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Some Applications Related to Differential Inclusions Based on the Use of a Weighted Space

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Abstract

In this paper, we present an existence theorem for the problem of discontinuous dynamical system related to ordinary differential inclusion, based on the use of the concepts related to weighted spaces introduced by Górka and Rybka, without using any fixed point theorem. The solution concept in this theorem is considered to belong to the weighted space. For comparison with the classical case and as an application of the theorem, we give an example problem that has such a solution but no continuously differentiable solution.

1. Introduction

In the mathematical modeling of systems with dynamic behavior in various fields of the real-world and in the qualitative and numerical analysis of these systems, differential equations with initial or boundary conditions and the existence and uniqueness of solutions and numerical approach techniques to solutions for these equations appear as important mathematical tools (see, e.g. [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]). (Ordinary) differential inclusions, which are generalized forms of ordinary differential equations and started to be studied after the advances in right-side discontinuous differential equations and solution methods for the problems related to these equations in the 1960s, have a similarly important role in applied mathematics since using directly in modeling and especially in the necessary and sufficient results of optimal control problems of discontinuous systems (see, e.g. [12], [13], [2], [14], [15], [16]). In the literature, differential inclusions specific to various fields such as engineering, biology, economics, and special types of differential inclusions that arise from the use of different notions are also encountered (see, e.g. [17], [7], [14], [16]). Fuzzy differential inclusions, measure differential inclusions, Volterra differential inclusions, and impulsive differential inclusions are a few of them.

Górka and Rybka in [18] obtained some results about the existence of a solution for ordinary differential equation with an initial condition based on the use of the weighted space equipped with the weighted norm. Here, they used Banach fixed point theorem under the boundedness assumption and the assumption of a special type of Lipschitz continuity (with $l(t)/t$ depending upon t).

In this paper, we present some results about the existence of a global solution for the discontinuous differential inclusion with an initial condition, based on the use of the weighted space, without boundedness assumption in nonconvex case. For this purpose, we construct a sequence, based on the uses of the weighted norm and approximations mentioned in [13], without using any fixed point theorem to derive the solution.

Since our results are true for the discontinuous ordinary differential equations as well, these results can be applied to the system described by the differential equation in [18] without boundedness assumption. In addition, an illustrative example satisfying the assumptions mentioned in the results is also given in this paper.

2. Preliminaries

For unexplained terminology and the basic results on the weighted spaces and differential inclusion theory we refer to [17], [19], [18], [15], [16]. For a fixed $b>0$, $C([0,b],\mathbb{R}^n)$ denotes Banach space of all continuous functions $g:[0,b]\to\mathbb{R}^n$ with the

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supremum norm $\|g\|_\infty = \sup_{t\in[0,b]} |g(t)|$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Let us fixed $\alpha \in]0,1]$ and $r \in \mathbb{R}^n$. Now, let $g \in C([0,b],\mathbb{R}^n)$ and we put

$$|g|_{r,\alpha} := \sup_{t\in]0,b]} \frac{|g(t)-r|}{t^\alpha}.$$

The collection of all functions $g \in C([0,b],\mathbb{R}^n)$ satisfying $|g|_{r,\alpha} < \infty$ is denoted by $W_{r,\alpha} = W_{r,\alpha}([0,b],\mathbb{R}^n)$. It is clear that $g(0) = r$ whenever $g \in W_{r,\alpha}$, and that $|\cdot|_{0,\alpha}$ is a norm when $r = 0$. Note that the function ρ defined as

$$\rho(g,h) = |g-h|_{0,\alpha} \text{ for } g,h \in W_{r,\alpha}.$$

is a metric on $W_{r,\alpha}$. Moreover, $(W_{r,\alpha},\rho)$ is a complete metric space (for details, see [18]).
Let S be a subset of $\mathbb{R}^n$ and $\Phi : S \to \mathbb{R}^n$ a set-valued map. For $a \in \mathbb{R}^n$, we denote the projection of a onto S by $\pi(a,S)$, that is, $\pi(a,S) = \{s \in S : |a-s| = d(x,S)\}$ where $d(a,S) = \inf\{|a-s| : s \in S\}$. If $\pi(a,S)$ is nonempty, then each element of it is called the closest point in S to a. It is known that $\pi(a,S)$ is nonempty and compact if S is closed (for details, see [20], [15]). A single-valued function $\phi : S \to \mathbb{R}^n$ is said to be a measurable selection from Φ if ϕ is measurable in the usual sense and $\phi(s) \in \Phi(s)$ for all s in S. Let E and Z be nonempty bounded subsets of $\mathbb{R}^n$. The ball of radius δ around E is defined as

$$O_\delta(E) = \left\{ r \in \mathbb{R}^n : d(E,r) = \inf_{x\in E} |x-r| < \delta \right\}.$$

The Hausdorff distance between E and Z is defined as

$$d_H(E,Z) = \inf\{\delta > 0 : O_\delta(E) \supseteq Z, O_\delta(Z) \supseteq E\}.$$

Note that the existence of corresponding finite $\delta > 0$ follows from the boundedness of sets E,Z.
Let $m \geq 1$. let $\mathscr{L}$ and $\mathscr{B}^m$ be the collection of Lebesgue measurable subsets of $[0,b]$ and Borel subsets of $\mathbb{R}^m$, respectively. The smallest $\sigma-$algebra of subsets of $[0,b]\times\mathbb{R}^m$ generated by Cartesian products of sets in $\mathscr{L}$ and $\mathscr{B}^m$ is denoted by $\mathscr{L}\times\mathscr{B}^m$. By L^1_m and $\|\cdot\|_1$, we denote the space of all Lebesgue integrable functions from $[0,b]$ into $\mathbb{R}^m$ and the norm on L^1_m as usual, respectively. Let $\Psi : [0,b]\times\mathbb{R}^m \to \mathbb{R}^n$ a set-valued map. Ψ is said to be $\mathscr{L}\times\mathscr{B}^m$-measurable if the set $\Psi^{-1}(V)$ lies in $\mathscr{L}\times\mathscr{B}^m$ for all open subset V of $\mathbb{R}^n$. We say that Ψ is w-integrably bounded (with η) if there exists a non-negative function $\eta \in L^1_1$ with $av(\eta,b) < \infty$ satisfying $\Psi(s,y) \subseteq \eta(s)B$ for all $(s,y) \in [0,b]\times\mathbb{R}^m$, where $av(\eta,b) := \sup_{t\in]0,b]} \frac{1}{t}\int_0^t \eta(s)\,ds$ and B is the closed unit ball of $\mathbb{R}^n$. We say that Ψ satisfies the w-Kamke-type Lipschitz condition (with ℓ) if there exists a non-negative function $\ell \in L^1_1$ satisfying $av(\ell,b) < 1$ and

$$d_H(\Psi(t,y),\Psi(t,x)) \leq \frac{\ell(t)}{t}|y-x|$$

for any (t,x) and (t,y) in $]0,b]\times\mathbb{R}^n$.
We now consider the following Cauchy problem related to a discontinuous differential inclusion,

$$\dot{x}(t) \in F(t,x(t)),\, x(0) = r \tag{2.1}$$

where $F : [0,b]\times\mathbb{R}^n \to \mathbb{R}^n$ is a given set-valued map.
We say that the absolutely continuous function $x \in W_{r,\alpha}([0,b],\mathbb{R}^n)$ satisfying the initial condition r and the differential inclusion in (2.1) a.e. on $[0,b]$ is a (global) solution of the problem.
Throughout this paper, "a.e. on $[0,b]$" is denoted by "a.e" briefly. AC denotes the space of absolutely continuous functions from $[0,b]$ to $\mathbb{R}^n$. For any $g(\cdot) \in AC$, the function ϑ_g define by $\vartheta_g(t) = d(\dot{g}(t),F(t,g(t)))$ a.e.

