty,Q}(\mathbf{y})=0, & \mathbf{y}\in\mathbb{G}_n\setminus\{(a,b)\}\\ 0, & \mathbf{y}=(a,b) \end{cases}.$$

Proof. We may prove this theorem by letting $\mathrm{p}\to\infty$ in Equations (4.7), (4.8), (4.9), (4.10) and invoking continuity (cf. Corollary 4.2). However, for completeness we compute formally. We let:

$$A=\frac{1}{2(n+1)}(1-n\xi)\text{and}\qquad\qquad B=\frac{1}{2(n+1)}(1+n\xi)$$

and, suppressing arguments and subscripts, compute:

$$\begin{aligned} Y_1f&=\mu c(n+1)(y_1-a)^n g^{A-1}h^{B-1}(Ah+Bg)\\ Y_2f&=\omega c(n+1)(y_1-a)^n g^{A-1}h^{B-1}(Ah-Bg)\\ \|\nabla_0 f\|^2&=2\mu^2c^2(n+1)^2(y_1-a)^{2n}g^{A+B-1}h^{A+B-1}\left(A^2+B^2\right)\\ Y_1\|\nabla_0 f\|^2&=4\mu^2c^2(n+1)^2(y_1-a)^{2n-1}g^{A+B-2}h^{A+B-2}(A^2+B^2)\\ &\quad\times\left(ngh+\mu^2c^2(n+1)(y_1-a)^{2n+2}(A+B-1)\right)\\ Y_2\|\nabla_0 f\|^2&=-4\omega^2\mu^2c^3(n+1)^4(y_1-a)^{3n}(y_2-b)\big(A^2+B^2\big)(A+B-1)\\ &\quad\times g^{A+B-2}h^{A+B-2} \end{aligned}$$

so that:

$$\begin{aligned} \Delta_\infty f&=Y_1\|\nabla_0 f\|^2Y_1f+Y_2\|\nabla_0 f\|^2Y_2f\\ &=4\mu^3c^3(n+1)^3(A^2+B^2)(y_1-a)^{3n-1}g^{2A+B-3}h^{A+2B-3}\\ &\times\Big((Ah+Bg)\big(ngh+\mu^2c^2(n+1)(A+B-1)(y_1-a)^{2n+2}\big)\\ &+\omega\mu c(n+1)^2(y_1-a)^{n+1}(y_2-b)(A+B-1)(Ah-Bg)\Big)\\ &=4\xi\omega\mu^3c^3n^2(n+1)^3(y_1-a)^{3n-1}(y_2-b)g^{2A+B-2}h^{A+2B-2}(A^2+B^2). \end{aligned}$$

We also compute:

$$\begin{aligned} Q[Y_1,Y_2]\left(\|\nabla_0 f\|^2\right)f&=Qg^Ah^B\left(cn(y_1-a)^{n-1}\frac{\partial}{\partial y_2}\|\nabla_0 f\|^2\right)\\ &=-4\xi\omega\mu^3c^3n^2(n+1)^3(y_1-a)^{3n-1}(y_2-b)\big(A^2+B^2\big)\\ &\times g^{A+B-2}h^{A+B-2} \end{aligned}$$

The theorem follows. ∎

5.2.2. *Case II: $L + M + N = 0$.* Considering equation 4.11 and formally letting $\mathrm{p} \to \infty$ yields:

$$f_{\infty,Q}(y_1, y_2) = g(y_1, y_2)^{\frac{1}{2(n+1)}(1-n\xi)} h(y_1, y_2)^{\frac{1}{2(n+1)}(1+n\xi)},$$

where we recall the functions $g(y_1, y_2)$ and $h(y_1, y_2)$ are given by:

$$g(y_1, y_2) = \xi c(y_1 - a)^{n+1} + Q(n+1)(y_2 - b)$$
$$h(y_1, y_2) = \xi c(y_1 - a)^{n+1} - Q(n+1)(y_2 - b).$$

We have the following theorem.

Theorem 5.12. *The function $f_{\infty,Q}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \mathcal{G}_{\infty,Q} f_{\infty,Q}(\mathbf{y}) = 0, & \mathbf{y} \in \mathbb{G}_n \setminus \{(a,b)\} \\ 0, & \mathbf{y} = (a,b) \end{cases}.$$

Proof. The proof of Theorem 5.12 is similar to that of Theorem 5.11 and omitted. □

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Research Article

Estimate of the spectral radii of Bessel multipliers and consequences

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ABSTRACT. Bessel multipliers are operators defined from two Bessel sequences of elements of a Hilbert space and a complex sequence, and have frame multipliers as particular cases. In this paper, an estimate of the spectral radius of a Bessel multiplier is provided involving the cross Gram operator of the two sequences. As an upshot, it is possible to individuate some regions of the complex plane, where the spectrum of a multiplier of dual frames is contained.

Keywords: Bessel multipliers, dual frames, spectral radius, spectrum.

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1. INTRODUCTION

Bessel multipliers, as introduced in [2], are operators in a Hilbert space which have been extensively studied [5, 11, 24, 25], occur in various fields of applications [4, 14, 21] and include the class of frame multipliers [9, 10, 12, 19, 22]. Recently, in [10], given a frame multiplier some regions of the complex plane containing the spectrum have been identified. In order to present the main contributions of this paper, which follows the line of [10], we need to give some definitions and preliminary results.

A *Bessel sequence* of a separable Hilbert space $\mathcal{H}$ (with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$) is a sequence $\varphi = \{\varphi_n\}_{n \in \mathbb{N}}$ of elements of $\mathcal{H}$ such that

$$\sum_{n \in \mathbb{N}} |\langle f, \varphi_n \rangle|^2 \leq B_\varphi \|f\|^2, \qquad \forall f \in \mathcal{H}$$

for some $B_\varphi > 0$ (called a *Bessel bound* of φ). A sequence $\varphi = \{\varphi_n\}_{n \in \mathbb{N}}$ is a *frame* for $\mathcal{H}$ if there exist $A_\varphi, B_\varphi > 0$ such that

$$A_\varphi \|f\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f, \varphi_n \rangle|^2 \leq B_\varphi \|f\|^2, \qquad \forall f \in \mathcal{H}. \tag{1.1}$$

Given two Bessel sequences $\varphi = \{\varphi_n\}_{n \in \mathbb{N}}, \psi = \{\psi_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}$ and $m = \{m_n\}_{n \in \mathbb{N}}$ a bounded complex sequence (in short, $m \in \ell^\infty(\mathbb{N})$) it is possible to define a bounded operator $M_{m,\varphi,\psi}$ on $\mathcal{H}$ in the following way

$$M_{m,\varphi,\psi} f = \sum_{n \in \mathbb{N}} m_n \langle f, \psi_n \rangle \varphi_n, \qquad f \in \mathcal{H}.$$

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This operator is said the *Bessel multiplier* of φ, ψ with *symbol* m. It thus consists of three processes: analysis through the sequence ψ, multiplication of the analysis coefficients by m and synthesis processes by φ. When φ and ψ are frames, $M_{m,\varphi,\psi}$ is called a *frame multiplier*.

Since a Bessel multiplier is a bounded operator, its spectrum is contained in some disk and, more precisely, the following bound has been given.

Proposition 1.1 ([10, Proposition 1]). *The spectrum $\sigma(M_{m,\varphi,\psi})$ of any Bessel multiplier $M_{m,\varphi,\psi}$ is contained in the closed disk centered the origin with radius $\sup_n |m_n| B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$, where B_φ and B_ψ are Bessel bounds of φ and ψ, respectively.*

A special case occurs when φ and ψ are *dual frames*, i.e. two frames satisfying the condition[1]

$$f = \sum_{n\in\mathbb{N}} \langle f, \psi_n \rangle \varphi_n, \qquad \forall f \in \mathcal{H}. \tag{1.2}$$

In this setting, it was possible to find more precise regions where the spectra are contained, as stated in the following result.

Proposition 1.2 ([10, Propositions 2 and 3]). *Let φ, ψ be dual frames for $\mathcal{H}$ with upper bounds B_φ, B_ψ, respectively, and let $m \in \ell^\infty(\mathbb{N})$.*

(1) If m is contained in the disk of center μ with radius R, then $\sigma(M_{m,\varphi,\psi})$ is contained in the disk of center μ with radius $RB_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$.
(2) If m is a real sequence, then $\sigma(M_{m,\varphi,\psi})$ is contained in the disk of center

$$\frac{1}{2}(\sup_n m_n + \inf_n m_m)$$

with radius

$$\frac{1}{2}(\sup_n m_n - \inf_n m_m) B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}.$$

(3) If ψ is the canonical dual[2] of φ, then $\sigma(M_{m,\varphi,\psi})$ is contained in the closed convex hull of m.

One of the two main results of this paper, which is right below, gives an estimate of the spectral radius of a Bessel multiplier in terms of the cross Gram operator $G_{\varphi,\psi}$ [3] of φ and ψ which is recalled in Section 2. A direct consequence, contained in the statement, is an improvement of Proposition 1.1.

