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BIPOLAR SOFT CONTINUITY ON BIPOLAR SOFT TOPOLOGICAL SPACES

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Abstract. The amazing idea of soft sets was first claimed by Molodtsov [18], a new mathematical tool for dealing with uncertainties free from the other theories' limitations. After the advent of soft set theory, bipolar soft sets, as a generalization of soft sets, a new model of uncertain information, were introduced by Shabir and Naz [21]. The primary purpose of this paper is to introduce and investigate the structures of bipolar soft continuity, bipolar soft openness, bipolar soft closedness and bipolar soft homeomorphism.

1. Introduction

Mainly since the problems in many vital areas of our lives, such as economics, environment, and engineering, cannot be solved due to the inherent difficulties of conventional methods, many theories have been put forward to combat these problems. In 1999, Molodtsov [18] introduced a practical theory called soft set theory which is free from the other theories. The amazing idea of soft sets was given as a new mathematical tool for dealing with uncertainties free from the other theories' limitations. At present, many studies on soft set theory have been carried out different areas by some researchers [1, 5, 6, 9, 13, 16, 2, 3].

After the advent of soft set theory, bipolar soft sets, a new model of uncertain information, were introduced by Shabir and Naz [21]. It is known that the structure of a bipolar soft set consists of two mappings. Since these mappings explain both positive information and opposite approximation, the idea of the bipolar soft set has recently gained momentum among many researchers. Aslam et al. [4] combined the concept of a bipolar fuzzy set and a soft set. In addition, they introduced the notion of bipolar fuzzy soft set and studied fundamental properties. Naz and Shabir [22] gave algebraic structures of bipolar fuzzy soft sets. Hayat et al. [14] applied the concept of bipolar soft sets to hemirings. Karaaslan and Karataş [15] redefined the idea of bipolar soft set and bipolar soft set operations as more functional than Shabir

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and Naz's definition and operations. In 2017, Shabir and Bakhtawar [23] employed the notion of bipolar soft sets to give the concept of bipolar soft topological spaces, which are extensions of soft topological space. They gave some new important structures in bipolar soft topological spaces, such as bipolar soft connected spaces, bipolar soft disconnected spaces and bipolar soft compact spaces. Since the concept of bipolar soft topology has great importance, the topological structures of bipolar soft sets have been studied by many authors [7, 19, 20]. Gunduz et al. [10] recently defined a new bipolar soft point. By using the bipolar soft point, Gunduz et al. [12] examined some important properties of bipolar soft functions.

Since functions are one of the most critical concepts in mathematics, they have many applications. Many studies have been done on soft functions in soft topological spaces in [11, 17]. The concept of bipolar soft functions was defined in [8] as the generalization of soft functions and given the notion of bipolar soft image and inverse image.

The primary purpose of this paper is to introduce and investigate the structures of bipolar soft continuity, bipolar soft openness, bipolar soft closedness and bipolar soft homeomorphism and show these examples.

2. Preliminaries

Throughout this section, the symbols U, $\widetilde{E} = E \cup \neg E$ and $P(U)$ denote the initial universe, a set of parameters, and the power set of U, respectively.

Definition 2.1. [16] *Let $E = \{e_i : i = 1, 2, ..., n\}$ be a set of parameters. The not set of E, denoted by $\neg E$, is defined by $\neg E = \{\neg e_i : i = 1, 2, ..., n\}$, where $\neg e_i =$ not e_i for all i.*

Definition 2.2. [21] *A bipolar soft set $\left(F, \widetilde{E}\right)$ on U is defined as*

$$F_{\widetilde{E}} = \{(e_i, F(e_i), F(\neg e_i)) : e_i \in E\},$$

where $F : \widetilde{E} \to P(U)$ such that $F(e) \cap F(\neg e) = \varnothing$, for each $e \in E$.

In this paper, a bipolar soft set denoted by $F_{\widetilde{E}}$ instead of $\left(F, \widetilde{E}\right)$. The collection of all bipolar soft sets on U is denoted by $BS\left(U_{\widetilde{E}}\right)$.

Definition 2.3. [21] *Let $F_{\widetilde{E}}, G_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then,*

1. $F_{\widetilde{E}} \widetilde{\subseteq} G_{\widetilde{E}}$, if $F(e) \subset G(e)$ and $G(\neg e) \subset F(\neg e)$, for each $e \in E$.
2. $F_{\widetilde{E}} = G_{\widetilde{E}}$, if $F_{\widetilde{E}} \widetilde{\subseteq} G_{\widetilde{E}}$ and $G_{\widetilde{E}} \widetilde{\subseteq} F_{\widetilde{E}}$.
3. $F_{\widetilde{E}} \widetilde{\cup} G_{\widetilde{E}} = H_{\widetilde{E}}$ where $H(e) = (F \cup G)(e) = F(e) \cup G(e)$ and $H(\neg e) = (F \cup G)(\neg e) = F(\neg e) \cap G(\neg e)$, for each $e \in E$.
4. $F_{\widetilde{E}} \widetilde{\cap} G_{\widetilde{E}} = Z_{\widetilde{E}}$ where $Z(e) = (F \cap G)(e) = F(e) \cap G(e)$ and $Z(\neg e) = (F \cap G)(\neg e) = F(\neg e) \cup G(\neg e)$, for each $e \in E$.

Definition 2.4. [21] *The bipolar soft complement of $F_{\widetilde{E}}$, denoted by $F^c_{\widetilde{E}}$, where $F^c : \widetilde{E} \to P(U)$ is a mapping defined by $F^c(e) = F(\neg e)$ and $F^c(\neg e) = F(e)$, for each $e \in E$.*

Definition 2.5. [21] *If $F(e) = \varnothing$ and $F(\neg e) = U$ for each $e \in E$,$\Phi_{\widetilde{E}}$ is called a null bipolar soft set. Also, if $F(e) = U$ and $F(\neg e) = \varnothing$ for each $e \in E$, $\widetilde{U}_{\widetilde{E}}$ is called an absolute bipolar soft set.*

Symbolically, $\Phi_{\widetilde{E}} = \{(e_i, \varnothing, U) : e_i \in E\}$ and $\widetilde{U}_{\widetilde{E}} = \{(e_i, U, \varnothing) : e_i \in E\}$.

Definition 2.6. [23] *Let $\widetilde{\tau} \subset BS\left(U_{\widetilde{E}}\right)$. $\widetilde{\tau}$ is said to be a bipolar soft topology on U, if $\widetilde{\tau}$ confirms the following conditions:*

(1) $\Phi_{\widetilde{E}}, \widetilde{U}_{\widetilde{E}} \in \widetilde{\tau}$.

(2) The union of any number of bipolar soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

(3) The intersection of any two bipolar soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

$\left(U, \widetilde{\tau}, \widetilde{E}\right)$ is called bipolar soft topological space and the family of all bipolar soft topological space on U denoted as $BSTS$.

Definition 2.7. [23] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a bipolar soft topological space on U and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. $F_{\widetilde{E}}$ is called*

(1) a bipolar soft open set, if it belongs to $\widetilde{\tau}$.

(2) a bipolar soft closed set, if $F^c_{\widetilde{E}}$ belongs to $\widetilde{\tau}$.

Definition 2.8. [7] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$ and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, the bipolar soft interior of $F_{\widetilde{E}}$, denoted by $F^{\circ}_{\widetilde{E}}$, is the union of all bipolar soft open subsets of $F_{\widetilde{E}}$.*

Definition 2.9. [7] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$ and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, the bipolar soft closure of $F_{\widetilde{E}}$, denoted by $\overline{F_{\widetilde{E}}}$, is the intersection of all bipolar soft closed sets containing $F_{\widetilde{E}}$.*

Theorem 2.1. [7] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$ and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, $\left[\overline{F_{\widetilde{E}}}\right]^c = \left(F^c_{\widetilde{E}}\right)^{\circ}$.*

Theorem 2.2. [7] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$ and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, $\overline{\left[F^c_{\widetilde{E}}\right]} = \left[F^{\circ}_{\widetilde{E}}\right]^c$.*

3. Continuous functions on bipolar soft topological spaces

We mention, in this section, some important concepts, such as bipolar soft image and bipolar soft pre-image for the bipolar soft sets on U whose set of parameters is a subset of E. In addition, we recall the relationship between the image and the inverse image of bipolar soft sets. This is followed by the definition of bipolar soft continuous function associated with some of its results. Later, we will give a detailed investigation of bipolar soft continuous functions.

E and E', respectively, stand for the sets of parameters of U and V; $\varnothing \neq E_1, E_2, E_3 \subset E$ and $\varnothing \neq E'_1, E'_2 \subset E'$.

Definition 3.1. [8] *Let $f : U \to V$ be an injective function, $\varphi : E \to E'$ and $\vartheta : \neg E \to \neg E'$ be two functions where $\vartheta(\neg e) = \neg\varphi(e)$, for all $\neg e \in \neg E$. Then, $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ is called a bipolar soft function.*

Definition 3.2. [8] *Let $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ be a bipolar soft function and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, the image of $F_{\widetilde{E}}$ under $\psi_{f\varphi\vartheta}$,*

$$\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = \left(\psi_{f\varphi\vartheta}\left(F(e)\right), \psi_{f\varphi\vartheta}\left(F(\neg e)\right), E'\right)$$

is defined as follows: for all $e' \in E'$,

$$\psi_{f\varphi\vartheta}(F)(e') = \begin{cases} f\left(\underset{e\in\ \varphi^{-1}(e')\cap E_1}{\cup} F(e)\right), & \text{if } \varphi^{-1}(e')\cap E_1 \neq \varnothing, \\ \varnothing, & \text{otherwise.} \end{cases}$$

and

$$\psi_{f\varphi\vartheta}(F)(\neg e') = \begin{cases} f\left(\underset{\neg e\in\ \vartheta^{-1}(\neg e')\cap\neg E_1}{\cap} F(\neg e)\right), & \text{if } \vartheta^{-1}(\neg e')\cap \neg E_1 \neq \varnothing, \\ V, & \text{otherwise.} \end{cases}$$

Remark. *The condition that the function f is an injective function is essential.*

Proposition 3.1. *Let $\psi_{f\varphi\vartheta} : BS\left(U_{\tilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ be a bipolar soft function and $F_{\widetilde{E}} \in BS\left(U_{\tilde{E}}\right)$. Then, $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)$ is a bipolar soft set in $BS\left(V_{\widetilde{E'}}\right)$.*

Proof. For all $e' \in E'$,

$$\begin{aligned} \psi_{f\varphi\vartheta}(F)(e') \cap \psi_{f\varphi\vartheta}(F)(\neg e') &= f\left(\underset{e\in\ \varphi^{-1}(e')\cap E}{\cup} F(e)\right) \cap f\left(\underset{\neg e\in\ \vartheta^{-1}(\neg e')\cap\neg E}{\cap} F(\neg e)\right) \\ &= \varnothing \end{aligned}$$

Then, $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)$ is a bipolar soft set in $BS\left(V_{\widetilde{E'}}\right)$. □

Definition 3.3. [8] *Let $\psi_{f\varphi\vartheta} : BS\left(U_{\tilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ be a bipolar soft function. Then,*

(1) If f and φ are surjective functions, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft surjective function.
(2) If f and φ are injective functions, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft injective function.
(3) If f and φ are bijective functions, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft bijective function.

Remark. [8] *It is clear that $\psi_{f\varphi\vartheta}$ is a bipolar soft surjective if and only if $\psi_{f\varphi\vartheta}\left(\widetilde{U}_{\widetilde{E}}\right) = \widetilde{V}_{\widetilde{E'}}$.*

Remark. [8] *If the bipolar soft sets have the same set of parameters, for each $F_{\widetilde{E_1}}, G_{\widetilde{E_1}} \in BS\left(U_{\tilde{E}}\right)$, when $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right) = \psi_{f\varphi\vartheta}\left(G_{\widetilde{E_1}}\right)$, we obtain $F_{\widetilde{E_1}} = G_{\widetilde{E_1}}$, i.e. $\psi_{f\varphi\vartheta}$ is a bipolar soft injective function.*

Theorem 3.2. [8] *Let $\psi_{f\varphi\vartheta} : BS\left(U_{\tilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ be a bipolar soft function. If $F_{\widetilde{E_1}}, G_{\widetilde{E_2}} \in BS\left(U_{\tilde{E}}\right)$, then*

(1) $\psi_{f\varphi\vartheta}\left(\Phi_{\widetilde{E}}\right) \widetilde{\supseteq} \left(\Phi_{\widetilde{E'}}\right)$. If f is surjective, then the equality holds.
(2) $\psi_{f\varphi\vartheta}\left(\widetilde{U}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\widetilde{V}_{\widetilde{E'}}\right)$.
(3) $F_{\widetilde{E_1}} \widetilde{\subseteq} G_{\widetilde{E_2}} \Rightarrow \psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right) \subseteq \psi_{f\varphi\vartheta}\left(G_{\widetilde{E_2}}\right)$.
(4) $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}} \widetilde{\cup} G_{\widetilde{E_2}}\right) = \psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right) \widetilde{\cup}\, \psi_{f\varphi\vartheta}\left(G_{\widetilde{E_2}}\right)$.
(5) $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}} \widetilde{\cap} G_{\widetilde{E_2}}\right) \subseteq \psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right) \widetilde{\cap}\, \psi_{f\varphi\vartheta}\left(G_{\widetilde{E_2}}\right)$. If $\psi_{f\varphi\vartheta}$ is a bipolar soft injective function, then the equality holds.

Definition 3.4. [8] *Let* $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ *be a bipolar soft function. The inverse image of the bipolar soft set* $H_{\widetilde{E_1'}} \in BS\left(V_{\widetilde{E'}}\right)$ *under* $\psi_{f\varphi\vartheta}$,

$$\psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}}\right) = \left(\psi_{f\varphi\vartheta}^{-1}\left(H\left(e\right)\right), \psi_{f\varphi\vartheta}^{-1}\left(H\left(\neg e\right)\right), E\right)$$

is given as follows: for all $e \in E$,

$$\begin{aligned} \psi_{f\varphi\vartheta}^{-1}\left(H\left(e\right)\right) &= \begin{cases} f^{-1}\left(H\left(\varphi\left(e\right)\right)\right), & \text{if } \varphi\left(e\right) \in E_1', \\ \varnothing, & \text{if } \varphi\left(e\right) \notin E_1'. \end{cases} \\ \psi_{f\varphi\vartheta}^{-1}\left(H\left(\neg e\right)\right) &= \begin{cases} f^{-1}\left(H\left(\vartheta\left(\neg e\right)\right)\right), & \text{if } \vartheta\left(\neg e\right) \in \neg E_1', \\ U, & \text{if } \vartheta\left(\neg e\right) \notin \neg E_1'. \end{cases} \end{aligned}$$

Proposition 3.3. *Let* $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ *be a bipolar soft function and* $H_{\widetilde{E_1'}} \in BS\left(V_{\widetilde{E'}}\right)$. *Then,* $\psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}}\right)$ *is a bipolar soft set in* $BS\left(U_{\widetilde{E}}\right)$.

