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Research Article

Maximal extensions of a linear functional

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ABSTRACT. Extensions of a positive hermitian linear functional ω, defined on a dense *-subalgebra $\mathfrak{A}_o$ of a topological *-algebra $\mathfrak{A}[\tau]$ are analyzed. It turns out that their maximal extensions as linear functionals or hermitian linear functionals are everywhere defined. The situation however changes deeply if one looks for positive extensions. The case of fully positive and widely positive extensions considered in [2] is revisited from this point of view. Examples mostly taken from the theory of integration are discussed.

Keywords: Positive linear functionals, topological *-algebras, extension of linear functionals.

2020 Mathematics Subject Classification: 46H05, 46H35.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we continue the analysis, undertaken in [2], [3] of the possibility of extending a positive hermitian linear functional ω, defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra (in general, without unit), with topology τ and continuous involution *, to some elements of $\mathfrak{A}$. Moreover, we resume the notion of positive regular slight extension that closely reminds the construction of the Lebesgue integral or Segal's construction of noncommutative integration [16].

If we take, for instance, as $\mathfrak{A}$ the *-algebra of Lebesgue measurable functions on a compact interval X of $\mathbb{R}$ with the topology of convergence in measure, and $\mathfrak{A}_0 := C(X)$ is the *-algebra of continuous functions on X, then the Lebesgue integral ω_L provides an extension of the Riemann integral on $\mathfrak{A}_0$, which we regard as a positive linear functional on $\mathfrak{A}_0$. This extension is not unique as the literature on Integration Theory shows (think of Denjoy, Perron or Henstock-Kurzweil integrals see e.g. [7, 8, 1]). Thus, in an abstract set-up it makes sense to consider extensions enjoying appropriate properties. As in [2] and [3], the starting point is the notion of *slight extension*, which is treated for general linear maps in Köthe's book [6]. As application of the developed ideas, we report interesting results concerning infinite sums (see [2]).

In this paper, after showing that maximal extensions of linear functionals are necessarily everywhere defined, we revisit widely positive, fully positive and absolute convergent extensions already discussed in [2] and prove several new features that emerge from the discussion. Applications to extensions of Riemann integral on continuous functions are also examined.

We will adopt the following definitions and terminology. If $\mathfrak{A}$ is an arbitrary *-algebra, we put

$$\mathfrak{A}_h = \{b \in \mathfrak{A} : b = b^*\}, \quad \mathcal{P}(\mathfrak{A}) = \left\{ \sum_{i=1}^{n} a_i^* a_i : a_i \in \mathfrak{A} \right\}.$$

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Elements of $\mathfrak{A}_h$ are called *self-adjoint*; elements of $\mathcal{P}(\mathfrak{A})$ are called *positive*. Clearly, $\mathcal{P}(\mathfrak{A}) \subseteq \mathfrak{A}_h$. A linear functional ω, defined on a subspace $D(\omega)$ of $\mathfrak{A}$, is called

hermitian: if $a \in D(\omega) \Leftrightarrow a^* \in D(\omega)$ and $\omega(a^*) = \overline{\omega(a)}$, for every $a \in D(\omega)$;
positive: if $\omega(b) \geq 0$, for every $b \in D(\omega) \cap \mathcal{P}(\mathfrak{A})$.

Throughout this paper, we denote by ω a positive hermitian linear functional defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra $\mathfrak{A}[\tau]$, with continuous involution *.

2. Maximal extensions

The problem of finding extensions of ω to larger subspaces of $\mathfrak{A}$ has, in some situations, easy solutions, namely when ω is τ-continuous or equivalently, closable [3, 17, 19], as discussed in the Appendix. For this reason, we will only consider the case of nonclosable i.e., discontinuous ω and we denote by G_ω the graph of ω:

$$G_\omega = \{(a, \omega(a)) \in \mathfrak{A}_0 \times \mathbb{C};\ a \in \mathfrak{A}_0\}.$$

The linear functional ω is closable if $\overline{G_\omega}$, the closure of G_ω, does not contain couples $(0, \ell)$ with $\ell \neq 0$. It turns out that a linear functional is closable if and only if it is continuous (see the Appendix). Let $\mathcal{S}_\omega$ denote the collection of all subspaces H of $\mathfrak{A} \times \mathbb{C}$ such that

(g1) $G_\omega \subseteq H \subseteq \overline{G_\omega}$;
(g2) $(0, \ell) \in H$ if, and only if, $\ell = 0$.

If ω is nonclosable, i.e. $\overline{G_\omega}$ contains pairs $(0, \ell)$ with $\ell \neq 0$, then $\overline{G_\omega} \notin \mathcal{S}_\omega$. To every $H \in \mathcal{S}_\omega$, there corresponds an extension ω_H, to be called a *slight* extension of ω, defined on

$$D(\omega_H) = \{a \in \mathfrak{A} : (a, \ell) \in H\}$$

by

$$\omega_H(a) = \ell,$$

where, from (g2), ℓ is the unique complex number such that $(a, \ell) \in H$. Moreover, by applying Zorn's lemma to the family $\mathcal{S}_\omega$, one proves that ω *admits a maximal slight extension*.

Remark 2.1. *The construction relies on the fact that if $a \notin \mathfrak{A}_0$ and $(a, \ell) \in \overline{G_\omega}$, then $H := G_\omega \oplus \langle (a, \ell) \rangle \in \mathcal{S}_\omega$ and we can construct the extension ω_H. Now if $a' \notin D(\omega_H)$ and $(a', \ell') \in \overline{G_\omega}$, then $H' := H \oplus \langle (a', \ell') \rangle \in \mathcal{S}_\omega$ and we can construct a new extension $\omega_{H'}$. Continuing in this way, at the end (i.e. invoking Zorn's lemma), we will find a maximal extension of ω.*

Using the same notations of Köthe's book [6], we put

$$\mathcal{K}_\omega := \{a \in \mathfrak{A} : (a, \ell) \in \overline{G_\omega},\ \text{for some } \ell \in \mathbb{C}\}.$$

The following propositions hold [2, 3, 6].

Proposition 2.1. *Let ω be nonclosable. If there exists $m \in \mathbb{C}$ such that $(a, m) \in \overline{G_\omega}$, then $(a, \ell) \in \overline{G_\omega}$ for every $\ell \in \mathbb{C}$, hence $\overline{G_\omega} = \mathcal{K}_\omega \times \mathbb{C}$.*

From this (see Remark 2.1), follows the next:

Proposition 2.2. *If ω is nonclosable and $\mathfrak{A}_0$ is a proper subspace of $\mathcal{K}_\omega$, then ω admits infinitely many maximal extensions.*

Furthermore,

Proposition 2.3. *For every maximal extension $\breve{\omega}$ of ω, $D(\breve{\omega}) = \mathcal{K}_\omega$.*

Corollary 2.1. *An extension $\breve{\omega}$ is maximal if and only if $D(\breve{\omega}) = \mathcal{K}_\omega$.*

Remark 2.2. *A stronger consequence actually comes from previous results. If $\widehat{\omega}$ is an extension of ω, and $a \in \mathcal{K}_\omega \setminus D(\widehat{\omega})$, then for any fixed $\ell \in \mathbb{C}$ there exists a maximal hermitian extension $\breve{\omega}$ of $\widehat{\omega}$ such that $\breve{\omega}(a) = \ell$; so we can choose arbitrarily the value that an extension takes at a.*

To construct hermitian extensions (see [2, 3]), we define $\mathcal{H}_\omega$ as the collection of all subspaces $H \in \mathcal{S}_\omega$ for which the following additional condition holds

(h3) $(a, \ell) \in H$ if and only if $(a^*, \overline{\ell}) \in H$,

and then we proceed like in the case of the construction of slight extensions. In [2], it is proved that all maximal hermitian extensions share the same domain. More precisely:

Proposition 2.4. *Every ω admits a maximal hermitian extension $\breve{\omega}$ which is, at once, a maximal extension so $D(\breve{\omega}) = \mathcal{K}_\omega$. Moreover, if ω is nonclosable and $\mathfrak{A}_0$ is a proper subspace of $\mathcal{K}_\omega$, then ω admits infinitely many maximal hermitian extensions.*

Remark 2.3. *Let ω_1 be a hermitian extension of ω and let $a \in \mathcal{K}_\omega \setminus D(\omega_1)$. If we want to extend ω_1, so that ω_2 is a hermitian extension of ω_1, in general we cannot choose arbitrarily the value $\omega_2(a)$. Indeed let $a = b + ic$ with $b = b^*$ $c = c^*$, and suppose $b \notin D(\omega_1)$. Then we can choose arbitrarily the real value $\ell_1 \in \mathbb{R}$ so that $\omega_2(b) = \ell_1$. Now if $c \notin \operatorname{span}\{D(\omega_1), b\}$, then we can choose arbitrarily the real value $\ell_2 = \omega_2(c)$, but if $c \in \operatorname{span}\{D(\omega_1), b\}$, then the value ℓ_2 is already fixed. The same argument can be made in the case $c \notin D(\omega_1)$.*

3. Some interesting situations

3.1. **Extensions of the Riemann integral.** Let $X = [0, 1]$, $\mathfrak{A}$ be the *-algebra of Lebesgue measurable functions on X, τ be the topology of convergence in measure, $\mathfrak{A}_0 = C(X)$ be the *-algebra of all continuous functions on X and ω be the Riemann integral i.e.

$$\omega(f) := \int_0^1 f(x)\, dx.$$

It is well-known that the Riemann integral is nonclosable. To see this, let us consider the sequence

$$h_n(x) := \begin{cases} 2n(1 - nx) & \text{if } 0 \le x \le 1/n, \\ 0 & \text{if } 1/n < x \le 1. \end{cases} \tag{3.1}$$

Then $h_n \to 0$ almost everywhere and hence in measure, but

$$\int_0^1 h_n(x) dx = 1, \ \forall\, n \ge 1.$$

Recall that we have defined

$$G_\omega = \{(a, \omega(a)) \in \mathfrak{A}_0 \times \mathbb{C};\ a \in \mathfrak{A}_0\}.$$

We will prove the following:

Theorem 3.1. *Let ω be the Riemann integral on a compact interval $I \subseteq \mathbb{R}$. Given $g \in \mathfrak{A}$, then $(g, \ell) \in \overline{G_\omega}$, for every $\ell \in \mathbb{C}$. Hence $\overline{G_\omega} = \mathfrak{A} \times \mathbb{C}$, thus $\mathcal{K}_\omega = \mathfrak{A}$.*

Proof. We can suppose, without loss of generality, that $I = [0, 1]$ so we can use the previous sequence (3.1). As $\mathfrak{A}_0$ is dense in $\mathfrak{A}$, then there exists a sequence $(f_n) \subseteq \mathfrak{A}_0$, such that $f_n \to g$, and we put $\lambda_n := \omega(f_n)$. Fixed $\ell \in \mathbb{C}$, let $\alpha_n := \ell - \lambda_n$. Then $\alpha_n h_n \to 0$, $f_n + \alpha_n h_n \to g$ and $\omega(f_n + \alpha_n h_n) = \omega(f_n) + \alpha_n \omega(h_n) = \lambda_n + \ell - \lambda_n = \ell$, $\forall n$. Then $(g, \ell) \in \overline{G_\omega}$, hence $\overline{G_\omega} = \mathfrak{A} \times \mathbb{C}$. □

From Proposition 2.2, Proposition 2.3, Proposition 2.4 and Theorem 3.1, it follows the next

Theorem 3.2. *Let ω be the Riemann integral on a compact interval $I \subseteq \mathbb{R}$, then ω admits infinitely many maximal hermitian extensions with domain the whole algebra and any maximal hermitian extension of the Riemann integral has the whole algebra $\mathfrak{A}$ as domain.*

Example 3.1. *Let ω be the Riemann integral on $[0,1]$, h_n given by (3.1) and let $c(x): \ [0,1] \to \mathbb{C}$ be the following function:*

$$c(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1/x & \text{otherwise.} \end{cases}$$

Then $c \in \mathfrak{A} \setminus D(\omega)$.
Let $f_n(x): \ [0,1] \to \mathbb{C}, \ n \geq 1$, be the following sequence of functions:

$$f_n(x) := \begin{cases} n^2 x & \text{if } 0 \leq x \leq 1/n, \\ 1/x & \text{otherwise.} \end{cases}$$

Then $f_n \to c$ pointwise and hence in measure, $f_n \in \mathfrak{A}_0, \ \forall\, n \geq 1$ and

$$\omega(f_n) = \int_0^1 f_n(x)dx = 1/2 + \log(n).$$

Fixed $\ell \in \mathbb{C}$, let $\alpha_n := \ell - \omega(f_n) = \ell - (1/2 + \log(n))$. Then $\alpha_n h_n \to 0$, $f_n + \alpha_n h_n \to c$, and $\omega(f_n + \alpha_n h_n) = \omega(f_n) + \alpha_n \omega(h_n) = \ell$, $\forall n$. Since $f_n + \alpha_n h_n \in \mathfrak{A}_0$ $\forall n$, then $(c, \ell) \in \overline{G_\omega}$ and so, for any $\ell \in \mathbb{C}$, there exists a maximal hermitian extension $\widehat{\omega}$ of ω such that $\widehat{\omega}(c) = \ell$.

Remark 3.4. *The previous Example 3.1 first shows explicitly the construction used in Theorem 3.1, pointing out that any $a \in \mathfrak{A}$ is in $\mathcal{K}_\omega$; then, by Remark 2.2, it shows that, if $a \notin D(\omega)$, then $\forall \ell \in \mathbb{C}$ there exists a maximal hermitian extension $\breve{\omega}$ of ω such that $\breve{\omega}(a) = l$. We note that even if $c(x) \in \mathcal{P}(\mathfrak{A})$, we can choose $\ell < 0$. This shows that the previous construction could be inappropriate for most useful situations. As we will see later, we will be able to construct maximal positive extensions of the Riemann integral but, it is possible to prove that there are not positive extensions $\widehat{\omega}$ of the Riemann integral such that the function $c(x)$ is in the domain of $\widehat{\omega}$ (see Example 5.2).*

3.2. **The case of infinite sums.** Let $\mathfrak{S}$ denote the complex vector space of all infinite sequences of complex numbers. $\mathfrak{S}$ is a *-algebra if the product $\mathbf{a} \cdot \mathbf{b}$ of two sequences $\mathbf{a} = (a_k)$, $\mathbf{b} = (b_k)$, $k \geq 1$, is defined componentwise and the involution by $\mathbf{a}^* = (\overline{a_k})$. Let us endow $\mathfrak{S}$ with the topology defined by the set of seminorms

$$p_k(\mathbf{a}) = |a_k|, \quad \mathbf{a} = (a_k) \in \mathfrak{S}.$$

Let $\mathfrak{S}_0$ denote the *-subalgebra of $\mathfrak{S}$ consisting of all *finite* sequences in the sense that $\mathbf{a} = (a_k) \in \mathfrak{S}_0$ if, and only if, there exists $N \in \mathbb{N}$ such that $a_k = 0$ if $k > N$. We define

$$\omega(\mathbf{a}) = \sum_{k=1}^{\infty} a_k, \quad \mathbf{a} = (a_k) \in \mathfrak{S}_0.$$

The symbol of series is only *graphic* since all sums are finite. This functional, which is obviously positive hermitian, is nonclosable. To see this, let us consider the sequence of sequences $(\mathbf{a}_n) = ((a_{n,k})) \subseteq \mathfrak{S}_0$ with, for $n \geq 1$,

$$a_{n,k} := \delta_{n,k} \quad \text{(the Kronecker delta).}$$

For fixed k, clearly $\lim_{n\to\infty} a_{n,k} = 0$. Hence $\mathbf{a}_n \to \mathbf{0}$ as $n \to \infty$ and, applying ω, we get

$$\omega(\mathbf{a}_n) = \omega((a_{n,k})) = 1, \quad \forall n \geq 1.$$

We observe that any convergent series which converges to $l \in \mathbb{C}$, can be "rewritten" as a sequence of sequences $(\mathbf{a}_n) \subseteq \mathfrak{S}_0$, with $\mathbf{a}_n \to \mathbf{0}$ and $\omega(\mathbf{a}_n) \to l$, as $n \to \infty$. Indeed, given the series $c_1 + c_2 + c_3 \dots$ converging to l, we define $(\mathbf{a}_n) = ((a_{n,k}))$, for $n \geq 1$, as follows:

$$a_{n,k} := \begin{cases} c_{n+1-k} & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

Clearly $((a_{n,k})) \subseteq \mathfrak{S}_0$ and $\omega((a_{n,k})) = c_1 + c_2 + \cdots + c_n$. Since the series is convergent, for fixed k, $a_{n,k} \to 0$ as $n \to \infty$ and, finally, $\omega((a_{n,k})) \to l$ as $n \to \infty$.

The next proposition shows that in this case $\mathcal{K}_\omega$ is not a proper subset of the algebra.

Proposition 3.5. *Let $\mathfrak{S}$ and ω be as above. Then $\mathfrak{S}_0$ is a dense subalgebra of $\mathfrak{S}$ and $\mathcal{K}_\omega = \mathfrak{S}$.*

Proof. See Proposition 4.2 of [2]. □

Now, it seems interesting to us to show another example in which $\mathcal{K}_\omega$ coincides with the entire algebra $\mathfrak{A}$. Starting with a subalgebra of $\mathfrak{S}$ and changing the topology with a finer one, we will find a new topological *-algebra $\mathfrak{S}_1$. Then, taking the closure of $\mathfrak{S}_0$ in $\mathfrak{S}_1$, we will obtain the required algebra $\mathfrak{A} \subseteq \mathfrak{S}_1$.

We point out that in the following we will adopt notations that are not the usual ones. Let us consider the subalgebra $\mathfrak{S}_1 \subseteq \mathfrak{S}$ of all bounded sequences $x = (x_k)$, endowed with the norm

$$\|x\|_\infty = \sup_k |x_k|.$$

Then $\mathfrak{S}_1$ is a topological (precisely, a Banach) *-algebra with $\mathfrak{S}_0 \subseteq \mathfrak{S}_1$. In [2], it is shown first that the closure of $\mathfrak{S}_0$ in $\mathfrak{S}_1$ is the algebra $\mathfrak{A} := \{(c_k) \in \mathfrak{S}_1 \ : \ |c_k| \to 0 \text{ as } k \to \infty\}$; then it is shown that ω is a nonclosable positive hermitian linear functional defined on $\mathfrak{S}_0$ (a dense *-subalgebra of the topological *-algebra $\mathfrak{A}$); finally it is shown (see Proposition 4.10 of [2]) that, even in this case, $\mathcal{K}_\omega = \mathfrak{A}$.

4. The domain of maximal extensions

As seen in Theorem 3.2 for the case of the Riemann integral and in Proposition 3.5 for the infinite sums, all maximal extensions of a nonclosable linear functional have the same domain. The following theorem generalizes this statement to the abstract case.

Theorem 4.3. *Let ω be nonclosable. Then $\overline{G_\omega} = \mathfrak{A} \times \mathbb{C}$. Hence $\mathcal{K}_\omega = \mathfrak{A}$.*

Proof. As ω is nonclosable, then there exists a net $(a_\alpha)_{\alpha \in \Gamma} \subseteq \mathfrak{A}_0$, such that $a_\alpha \xrightarrow{\tau} 0$ and $\omega(a_\alpha) \to l \in \mathbb{C}$, with $l \neq 0$. Now let $b \in \mathfrak{A}$. Since $\mathfrak{A}_0$ is a dense subalgebra of $\mathfrak{A}$, there exists a net $(b_\alpha)_{\alpha \in \Gamma} \subseteq \mathfrak{A}_0$, such that $b_\alpha \xrightarrow{\tau} b$: indeed, since $\mathfrak{A}$ is a topological vector space, we can choose as unique set of indices Γ, the class of all neighbourhoods of 0, directed by inclusion. Now, since $l \neq 0$, there exist subnets $(a_\gamma) \subseteq (a_\alpha)$, $(b_\gamma) \subseteq (b_\alpha)$, $\gamma \in \Gamma_1 \subseteq \Gamma$, such that:

- $a_\gamma \xrightarrow{\tau} 0$;
- $\omega(a_\gamma) \neq 0, \ \forall \gamma$;
- $b_\gamma \xrightarrow{\tau} b$.

Let $\lambda \in \mathbb{C}$ and, for each γ, let $\lambda'_\gamma := \lambda - \omega(b_\gamma) \in \mathbb{C}$. We assert that there exists a monotone function $h : \Gamma_1 \to \Gamma_1$, such that

$$\lambda'_\gamma \cdot a_{h(\gamma)} \xrightarrow{\tau} 0.$$

Indeed if $\lambda'_\gamma = 0$ we put $h(\gamma) = \gamma$; otherwise, for every neighbourhood U of the origin, $U' := 1/\lambda'_\gamma U$ is still a neighbourhood of the origin, so there exists $\gamma' \geq \gamma$ such that $a_{\gamma'} \in U'$, and therefore $\lambda'_\gamma \cdot a_{\gamma'} \in U$. Now, since $a_\gamma \xrightarrow{\tau} 0$, put $h(\gamma) := \gamma'$, we have $\lambda'_\gamma \cdot a_{h(\gamma)} \xrightarrow{\tau} 0$. Since, obviously, $\omega(a_{h(\gamma)}) \to l$ then:

$$\frac{\lambda'_\gamma}{\omega(a_{h(\gamma)})} \cdot a_{h(\gamma)} \to 0.$$

Hence:

$$\frac{\lambda'_\gamma}{\omega(a_{h(\gamma)})} \cdot a_{h(\gamma)} + b_\gamma \to b \tag{4.2}$$

and

$$\omega\left(\frac{\lambda'_\gamma}{\omega(a_{h(\gamma)})} \cdot a_{h(\gamma)} + b_\gamma\right) = \frac{\lambda - \omega(b_\gamma)}{\omega(a_{h(\gamma)})} \cdot \omega(a_{h(\gamma)}) + \omega(b_\gamma) = \lambda, \ \forall\gamma. \tag{4.3}$$

Therefore (3.1) and (4.2) imply that $\forall b \in \mathfrak{A}$, and $\forall \lambda \in \mathbb{C}$, $(b, \lambda) \in \overline{G_\omega}$, from which the statement follows. □

By Proposition 2.4, the analogue of Theorem 3.2 is the following:

Theorem 4.4. *Let ω be nonclosable. Then ω admits infinitely many maximal hermitian extensions with domain the whole algebra $\mathfrak{A}$ and any maximal hermitian extension of ω has the whole algebra $\mathfrak{A}$ as domain.*

5. Widely positive and fully positive extensions

We have proved that all maximal extensions of a nonclosable linear functional ω are defined on the whole algebra $\mathfrak{A}$. This leads to a significant simplification on the the notion of *widely positive* and *fully positive* extension introduced in [2]. By Theorem 4.3, the definitions can be lightened and, in this way, several new developments emerge.

