FUNDAMENTAL JOURNAL OF MATHEMATICS AND APPLICATIONS

VOLUME VI ISSUE III

FUJMA

www.dergipark.org.tr/en/pub/fujma

ISSN 2645-8845

VOLUME 6 ISSUE 3
ISSN 2645-8845

September 2023
www.dergipark.org.tr/en/pub/fujma

FUNDAMENTAL JOURNAL OF MATHEMATICS AND APPLICATIONS

Editors in Chief

Fuat Usta
Düzce University, Türkiye
fuatusta@duzce.edu.tr

Serkan Aracı
Hasan Kalyoncu University, Türkiye
serkan.araci@hku.edu.tr

Editorial Board

Ulrich Abel
Technische Hochschule Mittelhessen,
Germany

Praveen Agarwal
Anand International College of Engineering,
Jaipur, India

Martin Bohner
Missouri University of Science and Technology,
USA

Marcelo Moreira Cavalcanti
Universidade Estadual de Maringá, Maringá,
Brazil

Clemente Cesarano
International Telematic University Uninettuno,
Italy

Danilo Costarelli
University of Perugia,
Italy

Tom Cuchta
Marshall University,
USA

Cristina Flaut
Ovidius University, Constanta,
Romania

Naoyuki Ishimura
Chuo University,
Japan

Alexander Kasprzyk
University of Nottingham,
United Kingdom

Taekyun Kim
Kwangwoon University,
South Korea

Sunil Kumar
National Institute of Technology,
Jamshedpur, Jharkhand, India

Mircea Merca
University Politehnica of Bucharest,
Romania

Kottakaran Sooppy Nisar
Prince Sattam bin Abdulaziz University,
Saudi Arabia

Samad Noeiaghdam
Irkutsk National Research Technical University,
Russia

Georgia Irina Oros
University of Oradea,
Romania

Stojan Radenovic
University of Belgrade,
Serbia

Communications Editor

Bilal Khan
East China Normal University, China

Language Editor

Mohammadreza Valizadeh
Hasan Kalyoncu University, Türkiye

Technical Editor

Kamal Najafi Ghazikandi
Düzce University, Türkiye

Editorial Secretariat

Muhammet Civelek
Düzce University, Türkiye

Contents

Fundamental Journal of Mathematics and Applications, 6 (3) (2023) 137-146
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1261409

A New Smoothing Algorithm to Solve a System of Nonlinear Inequalities

Nurullah Yılmaz[1*] and Ayşegül Kayacan[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Süleyman Demirel University, Isparta, Türkiye*
[*]*Corresponding author*

Article Info

Keywords: *Iterative methods, Smoothing, System of non-linear inequalities.*
2010 AMS: *65D10, 90C53*
Received: *9 March 2023*
Revised: *5 June 2023*
Accepted: *19 June 2023*
First Online: *24 August 2023*
Published: *30 September 2023*

Abstract

In this study, the system of nonlinear inequalities (SNI) problem is investigated. First, a SNI is reformulated as a system of nonsmooth and nonlinear equations (SNNE). Second, a new smoothing technique for the "max" function is proposed and the smoothing technique is employed for each element of the SNNE. Then, a new smoothing algorithm is developed in order to solve SNNE by combining the smoothing technique with the iterative method. The new algorithm is applied to some numerical examples to show the efficiency of our algorithm.

1. Introduction

In this paper, the following system of non-linear inequalities is considered:

$$H(x) \leq 0, \tag{1.1}$$

where, $H(x) := (h_1(x), h_2(x), \ldots, h_n(x))^T$ with $h_i : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable for any $i \in \{1, \ldots, n\}$ [1,2]. The SNI has been emerged in many real-world applications such as image restoration problems, data analysis, supply chain problems, computer aided design problems, compressive sensing problems and set separation problems [2–4]. In recent years, motion control involving two–joint planar robotic manipulator are modelled as SNI in [5] and it is faced a problem of SNI in process of designing parallel manipulator for aliquoting of biomaterials [6]. Depending on all of these practical applications, SNI has been extensively studied over the years [7–10].

There are many interesting methods have been proposed to solve the problem (1.1) such as modified Newton methods [11], smoothing Newton methods [1,2,8], Broyden-like methods [12], Conjugate Gradient methods [13,14], trust-region method [15], homotopy method [16] and etc. Although there is a parameter that must be tune to construct a smoothing function, among the all methods the smoothing Newton methods come into prominance due to their excellent numerical performance [17]. Smoothing Newton methods developed by modifiying the line search strategies such as exact, inexact, bactracking and etc. types line search techniques [18, 19]. One of the main tool of smoothing Newton method is smoothing functions. The smoothing function is defined as follows:

Definition 1.1. *[20] A function $\tilde{H} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is called a smoothing function of a non-smooth function $H : \mathbb{R}^n \to \mathbb{R}^m$ if, for any $\varepsilon > 0$, $\tilde{H}(x, \varepsilon)$ is continuously differentiable and*

$$\lim_{z \to x, \varepsilon \downarrow 0} \tilde{H}(z, \varepsilon) = H(x)$$

for any $x \in R^n$.

Email addresses and ORCID numbers: **nurullahyilmaz@sdu.edu.tr, 0000-0001-6429-7518 (N. Yilmaz), aysegulkayacan98@gmail.com, 0000-0002-6285-5771 (A. Kayacan)**
Cite as: N. Yılmaz, A. Kayacan, *A new smoothing algorithm to solve a system of nonlinear inequalities*, Fundam. J. Math. Appl., 6(3) (2023), 137-146

Smoothing functions have been studied by many scholars [21–24] and they have been applied to solve many interesting non-smooth problems over the years [25–27]. The comprehensive overview on smoothing approaches can be found in [20, 28, 29]. Among the smoothing functions, the smoothing function studied in [30] distinguishes itself from the others due to different structure, formulations and useful properties.
In this study, we propose a new smoothing function inspiring from the smoothing methods given in [30] based on the problem (1.1). By the help of this smoothing approach, we design a family of smooth equations which is surrogate for the original problem (1.1). A practical and user-friendly algorithm is developed to solve the surrogate system. The algorithm is implemented to some test problems in the literature in order to demonstrate the numerical performance of it. The comparison with corner stone studies has been presented in order to show the superiority of our algorithm.
Throughout the paper, $\mathbb{R}^{n_+}$ denotes the non-negative part of $\mathbb{R}^n$, I denotes $n \times n$ identity matrix. For any vector $u \in \mathbb{R}^n$, u^T denotes the transpose of u and the Euclidean norm of u is denoted by $\|u\|$. In the following section, we present the new smoothing technique and a new formulation of system of non-linear inequalities. In Section 3, some numerical experiments are illustrated and comparison with the other methods is presented. Some concluding remarks are given in the last section.

2. Main Results

In the first subsection, we re-formulate the SNI (2.9) as a system of nonsmooth and nonlinear equations, then we propose a new smoothing technique to make smooth the each element of reformulated problem. In the next, we propose an algorithm to solve smoothed reformulated problem.

2.1. Smoothing Techniques

Let us define the function for any $y \in \mathbb{R}^n$

$$y_+ := \left(\max\{0, y_1\}, \ldots, \max\{0, y_n\}\right)^T. \tag{2.1}$$

Then, problem (1.1) is reformulated as the following system of nonlinear equations:

$$H(x)_+ = 0. \tag{2.2}$$

By considering the smoothing techniques proposed in our another study [30] and adapt it for "max" function. For any $t \in \mathbb{R}$, define the function $\phi(t) = \max\{t, 0\}$ which is also stated as

$$\phi(t) = \begin{cases} 0, & t \leq 0, \\ t, & t > 0. \end{cases}$$

The function $\phi(t)$ is re-stated by the help of indicator function as

$$\phi(t) = t\psi(t), \tag{2.3}$$

where,

$$\psi(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases} \tag{2.4}$$

The function defined in (2.4) is not smooth. We propose to use the following smoothing function:

$$\psi^{l_j}(t, \varepsilon) = \begin{cases} 0, & t < -\varepsilon, \\ D_j(t, \varepsilon), & -\varepsilon \leq t \leq \varepsilon, \\ 1, & t > \varepsilon, \end{cases} \tag{2.5}$$

where,

$$D_1(t, \varepsilon) = \frac{-1}{4\varepsilon^3}t^3 + \frac{3}{4\varepsilon}t + \frac{1}{2},$$

and

$$D_2(t, \varepsilon) = \frac{3}{16\varepsilon^5}t^5 - \frac{10}{16\varepsilon^3}t^3 + \frac{15}{16\varepsilon}t + \frac{1}{2},$$

for $j = 1, 2$. Based on the smoothing functions of indicator function, the smoothing function of $\phi(t)$ in (2.3) is defined as

$$\phi^{l_j}(t, \varepsilon) = t\psi^{l_j}(t, \varepsilon), \tag{2.6}$$

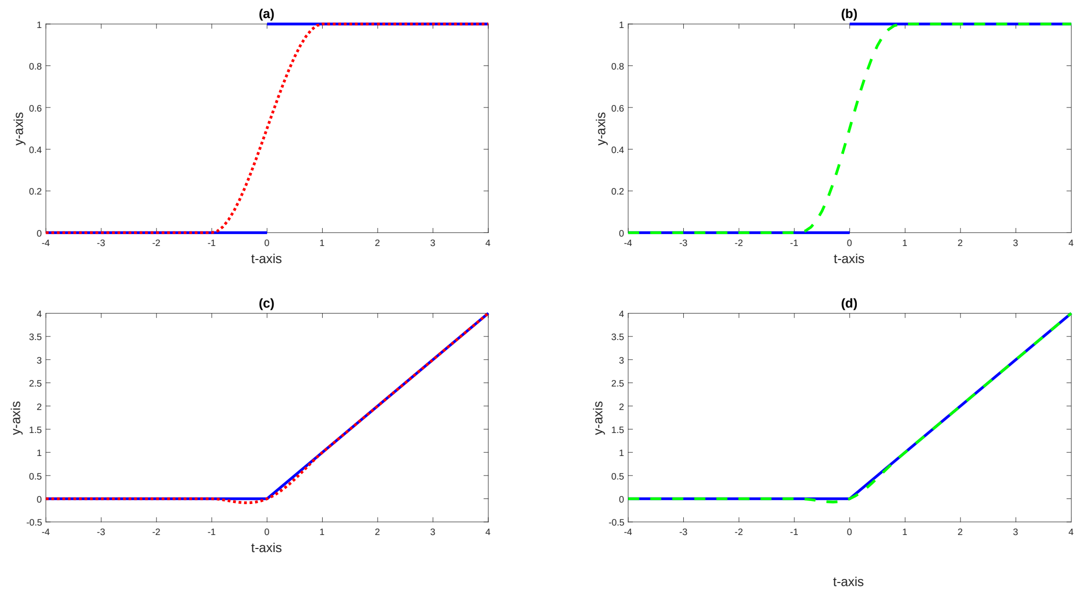

Figure 2.1: The graphics of smoothing functions of indicator functions and smoothing functions of max function.

for $j=1,2$.
We illustrate all the smoothing process by using graphs. The graphs of $\psi(t)$ and $\psi^{l_j}(t,\varepsilon)$ is illustrated in Fig. 2.1 (a) and (b). In Fig. 2.1 (a), the blue and solid graph indicates the function $\psi(t)$ and the red and dotted one indicates $\psi^{l_1}(t,\varepsilon)$ and, Fig. 2.1 (b) the blue and solid graph again indicates the function $\psi(t)$ and the green and dashed one indicates $\psi^{l_2}(t,\varepsilon)$ for $\varepsilon=1$.
The graphs of $\phi(t)$ and $\phi^{l_j}(t,\varepsilon)$ is illustrated in Fig. 2.1 (c) and (d). In Fig. 2.1 (c), the blue and solid graph indicates the function $\phi(t)$ and the red and dotted one indicates $\phi^{l_1}(t,\varepsilon)$ and, Fig. 2.1 (d) the blue and solid graph again indicates the function $\psi(t)$ and the green and dashed one indicates $\phi^{l_2}(t,\varepsilon)$ for $\varepsilon=1$.

Lemma 2.1. *Let $\phi^{l_j}:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ be defined as in (2.6) then,*

i. ϕ^{l_j} is continuously differentiable at $t\in\mathbb{R}$,
ii. $\lim_{\varepsilon\to 0}\phi^{l_j}(t,\varepsilon)=\phi(t)$ for any $t\in\mathbb{R}$,

for any $\varepsilon>0$ and $j=1,2$. Moreover ϕ^{l_2} is second order continuously differentiable.

Proof. i. The derivative of the smoothing functions ϕ^{l_j} are

$$\frac{d}{dt}\left(\phi^{l_j}(t,\varepsilon)\right)=\psi^{l_j}(t,\varepsilon)+t\frac{d}{dt}\left(\psi^{l_j}(t,\varepsilon)\right),$$

where,

$$\frac{d}{dt}\left(\psi^{l_j}(t,\varepsilon)\right)=\begin{cases}0, & t<-\varepsilon,\\ \frac{d}{dt}(D_j(t,\varepsilon)), & -\varepsilon\le t\le\varepsilon,\\ 0, & t>\varepsilon,\end{cases}$$

with

$$\frac{d}{dt}(D_1(t,\varepsilon))=\frac{-3}{4\varepsilon^3}t^2+\frac{3}{4\varepsilon},$$

and

$$\frac{d}{dt}(D_2(t,\varepsilon))=\frac{15}{16\varepsilon^5}t^4-\frac{30}{16\varepsilon^3}t^2+\frac{15}{16\varepsilon},$$

and

$$\frac{d^2}{dt^2}(D_2(t,\varepsilon))=\frac{15}{4\varepsilon^5}t^3-\frac{15}{4\varepsilon^3}t,$$

for $j=1,2$.

ii. For any $\varepsilon > 0$ and $t \notin I = [-\varepsilon, \varepsilon]$, then $\phi_{l_j}(t,\varepsilon) = \phi(t)$ for $j = 1,2$. Assume that $t \in [-\varepsilon, 0]$, since $\psi(t) = 0$ we have

$$\begin{aligned} 0 \le \phi(t) - \phi^{l_j}(t,\varepsilon) &= t\psi(t) - t\psi^{l_j}(t,\varepsilon) \\ &\le -t\psi^{l_j}(t,\varepsilon) \\ &\le \frac{\varepsilon}{2}. \end{aligned}$$

Now, let $t \in [0, \varepsilon]$ then, we have

$$\begin{aligned} 0 \le \phi(t) - \phi^{l_j}(t,\varepsilon) &= t\psi(t) - t\psi^{l_j}(t,\varepsilon) \\ &\le t\left(1 - \psi^{l_j}(t,\varepsilon)\right) \\ &\le \frac{\varepsilon}{2}. \end{aligned}$$

From the above results the $\phi^{l_j}(t,\varepsilon) \to \phi(t)$ as $\varepsilon \to 0$ for $j = 1,2$.
The proof is completed. □

By considering the functions $\phi^{l_j}(t,\varepsilon)$ instead of $\phi(t)$, the corresponding smoothing function of $\phi\left(h_i(x)\right)$ is obtained, for $j = 1,2$ and $i = 1,2,\ldots,n$. The resulting smoothing approximation of $H(x)_+$ is stated as a system of smooth nonlinear equations by

$$\tilde{H}(x,\varepsilon) = 0, \tag{2.7}$$

where

$$\tilde{H}(x,\varepsilon) = \begin{bmatrix} \phi_1^{l_j}(x,\varepsilon) \\ \phi_2^{l_j}(x,\varepsilon) \\ \vdots \\ \phi_n^{l_j}(x,\varepsilon) \end{bmatrix},$$

and $\phi_1^{l_j}(x,\varepsilon) = \phi^{l_j}(h_1(x),\varepsilon)$, $\phi_2^{l_j}(x,\varepsilon) = \phi^{l_j}(h_2(x),\varepsilon), \ldots, \phi_n^{l_j}(x,\varepsilon) = \phi^{l_j}(h_n(x),\varepsilon)$ for $\varepsilon > 0$.

Theorem 2.2. *Assume the functions $H(x)_+$ and $\tilde{H}(x,\varepsilon)$ be stated as in (1.1) and (2.7), respectively. Then, we obtain*

$$\|H(x)_+ - \tilde{H}(x,\varepsilon)\| \le \frac{\varepsilon}{2}\sqrt{n}.$$

Proof. For any $\varepsilon > 0$,

$$\begin{aligned} \|H(x)_+ - \tilde{H}(x,\varepsilon)\|^2 &= \sum_{i=1}^{n} \left|\phi_i(x) - \phi_i^{l_j}(x,\varepsilon)\right|^2 \\ &\le \sum_{i=1}^{n} \left(\frac{\varepsilon}{2}\right)^2 \\ &= n\frac{\varepsilon^2}{4} \end{aligned}$$

for $j = 1,2$. This completes the proof. □

Theorem 2.3. *The function $\tilde{H}(x,\varepsilon)$ is continuously differentiable and the Jacobian of $\tilde{H}(x,\varepsilon)$ is obtained as*

$$\tilde{H}'(x,\varepsilon) = \begin{bmatrix} \frac{\partial \phi_1^{l_j}(x,\varepsilon)}{\partial x_1} & \frac{\partial \phi_1^{l_j}(x,\varepsilon)}{\partial x_2} & \ldots & \frac{\partial \phi_1^{l_j}(x,\varepsilon)}{\partial x_n} \\ \frac{\partial \phi_2^{l_j}(x,\varepsilon)}{\partial x_1} & \frac{\partial \phi_2^{l_j}(x,\varepsilon)}{\partial x_2} & \ldots & \frac{\partial \phi_2^{l_j}(x,\varepsilon)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n^{l_j}(x,\varepsilon)}{\partial x_1} & \frac{\partial \phi_n^{l_j}(x,\varepsilon)}{\partial x_2} & \ldots & \frac{\partial \phi_n^{l_j}(x,\varepsilon)}{\partial x_n} \end{bmatrix}, \tag{2.8}$$

for any $\varepsilon > 0$.

2.2. Algorithm

Define the following function

$$G(x,\lambda)=\frac{\lambda}{2}\|H(x)_+\|^2,$$

and its smooth approximation

$$\tilde{G}(x,\lambda,\varepsilon)=\frac{\lambda}{2}\|\tilde{H}(x,\varepsilon)\|^2,$$

where $\lambda,\varepsilon>0$.

Assumption 1. *$\tilde{H}'(x,\varepsilon)$ is invertible for any $x\in\mathbb{R}^n$.*

Theorem 2.4. *Assume that the Assumption 1 is hold. Then, $\nabla\tilde{G}(x,\lambda,\varepsilon)=0$ if and only if $\tilde{H}(x,\varepsilon)=0$.*

Proof. Let $\nabla\tilde{G}(x,\lambda,\varepsilon)=0$ for any $\lambda,\varepsilon>0$. Then, we have that

$$\nabla\tilde{G}(x,\lambda,\varepsilon)=\lambda[\tilde{H}'(x,\varepsilon)]^T\tilde{H}(x,\varepsilon)=0. \tag{2.9}$$

There are two different cases in solving (2.9). At the first one is $\tilde{H}(x,\varepsilon)=0$ in which the proof is directly obtained. At the second one, $\tilde{H}(x,\varepsilon)\neq 0$ and $\tilde{H}'(x,\varepsilon)=0$ are obtained. In this case, $\tilde{H}'(x,\varepsilon)=0$ contradicts the Assumption 1. □

It should be stated that the Assumption 1 is necessary to guarantee the equivalence of problems 2.2 and 2.10.
By considering Theorem 2.4, the following optimization problem

$$\min_{x\in\mathbb{R}^n}\tilde{G}(x,\lambda,\varepsilon) \tag{2.10}$$

can be considered as a surrogate problem for $\tilde{H}(x,\varepsilon)=0$.

Remark 2.5. *It should be stated that the Assumption 1 is necessary to guarantee the equivalence of problems 2.2 and 2.10.*

Theorem 2.6. *Let x^* and $\bar{x}$ are local minimizers of $G(x,\lambda)$ and $\tilde{G}(x,\lambda,\varepsilon)$, respectively. Then,*

$$0\leq G(x^*,\lambda)-\tilde{G}(\bar{x},\lambda,\varepsilon)\leq n\lambda\frac{\varepsilon^2}{8}.$$

Proof. From Theorem 2.2 we obtain

$$\begin{aligned}
0\leq G(x^*,\lambda)-\tilde{G}(x^*,\lambda,\varepsilon) &\leq \frac{\lambda}{2}\left(\|H(x^*)\|^2-\left\|\tilde{H}(\bar{x},\varepsilon)\right\|^2\right)\\
&\leq \frac{\lambda}{2}\left\|H(x^*)-\tilde{H}(\bar{x},\varepsilon)\right\|^2\\
&\leq \frac{\lambda}{2}\left\|H(\bar{x})-\tilde{H}(\bar{x},\varepsilon)\right\|^2\\
&\leq n\lambda\frac{\varepsilon^2}{8}.
\end{aligned}$$

□

It is easy to see that $\tilde{G}(\bar{x},\lambda,\varepsilon)\to G(x^*,\lambda)$ as $\varepsilon\to 0$. We now give the following definition.

Definition 2.7. *A point x is called as $\tau-$approximate solution for (1.1) if the condition*

$$\|H(x)_+\|<\tau$$

is hold.

The following algorithm is proposed in order to solve (2.10), numerically.

Algorithm 1:

1 Choose the starting point x^0 and tolerance parameter $\tau=10^{-4}$. Select the parameters $\lambda^0>0$, $\varepsilon^0>0$, $L>1$, $0<\eta<1$ and let $k=0$.
2 Solve the problem (2.10) by using x^k as a starting point. Let x^{k+1} be the optimal solution.
3 If x^{k+1} is $\tau-$approximate solution, then stop. Otherwise, update the parameters $\varepsilon^{k+1}=\eta\varepsilon^k$, $\lambda^{k+1}=L\lambda^k$ and $k=k+1$, then go to Step 2.

It should be noted that Quasi Newton method is used in Step 1. Now, we prove the convergence of the Algorithm I. First, we define the level set as

$$\mathscr{L}(x^0)=\left\{x\in\mathbb{R}^n:\|H(x)_+\|^2\leq\|H(x^0)_+\|^2\right\}$$

for a stating point x^0.

Assumption 2. *For any $\varepsilon > 0$ and for a starting point x^0, the set*

$$\mathscr{L}_\varepsilon(x^0) = \left\{x \in \mathbb{R}^n : \|\tilde{H}(x,\varepsilon)\|^2 \leq \|\tilde{H}(x^0,\varepsilon)\|^2\right\}$$

is bounded.

Theorem 2.8. *Assume that Assumptions 1 and 2 are hold. Then, a sequence $\{x^k\}$ generated by* Algorithm *I converges to the optimal solution of the problem (2.2).*

Proof. It can be seen that $x^k \in \mathscr{L}_{\varepsilon^k}(x^0)$ for all $k \geq 0$. Since, $\mathscr{L}_{\varepsilon^k}(x^0)$ is bounded then there exists set $K \subset \mathbb{N}$ such that $\{x^k\}$ has a limit point for $k \in K$. Assume that $\bar{x}$ is a limit point of $\{x^k\}$. We have to show that $\bar{x}$ is the optimal solution for (2.2). Thus, it suffices to show that $\bar{x} \in \mathscr{L}(x^0)$ and $\|F(\bar{x})\|^2 \leq \inf_{x \in \mathscr{L}(x^0)} \|H(x)\|^2$.
Let us consider the contrary that $\bar{x} \notin \mathscr{L}(x^0)$, i.e. for sufficiently large $k \in K$, there exist $\beta_0 > \|H(x^0)_+\|^2$ and $i_0 \in \{1,2,\ldots,n\}$ such that

$$h_{i_0}^2(x^k) \geq \beta_0 > 0.$$

Since x^k is the global minimum according $k-$th values of the parameters ε^k and λ^k, for any $x \in \mathscr{L}(x^0)$ we have

$$\tilde{G}(x^0,\lambda^k,\varepsilon^k) \geq \tilde{G}(x^k,\lambda^k,\varepsilon^k) = \frac{\lambda^k}{2}\left(\left(\phi_{i_0}(x^k,\varepsilon)\right)^2 + \sum_{\substack{i=1\\ i\neq i_0}}^{n}\left(\phi_i(x^k,\varepsilon)\right)^2\right) \geq \frac{\lambda^k}{2}\beta_0.$$

If $k \to \infty$ then, $\lambda^k \to \infty$ and $\lim_{k\to\infty} \tilde{G}(x^k,\lambda^k,\varepsilon^k) = \infty$. It contradicts with the boundedness of the $\mathscr{L}_\varepsilon(x^0)$. Moreover, we have

$$\tilde{G}(x^k,\lambda^k,\varepsilon^k) \leq \tilde{G}(x,\lambda^k,\varepsilon^k)$$

for any $x \in \mathscr{L}(x^0)$. When $k \to \infty$, we have $\tilde{G}(\bar{x},\lambda,\varepsilon) \leq \tilde{G}(x,\lambda,\varepsilon)$. □

3. Numerical Results

In this section, we implement the Algorithm I to some test problems in order to evaluate the efficiency of Algorithm I. We compare our numerical results with the methods given in [2,7,8,16]. The numerical experiments have been performed on a PC with Intel Core i5-1035G1 CPU 1.00 GHz and 8GB RAM. The operating system is Windows 10 and the implementations have been done in MATLAB. At the algorithm, the parameters are taken as $\varepsilon_0 = 10^{-1}$ and $\eta = 0.1$. It is accepted that the problem is solved, if the accuracy 10^{-4} with respect to function value is obtained. The "fminunc" function is used as solver. The numerical test problems 1 to 7 are of the form (1.1) and the details are presented as follows:

Problem 1. *[1,2] Consider the function $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$H(x) = \begin{bmatrix} \sin(x_1) \\ \cos(x_2) \end{bmatrix}.$$

Problem 2. *[1,2] Consider the function $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that*

$$H(x) = \begin{bmatrix} (x_1-0.5)^2 + (x_2-1)^2 - 0.25 \\ -(x_1-0.5)^2 - (x_1-1.1)^2 + x_2^2 - 0.26 \\ x_2 + x_3^2 - 1 \end{bmatrix}.$$

Problem 3. *[1,2] Consider the function $H : \mathbb{R}^6 \to \mathbb{R}^6$ such that*

$$H(x) = \begin{bmatrix} \sin(x_1) + \varepsilon \\ -\cos(x_2) + \varepsilon \\ x_1 - 3\pi + x_3^2 + \varepsilon \\ x_2 - \pi/2 + x_4^2 + \varepsilon \\ -x_1 - \pi + x_5^2 + \varepsilon \\ -x_2 - \pi/2 + x_6^2 + \varepsilon \end{bmatrix},$$

where $\varepsilon = 10^{-5}$.