Proof. For all $e \in E$,

$$\begin{aligned} \psi_{f\varphi\vartheta}^{-1}\left(H\left(e\right)\right) \cap \psi_{f\varphi\vartheta}^{-1}\left(H\left(\neg e\right)\right) &= f^{-1}\left(H\left(\varphi\left(e\right)\right)\right) \cap f^{-1}\left(H\left(\vartheta\left(\neg e\right)\right)\right) \\ &= f^{-1}\left(\left(H\left(\varphi\left(e\right)\right) \cap H\left(\vartheta\left(\neg e\right)\right)\right)\right) \\ &= f^{-1}\left(\varnothing\right) \\ &= \varnothing \end{aligned}$$

Thus, $\psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}}\right)$ is a bipolar soft set in $BS\left(U_{\widetilde{E}}\right)$. □

Remark. *Although the image of the inverse image of a set for any classical function is a subset of this set, this is not true in bipolar soft functions. A condition must be added to enable this property.*

Theorem 3.4. [8] *Let* $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ *be a bipolar soft function. If* $H_{\widetilde{E_1'}}, Q_{\widetilde{E_2'}} \in BS\left(V_{\widetilde{E'}}\right)$, *then*

(1) $\psi_{f\varphi\vartheta}^{-1}\left(\widetilde{V}_{\widetilde{E'}}\right) = \widetilde{U}_{\widetilde{E}}$.
(2) $\psi_{f\varphi\vartheta}^{-1}\left(\Phi_{\widetilde{E'}}\right) = \Phi_{\widetilde{E}}$.
(3) $H_{\widetilde{E_1'}} \widetilde{\subseteq} Q_{\widetilde{E_2'}} \Rightarrow \psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}}\right) \widetilde{\subseteq} \psi_{f\varphi\vartheta}^{-1}\left(Q_{\widetilde{E_2'}}\right)$.
(4) $\psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}} \widetilde{\cup} Q_{\widetilde{E_2'}}\right) = \psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}}\right) \widetilde{\cup} \psi_{f\varphi\vartheta}^{-1}\left(Q_{\widetilde{E_2'}}\right)$.
(5) $\psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}} \widetilde{\cap} Q_{\widetilde{E_2'}}\right) = \psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}}\right) \widetilde{\cap} \psi_{f\varphi\vartheta}^{-1}\left(Q_{\widetilde{E_2'}}\right)$.
(6) $\psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}}^{c}\right) = \left(\psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E_1'}}\right)\right)^{c}$.

Now, we consider the relationships between the image and inverse image of bipolar soft sets.

Theorem 3.5. [8] *Let* $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ *be a bipolar soft function,* $F_{\widetilde{E_1}} \in BS\left(U_{\widetilde{E}}\right)$ *and* $H_{\widetilde{E'}} \in BS\left(V_{\widetilde{E'}}\right)$. *Then,*

(1) $F_{\widetilde{E_1}} \widetilde{\subseteq}\ \psi_{f\varphi\vartheta}^{-1}\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right)\right)$. If $E_1 = E$ and $\psi_{f\varphi\phi}$ is a bipolar soft injective function, then the equality holds.

(2) If f is a surjective function, then $\psi_{f\varphi\vartheta}\left(\psi_{f\varphi\vartheta}^{-1}H_{\widetilde{E'}}\right)\widetilde{\subseteq}H_{\widetilde{E'}}$. If $\psi_{f\varphi\vartheta}$ is a bipolar soft surjective function, the equality holds.

Now, we consider the concept of bipolar soft point followed by some relations between them.

Definition 3.5. [10] *A bipolar soft subset $F_{\widetilde{E}}$ of $\widetilde{U}_{\widetilde{E}}$ is called a bipolar soft point if there exist $x, y \in U$,($x \neq y$ need not be true) $e \in E$ satisfying*

$$x_e^y(e_1) = \begin{cases} \varnothing, & \text{if } e \neq e_1, \\ \{x\}, & \text{if } e = e_1. \end{cases}$$

and

$$x_e^y(\neg e_1) = \begin{cases} U, & \text{if } e \neq e_1, \\ U - \{x, y\}, & \text{if } e = e_1. \end{cases}$$

The bipolar soft point will be shortly denoted by x_e^y.

Definition 3.6. *Let x_e^y and $x_{1e_1}^{y_1}$ be two bipolar soft points over U. Then, x_e^y and $x_{1e_1}^{y_1}$ are called different bipolar soft points, if $x \neq x_1$ or $e \neq e_1$.*

Definition 3.7. [20] *Let x_e^y be a bipolar soft point and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. We said that x_e^y belongs to the bipolar soft set $F_{\widetilde{E}}$, denoted by $x_e^y \widetilde{\in} F_{\widetilde{E}}$, if $x_e^y(e) = \{x\} \subset F(e)$ and $x_e^y(\neg e) \supset F(\neg e)$.*

Remark. *Every bipolar soft set can be written as a union of its bipolar soft points.*

Example 3.1. *Let $U = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2\}$ and*

$$F_{\widetilde{E}} = \{(e_1, \{x_1, x_2\}, \{x_4\}), (e_2, \{x_2, x_3\}, \{x_1, x_4\})\}.$$

Then, we can write $F_{\widetilde{E}}$ as a union of some bipolar soft points. Indeed, for $e_1, e_2 \in E$,

$$\begin{aligned} F(e_1) &= \left(x_{1e_1}^{x_2} \cup x_{1e_1}^{x_3} \cup x_{2e_1}^{x_1} \cup x_{2e_1}^{x_3}\right)(e_1), \\ F(\neg e_1) &= \left(x_{1e_1}^{x_2} \cap x_{1e_1}^{x_3} \cap x_{2e_1}^{x_1} \cap x_{2e_1}^{x_3}\right)(\neg e_1), \\ F(e_2) &= \left(x_{2e_2}^{x_3} \cup x_{3e_2}^{x_2}\right)(e_2), \\ F(\neg e_2) &= \left(x_{2e_2}^{x_3} \cap x_{3e_2}^{x_2}\right)(\neg e_2), \end{aligned}$$

where

$$\begin{aligned} x_{1e_1}^{x_2}(e_1) &= \{x_1\},\ x_{1e_1}^{x_2}(\neg e_1) = \{x_3, x_4\}, \\ x_{1e_1}^{x_3}(e_1) &= \{x_1\},\ x_{1e_1}^{x_3}(\neg e_1) = \{x_2, x_4\}, \\ x_{2e_1}^{x_1}(e_1) &= \{x_2\},\ x_{2e_1}^{x_1}(\neg e_1) = \{x_3, x_4\}, \\ x_{2e_1}^{x_3}(e_1) &= \{x_2\},\ x_{2e_1}^{x_3}(\neg e_1) = \{x_1, x_4\}, \\ x_{2e_2}^{x_3}(e_2) &= \{x_2\},\ x_{2e_2}^{x_3}(\neg e_2) = \{x_1, x_4\}, \\ x_{3e_2}^{x_2}(e_2) &= \{x_3\},\ x_{3e_2}^{x_2}(\neg e_2) = \{x_1, x_4\}. \end{aligned}$$

Definition 3.8. [20] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$, $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$ and x_e^y be a bipolar soft point in U. Then, $F_{\widetilde{E}}$ is said to be a bipolar soft neighbourhood of x_e^y, if there exists a bipolar soft open set $G_{\widetilde{E}}$ such that $x_e^y \widetilde{\in} G_{\widetilde{E}} \widetilde{\subseteq} F_{\widetilde{E}}$.*

Definition 3.9. *Let* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *and* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be two bipolar soft topological spaces over* U *and* V, *respectively and* $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be a function. For each bipolar soft neighbourhood* $H_{\widetilde{E'}}$ *of* $\psi_{f\varphi\vartheta}\left(x_e^y\right)$, *if there exists a bipolar soft neighbourhood* $F_{\widetilde{E}}$ *of* x_e^y *such that* $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\widetilde{\subseteq}H_{\widetilde{E'}}$, *then* $\psi_{f\varphi\vartheta}$ *is called a bipolar soft continuous function at* x_e^y.

Moreover, $\psi_{f\varphi\vartheta}$ is called bipolar soft continuous function on U if $\psi_{f\varphi\vartheta}$ is a bipolar soft continuous function for all x_e^y.

Theorem 3.6. *Let* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *and* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be two bipolar soft topological spaces over* U *and* V, *respectively and* $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be a mapping. Then, the following conditions are equivalent:*

(1) $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ is a bipolar soft continuous function,

(2) For each $G_{\widetilde{E'}}\in\widetilde{\tau'}$, $\psi^{-1}_{f\varphi\vartheta}\left(G_{\widetilde{E'}}\right)\in\widetilde{\tau}$,

(3) For each bipolar soft closed set $H_{\widetilde{E'}}$ over V, $\psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'}}\right)$ is a bipolar soft closed set over U,

(4) For each $F_{\widetilde{E}}\in BS\left(U_{\widetilde{E}}\right)$, $\psi_{f\varphi\vartheta}\left(\overline{F_{\widetilde{E}}}\right)\widetilde{\subseteq}\overline{\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)}$,

(5) For each $D_{\widetilde{E'}}\in BS\left(V_{\widetilde{E'}}\right)$, $\overline{\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)}\widetilde{\subseteq}\ \psi^{-1}_{f\varphi\vartheta}\left(\overline{D_{\widetilde{E'}}}\right)$,

(6) For each $D_{\widetilde{E'}}\in BS\left(V_{\widetilde{E'}}\right)$, $\psi^{-1}_{f\varphi\vartheta}\left(D^{\circ}_{\widetilde{E'}}\right)\widetilde{\subseteq}\left(\psi_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^{\circ}$.

Proof. (1) $\Rightarrow$ (2) Let $G_{\widetilde{E'}}\in\widetilde{\tau'}$ and $x_e^y\widetilde{\in}\psi^{-1}_{f\varphi\vartheta}\left(G_{\widetilde{E'}}\right)$. Then, $\psi_{f\varphi\vartheta}\left(x_e^y\right)\widetilde{\in}G_{\widetilde{E'}}$. Since $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ is a bipolar soft continuous mapping, there is $x_e^y\widetilde{\in}F_{\widetilde{E}}\widetilde{\in}\widetilde{\tau}$ such that $\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)\widetilde{\subseteq}G_{\widetilde{E'}}$. Hence, $x_e^y\ \widetilde{\in}F_{\widetilde{E}}\widetilde{\subseteq}\psi^{-1}_{f\varphi\vartheta}\left(G_{\widetilde{E'}}\right)$. This implies that $\psi^{-1}_{f\varphi\vartheta}\left(G_{\widetilde{E'}}\right)$ is a bipolar soft open set over U.

(2) $\Rightarrow$ (1) Let x_e^y be any bipolar soft point and $\left(\psi_{f\varphi\vartheta}\right)\left(x_e^y\right)\widetilde{\in}G_{\widetilde{E'}}$ be an arbitrary bipolar soft neighbourhood of $\psi_{f\varphi\vartheta}\left(x_e^y\right)$. Then, $x_e^y\ \widetilde{\in}\psi^{-1}_{f\varphi\vartheta}\left(G_{\widetilde{E'}}\right)$ is a bipolar soft neighbourhood and $\left(\psi_{f\varphi\vartheta}\right)\left(\psi^{-1}_{f\varphi\vartheta}\left(G_{\widetilde{E'}}\right)\right)\widetilde{\subseteq}G_{\widetilde{E'}}$.

(2) $\Rightarrow$ (3) From the definition of complement of bipolar soft set, it is obtained.

(3) $\Rightarrow$ (4) Let $F_{\widetilde{E}}\in BS\left(U_{\widetilde{E}}\right)$. Since

$$\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)\widetilde{\subseteq}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)},$$

$F_{\widetilde{E}}\widetilde{\subseteq}\psi^{-1}_{f\varphi\vartheta}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)}$ is obtained. By part (3), since $\psi^{-1}_{f\varphi\vartheta}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)}$ is a bipolar soft closed set over U, $\overline{F_{\widetilde{E}}}\widetilde{\subseteq}\psi^{-1}_{f\varphi\vartheta}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)}$. Thus, $\left(\psi_{f\varphi\vartheta}\right)\left(\overline{F_{\widetilde{E}}}\right)\widetilde{\subseteq}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)}$ is satisfied.

(4) $\Rightarrow$ (5) Let $D_{\widetilde{E'}}\in BS\left(V_{\widetilde{E'}}\right)$ and $\left(\psi_{f\varphi\phi}\right)^{-1}\left(D_{\widetilde{E'}}\right)=F_{\widetilde{E}}$. By part (4),

$$\begin{aligned}\left(\psi_{f\varphi\vartheta}\right)\left(\overline{F_{\widetilde{E}}}\right) &= \left(\psi_{f\varphi\vartheta}\right)\left(\overline{\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)}\right)\\ &\widetilde{\subseteq}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)}\widetilde{\subseteq}\overline{D_{\widetilde{E'}}}.\end{aligned}$$

Then, $\left(\overline{\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)}\right)=\overline{\left(F_{\widetilde{E}}\right)}\widetilde{\subseteq}\psi^{-1}_{f\varphi\vartheta}\left(\overline{D_{\widetilde{E'}}}\right)$ is obtained.