Definition 5.1. *Given ω, we define $\mathcal{P}_\omega$ as the collection of all subspaces $K \in \mathcal{H}_\omega$ satisfying the following additional condition*

(p4) *$(a, \ell) \in K$ and $a \in \mathcal{P}(\mathfrak{A})$, implies $\ell \geq 0$.*

Since ω is positive, then $\mathcal{P}_\omega \neq \emptyset$ and $G_\omega \subseteq K \subseteq \mathfrak{A}$ for every $K \in \mathcal{P}_\omega$. To every $K \in \mathcal{P}_\omega$, there corresponds a hermitian extension ω_K of ω, defined on

$$D(\omega_K) = \{a \in \mathfrak{A} : (a, \ell) \in K\}$$

by

$$\omega_K(a) = \ell, \quad a \in D(\omega_K),$$

where, from $(g2)$ of Section 2, ℓ is the unique complex number such that $(a, \ell) \in K$. By (p4), ω_K is a positive hermitian extension of ω. We observe that $\mathfrak{A}_0 \subseteq D(\omega_K) \subseteq \mathfrak{A}$ as vector spaces. Since $\mathcal{P}_\omega$ satisfies the assumptions of Zorn's lemma, we have the following:

Theorem 5.5. *Every positive hermitian linear functional ω admits a maximal positive hermitian extension.*

Definition 5.2. *Let $\widehat{\omega}$ be an extension of ω defined on the domain $D(\widehat{\omega})$ with $\mathfrak{A}_0 \subseteq D(\widehat{\omega}) \subseteq \mathfrak{A}$. We say that $\widehat{\omega}$ is* fully positive *if $\widehat{\omega}$ is positive and $D(\widehat{\omega}) \supseteq \mathcal{P}(\mathfrak{A})$.*

For $a, b \in \mathfrak{A}_h$, we define

$$a \leq b \Leftrightarrow b - a \in \mathcal{P}(\mathfrak{A}).$$

Remark 5.5. *Let $\widehat{\omega}$ be a hermitian extension of ω, $a \in D(\widehat{\omega})$ and $c \in \mathfrak{A}_h$. If $b := \pm(a-c) \in \mathfrak{A}_h$, then $\widehat{\omega}(a) \in \mathbb{R}$. Indeed if $b \in \mathfrak{A}_h$, then $a = c \pm b \in \mathfrak{A}_h$ and so, by the hermiticity of $\widehat{\omega}$, $\widehat{\omega}(a) \in \mathbb{R}$. Moreover if $\widehat{\omega}$ is a positive hermitian extension of ω and $a, c \in D(\widehat{\omega}) \cap \mathfrak{A}_h$ with $a \geq c$, put $b := a - c$, then $b \in \mathcal{P}(\mathfrak{A}) \cap D(\widehat{\omega})$, so $\widehat{\omega}(a) = \widehat{\omega}(c) + \widehat{\omega}(b) \geq \widehat{\omega}(c)$. Hence $\widehat{\omega}$ is monotone on $D(\widehat{\omega}) \cap \mathfrak{A}_h$.*

If $\widehat{\omega}$ is a positive hermitian extension of ω and $c \in \mathfrak{A}_h$, then by Remark 5.5, we can introduce (see [2]) the following notations that will use both to characterize the elements for which it is possible to find a positive hermitian extension and, given such an element, the values this extension may assume.

$$\mu_{c,\widehat{\omega}} := \inf \{\widehat{\omega}(a) : \ a \in D(\widehat{\omega}), \, a \geq c\},$$

where we put $\mu_{c,\widehat{\omega}} := +\infty$ if the set in the right hand side of the definition is the empty set;

$$\lambda_{c,\widehat{\omega}} := \sup \{\widehat{\omega}(a) : \ a \in D(\widehat{\omega}), \, a \leq c\}.$$

Definition 5.3. *Let $\widehat{\omega}$ be a positive hermitian extension of ω and let*

$$\mathcal{K}^{\ddagger}_{\widehat{\omega}} := \{c \in \mathcal{P}(\mathfrak{A}) : \lambda_{c,\widehat{\omega}} \text{ is finite}\}. \tag{5.4}$$

We say that $\widehat{\omega}$ is widely positive *if $\widehat{\omega}$ is positive and $D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) = \mathcal{K}^{\ddagger}_{\omega}$.*

The following statements hold (see [2]).

Lemma 5.1. *Let $\widehat{\omega}$ be a positive hermitian extension of ω and let $c \in \mathcal{P}(\mathfrak{A})$. Then $0 \leq \lambda_{c,\widehat{\omega}} \leq \mu_{c,\widehat{\omega}}$.*

Theorem 5.6. *Let ω be nonclosable, $\widehat{\omega}$ a positive hermitian extension of ω and $c \in \mathcal{K}^{\ddagger}_{\widehat{\omega}}$ with $c \notin D(\widehat{\omega})$. Then, $\forall \, \gamma \in \mathbb{R}$ such that $\lambda_{c,\widehat{\omega}} \leq \gamma \leq \mu_{c,\widehat{\omega}}$, there exists a positive hermitian extension ω_1 of $\widehat{\omega}$, such that $c \in D(\omega_1)$ and $\omega_1(c) = \gamma$.*

Theorem 5.7. *Let $c \in \mathcal{P}(\mathfrak{A}) \setminus \mathcal{K}^{\ddagger}_{\omega}$. Then there is no positive hermitian extension $\widehat{\omega}$ of ω such that $c \in D(\widehat{\omega})$.*

In the following examples, we use the notation introduced in Section 3.1.

Example 5.2. *Let us consider again the function $c(x) : \ [0,1] \to \mathbb{C}$*

$$c(x) := \begin{cases} 0 & \text{if } \ x = 0 \\ 1/x & \text{otherwise} \end{cases},$$

and let, like in Example 3.1, $f_n(x) : \ [0,1] \to \mathbb{C}, \ n \geq 1$ be the sequence

$$f_n(x) := \begin{cases} n^2 \, x & \text{if } \ 0 \leq x \leq 1/n, \\ 1/x & \text{otherwise.} \end{cases}$$

If ω is the Riemann integral on $[0,1]$, then $\omega(f_n) = 1/2 + \log(n)$ and $0 \leq f_n(x) \leq c(x), \forall x \in [0,1], \forall n \geq 1$. Since $\omega(f_n) \to +\infty$, as $n \to \infty$ then, by definition, $\lambda_{c,\omega} = +\infty$, so there is no positive hermitian extension ω' of Riemann integral such that $c \in D(\omega')$.

Remark 5.6. *The previous Example 3.1 shows that if we impose to an extension $\widehat{\omega}$ the constraint to be positive, differently from the case of Theorem 3.2, the domain of the extension is, in general, a proper subset of the algebra $\mathfrak{A}$: $D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) \subsetneq \mathcal{P}(\mathfrak{A})$. In particular, the following result holds true.*

Theorem 5.8. *There are no fully positive extensions of the Riemann integral.*

Remark 5.7. *We note that in the case of the Riemann integral with $c \in \mathfrak{A}_h$, $\lambda_{c,\omega}$ and $\mu_{c,\omega}$ correspond to the lower and upper Riemann integral, respectively.*

Let us now consider the following function $c_1(x) : [0,1] \to \mathbb{C}$:

$$c_1(x) := \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/x & \text{otherwise.} \end{cases}$$

From the density of $\mathbb{Q}$ follows that $\lambda_{c_1,\omega} = 0$, so $c_1 \in \mathcal{K}_\omega^\ddagger$; but, since of course c_1 is not Lebesgue integrable, then we have proved the following:

Theorem 5.9. *The Lebesgue integral (as an extension of the Riemann integral on $\mathfrak{A}_0$) is not widely positive.*

At this point one might ask whether there exists any extension of the Lebesgue integral which is widely positive. From [2] (see Remark 3.15) it follows that, if ω_1, ω_2 are positive hermitian extensions of the Riemann integral ω, with $D(\omega_1) \subseteq D(\omega_2)$, then $\mathcal{K}_{\omega_2}^\ddagger \subseteq \mathcal{K}_{\omega_1}^\ddagger \subseteq \mathcal{K}_\omega^\ddagger$. Hence given ω_1 the Lebesgue integral on $[0,1]$, since the previous function $c_1 \in \mathcal{K}_\omega^\ddagger$, if we prove that $c_1 \notin \mathcal{K}_{\omega_1}^\ddagger$, then we will have proved the following:

Theorem 5.10. *There are no widely positive extensions of the Lebesgue integral, considered as an extension of the the Riemann integral on $\mathfrak{A}_0$.*

Proof. Let ω_1 be the Lebesgue integral and let $g_n(x) : [0,1] \to \mathbb{C},\ n \geq 1$ be the sequence

$$g_n(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq 1/n \ \vee \ x \in \mathbb{Q}, \\ 1/x & \text{otherwise.} \end{cases}$$

Then $\omega_1(g_n) = \log(n)$ and $0 \leq g_n(x) \leq c_1(x)$, $\forall x \in [0,1]$, $\forall n \geq 1$. Since $\omega_1(g_n) \to +\infty$ as $n \to \infty$, by definition, $\lambda_{c_1,\omega_1} = +\infty$, and so $c_1 \notin \mathcal{K}_{\omega_1}^\ddagger$. From this, the statement follows. □

Corollary 5.2. *There are no widely positive extensions of the Henstock-Kurzweil integral.*

On the other hand since the Lebesgue integral belongs to the family $\mathcal{P}_\omega$ of Definition 5.1, by Zorn's lemma, there exists $\breve{\omega}$, a maximal positive hermitian extension of the Riemann integral that is actually a positive hermitian extension of the Lebesgue integral. Hence the existence of $\breve{\omega}$ shows that even if a positive hermitian linear functional is a maximal extension, it is not necessarily widely positive. In other words, we have proved the following:

Proposition 5.6. *Given a positive hermitian linear functional ω, there are maximal positive hermitian extensions of ω that are not widely positive.*

Now, we want to analyse the case where we start from the the Lebesgue integral.
Notation: From now on, ω is the Lebesgue integral on a compact interval I of $\mathbb{R}$ and $\mathfrak{A}_0 := L^\infty(I) \subseteq \mathfrak{A}$ is the algebra of all measurable functions which are essentially bounded on I.

Theorem 5.11. *Let $I \subseteq \mathbb{R}$ be a compact interval. Then the Lebesgue integral ω on I is widely positive. Hence any positive hermitian extension of the Lebesgue integral is widely positive.*

Proof. By Lemma 3.19 of [2], we will just prove that $\mathcal{K}_\omega^\ddagger \subseteq D(\omega) \cap \mathcal{P}(\mathfrak{A})$. Let $c \in \mathcal{K}_\omega^\ddagger$, then $\lambda_0 := \lambda_{c,\omega} < +\infty$. Since $\mathcal{K}_\omega^\ddagger \subseteq \mathcal{P}(\mathfrak{A})$ and c is measurable, then c is the limit of a sequence (b_n) of simple functions such that $b_n \geq 0$, with (b_n) increasing and $b_n \leq c$, $\forall n \geq 1$. Since simple functions on I are Lebesgue integrable, then $\forall n \geq 1,\ b_n \in D(\omega)$, with $\omega(b_n) \leq \omega(b_{n+1})$. Then the limit $\bar{\lambda} := \lim_n \omega(b_n)$ exists and, by definition of $\lambda_{c,\omega}$, $\bar{\lambda} \leq \lambda_0 < +\infty$. Hence

$$\liminf_n \omega(b_n) = \lim_n \omega(b_n) < +\infty;$$

so, by Fatou's lemma,

$$\omega(c) = \omega(\lim_n b_n) = \omega(\liminf_n b_n) \leq \liminf_n \omega(b_n) = \lim_n \omega(b_n) = \bar{\lambda} < +\infty.$$

Hence c is Lebesgue integrable with $\omega(c) \leq \bar{\lambda}$. □

From Example 5.2, we have the next

Corollary 5.3. *There are no fully positive extensions of the Lebesgue integral. In particular, the Lebesgue integral is not fully positive.*

From Corollary 5.3 and from Theorem 5.11, it follows the next interesting:

Remark 5.8. *Let $\widehat{\omega}$ be an extension of the Lebesgue integral and let $a \in D(\widehat{\omega}) \setminus \mathfrak{A}_0$, then $a \notin \mathcal{P}(\mathfrak{A})$.*

Finally, we recall (see [3]) that the Henstock-Kurzweil integral is a positive extension of the Lebesgue integral that is not maximal, so (see [2]) we have:

Theorem 5.12. *There exists a maximal positive hermitian extension of the Henstock-Kurzweil integral.*

Returning to the general case, let $\mathfrak{A}$ be a *-algebra. We say that $\mathfrak{A}$ has the property (D) if, for every $a \in \mathfrak{A}_h$, there exists a unique pair (a_+, a_-) of elements of $\mathfrak{A}$, with $a_+, a_- \in \mathcal{P}(\mathfrak{A})$ such that

(D1) $a = a_+ - a_-$;
(D2) $a_+a_- = a_-a_+ = 0$;
(D3) $(\lambda a)_+ = \lambda a_+, \quad \forall a \in \mathfrak{A}_h, \lambda \in \mathbb{R}^+$;

then we put

$$|a| := a_+ + a_-.$$

If $\mathfrak{A}$ has the property (D), one has:

$$|a| \in \mathcal{P}(\mathfrak{A}), \quad \forall a \in \mathfrak{A}_h.$$

We remind that a positive hermitian linear functional $\bar{\omega}$ defined on a subspace of $\mathfrak{A}$ is called *absolutely convergent* if for all $a \in D(\bar{\omega}) \cap \mathfrak{A}_h$, $a_+, a_- \in D(\bar{\omega})$, and so $|a| \in D(\bar{\omega})$.

Several examples that guarantee the existence of absolutely convergent extensions, are given in [20]. Now, we state the following theorem and corollary (see [2]).

Theorem 5.13. *Let $\breve{\omega}$ be an absolutely convergent extension of ω. If $\breve{\omega}$ is widely positive, then $\breve{\omega}$ is a maximal absolutely convergent extension of ω.*

Corollary 5.4. *Let $\breve{\omega}$ be an absolutely convergent extension of ω. If $\breve{\omega}$ is fully positive, then $\breve{\omega}$ is a maximal absolutely convergent extension of ω.*

Concluding remark

The problem of extending the Riemann integral defined on continuous functions is probably as old as the Riemann integral itself. In [2] , [3] and in the present paper, this question has been cast into an abstract framework, looking for extensions of a positive hermitian linear functional ω, defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra (in general, without unit), with topology τ and continuous involution *, to a larger family of elements of $\mathfrak{A}$. Several particular cases have been discussed in those papers; among them *positive regular slight extension* arouse interest since it closely reminds the construction of the Lebesgue integral or Segal's construction of noncommutative integration [16]. We have first proved that there are maximal extension of the Riemann integral defined on the whole *-algebra of Lebesgue measurable functions on a compact interval, and then this result has been shown to hold also in the abstract case for certain functional. Of course there is a price to pay for this: for instance several familiar properties of the integral are missing for this maximal everywhere defined extension (e.g., positivity). In the end, our reader may legitimately wonder how does this extension of the integral work. This aspect is matter of further investigations.

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APPENDIX A. A DIFFERENT APPROACH

The proof of Theorem 4.3 might be modified using a different approach [13]. The starting point is observing that the closability and continuity are equivalent. Let V be a complex topological vector space with topology τ (for short, $V[\tau]$). Let ω be a nonzero linear functional defined on V. We collect some elementary (and well-known) facts.

Lemma A.2. *The following statement hold:*

(a) *The range $\omega(V)$ coincides with $\mathbb{C}$.*
(b) *The kernel $Ker\ \omega$ of ω is a proper maximal subspace of V.*
(c) *$Ker\ \omega$ is either closed or dense in $V[\tau]$.*
(d) *ω is continuous if, and only if, $Ker\ \omega$ is closed in $V[\tau]$.*
(e) *If θ is another nonzero linear functionals on V, $Ker\ \omega = Ker\ \theta$ if, and only if, θ is a multiple of ω.*

We consider the graph of ω, i.e.,

$$G_\omega = \{(x, \omega(x)) \in V \times \mathbb{C}\}.$$

The functional ω is said to be closable if one of the two equivalent statements which follow is satisfied.

- If $x_\alpha \to 0$ w.r. to τ and $\omega(a_\alpha) \to \ell$, then $\ell = 0$.
- $\overline{G_\omega}$, the closure of G_ω, does not contain couples $(0, \ell)$ with $\ell \neq 0$.

It turns out that in the case of linear functionals closability and continuity are equivalent. As a consequence, a discontinuous linear functional is *never* closable.

Remark A.9. *So far, we have considered the case of functionals that are everywhere defined on V. Suppose that Y is a dense subspace of $V[\tau]$, and let ω be a linear functional on Y. As stated before, if ω is closable in Y it is continuous on Y and then it extends by continuity to the whole space V and, of course, the extension is continuous.*

On $V \oplus \mathbb{C}$ define $\Omega(a, \ell) = \omega(a) - \ell$. It is easily seen that Ω is a linear functional on $V \oplus \mathbb{C}$.

Lemma A.3. *Ω is continuous on $V \oplus \mathbb{C}$ if, and only if, ω is continuous on V.*

Proof. Let Ω be continuous and let $(x_\alpha)_{\alpha\in\Gamma} \subseteq V$ be a net converging to $x \in V$. Then $\omega(x_\alpha) = \Omega(x_\alpha, 0) \to \Omega(x, 0) = \omega(x)$; i.e., ω is continuous. Conversely, assume that ω is continuous and that $(x_\alpha, \lambda_\alpha) \to (x, \lambda)$. Then $x_\alpha \to x$ and $\lambda_\alpha \to \lambda$. Hence,

$$\Omega(x_\alpha, \lambda\alpha) = \omega(x_\alpha) - \lambda\alpha \to \omega(x) - \lambda = \Omega(x, \lambda).$$

$\square$

Proposition A.7. *If ω, defined on V, is discontinuous, then G_ω is dense in $V \oplus \mathbb{C}$.*

Proof. By Lemma A.3, Ω is linear and discontinuous then its kernel is dense in $V \oplus \mathbb{C}$. It is easily seen that $Ker\, \Omega = G_\omega$. Thus $\overline{G_\omega} = V \oplus \mathbb{C}$. $\square$

Proposition A.8. *Let V be a vector space and W a proper subspace of V. Then, for every $x \in V \setminus W$, there exists a linear functional ω on V such that $\omega(y) = 0$ for every $y \in W$ and $\omega(x) = 1$.*

Proof. Let $x \in V \setminus W$ then the span $\mathbb{C} \cdot x$ is a subspace of V with $W \cap \mathbb{C} \cdot x = \{0\}$. Then on $W \oplus \mathbb{C} \cdot x$ we can define $\omega(y + \lambda x) = \lambda$; then, $\omega(y) = 0$, for every $y \in W$ and $\omega(x) = 1$. If $\{e_j\}$ is a Hamel basis of W, we can find linearly independent vectors $\{h_k\}$ of $V \setminus W \oplus \mathbb{C} \cdot x$ such that $\{e_j\} \cup \{h_k\} \cup \{x\}$ is a Hamel basis for V. It is now sufficient to define $\omega(h_k) = 0$, for every k. □

Let V_0 be a dense subspace of $V[\tau]$ and ω a linear functional defined on V_0.

A linear functional $\widehat{\omega}$ defined on a vector subspace $D(\widehat{\omega})$ of V is called an *extension* of ω if

$$V_0 \subseteq D(\widehat{\omega})$$
$$\widehat{\omega}(x) = \omega(x), \ \forall x \in V_0.$$

In this case, we write $\omega \subseteq \widehat{\omega}$. It is clear that $\omega \subseteq \widehat{\omega}$ if, and only if, $G_\omega \subseteq G_{\widehat{\omega}}$. If ω is continuous on V_0, it has a unique continuous extension to V. In what follows, we will consider the case when ω is discontinuous (equivalently, nonclosable) in V_0.

As in Section 2, an extension $\widehat{\omega}$ of ω is a *slight extension* if $G_{\widehat{\omega}} \subseteq \overline{G_\omega}$. We denote by $\mathcal{S}_\omega$ the collection of all subspaces H of $V \times \mathbb{C}$ such that

(g1) $G_\omega \subseteq H \subseteq \overline{G_\omega}$;
(g2) $(0, \ell) \in H$ if, and only if, $\ell = 0$.

$\mathcal{S}_\omega$ is nonempty, since it contains G_ω and each $H \in \mathcal{S}_\omega$ defines an extension ω_H as follows

$$D(\omega_H) = \{x \in V : (x, \ell) \in H\}$$
$$\omega_H(x) = \ell.$$

It is clear on the other hand that every slight extension $\widehat{\omega}$ of ω defines a subspace $H' \in \mathcal{S}_\omega$. Namely, $H' = G(\widehat{\omega})$. By Proposition A.7, it follows that $\overline{G_\omega} = V_0 \oplus \mathbb{C}$. The density of V_0 in V implies that $V_0 \oplus \mathbb{C}$ is dense in $V \oplus \mathbb{C}$. Then, we conclude that $\overline{G_\omega} = V \oplus \mathbb{C}$. Therefore, the set

$$\mathcal{K}_\omega = \{x \in V : (x, \ell) \in \overline{G_\omega}, \text{ for some } \ell \in \mathbb{C}\}$$

coincides evidently with V. From these considerations, it follows also that *every extension $\widehat{\omega}$ of ω is a slight* one.

As discussed in Section 2, the existence of maximal extensions of ω can be proved by using Zorn's lemma. Let $\breve{\omega}$ denote a maximal extension of ω. Then as proved in [2] $D(\breve{\omega}) = \mathcal{K}_\omega$. Thus, in conclusion,

Proposition A.9. *An extension $\breve{\omega}$ of ω is maximal if, and only if, $D(\breve{\omega}) = V$.*

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Research Article

On the eigenvalue-separation properties of real tridiagonal matrices

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ABSTRACT. In this paper, we give a simple sufficient condition for the eigenvalue-separation properties of real tridiagonal matrices T. This result is much more than the statement that the pertinent eigenvalues are distinct. Its derivation is based on recurrence formulae satisfied by the polynomials made up by the minors of the characteristic polynomial $\det(xE - T)$ that are proven to form a Sturm sequence. This is a new result, and it proves the simple spectrum property of a symmetric tridiagonal matrix studied in a Grünbaum paper. Two numerical examples underpin the theoretical findings. The style of the paper is expository in order to address a large readership.

Keywords: Characteristic polynomial, distinct eigenvalues, eigenvalue-separation properties, minors of determinant, Sturm sequence, tridiagonal matrix.

2020 Mathematics Subject Classification: 15A15, 15A18, 15B05, 65F15, 65F22.

1. INTRODUCTION

In [2, p.30], Grünbaum assumes that the roots of the characteristic polynomial of a special symmetric tridiagonal matrix are distinct. In this paper, we give a simple sufficient condition for this. More precisely, we are able to show that the minors of the characteristic polynomial of a real tridiagonal matrix satisfy a Sturm sequence provided that the products of corresponding entries above and below the diagonal are positive. In the case of a symmetric tridiagonal matrix, this condition means that all entries above and below the diagonal are different from zero. As a consequence, we obtain the eigenvalue-separation properties of Sturm sequences which is much more that the statement that the eigenvalues are distinct.

The paper is structured as follows. In Section 2, we take over a lemma on the Sturm sequence from [4] as the main tool to derive a sufficient condition for the eigenvalue-separation properties of tridiagonal matrices. Then, in Section 3, the lemma on Sturm sequences from Section 2 is applied to obtain the eigenvalue-separation properties of tridiagonal matrices provided that the products of corresponding entries above and below the diagonal are positive. Section 4 contains two numerical examples that underpin the theoretical findings, one with nonsymmetric and one with symmetric tridiagonal matrix. In Section 5, the conclusion is given. The non-cited references [1] and [3] are given because they also contain sections on Sturm sequences so that they may be of interest to the reader in the context of the treated subject.

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2. Preliminaries on Sturm Sequence

In [4, Section 10.3, pp. 194-200], it is shown that for every linear symmetric mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ a basis of n orthonormal vectors associated with Lanczos polynomials can be constructed such that the mapping A in this basis is represented by a symmetric tridiagonal matrix T and further that the minors of the characteristic polynomial $p(x) = det(x\,E - T)$ form a *Sturm sequence* having interesting eigenvalue-separation properties.

In this section, we take over the results derived [4] as the main tool to be used in Section 3. We start with a sequence of polynomials $p_0, p_1, \ldots, p_m$ with real β_j, γ_j that fulfill the recursion formulae

(2.1) $$p_0(x) = 1,\; p_1(x) = x - \beta_0,\; p_{j+1}(x) = (x - \beta_j)\,p_j(x) - \gamma_j\,p_{j-1}(x),\; \gamma_j > 0,$$

$j = 1, \ldots, m-1$ for $x \in \mathbb{R}$. These polynomials then form a *Sturm sequence*, also called *Sturm chain*, allowing far-reaching assertions about the position and separation properties of their zeros as stated in the following lemma.