Proposition 2.1. *(see, [15]) Suppose that a sequence $\{\phi_n\}$in $L^1([0,b],\mathbb{R}^n)$ converges to a function $\phi \in L^1([0,b],\mathbb{R}^n)$ in $\|\cdot\|_1$. Then there exists a subsequence of $\{\phi_n\}$ that converges pointwise to ϕ a.e.*

3. Main results

Theorem 3.1. *Let F be the $\mathscr{L}\times\mathscr{B}^m$-measurable set-valued map with nonempty closed values satisfying w-Kamke-type Lipschitz condition (with ℓ). Then for any $g \in W_{r,1}\cap AC$ satisfying $\vartheta_g \in L^1_1$ and $av(\vartheta_g,b) < \infty$, there exists a solution of the problem (2.1) in $B_\delta(g)$. Here, δ is a positive number satisfying $\delta < \frac{av(\vartheta_g,b)}{1-av(\ell,b)}$ and $B_\delta(g)$ is the open ball of $(W_{r,1},\rho)$ with radius δ.*

Proof. The main idea in the proof of this theorem would be to construct a Cauchy sequence $\{g_n\}$ (approximations) in the complete $(W_{r,1},\rho)$. Here, it will be determined on the basis of choosing $\dot{g}_n(t)$ as the closest point in $F(t,g_{n-1}(t))$ to $\dot{g}_{n-1}(t)$, and the desired solution will be obtained with the limit of the sequence. With this goal, let $g_0 = g \in W_{r,1}$. By using Proposition 2.3.2

in [16] and Corollary 8.2.13 in [17] together, it can be easily observed that there exists a measurable selection $\phi_0 = \phi_0(g_0(\cdot))$ from $\pi(\dot{g}_0(\cdot), F(\cdot, g_0(\cdot)))$. Since the inequality $|\phi_0(t)| \leq |\dot{g}_0(t)| + \vartheta_{g_0}(t)$ holds a.e. and $\vartheta_{g_0} \in L_1^1$, we get $\phi_0 \in L_n^1$. Thus we can define an operator I_0 for $t \in [0,b]$ as

$$I_0(t) = g_0(0) + \int_0^t \phi_0(s)ds.$$

Now put $g_1 = I_0$. It is clear that $g_1 \in AC$. Then $\dot{g}_1 = \phi_0$ and $|\dot{g}_1 - \dot{g}_0| = \vartheta_{g_0}$ a.e. It follows from the relation

$$|g_1(t) - g_0(t)| \leq \int_0^t |\dot{g}_1(s) - \dot{g}_0(s)|\, ds = \int_0^t \vartheta_{g_0}(s)ds \tag{3.1}$$

that $(|g_1(t) - g_0(0)|/t) \leq av\left(\vartheta_{g_0}, b\right) + |g_0|_{r,1}$ a.e. As $av\left(\vartheta_{g_0}, b\right) < \infty$ then $g_1 \in W_{r,1}$. Moreover, by using the above inequalities, the basic properties of the Hausdorff distance notion and the Lipschitz condition, we have

$$\begin{aligned}
\vartheta_{g_1}(t) &\leq |\dot{g}_1(t) - \dot{g}_0(t)| + \vartheta_{g_0}(t) + d_H(F(t, g_0(t)), F(t, g_0(t))) \\
&\leq 2\vartheta_{g_0}(t) + (\ell(t)/t)\,|g_1(t) - g_0(t)| \\
&\leq 2\vartheta_{g_0}(t) + (\ell(t)/t) \int_0^t \vartheta_{g_0}(s)ds \\
&\leq 2\vartheta_{g_0}(t) + \ell(t)\, av\left(\vartheta_{g_0}, b\right) \text{ a.e.}
\end{aligned}$$

So it can easily be concluded that $\vartheta_{g_1} \in L_1^1$ and $av(\vartheta_{g_1}, b) < \infty$.
In this way, by defining $g_n := I_{n-1}$ and ϕ_n and using induction on $n = 1, 2, ...$, we get a sequence $\{g_n\}$ in $W_{r,1}$. Let us prove that the sequence $\{g_n\}$ is Cauchy in $W_{r,1}$. By definition of $\{g_n\}$ and $\{\phi_n\}$, for $n = 0, 2, ...$ we get

$$\dot{g}_{n+1} = \phi_n,\ |\dot{g}_{n+1} - \dot{g}_n| = \vartheta_{g_n} \text{ a.e.}$$

From the equality $d(\dot{g}_n(t), F(t, g_{n-1}(t))) = 0$ a.e. for $n = 1, 2, ...,$

$$\begin{aligned}
\vartheta_{g_n} &\leq d(\dot{g}_n(t), F(t, g_{n-1}(t))) + d_H(F(t, g_{n-1}(t)), F(t, g_n(t))) \\
&\leq (\ell(t)/t)\,|g_n(t) - g_{n-1}(t)| \text{ a.e.}
\end{aligned} \tag{3.2}$$

Taking integral from both sides, we have

$$|g_{n+1}(t) - g_n(t)| \leq \left(\sup_{s \in]0,b]} \frac{|g_n(s) - g_{n-1}(s)|}{s}\right) \int_0^t \ell(s)\, ds. \tag{3.3}$$

Therefore, it can be easily verified that

$$\rho(g_{n+1}, g_n) \leq av(\ell, b)\, \rho(u_n, u_{n-1}). \tag{3.4}$$

Note that the last inequality implies

$$\rho(g_{n+1}, g_n) \leq (av(\ell, b))^n \rho(g_1, g_0). \tag{3.5}$$

From here, we derive that

$$\begin{aligned}
|g_n|_{x_0,1} &\leq \rho(g_n, g_0) + |g_0|_{x_0,1} \\
&\leq \rho(g_n, g_{n-1}) + ... + \rho(g_1, g_0) + |g_0|_{x_0,1} < \infty.
\end{aligned}$$

Thus $g_n \in W_{r,1}$. In addition, the relations $\rho(g_1, g_0) \leq \vartheta_{g_0}$(as a result of (3.1)) and (3.5) implies that,

$$\begin{aligned}
\rho(g_n, g_0) &\leq \rho(g_n, g_{n-1}) + ... + \rho(g_1, g_0) \\
&\leq \vartheta_{g_0} \sum_{i=0}^{n-1} (av(\ell, b))^i.
\end{aligned} \tag{3.6}$$

As $av(\ell, b) < 1$, the relation (3.5) implies that the sequence $\{g_n\}$ is Cauchy, $W_{r,1}$ being complete, it converges uniformly to some function $y \in W_{r,1}$. Taking into account (3.2) and (3.3), we get

$$\|\phi_n - \phi_{n-1}\|_1 \leq \rho(g_n, g_{n-1}) \int_0^b \ell(s)\, ds,$$

so that $\{\phi_n\}$ is a Cauchy sequence in L_n^1. Let ϕ be the limit of $\{\phi_n\}$. One can easily have $y(t) = r + \int_0^t \phi(s)\, ds$. Moreover,

$$\begin{aligned}
\vartheta_{g_{n+1}} &\leq |\dot{g}_{n+1}(t) - \dot{y}(t)| + d(\dot{y}(t), F(t, y(t))) + d_H(F(t, g_{n+1}(t)), F(t, y(t))) \\
&\leq |\phi_n(t) - \dot{y}(t)| + \vartheta_y(t) + (\ell(t)/t)\,|g_{n+1}(t) - y(t)| \text{ a.e.}
\end{aligned}$$