Problem 4. *[16] Consider $H:\mathbb{R}^2\to\mathbb{R}^2$ such that*

$$H(x)=\begin{bmatrix}-x_1^2-x_2^2-x_1+1+\varepsilon\\ -x_1^2-x_2^2+2x_2+2+\varepsilon\end{bmatrix},$$

where $\varepsilon=10^{-7}$.

Problem 5. *[16] Consider $H:\mathbb{R}^5\to\mathbb{R}^5$ such that*

$$H(x)=\begin{bmatrix}\sin(x_1)+\varepsilon\\ -\cos(x_2)+\varepsilon\\ 2x_1-x_1^2+x_3^2+\varepsilon\\ 2x_2-x_2^2+x_4^2+\varepsilon\\ x_1-x_2+x_5^2+\varepsilon\end{bmatrix},$$

where $\varepsilon=10^{-7}$.

Problem 6. *[16] Consider $H:\mathbb{R}^3\to\mathbb{R}^3$ such that*

$$H(x)=\begin{bmatrix}x_1(x_1-2)(x_1-3)+\varepsilon\\ x_2^2-3x_2+2+\varepsilon\\ x_1^2-x_2+x_3^2+\varepsilon\end{bmatrix},$$

where $\varepsilon=10^{-7}$.

Problem 7. *[8] Consider $H:\mathbb{R}^2\to\mathbb{R}^2$ such that*

$$H(x)=\begin{bmatrix}x_1^2+x_2^2-1+\varepsilon\\ -x_1^2-x_2^2+(0.999)^2+\varepsilon\end{bmatrix},$$

where $\varepsilon=10^{-5}$.

We apply our algorithm also to test problems of the following form:

$$H(x)=\begin{cases}h_I(x)\leq 0,\\ h_J(x)=0,\end{cases}\tag{3.1}$$

where $I=\{1,2,\ldots,m\}$ and $J=\{m+,m+2,\ldots,n\}$. The function $h_I:\mathbb{R}^n\to\mathbb{R}^m$ is defined as

$$h_I=\begin{bmatrix}h_1(x)\\ h_2(x)\\ \vdots\\ h_m,\end{bmatrix}$$

and $h_J:\mathbb{R}^n\to\mathbb{R}^{n-m}$ is defined as

$$h_J=\begin{bmatrix}h_m(x)\\ h_{m+1}(x)\\ \vdots\\ h_n,\end{bmatrix}$$

where the component functions $f_i, f_j : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable. In other words, the function $H : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$H(x) = \begin{bmatrix} h_I(x) \\ \\ h_J(x) \end{bmatrix}.$$

It is easy to see that when the set J is empty, the system (3.1) corresponds to the system (1.1). Now consider the following test problems which are of the form (3.1).

Problem 8. *[8] Consider $H : \mathbb{R}^5 \to \mathbb{R}^5$ such that*

$$h_I(x) = \begin{bmatrix} x_1 + x_3 - 1.6 + \varepsilon \\ \\ 1.333x_2 + x_4 - 3 + \varepsilon \\ \\ -x_3 - x_4 + x_5 + \varepsilon \end{bmatrix} \qquad \text{and} \qquad h_J(x) = \begin{bmatrix} x_1^2 + x_3^2 - 1.25, \\ \\ x_2^{1.5} + 1.5x_4 - 3 \end{bmatrix},$$

where $\varepsilon = 10^{-5}$.

Problem 9. *[8] Consider $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that*

$$h_I(x) = \begin{bmatrix} x_1 + x_2 e^{0.8x_3} + e^{1.6} + \varepsilon \\ \\ x_1^2 + x_2^2 + x_3^2 - 5.2675 + \varepsilon \end{bmatrix} \qquad \text{and} \qquad h_J(x) = \left[x_1 + x_2 + x_3 - 0.2605\right],$$

where $\varepsilon = 10^{-5}$.

Problem 10. *[8] Consider $H : \mathbb{R}^2 \to \mathbb{R}^3$ such that*

$$h_I(x) = \left[0.8 - e^{x_1 + x_2} + \varepsilon\right] \qquad \text{and} \qquad h_J(x) = \begin{bmatrix} 1.21e^{x_1} + e^{x_2} - 2.2 \\ \\ x_1^2 + x_2^2 + x_2 - 0.1135 \end{bmatrix},$$

where $\varepsilon = 10^{-5}$.

Problem 11. *[8] Consider $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that*

$$h_I(x) = \left[x_1^2 + x_2^2 + x_3^2 - 10000 + \varepsilon\right] \qquad \text{and} \qquad h_J(x) = \begin{bmatrix} x_1 - 0.7\sin(x_1) - 0.2\cos(x_2) \\ \\ x_2 - 0.7\cos(x_1) + 0.2\sin(x_2) \end{bmatrix},$$

where $\varepsilon = 10^{-5}$.

Table 1: The numerical results

PN	SP	FSP	TIN	TFE	SFV	FV	Time
1	$(0,1)$	$(3.2471e-06, 0.3155)$	2	12	$2.9236e-06$	$1.0544e-11$	0.2738
2	$(1,1,1)$	$(0.8759, 0.6721, 0.57163)$	14	72	$6.2936e-07$	$3.3391e-08$	0.3742
3	$(0,1,1,1,1,1)$	$(0.0000, 0.3156, 1.0000, 1.0000, 1.0000, 1.0000)$	2	28	$7.1189e-05$	$9.2784e-08$	0.3744
4	$(1,1)$	$(2,1)$	1	6	0	0	0.3342
5	$(-1,0,1,2,2)$	$(-1.3024, -0.3418, 1, 0.8945, 0.9801)$	11	90	$4.4538e-08$	$1.8001e-11$	0.3986
6	$(1,2,2)$	$(-0.5814, 2.0039, 0.8772)$	2	16	$5.2416e-05$	$1.5545e-05$	0.3118
7	$(0,5)$	$(-2.0903e-09, 0.9994)$	6	36	$7.4149e-05$	0	0.3498
8	$(0.5,2,1,0,0)$	$(0.5018, 2.0535, 0.9991, 0.0382, 0)$	4	48	$7.3160e-09$	$7.275e-09$	0.3065
9	$(-1,1,1)$	$(-0.8353, -0.8601, 1.9564)$	16	84	$5.9864e-06$	$1.5566e-05$	0.3715
10	$(0,0,0)$	$(-0.1015, 0.0992, 0)$	4	32	$4.0599e-05$	$4.0599e-05$	0.3504
11	$(0,1,0)$	$(0.4968, 0.6296, 0)$	6	36	$3.0136e-06$	$3.0136e-06$	0.3166

The results of the numerical experiments are presented in Table 1. In the Table 1, the problem number (PN), the starting point (SP), total iteration number (TIN), total function evaluation (TFE), founded solution point (FSP), the norm value of each smoothed problem $\|\tilde{H}(x,\varepsilon)\|$ in the problem (SFV), the norm value of each problem $\|H(x)\|$ in the problem (FV) and total CPU time (Time) are reported. The satisfactory results are obtained for all test problems. Since, our smoothing functions has the same value with the original function at the same location, the minimum value is obtained with lower number of iteration. This property is the main advantage of our method.

The results are compared with the results obtained from the methods suggested in [2,7,8,16] in terms of "TIN" and "TFE". The results are presented in Table 2 and it is observed that all of the test problems are successfully solved by using our algorithm. Moreover our algorithm presents better results in 82% of all the test problems than other methods in terms of "TIN".

Table 2: The comparison of the numerical results with the competing algorithms

	Algorithm I		Algorithm 3.1 in [2]		Algorithm in [16]		Algorithm 2.1 in [8]		Algorithm 3.1 in [7]	
PN	TIN	TFE	TIN	TFE	TIN	TFE	TIN	TFE	TIN	TFE
1	2	12	3	-	-	-	-	-	3	3
2	14	72	4	-	-	-	-	-	-	-
3	2	28	5	-	-	-	-	-	6	8
4	1	6	-	-	2	-	-	-	-	-
5	11	90	-	-	14	-	-	-	-	-
6	2	16	-	-	18	-	-	-	-	-
7	6	36	-	-	-	-	8	12	8	9
8	4	48	-	-	-	-	5	6	4	4
9	16	84	-	-	-	-	24	39	5	5
10	4	32	-	-	-	-	6	8	4	4
11	6	36	-	-	-	-	10	16	9	14

4. Conclusion

A new algorithm with the new smoothing approach is proposed to solve SNI and the convergence of the algorithm is theoretically presented. The efficiency of our algorithm is illustrated on test problems in the literature. The superiority of our method among the similar algorithms is proved numerically by considering Table 2. According to the comparison of the results with the other methods, it is shown that the Algorithm I has many advantages in terms of computational costs. On the other hand, this study presents a methodology to solve these kinds of problems.
For future works, the proposed smoothing approach can also be applied to other non-smooth problems such as min-max, complementarity, exact penalty, l_1 signal reconstruction and etc. Furthermore, the smoothing function can be used along with the other algorithms such as Newton type and Conjugate gradient algorithms, and related numerical performance can be investigated accordingly.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: The authors contributed equally to the writing of this paper.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Copyright Statement: Authors own the copyright of their work published in the journal and their work is published under the CC BY-NC 4.0 license.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

References

[1] Z. H. Huang, Y. Zhang, W. Wu, *A smoothing-type algorithm for solving system of inequalities*, J. Comput. Appl. Math., **220** (2008), 355–363.
[2] Y. Liuyang, C. Fei, W. Zhongping, L. Weigang, W. Wenbo, *A nonmonotone smoothing Newton method for system of nonlinear inequalities based on a new smoothing function*, Comput Appl. Math., **134** (2019), 38–91.
[3] C. Wu, J. Wang, J. H. Alcantara and J. S. Chen, *Smoothing strategy along with conjugate gradient algorithm for signal seconstruction*, J. Sci. Comput. **87** (2021), 21.
[4] Y. Xiao, Q. Wang, Q. Hu, *Non-smooth equations based method for ℓ_1-norm problems with applications to compressed sensing*, Nonlinear Anal. Theory Methods Appl., **74**(11) (2011), 3570–3577.
[5] A. S. Halilu, A. Majumder, M. Y. Waziri, K. Ahmed, A.M Awwal, *Motion control of the two joint planar robotic manipulators through accelerated Dai–Liao method for solving system of nonlinear equations*, Eng. Comput., **39**(5) (2022), 1802–1840.
[6] D. Malyshev, L. Rybak, G. Carbone, T. Semenenko, A. Nozdracheva, *Optimal design of a parallel manipulator for aliquoting of biomaterials considering workspace and singularity zones*, Appl. Sci., **12**(4) (2022), 2070.
[7] Y. Zhang, Z.-H. Huang, *A nonmonotone smoothing-type algorithm for solving a system of equalities and inequalities*, J. Comput. Appl. Math., **233** (2010) 2312–2321.
[8] J. S. Chen, C. H. Ko, Y. D. Liu, S. P. Wang, *New smoothing functions for solving a system of equalities and inequalities*, Pacific J. Optim, **12**(1) (2016), 185-206.
[9] G. Yuan, T. Li, W. Hu, *A conjugate gradient algorithm for large-scale nonlinear equations and image restoration problems*, Appl. Numer. Math., **147** (2020), 129–141.

[10] R. Li, M. Cao, G. Zhou, *A new adaptive accelerated Levenberg–Marquardt method for solving nonlinear equations and its applications in supply chain problems*, Symmetry, **15**(3) (2023), 588.
[11] D.Q. Mayne, E. Polak, A. J. Heunis, *Solving nonlinear inequalities in a finite number of iterations*, J. Optim. Theory Appl., **33**(2) (1981), 207–221.
[12] H. Che, M. Li, *A smoothing and regularization Broyden-like method for nonlinear inequalities*, J. Appl. Math. Comput., **41** (2013) 209–227.
[13] A. B. Abubakar, P. Kumam, *A descent Dai-Liao conjugate gradient method for nonlinear equations*, Numer. Algorithms, 81 (2019), 197-210.
[14] M. Y. Waziri, K. Ahmed, K. *Two descent Dai-Yuan conjugate gradient methods for systems of monotone nonlinear equations*, J. Sci. Comput., **90** (2022), 1-53.
[15] Z. Saeidian, M. R. Peyghami, M. Habibi, S. Ghasemi, *A new trust-region method for solving systems of equalities and inequalities*, Comp. Appl. Math. **36** (2017), 769 70.
[16] X. Fan, Q. Yan, *Solving system of inequalities via a smoothing homotopy method*, Numer Algorithms **82** (2019), 719–728.
[17] L. Qi, D. Sun, *Smoothing functions and smoothing Newton method for complementarity and variational inequality problems*, J. Optim. Theory Appl. **113** (2002), 121–147.
[18] S. Huang, Z. Wan, X. H. Chen, *A new nonmonotone line search technique for unconstrained optimization*, Numer. Algorithms **68** (4) (2015), 671–689.
[19] H. C. Zhang, W. W. Hager, *A nonmonotone line search technique and its application to unconstrained optimization*, SIAM J. Optim., **14**(4) (2004), 1043–756.
[20] X. Chen, *Smoothing methods for nonsmooth, nonconvex minimization*, Math. Program. Ser. (B), **134** (2012), 71–99.
[21] D. Bertsekas, *Nondifferentiable optimization via approximation*, Math Programmming Study, **3** (1975) 1–25.
[22] I. Zang, *A smooting out technique for min-max optimization*, Math. Programm., **19** (1980), 61–77.
[23] C. Chen, O.L. Mangasarian, *A class of smoothing functions for nonlinear and mixed complementarity problem*, Comput. Optim. Appl., **5** (1996) 97-138.
[24] A. E. Xavier, *Optimal covering of plane domains by circles via hyperbolic smoothing*, J. Global Optim., **31** (2005), 493–504.
[25] A. M. Bagirov, A. Al Nuamiat, N. Sultanova, *Hyperbolic smoothing functions for nonsmooth minimization*, Optimization, **62** (6) (2013), 759-782.
[26] A. Sahiner, G. Kapusuz, N. Yilmaz, *A new smoothing approach to exact penalty functions for inequality constrained optimization problems*, Numer. Algebra Control Optim., **6** (2) (2016) 161–173.
[27] N. Yilmaz, A. Sahiner, *On a new smoothing technique for non-smooth, non-convex optimization*, Numer. Algebra Control Optim., 10 (2020), 317–330.
[28] C. Grossmann, *Smoothing techniques for exact penalty function methods*, *Contemporary Mathematics: A Panaroma of Mathematics Pure and Applied*, American Mathematical Society, Rhode Island, **658** (2016) 249–265.
[29] L. Qi, P. Tseng, *On almost smooth functions and piecewise smooth functions*, Nonlinear Anal. **67** (2007) 773–794.
[30] N. Yilmaz, A. Sahiner, *New smoothing approximations to piecewise smooth functions and applications*, Numer. Func. Anal. Optim., **40** (2019), 513–534.

Fundamental Journal of Mathematics and Applications, 6 (3) (2023) 147-156
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: *www.dergipark.org.tr/en/pub/fujma*
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1299982

Set-Valued Stabilization of Reaction-Diffusion Model by Chemotherapy and or Radiotherapy

Amine Moustafid[1]

[1] *Department of Mathematics and Informatics, Faculty of Science Ain Chock , Hassan II University, Casablanca, Morocco*

Article Info

Keywords: *Chemotherapy, Radiotherapy, Reaction-diffusion equation, Set-valued analysis, Viability theory*
2010 AMS: *93B52, 93C20, 93D15*
Received: *20 May 2023*
Revised: *8 September 2023*
Accepted: *14 September 2023*
First Online: *19 September 2023*
Published: *30 September 2023*

Abstract

This paper aims to control partial differential equations, modeling cancer chemotherapy and or radiotherapy, so in order to asymptotically stabilize the tumor density. Viability kernel of general model on set of initial condition is used to solve the control problem, and characterize the control solution as regulation law of regulation map. Three models from the literature are considered to simulate the results. The first model includes chemotherapy effect on logistic tumor proliferation, while the second one demonstrates radiotherapy effect on exponential tumor increasing, whereas the third one models the effects of the combination of chemotherapy and radiotherapy on Gompertzian tumor growth.

1. Introduction

Partial differential equations are usually used in cancer modeling to describe the tumors progression, and they provide a valuable tool for cancer researchers to understand and predict the behavior of tumors, and to develop new therapies to combat the cancer disease.
The reaction-diffusion equations used in [1, 14] to model cancer, are unified in the general partial derivatives equation

$$u_t = \nabla \cdot (D(x)\nabla u) + f(u,c), (x,t) \in \Omega \times [0,T], \tag{1.1a}$$

where $t \in [0,T] \subset \mathbb{R}_+$ $(0 < T < \infty)$ and $x \in \Omega \subset \mathbb{R}^m$ $(m \in \mathbb{N}^*)$ denote time and position dependent of therapy by the vector valued function $c(x,t) \in \mathbb{R}^\ell$ $(\ell \in \mathbb{N}^*)$ on the tumor density by the scalar valued function $u(x,t) \in \mathbb{R}$,

$$c : \bar{\Omega} \times [0,T] \to \mathbb{R}^\ell \text{ and } u : \bar{\Omega} \times [0,T] \to \mathbb{R},$$

while u_t and ∇ denote the temporal derivative $\partial_t u$ and the spatial gradient operator $(\partial/\partial x_i)_{1\le i\le m}$, respectively, furthermore $D(x) \in L^\infty(\Omega)$ is the spatially varying scalar diffusion coefficient, as well as the reaction-control term $f(u,c)$ is given.
The partial differential equation (1.1a) is supplied with the boundary condition

$$n \cdot \nabla u = 0, (x,t) \in \partial\Omega \times [0,T], \tag{1.1b}$$

where n is the normal vector on the boundary $\partial\Omega$, and augmented by the initial condition

$$u(x,0) = u_0(x), \text{for almost all } x \in \Omega, \text{where the initial state } u_0 \in L^2(\Omega). \tag{1.1c}$$

Several studies have been conducted on the system (1.1). [1] Uses Crank-Nicolson scheme to solve three models of glioma. [2] Presents an alternative fractional differential equation, to investigate the concentration of glioblastomas by using the theory of

Email address and ORCID number: **a.moustafid@gmail.com, 0000-0002-9121-0745 (A. Moustafid)**
Cite as: A. Moustafid, *Set-valued stabilization of reaction-diffusion model by chemotherapy and or radiotherapy*, Fundam. J. Math. Appl., 6(3) (2023), 147-156.

fractional calculus. [3] Introduces a weighted parameter diffusion of brain glioma. [4] Uses statistical procedures to estimate intra- and inter-patient heterogeneity for tumor growth model. [5] Investigates the stability of the Fisher-Stefan equation. [6] Analyzes a spectral regularization of a time-reversed model problem. [7] Solves a constrained optimization problem to calibrate tumor growth model. [8] Introduces and studies model to explain the dynamics of cancer propagation. [9] Presents numerical scheme driven from Fibonacci wavelets to solve Burgess model of brain tumor growth. [10] Focuses on the Fisher-KPP model of tumor growth. [11] Investigates the bang-bang property under spatio-temporal controls. [12] Uses equation to simulate the growth of the glioblastoma and radiotherapy prevention. [13] Solves equation governing tumor growth in human brain under radiotherapy. [14] Derives a nonlinear conjugate gradient method for identifying treatment parameter of brain tumors under therapy.
The originality of this paper is that the set-valued methods developed to feedback stabilize the tumor density $u(t)$,

$$u(t) \to 0, \text{ when } t \to \infty,$$

subject to the odes [15, 16, 17]

$$\begin{aligned} \dot{u}(t) &= f(t,u,c), u(0) = u_0, \\ \dot{c}(t) &= g(c), \end{aligned}$$

and to the odes [18, 19, 20, 21, 22, 23]

$$\begin{aligned} \dot{u}(t) &= f(u,v,c), u(0) = u_0, \\ \dot{v}(t) &= g(u,v,c), v(0) = v_0, \end{aligned}$$

where the time dependent vector $v = v(t) \in \mathbb{R}^k$ ($k \in \mathbb{N}^*$) denotes densities of interactive cells with the tumor density $u(t)$; will be adapted here to stabilize the tumor density $u(x,t)$, subject to the partial differential equation (1.1), and applied on the following models.

Model	Aim	Reference
$u_t = \nabla \cdot (D(x)\nabla u) + \rho u(1-u) - c(x,t)u$	Develops a gradient based algorithm to optimize chemotherapy.	[24]
$u_t = \nabla \cdot (D(x)\nabla u) + \rho u - c(x,t)u$	Extends Swanson's equation to consider the radiotherapy effect.	[25]
$u_t = \nabla \cdot (D(x)\nabla u) - \rho u \ln u - c(x,t)u$	Shows the effects of combined radiotherapy with chemotherapy.	[26]

Table 1: Samples of reaction-diffusion equations with: logistic, exponential, and Gompertz growth laws, under chemotherapy and or radiotherapy.

The structure of this paper is: section 1 introduced preceding papers [1, 14] on cancer modeling and analysis, section 2 proposes to solve the associated stabilizing problem in the viability framework, section 3 applies the obtained results to therapeutic models implying chemotherapy and or radiotherapy, and section 4 concludes by the effectiveness of the combination of chemotherapy and radiotherapy.

2. Problem statement and viability approach

This section considers the general distributed control system (2.1) and associates the corresponding control problem 1, gives the definition 2.1 to the viability property of the state solution, sets the closed subsets (2.4) to express the problem 1 in the viability sens by the proposition 2.2, characterizes the viability property by the regulation map (2.6) in the corollary 2.4, recalls the lemma 2.5 for the contingent cone calculus, and gets useful expression of the regulation law (2.8) in the corollary 2.6 by (2.10).

- State equation

$$u_t = Au + f(u,c), \tag{2.1a}$$

where A is an elliptic differential operator and f is a function from $\mathbb{R} \times \mathscr{U}$ to $\mathbb{R}$, where $\mathscr{U}$ is a Hilbert space of controls.
- State-dependent feedback controls

$$c(x,t) \in U(x,u(x,t)), \text{ for almost all } (x,t) \in \Omega \times [0,T], \tag{2.1b}$$

where U is a multifunction from $\Omega \times \mathbb{R}$ to $\mathscr{U}$.
- Dirichlet constraint

$$u(x,t)|_{\partial\Omega} = 0, \text{ for all } t \in [0,T]. \tag{2.1c}$$

- Neumann constraint

$$n \cdot \nabla u(x,t)|_{\partial\Omega} = 0, \text{ for all } t \in [0,T]. \tag{2.1d}$$

- Onset state

$$u(x,0) = u_0(x), \text{ for almost all } x \in \Omega. \tag{2.1e}$$

Problem 1. *Find a control $c \in L^2(\Omega, \mathscr{U})$ rending the tumor density $u \in L^2(\Omega)$ asymptotically stable, i.e.,*

$$u(x,t) \to 0, \text{ when } t \to \infty, \text{ for almost all } x \in \Omega. \tag{2.2}$$

Definition 2.1. *System (2.1) is viable in a closed subset $K \subset L^2(\Omega)$, if for any initial state $u_0 \in K$ the system (2.1) admits viable solution u in K, i.e.,*

$$\forall t \geq 0, u(\cdot,t) \in K. \tag{2.3}$$

Let be the subsets

$$K_\alpha := \{u \in L^2(\Omega), u \geq 0 \text{ and } \psi_\alpha(u) \leq 0, \text{ almost everywhere}\}, \tag{2.4a}$$

where the function ψ_α expression is

$$\psi_\alpha(u)(x,t) = u(x,t) - u_0(x)\exp(-\alpha t), \tag{2.4b}$$

and parameter α is as follows

$$\alpha \in \mathbb{R}_+^*. \tag{2.4c}$$

Proposition 2.2. *If $c \in L^2(\Omega, \mathscr{U})$ rends viable the distributed control system (2.1) in a subset (2.4), then c is solution to the problem 1.*

Proof. For almost all $x \in \Omega$

$$\begin{aligned} \psi_\alpha(u)(x,0) &= u(x,0) - u_0(x)\exp(-\alpha \times 0) \\ &= u(x,0) - u_0(x) \\ &= 0, \end{aligned}$$

which imlplies that $u_0 \in K_\alpha$.
And for all $t \geq 0$

$$\begin{aligned} u \in K_\alpha \quad &\Longrightarrow \quad u(x,t) - u_0(x)\exp(-\alpha t) \leq 0 \\ &\Longrightarrow \quad u(x,t) \leq u_0(x)\exp(-\alpha t) \\ &\Longrightarrow \quad \lim_{t \to \infty} u(x,t) = 0. \end{aligned}$$

The initial state u_0 in (2.1e) is supposed regular enough (for example upper bounded) so that $u_0(x)\exp(-\alpha t)$ admits null limit. □

Remark 2.3. *The non-negative reel parameter α is introduced to further reduce the tumor density $u(x,t)$.*

Assumption 1. *Let introduce the following hypotheses for the next corollary.*

(H0): *The subset Ω is bounded open such that the trace operators*

$$\begin{aligned} \gamma\colon H^1(\Omega) &\to H^{1/2}(\partial\Omega) \\ u(x) &\mapsto u(x)|_{\partial\Omega} \end{aligned}$$

and

$$\begin{aligned} \delta\colon H^1(\Omega) &\to H^{-1/2}(\partial\Omega) \\ u(x) &\mapsto n \cdot \nabla u(x)|_{\partial\Omega} \end{aligned}$$

are surjective continuous linear.