Lemma 2.1 (Lemma on Sturm sequence)**.** *For every $j = 1, \ldots, m$, all the zeros of the polynomial p_j in (2.1) are real as well as pairwise distinct and can be arranged according to*

(P1) $$-\infty < \lambda_{jj} < \lambda_{j,j-1} < \cdots < \lambda_{j2} < \lambda_{j1} < +\infty.$$

For every $j = 1, \ldots, m-1$ and every zero λ of p_j, the relations

(P2) $$p_j(\lambda) = 0, \quad p_{j+1}(\lambda)\,p_{j-1}(\lambda) < 0$$

hold. If one sets formally $\lambda_{j,j+1} = -\infty$ and $\lambda_{j0} = +\infty$ (that are the left and right boundaries in (P1)), then

(P3) $$(-1)^k\,p_j(x) > 0, \quad \lambda_{j,k+1} < x < \lambda_{jk},\; k = 0, \ldots, j,\; j = 0, \ldots, m$$

and, for every j=0,1, ...,m-1, there is just one zero $\lambda_{j+1,k}$ of p_{j+1} in the interval

(P4) $$\lambda_{jk} < \lambda_{j+1,k} < \lambda_{j,k-1},\; k = 1, \ldots, j+1.$$

Proof. See [4, Section 10.3, Formula (27), p.200], where the text was taken over with minor changes for the reason of clarity. □

We note that the necessity of the condition $\gamma_j > 0$ in the definition of the Sturm sequence is not obvious, but is seen during the proof of Lemma 2.1 in the above-cited book. We remark further that the proof does not depend on the assumption that $p_j(x)$ is a minor of $p_m(x) = det(x\,E - T)$ with some matrix T. But, in Section 3, we will use polynomials $p_j(x)$ that are minors of $p_m(x) = det(x\,E - T)$, where T is a real tridiagonal matrix.

We mention that with the definitions $\lambda_{j,j+1} = -\infty$ and $\lambda_{j0} = +\infty$, property (P4) reads for $k = j+1$ as follows

$$(P4)_{k=j+1} \qquad -\infty < \lambda_{j+1,j+1} < \lambda_{jj}$$

and for $k = 1$ as follows

$$(P4)_{k=1} \qquad \lambda_{j1} < \lambda_{j+1,j} < +\infty.$$

We want to point out that if we would not have introduced the definitions $\lambda_{j,j+1} = -\infty$ and $\lambda_{j0} = +\infty$, then instead of $(P4)_{k=j+1}$, we would have

$$(P4)'_{k=j+1} \qquad \lambda_{j+1,j+1} < \lambda_{jj}$$

which is equivalent to $(P4)_{k=j+1}$ since we have trivially $-\infty < \lambda_{j-1,j+1}$; and instead of $(P4)_{k=1}$, we would have

$$(P4)'_{k=1} \qquad \lambda_{j1} < \lambda_{j+1,j}$$

which is equivalent to $(P4)_{k=1}$ since we have trivially $\lambda_{j-1,j} < +\infty$.

3. Application to Real Tridiagonal Matrices

This section is the core of the present paper. Its results are obtained by applying Lemma 2.1 on Sturm sequences to real tridiagonal matrices. We start with the real tridiagonal matrix

$$T = tridiag[a_{i-1}, c_i, d_i]_{i=1,\ldots,N+1} \in \mathbb{R}^{(N+1)\times(N+1)}$$

with

$$\begin{array}{ll} a_0 = a_{N+1} = 0, & a_i \in \mathbb{R},\ i = 1,\ldots,N, \\ d_0 = d_{N+1} = 0, & d_i \in \mathbb{R},\ i = 1,\ldots,N, \end{array} \quad c_i \in \mathbb{R},\ i = 1,\ldots,N+1,$$

or, written in full,

$$T = \begin{bmatrix} c_1 & d_1 & & & & \\ a_1 & c_2 & d_2 & & & \\ & a_2 & c_3 & d_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{N-1} & c_N & d_N \\ & & & & a_N & c_{N+1} \end{bmatrix}. \tag{3.2}$$

Such a matrix for the symmetric case $d_i = a_i$, $i = 1,\ldots,N$ with the special diagonal entries $c_i = b_i - a_i - a_{i-1}$, $i = 1,\ldots,N+1$, where b_i are real elements is studied in [2, p.27].

Here, we want to apply Lemma 2.1 on the Sturm sequence. For this, we change the denotations of the entries of matrix T as follows. With

$$\gamma'_0 = \gamma'_m = 0,$$

we set

$$T = tridiag[\gamma'_{i-1}, \beta_{i-1}, \delta'_i]_{i=1,\ldots,m}$$

or, written out,

$$T = \begin{bmatrix} \beta_0 & \delta'_1 & & & & \\ \gamma'_1 & \beta_1 & \delta'_2 & & & \\ & \gamma'_2 & \beta_2 & \delta'_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma'_{m-2} & \beta_{m-2} & \delta'_{m-1} \\ & & & & \gamma'_{m-1} & \beta_{m-1} \end{bmatrix}. \tag{3.3}$$

Comparison of (3.2) and (3.3) leads to

$$\beta_{i-1} = c_i,\ i = 1,\ldots,m(= N+1), \tag{3.4}$$

$$\begin{array}{rcl} \gamma'_{i-1} & = & a_{i-1},\ i = 2,\ldots,m(= N+1), \\ \delta'_{i-1} & = & d_{i-1},\ i = 2,\ldots,m(= N+1). \end{array} \tag{3.5}$$

Now, we introduce the condition

$$\begin{array}{l} a_j \neq 0,\ j = 1,\ldots,N(= m-1), \\ d_j \neq 0,\ j = 1,\ldots,N(= m-1). \end{array}$$

This entails

$$\begin{array}{l} \gamma'_j = a_j \neq 0,\ j = 1,\ldots,m-1(= N), \\ \delta'_j = d_j \neq 0,\ j = 1,\ldots,m-1(= N). \end{array}$$

Next, we define

$$\gamma_j := \gamma'_j\,\delta'_j,\ j = 1,\ldots,m-1(= N). \tag{3.6}$$

Further, we make the restriction

$$\gamma_j > 0,\ j = 1, \ldots, m-1. \tag{3.7}$$

We choose these γ_j in the Sturm sequence (2.1). Further, more generally than in [4, Section 10.3, Formula (25), p. 199], we define

$$p_j(x) = \det \begin{bmatrix} x-\beta_0 & -\delta_1' & & & & \\ -\gamma_1' & x-\beta_1 & -\delta_2' & & & \\ & -\gamma_2' & x-\beta_2 & -\delta_3' & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\gamma_{j-2}' & x-\beta_{j-2} & -\delta_{j-1}' \\ & & & & -\gamma_{j-1}' & x-\beta_{j-1} \end{bmatrix}, \tag{3.8}$$

$j = 1, \ldots, m$ so that $p_j(x)$ for $j = 1, \ldots, m$ are the minors of

$$p_m(x) = det(x\,E - T)$$

with the identity matrix E and matrix T in (3.3).

Theorem 3.1 (Theorem on Eigenvalue-Separation Properties)**.** *Let the real tridiagonal matrix T in* (3.2) *be given. Further, let this matrix be rewritten in the form* (3.3) *with the entries defined in* (3.4), (3.5)*, and* (3.6)*. Then, if the condition* (3.7) *is satisfied, the polynomials defined by* (3.8) *fulfill the recursion formulae* (2.1)*.*

Proof. The proof is done by mathematical induction.
Base case $(j = 1)$: For $j = 1$, we have

$$\begin{aligned} p_{1+1}(x) = p_2(x) &= \det \begin{bmatrix} x-\beta_0 & -\delta_1' \\ -\gamma_1' & x-\beta_1 \end{bmatrix} = \begin{vmatrix} x-\beta_0 & -\delta_1' \\ -\gamma_1' & x-\beta_1 \end{vmatrix} \\ &= (x-\beta_0)\,(x-\beta_1) - \gamma_1'\,\delta_1' \\ &= (x-\beta_1)\,\underbrace{(x-\beta_0)}_{=p_1(x)} - \gamma_1\,\underbrace{1}_{=p_0(x)} \\ &= (x-\beta_1)\,p_1(x) - \gamma_1\,p_0(x) \end{aligned}$$

so that the recursion formula $p_{j+1}(x) = (x-\beta_j)\,p_j(x) - \gamma_j\,p_{j-1}(x)$ in (2.1) is proven for $j = 1$.

Before we continue with the induction step, we add two further base steps in order to obtain more insight into the process and to prepare the induction step.
Additional base case $(j = 2)$: For $j = 2$, we get

$$p_{2+1}(x) = p_3(x) = \begin{vmatrix} x-\beta_0 & -\delta_1' & 0 \\ -\gamma_1' & x-\beta_1 & -\delta_2' \\ 0 & -\gamma_2' & x-\beta_2 \end{vmatrix}.$$

Expansion of this determinant along the last row leads to

$$\begin{aligned} p_{2+1}(x) &= (-\gamma_2')\,(-1)^{3+2} \begin{vmatrix} x-\beta_0 & 0 \\ -\gamma_1' & -\delta_2' \end{vmatrix} + (x-\beta_2)\,(-1)^{3+3} \begin{vmatrix} x-\beta_0 & -\delta_1' \\ -\gamma_1' & x-\beta_1 \end{vmatrix} \\ &= \gamma_2'\,[(x-\beta_0)\,(-\delta_2')] + (x-\beta_2)\,p_2(x) \\ &= (x-\beta_2)\,p_2(x) - \gamma_2'\,\delta_2'\,(x-\beta_0) \\ &= (x-\beta_2)\,p_2(x) - \gamma_2\,(x-\beta_1) \\ &= (x-\beta_2)\,p_2(x) - \gamma_2\,(x-\beta_{2-1}) \end{aligned}$$

so that the recursion formula $p_{j+1}(x) = (x - \beta_j)\, p_j(x) - \gamma_j\, p_{j-1}(x)$ in (2.1) is proven for $j = 2$.
Additional base case $(j = 3)$: For $j = 3$, we obtain

$$p_{3+1}(x) = p_4(x) = \begin{vmatrix} x-\beta_0 & -\delta_1' & 0 & 0 \\ -\gamma_1' & x-\beta_1 & -\delta_2' & 0 \\ 0 & -\gamma_2' & x-\beta_2 & -\delta_3' \\ 0 & 0 & -\gamma_3' & x-\beta_3 \end{vmatrix}.$$

Expansion of this determinant along the last row leads to

$$\begin{aligned} p_{3+1}(x) &= (-\gamma_3')\,(-1)^{4+3} \begin{vmatrix} x-\beta_0 & -\delta_1' & 0 \\ -\gamma_1' & x-\beta_1 & 0 \\ 0 & -\gamma_2' & -\delta_3' \end{vmatrix} \\ &\quad + (x-\beta_3)\,(-1)^{4+4} \underbrace{\begin{vmatrix} x-\beta_0 & -\delta_1' & 0 \\ -\gamma_1' & x-\beta_1 & -\delta_2' \\ 0 & -\gamma_2' & x-\beta_2 \end{vmatrix}}_{=p_3(x)} \\ &= \gamma_3' \left\{ (-\gamma_2')\,(-1)^{3+1} \begin{vmatrix} x-\beta_0 & 0 \\ -\gamma_1' & 0 \end{vmatrix} + (-\delta_3')\,(-1)^{3+3} \begin{vmatrix} x-\beta_0 & -\delta_1' \\ -\gamma_1' & x-\beta_1 \end{vmatrix} \right\} \\ &\quad + (x-\beta_3)\, p_3(x) \\ &= \gamma_3' \,[\, 0 + (-\delta_3')\, p_2(x)\,] + (x-\beta_3)\, p_3(x) = (x-\beta_3)\, p_3(x) - \gamma_3'\, \delta_3'\, p_2(x) \\ &= (x-\beta_3)\, p_3(x) - \gamma_3\, (x - \beta_{3-1}) \end{aligned}$$

so that the recursion formula $p_{j+1}(x) = (x - \beta_j)\, p_j(x) - \gamma_j\, p_{j-1}(x)$ in (2.1) is proven for $j = 3$.
Induction step: Assume that the recursion formula

$$p_{j+1}(x) = (x - \beta_j)\, p_j(x) - \gamma_j\, p_{j-1}(x)$$

is proven for an index $j \geq 4$ (for $j \in \{1, 2, 3\}$, we have already checked the validity). Then, we have to show

$$p_{j+2}(x) = (x - \beta_{j+1})\, p_{j+1}(x) - \gamma_{j+1}\, p_j(x).$$

Now, according to (3.8), we have

$$p_{j+2}(x) = \begin{vmatrix} x-\beta_0 & -\delta_1' & & & & & 0 \\ -\gamma_1' & x-\beta_1 & -\delta_2' & & & & 0 \\ & -\gamma_2' & x-\beta_2 & -\delta_3' & & & 0 \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & -\gamma_{j-1}' & x-\beta_{j-1} & -\delta_j' & 0 \\ & & & & -\gamma_j' & x-\beta_j & -\delta_{j+1}' \\ 0 & 0 & 0 & \cdots & 0 & -\gamma_{j+1}' & x-\beta_{j+1} \end{vmatrix}.$$

Expansion along the last row gives

$$
rcl p_{j+2}(x) = (-\gamma'_{j+1})\,(-1)^{(j+2)+(j+1)}
\begin{vmatrix}
x-\beta_0 & -\delta'_1 & & & 0 \\
-\gamma'_1 & x-\beta_1 & -\delta'_2 & & 0 \\
 & -\gamma'_2 & x-\beta_2 & -\delta'_3 & 0 \\
 & \ddots & \ddots & \ddots & \\
 & & -\gamma'_{j-1} & x-\beta_{j-1} & 0 \\
0 & 0 & 0 & -\gamma'_j & -\delta'_{j+1}
\end{vmatrix}
$$

$$
+ (x-\beta_{j+1})\,(-1)^{(j+2)+(j+2)}
\underbrace{\begin{vmatrix}
x-\beta_0 & -\delta'_1 & & & \\
-\gamma'_1 & x-\beta_1 & -\delta'_2 & & \\
 & -\gamma'_2 & x-\beta_2 & -\delta'_3 & \\
 & \ddots & \ddots & \ddots & \\
 & & -\gamma'_{j-1} & x-\beta_{j-1} & -\delta'_j \\
 & & 0 & -\gamma'_j & x-\beta_j
\end{vmatrix}}_{=p_{j+1}(x)}
$$

$$
= \gamma'_{j+1} \left\{ (-\gamma'_j)\,(-1)^{(j+1)+j}
\begin{vmatrix}
x-\beta_0 & -\delta'_1 & 0 & 0 & 0 \\
-\gamma'_1 & x-\beta_1 & -\delta'_2 & 0 & 0 \\
 & -\gamma'_2 & x-\beta_2 & -\delta'_3 & 0 \\
 & \ddots & \ddots & \ddots & \\
0 & 0 & 0 & -\delta'_{j-1} & 0
\end{vmatrix}\right.
$$

$$
\left. + (-\delta'_{j+1})\,(-1)^{(j+1)+(j+1)}
\begin{vmatrix}
x-\beta_0 & -\delta'_1 & & & 0 \\
-\gamma'_1 & x-\beta_1 & -\delta'_2 & & 0 \\
 & -\gamma'_2 & x-\beta_2 & -\delta'_3 & 0 \\
 & \ddots & \ddots & \ddots & \\
 & & -\gamma'_{j-2} & x-\beta_{j-2} & -\delta'_{j-1} \\
 & & 0 & -\gamma'_{j-1} & x-\beta_{j-1}
\end{vmatrix} \right\}
$$

$$
\begin{aligned}
&+ (x-\beta_{j+1})\,p_{j+1}(x) \\
&= \gamma'_{j+1}\,\{\,\gamma'_j \cdot 0 - \delta'_{j+1}\,p_j(x)\,\} + (x-\beta_{j+1})\,p_{j+1(x)} \\
&= (x-\beta_{j+1})\,p_{j+1}(x) - \gamma_{j+1}\,p_j(x)
\end{aligned}
$$

so that, indeed, the recursion formula $p_{j+2}(x) = (x-\beta_{j+1})\,p_{j+1}(x) - \gamma_{j+1}\,p_j(x)$ in (2.1) is proven, which was to be shown. This ends the proof of Theorem 3.1. □

From Theorem 3.1, we obtain the following important consequences.

Consequence 3.1 (Simple sufficient condition for eigenvalue-separation properties of real tridiagonal matrices). *Let the real tridiagonal matrix T be given by (3.2) with elements a_i satisfying $a_0 = a_{N+1} = 0$ and real $a_i,\ i = 1,\dots,N$ as well as $d_0 = d_{N+1} = 0$ and real elements $d_i,\ i = 1,\dots,N$ with the condition that $a_i\,d_i > 0,\ i = 1,\dots,N$. Since the polynomials defined in (3.8) fulfill the recursion formulae in (2.1), the eigenvalues of matrix T are real as well as pairwise distinct, and they can be arranged such that the properties (P1) - (P4) in Lemma 2.1 on Sturm sequences are valid.*

Proof. We need only mention that, in this special case, $\gamma_j = \gamma_j' \, \delta_j' = a_i \, d_j > 0, \; j = 0, \ldots, m-1 = N$. $\square$

Consequence 3.2 (Simple sufficient condition for eigenvalue-separation properties of symmetric tridiagonal matrices). *Let the symmetric tridiagonal matrix T be given by (3.2) with $a_0 = a_{N+1} = 0$ satisfying elements $d_i = a_i \neq 0, \; j = 1, \ldots, N$. Since the polynomials defined in (3.8) fulfill the recursion formulae in (2.1), the eigenvalues of matrix T are real as well as pairwise distinct, and they can be arranges such that the properties (P1) - (P4) in Lemma 2.1 on Sturm sequences are valid.*

Proof. We need only mention that, in this case, $\gamma_j = {\gamma_j'}^2 = a_i^2 > 0, \; j = 0, \ldots, m-1 = N$. $\square$

Remark 3.1.

(R1) *This is much more than the statement that the eigenvalues of T are distinct.*
(R2) *The obtained result holds, in particular, in the case when $d_i = a_i$, $i = 1, \ldots, N$ and $c_i = b_i - a_i - a_{i-1}$, $i = 1, \ldots, N+1$ with real elements b_i studied in* [2, p.27]. *Whereas Grünbaum assumes that the pertinent eigenvalues are distinct, in this paper, under mild conditions it could be proven that they are distinct.*
(R3) *We remind the reader that the eigenvectors of symmetric matrices with distinct eigenvalues are pairwise orthogonal.*

4. Numerical Example

In this section, we present two numerical examples. The first one treats a real nonsymmetric tridiagonal matrix and the second one a symmetric tridiagonal matrix.

4.1. **Numerical Example 1.** As the first numerical example, we choose

$$T = T_4 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$$

so that $m = 4$ as well as $a_i = -1, \; i = 1, 2, 3 (= N)$ and $d_i = -2, \; i = 1, 2, 3 (= N)$ or $\gamma_j' = -1, \; j = 1, 2, 3 (= m-1)$ and $\delta_j' = -2, \; j = 1, 2, 3 (= m-1)$ as well as $c_i = 0, \; i = 1, 2, 3, 4 (= N+1)$. Further, $\beta_j = 0, \; j = 0, 1, 2, 3 (= m-1)$, $\gamma_j = \gamma_j' \, \delta_j' = 2 > 0, \; j = 1, 2, 3 (= m-1)$. Let $E_i \in \mathbb{R}^{i \times i}$ be the $i \times i$-identity matrix for $i = 2, 3, 4$. Herewith,

$$p_4(x) = (x \, E_4 - T_4).$$

Further, let

$$T_3 = \begin{bmatrix} 0 & -2 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

and

$$T_2 = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} \in \mathbb{R}^{2 \times 2}.$$

Then,

$$p_j(x) = (x \, E_j - T_j), \quad j = 4, 3, 2.$$

For the eigenvalues

$$\lambda_{j,k}, \; k = 1, \ldots, j$$

of T_j for $j = (m = 4), 3, 2$, we obtain

$$\lambda_4 := \begin{bmatrix} \lambda_{4,1} \\ \lambda_{4,2} \\ \lambda_{4,3} \\ \lambda_{4,4} \end{bmatrix} = \begin{bmatrix} 2.2882 \\ 0.8740 \\ -0.8740 \\ -2.2882 \end{bmatrix}, \quad \lambda_3 := \begin{bmatrix} \lambda_{3,1} \\ \lambda_{3,2} \\ \lambda_{3,3} \end{bmatrix} = \begin{bmatrix} 2.0000 \\ -0.0000 \\ -2.0000 \end{bmatrix}$$

and

$$\lambda_2 := \begin{bmatrix} \lambda_{2,1} \\ \lambda_{2,2} \end{bmatrix} = \begin{bmatrix} 1.4142 \\ -1.4142 \end{bmatrix},$$

where the numbering of the vector components is such that

$$-\infty < \lambda_{j,j} < \lambda_{j,j-1} < \cdots < \lambda_{j,2} < \lambda_{j,1} < +\infty$$

for $j = 4, 3, 2$. Therefore, (P1) is satisfied.

Further, we check (P2). The cases $j = 4$ and $j = 1$ are left to the reader. For $j = 3$, we obtain

$$(P2)_{k=1}: \quad p_3(\lambda_{3,1}) = 0\,, \quad p_4(\lambda_{3,1})\, p_2(\lambda_{3,1}) = -8 < 0,$$

$$(P2)_{k=2}: \quad p_3(\lambda_{3,2}) = 0\,, \quad p_4(\lambda_{3,2})\, p_2(\lambda_{3,2}) = -8 < 0,$$

$$(P2)_{k=3}: \quad p_3(\lambda_{3,3}) = 0\,, \quad p_4(\lambda_{3,3})\, p_2(\lambda_{3,3}) = -8 < 0.$$

Moreover, for $j = 2$,

$$(P2)_{k=1}: \quad p_2(\lambda_{2,1}) = 0, \quad p_3(\lambda_{2,1})\, p_1(\lambda_{2,1}) = -4 < 0,$$

$$(P2)_{k=2}: \quad p_2(\lambda_{2,2}) = 0, \quad p_3(\lambda_{2,2})\, p_1(\lambda_{2,2}) = -4 < 0.$$

Thus, (P2) is numerically underpinned. Next, we check (P3). The cases $j = 4$ and $j = 1$ are left to the reader. For $j = 3$, we obtain

$$(P3)_{k=1}: \quad (-1)^k\, p_3(x) = \underbrace{(-1)^k}_{<0}\, \underbrace{(x - \lambda_{3,1})}_{<0}\, \underbrace{(x - \lambda_{3,2})}_{>0}\, \underbrace{(x - \lambda_{3,3})}_{>0} = 3 > 0$$

for

$$x = \frac{\lambda_{3,2} + \lambda_{3,1}}{2} \doteq 1$$

so that

$$\lambda_{3,2} < x < \lambda_{3,1}.$$

Further,

$$(P3)_{k=2}: \quad (-1)^k\, p_3(x) = \underbrace{(-1)^k}_{>0}\, \underbrace{(x - \lambda_{3,1})}_{<0}\, \underbrace{(x - \lambda_{3,2})}_{<0}\, \underbrace{(x - \lambda_{3,3})}_{>0} = 3 > 0$$

for

$$x = \frac{\lambda_{3,3} + \lambda_{3,2}}{2} = -1$$

so that

$$\lambda_{3,3} < x < \lambda_{3,2}.$$

Moreover, for $j = 2$,

$$(P3)_{k=1}: \quad (-1)^k\, p_2(x) = \underbrace{(-)^k}_{<0}\, \underbrace{(x - \lambda_{2,1})}_{<0}\, \underbrace{(x - \lambda_{2,2})}_{>0} = 2 > 0$$

for

$$x = \frac{\lambda_{2,2} + \lambda_{2,1}}{2} = 0$$

so that

$$\lambda_{2,2} < x < \lambda_{2,1}.$$

On the whole, (P3) is numerically underpinned. Finally, we check (P4). The cases $j = 1$ and $j = 4$ are left to the reader. For $j = 3$, we obtain

$$(P4)_{k=1}: \quad \lambda_{3,1} \doteq 2.0000 < 2.2882 \doteq \lambda_{4,1} < \lambda_{3,0} = +\infty,$$

and $\lambda_{4,1}$ is the only component of λ_4 satisfying the above inequality. Further,

$$(P4)_{k=2}: \quad \lambda_{3,2} \doteq -0.0000 < 0.8740 \doteq \lambda_{4,2} < \lambda_{3,1} \doteq 2.0000,$$

and $\lambda_{4,2}$ is the only component of λ_4 satisfying the above inequality. Finally,

$$(P4)_{k=3}: \quad \lambda_{3,3} \doteq -2.0000 < -0.8740 \doteq \lambda_{4,3} < \lambda_{3,2} \doteq -0.0000,$$

and $\lambda_{4,3}$ is the only component of λ_4 satisfying the above inequality.