Thus,

$$\left|\vartheta_{g_{n+1}}(t)-\vartheta_y(t)\right| \leq \phi(t)\rho(g_{n+1},y)+|\phi_n(t)-\dot{y}(t)| \text{ a.e.} \tag{3.7}$$

From Proposition 2.1, there exists a subsequence $\{\phi_{n_k}(t)\}$ converging to $\phi(t)$ a.e. Replacing ϕ_n with the ϕ_{n_k} in (3.7), we derive that $\vartheta_y(t)=\lim_{k\to\infty}\vartheta_{g_{n_k+1}}(t)$ a.e. By the inequality (3.2) one can get,

$$\vartheta_y(t) \leq \ell(t)\lim_{k\to\infty}\rho\left(g_{n_k+1},g_{n_k}\right) \text{ a.e.}$$

which implies $\vartheta_y(t)=0$ a.e. From here, we conclude that $y\in W_{r,1}$ is a solution. Moreover, by using (3.6), one can easily have $\rho(y,g)<\delta$. □

Remark that the following Corollary is a consequence of Theorem 3.1 for $g\equiv r$.

Corollary 3.2. *Let F be the $\mathscr{L}\times\mathscr{B}^m$-measurable set-valued map with nonempty closed values satisfying w-Kamke-type Lipschitz condition (with ℓ). If $\vartheta_r\in L_1^1$ and $av(\vartheta_r,b)<\infty$, then the problem (2.1) has at least one solution (in $W_{r,1}$).*

Corollary 3.3. *Let F be the $\mathscr{L}\times\mathscr{B}^m$-measurable set-valued map with nonempty closed values satisfying w-Kamke-type Lipschitz condition (with ℓ). Suppose further that F is w-integrably bounded (with η). Then the problem (2.1) has at least one solution (in $W_{r,1}$).*

Proof. Let the function $h^*\equiv(h_1,h_2,...h_n):[0,T]\to\mathbb{R}^n$ defined by $h_i(t)=r_i+\int_0^t\eta(s)ds$. We choose $g\equiv h^*$. By hypotheses we get $\dot{g}=(\eta,\eta,...\eta)$, $\vartheta_g(t)\leq(1+\sqrt{n})\,\eta(t)$ a.e. and $g\in W_{r,1}\cap AC$. Thus, $\vartheta_g\in L^1$ and $av(\vartheta_g,b)<\infty$. By Theorem 3.1, we have desired conclusion. □

Remark 3.4. *Let $h:[0,b]\times\mathbb{R}^n\to\mathbb{R}^n$ be single-valued function. Consider F as set-valued map with value $F(s,z)=\{h(s,z)\}$. Then the problem (2.1) turns into the following Cauchy problem related to a discontinuous differential equation:*

$$\dot{z}(s)=h(s,z(s)),\ z(0)=r. \tag{3.8}$$

It is known that the uniqueness and existence results for the problem (2.1) can be obtained from hypotheses of Theorem 2.6 and Theorem 3.1 in [18]. Note that hypotheses of Corollary 3.2 are similar to these hypotheses except for the boundedness hypothesis (that is, for every $c>0$ there exists a non-negative function $m_c\in L_1^1$ such that $|z|<c$ implies $|h(s,z)|\leq m_c(s)$ for a.e.). It follows from Corollary 3.2 that the following existence result still holds without boundedness assumption. The uniqueness result can be obtained easily with the same proof in [18].

Corollary 3.5. *Let $h:[0,b]\times\mathbb{R}^n\to\mathbb{R}^n$ be the function satisfying the following:*

(a) h is $\mathscr{L}\times\mathscr{B}^m$-measurable,
(b) there exists a non-negative function $\ell\in L_1^1$ with $av(\ell,b)<1$ satisfying

$$|h(s,y)-h(t,z)|\leq(\ell(s)/s)\,|y-z|$$

for any (s,y) and (s,z) in $]0,b]\times\mathbb{R}^n$,
(c) $|h(\cdot,r)|\in L_1^1$ and $av(|h(\cdot,r)|,b)<\infty$.

Then the problem (3.8) has a unique solution (in $W_{r,1}$).

Example 3.6. *Let $r>0$, $b\in\left]\frac{r}{2},2r\right[$ and consider the following problem:*

$$h(s,z) = \begin{cases} \frac{2}{2s+r}z & s>\frac{r}{2} \\ 0 & 0\leq s\leq\frac{r}{2} \end{cases},\ s\in[0,b]$$

$$\dot{z}(s) = h(s,z),\ z(0)=r.$$

The problem has no a continuously differentiable solution. It can be easily verified that h is $\mathscr{L}\times\mathscr{B}^m$-measurable, and that h satisfies the Lipschitz condition with l (defined by $l(s)=\frac{s}{r}$) given in Corollary 3.5. Moreover, $|h(\cdot,r)|$ is Riemann integrable on $[0,b]$ and $av(|h(\cdot,r)|,b)<\infty$. As the hypotheses of Corollary 3.2 are satisfied, the problem has a unique solution (in $W_{r,1}$). Note that the solution is the function $z:[0,b]\to\mathbb{R}$ defined by $z(s)=s+\frac{r}{2}$ if $\frac{r}{2}<s\leq b$ and $z(s)=r$ if $0\leq s\leq\frac{r}{2}$.

4. Conclusion

In this paper, an existence result for the discontinuous differential inclusion with an initial condition, where the solution lies in the weighted space, is given in Theorem 3.1. Here, unlike the classical existence results, the concepts related to the weighted space and the topology of this space are used in the nonconvex and unbounded case. As a consequence of the theorem, the existence result of the differential equations in [18] is generalized to differential inclusions without boundedness assumption. In addition, in the proof of the theorem, the approximations mentioned in [13] is used to be members of the weighted space. Considering recent studies using similar approximations in various fields related to differential inclusion theory (see, e.g. [21], [22], [23]), this paper will contribute to the theory by providing the generalized results based on the use of the concepts and the approximations related to the weighted space.

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A New Generalization of Szász-Mirakjan Kantorovich Operators for Better Error Estimation

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Abstract

In this article, we construct a new sequence of Szász-Mirakjan-Kantorovich operators denoted as $K_{n,\gamma}(f;x)$, which depending on a parameter γ. We prove direct and local approximation properties of $K_{n,\gamma}(f;x)$. We obtain that, if $\gamma > 1$, then the operators $K_{n,\gamma}(f;x)$ provide better approximation results than classical case for all $x \in [0,\infty)$. Furthermore, we investigate the approximation results of $K_{n,\gamma}(f;x)$, graphically and numerically. Moreover, we introduce new operators from $K_{n,\gamma}(f;x)$ that preserve affine functions and bivariate case of $K_{n,\gamma}(f;x)$. Then, we study their approximation properties and also illustrate the convergence of these operators comparing with their classical cases.