Theorem 1.1. *Let φ, ψ be Bessel sequences of $\mathcal{H}$ with cross Gram operator $G_{\varphi,\psi}$ and let $m \in \ell^\infty(\mathbb{N})$. Let M_m be the multiplication operator by m on $\ell^2(\mathbb{N})$. Then, $M_{m,\varphi,\psi}$ and $M_m G_{\varphi,\psi}$ have the same spectral radius*

$$r(M_{m,\varphi,\psi}) = r(M_m G_{\varphi,\psi}). \tag{1.3}$$

In particular, the following bound holds

$$r(M_{m,\varphi,\psi}) \le \sup_n |m_n| \, \|G_{\varphi,\psi}\|. \tag{1.4}$$

Therefore, the spectrum of any Bessel multiplier $M_{m,\varphi,\psi}$ is contained in the closed disk centered the origin with radius $\sup_n |m_n| \, \|G_{\varphi,\psi}\|$.

[1] or, equivalently, the condition $f = \sum_{n\in\mathbb{N}} \langle f, \varphi_n \rangle \psi_n$ for every $f \in \mathcal{H}$.

[2] among all the dual frames of φ there is a special one called the canonical dual; the definition is given in Section 2.

The second main result, i.e. Theorem 3.2, concerns dual frames and is the counterpart of Proposition 1.2 in which the cross Gram operator is involved again. Both in Theorems 1.1 and 3.2, the constant $B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$ appearing in Propositions 1.1 and 1.2 is substituted by the norm $\|G_{\varphi,\psi}\|$ of $G_{\varphi,\psi}$. Since the inequality $\|G_{\varphi,\psi}\| \leq B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$ always holds, Theorems 1.1 and 3.2 improve in fact Propositions 1.1 and 1.2. In connections to the main results, throughout the paper we will discuss some remarks and examples.

2. Preliminaries

We denote by $\ell^2(\mathbb{N})$ (respectively, $\ell^\infty(\mathbb{N})$) the usual spaces of square summable (respectively, bounded) complex sequences indexed by $\mathbb{N}$.
Given two Bessel sequences φ and ψ of $\mathcal{H}$ the following operators can be defined (see [3, 6]):

- $C_\varphi : \mathcal{H} \to \ell^2(\mathbb{N})$, defined by $C_\varphi f = \{\langle f, \varphi_n \rangle\}$, is the *analysis operator* of φ.
- $D_\varphi : \ell^2(\mathbb{N}) \to \mathcal{H}$, defined by $D_\varphi \{c_n\} = \sum_{n\in\mathbb{N}} c_n \varphi_n$, is the *synthesis operator* of φ.
- $S_\varphi : \mathcal{H} \to \mathcal{H}$, $S_\varphi = D_\varphi C_\varphi$ is called the *frame operator* of φ; the action of S_φ is

$$S_\varphi f = \sum_{n\in\mathbb{N}} \langle f, \varphi_n \rangle \varphi_n, \qquad f \in \mathcal{H}.$$

- $G_{\varphi,\psi} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $G_{\varphi,\psi} = C_\psi D_\varphi$, is the *cross Gram operator* of φ and ψ which acts as $G_{\varphi,\psi}\{c_n\} = \{d_k\}$, where $d_k = \sum_{n\in\mathbb{N}} c_n \langle \varphi_n, \psi_k \rangle$. In other words, $G_{\varphi,\psi}$ can be associated to the matrix $(\langle \varphi_n, \psi_k \rangle)_{n,k\in\mathbb{N}}$.

Moreover, C_φ and D_φ are one the adjoint of the other one, $C_\varphi = D_\varphi^*$, and $\|C_\varphi\| = \|D_\varphi\| \leq B_\varphi^{\frac{1}{2}}$, where B_φ is a Bessel bound of φ. Consequently, S_φ is a positive self-adjoint operator and it is also invertible with bounded inverse S_φ^{-1} on $\mathcal{H}$. We recall that in the introduction we gave the definition of dual frames. A frame φ always has a dual frame, namely the sequence $\{S_\varphi^{-1}\varphi_n\}_{n\in\mathbb{N}}$, which is the so-called *canonical dual* of φ.

Finally, we note that, introducing the operators D_φ and C_ψ, it is possible to write $M_{m,\varphi,\psi} = D_\varphi M_m C_\psi$, where M_m is the multiplication operator by m on $\ell^2(\mathbb{N})$, defined by $M_m\{c_n\} = \{m_n c_n\}$ for $\{c_n\} \in \ell^2(\mathbb{N})$.

3. Proofs of the main results

Theorem 1.1 concerns the spectral radius of a Bessel multiplier. For a bounded operator $T : \mathcal{H} \to \mathcal{H}$, we write $\sigma(T)$ for the spectrum and $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$ for the *spectral radius* (see, for instance, [7, 20, 23]). The spectral radius represents then the radius of the smallest disk centered in the origin and containing the spectrum. Propositions 1.1 and 1.2 can be restated in terms of spectral radius. For example, we can say that for any Bessel multiplier $M_{m,\varphi,\psi}$, we have $r(M_{m,\varphi,\psi}) \leq \sup_n |m_n| B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}$.

For the proof of Theorem 1.1 below, we are going to use some classical results about the spectral radius (see e.g. [7, Proposition 3.8]): for every bounded operator $T : \mathcal{H} \to \mathcal{H}$, we have

$$r(T) = \lim_{N\to+\infty} \|T^N\|^{\frac{1}{N}} \tag{3.5}$$

and

$$r(T) \leq \|T\|. \tag{3.6}$$

Proof of Theorem 1.1. If $B_\varphi = 0$, $B_\psi = 0$ or $m \equiv 0$, then (1.4) trivially holds, because both the operators $M_{m,\varphi,\psi}$ and $M_m G_{\varphi,\psi}$ are null. So we can assume that $B_\varphi, B_\psi > 0$ and that m is not

identically null.
Since $M_{m,\varphi,\psi} = D_\varphi M_m C_\psi$ and $G_{\varphi,\psi} = C_\psi D_\varphi$, then for $N \geq 2$ we have

$$M^N_{m,\varphi,\psi} = (D_\varphi M_m C_\psi)^N = D_\varphi(M_m C_\psi D_\varphi)^{N-1} M_m C_\psi = D_\varphi(M_m G_{\varphi,\psi})^{N-1} M_m C_\psi.$$

Therefore,

$$\|M^N_{m,\varphi,\psi}\| \leq \|(M_m G_{\varphi,\psi})^{N-1}\| \|M_m\| \|C_\psi\| \|D_\varphi\|.$$

Thus, by (3.5),

$$\begin{aligned} r(M_{m,\varphi,\psi}) &= \lim_{N\to+\infty} \|M^N_{m,\varphi,\psi}\|^{\frac{1}{N}} \leq \lim_{N\to+\infty} (\|(M_m G_{\varphi,\psi})^{N-1}\| \|M_m\| \|C_\psi\| \|D_\varphi\|)^{\frac{1}{N}} \\ &= \lim_{N\to+\infty} \|(M_m G_{\varphi,\psi})^{N-1}\|^{\frac{1}{N}} \lim_{N\to+\infty} (\|M_m\| \|C_\psi\| \|D_\varphi\|)^{\frac{1}{N}} = r(M_m G_{\varphi,\psi}). \end{aligned}$$

For the reverse inequality, we observe that

$$(M_m G_{\varphi,\psi})^{N+1} = M_m C_\psi D_\varphi (M_m G_{\varphi,\psi})^{N-1} M_m C_\psi D_\varphi = M_m C_\psi M^N_{m,\varphi,\psi} D_\varphi.$$

Hence, with an analog calculation as before, we find that $r(M_m G_{\varphi,\psi}) \leq r(M_{m,\varphi,\psi})$, so in conclusion (1.3) is proved. Lastly, (1.4) holds because by (1.3) and (3.6), we have

$$r(M_{m,\varphi,\psi}) = r(M_m G_{\varphi,\psi}) \leq \|M_m G_{\varphi,\psi}\| \leq \|M_m\| \|G_{\varphi,\psi}\| = \sup_n |m_n| \, \|G_{\varphi,\psi}\|. \qquad \square$$

Remark 3.1. *(i) Inequality (1.4) may be strict. In fact, let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$, $\varphi = \{e_n\}$, $\psi = \{\frac{1}{2}e_1, 2e_2, \frac{1}{2}e_3, 2e_4, \dots\}$ and $m = \{2, \frac{1}{2}, 2, \frac{1}{2}, \dots\}$. A trivial calculation shows that $M_{m,\varphi,\psi}$ is the identity operator, so $r(M_{m,\varphi,\psi}) = 1$, while $\sup_n |m_n| \, \|G_{\varphi,\psi}\| = 4$.*

(ii) A Riesz basis φ for $\mathcal{H}$ is the image of an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ through an bounded operator with bounded inverse defined on $\mathcal{H}$ [6]. A Riesz basis φ is, in particular, a frame for $\mathcal{H}$ and it has a unique dual ψ (the canonical one) which is a Riesz basis too. Moreover,

$$\langle \varphi_n, \psi_k \rangle = \delta_{n,k} = \begin{cases} 1 & n = k \\ 0 & n \neq k. \end{cases}$$

Therefore, if φ is a Riesz basis for $\mathcal{H}$ and ψ is its canonical dual, then $G_{\varphi,\psi}$ is the identity operator on $\ell^2(\mathbb{N})$ and so $\|G_{\varphi,\psi}\| = 1$. Anyway, for this choice of φ, ψ, (1.4) is an immediate consequence of the fact that $\sigma(M_{m,\varphi,\psi})$ is the closure of $\{m_n : n \in \mathbb{N}\}$ (see [9, Proposition 4]).