For $j = 2$, we obtain

$$(P4)_{k=1}: \quad \lambda_{2,1} \doteq 1.4142 < 2.0000 \doteq \lambda_{3,1} < \lambda_{2,0} = +\infty,$$

and $\lambda_{3,1}$ is the only component of λ_3 satisfying the above inequality. Further,

$$(P4)_{k=2}: \quad \lambda_{2,2} \doteq -1.4142 < -0.0000 \doteq \lambda_{3,2} < \lambda_{2,1} \doteq 1.4142,$$

and $\lambda_{3,2}$ is the only component of λ_3 satisfying the above inequality.

Therefore, (P4) is numerically underpinned.

Remark 4.2. *The computations of the eigenvalues were done by the Matlab routine eig.m.*

4.2. **Numerical Example 2.** As the second numerical example, we choose

$$T = T_4 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

so that $m = 4$ as well as $a_i = -1 \neq 0, \ i = 1, 2, 3 (= N)$ or $\gamma'_j = -1, \ j = 1, 2, 3 (= m - 1)$ and $c_i = 0, \ i = 1, 2, 3, 4 (= N + 1)$. Further, $\beta_j = 0, \ j = 0, 1, 2, 3 (= m - 1)$, $\gamma_j = {\gamma'_j}^2 = 1 > 0, \ j = 1, 2, 3 (= m - 1)$. Let $E_i \in \mathbb{R}^{i\times i}$ be the $i \times i$-identity matrix for $i = 2, 3, 4$. Herewith,

$$p_4(x) = (x\, E_4 - T_4).$$

Further, let

$$T_3 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

and

$$T_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Then,

$$p_j(x) = (x\, E_j - T_j), \quad j = 4, 3, 2.$$

For the eigenvalues

$$\lambda_{j,k}, \ k = 1, \ldots, j$$

of T_j for $j = (m = 4), 3, 2$, we obtain

$$\lambda_4 := \begin{bmatrix} \lambda_{4,1} \\ \lambda_{4,2} \\ \lambda_{4,3} \\ \lambda_{4,4} \end{bmatrix} = \begin{bmatrix} 1.6180 \\ 0.6180 \\ -0.6180 \\ -1.6180 \end{bmatrix}, \quad \lambda_3 := \begin{bmatrix} \lambda_{3,1} \\ \lambda_{3,2} \\ \lambda_{3,3} \end{bmatrix} = \begin{bmatrix} 1.4142 \\ 0.0000 \\ -1.4142 \end{bmatrix}$$

and

$$\lambda_2 := \begin{bmatrix} \lambda_{2,1} \\ \lambda_{2,2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where the numbering of the vector components is such that

$$-\infty < \lambda_{j,j} < \lambda_{j,j-1} < \cdots < \lambda_{j,2} < \lambda_{j,1} < +\infty$$

for $j = 4, 3, 2$. Therefore, (P1) is satisfied. Further, we check (P2). The cases $j = 4$ and $j = 1$ are left to the reader. For $j = 3$, we obtain

$$(P2)_{k=1}: \quad p_3(\lambda_{3,1}) = 0\,, \quad p_4(\lambda_{3,1})\, p_2(\lambda_{3,1}) = -1 < 0,$$

$$(P2)_{k=2}: \quad p_3(\lambda_{3,2}) = 0\,, \quad p_4(\lambda_{3,2})\, p_2(\lambda_{3,2}) = -1 < 0,$$

$$(P2)_{k=3}: \quad p_3(\lambda_{3,3}) = 0\,, \quad p_4(\lambda_{3,3})\, p_2(\lambda_{3,3}) = -1 < 0.$$

Moreover, for $j = 2$,

$$(P2)_{k=1}: \quad p_2(\lambda_{2,1}) = 0, \quad p_3(\lambda_{2,1})\, p_1(\lambda_{2,1}) = -1 < 0,$$

$$(P2)_{k=2}: \quad p_2(\lambda_{2,2}) = 0, \quad p_3(\lambda_{2,2})\, p_1(\lambda_{2,2}) = -1 < 0.$$

Thus, (P2) is numerically underpinned. Next, we check (P3). The cases $j = 4$ and $j = 1$ are left to the reader. For $j = 3$, we obtain

$$(P3)_{k=1}: \quad (-1)^k\, p_3(x) = \underbrace{(-1)^k}_{<0}\, \underbrace{(x - \lambda_{3,1})}_{<0}\, \underbrace{(x - \lambda_{3,2})}_{>0}\, \underbrace{(x - \lambda_{3,3})}_{>0} \doteq 1.0607 > 0$$

for

$$x = \frac{\lambda_{3,2} + \lambda_{3,1}}{2} \doteq 0.7071$$

so that

$$\lambda_{3,2} < x < \lambda_{3,1}.$$

Further,

$$(P3)_{k=2}: \quad (-1)^k\, p_3(x) = \underbrace{(-1)^k}_{>0}\, \underbrace{(x - \lambda_{3,1})}_{<0}\, \underbrace{(x - \lambda_{3,2})}_{<0}\, \underbrace{(x - \lambda_{3,3})}_{>0} \doteq 1.0607 > 0$$

for

$$x = \frac{\lambda_{3,3} + \lambda_{3,2}}{2} \doteq -0.7071$$

so that

$$\lambda_{3,3} < x < \lambda_{3,2}.$$

Moreover, for $j = 2$,

$$(P3)_{k=1}: \quad (-1)^k\, p_2(x) = \underbrace{(-)^k}_{<0}\, \underbrace{(x - \lambda_{2,1})}_{<0}\, \underbrace{(x - \lambda_{2,2})}_{>0} = 1 > 0$$

for

$$x = \frac{\lambda_{2,2} + \lambda_{2,1}}{2} = 0$$

so that

$$\lambda_{2,2} < x < \lambda_{2,1}.$$

On the whole, (P3) is numerically underpinned. Finally, we check (P4). The cases $j = 1$ and $j = 4$ are left to the reader. For $j = 3$, we obtain

$$(P4)_{k=1}: \quad \lambda_{3,1} \doteq 1.4142 < 1.6180 \doteq \lambda_{4,1} < \lambda_{3,0} = +\infty,$$

and $\lambda_{4,1}$ is the only component of λ_4 satisfying the above inequality. Further,

$$(P4)_{k=2}: \quad \lambda_{3,2} \doteq 0.0000 < 0.6180 \doteq \lambda_{4,2} < \lambda_{3,1} \doteq 1.4142,$$

and $\lambda_{4,2}$ is the only component of λ_4 satisfying the above inequality. Finally,

$$(P4)_{k=3}: \quad \lambda_{3,3} \doteq -1.4142 < -0.6180 \doteq \lambda_{4,3} < \lambda_{3,2} \doteq 0.0000,$$

and $\lambda_{4,3}$ is the only component of λ_4 satisfying the above inequality.

For $j = 2$, we obtain

$$(P4)_{k=1}: \quad \lambda_{2,1} = 1 < 1.4142 \doteq \lambda_{3,1} < \lambda_{2,0} = +\infty,$$

and $\lambda_{3,1}$ is the only component of λ_3 satisfying the above inequality. Further,

$$(P4)_{k=2}: \quad \lambda_{2,2} = -1 < 0.0000 \doteq \lambda_{3,2} < \lambda_{2,1} = 1,$$

and $\lambda_{3,2}$ is the only component of λ_3 satisfying the above inequality. Therefore, (P4) is numerically underpinned.

Remark 4.3. *Again, the computations of the eigenvalues were done by the Matlab routine eig.m.*

5. Conclusion

In this paper, as the main new result, we could show that the eigenvalues of a real tridiagonal matrix have the eigenvalue-separation properties (P1) - (P4) of Lemma 2.1 provided that the products of corresponding entries above and below the diagonal are positive. In the special case of a symmetric tridiagonal matrix, this turns into the simple sufficient condition that all entries above and below the diagonal are different from zero. This applies, in particular, to the special matrix studied by Grünbaum who assumed that the eigenvalues are distinct whereas here this could be proven. The eigenvalue-separation properties are much more than the property that its eigenvalues are just distinct. A further interesting point is that the elements γ_j in (2.1) are independent of the diagonal entries β_j so that the sufficient condition $\gamma_j > 0$ depends only on the entries under and above the diagonal, not on the diagonal entries.

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Research Article

Systems of left translates and oblique duals on the Heisenberg group

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ABSTRACT. In this paper, we characterize the system of left translates $\{L_{(2k,l,m)}g : k,l,m \in \mathbb{Z}\}$, $g \in L^2(\mathbb{H})$, to be a frame sequence or a Riesz sequence in terms of the twisted translates of the corresponding function g^λ. Here, $\mathbb{H}$ denotes the Heisenberg group and g^λ the inverse Fourier transform of g with respect to the central variable. This type of characterization for a Riesz sequence allows us to find some concrete examples. We also study the structure of the oblique dual of the system of left translates $\{L_{(2k,l,m)}g : k,l,m \in \mathbb{Z}\}$ on $\mathbb{H}$. This result is also illustrated with an example.

Keywords: B-splines, Heisenberg group, Gramian, Hilbert-Schmidt operator, Riesz sequence, moment problem, oblique dual, Weyl transform.

2020 Mathematics Subject Classification: 42C15, 41A15, 43A30.

1. INTRODUCTION

A closed subspace $V \subset L^2(\mathbb{R})$ is said to be a shift-invariant space if $f \in V \Rightarrow \mathcal{T}_k f \in V$ for any $k \in \mathbb{Z}$, where $\mathcal{T}_x f(y) = f(y-x)$ denotes the translation operator. These spaces appear in the study of multiresolution analyses in order to construct wavelets. We refer to [17, 18] in this context. For $\phi \in L^2(\mathbb{R})$, the shift-invariant space $V(\phi) = \overline{\operatorname{span}\{\mathcal{T}_k \phi : k \in \mathbb{Z}\}}$ is called a principal shift-invariant space. Shift-invariant spaces are broadly applied in various fields such as approximation theory, mathematical sampling theory, communication engineering, and so on. Apart from this, shift-invariant spaces have also been explored in various group settings.

In [6], Bownik obtained a characterization of shift-invariant spaces on $\mathbb{R}^n$ by using range functions. He derived equivalent conditions for a system of translates to be a frame sequence or a Riesz sequence. Later, these results were studied on locally compact abelian groups in [7, 8, 15, 16] and on non-abelian compact groups in [14, 20].

In recent years, problems in connection with frames, Riesz bases, wavelets, and shift-invariant spaces on non-abelian groups, nilpotent Lie groups, especially the Heisenberg group, have drawn the attention of several researchers globally (see, for example, [2, 3, 4, 5, 11, 19] in this context).

In [12], Das et al. obtained characterization results for a shift-invariant system to be a frame sequence or a Riesz sequence in terms of the Gramian and the dual Gramian, respectively, on the Heisenberg group. Although the characterization results mentioned in this paper are interesting from the theoretical point of view, they are not useful in obtaining concrete Riesz sequences of system of translates. In this paper, we attempt to overcome this difficulty and try

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to obtain a characterization for the system of left translates on the Heisenberg group to form a frame sequence or a Riesz sequence. This is done with the help of deriving such characterizations for λ-twisted translates on $\mathbb{R}^2$. Apart from this, we also study the problem of obtaining oblique dual for a system of left translates on the Heisenberg group.

The structure of this paper is as follows. After introducing some background information about frames and the Heisenberg group in Section 2, we consider systems of left translates and their relation to frame and Riesz sequences on the Heisenberg group in Section 3. Obliques duals of these systems of left translates are then investigated in Section 4.

2. Background

To proceed, we require the following definitions and results from frame theory and harmonic analysis on the Heisenberg group. In the former case, most of these can be found in, for instance, [9], and in the latter case in, i.e., [13, 21].

$0 \neq \mathcal{H}$ always denotes a separable Hilbert space.

Definition 2.1. *A sequence $\{f_k : k \in \mathbb{N}\} \subset \mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ satisfying*

$$A\|f\|^2 \leq \sum_{k\in\mathbb{N}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \qquad \forall f \in \mathcal{H}. \tag{2.1}$$

If $\{f_k : k \in \mathbb{N}\}$ is a frame for $\overline{\operatorname{span}\{f_k : k \in \mathbb{N}\}}$, then it is called a frame sequence.

A sequence $\{f_k : k \in \mathbb{N}\} \subset \mathcal{H}$ satisfying only the upper bound in the frame condition (2.1) is called a Bessel sequence.

Definition 2.2. *A sequence of the form $\{Ue_k : k \in \mathbb{N}\}$, where $\{e_k : k \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{H}$ and U is a bounded invertible operator on $\mathcal{H}$, is called a Riesz basis. If $\{f_k : k \in \mathbb{N}\}$ is a Riesz basis for $\overline{\operatorname{span}\{f_k : k \in \mathbb{N}\}}$, then it is called a Riesz sequence.*

Equivalently, $\{f_k : k \in \mathbb{N}\}$ is said to be a Riesz sequence if there exist constants $A, B > 0$ such that

$$A\|\{c_k\}\|^2_{\ell^2(\mathbb{N})} \leq \left\| \sum_{k\in\mathbb{N}} c_k f_k \right\|^2 \leq B\|\{c_k\}\|^2_{\ell^2(\mathbb{N})}$$

for all finite sequences $\{c_k\} \in \ell^2(\mathbb{N})$.

Theorem 2.1. *Let $h \in L^2(\mathbb{R})$. The system $\{T_k h : k \in \mathbb{Z}\}$ is a Riesz sequence with bounds $A, B > 0$ iff*

$$A \leq \sum_{k\in\mathbb{Z}} |\widehat{h}(\lambda + k)|^2 \leq B \quad \text{for a.e. } \lambda \in (0, 1].$$

Definition 2.3. *The Gramian G associated with a Bessel sequence $\{f_k : k \in \mathbb{N}\}$ is a bounded operator on $\ell^2(\mathbb{N})$ defined by*

$$G\{c_k\} := \left\{ \sum_{k\in\mathbb{N}} \langle f_k, f_j\rangle c_k \right\}_{j\in\mathbb{N}}.$$

It is well known that $\{f_k : k \in \mathbb{N}\}$ is a Riesz sequence with bounds $A, B > 0$ iff

$$A\|\{c_k\}\|^2_{\ell^2(\mathbb{N})} \leq \big\langle G\{c_k\}, \{c_k\}\big\rangle \leq B\|\{c_k\}\|^2_{\ell^2(\mathbb{N})}.$$

Definition 2.4. *Let $\{f_k : k \in \mathbb{N}\}$ be a Riesz sequence in $\mathcal{H}$. If*

$$f = \sum_{k\in\mathbb{N}} \langle f, g_k\rangle f_k, \quad \forall f \in \overline{\operatorname{span}\{f_k : k \in \mathbb{Z}\}}$$

for some $\{g_k : k \in \mathbb{N}\} \subset \mathcal{H}$, *then* $\{g_k : k \in \mathbb{N}\}$ *is called a generalized dual generator of* $\{f_k : k \in \mathbb{N}\}$. *In addition, if* $\{g_k : k \in \mathbb{N}\}$ *is a frame sequence, then* $\{g_k : k \in \mathbb{N}\}$ *is called an oblique dual generator of* $\{f_k : k \in \mathbb{N}\}$.

Definition 2.5. *Let* $\{f_k : k \in J\}$ *be a countable collection of elements in* $\mathcal{H}$ *and* $\{\alpha_k\}_{k\in J} \in \ell^2(J)$. *Consider the system of equations*

$$\langle f, f_k\rangle = \alpha_k, \quad \forall\, k \in J. \tag{2.2}$$

Finding such an $f \in \mathcal{H}$ *from* (2.2) *is known as the moment problem.*

A moment problem may not have any solution at all or may have infinitely many solutions. But if $\{f_k : k \in J\}$ is a Riesz sequence, then the moment problem has a unique solution $f \in \overline{\operatorname{span}\{f_k : k \in J\}}$. For the existence of a solution of a moment problem, one has the following result.

Lemma 2.1 ([10])**.** *Let* $\{f_k : k = 1, 2, \cdots, N\}$ *be a finite collection of vectors in* $\mathcal{H}$. *Consider the moment problem*

$$\langle f, f_k\rangle = \delta_{k,1}\ , \quad k = 1, 2, \cdots, N.$$

Then the following statements are equivalent:

(i) *The moment problem has a solution* $f \in \mathcal{H}$.

(ii) $\sum\limits_{k=1}^{N} c_k f_k = 0$, *for some* $\{c_k\}$ *implies* $c_1 = 0$.

(iii) $f_1 \notin \operatorname{span}\{f_2, f_3, \cdots, f_N\}$.

Definition 2.6. *A closed subspace* $V \subset L^2(\mathbb{R})$ *is called a shift-invariant space if* $f \in V \Rightarrow T_k f \in V$ *for any* $k \in \mathbb{Z}$, *where* T_x *denotes the translation operator* $T_x f(y) = f(y - x)$. *In particular, if* $\phi \in L^2(\mathbb{R})$, *then* $V(\phi) = \overline{\operatorname{span}}\ \{T_k\phi : k \in \mathbb{Z}\}$ *is called a principal shift-invariant space.*

For a study of frames, Riesz basis on $\mathcal{H}$, and shift-invariant spaces on $L^2(\mathbb{R})$, we refer to [9].

Definition 2.7. *Let* χ *denote the characteristic function of* $[0, 1]$. *For* $n \in \mathbb{N}$, *set*

$$\begin{aligned} &B_1 : [0, 1] \to [0, 1], \quad x \mapsto \chi(x);\\ &B_n := B_{n-1} * B_1, \quad n \geq 2, \quad n \in \mathbb{N}. \end{aligned} \tag{2.3}$$

Then, B_n *is called a (cardinal) polynomial B-spline of order* n.

For more and detailed information about B-splines and their applications, the interested reader may wish to consult any of the many references regarding B-splines.

The next stated result shows that cardinal B-splines form principal shift-invariant spaces.

Theorem 2.2 ([9, Theorem 9.2.6])**.** *For each* $n \in \mathbb{N}$, *the sequence* $\{T_k B_n\}_{k\in\mathbb{Z}}$ *is a Riesz sequence.*

The Heisenberg group $\mathbb{H}$ is a nilpotent Lie group whose underlying manifold is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ endowed with a group operation defined by

$$(x, y, t)(x', y', t') := (x + x', y + y', t + t' + \tfrac{1}{2}(x'y - y'x)),$$

and where Haar measure is Lebesgue measure $dx\, dy\, dt$ on $\mathbb{R}^3$. By the Stone–von Neumann theorem, every infinite dimensional irreducible unitary representation on $\mathbb{H}$ is unitarily equivalent to the representation π_λ given by

$$\pi_\lambda(x, y, t)\phi(\xi) = e^{2\pi i\lambda t} e^{2\pi i\lambda(x\xi + \frac{1}{2}xy)}\phi(\xi + y)$$

for $\phi \in L^2(\mathbb{R})$ and $\lambda \in \mathbb{R}^\times := \mathbb{R}\setminus\{0\}$. This representation π_λ is called the Schrödinger representation of the Heisenberg group. For $f, g \in L^1(\mathbb{H})$, the group convolution of f and g is defined by

$$f * g(x,y,t) := \int_{\mathbb{H}} f\big((x,y,t)(u,v,s)^{-1}\big) g(u,v,s)\, du\, dv\, ds. \tag{2.4}$$

Under this group convolution, $L^1(\mathbb{H})$ becomes a non-commutative Banach algebra. The group $Fourier$ transform of $f \in L^1(\mathbb{H})$ is defined by

$$\widehat{f}(\lambda) = \int_{\mathbb{H}} f(x,y,t)\, \pi_\lambda(x,y,t)\, dxdydt, \quad \lambda \in \mathbb{R}^\times, \tag{2.5}$$

where the integral is a $Bochner$ integral acting on the $Hilbert$ space $L^2(\mathbb{R})$. The group Fourier transform is an isometric isomorphism between $L^2(\mathbb{H})$ and $L^2(\mathbb{R}^\times, \mathcal{B}_2; d\mu)$, where $d\mu(\lambda)$ denotes Plancherel measure $|\lambda| d\lambda$ and $\mathcal{B}_2$ is the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R})$ with inner product given by $(T,S) := \operatorname{tr}(TS^*)$. Thus, we can write (2.5) as

$$\widehat{f}(\lambda) = \int_{\mathbb{R}^2} f^\lambda(x,y)\pi_\lambda(x,y,0)\, dxdy\ ,$$

where

$$f^\lambda(x,y) := \int_{\mathbb{R}} f(x,y,t) e^{2\pi i \lambda t}\, dt.$$

Note that the function $f^\lambda(x,y)$ is the inverse Fourier transform of f with respect to the t variable. For $g \in L^1(\mathbb{R}^2)$, let

$$W_\lambda(g) := \int_{\mathbb{R}^2} g(x,y)\pi_\lambda(x,y,0)\, dxdy, \ \text{ for } \lambda \in \mathbb{R}^\times.$$

Using this operator, we can rewrite $\widehat{f}(\lambda)$ as $W_\lambda(f^\lambda)$. When $f, g \in L^2(\mathbb{H})$, one can show that $f^\lambda, g^\lambda \in L^2(\mathbb{R}^2)$ and W_λ satisfies

$$\big\langle f^\lambda, g^\lambda \big\rangle_{L^2(\mathbb{R}^2)} = |\lambda| \big\langle W_\lambda(f^\lambda), W_\lambda(g^\lambda) \big\rangle_{\mathcal{B}_2}. \tag{2.6}$$

Now, define $\tau : L^2(\mathbb{H}) \to L^2((0,1], \ell^2(\mathbb{Z}, \mathcal{B}_2))$ by

$$\tau f(\lambda) := \{|\lambda - r|^{1/2} \widehat{f}(\lambda - r)\}_{r\in\mathbb{Z}}, \quad \forall\, f \in L^2(\mathbb{H}),\ \lambda \in (0,1].$$

Then, τ is an isometric isomorphism between $L^2(\mathbb{H})$ and $L^2((0,1], \ell^2(\mathbb{Z}, \mathcal{B}_2))$ (see [11, 12] in this context). For $(u,v,s) \in \mathbb{H}$, the left translation operator $L_{(u,v,s)}$ is defined by

$$L_{(u,v,s)} f(x,y,t) := f((u,v,s)^{-1}(x,y,t)), \quad \forall\, (x,y,t) \in \mathbb{H},$$

which is a unitary operator on $L^2(\mathbb{H})$. Using the definitions of the left translation operator and the convolution, one can show that

$$L_{(u,v,s)}(f * g) = (L_{(u,v,s)} f) * g. \tag{2.7}$$

For $(u,v) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}^\times$, the λ-twisted translation operator $(T^t_{(u,v)})^\lambda$ is defined by

$$(T^t_{(u,v)})^\lambda F(x,y) := e^{\pi i \lambda (vx - uy)} F(x-u, y-v), \quad \forall\, (x,y) \in \mathbb{R}^2,$$

which is also a unitary operator on $L^2(\mathbb{R}^2)$. It is easy to see that

$$(L_{(u,v,s)} f)^\lambda = e^{2\pi i s \lambda} (T^t_{(u,v)})^\lambda f^\lambda. \tag{2.8}$$

For further properties of λ-twisted translation, we refer to [19].

Recall that for a locally compact group G, a lattice Γ in G is defined to be a discrete subgroup of G which is co-compact. The standard lattice in $\mathbb{H}$ is taken to be $\Gamma := \{(2k, l, m) : k, l, m \in \mathbb{Z}\}$. For a study of analysis on the Heisenberg group we refer to [13, 21].

3. System of left translates as a frame sequence and a Riesz sequence

Let $g \in L^2(\mathbb{H})$. In this section, we wish to obtain characterization results for the system $\{L_{(2k,l,m)}g : k, l, m \in \mathbb{Z}\}$ to form a frame sequence or a Riesz sequence in terms of the λ-twisted translations g^λ of g.

