Universal Journal of Mathematics and Applications

VOLUME VI
ISSUE III

ISSN 2619-9653

http://dergipark.gov.tr/ujma

VOLUME VI ISSUE III
ISSN 2619-9653

September 2023
http://dergipark.gov.tr/ujma

UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

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Universal Journal of Mathematics and Applications, 6 (3) (2023) 91-99
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1297362

On Multi-G-Metric Spaces

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Article Info

Keywords: *Generalized multi-metric space, Mset, Multi-metric space*
2010 AMS: *54A05, 54E35, 08A05*
Received: *15 May 2023*
Accepted: *29 July 2023*
Available online: *18 September 2023*

Abstract

Multisets have many applications in a variety of fields today, including computer science, medicine, banking, engineering, information storage, and information analysis. In this paper, we present a new generalized multi-G-metric space, a multi-G-metric space. We investigate some of its fundamental details, connections, and topological characteristics.

1. Introduction

Several branches of modern mathematics have developed that goes against a basic tenet of conventional mathematical theory. The underlying premise of traditional mathematics is that all mathematical objects are unique. As a result, there are two options between two numbers: they could be equal or dissimilar. In reality and in science, this is not the case. In the physical world, there seems to be a lot of repetition. For instance, between numerous DNA strands, many water molecules, or many hydrogen atoms. Even though they are independent objects, coins, electrons, and grains of sand with the same value and year appear to be the same.

Assuming that mathematical objects are not repeated, the classical set theory states that a particular element may only be written once in a set. That is, there is only an equal or different relationship between any two mathematical objects.

In reality and in science, this is not the case. In the physical world, it has been noted that there is too much repetition. For instance, between hydrogen atoms, water molecules, DNA strands, and so forth. Any two physical items can have one of three connections as a result: they can be distinct, distinct but the same, or overlap and be the same. They are unambiguously the same or equal if they cannot be distinguished, and they are the same and identical if they physically overlap. An organized collection of various items is referred to as a set in classical set theory. A multi-set is one that permits any object to be repeated within it (mset or bag for short). A multi-set is so distinct from a set. To describe this structure, it is appropriate to make the distinction between the sets a, b, and c and the collections a, a, a, b, c, and c. When viewed as a set, the second is identical to the first. However, with the latter, some components are purposefully used more than once. A mset is a group of components created with a specific multiplicity. $\{k_1/x_2, k_2/x_2, \ldots, k_n/x_n\}$ can be written so that the multiple set x_i is found k_i times. Here k_i is an integer [1–7].

The metric idea, which is basic to mathematics and has a variety of diverse applications, plays a crucial role in topology and analysis. This idea has been examined in relation to a number of generalizations, including G-metric spaces [8], fuzzy metric spaces [9] and cone metric spaces [10].

Some of the key ideas and findings of cluster analysis can be extended to the arrangement of numerous clusters. Recently, research on many sets in mathematics has begun. Das and Roy [11] described one of these research in 2021. They began by defining the idea of multi-real numbers and studying their fundamental characteristics in this study. The idea of numerous metrics on different sets is introduced and its fundamental characteristics are investigated in this study at the same time. The topological characteristics of other metric spaces were then researched using the findings of this study [12]. Through this study, we also hope to contribute to this expansion. As a generalization of a multi-metric space, multi-G-metric spaces are used to explore the fundamental characteristics of multiple G-metrics. The multi-G-metric topology produced with the aid of multiple G-metrics will also be specified, and its fundamental characteristics have been investigated. After that, the ideas of multi-G-convergence and multi-G-Cauchy are introduced, and a few of their characteristics are examined.

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This paper is supported by Ege University Scientific Research Projects Coordination Unit. Project Number: FM-YLT-2022-23913
Cite as: E. Mutlu, A. Çaksu Güler, *On multi-G-metric spaces*, Univers. J. Math. Appl., 6(3) (2023), 91-99.

2. Preliminaries

Definition 2.1 ([13]). *The function CountM, often known as C_M, is defined as $C_M : X \to N$, where N stands for the set of non-negative integers. A mset M selected from the set X is represented by this function. The number of times the element x appears in the M mset is represented here by $C_M(x)$. We write the mset M as $M = \{m_1/x_1, m_2/x_2, ...m_n/x_n\}$, where m_i is the number of times the element x_i appears in the mset M denoted by $x_i \in_i^m M, i = 1,2,3,...n$. But, elements that are omitted from the M mset have zero counts.*

Definition 2.2 ([13]). *Let M and N be two msets drawn from a set X. Then, the followings are defined:*

(1) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$,
(2) $M \subset N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$,
(3) $P = M \cup N$ if $C_P(x) = Max\{C_M(x), C_N(x)\}$ for all $x \in X$,
(4) $P = M \cap N$ if $C_P(x) = Min\{C_M(x), C_N(x)\}$ for all $x \in X$.

Definition 2.3 ([13]). *The power set of an mset is denoted by $P^*(M)$ and it is an ordinary set whose members are sub msets of M. An mset M is said to be an empty mset (multi-empty set) if for all $x \in X$, $C_M(x)$ = 0.*

Definition 2.4 ([13]). *Let the mset space $[X]^\omega$ be the set of all msets whose elements are in X such that no element in the mset occurs more than ω times. Let $M \in [X]^\omega$ and $\tau \subseteq P^*(M)$. Then τ is called a mset topology (M-topology) of M if τ satisfies the following properties,*

(1) The mset M and multi empty set are in τ,
(2) The mset union of the elements of any subcollection of τ is in τ,
(3) The mset intersection of the element of any finite subcollection of τ is in τ, Mathematically an mset topological space is an ordered pair (M, τ) consisting of an mset $M \in [X]^\omega$ and a mset topology $\tau \subseteq P^*(M)$ on M.

Definition 2.5 ([11]). *Let M be a mset over the universal set X. The mapping $P_x^l : X \to \mathbb{N}$ such that $P_x^l(x) = l$ where $l \leq C_M(x)$ defines a multi-point of M, where x and l are the base and multiplicity of the multi-point P_x^l, respectively. M_{pt} denotes the collection of all multi-points in a mset M.*

Definition 2.6 ([11]). *The mset produced by a set N of multi-points is represented by the symbol $MS(N)$, and its definition is given by the formula $C_{MS(N)}(x) = sup\{l : P_x^l \in N\}$. The collection of its multi-points can be used to create a mset. If M_{pt} stands for the collection of all multi points of M, then $C_M(x) = sup\{l : P_x^l \in M_{pt}\}$ and $M = MS(M_{pt})$ are obvious conclusions.*

Definition 2.7 ([11]). *Let M be a mset over the universal set X.*

(1) The elementary union between two collections of multi points C and D is denoted by $C \sqcup D$ and is defined as $C \sqcup D = \{P_x^k : P_x^l \in C, P_x^m \in D \text{ and } k = max\{l,m\}\}$.
(2) The elementary intersection between two collections of multi points C and D is denoted by $C \sqcap D$ and is defined as $C \sqcap D = \{P_x^k : P_x^l \in C, P_x^m \in D \text{ and } k = min\{l,m\}\}$.
(3) For two collection of multi points C and D, C is said to be an elementary subset of D, denoted by $C \sqsubset D$, iff $P_x^l \in C$ there exists $m \geq l$ such that $P_x^m \in D$.

Theorem 2.8 ([11]). *Let M be a mset over the universal set X.*

(1) For two collections of multi-points C and D, $C \cup D \supset C \sqcup D$.
(2) For a collection N of multi-points, $[MS(N)]_{pt} \supset N$.
(3) For two msets A and B, $A \subset B$ iff $A_{pt} \subset B_{pt}$.
(4) For two collections of multi-points C and D, $MS(C \sqcap D) = MS(C) \cap MS(D)$.

Definition 2.9 ([11]). *$m\mathbb{R}^+$ denotes the mset over $\mathbb{R}^+$ (set of non-negative real numbers) having a multiplicity of each element equal to $\omega \in \mathbb{N}$. The members of $(m\mathbb{R}^+)_{pt}$ will be called non-negative multi-real points.*

Definition 2.10 ([11]). *Let P_a^i and P_b^j be two multi real points of $(m\mathbb{R}^+)_{pt}$.*

(1) $P_a^i > P_b^j$ if $a > b$ or $P_a^i > P_b^j$ if $i > j$ when $a = b$.
(2) $P_a^i + P_b^j = P_{a+b}^k$ where $k = Max\{i, j\}$.
(3)

$$P_a^i \times P_b^j = \begin{cases} P_0^1, & if \ either \ P_a^i \ or \ P_b^j equal \ to \ P_0^1, \\ P_{ab}^k, & otherwise \quad where \quad k = Max\{i, j\}. \end{cases}$$

Definition 2.11 ([12]). *The subtraction of two multi real points in $m\mathbb{R}^+$ is defined as follows:*

$$P_a^i - P_b^j = \begin{cases} P_0^1, & if \ P_a^i = P_b^j, \\ P_{a-b}^k, & if \ P_a^i > P_b^j \quad where \quad k = min\{i, j\}. \end{cases}$$

Definition 2.12. *The division of two multi real points in $m\mathbb{R}^+$ is defined as follows:*

$$P_a^i / P_b^j = \begin{cases} P_1^1, & if \ P_a^i = P_b^j, \\ P_{a/b}^k, & if \ P_a^i \neq P_b^j \quad where \quad k = Max\{i, j\}. \end{cases}$$

Definition 2.13. *We define maximum of two multi-real points in $m\mathbb{R}^+$ as follows:*

$$max\{P_a^i, P_b^j\} = \begin{cases} P_a^i, & if\ P_a^i > P_b^j, \\ P_b^j, & otherwise. \end{cases}$$

Definition 2.14 ([11]). *Let's say that $d : M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ (M being a multi set over a universal set X with multiplicity of any member at most equal to ω) be a mapping that meets the following requirements:*

(md_1) $md(P_x^l, P_y^m) > P_0^1$ *for all* $P_x^l, P_y^m \in M_{pt}$ *and* $P_x^l \neq P_y^m$,
(md_2) $md(P_x^l, P_y^m) = P_0^1$ *iff* $P_x^l = P_y^m$,
(md_3) $md(P_x^l, P_y^m) = md(P_y^m, P_x^l)$,
(md_4) $md(P_x^l, P_y^m) + md(P_y^m, P_z^n) \geq md(P_x^l, P_z^n)$, *for all* $P_x^l, P_y^m, P_z^n \in M_{pt}$,
(md_5) *For* $l \neq m$, $md(P_x^l, P_y^m) = P_0^k$ *iff* $x = y$ *and* $k = Max\{l, m\}$.

Then, (M, md) is referred to a multi-metric (or an M-metric) space and md is said to be a multi-metric on M.

Definition 2.15. *Let (M, md) be a multi-metric space. Let $\{P_{x_n}^{l_n}\}$ be a sequence of multi-points in M. The sequence $\{P_{x_n}^{l_n}\}_n$ is said to convergence to $P_x^l \in M_{pt}$, if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $md(P_{x_n}^{l_n}, P_x^l) < P_\varepsilon^1$, $\forall n \geq n_0$ i.e. $n \geq n_0$ then the sequence $\{P_{x_n}^{l_n}\}$ is multi convergent (md-convergent) to P_x^l and written as $\{P_{x_n}^{l_n}\} \to P_x^l$.*

Definition 2.16. *Let (M, md) be a multi-metric space. Let $\{P_{x_n}^{l_n}\}_n$ be a sequence of multi-points in M. The sequence $\{P_{x_n}^{l_n}\}_n$ is said to be multi-Cauchy (md-Cauchy) if every $P_\varepsilon^1 > P_0^1$, there exists a $n_0 \in \mathbb{N}$ such that $md(P_{x_n}^{l_n}, P_{x_m}^{l_m}) < P_\varepsilon^1$ for all $m, n \geq n_0$.*

Definition 2.17. *A multi-metric space (M, md) is said to be md-complete if every md-Cauchy sequence in (M, md) is md-convergent in (M, md).*

Definition 2.18 ([8]). *Let U be a nonempty set, and let $G : U \times U \times U \to \mathbb{R}^+$ be a function satisfying the following conditions:*

(G_1) $G(x, y, z) = 0$ *if* $x = y = z$,
(G_2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,
(G_3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$,
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$,
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in U$,

then the function G is called a generalized metric, or, more specifically, a G-metric on U, and the pair (U, G) is a G-metric space.

3. Multi G-Metric Spaces

The concept of multi-G-metric space is defined and its fundamental characteristics are determined in this section. Also, we investigate any relationships that may exist between multi-metric and multi-G-metric.

Definition 3.1. *Assume that X is a non-empty set and that M is a multi-set over X with multiplicity of any element approximately equal to ω. A mapping $mG : M_{pt} \times M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ is said to be a multi generalized metric or multi G-metric on M if mG satisfies the following conditions:*

(mG_1) $mG(P_x^l, P_x^l, P_y^m) > P_0^1$ *for all* $P_x^l, P_y^m \in M_{pt}$ *with* $P_x^l \neq P_y^m$,
(mG_2) $mG(P_x^l, P_y^m, P_z^n) = P_0^1$ *if* $P_x^l = P_y^m = P_z^n$
(mG_3) $mG(P_x^l, P_y^m, P_z^n) = mG(P_x^l, P_z^n, P_y^m) = mG(P_y^m, P_z^n, P_x^l) = \ldots$,
(mG_4) $mG(P_x^l, P_x^l, P_y^m) \leq mG(P_x^l, P_y^m, P_z^n)$ *for all* $P_x^l, P_y^m, P_z^n \in M_{pt}$ *with* $P_y^m \neq P_z^n$,
(mG_5) $mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^k, P_a^k) + mG(P_a^k, P_y^m, P_z^n)$ *for all* $P_x^l, P_y^m, P_z^n, P_a^k \in M_{pt}$,
(mG_6) *For at least two of the l, m, n variables are different,* $mG(P_x^l, P_y^m, P_z^n) = P_0^r$ *iff* $x = y = z$ *and* $r = max\{l, m, n\}$.

Then (M, mG) is said to be a multi G-metric $(m\text{-}g\text{-}metric)$ space.

Example 3.2. *Assume that X is a non-empty set and that M is a multi-set over X with multiplicity of any element approximately equal to ω. A mapping $mG : M_{pt} \times M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ are defined by*

$$mG(P_x^l, P_y^m, P_z^n) = \begin{cases} P_0^1, & if\ all\ of\ the\ variables\ P_x^l, P_y^m, P_z^n are\ equal, \\ P_2^k, & if\ all\ of\ the\ variables\ x, y, z\ are\ different, k = max\{l, m, n\}, \\ P_1^k, & if\ two\ of\ the\ variables\ P_x^l, P_y^m, P_z^n\ are\ equal, and\ theremaining\ one\ is\ distinct\ and\ k = max\{l, m, n\}, \\ P_0^k, & if\ x = y = z\ and\ for\ at\ least\ two\ of\ the\ l, m, n\ variables\ are\ different,\ k = max\{l, m, n\}. \end{cases}$$

Then mG satisfies all the multi-G-metric axioms.

Example 3.3. *Assume that (M, md) is multi-metric space. A mapping $mG : M_{pt} \times M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ is defined by $mG(P_x^l, P_y^m, P_z^n) = max\{md(P_x^l, P_y^m),\ md(P_y^m, P_n^z),\ md(P_x^l, P_z^n)\}$. Then mG satisfies all the multi-G-metric axioms.*

Definition 3.4. *A multi G-metric space (M, mG) is said to be symmetric if $mG(P_x^l, P_y^m, P_y^m) = mG(P_x^l, P_x^l, P_y^m)$ for any $P_x^l, P_y^m \in M_{pt}$.*

Proposition 3.5. *Assume that X is a non-empty set and that M is a mset over X with a multiplicity of any element approximately equal to ω. Let mG be a multi-G-metric. Then, the following hold for all $P_x^l, P_y^m, P_z^n, P_a^r \in M_{pt}$.*

(1) If $mG(P_x^l, P_y^m, P_z^n) = P_0^1$ *then* $P_x^l = P_y^m = P_z^n$.
(2) $mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_x^l, P_y^m) + mG(P_x^l, P_x^l, P_z^n)$.
(3) $mG(P_x^l, P_y^m, P_y^m) \leq P_2^1 mG(P_y^m, P_x^l, P_x^l)$.
(4) $mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n)$.
(5) $mG(P_x^l, P_y^m, P_z^n) \leq P_{2/3}^1(mG(P_x^l, P_y^m, P_a^r) + mG^(P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n))$.
(6) $mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_a^r) + mG(P_y^m, P_a^r, P_a^r) + mG(P_z^n, P_a^r, P_a^r)$.

Proof. (1) By the definition of multi-G-metric, it is clear.
(2) **Case 1:** Let $P_x^l \neq P_y^m \neq P_z^n$. Then we have

$$mG(P_x^l, P_x^l, P_y^m) + mG(P_x^l, P_x^l, P_z^n) \geq mG(P_y^m, P_x^l, P_z^n) = mG(P_x^l, P_y^m, P_z^n)$$

from (mG_3) and (mG_5).
Case 2: Let $P_x^l = P_y^m \neq P_z^n$. Then we have

$$mG(P_x^l, P_x^l, P_y^m) + mG(P_x^l, P_x^l, P_z^n) = P_0^1 + mG(P_x^l, P_x^l, P_z^n) = P_0^1 + mG(P_x^l, P_y^m, P_z^n) \geq mG(P_x^l, P_y^m, P_z^n)$$

from (mG_2), (mG_4).
Case 3: Let $x = y = z$. The proof is clear.
Other cases' proofs are produced in a similar way.
(3) We know that $mG(P_x^l, P_y^m, P_y^m) \leq mG(P_x^l, P_x^l, P_y^m) + mG(P_x^l, P_x^l, P_y^m) = P_2^1 mG(P_x^l, P_x^l, P_y^m)$ by (2). Then we obtain

$$mG(P_x^l, P_y^m, P_y^m) \leq P_2^1 mG(P_y^m, P_x^l, P_x^l)$$

from (mG_3).
(4) **Case 1:** Let $P_x^l \neq P_z^n$. Thus we have

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_a^r, P_a^r, P_x^l) + mG(P_a^r, P_y^m, P_z^n) \leq mG(P_a^r, P_x^l, P_z^n) + mG(P_a^r, P_y^m, P_z^n)$$

from (mG_5),(mG_3),(mG_4) respectively. So, we get

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n)$$

by (mG_3).
Case 2: Let $P_x^l = P_z^n$ and $P_y^m \neq P_a^r$. Then, we have

$$mG(P_x^l, P_y^m, P_z^n) = mG(P_x^l, P_y^m, P_x^l) = mG(P_x^l, P_x^l, P_y^m) \leq mG(P_x^l, P_y^m, P_a^r)$$

from (mG_3), (mG_4). Therefore we obtain

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_a^r, P_y^m, P_x^l) + mG(P_x^l, P_a^r, P_x^l) = mG(P_a^r, P_y^m, P_z^n) + mG(P_x^l, P_a^r, P_z^n)$$

by (mG_3).
Case 3: Let $x = y = z$ and $P_y^m = P_a^r$. Then it is obvious.
Case4: Let $x = y = z = a$,then we have

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_a^r) + mG(P_a^r, P_y^m, P_z^n) = mG(P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n).$$

(5) By using (4) and (mG_3), we get

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_z^n + mG(P_a^r, P_y^m, P_z^n)$$

and

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_x^l, P_a^r, P_y^m) + mG(P_z^n, P_a^r, P_x^l)$$

and

$$mG(P_x^l, P_y^m, P_z^n) \leq mG(P_y^m, P_a^r, P_x^l) + mG(P_z^n, P_a^r, P_x^l).$$

Thus we get

$$P_3^1 mG(P_x^l, P_y^m, P_z^n) \leq P_2^1(mG(p_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n) + mG(P_a^r, P_x^l, P_y^m))$$

from (mG_3). So, we obtain $mG(P_x^l, P_y^m, P_z^n) \leq P_{2/3}^1(mG(P_x^l, P_a^r, P_z^n) + mG(P_a^r, P_y^m, P_z^n) + mG(P_a^r, P_x^l, P_z^n))$.