(iii) Since $G_{\varphi,\psi} = C_\psi D_\varphi$, we always have

$$\|G_{\varphi,\psi}\| \leq \|C_\psi\| \|D_\varphi\| \leq B_\varphi^{\frac{1}{2}} B_\psi^{\frac{1}{2}}. \tag{3.7}$$

Therefore, Theorem 1.1 is finer than Proposition 1.1. Moreover, if $\varphi = \psi$, then $G_{\varphi,\varphi} = C_\varphi D_\varphi = D^*_\varphi D_\varphi$ is a positive self-adjoint operator, so $\|G_{\varphi,\varphi}\| = \|D_\varphi\|^2$.

Besides with (3.7) it is possible to estimate the norm of $G_{\varphi,\psi}$ with some other considerations which we discuss below.

Remark 3.2. *(i) An estimate of $\|G_{\varphi,\psi}\|$ can be given if*

$$\sup_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} |\langle \varphi_n, \psi_k \rangle| \leq \Gamma_1 \quad \text{and} \quad \sup_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} |\langle \varphi_n, \psi_k \rangle| \leq \Gamma_2. \tag{3.8}$$

Indeed, by Schur test (see for instance [15, Lemma 6.2.1]), we have $\|G_{\varphi,\psi}\| \leq \Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}}$.

(ii) Let φ, ψ be Bessel sequences of $\mathcal{H}$ such that $\langle \varphi_n, \psi_k \rangle = 0$ for $n, k \in \mathbb{N}$ with $|n - k| > d$. As particular case of the previous remark, if $\sum_{i=-d}^{d} \sup_n |\langle \varphi_n, \psi_{n+i} \rangle| \leq \Gamma$ (where, with an abuse of notation, we mean $\psi_{-d+1}, \dots, \psi_{-1}, \psi_0 = 0$), then $\|G_{\varphi,\psi}\| \leq \Gamma$.

(iii) Another use of conditions (3.8) can be made in the context of localized frames [1, 8, 16, 17].

In what follows, we give another example where in particular it is possible to exactly calculate the norm of the cross Gram operator.

Example 3.1. *Let $\mathcal{G}$ be a countable locally compact abelian group equipped with the discrete topology. We write the group operation of $\mathcal{G}$ in the additive notation and we denote by $\widehat{\mathcal{G}}$ the dual group of $\mathcal{G}$ (i.e. the multiplicative group of the characters on $\mathcal{G}$). Since $\mathcal{G}$ is discrete, then $\widehat{\mathcal{G}}$ is compact (see* [13, Proposition 4.4]*). Moreover, we will choose the Haar measure on $\mathcal{G}$ to be the counting measure; hence by* [13, Proposition 4.24], $|\widehat{\mathcal{G}}| = 1$.

Let τ be a unitary representation of $\mathcal{G}$ on $\mathcal{H}$. In particular, let us assume that τ is dual integrable [18]*, i.e. there exist a Haar measure $d\xi$ and a function $[\cdot,\cdot] : \mathcal{H}\times\mathcal{H} \to L^1(\widehat{\mathcal{G}}, d\xi)$ such that*

$$\langle \chi, \tau_g \eta\rangle = \int_{\widehat{\mathcal{G}}} [\chi,\eta](\xi) e_{-g}(\xi) d\xi, \qquad \forall g \in \mathcal{G}, \chi,\eta \in \mathcal{H}, \tag{3.9}$$

where $e_{-g}(x)$ is the character induced by $-g$, namely $e_{-g}(\xi) = e^{-2\pi i (g,\xi)}$, and $(\cdot,\cdot)$ is the duality between $\mathcal{G}$ and $\widehat{\mathcal{G}}$. The function $[\cdot,\cdot]$ is called the bracket function. Classical examples (treated for instance in [6, 15]*) of this framework are*

- $\mathcal{G} = \mathbb{Z}^d$, $\widehat{\mathcal{G}} = \mathbb{T}^n$, $\mathcal{H} = L^2(\mathbb{R})$, $(\tau_k f)(x) = (T_k f)(x) = f(x-k)$ *for* $k \in \mathbb{Z}^d$ *and*
$$[\chi,\eta](\xi) = \sum_{k\in\mathbb{Z}^d} \hat{\chi}(\xi+k)\overline{\hat{\eta}(\xi+k)}, \qquad \xi \in \mathbb{R}^d, \chi,\eta \in L^2(\mathbb{R}^d),$$
where $\hat{\chi}$ and $\hat{\eta}$ are the Fourier transforms of χ and η, respectively;
- $\mathcal{G} = \mathbb{Z}^d \times \mathbb{Z}^d$, $\widehat{\mathcal{G}} = \mathbb{T}^n$, $\mathcal{H} = L^2(\mathbb{R})$, $(\tau_{(k,l)} f)(x) = (T_k M_l f)(x) = e^{2\pi i l\cdot x} f(x-k)$ *for* $(k,l) \in \mathbb{Z}^d \times \mathbb{Z}^d$ *and*
$$[\chi,\eta](x,\xi) = Z\chi(x,\xi)\overline{Z\eta(x,\xi)}, \qquad x,\xi \in \mathbb{R}^d, \chi,\eta \in L^2(\mathbb{R}^d),$$
where $Z\chi(x,\xi) = \sum_{k\in\mathbb{Z}^d} e^{-2\pi i k\cdot\xi}\chi(x-k)$ is the Zak transform of $\chi \in L^2(\mathbb{R}^d)$.

After introducing this setting, we now consider two special sequences[3]*. More precisely, let $\chi,\eta \in \mathcal{H}$ be such that $\varphi = \{T_g\chi\}_{g\in\mathcal{G}}$ and $\psi = \{T_g\eta\}_{g\in\mathcal{G}}$ are Bessel sequences*[4] *of $\mathcal{H}$. As we are going to see, the norm $\|G_{\varphi,\psi}\|$ can be exactly calculated in terms of $[\chi,\eta]$. Indeed, for any complex sequences $\{c_g\}_{g\in\mathcal{G}}, \{d_g\}_{g\in\mathcal{G}} \in \ell^2(\mathcal{G})$, we have*

$$\begin{aligned}
\langle G_{\varphi,\psi}\{c_g\},\{d_g\}\rangle &= \langle C_\psi D_\varphi\{c_g\},\{d_g\}\rangle = \langle D_\varphi\{c_g\}, D_\psi\{d_g\}\rangle \\
&= \left\langle \sum_{g\in\mathcal{G}} c_g T_g\chi, \sum_{g\in\mathcal{G}} d_g T_g\eta \right\rangle = \sum_{g,h\in\mathcal{G}} c_g\overline{d_h}\langle T_g\chi, T_h\eta\rangle \\
&= \sum_{g,h\in\mathcal{G}} c_g\overline{d_h}\langle \chi, T_{h-g}\eta\rangle = \sum_{g,h\in\mathcal{G}} c_g\overline{d_h}\int_{\widehat{\mathcal{G}}}[\chi,\eta](\xi)e_{g-h}(\xi)d\xi \\
&= \int_{\widehat{\mathcal{G}}}[\chi,\eta](\xi)\sum_{g,h\in\mathcal{G}} c_g\overline{d_h}e_{g-h}(\xi)d\xi \\
&= \int_{\widehat{\mathcal{G}}}[\chi,\eta](\xi)\sum_{g\in\mathcal{G}} c_g e_g(\xi)\overline{\sum_{h\in\mathcal{G}} d_h e_h(\xi)}d\xi.
\end{aligned} \tag{3.10}$$

[3] In this example, the sequences are indexed by the countable set $\mathcal{G}$ in contrast to the setting of the rest of the paper. However, this does not change the validity of Theorems 1.1 and 3.2 since the series defining a multiplier is unconditionally convergent so the ordering of a Bessel sequence is not relevant (see [6, 15]).

[4] This happen if and only if $[\chi,\chi]$ and $[\eta,\eta]$ are bounded above a.e. in $\widehat{\mathcal{G}}$, see [18, Section 5].

By the Pontrjagin duality theorem and by [13, Corollary 4.26], $\{e_g\}_{g\in\mathcal{G}}$ *is an orthonormal basis of* $L^2(\widehat{\mathcal{G}}, d\xi)$. *This fact, together with* (3.10)*, implies that the Gram operator* $G_{\varphi,\psi}$ *can be reduced to the multiplication operator by* $[\chi,\eta]$ *on* $L^2(\widehat{\mathcal{G}}, d\xi)$. *Hence, we conclude that*

$$\|G_{\varphi,\psi}\| = \sup_{\{c_g\},\{d_g\}\neq 0} \frac{|\langle G_{\varphi,\psi}\{c_g\},\{d_g\}\rangle|}{\|\{c_g\}\|\|\{d_g\}\|} = \sup_{\xi\in\widehat{\mathcal{G}}} |[\chi,\eta](\xi)|,$$

i.e. the essential supremum of $[\chi,\eta]$ *(see* [20, Example 2.11 - Ch. III]*). Thus, by Theorem* 1.1*, given a bounded complex sequence* $m = \{m_g\}_{g\in\mathcal{G}}$ *we have*

$$r(M_{m,\varphi,\psi}) \leq \sup_{g\in\mathcal{G}} |m_g| \sup_{\xi\in\widehat{\mathcal{G}}} |[\chi,\eta](\xi)|.$$

We now move to prove the result for dual frames. In particular, it provides regions containing the spectrum which are smaller than the disk of Theorem 1.1.

Theorem 3.2. *Let* φ,ψ *be dual frames for* $\mathcal{H}$ *and let* $m\in\ell^\infty(\mathbb{N})$.