From Corollary 3 of [12], we know that $\{L_{(2k,l,m)}g : k, l, m \in \mathbb{Z}\}$ is a frame sequence with bounds $A, B > 0$ iff

(3.1)
$$A\|\Phi(\lambda)\|^2 \leq \sum_{k,l\in\mathbb{Z}} |\langle \Phi(\lambda), \tau(L_{(2k,l,0)}g)(\lambda)\rangle|^2 \leq B\|\Phi(\lambda)\|^2, \ \forall\, \Phi(\lambda) \in J(\lambda), \text{ for a.e. } \lambda \in (0, 1],$$

where $J(\lambda) := \overline{\operatorname{span}\{\tau(L_{(2k,l,0)}g)(\lambda) : k, l \in \mathbb{Z}\}}$. In order to prove that $\{L_{(2k,l,m)}g : k, l, m \in \mathbb{Z}\}$ is a frame sequence, it suffices to consider the class $\operatorname{span}\{\tau(L_{(2k,l,0)}g)(\lambda) : k, l \in \mathbb{Z}\}$ instead of $J(\lambda)$. Thus, the required condition for the verification of frame sequence reduces to the following two inequalities:

For any finite $\mathcal{F} \subset \mathbb{Z}^2$ and any finite sequence $\{\alpha_{k,l}\} \in \ell^2(\mathbb{Z}^2)$,

$$\begin{aligned} & A\Big\| \sum_{(k,l)\in\mathcal{F}} \alpha_{k,l}\tau(L_{(2k,l,0)}g)(\lambda)\Big\|^2_{\ell^2(\mathbb{Z},\mathcal{B}_2)} \\ \leq & \sum_{k,l\in\mathbb{Z}} \Big|\Big\langle \sum_{(k',l')\in\mathcal{F}} \alpha_{k',l'}\tau(L_{(2k',l',0)}g)(\lambda), \tau(L_{(2k,l,0)}g)(\lambda)\Big\rangle\Big|^2 \\ \leq & B\Big\| \sum_{(k,l)\in\mathcal{F}} \alpha_{k,l}\tau(L_{(2k,l,0)}g)(\lambda)\Big\|^2_{\ell^2(\mathbb{Z},\mathcal{B}_2)}, \text{ a.e. } \lambda \in (0, 1]. \end{aligned}$$

Now, for $k, k', l, l' \in \mathbb{Z}$,

$$\begin{aligned} & \langle \tau(L_{(2k',l',0)}g)(\lambda), \tau(L_{(2k,l,0)}g)(\lambda)\rangle_{\ell^2(\mathbb{Z},\mathcal{B}_2)} \\ = & \sum_{r\in\mathbb{Z}} |\lambda - r| \left\langle \widehat{L_{(2k',l',0)}g}(\lambda - r), \widehat{L_{(2k,l,0)}g}(\lambda - r)\right\rangle_{\mathcal{B}_2} \\ = & \sum_{r\in\mathbb{Z}} |\lambda - r| \left\langle W_{\lambda-r}\big((L_{(2k',l',0)}g)^{\lambda-r}\big), W_{\lambda-r}\big((L_{(2k,l,0)}g)^{\lambda-r}\big)\right\rangle_{\mathcal{B}_2} \\ = & \sum_{r\in\mathbb{Z}} \left\langle (L_{(2k',l',0)}g)^{\lambda-r}, (L_{(2k,l,0)}g)^{\lambda-r}\right\rangle_{L^2(\mathbb{R}^2)} \\ = & \sum_{r\in\mathbb{Z}} \left\langle (T^t_{(2k',l')})^{\lambda-r} g^{\lambda-r}, (T^t_{(2k,l)})^{\lambda-r} g^{\lambda-r}\right\rangle_{L^2(\mathbb{R}^2)} \end{aligned}$$

$$= \sum_{r\in\mathbb{Z}} e^{\pi i(\lambda-r)(kl'-lk')} \left\langle \big(T^t_{(2(k'-k),l'-l)}\big)^{\lambda-r} g^{\lambda-r}, g^{\lambda-r}\right\rangle_{L^2(\mathbb{R}^2)}, \tag{3.2}$$

where we used (2.6) and (2.8).

In the following theorem, we state a condition for the system $\{L_{(2k,l,m)}g : k, l, m \in \mathbb{Z}\}$ to be a frame sequence in terms of the λ-twisted translates of g^λ on $\mathbb{R}^2$.

Theorem 3.3. *Let $g \in L^2(\mathbb{H})$. Then, the system $\{L_{(2k,l,m)}g : k,l,m \in \mathbb{Z}\}$ is a frame sequence with bounds $A, B > 0$ iff*

$$
\begin{aligned}
&A\sum_{r\in\mathbb{Z}}\Big\|\sum_{(k',l')\in\mathcal{F}}\alpha_{k',l'}(T^t_{(2k',l')})^{\lambda-r}g^{\lambda-r}\Big\|^2_{L^2(\mathbb{R}^2)}\\
\le&\sum_{k,l\in\mathbb{Z}}\Big|\sum_{(k',l')\in\mathcal{F},r\in\mathbb{Z}}\alpha_{k',l'}e^{\pi i(\lambda-r)(kl'-lk')}\Big\langle(T^t_{(2(k'-k),l'-l)})^{\lambda-r}g^{\lambda-r},g^{\lambda-r}\Big\rangle_{L^2(\mathbb{R}^2)}\Big|^2\\
\le& B\sum_{r\in\mathbb{Z}}\Big\|\sum_{(k',l')\in\mathcal{F}}\alpha_{k',l'}(T^t_{(2k',l')})^{\lambda-r}g^{\lambda-r}\Big\|^2_{L^2(\mathbb{R}^2)},\quad a.e.\ \lambda\in(0,1]
\end{aligned}
$$

for any finite $\mathcal{F}\subset\mathbb{Z}^2$ and any finite sequence $\{\alpha_{k,l}\}\in\ell^2(\mathbb{Z}^2)$.

Proof. The system $\{L_{(2k,l,m)}g : k,l,m \in \mathbb{Z}\}$ is a frame sequence with bounds $A,B>0$ iff (3.1) holds. Consider $\Phi(\lambda) := \sum_{(k,l)\in\mathcal{F}}\alpha_{k,l}\tau(L_{(2k,l,0)}g)(\lambda)$, for some finite $\mathcal{F}\subset\mathbb{Z}^2$ and a finite sequence $\{\alpha_{k,l}\}\in\ell^2(\mathbb{Z}^2)$. Then,

$$
\begin{aligned}
\|\Phi(\lambda)\|^2_{\ell^2(\mathbb{Z},\mathcal{B}_2)} &= \Big\|\sum_{(k,l)\in\mathcal{F}}\alpha_{k,l}\tau(L_{(2k,l,0)}g)(\lambda)\Big\|^2_{\ell^2(\mathbb{Z},\mathcal{B}_2)}\\
&= \Big\|\Big\{|\lambda-r|^{1/2}\sum_{(k,l)\in\mathcal{F}}\alpha_{k,l}\widehat{L_{(2k,l,0)}g}(\lambda-r)\Big\}_{r\in\mathbb{Z}}\Big\|^2_{\ell^2(\mathbb{Z},\mathcal{B}_2)}\\
&= \sum_{r\in\mathbb{Z}}|\lambda-r|\Big\|\sum_{(k,l)\in\mathcal{F}}\alpha_{k,l}\widehat{L_{(2k,l,0)}g}(\lambda-r)\Big\|^2_{\mathcal{B}_2}\\
&= \sum_{r\in\mathbb{Z}}|\lambda-r|\Big\|\sum_{(k,l)\in\mathcal{F}}\alpha_{k,l}W_{\lambda-r}\big((L_{(2k,l,0)}g)^{\lambda-r}\big)\Big\|^2_{\mathcal{B}_2}.
\end{aligned}
$$

Employing (2.6) and (2.8), yields

$$
\begin{aligned}
\|\Phi(\lambda)\|^2_{\ell^2(\mathbb{Z},\mathcal{B}_2)} &= \sum_{r\in\mathbb{Z}}\Big\|\sum_{(k,l)\in\mathcal{F}}\alpha_{k,l}(L_{(2k,l,0)}g)^{\lambda-r}\Big\|^2_{L^2(\mathbb{R}^2)}\\
&= \sum_{r\in\mathbb{Z}}\Big\|\sum_{(k,l)\in\mathcal{F}}\alpha_{k,l}(T^t_{(2k,l)})^{\lambda-r}g^{\lambda-r}\Big\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}\tag{3.3}
$$

On the other hand,

$$
\langle\Phi(\lambda),\tau(L_{(2k,l,0)}g)(\lambda)\rangle = \sum_{(k',l')\in\mathcal{F}}\alpha_{k',l'}\langle\tau(L_{(2k',l',0)}g)(\lambda),\tau(L_{(2k,l,0)}g)(\lambda)\rangle \tag{3.4}
$$

for $k,l\in\mathbb{Z}$. Using (3.2) in (3.4), we obtain

$$
\langle\Phi(\lambda),\tau(L_{(2k,l,0)}g)(\lambda)\rangle = \sum_{(k',l')\in\mathcal{F},r\in\mathbb{Z}}\alpha_{k',l'}e^{\pi i(\lambda-r)(kl'-lk')}\Big\langle\big(T^t_{(2(k'-k),l'-l)}\big)^{\lambda-r}g^{\lambda-r},g^{\lambda-r}\Big\rangle_{L^2(\mathbb{R}^2)}. \tag{3.5}
$$

Employing (3.3) and (3.5) in (3.1), the required result follows. $\square$

Next, we aim to characterize the system of left translates $\{L_{(2k,l,m)}g : k,l,m \in \mathbb{Z}\}$ to be a Riesz sequence, again in terms of λ-twisted translates of g^λ. To this end, we consider the Gramian associated with the system $\{\tau(L_{(2k,l,0)}g)(\lambda) : k,l \in \mathbb{Z}\}$ and obtain an equivalent condition for a Riesz sequence.

First, consider the Gramian associated with the system $\{\tau(L_{(2k,l,0)}g)(\lambda) : k,l \in \mathbb{Z}\}$. For $g \in L^2(\mathbb{H})$ and $\lambda \in (0,1]$, the Gramian of $\{\tau(L_{(2k,l,0)}g)(\lambda) : k,l \in \mathbb{Z}\}$ is defined by

$$G(\lambda) := H(\lambda)^* H(\lambda) : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2),$$

where $H(\lambda) : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}, \mathcal{B}_2)$ is given by

$$H(\lambda)\big(\{c_{k,l}\}\big) := \sum_{k,l\in\mathbb{Z}} c_{k,l}\tau(L_{(2k,l,0)}g)(\lambda).$$

We obtain the following

Theorem 3.4. *The system $\{L_{(2k,l,m)}g : k,l,m \in \mathbb{Z}\}$ is a Riesz sequence iff there exists $A, B > 0$ such that*

$$\begin{aligned} A\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)} \le& \sum_{k,l,k',l'\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} c_{k,l}\bar{c}_{k',l'} e^{2\pi i(\lambda-r)(lk'-kl')} \big\langle \big(T^t_{(2(k-k'),l-l')}\big)^{\lambda-r} g^{\lambda-r}, g^{\lambda-r}\big\rangle_{L^2(\mathbb{R}^2)} \\ \le& B\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)} \end{aligned} \tag{3.6}$$

for a.e. $\lambda \in (0,1]$ and for all $\{c_{k,l}\} \in \ell^2(\mathbb{Z}^2)$.

Proof. By Theorem 6 of [12], the system $\{L_{(2k,l,m)}g : k,l,m \in \mathbb{Z}\}$ is a Riesz sequence iff there exist $A, B > 0$ such that

$$A\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)} \le \langle G(\lambda)\{c_{k,l}\}, \{c_{k,l}\}\rangle_{\ell^2(\mathbb{Z}^2)} \le B\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)} \tag{3.7}$$

for a.e. $\lambda \in (0,1]$ and for all $\{c_{k,l}\} \in \ell^2(\mathbb{Z}^2)$. But

$$\begin{aligned} \langle G(\lambda)\{c_{k,l}\}, \{c_{k,l}\}\rangle_{\ell^2(\mathbb{Z}^2)} &= \big\|H(\lambda)\big(\{c_{k,l}\}\big)\big\|^2_{\ell^2(\mathbb{Z},\mathcal{B}_2)} \\ &= \Big\|\sum_{k,l\in\mathbb{Z}} c_{k,l}\tau(L_{(2k,l,0)}g)(\lambda)\Big\|^2_{\ell^2(\mathbb{Z},\mathcal{B}_2)} \\ &= \sum_{k,l,k',l'\in\mathbb{Z}} c_{k,l}\bar{c}_{k',l'} \big\langle \tau(L_{(2k,l,0)}g)(\lambda), \tau(L_{(2k',l',0)}g)(\lambda)\big\rangle_{\ell^2(\mathbb{Z},\mathcal{B}_2)} \\ &= \sum_{k,l,k',l'\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} c_{k,l}\bar{c}_{k',l'} e^{2\pi i(\lambda-r)(lk'-kl')} \big\langle \big(T^t_{(2(k-k'),l-l')}\big)^{\lambda-r} g^{\lambda-r}, g^{\lambda-r}\big\rangle_{L^2(\mathbb{R}^2)} \end{aligned} \tag{3.8}$$

by using (3.2). Employing (3.8) in (3.7), we obtain (3.6). $\square$

Example 3.1. *Let $\phi(x,y,t) := \chi_{[0,2]}(x)\chi_{[0,2]}(y)h(t)$, where $\chi_{[0,2]}$ denotes the characteristic function on $[0,2]$ and $h \in L^2(\mathbb{R})$ is given by $\widehat{h}(\lambda) = \chi_{[0,p]}(\lambda)$, for $\mathbb{N} \ni p \ge 3$. Then, $\|\phi\|^2_{L^2(\mathbb{H})} = 4\,\|h\|^2_{L^2(\mathbb{R})}$.*

Furthermore, $\phi^\lambda(x,y) = \chi_{[0,2]}(x)\chi_{[0,2]}(y)\widehat{h}(-\lambda)$. *Now,*

$$\left\langle \left(T^t_{(2k,l)}\right)^\lambda \phi^\lambda, \phi^\lambda \right\rangle = \int_{\mathbb{R}^2} e^{\pi i\lambda(lx-2ky)}\phi^\lambda(x-2k, y-l)\overline{\phi^\lambda(x,y)}\,dx\,dy$$
$$= \overline{\widehat{h}(-\lambda)} \int_0^2 \int_0^2 e^{\pi i\lambda(lx-2ky)}\phi^\lambda(x-2k,y-l)\,dy\,dx$$
$$= \overline{\widehat{h}(-\lambda)} \int_{-2k}^{2-2k} \int_{-l}^{2-l} e^{\pi i\lambda(lx-2ky)}\phi^\lambda(x,y)\,dy\,dx$$
$$= |\widehat{h}(-\lambda)|^2 \int_{[-2k,2-2k]\cap[0,2]} \int_{[-l,2-l]\cap[0,2]} e^{\pi i\lambda(lx-2ky)}\,dy\,dx. \tag{3.9}$$

For $\lambda \in (0,1]$ *and* $\{c_{k,l}\} \in \ell^2(\mathbb{Z}^2)$*, consider the middle term in (3.6) which is* $\langle G(\lambda)\{c_{k,l}\}, \{c_{k,l}\}\rangle_{\ell^2(\mathbb{Z}^2)}$*. It follows from (3.9) that only* $k' = k$ *and* $l' = l-1,\ l,\ l+1$ *will contribute to the sum over* $k', l' \in \mathbb{Z}$*. Thus, we have*

$$\langle G(\lambda)\{c_{k,l}\}, \{c_{k,l}\}\rangle_{\ell^2(\mathbb{Z}^2)} = M_1 + M_2 + M_3,$$

where

$$M_1 := \sum_{r\in\mathbb{Z}} \sum_{k,l\in\mathbb{Z}} c_{k,l}\overline{c_{k,l-1}} e^{2\pi i(\lambda-r)k} \left\langle \left(T^t_{(0,1)}\right)^{\lambda-r} \phi^{\lambda-r}, \phi^{\lambda-r} \right\rangle, \tag{3.10}$$

$$M_2 := \sum_{r\in\mathbb{Z}} \sum_{k,l\in\mathbb{Z}} c_{k,l}\overline{c_{k,l+1}} e^{-2\pi i(\lambda-r)k} \left\langle \left(T^t_{(0,-1)}\right)^{\lambda-r} \phi^{\lambda-r}, \phi^{\lambda-r} \right\rangle$$

and

$$M_3 := \sum_{k,l\in\mathbb{Z}} |c_{k,l}|^2 \sum_{r\in\mathbb{Z}} \left\|\phi^{\lambda-r}\right\|^2_{L^2(\mathbb{R}^2)}.$$

We observe that $M_2 = \overline{M_1}$*. Hence,* $\langle G(\lambda)\{c_{k,l}\}, \{c_{k,l}\}\rangle_{\ell^2(\mathbb{Z}^2)} = 2\,\mathrm{Re}\,(M_1) + M_3$*. But* $\mathrm{Re}(M_1) \le |M_1|$*. Applying the Cauchy-Schwarz inequality in (3.10), we obtain* $\mathrm{Re}(M_1) \le \|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)} I_{1,\lambda}$*, where*

$$I_{1,\lambda} := \left| \sum_{r\in\mathbb{Z}} \left\langle \left(T^t_{(0,1)}\right)^{\lambda-r} \phi^{\lambda-r}, \phi^{\lambda-r} \right\rangle \right|$$
$$= \left| \sum_{r\in\mathbb{Z}} |\widehat{h}(-(\lambda-r))|^2 \int_0^2 e^{\pi i(\lambda-r)x}\,dx \right|$$
$$= \left| \sum_{r=1}^{p} \int_0^2 e^{\pi i(\lambda-r)x}\,dx \right|.$$

But,

$$\int_0^2 e^{\pi i(\lambda-r)x}\,dx = 2e^{\pi i(\lambda-r)}\,\mathrm{sinc}(\lambda-r).$$

Hence,

$$I_{1,\lambda} \le 2\sum_{r=1}^{p} |\,\mathrm{sinc}(\lambda-r)| \le 2\sum_{r=1}^{p} 1 = 2p.$$

As $\|\phi^{\lambda-r}\|^2_{L^2(\mathbb{R}^2)} = |\widehat{h}(-(\lambda-r))|^2$,

$$M_3 = 2\left[\sum_{r\in\mathbb{Z}} |\widehat{h}(-(\lambda-r))|^2\right] \|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)} = 2p\,\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)}\,.$$

Therefore,

$$\langle G(\lambda)\{c_{k,l}\},\{c_{k,l}\}\rangle_{\ell^2(\mathbb{Z}^2)} \le 6p\,\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)}\,.$$

On the other hand, $\operatorname{Re}(M_1) \ge -|M_1|$ *leads to*

$$\langle G(\lambda)\{c_{k,l}\},\{c_{k,l}\}\rangle_{\ell^2(\mathbb{Z}^2)} \ge 2\left[p - 2\left|\sum_{r=1}^{p} e^{\pi i(\lambda-r)}\operatorname{sinc}(\lambda-r)\right|\right] \|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)}\,.$$

Now,

$$\begin{aligned}
&p - 2\left|\sum_{r=1}^{p} e^{\pi i(\lambda-r)}\operatorname{sinc}(\lambda-r)\right| \\
=&p-2\left|\left(\sum_{r=1}^{p}\cos\left(\pi(\lambda-r)\right)\operatorname{sinc}(\lambda-r)\right) + i\left(\sum_{r=1}^{p}\sin\left(\pi(\lambda-r)\right)\operatorname{sinc}(\lambda-r)\right)\right| \\
=&:p - A_p(\lambda). \qquad (3.11)
\end{aligned}$$

Employing some properties of the digamma function [1, Section 6.3]

$$\psi^{(0)}(z) := \frac{d}{dz}\log\Gamma(z), \quad \operatorname{Re} z > 0,$$

we deduce that

$$\begin{aligned}
&\sum_{r=1}^{p}\cos\left(\pi(\lambda-r)\right)\operatorname{sinc}(\lambda-r) \\
&= -\sum_{r=1}^{p}\frac{\cos(\pi\lambda)\sin(\pi\lambda)}{\pi(r-\lambda)} \\
&= -\left(\frac{\cos(\pi\lambda)\sin(\pi\lambda)}{\pi(1-\lambda)} + \frac{\sin(\pi\lambda)\cos(\pi\lambda)(\psi^{(0)}(p-\lambda+1)-\psi^{(0)}(2-\lambda))}{\pi}\right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{r=1}^{p}\sin\left(\pi(\lambda-r)\right)\operatorname{sinc}(\lambda-r) &= \sum_{r=1}^{p}\frac{\sin^2(\pi\lambda)}{\pi(r-\lambda)} \\
&= \frac{\sin^2(\pi\lambda)}{\pi(1-\lambda)} + \frac{\sin^2(\pi\lambda)(\psi^{(0)}(p-\lambda+1)-\psi^{(0)}(2-\lambda))}{\pi}.
\end{aligned}$$

Hence,

$$\begin{aligned}
A_p(\lambda) &= 2\frac{\sin(\pi\lambda)}{\pi(1-\lambda)}\left(1-(1-\lambda)\psi^{(0)}(2-\lambda)+(1-\lambda)\psi^{(0)}(p-\lambda+1)\right) \\
&= 2\operatorname{sinc}(1-\lambda)\left[1+(1-\lambda)\left(\psi^{(0)}(p-\lambda+1)-\psi^{(0)}(2-\lambda)\right)\right],
\end{aligned}$$

where we used that $\sin(\pi\lambda) = \sin\pi(1-\lambda)$.

The goal is to find those values of p for which $p - 2A_p(\lambda) > 0$, for all $\lambda \in (0, 1]$. As the digamma function is monotone increasing and positive for integer arguments ≥ 2 and as

$$\lim_{\lambda\to 0+} A_p(\lambda) = 0 \quad \textit{and} \quad A_p(1) = 2, \tag{3.12}$$

we show that $p - A_p(\lambda)$ has a unique positive minimum at $\lambda_0 \in (0, 1)$ whose value is strictly positive for $p \geq 3$ and that $p + 1 - A_{p+1}(\lambda) > p - A_p(\lambda)$, for all $\lambda \in (0, 1)$ and $p \geq 3$. To establish the latter, note that

$$\begin{aligned}
p + 1 - A_{p+1}(\lambda) &= p + 1 - (2\operatorname{sinc}(1-\lambda))\big[1 + (1-\lambda)\big(\psi^{(0)}(p+1-\lambda+1) - \psi^{(0)}(2-\lambda)\big]\\
&= p + 1 - (2\operatorname{sinc}(1-\lambda))\big[1 + (1-\lambda)\big(\psi^{(0)}(p+1-\lambda) + \frac{1}{p+1-\lambda} - \psi^{(0)}(2-\lambda)\big]\\
&= p - A_p(\lambda) + 1 - \frac{2(1-\lambda)}{p+1-\lambda}\operatorname{sinc}(1-\lambda)\\
&> p - A_p(\lambda), \quad \textit{for } p \geq 3.
\end{aligned}$$

Hence, it suffices to show that $3 - A_3(\lambda)$ has a unique minimum value for a $\lambda \in [0, 1]$. To this end, we remark that

$$\begin{aligned}
3 - A_3(\lambda) &= 3 - (2\operatorname{sinc}(1-\lambda))\big[1 + (1-\lambda)\big(\psi^{(0)}(4-\lambda+1) - \psi^{(0)}(2-\lambda)\big]\\
&= 3 - (2\operatorname{sinc}(1-\lambda))\left[3 - \frac{2}{3-\lambda} - \frac{1}{2-\lambda}\right] =: \Psi(\lambda).
\end{aligned}$$

Differentiation of Ψ with respect to λ yields

$$\Psi'(\lambda) = \frac{2\pi(\lambda-3)(\lambda-2)(\lambda-1)(3(\lambda-4)\lambda+11)\cos(\pi\lambda) - 2(3(\lambda-4)\lambda((\lambda-4)\lambda+8)+49)\sin(\pi\lambda)}{\pi(\lambda-3)^2(\lambda-2)^2(\lambda-1)^2}.$$

Numerically solving $\Psi'(\lambda) = 0$, $0 < \lambda < 1$, produces an unique zero at $\lambda_0 \approx 0.762714$. As $\Psi''(\lambda_0) \approx 12.8421$ and because of equations. 3.12, the point $(\lambda_0, 3 - A_3(\lambda_0)) \approx (0.762714, 0.638135)$ is the unique global minimum of $3 - A_3(\lambda)$ on $[0, 1]$. Therefore, the right-hand side of (3.11) is strictly positive. Hence, by Theorem 3.4, we conclude that the shift-invariant system $\{L_{(2k,l,m)}\phi : k, l, m \in \mathbb{Z}\}$ forms a Riesz sequence for each $p \geq 3$.