(5) $\Rightarrow$ (6) Let $D_{\widetilde{E'}} \in BS\left(V_{\widetilde{E'}}\right)$. Substituting $D^c_{\widetilde{E'}}$ for condition in (5). Then, $\overline{\psi^{-1}_{f\varphi\vartheta}\left(D^c_{\widetilde{E'}}\right)} \widetilde{\subseteq} \psi^{-1}_{f\varphi\vartheta}\left(\overline{D^c_{\widetilde{E'}}}\right)$. Since $D^\circ_{\widetilde{E'}} = \left(\overline{D^c_{\widetilde{E'}}}\right)^c$, then

$$\begin{array}{rcl} \psi^{-1}_{f\varphi\vartheta}\left(D^\circ_{\widetilde{E'}}\right) & = & \psi^{-1}_{f\varphi\vartheta}\left(\left(\overline{D^c_{\widetilde{E'}}}\right)^c\right) \\ & = & \left(\left(\psi^{-1}_{f\varphi\vartheta}\left(\overline{D^c_{\widetilde{E'}}}\right)\right)\right)^c \\ & \widetilde{\subseteq} & \left(\overline{\psi^{-1}_{f\varphi\vartheta}\left(D^c_{\widetilde{E'}}\right)}\right)^c \\ & = & \left(\overline{\left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^c}\right)^c \\ & = & \left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^\circ. \end{array}$$

(6) $\Rightarrow$ (2) Let $D_{\widetilde{E'}} \in \widetilde{\tau'}$. Since

$$\begin{array}{rcl} \left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^\circ & \widetilde{\subseteq} & \left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right) \\ & = & \psi^{-1}_{f\varphi\vartheta}\left(\left(D_{\widetilde{E'}}\right)^\circ\right) \\ & \widetilde{\subseteq} & \left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^\circ, \end{array}$$

then $\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right) \in \widetilde{\tau}$. $\square$

Example 3.2. *Let $U = \{x_1, x_2, x_3, x_4\}$, $V = \{y_1, y_2, y_3, y_4\}$ be two sets and $E = E' = \{e_1, e_2\}$ be two sets of parameters. Then,*

$$\widetilde{\tau} = \left\{\Phi_{\widetilde{E}}, \widetilde{U}_{\widetilde{E}}, F_{1_{\widetilde{E}}}, F_{2_{\widetilde{E}}}, F_{3_{\widetilde{E}}}, F_{4_{\widetilde{E}}}\right\}$$

is a bipolar soft topology over U and

$$\widetilde{\tau'} = \left\{\Phi_{\widetilde{E}}, \widetilde{V}_{\widetilde{E}}, G_{1_{\widetilde{E}}}, G_{2_{\widetilde{E}}}, G_{3_{\widetilde{E}}}, G_{4_{\widetilde{E}}}\right\}$$

is a bipolar soft topology over V, where

$$\begin{array}{rcl} F_{1_{\widetilde{E}}} & = & \{(e_1, \{x_1, x_2\}, \{x_3\}), (e_2, \{x_1, x_3\}, \{x_2, x_4\})\}, \\ F_{2_{\widetilde{E}}} & = & \{(e_1, \{x_1, x_4\}, \{x_2, x_3\}), (e_2, \{x_2, x_3\}, \{x_4\})\}, \\ F_{3_{\widetilde{E}}} & = & \{(e_1, \{x_1, x_2, x_4\}, \{x_3\}), (e_2, \{x_1, x_2, x_3\}, \{x_4\})\}, \\ F_{4_{\widetilde{E}}} & = & \{(e_1, \{x_1\}, \{x_2, x_3\}), (e_2, \{x_3\}, \{x_2, x_4\})\} \end{array}$$

and

$$\begin{array}{rcl} G_{1_{\widetilde{E}}} & = & \{(e_1, \{y_1, y_2\}, \{y_3\}), (e_2, \{y_1, y_3\}, \{y_2, y_4\})\}, \\ G_{2_{\widetilde{E}}} & = & \{(e_1, \{y_1, y_4\}, \{y_2, y_3\}), (e_2, \{y_2, y_3\}, \{y_4\})\}, \\ G_{3_{\widetilde{E}}} & = & \{(e_1, \{y_1, y_2, y_4\}, \{y_3\}), (e_2, \{y_1, y_2, y_3\}, \{y_4\})\}, \\ G_{4_{\widetilde{E}}} & = & \{(e_1, \{y_1\}, \{y_2, y_3\}), (e_2, \{y_3\}, \{y_2, y_4\})\}. \end{array}$$

Let $f : U \to V$ be a function defined as $f(x_i) = y_i$, $i = 1, 2, 3, 4$, the function $\varphi : E \to E$ be defined as $\varphi(e_i) = e_i$ and the function $\vartheta : \neg E \to \neg E$ be defined as $\vartheta(\neg e_i) = \neg\varphi(e_i)$, $i = 1, 2$. Then, $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ is a bipolar soft continuous function.

Theorem 3.7. *Let* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft continuous function. Then, the functions* $f_\varphi : (U, \tau, E) \to (V, \tau', E')$ *and* $f_\varphi : (U, \tau, \neg E) \to (V, \tau', \neg E')$ *are soft continuous functions.*

Proof. The proof is clear. □

Remark. *If* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *is a bipolar soft continuous function, then* $f_e : (U, \tau_e) \to \left(V, \tau'_{\varphi(e)}\right)$ *and* $f_e : (U, \tau_{\neg e}) \to \left(V, \tau'_{\neg\varphi(e)}\right)$ *are continuous function on topological spaces, for each* $e \in E$.

Example 3.3. *Consider the bipolar soft function defined previous example. Then, for the parameter* $e_i \in E, i = 1, 2,$

$$\begin{aligned}
\tau_{e_1} &= \{\varnothing, U, \{x_1, x_2\}, \{x_1, x_4\}, \{x_1, x_2, x_4\}, \{x_1\}\},\ \tau_{\neg e_1} = \{\varnothing, U, \{x_3\}, \{x_2, x_3\}\},\\
\tau_{e_2} &= \{\varnothing, U, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_3\}\},\ \tau_{\neg e_2} = \{\varnothing, U, \{x_2, x_4\}, \{x_4\}\},\\
\tau'_{e_1} &= \{\varnothing, V, \{y_1\}, \{y_1, y_2\}, \{y_1, y_4\}, \{y_1, y_2, y_4\}\},\ \tau'_{\neg e_1} = \{\varnothing, V, \{y_3\}, \{y_2, y_3\}\},\\
\tau'_{e_2} &= \{\varnothing, V, \{y_3\}, \{y_1, y_3\}, \{y_2, y_3\}, \{y_1, y_2, y_3\}\},\ \tau'_{\neg e_2} = \{\varnothing, V, \{y_4\}, \{y_2, y_4\}\}.
\end{aligned}$$

Therefore, $f_{e_i} : (U, \tau_{e_i}) \to \left(V, \tau'_{e_i}\right)$ *and* $f_{e_i} : (U, \tau_{\neg e_i}) \to \left(V, \tau'_{\neg e_i}\right)$ *are continuous functions on topological spaces, for* $i = 1, 2$.

Example 3.4. *Let* $U = \{x_1, x_2, x_3\}$, $V = \{y_1, y_2, y_3\}$ *and* $E = E' = \{e_1, e_2\}$. *Then,* $\widetilde{\tau} = \left\{\Phi_{\widetilde{E}}, \widetilde{U}_{\widetilde{E}}, F1_{\widetilde{E}}, F2_{\widetilde{E}}, F3_{\widetilde{E}}, F4_{\widetilde{E}}\right\}$ *is a bipolar soft topology over* U *and* $\widetilde{\tau'} = \left\{\Phi_{\widetilde{E}}, \widetilde{V}_{\widetilde{E}}, G_{\widetilde{E}}\right\}$ *is a bipolar soft topology over* V, *where*

$$\begin{aligned}
F1_{\widetilde{E}} &= \{(e_1, \{x_1, x_3\}, \{x_2\}), (e_2, \{x_1\}, \{x_2, x_3\})\},\\
F2_{\widetilde{E}} &= \{(e_1, \{x_2, x_3\}, \{x_1\}), (e_2, \{x_1, x_3\}, \{x_2\})\},\\
F3_{\widetilde{E}} &= \{(e_1, U, \varnothing), (e_2, \{x_1, x_3\}, \{x_2\})\},\\
F4_{\widetilde{E}} &= \{(e_1, \{x_3\}, \{x_1, x_2\}), (e_2, \{x_1\}, \{x_2, x_3\})\},
\end{aligned}$$

and

$$G_{\widetilde{E}} = \{(e_1, \{y_1\}, \{y_2, y_3\}), (e_2, \{y_1, y_2\}, \{y_3\})\}.$$

Let $f : U \to V$ *be a function defined as* $f(x_1) = y_2$, $f(x_2) = y_1$, $f(x_3) = y_3$, *the function* $\varphi : E \to E$ *be defined as* $\varphi(e_i) = e_i$ *and the function* $\vartheta : \neg E \to \neg E$ *be defined as* $\vartheta(\neg e_i) = \neg\varphi(e_i)$, $i = 1, 2$. *Since* $\psi^{-1}_{f\varphi\vartheta}\left(G_{\widetilde{E}}\right) \notin \widetilde{\tau}$, $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *is not a bipolar soft continuous function. Here,*

$$\begin{aligned}
\psi^{-1}_{f\varphi\vartheta}(G)(e_1) &= f^{-1}(G(\varphi(e_1))) = \{x_2\},\\
\psi^{-1}_{f\varphi\vartheta}(G)(\neg e_1) &= f^{-1}(G(\vartheta(\neg e_1))) = \{x_1, x_3\},\\
\psi^{-1}_{f\varphi\vartheta}(G)(e_2) &= f^{-1}(G(\varphi(e_2))) = \{x_1, x_2\},\\
\psi^{-1}_{f\varphi\vartheta}(G)(\neg e_2) &= f^{-1}(G(\vartheta(\neg e_2))) = \{x_3\}.
\end{aligned}$$

Theorem 3.8. *Let* $\left(U, \widetilde{\tau}, \widetilde{E}\right), \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *and* $\left(W, \widetilde{\tau'}, \widetilde{E^*}\right)$ *be bipolar soft topological spaces over* U, V *and* W, *respectively. If* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *and* $\omega_{g\varphi_1\vartheta_1} : \left(V, \widetilde{\tau'}, \widetilde{E'}\right) \to \left(W, \widetilde{\tau'}, \widetilde{E^*}\right)$ *are bipolar soft continuous functions, then* $\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right) : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(W, \widetilde{\tau'}, \widetilde{E^*}\right)$ *is a bipolar soft continuous function.*

Proof. Let $K_{\widetilde{E^*}} \in \widetilde{\tau^*}$. Let us show that $\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)^{-1} K_{\widetilde{E^*}} \in \widetilde{\tau}$. Since

$$\begin{aligned}
\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)^{-1}(K)(e) &= f^{-1}\left(g^{-1}\left(K\left(\left(\varphi_1 \circ \varphi\right)(e)\right)\right)\right), \\
\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)^{-1}(K)(\neg e) &= f^{-1}\left(g^{-1}\left(K\left(\left(\vartheta_1 \circ \vartheta\right)(\neg e)\right)\right)\right),
\end{aligned}$$

and $\omega_{g\varphi_1\vartheta_1}$ is a bipolar soft continuous function, then $g^{-1}\left(K\left(\left(\varphi_1 \circ \varphi\right)(e)\right)\right) \in \widetilde{\tau'}$. Also, since $\psi_{f\varphi\vartheta}$ is a bipolar soft continuous function, then $f^{-1}\left(g^{-1}\left(K\left(\left(\varphi_1 \circ \varphi\right)(e)\right)\right)\right) \in \widetilde{\tau}$. That is,

$$\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)^{-1} K_{\widetilde{E^*}} \in \widetilde{\tau}$$

and $\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)$ is a bipolar soft continuous function. □

Definition 3.10. *Let* $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ *and* $\left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be two bipolar soft topological spaces and* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft function.*

(1) If the image $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) \in \widetilde{\tau'}$ for any $F_{\widetilde{E}} \in \widetilde{\tau}$, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft open function,

(2) If the image $\psi_{f\varphi\vartheta}\left(G_{\widetilde{E}}\right)$ is a bipolar soft closed set in V for any bipolar soft closed set $G_{\widetilde{E}}$ in U, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft closed function.

Proposition 3.9. *Let* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft open (closed) function. Then, the functions* $f_\varphi : (U, \tau, E) \to (V, \tau', E')$ *and* $f_\varphi : (U, \tau, \neg E) \to (V, \tau', \neg E')$ *are soft open (closed) functions.*

Proof. The proof is obtained from the definition of bipolar soft open (closed) function. □

Proposition 3.10. *If* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft open (closed) function, for each* $e \in E$, *then* $f_e : (U, \tau_e) \to \left(V, \tau'_{\varphi(e)}\right)$ *is a open (closed) function on topological spaces, for each* $e \in E$.

Proof. The proof of the proposition is straightforward. □

Example 3.5. *Let* $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$ *and* $E = E' = \{e_1, e_2\}$. *Then* $\widetilde{\tau} = \left\{\Phi_{\widetilde{E}}, \widetilde{U}_{\widetilde{E}}, F_{\widetilde{E}}\right\}$ *is a bipolar soft topology over* U *and* $\widetilde{\tau'} = \left\{\Phi_{\widetilde{E}}, \widetilde{V}_{\widetilde{E}}, G_{1_{\widetilde{E}}}, G_{2_{\widetilde{E}}}, G_{3_{\widetilde{E}}}, G_{4_{\widetilde{E}}}\right\}$ *is a bipolar soft topology over* V, *where*

$$F_{\widetilde{E}} = \left\{\left(e_1, \{x_1\}, \{x_2\}\right), \left(e_2, \{x_2\}, \varnothing\right)\right\},$$

and

$$\begin{aligned}
G_{1_{\widetilde{E}}} &= \left\{\left(e_1, \{y_1\}, \{y_2\}\right), \left(e_2, \{y_2\}, \varnothing\right)\right\}, \\
G_{2_{\widetilde{E}}} &= \left\{\left(e_1, \{y_1\}, \{y_2\}\right), \left(e_2, \{y_1\}, \varnothing\right)\right\}, \\
G_{3_{\widetilde{E}}} &= \left\{\left(e_1, \{y_1\}, \{y_2\}\right), \left(e_2, \{y_1, y_2\}, \varnothing\right)\right\}, \\
G_{4_{\widetilde{E}}} &= \left\{\left(e_1, \{y_1\}, \{y_2\}\right), \left(e_2, \varnothing, \varnothing\right)\right\}.
\end{aligned}$$

Let $f : U \to V$ *be a function defined as* $f(x_1) = y_1$, $f(x_2) = y_2$, *the function* $\varphi : E \to E$ *be defined as* $\varphi(e_i) = e_i$ *and the function* $\vartheta : \neg E \to \neg E$ *be defined* *as* $\vartheta(\neg e_i) = \neg\varphi(e_i)$, $i = 1,2$. *Then, since* $\psi^{-1}_{f\varphi\vartheta}\left(G_{2_{\widetilde{E}}}\right) \notin \widetilde{\tau}$, $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *is not a bipolar soft continuous function. Here,*

$$\begin{aligned}
\psi^{-1}_{f\varphi\vartheta}(G_2)(e_1) &= f^{-1}\left(G_2^+(\varphi(e_1))\right) = \{x_1\}, \\
\psi^{-1}_{f\varphi\vartheta}(G_2)(\neg e_1) &= f^{-1}\left(G_2^-(\vartheta(\neg e_1))\right) = \{x_2\}, \\
\psi^{-1}_{f\varphi\vartheta}(G_2)(e_2) &= f^{-1}\left(G_2^+(\varphi(e_2))\right) = \{x_1\}, \\
\psi^{-1}_{f\varphi\vartheta}(G_2)(\neg e_2) &= f^{-1}\left(G_2^-(\vartheta(\neg e_2))\right) = \varnothing.
\end{aligned}$$

Since $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = G_{1_{\widetilde{E}}} \in \widetilde{\tau'}$, $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *is a bipolar soft open function.*

Theorem 3.11. *Let* $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ *and* $\left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be two bipolar soft topological spaces and* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft function.*

(1) $\psi_{f\varphi\vartheta}$ is a bipolar soft open function if and only if for any $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$,

$$\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ}.$$

(2) $\psi_{f\varphi\vartheta}$ is a bipolar soft closed function if and only if for any $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$,

$$\overline{\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)} \widetilde{\subseteq} \psi_{f\varphi\vartheta}\overline{\left(F_{\widetilde{E}}\right)}.$$

Proof. (1) Let $\psi_{f\varphi\vartheta}$ be a bipolar soft open function and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, $F^{\circ}_{\widetilde{E}} \in \widetilde{\tau}$ and $F^{\circ}_{\widetilde{E}} \widetilde{\subseteq} F_{\widetilde{E}}$. Since $\psi_{f\varphi\vartheta}$ is a bipolar soft open function, $\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \in \widetilde{\tau'}$ and $\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)$. Thus,

$$\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ}$$

is obtained.