(H1): *The single-valued map f is continuous affine to c and linear growth to c,*

$$\sup_u |f(u,c)| \leq \varsigma(\|c\|+1). \tag{2.5}$$

(H2): *The set-valued map* $\mathbb{U} : L^2(\Omega) \rightsquigarrow L^2(\Omega, \mathscr{U})$ *defined by*

$$\mathbb{U}(u(\cdot)) := \{c(\cdot) \in L^2(\Omega, \mathscr{U}), c(x) \in U(x, u(x)), \textit{ for almost all } x \in \Omega\},$$

is bounded and upper semicontinuous with closed convex images.

Corollary 2.4 ([27, Corollary 13.4.2])**.** *Let* $K \subset L^2(\Omega)$ *be a closed subset such that some* $\|u_0\|_{H_0^1(\Omega)} \leq \kappa$ *belongs to its interior in* $L^2(\Omega)$*. Then* K *is viable when and only when the regulation map*

$$R_K(u) := \{c \in \mathbb{U}(u), Au + f(u,c) \in T_K(u)\}, \tag{2.6}$$

where $T_K(u)$ *is the contingent cone, enjoys non-emptiness property, in the sens that*

$$\forall u \in K, R_K(u) \neq \emptyset, \tag{2.7}$$

and viable solution u *is given by regulation law* c*,*

$$c(x,t) \in R_K(u(x,t)), \textit{ for almost all } (x,t) \in \Omega \times [0,T]. \tag{2.8}$$

Lemma 2.5 ([16, Lemma 3.3])**.** *The belonging of directions* $\bar{u}$ *in* $T_{K_\alpha}(u)$ *is characterized by both inequalities*

$$\bar{u} \geq 0 \quad \textit{if} \quad u(x,t) = 0, \tag{2.9a}$$

$$D\psi_\alpha(u(x,t))\bar{u} \leq 0 \quad \textit{if} \quad \psi_\alpha(u(x,t)) = 0, \tag{2.9b}$$

where D *denotes the differential operator.*

Corollary 2.6. *The regulation law* c *of the regulation map* R_{K_α} *is characterized by*

$$Au(x,t) + f(u(x,t), c(x,t)) \geq 0 \quad \textit{if} \quad u(x,t) = 0, \tag{2.10a}$$

$$D\psi_\alpha(u(x,t))(Au(x,t) + f(u(x,t), c(x,t)) \leq 0 \quad \textit{if} \quad \psi_\alpha(u(x,t)) = 0. \tag{2.10b}$$

Proof. By definition of the regulation map R_{K_α}

$$c(x,t) \in R_{K_\alpha}(u(x,t)),$$

if and only if

$$Au(x,t) + f(u(x,t), c(x,t)) \in T_{K_\alpha}(u(x,t)),$$

and by characterization (2.9) of contingent cone $T_{K_\alpha}(u(x,t))$ of the subsets K_α at $u(x,t)$

$$\begin{aligned} Au(x,t) + f(u(x,t), c(x,t)) \geq 0 \quad &\text{if} \quad u(x,t) = 0, \\ D\psi_\alpha(u(x,t))(Au(x,t) + f(u(x,t), c(x,t)) \leq 0 \quad &\text{if} \quad \psi_\alpha(u(x,t)) = 0. \end{aligned}$$

□

3. Therapy application

This section is a numerical application of the theoretical results in the previous section 2, for chemotherapy and or radiotherapy, we consider the three representative models in the Table 1, with the regular domain $\Omega =]0,1[$ for the hypothesis $(H0)$, and the Neumann boundary condition (2.1d) (even (1.1b)), without any state-constraint controls (2.1b) (no need to check hypothesis $(H2)$). For all given simulations the parameter $\alpha = 1$ (2.4c).

3.1. Chemotherapy

Let consider the numerical model [24]

$$u_t = \nabla \cdot (0.65 \nabla u) + 0.012(1-u)u - c(x,t)u, \quad (x,t) \in]0,1[\times [0,T], \tag{3.1a}$$

$$n \cdot \nabla u = 0, \quad (x,t) \in \{0,1\} \times [0,T], \tag{3.1b}$$

$$u(x,0) = 0.5 + 0.5\cos^2(\pi x), \quad \text{for almost all } x \in]0,1[, \tag{3.1c}$$

for $(H1)$ hypothesis

$$\begin{aligned} \sup_{u \in K_1} |f(u,c)| &= \sup_{u \in K_1} |0.012(1-u)u - cu| \\ &\leq 1 + \|c\| \max_{x \in]0,1[} u(x,0) \\ &\leq \|c\| + 1, \end{aligned}$$

as for (2.10a)

$$\nabla\cdot(0.65\nabla u(x,t))+0.012(1-u(x,t))u(x,t)-c(x,t)u(x,t)=0 \text{ if } u(x,t)=0,$$

only (2.10b) is used to simulate chemotherapy control $c(x,t)$ in Figure 3.3.

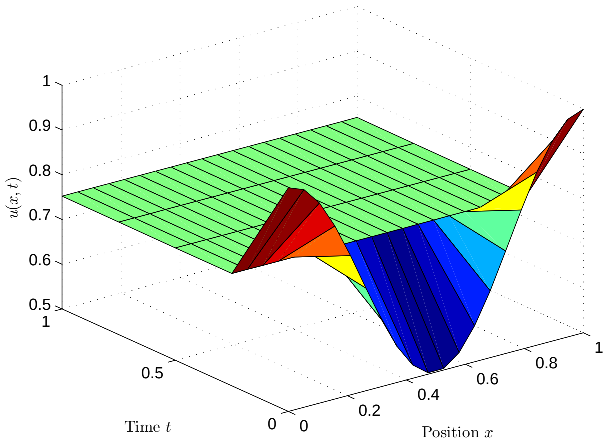

Figure 3.1: Spatio-temporal tumor evolution $u(x,t)$ subject to (3.1).

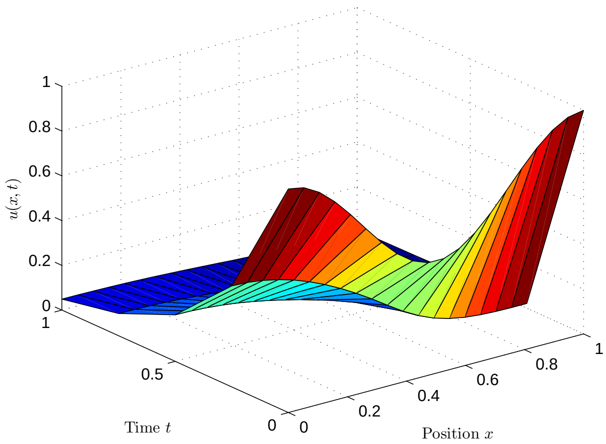

Figure 3.2: Spatio-temporal tumor response $u(x,t)$ subject to (3.1) by spatio-temporal chemotherapy control $c(x,t)$ in Figure 3.3.

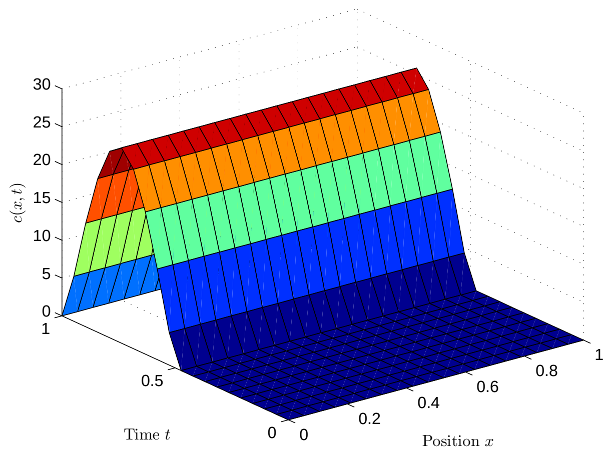

Figure 3.3: Spatio-temporal chemotherapy control $c(x,t)$ given by (2.10b).

3.2. Radiotherapy

Let consider the numerical model [25]

$$\begin{aligned} u_t &= \nabla\cdot(1.43\nabla u)+16.25u-c(x,t)u, \quad (x,t)\in]0,1[\times[0,T], &(3.2a)\\ n\cdot\nabla u &= 0, \quad (x,t)\in\{0,1\}\times[0,T], &(3.2b)\\ u(x,0) &= 20^3\exp(-100x^2), \quad \text{for almost all } x\in]0,1[, &(3.2c)\end{aligned}$$

as for $(H1)$ hypothesis

$$\begin{aligned} \sup_{u\in K_1}|f(u,c)| &= \sup_{u\in K_1}|16.25u-cu| \\ &\le 16.25\max_{x\in]0,1[}u(x,0)+\|c\|\max_{x\in]0,1[}u(x,0) \\ &\le 16.25\times 20^3(\|c\|+1),\end{aligned}$$

as for (2.10a)

$$\nabla\cdot(1.43\nabla u(x,t))+16.25u(x,t)-c(x,t)u(x,t)=0 \text{ if } u(x,t)=0,$$

only (2.10b) is used to simulate radiotherapy control $c(x,t)$ in Figure 3.6.

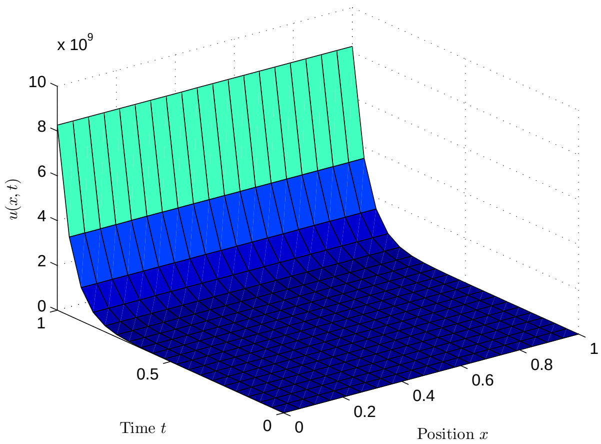

Figure 3.4: Spatio-temporal tumor evolution $u(x,t)$ subject to (3.2).

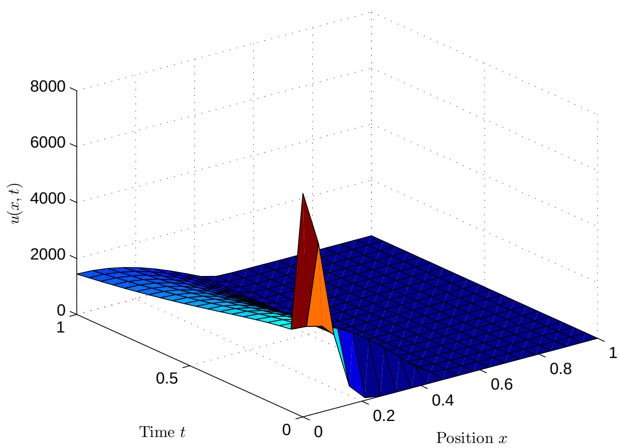

Figure 3.5: Spatio-temporal tumor response $u(x,t)$ subject to (3.2) by spatio-temporal radiotherapy control $c(x,t)$ in Figure 3.6.

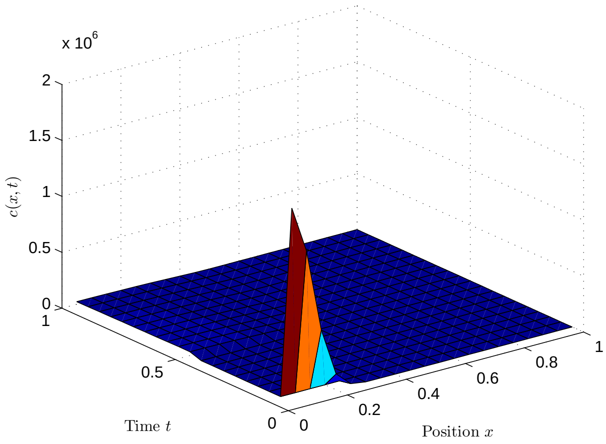

Figure 3.6: Spatio-temporal radiotherapy control $c(x,t)$ given by (2.10b).

3.3. Combination therapy

Let consider the numerical model [26]

$$\begin{aligned} u_t &= \nabla\cdot(\nabla u) - 0.012u\ln(u) - 0.0552c_1(x,t)u - c_2(x,t)u, \quad (x,t)\in]0,1[\times[0,T], && (3.3a)\\ n\cdot\nabla u &= 0, \quad (x,t)\in\{0,1\}\times[0,T], && (3.3b)\\ u(x,0) &= 1.25\exp\left(\frac{-x^2}{0.65}\right), \quad \text{for almost all } x\in]0,1[, && (3.3c) \end{aligned}$$

as for $(H1)$ hypothesis

$$\begin{aligned} \sup_{u\in K_1}|f(u,c)| &= \sup_{u\in K_1}|-0.012u\ln(u) - 0.0552c_1u - c_2u| \\ &\leq 1 + \|c\| \max_{x\in]0,1[} u(x,0) \\ &\leq 1.25(\|c\|+1), \end{aligned}$$

as for (2.10a)

$$\nabla\cdot(\nabla u(x,t)) - 0.012u(x,t)\ln(u(x,t)) - 0.0552c_1(x,t)u(x,t) - c_2(x,t)u(x,t) = 0 \text{ if } u(x,t)=0, (\lim_{u\to 0^+} u\ln u = 0)$$

only (2.10b) is used to simulate chemotherapy control $0.0552c_1(x,t)$ in Figure 3.9 and radiotherapy control $c_2(x,t)$ in Figure 3.10.

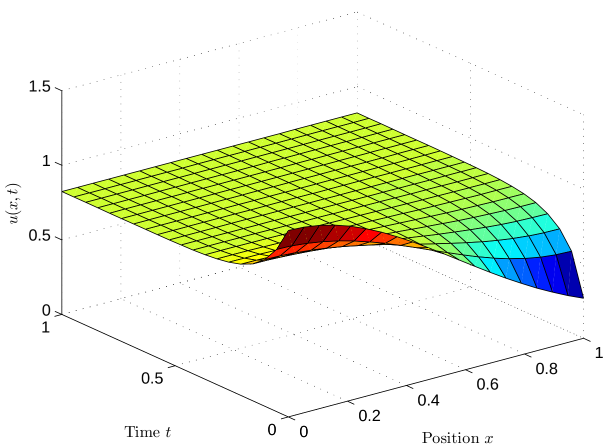

Figure 3.7: Spatio-temporal tumor evolution $u(x,t)$ subject to (3.3).

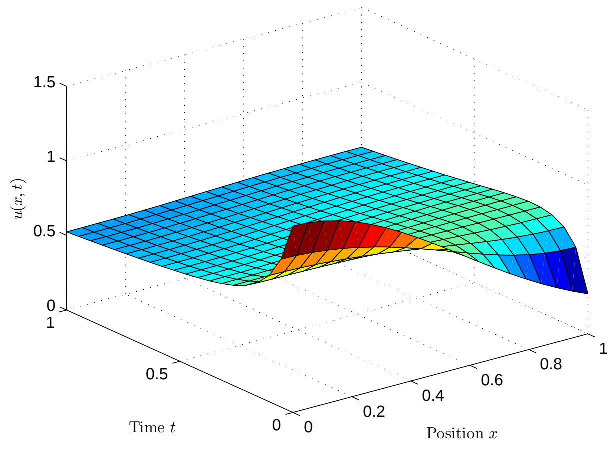

Figure 3.8: Spatio-temporal tumor response $u(x,t)$ subject to (3.3) by spatio-temporal chemotherapy control $0.0552c_1(x,t)$ in Figure 3.9 combined with radiotherapy control $c_2(x,t)$ in Figure 3.10.

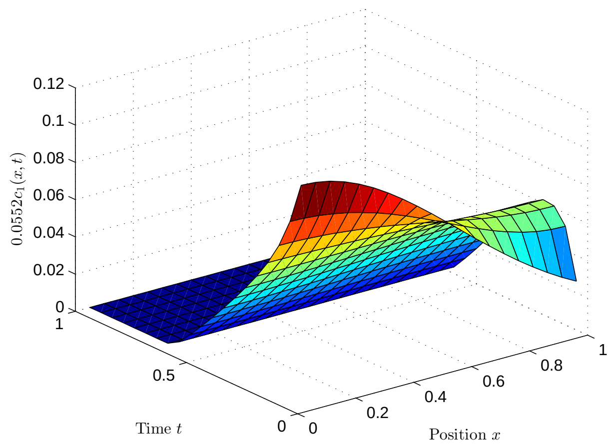

Figure 3.9: Spatio-temporal chemotherapy control $0.0552c_1(x,t)$, where c_1 is given by (2.10b).

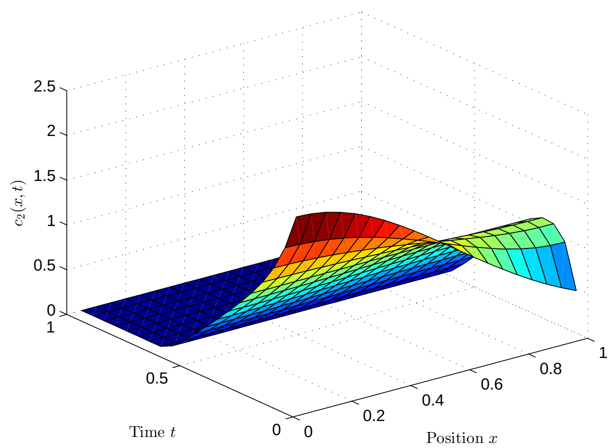

Figure 3.10: Spatio-temporal radiotherapy control $c_2(x,t)$ given by (2.10b).

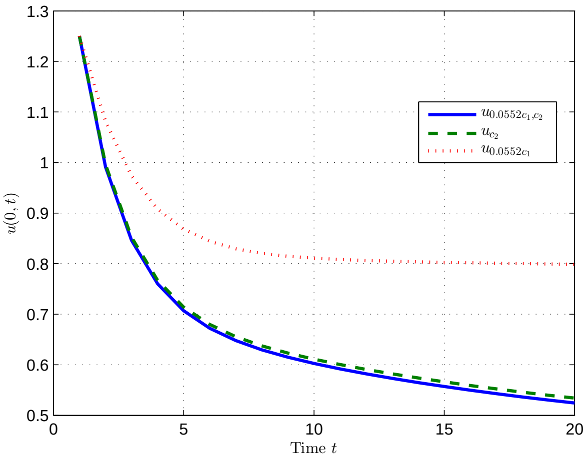

Figure 3.11: Temporal tumor response $u(0,t)$ subject to (3.3) by temporal chemotherapy control $0.0552c_1(0,t)$ in Figure 3.9 and or radiotherapy control $c_2(0,t)$ in Figure 3.10.

4. Conclusion

The control problem 1 is associated to the general system (1.1) in order to stabilize tumor density (2.2). Proposition 2.2 proves that any control rending the system (2.1) viable in a subset of the type (2.4) is a solution to the problem 1. Such control is obtained as the regulation law (2.10) of the regulation map (2.6). Simulation results in Figures 3.2, 3.5 and 3.8 show the effectiveness to stabilize the tumor densities, by the regulation laws in Figures 3.3, 3.6 and 3.9-3.10, on the models (3.1), (3.2) and (3.3) in Figures 3.1, 3.4 and 3.7 respectively. Furthermore, Figure 3.11 shows that chemotherapy and radiotherapy combination $0.0552c_1(x,t)+c_2(x,t)$, is more effective to reduce tumor density than using either treatment alone $0.0552c_1(x,t)$ or $c_2(x,t)$. Overall, this paper is a contribution to the field of cancer treatment, and proposes an effective control to stabilize the class of cancer models (2.1).

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: The authors contributed equally to the writing of this paper.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Copyright Statement: Authors own the copyright of their work published in the journal and their work is published under the CC BY-NC 4.0 license.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

References

[1] V. Noviantri, T. Tomy, A. Chowanda, *Linear and nonlinear model of brain tumor growth simulation using finite difference method*, Procedia Comput. Sci., **179** (2021), 297-304.
[2] F. Mohd, H. Sulaiman, N. A. Alias, *Modified Swanson's equation to detect the growth of glioblastomas multiforme (GBM) tumour*, Int. J. Adv. Res. Eng. Innıv., **3**(1) (2021), 1-18.
[3] Y. Zhang, P. X. Liu, W. Hou, *Modeling of glioma growth using modified reaction-diffusion equation on brain MR images*, Comput. Methods Programs Biomed., **227** (2022), 107233.
[4] N. Henscheid, *Generating patient-specific virtual tumor populations with reaction-diffusion models and molecular imaging data*, Math. Biosci. Eng., **17**(6) (2020), 6531-6556.
[5] A. K. Y. Tam, M. J. Simpson, *Pattern formation and front stability for a moving-boundary model of biological invasion and recession*, Phys. D: Nonlinear Phenom., **444** (2023), 133593.
[6] V. A. Khoa, *Convergence of a spectral regularization of a time-reversed reaction-diffusion problem with high-order Sobolev-Gevrey smoothness*, J. Math. Anal. Appl., **518**(1) (2023), 126666.
[7] B. Liang, L. Lozenski, U. Villa, D. Faghihi, *PDE-constrained optimization formulation for tumor growth model calibration*, ArXiv Preprint ArXiv:2302.06445, (2023).

[8] K. N. Soltanov, *Behavior of solutions nonlinear reaction-diffusion PDE's relation to dynamics of propagation of cancer*, ArXiv Preprint ArXiv:2108.03023, (2021).

[9] N. A. Nayied, F. A. Shah, K. S. Nisar, M. A. Khanday, S. Habeeb, *Numerical assessment of the brain tumor growth model via fibonacci and haar wavelets*, Fractals, **31**(2) (2023), 2340017.

[10] R. Everett, K. B. Flores, N. Henscheid, J. Lagergren, K. Larripa, D. Li, T. J. Nardini, P.T.T. Nguyen, E. B. Pitman, E. M. Rutter, *A tutorial review of mathematical techniques for quantifying tumor heterogeneity*, Math. Biosci. Eng., **17**(4) (2020), 3660-3709.

[11] I. Mazari, *The bang-bang property in some parabolic bilinear optimal control problems via two-scale asymptotic expansions*, J. Funct. Anal., **284**(10) (2023), 109855.

[12] N. Moshtaghi-Kashanian, H. Niroomand-Oscuii, N. Meghdadi, *Simulating glioblastoma growth consisting both visible and invisible parts of the tumor using a diffusion–reaction model followed by resection and radiotherapy*, Acta Neurol. Belg., **120** (2020), 629-637.

[13] M. S. Iqbal, N. Ahmed, R. Naeem, A. Akgül, A. Razzaque, M. Inc, H. Khurshid, *Dynamical behavior of cancer cell densities in two dimensional domain by the representation theory of solitons*, Phys. Lett. A, **463** (2023), 128670.

[14] G. Baravdish, B. T. Johansson, O. Svensson, W. Ssebunjo, *Identifying a response parameter in a model of brain tumor evolution under therapy*, IMA J. Appl. Math., (2023), 1-27.

[15] A. Moustafid, *General chemotherapy protocols*, J. Appl. Dyn. Syst. Control, **4**(2) (2021), 18-25.

[16] A. Moustafid, *Set-valued control of cancer by combination chemotherapy*, J. Math. Sci. Model.,**6**(1) (2023), 7-16.

[17] A. Moustafid, *Viability control of chemo-immunotherapy and radiotherapy by set-valued analysis*, Int. J. Inform. Appl. Math., **6**(1) (2023), 40-60.

[18] A. Moustafid, *Set-valued analysis of anti-angiogenic therapy and radiotherapy*, Math. Model. Numer. Simul. Appl., **2**(3) (2022), 187-196.

[19] K. Kassara, A. Moustafid, *Angiogenesis inhibition and tumor-immune interactions with chemotherapy by a control set-valued method*, Math. Biosci., **231**(2) (2011), 135-143.

[20] A. Moustafid, *General anti-angiogenic therapy protocols with chemotherapy*, Int. J. Math. Model. Comput., **11**(3) (2021), 1-14.

[21] A. Moustafid, *Feedback protocols for anti-angiogenic therapy in the treatment of cancer tumors by chemotherapy*, Int. J. Optim. Appl., **2** (2022), 17-24.

[22] L. Boujallal, M. Elhia, O. Balatif, *A novel control set-valued approach with application to epidemic models*, J. Appl. Math. Comput., **65** (2021), 295-319.

[23] K. Kassara, *A unified set-valued approach to control immunotherapy*, SIAM J. Control Optim., **48**(2) (2009), 909-924.

[24] M. Yousefnezhad, C. Y. Kao, S. A. Mohammadi, *Optimal chemotherapy for brain tumor growth in a reaction-diffusion model*, SIAM J. Appl. Math., **81**(3) (2021), 1077-1097.

[25] R. Rockne, E. C. Alvord, J. K. Rockhill, K. R. Swanson, *A mathematical model for brain tumor response to radiation therapy*, J. Math. Biol., **58** (2009), 561-578.

[26] G. B. Powathil, *Modeling of Brain Tumors: Effects of Microenvironment and Associated Therapeutic Strategies*, (2009), http://hdl.handle.net/10012/4738.

[27] J. Aubin, *Viability Theory*, Springer Science & Business Media, (2009).