The following result shows that one can obtain more examples of Riesz sequences of left translates on $\mathbb{H}$ from the Riesz sequence of classical translates on $\mathbb{R}$.

Proposition 3.1. *Let $h \in L^2(\mathbb{R})$. Define $\phi(x, y, t) := \chi_{[0,2]}(x)\chi_{[0,1]}(y)h(t)$. Then, the system $\{L_{(2k,l,m)}\phi : k, l, m \in \mathbb{Z}\}$ is a Riesz sequence in $L^2(\mathbb{H})$ with bounds $A, B > 0$ iff the system $\{T_r h : r \in \mathbb{Z}\}$ is a Riesz sequence in $L^2(\mathbb{R})$ with bounds $\frac{1}{2}A$ and $\frac{1}{2}B$.*

Proof. We have $\phi^\lambda(x, y) = \chi_{[0,2]}(x)\chi_{[0,1]}(y)\widehat{h}(-\lambda)$. Now,

$$\begin{aligned}
\left\langle \left(T^t_{(2k,l)}\right)^\lambda \phi^\lambda, \phi^\lambda \right\rangle &= \int_{\mathbb{R}^2} e^{\pi i\lambda(lx-2ky)}\phi^\lambda(x-2k, y-l)\overline{\phi^\lambda(x, y)}\, dx\, dy\\
&= \overline{\widehat{h}(-\lambda)} \int_0^2\int_0^1 e^{\pi i\lambda(lx-2ky)}\phi^\lambda(x-2k, y-l)\, dy\, dx\\
&= \overline{\widehat{h}(-\lambda)} \int_{-2k}^{2-2k}\int_{-l}^{1-l} e^{\pi i\lambda(lx-2ky)}\phi^\lambda(x, y)\, dy\, dx\\
&= |\widehat{h}(-\lambda)|^2 \int_{[-2k,2-2k]\cap[0,2]}\int_{[-l,1-l]\cap[0,1]} e^{\pi i\lambda(lx-2ky)}\, dy\, dx,
\end{aligned}$$

which in turn implies that $\langle (T^t_{(2k,l)})^\lambda \phi^\lambda, \phi^\lambda \rangle = 0$, $\forall\ (k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Moreover, for $(k,l) = (0,0)$, $\langle (T^t_{(2k,l)})^\lambda \phi^\lambda, \phi^\lambda \rangle = 2|\widehat{h}(-\lambda)|^2$. For $\{c_{k,l}\} \in \ell^2(\mathbb{Z}^2)$, the middle term in (3.6) becomes

$$\begin{aligned}\langle G(\lambda)\{c_{k,l}\}, \{c_{k,l}\}\rangle_{\ell^2(\mathbb{Z}^2)} &= \sum_{k,l\in\mathbb{Z}} |c_{k,l}|^2 \sum_{r\in\mathbb{Z}} 2|\widehat{h}(-(\lambda - r))|^2 \\ &= 2\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)} \sum_{r\in\mathbb{Z}} |\widehat{h}(-(\lambda - r))|^2.\end{aligned}$$

From Theorem 3.4, the system $\{L_{(2k,l,m)}\phi : k,l,m \in \mathbb{Z}\}$ is a Riesz sequence with bounds $A, B > 0$ iff

$$A\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)} \le 2\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)} \sum_{r\in\mathbb{Z}} |\widehat{h}(-(\lambda - r))|^2 \le B\|\{c_{k,l}\}\|^2_{\ell^2(\mathbb{Z}^2)}$$

for a.e. $\lambda \in (0,1]$, which is equivalent to

$$\frac{A}{2} \le \sum_{r\in\mathbb{Z}} |\widehat{h}(-(\lambda - r))|^2 \le \frac{B}{2}$$

for a.e. $\lambda \in (0,1]$. Hence, the required result follows from Theorem 2.1. $\square$

Example 3.2. *Let $\phi(x,y,t) := \chi_{[0,2]}(x)\chi_{[0,1]}(y)B_n(t)$, where B_n denotes the cardinal polynomial B-spline of order n. It is well known that $\{T_r B_n : r \in \mathbb{Z}\}$ is a Riesz sequence in $L^2(\mathbb{R})$, for each $n \in \mathbb{N}$. Hence, it follows from Proposition 3.1 that $\{L_{(2k,l,m)}\phi : k,l,m \in \mathbb{Z}\}$ is a Riesz sequence.*

4. Oblique dual of the system of left translates

In this section, we investigate the structure of an oblique dual of the system of left translates $\{L_{(2k,l,m)}\phi : k,l,m \in \mathbb{Z}\}$.

Lemma 4.2. *Assume that $\phi, \widetilde{\phi} \in L^2(\mathbb{H})$ have compact support and $\{L_{(2k,l,m)}\phi : k,l,m \in \mathbb{Z}\}$ and $\{L_{(2k,l,m)}\widetilde{\phi} : k,l,m \in \mathbb{Z}\}$ form Riesz sequences. Then, the following statements are equivalent:*

(i) $f = \sum_{k,l,m\in\mathbb{Z}} \langle f, L_{(2k,l,m)}\widetilde{\phi}\rangle L_{(2k,l,m)}\phi, \quad \forall\ f \in V := \overline{\operatorname{span}\{L_{(2k,l,m)}\phi : k,l,m \in \mathbb{Z}\}}.$

(ii) $\langle \phi, L_{(2k,l,m)}\widetilde{\phi}\rangle = \delta_{(k,l,m),(0,0,0)}, \quad \forall\ (k,l,m) \in \mathbb{Z}^3.$

Proof. The proof of this lemma is similar to the proof of Lemma 2.1 in [10]. However, for the sake of completeness, we provide the proof. Suppose that (i) holds. As (i) is true for $f = \phi$, we have

$$\phi = \sum_{k,l,m\in\mathbb{Z}} \langle \phi, L_{(2k,l,m)}\widetilde{\phi}\rangle L_{(2k,l,m)}\phi,$$

which leads to

$$[\langle \phi, L_{(2k,l,m)}\widetilde{\phi}\rangle - 1]\phi \ + \sum_{\substack{k,l,m\in\mathbb{Z}\\ (k,l,m)\neq(0,0,0)}} \langle \phi, L_{(2k,l,m)}\widetilde{\phi}\rangle L_{(2k,l,m)}\phi = 0.$$

As $\{L_{(2k,l,m)}\phi : k,l,m \in \mathbb{Z}\}$ is a Riesz sequence, we know that $\langle \phi, L_{(2k,l,m)}\widetilde{\phi}\rangle = \delta_{(k,l,m),(0,0,0)}$, $\forall\ (k,l,m) \in \mathbb{Z}^3$, which is (ii).

Conversely, suppose (ii) holds. Let $f \in V$. Then $f = \sum_{k,l,m\in\mathbb{Z}} c_{k,l,m} L_{(2k,l,m)}\phi$ for some coefficients $\{c_{k,l,m}\}$. Now,

$$\begin{aligned}\langle f, L_{(2k,l,m)}\widetilde{\phi}\rangle &= \sum_{k',l',m'\in\mathbb{Z}} c_{k',l',m'} \langle L_{(2k',l',m')}\phi, L_{(2k,l,m)}\widetilde{\phi}\rangle \\ &= \sum_{k',l',m'\in\mathbb{Z}} c_{k',l',m'} \langle \phi, L_{(2(k-k'),l-l',m-m'+(k'l-l'k))}\widetilde{\phi}\rangle \\ &= \sum_{k',l',m'\in\mathbb{Z}} c_{k',l',m'} \delta_{(k-k',l-l',m-m'+(k'l-l'k)),(0,0,0)} \\ &= c_{k,l,m},\end{aligned}$$

from which (i) follows. □

Theorem 4.5. *Let $\phi \in L^2(\mathbb{H})$ be supported in $[0,2n]\times[0,n]\times[0,M]$ for some $M,n\in\mathbb{N}$. Also assume that the system $\{L_{(2k,l,m)}\phi : k,l,m\in\mathbb{Z}\}$ forms a Riesz sequence. Then, the following statements are equivalent:*

- (i) *The system $\{L_{(2k,l,m)}\phi : k,l,m\in\mathbb{Z}\}$ has a generalized dual $\{L_{(2k,l,m)}\widetilde{\phi} : k,l,m\in\mathbb{Z}\}$ with* $\operatorname{supp}\widetilde{\phi}\subset Q$*, where $Q := [0,2]\times[0,1]\times[0,1]$.*
- (ii) *If $\sum_{(k,l,m)\in A} c_{k,l,m} L_{(2k,l,m)}\phi(x,y,t) = 0$, for all $(x,y,t)\in Q$ and for some coefficients $\{c_{k,l,m}\}$, then $c_{0,0,0}=0$, where $A := \{-(n-1)\le k,l\le 0, -M-n+1<m<n\}$.*
- (iii) $\phi|_Q \notin \operatorname{span}\{(L_{(2k,l,m)}\phi)|_Q : (k,l,m)\in A\setminus\{(0,0,0)\}\}$.

In case that any one of the above conditions is satisfied, the generalized duals $\{L_{(2k,l,m)}\widetilde{\phi} : k,l,m\in\mathbb{Z}\}$ form orthogonal sequences and they are oblique duals of $\{L_{(2k,l,m)}\phi : k,l,m\in\mathbb{Z}\}$. One can choose $\widetilde{\phi}$ to be of the form

$$\widetilde{\phi} = \left[\sum_{(k,l,m)\in A} d_{k,l,m} L_{(2k,l,m)}\phi\right]\chi_Q$$

for some coefficients $\{d_{k,l,m}\}$. Here, χ_Q denotes the characteristic function of Q.

Proof. The idea of the proof is similar to that of Theorem 3.1 of [10]. Here, we provide the main steps in the proof.

Let $\widetilde{\phi}\in L^2(\mathbb{H})$ be such that $\operatorname{supp}\widetilde{\phi}\subset Q$. Then,

$$\begin{aligned}&\langle L_{(2k,l,m)}\phi, \widetilde{\phi}\rangle \\ &= \int_Q L_{(2k,l,m)}\phi(x,y,t)\overline{\widetilde{\phi}}(x,y,t)\,dxdydt \\ &= \int_{-2k}^{2(1-k)}\int_{-l}^{1-l}\int_{-m+\frac{1}{2}(2ky-lx)}^{1-m+\frac{1}{2}(-lx+2ky)} \phi(x,y,t)\overline{\widetilde{\phi}}(x+2k,y+l,t+m-\frac{1}{2}(-lx+2ky))\,dtdydx,\end{aligned}$$

by applying a change of variables. Further, using $\operatorname{supp}\phi\subset[0,2n]\times[0,n]\times[0,M]$, we obtain $\langle L_{(2k,l,m)}\phi,\widetilde{\phi}\rangle = 0,\ \forall\,(k,l,m)\in A^c$.

Assume that (i) holds. Then, by Lemma 4.2, we have that $\langle\phi, L_{(2k,l,m)}\widetilde{\phi}\rangle = \delta_{(k,l,m),(0,0,0)}$, $\forall\,(k,l,m)\in\mathbb{Z}^3$. Hence, we obtain the moment problem

$$\langle L_{(2k,l,m)}\phi,\widetilde{\phi}\rangle = \delta_{(k,l,m),(0,0,0)}$$

for $(k,l,m) \in A$. Now, condition (i) is equivalent to the existence of a solution of the moment problem. By Lemma 2.1, the existence of a solution of the moment problem is equivalent to conditions (ii) and (iii). Moreover, if (i) is true, then $\operatorname{supp}\widetilde{\phi} \subset Q$ leads to the fact that the system $\{L_{(2k,l,m)}\widetilde{\phi} : k,l,m \in \mathbb{Z}\}$ is an orthogonal sequence. $\square$

Example 4.3. *Let $\phi(x,y,t) := \chi_{[0,2]}(x)\chi_{[0,1]}(y)B_3(t)$, where B_3 is the cardinal polynomial B-spline of order 3, given by*

$$B_3(t) = \begin{cases} \frac{1}{2}t^2, & t \in [0,1]; \\ -t^2 + 3t - \frac{3}{2}, & t \in [1,2]; \\ \frac{1}{2}t^2 - 3t + \frac{9}{2}, & t \in [2,3]; \\ 0, & \textit{otherwise}. \end{cases}$$

Thus, it follows from Example 3.2 that $\{L_{(2k,l,m)}\phi : k,l,m \in \mathbb{Z}\}$ is a Riesz sequence. We know that $\operatorname{supp}\widetilde{\phi} \subset Q$. Consider

$$\begin{aligned} &\langle L_{(2k,l,m)}\phi, \widetilde{\phi}\rangle \\ &= \int_0^2 \int_0^1 \int_0^1 \phi(x-2k, y-l, t-m+\tfrac{1}{2}(2ky-lx))\overline{\widetilde{\phi}(x,y,t)}\, dt\, dy\, dx \\ &= \int_{[-2k,2-2k]\cap[0,2]} \int_{[-l,1-l]\cap[0,1]} \int_0^1 B_3(t-m+\tfrac{1}{2}(2ky-lx))\overline{\widetilde{\phi}(x+2k,y+l,t)}\, dt\, dy\, dx. \end{aligned}$$

Hence, for $(k,l) \neq (0,0)$, $\langle L_{(2k,l,m)}\phi, \widetilde{\phi}\rangle = 0$.

For $(k,l) = (0,0)$, we have

$$\langle L_{(0,0,m)}\phi, \widetilde{\phi}\rangle = \int_0^2 \int_0^1 \int_{[-m,1-m]\cap[0,3]} B_3(t)\overline{\widetilde{\phi}(x.y,t+m)}\, dt\, dy\, dx,$$

which shows that $-2 \le m \le 0$. Define $\Lambda := \{(0,0,-2),(0,0,-1),(0,0,0)\}$. Then, $\langle L_{(0,0,m)}\phi, \widetilde{\phi}\rangle = 0$, $\forall\ (k,l,m) \notin \Lambda$. Furthermore, it is easy to show that $\{\phi|_Q, \left(L_{(0,0,-1)}\phi\right)|_Q, \left(L_{(0,0,-2)}\phi\right)|_Q\}$ is a linearly independent set. Thus, by Theorem 4.5, an oblique dual of ϕ is given by

$$\widetilde{\phi} = \left[\sum_{m=-2,-1,0} d_m L_{(0,0,m)}\phi\right]\chi_Q, \tag{4.1}$$

satisfying the moment problem

$$\langle L_{(0,0,m)}\phi, \widetilde{\phi}\rangle = \delta_{0,m} \tag{4.2}$$

for $m = -2,-1,0$.

Next, we proceed to solve the above moment problem. Substituting (4.1) in (4.2), we get the following equations

$$\begin{aligned} \sum_{m=-2,-1,0} \overline{d_m}\ \langle \phi, L_{(0,0,m)}\phi \cdot \chi_Q\rangle &= 1, \\ \sum_{m=-2,-1,0} \overline{d_m}\ \langle L_{(0,0,-1)}\phi, L_{(0,0,m)}\phi \cdot \chi_Q\rangle &= 0, \\ \sum_{m=-2,-1,0} \overline{d_m}\ \langle L_{(0,0,-2)}\phi, L_{(0,0,m)}\phi \cdot \chi_Q\rangle &= 0. \end{aligned}$$

Upon simplification, we obtain

$$\begin{aligned} 6d_0 + 13d_{-1} + d_{-2} &= 60, \\ d_0 + \frac{54}{13}d_{-1} + d_{-2} &= 0, \\ d_0 + 13d_{-1} + 6d_{-2} &= 0. \end{aligned}$$

Solving these equations and then substituting back into (4.1), *yields*

$$\widetilde{\phi}(x,y,t) = \tfrac{3}{2}(40t^2 - 36t + 5)\chi_Q(x,y,t).$$

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Research Article

On a new approach in the space of measurable functions

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ABSTRACT. In this paper, we present a new modulus of continuity for locally integrable function spaces which is effected by the natural structure of the L_p space. After basic properties of it are expressed, we provide a quantitative type theorem for the rate of convergence of convolution type integral operators and iterates of them. Moreover, we state their global smoothness preservation property including the new modulus of continuity. Finally, the obtained results are performed to the Gauss-Weierstrass operators.

Keywords: Convolution type integral operators, measurable functions, weighted modulus of continuity.

2020 Mathematics Subject Classification: 41A36, 41A25.

1. INTRODUCTION

The study of quantitative type theorems for an approximation process is one of the research topics in Approximation Theory. Quantitative type theorems are significant tools to identify not only the convergence of a sequence of operators into an identity operator but also the rate of this convergence in a unique theorem. On the other hand, the modulus of continuity represents considerable tools for obtaining quantitative estimates of the error of approximation for positive processes. They can be defined in more special functions related to a wide class of function space. Gadjiev et al. in [27], motivated us to write this paper, was presented a new approximation process. The authors took into account weighted local integrable function space which contains classical $L_p(\mathbb{R})$ space and obtained the Korovkin type theorem on this space. Thus, some results regarding the Korovkin type approximation theorem in the space $L_p[a,b]$ of the Lebesgue integrable functions on a compact interval are generalized the results on unbounded intervals. Also, in [25], rates of A- statistical convergence of operators in the space of locally integrable functions are handled. The main advantage in considering weighted local integrable functions space, any function that is bounded with respect to the corresponding norm of the space, can be unbounded for the usual L_p norm. This allows us to widen the class of functions for which we consider the above approximation problems. In fact, in the literature, approximation results have been primarily considered either $L_p[a,b]$ or $L_p(\mathbb{R})$ space (see [17], [24], [28] and [26]), for a more general space of functions, for instance Orlicz spaces, see [23, 4].

On the other hand, Mellin transformations play major roles not only in mathematics but also in engineering, computer science, physics, etc. Their significance arises from their applications to real-life problems. For example, they are concerned with signal processing problems as in the classical Shannon Sampling Theorem, but exponentially spaced (see e.g., [18], [19], [21],

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[15]). Comprehensive approach to Mellin transforms and connections with the Mellin convolution operators were improved in [20]. The singular integrals of Mellin convolution type were first introduced by W. Kolbe and R. J. Nessel in [29]. Butzer and Jansche [20] comprehensively analyzed Mellin transformation. They defined the Mellin convolution and gained some significant results. It plays a prominent role in the Mellin analysis, like the conventional convolution operators in the Fourier analysis. These convolution integrals are used to characterize the behavior of solutions of certain boundary value problems in wedge-shaped regions. In [31], quantitative theorems on linear approximation processes of convolution operators in Banach spaces are given. Butzer and Jansche [20] extensively studied them, in connection with the L_p convergence. Later, Bardaro and Mantellini [11] concerned Mellin convolution operators of type

$$(T_w f)(s) = \int_0^\infty K_w(t) f(ts) \frac{dt}{t}, \quad s \in \mathbb{R}^+,$$

where f belongs to domain of the operator T_w and $K_w : (0,\infty) \to \mathbb{R}$ is a set of the kernels. Compared with the usual classical convolution, the translation operator is replaced by a dilation operator, and Lebesgue measure by the Haar measure $\mu = dt/t$ of the multiplicative group $\mathbb{R}^+$. This makes fully independent the operator from the classical convolution operators over the line group. We will denote by $L_p(\mu, \mathbb{R}^+) = L_p(\mu)$, $1 \leq p < +\infty$, the Lebesgue spaces with respect to the measure μ and by $L_\infty(\mu)$ the space of all the essentially bounded functions. We will denote by $\|f\|_p$ and $\|f\|_\infty$ the corresponding norms.

Mamedov [30] developed the approximation theory by Mellin convolution operators T_w by considering the logarithmic Taylor formula, Mellin derivatives, logarithmic uniform continuity and logarithmic moment of kernel function K_w, which makes probable us to have better order of approximation. In [8] and [9], the authors introduced a suitable linear combination of Mellin type operators to accelerate convergence. A crucial contribution for the Voronovskaja type results for singular integral operators of Mellin convolution given by C. Bardaro and I. Mantellini in [14], [11] and [7]. Another approach to gain better approximation order, Bardaro and Mantellini [12] considered linear combinations of Mellin type operators using the iterated kernels. Also, in [10], [16] and [5], some contributions on sampling operators in Mellin-Lebesgue spaces were obtained recently. Angeloni and Vinti, both in [2] and [3], studied Mellin integral operators in the space of functions of bounded variation in the multidimensional setting via the notion of variation for multivariate functions. In the recent past, in [32], Ozsarac et al. defined a new generalization of Mellin convolution operators that preserve logarithmic functions, and investigated the weighted approximation properties of the operators.

Pointwise convergence for type linear singular integrals in periodic case or in the line group was thoroughly worked in the classical book by P.L. Butzer and R.J. Nessel [22], where in particular an almost everywhere convergence is acquired using the notion of the Lebesgue point of a function $f \in L_p$, $1 \leq p \leq +\infty$. Also, in [13], pointwise convergence theorems for nonlinear Mellin convolution operators are verified.

In [27], even if, for what concerns Korovkin type results on this concept have been proved, quantitative type results of approximation have not been yet studied. In this paper, we tackle the above problems for convolution type operators. Such operators are studied by many mathematicians due to their various application in different domains of mathematics and physics (see [22]).

For this intention, in this paper, with the motivation of [6], we present a new modulus of continuity whose structure is compatible with the nature of the locally integrable function space to measure the rate of convergence. Also, the global smoothness preservation property of the convolution-type operators is proved. This property is also used to obtain a quantitative type theorem for the convolution type operator with an iterated kernel instead of a basic kernel.

Now, we express the notion of locally integrable function in Mellin setting. In the course of this paper, we will use the weight function ω defined by $\omega(x) = 1 + \log^2 x$, $x \in \mathbb{R}^+$. Then, we will denote by $X_{p,\omega}(loc)$ the space of all locally integrable functions, that is the space of all measurable functions f satisfying the inequality

$$\left(\frac{1}{2\log h} \int_{x/h}^{xh} |f(s)|^p \frac{ds}{s} \right)^{1/p} \leq M_f \omega(x), x \in \mathbb{R}^+,$$

where M_f is a positive constant which depends on the function f, $p > 1$ and $h > 1$ is any positive constant.

To simplify statement, we need the followings. For any real numbers a and b ($a < b$), we write

$$\|f; X_p(a,b)\| = \left(\frac{1}{\log \frac{b}{a}} \int_a^b |f(s)|^p \frac{ds}{s} \right)^{1/p}.$$

$X_{p,\omega}(loc)$ is a linear normed space with the norm

$$\begin{aligned} \|f\|_{X_{p,\omega}} &= \sup_{x \in \mathbb{R}^+} \frac{\left(\frac{1}{2\log h} \int_{x/h}^{xh} |f(s)|^p \frac{ds}{s} \right)^{1/p}}{\omega(x)} \\ &= \sup_{x \in \mathbb{R}^+} \frac{\|f; X_p(xh, x/h)\|}{\omega(x)}. \end{aligned} \tag{1.1}$$

It is clear that

$$L_p(\mathbb{R}^+) \subset X_{p,\omega}(loc),$$

where $L_p(\mathbb{R}^+)$ is the Lebesgue space with respect to the measure ds/s. Let $X_{p,\omega}^{k_f}(loc)$ be the subspace of all functions $f \in X_{p,\omega}(loc)$ for which there exists a constant k_f such that

$$\lim_{x \to \infty} \frac{\|f - k_f \omega; X_{p,\omega}(xh, x/h)\|}{\omega(x)} = 0.$$

In the case of $k_f = 0$, we will write $X_{p,\omega}^0(loc)$.