(6) From (mG_5), (2) and (mG_3), we have

$$mG(P_x^l,P_y^m,P_z^n) \leq mG(P_x^l,P_a^r,P_a^r)+mG(P_a^r,P_y^m,P_z^n) \leq mG(P_x^l,P_a^r,P_a^r)+mG(P_y^m,P_a^r,P_a^r)+mG(P_z^n,P_a^r,P_a^r)$$

and

$$mG(P_x^l,P_y^m,P_z^n) \leq mG(P_y^m,P_a^r,P_a^r)+mG(P_a^r,P_x^l,P_z^n) \leq mG(P_y^m,P_a^r,P_a^r)+mG(P_x^l,P_a^r,P_a^r)+mG(P_z^n,P_a^r,P_a^r)$$

and

$$mG(P_x^l,P_y^m,P_z^n) \leq mG(P_z^n,P_a^r,P_a^r)+mG(P_a^r,P_x^l,P_y^m) \leq mG(P_z^n,P_a^r,P_a^r)+mG(P_y^m,P_a^r,P_a^r)+mG(P_x^l,P_a^r,P_a^r).$$

So we obtain

$$mG(P_x^l,P_y^m,P_z^n) \leq mG(P_x^l,P_a^r,P_a^r)+mG(P_y^m,P_a^r,P_a^r)+mG(P_z^n,P_a^r,P_a^r).$$

□

Proposition 3.6. *Assume that X is a non-empty set and that M is a multi-set over X with multiplicity of any element approximately equal to ω. Let (M,mG) be a multi G-metric space; then the followings are equivalent:*

(1) (M,mG) is symmetric,
(2) $mG(P_x^l,P_y^m,P_y^m) \leq mG(P_x^l,P_y^m,P_a^r)$ for all $P_x^l,P_y^m,P_a^r \in M_{pt}$,
(3) $mG(P_x^l,P_y^m,P_z^n) \leq mG(P_x^l,P_y^m,P_a^r)$ + $mG(P_z^n,P_y^l,P_b^s)$ for all $P_x^l,P_y^m,P_z^n,P_a^r,P_b^s \in M_{pt}$.

Proof. It is obvious from (mG_3), (mG_4) and Proposition 3.5. □

Example 3.7. *Let X be a nonempty set and M be a mset over X having multiplicity of any element almost equal to ω. (M,mG) is an multi-G-metric space. Then mG_1 is multi G-metric on M where $mG_1(P_x^l,P_y^m,P_z^n)=min\{P_k^t,mG(P_x^l,P_y^m,P_z^n)\}$ such that $P_k^t > P_1^0$.*

Proposition 3.8. *Let (M,md) be a multi-metric space. Then $mG_s(d)$ and $mG_m(d)$ expressed as follows define multi G-metrics on X.*

(1) $mG_s(d)(P_x^l,P_y^m,P_z^n)=P_{1/3}^1(md(P_x^l,P_y^m)+md(P_y^m,P_z^n)+md(P_x^l,P_z^n))$.
(2) $mG_m(d)(P_x^l,P_y^m,P_z^n)=\max\{md(P_x^l,P_y^m),md(P_y^m,P_z^n),md(P_x^l,P_z^n)\}$.

Proof.(mG_1) -(mG_3) It is obvious.

(mG_4) **Case 1:** $P_x^l=P_y^m$. Thus $mG_s(d)(P_x^l,P_x^l,P_y^m)$=$P_0^1 \leq mG_s(d)(P_x^l,P_y^m,P_z^n)$.

Case 2:

(a) $x=y$, $y\neq z$ and $l\neq m$. Then

$$mG_s(d)(P_x^l,P_x^l,P_x^m)=P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_x^m)+md(P_x^l,P_x^m))=P_{1/3}^1(P_0^1+P_2^1 md(P_x^l,P_x^m)) \leq mG_s(d)(P_x^l,P_x^m,P_z^n).$$

(b) $x=y=z, m\neq n$ and $l\neq m$. Then

$$mG_s(d)(P_x^l,P_x^l,P_x^m)=P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_x^m)+md(P_x^l,P_x^m))=P_{1/3}^1(P_0^1+P_2^1 md(P_x^l,P_x^m))=P_{1/3}^1 P_0^k \leq mG_s(d)(P_x^l,P_x^m,P_x^n)$$

where $k=max\{l,m\}$.

Case 3:

(a) $x\neq y$, $y\neq z$ and $x\neq z$. By using (md_4), we have

$$P_2^1 md(P_x^l,P_y^m) \leq md(P_x^l,P_y^m)+md(P_x^l,P_z^n)+md(P_z^n,P_y^m).$$

Then

$$mG_s(d)(P_x^l,P_x^l,P_y^m)=P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_y^m)+md(P_x^l,P_y^m)) \leq P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_y^m)+md(P_z^n,P_y^m)+md(P_x^l,P_y^m)).$$

(b) $x\neq y, y\neq z$ and $P_x^l=P_z^n$. By using (mG_4), we have $mG_s(d)(P_x^l,P_x^l,P_y^m)$= $mG_s(d)(P_x^l,P_y^m,P_x^l)$.

(c) $x\neq y, y\neq z, x=z$ and $l\neq n$. By using (md_4), we have

$$md(P_x^l,P_y^m) \leq md(P_x^l,P_x^n)+md(P_x^n,P_y^m).$$

Then

$$mG_s(d)(P_x^l,P_x^l,P_y^m)=P_{1/3}^1(md(P_x^l,P_x^l)+md(P_x^l,P_y^m)+md(P_x^l,P_y^m))=P_{1/3}^1(P_0^1+P_2^1 md(P_x^l,P_x^m)) \leq mG_s(d)(P_x^l,P_x^m,P_x^n).$$

The proofs of other cases are done in a similar way.

(mG_5) From (md_4) we get $md(P_x^l,P_y^m) \leq md(P_x^l,P_a^r)+md(P_a^r,P_y^m)$ and $md(P_x^l,P_z^n) \leq md(P_x^l,P_a^r)+md(P_a^r,P_z^n)$. Therefore

$$\begin{aligned} mG_s(d)(P_z^l,P_y^m,P_z^n) &= P_{1/3}^1(md(P_x^l,P_y^m)+md(P_x^l,P_z^n)+md(P_z^n,P_y^m)) \\ &\leq P_{1/3}^1(md(P_x^l,P_a^r)+md(P_a^r,P_y^m)+md(P_x^l,P_a^r)+md(P_x^l,P_a^r)+md(P_a^r,P_z^n)+md(P_y^m,P_z^n)) \\ &= mG_s(d)(P_x^l,P_a^r,P_a^r)+mG_s(d)(P_a^r,P_y^m,P_z^n). \end{aligned}$$

(mG_6) It is obvious from md_6.

□

Proposition 3.9. *Let (M,mG) be a multi-G-metric space.Then md_G defined a multi-metric on M following holds:*

$$md_G(P_x^l,P_y^m) = mG(P_x^l,P_y^m,P_y^m) + mG(P_x^l,P_x^l,P_y^m).$$

Proof. The proofs of (md_1), (md_3) and (md_4) obviously follow from (mG_1), (mG_3), (mG_4) respectively.

(md_2) Let $md_G(P_z^l,P_y^m) = P_0^1$. Assume that $P_x^l \neq P_y^m$. Since $mG(P_x^l,P_y^m,P_y^m) + mG(P_x^l,P_x^l,P_y^m) = P_0^1$. We would have $mG(P_x^l,P_y^m,P_y^m) \leq P_0^1$ by Proposition 3.5. This contradicts to (mG_1). Hence our assumption is not true. That is $P_x^l = P_y^m$. The converse is clear.

(md_5) Let $md_G(P_x^l,P_y^m) = P_0^1$ for $l \neq m$. Thus we get $mG(P_x^l,P_y^m,P_z^n) + mG(P_x^l,P_x^l,P_y^m) = P_0^k$. Then, we get

$$mG(P_x^l,P_y^m,P_y^m) = mG(P_x^l,P_x^l,P_y^m) = P_0^k \quad \text{and} \quad x = y, \quad k = \max\{l,m\}.$$

Conversely, let $x = y$ and $k = \max\{l,m\}$. Thus we have

$$md_G(P_x^l,P_y^m) = mG(P_x^l,P_y^m,P_y^m) + mG(P_x^l,P_x^l,P_y^m) = mG(P_x^l,P_x^m,P_x^m) + mG(P_x^l,P_x^l,P_x^m) = P_0^k.$$

□

Proposition 3.10. *Let (M,mG) be a multi-G-metric space.The function $md : M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ defined by $md_G(P_x^l,P_y^m) = mG(P_x^l,P_y^m,P_y^m)$ satisfies the following properties.*

1. *If $md_G(P_x^l,P_y^m) = P_0^1$ if and only if $P_x^l = P_y^m$.*
2. *$md_G(P_x^l,P_y^m) \leq md_G(P_x^l,P_z^n)$+ $md_G(P_y^m,P_z^n)$ for all $P_x^l,P_y^m,P_z^n \in M_{pt}$.*
3. *For at least two of the l, m variables are different, $md_G(P_x^l,P_y^m) = P_0^k$ iff $x = y$ and $k = max\{l,m\}$.*

Proof. (1) Let $md_G(P_z^l,P_y^m) = P_0^1$. By hypothesis, $mG(P_x^l,P_y^m,P_y^m) = P_0^1$. From Proposition 3.5, we get $P_x^l = P_y^m$. The converse is clear.

(2) From (mG_5), we get $md_G(P_x^l,P_y^m) = mG(P_x^l,P_y^m,P_y^m) \leq mG(P_x^l,P_z^n,P_z^n)$+ $mG(P_z^n,P_x^m,P_y^m)$=$md_G(P_x^l,P_z^n)$ +$md_G(P_z^n,P_y^m)$.

(3) Let $md_G(P_x^l,P_y^m) = P_0^k$. Then, we have $mG(P_x^l,P_y^m,P_y^m) = P_0^k$. By (mG_6), we have $x = y$ and $k = max\{l,m\}$. Conversely, let $x = y$ and $k = \max\{l,m\}$. Thus, we have $mG(P_x^l,P_y^m,P_y^m)$ =$md_G(P_x^l,P_y^m) = P_0^k$.

□

4. Some topological properties of multi-G-metric spaces

In this section, we establish a few topological concepts on multi-G-metric spaces and explore a few of their associated characteristics.

Definition 4.1. *Let (M,mG) be a multi-G- metric space. For $P_a^l \in M_{pt}$ and $P_r^1 > P_0^1$ the mG-open ball with centre P_a^l and radius P_r^1 is defined by*

$$B_{mG}(P_a^l,P_r^1) = \{P_y^m \in M_{pt} : mG(P_x^l,P_y^m,P_y^m) < P_r^1\}.$$

The mset $MS[B_{mG}(P_a^l,P_r^1)]$ will be called a multi-open ball with centre P_a^l and radius $P_r^1 > P_0^1$.

Proposition 4.2. *Let (M,mG) be a multi-G- metric space. Let $P_x^l \in M_{pt}$ and $P_r^1 > P_0^1$. If $mG(P_x^l,P_y^m,P_z^n) < P_r^1$ then $P_x^l,P_y^m \in B_{mG}(P_x^l,P_r^1)$.*

Proof. It is obvious from (mG_{m4}). □

Example 4.3. *Consider the multi-G-metric space. (M,mG) given in Example 3.2. Then we have*

$$B_{mG}(P_a^l,P_r^1) = \begin{cases} M_{pt}, & if P_r^1 > P_1^1, \\ \{P_a^n : 1 \leq n \leq \omega\}, & if P_r^1 \leq P_1^1, \end{cases}$$

for any $P_a^l \in M_{pt}$.

Definition 4.4. *Let (M,mG) be a multi-G-metric space. For $P_a^l \in M_{pt}$ and $P_r^1 > P_0^1$ the mG-closed ball with P_a^l and radius P_1^r is defined by*

$$B_{mG}[P_a^l,P_r^1] = \{P_y^m \in M_{pt} : mG(P_x^l,P_y^m,P_y^m) < P_r^1\}.$$

$MS[B_{mG}[P_a^l,P_r^1]]$ will be called a multi-closed ball with center P_a^l and radius $P_r^1 > P_0^1$. The empty mset $\emptyset$ is separately considered as multi-G-open in (M,mG).

Definition 4.5. *Let (M,mG) be a multi-G- metric space. Then, O_M, a collection of multi-points of M, is said to be mG-open if for each $P_x^l \in O_M$ there exists an mG-open ball $B_{mG}(P_a^l,P_r^1)$ with center at P_a^l and radius $P_r^1 > P_0^1$ such that $B_{mG}(P_a^l,P_r^1) \subseteq O_M$. $N \subset M$ is multi-mG-open iff there exists a collection O of multi points of N such that O is mG-open and $MS(O) = N$.*

Proposition 4.6. *Every mG-open ball is mG-open in a multi-G-metric space.*

Proof. Let $P_y^m \in B_{mG}(P_x^l, P_r^1)$. Suppose $P_z^n \in B_{mG}(P_y^m, P_s^1)$. Then, we have $mG(P_z^n, P_z^n, P_y^m) < P_s^1$. By (mG_5) we get

$$mG(P_z^n, P_z^n, P_x^l) \leq mG(P_x^l, P_y^m, P_y^m) + mG(P_y^m, P_z^n, P_z^n) < mG(P_x^l, P_y^m, P_y^m) + P_s^1.$$

Let $P_s^1 = P_r^1 - mG(P_x^l, P_y^m, P_y^m)$. We obtain

$$mG(P_z^n, P_z^n, P_x^l) < P_r^1.$$

Hence $P_z^n \in B_{mG}(P_x^l, P_r^1)$. □

Theorem 4.7. *Every multi-G-metric space generates a multi-G-topology as follows:*

$$\tau_{mG} = \{N : for\ every\ P_a^l \in N_{pt},\ there\ exists\ aP_r^1 such\ that\ B_{mG}(P_a^l, P_r^1) \subseteq O_N\ and\ MS(O_N) = N\}.$$

This topology is said to be multi-topology produced by multi-G-metric.

Proof. (1) Let M_{pt} be the collection of for all multi-points in (M, mG) multi-G-metric space. Then M_{pt} is mG-open. Hence, $M = MS(M_{pt})$ is multi-G-open.

(2) Let N_i $i = 1,2$ be two multi-G-open sets in (M, mG). Then there exists O_{N_i} such that $N_i = MS(O_{N_i})$ and O_{N_i} is mG-open set of multi-points in (M, mG). Let $P_x^l \in O_{N_1} \cap O_{N_2}$. Then, there exist $P_r^1, P_s^1 > P_0^1$ such that $B_{mG}(P_x^l, P_r^1) \subset O_{N_1}$ and $B_{mG}(P_x^l, P_s^1) \subset O_{N_2}$. Let $t = min\{r, s\}$. Then, we have $B_{mG}(P_x^l, P_1^t) \subset B_{mG}(P_x^l, P_1^r) \subset O_{N_1}$ and $B_{mG}(P_x^l, P_t^1) \subset B_{mG}(P_x^l, P_s^1) \subset O_{N_2}$. Therefore, we have $B_{mG}(P_x^l, P_t^1) \subset O_{N_1} \sqcap O_{N_2}$ and $O_{N_1} \sqcap O_{N_2}$ is mG-open. Since from Theorem 2.8, $N_1 \cap N_2 = MS(O_{N_1}) \cap MS(O_{N_2}) = MS(O_{N_1} \sqcap O_{N_2})$. Hence, $N_1 \cap N_2$ is multi-G-open.

(3) The proof can be done in a similar way (2).

□

Definition 4.8. *Let (M, mG) be a multi-G- metric space. and N_{mG} be a mset in this G-multi metric space. Then N_{mG} is said to be multi-closed if its complement N_{mG}^c is multi-open in this multi-G-metric space.*

Theorem 4.9. *Let (M, mG) be a multi-G- metric space. The followings are held:*

(1) The multi-empty set is multi-closed,
(2) The absolute mset M is multi closed,
(3) Arbitrary intersection of multi-closed sets is multi-closed,
(4) Finite union of multi-closed sets is multi-closed.

Proof. The proofs are obvious from Theorem 4.7 and Definition 4.8. □

Definition 4.10. *Let (M, mG) be a multi-G-metric space. Let $\{P_{x_n}^{l_n}\}$ be a sequence of multi-points in M. The sequence $\{P_{x_n}^{l_n}\}_n$ is said to multi G-convergent (mG-convergent) to $P_x^l \in M_{pt}$, if for every $P_\varepsilon^1 > P_0^1$, there exists $n_0 \in \mathbb{N}$ such that $mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) < P_\varepsilon^1$, $\forall n \geq n_0$ i.e. $n \geq n_0$ implies $\{P_{x_n}^{l_n}\} \in MS(B_{mG}(P_x^l, P_\varepsilon^1))$. We denote the sequence $\{P_{x_n}^{l_n}\}$ is multi G-convergent to P_x^l and written as $\{P_{x_n}^{l_n}\} \to P_x^l$.*

Proposition 4.11. *In a multi-G- metric space, a sequence of multi-points multi G-converges at most one multi-point of the space.*

Proof. The proof is easily obtained from Definition 4.10. □

Proposition 4.12. *Let (M, mG) be a multi-G- metric space. For the sequence $\{P_{x_n}^{l_n}\}_n \subset M_{pt}$ and a point $P_x^l \in M_{pt}$ the followings are equivalent:*

(1) $\{P_{x_n}^{l_n}\}$ is mG-convergent to P_x^l,
(2) $md_G(P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as $n \to \infty$,
(3) $mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as $n \to \infty$,
(4) $mG(P_{x_n}^{l_n}, P_x^l, P_x^l) \to P_0^1$ as as $n \to \infty$,
(5) $mG(P_{x_m}^{l_m}, P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as m, $n \to \infty$.

Proof. (1) $\Rightarrow$(2) It is obvious from Proposition 3.5.

(2) $\Rightarrow$(3) Let $md_G(P_{x_n}^{l_n}, P_x^l) \to P_0^1$. Then, for each $P_\varepsilon^1 > P_0^1$, there exists a natural number n_0 such that $md(P_{x_n}^{l_n}, P_x^l) < P_\varepsilon^1$ whenever $n \geq n_0$. By Proposition 3.9, we have $mG(P_{x_n}^{l_n}, P_x^l, P_x^l) + mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) = md_G(P_{x_n}^{l_n}, P_x^l) < P_\varepsilon^1$. Thus, we obtain $mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) \to P_0^1$.

(3) $\Rightarrow$(4) It is clear since $mG(P_{x_n}^{l_n}, P_x^l, P_x^l) \leq P_2^1 mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l)$ by Proposition 3.5.

(4) $\Rightarrow$(5) It is follows from Proposition 3.5 since $mG((P_{x_m}^{l_m}, P_{x_n}^{l_n}, P_x^l) \leq mG(P_{x_m}^{l_m}, P_{x_m}^{l_m}, P_{x_n}^{l_n}) + mG(P_{x_m}^{l_m}, P_{x_m}^{l_m}, P_x^l)$.

(5) $\Rightarrow$(2) Let $mG(P_{x_m}^{l_m}, P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as m, $n \to \infty$. Since

$$md_G(P_{x_n}^{l_n}, P_x^l) = mG(P_{x_n}^{l_n}, P_x^l, P_x^l) + mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) \leq P_2^1 mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) + mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l) = P_3^1 mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_x^l)$$

from Proposition 3.9 and Proposition 3.5, we have $md_G(P_{x_n}^{l_n}, P_x^l) \to P_0^1$ as $n \to \infty$.

□

Definition 4.13. *Let (M,mG) be a multi-G-metric space. Let $\{P_{x_n}^{l_n}\}_n$ be a sequence of multi points in M. Then the sequences $\{P_{x_n}^{l_n}\}_n$ is said to be multi G-bounded if there exists a positive multi real point $P_a^1 > P_0^1$ such that*

$$mG(P_{x_n}^{l_n}, P_{x_n}^{l_n}, P_{x_m}^{l_m}) \leq P_a^1$$

for each m, $n \in \mathbb{N}$.