1. Introduction

The Weierstrass approximation theorem is a fundamental result in mathematical analysis which states that any continuous function on a closed interval can be uniformly approximated by a polynomial function (see [1]). Bernstein provides a simple and constructive proof to the Weierstrass Approximation Theorem for $C[0,1]$, where $C[0,1]$ is the set of all continuous functions (see [2], [3]). Because of the importance of the Bernstein Operators, many researchers lead to the discovery of their numerous generalizations such as [4], [5], [6], [7], [8], [9], [10]. For a function f belonging to the space $C[0,\infty)$, the Szász-Mirakjan operators are introduced by

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where,

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \tag{1.2}$$

for any $x \in [0,\infty)$, in [11] and [12]. However, this kind of operators do not suitable for discontinuous functions. P. L. Butzer introduced the Kantorovich type Szász-Mirakjan operators for Lebesque-integrable function space, in [13] as:

$$S_n(f;x) = n\sum_{k=0}^{\infty} s_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \tag{1.3}$$

where $s_{n,k}(x)$ is defined in (1.2). Szász-Mirakjan operators, Kantorovich type Szász-Mirakjan operators and some of their generalizations have been the subject of extensive research by various scholars as documented [14], [15], [16], [17], [18], [19], [20]. For further developments in this area, interested readers are encouraged to explore the insights provided in [21], [22],

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[23], [24], [25], [26].
In this article, we introduce a new family of Kantorovich type Szász-Mirakjan operators $K_{n,\gamma}$ as:

$$K_{n,\gamma}(f;x) := \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt, \tag{1.4}$$

where $s_{n,k}(x)$ is given in (1.2) and $\gamma \in \mathbb{R}^+$. Note that, $K_{n,\gamma}$ are positive and linear. One can easily obtain that, in (1.4) the classical Százs-Mirakjan Kantorovich operators can be produced, by choosing $\gamma = 1$. We observe that, the error estimation of $K_{n,\gamma}$ decreases by increasing the value γ. Therefore, in cases where we choose the γ value greater than 1, it can be seen that the error estimation is less than the classical case. Therefore, this modification gives better approximation results than classical one, when $\gamma > 1$. It should be stressed out that this kind of Kantorovich type operators for the Bernstein polynomials was defined and studied in [27] by Özarslan, and Duman.
After giving geometric properties and significant results of $K_{n,\gamma}$ in Section 2, direct and local approximation properties, theoretical proofs, and numerical examples for better error estimations are given for these operators in Section 3. Then, applying slight modification to the operators $K_{n,\gamma}$, a new family of these operators is introduced, that preserves affine functions. In Section 5, bivariate cases of these operators are introduced and studied.

2. Some basic results

In this part, we provide some geometric properties of $K_{n,\gamma}$ and significant results of $K_{n,\gamma}$ which will be used in the next sections.

Theorem 2.1. *Let $0 \le \gamma < \infty$ and $n \in \mathbb{N}$, then,*

1. *If the function f is increasing (or decreasing) on $[0,\infty)$, then $K_{n,\gamma}(f;x)$ is also increasing (or decreasing) on $[0,\infty)$.*
2. *If the function f is convex (or concave) function on $[0,\infty)$, then $K_{n,\gamma}(f;x)$ is also convex (or concave) on $[0,\infty)$.*

Proof. 1. Taking the first derivative of $K_{n,\gamma}(f;x)$ we get,

$$\begin{aligned}
\left(K_{n,\gamma}\right)'(f;x) &= \sum_{k=0}^{\infty} s'_{n,k}(x) \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt \\
&= \sum_{k=0}^{\infty} \left[-ne^{-nx}\frac{(nx)^k}{k!} + e^{-nx}\frac{nk(nx)^{k-1}}{k!}\right] \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt \\
&= \sum_{k=1}^{\infty} \left[ne^{-nx}\frac{k(nx)^{k-1}}{k!}\right] \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt - n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt \\
&= n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} \int_0^1 f\left(\frac{k+1+t^\gamma}{n}\right) dt - n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt \\
&= n\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} \left[\int_0^1 f\left(\frac{k+1+t^\gamma}{n}\right) dt - \int_0^1 f\left(\frac{k+t^\gamma}{n}\right) dt\right] \\
&= n\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \Delta_h^1 f\left(\frac{k+t^\gamma}{n}\right) dt,
\end{aligned} \tag{2.1}$$

where $h = \frac{1}{n}$ and $n = 1,2,....$
For an increasing function f on $[0,\infty)$, we have

$$\Delta_h^1 f\left(\frac{k+t^\gamma}{n}\right) = f\left(\frac{k+1+t^\gamma}{n}\right) - f\left(\frac{k+t^\gamma}{n}\right) \ge 0, \tag{2.2}$$

where $k = 0,1,...$ and $t \in [0,1]$. Therefore, combining (2.1) and (2.2), we obtain

$$\left(K_{n,\gamma}\right)'(f;x) \ge 0 \text{ for each } x \in [0,\infty).$$

In other words, $K_{n,\gamma}(f;x)$ is increasing on $[0,\infty)$.

2. Similarly, the second derivative of $K_{n,\gamma}(f;x)$ is

$$\left(K_{n,\gamma}\right)''(f;x) = n^2 \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \Delta_h^2 f\left(\frac{k+t^\gamma}{n}\right) dt \tag{2.3}$$

where $h = \frac{1}{n}$ for $n = 1,2,3,....$ Let f is convex on $[0,\infty)$, then for any $k = 0,1,...$ we have

$$0 \le \frac{k+t^\gamma}{n} \le \frac{k+t^\gamma+1}{n} \le \frac{k+t^\gamma+2}{n} \le 1.$$

Theorem 3.2.2 in [[28], p.59] implies that

$$\Delta_h^2 f\left(\frac{k+t^\gamma}{n}\right) \geq 0. \tag{2.4}$$

Therefore, combining (2.3) and (2.4) we get,

$$\left(K_{n,\gamma}\right)''(f;x) \geq 0,$$

$\forall x \in [0,\infty)$. As a conclusion, $K_{n,\gamma}(f;x)$ is convex on $[0,\infty)$. □

Lemma 2.2. *Recall the first 3 moments of (1.1)*

1. $S_n(1;x) = 1$
2. $S_n(t;x) = x$
3. $S_n(t^2;x) = x^2 + \frac{x}{n}$.

Lemma 2.3. *Let $\gamma \in (0,\infty)$ and $x \in [0,\infty)$, then*

1. $K_{n,\gamma}(1;x) = 1$
2. $K_{n,\gamma}(t;x) = x + \frac{1}{(\gamma+1)n}$
3. $K_{n,\gamma}(t^2;x) = x^2 + \frac{(\gamma+3)x}{(\gamma+1)n} + \frac{1}{(2\gamma+1)n^2}$.