2. Definition of New Weighted Modulus of Continuity

In this part, to obtain the rate of convergence of approximation, we introduce a new type weighted modulus of continuity for function $f \in X_{p,\omega}(loc)$. Firstly, the new weighted modulus of continuity has some properties that are similar to the properties of the classical modulus of continuity. Using the weighted modulus of continuity, we obtain estimates of approximation of function $f \in X_{p,\omega}(loc)$ with respect to weighted norm. For each $f \in X_{p,\omega}(loc)$, we set

$$\Omega_{X,\omega}(f;\delta) = \sup_{|\log s|\leq\delta}\sup_{x\in\mathbb{R}^+}\frac{\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh}|f(ts)-f(t)|^p\,\frac{dt}{t}\right)^{1/p}}{\omega(x)\,\omega(s)}$$

$$(2.2)\qquad = \sup_{|\log s|\leq\delta}\sup_{x\in\mathbb{R}^+}\frac{\|f(ts)-f(t);X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)},$$

where $\delta > 0$. It is clear that $\Omega_{X,\omega}(f;\delta)$ is a non-negative and non-decreasing function. First, we show that $\Omega_{X,\omega}$ is bounded.

Lemma 2.1. *For any $f \in X_{p,\omega}(loc)$ and $\delta > 0$, we have*

$$\Omega_{X,\omega}(f;\delta) \leq 3\,\|f\|_{X_{p,\omega}}.$$

Proof. Using the inequality $\omega(xs) \leq 2\omega(x)\,\omega(s)$, we obtain by (1.1)

$$\begin{aligned}\Omega_{X,\omega}(f;\delta) &\leq \sup_{|\log s|\leq\delta}\sup_{x\in\mathbb{R}^+}\frac{\|f(\cdot s);X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)}\\ &+ \sup_{|\log s|\leq\delta}\sup_{x\in\mathbb{R}^+}\frac{\|f;X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)}\\ &\leq 3\,\|f\|_{X_{p,\omega}}.\end{aligned}$$

□

Lemma 2.2. *For any non-negative real numbers λ and δ, the following relation*

$$(2.3)\qquad \Omega_{X,\omega}(f;\lambda\delta) \leq (1+\lambda)\,\Omega_{X,\omega}(f;\delta)$$

holds.

Proof. We take into account $\delta > 0$. For any positive integer n, we may write

$$\begin{aligned}\Omega_{X,\omega}(f;n\delta) &= \sup_{|\log s|\leq n\delta}\sup_{x\in\mathbb{R}^+}\frac{\|f(ts)-f(t);X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)}\\ &= \sup_{|\log s|\leq\delta}\sup_{x\in\mathbb{R}^+}\frac{\|f(ts^n)-f(t);X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s^n)}\\ &\leq \sup_{|\log s|\leq\delta}\sup_{x\in\mathbb{R}^+}\sum_{k=1}^{n}\frac{\left\|f\left(ts^k\right)-f\left(ts^{k-1}\right);X_{p,\omega}(xh,x/h)\right\|}{\omega(x)\,\omega(s^n)}\\ &= \sup_{|\log s|\leq\delta}\sup_{x\in\mathbb{R}^+}\sum_{k=1}^{n}\frac{\left\|f\left(ts^k\right)-f\left(ts^{k-1}\right);X_{p,\omega}(xh,x/h)\right\|}{\omega(x)\,\omega(s)}\\ &\times\frac{\omega(s)}{\omega(s^n)}\\ &\leq n\Omega_{X,\omega}(f;\delta).\end{aligned}$$

Since $\Omega_{X,\omega}(f;\delta)$ is non-decreasing function of δ, the inequality

$$\Omega_{X,\omega}(f;\lambda\delta) \leq \Omega_{X,\omega}(f;([\lambda]+1)\,\delta) \leq (\lambda+1)\,\Omega_{X,\omega}(f;\delta)$$

holds for $\lambda > 0$, where $[\cdot]$ means the integer part. □

Theorem 2.1. *If $f \in X_{p,\omega}^{k_f}(loc)$, then $\lim_{\delta\to 0}\Omega_{X,\omega}(f;\delta) = 0$.*

Proof. Because of $f \in X^{k_f}_{p,\omega}(loc)$, $\lim\limits_{x\to\infty} \frac{\|f-k_f\omega;X_{p,\omega}(xh,x/h)\|}{\omega(x)} = 0$, for all $\varepsilon > 0$, there exists a positive real number x_0 such that for all $x > x_0$

$$\|f - k_f\omega; X_{p,\omega}(xh, x/h)\| < \varepsilon\omega(x).$$

Let $x_1 > x_0 + \delta$. Let us divide the norm into two parts. Then

$$\begin{aligned}
\Omega_{X,\omega}(f;\delta) &= \sup_{|\log s|\le\delta}\sup_{x\in\mathbb{R}^+} \frac{\|f(ts) - f(t); X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)} \\
&\le \sup_{|\log s|\le\delta}\sup_{0<x\le x_1} \frac{\|f(ts) - f(t); X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)} \\
&\quad+ \sup_{|\log s|\le\delta}\sup_{x>x_1} \frac{\|f(ts) - f(t); X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)} \\
&\le \omega_X(f;\delta) + \sup_{|\log s|\le\delta}\sup_{x>x_1} \frac{\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(ts) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p}}{\omega(x)\,\omega(s)} \\
&\quad+ \sup_{|\log s|\le\delta}\sup_{x>x_1} \frac{\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(t) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p}}{\omega(x)\,\omega(s)},
\end{aligned}$$

where

$$\omega_X(f;\delta) = \sup_{|\log s|\le\delta}\sup_{|x|\le x_1} \|f(ts) - f(t); X_{p,\omega}(xh,x/h)\|.$$

It is shown that in [20, page:340], for each $\varepsilon > 0$, there exists $h > 1$ such that for all $0 < s < x_1$. Then for $x > x_1$ and $|\log s| \le \delta$, with the elementary calculations, we get

$$\begin{aligned}
\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(ts) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p} &\le \left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(ts) - k_f\omega(ts)|^p \frac{dt}{t}\right)^{1/p} \\
&\quad+ k_f\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |\omega(ts) - \omega(t)|^p \frac{dt}{t}\right)^{1/p} \\
&\le \left(\frac{1}{2\log h}\int\limits_{xs/h}^{xsh} |f(t) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p} \\
&\quad+ 4k_f|\log s|\left(|\log x| + \log h + |\log s|\right).
\end{aligned}$$

For $x > x_1$ and $|\log s| \le \delta$, we obtain

$$\Omega_{X,\omega}(f;\delta) \le \omega_X(f;\delta) + \sup_{|\log s|\le\delta}\sup_{|x|>x_1} \frac{\left(\frac{1}{2\log h}\int\limits_{xs/h}^{xsh} |f(t) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p}}{\omega(x)\,\omega(s)}$$
$$+ \sup_{|\log s|\le\delta}\sup_{|x|>x_1} \frac{4k_f |\log s| \left(|\log x| + \log h + |\log s|\right)}{\omega(x)\,\omega(s)}$$
$$+ \sup_{|\log s|\le\delta}\sup_{|x|>x_1} \frac{\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(t) - k_f\omega(t)| \frac{dt}{t}\right)^{1/p}}{\omega(x)} \frac{1}{\omega(s)}$$

and

$$\Omega_{X,\omega}(f;\delta) \le \omega_X(f;\delta) + \varepsilon + 4k_f\delta(1 + \delta + \log h) + \varepsilon.$$

As $[x_1/h, x_1 h]$ is compact interval, we get $\lim\limits_{\delta\to 0}\omega_X(f;\delta) = 0$. Therefore, we have $\lim\limits_{\delta\to 0}\Omega_{X,\omega}(f;\delta) < 2\varepsilon$. Since the inequality is true for each $\varepsilon > 0$, desired result is attained. □

3. Approximation Properties

Let $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a kernel function homogenous degree 0, i.e.

$$K(\lambda s, \lambda t) = K(s,t)$$

for every $\lambda, s, t > 0$. We will assume that K is globally measurable $K(s,.) \in L_1(\mathbb{R}^+)$ with

$$\int\limits_{\mathbb{R}^+} K(s,t)\frac{dt}{t} = 1, \quad s \in \mathbb{R}^+.$$

For a given $j \in \mathbb{N}$, we define logarithmic and absolute logarithmic moment of order j of the function K, respectively by

$$m_j(K) := \int\limits_{\mathbb{R}^+} K(s,t)\log^j\left(\frac{t}{s}\right)\frac{dt}{t} \tag{3.4}$$

and

$$M_j(K) := \int\limits_{\mathbb{R}^+} |K(s,t)| \left|\log^j\left(\frac{t}{s}\right)\right| \frac{dt}{t}. \tag{3.5}$$

Also, we define Mellin Fejer kernel (K_w) for all $w > 0$ generated by K putting

$$K_w(s,t) = wK(s^w, t^w), \quad s,t \in \mathbb{R}.$$

It is easy to see that

$$\int\limits_{\mathbb{R}^+} K_w(s,t)\frac{dt}{t} = 1. \tag{3.6}$$

Let us regard the convolution type singular integral operator

$$(T_w f)(s) = \int\limits_{\mathbb{R}^+} K_w(s,t) f(t)\frac{dt}{t} \tag{3.7}$$

for every $f : \mathbb{R}^+ \to \mathbb{R}$ in the domain of the operators T_w.

Lemma 3.3. *T_w be defined by (3.7). If $f \in X_{p,\omega}\left(loc\right)$, then we have*

$$\left\|T_w f\right\|_{X_{p,\omega}} \leq 2\left(M_0\left(K_w\right) + M_2\left(K_w\right)\right)\left\|f\right\|_{X_{p,\omega}}.$$

Proof. Taking into account the operator defined by (3.7), we can write

$$\begin{aligned}
\left\|T_w f\right\|_{X_{p,\omega}} &= \sup_{x\in\mathbb{R}^+} \frac{\left\|T_w f; X_{p,\omega}\left(xh, x/h\right)\right\|}{\omega\left(x\right)} \\
&= \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \left(\frac{1}{2\log h}\int_{x/h}^{xh}\left|\int_{\mathbb{R}^+} K_w\left(s,t\right) f\left(t\right)\frac{dt}{t}\right|^p \frac{ds}{s}\right)^{1/p} \\
&= \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \left(\frac{1}{2\log h}\int_{x/h}^{xh}\left|\int_{\mathbb{R}^+} K_w\left(1,t\right) f\left(ts\right)\frac{dt}{t}\right|^p \frac{ds}{s}\right)^{1/p}.
\end{aligned}$$

From Minkowski inequality, we obtain

$$\begin{aligned}
\left\|T_w f\right\|_{X_{p,\omega}} &\leq \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \int_{x\in\mathbb{R}^+} \left(\frac{1}{2\log h}\int_{x/h}^{xh}\left|f\left(ts\right)\right|^p \frac{ds}{s}\right)^{1/p} \left|K_w\left(1,t\right)\right|\frac{dt}{t} \\
&= \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \int_{x\in\mathbb{R}^+} \left(\frac{1}{2\log h}\int_{xt/h}^{xht}\left|f\left(s\right)\right|^p \frac{ds}{s}\right)^{1/p} \left|K_w\left(1,t\right)\right|\frac{dt}{t} \\
&\leq \left\|f\right\|_{X_{p,\omega}} \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \int_{\mathbb{R}^+} \omega\left(tx\right)\left|K_w\left(1,t\right)\right|\frac{dt}{t} \\
&\leq 2\left\|f\right\|_{X_{p,\omega}} \int_{\mathbb{R}^+} \omega\left(t\right)\left|K_w\left(1,t\right)\right|\frac{dt}{t} \\
&= 2\left\|f\right\|_{X_{p,\omega}} \left(1 + \int_{\mathbb{R}^+} \log^2 t\left|K_w\left(1,t\right)\right|\frac{dt}{t}\right).
\end{aligned}$$

From (3.5) for $j = 2$, we get desired result. $\square$

Our main results are following:

Theorem 3.2. *Let T_w be defined by (3.7) and $\Omega_{X,\omega}\left(f;\delta\right)$ be defined (2.2). If $f \in X_{p,\omega}\left(loc\right)$, then we have*

$$\left\|T_w f - f\right\|_{X_{p,\omega}} \leq P_w \Omega_{X,\omega}\left(f; \left(M_2\left(K_w\right)\right)^{1/2}\right),$$

where $P_w := 1 + M_2\left(K_w\right) + \sqrt{2}\sqrt{1 + M_4\left(K_w\right)}$.

Proof. We attain

$$\left(T_w f\right)\left(s\right) - f\left(s\right) = \int_{\mathbb{R}^+} K_w\left(s,t\right)\left(f\left(t\right) - f\left(s\right)\right)\frac{dt}{t}.$$

We conclude

$$\begin{aligned}
\|T_w f - f\|_{X_{p,\omega}} &= \sup_{x\in\mathbb{R}^+} \frac{\|T_w f - f; X_{p,\omega}(xh, x/h)\|}{\omega(x)} \\
&= \sup_{x\in\mathbb{R}^+} \frac{1}{\omega(x)} \left(\frac{1}{2\log h} \int_{x/h}^{xh} \left| \int_{\mathbb{R}^+} (f(t) - f(s)) K_w(s,t) \frac{dt}{t} \right|^p \frac{ds}{s} \right)^{1/p} \\
&\leq \int_{\mathbb{R}^+} \sup_{x\in\mathbb{R}^+} \frac{1}{\omega(x)} \left(\frac{1}{2\log h} \int_{x/h}^{xh} |(f(ts) - f(s))|^p \frac{ds}{s} \right)^{1/p} |K_w(1,t)| \frac{dt}{t} \\
&= \int_{\mathbb{R}^+} \Omega_{X,\omega}(f; |\log t|)\, \omega(t) |K_w(1,t)| \frac{dt}{t}.
\end{aligned}$$

From (2.3), for any $\delta > 0$, we can write

$$\|T_w f - f\|_{X_{p,\omega}} \leq \Omega_{X,\omega}(f;\delta) \int_{\mathbb{R}^+} \left(1 + \frac{|\log t|}{\delta}\right) \omega(t) |K_w(1,t)| \frac{dt}{t}.$$

Using Cauchy-Schwarz inequality and (3.5), we obtain

$$\begin{aligned}
&\|T_w f - f\|_{X_{p,\omega}} \\
\leq &\Omega_{X,\omega}(f;\delta) \left(1 + M_2(K_w) + \frac{1}{\delta} \left(\int_{\mathbb{R}} \log^2 t\, |K_w(1,t)| \frac{dt}{t} \right)^{1/2} \left(\int_{\mathbb{R}} \omega^2(t) |K_w(1,t)| \frac{dt}{t} \right)^{1/2} \right) \\
= &\Omega_{X,\omega}(f;\delta) \left(1 + M_2(K_w) + \frac{\sqrt{2}}{\delta} (M_2(K_w))^{1/2} \sqrt{1 + M_4(K_w)} \right).
\end{aligned}$$

If we choose $\delta = (M_2(K_w))^{1/2}$, then we have desired result. □

The global smoothness preservation property of the operator $T_w f$ is following:

Theorem 3.3. *Let T_w be defined by (3.7) and let $\Omega_{X,\omega}(f;\delta)$ be defined (2.2). If $f \in X_{p,\omega}(loc)$ and $\delta > 0$, then we get*

$$\Omega_{X,\omega}(T_w f;\delta) \leq 2(M_0(K_w) + M_2(K_w))\, \Omega_{X,\omega}(f;\delta).$$

Proof. We have

$$\begin{aligned}
J_t(x) :&= \frac{\|T_w f(\cdot z) - T_w f(\cdot); X_{p,\omega}(xh, x/h)\|}{\omega(x)\omega(z)} \\
&= \left(\frac{1}{2\log h} \int_{x/h}^{xh} \frac{|T_w f(uz) - T_w f(u)|^p}{\omega(x)\omega(z)} \frac{du}{u} \right)^{1/p} \\
&= \left(\frac{1}{2\log h} \int_{x/h}^{xh} \left| \int_{\mathbb{R}^+} \frac{[f(uzt) - f(tu)]}{\omega(x)\omega(z)} K_w(1,t) \frac{dt}{t} \right|^p \frac{du}{u} \right)^{1/p}
\end{aligned}$$

$$\leq \int_{\mathbb{R}^+} \left(\frac{1}{2\log h} \int_{x/h}^{xh} \left| \frac{[f(uzt) - f(ut)]}{\omega(x)\,\omega(z)} \right|^p \frac{du}{u} \right)^{1/p} |K_w(1,t)| \frac{dt}{t}.$$

Using the inequality $\omega(xt) \leq 2\omega(x)\,\omega(t)$, Minkowski's integral inequality for two dimensional spaces and identity (3.5) for $j = 2$, we gain

$$J_t(x) \leq \int_{\mathbb{R}^+} \left(\frac{1}{2\log h} \int_{x/h}^{xh} \left| \frac{[f(uzt) - f(ut)]}{\omega(x)\,\omega(z)} \right|^p \frac{du}{u} \right)^{1/p} |K_w(1,t)| \frac{dt}{t}$$

$$= \int_{\mathbb{R}^+} \left(\frac{1}{2\log h} \int_{tx/h}^{txh} \left| \frac{\omega(xt)}{\omega(x)\,\omega(xt)\,\omega(z)} [f(vz) - f(v)] \right|^p \frac{dv}{v} \right)^{1/p} |K_w(1,t)| \frac{dt}{t}.$$

Then, we have

$$\Omega_{X,\omega}(T_w f;\delta) \leq 2\Omega_{X,\omega}(f;\delta) \int_{\mathbb{R}^+} \omega(t)\,|K_w(1,t)| \frac{dt}{t}$$

$$= 2\left(M_0(K_w) + M_2(K_w)\right)\Omega_{X,\omega}(f;\delta).$$

Hence, the proof is fulfilled. □

4. Iterations of T_w

Given the function K we define for every $n \in \mathbb{N}$ the iterated kernel of order n of K as in [12], in the following way: for $n = 2$,

$$K^2(s,t) := \int_{\mathbb{R}^+} K(s,z)\,K(z,t) \frac{dz}{z}$$

and for $n > 2$

$$K^n(s,t) := \int_{\mathbb{R}^+} K(s,z)\,K^{n-1}(z,t) \frac{dz}{z}.$$

Similarly, endowed the function K_w, we define for every $n \in \mathbb{N}$, the iterated kernel of order n of K_w in the following way: for $n = 2$,

$$K_w^2(s,t) := \int_{\mathbb{R}^+} K_w(s,z)\,K_w(z,t) \frac{dz}{z}$$

and for $n > 2$

$$K_w^n(s,t) := \int_{\mathbb{R}^+} K_w(s,z)\,K_w^{n-1}(z,t) \frac{dz}{z}.$$

We gain for every $n \in \mathbb{N}$

$$\int_{\mathbb{R}^+} K_w^n(s,t) \frac{dt}{t} = 1.$$

Also, we have

$$m_j(K_w^n) = \frac{1}{w^j} m_j(K^n)$$

and

$$M_j\left(K_w^n\right)=\frac{1}{w^j}M_j\left(K^n\right),\ \ j\in\mathbb{N}.$$

In the same method, let us consider $n-$iterations of T_w defined by

$$\left(T_w^n f\right)(s)=\int\limits_{\mathbb{R}^+}K_w^n(s,t)\,f(t)\,\frac{dt}{t}. \tag{4.8}$$

We start with the following

Theorem 4.4. *T_w^n be defined by (4.8). If $f\in X_{p,\omega}(loc)$, then we have, for every $n\in\mathbb{N}$*

$$\left\|T_w^n f-f\right\|_{X_{p,\omega}}\leq P_w\left[\sum_{k=0}^{n-1}\left[2\left(1+M_2\left(K_w\right)\right)\right]^k\right]\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right),$$

where P_w is as in Theorem 3.2.

Proof. For $n=2$, we obtain

$$T_w^2 f(u)-f(u)=T_w\left(T_w f\right)(u)-\left(T_w f\right)(u)+\left(T_w f\right)(u)-f(u).$$

Using Theorem 3.2, we have

$$\left\|T_w^2 f-f\right\|_{X_{p,\omega}}\leq P_w\Omega_{X,\omega}\left(T_w f;\left(M_2\left(K_w\right)\right)^{1/2}\right)+P_w\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right).$$

Using Theorem 3.3, we achieve

$$\left\|T_w^2 f-f\right\|_{X_{p,\omega}}\leq P_w\left[2\left(1+M_2\left(K_w\right)\right)+1\right]\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right).$$

By induction, we gain

$$\left\|T_w^n f-f\right\|_{X_{p,\omega}}\leq P_w\left[\sum_{k=0}^{n-1}\left[2\left(1+M_2\left(K_w\right)\right)\right]^k\right]\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right).$$

□

Now, we can denote following result which expresses the difference of $n-$iterations and itself of T_w.

Corollary 4.1. *T_w^n be defined by (4.8) and T_w be defined by (3.7). If $f\in X_{p,\omega}(loc)$, then we get, for every $n\in\mathbb{N}$*

$$\left\|T_w^n f-T_w f\right\|_{X_{p,\omega}}\leq P_w\left[\sum_{k=1}^{n-1}\left[2\left(1+M_2\left(K_w\right)\right)\right]^k\right]\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right),$$

where P_w is as in Theorem 3.2.

5. Application

This section is allocated to some example. The results obtained in the previous sections can be applied to the Gauss-Weierstrass operators. The recent results related to the Gauss-Weierstrass operators also can be found in [1]. Let $K:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ be a function defined by

$$K(s,t)=\frac{1}{2\sqrt{\pi}}\exp\left(-\left(\frac{1}{2}\log\frac{t}{s}\right)^2\right)$$

(See [22]). It is easy to check that

$$\int_{\mathbb{R}} K(s,t)\,\frac{dt}{t} = 1.$$

The Mellin-Fejer kernel generated by K is given by

$$K_w(s,t) = \frac{w}{2\sqrt{\pi}} \exp\left(-\left(\frac{w}{2} \log \frac{t}{s} \right)^2 \right).$$

The corresponding Mellin-Gauss-Weierstrass operator is given by

$$(G_w f)(s) = \frac{w}{2\sqrt{\pi}} \int_{\mathbb{R}^+} \exp\left(-\left(\frac{w}{2} \log \frac{t}{s} \right)^2 \right) f(t)\,\frac{dt}{t}.$$

If j is even, we get the moment of order $2j$ of the function G_w

(5.9) $$m_j(K) = M_j(K) = 2^{j/2}(j-1)!!,$$

where in the case $n!! = 3.5...n$ with n is odd. For the n-iterated kernels, we have the formula

$$G_w^n(s,t) = \frac{w}{2\sqrt{n}\sqrt{\pi}} \exp\left(-\left(\frac{w}{2\sqrt{n}} \log \frac{s}{t} \right)^2 \right)$$

(see [12]). We have by Theorem 3.2 and (5.9), the following:

Corollary 5.2. *Let $\Omega_{X,\omega}(f;\delta)$ be defined (2.2). If $f \in X_{p,\omega}(loc)$, then we get*

$$\|G_w f - f\|_{X_{p,\omega}} \le P_w \Omega_{X,\omega}\left(f; \frac{\sqrt{2}}{w} \right),$$

where $P_w := 1 + \frac{2}{w^2} + \sqrt{2}\sqrt{1 + \frac{12}{w^4}}$.

We have by Theorem 4.4 and (5.9), the following:

Corollary 5.3. *If $f \in X_{p,\omega}(loc)$, then we get*

$$\|G_w^n f - f\|_{X_{p,\omega}} \le \left(1 + \frac{2}{w^2} + \sqrt{2}\sqrt{1 + \frac{12}{w^4}} \right) \left[\sum_{k=0}^{n-1} \left[2\left(1 + \frac{2}{w^2} \right) \right]^k \right] \Omega_{p,\omega}\left(f; \frac{\sqrt{2}}{w} \right).$$

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Research Article

Some additive reverses of Callebaut and Hölder inequalities for isotonic functionals

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ABSTRACT. In this paper, we obtain some reverses of Callebaut and Hölder inequalities for isotonic functionals via a reverse of Young's inequality we have established recently. Applications for integrals and n-tuples of real numbers are provided as well.

Keywords: Isotonic functionals, Hölder's inequality, Schwarz's inequality, Callebaut's inequality, integral inequalities, discrete inequalities.