Conversely, let $F_{\widetilde{E}} \in \widetilde{\tau}$. Then, $F_{\widetilde{E}} = F^{\circ}_{\widetilde{E}}$. From the condition of theorem,

$$\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ}.$$

Hence,

$$\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = \psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ} \widetilde{\subseteq} \psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right).$$

It is clear that $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ}$, i.e. $\psi_{f\varphi\vartheta}$ is a bipolar soft open function.

(2) Let $\psi_{f\varphi\vartheta}$ be a bipolar soft closed function and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Since $\psi_{f\varphi\vartheta}$ is a bipolar soft closed function, $\psi_{f\varphi\vartheta}\left(\overline{F_{\widetilde{E}}}\right)$ is a bipolar soft closed set in V and $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) \widetilde{\subseteq} \psi_{f\varphi\vartheta}\left(\overline{F_{\widetilde{E}}}\right)$. Henceforth $\overline{\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)} \widetilde{\subseteq} \psi_{f\varphi\vartheta}\left(\overline{F_{\widetilde{E}}}\right)$ is obtained.

Conversely, now let $F_{\widetilde{E}}$ be any bipolar soft closed set over U. Then, $F_{\widetilde{E}} = \overline{F_{\widetilde{E}}}$. From the condition of theorem, $\overline{\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)} \widetilde{\subseteq} \psi_{f\varphi\vartheta}\overline{\left(F_{\widetilde{E}}\right)} = \psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) \widetilde{\subseteq} \overline{\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)}$. It is clear that $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = \overline{\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)}$, i.e. $\psi_{f\varphi\vartheta}$ is a bipolar soft closed function. $\square$

Definition 3.11. *Let* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *and* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be two bipolar soft topological spaces and* $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be a bipolar soft function.* $\psi_{f\varphi\phi}$ *is called a bipolar soft homeomorphism from* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *to* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$*, if* $\psi_{f\varphi\vartheta}$ *is both bipolar soft bijective and bipolar soft continuous,* $\psi^{-1}_{f\varphi\vartheta}$ *is a bipolar soft continuous function.*

Theorem 3.12. *Let* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *and* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be two bipolar soft topological spaces and* $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be a bipolar soft function. Then, the following conditions are equivalent:*

(1) $\psi_{f\varphi\vartheta}$ is a bipolar soft homeomorphism,
(2) $\psi_{f\varphi\vartheta}$ is both a bipolar soft continuous and bipolar soft closed function,
(3) $\psi_{f\varphi\vartheta}$ is both a bipolar soft continuous and bipolar soft open function.

Proof. The proof is clear. □

4. Conclusion

Since we have defined the concepts of bipolar soft continuity, bipolar soft openness, bipolar soft closedness and bipolar soft homeomorphism, this paper contributes to the topology field from a bipolar view. Theorems and examples support the concepts given. In forthcoming works, we aim to give the concept bipolar infra soft topology and examine some structures such as connectedness and different separation axioms.

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DISTRIBUTION OF EIGENVALUES OF A PERTURBED DIFFERENTIATION OPERATOR ON THE INTERVAL

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Abstract. In this paper, we construct a characteristic determinant of the spectral problem of a first-order differential equation on an interval with an integral perturbation in the boundary value condition, which is an entire analytic function of the spectral parameter. Based on the formula for the characteristic determinant, conclusions are drawn about the asymptotic behavior of the spectrum of the perturbed spectral problem depending on the modulus of continuity of the subinteral function.

1. Introduction

Works [1, 2, 3, 4] are devoted to studies of zeros of entire functions with an integral representation. Sometimes entire functions coincide with quasi-polynomials, zeros of which were investigated in papers [5, 6]. Connection between the zeros of quasi-polynomials and spectral problems is reflected in papers [7, 8, 9, 10, 11, 12]. Eigenvalue problems for some classes of differential operators on an interval are reduced to a similar problem. In particular, spectral problem for a first-order equation on an interval with a spectral parameter in a boundary-value condition with integral perturbation leads to the studied problem [13].

Asymptotic properties of entire functions with a given law of distribution of roots were deeply investigated in the doctoral dissertation of V.B. Sherstyukov, on its basis, the paper [14] was published.

The questions on location of the zeros of an entire function: on one ray, on a straight line, on several rays, in an angle or arbitrarily in the complex plane were studied in the works [1], [3], [9], [11] and [15].

Meramorphic functions of completely regular growth in the upper half-plane with respect to the growth function have been studied in one of the last works of K.G. Malyutin and M.V. Cabanco [16]. In the paper of Rabha W., Ibrahim,

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Ibtisam Aldawish [17], a new symmetric differential operator associated with a special class of meromorphic - multivalent functions in a punctured unit disc is presented. This study explories some of its geometric properties. A new class of holomorphic functions related by a symmetric differential operator is considered.

The paper is devoted to construction of a characteristic determinant of the spectral problem for the differentiation operator on an interval with an integral perturbation in the boundary value condition, which is an entire holomorphic function, where the integrand function has continuity property. Based on the formula for the characteristic determinant, conclusions are established about asymptotics of the spectrum of the perturbed spectral problem depending on the continuity modulus of the integrand function.The considered problem belongs to the nonlocal type of spectral problems. Such problems have been studied many times before. Among the recent publications, we note works [18, 19, 20, 21]. The main fundamental feature of such problems is their non-self-adjointness. This causes the main difficulties in their study.

1.1. **Problem Statement .** In the space $W_2^1(-1,1)$ we consider the following problem on eigenvalues of the operator:

$$L_1 y \equiv y'(t) = \lambda y(t), \quad -1 \le t \le 1 \tag{1.1}$$

with boundary value condition

$$y(-1) - y(1) = \int_{-1}^{1} y(t) \cdot \Phi(t) dt, \tag{1.2}$$

where $\Phi(t)$ is a continuous function on the interval $[-1,1]$ and $\Phi(-1) = \Phi(1) = 1$, λ is a complex number, spectral parameter.

It is required to find those complex value of λ, in which the operator equation (1.1) has a nonzero solution.

2. Main Results

We introduce the general solution of the equation (1.1) by the formula $y(t) = Ce^{\lambda t}$, $\forall C > 0$, and satisfing the boundary value condition (1.2), we obtain the characteristical determinant of the problem (1.1) – (1.2):

$$\Delta_1(\lambda) = e^{-\lambda} - e^{\lambda} - \int_{-1}^{1} e^{\lambda t} \cdot \Phi(t) dt, \tag{2.1}$$

which is an entire analytical function of the variable $\lambda = x + iy$, $Re\lambda = x$, $Im\lambda = y$, $i = \sqrt{-1}$.

If the function $\Phi(t) \equiv 0$, then we get that $\Delta_0(\lambda) = e^{-\lambda} - e^{\lambda}$ is a characteristical determinant of the following spectral problem:

$$L_0 y \equiv y'(t) = \lambda y(t), \quad -1 \le t \le 1, \quad y(-1) = y(1). \tag{2.2}$$

The numbers $\lambda_n^0 = in\pi$, $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$, are eigenvalues, moreover, $\forall C > 0$, $y_{n0}^0 = C \cdot e^{in\pi t}$ are eigen functions of the operator L_0, which forms a complete orthonormal system in $L_2(-1,1)$, and forms a basis in $L_2(-1,1)$.

In our case, the function $\Phi(t)$ is continuous on the interval $[-1,1]$. Due to well-known Rouche's theorem [22], we introduce the function in (2.1):

$$\Delta_1(\lambda)=\Delta_0(\lambda)-f(\lambda),\ \text{where}\ \Delta_0(\lambda)=e^{-\lambda}-e^{\lambda},$$

$$f(\lambda)=\int\limits_{-1}^{1}e^{\lambda t}\cdot\Phi(t)dt,$$

where all of these functions are entire analytical functions. We estimate the function $\Delta_0(\lambda)$ from below:

$$|\Delta_0(\lambda)|\geq e^{|\lambda|}-e^{-|\lambda|}\geq e^{x}-e^{-x}.$$

Distribution of zeros of the entire function $f(\lambda)$ is investigated separately. We split the interval $[-1,1]$ into $2m$ equal parts. Then the function $f(\lambda)$ takes the following form:

$$f(\lambda)=\int\limits_{-1}^{1}e^{\lambda t}\cdot\Phi(t)dt=\int\limits_{-1}^{\frac{-2(m-1)}{2m}}e^{\lambda t}\cdot\Phi(t)dt+\int\limits_{\frac{-2(m-1)}{2m}}^{\frac{2(2-m)}{2m}}e^{\lambda t}\cdot\Phi(t)dt$$
$$+\int\limits_{\frac{2(2-m)}{2m}}^{\frac{2(3-m)}{2m}}e^{\lambda t}\cdot\Phi(t)dt+\ldots+\int\limits_{\frac{2(m-1)}{2m}}^{1}e^{\lambda t}\cdot\Phi(t)dt=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\Phi(t)dt.$$

We transform the function $f(\lambda)$:

$$f(\lambda)=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\Phi(t)dt=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\left[\Phi(t)-\Phi(\frac{p}{m})+\Phi(\frac{p}{m})\right]dt$$
$$=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\Phi(\frac{p}{m})dt+\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\left[\Phi(t)-\Phi(\frac{p}{m})\right]dt.$$

Let us show that $f(\lambda)$ does not have zeros outside the domain ($|x|\leq nrw(\frac{1}{n})$, for some n). Due to the Rouche's theorem [22], we introduce the designation

$$h(\lambda)=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\Phi(\frac{p}{m})dt,$$

and

$$G(\lambda)=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\left(\Phi(t)-\Phi(\frac{p}{m})\right)dt.$$

Let $Re\lambda > 0$. We compute the integrals included in the function $h(\lambda)$.

$$h(\lambda) = \sum_{p=-m+1}^{m} \Phi(\frac{p}{m})\frac{1}{\lambda}\left(e^{\lambda\frac{p}{m}} - e^{\lambda\frac{p-1}{m}}\right) = \frac{1}{\lambda}\left[\Phi(-1+\frac{1}{m})\left(e^{\lambda(-1+\frac{1}{m})} - e^{-\lambda}\right)\right.$$
$$+ \Phi(-1+\frac{2}{m})\left(e^{\lambda(-1+\frac{2}{m})} - e^{\lambda(-1+\frac{1}{m})}\right) + \ldots + \Phi(1-\frac{2}{m})\left(e^{\lambda(1-\frac{2}{m})} - e^{\lambda(1-\frac{3}{m})}\right)$$
$$\left.+\Phi(1-\frac{1}{m})\left(e^{\lambda(1-\frac{1}{m})} - e^{\lambda(1-\frac{2}{m})}\right)\right] = \frac{1}{\lambda}\left[e^{\lambda(-1+\frac{1}{m})}\left(\Phi(-1+\frac{1}{m}) - \Phi(-1+\frac{2}{m})\right)\right.$$
$$+e^{\lambda(-1+\frac{2}{m})}\left(\Phi(-1+\frac{2}{m}) - \Phi(-1+\frac{3}{m})\right)+e^{\lambda(-1+\frac{3}{m})}\left(\Phi(-1+\frac{3}{m}) - \Phi(-1+\frac{4}{m})\right)$$
$$\left.- \Phi(-1+\frac{1}{m})e^{-\lambda} + \ldots + e^{\lambda(1-\frac{1}{m})}\left(\Phi(1-\frac{1}{m}) - \Phi(1)\right) + \Phi(1)e^{\lambda}\right].$$

Grouping the exponents in pairs, we have

$$h(\lambda) = \frac{1}{\lambda}\left[e^{\lambda(-1+\frac{1}{m})}\left(\Phi(-1+\frac{1}{m}) - \Phi(-1+\frac{2}{m})\right) - \Phi(-1+\frac{1}{m})e^{-\lambda}\right.$$
$$\left.+ \Phi(-1)e^{-\lambda} - \Phi(-1)e^{-\lambda} + \ldots + e^{\lambda(1-\frac{1}{m})}\left(\Phi(1-\frac{1}{m}) - \Phi(1)\right) + \Phi(1)e^{\lambda}\right]$$
$$= \frac{1}{\lambda}\left[e^{\lambda(-1+\frac{1}{m})}\left(\Phi(-1+\frac{1}{m}) - \Phi(-1+\frac{2}{m})\right) + e^{-\lambda}\left(\Phi(-1) - \Phi(-1+\frac{1}{m})\right)\right.$$
$$\left.- \Phi(-1)e^{-\lambda} + \ldots + e^{\lambda(1-\frac{1}{m})}\left(\Phi(1-\frac{1}{m}) - \Phi(1)\right) + \Phi(1)e^{\lambda}\right].$$

We denote

$$h_1(\lambda) = \frac{1}{\lambda}\left[\Phi(1)e^{\lambda} - \Phi(-1)e^{-\lambda}\right] = \frac{1}{\lambda}\left[e^{\lambda} - e^{-\lambda}\right],$$

and

$$g(\lambda) = \frac{1}{\lambda}\left[e^{\lambda(-1+\frac{1}{m})}\left(\Phi(-1+\frac{1}{m}) - \Phi(-1+\frac{2}{m})\right)\right.$$
$$\left.+ e^{-\lambda}\left(\Phi(-1) - \Phi(-1+\frac{1}{m})\right) + \ldots + e^{\lambda(1-\frac{1}{m})}\left(\Phi(1-\frac{1}{m}) - \Phi(1)\right)\right]$$
$$= \frac{1}{\lambda}\sum_{p=-m+1}^{m} e^{\lambda(\frac{p-1}{m})}\left(\Phi(\frac{p-1}{m}) - \Phi(\frac{p}{m})\right).$$

Then

$$\mu(\lambda) = G(\lambda) + g(\lambda) =$$
$$= \sum_{p=-m+1}^{m}\left(\int_{\frac{p-1}{m}}^{\frac{p}{m}} e^{\lambda t}\cdot\left(\Phi(t) - \Phi(\frac{p}{m})\right)dt + \frac{e^{\lambda(\frac{p-1}{m})}}{\lambda}\left(\Phi(\frac{p-1}{m}) - \Phi(\frac{p}{m})\right)\right).$$

We estimate the function $h_1(\lambda)$ from below, at the same time as the remaining terms, that is, the function $\mu(\lambda)$ is estimated from above

$$|h_1(\lambda)| = \left|\frac{1}{\lambda}\left(e^{\lambda} - e^{-\lambda}\right)\right| \geq \frac{1}{|\lambda|}e^{\lambda} - \frac{1}{|\lambda|}\left|\underline{o}(e^{-\lambda})\right|. \tag{2.3}$$

We estimate the function $\mu(\lambda)$ from above:

$$|\mu(\lambda)| \leq \sum_{p=-m+1}^{m} \left[\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}} |e^{\lambda t}| \cdot \left| \Phi(t) - \Phi(\frac{p}{m}) \right| dt + \frac{e^{\lambda(\frac{p-1}{m})}}{|\lambda|} \left| \Phi(\frac{p-1}{m}) - \Phi(\frac{p}{m}) \right| \right]$$

$$\leq \sum_{p=-m+1}^{m} \left[\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}} e^{xt} \sup_{\frac{p-1}{m} \leq t \leq \frac{p}{m}} \left| \Phi(t) - \Phi(\frac{p}{m}) \right| dt + \frac{e^{x(\frac{p-1}{m})}}{|\lambda|} \sup_{|t-\tau| \leq \frac{1}{m}} |\Phi(t) - \Phi(\tau)| \right].$$

We introduce module of continuity of the function $\Phi(t)$ by the formula

$$w(\frac{1}{m}) = \sup_{|t-\tau|<\frac{1}{m}} |\Phi(t) - \Phi(\tau)|.$$

Then

$$|\mu(\lambda)| \leq \sum_{p=-m+1}^{m} \left[\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}} e^{xt} \cdot w(\frac{1}{m}) dt + \frac{e^{x(\frac{p-1}{m})}}{x} w(\frac{1}{m}) \right] \leq w(\frac{1}{m}) \frac{e^x - e^{-x}}{x}. \quad (2.4)$$

Therefore, due to (2.3), (2.4), we come to the estimation:

$$|f(\lambda)| \geq \frac{1}{|\lambda|} e^x - \frac{1}{|\lambda|} \left| \overline{\overline{o}}(\frac{1}{|\lambda}|) \right| - w(\frac{1}{m}) \left(\frac{e^x + e^{-x}}{x} \right).$$

Assuming that, $|\lambda| = r$, $m = [r]$, we have

$$|\lambda f(\lambda)| = |f_1(\lambda)| \geq e^x - \frac{e^x w(\frac{1}{r}) r}{x} - e^{-x} - \frac{e^{-x} r w(\frac{1}{r})}{x}. \quad (2.5)$$

For the final approval, we will choose n so that

$$\left| \frac{w(\frac{1}{r}) r}{x} \right| + e^{-2x} + e^{-2x} \cdot \frac{r w(\frac{1}{r})}{x} < \frac{1}{2}$$

as $x > nrw(\frac{1}{r})$. It is possible, since value of the left part of the last inequality is defined in the main first term.