Definition 4.14. *Let (M,mG) be a multi-G- metric space. Let $\{P_{x_n}^{l_n}\}_n$ be a sequence of multi-points in M. Then the sequence $\{P_{x_n}^{l_n}\}_n$ is said to be multi-G-Cauchy (mG-Cauchy) if every $P_\varepsilon^1 > P_0^1$, there exists a $n_0 \in \mathbb{N}$ such that $mG(P_{x_n}^{l_n}, P_{x_m}^{l_m}, P_{x_p}^{l_p}) < P_\varepsilon^1$ whenever $m,n,p \geq n_0$.*

Proposition 4.15. *Let (M,mG) be a multi-G- metric space. Then the followings are equivalent:*

(1) The sequence $\{P_{x_n}^{l_n}\}_n$ is mG-Cauchy.
(2) For every $P_\varepsilon^1 > P_0^1$, there exists a natural number n_0 such that $mG(P_{x_n}^{l_n}, P_{x_m}^{l_m}, P_{x_m}^{l_m}) < P_\varepsilon^1$ for all $n,m \geq n_0$.
(3) $\{P_{x_n}^{l_n}\}_n$ is a mG-Cauchy sequence in the multi-metric space (M, md_G).

Proof. (1) $\Rightarrow$(2) It is obvious by (mG_4),
(2) $\Leftrightarrow$(3) It is clear from Proposition 3.9,
(2) $\Rightarrow$(1) It is obvious by (mG_5) if it set $a = x_m$.

□

Corollary 4.16. *Every mG-convergent sequence in any multi-G-metric space is mG-Cauchy.*

Proof. It is obvious by (mG_5) and Proposition 4.12. □

Corollary 4.17. *Every mG-Cauchy sequence is multi-G-bounded.*

Proof. It is obvious by Definition 4.13 and Proposition 4.12. □

Definition 4.18. *A mG-multi metric space (M,mG) is said to be mG- complete if every mG-Cauchy sequence in (M,mG) is mG-convergent in (M,mG).*

Proposition 4.19. *A multi G-metric space (M,mG) is mG- complete if and only if (M, md_G) is a complete multi-metric space.*

Proof. It is obvious from Proposition 3.9 and Proposition 4.15 □

Proposition 4.20. *Let (M,mG) be a multi-G-metric space. Let $md : M_{pt} \times M_{pt} \to (m\mathbb{R}^+)_{pt}$ be the function defined by $m\delta_G(P_x^l, P_y^m) = max\{mG(P_x^l, P_y^m, P_y^m), mG(P_y^m, P_x^l, P_x^l)\}$. Thus the followings hold:*

(1) $(M, m\delta_G)$ is multi-metric space,
(2) $\{P_{x_n}^{l_n}\}_n$ is a mG-convergent to $P_x^l \in M_{pt}$ if and only if $\{P_{x_n}^{l_n}\}_n$ is a convergent to $P_x^l \in M_{pt}$ in $(M, m\delta_G)$,
(3) $\{P_{x_n}^{l_n}\}_n$ is a mG-Cauchy if and only if $\{P_{x_n}^{l_n}\}_n$ is Cauchy in $(M, m\delta_G)$,
(4) (M,mG) is a mG-complete if and only if $(M, m\delta_G)$ is multi-complete.

Proof. The proofs are clear from the definition of multi-G-metric and Proposition 4.12. □

5. Conclusion

We introduced and studied multi-G-metric spaces as generalizations of G-metric spaces and multi-metric spaces. We think this research will advance and increase future investigations into multi-topology and multi-metric systems by providing a broad framework for their practical applications.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

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Universal Journal of Mathematics and Applications, 6 (3) (2023) 100-105
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1323655

Local Antisymmetric Connectedness in Asymmetrically Normed Real Vector Spaces

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Article Info

Keywords: *Antisymmetry component, Asymmetrically normed real vector space, Antisymmetric path, Complementary graph, Connected graph, Local antisymmetric connectedness, Symmetrization metric, Symmetric pair, T_0-quasi-metric*
2010 AMS: *54D05, 05C10, 46B40, 54E35*
Received: *6 July 2023*
Accepted: *10 September 2023*
Available online: *18 September 2023*

Abstract

In this paper, some properties of locally antisymmetrically connected spaces which are the localized version of the antisymmetrically connected T_0-quasi-metric spaces constructed as the natural counterparts of connected complementary graphs, are presented in terms of asymmetric norms.

According to that, we investigated some different aspects and examples of local antisymmetric connectedness in the framework of asymmetrically normed real vector spaces.

Specifically, it is proved that the structures of antisymmetric connectedness and local antisymmetric connectedness coincide for the T_0-quasi-metrics induced by the asymmetric norms which associate the theory of quasi-metrics with functional analysis.

1. Introduction and Preliminaries

The structure of antisymmetric connectedness of a T_0-quasi-metric space was first described in [1]. This theory was especially discussed in terms of graph theory [2, 3] as a suitable counterpart of the connectedness for complementary graph. It is also observed that there were natural relationships between the theory of antisymmetrically connected T_0-quasi-metric spaces and the theory of connected complementary graphs, with the help of symmetry graphs introduced in [1].

On the other hand, it is useful to localize some topological properties (see [4]) as is well-known from topology. In the light of these considerations; following the structure of antisymmetric connectedness constructed lately, the "locality" status of antisymmetric connectedness was investigated in [5] under the name local antisymmetric connectedness.

As for this paper, many interesting properties of locally antisymmetrically connected spaces will be presented in the context of asymmetrically normed real vector spaces.

Therefore, the paper is organized in the following format:

Some necessary background material for the remaining of paper is first given in Section 1 via some references. Particularly, Section 1 mostly consists of the information about the antisymmetrically connected T_0-quasi-metric spaces, besides the other types of spaces peculiar to asymmetric topology.

In Section 2, the required propositions and examples about the locally antisymmetrically connected T_0-quasi-metric spaces are reminded via [5], in detail.

After recalling all preliminary information, as the main purpose of paper; in Section 3, we studied some properties of the theory of local antisymmetric connectedness in the framework of asymmetric norms. Indeed, it is known from [1] that the problem to determine the antisymmetry components of points in X turns out to be easier when it is formulated for a T_0-quasi-metric induced by the asymmetric norm of an asymmetrically normed real vector space which is introduced by Cobzaş [6] in Functional Analysis. In the light of this idea, locally

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Cite as: N. Javanshir, F. Yıldız, *Local antisymmetric connectedness in asymmetrically normed real vector spaces*, Univers. J. Math. Appl., 6(3) (2023), 100-105.

antisymmetrically connected T_0-quasi-metric spaces are investigated first time in the context of asymmetrically normed real vector spaces in detail.

Specifically, in Section 3 it is proved that the antisymmetric connectedness and local antisymmetric connectedness coincide for the T_0-quasi-metrics induced by the asymmetric norms.

Finally, a conclusion part together with the two questions that could be the subject of a future work is presented as the last section of paper.

Now let us recall some required notions and also examples from [1].

Definition 1.1. *Let X be a set. Then the function $\rho : X \times X \to [0,\infty)$ is called a* quasi-pseudometric *on X if*

(a) $\rho(x,x) = 0$ *whenever* $x \in X$,
(b) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ *whenever* $x,y,z \in X$.

We will say that ρ is a T_0-quasi-metric *provided that ρ also satisfies the following condition:*

For each $x,y \in X$,

$$\rho(x,y) = 0 = \rho(y,x) \ \text{implies that } x = y.$$

Remark 1.2. *If ρ is a T_0-quasi-metric on a set X then the function $\rho^{-1} : X \times X \to [0,\infty)$ defined by $\rho^{-1}(x,y) = \rho(y,x)$ whenever $x,y \in X$, is also a T_0-quasi-metric, called the* conjugate T_0-quasi-metric of ρ. *If $\rho = \rho^{-1}$ then ρ is a* metric. *In line with the usual notational conventions, we write*

$$\rho^s = \sup\{\rho, \rho^{-1}\} = \rho \vee \rho^{-1}$$

for the symmetrization metric of ρ.

The notation τ_{ρ^s} denotes the topology generated by the (symmetrization) metric ρ^s and it is called symmetrization topology of ρ.

An adequate background for the T_0-quasi-metric spaces can be obtained in the works [7–12]. Now, as for the main structures required for the paper:

Definition 1.3. *If (X,ρ) is a T_0-quasi-metric space then the pair $(x,y) \in X \times X$ is called*

(i) antisymmetric pair *whenever the condition $\rho(x,y) \neq \rho(y,x)$ is satisfied.*
(ii) symmetric pair *if it satisfies the condition $\rho(x,y) = \rho(y,x)$.*

Definition 1.4. *If (X,ρ) is a T_0-quasi-metric space then a finite sequence of the elements in X is called an* antisymmetric path *(*symmetric path*) $Q_{x,y} = (x = x_0, x_1, \dots, x_{n-1}, x_n = y)$, $n \in \mathbb{N}$, from the starting point x to the ending point y provided that all the pairs (x_i, x_{i+1}) are antisymmetric pairs (symmetric pairs) for $i \in \{0,1,\dots,n-1\}$.*

Now, let us recall the antisymmetric connectedness from [1]:

Definition 1.5. (i) *In a T_0-quasi-metric space (X,ρ), two points $x,y \in X$ will be called* antisymmetrically connected *if we have an antisymmetric path $Q_{x,y}$ starting at x and ending with y, or $x = y$.*

Obviously, the relation "antisymmetric connectedness" is an equivalence relation on the set X.

(ii) *The equivalence class of $x \in X$ according to the antisymmetric connectedness relation T_ρ is called the* antisymmetry component *of x, and the notation*

$$T_\rho(x) = \{y \in X : \text{there is an antisymmetric path } Q_{x,y} \text{ from } x \text{ to } y\}$$

will be used for it.
It is clear that $T_\rho(x)$ is the largest antisymmetrically connected subspace of X containing $x \in X$.

(iii) *If $T_\rho = X \times X$, that is $T_\rho(x) = X$ whenever $x \in X$, then (X,ρ) is called* antisymmetrically connected space.

Now, let us present a well-known antisymmetrically connected T_0-quasi-metric space as follows:

Example 1.6. *On the set $\mathbb{R}$ of the reals, take $\mu(x,y) = max\{x-y, 0\}$ whenever $x,y \in \mathbb{R}$. It is easy to verify that μ is a T_0-quasi-metric, called the* standard *T_0-quasi-metric on $\mathbb{R}$. Moreover, the space $(\mathbb{R},\mu)$ is antisymmetrically connected since $T_\mu(x) = \mathbb{R}$ for each $x \in \mathbb{R}$.*

From now on, we can turn our attention to some other notions and details, required for the paper. Then an opposite notion to that of "metric" can be recalled from [1]:

Definition 1.7. *A T_0-quasi-metric space (X,ρ) is called* antisymmetric *if*

$$\rho(x,y) \neq \rho(y,x) \text{ whenever } x \neq y$$

for all $x,y \in X$.

Therefore, by Definition 1.5 (iii) we have:

Proposition 1.8. *Each antisymmetric T_0-quasi-metric space will be antisymmetrically connected.*

Additionally, the dual notion of the antisymmetric connectedness can be recalled from [1], in the framework of T_0-quasi-metrics.

Definition 1.9. (i) *If (X,ρ) is a T_0-quasi-metric space then $x \in X$ is called* symmetrically connected to *$y \in X$ whenever there exists a symmetric path (see Definition 1.4) $Q_{x,y}$, from x to y.*

Trivially, the relation "symmetric connectedness" will be an equivalence relation on the points in X.

(ii) *The equivalence class of $x \in X$ according to the symmetric connectedness relation C_ρ is called* symmetry component of x*, and the notation*

$$C_\rho(x) = \{y \in X : \text{there is a symmetric path } Q_{x,y} \text{ from } x \text{ to } y\}$$

will be used for it. Obviously $C_\rho(x)$ is the largest symmetrically connected subspace of X containing $x \in X$.

(iii) *If $C_\rho = X \times X$, that is $C_\rho(x) = X$ whenever $x \in X$, then (X,ρ) is called* symmetrically connected space.

In the light of above considerations, the next proposition was established in [1] as Corollary 25, by using the following crucial result well-known from graph theory.

For any graph G, G is connected or $\overline{G}$ the complement of G is connected in terms of graph theory. (See [2,3])

Proposition 1.10. *If (X,ρ) is a T_0-quasi-metric space then (X,ρ) is antisymmetrically connected or symmetrically connected.*

Note here that even though we will not be interested in this theory for the remainder of Section 1, the detailed background on the theory of "symmetric connectedness" can be found in [1].

The next notions will be required for the remainder of paper.

Definition 1.11. *Let ρ be a T_0-quasi-metric on X and $x \in X$. In this case,*

(i) *The point x is called* antisymmetric *whenever (x,y) is antisymmetric pair for $y \in X \setminus \{x\}$,*

(ii) *The point x is called* symmetric *whenever (x,y) is symmetric pair for $y \in X$.*

Hence, the next proposition which completes Section 1 can be seen easily via Definition 1.11.

Proposition 1.12. *If (X,ρ) is a T_0-quasi-metric space then all points of X are antisymmetric if and only if the space (X,ρ) is antisymmetric.*

After giving the required information above, now we are in the position to recall from [5] the localized version of antisymmetrically connected spaces.

2. Local Antisymmetric Connectedness

The following notions and the all propositions are presented in [5].

Definition 2.1. *Let (X,ρ) be a T_0-quasi-metric space and $x_0 \in X$. Thus (X,ρ) is called* locally antisymmetrically connected at $x_0 \in X$ *if $T_\rho(x_0) \in \tau_{\rho^s}$.*

As mentioned in Section 1, τ_{ρ^s} denotes the symmetrization topology generated by the metric $\rho^s = \rho \vee \rho^{-1}$.

Definition 2.2. *A T_0-quasi-metric space (X,ρ) is called* locally antisymmetrically connected *if (X,ρ) is locally antisymmetrically connected at each point of X.*

Hence, we have obviously the next crucial characterization because of Definition 2.1 and Definition 2.2.

Proposition 2.3. *A T_0-quasi-metric space (X,ρ) is locally antisymmetrically connected if and only if $T_\rho(x)$ (the antisymmetry component of x) is τ_{ρ^s}-open for each $x \in X$.*

Example 2.4. *Consider the (bounded) Sorgenfrey T_0-quasi-metric space $(\mathbb{R},\upsilon)$ where $\upsilon(x,y) = \min\{x-y,1\}$ if $x \geq y$ and $\upsilon(x,y) = 1$ if $x < y$. It is easy to show that the space $(\mathbb{R},\upsilon)$ is antisymmetrically connected, but not antisymmetric. Also, all antisymmetry components $T_\upsilon(x)$ $(x \in \mathbb{R})$ in the space $(\mathbb{R},\upsilon)$ are $\mathbb{R}$, and so they are open w.r.t. the discrete topology τ_{υ^s} generated by the symmetrization metric (which is discrete metric) of υ. That is, the space $(\mathbb{R},\upsilon)$ is locally antisymmetrically connected by Proposition 2.3.*

The following propositions and the last result were proved in [5] by taking into account Proposition 2.3:

Proposition 2.5. *Let (X,ρ) be a T_0-quasi-metric space. If (X,ρ) is antisymmetrically connected then (X,ρ) is locally antisymmetrically connected.*

The converse of Proposition 2.5 is not true by virtue of the next example:

Example 2.6. *Consider a T_0-quasi-metric on the set $X = \{1,2,3,4\}$ by the matrix*

$$\mathbf{W} = \begin{pmatrix} 0 & 8 & 4 & 1 \\ 9 & 0 & 6 & 7 \\ 4 & 6 & 0 & 5 \\ 3 & 7 & 5 & 0 \end{pmatrix}$$

That is, $W = (w_{ij})$ where $w(i,j) = w_{ij}$ for $i,j \in X$. Clearly, w is a T_0-quasi-metric on X. Indeed, it satisfies the other conditions of Definition 1.1, so we just prove the triangle inequality as follows: $w(1,2) = 8 \leq 4+6 = w(1,3)+w(3,2)$, $w(1,2) = 8 \leq 1+7 = w(1,4)+w(4,2)$,
$w(1,3) = 4 \leq 8+6 = w(1,2)+w(2,3)$, $w(1,3) = 4 \leq 1+5 = w(1,4)+w(4,3)$,
$w(1,4) = 1 \leq 8+7 = w(1,2)+w(2,4)$, $w(1,4) = 1 \leq 4+5 = w(1,3)+w(3,4)$,
$w(2,3) = 6 \leq 9+4 = w(2,1)+w(1,3)$, $w(2,3) = 6 \leq 7+5 = w(2,4)+w(4,3)$,

$w(2,4)=7\leq 9+1=w(2,1)+w(1,4)$, $w(2,4)=7\leq 6+5=w(2,3)+w(3,4)$,

$\vdots$

Also, note that (X,w) is not antisymmetric since $w(4,3)=w(3,4)$, and moreover it is not antisymmetrically connected since there is no any antisymmetric path from 1 to 3. Despite these, it is locally antisymmetrically connected: Note that (X,w) is a finite T_0-quasi-metric space. Thus, the symmetrized topological space (X,τ_{w^s}) will be discrete since the unique topology which is T_1 on a finite set is discrete. Hence, (X,w) will be locally antisymmetrically connected by Proposition 2.3 as the antisymmetry components of all points in X are open w.r.t. the symmetrization topology τ_{w^s}.

Proposition 2.7. *If (X,ρ) is locally antisymmetrically connected T_0-quasi-metric space and the topological space (X,τ_{ρ^s}) is connected then (X,ρ) is antisymmetrically connected.*

Proposition 2.8. *If (X,ρ) is a T_0-quasi-metric space then (X,ρ) is symmetrically connected or locally antisymmetrically connected.*

Corollary 2.9. *If (X,ρ) is an antisymmetric space then (X,ρ) is locally antisymmetrically connected.*

3. Locally Antisymmetrically Connected Spaces in the Context of Asymmetric Norms

Asymmetrically normed real vector spaces in the sense of [6] are also investigated in [1] as a new approach to the theory of asymmetry measurement for T_0-quasi-metrics.

First of all, let us recall the notion of *asymmetric norm* from [6]:

Definition 3.1. *Let X be a real vector space equipped with a given map $\|\cdot|:X\to[0,\infty)$ satisfying the conditions:*

(a) $\|x|=\|-x|=0$ *if and only if* $x=\mathbf{0}$.
(b) $\|\lambda x|=\lambda\|x|$ *whenever* $\lambda\geq 0$ *and* $x\in X$.
(c) $\|x+y|\leq\|x|+\|y|$ *whenever* $x,y\in X$.

Then $\|\cdot|$ is called an asymmetric norm *and $(X,\|\cdot|)$ an* asymmetrically normed real vector space. *(In* (a), $\mathbf{0}$ denotes the zero vector of the vector space X.)

Obviously, an asymmetric norm induces a T_0-quasi-metric on X with the equality $\rho_{\|\cdot|}(x,y)=\|x-y|$ for each $x,y\in X$, where $(X,\|\cdot|)$ is an asymmetrically normed real vector space. But, naturally some T_0-quasi-metrics may not be induced by an asymmetric norm:

Example 3.2. *Consider the function υ on $\mathbb{R}$ as follows:*
$\upsilon(x,y)=\begin{cases}\min\{x-y,1\} & ; \ x\geq y\\ 1 & ; \ x<y\end{cases}$ *for each $x,y\in\mathbb{R}$. It is easy to show that υ is a T_0-quasi-metric, but it cannot be induced by an asymmetric norm.*

Incidentally, the notation $\rho_{\|\cdot|}$ will be used for the T_0-quasi-metric induced by the asymmetric norm $\|\cdot|$.

Moreover, the function $\|\cdot|^s=\|\cdot|\vee\|\cdot|^{-1}=\|\cdot\|$ describes the standard (symmetrization) norm on X, where $\|a|^{-1}=\|-a|$ for $a\in X$, and so $\rho^s_{\|\cdot|}=\rho_{\|\cdot|^s}=\rho_{\|\cdot\|}$.