2020 Mathematics Subject Classification: 26D15, 26D10.

1. INTRODUCTION

Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$.
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An isotonic linear functional $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
(A3) The mapping A is said to be normalised if $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [20] and [21]). For other inequalities for isotonic functionals, see [1], [4]-[19] and [22]-[25]. For related results, see [10, 11]

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$ in the second ($p_k \geq 0$, $k \in E$). As is known to all, the famous Young inequality for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality. We consider the function $f_\nu : [0, \infty) \to [0, \infty)$ defined for $\nu \in (0, 1)$

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by

$$f_{\nu}(x)=1-\nu+\nu x-x^{\nu}. \tag{1.2}$$

For $[m,M]\subset[0,\infty)$, define

$$\Delta_{\nu}(m,M):=\begin{cases} f_{\nu}(m), & M<1\\ \max\{f_{\nu}(m),f_{\nu}(M)\}, & m\le 1\le M\\ f_{\nu}(M), & 1<m\end{cases} \tag{1.3}$$

and

$$\delta_{\nu}(m,M):=\begin{cases} f_{\nu}(M), & M<1\\ 0, & m\le 1\le M\\ f_{\nu}(m), & 1<m\end{cases}. \tag{1.4}$$

In the recent paper [9], we obtained the following refinement and reverse for the additive Young's inequality:

$$\delta_{\nu}(m,M)\,a\le(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\le\Delta_{\nu}(m,M)\,a \tag{1.5}$$

for positive numbers a, b with $\frac{b}{a}\in[m,M]\subset(0,\infty)$ and $\nu\in[0,1]$, where $\Delta_{\nu}(m,M)$ and $\delta_{\nu}(m,M)$ are defined by (1.3) and (1.4), respectively.

Kittaneh and Manasrah [16], [17] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\le(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\le R\left(\sqrt{a}-\sqrt{b}\right)^{2}, \tag{1.6}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.6) to an identity. Using (1.5) and (1.6), we have the simpler, however coarser bounds:

$$\begin{aligned} & r\times\begin{cases}\left(1-\sqrt{M}\right)^{2}a, & M<1\\ 0, & m\le 1\le M\\ \left(\sqrt{m}-1\right)^{2}a, & 1<m\end{cases}\\ & \le(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\\ & \le R\times\begin{cases}\left(1-\sqrt{m}\right)^{2}a, & M<1\\ \max\left\{\left(1-\sqrt{m}\right)^{2},\left(\sqrt{M}-1\right)^{2}\right\}a, & m\le 1\le M\\ \left(\sqrt{M}-1\right)^{2}a, & 1<m\end{cases}. \end{aligned} \tag{1.7}$$

We recall that Specht's ratio is defined by [24]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)}, & h\in(0,1)\cup(1,\infty)\\ 1, & h=1\end{cases}. \tag{1.8}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$. The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\le(1-\nu)\,a+\nu b\le S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.9}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$. The second inequality in (1.3) is due to Tominaga [26], while the first one is due to Furuichi [15]. On making use of (1.5) and (1.9), we have the following lower and upper bounds in terms of Specht's ratio:

$$\begin{aligned} &\begin{cases} [S(M^r)-1]\,M^\nu a, & M<1 \\ 0, & m\le 1\le M \\ [S(m^r)-1]\,m^\nu a, & 1<m \end{cases} \\ &\le (1-\nu)\,a+\nu b-a^{1-\nu}b^\nu \\ &\le \begin{cases} [S(m)-1]\,m^\nu a, & M<1 \\ \max\{[S(m)-1]\,m^\nu, [S(M)-1]\,M^\nu\}\,a, & m\le 1\le M \\ [S(M)-1]\,M^\nu a, & 1<m \end{cases}. \end{aligned} \tag{1.10}$$

We consider the Kantorovich's constant defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{1.11}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\ge 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$. The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds.

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^\nu \le (1-\nu)\,a+\nu b \le K^R\left(\frac{a}{b}\right)a^{1-\nu}b^\nu, \tag{1.12}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$. The first inequality in (1.12) was obtained by Zou et al. in [27], while the second by Liao et al. [18]. By making use of (1.5) and (1.9), we have the following lower and upper bounds in terms of Kantorovich's constant:

$$\begin{aligned} &\begin{cases} [K^r(M)-1]\,M^\nu a, & M<1 \\ 0, & m\le 1\le M \\ [K^r(m)-1]\,m^\nu a, & 1<m \end{cases} \\ &\le (1-\nu)\,a+\nu b-a^{1-\nu}b^\nu \\ &\le \begin{cases} \left[K^R(m)-1\right]m^\nu a, & M<1 \\ \max\left\{\left[K^R(m)-1\right]m^\nu, \left[K^R(M)-1\right]M^\nu\right\}a, & m\le 1\le M \\ \left[K^R(M)-1\right]M^\nu a, & 1<m \end{cases}. \end{aligned} \tag{1.13}$$

In this paper, we obtain some reverses of Callebaut and Hölder inequalities for isotonic functionals via the reverse of Young's inequality obtained in (1.5). Applications for integrals and n-tuples of real numbers are provided as well.

2. Reverses of Callebaut's Inequality

The functional version of Callebaut's inequality states that

$$A^2(fg) \le A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right) \le A\left(f^2\right)A\left(g^2\right) \tag{2.14}$$

provided that f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}$, $fg \in L$ for some $\nu\in[0,1]$. For the discrete and integral versions in one real variable, see [3].

We start with the following result:

Theorem 2.1. *Let $A, B : L \to \mathbb{R}$ be two normalised isotonic functionals. If $f, g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, f^2, g^2, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and*

$$0 < m \leq \frac{f}{g} \leq M < \infty \tag{2.15}$$

for some constants m, M, then

$$\begin{aligned}(0 \leq) (1-\nu) A\left(f^2\right) B\left(g^2\right) + \nu A\left(g^2\right) B\left(f^2\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right) B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} A\left(f^2\right) B\left(g^2\right),\end{aligned} \tag{2.16}$$

where f_ν is defined by (1.2). In particular,

$$\begin{aligned}(0 \leq) A\left(f^2\right) A\left(g^2\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right) A\left(f^{2\nu}g^{2(1-\nu)}\right) \\ \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} A\left(f^2\right) A\left(g^2\right).\end{aligned} \tag{2.17}$$

Proof. For any $x, y \in E$, we have

$$m^2 \leq \frac{f^2(x)}{g^2(x)}, \frac{f^2(y)}{g^2(y)} \leq M^2.$$

Consider

$$a = \frac{f^2(x)}{g^2(x)}, \; b = \frac{f^2(y)}{g^2(y)},$$

then $\frac{b}{a} \in \left[\left(\frac{m}{M}\right)^2, \left(\frac{M}{m}\right)^2\right]$ and by the inequality (1.5), we have

$$\begin{aligned}(0 \leq) (1-\nu) \frac{f^2(x)}{g^2(x)} + \nu \frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu} \left(\frac{f^2(y)}{g^2(y)}\right)^{\nu} \\ \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} \frac{f^2(x)}{g^2(x)}\end{aligned} \tag{2.18}$$

for any $x, y \in E$. Now, if we multiply (2.18) by $g^2(x) g^2(y) > 0$ then we get

$$\begin{aligned}(1-\nu) g^2(y) f^2(x) + \nu f^2(y) g^2(x) - f^{2(1-\nu)}(x) g^{2\nu}(x) f^{2\nu}(y) g^{2(1-\nu)}(y) \\ \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} f^2(x) g^2(y)\end{aligned} \tag{2.19}$$

for any $x, y \in E$. Fix $y \in E$. Then by (2.19), we have in the order of L that

$$\begin{aligned}(1-\nu) g^2(y) f^2 + \nu f^2(y) g^2 - f^{2\nu}(y) g^{2(1-\nu)}(y) f^{2(1-\nu)} g^{2\nu} \\ \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} g^2(y) f^2.\end{aligned} \tag{2.20}$$

If we take the functional A in (2.19), then we get

$$\begin{aligned}(1-\nu) g^2(y) A\left(f^2\right) + \nu f^2(y) A\left(g^2\right) - f^{2\nu}(y) g^{2(1-\nu)}(y) A\left(f^{2(1-\nu)}g^{2\nu}\right) \\ \leq \max\left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} g^2(y) A\left(f^2\right)\end{aligned}$$

for any $y \in E$. This inequality can be written in the order of L as

$$(1-\nu) A\left(f^2\right) g^2+\nu A\left(g^2\right) f^2-A\left(f^{2(1-\nu)} g^{2 \nu}\right) f^{2 \nu} g^{2(1-\nu)} \leq \max \left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\} A\left(f^2\right) g^2. \tag{2.21}$$

Now, if we take the functional B in (2.21), then we get the desired result (2.16). □

Corollary 2.1. *Let $A, B : L \to \mathbb{R}$ be two normalised isotonic functionals. If $f, g : E \to \mathbb{R}$ are such that $f \geq 0$, $g > 0$, $f^2, g^2, fg \in L$ and the condition (2.15) holds true, then*

$$(0 \leq) \frac{1}{2}\left[A\left(f^2\right) B\left(g^2\right)+A\left(g^2\right) B\left(f^2\right)\right]-A(f g) B(f g) \leq \frac{1}{2}\left(\frac{M}{m}-1\right)^2 A\left(f^2\right) B\left(g^2\right). \tag{2.22}$$

In particular,

$$(0 \leq) A\left(f^2\right) A\left(g^2\right)-A^2(f g) \leq \frac{1}{2}\left(\frac{M}{m}-1\right)^2 A\left(f^2\right) A\left(g^2\right), \tag{2.23}$$

or, equivalently

$$(0 \leq) 1-\frac{A^2(f g)}{A\left(f^2\right) A\left(g^2\right)} \leq \frac{1}{2}\left(\frac{M}{m}-1\right)^2. \tag{2.24}$$

Proof. Observe that

$$f_{\frac{1}{2}}\left(\left(\frac{m}{M}\right)^2\right)=\frac{m^2+M^2}{2 M^2}-\frac{m}{M}=\frac{(M-m)^2}{2 M^2}$$

and

$$f_\nu\left(\left(\frac{M}{m}\right)^2\right)=\frac{m^2+M^2}{2 m^2}-\frac{M}{m}=\frac{(M-m)^2}{2 m^2}.$$

Therefore

$$\max \left\{f_\nu\left(\left(\frac{m}{M}\right)^2\right), f_\nu\left(\left(\frac{M}{m}\right)^2\right)\right\}=\frac{(M-m)^2}{2 m^2}=\frac{1}{2}\left(\frac{M}{m}-1\right)^2$$

and by (2.16), we get the desired result (2.22). □

Remark 2.1. *We observe that the inequality (2.23) can be written as*

$$A\left(f^2\right) A\left(g^2\right)\left[1-\frac{1}{2}\left(\frac{M}{m}-1\right)^2\right] \leq A^2(f g). \tag{2.25}$$

We observe that the function $\varphi : [1, \infty) \to \mathbb{R}$, $\varphi(t) = 1 - \frac{1}{2}(t-1)^2$ is positive for $t \in \left(1, 1+\sqrt{2}\right)$ and negative for $t \in [1, \infty)$. Therefore, the inequality (2.25) is of interest only in the case that $\frac{M}{m} \in \left(1, 1+\sqrt{2}\right)$.

On using the inequality (2.16) and (1.7), we get

$$(0 \leq)(1-\nu) A\left(f^2\right) B\left(g^2\right)+\nu A\left(g^2\right) B\left(f^2\right)-A\left(f^{2(1-\nu)} g^{2 \nu}\right) B\left(f^{2 \nu} g^{2(1-\nu)}\right) \leq R \max \left\{\left(1-\frac{m}{M}\right)^2,\left(\frac{M}{m}-1\right)^2\right\} A\left(f^2\right) B\left(g^2\right) \tag{2.26}$$

and since

$$\max\left\{\left(1-\frac{m}{M}\right)^2,\left(\frac{M}{m}-1\right)^2\right\}=\left(\frac{M}{m}-1\right)^2,$$

then we get from (2.26) that

$$(0\le)\,(1-\nu)\,A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \le R\left(\frac{M}{m}-1\right)^2 A\left(f^2\right)B\left(g^2\right) \tag{2.27}$$

provided $f\ge 0$, $g>0$, $f^2, g^2, f^{2(1-\nu)}g^{2\nu}, f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$.

On using the inequality (2.16) and (1.10), we get the following reverse of Callebaut's inequality in terms of Specht's ratio

$$(0\le)\,(1-\nu)\,A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \le \max\left\{\left[S\left(\left(\frac{m}{M}\right)^2\right)-1\right]\left(\frac{m}{M}\right)^{2\nu},\left[S\left(\left(\frac{M}{m}\right)^2\right)-1\right]\left(\frac{M}{m}\right)^{2\nu}\right\}A\left(f^2\right)B\left(g^2\right) \tag{2.28}$$

provided $f\ge 0$, $g>0$, $f^2, g^2, f^{2(1-\nu)}g^{2\nu}, f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$.

Finally, on using the inequality (2.16) and (1.13), we get the following reverse of Callebaut's inequality in terms of Kantorovich's constant

$$(0\le)\,(1-\nu)\,A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \le \max\left\{\left[K^R\left(\left(\frac{m}{M}\right)^2\right)-1\right]\left(\frac{m}{M}\right)^{2\nu},\left[K^R\left(\left(\frac{M}{m}\right)^2\right)-1\right]\left(\frac{M}{m}\right)^{2\nu}\right\} \times A\left(f^2\right)B\left(g^2\right) \tag{2.29}$$

provided $f\ge 0$, $g>0$, $f^2, g^2, f^{2(1-\nu)}g^{2\nu}, f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$.

3. Reverses of Hölder's Inequality

We have the following additive reverse of Hölder's inequality:

Theorem 3.2. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If f, $g:E\to\mathbb{R}$ are such that fg, f^p, $g^q\in L$ and*

$$0<m_1\le f\le M_1<\infty,\ 0<m_2\le g\le M_2<\infty, \tag{3.30}$$

then

$$(0\le)1-\frac{A(fg)}{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}} \le \max\left\{f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right),f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)\right\}, \tag{3.31}$$

where $f_{\frac{1}{p}}$ is defined by

$$f_{\frac{1}{p}}(x)=\frac{1}{q}+\frac{1}{p}x-x^{\frac{1}{p}}. \tag{3.32}$$

Proof. Observe that, by (3.30), we have

$$m_1^p \le A(f^p) \le M_1^p \text{ and } m_2^q \le A(g^q) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{A(f^p)} \le \left(\frac{M_1}{m_1}\right)^p \text{ and } \left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{A(g^q)} \le \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1} \le \frac{\frac{f^p}{A(f^p)}}{\frac{g^q}{A(g^q)}} \le \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q.$$

Using the inequality (1.5) for $b = \frac{f^p}{A(f^p)}$, $a = \frac{g^q}{A(g^q)}$, $\nu = \frac{1}{p}$, $M = \left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q$ and $m = \left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}$, we have

$$\begin{aligned}(3.33)\qquad & 0 \le \frac{1}{q}\frac{g^q}{A(g^q)} + \frac{1}{p}\frac{f^p}{A(f^p)} - \frac{fg}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\le \max\left\{ f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\} \frac{g^q}{A(g^q)}.\end{aligned}$$

If we take the functional A in (3.33), then we get

$$\begin{aligned} & 0 \le \frac{1}{q}\frac{A(g^q)}{A(g^q)} + \frac{1}{p}\frac{A(f^p)}{A(f^p)} - \frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\le \max\left\{ f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\} \frac{A(g^q)}{A(g^q)},\end{aligned}$$

which is equivalent to the desired result (3.30). □

The following reverse of Cauchy-Bunyakovsky-Schwarz inequality for isotonic functionals holds:

Corollary 3.2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional, $f, g : E \to \mathbb{R}$ are such that fg, f^2, $g^2 \in L$ and the condition (3.30) is valid, then*

$$(3.34)\qquad (0 \le)\, 1 - \frac{A(fg)}{[A(f^2)]^{1/2}[A(g^2)]^{1/2}} \le \frac{(M_1M_2 - m_1m_2)^2}{2m_1^2m_2^2}.$$

Proof. For $p = 2$, we have $f_{\frac{1}{2}}(x) = \frac{1+x}{2} - \sqrt{x}$, $x \ge 0$. Then

$$f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^2 \left(\frac{M_2}{m_2}\right)^2\right) = \frac{(M_1M_2 - m_1m_2)^2}{2m_1^2m_2^2}$$

and

$$f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{-2} \left(\frac{M_2}{m_2}\right)^{-2}\right) = \frac{(M_1M_2 - m_1m_2)^2}{2M_1^2M_2^2}$$

and since

$$\max\left\{ f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^2 \left(\frac{M_2}{m_2}\right)^2\right), f_{\frac{1}{2}}\left(\left(\frac{M_1}{m_1}\right)^{-2} \left(\frac{M_2}{m_2}\right)^{-2}\right)\right\} = \frac{(M_1M_2 - m_1m_2)^2}{2m_1^2m_2^2},$$

then by (3.31) we get the desired result (3.34). □

Using the inequality (3.34) and (1.7), we get

$$(0\leq)\,1-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \leq T\max\left\{\left(1-\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right)^2,\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^2\right\}, \tag{3.35}$$

where $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$. Since

$$\max\left\{\left(1-\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right)^2,\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^2\right\} =\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^2,$$

then by (3.35) we have the inequality

$$(0\leq)\,1-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}\leq T\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}-1\right)^2, \tag{3.36}$$

where $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, $f,g:E\to\mathbb{R}$ are such that $fg,f^p,g^q\in L$ and they satisfy the condition (3.30). Using the inequality (3.34) and (1.10), we get

$$(0\leq)1-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \leq\max\left\{\left[S\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right)-1\right]\left(\frac{M_1}{m_1}\right)^{-1}\left(\frac{M_2}{m_2}\right)^{-\frac{q}{p}},\right. \left.\left[S\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)-1\right]\left(\frac{M_1}{m_1}\right)\left(\frac{M_2}{m_2}\right)^{\frac{q}{p}}\right\} \tag{3.37}$$

provided $f,g:E\to\mathbb{R}$ are such that $fg,f^p,g^q\in L$ and they satisfy the condition (3.30). Using the inequality (3.34) and (1.13), we get

$$(0\leq)1-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \leq\max\left\{\left[K^T\left(\left[\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right)-1\right]\left(\frac{M_1}{m_1}\right)^{-1}\left(\frac{M_2}{m_2}\right)^{-\frac{q}{p}},\right. \left.\left[K^T\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\right)-1\right]\left(\frac{M_1}{m_1}\right)\left(\frac{M_2}{m_2}\right)^{\frac{q}{p}}\right\}, \tag{3.38}$$

where $T=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, $f,g:E\to\mathbb{R}$ are such that $fg,f^p,g^q\in L$ and they satisfy the condition (3.30).

4. Applications for Integrals

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)| w(x)\, d\mu(x) < \infty\}.$$

For simplicity of notation, we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.

Let f, g be μ-measurable functions with the property that there exists the constants $M, m > 0$ such that

$$0 < m \leq \frac{f}{g} \leq M < \infty \ \mu\text{-almost everywhere (a.e.) on } \Omega. \tag{4.39}$$

If $f^2, g^2 \in L_w(\Omega, \mu)$, then by (2.17) we have

$$\begin{aligned}(0 \leq) &\int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu - \int_\Omega wf^{2(1-s)} g^{2s} d\mu \int_\Omega wf^{2s} g^{2(1-s)} d\mu \\ &\leq \max\left\{f_s\left(\left(\frac{m}{M}\right)^2\right), f_s\left(\left(\frac{M}{m}\right)^2\right)\right\} \int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu\end{aligned} \tag{4.40}$$

for any $s \in [0, 1]$, where f_s is defined by (1.2), and, in particular,

$$(0 \leq)\, 1 - \frac{\left(\int_\Omega wfg d\mu\right)^2}{\int_\Omega wf^2 d\mu \int_\Omega wg^2 d\mu} \leq \frac{1}{2}\left(\frac{M}{m} - 1\right)^2. \tag{4.41}$$

Let f, g be μ-measurable functions with the property that there exists the constants m_1, M_1, m_2, M_2 such that

$$0 < m_1 \leq f \leq M_1 < \infty,\ 0 < m_2 \leq g \leq M_2 < \infty \ \mu\text{-a.e. on } \Omega. \tag{4.42}$$

Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (3.31) we have the following reverse of Hölder's inequality

$$\begin{aligned}(0 \leq) 1 &- \frac{\int_\Omega wfg d\mu}{\left(\int_\Omega wf^p d\mu\right)^{1/p} \left(\int_\Omega wg^q d\mu\right)^{1/q}} \\ &\leq \max\left\{f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\},\end{aligned} \tag{4.43}$$

where $f_{\frac{1}{p}}$ is defined by (3.32).

In particular, we have the reverse of Cauchy-Bunyakovsky-Schwarz inequality

$$(0 \leq)\, 1 - \frac{\int_\Omega wfg d\mu}{\left(\int_\Omega wf^2 d\mu\right)^{1/2} \left(\int_\Omega wg^2 d\mu\right)^{1/2}} \leq \frac{(M_1 M_2 - m_1 m_2)^2}{2m_1^2 m_2^2}. \tag{4.44}$$

From (3.36), we have, for $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, that

$$(0 \leq)\, 1 - \frac{\int_\Omega wfg d\mu}{\left(\int_\Omega wf^p d\mu\right)^{1/p} \left(\int_\Omega wg^q d\mu\right)^{1/q}} \leq T\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}} \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}} - 1\right)^2. \tag{4.45}$$

5. Applications for Real Numbers

We consider the n-tuples of positive numbers $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and the probability distribution $p = (p_1, ..., p_n)$, i.e. $p_i \geq 0$ for any $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$.

If there exist the constants $m, M > 0$ such that

$$0 < m \leq \frac{a_i}{b_i} \leq M < \infty \text{ for any } i \in \{1, ..., n\},$$

then by (4.40), for the counting discrete measure, we have

$$\begin{aligned}(5.46)\qquad (0 \leq) \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 - \sum_{i=1}^{n} p_i a_i^{2(1-s)} b_i^{2s} \sum_{i=1}^{n} p_i a_i^{2s} b_i^{2(1-s)} \\ \leq \max\left\{ f_s\left(\left(\frac{m}{M}\right)^2\right), f_s\left(\left(\frac{M}{m}\right)^2\right)\right\} \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2\end{aligned}$$

for any $s \in [0, 1]$, where f_s is defined by (1.2). In particular,

$$(5.47)\qquad (0 \leq)\, 1 - \frac{\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2}{\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2} \leq \frac{1}{2}\left(\frac{M}{m} - 1\right)^2.$$

If there exists the constants m_1, M_1, m_2, M_2 such that

$$(5.48)\qquad 0 < m_1 \leq a_i \leq M_1 < \infty,\ 0 < m_2 \leq b_i \leq M_2 < \infty \text{ for any } i \in \{1, ..., n\}$$

and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (4.43) we have the following reverse of Hölder's inequality

$$\begin{aligned}(5.49)\qquad (0 \leq) 1 - \frac{\sum_{i=1}^{n} p_i a_i b_i}{\left(\sum_{i=1}^{n} p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i b_i^q\right)^{1/q}} \\ \leq \max\left\{ f_{\frac{1}{p}}\left(\left[\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right]^{-1}\right), f_{\frac{1}{p}}\left(\left(\frac{M_1}{m_1}\right)^p \left(\frac{M_2}{m_2}\right)^q\right)\right\},\end{aligned}$$

where $f_{\frac{1}{p}}$ is defined by (3.32). In particular, we have the reverse of Cauchy-Bunyakovsky-Schwarz inequality

$$(5.50)\qquad (0 \leq)\, 1 - \frac{\sum_{i=1}^{n} p_i a_i b_i}{\left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i b_i^2\right)^{1/2}} \leq \frac{\left(M_1 M_2 - m_1 m_2\right)^2}{2 m_1^2 m_2^2}.$$

From (4.45), we have for $T = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, that

$$(5.51)\qquad (0 \leq)\, 1 - \frac{\sum_{i=1}^{n} p_i a_i b_i}{\left(\sum_{i=1}^{n} p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i b_i^q\right)^{1/q}} \leq T\left(\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}} \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}} - 1\right)^2$$

provided a and b satisfy the condition (5.48).

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