By the condition of the Rouche's theorem [22], defining the main part of the function $\Delta_1(\lambda)$, due to lower estimation of the function $\Delta_0(\lambda)$ and $f_1(\lambda)$ in (2.5), i.e. $|\Delta_0(\lambda)| > |f_1(\lambda)|$, we come to the following theorem:

Theorem 2.1. *If the function $\Phi(t)$ is continuous on the interval $[-1, 1]$ and satisfies the condition $\Phi(-1) = \Phi(1) = 1$, then all eigenvalues of the operator L_1 lie in the $|Re\lambda| < nrw(\frac{1}{r})$ at some n, where $\lambda = x+iy$, $Re\lambda = x$, and $w(\delta)$ is a continuity madule of the function $\Phi(t)$, $r = |\lambda|$.*

Remark. *If $\Phi(t)$ is continuous on $[-1, 1]$ and $\Phi(-1) = \Phi(1) = 1$, then all eigenvalues of the operator L_1 are lie in the $|Re\lambda| < nrw(\frac{1}{r})$ on the complex plane λ, which expands depending on properties of the continuity module $w(\delta)$ of the function $\Phi(t)$.*

Theorem 2.2. *Let $\Phi(t)$ be a continuous function on $[-1, 1]$ and $\Phi(-1) = \Phi(1) = 1$. Then set of zeros of the entire function $\Delta_1(\lambda)$ as $n \to \infty$, $\lambda_n = i\pi n + \underline{\underline{o}}(nw(\frac{1}{n}))$, where $w(h)$ is a continuity module of $\Phi(t)$.*

Proof. To prove Theorem 2.1 we used two functions $h_1(\lambda)$ and $\mu(\lambda)$, such that $f(\lambda) = h_1(\lambda) + \mu(\lambda)$. Zeros of the functions $\Delta_0(\lambda)$ and $h_1(\lambda)$ have the form $\lambda_n^0 = i\pi n$, $n = \pm 1, \pm 2, \ldots$. We consider a square T with sides 2ε and with a center at the point λ_n^0 on the complex plane λ. Assume that, sides of T are parallel to real and imaginary axes of the λ variable. Proof of Theorem 2.2 consists in choosing ε so that conditions of Rouche's Theorem [22] were satisfied for the functions $\Delta_0(\lambda)$, $h_1(\lambda)$ and $\mu(\lambda)$ on the sides of the square T. First, we consider the right half of the square T, that is, in the case $Re\lambda \geq 0$. Divide the side of the square T into two parts $0 \leq Re\lambda \leq C$ and $C \leq Re\lambda \leq \varepsilon$, where $C > 0$, which we will choose later.

2.1 Case. Let $0 \leq Re\lambda \leq C$. Since zeros of the functions $\Delta_0(\lambda)$ and $h_1(\lambda)$ are the same and these functions are equal to each other, therefore, it is enough to estimate the function $h_1(\lambda)$. Let's compare the modules of functions $h_1(\lambda)e^{-\lambda}$ and $\mu(\lambda)e^{-\lambda}$. Taking into account boundedness of the corresponding derivative, we obtain the following estimate:

$$|h_1(\lambda)e^{-\lambda}| = |h_1(\lambda)e^{-\lambda} - h_1(\lambda_n^0)e^{-\lambda_n^0}| = |\frac{d}{d\lambda}h_1(\lambda)e^{-\lambda}| \cdot |\lambda - \lambda_n^0| \geq \frac{C_1}{|\lambda|} \cdot \varepsilon.$$

Due to boundedness of modules of exponents, included in $\mu(\lambda)$, we write the inequalities

$$|\mu(\lambda)e^{-\lambda}| \leq C_2 w(\frac{1}{n}).$$

Therefore, to satisfy conditions of Rouche's Theorem it is enough to take ε from

$$\varepsilon = \underline{\underline{o}}(nw(\frac{1}{n})),$$

since module of λ behaves like $\lambda = n(1 + \overline{\overline{o}}(1))$.

2.2 Case. Let $C \leq Re\lambda \leq \varepsilon$. When $C > 0$, the module of $h_1(\lambda)$ is estimated by the module of one of the exponents included in $h_1(\lambda)$:

$$|h_1(\lambda)| = \left|\frac{e^{\lambda} - e^{-\lambda}}{\lambda}\right| \geq \frac{1}{2}\frac{e^x}{|\lambda|}.$$

We note that C must be chosen from the inequality $C > \ln\varphi$. Since modules of the exponents included in the function $\mu(\lambda)$ are bounded from above by the next exponent e^x, it is true that

$$|\mu(\lambda)| \leq e^x w(\frac{1}{n})C_3.$$

Hence, it follows that in order to satisfy the condition of Rouche's Theorem [22], it suffices to take ε from bound of the form

$$\varepsilon = \underline{\underline{o}}(nw(\frac{1}{n})).$$

Thus, Theorem 2.2 is completely proved. □

Remark. *One of the features of the considered problem is that the conjugate to (1.1) – (1.2) is the spectral problem for the loaded differential equation:*

$$L_1^* v = v'(t) + \Phi(t)v(1) = \overline{\lambda}v(t),$$

$$v(1) = v(-1).$$

In [23], eigenvalues of a loaded differential operator of the first order with general boundary value conditions on an interval were found, and in the papers [20], [24] and [25] questions on stability of basis properties of the root vectors of a loaded operator of multiple differentiation were studied in the space $L_2(0,1)$.

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NEUTROSOPHIC SEPERATION AXIOMS

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Abstract. This study is dedicated to make an attempt to define different types of separation axioms in neutrosophic topological spaces. The relationships among them are shown with a diagram and counterexamples. We also introduce some new terms such as introduced neutrosophic topology, neutrosophic regular space, neutrosophic normal space, neutrosophic subspace.

1. Introduction

Undoubtedly, the concept of separation axioms has always been an indispensable character in the world of topology. This concept formed the basis of many valuable researches in general topology. And, these researches played very important roles in many parts of real life and the findings of these researches came to life in many applications. But, as technology advances and the industry evolves, peoples needs have changed and general topology has become inadequate in real life. So, the impact of these findings on real life has diminished. Then,scientists went on to find different types of topological spaces and separation axioms occupied an important place in these topological spaces. In [17], Smarandache offered the concept of neutrosophic set. This idea became the leading actor in numerous studies as in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16]. Also, by using this new concept, Salma and Alblowi introduced the theory of neutrosophic topological space in [15]. In this study, we present different types of separation axioms in neutrosophic topological spaces as a new instrument for real life applications and new terms that we think benefit in other investigations.Throughout the paper, without any explanation, we use the symbols and definitions introduced in [13, 15, 17]. For the sake of shortness we use the notation N instead of neutrosophic.

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2. Some Required Definitions

In this section, we give newest predefined definitions that will be required in the next section. Also, some new definitions are given.

Definition 2.1. [4] *An N-point $x_{r,t,s}$ is said to be N-quasi-coincident (N-q-coincident, for short) with F, denoted by $x_{r,t,s}qF$ iff $x_{r,t,s} \subseteq F^c$. If $x_{r,t,s}$ is not N-quasi-coincident with F, we denote by $x_{r,t,s}\tilde{q}F$.*

Definition 2.2. [4] *An N-set F in an N-topological space (X, τ) is said to be an N-q-neighborhood of an N-point $x_{r,t,s}$ iff there exists an N-open set G such that $x_{r,t,s}qG \subset F$.*

Definition 2.3. [4] *An N-set G is said to be N-quasi-coincident (N-q-coincident, for short) with F, denoted by GqF iff $G \nsubseteq F^c$. If G is not N-quasi-coincident with F, we denote by $G\tilde{q}F$.*

Definition 2.4. *Consider that (X, τ) is an N-topological space and $Y \subseteq X$. Let H be an N-set over Y such that*

$$T_H(x) = \begin{cases} 1, & x \in Y \\ 0, & x \notin Y \end{cases}, \quad I_H(x) = \begin{cases} 1, & x \in Y \\ 0, & x \notin Y \end{cases}, \quad F_H(x) = \begin{cases} 0, & x \in Y \\ 1, & x \notin Y \end{cases}$$

Consider that $\tau_Y = \{H \cap F : F \in \tau\}$, then (Y, τ_Y) is called N-subspace of (X, τ). If $H \in \tau$ (resp. $H^c \in \tau$), then (Y, τ_Y) is called N-open (resp. closed) subspace of (X, τ).

Definition 2.5. [4] *An N-point $x_{r,t,s}$ is said to be an N-cluster point of an N-set F iff every N-open q-neighborhood G of $x_{r,t,s}$ is q-coincident with F. The union of all N-cluster points of F is called the N-closure of F and denoted by $\overline{F}$.*

Definition 2.6. [4] *Consider that f is a function from X to Y. Let A be an N-set in X with membership funtion $T_A(x)$, indeterminacy function $I_A(x)$ and non-membership function $F_A(x)$.The image of A under f, written as $f(A)$, is an N-subset of Y whose membership function, indeterminacy function and non-membership function are defined as*

$$T_{f(A)}(y) = \begin{cases} \sup_{z \in f^{-1}(y)}\{T_A(z)\} & \text{, if } f^{-1}(y) \text{ is not empty,} \\ 0 & \text{, if } f^{-1}(y) \text{ is empty,} \end{cases}$$

$$I_{f(A)}(y) = \begin{cases} \sup_{z \in f^{-1}(y)}\{I_A(z)\} & \text{, if } f^{-1}(y) \text{ is not empty,} \\ 0 & \text{, if } f^{-1}(y) \text{ is empty,} \end{cases}$$

$$F_{f(A)}(y) = \begin{cases} \inf_{z \in f^{-1}(y)}\{F_A(z)\} & \text{, if } f^{-1}(y) \text{ is not empty,} \\ 1 & \text{, if } f^{-1}(y) \text{ is empty,} \end{cases}$$

for all y in Y, where $f^{-1}(y) = \{x : f(x) = y\}$, respectively.

If f is a bijective function from X to Y, then it is an invertible N-function. Conversely, consider that B is an N-set in Y with membership funtion $T_B(y)$, indeterminacy function $I_B(y)$ and non-membership function $F_B(y)$. Then, the inverse image of B under f, written as $f^{-1}(B)$, is an N-subset of X whose membership function, indeterminacy function and non-membership function are defined as $T_{f^{-1}(B)}(x) = T_B(f(x))$, $I_{f^{-1}(B)}(x) = I_B(f(x))$ and $F_{f^{-1}(B)}(x) = F_B(f(x))$ for all x in X, respectively.

Definition 2.7. *Consider that* (X, τ), (Y, δ) *are N-topological spaces and* $f : (X, \tau) \to (Y, \delta)$ *is an N-function. The function* f *is said to be N-continuous, if* $f^{-1}(G) \in \tau$ *for any* $G \in \delta$.

Definition 2.8. *Consider that* (X, τ), (Y, δ) *are N-topological spaces and* $f : (X, \tau) \to (Y, \delta)$ *is an N-function. The function* f *is said to be N-open, if* $f(F) \in \delta$ *for any* $F \in \tau$.

3. N-T_i-Spaces (i=0, 1, 2)

In this section, we present different types of separation axioms and investigate their properties. Also, the relationships among them are shown with a diagram and counterexamples. Additionally, we analyze their characteristics in N-topological subspaces.

Definition 3.1. *An N-topological space* (X, τ) *is said to be an N-*T_0*-space if for every pair of N-points* $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, *whose supports are different, there exist N-open sets* F, G *such that* $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ *or* $x_{\alpha,\beta,\gamma} \in G^c$, $y_{\alpha',\beta',\gamma'} \in G$.

Theorem 3.1. *Consider that* (X, τ) *is an N-topological space, then* (X, τ) *is N-*T_0*-space iff, for any two N-points,* $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, *whose supports are different,* $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ *or* $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}y_{\alpha',\beta',\gamma'}$.

Proof. Consider that (X, τ) is an N-T_0-space and $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$ are two N-points with different supports. Then, there exist N-open sets F, G such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ or $x_{\alpha,\beta,\gamma} \in G^c$, $y_{\alpha',\beta',\gamma'} \in G$. This implies that $y_{\alpha',\beta',\gamma'}\tilde{q}F$ or $x_{\alpha,\beta,\gamma}\tilde{q}G$. So, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ or $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}y_{\alpha',\beta',\gamma'}$. Let (X, τ) be an N-topological space such that, for any two N-points $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$ with different supports, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ or $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}y_{\alpha',\beta',\gamma'}$. Then, there exists an N-open set F such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'}\tilde{q}F$ or there exists an N-open set G such that $y_{\alpha',\beta',\gamma'} \in G$, $x_{\alpha,\beta,\gamma}\tilde{q}G$. This implies that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ or $x_{\alpha,\beta,\gamma} \in G^c$, $y_{\alpha',\beta',\gamma'} \in G$. Therefore, (X, τ) is an N-T_0-space. □

Definition 3.2. *An N-topological space* (X, τ) *is said to be an N-*T_1*-space if for every pair of N-points* $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, *whose supports are different, there exist N-open sets* F, G *such that* $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ *and* $x_{\alpha,\beta,\gamma} \in G^c$, $y_{\alpha',\beta',\gamma'} \in G$.

Theorem 3.2. *Consider that* (X, τ) *is an N-topological space, then* (X, τ) *is N-*T_1*- space iff, for any two N-points,* $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, *whose supports are different,* $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ *and* $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}y_{\alpha',\beta',\gamma'}$.