(1) *If* m *is contained in the disk of center* μ *with radius* R*, then* $\sigma(M_{m,\varphi,\psi})$ *is contained in the disk of center* μ *with radius* $R\|G_{\varphi,\psi}\|$.
(2) *If* m *is real, then* $\sigma(M_{m,\varphi,\psi})$ *is contained in the disk of center*
$$\frac{1}{2}(\sup_n m_n + \inf_n m_m)$$
with radius
$$\frac{1}{2}(\sup_n m_n - \inf_n m_m)\|G_{\varphi,\psi}\|.$$
(3) *If* ψ *is the canonical dual of* φ*, then* $\sigma(M_{m,\varphi,\psi})$ *is contained in the closed convex hull of* m.

Proof. To prove statement (i), let us consider a disk of center μ with radius R containing the sequence m. By (1.2), we have

$$M_{m,\varphi,\psi} - \mu I = \sum_{n\in\mathbb{N}} (m_n - \mu)\langle f,\psi_n\rangle\varphi_n = M_{m-\mu,\varphi,\psi},$$

where with $m-\mu$ we mean the sequence $\{m_n - \mu\}$. Therefore applying (1.4) to $M_{m-\mu,\varphi,\psi}$, we obtain

$$r(M_{m,\varphi,\psi} - \mu I) \leq \sup_n |m_n - \mu|\,\|G_{\varphi,\psi}\| \leq R\|G_{\varphi,\psi}\|,$$

which means that $\sigma(M_{m,\varphi,\psi})$ is contained in the disk of center μ with radius $R\|G_{\varphi,\psi}\|$, because $\sigma(M_{m,\varphi,\psi}) = \{\lambda+\mu : \lambda\in\sigma(M_{m-\mu,\varphi,\psi})\}$.

Statement (ii) is a consequence of (i) taking $\mu = \frac{1}{2}(\sup_n m_n + \inf_n m_m)$ and $R = \frac{1}{2}(\sup_n m_n - \inf_n m_m)$. Finally, statement (iii) was proved in [10, Proposition 2]. □

By (3.7), we can make a similar observation of Remark 3.1(iii), that is Theorem 3.2 is stronger than Proposition 1.2. We conclude with a comment for the case of a frame and its canonical dual.

Remark 3.3. *Let* φ *and* ψ *be dual frames. Making use of inequality* (1.4) *(taking* $m_n = 1$ *for every* $n\in\mathbb{N}$*), we find that* $\|G_{\varphi,\psi}\| \geq 1$.
If, in particular, ψ *is the canonical dual of* φ*, then*

$$\langle\varphi_n,\psi_k\rangle = \langle\varphi_n, S_\varphi^{-1}\varphi_k\rangle = \langle S_\varphi^{-\frac{1}{2}}\varphi_n, S_\varphi^{-\frac{1}{2}}\varphi_k\rangle.$$

In other words, $G_{\varphi,\psi}$ *is equals to the Gram operator of the canonical tight frame* $\chi := S_\varphi^{-\frac{1}{2}}\varphi$ *of* φ*, which is a Parseval frame (i.e. it satisfies condition* (1.1) *with* $A_\chi = B_\chi = 1$*, see* [6, Theorem 6.1.1]*). Thus,* *for the initial observation and for Remark 3.1 (iii), if* φ *is a frame,* ψ *is its canonical dual and* $\chi = S_\varphi^{-\frac{1}{2}}\varphi$*, then we have* $1 \leq \|G_{\varphi,\psi}\| = \|D_\chi\|^2 \leq B_\chi = 1$*, so* $\|G_{\varphi,\psi}\| = 1$.

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Research Article

Some recent and new fixed point results on orthogonal metric-like space

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ABSTRACT. In this paper, we give some recent and new fixed point results for some contraction mappings on $O-$complete metric-like space and also we give illustrative examples. At the end, we give an application to show the existence of a solution of a differential equation.

Keywords: Fixed point, orthogonal metric-like space, Geraghty contraction.

2020 Mathematics Subject Classification: 54H25, 47H10.

1. INTRODUCTION AND PRELIMINARIES

The Banach contraction principle [7] is one of the fundamental foundations of the metrical fixed theory. Many authors have come up with generalizations, extensions and applications of this principle. They have studied many aspects of the Banach contraction principle and have further developed their findings. One of the most popular topics is studying new classes of spaces and their fundamental properties (see [9, 10, 11, 18, 20, 27]).

Amini-Harandi [6] introduced the concept of metric-like space and gave some fixed point theorems on complete metric-like space. Later, some authors worked on fixed point theorems for various new types of contraction conditions on the metric-like space. For more details, we refer to [1, 5, 16, 21].

We start by recalling some definitions and lemmas about metric-like space.

Definition 1.1 ([6]). *Let X be any non-empty set. A function $\rho : X \times X \to [0, \infty)$ is said to be a metric-like on X if for any $a, b, c \in X$ the following conditions are satisfied:*

(σ_1) $\rho(a, b) = 0 \Rightarrow a = b$,
(σ_2) $\rho(a, b) = \rho(b, a)$,
(σ_3) $\rho(a, b) \leq \rho(a, c) + \rho(c, b)$.

The pair (X, ρ) is called a metric-like space. Each metric-like ρ on X generates a T_0 topology τ_p on X which has as a base the family open ρ-balls

$$\{B_\rho(a, \varepsilon) : a \in X, \varepsilon > 0\},$$

where

$$B_\rho(a, \varepsilon) = \{b \in X : |\rho(a, b) - \rho(a, a)| < \varepsilon\}$$

for all $a \in X$ and $\varepsilon > 0$.

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Definition 1.2 ([6]). *(i) A sequence $\{a_n\}$ in a metric-like space (X, ρ) converges to a point $a \in X$ if and only if $\rho(a, a) = \lim_{n\to\infty} \rho(a, a_n)$.*

(ii) A sequence $\{a_n\}$ in a metric-like space (X, ρ) is called a Cauchy sequence if $\lim_{n,m\to\infty} \rho(a_n, a_m)$ exists (and is finite) .

(iii) A metric-like space (X, ρ) is said to be complete if every Cauchy sequence $\{a_n\}$ in X converges, with respect to τ_p, to a point $a \in X$ such that

$$\lim_{n\to\infty} \rho(a, a_n) = \rho(a, a) = \lim_{n,m\to\infty} \rho(a_n, a_m).$$

Lemma 1.1 ([16]). *Let (X, ρ) be a metric-like space. Let $\{a_n\}$ be a sequence in X such that $a_n \to a$, where $a \in X$ and $\rho(a, a) = 0$. Then for all $b \in X$, we have*

$$\lim_{n\to\infty} \rho(a_n, b) = \rho(a, b).$$

Recently, Gordji et al. [12] extended the literature on metric spaces by introducing the notion of orthogonality. In [12], they proved the Banach fixed point theorem using the orthogonality such as:

Theorem 1.1 ([12]). *Let $(X, \rho, \perp)$ be an O-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp$-continuous, $\perp$-contraction (with Lipschitz constant λ) and $\perp$-preserving. Then f has a unique fixed point x^* in X and is a Picard operator, that is, $\lim f^n(x) = x^*$ for all $x \in X$.*

Also, they show that this theorem is a real extension of Banach's contraction principle.

Corollary 1.1 ([12]). *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that, for some $\lambda \in (0, 1]$,*

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. Then f has a unique fixed point in X.

There are several applications of this new idea of orthogonal sets and also numerous forms of orthogonality. We refer the reader to ([2, 3, 4, 8, 13, 14, 19, 22, 23, 24, 25]) for more details.

In this paper, we give some recent and new results for some contraction mappings on $O-$complete metric-like space and also we give illustrative examples. At the end, we give an application to show the existence of a solution of a differential equation. In order to do this, we will first recall some basic definitions and notations of the orthogonality.

Definition 1.3 ([12]). *Let X be a non-empty set and $\perp$ be a binary relation defined on X. If binary relation $\perp$ fulfills the following criteria:*

$$\exists a_0 (\forall b \in X, b \perp a_0) \text{ or } (\forall b \in X, a_0 \perp b),$$

then pair $(X, \perp)$ known as an orthogonal set. The element a_0 is called an orthogonal element. We denote this O-set or orthogonal set by $(X, \perp)$.