Note also that each norm is an asymmetric norm. However we have:

Example 3.3. (i) *If we take the function $\|x|=x\vee 0$ on $\mathbb{R}$, then $\|\cdot|$ satisfies the above conditions and gives an asymmetric norm, not norm on $\mathbb{R}$.*
(ii) *The function $\|\cdot|$ described by the equality $\|(x,y)|=x\vee y\vee 0$ on $\mathbb{R}^2$, where $x,y\in\mathbb{R}$, satisfies the above conditions and thus, it is an asymmetric norm which is not norm on $\mathbb{R}^2$.*

Now we are in the position to present some new considerations peculiar to asymmetrically normed real vector spaces.

Lemma 3.4. *Let $(X,\|\cdot|)$ be an asymmetrically normed real vector space. If $(X,\rho_{\|\cdot|})$ has an antisymmetric point, then $\mathbf{0}$ is an antisymmetric point.*

Proof. Let $a\in X$ be an antisymmetric point, and $b\in X$. Thus $a-b\in X$ and so by assumption, we have $\rho_{\|\cdot|}(a,a-b)\neq\rho_{\|\cdot|}(a-b,a)$ by Definition 1.11 (i). This means that $\|b|\neq\|-b|$ by the definition of induced T_0-quasi-metric $\rho_{\|\cdot|}$. That is, $\rho_{\|\cdot|}(b,\mathbf{0})\neq\rho_{\|\cdot|}(\mathbf{0},b)$, and so $\mathbf{0}$ is an antisymmetric point. □

Proposition 3.5. *Let $(X,\|\cdot|)$ be an asymmetrically normed real vector space. If $\mathbf{0}$ is antisymmetric point then each point in X is an antisymmetric point in $(X,\rho_{\|\cdot|})$.*

Proof. Suppose that $\mathbf{0}$ is antisymmetric point and $a\in X$. In order to show that a is antisymmetric, let us take $b\in X$. In this case, $a-b\in X$ and so, $\rho_{\|\cdot|}(\mathbf{0},a-b)\neq\rho_{\|\cdot|}(a-b,\mathbf{0})$ since $\mathbf{0}$ is antisymmetric point. That is, $\|b-a|\neq\|a-b|$. Thus, $\rho_{\|\cdot|}(a,b)\neq\rho_{\|\cdot|}(b,a)$, and the point a will be antisymmetric. □

Therefore, with Lemma 3.4 and Proposition 3.5 the following characterization will be clear taking into account Proposition 1.12.

Corollary 3.6. *Let $(X,\|\cdot|)$ be an asymmetrically normed real vector space. The T_0-quasi-metric space $(X,\rho_{\|\cdot|})$ has an antisymmetric point if and only if $(X,\rho_{\|\cdot|})$ is antisymmetric space.*

Finally, by virtue of Corollary 2.9 and Corollary 3.6 we have the next result, trivially.

Corollary 3.7. *If the T_0-quasi-metric space $(X,\rho_{\|\cdot|})$ has an antisymmetric point then $(X,\rho_{\|\cdot|})$ is locally antisymmetrically connected.*

The following equality will be very useful for the remaining of paper.

Lemma 3.8. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. Then $T_{\rho_{\|\cdot|}}(x) = T_{\rho_{\|\cdot|}}(\mathbf{0}) + x$, whenever $x \in X$.*

Proof. First of all, let us take $\rho = \rho_{\|\cdot|}$ for the simplicity in the proof.

Assume that $y \in T_\rho(x)$. Then there exists an antisymmetric path $Q_{x,y} = (x_0, x_1, \ldots, x_n)$ from x to y, where $x_0 = x$, $x_n = y$. Define the path $Q_{\mathbf{0},y-x}$ as $(x_0 - x, x_1 - x, \ldots, x_n - x)$. Then $Q_{\mathbf{0},y-x}$ is an antisymmetric path from $\mathbf{0}$ to $y - x$. Thus $y - x \in T_\rho(\mathbf{0})$. Therefore $y \in T_\rho(\mathbf{0}) + x$ and $T_\rho(x) \subseteq T_\rho(\mathbf{0}) + x$.

For the converse part, let $y \in T_\rho(\mathbf{0}) + x$. Then there exists $t \in T_\rho(\mathbf{0})$ with $y = t + x$. Furthermore, for some $n \in \mathbb{N}$ there is an antisymmetric path $Q_{\mathbf{0},t} = (\mathbf{0}, x_1 \ldots, t)$ from $\mathbf{0}$ to t. Then define $Q_{x,x+t}$ as the path $(x, x + x_1, \ldots, x + t)$. Obviously $Q_{x,x+t}$ is an antisymmetric path from x to $x + t$. Therefore $y = t + x \in T_\rho(x)$ and we established that $T_\rho(\mathbf{0}) + x \subseteq T_\rho(x)$. □

At this stage, we have the following characterizations with the help of Lemma 3.8:

Proposition 3.9. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. In this case, $T_{\rho_{\|\cdot|}}(\mathbf{0})$ is τ_{ρ^s}-open if and only if for each $x \in X$, the component $T_{\rho_{\|\cdot|}}(x)$ is τ_{ρ^s}-open.*

Proof. If $z \in T_{\rho_{\|\cdot|}}(x)$ then $z - x \in T_{\rho_{\|\cdot|}}(\mathbf{0})$ by Lemma 3.8. In addition, since $T_{\rho_{\|\cdot|}}(\mathbf{0})$ is τ_{ρ^s}-open, there exists $\varepsilon > 0$ such that $B_{\rho^s_{\|\cdot|}}(z - x, \varepsilon) \subseteq T_{\rho_{\|\cdot|}}(\mathbf{0})$. Therefore, in a similar manner it is easy to verify that $B_{\rho^s_{\|\cdot|}}(z, \varepsilon) \subseteq T_{\rho_{\|\cdot|}}(x)$ with the help of the fact that $\rho^s_{\|\cdot|}(x,y) = \|x - y\| = \rho_{\|\cdot|^s}(x,y)$. Finally, $T_{\rho_{\|\cdot|}}(x)$ will be τ_{ρ^s}-open.

The converse part is clear. □

Proposition 3.10. *For an asymmetrically normed real vector space $(X, \|\cdot|)$, $T_{\rho_{\|\cdot|}}(\mathbf{0}) = X$ if and only if $T_{\rho_{\|\cdot|}}(x) = X$ for each $x \in X$.*

Proof. Straightforward. □

Incidentally, the following characterization will also be obvious via Proposition 2.3 and Proposition 3.9.

Corollary 3.11. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space. In this case, $T_{\rho_{\|\cdot|}}(\mathbf{0})$ is τ_{ρ^s}-open that is $(X, \rho_{\|\cdot|})$ is locally antisymmetrically connected at the point $\mathbf{0}$ if and only if $(X, \rho_{\|\cdot|})$ is locally antisymmetrically connected.*

Now, let us present the main theorem related to local antisymmetric connectedness, in the context of asymmetrically normed real vector spaces.

Theorem 3.12. *For an asymmetrically normed real vector space $(X, \|\cdot|)$, the T_0-quasi-metric space $(X, \rho_{\|\cdot|})$ is locally antisymmetrically connected if and only if $(X, \rho_{\|\cdot|})$ is antisymmetrically connected.*

Proof. Suppose that $(X, \rho_{\|\cdot|})$ is locally antisymmetrically connected. Now, let us note that for any asymmetric norm $\|\cdot|$, the topology $\tau_{\|\cdot|^s} = \tau_{\rho^s_{\|\cdot|}} = \tau_{\|\cdot\|}$ generated by the symmetrization norm $\|\cdot|^s = \|\cdot| \vee \|\cdot|^{-1} = \|\cdot\|$, will be path-connected.

That is, the (normed) topological space $(X, \tau_{\|\cdot|^s})$ is path-connected, and so the same topological space $(X, \tau_{\rho^s_{\|\cdot|}})$ is connected. In this case, by Proposition 2.7, the T_0-quasi-metric space $(X, \rho_{\|\cdot|})$ will be antisymmetrically connected.

Conversely, the assertion is clear due to Proposition 2.5. □

Let us also recall a crucial proposition proved in [1, Proposition 58], as follows:

Proposition 3.13. *Each asymmetrically normed real vector space that is not normed is antisymmetrically connected.*

Consequently, we may state the next result by Theorem 3.12 and Proposition 3.13:

Corollary 3.14. *Each asymmetrically normed real vector space that is not normed is locally antisymmetrically connected.*

Even if a space with the T_0-quasi-metric induced by an asymmetric norm is locally antisymmetrically connected, its subspace need not be locally antisymmetrically connected in the context of asymmetrically normed real vector spaces. Indeed, we have the following example for this fact, moreover even when the subspace is dense w.r.t. the symmetrization topology on the space.

Example 3.15. *Consider the plane $\mathbb{R}^2$ with the T_0-quasi metric ρ induced by the maximum asymmetric norm $\|(x,y)| = x \vee y \vee 0$ (see Example 3.3* (ii)). It is easy to see that the space $(\mathbb{R}^2, \rho)$, where $\rho = \rho_{\|\cdot|}$, is locally antisymmetrically connected from Corollary 3.14 since $\|\cdot|$ is not a norm.

Now take the subset $C = \{(x, -x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$. It is easy to show that this set is not dense w.r.t. the topology τ_{ρ^s} generated by symmetrization metric $\rho^s = \rho^s_{\|\cdot|}$, which is the Euclidean topology on $\mathbb{R}^2$. Moreover, the subspace (C, ρ_C) is a metric space, and so $T_\rho((a,b)) = \{(a,b)\}$ for all $(a,b) \in C$ since all points of the subset C are symmetric. In addition, the topology $\tau_{\rho^s_C}$ generated by the restricted symmetrization metric $\rho^s_C = (\rho^s)_C$ on C, is homeomorphic to the usual real line topology. Thus, the sets $T_\rho((a,b)) = \{(a,b)\}$ are not open w.r.t. the restricted topology $(\tau_{\rho^s})_C = \tau_{\rho^s_C}$.

Finally, the subspace (C, ρ_C) cannot be locally antisymmetrically connected.

4. Conclusion

After the theory of antisymmetrically connected T_0-quasi-metric spaces has been constructed as the suitable counterpart of connected complementary graphs in graph theory, the authors defined and studied in [5] the localized form of antisymmetric connectedness, in the context of quasi-metrics. According to that various topological characterizations of local antisymmetric connectedness for T_0-quasi-metric spaces are mentioned with the help of metrics, particularly.

Following these ideas, in this paper the theory of local antisymmetric connectedness is investigated first-time in the context of asymmetric norms, as a different approach to the asymmetric structure of a T_0-quasi-metric not metric. Thus, some relationships between the theories of antisymmetric connectedness and local antisymmetric connectedness are discussed through various propositions, results and examples in the environment of asymmetrically normed real vector spaces.

As the future work; it is very natural to observe the following questions.

How does the local antisymmetric connectedness behaves for subspaces, superspaces, products and unions in the context of T_0-quasi-metrics?

Is the image of locally antisymmetrically connected spaces preserved under an isometric isomorphism ?

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

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Universal Journal of Mathematics and Applications, 6 (3) (2023) 106-113
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1339603

Hybrid Numbers with Fibonacci and Lucas Hybrid Number Coefficients

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Article Info

Keywords: *Binet formula, Fibonacci hybrid numbers, Generating function, Lucas hybrid numbers, Matrix method*
2010 AMS: *05A15, 11B39, 11R52*
Received: *8 August 2023*
Accepted: *17 September 2023*
Available online: *18 September 2023*

Abstract

In this paper, we introduce hybrid numbers with Fibonacci and Lucas hybrid number coefficients. We give the Binet formulas, generating functions, exponential generating functions for these numbers. Then we define an associate matrix for these numbers. In addition, using this matrix, we present two different versions of Cassini identity of these numbers.

1. Introduction

Recently, in [1], Özdemir defined the set of hybrid numbers which contains complex, dual and hyperbolic numbers as

$$\mathbb{K}=\left\{a+b\mathbf{i}+c\varepsilon+d\mathbf{h}: a,b,c,d\in\mathbb{R}, \mathbf{i}^2=-1, \varepsilon^2=0, \mathbf{h}^2=1, \mathbf{ih}=-\mathbf{hi}=\varepsilon+\mathbf{i}\right\}.$$

This number system is a generalization of complex $\left(\mathbf{i}^2=-1\right)$, hyperbolic $\left(\mathbf{h}^2=1\right)$ and dual number $\left(\varepsilon^2=0\right)$ systems. Here, $\mathbf{i}$ is complex unit, ε is dual unit and $\mathbf{h}$ is hyperbolic unit. We call these units as hybrid units. In the last few years, researchers from many different fields have taken this number system and used it in various fields of applied sciences. For some applications of hybrid numbers we refer the reader to [2, 3]. There is no doubt that this number system will be studied by other applied science researchers in the near future.
The conjugate of a hybrid number $K=a+b\mathbf{i}+c\varepsilon+d\mathbf{h}$ is defined by

$$\overline{K}=a-b\mathbf{i}-c\varepsilon-d\mathbf{h}.$$

From the definition of hybrid numbers, the multiplication table of the hybrid units is given by the following table:

$\bullet$	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$\mathbf{h}+\mathbf{1}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

Table 1: The Multiplication Table for Hybrid Units

This table shows that the multiplication of hybrid numbers is not commutative. Using the above datas, Özdemir [1] investigated various algebraic and geometric properties of hybrid numbers. For instance, he defined a ring isomorphism between the hybrid number ring $\mathbb{K}$ and the ring of real 2×2 matrices $\mathbb{M}_{2\times 2}$. This map is $\varphi:\mathbb{K}\to\mathbb{M}_{2\times 2}$ where

$$\varphi\left(a+b\mathbf{i}+c\varepsilon+d\mathbf{h}\right)=\begin{bmatrix} a+c & b-c+d \\ c-b+d & a-c \end{bmatrix}. \tag{1.1}$$

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Cite as: E. Polatlı, *Hybrid numbers with Fibonacci and Lucas hybrid number coefficients*, Univers. J. Math. Appl., 6(3) (2023), 106-113.

We refer the reader to [1] for more details and properties about hybrid numbers.
The well-known Fibonacci and Lucas sequences are defined as (for $n \geq 0$)

$$F_{n+2} = F_{n+1} + F_n$$

and

$$L_{n+2} = L_{n+1} + L_n$$

where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$. Note that for $n \geq 1$, $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ and $L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{n+1}$.
In [4], the authors introduced the Fibonacci hybrid numbers and derived some combinatorial properties of these numbers. For $n \geq 0$, they defined the nth Fibonacci hybrid and nth Lucas hybrid numbers as

$$FH_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\varepsilon + F_{n+3}\mathbf{h}$$

and

$$LH_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\varepsilon + L_{n+3}\mathbf{h}$$

where $FH_0 = \mathbf{i} + \varepsilon + 2\mathbf{h}$, $FH_1 = 1 + \mathbf{i} + 2\varepsilon + 3\mathbf{h}$, $LH_0 = 2 + \mathbf{i} + 3\varepsilon + 4\mathbf{h}$ and $LH_1 = 1 + 3\mathbf{i} + 4\varepsilon + 7\mathbf{h}$. They also gave the Binet formulas of these hybrid numbers as

$$FH_n = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta}$$

and

$$LH_n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n,$$

respectively, where $\underline{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\varepsilon + \alpha^3\mathbf{h}$, $\underline{\beta} = 1 + \beta\mathbf{i} + \beta^2\varepsilon + \beta^3\mathbf{h}$, $\alpha = \left(1+\sqrt{5}\right)/2$ and $\beta = \left(1-\sqrt{5}\right)/2$.
Hybrid number sequences have been studied by many researchers. For instance, in [5], Cerda-Morales studied generalized hybrid Fibonacci numbers and their properties. In [6], using an associate matrix, Irmak gave various identities about Fibonacci and Lucas quaternions by matrix methods. In [7], Kızılateş investigated the q-Fibonacci and the q-Lucas hybrid numbers and gave some algebraic properties of these numbers. In [8], the same author introduced the Horadam hybrid polynomials called Horadam hybrinomials. Liana et al. studied Pell hybrinomials in [9]. In [10–15], Liana and Wloch introduced several hybrid number sequences and polynomials and gave various properties of them. In [16], Şentürk et al. studied Horadam hybrid numbers and obtained various properties. In [17], the author examined the ratios of Fibonacci hybrid and Lucas hybrid numbers. Karaca and Yılmaz [18] gave some fundamental definitions and theorems about harmonic complex numbers and harmonic hybrid Fibonacci numbers in detail. Moreover, they examined some algebraic properties such as Binet-like-formula, partial sums related to these sequences.
In this paper, motivated by the above papers, we introduce hybrid numbers with Fibonacci and Lucas hybrid number coefficients. We give the Binet formulas, generating functions, exponential generating functions for these numbers. Then we define an associate matrix for these numbers. Finally, using this matrix, we present two different versions of Cassini identity of these numbers.

2. Main Results

In this section, we define hybrid numbers with Fibonacci and Lucas hybrid number coefficients. Then we give Binet formulas, generating functions, exponential generating functions, and some summation formulas for these numbers.

Definition 2.1. *For $n \geq 0$, the nth term of hybrid number with Fibonacci hybrid number coefficients is given by*

$$\mathbb{F}_n = FH_n + FH_{n+1}\mathbf{i} + FH_{n+2}\varepsilon + FH_{n+3}\mathbf{h}. \tag{2.1}$$

Definition 2.2. *For $n \geq 0$, the nth term of hybrid numbers with Lucas hybrid number coefficients is given by*

$$\mathbb{L}_n = LH_n + LH_{n+1}\mathbf{i} + LH_{n+2}\varepsilon + LH_{n+3}\mathbf{h}. \tag{2.2}$$

Remark 2.3. *If we expand the definitions of $\mathbb{F}_n$ and $\mathbb{L}_n$, we get*

$$\mathbb{F}_n = F_n - F_{n+2} + 2F_{n+3} + F_{n+6} + 2F_{n+1}\mathbf{i} + 2F_{n+2}\varepsilon + 2F_{n+3}\mathbf{h}$$

and

$$\mathbb{L}_n = L_n - L_{n+2} + 2L_{n+3} + L_{n+6} + 2L_{n+1}\mathbf{i} + 2L_{n+2}\varepsilon + 2L_{n+3}\mathbf{h},$$

respectively.

For $n \geq 0$, it is clear that

$$\mathbb{F}_{n+2} = \mathbb{F}_{n+1} + \mathbb{F}_n \tag{2.3}$$

and

$$\mathbb{L}_{n+2} = \mathbb{L}_{n+1} + \mathbb{L}_n. \tag{2.4}$$

Theorem 2.4. *For $n \geq 0$, Binet formulas of hybrid numbers with Fibonacci and Lucas hybrid number coefficients are given by*

$$\mathbb{F}_n = \frac{(\underline{\alpha})^2 \alpha^n - \left(\underline{\beta}\right)^2 \beta^n}{\alpha - \beta} \tag{2.5}$$

and

$$\mathbb{L}_n = (\underline{\alpha})^2 \alpha^n + \left(\underline{\beta}\right)^2 \beta^n, \tag{2.6}$$

respectively, where $\underline{\alpha} = 1 + \alpha \mathbf{i} + \alpha^2 \varepsilon + \alpha^3 \mathbf{h}$, $\underline{\beta} = 1 + \beta \mathbf{i} + \beta^2 \varepsilon + \beta^3 \mathbf{h}$, $\alpha = \left(1+\sqrt{5}\right)/2$ *and* $\beta = \left(1-\sqrt{5}\right)/2$.