Proof. For each $\gamma \in (0,\infty)$ and $x \in [0,\infty)$, using Lemma 2.2, we get

1.

$$K_{n,\gamma}(1;x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 dt = 1.$$

2.

$$\begin{aligned}
K_{n,\gamma}(t;x) &= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \frac{k+t^\gamma}{n} dt \\
&= \sum_{k=0}^{\infty} \frac{k}{n} s_{n,k}(x) \int_0^1 dt + \frac{1}{n} \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 t^\gamma dt \\
&= \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k}{n} + \frac{1}{\gamma+1} \frac{1}{n} \sum_{k=0}^{\infty} s_{n,k}(x) \\
&= x + \frac{1}{(\gamma+1)n}.
\end{aligned}$$

3.

$$\begin{aligned}
K_{n,\gamma}(t^2;x) &= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \left(\frac{k+t^\gamma}{n}\right)^2 dt \\
&= \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \left(\frac{k^2+2kt^\gamma+t^{2\gamma}}{n^2}\right) dt \\
&= \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k^2}{n^2} \int_0^1 dt + \frac{2}{n} \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k}{n} \int_0^1 t^\gamma dt + \frac{1}{n^2} \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 t^{2\gamma} dt \\
&= \sum_{k=0}^{\infty} s_{n,k}(x) \frac{k^2}{n^2} + \frac{2}{(\gamma+1)n} \sum_{k=0}^{\infty} \frac{k}{n} s_{n,k}(x) + \frac{1}{(2\gamma+1)n^2} \sum_{k=0}^{\infty} s_{n,k}(x) \\
&= x^2 + \frac{(\gamma+3)}{(\gamma+1)n} x + \frac{1}{(2\gamma+1)n^2}.
\end{aligned}$$

□

In the following lemma, we establish the connection between the moments of the operators $K_{n,\gamma}$ and S_n. Consequently, we can compute higher-order moments of $K_{n,\gamma}$ by utilizing classical Szász-Mirakjan operators S_n.

Lemma 2.4. *Consider $n \in \mathbb{N}$, $x \in [0,\infty)$, and $\gamma \in (0,\infty)$, we get*

$$K_{n,\gamma}(t^m;x) = \frac{1}{n^m}\sum_{i=0}^{m}\binom{m}{i}\frac{n^i}{\gamma(m-i)+1}S_n(t^i;x) \tag{2.5}$$

where S_n is defined in (1.1).

Proof. From (1.4), we get

$$\begin{aligned}
K_{n,\gamma}(t^m;x) &= \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left(\frac{k+t^\gamma}{n}\right)^m dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 (k+t^\gamma)^m dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 \sum_{i=0}^{m}\binom{m}{i}k^i t^{\gamma(m-i)}dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\sum_{i=0}^{m}\binom{m}{i}k^i\int_0^1 t^{\gamma(m-i)}dt \\
&= \frac{1}{n^m}\sum_{k=0}^{\infty} s_{n,k}(x)\sum_{i=0}^{m}\binom{m}{i}k^i\frac{1}{\gamma(m-i)+1} \\
&= \frac{1}{n^m}\sum_{i=0}^{m}\binom{m}{i}\frac{n^i}{\gamma(m-i)+1}\sum_{k=0}^{\infty} s_{n,k}(x)\frac{k^i}{n^i} \\
&= \frac{1}{n^m}\sum_{i=0}^{m}\binom{m}{i}\frac{n^i}{\gamma(m-i)+1}S_n(t^i;x).
\end{aligned}$$

□

Corollary 2.5. *We obtain,*

1. $K_{n,\gamma}(t-x;x) = \dfrac{1}{(\gamma+1)n}$
2. $K_{n,\gamma}((t-x)^2;x) = \frac{x}{n} + \dfrac{1}{(2\gamma+1)n^2}$.

3. Direct and local approximation properties of $K_{n,\gamma}$

We now turn our focus to direct and local approximation properties of $K_{n,\gamma}$. To begin, let's remember that $C_B[0,\infty)$ signifies the set of all real-valued functions f on $[0,\infty)$ that are both uniformly bounded and continuous. We measure the norm of such functions using $\|.\|$ defined as:

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

Theorem 3.1. *For any $A \in \mathbb{R}^+$, let $f \in C_B[0,A]$, and $\gamma \in (0,\infty)$, then $K_{n,\gamma}(f;x)$ is uniformly convergent to $f(x)$ on $[0,A]$.*

Proof. According to the Bohman-Korovkin Theorem (see [30]), it suffices to establish that

$$\lim_{n\to\infty}\sup_{x\in[0,A]}\left|K_{n,\gamma}(t^i;x) - t^i\right| = 0, \tag{3.1}$$

for $i = 0,1,2$. As a result of Lemma 2.3, one can easily see that (3.1) is hold for $i = 0,1,2$. So, the proof is done. □

For each $A \in \mathbb{R}^+$, the operators $K_{n,\gamma}$ on $C_B[0,A]$ satisfies $K_{n,\gamma}(1;x) = 1$. Therefore, for all $\varepsilon > 0$ we get

$$K_{n,\gamma}(f;x) \le \varepsilon + \frac{2\|f\|}{\delta^2}K_{n,\gamma}((t-x)^2;x)$$

where $x \in [0,A]$. Here δ comes from the uniform continuity of f. Therefore, the order of approximation of $K_{n,\gamma}(f;x)$ to f is much better controlled by the term $K_{n,\gamma}((t-x)^2;x)$.
Let $A \in (0,\infty)$ and $f \in C_B[0,A]$. If we consider $\gamma > 0$ and $x \in [0,A]$ such that

$$K_{n,\gamma}((t-x)^2;x) \leq K_n((t-x)^2;x). \tag{3.2}$$

We can compare how well the operators $K_{n,\gamma}(f;x)$ and $K_n(f;x)$ approximate the function f. From Lemma 2.3 and equation (3.2),

$$\begin{aligned} K_{n,\gamma}((t-x)^2;x) &\leq K_n((t-x)^2;x) \\ \frac{x}{n}+\frac{1}{(2\gamma+1)n^2} &\leq \frac{x}{n}+\frac{1}{3n^2} \\ \frac{1}{(2\gamma+1)n^2} &\leq \frac{1}{3n^2} \\ 1 &\leq \gamma. \end{aligned}$$

Hence, for every $\gamma > 1$, the accuracy of the approximation $K_{n,\gamma}(f;x)$ to $f(x)$ outperforms that of the classical Szász-Mirakjan Kantorovich operators for any $f \in C_B[0,A]$ and $x \in [0,A]$. Additionally, the approximation error of $K_{n,\gamma}(f;x)$ to $f(x)$ diminishes with increasing γ.
Now, we give some graphical and numerical results to illustrate that we have better error estimation by increasing the value γ.

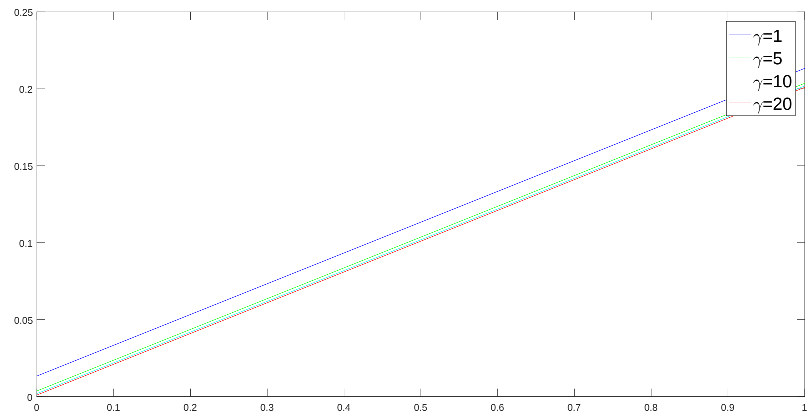

Figure 1: Error of approximation $K_{n,\gamma}((t-x)^2;x)$ for, $\gamma=1$, $\gamma=5$, $\gamma=10$, $\gamma=20$, when $n=5$ and $x \in [0,1]$.

x	$K_{5,1}((t-x)^2;x)$	$K_{5,5}((t-x)^2;x)$	$K_{5,10}((t-x)^2;x)$	$K_{5,20}((t-x)^2;x)$
0.00	0.0133	0.0036	0.0019	0.0010
0.10	0.0333	0.0236	0.0219	0.0210
0.20	0.0533	0.0436	0.0419	0.0410
0.30	0.0733	0.0636	0.0619	0.0610
0.40	0.0733	0.0836	0.0819	0.0810
0.50	0.1133	0.1036	0.1019	0.1010
0.60	0.1333	0.1236	0.1219	0.1210
0.70	0.1533	0.1436	0.1419	0.1410
0.80	0.1733	0.1636	0.1619	0.1610
0.90	0.1933	0.1836	0.1819	0.1810
1.00	0.2133	0.2036	0.2019	0.2010

Table 1: Table captions the different values of x.