Proof. The proof of this theorem is similar to that of above theorem. So, it is omitted. □

Theorem 3.3. *Consider that* (X, τ) *is an N-topological space. If every N-point* $x_{\alpha,\beta,\gamma}$ *is N-closed in* (X, τ), *then* (X, τ) *is an N-*T_1*-space.*

Proof. Consider any two N-points $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$ in (X, τ) such that $x_{\alpha,\beta,\gamma}\tilde{q}y_{\alpha',\beta',\gamma'}$. Then, $x_{\alpha,\beta,\gamma} \subset (y_{\alpha',\beta',\gamma'})^c$ and $y_{\alpha',\beta',\gamma'} \subset (x_{\alpha,\beta,\gamma})^c$ where $(y_{\alpha',\beta',\gamma'})^c$ and $(x_{\alpha,\beta,\gamma})^c$ are N-open sets in (X, τ). Since, $x_{\alpha,\beta,\gamma}\tilde{q}(x_{\alpha,\beta,\gamma})^c$ and $y_{\alpha',\beta',\gamma'}\tilde{q}(y_{\alpha',\beta',\gamma'})^c$, (X, τ) is an N-T_1-space. □

The converse statement is not always true as seen in the example below.

Example 3.1. *Consider the set $X = \{x, y\}$ and the family $\tau = \{\{x_{\alpha,\alpha,1-\alpha}, y_{\beta,\beta,1-\beta}\} : \alpha, \beta \in [0,1]\}$. Then, τ is an N-topology over X. It is easily seen that (X, τ) is an N-T_1-space. But, the N-point $x_{0,2,0,2,0,7}$ is not closed in τ. Because, $x_{0,2,0,2,0,7} \neq \overline{x_{0,2,0,2,0,7}}$.*

Definition 3.3. *An N-topological space τ is said to be an N-T_2-space, if, for every pair of N-points $x_{\alpha,\beta,\gamma}$, $y_{\alpha',\beta',\gamma'}$, whose supports are different, there exists N-open sets F, G such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$, $y_{\alpha',\beta',\gamma'} \in G$, $x_{\alpha,\beta,\gamma} \in G^c$ and $F\tilde{q}G$.*

For an N-topological space (X, τ) we have the following diagram:

$$\begin{array}{c} \text{N-}T_2\text{-space} \\ \downarrow \\ \text{N-}T_1\text{-space} \\ \downarrow \\ \text{N-}T_0\text{-space} \end{array}$$

Converse statements may not be true as shown in the examples below;

Example 3.2. *Consider the set $X = \{x, y\}$ and the family*

$$\tau = \{\{x_{\alpha,\alpha,1-\alpha}, y_{\beta,\beta,1-\beta}\} : \alpha \in [0,1], \beta \in [0,1)\}.$$

Then, τ is an N-topology over X. It is easily seen that (X, τ) is an N-T_0-space. But, it is not an N-T_1-space. Because, $x_{1,1,0}$ and $y_{1,1,0}$ are N-points in (X, τ) with different supports and the only N-open set that contains $y_{1,1,0}$ is 1_X.

Example 3.3. *Consider that $X = N$ is the set of naturel numbers. For any $n \in N$, $n_{1,1,0}$ is an N-point. Clearly, there is a one-to-one compatibility between N and $\{n_{1,1,0} : n \in N\}$. Then, we can define a cofinite topology on $\{n_{1,1,0} : n \in N\}$. That is, an N-set F is N-open iff it is constituted by discarding a finite number of elements from $\{n_{1,1,0} : n \in N\}$. Hence, this cofinite topological space is an N-T_1-space. But, it is not an N-T_2-space.*

Theorem 3.4. *An N-subspace (Y, τ_Y) of an N-T_i-space (X, τ) is an N-T_i-space $(i = 0, 1, 2)$.*

Proof. (Case $i = 0$) Consider that (X, τ) is an N-T_0-space and (Y, τ_Y) is an N-subspace of (X, τ). Take any two N-points $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ in (Y, τ_Y) with different supports. Then, $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ are also N-points in (X, τ). Since (X, τ) is an N-T_0-space, there exist N-sets F and G such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in F^c$ or $y_{\alpha',\beta',\gamma'} \in G$, $x_{\alpha,\beta,\gamma} \in G^c$. Consider an N-set H as given in Definition 2.4. Then, $F \cap H$ and $G \cap H$ are N-open sets in (Y, τ_Y) such that $x_{\alpha,\beta,\gamma} \in F \cap H$, $y_{\alpha',\beta',\gamma'} \in (F \cap H)^c$ or or $y_{\alpha',\beta',\gamma'} \in G \cap H$, $x_{\alpha,\beta,\gamma} \in (G \cap H)^c$. This implies that (Y, τ_Y) is N-T_0. □

In the other cases in which $i = 1$ and $i = 2$, we can make the proofs in similar ways. So, they are omitted.

4. N-R_i-Spaces (i=0, 1)

In this section, we introduce N-R_0 and N-R_1 spaces. Their connections with N-T_1 and N-T_2 spaces are investigated. Also, we define the concept of N-topological space induced by a topological space and some implications are given in induced N-topological spaces. Additionally, it is shown that inverse statements of these implications are not always true with counter examples.

Definition 4.1. *An N-topological space (X,τ) is said to be an N- R_0-space iff, for any two N-points $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$, if $x_{\alpha,\beta,\gamma}\tilde{q}y_{\alpha',\beta',\gamma'}$ then $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$.*

Definition 4.2. *An N-topological space (X,τ) is said to be an N-R_1 -space iff, for any two N-points $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ then $x_{\alpha,\beta,\gamma}$, if $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$ then there exists two N-open sets F and G in (X,τ) such that $x_{\alpha,\beta,\gamma}\in F$, $y_{\alpha',\beta',\gamma'}\in G$ and $F\tilde{q}G$.*

Theorem 4.1. *Every N-T_i-space (X,τ) is an N-R_{i-1}-space $(i=1,2)$.*

Proof. Obvious. □

Theorem 4.2. *An N-topological space (X,τ) is an N-T_i-space iff it is N-T_{i-1} and N-R_{i-1} $(i=1,2)$.*

Proof. (Case $i=2$) The necessity is obvious from Theorem 4.1 and the diagram given after Definition 3.3. Consider an N-topological space (X,τ) which is N-T_1 and N- R_1. Take two N-points $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ in (X,τ) with different supports. Then, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$. Since (X,τ) is N-T_1, $x_{\alpha,\beta,\gamma}\tilde{q}\overline{y_{\alpha',\beta',\gamma'}}$. Then, there exists two N-open sets F and G in (X,τ) such that $x_{\alpha,\beta,\gamma}\in F$, $y_{\alpha',\beta',\gamma'}\in G$ and $F\tilde{q}G$ for (X,τ) is N-R_1. Hence, (X,τ) is N-T_2. In the cases in which $i=1$, we can make the proof in similar way. So, it is omitted. □

Definition 4.3. *Consider that (X,τ) is a topological space and A is a subset of X. Let the N-set X_A be whose is with membership function $T_{X_A}(x)$, indeterminacy function $I_{X_A}(x)$ and non-membership function $F_{X_A}(x)$ defined as follows;*

$$T_{X_A}(x)=\begin{cases}1 & ,\text{ if } x\in A,\\ 0 & ,\text{ if } x\notin A,\end{cases}$$

$$I_{X_A}(x)=\begin{cases}1 & ,\text{ if } x\in A,\\ 0 & ,\text{ if } x\notin A,\end{cases}$$

$$F_{X_A}(x)=\begin{cases}1 & ,\text{ if } x\in A,\\ 0 & ,\text{ if } x\notin A,\end{cases}$$

X_A is called an N-set induced by A and the family $\delta_\tau=\{X_A : A\in X\}$ is called an N-topology over X induced by τ.

Theorem 4.3. *Consider that (X,τ) is a topological space and (X,δ_τ) is an N-topological space, where δ_τ is an N-topology induced by τ. If (X,δ_τ) is N-T_0-space then (X,τ) is a T_0-space.*

Proof. Take any two distinct points $x,y\in X$. Then, $x_{1,1,0}$ and $y_{1,1,0}$ are two N-points in (X,δ_τ) with different supports. Since (X,δ_τ) is N-T_0-space, there exist N-sets F, G such that $x_{1,1,0}\in F$, $y_{1,1,0}\in F^c$ or $y_{1,1,0}\in G$ and $x_{1,1,0}\in G^c$. Then there exists $0_F\in\tau$ such that $x\in 0_F$, $y\notin 0_F$, where $F=X_{0_F}\in\delta_\tau$ or there exists $0_G\in\tau$ such that $y\in 0_G$,$x\notin 0_G$, where $G=X_{0_G}\in\delta_\tau$. Hence, (X,τ) is a T_0-space. □

Theorem 4.4. *Consider that (X,τ) is a topological space and (X,δ_τ) is an N-topological space, where δ_τ is an N-topology induced by τ. If (X,δ_τ) is N-T_1-space then (X,τ) is a T_1-space.*

Proof. The proof is similar to that of above theorem. □

The converse statements may not be true as seen in then following examples.

Example 4.1. *Consider that $X = \{x, y, z\}$. Then, the family $\tau = \{\emptyset, X, \{x\}, \{z\}, \{x, z\}\}$ is a topology over X and $\delta_\tau = \{0_X, 1_X, x_{1,1,0}, z_{1,1,0}, \{x_{1,1,0}, z_{1,1,0}\}\}$ is an N-topology over X induced by τ. Then, (X, τ) is a T_0-space. But, (X, δ_τ) is not N-T_0-space.*

Example 4.2. *Consider that $X = \{x\}$. Then, the family $\tau = \{\emptyset, X\}$ is a topology over X and $\delta_\tau = \{0_X, 1_X\}$ is an N-topology over X induced by τ. Then, (X, τ) is a T_1-space. But, (X, δ_τ) is not N-T_1-space.*

5. N-regular, N-normal and N-T_i-Spaces (i=3, 4)

In this section, we first introduce N-regular spaces and N-normal spaces. Some of their characteristics are given and the relationships with N-R_0 and N-R_1 spaces are investigated. Then, we introduce N-T_3 spaces, N-T_4 spaces and examine their relations.

Definition 5.1. *An N-topological space (X, τ) is said to be an N-regular (N-R_2-space, for short) space iff, for any N-points $x_{\alpha,\beta,\gamma}$ and any N-closed set H in (X, τ) such that $x_{\alpha,\beta,\gamma} \tilde{q} H$, there exists two N-open sets F and G in (X, τ) such that $x_{\alpha,\beta,\gamma} \in F$, $H \subset G$ and $F \tilde{q} G$.*

Definition 5.2. *An N-topological space (X, τ) is said to be an N-normal (N-R_3-space, for short) space iff, for any two N-closed sets H and K in (X, τ) such that $H \tilde{q} K$, there exists two N-open sets F and G in (X, τ) such that $H \subset F$, $K \subset G$ and $F \tilde{q} G$.*

Theorem 5.1. *Consider that (X, τ) and (Y, δ) are N-topological spaces and $f : (X, \tau) \to (Y, \delta)$ is an N-function which is bijective, N-continuous and N-open. If (X, τ) is N-normal, then (Y, δ) is also N-normal.*

Proof. Consider that F and G is N-closed sets in (Y, δ) such that $F \tilde{q} G$. Since f is N-continuous, $f^{-1}(F)$ and $f^{-1}(G)$ are also N-closed sets in (X, τ) and $f^{-1}(F) \tilde{q} f^{-1}(G)$. Then, there exists N-open sets K and L such that $f^{-1}(F) \subset K$, $f^{-1}(G) \subset L$ and $K \tilde{q} L$. It follows that $F \subset f(f^{-1}(F)) \subset f(K)$, $G \subset f(f^{-1}(G)) \subset f(L)$ and $f(K) \tilde{q} f(L)$. Since, f is N-open, $f(K)$ and $f(L)$ are N-open sets such that $F \subset f(K)$, $G \subset f(L)$ and $f(K) \tilde{q} f(L)$. Hence, (Y, δ) is N-normal. □

Theorem 5.2. *Consider that (X, τ) and (Y, δ) are N-topological spaces and $f : (X, \tau) \to (Y, \delta)$ is an N-function which is bijective, N-continuous and N-open. If (X, τ) is N-regular, then (Y, δ) is also N-regular.*

Proof. It is similar. □

Theorem 5.3. *Consider that (X, τ) is an N-topological space and $x_{\alpha,\beta,\gamma}$ is any N-point in (X, τ). Then, (X, τ) is an N-R_2-space iff, for every N-open set F such that $x_{\alpha,\beta,\gamma} \in F$, there exists an N-open set G such that $x_{\alpha,\beta,\gamma} \in G$ and $\overline{G} \subset F$.*

Proof. Consider that (X, τ) is an N-R_2-space and $x_{\alpha,\beta,\gamma}$ is any N-point in (X, τ). Let an N-open set F be in (X, τ) such that $x_{\alpha,\beta,\gamma} \in F$. Then, F^c is an N-closed set in (X, τ). It is clear that $F \tilde{q} F^c$. Since $x_{\alpha,\beta,\gamma} \in F$, $x_{\alpha,\beta,\gamma} \tilde{q} F^c$. There exist N-open sets G and H such that $x_{\alpha,\beta,\gamma} \in G$, $F^c \subset H$ and $G \tilde{q} H$. This implies that $G \subset H^c$. Since H^c is an N-closed set in (X, τ), $G \subset H^c$. Conversely, let $x_{\alpha,\beta,\gamma}$ be an N-point in (X, τ) and F be an N-closed set such that $x_{\alpha,\beta,\gamma} \tilde{q} F$. Then, $x_{\alpha,\beta,\gamma} \in F^c$ and F^c

is an N-open set in (X, τ). From our hypothesis, there exists an N-open set G such that $x_{\alpha,\beta,\gamma} \in G$ and $\overline{G} \subset F^c$. So, $F \subset (\overline{G})^c$ and $G\tilde{q}(\overline{G})^c$. Clearly, $G\tilde{q}(\overline{G})^c$. This implies that (X, τ) is an N-R_2-space. □

Theorem 5.4. *Consider that (X, τ) is an N-topological space. Then, (X, τ) is an N-R_2-space iff, for every N-point $x_{\alpha,\beta,\gamma}$ and N-closed set F in (X, τ) such that $x_{\alpha,\beta,\gamma}\tilde{q}F$, there exist N-open sets G and H such that $x_{\alpha,\beta,\gamma} \in G$, $F \subset H$ and $\overline{G}\tilde{q}\overline{H}$.*

Proof. Consider that (X, τ) is an N-R_2-space. Take an N-point $x_{\alpha,\beta,\gamma}$ and an N-closed set F in (X, τ) such that $x_{\alpha,\beta,\gamma}\tilde{q}F$. Then, there exist N-open sets G and H such that $x_{\alpha,\beta,\gamma} \in G$, $F \subset H$ and $G\tilde{q}H$. So, $H \subset G^c$. Since G^c is N-closed in (X, τ), $\overline{H} \subset G^c$. Clearly, $\overline{H}\tilde{q}G$. It is easily seen that $x_{\alpha,\beta,\gamma}\tilde{q}H$. Since (X, τ) is an N-R_2-space, there exist N-open sets K and L such that $x_{\alpha,\beta,\gamma} \in K$, $\overline{H} \subset L$ and $K\tilde{q}L$. So, $K \subset L^c$. Since L^c is N-closed in (X, τ), $\overline{K} \subset L^c$. Clearly, $\overline{K}\tilde{q}L$. Therefore, $\overline{K}\tilde{q}\overline{H}$.
The proof of the converse statement is obvious. So, it is omitted. □