Proof. Using the Binet formula of hybrid Fibonacci numbers, we have

$$\begin{aligned}
\mathbb{F}_n &= \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta} + \frac{\underline{\alpha}\alpha^{n+1} - \underline{\beta}\beta^{n+1}}{\alpha - \beta}\mathbf{i} + \frac{\underline{\alpha}\alpha^{n+2} - \underline{\beta}\beta^{n+2}}{\alpha - \beta}\,\varepsilon + \frac{\underline{\alpha}\alpha^{n+3} - \underline{\beta}\beta^{n+3}}{\alpha - \beta}\mathbf{h} \\
&= \frac{\underline{\alpha}\alpha^n \left(1 + \alpha\mathbf{i} + \alpha^2\varepsilon + \alpha^3\mathbf{h}\right) - \underline{\beta}\beta^n \left(1 + \beta\mathbf{i} + \beta^2\varepsilon + \beta^3\,\mathbf{h}\right)}{\alpha - \beta} \\
&= \frac{(\underline{\alpha})^2 \alpha^n - \left(\underline{\beta}\right)^2 \beta^n}{\alpha - \beta}.
\end{aligned}$$

By a similar calculation, we obtain

$$\mathbb{L}_n = (\underline{\alpha})^2 \alpha^n + \left(\underline{\beta}\right)^2 \beta^n$$

□

Theorem 2.5. *The generating functions of hybrid numbers with Fibonacci and Lucas hybrid number coefficients are*

$$F(x) = \sum_{n \geq 0} \mathbb{F}_n x^n = \frac{11 + 7x + 2\mathbf{i} + 2(1+x)\varepsilon + (4+2x)\mathbf{h}}{1 - x - x^2}$$

and

$$L(x) = \sum_{n \geq 0} \mathbb{L}_n x^n = \frac{25 + 15x + 2(1+2x)\mathbf{i} + 2(3+x)\varepsilon + 2(4+3x)\,\mathbf{h}}{1 - x - x^2},$$

recpectively.

Proof. By taking the generating function of both sides of equation (2.1), we directly have

$$\begin{aligned}
\sum_{n \geq 0} \mathbb{F}_n x^n &= \sum_{n \geq 0} FH_n x^n + \left(\sum_{n \geq 0} FH_{n+1} x^n\right)\mathbf{i} + \left(\sum_{n \geq 0} FH_{n+2} x^n\right)\varepsilon + \left(\sum_{n \geq 0} FH_{n+3} x^n\right)\mathbf{h} \\
&= \frac{11 + 7x + 2\mathbf{i} + 2(1+x)\varepsilon + (4+2x)\mathbf{h}}{1 - x - x^2}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\sum_{n \geq 0} \mathbb{L}_n x^n &= \sum_{n \geq 0} LH_n x^n + \left(\sum_{n \geq 0} LH_{n+1} x^n\right)\mathbf{i} + \left(\sum_{n \geq 0} LH_{n+2} x^n\right)\varepsilon + \left(\sum_{n \geq 0} LH_{n+3} x^n\right)\mathbf{h} \\
&= \frac{25 + 15x + 2(1+2x)\mathbf{i} + 2(3+x)\,\varepsilon + 2(4+3x)\mathbf{h}}{1 - x - x^2}.
\end{aligned}$$

□

Theorem 2.6. *For $m, n \in \mathbb{Z}$, generating functions of $\mathbb{F}_{n+m}$ and $\mathbb{L}_{n+m}$ are*

$$\sum_{n \geq 0} \mathbb{F}_{n+m} x^n = \frac{FH_m + FH_{m-1}x}{1 - x - x^2} + \left(\frac{FH_{m+1} + FH_m x}{1 - x - x^2}\right)\mathbf{i} + \left(\frac{FH_{m+2} + FH_{m+1}x}{1 - x - x^2}\right)\varepsilon + \left(\frac{FH_{m+3} + FH_{m+2}x}{1 - x - x^2}\right)\mathbf{h}$$

and

$$\sum_{n \geq 0} \mathbb{L}_{n+m} x^n = \frac{LH_m + LH_{m-1}x}{1 - x - x^2} + \left(\frac{LH_{m+1} + LH_m x}{1 - x - x^2}\right)\mathbf{i} + \left(\frac{LH_{m+2} + LH_{m+1}x}{1 - x - x^2}\right)\varepsilon + \left(\frac{LH_{m+3} + LH_{m+2}x}{1 - x - x^2}\right)\mathbf{h},$$

respectively.

Proof. By the virtue of generating function of Fibonacci hybrid sequence given in [4], we have

$$\begin{aligned}\sum_{n\geq 0}\mathbb{F}_{n+m}x^n &= \sum_{n\geq 0}FH_{n+m}x^n+\left(\sum_{n\geq 0}FH_{n+m+1}x^n\right)\mathbf{i}+\left(\sum_{n\geq 0}FH_{n+m+2}x^n\right)\varepsilon+\left(\sum_{n\geq 0}FH_{n+m+3}x^n\right)\mathbf{h}\\ &= \frac{FH_m+FH_{m-1}x}{1-x-x^2}+\left(\frac{FH_{m+1}+FH_mx}{1-x-x^2}\right)\mathbf{i}+\left(\frac{FH_{m+2}+FH_{m+1}x}{1-x-x^2}\right)\varepsilon+\left(\frac{FH_{m+3}+FH_{m+2}x}{1-x-x^2}\right)\mathbf{h}.\end{aligned}$$

□

Theorem 2.7. *Exponential generating functions of $\mathbb{F}_n$ and $\mathbb{L}_n$ are given by*

$$\sum_{n\geq 0}\mathbb{F}_n\frac{x^n}{n!}=\frac{(\underline{\alpha})^2e^{\alpha x}-\left(\underline{\beta}\right)^2e^{\beta x}}{\alpha-\beta}$$

and

$$\sum_{n\geq 0}\mathbb{L}_n\frac{x^n}{n!}=(\underline{\alpha})^2e^{\alpha x}+\left(\underline{\beta}\right)^2e^{\beta x},$$

respectively.

Proof. Using equation (2.5) and equation (2.6), we get

$$\begin{aligned}\sum_{n\geq 0}\mathbb{F}_n\frac{x^n}{n!} &= \sum_{n\geq 0}\left(\frac{(\underline{\alpha})^2\alpha^n-\left(\underline{\beta}\right)^2\beta^n}{\alpha-\beta}\right)\frac{x^n}{n!}\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}\sum_{n\geq 0}\frac{(\alpha x)^n}{n!}-\frac{\left(\underline{\beta}\right)^2}{\alpha-\beta}\sum_{n\geq 0}\frac{(\beta x)^n}{n!}\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}e^{\alpha x}-\frac{\left(\underline{\beta}\right)^2}{\alpha-\beta}e^{\beta x}\\ &= \frac{(\underline{\alpha})^2e^{\alpha x}-\left(\underline{\beta}\right)^2e^{\beta x}}{\alpha-\beta}\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 0}\mathbb{L}_n\frac{x^n}{n!} &= \sum_{n\geq 0}\left((\underline{\alpha})^2\alpha^n+\left(\underline{\beta}\right)^2\beta^n\right)\frac{x^n}{n!}\\ &= (\underline{\alpha})^2\sum_{n\geq 0}\frac{(\alpha x)^n}{n!}+\left(\underline{\beta}\right)^2\sum_{n\geq 0}\frac{(\beta x)^n}{n!}\\ &= (\underline{\alpha})^2e^{\alpha x}+\left(\underline{\beta}\right)^2e^{\beta x}\end{aligned}$$

as desired. □

Now we give some summation formulas containing $\mathbb{F}_n$ and $\mathbb{L}_n$.

Proposition 2.8. *The following formulas containing $\mathbb{F}_n$ and $\mathbb{L}_n$ are hold:*

(i) $\sum_{k=0}^{n}\mathbb{F}_k=\mathbb{F}_{n+2}-(18+2\mathbf{i}+4\,\varepsilon+6\mathbf{h})$,
(ii) $\sum_{k=0}^{n}\mathbb{L}_k=\mathbb{L}_{n+2}-(40+6\mathbf{i}+8\,\varepsilon+14\mathbf{h})$,
(iii) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{F}_k=\mathbb{F}_{2n}$,
(iv) $\sum_{k=0}^{n}\binom{n}{k}\mathbb{L}_k=\mathbb{L}_{2n}$.

Proof. We give only the proofs of (i) and (iii). The others can be done in a similar way.

(i) From the equation (2.3), we can write the following equations:

$$\begin{aligned}\mathbb{F}_0 &= \mathbb{F}_2-\mathbb{F}_1,\\ \mathbb{F}_1 &= \mathbb{F}_3-\mathbb{F}_2,\\ \mathbb{F}_2 &= \mathbb{F}_4-\mathbb{F}_3,\\ &\vdots\\ \mathbb{F}_{n-1} &= \mathbb{F}_{n+1}-\mathbb{F}_n,\\ \mathbb{F}_n &= \mathbb{F}_{n+2}-\mathbb{F}_{n+1}.\end{aligned}$$

If we add the above equations side by side, then we obtain

$$\begin{aligned}\sum_{k=0}^{n}\mathbb{F}_k &= \mathbb{F}_{n+2}-\mathbb{F}_1\\ &= \mathbb{F}_{n+2}-(18+2\mathbf{i}+4\,\varepsilon+6\mathbf{h}).\end{aligned}$$

(iii) With the help of the equation (2.5) and binomial theorem, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\mathbb{F}_k &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{(\underline{\alpha})^2\alpha^k-\left(\underline{\beta}\right)^2\beta^k}{\alpha-\beta}\right)\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\alpha^k-\frac{\left(\underline{\beta}\right)^2}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\beta^k\\ &= \frac{(\underline{\alpha})^2}{\alpha-\beta}(1+\alpha)^n-\frac{\left(\underline{\beta}\right)^2}{\alpha-\beta}(1+\beta)^n\\ &= \frac{(\underline{\alpha})^2\alpha^{2n}-\left(\underline{\beta}\right)^2\beta^{2n}}{\alpha-\beta} \qquad (\text{since } 1+\alpha=\alpha^2 \text{ and } 1+\beta=\beta^2)\\ &= \mathbb{F}_{2n}.\end{aligned}$$

□

3. A Matrix Approach For Hybrid Numbers with Fibonacci and Lucas Hybrid Number Coefficients

Firstly, let us consider the following matrix:

$$Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}.$$

This Q-matrix was studied by Charles H. King [19] in 1960 for his Master's thesis. It is well-known that

$$Q^n=\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}.$$

In 1963, Hoggatt and Ruggles [20] introduced the following R-matrix:

$$R=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}.$$

It is easily seen that

$$RQ^n=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}=\begin{pmatrix}L_{n+1} & L_n\\ L_n & L_{n-1}\end{pmatrix}.$$

Now, motivated by [6], we define an associate matrix as

$$A=\begin{pmatrix}18+2\mathbf{i}+4\varepsilon+6\mathbf{h} & 11+2\mathbf{i}+2\,\varepsilon+4\mathbf{h}\\ 11+2\mathbf{i}+2\varepsilon+4\mathbf{h} & 7+2\varepsilon+2\mathbf{h}\end{pmatrix}.$$

Then we can easily see that

$$Q^nA=\begin{pmatrix}\mathbb{F}_{n+1} & \mathbb{F}_n\\ \mathbb{F}_n & \mathbb{F}_{n-1}\end{pmatrix} \tag{3.1}$$

and

$$RQ^nA=\begin{pmatrix}\mathbb{L}_{n+1} & \mathbb{L}_n\\ \mathbb{L}_n & \mathbb{L}_{n-1}\end{pmatrix}. \tag{3.2}$$

Theorem 3.1 (First Type of Cassini Identity)**.** *For $n\geq 1$, we have*

$$\mathbb{F}_{n-1}\mathbb{F}_{n+1}-\mathbb{F}_n^2=(-1)^n(1-34\mathbf{i}+12\varepsilon-6\mathbf{h})$$

and

$$\mathbb{L}_{n-1}\mathbb{L}_{n+1}-\mathbb{L}_n^2=5(-1)^{n+1}(1-34\mathbf{i}+12\varepsilon-6\mathbf{h}),$$

respectively.

Proof. By using matrices (3.1) and (3.2), we have

$$\begin{aligned}\mathbb{F}_{n-1}\mathbb{F}_{n+1}-\mathbb{F}_n^2 &= \begin{vmatrix}\mathbb{F}_{n+1} & \mathbb{F}_n\\ \mathbb{F}_n & \mathbb{F}_{n-1}\end{vmatrix}\\ &= |Q^nA|\\ &= (-1)^n\left[(7+2\varepsilon+2\mathbf{h})(18+2\mathbf{i}+4\varepsilon+6\mathbf{h})-(11+2\mathbf{i}+2\varepsilon+4\mathbf{h})^2\right]\\ &= (-1)^n(1-34\mathbf{i}+12\varepsilon-6\mathbf{h})\end{aligned}$$

and

$$\begin{aligned}\mathbb{L}_{n-1}\mathbb{L}_{n+1}-\mathbb{L}_n^2 &= \begin{vmatrix}\mathbb{L}_{n+1} & \mathbb{L}_n\\ \mathbb{L}_n & \mathbb{L}_{n-1}\end{vmatrix}\\ &= |RQ^nA|\\ &= 5(-1)^{n+1}\left[(7+2\varepsilon+2\mathbf{h})(18+2\mathbf{i}+4\varepsilon+6\mathbf{h})-(11+2\mathbf{i}+2\varepsilon+4\mathbf{h})^2\right]\\ &= 5(-1)^{n+1}(1-34\mathbf{i}+12\varepsilon-6\mathbf{h}).\end{aligned}$$

□

Theorem 3.2 (Second Type of Cassini Identity)**.** *For $n \geq 1$, we have*

$$\mathbb{F}_{n+1}\mathbb{F}_{n-1}-\mathbb{F}_n^2 = (-1)^n(1-26\mathbf{i}+28\varepsilon-14\mathbf{h})$$

and

$$\mathbb{L}_{n+1}\mathbb{L}_{n-1}-\mathbb{L}_n^2 = 5(-1)^{n+1}(1-26\mathbf{i}+28\varepsilon-14\mathbf{h}).$$

Proof. Again, by using matrices (3.1) and (3.2), we obtain

$$\begin{aligned}\mathbb{F}_{n+1}\mathbb{F}_{n-1}-\mathbb{F}_n^2 &= (-1)^n\left[(18+2\mathbf{i}+4\varepsilon+6\mathbf{h})(7+2\varepsilon+2\mathbf{h})-(11+2\mathbf{i}+2\,\varepsilon+4\mathbf{h})^2\right]\\ &= (-1)^n(1-26\mathbf{i}+28\varepsilon-14\,\mathbf{h})\end{aligned}$$

and

$$\begin{aligned}\mathbb{L}_{n+1}\mathbb{L}_{n-1}-\mathbb{L}_n^2 &= 5(-1)^{n+1}\left[(18+2\mathbf{i}+4\varepsilon+6\mathbf{h})(7+2\varepsilon+2\mathbf{h})-(11+2\mathbf{i}+2\varepsilon+4\mathbf{h})^2\right]\\ &= 5(-1)^{n+1}(1-26\mathbf{i}+28\varepsilon-14\mathbf{h}),\end{aligned}$$

respectively. □

Now, let us define the conjugate matrix of A as

$$\overline{A} = \begin{pmatrix}18-2\mathbf{i}-4\varepsilon-6\mathbf{h} & 11-2\mathbf{i}-2\,\varepsilon-4\mathbf{h}\\ 11-2\mathbf{i}-2\varepsilon-4\mathbf{h} & 7-2\varepsilon-2\mathbf{h}\end{pmatrix}. \tag{3.3}$$

Thus, using the matrix $\overline{A}$, we can give two types of Cassini identity for the conjugate hybrid numbers with Fibonacci and Lucas hybrid number coefficient respectively. Note that

$$\begin{aligned}\overline{A}Q^n &= \begin{pmatrix}18-2\mathbf{i}-4\varepsilon-6\mathbf{h} & 11-2\mathbf{i}-2\,\varepsilon-4\mathbf{h}\\ 11-2\mathbf{i}-2\varepsilon-4\mathbf{h} & 7-2\varepsilon-2\mathbf{h}\end{pmatrix}\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}\\ &= \begin{pmatrix}\overline{\mathbb{F}}_{n+1} & \overline{\mathbb{F}}_n\\ \overline{\mathbb{F}}_n & \overline{\mathbb{F}}_{n-1}\end{pmatrix}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\overline{A}RQ^n &= \begin{pmatrix}18-2\mathbf{i}-4\varepsilon-6\mathbf{h} & 11-2\mathbf{i}-2\,\varepsilon-4\mathbf{h}\\ 11-2\mathbf{i}-2\varepsilon-4\mathbf{h} & 7-2\varepsilon-2\mathbf{h}\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}\\ &= \begin{pmatrix}\overline{\mathbb{L}}_{n+1} & \overline{\mathbb{L}}_n\\ \overline{\mathbb{L}}_n & \overline{\mathbb{L}}_{n-1}\end{pmatrix}.\end{aligned} \tag{3.5}$$

Theorem 3.3. *For $n \geq 1$, we have*

(i) $\overline{\mathbb{F}}_{n-1}\overline{\mathbb{F}}_{n+1}-\left(\overline{\mathbb{F}}_n\right)^2 = (-1)^n(1+26\mathbf{i}-28\varepsilon+14\mathbf{h})$,
(ii) $\overline{\mathbb{F}}_{n+1}\overline{\mathbb{F}}_{n-1}-\left(\overline{\mathbb{F}}_n\right)^2 = (-1)^n(1+34\mathbf{i}-12\varepsilon+6\mathbf{h})$,
(iii) $\overline{\mathbb{L}}_{n-1}\overline{\mathbb{L}}_{n+1}-\left(\overline{\mathbb{L}}_n\right)^2 = 5(-1)^{n+1}(1+26\mathbf{i}-28\varepsilon+14\mathbf{h})$,
(iv) $\overline{\mathbb{L}}_{n+1}\overline{\mathbb{L}}_{n-1}-\left(\overline{\mathbb{L}}_n\right)^2 = 5(-1)^{n+1}(1+34\mathbf{i}-12\varepsilon+6\mathbf{h})$.

Proof. We give only the proofs of (i) and (iii). The others can be done in a similar way.

(i) By using (3.4), we have

$$
\begin{aligned}
\overline{\mathbb{F}}_{n-1}\overline{\mathbb{F}}_{n+1}-\left(\overline{\mathbb{F}}_{n}\right)^{2} &= \begin{vmatrix} \overline{\mathbb{F}}_{n+1} & \overline{\mathbb{F}}_{n} \\ \overline{\mathbb{F}}_{n} & \overline{\mathbb{F}}_{n-1} \end{vmatrix} \\
&= \left|\overline{A}Q^{n}\right| \\
&= (-1)^{n}\left[(7-2\varepsilon-2\mathbf{h})(18-2\mathbf{i}-4\varepsilon-6\mathbf{h})-(11-2\mathbf{i}-2\varepsilon-4\mathbf{h})^{2}\right] \\
&= (-1)^{n}(1+26\mathbf{i}-28\varepsilon+14\mathbf{h}).
\end{aligned}
$$

(iii) With the help of the (3.5), we obtain

$$
\begin{aligned}
\overline{\mathbb{L}}_{n-1}\overline{\mathbb{L}}_{n+1}-\left(\overline{\mathbb{L}}_{n}\right)^{2} &= \begin{vmatrix} \overline{\mathbb{L}}_{n+1} & \overline{\mathbb{L}}_{n} \\ \overline{\mathbb{L}}_{n} & \overline{\mathbb{L}}_{n-1} \end{vmatrix} \\
&= \left|\overline{A}RQ^{n}\right| \\
&= 5(-1)^{n+1}\left[(7-2\varepsilon-2\mathbf{h})(18-2\mathbf{i}-4\varepsilon-6\mathbf{h})-(11-2\mathbf{i}-2\varepsilon-4\mathbf{h})^{2}\right] \\
&= 5(-1)^{n+1}(1+26\mathbf{i}-28\varepsilon+14\mathbf{h}).
\end{aligned}
$$

□

Remark 3.4. *This paper is revised version of the preprint* [21].

4. Conclusion

In this paper, we have introduced hybrid numbers with Fibonacci and Lucas hybrid number coefficients. We have given the Binet formulas, generating functions, exponential generating functions, some summation formulas for these numbers. Then we have defined an associate matrix for these numbers. Using this matrix, we have given two different versions of Cassini identitiy of these numbers. For the interest of the readers of our paper, the results given here have the potential to motivate further researchers of the subject of the higher order hybrid numbers with Fibonacci and Lucas hybrid number coefficients.