Now, let's delve into the local approximation properties of $K_{n,\gamma}$. Recall that, in the case of $f \in C_B[0,\infty)$, the modulus of continuity (see [29]) is

$$w(f;\delta) = sup_{0<h\leq\delta} sup_{x\in[0,\infty)} |f(x+h)-f(x)|.$$

Theorem 3.2. *Let $f \in C_B[0,\infty)$, and $\gamma \in (0,\infty)$, we obtain,*

$$\left|K_{n,\gamma}(f;x)-f(x)\right| \leq 2\omega\left(f;\sqrt{\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}}\right)$$

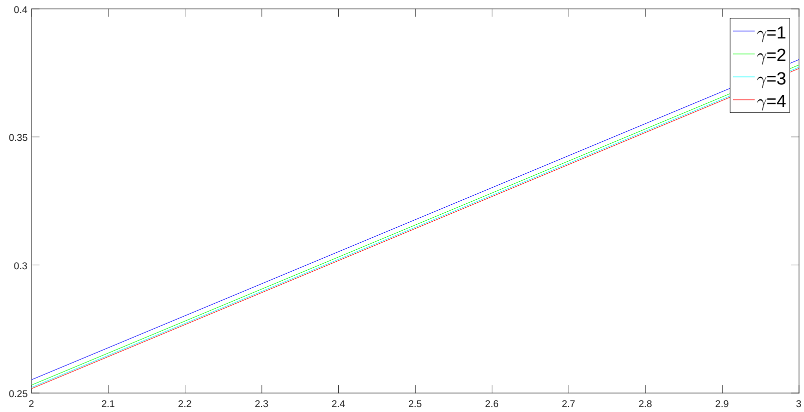

Figure 2: Error of approximation $K_{n,\gamma}((t-x)^2;x)$ for, $\gamma=1$, $\gamma=2$, $\gamma=3$, $\gamma=4$, when $n=8$ and $x\in[2,3]$.

x	$K_{8,1}((t-x)^2;x)$	$K_{8,2}((t-x)^2;x)$	$K_{8,3}((t-x)^2;x)$	$K_{8,4}((t-x)^2;x)$
2.00	0.2552	0.2531	0.2522	0.2517
2.10	0.2677	0.2656	0.2647	0.2642
2.20	0.2802	0.2781	0.2772	0.2767
2.30	0.2927	0.2906	0.2897	0.2892
2.40	0.3052	0.3031	0.3022	0.3017
2.50	0.3177	0.3156	0.3147	0.3142
2.60	0.3302	0.3281	0.3272	0.3267
2.70	0.3427	0.3406	0.3397	0.3392
2.80	0.3552	0.3531	0.3522	0.3517
2.90	0.3677	0.3656	0.3647	0.3642
3.00	0.3802	0.3781	0.3772	0.3767

Table 2: Table captions the different values of x.

for all $x\in[0,\infty)$.

Proof. According to the positivity of $K_{n,\gamma}$ and the equality $K_{n,\gamma}(1;x)=1$, we have,

$$\left|K_{n,\gamma}(f;x)-f(x)\right|\leq\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^1\left|f\left(\frac{k+t^{\gamma}}{n}\right)-f(x)\right|dt. \tag{3.3}$$

Applying the property of the modulus of continuity, which is

$$|f(\zeta)-f(\lambda)|\leq\left(1+\frac{|\zeta-\lambda|}{\delta}\right)\omega(f;\delta)$$

to (3.3). We obtain,

$$\left|K_{n,\gamma}(f;x)-f(x)\right|\leq\omega(f;\delta)\left(1+\frac{1}{\delta}\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^1\left|\frac{k+t^{\gamma}}{n}-x\right|dt\right).$$

Applying Cauchy-Schwarz inequality,

$$\begin{aligned}\left|K_{n,\gamma}(f;x)-f(x)\right| &\leq \omega(f;\delta)\left(1+\frac{1}{\delta}\sqrt{\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^1\left(\frac{k+t^{\gamma}}{n}-x\right)^2dt}\right)\\ &= \omega(f;\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}}\right).\end{aligned}$$

Choosing $\delta=\sqrt{\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}}$, we have the desired result. □

Lemma 3.3. *Let $\gamma \in (0,\infty)$, then for each $f \in C_B[0,\infty)$, we get*

$$\|K_{n,\gamma}(f;\cdot)\| \leq \|f\| \tag{3.4}$$

where $\|.\|$ denotes the uniform norm of $C_B[0,\infty)$.

The Peetre-K functional is

$$K_2(f;\delta) := inf_{\tau\in\varpi^2[0,\infty)}\left\{\|f-\tau\|+\delta\|\tau''\|\right\},(\delta>0)$$

where $\varpi^2[0,\infty)=\{\tau\in C_B[0,\infty):\tau',\tau''\in C_B[0,\infty)\}$.Furthermore, $\exists\, C>0$ (See [31]) such that

$$K_2(f;\delta)\leq C\omega_2(f;\sqrt{\delta})$$

where $\omega_2(f;\sqrt{\delta})$ is the modulus of smoothness for $f\in C_B[0,\infty)$ defined as

$$\omega_2(f;\sqrt{\delta}) = sup_{0<h\leq\delta}sup_{x\in[0,\infty)}\left|f(x+2h)-2f(x+h)+f(x)\right|.$$

Theorem 3.4. *Assume that $n\in\mathbb{N}$, $\gamma\in(0,\infty)$ and $f\in C_B[0,\infty)$. Then, $\exists C\in\mathbb{R}^+$ such that*

$$|K_{n,\gamma}(f;x)-f(x)|\leq C\omega_2\left(f;\frac{1}{2}\sqrt{\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}+\left(\frac{1}{(\gamma+1)n}\right)^2}\right)+\omega\left(f;\frac{1}{(\gamma+1)n}\right)$$

$\forall x\in[0,\infty)$.