Fundamental Journal of Mathematics and Applications, 6 (3) (2023) 157-169
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1315178

Helicoidal Surfaces with Prescribed Curvatures in Some Conformally Flat Pseudo-Spaces of Dimensional Three

Büşra Sancı[1] and Fırat Yerlikaya[2*]

[1]*Institute of Graduate Education, Ondokuz Mayıs University, Samsun, Turkey*
[2]*Department of Mathematics, Faculty of Science, Ondokuz Mayıs University, Samsun, Turkey*
[*]*Corresponding author*

Article Info

Keywords: *Conformally flat pseudo-metric, conformally flat pseudo-space, Extrinsic curvature, Helicoidal surface*
2010 AMS: *53C18, 53C21, 53C42*
Received: *15 June 2023*
Revised: *6 September 2023*
Accepted: *13 September 2023*
First Online: *28 September 2023*
Published: *30 September 2023*

Abstract

In this work, we consider the problem of finding explicit parametrizations for non-degenerate helicoidal surfaces with prescribed curvatures in some conformally flat pseudo-spaces with conformal pseudo-metrics whose conformal factors are related to three types of generic cylindrical functions. In the first two, we get a two-parameter family of these surfaces with prescribed extrinsic curvature or mean curvature given by smooth functions. In the last one, we discover a one-parameter family when both curvatures of these surfaces are zero; however, we find a two-parameter family when either one of those curvatures is zero. Also, we support them with examples.

1. Introduction

Mathematics is a branch of science that examines the properties of both abstract shapes and measurable quantities through equivalences. When the abstract shape is a smooth manifold, the term *diffeomorphism* refers to the idea that two manifolds are equivalent in terms of differentiability. *Isometry* is a concept that expresses metrical equivalence between two Riemannian manifolds. The angle and distance between two directions are both preserved in the concept of isometry. Furthermore, the term *conformal* is a more general concept than isometry, in which only the angle is preserved but not the distance. Let g and $\bar{g}$ denote the metrics of two Riemannian (or pseudo-Riemannian) manifolds, respectively. These Riemannian manifolds are said to be conformally equivalent if there exists a differentiable function λ, known as a conformal factor, that provides the equality

$$g = \frac{1}{\lambda^2}\bar{g}.$$

It is common knowledge that when the metric $\bar{g}$ is the Euclidean metric, the Euclidean space R^3 is conformally equivalent to the sphere S^3 (or the hyperbolic space H^3) through suitable conformal factors. The Minkowski space R^3_1 and the de Sitter S^3_1 (or the anti-de Sitter H^3_1) are conformally equivalent in accordance with the Minkowski metric $\bar{g}$. Conformally flat spaces with a bounded conformal factor have gained more attention in recent years. Keep in mind that selecting such a conformal factor results in the metric becoming complete, and as a result, the space in question is referred to as a complete Riemannian manifold.
Particular types of special surfaces, such as rotational and helicoidal surfaces, are surveyed in conformally flat spaces. The construction of a helicoidal surface, in contrast to that of a rotational surface, is performed using screw motions, which include a translation in addition to a rotation around an axis. The problem of finding the explicit parameterization of helicoidal

Email addresses and ORCID numbers: **busramatematik@gmail.com, 0000-0001-6815-889X (B. Sancı), firat.yerlikaya@omu.edu.tr, 0000-0003-2360-1522 (F. Yerlikaya)**
Cite as: B. Sancı, F. Yerlikaya, *Helicoidal surfaces with prescribed curvatures in some conformally flat pseudo-spaces of dimensional three*, Fundam. J. Math. Appl., 6(3) (2023), 157-169

surfaces are done in a wide variety of spaces. It should not be surprising that the properties of such surfaces remain unchanged under screw motions. So, it is important to determine the proper conformal factor if you want to conduct surveys of the above-mentioned surfaces in conformally flat spaces. A function f is said to be invariant under a transformation T of space into itself if the property $f(Tx) = f(x)$ holds for all x. If the conformal factor λ is a function satisfying this condition, then it makes sense to think of such surfaces in conformally flat spaces. An estimation for this kind of function can be found in the cartesian equation of standard geometric shapes like the sphere and the cylinder. Unlike the spherical function, which is only invariant under rotational symmetry, the cylindrical function is invariant under both rotational symmetry and translational symmetry. See [1, 2] for information on surveys carried out in the context of the spherical function. Refer to [3, 4, 5, 6, 7, 8] for the other one.

In a recent survey, Yerlikaya introduces the conformally flat pseudo-space of dimensional three and presents a non-degenerate surface's curvatures in this space. After investigating the presence of helicoidal surfaces in some conformally flat pseudo-spaces, the author works on the problem of determining the explicit parametrization of those surfaces. The author means by "some pseudo-spaces" that it is a conformally flat pseudo-space with a pseudo-metric that corresponds to the determined conformal factor, where the author preferred such a way that this pseudo-metric is a solution to the famous Einstein field equation. The process for determining such a conformal factor, which was just explained as being significant, is carried out in accordance with the causal character of the axis of helicoidal surfaces (see [8] for detail). In this study, we discuss the same problem for generic cylindrical conformal factors, taking the causal character of the axis into account once again.

2. Preliminaries

2.1. Basic Notations

Equipped the Minkowski space $\mathbb{R}^3_1$ with a conformally flat pseudo-metric specified by the angle-bracket notation

$$\langle w_1, w_2 \rangle_{g_\lambda} = \frac{1}{\lambda^2(p)} \langle w_1, w_2 \rangle_L, \quad \forall w_1, w_2 \in T_p\mathbb{R}^3_1, \ \forall p \in \mathbb{R}^3_1,$$

the resulting space is said to be the complete pseudo-Riemannian manifold if the conformal factor λ is bounded. From now on, unless otherwise stated, we shall refer to this pseudo-manifold as the conformally flat pseudo-space and represented by $\mathbb{F}^1_3$. Here, note that the pseudo-metric $\langle , \rangle_L$ is the Minkowski metric whose coefficients are

$$\langle e_1, e_1 \rangle_L = -1, \ \langle e_i, e_j \rangle_L = 1 \quad \text{for} \ 2 \leq i = j \leq 3,$$

$$\langle e_i, e_j \rangle_L = 0, \ \text{for} \ 1 \leq i \neq j \leq 3.$$

On the other hand, let M be a non-degenerate surface in $\mathbb{F}^1_3$. Given the Gaussian curvature K and the mean curvature H of the surface M in $\mathbb{R}^3_1$, both the extrinsic and mean curvatures of M are calculated as

$$\widetilde{K}_E = \varepsilon h^2 - 2H\lambda h + \lambda^2 K \tag{2.1}$$

and

$$\widetilde{H} = \lambda H - \varepsilon h, \quad \varepsilon = \langle N, N \rangle \tag{2.2}$$

respectively. For the unit normal vector field N of M, the function h holds $h = h(s,t) = \sum_{j=1}^{3} N^j \frac{\partial \lambda}{\partial x_j}$, where x_j for $1 \leq j \leq 3$ is an usual coordinate system of $\mathbb{R}^3_1$ [8].

2.2. The Process for Determining Proper Conformal Factors

Conformally flat spaces acquire special qualities by means of their conformal factors. If a space with a conformal factor λ has a transformation T that satisfies the equation $\lambda(Tx) = \lambda(x)$ for all x, then the space is invariant under the transformation T along an axis, as stated in the introduction. Thereby, it makes sense in this space to consider surfaces generated by the transformation T, because the properties of such surfaces remain unchanged under the one T. The infinitesimal isometry group, which consists of both the rotation and translation transformations of space, is represented by its Killing vector field. In [8], the author performs a calculation to determine the Killing vector field for conformal factors that correspond to the causal character of the axis.

Since we will be discussing helicoidal surfaces in this work, we consider that the transformation T must be both rotational and translational. The type of function that is invariant under both rotational and translational transformations is of the cylindrical type.

The spacelike axis: In this case, we take the conformal factor λ_h in

$$\lambda_h : \mathbb{R}_1^3 \to \mathbb{R}, \quad (x_1, x_2, x_3) \to \lambda_h (x_1, x_2, x_3) = \sqrt{x_2^2 - x_1^2},$$

which is directly related to a Lorentzian hyperbolic cylinder with x_3-axis in $\mathbb{R}_1^3$. In [8], taking the procedure of determining the Killing vector field into account, for the determined conformal factor above, we compute the Killing vector field V in $\left(\mathbb{F}_3^1\right)_{\lambda_h}$ as

$$V = c_1 \left(x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \right) + c_2 \frac{\partial}{\partial x_3}$$

for some constants c_1 and c_2. This means that the corresponding isometry group consists of the translation along the x_3 axis given by

$$T (x_1, x_2, x_3) = (x_1, x_2, x_3 + t) \tag{2.3}$$

and the rotation around the x_3 axis given by

$$R (x_1, x_2, x_3) = (x_1 \cosh t + x_2 \sinh t, x_1 \sinh t + x_2 \cosh t, x_3), \tag{2.4}$$

where $(x_1, x_2, x_3) \in \mathbb{R}_1^3$ and $t \in \mathbb{R}$.

Remark 2.1. *When considering the conformal factor λ_h, it is evident that Eqs.* (2.3) *and* (2.4) *both satisfy the function invariance mentioned in the introduction.*

So, we can maintain the following process.

When applying the profile curve $\gamma(s) = (0, s, n(s))$ to Eq. (2.4) together with Eq. (2.3), i.e.,

$$\begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ s \\ n(s) \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix},$$

the resulting surface is said to be a helicoidal surface in $\left(\mathbb{F}_3^1\right)_{\lambda_h}$ with the spacelike axis of rotation, and so its parametrization is represented by

$$X : I \times \mathbb{R} \to \left(\mathbb{F}_3^1\right)_{\lambda_h}; (s,t) \to X(s,t) = \Big(s \sinh t, s \cosh t, n(s) + ct \Big), \tag{2.5}$$

where $n(s)$ is a smooth function and c is a constant.

The timelike axis: In this case, we determine the conformal factor λ_c as

$$\lambda_c : \mathbb{R}_1^3 \to \mathbb{R}, \quad (x_1, x_2, x_3) \to \lambda_h (x_1, x_2, x_3) = \sqrt{x_2^2 + x_3^2}.$$

This is related to a Lorentzian circular cylinder with x_1-axis in $\mathbb{R}_1^3$. Similarly, we compute the Killing vector field V in $\left(\mathbb{F}_3^1\right)_{\lambda_c}$ as

$$V = c_1 \frac{\partial}{\partial x_1} + c_2 \left(-x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} \right)$$

for some constants c_1 and c_2. This means that the corresponding isometry group consists of the translation along the x_1 axis given by

$$T (x_1, x_2, x_3) = (x_1 + t, x_2, x_3) \tag{2.6}$$

and the rotation around the x_1 axis given by

$$R (x_1, x_2, x_3) = (x_1, x_2 \cos t - x_3 \sin t, x_2 \sin t + x_3 \cos t), \tag{2.7}$$

where $(x_1, x_2, x_3) \in \mathbb{R}_1^3$ and $t \in \mathbb{R}$.

Remark 2.2. *When considering the conformal factor λ_c, it is evident that Eqs.* (2.6) *and* (2.7) *both satisfy the function invariance mentioned in the introduction.*

In a manner that is analogous to the process that was just outlined, using Eq. (2.7) together with Eq. (2.6), we get a helicoidal surface in $\left(\mathbb{F}_3^1\right)_{\lambda_c}$ with the timelike axis of rotation, and so its parametrization is represented by

$$X : I\times\mathbb{R}\to\left(\mathbb{F}_3^1\right)_{\lambda_c};(s,t)\to X(s,t)=\Big(n(s)+ct, s\cos t, s\sin t\Big). \tag{2.8}$$

The lightlike axis: In this case, we determine the conformal factor λ_p as

$$\lambda_p:\mathbb{R}_1^3\to\mathbb{R},\quad (x_1,x_2,x_3)\to\lambda_p(x_1,x_2,x_3)=\sqrt{x_2-x_1}.$$

This is related to a Lorentzian parabolic cylinder with $(1,1,0)$-axis in $\mathbb{R}_1^3$. Thus, we compute the Killing vector field V in $\left(\mathbb{F}_3^1\right)_{\lambda_p}$ as

$$V=c_1\left(-x_3\frac{\partial}{\partial x_1}-x_3\frac{\partial}{\partial x_2}+(-x_1+x_2)\frac{\partial}{\partial x_3}\right)+c_2\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right).$$

for some constants c_1 and c_2. This means that the corresponding isometry group consists of the translation along the $(1,1,0)$ axis given by

$$T(x_1,x_2,x_3)=(x_1+t,x_2+t,x_3) \tag{2.9}$$

and the rotation around the $(1,1,0)$ axis given by

$$R(x_1,x_2,x_3)=\Big(\left(1+t^2/2\right)x_1-\left(t^2/2\right)x_2+tx_3,\left(t^2/2\right)x_1+\left(1-t^2/2\right)x_2+tx_3,tx_1-tx_2+x_3\Big), \tag{2.10}$$

where $(x_1,x_2,x_3)\in\mathbb{R}_1^3$ and $t\in\mathbb{R}$.

Remark 2.3. *When considering the conformal factor λ_p, it is evident that Eqs.* (2.9) *and* (2.10) *both satisfy the function invariance mentioned in the introduction.*

Similarly, from Eqs. (2.10) and (2.9), we have a helicoidal surface in $\left(\mathbb{F}_3^1\right)_{\lambda_p}$ with the lightlike axis of rotation generated by the profile curve $\gamma(s)=(s,n(s),0)$. A parametrization of this surface is as follows

$$X : I\times\mathbb{R}\to\left(\mathbb{F}_3^1\right)_{\lambda_p};(s,t)\to X(s,t)=\Big(\left(1+t^2/2\right)s-\left(t^2/2\right)n(s)+ct,\left(t^2/2\right)s+\left(1-t^2/2\right)n(s)+ct,(s-n(s))t\Big). \tag{2.11}$$

3. Results

Firstly, let's take a look at the helicoidal surface with the spacelike axis of rotation given by Eq. (2.5). For this surface, we have

$$K(s)=\frac{s^3n'n''+c^2}{\left[-c^2+s^2\left(1+n'^2\right)\right]\left|-c^2+s^2\left(1+n'^2\right)\right|} \tag{3.1}$$

and

$$H(s)=-\frac{s^2n'^3+n'\left(-2c^2+s^2\right)+sn''\left(-c^2+s^2\right)}{2\left[-c^2+s^2\left(1+n'^2\right)\right]\left|-c^2+s^2\left(1+n'^2\right)\right|^{\frac{1}{2}}}, \tag{3.2}$$

where the first one is the Gaussian curvature and the other one is the mean curvature. Assuming that $EG-F^2=c^2-s^2\left(1+n'^2\right)<0$, meanings that this surface is timelike, Eqs. (3.1) and (3.2) yield

$$K(s)=\frac{s^3n'n''+c^2}{\left[-c^2+s^2\left(1+n'^2\right)\right]^2} \tag{3.3}$$

and

$$H(s)=-\frac{s^2n'^3+n'\left(-2c^2+s^2\right)+sn''\left(-c^2+s^2\right)}{2\left[-c^2+s^2\left(1+n'^2\right)\right]^{\frac{3}{2}}}, \tag{3.4}$$

respectively. For the determined conformal factor λ_h, we have

$$\sum_{j=1}^{3} N^j \lambda_{,j} = \frac{sn'}{\sqrt{-c^2+s^2\left(1+n'^2\right)}}. \tag{3.5}$$

Inserting Eqs. (3.3), (3.4) and (3.5) into Eqs. (2.1) and (2.2), we get

$$\widetilde{K}_{E_{\lambda_h}} = \frac{s^2\left(sn'n''\left(-c^2+2s^2\right)+2s^2n'^4+n'^2\left(2s^2-3c^2\right)+c^2\right)}{\left[-c^2+s^2\left(1+n'^2\right)\right]^2} \tag{3.6}$$

and

$$\widetilde{H}_{\lambda_h} = -\frac{s\left(sn''\left(-c^2+s^2\right)+3s^2n'^3+n'\left(3s^2-4c^2\right)\right)}{2\left[-c^2+s^2\left(1+n'^2\right)\right]^{\frac{3}{2}}}. \tag{3.7}$$

In order to find a solution to Eq. (3.6), we turn Eq. (3.6) into

$$A'(s)+\left(\frac{5}{s}+\frac{4s}{c^2-2s^2}\right)A(s) = \frac{4}{c^2-2s^2}+\frac{2}{s^2}\widetilde{K}_{E_{\lambda_h}}(s), \tag{3.8}$$

where

$$A(s) = \frac{sn'^2-s}{-c^2+s^2\left(1+n'^2\right)}. \tag{3.9}$$

The general solution to Eq. (3.8) becomes

$$A(s) = \frac{c^2-2s^2}{s^5}\left[\int \frac{s^5}{c^2-2s^2}\left\{\frac{4}{c^2-2s^2}+\frac{2}{s^2}\widetilde{K}_{E_{\lambda_h}}(s)\right\}ds+c_1\right], \tag{3.10}$$

where c_1 is constant. When we compare Eq. (3.9) with Eq. (3.10), we obtain

$$\left[s^2\left(s^4-\psi(s)\right)\right]n'^2(s) = s^6+\left(s^2-c^2\right)\psi(s), \tag{3.11}$$

where

$$\psi(s) = \left(c^2-2s^2\right)\left[\int \frac{s^5}{c^2-2s^2}\left\{\frac{4}{c^2-2s^2}+\frac{2}{s^2}\widetilde{K}_{E_{\lambda_h}}(s)\right\}ds+c_1\right]. \tag{3.12}$$

The next theorem can now be established:

Theorem 3.1. *Let $\gamma(s) = (0,s,n(s))$ be a profile curve of the timelike helicoidal surface (2.5) in $\left(\mathbb{F}_3^1\right)_{\lambda_h}$. Assuming its extrinsic curvature at the point $(0,s,n(s))$ is represented by $\widetilde{K}_{E_{\lambda_h}}(s)$, there exists an open subinterval $\widetilde{I} \subset I$ concerning c_1 such that the function $n(s)$ is*

$$n(s) = \pm\int \frac{\sqrt{\left|s^6+\left(s^2-c^2\right)\psi(s)\right|}}{|s|\sqrt{\left|s^4-\psi(s)\right|}}ds+c_2, \tag{3.13}$$

where $\psi(s)$ is given by Eq.(3.12) and c_2 is a constant. Also, for the designated constant c_1 and some constants c and c_2, there exists the two-parameter family of curves such that

$$\gamma(s;\widetilde{K}_{E_{\lambda_h}}(s),c,c_1,c_2) = \left(0,s,\pm\int \frac{\sqrt{\left|s^6+\left(s^2-c^2\right)\psi(s)\right|}}{|s|\sqrt{s^4-\psi(s)}}ds+c_2\right),\quad s\in\widetilde{I}\cap\left(\mathbb{R}\backslash\left(-c/\sqrt{2},c/\sqrt{2}\right)\right) \tag{3.14}$$

Proof. For the known function $\widetilde{K}_{E_{\lambda_h}} : I \to \mathrm{R}$ referred to as the extrinsic curvature of the helicoidal surface (2.5), we have Eq. (3.11). For an arbitrary number c_1, we establish a function $\mathscr{F}(s, c_1) = s^4 - \psi$ on the product of sub-intervals containing the numbers s_0 and $\widetilde{c_1}$ which satisfy the equality

$$\widetilde{c_1} = -\left(\int \frac{s^5}{c^2 - 2s^2}\left\{\frac{4}{c^2 - 2s^2} + \frac{2}{s^2}\widetilde{K}_{E_{\lambda_h}}(s)\right\}ds\right)(s_0).$$

Based on the function $\mathscr{F}$ to be continuous, we find a product set $\widetilde{I} \times J$ where Eq. (3.11) turns into

$$n'^2(s) = \frac{s^6 + (s^2 - c^2)\,\psi(s)}{s^2\left(s^4 - \psi(s)\right)},$$

which is the equation whose integration gives Eq. (3.13). Combining Eqs. (3.9) and (3.10), we obtain Eq. (3.14). □

Theorem 3.2. *Let c and c_2 be arbitrary constants. Thus, for any c_1 and a smooth function $\widetilde{K}_{E_{\lambda_h}}$, setting an open subinterval $\widetilde{I}$ of I in which the function $n(s)$ given by Eq. (3.13) is defined, we can construct the two-parameter family of timelike helicoidal surfaces defined on $\widetilde{I} \times \mathbb{R} \subset \mathbb{R}^2$, with the extrinsic curvature $\widetilde{K}_{E_{\lambda_h}}(s)$, with the profile curve $\gamma(s; \widetilde{K}_{E_{\lambda_h}}(s), c, c_1, c_2)$, $s \in \widetilde{I}$.*

Proof. Eq. (3.13) is a solution to Eq. (3.8), which implies the requirement that concludes the proof. When considering the sub-intervals that are stated in the proof of Theorem (3.1), for two numbers $c_1 \in J$, $c_2 \in \mathbb{R}$, a function $\widetilde{K}_{E_{\lambda_h}}(s)$ and $c \in \mathbb{R}$, we obtain the desired family defined on $\widetilde{I} \times \mathbb{R} \subset \mathbb{R}^2$. □

Now considering Eq. (3.7), we constitute

$$B'(s) + \frac{4}{s}B(s) = -\frac{2}{s^2}\widetilde{H}_{\lambda_h}(s), \qquad s \neq 0 \tag{3.15}$$

where

$$B(s) = \frac{n'}{\sqrt{-c^2 + s^2(1 + n'^2)}}, \tag{3.16}$$

which implies that the general solution to Eq. (3.15) becomes

$$B(s) = -\frac{1}{s^4}\left[\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right], \tag{3.17}$$

where c_1 is constant. With Eqs. (3.16) and (3.17), we have

$$\left[s^2\left(s^6 - \left(\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right)^2\right)\right]n'^2(s) = (s^2 - c^2)\left(\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right)^2.$$

Theorem 3.3. *Let $\gamma(s) = (0, s, n(s))$ be a profile curve of the timelike helicoidal surface (2.5) in $\left(\mathbb{F}_3^1\right)_{\lambda_h}$. Assuming its mean curvature at the point $(0, s, n(s))$ is represented by $\widetilde{H}_{\lambda_h}(s)$, there exists an open subinterval $\widetilde{I} \subset I$ relating to c_1 such that the function $n(s)$ is*

$$n(s) = \pm\int \frac{\sqrt{|s^2 - c^2|}\left|\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right|}{|s|\left(\left|s^6 - \left(\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right)^2\right|\right)^{\frac{1}{2}}}ds + c_2, \tag{3.18}$$

where c_2 is a constant. Also, for the designated constant c_1 and some constants c and c_2, there exists the two-parameter family of curves such that

$$\gamma(s; \widetilde{H}_{\lambda_h}(s), c, c_1, c_2) = \left(0, s, \int \frac{\sqrt{s^2 - c^2}\left|\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right|}{|s|\left(s^6 - \left(\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right)^2\right)^{\frac{1}{2}}}ds + c_2\right), \; s \in \widetilde{I} \cap (\mathbb{R} \backslash (-c, c)). \tag{3.19}$$

Proof. Define the function $\mathscr{F}$ to be $(s,c_1) \to s^6 - \left(\int 2s^2 \widetilde{H}_{\lambda_h}(s)\, ds + c_1\right)^2$. So, the technique for proving that Eqs. (3.18) and (3.19) can be obtained for the known function $\widetilde{H}_{\lambda_h}$ referred to as the mean curvature of the helicoidal surface (2.5) is the same as that of Theorem (3.1). $\square$

Example 3.4. *Let the mean curvature be $\widetilde{H}_{\lambda_h}(s) = -\frac{1}{s}$, which implies that the function $\mathscr{F}$ amounts $s^6 - (s^2 + c_1)^2$. So, we get the inequality $s^2(1-s) < c_1 < s^2(1+s)$ such that $\mathscr{F}$ is positive. Establish two functions $f(s) := s^2(1+s)$ and $g(s) := s^2(1-s)$. For $c_1 = 0$, consider the interval $\left(11/10, \sqrt{2}\right)$. Thus, the number c_1 falls in the interval $\left(2-\sqrt{8}, 2+\sqrt{8}\right)$. In this way, we determine the sub-interval $\widetilde{I}$ of I to be $\left(11/10, \sqrt{2}\right)$ in accordance with the positivity of $\mathscr{F}$. Ultimately, for $c_2 = 0$ and $c = 1$, Eq. (3.19) lead into*

$$\gamma(s) = \Big(0, s, \ln s\Big), \qquad s \in \left(11/10, \sqrt{2}\right).$$

Replacing the last one into Eq. (2.5), we get

$$X(s,t) = \Big(s \sinh t, s \cosh t, \ln s + t\Big),$$

which is the parametrization of a timelike helicoidal surface.