Article Information

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: The article has a single author. The author has read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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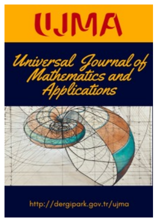
Universal Journal of Mathematics and Applications, 6 (3) (2023) 114-121
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1330866



Conchoidal Surfaces in Euclidean 3-space Satisfying $\Delta x_i = \lambda_i x_i$

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Article Info

Keywords: *Conchoid, Gaussian curvature, Laplace operator, Mean curvature*
2010 AMS: *53C40, 53C42*
Received: *21 July 2023*
Accepted: *18 September*
Available online: *21 September*

Abstract

In this paper, we study the conchodial surfaces in 3-dimensional Euclidean space with the condition $\Delta x_i = \lambda_i x_i$ where Δ denotes the Laplace operator with respect to the first fundamental form. We obtain the classification theorem for these surfaces satisfying under this condition. Furthermore, we have give some special cases for the classification theorem by given the radius function $r(u,v)$ with respect to the parameter u and v.

1. Introduction

The invention of the conchoid was attributed to Greek mathematician Nicomedes by Pappus and other classical authors in the second century BC. Based on the oldest data, the conchoid curve was designed by Nicomedes as a result of the problem of dividing an angle into three equal parts, which has been a problem for many mathematicians for many years. The word conchoid is derived from the Greek word "conch", which means crustacean, and is also referred to as mussel shell shape in the literature.

This curve became a favourite of many mathematicians in the 17th century as an example of new methods in analytical geometry and calculus. For this reason, Newton suggested that it should be treated as a 'standard' curve [1]. In 1837, Pierre Wantzel showed that an arbitrary angle is not divisible by three in the classical way, and therefore conchoid curves were obtained, which can be examples of many curves. The best known of these curves are Hippias' quadratrix curve, Nicomedes' conchoid, Pascal's limachon and cycloid curves.

The conchoid structure is usually best applied to curves in the Euclidean plane $\mathbb{E}^2$ [2]. A conchoid curve is obtained by using a planar curve, a fixed point and a fixed distance. The set of points on the line at a fixed distance from a moving point on a planar curve gives the conchoid of this planar curve [3]. In [2], the concept of a conchoidal curve is generalized to the concept of a conchoidal transformation of two curves, and when one of the two curves is a circle, the conchoidal transformation becomes a classical conchoidal curve. It is known that conchoid curves have many applications. In particular, they have been used in the construction of buildings and structures and are also used in physics, astronomy, optics, electromagnetic research, biology and medical engineering applications(see, [3]- [5]).

The conchoid transformation has been applied to surfaces in Euclidean 3-space in ([6]- [11]) in order to construct new classes of surfaces and making them accessible to the algorithms implemented in CAGD systems. The concept of conchoid surface is also based on the concept of curve and studies on conchoid surfaces of quadrics, conchoid surfaces of sphere, conchoid surfaces of ruled surfaces have been carried out. In addition, in [12] conchoid curves and surfaces in 3-dimensional Euclidean space are considered and the curvatures that determine the geometric properties of these curves and surfaces are calculated. Also in ([13]) the authors computed the types of spacelike conchoid curves in the Minkowski plane and in ([14]) the authors examined the condition which is the conchoidal surface and the surface of revolution given with a conchoid curve to be a Bonnet surface in Euclidean 3-space. The latest studies in Euclidean 3-space is conchoidal twisted surface which isformed by the synchronized anti-symmetric rotation matrix of a planar conchoidal curve ([15]).

This paper is organised as follows: In section 2 we give some basic concepts of the surfaces in $\mathbb{E}^3$ and also surfaces satisfying the condition $\Delta x_i = \lambda_i x_i$. In section 3 we consider conchoidal surfaces in $\mathbb{E}^3$and we gave the results of Gaussian and mean curvature of these surfaces with respect to the given paper in [12]. In the final section we consider conchoidal surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta x_i = \lambda_i x_i$. We obtain the classification theorem for these surfaces satisfying under this condition. Furthermore, we have give some special cases for the classification theorem by given the radius function $r(u,v)$ with respect to the parameter u and v.

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Cite as: B. Bulca, T Dirim, *Conchoidal Surfaces in Euclidean 3-space Satisfying* $\Delta x_i = \lambda_i x_i$, Univers. J. Math. Appl., 6(3) (2023), 114-121.

2. Basic Concepts

2.1. Surfaces in $\mathbb{E}^3$

Let M be a smooth surface in $\mathbb{E}^3$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$. The unit normal vector field or surface normal N is defined by

$$N(u,v) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(u,v)$$

at those points $(u,v) \in D$ at which $X_u \times X_v$ does not vanish, i.e., X is regular.
Let $X : D \subset \mathbb{E}^2 \to \mathbb{E}^3$ be a regular patch. Then the Gaussian curvature and mean curvature of the surface are given by the formulas

$$K = \frac{eg - f^2}{EG - F^2}$$

and

$$H = \frac{eG + gE - 2fF}{2(EG - F^2)}$$

where

$$E = \langle X_u, X_u \rangle,$$
$$F = \langle X_u, X_v \rangle,$$
$$G = \langle X_v, X_v \rangle$$

and

$$e = \langle X_{uu}, N \rangle,$$
$$f = \langle X_{uv}, N \rangle,$$
$$g = \langle X_{vv}, N \rangle$$

are the coefficients of first and second fundamental form of the surface respectively. Recall that a surface M is said to be flat and minimal if its Gaussian curvature and mean curvature vanishes respectively [22, 23].

2.2. Surfaces satisfying $\Delta x_i = \lambda_i x_i$

The definition of submanifolds of finite type was introduced by B.Y. Chen in the late 1970s in order to understand the total mean mean curvature for general Euclidean submanifolds. So, the author introduced the notions of order and type for Euclidean submanifolds. By applying such notions, he introduced the notions of finite type submanifolds an finite type maps. The family of finite-type submanifolds is quite large. The most important and widely known; minimal submanifolds in Euclidean space, minimal submanifolds on hyperspheres and all parallel submanifolds [24].
Let u_i, u_j be a local coordinate system of M. For the array g_{ij} $(i, j = 1, 2)$ consisting of components of the induced metric on M, we denote by $g^{ij} = (g_{ij})^{-1}$ the inverse matrix of the array g_{ij}. Then the Laplacian operator Δ of the induced metric on M is given

$$\Delta = -\frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j} \frac{\partial}{\partial u_i} \left(\sqrt{\det(g_{ij})} g^{ij} \frac{\partial}{\partial u_i} \right).$$

An isometric immersion $x : M \to \mathbb{E}^m$ of a submanifold M in Euclidean m-space $\mathbb{E}^m$ is said to be of finite type if x identified with the position vector field of M in $\mathbb{E}^m$ can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is;

$$x = x_0 + \sum_{i=1}^{k} x_i$$

where x_0 is a constant map, x_1, x_2, ..., x_k non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \le i \le k$. If λ_1, λ_2, ..., λ_k are different, then M is said to be of k-type.
Similarly, a smooth map φ of an n-dimensional Riemannian manifold M of $\mathbb{E}^m$ is said to be of finite type if φ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of Δ (see, [24], [25]).
It is well known the Beltrami formula [24] ;

$$\Delta \overrightarrow{x} = -2\overrightarrow{H}$$

which shows, in particular, that M is minimal surface in $\mathbb{R}^3$ if and only if its coordinate functions are harmonic. Moreover, T. Takahashi [26] states that minimal surfaces and spheres are the only surfaces in $\mathbb{R}^3$ satisfying the condition

$$\Delta \overrightarrow{x} = \lambda \overrightarrow{x}, \quad \lambda \in \mathbb{R}.$$

On the other hand Garay [16] determined the complete surfaces of revolution in $\mathbb{R}^3$ whose component functions are eigenfunctions of their Laplace operator i.e.

$$\Delta x_i = \lambda_i x_i \ \ \lambda_i \in \mathbb{R} \tag{2.1}$$

Later Lopez [17] studied the hypersurfaces in $\mathbb{R}^{n+1}$, Bekkar and Zoubir [18] classified the surfaces of revolution with non zero Gaussian curvature in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski spaces and Bekkar and Senoussi [19] studied the factorable surfaces in the 3-dimensional Euclidean and Minkowski space under the condition (2.1). Also Difi et al. [20] studied the translation-factorable surfaces in 3-dimensional Euclidean and Lorentzian spaces satisfying the condition $\Delta x_i = \lambda_i x_i$. Zoubi et al. [21] gave a classification of surfaces of coordinate finite type in the Lorentz–Minkowski 3-Space.
In this paper we classify the conchoidal surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ satisfying the condition $\Delta x_i = \lambda_i x_i$ where $\lambda_i \in \mathbb{R}$.

3. Conchoidal Surfaces in $\mathbb{E}^3$

In this section some results on conchoid surfaces are given. Gaussian and mean curvatures of conchoid surfaces given in 3-dimensional Euclidean space have been investigated in the paper Bulca et al. [12].
The conchoid surface M_d of a given surface M with respect to a point O is roughly speaking the surface obtained by increasing the radius function of M with respect to O by a constant d. Consider $M \subset \mathbb{R}^3$ be a regular surface, distance $d \in \mathbb{R}$, with respect to a given fixed point $O = (0,0,0) \in \mathbb{R}^3$. Let M be represented by polar representation

$$x(u,v) = r(u,v)\rho(u,v) \tag{3.1}$$

with $\|\rho(u,v)\| = 1$. Taking into account parametrization $\rho(u,v) = (\cos u\cos v, \sin u\cos v, \sin v)$ of the unit sphere S^2, so $\rho(u,v)$ is called spherical part of $x(u,v)$ and $r(u,v)$ its radius function. The conchoidal surface M_d of M at distance d parametrized by

$$x_d(u,v) = (r(u,v) \pm d)\rho(u,v) \tag{3.2}$$

(see, [9]).

Theorem 3.1. *([12])Let M be a regular surface given with the parametrization (3.1). Then the Gaussian and mean curvature of M becomes*

$$K = -\frac{\delta^2(u,v) - \psi(u,v)\xi(u,v)\cos^2 v}{r^2\left((r^2+r_v^2)\cos^2 v + r_u^2\right)^2} \tag{3.3}$$

and

$$H = -\frac{\cos v(r^2+r_v^2)\psi(u,v) + \cos v(r^2\cos^2 v + r_u^2)\xi(u,v) + 2r_u r_v\delta(u,v)}{2r^2\left((r^2+r_v^2)\cos^2 v + r_u^2\right)^{3/2}} \tag{3.4}$$

respectively,where

$$\begin{aligned}
\delta(u,v) &= rr_{uv}\cos v - 2r_u r_v\cos v + rr_u\sin v,\\
\psi(u,v) &= 2r_u^2 + rr_v\sin v\cos v + r^2\cos^2 v - rr_{uu},\\
\xi(u,v) &= 2r_v^2 + r^2 - rr_{vv}
\end{aligned}$$

are the differentiable functions.

Corollary 3.2. *([12])Let M be a regular surface given with the parametrization (3.1).*
i) If the radius function $r(u,v)$ be an u-parameter function then the Gaussian and mean curvature of M

$$\begin{aligned}
K &= \frac{\cos^2 v\left(2r_u^2 + r^2\cos^2 v - rr_{uu}\right) - r_u^2\sin^2 v}{(r^2\cos^2 v + r_u^2)^2},\\
H &= -\frac{\cos v\left(3r_u^2 + 2r^2\cos^2 v - rr_{uu}\right)}{2\left(r^2\cos^2 v + r_u^2\right)^{3/2}}
\end{aligned}$$

ii) If the radius function $r(u,v)$ be a v-parameter function then the Gaussian and mean curvature of M

$$\begin{aligned}
K &= \frac{(r_v\sin v + r\cos v)\left(2r_v^2 + r^2 - rr_{vv}\right)}{r\cos v(r^2+r_v^2)^2},\\
H &= -\frac{(r_v\sin v + r\cos v)\left(r^2+r_v^2\right) + r\cos v\left(2r_v^2 + r^2 - rr_{vv}\right)}{2r\cos v\left(r^2+r_v^2\right)^{3/2}}
\end{aligned}$$

Theorem 3.3. *([12])Let M_d be a conchoidal surface of M given with the parametrization (3.2). Then the Gaussian and mean curvature of M_d becomes*

$$\widetilde{K} = -\frac{\widetilde{\delta}^2(u,v) - \widetilde{\psi}(u,v)\widetilde{\xi}(u,v)\cos^2 v}{(r\pm d)^2\left(((r\pm d)^2 + r_v^2)\cos^2 v + r_u^2\right)^2}$$

and

$$\widetilde{H} = -\frac{\widetilde{\xi}(u,v)\cos v\left((r\pm d)^2\cos^2 v + r_u^2\right) + \widetilde{\psi}(u,v)\cos v\left((r\pm d)^2 + r_v^2\right) + 2r_u r_v\widetilde{\delta}(u,v)}{2(r\pm d)^2\left(((r\pm d)^2 + r_v^2)\cos^2 v + r_u^2\right)^{3/2}}$$

respectively, where

$$\widetilde{\delta}(u,v) = rr_{uv}\cos v - 2r_u r_v \cos v + rr_u \sin v,$$
$$\widetilde{\psi}(u,v) = 2r_u^2 + rr_v \sin v \cos v + r^2\cos^2 v - rr_{uu},$$
$$\widetilde{\xi}(u,v) = 2r_v^2 + r^2 - rr_{vv}$$

are the differentiable functions.

Corollary 3.4. *([12])Let M_d be a regular surface given with the parametrization (3.2).*
i) If the radius function $r(u,v)$ be an u-parameter function then the Gaussian and mean curvature of M_d

$$\widetilde{K} = \frac{\cos^2 v\left(2r_u^2 + (r\pm d)^2\cos^2 v - (r\pm d)r_{uu}\right) - r_u^2\sin^2 v}{((r\pm d)^2\cos^2 v + r_u^2)^2},$$
$$\widetilde{H} = -\frac{\cos v\left(3r_u^2 + 2(r\pm d)^2\cos^2 v - (r\pm d)^2 r_{uu}\right)}{2\left((r\pm d)^2\cos^2 v + r_u^2\right)^{3/2}}$$

ii) If the radius function $r(u,v)$ be a v-parameter function then the Gaussian and mean curvature of M_d

$$\widetilde{K} = \frac{(r_v\sin v + (r\pm d)\cos v)\left(2r_v^2 + (r\pm d)^2 - (r\pm d)r_{vv}\right)}{(r\pm d)\cos v((r\pm d)^2 + r_v^2)^2},$$
$$\widetilde{H} = -\frac{(r_v\sin v + (r\pm d)\cos v)\left((r\pm d)^2 + r_v^2\right) + (r\pm d)\cos v\left(2r_v^2 + (r\pm d)^2 - (r\pm d)r_{vv}\right)}{2(r\pm d)\cos v\left((r\pm d)^2 + r_v^2\right)^{3/2}}.$$

4. Conchoidal Surfaces in Euclidean 3-space Satisfying $\Delta x_i = \lambda_i x_i$

In this section we consider a conchoidal surfaces given with the parametrization (3.2) which is satisfying the condition $\Delta x_i = \lambda_i x_i$. Firstly we consider the polar representation of the surfaces M given with the parametrization (3.1). The coefficients of the first fundamental form and the unit normal vector field of the surface M are:

$$E = r^2\cos^2 v + r_u^2,$$
$$F = r_u r_v,$$
$$G = r^2 + r_v^2,$$

and

$$N = \frac{(r_v\cos u\cos v\sin v + r\cos u\cos^2 v + r_u\sin u, r_v\sin u\cos v\sin v + r\sin u\cos^2 v - r_u\cos u, -r_v\cos^2 v + r\cos v\sin v)}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}}. \tag{4.1}$$

Further, the coefficients of the second fundamental form as follows;

$$e = -\frac{\cos v\left(2r_u^2 + rr_v\sin v\cos v + r^2\cos^2 v - rr_{uu}\right)}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}},$$
$$f = \frac{rr_{uv}\cos v - 2r_u r_v\cos v + rr_u\sin v}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}},$$
$$g = -\frac{\cos v\left(2r_v^2 + r^2 - rr_{vv}\right)}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}}.$$

The Laplacian Δ of M is given by with respect to the Beltrami formula is $\Delta x = -2\overrightarrow{H}$. So if we use this formula we can obtain,

$$\begin{aligned} \Delta x_1 &= -2Hn_1 \\ \Delta x_2 &= -2Hn_2 \\ \Delta x_3 &= -2Hn_3 \end{aligned} \tag{4.2}$$

where H and n_i are defined in (3.4) and (4.1) respectively. If the the polar representation of the surfaces M given with the parametrization (3.1) is constructed with component functions which are eigenfunctions of its Laplacian, we shall have that

$$\begin{aligned} &\Delta\left(r(u,v)\cos u\cos v\right) = \lambda_1 r(u,v)\cos u\cos v \\ &\Delta\left(r(u,v)\sin u\cos v\right) = \lambda_2 r(u,v)\sin u\cos v \\ &\Delta r(u,v)\sin v = \lambda_3 r(u,v)\sin v \end{aligned} \tag{4.3}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Using the equations (4.1),(4.2) and (4.3) we obtain

$$\begin{aligned} &-2H(r_v\cos u\cos v\sin v + r\cos u\cos^2 v + r_u\sin u) = \lambda_1 W\cos u\cos v, \\ &-2H(r_v\sin u\cos v\sin v + r\sin u\cos^2 v - r_u\cos u) = \lambda_2 W\sin u\cos v, \\ &-2H(-r_v\cos^2 v + r\sin v\cos v) = \lambda_3 W\sin v, \end{aligned} \tag{4.4}$$

where

$$W = r\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}$$

We distinguish two special cases according to whether this surface satisfying the condition given by (4.4)
Case 1. For the first case we suppose that the radius function $r(u,v)$ given with the parameter u. So, if the function $r = r(u)$ then the mean curvature of the surface M and the conditions of $\Delta x_i = \lambda_i x_i$ are

$$H = -\frac{\cos v\left(3r_u^2 + 2r^2\cos^2 v - rr_{uu}\right)}{2\left(r^2\cos^2 v + r_u^2\right)^{3/2}},$$

and

$$-2H(r\cos u\cos^2 v + r_u \sin u) = \lambda_1 W\cos u\cos v, \tag{4.5}$$

$$-2H(r\sin u\cos^2 v - r_u\cos u) = \lambda_2 W\sin u\cos v, \tag{4.6}$$

$$-2H(r\cos v) = \lambda_3 W. \tag{4.7}$$

Furthermore, we explore the classification of the surface M given with the parametrization (3.1) satisfying the equation (2.1);

1) Let $\lambda_3 = 0$ then the equation (4.5) gives rise to $H = 0$ which means that the surface is minimal. We get also by the equations (4.5) and (4.6) $\lambda_1 = \lambda_2 = 0$.
2) Let $\lambda_3 \neq 0$, so $H \neq 0$. We get four cases for these condition.

i) If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ then $H \neq 0$. So, from (4.5) we have

$$r\cos u\cos^2 v + r_u\sin u = 0. \tag{4.8}$$

The solution of the differential equation (4.8) we obtain the radius function

$$r(u) = \frac{C_1}{\sqrt{(\sin u)^{\cos 2v+1}}},$$

where C_1 is a real constant.
ii) If $\lambda_1 \neq 0$ and $\lambda_2 = 0$ then $H \neq 0$. So, from (4.6) we have

$$r\sin u\cos^2 v - r_u\cos u = 0. \tag{4.9}$$

The solution of the differential equation (4.9) we obtain the radius function

$$r(u) = \frac{C_2}{\sqrt{\cos(u)^{\cos 2v+1}}}$$

where C_2 is a real constant.
iii) If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Equations (4.5) and (4.6) imply that:

$$r\cos u\cos^2 v + r_u\sin u \neq 0,$$

$$r\sin u\cos^2 v - r_u\cos u \neq 0.$$

Also, the Equations (4.5) and (4.7) imply that,

$$(r\cos u\cos^2 v + r_u\sin u)\lambda_3 = \lambda_1 r\cos u\cos^2 v, \tag{4.10}$$

and the Equations (4.6) and (4.7) imply that;

$$(r\sin u\cos^2 v - r_u\cos u)\lambda_3 = \lambda_2 r\sin u\cos^2 v. \tag{4.11}$$

So, the solution of the differential equations (4.10) and (4.11) we obtain the radius function

$$r(u) = C_3\sqrt{\cos(u)^{\frac{(\cos(2v)+1)(\lambda_2-\lambda_3)}{\lambda_3}}}$$

or

$$r(u) = C_4\sqrt{\sin(u)^{\frac{(\cos(2v)+1)(\lambda_1-\lambda_3)}{\lambda_3}}}$$

where C_3, C_4 are real constants.
iv) If $\lambda_1 = 0$ and $\lambda_2 = 0$ then from the equations (4.5) and (4.6) we get,

$$r\cos u\cos^2 v + r_u\sin u = 0,$$

$$r\sin u\cos^2 v - r_u\cos u = 0$$

The solution of these differential equations we obtain $H = 0$. So this is a contradiction.