Proof. Let

$$K^*_{n,\gamma}(f;x) := K_{n,\gamma}(f;x)+f(x)-f\left(x+\frac{1}{(\gamma+1)n}\right). \tag{3.5}$$

From Lemma 2.3, we obtain

$$K^*_{n,\gamma}(1;x)=1,$$

and

$$K^*_{n,\gamma}(t-x;x)=0.$$

Now, assume that $\tau\in\varpi^2[0,\infty)$. By the Taylor's expansion,

$$\tau(t)=\tau(x)+(t-x)\tau'(x)+\int_x^t(t-u)\tau''(u)du. \tag{3.6}$$

Applying the operators $K^*_{n,\gamma}$ for both sides, we get

$$\begin{aligned} K^*_{n,\gamma}(\tau;x) &= \tau(x)+K^*_{n,\gamma}\left(\int_x^t(t-u)\tau''(u)du;x\right) \\ &= \tau(x)+K_{n,\gamma}\left(\int_x^t(t-u)\tau''(u)du;x\right)-\int_x^{x+\frac{1}{(\gamma+1)n}}\left(x+\frac{1}{(\gamma+1)n}-u\right)\tau''(u)du. \end{aligned}$$

Hence;

$$K^*_{n,\gamma}(\tau;x)-\tau(x)=K_{n,\gamma}\left(\int_x^t(t-u)\tau''(u)du;x\right)-\int_x^{x+\frac{1}{(\gamma+1)n}}\left(x+\frac{1}{(\gamma+1)n}-u\right)\tau''(u)du.$$

By using above equation, we get

$$\begin{aligned} |K^*_{n,\gamma}(\tau;x)-\tau(x)| &\leq \left|K_{n,\gamma}\left(\int_x^t(t-u)\tau''(u)du;x\right)\right|+\left|\int_x^{x+\frac{1}{(\gamma+1)n}}(x+\frac{1}{(\gamma+1)n}-u)\tau''(u)du\right| \\ &\leq K_{n,\gamma}\left(\left|\int_x^t(t-u)\tau''(u)du\right|;x\right)+\int_x^{x+\frac{1}{(\gamma+1)n}}\left|x+\frac{1}{(\gamma+1)n}-u\right|\left|\tau''(u)\right|du \\ &\leq K_{n,\gamma}\left(\left|\int_x^t|(t-u)|du\right|;x\right)\|\tau''\|+\int_x^{x+\frac{1}{(\gamma+1)n}}\left|x+\frac{1}{(\gamma+1)n}-u\right|du\|\tau''\| \\ &\leq K_{n,\gamma}((t-x)^2;x)\|\tau''\|+\left(x+\frac{1}{(\gamma+1)n}-x\right)^2\|\tau''\| \\ &= \left[\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}+\left(\frac{1}{(\gamma+1)n}\right)^2\right]\|\tau''\|. \end{aligned}$$

So, we have

$$\left|K^*_{n,\gamma}(\tau;x)-\tau(x)\right| \le \left[\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}+\left(\frac{1}{(\gamma+1)n}\right)^2\right]\|\tau''\|. \tag{3.7}$$

Also, from Lemma 3.3 and the equation (3.5), we get

$$\left|K^*_{n,\gamma}(f;\cdot)\right| \le 3\|f\| \tag{3.8}$$

for all $f \in C_B[0,\infty)$ and $x \in [0,\infty)$.

For $f \in C_B[0,\infty)$ and $\tau \in \varpi^2[0,\infty)$, using (3.7) and (3.8), we observe that

$$\begin{aligned}
\left|K_{n,\gamma}(f;x)-f(x)\right| &= \left|K^*_{n,\gamma}(f;x)-f(x)+f\left(x+\frac{1}{(\gamma+1)n}\right)-f(x)\right| \\
&= \left|K^*_{n,\gamma}(f;x)-K^*_{n,\gamma}(\tau;x)+K^*_{n,\gamma}(\tau;x)-\tau(x)+\tau(x)-f(x)+f\left(x+\frac{1}{(\gamma+1)n}\right)-f(x)\right| \\
&\le \left|K^*_{n,\gamma}(f;x)-K^*_{n,\gamma}(\tau;x)\right|+\left|K^*_{n,\gamma}(\tau;x)-\tau(x)\right|+|\tau(x)-f(x)|+\left|f\left(x+\frac{1}{(\gamma+1)n}\right)-f(x)\right| \\
&\le 4\|f-\tau\|+\left[\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}+\left(\frac{1}{(\gamma+1)n}\right)^2\right]\|\tau''\|+\omega\left(f;\frac{1}{(\gamma+1)n}\right).
\end{aligned}$$

Hence, by taking the infimum on the right-hand side over all $\tau \in \varpi^2[0,\infty)$, we obtain:

$$\begin{aligned}
\left|K_{n,\gamma}(f;x)-f(x)\right| &\le 4K_2\left(f;\frac{\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}+\left(\frac{1}{(\gamma+1)n}\right)^2}{4}\right)+\omega\left(f;\frac{1}{(\gamma+1)n}\right) \\
&= C\omega_2\left(f;\frac{1}{2}\sqrt{\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}+\left(\frac{1}{(\gamma+1)n}\right)^2}\right)+\omega\left(f;\frac{1}{(\gamma+1)n}\right).
\end{aligned}$$

So, the proof is completed. □

Recall that the usual Lipschitz class for $0<a\le 1$ and $M>0$ is

$$Lip_M(a) := \{f \in C_B[0,\infty) : |f(\rho)-f(\sigma)| \le M|\rho-\sigma|^a\}$$

$\forall \rho,\sigma \in [0,\infty)$.

Theorem 3.5. *For every $f \in Lip_M(a)$, we have*

$$\left|K_{n,\gamma}(f;x)-f(x)\right| \le M\left[\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}\right]^{\frac{a}{2}}.$$

Proof. Assume that $f \in Lip_M(a)$, then,

$$\begin{aligned}
\left|K_{n,\gamma}(f;x)-f(x)\right| &\le \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left|f\left(\frac{k+t^\gamma}{n}\right)-f(x)\right|dt \\
&\le M\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left|\frac{k+t^\gamma}{n}-x\right|^a dt.
\end{aligned}$$

Utilizing Hölder's inequality, we obtain

$$\begin{aligned}
\left|K_{n,\gamma}(f;x)-f(x)\right| &\le M\left[\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1\left(\frac{k+t^\gamma}{n}-x\right)^2 dt\right]^{\frac{a}{2}}\left[\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 dt\right]^{\frac{2-a}{2}} \\
&= M\left[\frac{x}{n}+\frac{1}{(2\gamma+1)n^2}\right]^{\frac{a}{2}}.
\end{aligned}$$

Therefore, the proof is completed. □

Now, we present graphical and numerical results to illustrate the convergence of $K_{n,\gamma}(f;x)$ to certain functions $f(x)$. Additionally, we compare the newly defined operators $K_{n,\gamma}(f;x)$ with the classical Szász-Mirakjan Kantorovich operators $K_n(f;x)$ for different values of γ and n. As anticipated, the results of these comparisons consistently demonstrate that, for any γ chosen to be greater than 1, the approximation of $K_{n,\gamma}(f;x)$ to $f(x)$ surpasses that of $K_n(f;x)$. Moreover, as the value of γ increases, the convergence of the operators $K_{n,\gamma}(f;x)$ to the functions $f(x)$ improves.

In the following Figure 3, we compare the approximation of the operators $K_{20,1}(f;x)$, $K_{20,4}(f;x)$, $K_{20,16}(f;x)$ to

$$f(x) = \begin{cases} 1+x(x-1)(x-2), & 0\leq x\leq 2 \\ 1, & otherwise. \end{cases}$$

Here, $K_{20,1}(f;x)$ is the classical Százs-Mirakjan Kantorovich operators. Then, the graphics show that choosing $\gamma>1$ we get better approximation results to the function. Furthermore Figure 4 gives that the graphics of the error of approximation and Table 3 shows the numerical results of the error of approximation of these operators.