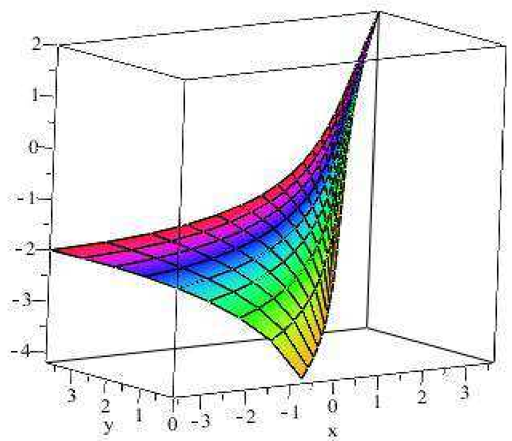

Figure 3.1: The graphic belongs to a timelike helicoidal surface of spacelike axis of rotation with $\widetilde{H}_{\lambda_h}(s) = -\frac{1}{s}$

Theorem 3.5. *Let c and c_2 be arbitrary constants. Thus, for any c_1 and a smooth function $\widetilde{H}_{\lambda_h}$, setting an open subinterval $\widetilde{I}$ of I in which the function $n(s)$ given by Eq. (3.18) is defined, we can construct the two-parameter family of timelike helicoidal surfaces defined on $\widetilde{I} \times \mathbb{R} \subset \mathbb{R}^2$, with the mean curvature $\widetilde{H}_{\lambda_h}(s)$, with the profile curve $\gamma(s; \widetilde{H}_{\lambda_h}(s), c, c_1, c_2)$, $s \in \widetilde{I}$.*

Proof. The process for proving that the desired family can be construct is the same as that of Theorem (3.2). $\square$

Remark 3.6. *It is important to remember that analogous outcomes may be obtained if we pick $EG - F^2 = c^2 - s^2\left(1 + n'^2\right) > 0$, meanings that the helicoidal surface in $\left(\mathbb{F}_3^1\right)_{\lambda_h}$ is spacelike.*

Now, we consider the helicoidal surface given by Eq. (2.8). For this surface, taking $EG - F^2 = -c^2 + s^2\left(1 - n'^2\right) > 0$ into account, we have

$$K(s) = \frac{-s^3 n' n'' + c^2}{\left[-c^2 + s^2\left(1 - n'^2\right)\right]^2} \tag{3.20}$$

and

$$H(s)=\frac{s^2n'^3+n'\left(2c^2-s^2\right)-sn''\left(-c^2+s^2\right)}{2\left[-c^2+s^2\left(1-n'^2\right)\right]^{\frac{3}{2}}}. \tag{3.21}$$

Also, we get

$$\sum_{j=1}^{3}N^j\lambda_{,j}=-\frac{sn'}{\sqrt{-c^2+s^2\left(1-n'^2\right)}}. \tag{3.22}$$

Considering Eqs. (3.20), (3.21) and (3.22) together, from Eqs. (2.1) and (2.2), we obtain

$$\widetilde{K}_{E_{\lambda_h}}=\frac{s^2\left(sn'n''\left(c^2-2s^2\right)+2s^2n'^4+n'^2\left(3c^2-2s^2\right)+c^2\right)}{\left[-c^2+s^2\left(1-n'^2\right)\right]^2} \tag{3.23}$$

and

$$\widetilde{H}_{\lambda_h}=\frac{s\left(sn''\left(c^2-s^2\right)+3s^2n'^3+n'\left(4c^2-3s^2\right)\right)}{2\left[-c^2+s^2\left(1-n'^2\right)\right]^{\frac{3}{2}}}. \tag{3.24}$$

In a similar way, Eq. (3.23) turns into

$$A'(s)+\left(\frac{5}{s}-\frac{4s}{2s^2-c^2}\right)A(s)=\frac{4}{2s^2-c^2}-\frac{2}{s^2}\widetilde{K}_{E_{\lambda_c}}(s), \tag{3.25}$$

where

$$A(s)=\frac{sn'^2+s}{-c^2+s^2\left(1-n'^2\right)}. \tag{3.26}$$

The general solution to Eq. (3.25) becomes

$$A(s)=\frac{2s^2-c^2}{s^5}\left[\int\frac{s^5}{2s^2-c^2}\left\{\frac{4}{2s^2-c^2}-\frac{2}{s^2}\widetilde{K}_{E_{\lambda_c}}(s)\right\}ds+c_1\right], \tag{3.27}$$

where c_1 is constant. Comparing Eqs. (3.26) with (3.27), we have

$$\left[s^2\left(s^4+\psi(s)\right)\right]n'^2(s)=-s^6+\left(s^2-c^2\right)\psi(s), \tag{3.28}$$

where

$$\psi(s)=\left(2s^2-c^2\right)\left[\int\frac{s^5}{2s^2-c^2}\left\{\frac{4}{2s^2-c^2}-\frac{2}{s^2}\widetilde{K}_{E_{\lambda_c}}(s)\right\}ds+c_1\right].$$

In light of the assumed sign of $EG-F^2$, it follows that $s^4+\psi(s)=\frac{s^4\left(2s^2-c^2\right)}{-c^2+s^2-s^2n'^2}>0$. Thus, from (3.28), we can write

$$n(s)=\pm\int\frac{\sqrt{\left|\left(s^2-c^2\right)\psi(s)-s^6\right|}}{|s|\sqrt{s^4+\psi(s)}}ds+c_2 \tag{3.29}$$

Theorem 3.7. *Let $\gamma(s)=(n(s),s,0)$ be a profile curve of the spacelike helicoidal surface (2.8) in $\left(\mathbb{F}_3^1\right)_{\lambda_c}$. Assuming its extrinsic curvature at the point $(n(s),s,0)$ is represented by $\widetilde{K}_{E_{\lambda_c}}(s)$, for some constant c, c_1 and c_2, there exists the two-parameter family of the spacelike helicoidal surface constituted by curves*

$$\gamma(s;\widetilde{K}_{E_{\lambda_c}}(s),c,c_1,c_2)=\left(\pm\int\frac{\sqrt{\left|\left(s^2-c^2\right)\psi(s)-s^6\right|}}{|s|\sqrt{s^4+\psi(s)}}ds+c_2,s,0\right),\quad s\in\mathbb{R}\backslash(-c,c).$$

Inversely, let c and c_2 be arbitrary constants. Thus, for any c_1 and a smooth function $\widetilde{K}_{E_{\lambda_c}}(s)$, we can construct the two-parameter family of spacelike helicoidal surfaces defined on $\widetilde{I}\times\mathbb{R}\subset\mathbb{R}^2$, with the extrinsic curvature $\widetilde{K}_{E_{\lambda_c}}(s)$, with the profile curve $\gamma(s;\widetilde{K}_{E_{\lambda_c}}(s),c,c_1,c_2)$, $s\in\widetilde{I}$.

Proof. For the known function $\widetilde{K}_{E_{\lambda_c}}$, it is seen that the function $n(s)$ takes Eq. (3.29), which means that concludes the necessity. By defining the function $\mathscr{F}$ to be

$$(s,c_1)\to s^4+\left(2s^2-c^2\right)\left[\int\frac{s^5}{2s^2-c^2}\left\{\frac{4}{2s^2-c^2}-\frac{2}{s^2}\widetilde{K}_{E_{\lambda_c}}(s)\right\}ds+c_1\right],$$

it is possible to perform the inverse of the proof in a manner similar to Theorem (3.1). □

Example 3.8. *Let the extrinsic curvature be $\widetilde{K}_{E_{\lambda_c}}(s)=\frac{2s^2\left(6s^2+5\right)}{\left(2s^2+1\right)^2}$. For $c_1=0$, we find $\psi(s)=-\frac{s^2\left(3s^4+1\right)}{2s^2+1}$. For c_1, using Eq. (3.29), we get the profile curve to be $\gamma(s)=\left(\sqrt{3}s+\sqrt{2}+t,s,0\right)$. Thus, we write the parametrization of the corresponding helicoidal surface as*

$$X(s,t)=\left(\sqrt{3}s+\sqrt{2}+t,s\cos t,s\sin t\right).$$

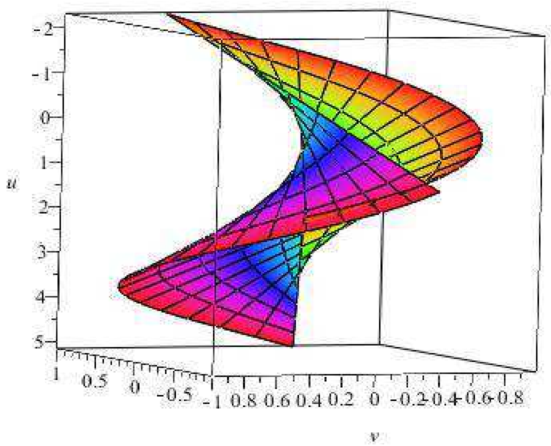

Figure 3.2: The graphic belongs to a spacelike helicoidal surface of timelike axis of rotation with $\widetilde{K}_{E_{\lambda_c}}(s)=\frac{2s^2\left(6s^2+5\right)}{\left(2s^2+1\right)^2}$

In a similar way, from Eq. (3.24), we write

$$B'(s)+\frac{4}{s}B(s)=-\frac{2}{s^2}\widetilde{H}_{\lambda_c}(s),\qquad s\neq 0 \tag{3.30}$$

where

$$B(s)=\frac{n'}{\sqrt{-c^2+s^2\left(1-n'^2\right)}}. \tag{3.31}$$

The general solution to Eq. (3.30) becomes

$$B(s)=-\frac{1}{s^4}\left[\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right], \tag{3.32}$$

where c_1 is constant. From Eq. (3.31) and Eq. (3.32), we obtain

$$n(s)=\pm\int\frac{\sqrt{s^2-c^2}\left|\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right|}{|s|\left(s^6+\left(\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right)^2\right)^{\frac{1}{2}}}ds+c_2,$$

Theorem 3.9. *Let $\gamma(s)=(n(s),s,0)$ be a profile curve of the spacelike helicoidal surface (2.8) in $\left(\mathbb{F}_3^1\right)_{\lambda_c}$. Assuming its mean curvature at the point $(n(s),s,0)$ is represented by $\widetilde{H}_{\lambda_c}(s)$, for some constant c, c_1 and c_2, there exists the two-parameter family of the spacelike helicoidal surface constituted by curves*

$$\gamma(s;\widetilde{H}_{\lambda_c}(s),c,c_1,c_2)=\left(\int\frac{\sqrt{s^2-c^2}\left|\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right|}{|s|\left(s^6+\left(\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right)^2\right)^{\frac{1}{2}}}ds+c_2,s,0\right),\quad s\in\mathbb{R}\backslash(-c,c)$$

Inversely, let c and c_2 be arbitrary constants. Thus, for any c_1 and a smooth function $\widetilde{H}_{\lambda_c}(s)$, we can construct the two-parameter family of spacelike helicoidal surfaces defined on $\widetilde{I}\times\mathbb{R}\subset\mathbb{R}^2$, with the mean curvature $\widetilde{H}_{\lambda_c}(s)$, with the profile curve $\gamma(s;\widetilde{H}_{\lambda_c}(s),c,c_1,c_2)$, $s\in\widetilde{I}$.

Proof. If the function $\widetilde{H}_{\lambda_c}(s)$ is known and by defining the function $\mathscr{F}$ to be

$$(s,c_1)\to s^6+\left(\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right)^2,$$

then the proof is reduced to nothing more than the proof of Theorem (3.7). □

Remark 3.10. *It is important to remember that analogous outcomes may be obtained if we pick $EG-F^2=-c^2+s^2\left(1-n'^2\right)<0$, meanings that the helicoidal surface in $\left(\mathbb{F}_3^1\right)_{\lambda_c}$ is timelike.*

Finally, for the helicoidal surface given by Eq. (2.11), we have

$$K(s)=\frac{n''(n-s)^3+c^2(1-n')^3}{(1-n')\left[(n-s)^2(n'+1)+c^2(1-n')\right]^2}\tag{3.33}$$

and

$$H(s)=-\frac{n''(n-s)^3+2c^2(1-n')^3+(n-s)^2(n'+1)(1-n')^2}{2\left[(1-n')\left((n-s)^2(n'+1)+c^2(1-n')\right)\right]^{\frac{3}{2}}}.\tag{3.34}$$

Assuming $EG-F^2<0$, observe that $1-n'>0$ and taking the conformal factor λ_p into account, we compute as

$$\sum_{j=1}^{3}N^j\lambda_{,j}=\frac{\sqrt{n-s}\,(1-n')}{2\sqrt{(1-n')\left((n-s)^2(n'+1)+c^2(1-n')\right)}}.\tag{3.35}$$

Inserting Eqs. (3.33), (3.34) and (3.35) into Eqs. (2.1) and (2.2), we get

$$\widetilde{K}_{E_{\lambda_p}}=\frac{3(n-s)\left(2n''(n-s)^3+(n-s)^2(n'+1)(1-n')^2+3c^2(1-n')^3\right)}{4(1-n')\left[(n-s)^2(n'+1)+c^2(1-n')\right]^2}\tag{3.36}$$

and

$$\widetilde{H}_{\lambda_p}=-\frac{\sqrt{n-s}\left(n''(n-s)^3+2(n-s)^2(n'+1)(1-n')^2+3c^2(1-n')^3\right)}{2\left[(1-n')\left((n-s)^2(n'+1)+c^2(1-n')\right)\right]^{\frac{3}{2}}}.\tag{3.37}$$

It is difficult to determine the general solution to Eqs. (3.36) and (3.37) unless in a few specific cases.

First, we look at the case where both $\widetilde{K}_{E_{\lambda_p}}$ and $\widetilde{H}_{\lambda_p}$ amount mutually to zero. Thus, by using $n(s) \neq s$, Eqs. (3.36) and (3.37) turn into

$$2n''(n-s)^3 + (n-s)^2 (n'+1)(1-n')^2 + 3c^2 (1-n')^3 = 0 \tag{3.38}$$

$$n''(n-s)^3 + 2(n-s)^2 (n'+1)(1-n')^2 + 3c^2 (1-n')^3 = 0. \tag{3.39}$$

By Eqs. (3.38) and (3.39) , we obtain

$$(n-s)^2 (1+n') + c^2 (1-n') = 0. \tag{3.40}$$

Assigned $n(s) - s = p(s)$, we turn Eq. (3.40) into

$$\left(p^2 - c^2\right) p' + 2p^2 = 0. \tag{3.41}$$

Then, the general solution to Eq. (3.41) is

$$p(s) = -(s+c_1) \pm \sqrt{(s+c_1)^2 - c^2},$$

where c_1 is an integration constant. From $n(s) - s = p(s)$, we find

$$n(s) = c_1 \pm \sqrt{(s+c_1)^2 - c^2}, \qquad c_1 \in \mathbb{R}.$$

Hence, we construct a one-parameter family of curves

$$\gamma(s;n(s),c,c_1) = \left(s, c_1 \pm \sqrt{(s+c_1)^2 - c^2}, 0\right).$$

As a result, with Eq. (2.11), the helicoidal surface turns into

$$X(s,t) = \left(\left(1+\frac{t^2}{2}\right)s - \frac{t^2}{2}\left(c_1 \pm \sqrt{(s+c_1)^2 - c^2}\right) + ct, \frac{t^2}{2}s + \left(1-\frac{t^2}{2}\right)\left(c_1 \pm \sqrt{(s+c_1)^2 - c^2}\right) + ct, \left(\left(s - c_1 \mp \sqrt{(s+c_1)^2 - c^2}\right)\right)t\right).$$

We will talk about what Eq. (3.36) turns into when the extrinsic curvature $\widetilde{K}_{E_{\lambda_p}}$ is zero, which is the situation in which Eq. (3.38) only is valid. In that case, we turn Eq. (3.36) into

$$2p^3 p'' + \left(p^2 - 3c^2\right) p'^3 + 2p^2 p'^2 = 0, \tag{3.42}$$

where $n(s) - s = p(s)$. Assigned $p'(s)$ to $\widetilde{p}(s)$, Eq. (3.42) turns into

$$\frac{1}{\widetilde{p}}\left(\frac{1}{\widetilde{p}}\right)' - \frac{1}{p}\frac{1}{\widetilde{p}} - \frac{p^2 - 3c^2}{2p^3} = 0. \tag{3.43}$$

If we put $\frac{1}{\widetilde{p}} = w(p)$, Eq. (3.43) turns into

$$\frac{dw}{dp} - \frac{1}{p}w - \frac{p^2 - 3c^2}{2p^3} = 0. \tag{3.44}$$

The general solution to Eq. (3.44) is

$$w(p) = \frac{2c_1 p^3 - p^2 + c^2}{2p^2},$$

where c_1 is a constant. Thus, the function $n = n(s)$ supplies the equality

$$c_1 n^3 - (3c_1 s + 1) n^2 + \left(3c_1 s^2 - 2c_2\right) n - c_1 s^3 + 2c_2 s + s^2 - c^2 = 0, \tag{3.45}$$

where c_2 is an integration constant.

Example 3.11. *For $c_1 = 0$, from Eq. (3.45), we find $n(s) = -c_2 \pm \sqrt{(s+c_2)^2 - c^2}, s \in (-\infty, -c-c_2) \cup (c-c_2, \infty)$. Using Eq. (2.11), we write*

$$X(s,t) = \left(\left(1+\frac{t^2}{2}\right) s - \frac{t^2}{2}\left(-c_2 \pm \sqrt{(s+c_2)^2 - c^2}\right) + ct, \frac{t^2}{2}s + \left(1-\frac{t^2}{2}\right)\left(-c_2 \pm \sqrt{(s+c_2)^2 - c^2}\right) + ct, \left(\left(s+c_2 \mp \sqrt{(s+c_2)^2 - c^2}\right)\right) t \right).$$

We now plot it putting for $c_2 = 0$ and $c = 3$.

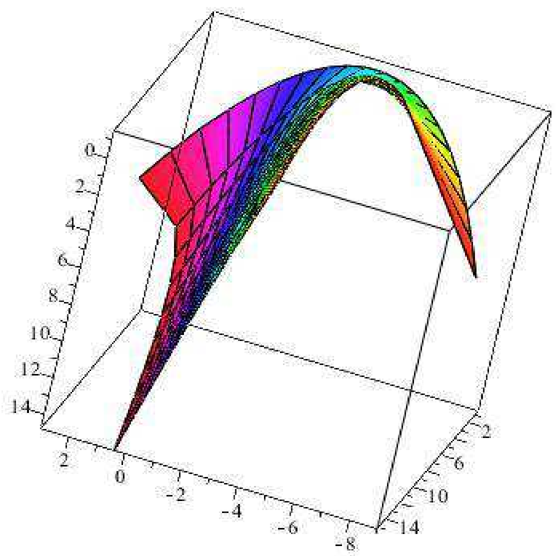

Figure 3.3: The graphic belongs to a minimal timelike helicoidal surface of lightlike axes of rotation with $\widetilde{K}_{E_{\lambda_p}} = 0$

Finally, we take an interest in Eq. (3.39), which requires a timelike helicoidal minimal surface. Similarly, the function $n(s)$ provides

$$\begin{aligned} 2c_1 n^6 - 12c_1 s n^5 + 30c_1 s^2 n^4 - 40c_1 s^3 n^3 + 5\left(6c_1 s^4 - 1\right) n^2 \\ -2\left(6c_1 s^5 + 5c_2\right) n + 2c_1 s^6 + 5s^2 + 10c_2 s - 5c^2 = 0. \end{aligned} \tag{3.46}$$

Example 3.12. *For $c_1 = 0$, from Eq. (3.46), we get $n(s) = -c_2 \pm \sqrt{(s+c_2)^2 - 5c^2}, s \in \left(-\infty, -\sqrt{5}c - c_2\right) \cup \left(\sqrt{5}c - c_2, \infty\right)$, in which by Eq. (2.11), the parametric form of a timelike helicoidal minimal surface turns into*

$$\begin{aligned} X(s,t) = \Bigg(& \left(1+\frac{t^2}{2}\right) s - \frac{t^2}{2}\left(-c_2 \pm \sqrt{(s+c_2)^2 - 5c^2}\right) + ct, \\ & \frac{t^2}{2}s + \left(1-\frac{t^2}{2}\right)\left(-c_2 \pm \sqrt{(s+c_2)^2 - 5c^2}\right) + ct, \\ & \left(\left(s+c_2 \mp \sqrt{(s+c_2)^2 - 5c^2}\right)\right) t \Bigg). \end{aligned}$$

We now plot it putting for $c_2 = 0$ and $c = 3$.

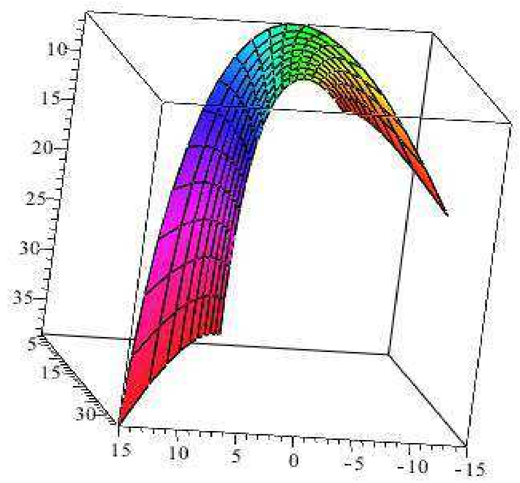

Figure 3.4: The graphic belongs to a minimal timelike helicoidal surface of lightlike axes of rotation with $\widetilde{K}_{E_{\lambda_p}} = 0$

Remark 3.13. *Observe that both helicoidal surfaces mentioned above, say Examples* (3.11) *and* (3.12) *satisfy both* $\widetilde{K}_{E_{\lambda_p}} = 0$ *and* $\widetilde{H}_{\lambda_p} = 0$.

Remark 3.14. *It is important to remember that analogous outcomes may be obtained if we pick*

$$EG - F^2 = \left(n' - 1\right)\left(\left(n - s\right)^2 \left(n' + 1\right) - c^2 \left(n' - 1\right)\right) > 0.$$

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: The authors contributed equally to the writing of this paper.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Copyright Statement: Authors own the copyright of their work published in the journal and their work is published under the CC BY-NC 4.0 license.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

References

[1] A. V. Corro, R. Pina, M. A. Souza, *Surfaces of rotation with constant extrinsic curvature in a conformally flat 3-space*, Results Math., **60** (2011), 225–234.
[2] A. V. Corro, M. A. Souza, R. Pina, *Classes of Weingarten surfaces in* $\mathrm{S}^2 \times \mathrm{R}$, Houston J. Math., **46** (2020), 651-654.
[3] B. P. Lima, P. A. Souza, B. M. Vieira, *Helicoidal hypersurfaces and graphs in conformally flat spaces*, Results Math., **77** (2022), 1-16.
[4] K. O. Araujo, N. Cui, R. S. Pina, *Helicoidal minimal surfaces in a conformally flat 3-space*, Bull. Korean Math., **53**(2) (2016), 531-540.
[5] K. O. Araujo, A. V. Corro, R. S. Pina, M. A. Souza, *Complete surfaces with zero curvatures in conformally flat spaces*, Publ. Math. Debrecen., **96**(3-4) (2020), 363-376.
[6] C. W. Lee, J. W. Lee, D. W. Yoon, *On helicoidal surfaces in a conformally flat 3-space*, Mediterr. J. Math., **14** (2017), 1-9.
[7] C. W. Lee, J. W. Lee, D. W. Yoon, *Helicoidal surfaces with prescribed curvatures in a conformally flat 3-space*, Georgian Math. J., **28**(5) (2021), 755-763.
[8] F. Yerlikaya, *Helicoidal surfaces in some conformally flat pseudo-spaces of dimensional three*, Int. J. Geom. Methods Mod., **20**(3) (2023), 2350052-76

Fundamental Journal of Mathematics and Applications, 6 (3) (2023) 170-176
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1248680

Identification of the Solely Time-Dependent Zero-Order Coefficient in a Linear Bi-Flux Diffusion Equation from an Integral Measurement

İbrahim Tekin[1*] **and Mehmet Akif Çetin**[2]

[1]*Department of Fundamental Sciences, Rafet Kayis Faculty of Engineering, Alanya Alaaddin Keykubat University, 07425 Antalya, Türkiye*
[2]*ALTSO Vocational School, Alanya Alaaddin Keykubat University, 07425 Antalya, Türkiye*
[*]*Corresponding author*

Article Info

Keywords: *Bi-flux equation, Fourier method, Inverse problem*
2010 AMS: *35R30, 35A02*
Received: *7 February 2023*
Revised: *12 July 2023*
Accepted: *28 September 2023*
First Online: *29 September 2023*
Published: *30 September 2023*

Abstract

Bi-flux diffusion equation, can be easily affected by the existence of external factors, is known as an anomalous diffusion. In this paper, the inverse problem (IP) of determining the solely time-dependent zero-order coefficient in a linear Bi-flux diffusion equation with initial and homogeneous boundary conditions from an integral additional specification of the energy is considered. The unique solvability of the inverse problem is demonstrated by using the contraction principle for sufficiently small times.

1. Introduction

The classical diffusion model is contingent on Fick's law and, this generally describes some practical problems such as heat and mass diffusion, bacterial infection diffusion, and predator-prey models, [1, 2, 3]. However, a number of sensitive particle systems, for example population systems arise in biology, are known as an anomalous diffusion. Because the diffusion motion of particles can be easily affected by the existence of the disturbing exogenous agents. To model this anomalous diffusion, the Bi-flux approach to diffusion problems was introduced in [4, 5, 6, 7] to deal with bizarre evolutionary processes by using a discrete formulation. The theory leads the flux to bifurcate into two separate currents corresponding to two independent potentials. The 1st and 2nd flows derive from the classical Fick potential and a new potential, respectively. From this theory the Bi-flux diffusion equation is derived as:

$$Z_\tau(\chi,\tau)+\beta(1-\beta)RZ_{\chi\chi\chi\chi}(\chi,\tau)-\beta DZ_{\chi\chi}(\chi,\tau)=H(\chi,\tau,Z), \tag{1.1}$$

where $Z(\chi,\tau)$ is the particle concentration, D is the diffusion coefficient, R is the reaction coefficient, $H(\chi,\tau,Z)$ linear or nonlinear reaction function, and β $(0\le\beta\le 1)$ is the fraction of particles displaced regarding to the Fick's law.
If $\beta=0$, the homogeneous equation (1.1) $(H(\chi,\tau,Z)=0)$ reduces to a stationary equation and $Z(\chi,\tau)=$constant. If $\beta=1$, the equation (1.1) reduces to the classical diffusion model which arises in many physical processes such as thermodynamics [8], predator-prey problems [2], bio-heat conduction [9], heat exchangers [10], and mass transport in groundwater [11]. If the diffusion coefficient $D=0$, the equation (1.1) is Mullin's equation which arises in thermal grooving by surface diffusion [12]. If the reaction function $H(\chi,\tau,Z)$ is linear and in the form $H(\chi,\tau,Z)=k(\tau)Z(\chi,\tau)+h(\chi,\tau)$, then the eq. (1.1) can be rewritten as

Email addresses and ORCID numbers: **ibrahim.tekin@alanya.edu.tr, 0000-0001-6725-5663 (I. Tekin), akif.cetin@alanya.edu.tr, 0000-0002-4991-5098 (M.A. Çetin)**
Cite as: İ. Tekin, M.A. Çetin, ***Identification of the solely time-dependent zero-order coefficient in a linear bi-flux diffusion equation from an integral measurement*****, Fundam. J. Math. Appl., 6(3) (2023), 170-176**

$$Z_\tau(\chi,\tau)+\beta(1-\beta)RZ_{\chi\chi\chi\chi}(\chi,\tau)-\beta DZ_{\chi\chi}(\chi,\tau)=k(\tau)Z(\chi,\tau)+h(\chi,\tau). \tag{1.2}$$

In this work we consider an IP of recovering the time-dependent zero-order coefficient $k(\tau)$ in a linear Bi-flux diffusion equation (1.2) together with the particle concentration $Z(\chi,\tau)$ in the rectangle domain $\Pi_T=\{(\chi,\tau):(\chi,\tau)\in[0,1]\times[0,T]\}$ for some fixed $T>0$, subject to the initial condition (IC)

$$Z(\chi,0)=z_0(\chi),\ \chi\in[0,1], \tag{1.3}$$

and the boundary conditions (BCs)

$$Z(0,\tau)=Z(1,\tau)=Z_{\chi\chi}(0,\tau)=Z_{\chi\chi}(1,\tau)=0,\ \tau\in[0,T], \tag{1.4}$$

and the additional condition (AC)

$$\int_0^1 Z(\chi,\tau)d\chi=m(\tau),\ \tau\in[0,T], \tag{1.5}$$

where, D and R are the positive diffusion and reaction coefficients respectively, $0<\beta<1$, $h(\chi,\tau)$ is the source term, $z_0(\chi)$ is the initial particle concentration, and $m(\tau)$ is the extra integral measurement, which often refers specification of the energy/mass, to obtain the solution of the IP.