Case 2. For the second case we suppose that the radius function $r(u,v)$ given with the parameter v. So, if the function $r=r(v)$ then the mean curvature of the surface M and the conditions of $\Delta x_i=\lambda_i x_i$ are

$$H=-\frac{(r_v\sin v+r\cos v)\left(r^2+r_v^2\right)+r\cos v\left(2r_v^2+r^2-rr_{vv}\right)}{2r\cos v\left(r^2+r_v^2\right)^{3/2}}$$

and

$$-2H(r_v\sin v+r\cos v)=\lambda_1 W \tag{4.12}$$
$$-2H(r_v\sin v+r\cos v)=\lambda_2 W \tag{4.13}$$
$$-2H(-r_v\cos^2 v+r\sin v\cos v)=\lambda_3 W\sin v \tag{4.14}$$

Furthermore, we explore the classification of the surface M given with the parametrization (3.1) satisfying (2.1);

1) Let $\lambda_3=0$. We get two cases for these condition. i) If $\lambda_3=0$ then the equation (4.12) gives rise to $H=0$ which means that the surface is minimal. We get also by the equations (4.12) and (4.13) $\lambda_1=\lambda_2=0$.
   ii) If $-r_v\cos^2 v+r\sin v\cos v=0$ then the solution of this differential equation we obtain the radius function
   $$r(v)=\frac{C_5}{\cos v} \tag{4.15}$$
   where C_5 is a real constant. For the radius function given with (4.15) one can get $H\neq 0$, so we obtain $\lambda_1=\lambda_2=\dfrac{1}{C_5^2}$.
2) Let $\lambda_3\neq 0$, so $H\neq 0$ and $(-r_v\cos^2 v+r\sin v\cos v)\neq 0$. For the equations (4.12) and (4.13) we get $\lambda_1=\lambda_2$ three cases for these condition.
   i) If $\lambda_1=0$ and $\lambda_2\neq 0$ (or $\lambda_1\neq 0$ and $\lambda_2=0$). So, from (4.12) and (4.13) we have
   $$r_v\sin v+r\cos v=0. \tag{4.16}$$
   The solution of the differential equation (4.16) we obtain the radius function
   $$r(v)=\frac{C_6}{\sin v},$$
   where C_6 is a real constant. For this radius function we get $H=0$, so this is a contradiction.
   ii)If $\lambda_1=\lambda_2\neq 0$ Then the Equations (4.12) and (4.14) imply that,
   $$\lambda_3\sin v(r_v\sin v+r\cos v)=\lambda_1\cos v(-r_v\cos v+r\sin v), \tag{4.17}$$
   So, the solution of the differential equation (4.17) we obtain the radius function
   $$r(v)=\frac{\sqrt{2}C_7}{\sqrt{\lambda_3(1-\cos(2v))+\lambda_1(1+\cos(2v))}}.$$
   iii) If $\lambda_1=\lambda_2=0$ then from the equations (4.12) we get $H=0$ or $r_v\sin v+r\cos v=0$.So this is a contradiction.

Theorem 4.1. *Let M be surface given with the parametrization (3.1) in $\mathbb{E}^3$. If the radius function $r(u,v)$ given with the parameter u, then M satisfies $\Delta r_i=\lambda_i r_i$, $(i=1,2,3)$ if and only if the following statements hold:*
i) M has zero mean curvature,
ii) The radius function $r=r(u)$ is

$$r(u)=\frac{C_1}{\sqrt{(\sin u)^{\cos 2v+1}}}\ \textit{or}\ r(u)=\frac{C_2}{\sqrt{\cos(u)^{\cos 2v+1}}},$$

iii) The radius function $r=r(u)$ is

$$r(u)=C_3\sqrt{\cos(u)^{\dfrac{(\cos(2v)+1)(\lambda_2-\lambda_3)}{\lambda_3}}}\quad \textit{or}\ r(u)=C_4\sqrt{\sin(u)^{\dfrac{(\cos(2v)+1)(\lambda_1-\lambda_3)}{\lambda_3}}}.$$

Theorem 4.2. *Let M be surface given with the parametrization (3.1) in $\mathbb{E}^3$. If the radius function $r(u,v)$ given with the parameter v, then M satisfies $\Delta r_i=\lambda_i r_i$, $(i=1,2,3)$ if and only if the following statements hold:*
i) M has zero mean curvature,
ii) The radius function $r=r(v)$ is

$$r(v)=\frac{C_5}{\cos v}\ \textit{or}\ r(v)=\frac{C_6}{\sin v},$$

iii) The radius function $r=r(v)$ is

$$r(v)=\frac{\sqrt{2}C_7}{\sqrt{\lambda_3(1-\cos(2v))+\lambda_1(1+\cos(2v))}}.$$

Using the similar way we obtain the conchoidal surface M_d of M at distance d given with the parametrization (3.2) satisfying the condition $\Delta x_i = \lambda_i x_i$.

Theorem 4.3. *Let M_d be conchodial surface given with the parametrization (3.2) in $\mathbb{E}^3$. If the radius function $r(u,v)$ given with the parameter u, then M_d satisfies $\Delta r_i = \lambda_i r_i$, $(i = 1,2,3)$ if and only if the following statements hold:*
i) M_d has zero mean curvature,
ii) The radius function $r = r(u)$ is

$$r(u) = \pm d + \frac{C_1}{\sqrt{(\sin u)^{\cos 2v+1}}} \text{ or } r(u) = \pm d + \frac{C_2}{\sqrt{\cos(u)^{\cos 2v+1}}},$$

iii) The radius function $r = r(u)$ is

$$r(u) = \pm d + C_3 \sqrt{\cos(u)^{\frac{(\cos(2v)+1)(\lambda_2-\lambda_3)}{\lambda_3}}} \text{ or } r(u) = \pm d + C_4 \sqrt{\sin(u)^{\frac{(\cos(2v)+1)(\lambda_1-\lambda_3)}{\lambda_3}}}.$$

Theorem 4.4. *Let M_d be conchodial surface given with the parametrization (3.2) in $\mathbb{E}^3$. If the radius function $r(u,v)$ given with the parameter v, then M_d satisfies $\Delta r_i = \lambda_i r_i$, $(i = 1,2,3)$ if and only if the following statements hold:*
i) M_d has zero mean curvature,
ii) The radius function $r = r(v)$ is

$$r(v) = \pm d + \frac{C_5}{\cos v} \text{ or } r(v) = \pm d + \frac{C_6}{\sin v},$$

iii) The radius function $r = r(v)$ is

$$r(v) = \pm d + \frac{\sqrt{2}C_7}{\sqrt{\lambda_3(1-\cos(2v)) + \lambda_1(1+\cos(2v))}}.$$

5. Conclusion

In this study, we study the conchodial surfaces in 3-dimensional Euclidean space with the condition $\Delta x_i = \lambda_i x_i$ where Δ denotes the Laplace operator with respect to the first fundamental form. We give a result for this condition for the special cases of radius function $r(u,v)$. In future studies, this problem can be done for the general solution for radius function. It is possible to consider these kind of surfaces in the other spaces or higher dimensional Euclidean spaces.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

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Universal Journal of Mathematics and Applications, 6 (3) (2023) 122-129
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1288015

On the Study of Semilinear Fractional Differential Equations Involving Atangana-Baleanu-Caputo Derivative

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Article Info

Keywords: *Atangana-Baleanu-Caputo derivative, Fractional semi-linear differential equation, Krasnoselskii fixed point theorem, Resolvent family*
2010 AMS: *34A12, 35B50, 32G34*
Received: *26 April 2023*
Accepted: *23 September 2023*
Available online: *24 September 2023*

Abstract

This work aims to study the existing results of mild solutions for a semi-linear Atangana-Baleanu-Caputo fractional differential equation with order $0<\theta<1$ in an arbitrary Banach space. We rely on some arguments to present the mild solution to our problem in terms of an θ-resolvent family. Then we study the existence of this mild solution by using Krasnoselskii's fixed point theorem. Finally, we give an example to prove our results.

1. Introduction

Fractional calculus has been developed intensively since the first conference on this area in 1974 [1]. Then, it gained popularity and significant consideration mainly due to the numerous applications in various fields of applied sciences and engineering. Its purpose is to extend fractional order derivation or integration using non-integer orders. the fractional calculus has been used in mechanics since 1930 and in electrochemistry since 1960. There are some examples of current applications of fractional calculus: fluid circulation, chemical physics, probability and statistics, viscoelasticity, dynamic processes in structures, optics and processing of signals, etc. See [2]- [5]. several mathematicians and physicists studied differential operators and fractional order systems. In [6], the authors studied the existence and uniqueness results to the linear and nonlinear proposed fractional differential equations involving the Atangana–Baleanu fractional derivative. In [7], optimal control for a fractional-order nonlinear mathematical model of cancer treatment is presented and the fractional derivative is defined in the Atangana–Baleanu Caputo sense. The advantage of the ABC-fractional derivative is that it is non-local and has a non-singular kernel. Therefore, it has many applications to demonstrate different problems including the fractional epidemiological model [8], such as, free motion of a coupled oscillator [9], coronavirus and smoking models [10, 11], etc. For more details on the theory of nonlinear ABC-fractional derivative. See [12]- [16].
In what follows, we discuss the existence of the mild solution of the following Atangana-Baleanu-Caputo fractional semi-linear differential equation

$$\begin{cases} {}^{ABC}D^{\theta}_{0_+}\left(w(t)-Q(t,{}^{AB}I^{\theta}_{0^+}w(t))\right)=A\left(w(t)-Q(t,{}^{AB}I^{\theta}_{0^+}w(t))\right)+Y(t,{}^{AB}I^{\theta}_{0^+}w(t),t\in[0,T] \\ w(0)=w_0,\quad w_0\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $0<\theta<1$, ${}^{ABC}D^{\theta}_{0_+}(.)$ is the Atangana-Baleanu-Caputo fractional derivative of order θ, $A:D(A)\subset X\to X$ is the infinitesimal generator of an θ-resolvent family $\{T_\theta(t)\}_{t\geq 0}$, $\{S_\theta(t)\}_{t\geq 0}$ is solution operator defined on the Banach space $(X,\|\,.\,\|)$, $Q\in C(J\times X,X)$, and $Y\in C(J\times X,X)$.
To the best of our knowledge, this is the first time that the problem (1.1) is being studied.

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Cite as: S. Zerbib, K. Hilal, A. Kajouni, *On the study of semilinear fractional differential equations involving Atangana-Baleanu-Caputo derivative*, Univers. J. Math. Appl., 6(3) (2023), 122-129.

2. Preliminaries

This section will be devoted to some definitions and lemmas on which we base ourselves to study our problem.
Let $J=[0,T]$ be a finite interval of $\mathbb{R}$. We denote by $C(J,\mathbb{R})$ the Banach space of continuous functions with the norm $\|\Psi\| = max\{|\Psi(t)| : t \in J\}$.

Definition 2.1. *[16]. Let $m \in [1,\infty)$ and B be an open subset of $\mathbb{R}$, the Sobolev space $H^m(B)$ is defined as*

$$H^m(B) = \left\{\Phi \in L^2(B) : D^{\delta}\Phi \in L^2(B), \forall|\delta| \leq m\right\}.$$

Lemma 2.2. *(Holder Inequality). Let $\Lambda \subset \mathbb{R}$ and $p,q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\Phi \in L^p(\Lambda,\mathbb{R})$, $\varphi \in L^q(\Lambda,\mathbb{R})$, then $\Phi\varphi \in L^1(\Lambda,\mathbb{R})$, and*

$$\| \Phi\varphi \|_{L^1(\Lambda,\mathbb{R})} \leq \| \Phi \|_{L^p(\Lambda,\mathbb{R})} \| \varphi \|_{L^q(\Lambda,\mathbb{R})}.$$

Definition 2.3. *[17]. The left-sided Riemann-Liouville fractional integral of order $n-1<\theta<n$ of a function Φ, such that $n=[\theta]+1$ is given by*

$$I^{\theta}_{0^+}\Phi(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\Phi(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt, z>0.$$

Definition 2.4. *[6]. We define the left-sided Atangana-Baleanu fractional integral of order $0<\theta<1$ of a function Φ, as follows*

$${}^{AB}I^{\theta}_{0^+}\Phi(t) = \frac{1-\theta}{B(\theta)}\Phi(t) + \frac{\theta}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\Phi(s)ds,$$

where $B(\beta) = 1-\beta+\frac{\beta}{\Gamma(\beta)}$ is a normalization function such that $B(0)=B(1)=1$.

Definition 2.5. *[6]. Let $0<\theta<1$ and $\Phi \in H^1(0,T)$. We define the left-sided Atangana-Baleanu fractional derivative of Φ of order θ in Riemann-Liouville sense as follows*

$${}^{ABR}D^{\theta}_{0_+}\Phi(t) = \frac{B(\theta)}{1-\theta}\frac{d}{dt}\int_0^t E_{\theta}\left[-\gamma(t-s)^{\theta}\right]\Phi(s)ds,$$

where $\gamma = \frac{\theta}{1-\theta}$ and E_{θ} is one parameter Mittag-Leffler function defined by [18].

$$E_{\theta}(\lambda) = \sum_{n=0}^{n=\infty}\frac{\lambda^n}{\Gamma(n\theta+1)}.$$

Definition 2.6. *[19]. Let $0<\theta<1$ and $\Phi \in H^1(0,T)$. We define the left-sided Atangana-Baleanu-Caputo fractional derivative of the function Φ of order θ as follows*

$${}^{ABC}D^{\theta}_{0_+}\Phi(t) = \frac{B(\theta)}{1-\theta}\int_0^t E_{\theta}(-\gamma(t-s)^{\theta})\Phi'(s)ds.$$

Lemma 2.7. *[19]. Let $0<\theta<1$, then we have*
${}^{AB}I^{\theta}_{0^+}({}^{ABC}D^{\theta}_{0^+}\Phi(t)) = \Phi(t)-\Phi(0)$.

Definition 2.8. *[20]. We denote by $\rho(A) = \{\beta \in \mathbb{C}; (\beta - A) : D(A) \to X \quad is \quad bijective\}$ the resolvent set. The resolvent $R(\beta,A) := (\beta-A)^{-1}$, $\beta \in \rho(A)$, is a bounded operator on X.*

Definition 2.9. *[20]. We say that A is a sectorial operator if the following conditions satisfies*

i) A is linear and closed operator.
ii) there exist constants $M>0$, $v \in \mathbb{R}$, and $\beta \in [\frac{\pi}{2},\pi]$, such that $\Sigma_{(\beta,v)} = \{\lambda \in \mathbb{C}; \lambda \neq v, |\lambda - v| < \beta\} \subset \rho(A)$
iii) $\| R(\lambda,A) \| \leq \frac{M}{|\lambda-v|}$, $\lambda \in \Sigma_{(\beta,v)}$.

Theorem 2.10. *[21]. Let S be a convex, closed, and nonempty subset of the Banach algebra X. Suppose that $P,F : S \to X$ are two operators such that:*

a) $Pw+Fv \in S$ for all $w,v \in S$.
b) P is a contraction on S.
c) F completely continuous on S.
Then, the operator $w = Pw+Fw$ has a solution in S.

3. Main Results

In this section, we give and prove an existence theorem of the mild solution to the ABC fractional semi-linear differential equation (1.1). The first, we give the following remark on which we will rely to prove our major results.

Remark 3.1. *To give the mild solution of the problem (1.1), we rely on the same arguments that the authors used in [22], [23] to determine the solution to the following Cauchy problem*

$$\begin{cases} {}^{ABC}D_t^{\theta}u(t) = Au(t) + g(t), \quad t \in [0,T], 0 < \theta < 1 \\ u(0) = u_0 \in X. \end{cases} \tag{3.1}$$

The problem (3.1) has a mild solution given by

$$u(t) = \chi T_\theta(t)u_0 + \frac{\chi\varphi(1-\theta)}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}g(s)ds + \frac{\theta\chi^2}{B(\theta)}\int_0^t S_\theta(t-s)g(s)ds,$$

where χ and φ are linear operators such that:
$\chi = \zeta(\zeta I - A)^{-1}$ *and* $\varphi = -\gamma A(\zeta I - A)^{-1}$ *with* $\zeta = \frac{B(\theta)}{1-\theta}$, $\gamma = \frac{\theta}{1-\theta}$, *and*

$$T_\theta(t) = E_\theta(-\varphi t^\theta) = \frac{1}{2\pi i}\int_\Gamma e^{t\tau}\tau^{(\theta-1)}(\tau^\theta I - \varphi)^{-1}d\tau$$

$$S_\theta(t) = t^{\theta-1}E_{\theta,\theta}(-\varphi t^\theta) = \frac{1}{2\pi i}\int_\Gamma e^{t\tau}(\tau^\theta I - \varphi)^{-1}d\tau,$$

where Γ is a certain path lying on $\Sigma_{(\beta,v)}$ and $g \in C(J,X)$. See [24].

Based on the above arguments, we give the following definition

Definition 3.2. *Let $Q \in C(J\times X, X)$, $J = [0,T]$ and $Y \in C(J\times X, X)$. Then the problem (1.1) admits a mild solution given by*

$$\begin{aligned} w(t) &= \chi T_\theta(t)(w_0 - Q(0, {}^{AB}I^\theta w_0)) + \frac{\chi\varphi(1-\theta)}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}Y(s, {}^{AB}I^\theta w(s))ds \\ &\quad + \frac{\theta\chi^2}{B(\theta)}\int_0^t S_\theta(t-s)Y(s, {}^{AB}I^\theta w(s))ds, \end{aligned}$$

where $\chi = \zeta(\zeta I - A)^{-1}$ and $\varphi = -\gamma A(\zeta I - A)^{-1}$ with $\zeta = \frac{B(\theta)}{1-\theta}$ and $\gamma = \frac{\theta}{1-\theta}$.