Theorem 5.5. *Consider that (X, τ) is an N-topological space and F is any N-closed set in (X, τ). Then, (X, τ) is an N-R_3-space iff, for every N-open set G such that $F \subset G$, there exists an N-open set H such that $F \subset H$ and $\overline{H} \subset G$.*

Proof. The proof is analogous to that of Theorem 5.3. □

Theorem 5.6. *Consider that (X, τ) is an N-topological space. Then, (X, τ) is an N-R_3-space iff, for every N-closed sets F, G in (X, τ) such that $F\tilde{q}G$, there exist N-open sets K and H such that $F \subset K$, $G \subset H$ and $\overline{K}\tilde{q}\overline{H}$.*

Proof. The proof is analogous to that of Theorem 5.4. □

Theorem 5.7. *Consider that (X, τ) is an N-topological space. If (X, τ) is an N-R_2 then it is an N-R_1-space.*

Proof. It is obvious. □

Theorem 5.8. *Consider that (X, τ) is an N-topological space. If (X, τ) is an N-R_3 and N-R_0- space then it is an N-R_2-space.*

Proof. Consider that (X, τ) is an N-R_3 and N-R_0-space. Take an N-point $x_{\alpha,\beta,\gamma}$ and an N-closed set F in (X, τ) such that $x_{\alpha,\beta,\gamma}\tilde{q}F$. Since (X, τ) is an N-R_0-space, $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}F$. Since (X, τ) is an N-R_3-space, there exist N-open sets G and H such that $\overline{x_{\alpha,\beta,\gamma}} \subset G$, $F \subset H$ and $G\tilde{q}H$. Hence, (X, τ) is an N-R_2-space. □

Corollary 5.9. *Let (X, τ) be an N-topological space. If (X, τ) is an N-R_3 and N-R_0-space then it is an N-R_1-space.*

Proof. It follows from Theorem 5.7 and Theorem 5.8. □

Definition 5.3. *An N-topological space (X, τ) is said to be an N-T_3-space iff, it is both an N-R_2 and N-T_1-space.*

Definition 5.4. *An N-topological space (X, τ) is said to be an N-T_4-space iff, it is both an N-R_3 and N-T_1-space.*

Theorem 5.10. *Consider that (X, τ) is an N-topological space. If (X, τ) is an N-T_4 - space then it is an N-T_3-space.*

Proof. Consider that (X, τ) is an N-T_4-space. Then, it is both N-R_3 and N-T_1. From Theorem 4.1, it is N-R_0. Take an N-point $x_{\alpha,\beta,\gamma}$ and an N-closed set F such that $x_{\alpha,\beta,\gamma}\tilde{q}F$. This implies that $\overline{x_{\alpha,\beta,\gamma}}\tilde{q}F$. Then, there exist N-open sets G and H such that $\overline{x_{\alpha,\beta,\gamma}} \subset G$, $F \subset H$ and $G\tilde{q}H$. Thus, (X, τ) is N-R_2. Hence, we obtain the result. □

Theorem 5.11. *Consider that (X, τ) is an N-topological space. If (X, τ) is an N-T_3-space then it is an N-T_2-space.*

Proof. It follows from Theorem 5.7 and Theorem 4.2. □

Theorem 5.12. *An N-subspace (Y, τ_Y) of an N-T_3-space (X, τ) is N-T_3.*

Proof. Consider that (X, τ) is an N-T_3-space, $Y \subseteq X$ and (Y, τ_Y) is an N-subspace as described in Definition 2.4. Let $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ in (Y, τ_Y) be N-points in (Y, τ_Y) such that $x_{\alpha,\beta,\gamma}\tilde{q}y_{\alpha',\beta',\gamma'}$. It is obvious that $x_{\alpha,\beta,\gamma}$ and $y_{\alpha',\beta',\gamma'}$ are also N-points in (X, τ). Since (X, τ) is an N-T_1-space, there exists N-open sets F and G in (X, τ) such that $x_{\alpha,\beta,\gamma} \in F$, $y_{\alpha',\beta',\gamma'} \in G$ and $F\tilde{q}G$. Then, there exists N- open sets H and K in (Y, τ_Y) such that $H = F \cap Y$ and $K = G \cap Y$. Clearly, $x_{\alpha,\beta,\gamma} \in H$, $y_{\alpha',\beta',\gamma'} \in K$ and $H\tilde{q}K$. This implies that (Y, τ_Y) is N-T_1. Now, we must show that (Y, τ_Y) is also an N-regular space. Let G be an N-closed set in (Y, τ_Y) and $x_{\alpha,\beta,\gamma}$ be an N-point in (Y, τ_Y) such that $x_{\alpha,\beta,\gamma}\tilde{q}G$. It is obvious that $x_{\alpha,\beta,\gamma}$ is also an N-point in (X, τ) and there exists a N-closed set F in (X, τ), $G = F \cap Y$. It is obvious that $x_{\alpha,\beta,\gamma}\tilde{q}F$. Since (X, τ) is a N-regular space, there exists N-open sets H and L in (X, τ) such that $x_{\alpha,\beta,\gamma} \in H$, $F \subset L$ and $H\tilde{q}L$. Then, there exists N-open sets K and M in (Y, τ_Y) such that $K = H \cap Y$ and $M = L \cap Y$. Clearly, $x_{\alpha,\beta,\gamma} \in K$, $G \subset M$ and $K\tilde{q}M$. This implies that (Y, τ_Y) is N-regular. Hence, (Y, τ_Y) is a N-T_3-space. □

Theorem 5.13. *An N-subspace (Y, τ_Y) of a N-T_4-space (X, τ) is N-T_4.*

Proof. The proof is similar to that of Theorem 5.12. □

6. Conclusion

Thus, we have brought a new perspective to the world of topology on separation axioms in N-topological spaces. In addition, we have given a new definition for N-subspace that we think will benefit the other mathematical studies especially in topology. It is our wish that the new terms and concepts we offer will help other scientists around the world to create new fields of work and make inventions that will benefit people. Besides, among our expectations, this study will pave the way for studies in the fields of statistics, medicine, economics, engineering and many different sciences, and to minimize the problems people face in their daily lives.

Conflict of interests

The authors declare that there is no conflict of interests.

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ABOUT SEQUENTIALLY OPEN AND CLOSED SUBSETS IN PRODUCT SPACES

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Abstract. We remind two facts for topological spaces. The one is that in a Hausdorff space X each convergent sequence has a unique limit. This allows us to have a function from the set of all convergent sequences in X to X. Another is that in the first countable spaces, some topological objects such as open subsets, closed subsets, closures and interiors of the sets, continuous functions and many others can be defined in terms of convergent sequences.

In this paper we compare these notions with their sequential versions in topological spaces. We will take the product spaces into account and give some results.

1. Introduction

Convergent sequences are important not only in pure mathematics but also in some others such as information theory, biological science and dynamical systems.

The convergent sequences enable us to give sequential definitions of open and closed subsets; and then to do these for some other topological concepts defined in terms of open and closed subsets. For example continuous maps, connectedness and compactness are among those notions. Sequential definitions of topological objects give us a relief in some proofs and solutions of the problems. Hence many authors have been in afford to find the sequential definitions of some topological objects.

In addition to the convergent sequences, in the literature there exist some varieties of other different types of convergences. The readers are referred for example to a large number of the works [7], Posner [24], Iwinski [17], Srinivasan [25], Antoni [2], Antoni and Salat [3], Spigel and Krupnik [26], Öztürk [27], Savaş and Das [28], Savaş [29], Borsik and Salat [5], [4] [13], Di Maio and Kočinac [19].

Connor and Grosse-Erdmann in [14] replacing the sequential convergence with a function defined on a subspace of the real sequences introduced G-methods. Then following this, Çakallı studied G- continuity in [10] (see also [15] and [11] for some other types of continuities), G-compactness in [12] and the G-connectedness in [9]

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(see also [8]). Mucuk and Şahan in [23] considered the notions of G-open subsets and G-neighbourhoods together with some extra properties of G-continuities.

Lin and Liu in [18] extended G-methods to arbitrary sets rather than topological spaces and presented G-hulls, G-closures, G-kernels and G-interiors. Mucuk and Çakallı recently improved G-connectedness in [21] and G-compactness in [22] for the topological groups with operations which generalises topological groups [6]. The authors in the paper [1] extend these ideas to the direction of neutrosophic topological spaces. We refer [20] and [16] for some sequential definitions and discussions.

In this paper we give an exposition of sequential definitions of some topological notions in product spaces.

We acknowledge that this paper forms some parts of thesis [30].

2. Preliminaries

Let X be a topological space. We use the boldface letters $\mathbf{x}$, $\mathbf{y}$, ... to denote the sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, ... of the terms in X; and $s(X)$ and $c(X)$ respectively the sets of all sequences and convergent sequences in X. A sequence $\mathbf{x} = (x_n)$ is said to be *convergent* to $\ell \in X$ when any open neighbourhood U of $a \in X$ includes almost all terms of $\mathbf{x}$, that means, except for a finite number of terms, all terms stay in U.

Let A be a subset of X and $x \in X$. The point $x \in X$ is said to be in the *sequentially hull* of A if there exists a sequence $\mathbf{x} = (x_n)$ in A with limit x. The sequentially hull of A is denoted by $[A]^s$ and A is said to be *sequentially closed* if $[A]^s \subseteq A$. Hence A is not sequentially closed whenever there exists a sequence $\mathbf{x} = (x_n)$ in A with a limit ℓ which is not in A.

We note that for $a \in A$, the constant sequence $\mathbf{a} = (a, a, \dots)$ has limit a and therefore we have that $A \subseteq [A]^s$. Hence A is sequentially-closed if and only if $[A]^s = A$. A subset $A \subseteq X$ is called *sequentially open* if $X \setminus A$ is sequentially closed. A subset $U \subseteq X$ is a *sequentially neighborhood* of a if there exists a sequentially open subset A of X such that $a \in A \subseteq U$.

The *sequentially closure* of A, denoted by $\overline{A}^s$, is defined to be the intersection of all sequentially closed subsets containing A, which is also a sequentially closed subset, because the intersection of squentially closed subsets is also sequentially closed. If $A \subseteq K$ and K is a sequentially closed subset, then $[A]^s \subseteq [K]^s \subseteq K$. Taking the intersection of all sequentially closed subsets including A, we conclude that $[A]^s \subseteq \overline{A}^s$.

We remind that a point a in first countable space X is an interior point of the subset A if any sequence $\mathbf{x} = (x_n)$ converging to a is almost in A. Therefore we define a point a in any topological space to be sequential interior point of A and write $a \in A^{0s}$ whenever any sequence $\mathbf{x} = (x_n)$ with limit a is almost in A or equivalently there is no any sequence $\mathbf{x} = (x_n)$ in $X \setminus A$ with limit a.

We say that A is sequentially open if $A \subseteq A^{0s}$. By the fact that the constant sequence $(x_n) = (a, a, \dots)$ converges to a, one can see that $A^{0s} \subseteq A$ and therefore A is sequentially open when $A \subseteq A^{0s}$ or equivalently $A^{0s} = A$.

3. Main Results

Let $X \times Y$ be the product space and $A \times B$ a subset of $X \times Y$. A point (x, y) of $X \times Y$ is said to be in the hull of $A \times B$ if there exists a sequence (a_n, b_n) in $A \times B$ with limit (x, y). The set of all hull points of $A \times B$ is denoted by $[A \times B]^s$. The subset $A \times B$ is *sequentially closed* if $[A \times B]^s \subseteq A \times B$. We can check that for the subsets A and B in $X \times Y$, we have $[A \times B]^s = [A]^s \times [B]^s$

Theorem 3.1. *For a topological space X; and the subsets $A, B \subseteq X$, we have the following*

(i) $[A \cap B]^s \subseteq [A]^s \cap [B]^s$.

(ii) $[A \cup B]^s = [A]^s \cup [B]^s$.

Proof. (i) For an $x \in [A \cap B]_G$, there exists a sequence $\mathbf{x} = (x_n)$ of the terms in $A \cap B$ with the limit x. Hence the sequence $\mathbf{x}$ is in both A and B; and therefore $x \in [A]_G$ and $x \in \cap[B]_G$, which means $x \in [A]^s \cup [B]^s$.

(ii) If $x \in [A \cup B]^s$, then there exists a sequence $\mathbf{x} = (x_n)$ in $A \cup B$ with limit x. Hence we can choose either a subsequence $\mathbf{a} = (a_n)$ in A or a subsequence $\mathbf{b} = (b_n)$ in B with limit x. Otherwise the sequence $\mathbf{x} = (x_n)$ is almost in $X \setminus A$ and $X \setminus B$; and therefore $\mathbf{x} = (x_n)$ is almost in $X \setminus A) \cap X \setminus B = X \setminus (A \cup B)$. This concludes that $x \in [A]^s \cup [B]^s$.

Let $x \in [A]^s \cup [B]^s$. Then either there exists a sequence $\mathbf{a} = (a_n)$ in A or a sequence $\mathbf{b} = (b_n)$ in B with limit x. Hence we can choose a sequence $\mathbf{x} = (x_n)$ in $A \cup B$ with limit x; and therefore $x \in [A \cup B]_G$. □

As a result of this theorem we can say that the finite intersections and unions of sequentially closed subsets are also sequentially closed.

Theorem 3.2. *For a topological space X and subsets $A, B \subseteq X$, we have the following:*

(a) $A \times B \subseteq [A \times B]^s \subseteq \overline{A \times B}^s$;

(b) $A \times B$ is sequentially closed if and only if $[A \times B]^s \subseteq A \times B$;

(c) $A \times B$ is sequentially closed if and only if $[A \times B]^s = A \times B$;

(d) If A and B are closed, then it is $A \times B$ is sequentially closed.

(e) $A \times B$ is sequentially closed if and only each convergence sequence in $A \times B$ has a limit in $A \times B$.

Proof. (a) For any point $(a, b) \in A \times B$, the constant sequence $(a_n, b_n) = ((a, b), (a, b), \dots)$ convergences to (a, b). Hence $(a, b) \in [A \times B]^s$. Further if $(x, y) \in [A \times B]^s$, there exists a sequence (a_n, b_n) in $A \times B$ which converges to (x, y). Hence $(x, y) \in \overline{A \times B}^s$.

(b) This is just the definition of a sequentially closed subset.

(c) This is a direct result of (a) and (b).

(d) If A and B are closed, then $A \times B$ is closed and therefore $\overline{A \times B}^s = A \times B$. Hence by (a) $[A \times B]^s = A \times B$, that means $A \times B$ is sequentially closed.

(e) This is obvious by the definition of a sequentially closed subset. □

Example 3.3. If $X \times Y$ has co-countable topology, then a sequence $(\mathbf{x}, \mathbf{y}) = (x_n, y_n)$ converges to (a, b) if and only if the terms are almost (a, b). Hence all subsets of $X \times Y$ are sequentially closed but not necessarily closed.