Definition 1.4 ([12]). *Let $(X, \perp)$ be an orthogonal set (O-set). Any two elements $a, b \in X$ such that $a \perp b$, then $a, b \in X$ are said to be orthogonally related.*

Definition 1.5 ([12]). *A sequence $\{a_n\}$ is called an orthogonal sequence (briefly O-sequence) if*

$$(\forall n \in \mathbb{N}, a_n \perp a_{n+1}) \text{ or } (\forall n \in \mathbb{N}, a_{n+1} \perp a_n).$$

Similarly, a Cauchy sequence $\{a_n\}$ is said to be a orthogonally Cauchy sequence (briefly Cauchy O-sequence) if

$$(\forall n \in \mathbb{N}, a_n \perp a_{n+1}) \text{ or } (\forall n \in \mathbb{N}, a_{n+1} \perp a_n).$$

Definition 1.6 ([12]). *Let $(X, \perp)$ be an orthogonal set and ρ be a metric on X. Then $(X, \perp, \rho)$ is called an orthogonal metric space (shortly O-metric space).*

Definition 1.7 ([12]). *Let $(X, \perp, \rho)$ be an orthogonal metric space. Then X is said to be a O-complete if every Cauchy O-sequence is converges in X.*

Definition 1.8 ([12]). *Let $(X, \perp, \rho)$ be an orthogonal metric space. A function $f : X \to X$ is said to be orthogonally continuous ($\perp$-continuous) at a if for each O-sequence $\{a_n\}$ converging to a implies $f(a_n) \to f(a)$ as $n \to \infty$. Also f is $\perp$-continuous on X if f is $\perp$-continuous at every $a \in X$.*

Definition 1.9 ([12]). *Let a pair $(X, \perp)$ be an O-set, where $X(\neq \emptyset)$ be a non-empty set and $\perp$ be a binary relation on set X. A mapping $f : X \to X$ is said to be $\perp$-preserving if $f(a) \perp f(b)$ whenever $a \perp b$ and weakly $\perp$-preserving if $f(a) \perp f(b)$ or $f(b) \perp f(a)$ whenever $a \perp b$.*

Definition 1.10 ([23]). *We say that an O-set is a transitive orthogonal set if $\perp$ is transitive.*

Definition 1.11 ([23]). *Let $(X, \perp)$ be an $O-$set. A path of length k in $\perp$ from a to b is a finite sequence $\{a_0, a_1, ..., a_k\} \subset X$ such that*

$$a_0 = a, a_k = b, a_i \perp a_{i+1} \text{ or } a_{i+1} \perp a_i$$

for all $i = 0, 1, ..., k-1$ and also $\lambda(a, b, \perp)$ be denoted as all path of length k in $\perp$ from a to b .

Before giving our main result, we want to remind some information about Geraghty contraction and also $(\hbar, \phi)$-contraction.

Let Λ be the family of all functions $\eta : [0, \infty) \to [0, 1)$ that satisfy the condition $\lim_{n\to\infty} \eta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

Furthermore, Ψ denotes the class of functions $\varpi : [0, \infty) \to [0, \infty)$ that satisfy the following conditions:

- ϖ is nondecreasing,
- ϖ is continuous,
- $\varpi(t) = 0$ if and ony if $t = 0$.

In [17], the authors proved the following particular result .

Theorem 1.2. *Let (X, ρ) be a complete metric-like space and $f : X \to X$ be a mapping. Suppose that there exists $\eta \in \Lambda$ such that*

$$\rho(fa, fb) \le \eta((\rho(a, b))\rho(a, b)$$

for all $a, b \in X$. Then f has a unique fixed point $u \in X$ with $\rho(u, u) = 0$.

Recently, in [1], the authors considered a new type of Geraghty contractions in the class of metric-like spaces and proved a fixed point theorem for this type of contractive mapping such that:

Theorem 1.3. *Let (X, ρ) be a complete metric-like space and $f : X \to X$ be a mapping. Suppose that there exists $\eta \in \Lambda$ such that*

$$\rho(fa, fb) \le \eta((F(a, b))F(a, b) \tag{1.1}$$

for all $a, b \in X$, where

$$F(a, b) = \rho(a, b) + |\rho(a, fa) - \rho(b, fb)| \,.$$

Then f has a unique fixed point $u \in X$ with $\rho(u, u) = 0$.

On the other hand, in [15], Jleli et al. introduced a family $\mathcal{H}$ of functions $\hbar : [0, +\infty[^3 \to [0, +\infty[$ satisfying the following conditions:

(H_1) $\max\{a, b\} \le \hbar(a, b, c)$ for all $a, b, c \in [0, +\infty[$;

(H_2) $\hbar(0,0,0) = 0$;
(H_3) $\hbar$ is continuous.

Some examples of functions belonging to $\mathcal{H}$ are given as follows:

(i) $\hbar(a,b,c) = a+b+c$ for all $a,b,c \in [0,+\infty[$;
(ii) $\hbar(a,b,c) = \max\{a,b\}+c$ for all $a,b,c \in [0,+\infty[$;
(iii) $\hbar(a,b,c) = a+b+ab+c$ for all $a,b,c \in [0,+\infty[$.

Using a function $\hbar \in \mathcal{H}$, the authors of [15] introduced the following notion of $(\hbar,\phi)$-contraction as:

Definition 1.12 ([15]). *Let (M,ρ) be a metric space, $\phi : M \to [0,+\infty[$ be a given function and $\hbar \in \mathcal{H}$. Then, $f : M \to M$ is called a $(\hbar,\phi)$-contraction with respect to the metric ρ if and only if*

$$\hbar(\rho(fa,fb),\phi(fa),\phi(fb)) \leq k\hbar(\rho(a,b),\phi(a),\phi(b)) \quad \text{for all } a,b \in M,$$

for some constant $k \in]0,1[$.

Now, we set

$$Z_\phi := \{a \in M : \phi(a) = 0\},$$
$$F f := \{a \in M : fa = a\}.$$

Furthermore, we say that f is a ϕ-Picard operator if and only if the following condition holds:

$$F_f \cap Z_\phi = \{\varsigma\} \text{ and } f^n a \to \varsigma, \text{ as } n \to +\infty \text{ for each } a \in M.$$

Theorem 1.4 ([15]). *Let (M,ρ) be a C.M.S, $\phi : M \to [0,+\infty[$ be a given function and $\hbar \in \mathcal{H}$. Suppose that the following conditions hold:*

(A_1) *ϕ is lower semi-continuous (l.s.c.);*
(A_2) *$f : M \to M$ is a $(\hbar,\phi)$-contraction with respect to the metric ρ.*

Then

(i) *$F_f \subset Z_\phi$;*
(ii) *f is a ϕ-Picard operator;*
(iii) *for all $a \in M$ and for all $n \in \mathbb{N}$, we have*

$$\rho(f^n a,\varsigma) \leq \frac{k^n}{1-k}\hbar(\rho(fa,a),\phi(fa),\phi(a)),$$

where $\{\varsigma\} = F_f \cap Z_\phi = F_f$.

2. Main Results

2.1. **A result for orthogonal ϖ_F−Geraghty contraction.** In this section, we give a definition of orthogonal ϖ_F−Geraghty contraction and we aim to obtain some results on O-complete metric-like space $(X,\perp,\rho)$.

Definition 2.13. *Let $(X,\perp,\rho)$ be an orthogonal metric-like space and $f : X \to X$ is a mapping. Then we say that f is orthogonal ϖ_F−Geraghty contraction if there exist $\varpi \in \Psi$ and $\eta \in \Lambda$ such that*

$$\varpi(\rho(fa,fb)) \leq \eta(\varpi(F(a,b)))\varpi(F(a,b)) \tag{2.2}$$

for all $a,b \in X$ with $a \perp b$, where

$$F(a,b) = \rho(a,b) + |\rho(a,fa) - \rho(b,fb)|.$$

Theorem 2.5. *Let $(X,\perp,\rho)$ be an O-complete metric-like space, a_0 is an orthogonal element and f is a self mapping on X satisfying the following conditions:*

(i) *$(X,\perp)$ is a transitive orthogonal set,*

(*ii*) f *is* $\perp$*-preserving,*
(*iii*) f *is orthogonal* ϖ_F *−Geraghty contraction,*
(*iv*) f *is* $\perp$*-continuous.*

Then, f *has an unique fixed point in* X*.*

Proof. From the definition of the orthogonality, it follows that $a_0 \perp f(a_0)$ or $f(a_0) \perp a_0$. Let

$$a_1 := fa_0, a_2 := fa_1 = f^2 a_0, \cdots, a_n := fa_{n-1} = f^n a_0$$

for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\rho(a_n, a_{n+1}) = 0$ for some n_0, so the proof is completed. Consequently, we assume that

$$\rho(a_n, a_{n+1}) \neq 0$$

for all n. Since f is $\perp$ −preserving, we have

$$a_n \perp a_{n+1} \text{ or } a_{n+1} \perp a_n.$$

This implies that $\{a_n\}$ is an O-sequence. Since f is orthogonal ϖ_F −Geraghty contraction, we have

$$\begin{aligned} \varpi(\rho(a_n, a_{n+1})) &= \varpi(\rho(fa_{n-1}, fa_n)) \\ &\leq \eta(\varpi(F(a_{n-1}, a_n)))\varpi(F(a_{n-1}, a_n)), \quad n \geq 1, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} F(a_{n-1}, a_n) &= \rho(a_{n-1}, a_n) + |\rho(a_{n-1}, fa_{n-1}) - \rho(a_n, fa_n)| \\ &= \rho(a_{n-1}, a_n) + |\rho(a_{n-1}, a_n) - \rho(a_n, a_{n+1})|\,. \end{aligned}$$

Take $\rho_n = \rho(a_{n-1}, a_n)$ and (2.3) becomes

$$\varpi(\rho_{n+1}) \leq \eta(\varpi(\rho_n + |\rho_n - \rho_{n+1}|))\varpi((\rho_n + |\rho_n - \rho_{n+1}|)). \tag{2.4}$$

Assume that there exists $n > 0$ such that $\rho_n \leq \rho_{n+1}$. From (2.4), we get

$$\varpi(\rho_{n+1}) \leq \eta(\varpi(\rho_{n+1}))\varpi(\rho_{n+1}) < \varpi(\rho_{n+1})$$

which is a contradiction. Thus for all $n > 0$, $\rho_n > \rho_{n+1}$. Hence, we deduce that the sequence $\{\rho_n\}$ is nonincreasing. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \rho_n = r.$$