Lemma 3.3. *[22]. If $A \in A^\theta(\beta_0, v_0)$ then $\| T_\theta(t) \| \le Me^{tv}$ and $\| S_\theta(t) \| \le Ce^{tv}(1+t^{\theta-1})$, for all $t > 0$, $v > v_0$.*

According to the Lemma above if we set $L_1 = \sup_{t\ge 0} \| T_\theta(t) \|$ and $L_2 = \sup_{t\ge 0} Ce^{tv}(1+t^{\theta-1})$. We get $\| T_\theta(t) \| \le L_1$ and $\| S_\theta(t) \| \le t^{\theta-1}L_2$. For more details see [22].
Next, we introduce the following assumptions:
(A_1) Both operators $T_\theta(t)$ and $S_\theta(t)$ are compact operators, $\forall t \in J$.
(A_2) There is a constant δ such that for each $p,q \in X$, and $t \in J$ we have: $|Y(t,p) - Y(t,q)| \le \delta|p-q|$.
(A_3) The function $Y(t,.) : X \to X$ is continuous, for all $t \in J$ and the function $Y(.,p) : X \to X$ is strongly measurable, $\forall p \in X$.
(A_4) There exists a constant $\alpha \in (0,\theta]$ and $h \in L^{\frac{1}{\alpha}}(J,\mathbb{R}^+)$, for all $p \in X$, and $t \in J$ we have

$$|Y(t,p)| \le h(t).$$

Define $S = \{v \in X, \|v\| \le R\}$, such that:

$$R = \| \chi \| \left(L_1(|w_0| + |Q(0, {}^{AB}I^\theta w_0)|) + (\frac{\| \varphi \| (1-\theta)}{|\Gamma(\theta)|} + L_2 \| \chi \|)\frac{T^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]} \right),$$

where $C = \frac{\theta-1}{1-\alpha} \in (-1,0)$.
It is easy to see that S is a convex, closed, and nonempty subset of the Banach algebra X. Define the operators $P : S \to X$ and $F : S \to X$, for each $t \in J$ by:

$$Pw(t) = \chi T_\theta(t)(w_0 - Q(0, {}^{AB}I^\theta w_0)) + \frac{\chi\varphi(1-\theta)}{B(\theta)\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}Y(s, {}^{AB}I^\theta w(s))ds.$$

$$Fw(t) = \frac{\theta\chi^2}{B(\theta)}\int_0^t S_\theta(t-s)Y(s, {}^{AB}I^\theta w(s))ds.$$

We consider the mapping $G : S \to X$ defined by

$$Gw(t) = Pw(t) + Fw(t), \quad t \in J.$$

Theorem 3.4. *If assumptions* (A_1)-(A_4) *hold. Then, the fractional semi-linear differential equation* (1.1) *admits a mild solution* $w \in X$ *provided:*

$$\frac{\| \chi \| \| \varphi \| (1-\theta) T^{\theta}}{|B(\theta)||\Gamma(\theta+1)|} \delta \Theta < 1, \tag{3.2}$$

where

$$\Theta = \frac{(1-\theta)}{B(\theta)} + \frac{T^{\theta}}{\Gamma(\theta+1)B(\theta)}. \tag{3.3}$$

Proof. To prove the Theorem 3.4 is equivalent to proving that the mapping G has a fixed point, we show that the operators P and F satisfy the conditions of the Theorem 2.10.
Before proceeding to the proof of the Theorem 3.4, we will need the following lemma

Lemma 3.5. *If there exists a constant* $\alpha \in (0, \theta]$ *and* $h \in L^{\frac{1}{\alpha}}([0,T], \mathbb{R}^+)$ *such that* $|Y(t,p)| \leq h(t)$ *for all* $p \in X$*, and almost all* $t \in [0,T]$*, we have the following inequality*

$$\int_0^t |(t-s)^{\theta-1} Y(s, {}^{AB}I^{\theta} w(s))| ds \leq \frac{T^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]}.$$

Proof. Assuming that all the conditions of the above lemma are satisfied. By a direct calculation, we get $(t-s)^{\theta-1} \in L^{\frac{1}{1-\alpha}}[0,t]$ for $t \in J$ and $\alpha \in (0,\theta]$. Then by using Lemma 2.2, we have

$$\begin{aligned}
\int_0^t |(t-s)^{\theta-1} Y(s, {}^{AB}I^{\theta} w(s))| ds &\leq \int_0^t |(t-s)^{\theta-1}||h(s)| ds \\
&\leq \left(\int_0^t (t-s)^{\frac{\theta-1}{1-\alpha}} ds \right)^{1-\alpha} \left(\int_0^t |h(s)|^{\frac{1}{\alpha}} ds \right)^{\alpha} \\
&\leq \frac{T^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]},
\end{aligned}$$

where $C = \frac{\theta-1}{1-\alpha} \in (-1,0)$. □

We now move on to continue the proof of the theorem, then the proof is as follows:
Step 1:
Let $w, v \in S$, then for all $t \in [0,T]$ according to the assumptions A_1, A_4, Lemma 3.3, and the Lemma 3.5, we have:

$$\begin{aligned}
&|(Pw(t) + Fv(t))| \\
&= \left| \chi T_{\theta}(t)(w_0 - Q(0, {}^{AB}I^{\theta} w_0)) + \frac{\chi \varphi (1-\theta)}{B(\theta)\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} Y(s, {}^{AB}I^{\theta} w(s)) ds + \frac{\theta \chi^2}{B(\theta)} \int_0^t S_{\theta}(t-s) Y(s, {}^{AB}I^{\theta} v(s)) ds \right| \\
&\leq \| \chi \| \| T_{\theta}(t) \| (|w_0| + |Q(0, {}^{AB}I^{\theta} w_0)|) + \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|} \int_0^t |(t-s)^{\theta-1}||Y(s, {}^{AB}I^{\theta} w(s))| ds \\
&+ \frac{\theta \| \chi^2 \|}{|B(\theta)|} \int_0^t \| S_{\theta}(t-s) \| |Y(s, {}^{AB}I^{\theta} v(s))| ds \\
&\leq \| \chi \| L_1 (|w_0| + |Q(0, {}^{AB}I^{\theta} w_0)|) + \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|} \int_0^t |(t-s)^{\theta-1}||Y(s, {}^{AB}I^{\theta} w(s))| ds + \frac{\| \chi^2 \|}{|B(\theta)|} L_2 \int_0^t |(t-s)^{\theta-1}||Y(s, {}^{AB}I^{\theta} v(s))| ds \\
&\leq \| \chi \| L_1 (|w_0| + |Q(0, {}^{AB}I^{\theta} w_0)|) + \left(\frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|} + \frac{L_2 \| \chi \|^2}{|B(\theta)|} \right) \frac{T^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]} \\
&\leq \| \chi \| \left\{ L_1 (|w_0| + |Q(0, {}^{AB}I^{\theta} w_0)|) + \left(\frac{\| \varphi \| (1-\theta)}{|\Gamma(\theta)|} + L_2 \| \chi \| \right) \frac{T^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]} \right\},
\end{aligned}$$

this gives:

$$\| (Pw(t) + Fv(t)) \| \leq R,$$

this implies, $(Pw(t) + Fv(t)) \in S$ for all $w, v \in S$.
Step 2: P is a contraction on S:
Let $w, v \in S$, then from assumption (A_2), we have

$$\begin{aligned}
|Pw(t) - Pv(t)| &= \left| \frac{\chi \varphi (1-\theta)}{B(\theta)\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} Y(s, {}^{AB}I^{\theta} w(s)) ds - \frac{\chi \varphi (1-\theta)}{B(\theta)\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} Y(s, {}^{AB}I^{\theta} v(s)) ds \right| \\
&\leq \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|} \int_0^t (t-s)^{\theta-1} |Y(s, {}^{AB}I^{\theta} w(s)) - Y(s, {}^{AB}I^{\theta} v(s))| ds \\
&\leq \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|} \delta \int_0^t (t-s)^{\theta-1} |{}^{AB}I^{\theta} w(s)) - {}^{AB}I^{\theta} v(s))| ds.
\end{aligned} \tag{3.4}$$

We have

$$
\begin{aligned}
\left|{}^{AB}I^{\theta}w(s)) - {}^{AB}I^{\theta}v(s))\right| &= \frac{(1-\theta)}{B(\theta)}|w(s)-v(s)| + \frac{\theta}{B(\theta)}\,{}^{RL}I^{\theta}|w(s)-v(s)| \\
&\leq \frac{(1-\theta)}{B(\theta)}|w(s)-v(s)| + \frac{1}{\Gamma(\theta)B(\theta)}\int_0^s (s-\tau)^{\theta-1}|w(\tau)-v(\tau)|d\tau \\
&\leq \| w-v \| \left(\frac{(1-\theta)}{B(\theta)} + \frac{1}{\Gamma(\theta)B(\theta)}\int_0^s (s-\tau)^{\theta-1}d\tau\right) \\
&\leq \| w-v \| \left(\frac{(1-\theta)}{B(\theta)} + \frac{T^{\theta}}{\Gamma(\theta+1)B(\theta)}\right) \\
&\leq \| w-v \| \Theta,
\end{aligned}
$$

where Θ is given by (3.3).
Therefore

$$
\begin{aligned}
(3.4) &\leq \frac{\| \chi \| \| \varphi \| (1-\theta)}{|B(\theta)||\Gamma(\theta)|}\delta\Theta \| w-v \| \int_0^t (t-s)^{\theta-1} \\
&\leq \frac{\| \chi \| \| \varphi \| (1-\theta)T^{\theta}}{|B(\theta)||\Gamma(\theta+1)|}\delta\Theta \| w-v \|,
\end{aligned}
$$

this implies that

$$
\| Pw(t) - Pv(t)) \| \leq \frac{\| \chi \| \| \varphi \| (1-\theta)T^{\theta}}{|B(\theta)||\Gamma(\theta+1)|}\delta\Theta \| w-v \| .
$$

Then, according to the condition (3.2) the operator P is a contraction on S.
Step 3: F is completely continuous:
i) F is continuous.
Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of S such that $w_n \to w$ as $n \to \infty$ in S. We prove that $Fw_n \to Fw$ as $n \to \infty$ in S. By using Lemma 3.3, we get

$$
\begin{aligned}
|Fw_n(t) - Fw(t)| &= \left|\frac{\theta\chi^2}{B(\theta)}\int_0^t S_{\theta}(t-s)Y(s,{}^{AB}I^{\theta}w_n(s))ds - \frac{\theta\chi^2}{B(\theta)}\int_0^t S_{\theta}(t-s)Y(s,{}^{AB}I^{\theta}w(s))ds\right| \\
&\leq \frac{\theta \| \chi^2 \|}{|B(\theta)|}\int_0^t \| S_{\theta}(t-s) \| \, |Y(s,{}^{AB}I^{\theta}w_n(s)) - Y(s,{}^{AB}I^{\theta}w(s))|ds \\
&\leq \frac{\theta \| \chi^2 \|}{|B(\theta)|}L_2\int_0^t (t-s)^{\theta-1}|Y(s,{}^{AB}I^{\theta}w_n(s)) - Y(s,{}^{AB}I^{\theta}w(s))|ds \\
&\leq \frac{\| \chi^2 \|}{|B(\theta)|}T^{\theta}L_2 \sup_{s\in[0,T]}|Y(s,{}^{AB}I^{\theta}w_n(s)) - Y(s,{}^{AB}I^{\theta}w(s))|ds.
\end{aligned}
$$

By using the assumption A_3 and Lebesgue dominated convergence theorem, we get:
$\|Fw_n - Fw\| \to 0$ as $n \to \infty$.
This implies that the operator $F : S \to X$ is continuous.
ii) $F(S) = \{Fw : w \in S\}$ is uniformly bounded.
Using assumptions (A_1) and (A_4), Lemma 3.3, and the Lemma 3.5, for any $w \in S$ and $t \in [0,T]$, we have:

$$
\begin{aligned}
|Fw(t)| &= \left|\frac{\theta\chi^2}{B(\theta)}\int_0^t S_{\theta}(t-s)Y(s,{}^{AB}I^{\theta}w(s))ds\right| \\
&\leq \frac{\| \chi^2 \|}{|B(\theta)|}L_2\int_0^t |(t-s)^{\theta-1}||Y(s,{}^{AB}I^{\theta}v(s))|ds \\
&\leq \frac{L_2 \| \chi \|^2 T^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]} .
\end{aligned}
$$

Therefore,

$$
\| Fw(t) \| \leq \frac{L_2 \| \chi \|^2 T^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}} \| h \|_{L^{\frac{1}{\alpha}}[0,t]},
$$

this proves that $F(S) = \{Fw : w \in S\}$ is uniformly bounded.
iii) $F(S)$ is equicontinuous

Let $t_1,t_2 \in J$ such that $t_1 < t_2$ and $w \in S$, then using assumptions (A_1) and (A_2), we have

$$\begin{aligned}
&|Fw(t_2)-Fw(t_1)|\\
&=\left|\frac{\theta\chi^2}{B(\theta)}\int_0^{t_2} S_\theta(t_2-s)Y(s,{}^{AB}I^\theta w(s))ds-\frac{\theta\chi^2}{B(\theta)}\int_0^{t_1} S_\theta(t_1-s)Y(s,{}^{AB}I^\theta w(s))ds\right|\\
&\leq \frac{\parallel\chi^2\parallel}{|B(\theta)|}\left|\int_0^{t_2}|S_\theta(t_2-s)||Y(s,{}^{AB}I^\theta w(s))|ds-\int_0^{t_1}|S_\theta(t_1-s)||Y(s,{}^{AB}I^\theta w(s))|ds\right|\\
&\leq \frac{\parallel\chi^2\parallel}{|B(\theta)|}L_2\left|\int_0^{t_1}(t_2-s)^{\theta-1}|Y(s,{}^{AB}I^\theta w(s))|ds+\int_{t_1}^{t_2}(t_2-s)^{\theta-1}|Y(s,{}^{AB}I^\theta w(s))|ds-\int_0^{t_1}(t_1-s)^{\theta-1}|Y(s,{}^{AB}I^\theta w(s))|ds\right|\\
&\leq \frac{\parallel\chi^2\parallel}{|B(\theta)|}L_2\left|\int_{t_1}^{t_2}(t_2-s)^{\theta-1}|Y(s,{}^{AB}I^\theta w(s))|ds\right|+\frac{\parallel\chi^2\parallel}{|B(\theta)|}L_2\left|\int_0^{t_1}((t_2-s)^{\theta-1}-(t_1-s)^{\theta-1})|Y(s,{}^{AB}I^\theta w(s))|ds\right|
\end{aligned}$$

If we set

$$I=\left|\int_{t_1}^{t_2}(t_2-s)^{\theta-1}|Y(s,{}^{AB}I^\theta w(s))|ds\right|,$$

$$J=\left|\int_0^{t_1}((t_2-s)^{\theta-1}-(t_1-s)^{\theta-1})|Y(s,{}^{AB}I^\theta w(s))|ds\right|,$$

and by using the arguments of lemma 3.5 we get

$$\begin{aligned}
I&=\left|\int_{t_1}^{t_2}(t_2-s)^{\theta-1}|Y(s,{}^{AB}I^\theta w(s))|ds\right|\\
&\leq \frac{(t_2-t_1)^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}}\parallel h\parallel_{L^{\frac{1}{\alpha}}[0,t]},
\end{aligned}$$

and

$$\begin{aligned}
J&=\left|\int_0^{t_1}((t_2-s)^{\theta-1}-(t_1-s)^{\theta-1})|Y(s,{}^{AB}I^\theta w(s))|ds\right|\\
&\leq \int_0^{t_1}((t_1-s)^{\theta-1}-(t_2-s)^{\theta-1})|h(s)|ds\\
&\leq \left(\int_0^{t_1}((t_1-s)^{\theta-1}-(t_2-s)^{\theta-1})^{\frac{1}{1-\alpha}}ds\right)^{1-\alpha}\left(\int_0^{t_1}|h(s)|^{\frac{1}{\alpha}}ds\right)^{\alpha}\\
&\leq \left(\int_0^{t_1}((t_1-s)^{C}-(t_2-s)^{C})ds\right)^{1-\alpha}\parallel h\parallel_{L^{\frac{1}{\alpha_1}}[0,t]}\\
&\leq \frac{\left(t_1^{1+C}-t_2^{1+C}+(t_2-t_1)^{1+C}\right)^{1-\alpha_1}}{(1+C)^{1-\alpha_1}}\parallel h\parallel_{L^{\frac{1}{\alpha}}[0,t]}\\
&\leq \frac{(t_2-t_1)^{(1+C)(1-\alpha)}}{(1+C)^{1-\alpha}}\parallel h\parallel_{L^{\frac{1}{\alpha}}[0,t]}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
|Fw(t_2)-Fw(t_1)|&\leq \frac{\parallel\chi^2\parallel}{|B(\theta)|}L_2(I+J)\\
&\leq \frac{2L_2\parallel\chi^2\parallel(t_2-t_1)^{(1+C)(1-\alpha)}}{|B(\theta)|(1+C)^{1-\alpha}}\parallel h\parallel_{L^{\frac{1}{\alpha}}[0,t]}.
\end{aligned}$$

Therefore, if $|t_1-t_2|\to 0$ then $|Fw(t_2)-Fw(t_1)|\to 0$, this implies that $F(S)$ is equicontinuous.
According to parts (ii), (iii), and Arzela-Ascoli theorem, it is deduced that $F(S)$ is relatively compact. And according to the part (i) deduce that it is completely continuous.
According to steps 1, 2, and 3, we notice that all the conditions of Theorem 2.10 hold. Then operator G admits a fixed point in S. This proves that the problem (1.1) admits a mild solution in $C(J,X)$. □

4. Example

This section is devoted to an illustrative example that shows the results of this work.
Let $X=L^2([0,1])$, $w(t)=w(.,t)$, $Y(t,{}^{AB}I_{0_+}^{\frac{1}{2}}w(.,t))=\frac{1}{19}(1+\cos({}^{AB}I_{0_+}^{\frac{1}{2}}w(.,t))$, $Q(t,{}^{AB}I_{0_+}^{\frac{1}{2}}w(.,t))=1+t^{AB}I_{0_+}^{\frac{1}{2}}w(.,t))$, $J=[0,1]$.

We Consider the following problem

$$\begin{cases} {}^{ABC}\partial_t^{\frac{1}{2}}\left(w(x,t)-(1+t^{AB}I_{0_+}^{\frac{1}{2}}w(x,t))\right)=\Delta\left(w(x,t)-(1+t^{AB}I_{0_+}^{\frac{1}{2}}w(x,t))\right) \\ +\frac{1}{19}(1+\cos({}^{AB}I_{0_+}^{\frac{1}{2}}w(x,t)), \quad t\in[0,1], x\in[0,1] \\ w(0,t)=w(1,t)=0, \quad t\in[0,1] \\ w(x,0)=w_0(x), \quad x\in[0,1], \end{cases} \tag{4.1}$$

where ${}^{ABC}\partial_t^{\frac{1}{2}}$ is the ABC-fractional partial derivative of order $\frac{1}{2}$, and $A : D(A) \subset X \longrightarrow X$ be an operator defined by

$$D(A) := H^2(0,1)\cap H^1(0,1) \quad \textit{and} \quad Au = \Delta u.$$

The operator A generates a uniformly bounded semi-group $T(t)_{t\geq 0}$ in X. See [20].
Let $v(t) =^{AB} I_{0^+}^{\frac{1}{2}}w(.,t)$, then $Y(t,v(t)) = \frac{1}{19}(1+\cos(v(t))$. It is clear that
$|Y(t,u(t))-Y(t,v(t))| \leq \frac{1}{19}|u-v|$ and $|Y(t,u(t))| \leq 1$. We take $h(t) = 1$, $(\parallel \chi \parallel, \parallel \varphi \parallel \leq 1)$, and $\delta = \frac{1}{19}$. then the assumptions A_2, A_3 and A_4 are satisfied.
Now we check for condition (3.2). We have $T = 1$, $\theta = \frac{1}{2}$, then after some calculations, we find

$$\begin{aligned} \frac{\parallel \chi \parallel\parallel \varphi \parallel (1-\theta)T^{\theta}}{|B(\theta)||\Gamma(\theta_1+1)|}\delta\Theta &\leq \frac{1-\frac{1}{2}}{19B(\frac{1}{2})\Gamma(\frac{1}{2}+1)}\left(\frac{1-\frac{1}{2}}{B(\frac{1}{2})}+\frac{1}{B(\frac{1}{2})\Gamma(\frac{1}{2}+1)}\right) \\ &\leq \frac{1}{19B(\frac{1}{2})\Gamma(\frac{3}{2})}\left(\frac{1}{B(\frac{1}{2})}+\frac{1}{B(\frac{1}{2})\Gamma(\frac{3}{2})}\right) \\ &\simeq 0,114 < 1. \end{aligned}$$

Then from the results above, deduce that the ABC-fractional semi-linear differential problem (4.1) has a mild solution w in $C([0,1]\times[0,1],X)$.

5. Conclusion

In this paper, we have studied the existence of the mild solutions of a fractional semi-linear differential equation involving Atangana-Baleanu-Caputo fractional derivative with order $0 < \theta < 1$ by using the Krasnoselskii fixed point theorem. In the end, an illustrative example is presented to demonstrate our results.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Not applicable.

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