The IPs for the classical diffusion equation ($\beta=1$) are satisfactorily studied theoretically and numerically. For instance, IPs of determining the time-dependent heat source are studied in [13, 14, 15, 16], and IPs of reconstructing the time-dependent reaction coefficient are investigated in [17, 18, 19, 20, 21] with different boundary conditions (local, non-local or non-classical conditions).

On contrary to the IPs for the classical diffusion equation, our aim is to study the Bi-flux diffusion equation to identify the time-dependent zero-order coefficient $k(\tau)$ along with the particle concentration $Z(\chi,\tau)$ theoretically, for the first time, in the rectangular domain, using the IC (1.3), homogeneous BCs (1.4) and the AC (1.5).

The paper is organized as follows. In Section 2, the equivalent IP formulation is derived. In Section 3, the equivalent IP reduced to a system of Volterra integral equations with respect to the unknown functions $Z(\chi,\tau)$, and $k(\tau)$. Then the theorem of the existence and uniqueness of the solution of the IP is phrased and proved by means of the contraction principle.

2. Mathematical formulation of the equivalent IP

In this section, first we will give the classical solution of the IP (1.2)-(1.5), and then reduce the considered IP to the equivalent problem.

Definition 2.1. *The classical solution of the IP (1.2)-(1.5) is the pair $\{Z(\chi,\tau),k(\tau)\}$ subject to the following properties:*

(i) $Z(\chi,\tau)\in C^{4,1}(D_T)$,
(ii) $k(\tau)\in C[0,T]$,
(iii) *the pair $\{Z(\chi,\tau),k(\tau)\}$ satisfies the eq. (1.2) and the conditions (1.3)-(1.5) in ordinary sense.*

$Z(\chi,\tau)\in C^{4,1}(\Pi_T)$ means that $Z(\chi,\tau)$ and its partial derivatives with respect to χ upto fourth and τ upto first order are continuous on Π_T.

Let us reduce the IP (1.2)-(1.5) to the equivalent inverse problem to prove the existence and uniqueness theorem for the IP.

Lemma 2.2. *Let $z_0(\chi)\in C^2[0,1]$, $m(\tau)\in C^1[0,T]$, $m(\tau)\neq 0$ for all $\tau\in[0,T]$ and*

$$z_0(0)=z_0(1)=z_0''(0)=z_0''(1)=0,$$

$$\int_0^1 z_0(\chi)d\chi=m(0),$$

are satisfied. Then the problem of finding the classical solution of the IP (1.2)-(1.5) is equivalent to the problem of finding the classical solution of the IP from the eq. (1.2) with the IC (1.3), the BCs (1.4) and the AC

$$m'(\tau)+\beta(1-\beta)R\int_0^1 Z_{\chi\chi\chi\chi}(\chi,\tau)d\chi-\beta D\int_0^1 Z_{\chi\chi}(\chi,\tau)d\chi=k(\tau)m(\tau)+\int_0^1 h(\chi,\tau)d\chi. \tag{2.1}$$

Proof. It is obvious that if the pair $\{Z(\chi,\tau),k(\tau)\}$ is the classical solution of the IP (1.2)-(1.5), then the AC (2.1) is also satisfied.

Let us show that the condition (1.5) is fulfilled if the pair $\{Z(\chi,\tau),k(\tau)\}$ is a classical solution of the IP (1.2)-(1.4) with the AC (2.1).

Integrating the eq. (1.2) with respect to χ from 0 to 1 yields

$$\int_0^1 Z_\tau(\chi,\tau)d\chi+\beta(1-\beta)R\int_0^1 Z_{\chi\chi\chi\chi}(\chi,\tau)d\chi-\beta D\int_0^1 Z_{\chi\chi}(\chi,\tau)d\chi=k(\tau)\int_0^1 Z(\chi,\tau)d\chi+\int_0^1 h(\chi,\tau)d\chi. \tag{2.2}$$

Considering the difference of the equations (2.2) and (2.1) gives the following initial value problem

$$\begin{cases} Y'(\tau)-k(\tau)Y(\tau)=0,\ \tau\in[0,T], \\ Y(0)=0, \end{cases} \tag{2.3}$$

where $Y(\tau)=\int_0^1 Z(\chi,\tau)d\chi-m(\tau)$. The Cauchy problem (2.3) has only a trivial solution $Y(\tau)=0$. i.e.

$$\int_0^1 Z(\chi,\tau)d\chi=m(\tau),\ \tau\in[0,T].$$

Thus the condition (1.5) is fulfilled. □

3. Unique Solvability of the IP

The elements of the system $\{y_n(\chi)\}_{n=1}^\infty=\left\{\sqrt{2}\sin(\mu_n\chi)\right\}_{n=1}^\infty$ are bi-orthonormal on $[0,1]$ with $\mu_n=n\pi$ for each $n=1,2,\ldots$, i.e.:

$$(y_n(\cdot),y_m(\cdot))=\int_0^1 y_n(\chi)y_m(\chi)d\chi=\begin{cases}1 & ,m=n \\ 0 & ,m\neq n\end{cases}.$$

Also the system $\{y_n(\chi)\}_{n=1}^\infty$ is complete and forms a Riesz basis in $L_2[0,1]$.
The following lemma is useful to prove the unique solvability of the IP:

Lemma 3.1. *Suppose that the assumptions*

A$_1$ $z_0(\chi)\in C^3[0,1],\ z_0^{(4)}(\chi)\in L_2[0,1],\ z_0(0)=z_0(1)=z_0''(0)=z_0''(1)=0,$
A$_2$ $h(\chi,\tau)\in C^{3,0}(\Pi_T),\ h_{\chi\chi\chi\chi}(\cdot,\tau)\in L_2[0,1],\ h(0,\tau)=h(1,\tau)=h_{\chi\chi}(0,\tau)=h_{\chi\chi}(1,\tau)=0,$

are satisfied. Then

$$\sum_{n=1}^\infty \mu_n^3|z_{0n}|\le c_1\left\|z_0^{(4)}\right\|_{L_2[0,1]},\ \sum_{n=1}^\infty \mu_n|z_{0n}|\le c_2\left\|z_0^{(4)}\right\|_{L_2[0,1]},$$

and

$$\sum_{n=1}^\infty \mu_n^3|h_n(\tau)|\le c_1\left\|h_{\chi\chi\chi\chi}(\cdot,\tau)\right\|_{L_2[0,1]},\ \sum_{n=1}^\infty \mu_n|h_n(\tau)|\le c_2\left\|h_{\chi\chi\chi\chi}(\cdot,\tau)\right\|_{L_2[0,1]},$$

hold. Here $z_{0n}=(z_0(\cdot),y_n(\cdot))$, $h_n(\tau)=(h(\cdot,\tau),y_n(\cdot))$, *and*

$$c_1=\left(\sum_{n=1}^\infty\frac{1}{\mu_n^2}\right)^{1/2}=\frac{1}{\sqrt{6}},\ c_2=\left(\sum_{n=1}^\infty\frac{1}{\mu_n^6}\right)^{1/2}=\frac{1}{\sqrt{945}}.$$

Proof. Let us prove first one of these estimates. The others can be proved analogously. By applying integration by parts four times we get

$$\mu_n^3 z_{0n}=\mu_n^3(z_0(\cdot),y_n(\cdot))=\mu_n^3\int_0^1 z_0(\chi)y_n(\chi)d\chi=\mu_n^3\frac{1}{\mu_n^4}\int_0^1 z_0^{(4)}(\chi)y_n(\chi)d\chi=\frac{1}{\mu_n}\int_0^1 z_0^{(4)}(\chi)y_n(\chi)d\chi=\frac{1}{\mu_n}\left(z_0^{(4)}(\cdot),y_n(\cdot)\right).$$

By using Cauchy-Schwartz and Bessel inequalities

$$\sum_{n=1}^\infty \mu_n^3|z_{0n}|=\sum_{n=1}^\infty\frac{1}{\mu_n}\left|\left(z_0^{(4)}(\cdot),y_n(\cdot)\right)\right|\le\left(\sum_{n=1}^\infty\frac{1}{\mu_n^2}\right)^{1/2}\left(\sum_{n=1}^\infty\left|\left(z_0^{(4)}(\cdot),y_n(\cdot)\right)\right|^2\right)^{1/2}\le c_1\left\|z_0^{(4)}\right\|_{L_2[0,1]}$$

with $c_1=\left(\sum_{n=1}^\infty\frac{1}{\mu_n^2}\right)^{1/2}=\frac{1}{\sqrt{6}}$. □

Since the system $\{y_n(\chi)\}_{n=1}^{\infty}$ forms a Riesz basis, we can seek the solution of the IP (1.2)-(1.4) and (2.1) as

$$Z(\chi,\tau)=\sum_{n=1}^{\infty} Z_n(\tau)y_n(\chi), \tag{3.1}$$

where

$$Z_n(\tau)=\int_0^1 Z(\chi,\tau)y_n(\chi)d\chi.$$

Applying Fourier method yields $Z_n(\tau)$, $n=1,2,\ldots$ satisfy the following initial-value problems (IVPs)

$$\begin{cases} Z_n'(\tau)+K_nZ_n(\tau)=H_n(\tau;Z,k), \\ Z_n(0)=z_{0n},\ n=1,2,\ldots, \end{cases} \tag{3.2}$$

where $K_n=\mu_n^4\beta(1-\beta)R+\mu_n^2\beta D$, $H_n(\tau;Z,k)=k(\tau)Z_n(\tau)+h_n(\tau)$.
The solution of the IVPs (3.2) are

$$Z_n(\tau)=z_{0n}\exp(-K_n\tau)+\int_0^{\tau} H_n(s;Z,k)\exp(-K_n(\tau-s))ds. \tag{3.3}$$

Substituting (3.3) into (3.1) we get the classical solution of the IP (1.2)-(1.4) (or the first component of the pair $\{Z(\chi,\tau),k(\tau)\}$) as

$$Z(\chi,\tau)=\sum_{n=1}^{\infty}\left[z_{0n}\exp(-K_n\tau)+\int_0^{\tau} H_n(s;Z,k)\exp(-K_n(\tau-s))ds\right]y_n(\chi). \tag{3.4}$$

To derive the equations for the unknown coefficient $k(\tau)$ consider

$$\int_0^1 Z_{\chi\chi\chi\chi}(\chi,\tau)dx=\int_0^1\sum_{n=1}^{\infty}\mu_n^4Z_n(\tau)y_n(\chi)d\chi,$$

and

$$\int_0^1 Z_{\chi\chi}(\chi,\tau)dx=-\int_0^1\sum_{n=1}^{\infty}\mu_n^2Z_n(\tau)y_n(\chi)d\chi,$$

into the AC (2.1). Then we get

$$k(\tau)=\frac{1}{m(\tau)}\left[m'(\tau)-h_{int}(\tau)\right.$$

$$\left.+\sqrt{2}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)K_n}{\mu_n}\left\{z_{0n}\exp(-K_n\tau)+\int_0^{\tau}H_n(s;Z,a)\exp(-K_n(\tau-s))ds\right\}\right], \tag{3.5}$$

where $h_{int}(\tau)=\int_0^1 h(\chi,\tau)d\chi$.
The equations (3.4) and (3.5) are Volterra type integral equations with regard to $Z(\chi,\tau)$, and $k(\tau)$ and form the system of integral equations. The inverse problem (1.2)-(1.4) and (2.1) reduced to the system of equations (3.4) and (3.5). Thus, we can conclude that solving the system of integral equations (3.4) and (3.5) and the inverse problem (1.2)-(1.4) and (2.1) are equivalent.
Before setting and proving the existence and uniqueness theorem of the solution of the system (3.4) and (3.5), let us give the following Banach spaces which will be used in the proof of the main theorem:

I

$$\begin{aligned} B_T \;=\; & \left\{Z(\chi,\tau)=\sum_{n=1}^{\infty}Z_n(\tau)y_n(\chi):\ Z_n(\tau)\in C[0,T],\right. \\ & \left. J_T(Z)=\left(\sum_{n=1}^{\infty}(\mu_n^4\|Z_n\|_{C[0,T]})^2\right)^{1/2}<+\infty\right\}, \end{aligned}$$

and the norm of $Z(\chi,\tau)$ defined as $\|Z\|_{B_T}\equiv\left(\sum_{n=1}^{\infty}(\mu_n^4\|Z_n\|_{C[0,T]})^2\right)^{1/2}$.

II $E_T = B_T \times C[0,T]$ is the space of the all pairs $\kappa(\chi,\tau) = \{Z(\chi,\tau), k(\tau)\}$ and the norm is

$$\|\kappa\|_{E_T} = \|Z\|_{B_T} + \|k\|_{C[0,T]}.$$

Theorem 3.2. *Let the assumptions of the Lemma 2.2 and 3.1 are satisfied. Then, the system (3.4) and (3.5) (or the IP (1.2)-(1.4) and (2.1)) has a unique solution for small T.*

Proof. Let $\kappa(\chi,\tau) = \{Z(\chi,\tau), k(\tau)\}$ is an arbitrary element belongs to $E_T = B_T \times C[0,T]$. Then the system of equations (3.4) and (3.5) we can be rewritten into the operator equation form as

$$\kappa = \Phi(\kappa)$$

where $\Phi(\kappa) \equiv \{\phi_1, \phi_2\}$ and

$$\phi_1(\kappa) = \sum_{n=1}^{\infty} \left[z_{0n} \exp(-K_n \tau) + \int_0^{\tau} H_n(s; Z, k) \exp(-K_n(\tau - s)) ds \right] y_n(\chi),$$

and

$$\phi_2(\kappa) = \frac{1}{m(\tau)} \left[m'(\tau) - h_{int}(\tau) \right.$$

$$\left. + \sqrt{2} \sum_{n=1}^{\infty} \frac{(1-(-1)^n) K_n}{\mu_n} \left\{ z_{0n} \exp(-K_n \tau) + \int_0^t H_n(s; Z, k) \exp(-K_n(\tau - s)) ds \right\} \right].$$

It is obvious that

$$K_n = \mu_n^4 \beta (1-\beta) R + \mu_n^2 \beta D > 0, \ n = 1, 2, \ldots.$$

Thus

$$\exp(-K_n \tau) \le 1, \text{ and } \exp(-K_n(\tau - s)) \le 1 \text{ for } 0 \le s \le \tau, \ 0 \le \tau \le T.$$

Let us prove that Φ is a contraction operator in two steps by considering these estimates.
I) Firstly prove that Φ is a continuous onto map on $E_T \to E_T$. It means that we require to prove $\phi_1(\kappa) \in B_T$ and $\phi_2(\kappa) \in C[0,T]$ for an arbitrary element $\kappa(x,\tau) = \{Z(\chi,\tau), k(\tau)\} \in E_T$ with $Z(\chi,\tau) \in B_T$, $k(\tau) \in C[0,T]$.
Let us verify that

$$J_T(\phi_1) = \left(\sum_{n=1}^{\infty} (\mu_n^4 \|(\phi_1)_n\|_{C[0,T]})^2 \right)^{1/2} < +\infty,$$

where $(\phi_1)_n(\tau) = RHS(Z_n(\tau))$. In other words let us show that $\phi_1(\kappa) \in B_T$. Under the assumptions of the Lemma 3.1 and Theorem 3.2, we obtain

$$J_T(\phi_1) \le 2 \left\| z_0^{(4)} \right\|_{L_2[0,1]} + 2T \|h_{\chi\chi\chi\chi}\|_{L_2[0,1]} + 2T \|k\|_{C[0,T]} \|Z\|_{B_T}. \tag{3.6}$$

Since $Z(\chi,\tau) \in B_T$, and $k(\tau) \in C[0,T]$, the norms $\|k\|_{C[0,T]}$, $\|Z\|_{B_T}$ are finite. Therefore, $J_T(\phi_1)$ is also finite. Thus $\phi_1(\kappa)$ belongs to the space B_T.
Now let's demonstrate that $\phi_2(\kappa) \in C[0,T]$. From the eq. (3.5), we obtain

$$\|\phi_2(\kappa)\|_{C[0,T]} \le \frac{1}{\alpha} \left[\|m'\|_{C[0,T]} + \|h_{int}\|_{C[0,T]} \right.$$

$$\left. + M \left(\left\| z_0^{(4)} \right\|_{L_2[0,1]} + T \|h_{\chi\chi\chi\chi}\|_{L_2[0,1]} + T \|k\|_{C[0,T]} \|Z\|_{B_T} \right) \right] \tag{3.7}$$

where $\min_{0 \le \tau \le T} |m(\tau)| \ge \alpha > 0$, $M = 2\sqrt{2}(c_1 \beta (1-\beta) R + c_2 \beta D)$.
As in the previous part we can conclude that $\|\phi_2(\kappa)\|_{C[0,T]}$ is bounded. So $\phi_2(\kappa) \in C[0,T]$.
From the eqs. (3.6) and (3.7), we can obtain

$$\|\Phi(\kappa)\|_{E_T} \le L_1(T) + L_2(T) \|k\|_{C[0,T]} \|Z\|_{B_T},$$

where

$$L_1(T) = \left(2+\frac{M}{\alpha}\right)\left\|z_0^{(4)}\right\|_{L_2[0,1]} + T\left(2+\frac{M}{\alpha}\right)\|h_{\chi\chi\chi\chi}\|_{L_2[0,1]} + \frac{1}{\alpha}\left[\|m'\|_{C[0,T]} + \|h_{int}\|_{C[0,T]}\right],$$

$$L_2(T) = T\left(2+\frac{M}{\alpha}\right).$$

Since $L_2(T)$ tends to zero as $T \to 0$, and $L_1(T)$ is a continuous function of T, there exists a sufficiently small $T > 0$ such that

$$(L_1(T)+1)^2 L_2(T) < 1.$$

Let us define a ball $A := \left\{\kappa \in E_T : \|\kappa\|_{E_T} \le L_1(T)+1\right\}$ for the fixed T. Then, for every $\kappa \in A$, we get

$$\begin{aligned}\|\Phi(\kappa)\|_{E_T} &\le L_1(T) + L_2(T)\|k\|_{C[0,T]}\|Z\|_{B_T}\\ &\le L_1(T) + L_2(T)(L_1(T)+1)^2 < L_1(T)+1.\end{aligned}$$

Thus, Φ is an onto continuous map on E_T.

II) This step our aim is to prove that the operator Φ is a contraction mapping operator. Assume that let z_1 and z_2 be any two elements of E_T. According to the definition of the space E_T yields $\|\Phi(\kappa_1)-\Phi(\kappa_2)\|_{E_T} = \|\phi_1(\kappa_1)-\phi_1(\kappa_2)\|_{B_T} + \|\phi_2(\kappa_1)-\phi_2(\kappa_2)\|_{C[0,T]}$. Here $\kappa_i = \left\{Z^i(\chi,\tau), k^i(\tau)\right\}$, $i=1,2$. consider the following differences

$$\phi_1(\kappa_1)-\phi_1(\kappa_2) = \sum_{n=1}^{\infty}\left[\int_0^\tau \left[H_n(s;Z^1,k^1) - H_n(s;Z^2,k^2)\right]\exp(-K_n(\tau-s))ds\right] y_n(\chi),$$

and

$$\phi_2(\kappa_1)-\phi_2(\kappa_2) = \frac{\sqrt{2}}{m(\tau)}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)K_n}{\mu_n}\int_0^\tau \left[H_n(s;Z^1,k^1) - H_n(s;Z^2,k^2)\right]\exp(-K_n(\tau-s))ds.$$

We can obtain from the last equations

$$\|\phi_1(\kappa_1)-\phi_1(\kappa_2)\|_{B_T} \le 2T\left(\left\|Z^2\right\|_{B_T}\left\|k^1-k^2\right\|_{C[0,T]} + \left\|k^1\right\|_{C[0,T]}\left\|Z^1-Z^2\right\|_{B_T}\right),$$

and

$$\|\phi_2(\kappa_1)-\phi_2(\kappa_2)\|_{C[0,T]} \le \frac{TM}{\alpha}\left(\left\|Z^2\right\|_{B_T}\left\|k^1-k^2\right\|_{C[0,T]} + \left\|k^1\right\|_{C[0,T]}\left\|Z^1-Z^2\right\|_{B_T}\right).$$

From the these inequalities it follows that

$$\|\Phi(\kappa_1)-\Phi(\kappa_2)\|_{E_T} \le L_2(T)C(k^1,Z^2)\|\kappa_1-\kappa_2\|_{E_T},$$

where $L_2(T) = T\left(2+\frac{M}{\alpha}\right)$, and $C(k^1,Z^2) = \max\left\{\left\|k^1\right\|_{C[0,T]}, \left\|Z^2\right\|_{B_T}\right\} = L_1(T)+1$.
Since $L_1(T)+1 \le (L_1(T)+1)^2$,

$$0 < L_2(T)C(k^1,Z^2) = L_2(T)(L_1(T)+1) \le L_2(T)(L_1(T)+1)^2 < 1,$$

i.e. $0 < L_2(T)C(k^1,Z^2) < 1$. Therefore, the operator Φ is contraction mapping operator.
Thus we can conclude that the operator Φ is contraction mapping and it is a continuous onto map on E_T. Then the solution of $\kappa = \Phi(\kappa)$ exists and unique regarding to Banach fixed point theorem. □

Example 3.3. *Consider the IP (1.2)-(1.5) with unknown smooth $k(\tau)$ and inputs:*

$$z_0(\chi) = \sin(\pi\chi),\ m(\tau) = \frac{2}{\pi}e^\tau,\ h(\chi,\tau) = \left[\left(1+\pi^4\beta(1-\beta)R + \pi^2\beta D\right)e^\tau - 1\right]\sin(\pi\chi).$$

These inputs satisfy the conditions of the Lemma 2.2 and Lemma 3.1, i.e.

$$z_0(\chi) \in C^3[0,1],\ z_0^{(4)}(\chi) \in L_2[0,1],\ z_0(0) = z_0(1) = z_0''(0) = z_0''(1) = 0,$$

$$h(\chi,\tau) \in C^{3,0}(\Pi_T),\ h_{\chi\chi\chi\chi}(\cdot,\tau) \in L_2[0,1],\ h(0,\tau) = h(1,\tau) = h_{\chi\chi}(0,\tau) = h_{\chi\chi}(1,\tau) = 0,$$

and

$$m(\tau) \in C^1[0,T], m(\tau) \ne 0 \textit{ for all } \tau \in [0,T] \textit{ and } \int_0^1 z_0(\chi)d\chi = m(0) = \frac{2}{\pi}.$$

According to the Theorem 3.2, the solution of the inverse problem (1.2)-(1.5) exists and unique and the solution is

$$\{Z(\chi,\tau), k(\tau)\} = \left\{e^\tau\sin(\pi\chi), e^{-\tau}\right\}.$$

4. Conclusion

The manuscript studies the IP of obtaining the solely time-dependent zero-order coefficient in a linear Bi-flux diffusion equation from an extra integral observation. The theorem of existence and uniqueness of the solution of the IP is proved for an adequately small time interval by applying the contraction mapping principle. This work is new and has never been studied theoretical or numerical before. The numerical method of the IP will be considered with a suitable scheme as a future work.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: The authors contributed equally to the writing of this paper.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Copyright Statement: Authors own the copyright of their work published in the journal and their work is published under the CC BY-NC 4.0 license.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

References

[1] S. Guenneau, D. Petiteau, M. Zerrad, C. Amra, T. Puvirajesinghe, *Transformed Fourier and Fick equations for the control of heat and mass diffusion*, AIP Adv., **5**(5) (2015), 053404.

[2] D. Shi, S. Zhang, *Unconditional superconvergence of the fully-discrete schemes for nonlinear prey-predator model*, Appl. Numer. Math., **172** (2022), 118-132.

[3] D. Shi, C. Li, *Superconvergence analysis of two-grid methods for bacteria equations*, Numer. Algorithms, **86**(1) (2021), 123-152.

[4] L. Bevilacqua, M. Jiang, A. Silva Neto, A.C.R. Galeão, *An evolutionary model of bi-flux diffusion processes*, J. Braz. Soc. Mech. Sci. Eng., **38**(5) (2016), 1421-1432.

[5] M. Jiang, L. Bevilacqua, A.J.S. Neto, A.C.R. Galeão, J. Zhu, *Bi-flux theory applied to the dispersion of particles in anisotropic substratum*, Appl. Math. Model., **64** (2018), 121-134.

[6] M. Jiang, L. Bevilacqua, J. Zhu, X. Yu, *Nonlinear Galerkin finite element methods for fourth-order Bi-flux diffusion model with nonlinear reaction term*, Comput. Appl. Math., **39**(3) (2020), 1-16.