Now, we shall prove that $r = 0$. Suppose that $r > 0$. From (2.2), we have

$$\varpi(\rho(a_n, a_{n+1})) \leq \eta(\varpi(F(a_{n-1}, a_n)))\varpi(F(a_{n-1}, a_n))$$

which implies

$$\varpi(\rho_{n+1}) \leq \eta(\varpi(2\rho_n - \rho_{n+1}))\varpi(2\rho_n - \rho_{n+1}).$$

Hence

$$\frac{\varpi(\rho_{n+1})}{\varpi(2\rho_n - \rho_{n+1})} \leq \eta(\varpi(2\rho_n - \rho_{n+1})) < 1.$$

This implies that $\lim_{n\to\infty} \eta(\varpi(2\rho_n - \rho_{n+1})) = 1$. Since $\eta \in \Lambda$, we have

$$\lim_{n\to\infty} \varpi(2\rho_n - \rho_{n+1}) = 0$$

which yields

$$r = \lim_{n\to\infty} \rho(a_n, a_{n+1}) = 0 \tag{2.5}$$

which is a contradiction. So $r = 0$. Now, we shall prove that $\{a_n\}$ is a Cauchy O-sequence. We will prove that

(2.6) $$\lim_{n,m\to\infty} \rho(a_n, a_m) = 0.$$

We argue by contradiction. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{a_{m(k)}\}$ and $\{a_{n(k)}\}$ of $\{a_n\}$ with $m(k) > n(k) > k$ such that for every k

(2.7) $$\rho(a_{m(k)}, a_{n(k)}) \geq \varepsilon.$$

Moreover corresponding to $n(k)$, we can choose $m(k)$ in such a way that is the smallest integer with $m(k) > n(k)$ and satisfying (2.7). Then

(2.8) $$\rho(a_{m(k)-1}, a_{n(k)}) < \varepsilon.$$

Using (2.7) and (2.8)

$$\begin{aligned}\varepsilon &\leq \rho(a_{m(k)}, a_{n(k)})\\ &\leq \rho(a_{m(k)}, a_{m(k)-1}) + \rho(a_{m(k)-1}, a_{n(k)})\\ &< \rho(a_{m(k)}, a_{m(k)-1}) + \varepsilon.\end{aligned}$$

By (2.5), we get

(2.9) $$\lim_{k\to\infty} \rho(a_{m(k)}, a_{n(k)}) = \varepsilon.$$

On the other hand, it is easy to see that

$$\left|\rho(a_{m(k)-1}, a_{n(k)-1}) - \rho(a_{m(k)}, a_{n(k)})\right| \leq \left|\rho(a_{m(k)-1}, a_{m(k)}) + \rho(a_{n(k)}, a_{n(k)-1})\right|.$$

Again by (2.5) and (2.9)

(2.10) $$\lim_{k\to\infty} \rho(a_{m(k)-1}, a_{n(k)-1}) = \varepsilon.$$

We go back to (2.2) to have

$$\begin{aligned}\varpi(\varepsilon) &\leq \varpi(\rho(a_{m(k)}, a_{n(k)}))\\ &= \varpi(\rho(fa_{m(k)-1}, fa_{n(k)-1}))\\ &\leq \eta(\varpi(F(a_{m(k)-1}, a_{n(k)-1})))\varpi(F(a_{m(k)-1}, a_{n(k)-1})),\end{aligned}$$

where

$$F(a_{m(k)-1}, a_{n(k)-1}) = \rho(a_{m(k)-1}, a_{n(k)-1}) + \left|\rho(a_{m(k)-1}, fa_{m(k)-1}) - \rho(a_{n(k)-1}, fa_{n(k)-1})\right|.$$

By (2.5) and (2.10)

(2.11) $$\lim_{k\to\infty} F(a_{m(k)-1}, a_{n(k)-1}) = \varepsilon.$$

We deduce

$$\lim_{k\to\infty} \eta\left(\varpi\left(F(a_{m(k)-1}, a_{n(k)-1})\right)\right) = 1.$$

Since $\eta \in \eta$, we have

$$\lim_{k\to\infty} F(a_{m(k)-1}, a_{n(k)-1}) = 0$$

which is a contradiction with respect to (2.11). Thus $\{a_n\}$ is a Cauchy O-sequence in (X, ρ). So there exists $u^* \in X$ such that

$$\lim_{n\to\infty} \rho(a_n, u^*) = \rho(u^*, u^*) = \lim_{n,m\to\infty} \rho(a_n, a_m).$$

By (2.6), we write

$$\lim_{n\to\infty} \rho(a_n, u^*) = \rho(u^*, u^*) = 0$$

because of O-completeness of X. Since f is $\perp-$continuous, we have

$$fu^* = f(\lim_{n\to\infty} fa_n) = \lim_{n\to\infty} a_{n+1} = u^*$$

and so u^* is a fixed point of f.

Now, we can show the uniqueness of the fixed point. We shall prove that such u^* verifying $\rho(u^*, u^*) = 0$ is the unique fixed point of f. We argue by contradiction. Assume that there exists $u^* \neq w^*$ so $\rho(u^*, w^*) > 0$ such that

$$u^* = fu^*,\ w^* = fw^*,\ \rho(u^*, u^*) = \rho(w^*, w^*) = 0.$$

Suppose that there exist two distinct fixed point u^* and w^*. Since $\lambda(a, b, \perp)$ is non-empty for all $a, b \in X$, there exists a path $\{z_0, z_1, ..., z_k\}$ of some finite lenght k in $\perp$ from u^* to w^* such that

$$u_0 = u^*, u_k = w^*, u_i \perp u_{i+1} \text{ or } u_{i+1} \perp u_i.$$

Since $(X, \perp)$ transitive orthogonal set, we get $u^* \perp w^*$ or $w^* \perp u^*$. Then, we have

$$\begin{aligned} F(u^*, w) &=^* \rho(u^*, w^*) + |\rho(u^*, fz) - \rho(w^*, fw^*)| \\ &= \rho(u^*, w^*) + |\rho(u^*, u^*) - \rho(w^*, w^*)| \\ &= \rho(u^*, w^*) \end{aligned}$$

and using this equality in (2.2), we get

$$\begin{aligned} \varpi(\rho(u^*, w^*)) &= \varpi(\rho(fu^*, fw^*)) \\ &\leq \eta(\varpi(F(u^*, w^*)))\varpi(F(u^*, w^*)) \\ &= \eta(\varpi(\rho(u^*, w^*)))\varpi(\rho(u^*, w^*)) \\ &< \varpi(\rho(u^*, w^*)) \end{aligned}$$

which is a contradiction. Thus there exists a unique $u^* \in X$ such that $u^* = fu^*$ with $\rho(u^*, u^*) = 0$. □

Example 2.1. *Let $X = [0, 1]$ and $\rho(a, b) = a + b$. Then (X, ρ) is $O-$complete metric-like space. Define a relation $\perp$ on X by*

$$a \perp b \iff ab \in \{a, b\}.$$

Define $f : X \to X$ by

$$fx = \begin{cases} 0, & x = 1 \\ \frac{x}{2} & x \neq 1 \end{cases}.$$

Then we can see that f is $\perp$-preserving and also $\perp$-continuous. Take $\varpi(t) = \frac{t}{2}$ and $\eta(\alpha) = \frac{1}{2}$, then it is clear that f is a orthogonal ϖ_F-Geraghty contraction and f has a fixed point $u = 0$ with $\sigma(u, u) = 0$.

In Theorem 2.5, if we get $\varpi(t) = t$, then we obtain the following corollary.

Corollary 2.2. *Let $(X, \perp, \rho)$ be an O-complete metric-like space with an orthogonal elements a_0 and f be a self mapping on X satisfying the following conditions:*

(i) *$(X, \perp)$ is a transitive orthogonal set,*
(ii) *f is $\perp$-preserving,*
(iii) *f is orthogonal Geraghty contraction such that*

$$\varpi\rho(fa, fb) \leq \eta((F(a, b)))F(a, b)$$

for all $a, b \in X$ with $a \perp b$, where

$$F(a, b) = \rho(a, b) + |\rho(a, fa) - \rho(b, fb)|,$$

(iv) *f is $\perp$-continuous.*

Then, f has a fixed point in X.

2.2. **A result for orthogonal $(\hbar,\phi)$-contraction.** Now, we give a definition of $(\hbar,\phi)$-contraction on orthogonal metric-like space and prove a fixed point theorem for this type contraction.

Definition 2.14. *Let (X,ρ) be a orthogonal metric-like space and $f : X \to X$ be a mapping. f is called a orthogonal $(\hbar,\phi)$-contraction if there exist $\hbar \in \mathcal{H}$ and $\phi : M \to [0,+\infty[$ s.t.*

$$\hbar(\rho(fa,fb),\phi(fa),\phi(fb))) \leq F(\hbar(\rho(a,b),\phi(a),\phi(b))) \tag{2.12}$$

for all $a,b \in M$ with $a \perp b$.