Theorem 3.4. Let $X \times Y$ be product topological spaces and let $\{A_i \times B_i \mid i \in I\}$ be a class of sets of $X \times Y$. Then we have the following

(a) $\bigcup_{i \in I} [A_i \times B_i]^s \subseteq [\bigcup_{i \in I} A_i \times B_i]^s$.

(b) $[\bigcap_{i \in I} A_i \times B_i]^s \subseteq \bigcap_{i \in I} [A_i \times B_i]^s$.

Proof. (a) If $(x, y) \in \bigcup_{i \in I} [A_i \times B_i]^s$, then $(x, y) \in [A_{i_0} \times B_{i_0}]^s$ for an $i_0 \in I$ and therefore there is a sequence (a_n, b_n) in $A_{i_0} \times B_{i_0}$ with limit (x, y). That means we have a sequence (a_n, b_n) in $\bigcup_{i \in I} A_i \times B_i$ and therefore $(x, y) \in [\bigcup_{i \in I} A_i \times B_i]^s$.

(b) For $(x,y) \in [\bigcap_{i\in I} A_i \times B_i]^s$, there exists a sequence (a_n, b_n) in $\bigcap_{i\in I} A_i \times B_i$ with limit (x,y). This means (a_n, b_n) is a sequence in each $A_i \times B_i$ for $i \in I$. Hence $(x,y) \in [A_i \times B_i]^s$, and therefore $(x,y) \in \bigcap_{i\in I}[A_i \times B_i]^s$.

□

Theorem 3.5. *For a topological space X and the subsets $A, B \subseteq X$, we have the following*
(i) $(A \cap B)^{0s} \; = \; A^{0s} \cap B^{0s}$.
(ii) $A^{0s} \cup B^{0s} \subseteq (A \cup B)^{0s}$.

Proof. (i) If $a \in (A \cap B)^{0s}$ and $\mathbf{x} = (x_n)$ is a sequence with limit a, then the sequence $\mathbf{x} = (x_n)$ is almost in $A \cap B$. Hence (x_n) is almost in both A and B and therefore $a \in A^0_G \cap B^0_G$.

On the other hand if $a \in A^{0^s} \cap A^{0^s}$ and (x_n) is a sequence with limit a, then (x_n) is almost in both A and B which means (x_n) is almost in $A \cap B$ and therefore $a \in (A \cap B)^{0^s}$

(ii) Let $a \in A^{0s} \cup B^{0s}$ and let the sequence $\mathbf{x} = (x_n)$ have the limit a. $a \in A^{0s}$ means that the sequence $\mathbf{x} = (x_n)$ is almost in A and similarly $a \in B^{0s}$ means that the sequence $\mathbf{x} = (x_n)$ is almost in B. Hence in both case the sequence is almost in $A \cup B$, which means that te sequence $\mathbf{x} = (x_n)$ is almost in A then $\mathbf{x}$ is almost either in A or in B; and therefore $a \in (A \cup B)^{0^s}$

□

As a result of Theorem 3.5 we can state that finite intersections and unions of sequentially open subsets are also sequentially open.

Theorem 3.6. *If X is a topological space and A is a subset $A \subseteq X$, then we have the following:*
(a) $(A \times B)^0 \subseteq (A \times B)^{0^s} \subseteq (A \times B)$;
(b) $A \times B$ is sequentially open if and only if $A \times B \subseteq (A \times B)^{0^s}$;
(c) $A \times B$ is sequentially open if and only if $A \times B = (A \times B)^{0^s}$;
(d) If A and B are respectively open in X and Y, then $(A \times B)$ is sequentially open.

Proof. (a) $(A \times B)^0$ is an open subset and therefore if $(a,b) \in (A \times B)^0$, then any sequence converging to (a,b) stays almost in $(A \times B)^0 \subset A \times B$. Hence $(a,b) \in (A \times B)^{0s}$. Moreover if $(a,b) \in (A \times B)^{0s}$, then any sequence converging to (a,b) becomes almost in $A \times B$. Since the constant sequence $(a_n, b_n) = ((a,b),(a,b),\dots)$ has limit (a,b) and therefore $(a,b) \in A \times B$.

(b) This is just the definition of sequentially open subset.

(c) This is a direct result of (a) and (b).

(d) If A and B are open, then $A \times B$ is open in $X \times Y$; and therefore $(A \times B)^0 = A \times B$. Hence by (a), we have that $A \times B = (A \times B)^{0s}$ which means $A \times B$ is sequentially open. □

Example 3.7. Let us consider $X \times Y$ with the co-countable topology. Then any subset $A \times B$ of $X \times Y$ is sequentially open but not necessarily open.

Theorem 3.8. *For a product topological space $X \times Y$, a subset $A \times B$ is sequentially open if and only if $X \times Y \setminus (A \times B)$ is sequentially closed.*

Proof. Assuming $A \times B \subseteq (A \times B)^{0s}$ we need to prove that $[X \times Y \setminus (A \times B)]^s \subseteq X \times Y \setminus (A \times B)$. For $(x,y) \in [X \times Y \setminus (A \times B)]^s$, there exists a sequence (x_n, y_n)

in $X \times Y \setminus A \times B$ with limit (x, y). Hence we have that $(x, y) \in X \times Y \setminus A \times B$. Otherwise if $(x, y) \in A \times B$, then by the assumption $A \times B \subseteq (A \times B)^{0s}$ we have $(x, y) \in (A \times B)^{0s}$ and therefore the sequence (x_n, y_n) is almost in $A \times B$. This is a contradiction since (x_n, y_n) is a sequence in $X \times Y \setminus A \times B$.

On the other hand assume $[X \times Y \setminus A \times B]^s \subseteq X \times Y \setminus A \times B$ and prove that $A \times B \subseteq (A \times B)^{0s}$. If $(a, b) \in A \times B$ and (x_n, y_n) is a sequence with limit (a, b), then the sequence (x_n, y_n) is almost in $A \times B$. Otherwise there exists a subsequence (x_{n_k}, y_{n_k}) of (x_n, y_n) of the terms of $X \times Y \setminus A \times B$ which has limit (a, b) and therefore $(a, b) \in [X \times Y \setminus A \times B]^s \subseteq X \times Y \setminus A \times B$ which means $(a, b) \in X \times Y \setminus A \times B$. This is a contradiction because $(a, b) \in A \times B$. $\square$

Theorem 3.9. Assume that $\{A_i \times B_i \mid i \in I\}$ is a class of the subsets in product space $X \times Y$. Then the following are satisfied.

(a) $(\bigcap_{i \in I} A_i \times B_i)^{0s} \subseteq \bigcap_{i \in I} (A_i \times B_i)^{0s}$.

(b) $\bigcup_{i \in I} (A_i \times B_i)^{0s} \subseteq (\bigcup_{i \in I} A_i \times B_i)^{0s}$.

Proof. (a) Assume that $(a, b) \in (\bigcap_{i \in I} A_i \times B_i)^{0s}$. We prove that $(a, b) \in \bigcap_{i \in I} (A_i \times B_i)^{0s}$. Let (a_n, b_n) be a sequence with limit (a, b). By assumption we have that the sequence (a_n, b_n) is almost in $\bigcap_{i \in I} A_i \times B_i$, and therefore in $A_i \times B_i$ for each $i \in I$. Hence $(a, b) \in (A_i \times B_i)^{0s}$ for each $i \in I$ and therefore $(a, b) \in \bigcap_{i \in I} (A_i \times B_i)^{0s}$.

(b) Assume $(a, b) \in \bigcup_{i \in I} (A_i \times B_i)^{0s}$ and (a_n, b_n) is a sequence with limit (a, b). By assumption $(a, b) \in (A_{i_0} \times B_{i_0})^{0s}$ for an $i_0 \in I$ and therefore the sequence (a_n, b_n) is almost in $A_{i_0} \times B_{i_0}$. That means the sequence (a_n, b_n) is almost in $(\bigcup_{i \in I} A_i \times B_i)$ and therefore $(a, b) \in (\bigcup_{i \in I} A_i \times B_i)^{0s}$. $\square$

4. Conclusion

We call a topological space X *sequentially connected* if it has no any sequentially open and closed proper subset. If X is not connected it has an open and closed proper subset $A \subseteq X$. Hence A is sequentially open and closed; and therefore X is not sequentially connected. Equivalently sequentially connected spaces are connected, but the converse is not always true. For example if X is uncountable set, then with co-countable X is connected but not sequentially connected, because all subsets of X a both re sequentially open and closed.

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ABEL'S CONVOLUTION FORMULAE THROUGH TAYLOR POLYNOMIALS

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Abstract. By making use of the Taylor polynomials, new proofs are presented for three binomial identities including Abel's convolution formula.

§1. **Introduction.** There are numerous identities in mathematical literature. Among them, Newton's binomial theorem is well known

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n.$$

Abel [1] (see [7, §3.1], for example) discovered the following deep generalizations of it with an extra λ-parameter:

$$\sum_{k=0}^{n} \binom{n}{k} x(x+k\lambda)^{k-1}(y-k\lambda)^{n-k} = (x+y)^n. \tag{1}$$

This convolution identity is fundamental in enumerative combinatorics and number theory. The reader can refer to [19] for a historical note. The known proofs can briefly be described as follows:

- Generating function method; see [9] and Chu [3].
- Series rearrangement and finite differences: Chu [4].
- The classical Lagrange expansion formula; see [17, §4.5].
- Lattice path combinatorics; see [15, §4.5] and [16, Appendix].
- The Cauchy residue method of integral representation; see [8, §2.1].
- Gould–Hsu Inverse series relations: Gould–Hsu [12] and Chu–Hsu [6, 2].
- Riordan arrays (which can trace back to Lagrange expansion); see [18].

The aim of this short article is to offer new and simple proofs for (1) and two other binomial identities via Taylor polynomials.

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§2. **Proof of** (1)**.** Denote by $P(y)$ the binomial sum in (1). Its mth derivative at $y=-x$ is determined by

$$\begin{aligned} P^{(m)}(-x) &= x\sum_{k=0}^{n-m}\frac{(n-k)!}{(n-k-m)!}\binom{n}{k}(x+k\lambda)^{k-1}(y-k\lambda)^{n-k-m}\big|_{y=-x} \\ &= \frac{n!\;x}{(n-m)!}\sum_{k=0}^{n-m}(-1)^{n-m-k}\binom{n-m}{k}(x+k\lambda)^{n-m-1}. \end{aligned} \tag{2}$$

To evaluate the last sum, we recall the difference operator Δ, which is defined for a function $f(y)$ at the point y by

$$\Delta f(y) = f(y+1) - f(y).$$

By applying n times of Δ, we have the nth difference

$$\Delta^n f(y) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(y+k).$$

In particular, when $f(y)$ is a polynomial of degree $m \le n$ with the leading coefficient c_m, then by induction, it is not hard to prove the important identity (see [13, Equation 5.42])

$$\Delta^n f(y) = n!\; c_m\; \chi(m=n), \tag{3}$$

where χ is the logical function given by $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

Therefore, the sum in (2) results in the $(n-m)$th difference of a polynomial of degree $n-m-1$. Consequently, $P^{(m)}(-x)$ vanishes for $0 \le m < n$ and $P^{(n)}(-x) = n!$.

Because $P(y)$ is a polynomial of degree n, we confirm Abel's identity (1) by expressing $P(y)$ in terms of the Taylor polynomial at $y=-x$ as follows:

$$P(y) = \sum_{m=0}^{n}\frac{(x+y)^m}{m!}P^{(m)}(-x) = (x+y)^n. \qquad \square$$

§3. **A binomial transformation.** Gould [11, Equation 1.10] recorded a binomial transformation which can be reproduced equivalently as

$$\sum_{k=0}^{n}\binom{x+1}{n-k}y^k = \sum_{i=0}^{n}\binom{x-i}{n-i}(1+y)^i. \tag{4}$$

Observing that both sides of the above equality are polynomials of degree n in y. Denote by $Q(y)$ the sum on the right–hand side. Its Maclaurin polynomial expression reads as

$$Q(y) = \sum_{k=0}^{n}\frac{y^k}{k!}Q^{(k)}(0).$$

Then we confirm (4) by computing the kth derivative of $Q(y)$ in the following manner

$$\begin{aligned} Q^{(k)}(0) &= k!\sum_{i=k}^{n}\binom{x-i}{n-i}\binom{i}{k} \\ &= k!(-1)^{n-k}\sum_{i=k}^{n}\binom{n-x-1}{n-i}\binom{-k-1}{i-k} \\ &= k!(-1)^{n-k}\binom{n-k-x-2}{n-k} = k!\binom{x+1}{n-k}, \end{aligned}$$

where the last step is justified by the Chu–Vandermonde convolution formula. $\square$

§4. **A binomial sum identity.** Let m and n be the two nonnegative integers with $m \le n$. There is an interesting binomial sum (see [20])

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(y+k\lambda)^m}{x+k} = \frac{(y-x\lambda)^m}{x\binom{x+n}{n}}. \tag{5}$$

Clearly, this is an identity between two polynomials of degree m in y. Let $R(y)$ stand for the sum on the left. Then its Taylor polynomial at $y = x\lambda$ is given by

$$R(y) = \sum_{j=0}^{m}\frac{(y-x\lambda)^j}{j!}R^{(j)}(x\lambda).$$

Evaluate the jth derivative by

$$R^{(j)}(x\lambda) = j!\binom{m}{j}\lambda^{m-j}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+k)^{m-j-1}.$$

When $0 \le j < m$, the last sum with respect to k is the nth difference of a polynomial of degree $m-j-1 < n$ that equals zero in view of (3). Instead, we have for $j = m$

$$R^{(m)}(x\lambda) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{m!}{x+k}.$$

Consequently, (5) will be confirmed if we can show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{x+k} = \frac{n!}{(x)_{n+1}}, \tag{6}$$

where the shifted factorial is defined by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n = 1, 2, \cdots.$$

In fact, it is routine to check that (6) follows from the partial fraction decomposition

$$\frac{n!}{(x)_{n+1}} = \sum_{k=0}^{n}\frac{A_k}{x+k}$$

with the connection coefficients being determined by

$$A_k = \lim_{x\to -k}\frac{n!(x+k)}{(x)_{n+1}} = \binom{n}{k}(-1)^k. \qquad \square$$

§5. **Two companion formulae.** For the formula (1), Abel [1] found also a companion one

$$\sum_{k=0}^{n}\binom{n}{k}x(x+k\lambda)^{k-1}(y-n\lambda)(y-k\lambda)^{n-k-1}=(x+y-n\lambda)(x+y)^{n-1}.$$

Besides, there exists a third one of Jensen type (cf. [14]) found by Gould [10]

$$\sum_{k=0}^{n}\binom{n}{k}(x+k\lambda)^{k}(y-k\lambda)^{n-k}=\sum_{m=0}^{n}\frac{n!}{m!}(x+y)^{m}\lambda^{n-m}.$$

Both of them reduce to the usual binomial theorem when $\lambda=0$. They can be proved by carrying out exactly the same procedure. The interested reader is encouraged to do it as an exercise.

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