[7] D. Shi, S. Zhang, *Unconditional superconvergence analysis for the nonlinear Bi-flux diffusion equation*, Appl. Math. Comput., **442** (2023), 127771.

[8] S. Guenneau, C. Amra, D. Veynante, *Transformation thermodynamics: cloaking and concentrating heat flux*, Opt. Express, **20**(7) (2012), 8207-8218.

[9] E.P. Scott, P.S. Robinson, T.E. Diller, *Development of methodologies for the estimation of blood perfusion using a minimally invasive thermal probe*, Meas. Sci. Technol., **9**(6) (1998), 888.

[10] L. Marin, L. Elliott, P.J. Heggs, D.B. Ingham, D. Lesnic, X. Wen, *Analysis of polygonal fins using the boundary element method*, Appl. Therm. Eng., **24**(8-9) (2004), 1321-1339.

[11] E.A. Sudicky, *The Laplace transform Galerkin technique: A time-continuous finite element theory and application to mass transport in groundwater*, Water Resour. Res., **25**(8) (1989), 1833-1846.

[12] M.I. Ismailov, *Direct and inverse problems for thermal grooving by surface diffusion with time dependent mullins coefficient*, Math. Model. Anal., **26**(1) (2021), 135-147.

[13] J.R. Cannon, *Determination of an unknown heat source from overspecified boundary data*, SIAM J. Numer. Anal., **5**(2) (1968), 275–286.

[14] A. Hasanov, *Simultaneous determination of source terms in a linear parabolic problem from the final overdetermination: weak solution approach*, J. Math. Anal. Appl., **330** (2007), 766-779

[15] A. Hazanee, D. Lesnic, M.I. Ismailov, N.B. Kerimov, *An inverse time-dependent source problem for the heat equation with a non-classical boundary condition*, Appl. Math. Model., **39**(20) (2015), 6258-6272.

[16] M.I. Ismailov, F. Kanca, D. Lesnic, *Determination of a time-dependent heat source under nonlocal boundary and integral overdetermination conditions*, Appl. Math. Comput., **218**(8) (2011), 4138-4146.

[17] J.R. Cannon, Y. Lin, S. Wang, *Determination of a control parameter in a parabolic partial differential equation*, J. Aust. Math. Soc. Series B, **33** (1991), 149-163.

[18] M.I. Ismailov, I. Tekin, S. Erkovan, *An inverse problem for finding the lowest term of a heat equation with Wentzell–Neumann boundary condition*, Inverse Probl. Sci. Eng., **27**(11) (2019), 1608-1634.

[19] M.I. Ismailov, F. Kanca, *An inverse coefficient problem for a parabolic equation in the case of nonlocal boundary and overdetermination conditions*, Math. Methods Appl. Sci., **34**(6) (2011), 692-702.

[20] N.B. Kerimov, M.I. Ismailov, *An inverse coefficient problem for the heat equation in the case of nonlocal boundary conditions*, J. Math. Anal. Appl., **396**(2) (2012), 546-554.

[21] L. Zhuo, D. Lesnic, M.I. Ismailov, İ. Tekin, S. Meng, *Determination of the time-dependent reaction coefficient and the heat flux in a nonlinear inverse heat conduction problem*, Int. J. Comput. Math., **96**(10) (2019), 2079-2099.

Fundamental Journal of Mathematics and Applications, 6 (3) (2023) 177-187
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1292885

Bisimplicial Commutative Algebras and Crossed Squares

Özgün Gürmen Alansal[1*] **and Erdal Ulualan**[1]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Kütahya Dumlupınar University, Kütahya, Turkiye*
[*] *Corresponding author*

Article Info

Keywords: *Bisimplicial algebras, Crossed modules, Moore complex*
2010 AMS: *18G30, 55U10*
Received: *5 May 2023*
Revised: *19 September 2023*
Accepted: *22 September 2023*
First Online: *29 September 2023*
Published: *30 September 2023*

Abstract

A simplicial commutative algebra with Moore complex of length 1 gives a crossed module structure over commutative algebras. In this study, we will give 2-dimensional version of this result by giving hypercrossed complex pairings for a bisimplicial algebra and its Moore bicomplex. We give a detailed calculation in low dimensions for these pairings to see their role in the structures of crossed squares and bisimplicial algebras. In this context, we prove that if the Moore bicomplex of a bisimplicial commutative algebra is of length 1, then it gives a crossed square structure over commutative algebras.

1. Introduction

The category of simplicial groups with Moore complex of length 1 is equivalent to the category of Whitehead's crossed modules [6]. This structure can be considered as an algebraic model for homotopy connected 2-types. Conduché [2] has proven 2-dimensional version of this result by giving the definition of a crossed module of length 2. He proved that the category of such objects are equivalent to the category of simplicial groups with Moore complex of length 2. The structure of a crossed square has been introduced by Guin-Walery and Loday [4]. This structure is a model for homotopy connected 3-types. The commutative algebra version of crossed modules has been defined by Porter in [11]. On the other hand crossed squares of commutative algebras has been investigated by Ellis [5]. Conduché also, [3], gave the close relationships among bisimplicial groups with crossed squares for the version of groups, and he proved that Loday's mapping cone complex of a crossed square gives a 2-crossed module.

Carrasco and Cegarra, [10], give a general version of the Dold-Kan theorem for the equivalence between simplicial groups and non-Abelian chain complexes. Porter in [12] has proven the equivalence between the category of n-types of simplicial groups and the category of crossed n-cubes. In [9], Gürmen-Alansal and Ulualan generalised these pairings for the Moore bicomplex in bisimplicial groups. It can be seen the role of these pairings for the relations among bisimplicial groups and crossed squares. Arvasi and Porter [13], using the Carrasco and Cegarra pairing operators for a Moore complex in a simplicial (commutative) algebra, and they have defined the functions $C_{\alpha,\beta}$ functions, and as an application, they proved that the category of 2-crossed modules of commutative algebras introduced by Grandjeán and Vale in [1] is equivalent to that of simplicial commutative algebras with Moore complex of length 2. Of course, this is the commutative algebra version of Conduché's result [2].

Our first aim in this work is to define the functions $C_{\alpha,\beta}$ for 2-dimensional simplicial algebras (or bisimplicial algebras) and second aim is to give the relationship between crossed squares and bisimplicial algebras by use of the functions $C_{\alpha,\beta}$.

2. Preliminaries

The simplicial set analogue has been studied in [8, 7, 13]. We give the following statements from [13]. Define the set $P(n)$ consisting of the pairs of elements in the form (α,β) from $S(n)$ with $\alpha\cap\beta=\emptyset$ and $\beta<\alpha$ where $\alpha=(i_r,\dots,i_1)$, $\beta=$

Email addresses and ORCID numbers: **ozgun.gurmen@dpu.edu.tr, 0000-0003-2851-986X (Ö. Gürmen Alansal), e.ulualan@dpu.edu.tr, 0000-0002-4823-8267 (E. Ulualan)**
Cite as: Ö.G. Alansal, E. Ulualan, ***Bisimplicial commutative algebras and crossed squares*****, Fundam. J. Math. Appl., 6(3) (2023), 177-187.**

$(j_s,\ldots,j_1)\in S(n)$. The k-linear morphisms are,

$$\{C_{\alpha,\beta}: NE_{n-\#\alpha}\otimes NE_{n-\#\beta}\to NE_n|(\alpha,\beta)\in P(n),\ 0\le n\}$$

given by composing:

$$\begin{aligned} C_{\alpha,\beta}(x_\alpha\otimes y_\beta) &= p\mu(s_\alpha\otimes s_\beta)(x_\alpha\otimes y_\beta)\\ &= p(s_\alpha(x_\alpha)s_\beta(x_\beta))\\ &= (1-s_{n-1}d_{n-1})\ldots(1-s_0d_0)(s_\alpha(x_\alpha)s_\beta(x_\beta)) \end{aligned}$$

where

$$s_\alpha = s_{i_r}\ldots s_{i_1}: NE_{n-\#\alpha}\to E_n,\ s_\beta = s_{j_s}\ldots s_{j_1}: NE_{n-\#\beta}\to E_n,$$

$p: E_n\to NE_n$ is given as composite projections $p=p_{n-1}\ldots p_0$ with

$$p_j = 1-s_jd_j \text{ for } j=0,1,\ldots,n-1$$

and $\mu: E_n\otimes E_n\to E_n$ denotes multiplication.
Arvasi and Porter in [13] studied the truncated simplicial algebras and their properties. By using $C_{\alpha,\beta}$ functions they then proved the following result:

Proposition 2.1. *Suppose that E is a simplicial algebra. We denote its Moore complex by NE. Then*

$$NE_2/\partial_3(NE_3\cap D_3)\xrightarrow{\overline{\partial_2}} NE_1\xrightarrow{\partial_1} NE_0$$

is a 2-crossed module of algebras with the Peiffer lifting map

$$\{-,-\}: NE_1\otimes NE_1\longrightarrow NE_2/\partial_3(NE_3\cap D_3)$$

given by $(x\otimes y)\mapsto\{x,y\}=C_{(0)(1)}(x\otimes y)+\partial_3(NE_3\cap D_3)=s_1(x)(s_1y-s_0y)+\partial_3(NE_3\cap D_3)$ for all $x,y\in NE_1$.

3. Hypercrossed Complex Pairings for Bisimplicial Algebras

In this section, we define the $C_{\alpha,\beta}$ functions in Moore bicomplex of a bisimplicial algebra. Let Δ be the category whose objects are the ordered sets $[n=\{0<1<2\cdots<n\}]$ and whose morphisms are non decreasing maps between them. Suppose $\Delta\times\Delta$ is the product category. Its objects are the pairs $([p],[q])$, the morphisms are the pairs of increasing maps. The functor $E_{.,.}:(\Delta\times\Delta)^{op}\to Alg$ can be considered as a bisimplicial algebra. Therefore, $E_{.,.}$ is equivalent to giving for each (p,q) an algebra $E_{p,q}$ and morphisms:

$$\begin{array}{llll} d_i^{h^{(pq)}}: E_{p,q}\to E_{p-1,q}; & s_i^{h^{(pq)}}: E_{p,q}\to E_{p+1,q}, & p\ge i\ge 0\\ d_j^{v^{(pq)}}: E_{p,q}\to E_{p,q-1}; & s_j^{v^{(pq)}}: E_{p,q}\to E_{p,q+1}, & q\ge j\ge 0 \end{array}$$

where the maps $d_j^{v^{(pq)}},s_j^{v^{(pq)}}$ commute with $d_i^{h^{(pq)}},s_i^{h^{(pq)}}$ and that the homomorphisms $d_j^{v^{(pq)}},s_j^{v^{(pq)}}$ respectively for $d_i^{h^{(pq)}},s_i^{h^{(pq)}}$. These maps satisfy the simplicial identities.
We consider of $d_j^{v^{(pq)}},s_j^{v^{(pq)}}$ as the vertical operators and $d_i^{h^{(pq)}},s_i^{h^{(pq)}}$ as the horizontal operators. If $E_{.,.}$ is a bisimplicial algebra, an element of $E_{p,q}$ can be thought as a product of a p-simplex and a q-simplex. Let **BiSimpAlg** be the category whose objects are bisimplicial algebras given by the functors $E_{.,.}:(\Delta\times\Delta)^{op}\to Alg$ and whose morphisms are natural transformations between the functors $E_{.,.}$ and $E'_{.,.}$.
The Moore bicomplex for a bisimplicial algebra is

$$NE_{n,m}=\bigcap_{(i,j)=(0,0)}^{(n-1,m-1)}\mathrm{Ker}d_i^{h^{(nm)}}\cap\mathrm{Ker}d_j^{v^{(nm)}}$$

with the boundary homomorphisms

$$\partial_i^{h^{(nm)}}: NE_{n,m}\longrightarrow NE_{n-1,m}$$

and

$$\partial_j^{v^{(nm)}}: NE_{n,m}\longrightarrow NE_{n,m-1}$$

obtained by the maps $d_i^{h^{(nm)}}$ and $d_j^{v^{(nm)}}$ where $0\leqslant j\leqslant m,\ 0\leqslant i\leqslant n,\ n,m\neq 0$.

We can denote this Moore bicomplex by Figure 3.1.

$$\begin{tikzcd}
\vdots & \vdots \arrow[d] & \vdots & \arrow[d] \\
\cdots \arrow[r] & NE_{1,2} \arrow[rr, "\partial_1^{h(12)}"'] \arrow[d, "\partial_2^{v(12)}"'] & & NE_{0,2} \arrow[d, "\partial_2^{v(02)}"] \\
\cdots \arrow[r] & NE_{1,1} \arrow[rr, "\partial_1^{h(11)}"'] \arrow[d, "\partial_1^{v(11)}"'] & & NE_{0,1} \arrow[d, "\partial_1^{v(01)}"] \\
\cdots \arrow[r] & NE_{1,0} \arrow[rr, "\partial_1^{h(10)}"'] & & NE_{0,0}.
\end{tikzcd}$$

Figure 3.1: Moore bicomplex

Now, we give the functions $C_{\alpha,\beta}$ for bisimplicial algebras.
Given $\underline{k}=(n,m)\in\mathbb{N}\times\mathbb{N}$. Let $S(\underline{k})=S(n)\times S(m)$ with the partial product order. Take $\underline{\alpha},\underline{\beta}\in S(\underline{k})$ where $\underline{\alpha}=(\alpha_1,\alpha_2)$, $\underline{\beta}=(\beta_1,\beta_2)$ for $\alpha_1,\beta_1\in S(n)$ and $\alpha_2,\beta_2\in S(m)$. The 2-dimensional case of the $C_{\alpha,\beta}$ functions given for any simplicial algebra [13] can be obtained as follows. We will need that the Pairings

$$\left\{C_{\underline{\alpha},\underline{\beta}}:NE_{\underline{k}-\#\underline{\alpha}}\times NE_{\underline{k}-\#\underline{\beta}}\longrightarrow NE_{\underline{k}}\mid \underline{\alpha}\neq\underline{\beta},\ \underline{\alpha},\underline{\beta}\in S(\underline{k})\right\}$$

are obtained by creating of the maps given in the diagram

$$\begin{tikzcd}
NE_{n-\#\alpha_1,m-\#\alpha_2}\otimes NE_{n-\#\beta_1,m-\#\beta_2} \arrow[r, "C_{\underline{\alpha},\underline{\beta}}"] \arrow[d, "(s_{\underline{\alpha}}\otimes s_{\underline{\beta}})"'] & NE_{n,m} \\
E_{n,m}\otimes E_{n,m} \arrow[r, "\mu"'] & E_{n,m} \arrow[u, "p"']
\end{tikzcd}$$

Figure 3.2: Construction of $C_{\underline{\alpha},\underline{\beta}}$

where $s_{\underline{\alpha}}:s^h_{\alpha_1}s^v_{\alpha_2}$, and where $s^h_{\alpha_1}=s^h_{i_r}\cdots s^h_{i_1}$ for $\alpha_1=(i_r,\cdots,i_1)\in S(n)$, similarly $s_{\underline{\beta}}:s^h_{\beta_1}s^v_{\beta_2}$, and where $s^h_{\beta_1}=s^h_{j_s}\cdots s^h_{j_1}$ for $\beta_1=(j_s,\cdots,j_1)\in S(n)$. We can define the maps similarly $s^v_{\alpha_2},s^v_{\beta_2}$ in $S(m)$. Note that $s^{(h,v)}_{\emptyset}=id$ is the identity map. By the composing the projections given below, the map p is defined as

$$p=\left(p^h_{n-1}...p^h_0\right)\left(p^v_{m-1}...p^v_0\right) \tag{3.1}$$

where $p^{(h,v)}_j(x)=x-s^{(h,v)}_j d^{(h,v)}_j(x)$, and μ is given by the multiplication.
Thus for $\underline{\alpha}=(\alpha_1,\alpha_2),\underline{\beta}=(\beta_1,\beta_2)\in S(n)\times S(m)$, it is obtained from the Figure 3.2 by composing the maps that

$$\begin{aligned}
C_{\underline{\alpha},\underline{\beta}}(x\otimes y) &= p\mu(s_{\underline{\alpha}}\otimes s_{\underline{\beta}})(x\otimes y)\\
&= p\mu(s^h_{\alpha_1}s^v_{\alpha_2}(x)\otimes s^h_{\beta_1}s^v_{\beta_2}(y))\\
&= p\left(s^h_{\alpha_1}s^v_{\alpha_2}(x)\cdot s^h_{\beta_1}s^v_{\beta_2}(y)\right)
\end{aligned}$$

for $x\in NE_{n-\#\alpha_1,m-\#\alpha_2}$ and $y\in NE_{n-\#\beta_1,m-\#\beta_2}$, where p is given by

$$\begin{aligned}
p:E_{n,m} &\rightarrow NE_{n,m}\\
a &\mapsto p^h_{n-1}\ldots p^h_1p^h_0p^v_{m-1}\ldots p^v_1p^v_0(a)=(1-s^h_{n-1}d^h_{n-1})\cdots(1-s^h_0d^h_0)(1-s^v_{m-1}d^v_{m-1})\cdots(1-s^v_0d^v_0)(a)
\end{aligned}$$

for all $a\in E_{n,m}$. Note that we obtain

$$C_{\underline{\alpha},\underline{\beta}}(x\otimes y)=C_{\underline{\beta},\underline{\alpha}}(y\otimes x)$$

for $x\in NE_{n-\#\alpha_1,m-\#\alpha_2}$ and $y\in NE_{n-\#\beta_1,m-\#\beta_2}$.

4. Low Dimensions Cases

4.1. The case $(n,m)=(0,1)$ or $(1,0)$.

Take $(n,m)=(0,1)$ or $(n,m)=(1,0)$. Firstly we calculate here $C_{\underline{\alpha},\underline{\beta}}$ functions whose codomain $NE_{0,1}$ or $NE_{1,0}$. Let $(n,m)=(0,1)$. So $S(n,m)=S(0)\times S(1)=\{(\emptyset,\emptyset),(\emptyset,(0))\}$, $\underline{\alpha}=(\emptyset,\emptyset)$ and $\underline{\beta}=(\emptyset,(0))$. Thus the function $C_{\underline{\alpha},\underline{\beta}}$ given by

$$C_{(\emptyset,\emptyset),(\emptyset,(0))}:NE_{0,1}\otimes NE_{0,0}\longrightarrow NE_{0,1}$$

is obtained by

$$\begin{aligned} C_{(\emptyset,\emptyset),(\emptyset,(0))}(x\otimes y) \quad &= p\mu(\overset{h}{s}_{\emptyset}\overset{v}{s}_{\emptyset}(x)\otimes \overset{h}{s}_{\emptyset}\overset{v}{s}_{(0)}(y))=p(id(x)\overset{v}{s}_{(0)}(y))\\ &=xs_0^{v(00)}(y)-s_0^{v(00)}d_0^{v(01)}(xs_0^{v(00)}(y))\\ &=xs_0^{v(00)}(y)-s_0^{v(00)}d_0^{v(01)}(x).s_0^{v(00)}(y)\\ &=xs_0^{v(00)}(y)\quad (\because x\in NE_{0,1}=\ker d_0^{v(01)}) \end{aligned}$$

for $x\in NE_{0,1}$, $y\in NE_{0,0}$.
Assume that $(n,m)=(1,0)$. After this, we take $S(1)\times S(0)=\{(\emptyset,\emptyset),((0),\emptyset)\}$. Let $\underline{\alpha}=(\emptyset,\emptyset)$ and $\underline{\beta}=((0),\emptyset)$. Then the function

$$C_{(\emptyset,\emptyset),((0),\emptyset)}:NE_{1,0}\otimes NE_{0,0}\longrightarrow NE_{1,0}$$

is defined as

$$C_{(\emptyset,\emptyset),((0),\emptyset)}(x\otimes y)=x(s_0^{h(00)}(y))$$

for $x\in NE_{1,0}$, $y\in NE_{00}$.

4.2. The case $(n,m)=(1,1)$.

Let $(n,m)=(1,1)$. Define the set

$$S(1)\times S(1)=\{(\emptyset,\emptyset),\ ((0),(0)),\ (\emptyset,(0)),\ ((0),\emptyset)\}.$$

1. For $\underline{\alpha}=(\emptyset,\emptyset)$, $\underline{\beta}=(\emptyset,(0))$, the function $C_{\underline{\alpha},\underline{\beta}}$ is from $NE_{1,1}\otimes NE_{1,0}$ to $NE_{1,1}$. The map can be defined by

$$C_{(\emptyset,\emptyset),(\emptyset,(0))}(x\otimes y)=xs_0^{v(10)}(y);\ x\in NE_{1,1},\ y\in NE_{1,0}.$$

2. For $\underline{\alpha}=(\emptyset,\emptyset)$, $\underline{\beta}=((0),\emptyset)$. Then, the function $C_{\underline{\alpha},\underline{\beta}}$ is from $NE_{1,1}\otimes NE_{0,1}$ to $NE_{1,1}$. The map can be calculated by

$$C_{(\emptyset,\emptyset),((0),\emptyset)}(x\otimes y)=xs_0^{h(01)}(y);\ x\in NE_{1,1},\ y\in NE_{0,1}.$$

3. For $\underline{\alpha}=(\emptyset,\emptyset)$, $\underline{\beta}=((0),(0))$, the map $C_{(\emptyset,\emptyset),((0),(0))}:NE_{1,1}\otimes NE_{0,0}\to NE_{1,1}$ is given by

$$C_{(\emptyset,\emptyset),((0),(0))}(x\otimes y)=x(s_0^{v(00)}s_0^{h(01)}(y));\ x\in NE_{1,1},\ y\in NE_{0,0}.$$

4. Take $\underline{\alpha}=((0),\emptyset)$ and $\underline{\beta}=(\emptyset,(0))$. Then the map

$$C_{((0),\emptyset),(\emptyset,(0))}:NE_{0,1}\otimes NE_{1,0}\to NE_{1,1}$$

can be calculated for any $x\in NE_{0,1}$, $y\in NE_{1,0}$ as

$$\begin{aligned} C_{((0),\emptyset)(\emptyset,(0))}(x\otimes y) \quad &= \quad p\mu(s_{\underline{\alpha}}\otimes s_{\underline{\beta}})(x\otimes y)\\ &= \quad p_0^hp_0^v(s_0^{h(01)}(x)s_0^{v(10)}(y))\\ &= \quad p_0^h\left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{v(10)}d_0^{v(11)}(s_0^{h(01)}(x)s_0^{v(10)}(y))\right)\\ &= \quad p_0^h\left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{v(10)}s_0^{h(00)}d_0^{v(01)}(x)s_0^{v(10)}(y)\right)\\ &= \quad p_0^h\left(s_0^{h(01)}(x)s_0^{v(10)}(y)\right)\quad (\because x\in\ker d_0^{v(01)})\\ &= \quad \left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{h(01)}d_0^{h(11)}(s_0^{h(01)}(x)s_0^{v(10)}(y))\right)\\ &= \quad \left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{h(01)}(x)s_0^{h(01)}d_0^{h(11)}s_0^{v(10)}(y)\right)\quad (\because d_0^{h(11)}s_0^{h(01)}=id)\\ &= \quad \left(s_0^{h(01)}(x)s_0^{v(10)}(y)-s_0^{h(01)}(x)s_0^{h(01)}s_0^{v(00)}d_0^{h(10)}(y)\right)\ (\because d_0^{h(11)}s_0^{v(10)}=s_0^{v(00)}d_0^{h(10)})\\ &= \quad \left(s_0^{h(01)}(x)s_0^{v(10)}(y)\right).\quad (\because y\in\ker d_0^{h(10)}=NE_{1,0}) \end{aligned}$$

5. For $\underline{\alpha} = ((0),\emptyset)$ and $\underline{\beta} = ((0),(0))$. Then the map

$$C_{((0),\emptyset),((0),(0))} : NE_{0,1} \otimes NE_{0,0} \to NE_{1,1}$$

can be given for any $x \in NE_{0,1}$, $y \in NE_{0,0}$ as

$$\begin{aligned}
C_{((0),\emptyset),((0),(0))}(x\otimes y) &= p\mu(s_{\underline{\alpha}}\otimes s_{\underline{\beta}})(x\otimes y) \\
&= p_0^h p_0^v (s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y)) \\
&= p_0^h \left(s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) - s_0^{v^{(10)}} d_0^{v^{(11)}} (s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y)) \right) \\
&= p_0^h \left(s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) - s_0^{v^{(10)}} d_0^{v^{(11)}} s_0^{h^{(01)}}(x) s_0^{v^{(10)}} d_0^{v^{(11)}} s_0^{h^{(01)}} s_0^{v^{(00)}}(y) \right) \\
&= p_0^h \left(s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) \right) \quad (\because x \in \ker d_0^{v^{(01)}}) \\
&= \left(s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) - s_0^{h^{(01)}} d_0^{h^{(11)}} (s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y)) \right) \\
&= \left(s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) - s_0^{h^{(01)}}(x) s_0^{h^{(01)}} s_0^{v^{(00)}}(y) \right) \quad (\because d_0^{h^{(11)}} s_0^{h^{(01)}} = id) \\
&= 0.
\end{aligned}$$

6. For $\underline{\alpha} = (\emptyset,(0))$ and $\underline{\beta} = ((0),(0))$, the map

$$C_{(\emptyset,(0)),((0),(0))} : NE_{1,0} \otimes NE_{0,0} \to NE_{1,1}$$

is the zero map as given in the previous step.