Lemma 2.2. *Let (X,ρ) be a orthogonal metric-like space and $f : X \to X$ be a $(\hbar,\phi)$-contraction. If $\{a_n\}$ is a sequence of Picard starting at $a_0 \in X$, then*

$$\lim_{n\to+\infty} \hbar(\rho(a_{n-1},a_n),\phi(a_{n-1}),\phi(a_n)) = 0,$$

and hence

$$\lim_{n\to+\infty} \rho(a_{n-1},a_n) = 0 \quad \textit{and} \quad \lim_{n\to+\infty} \phi(a_n) = 0.$$

Proof. By replacing the contradiction in [26, (3.2)] with contradiction (2.12) and following the proof of [26, Lemma 3.1], we immediately have the desired result. □

Theorem 2.6. *Let $(X,\perp,\rho)$ be an O-complete metric-like space and a_0 is an orthogonal element of X and f be a self-mapping on X such that:*
i) For all $a,b \in X$ with $a \perp b$

$$\hbar(\rho(fa,fb),\phi(fa),\phi(fb)) \leq k\hbar(\rho(a,b),\phi(a),\phi(b)) \tag{2.13}$$

for some $k \in (0,1)$, $\hbar \in \mathcal{H}$ and $\phi : X \to [0,\infty)$ is lower-semicontinuous function,
ii) f is $\perp$-preserving,
iii) $(X,\perp)$ is transitive orthogonal set.
Then f has a unique fixed point.

Proof. First, we shall prove the uniqueness. Suppose that a^*,b^* are two fixed point of f such that $a^* \neq b^*$. Since $\lambda(a,b,\perp)$ is non-empty for all $a,b \in X$, there exists a path $\{z_0,z_1,\cdots,z_k\}$ of some finite lenght k in $\perp$ from a to b such that

$$z_0 = a^*, z_k = b^*, z_{i+1} \perp z_i \text{ for all } i = 0,1,2,\cdots,k-1.$$

Since $(X,\perp)$ transitive orthogonal set, we get $a^* \perp b^*$ or $b^* \perp a^*$. From *i)*, we have

$$\begin{aligned}\hbar(\rho(a^*,b^*),\phi(a^*),\phi(b^*)) &= \hbar(\rho(fa^*,fb^*),\phi(fa^*),\phi(fb^*)) \\ &\leq k\hbar(\rho(a^*,b^*),\phi(a^*),\phi(b^*)) \\ &< \hbar(\rho(a^*,b^*),\phi(a^*),\phi(b^*))\end{aligned}$$

so, this is a contradiction. Then f has a unique fixed point.

Now, assume that $\varsigma \in X$ is a fixed point of f. Applying *(2.13)* with $a = b = \varsigma$, $a \perp b$, we have

$$\hbar(0,\phi(\varsigma),\phi(\varsigma)) \leq k\hbar(0,\phi(\varsigma),\phi(\varsigma)),$$

which implies (since $k \in (0,1)$) that

$$\hbar(0,\phi(\varsigma),\phi(\varsigma)) = 0. \tag{2.14}$$

Moreover, from *(H1)*, we have

$$\phi(\varsigma) \leq \hbar(0,\phi(\varsigma),\phi(\varsigma)). \tag{2.15}$$

Using (*2.14)* and (*2.15)*, we obtain $\phi(\varsigma) = 0$.

Now *ii)* from the definition of orthogonality, it follows that

$$a_0 \perp fa_0 \text{ or } fa_0 \perp a_0.$$

Let

$$a_1 = fa_0, a_2 = fa_1 = f^2 a_0, \cdots, a_n = fa_{n-1} = f^n a_0$$

for all $n \in \mathbb{N} \cup \{0\}$. If $a_{n^*} = a_{n^*+1}$ for some $n^* \in \mathbb{N} \cup \{0\}$, then

$$z = a_{n^*} = a_{n^*+1} = fa_{n^*} = fz$$

and z is a fixed point of f such that $\phi(z) = 0$. In fact by Lemma 2.2,

$$\hbar(\rho(a_{n^*-1}, a_{n^*}), \phi(a_{n^*-1}), \phi(a_{n^*})) = 0$$

and by the property *(H1)*, of the function $\hbar$, we have $\phi(z) = 0$. So, we assume that $a_n \neq a_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we have

$$a_n \perp a_{n+1} \text{ or } a_{n+1} \perp a_n.$$

This implies that $\{a_n\}$ is a O-sequence. Since f is an orthogonal $(\hbar, \phi)$-contraction, we have

$$\begin{aligned}
&\hbar(\rho(a_n, a_{n+1}), \phi(a_n), \phi(a_{n+1})) \\
=&\hbar(\rho(fa_{n-1}, fa_n), \phi(fa_{n-1}), \phi(fa_n)) \\
\leq&k\hbar(\rho(a_{n-1}, a_n), \phi(a_{n-1}), \phi(a_n)) \\
\leq&k^n\hbar(\rho(a_0, a_1), \phi(a_0), \phi(a_1)) \\
=&k^n\hbar(\rho(a_0, fa_0), \phi(a_0), \phi(fa_0)), n \in \mathbb{N} \cup \{0\}
\end{aligned}$$

which implies by property *(H1)* that for all $n \in \mathbb{N} \cup \{0\}$

$$\max\{\rho(a_n, a_{n+1}), \phi(a_n)\} \leq k^n\hbar(\rho(a_0, fa_0), \phi(a_0), \phi(fa_0)).$$

Then, we obtain

$$\rho(a_n, a_{n+1}) \leq k^n\hbar(\rho(a_0, fa_0), \phi(a_0), \phi(fa_0)), n \in \mathbb{N} \cup \{0\}$$

which implies that $\{a_n\}$ is a Cauchy O-sequence. Since X is O-complete then, there exists $a^* \in X$ such that $a_n \to a^*$ as $n \to \infty$.

Since f is orthogonal $(\hbar, \phi)$-contraction, taking into account that ϕ is lower-semicontinuous function, we have

$$0 \leq \phi(a^*) \leq \lim\inf_{n\to\infty} \phi(a_n) = 0,$$

that is, $\phi(a^*) = 0$. Now, show that a^* is a fixed point f.

If there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} = a^*$ or $fa_{n_k} = fa^*$ for all $k \in \mathbb{N}$, then a^* is a fixed point. Otherwise, we can assume that $a_n \neq a^*$ and $fa_n \neq fa^*$ for all $n \in \mathbb{N}$. So, using f is an $(\hbar, \phi)$-contraction, we deduce that for all $n \in \mathbb{N}$

$$\begin{aligned}
\hbar(\rho(fa_n, fa^*), \phi(fa_n), \phi(fa^*)) &\leq k\hbar(\rho(a_n, a^*), \phi(a_n), \phi(a^*)) \\
&< \hbar(\rho(a_n, a^*), \phi(a_n), \phi(a^*)),
\end{aligned}$$

and so,

$$\begin{aligned}
\rho(a^*, fa^*) &\leq \rho(a^*, a_{n+1}) + \rho(fa_n, fa^*) \\
&\leq \rho(a^*, a_{n+1}) + \hbar(\rho(fa_n, fa^*), \phi(fa_n), \phi(fa^*)) \\
&< \rho(a^*, a_{n+1}) + \hbar(\rho(a_n, a^*), \phi(a_n), \phi(a^*)) \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Finally, letting $n \to \infty$ in the above calculations and using that $\hbar$ is continuous in $(0, 0, 0)$, we deduce that

$$\rho(a^*, fa^*) \leq \hbar(0, 0, 0)$$

that is, $a^* = fa^*$. □

Remark 2.1. *In the Theorem2.6, if we assume that f is $\perp$-continuous, we have*

$$f\mu^* = f\left(\lim_{n\to\infty} \mu_n\right) = \lim_{n\to\infty} \mu_{n+1} = \mu^*$$

and μ^* is a fixed point of f.

2.3. **A result for rational $F-$contraction.** In this part, we modify definition of rational type $F-$contraction using orthogonality and then give some results for this type contraction on $O-$complete metric-like space. But firstly, we want to give some information about $F-$contraction.

Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

- $(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta, F(\alpha) < F(\beta)$,
- $(F2)$ for each sequence $\{a_n\}$ of positive numbers,

$$\lim_{n\to\infty} a_n = 0 \text{ if and only if } \lim_{n\to\infty} F(a_n) = -\infty,$$

- $(F3)$ there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 2.15 ([28]). *Let (X, ρ) be a metric space and $f : X \to X$ be a mapping. Given $F \in \mathcal{F}$, f is called as F-contraction if there exists $\tau > 0$ such that*

$$a, b \in M,\ \rho(fa, fb) > 0 \Rightarrow \tau + F(\rho(fa, fb)) \le F(\rho(a, b)).$$

Definition 2.16. *Let $(X, \perp, \rho)$ be an orthogonal metric-like space. We say that $f : X \to X$ is an orthogonal rational type $F-$contraction if there are $F \in \mathcal{F}$ and $\tau > 0$ such that the following condition holds:*

$$\forall a, b \in X \text{ with } a \perp b;\ \left[\rho(fa, fb) > 0 \Rightarrow \tau + F(\rho(fa, fb)) \le F(M(a, b))\right], \tag{2.16}$$

where

$$M(a, b) = \max\left\{\begin{array}{c} \rho(a, b), \rho(a, fa), \rho(b, fb), \\ \frac{\rho(a,fa)\rho(b,fb)}{1+\rho(a,b)}, \frac{\rho(a,fa)\rho(b,fb)}{1+\rho(fa,fb)} \end{array}\right\}.$$

Theorem 2.7. *Let $(X, \perp, \rho)$ be an $O-$complete orthogonal metric-like space, a_0 is an orthogonal element of X and f be a $\perp$ $-$preserving and $\perp$ $-$continuous mapping with satisfying (2.16). Also we assume that $(X, \perp)$ is a transitive orthogonal set, then, f has a unique fixed point in X.*