4.3. The case $(n,m) = (0,2)$ or $(2,0)$ and crossed modules

In this section, by considering (n,m)= $(2,0)$ and $(0,2)$, we can compute the possible non zero operators with codomain $NE_{2,0}$, $NE_{0,2}$ respectively. We give an application of these operators to the crossed modules.
For $(n,m) = (2,0)$. From the set

$$S(2)\times S(0) = \{((0),\emptyset),\ (\emptyset,\emptyset),\ ((1),\emptyset),\ ((1,0),\emptyset)\},$$

we can choose $\underline{\alpha} = ((1),\emptyset)$, $\underline{\beta} = ((0),\emptyset)$. Then $C_{\underline{\alpha},\underline{\beta}}$ is a map from $NE_{1,0}\otimes NE_{1,0}$ to $NE_{2,0}$. This map can be given by for $y,y' \in NE_{1,0}$

$$\begin{aligned}
C_{((1),\emptyset),((0),\emptyset)}(y\otimes y') &= p\mu(s_{\underline{\alpha}}\otimes s_{\underline{\beta}})(y\otimes y') \\
&= p_1^h p_0^h (s_1^{h^{(10)}}(y) s_0^{h^{(10)}}(y')) \\
&= s_1^{h^{(10)}}(y)(s_0^{h^{(10)}}(y') - s_1^{h^{(10)}}(y')) \in NE_{2,0}.
\end{aligned}$$

We have similarly

$$\begin{aligned}
\partial_2^{h^{(20)}}(C_{((1),\emptyset),((0),\emptyset)}(y\otimes y')) &= \partial_2^{h^{(20)}}(s_1^{h^{(10)}}(y)(s_0^{h^{(10)}}(y') - s_1^{h^{(10)}}(y'))) \\
&= y s_0^{h^{(00)}} d_1^{h^{(10)}}(y') - yy' \in NE_{1,0}.
\end{aligned}$$

Now suppose $(n,m) = (0,2)$. From the set

$$S(0)\times S(2) = \{(\emptyset,\emptyset),(\emptyset,(0)),(\emptyset,(1)),(\emptyset,(1,0))\},$$

we can find the functions $C_{\underline{\alpha},\underline{\beta}}$ with codomain $NE_{0,2}$. In this case, the only non zero operator $C_{\underline{\alpha},\underline{\beta}}$ can be calculated by choosing $\underline{\alpha} = (\emptyset,(1))$ and $\underline{\beta} = (\emptyset,(0))$. Therefore, this is a map from $NE_{0,1}\otimes NE_{0,1}$ to $NE_{0,2}$. For $x,x' \in NE_{0,1}$, we obtain

$$\begin{aligned}
C_{(\emptyset,(1)),(\emptyset,(0))}(x\otimes x') &= p\mu(s_{\underline{\alpha}}\otimes s_{\underline{\beta}})(x\otimes x') \\
&= p_1^v p_0^v (s_1^{v^{(01)}}(x) s_0^{v^{(01)}}(x')) \\
&= s_1^{v^{(01)}}(x)(s_0^{v^{(01)}}(x') - s_1^{v^{(01)}}(x')) \in NE_{0,2}.
\end{aligned}$$

We have also

$$\begin{aligned}
\partial_2^{v^{(02)}}(C_{(\emptyset,(1)),(\emptyset,(0))}(x\otimes x')) &= \partial_2^{v^{(02)}}(s_1^{v^{(01)}}(x)(s_0^{v^{(01)}}(x') - s_1^{v^{(01)}}(x'))) \\
&= x s_0^{v^{(00)}} d_1^{v^{(01)}}(x') - xx' \in NE_{0,1}.
\end{aligned}$$

Proposition 4.1. *Assume that $E_{*,*}$ is a bisimplicial algebra. Consider its Moore bicomplex $NE_{*,*}$. For $p+q \geqslant 2$, if $NE_{p,q} = \{0\}$, then the map*

$$\partial : NE_{0,1} \times NE_{1,0} \longrightarrow NE_{0,0}$$

given by

$$\partial(x,y) = d_1^{v(01)}(x) + d_1^{h(10)}(y)$$

for $x \in NE_{0,1}$, $y \in NE_{1,0}$ is a crossed module of commutative algebras. In particular, the maps $d_1^{v(01)}$ and $d_1^{h(10)}$ are crossed modules.

Proof. The action of $t \in NE_{0,0}$ on $(x,y) \in NE_{0,1} \times NE_{1,0}$ is given by

$$t \cdot (x,y) = \left((s_0^{v(00)} t)x, (s_0^{h(00)} t)y\right).$$

For this action we get

$$\begin{aligned} \partial(t \cdot (x,y)) &= \partial\left((s_0^{v(00)} t)x, (s_0^{h(00)} t)y\right) \\ &= d_1^{v(01)}((s_0^{v(00)} t)x + d_1^{h(10)}(s_0^{h(00)} t)y) \\ &= t(d_1^{v(01)}(x) + d_1^{h(10)}(y)) \\ &= t\partial(x,y) \end{aligned}$$

and this is the first axiom of the crossed module.
Now for $(x,y),(x',y') \in NE_{0,1} \times NE_{1,0}$, we obtain

$$\begin{aligned} \partial(x,y)\cdot(x',y') &= \left(s_0^{v(00)}(d_1^{v(01)}(x) + d_1^{h(10)}(y))x', s_0^{h(00)}(d_1^{v(01)}(x) + d_1^{h(10)}(y))y'\right) \\ &= \left(s_0^{v(00)} d_1^{v(01)}(x)x' + s_0^{v(00)} d_1^{h(10)}(y)x', s_0^{h(00)} d_1^{v(01)}(x)y' + s_0^{h(00)} d_1^{h(10)}(y)y'\right) \\ &= (s_0^{v(00)} d_1^{v(01)}(x)x' + d_1^{h(11)} s_0^{v(10)}(y)x', d_1^{v(11)} s_0^{h(01)}(x)y' + s_0^{h(00)} d_1^{h(10)}(y)y') \\ &\quad (\because s_0^{v(00)} d_1^{h(10)} = d_1^{h(11)} s_0^{v(10)}, s_0^{h(00)} d_1^{v(01)} = d_1^{v(11)} s_0^{h(01)}) \\ &= (s_0^{v(00)} d_1^{v(01)}(x)x', s_0^{h(00)} d_1^{h(10)}(y)y') \ (\because s_0^{h(01)}(x), s_0^{v(10)}(y) \in NE_{1,1} = \{0\}). \end{aligned}$$

Since $NE_{0,2} = \{0\}$, we obtain for $x, x' \in NE_{0,1}$

$$\partial_2^{v(02)}(C_{(\emptyset,(1)),(\emptyset,(0))}(x' \otimes x)) = x' s_0^{v(00)} d_1^{v(01)}(x) - x'x = 0$$

and therefore,

$$s_0^{v(00)} d_1^{v(01)}(x)x' = xx'.$$

Similarly, since $NE_{2,0} = \{0\}$, we obtain for $y, y' \in NE_{1,0}$

$$\partial_2^{h(20)}(C_{((1),\emptyset),((0),\emptyset)}(y' \otimes y)) = y' s_0^{h(00)} d_1^{h(10)}(y) - y'y = 0$$

and therefore,

$$s_0^{h(00)} d_1^{h(10)}(y)y' = yy'.$$

Thus, we have

$$\begin{aligned} \partial(x,y)\cdot(x',y') &= (xx', yy') \\ &= (x,y)(x',y') \end{aligned}$$

and this is the second axiom of crossed module. □

5. Crossed squares and Bisimplicial Algebras

If we take (n,m)= $(2,1)$ and $(1,2)$, we will define the possible non zero operators $C_{\underline{\alpha},\underline{\beta}}$ whose codomain $NE_{1,2}$ and $NE_{2,1}$ respectively. We give an application of these operators to the crossed squares.
Assume now that $(n,m) = (2,1)$. We think the set $S(2) \times S(1)$. We can choose appropriate pairs $\underline{\alpha}, \underline{\beta}$ from the set $S(2) \times S(1)$, we can compute similarly all the non zero maps with codomain $NE_{2,1}$. To get these maps, we take the possible $\underline{\alpha}, \underline{\beta}$ as follows.

1. $\underline{\alpha} = ((1),\emptyset)$, $\quad \underline{\beta} = ((0),\emptyset)$
2. $\underline{\alpha} = ((1),\emptyset)$, $\quad \underline{\beta} = (\emptyset,(0))$

3. $\underline{\alpha} = ((0),\emptyset)$, $\quad \underline{\beta} = (\emptyset,(0))$
4. $\underline{\alpha} = ((1),(0))$, $\quad \underline{\beta} = ((0),\emptyset)$
5. $\underline{\alpha} = ((0),(0))$, $\quad \underline{\beta} = ((1),\emptyset)$.

For $(\underline{\alpha},\underline{\beta})$, the necessary $C_{\underline{\alpha},\underline{\beta}}$ functions can be given as follows:

1. Take $\underline{\alpha} = ((1),\emptyset)$ and $\underline{\beta} = ((0),\emptyset)$, we have that the oparetor
$$C_{((1),\emptyset),((0),\emptyset)} : NE_{1,1} \otimes NE_{1,1} \longrightarrow NE_{2,1}.$$
This operator can be given by
$$C_{((1),\emptyset),((0),\emptyset)}(x \otimes y) = s_1^{h^{(11)}}(x)(s_0^{h^{(11)}}(y) - s_1^{h^{(11)}}(y)) \in NE_{2,1}$$
for $x,y \in NE_{1,1}$.
2. For $\underline{\alpha} = ((1),\emptyset), \underline{\beta} = (\emptyset,(0))$, we get the operator
$$C_{((1),\emptyset),(\emptyset,(0))} : NE_{1,1} \otimes NE_{2,0} \longrightarrow NE_{2,1}$$
defined by
$$C_{((1),\emptyset),(\emptyset,(0))}(x \otimes t) = s_1^{h^{(11)}}(x)s_0^{v^{(02)}}(t) \in NE_{2,1}$$
for $x \in NE_{1,1}$ and $t \in NE_{2,0}$.
3. For $\underline{\alpha} = ((0),\emptyset), \underline{\beta} = (\emptyset,(0))$, we get the operator
$$C_{((0),\emptyset),(\emptyset,(0))} : NE_{1,1} \otimes NE_{2,0} \longrightarrow NE_{2,1}$$
given by
$$C_{((0),\emptyset),(\emptyset,(0))}(x \otimes t) = s_0^{h^{(11)}}(x)s_0^{v^{(02)}}(t) \in NE_{2,1}$$
for $x \in NE_{1,1}$, $t \in NE_{2,0}$.
4. For $\underline{\alpha} = ((1),(0)), \underline{\beta} = ((0),\emptyset)$, we get the following operator
$$C_{((1),(0)),((0),\emptyset)} : NE_{1,0} \otimes NE_{1,1} \longrightarrow NE_{2,1}.$$
It is given by
$$C_{((1),(0)),((0),\emptyset)}(x \otimes y) = s_1^{h^{(11)}}s_0^{v^{(10)}}(x)s_0^{h^{(11)}}(y) \in NE_{2,1}$$
for $x \in NE_{1,0}$, $y \in NE_{1,1}$.
5. For $\underline{\alpha} = ((0),(0)), \underline{\beta} = ((1),\emptyset)$, we obtain the following operator
$$C_{((0),(0)),((1),\emptyset)} : NE_{1,0} \otimes NE_{1,1} \longrightarrow NE_{2,1}.$$
This can be defined as
$$C_{((0),(0)),((1),\emptyset)}(x \otimes y) = s_0^{h^{(11)}}s_0^{v^{(10)}}(x)s_1^{h^{(11)}}(y) \in NE_{2,1}$$
for $x \in NE_{1,0}$, $y \in NE_{1,1}$.

For $(n,m) = (1,2)$, we set
$$S(1) \times S(2) = \{(\emptyset,\emptyset),((0),(0)),(\emptyset,(0)),(\emptyset,(1)),((0),(1,0)),(\emptyset,(1,0)),((0),\emptyset),((0),(1))\}.$$
In the following calculations, if we take the appropriate pairs $\underline{\alpha},\underline{\beta}$ from the set $S(1)\times S(2)$, we will give all the non zero maps for $NE_{1,2}$. To get these maps, we can choose the possible $\underline{\alpha},\underline{\beta}$ from the set $S(1)\times S(2)$ as follows

1. $\underline{\alpha} = (\emptyset,(1))$, $\quad \underline{\beta} = (\emptyset,(0))$
2. $\underline{\alpha} = (\emptyset,(1))$, $\quad \underline{\beta} = ((0),\emptyset)$
3. $\underline{\alpha} = (\emptyset,(0))$, $\quad \underline{\beta} = ((0),\emptyset)$
4. $\underline{\alpha} = ((0),(1))$, $\quad \underline{\beta} = (\emptyset,(0))$
5. $\underline{\alpha} = ((0),(0))$, $\quad \underline{\beta} = (\emptyset,(1))$.

Now we compute the functions $C_{\underline{\alpha},\underline{\beta}}$ for these pairings $(\underline{\alpha},\underline{\beta})$.

1. For $\underline{\alpha} = (\emptyset,(1))$ and $\underline{\beta} = (\emptyset,(0))$, we obtain the operator
$$C_{(\emptyset,(1)),(\emptyset,(0))} : NE_{1,1} \otimes NE_{1,1} \longrightarrow NE_{1,2}.$$
This operator can be calculated by
$$C_{(\emptyset,(1)),(\emptyset,(0))}(x \otimes y) = s_1^{v^{(11)}}(x)(s_0^{v^{(11)}}(y) - s_1^{v^{(11)}}(y)) \in NE_{1,2}$$
for $x,y \in NE_{1,1}$.

2. For $\underline{\alpha} = (\emptyset,(1)), \underline{\beta} = ((0),\emptyset)$, we obtain the operator
$$C_{(\emptyset,(1)),((0),\emptyset)} : NE_{1,1} \otimes NE_{0,2} \longrightarrow NE_{1,2}$$
given by
$$C_{(\emptyset,(1)),((0),\emptyset)}(x\otimes t) = s_1^{v(11)}(x)s_0^{h(02)}(t) \in NE_{1,2}$$
for $x \in NE_{1,1}$ and $t \in NE_{0,2}$.
3. For $\underline{\alpha} = (\emptyset,(0)), \underline{\beta} = ((0),\emptyset)$, we have the following operator
$$C_{(\emptyset,(0)),((0),\emptyset)} : NE_{1,1} \otimes NE_{0,2} \longrightarrow NE_{1,2}$$
given by
$$C_{(\emptyset,(0)),((0),\emptyset)}(x\otimes t) = s_0^{v(11)}(x)s_0^{h(02)}(t) \in NE_{1,2}$$
for $x \in NE_{1,1}$ and $t \in NE_{0,2}$.
4. For $\underline{\alpha} = ((0),(1)), \underline{\beta} = (\emptyset,(0))$, we get the following operator
$$C_{((0),(1)),(\emptyset,(0))} : NE_{0,1} \otimes NE_{1,1} \longrightarrow NE_{1,2}$$
given by
$$C_{((0),(1)),(\emptyset,(0))}(x\otimes y) = s_0^{h(02)}s_1^{v(01)}(x)s_0^{v(11)}(y) \in NE_{1,2}$$
for $x \in NE_{0,1}$, $y \in NE_{1,1}$.
5. For $\underline{\alpha} = ((0),(0))$ and $\underline{\beta} = (\emptyset,(1))$, we get the following operator
$$C_{((0),(0)),(\emptyset,(1))} : NE_{0,1} \otimes NE_{1,1} \longrightarrow NE_{1,2}$$
given by
$$C_{((0),(0)),(\emptyset,(1))}(x\otimes y) = s_0^{h(02)}s_0^{v(01)}(x)s_1^{v(11)}(y) \in NE_{1,2}$$
for $x \in NE_{0,1}$ and $y \in NE_{1,1}$.

Thus, we can give the following result.

Proposition 5.1. *Let $E_{*,*}$ be a bisimplicial algebra with Moore bicomplex $NE_{*,*}$. If for $p \geqslant 2$ or $q \geqslant 2$, $NE_{p,q} = \{0\}$, then, Figure 5.1*

$$\begin{tikzcd}
NE_{1,1} \arrow[r, "\partial_1^{h(11)}"] \arrow[d, "\partial_1^{v(11)}"'] & NE_{0,1} \arrow[d, "\partial_1^{v(01)}"] \\
NE_{1,0} \arrow[r, "\partial_1^{h(10)}"'] & NE_{0,0}
\end{tikzcd}$$

Figure 5.1: Crossed square of Moore bicomplex

is a crossed square together with the h-map
$$h : NE_{0,1} \times NE_{1,0} \longrightarrow NE_{1,1}$$
given by
$$h(x,y) = C_{((0),\emptyset),(\emptyset,(0))}(x\otimes y) = s_0^{h(01)}(x)s_0^{v(10)}(y)$$
for all $x \in NE_{0,1}$ and $y \in NE_{1,0}$, where $((0),\emptyset),(\emptyset,(0)) \in S(1)\times S(1)$. (This result is the commutative algebra version of Conduché's result given in [3].)

Proof. Our purpose is to see the role of the functions $C_{\underline{\alpha},\underline{\beta}}$ in the structure.
Since $NE_{1,2} = NE_{2,0} = NE_{0,2} = NE_{2,1} = \{0\}$, the maps $\partial_1^{h(10)}, \partial_1^{v(01)}, \partial_1^{h(11)}$ and $\partial_1^{v(11)}$ are crossed modules.
An action of $x \in NE_{1,0}$ on $y \in NE_{0,1}$ is given by $x\cdot y = y\cdot x = s_0^{v(00)}d_1^{h(10)}(x)y$, similarly, the action of $a \in NE_{0,1}$ on $b \in NE_{1,0}$ is given by $a\cdot b = s_0^{h(00)}d_1^{v(01)}(a)b$.

For $y \in N_{0,1}$ and $x \in NE_{1,0}$, we obtain

$$\begin{aligned}
\partial_1^{h^{(11)}} h(y,x) &= \partial_1^{h^{(11)}}(s_0^{h^{(01)}}(y)s_0^{v^{(10)}}(x)) \\
&= y\partial_1^{h^{(11)}} s_0^{v^{(10)}}(x) \\
&= ys_0^{v^{(00)}} d_1^{h^{(10)}}(x) \\
&= y \cdot x.
\end{aligned}$$

We obtain similarly for $a \in NE_{0,1}$ and $b \in NE_{1,0}$

$$\begin{aligned}
\partial_1^{v^{(11)}} h(a,b) &= \partial_1^{v^{(11)}}(s_0^{h^{(01)}}(a)s_0^{v^{(10)}}(b)) \\
&= \partial_1^{v^{(11)}} s_0^{h^{(01)}}(a)bx \\
&= s_0^{h^{(00)}} d_1^{v^{(01)}}(a)b \\
&= a \cdot b.
\end{aligned}$$

Now we show that $h(x, \partial_1^{v^{(11)}} c) = x \cdot c$ for $x \in NE_{0,1}$ and $c \in NE_{1,1}$. For $s_0^{h^{(01)}}(x), c \in NE_{1,1}$, we obtain

$$\begin{aligned}
d_2^{v^{(12)}} C_{(\emptyset,(1)),(\emptyset,(0))}(s_0^{h^{(01)}}(x) \otimes c) &= d_2^{v^{(12)}}\left((s_1^{v^{(11)}}(s_0^{h^{(01)}}(x)))s_0^{v^{(11)}}(c) - s_1^{v^{(11)}}(c)\right) \in d_2^{v^{(12)}}(NE_{1,2}) \\
&= s_0^{h^{(01)}}(x)(s_0^{v^{(10)}} d_1^{v^{(11)}}(c) - c) \\
&= s_0^{h^{(01)}}(x)s_0^{v^{(10)}} d_1^{v^{(11)}}(c) - s_0^{h^{(01)}}(x)c = 0. \quad (\because NE_{1,2} = 0)
\end{aligned}$$

Thus, we have for $x \in NE_{0,1}$ and $c \in NE_{1,1}$,

$$\begin{aligned}
h(x, \partial_1^{v^{(11)}} c) &= s_0^{h^{(01)}}(x)s_0^{v^{(10)}} d_1^{v^{(11)}}(c) \\
&= s_0^{h^{(01)}}(x)c \quad (\because NE_{1,2} = 0) \\
&= x \cdot c.
\end{aligned}$$

For $a \in NE_{1,1}$ and $y \in NE_{1,0}$, we obtain

$$h(\partial_1^{h^{(11)}}(a), y) = s_0^{h^{(01)}} d_1^{h^{(11)}}(a)s_0^{v^{(10)}}(y).$$

For $s_0^{v^{(10)}}(y), a \in NE_{1,1}$, we obtain

$$\begin{aligned}
d_2^{h^{(21)}}\left(C_{((1),\emptyset),((0),\emptyset)}(a \otimes s_0^{v^{(10)}}(y))\right) &= d_2^{h^{(21)}}\left((s_1^{h^{(11)}} s_0^{v^{(10)}}(y))(s_0^{h^{(11)}}(a) - s_1^{h^{(11)}}(a))\right) \in d_2^{h^{(21)}}(NE_{2,1}) \\
&= s_0^{v^{(10)}}(y)(s_0^{h^{(01)}} d_1^{h^{(11)}}(a) - a) \\
&= s_0^{v^{(10)}}(y)s_0^{h^{(01)}} d_1^{h^{(11)}}(a) - s_0^{v^{(10)}}(y)a = 0. \quad (\because NE_{2,1} = 0)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
h(\partial_1^{h^{(11)}}(a), y) &= s_0^{h^{(01)}} d_1^{h^{(11)}}(a)s_0^{v^{(10)}}(y) \\
&= as_0^{v^{(10)}}(y) \\
&= a \cdot y.
\end{aligned}$$

We leave other crossed square axioms to the reader. □

Arvasi, in [14], proved that Loday's mapping cone complex

$$K \xrightarrow{(-\gamma,\gamma')} L \rtimes M \xrightarrow{\mu+\mu'} R$$

of the crossed square for commutative algebras in the Figure 5.2

$$\begin{tikzcd} K \arrow[r, "\gamma"] \arrow[d, "\gamma'"'] & L \arrow[d, "\mu"] \\ M \arrow[r, "\mu'"'] & R \end{tikzcd}$$

Figure 5.2: Crossed square

gives a 2-crossed module analogously to that given by Conduché in the group case [3].
Thus, we obtain the following result.
Let $E_{*,*}$ be a bisimplicial algebra with Moore bicomplex $NE_{*,*}$. If for $p \geqslant 2$ or $q \geqslant 2$, $NE_{p,q} = \{0\}$, then

$$NE_{1,1} \xrightarrow{(-\partial_1^{h(11)}, \partial_1^{v(11)})} NE_{0,1} \times NE_{1,0} \xrightarrow{\partial_1^{v(01)} + \partial_1^{h(10)}} NE_{0,0}$$

is a 2-crossed module together with Peiffer lifting map

$$\{-,-\} : (NE_{0,1} \times NE_{1,0}) \otimes (NE_{0,1} \times NE_{1,0}) \longrightarrow NE_{1,1}$$

given by

$$\{(x,y),(x',y')\} = C_{((0),\emptyset),(\emptyset,(0))}(x \otimes yy') = s_0^{h(01)}(x)s_0^{v(10)}(yy')$$

for all $x, x' \in NE_{0,1}$ and $y, y' \in NE_{1,0}$, where $((0),\emptyset),(\emptyset,(0)) \in S(1) \times S(1)$.

6. Conclusion

In this paper, we give the hypercrossed complex pairings for a Moore bicomplex of a bisimplicial algebra and we calculate in dimension 2 explicitly these pairings in the Moore bicomplex to see what the importance of these relations in the structures, for example crossed squares and 2-crossed modules. This idea can be extended to Lie algebra case. Defining these operators for bisimplicial Lie algebras and using these pairings the connection between bisimplicial Lie algebras and crossed squares over Lie algebras can be obtained similarly.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: The authors contributed equally to the writing of this paper.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Copyright Statement: Authors own the copyright of their work published in the journal and their work is published under the CC BY-NC 4.0 license.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

References

[1] A.R. Grandjeán, M.J. Vale, *2-Modulos cruzados an la cohomologia de andré-quillen*, Memorias de la Real Academia de Ciencias, **22** (1986), 1–28.
[2] D. Conduché, *Modules croisés cénéralisés de longueur 2*, J. Pure Appl. Algebra, **34** (1984), 155-178.
[3] D. Conduché, *Simplicial crossed modules and mapping cones*, Georgian Math. J., **10**(4) (2003), 623-636.
[4] D. Guin-Waléry, J-L. Loday, *Obsructioná l'excision en K-theories Algébrique*, Evanston conference on Algebraic K-Theory 1980, (Lectute Notes Mathematics) 854, 179-216, Berlin Heidelberg New York, Springer 1981.
[5] G.J. Ellis, *Higher dimensional crossed modules of algebras*, J. Pure Appl. Algebra, **52** (1988), 277-282.
[6] J.H.C. Whitehead, *Combinatorial homotopy II*, Bull. Amer. Math. Soc., **55** (1949), 453-496.
[7] L. Illusie, *Complex Cotangent et Deformations I, II*, Springer Lecture Notes in Mathematics 239 1971 II 283 1972.
[8] M. André, *Homologie des Algèbres Commutatives*, Die Grundlehren der Mathematischen Wissenschaften, 206 New York, Springer-Verlag, 1974.
[9] Ö. Gürmen Alansal, E. Ulualan, *Peiffer pairings in multisimplicial groups and crossed n-cubes and applications for bisimplicial groups*, Turkish J. Math., **45**(1) (2021), 360-386.
[10] P. Carrasco, A.M. Cegarra, *Group-theoretic algebraic models for homotopy types*, J. Pure Appl. Algebra, **75** (1991), 19-235.

[11] T. Porter, *Homology of commutative algebras and an invariant of simis and vasconceles*, J. Algebr., **99** (1986), 458-465.
[12] T. Porter, *n-type of simplicial groups and crossed n-cubes*, Topol., **32** (1993), 5-24.
[13] Z. Arvasi, T. Porter, *Higher dimensional Peiffer elements in simplicial commutative algebras*, Theory Appl. Categ., **3**(1) (1997), 1-23.
[14] Z. Arvasi, *Crossed squares and 2-crossed modules of commutative algebras*, Theory Appl. Categ., **3**(7) (1997), 160-181.