Proof. Using the definition of the orthogonality, we have $a_0 \perp f(a_0)$ or $f(a_0) \perp a_0$. Let

$$a_1 := fa_0, a_2 := fa_1 = f^2 a_0, \cdots, a_n := fa_{n-1} = f^n a_0$$

for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $\rho(a_n, a_{n+1}) = 0$ for some n_0, so the proof is completed. Consequently, we assume that

$$\rho(a_n, a_{n+1}) \ne 0$$

for all n. Thus, we get $\rho(a_n, a_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Then, we obtain

$$a_n \perp a_{n+1} \text{ or } a_{n+1} \perp a_n$$

from $\perp$ −preserving of f and then we say that $\{a_n\}$ is an $O-$sequence. From (2.16), for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
& F(\rho(a_n, a_{n+1})) \\
=& F(\rho(fa_{n-1}, fa_n)) \\
\leq& F(M(a_{n-1}, a_n)) - \tau \\
=& F\left(\max\left\{\begin{array}{c} \rho(a_{n-1}, a_n), \rho(a_{n-1}, fa_{n-1}), \rho(a_n, fa_n), \\ \frac{\rho(a_{n-1}, fa_{n-1})\rho(a_n, fa_n)}{1+\rho(a_{n-1}, a_n)}, \frac{\rho(a_{n-1}, fa_{n-1})\rho(a_n, fa_n)}{1+\rho(fa_{n-1}, fa_n)} \end{array}\right\}\right) - \tau \\
\leq& F(\rho(a_{n-1}, a_n)) - \tau. \qquad (2.17)
\end{aligned}
$$

Let $\alpha_n := \rho(a_n, a_{n+1})$ for all $n \in \mathbb{N}$ and from (2.17), we have

$$F(\alpha_n) \leq F(\alpha_{n-1}) - \tau \leq F(\alpha_{n-2}) - 2\tau \leq \cdots \leq F(\alpha_0) - n\tau. \qquad (2.18)$$

From (2.18), we get $\lim_{n\to\infty} F(\alpha_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} \alpha_n = 0. \qquad (2.19)$$

By the property (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0. \qquad (2.20)$$

By (2.18), we get

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -\alpha_n^k n\tau \leq 0 \qquad (2.21)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.21), we get

$$\lim_{n\to\infty} n\alpha_n^k = 0. \qquad (2.22)$$

From (2.22), there exits $n_1 \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_1$. So we have

$$\alpha_n \leq \frac{1}{n^{1/k}} \qquad (2.23)$$

for all $n \geq n_1$. In order to show that $\{a_n\}$ is a Cauchy $O-$sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.23), we have

$$
\begin{aligned}
\rho(a_n, a_m) &\leq \rho(a_n, a_{n+1}) + \rho(a_{n+1}, a_{n+2}) + \cdots + \rho(a_{m-1}, a_m) \\
&= a_n + a_{n+1} + \cdots + a_{m-1} \\
&= \sum_{i=n}^{m-1} a_i \\
&\leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}.
\end{aligned}
$$

Hence $\{a_n\}$ is a Cauchy $O-$sequence in M. Because of the $O-$completeness of X, we have $a^* \in M$ such that $a_n \to a^*$ as $n \to \infty$. Using $\perp -$ continuous of f, we have

$$fa^* = f(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} a_{n+1} = a^*$$

and so a^* is a fixed point of f.

Now, we can show the uniqueness of the fixed point. We shall prove that such a^* verifying $\rho(a^*, a^*) = 0$ is the unique fixed point of f. We argue by contradiction. Assume that there exists $a^* \neq w^*$ so $\rho(a^*, w^*) > 0$ such that

$$a^* = fa^*, \ w^* = fw^*, \ \rho(a^*, a^*) = \rho(w^*, w^*) = 0.$$

Suppose that there exist two distinct fixed point a^* and w^*. Since $\lambda(a, b, \perp)$ is non-empty for all $a, b \in X$, there exists a path $\{z_0, z_1, ..., z_k\}$ of some finite lenght k in $\perp$ from a^* to w^* such that

$$u_0 = a^*, u_k = w^*, u_i \perp u_{i+1} \text{ or } u_{i+1} \perp u_i.$$

Since $(X, \perp)$ transitive orthogonal set, we get $u^* \perp w^*$ or $w^* \perp u^*$. Then, we get

$$\begin{aligned} &\tau + F(\rho(a^*, w^*)) \\ &= \tau + F(\rho(fa^*, fw^*)) \\ &\leq F(M(a^*, w^*)) \\ &= F\left(\max\left\{\begin{array}{cc} \rho(a^*, w^*), \rho(a^*, fa^*), \rho(w^*, fw^*), \\ \frac{\rho(a^*, fa^*)\rho(w^*, fw^*)}{1+\rho(a^*, w^*)}, \frac{\rho(a^*, fa^*)\rho(w^*, fw^*)}{1+\rho(fa^*, fw^*)} \end{array}\right\}\right) \\ &= F(C(a^*, w^*)) \end{aligned}$$

which is a contradiction. Thus there exists a unique $a^* \in X$ such that $a^* = fa^*$ with $\rho(a^*, a^*) = 0$. □

Corollary 2.3. *Let $(X, \perp, \rho)$ be an $O-$complete metric-like space with an orthogonal elements a_0 and f be a $\perp-$preserving and $\perp-$continuous self mapping on M such that*

$$\forall a, b \in X \text{ with } a \perp b; \ [\rho(fa, fb) > 0 \Rightarrow \tau + F(\rho(fa, fb)) \leq F(\rho(a, b))].$$

Then, T has a unique fixed point in M.

3. APPLICATIONS

Recall that, for any $1 \leq p < \infty$, the space $Ł^p(X, F, \mu)$ (or $Ł^p(X)$) consists of all complex-valued measurable functions κ on the underlying space X satisfying

$$\int_M |\kappa(\wp)|^p \, d\mu(\wp),$$

where F is the σ-algebra of measurable sets and μ is the measure. When $p = 1$, the space $Ł^p(X)$ consists of all integrable functions κ on X and we define the $Ł^1$-norm of κ by

$$\|\kappa\|_1 = \int_M |\kappa(\wp)| \, d\mu(\wp).$$

In the section, using Theorem 2.7, we show the existence of a solution of the following differential equation:

$$\left\{\begin{array}{ll} u'(t) = \mathrm{f}(t, u(t)), & a.e. \ t \in I := [0, T] \\ u(0) = a, & a \geq 1, \end{array}\right. \tag{3.24}$$

where $\mathrm{f} : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:
(i) $\mathrm{f}(s, \eta) \geq 0$ for all $\eta \geq 0$ and $s \in I$;

(ii) for each $\wp, \gamma \in Ł^1(I)$ with $\wp(s)\gamma(s) \geq \wp(s)$ or $\wp(s)\gamma(s) \geq \gamma(s)$ for all $s \in I$, there exist $\kappa \in Ł^1(I)$ and $\tau > 0$ such that

$$|\mathrm{f}(s, \wp(s)) - \mathrm{f}(s, \gamma(s))| \leq \frac{\kappa(s)}{\left(1 + \tau\sqrt{\kappa(s)}\right)^2} |\wp(s) - \gamma(s)| \tag{3.25}$$

and

$$|\wp(s) - \gamma(s)| \leq \kappa(s) e^{A(s)}$$

for all $s \in I$, where $A(s) := \int_0^s |\kappa(w)| \, dw$.

Theorem 3.8. *Consider the differential equation (3.24). If (i) and (ii) are satisfied, then the differential equation (3.24) has a unique positive solution.*

Proof. Let $X = \{u \in C(I, \mathbb{R}) : u(t) > 0 \text{ for all } t \in I\}$. Define the orthogonality relation $\perp$ on X by

$$\wp \perp \gamma \Longleftrightarrow \wp(s)\gamma(s) \geq \wp(s) \text{ or } \wp(s)\gamma(s) \geq \gamma(s) \text{ for all } t \in I.$$

Since $A(t) = \int_0^t |\kappa(s)| \, ds$, we have $A'(t) = |\kappa(t)|$ for almost everywhere $t \in I$.

Define a mapping $d(\wp, \gamma) = \|\wp - \gamma\|_A = \sup_{t \in I} e^{-A(t)} |\wp(s) - \gamma(s)|$ for all $\wp, \gamma \in X$. Thus, (X, d) is a metric-like space and also a complete metric-like space. Define a mapping $\not\prec : X \to X$ by

$$(\not\prec\wp)(t) = a + \int_0^t \mathrm{f}(s, \wp(s)) \, ds.$$

Then, we see that $\not\prec$ is $\perp$-continuous.

Now, we show that $\not\prec$ is $\perp$- preserving. For each $\wp, \gamma \in X$ with $\wp \perp \gamma$ and $t \in I$, we have

$$(\not\prec\wp)(t) = a + \int_0^t \mathrm{f}(s, \wp(s)) \, ds \geq 1.$$

It follows that $[(\not\prec\wp)(t)][(\not\prec\gamma)(t)] \geq (\not\prec\gamma)(t)$ and so $(\not\prec\wp)(t) \perp (\not\prec\gamma)(t)$. Then $\not\prec$ is $\perp$-preserving.

Now, we can say that $\not\prec$ satisfies Corollary 2.3 with $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$. Hence the differential equation (3.24) has a unique positive solution. □

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