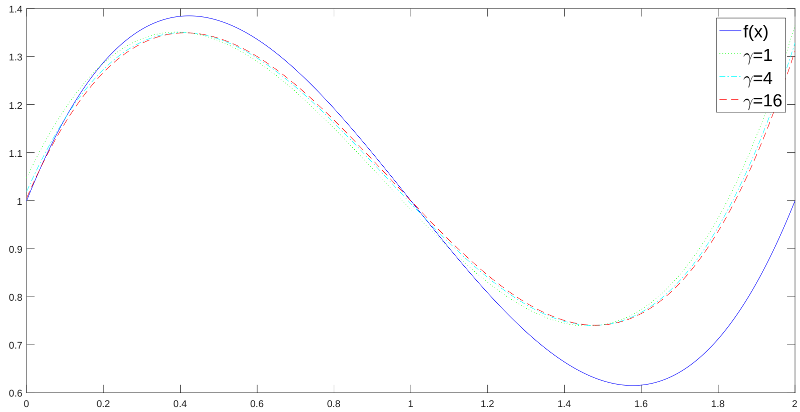

Figure 3: Approximation of $K_{n,\gamma}(f;x)$ to $f(x)$ for $\gamma=1$, $\gamma=4$ and $\gamma=16$ when $n=20$.

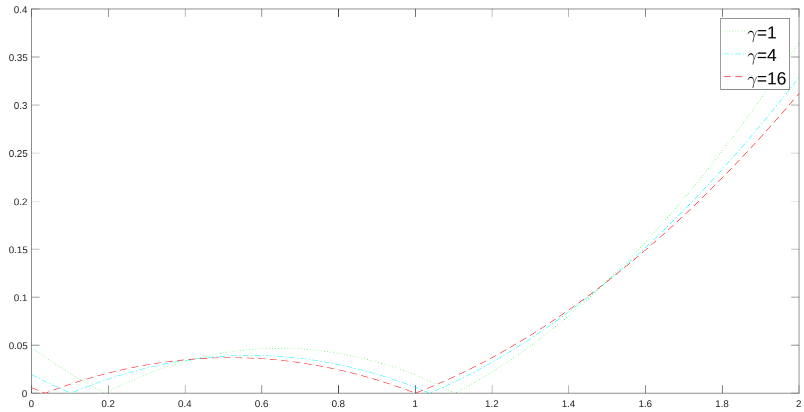

Figure 4: Error of approximation $\varepsilon_{n,\gamma}(f(x))=\left|K_{n,\gamma}(f;x)-f(x)\right|$ for $\gamma=1$, $\gamma=4$ and $\gamma=16$ when $n=20$.

Now, in Figure 5, we compare the approximation of the operators $K_{50,3}(f;x)$, $K_{100,3}(f;x)$, $K_{150,3}(f;x)$ to the function $f(x)$ where

$$f(x) = \begin{cases} (x-\frac{1}{2})(x-\frac{1}{4})(x-\frac{15}{4})(x-1)(x-2)(x-3)(x-4), & 0\leq x\leq 4 \\ 0, & x>4. \end{cases}$$

As expected, increasing the value of n we get better approximation results. Moreover, we give error of the approximation for these operators in Figure 6, graphically. And Table 4 shows the error of approximation, numerically.

x	$\mid K_{20,1}(f;x)-f(x)\mid$	$\mid K_{20,4}(f;x)-f(x)\mid$	$\mid K_{20,16}(f;x)-f(x)\mid$
0.0	0.0475	0.0192	0.0057
0.2	0.0017	0.0147	0.0209
0.4	0.0330	0.0341	0.0347
0.6	0.0462	0.0391	0.0359
0.8	0.0415	0.0298	0.0243
1.0	0.0187	0.0060	0.0000
1.2	0.0220	0.0322	0.0370
1.4	0.0808	0.0847	0.0867
1.6	0.1575	0.1517	0.1491
1.8	0.2523	0.2331	0.2242
2.0	0.3650	0.3288	0.3120

Table 3: Table captions the different values of x.

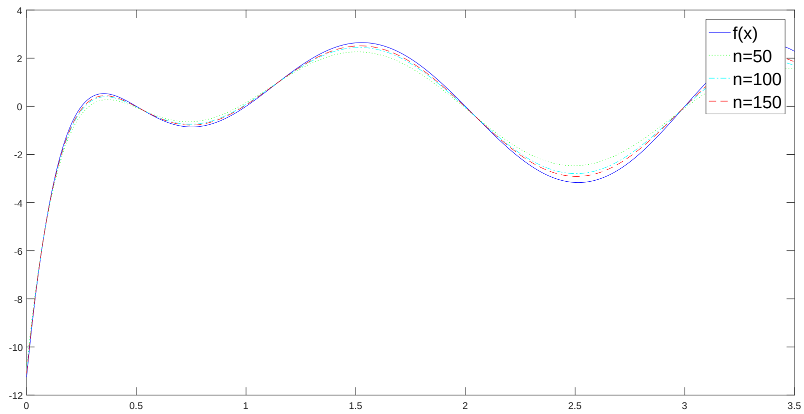

Figure 5: Approximation of $K_{n,\gamma}(f;x)$ to $f(x)$ for $n=50$, $n=100$ and $n=150$ when $\gamma=3$.

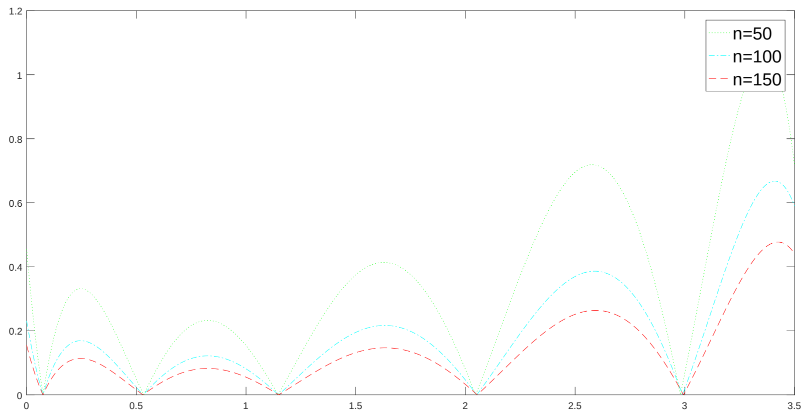

Figure 6: Error of approximation $\varepsilon_{n,\gamma}(f(x))=\left|K_{n,\gamma}(f;x)-f(x)\right|$ for $n=50$, $n=100$, and $n=150$ when $\gamma=3$.

4. A new modification of $K_{n,\gamma}$ for preserving affine functions

Classical Szász–Mirakjan–Kantorovich operators do not preserve affine functions. But in 2020, Bustamante modified these operators by a new technique (see [32]) and this new family of operators preserve affine functions. In this section, we apply this kind of modification to $K_{n,\gamma}(f;x)$ so that they preserves affine functions. We set,

$$A_{n,\gamma}(f;x)=\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^1 f\left(a_k\frac{k+t^{\gamma}}{n}\right)dt, \tag{4.1}$$

where $a_k=\dfrac{(\gamma+1)k}{[(\gamma+1)k+1]}$.

x	$\mid K_{50,3}(f;x)-f(x)\mid$	$\mid K_{100,3}(f;x)-f(x)\mid$	$\mid K_{150,3}(f;x)-f(x)\mid$
0.00	0.4545	0.2310	0.1549
0.35	0.2641	0.1341	0.0899
0.70	0.1873	0.0984	0.0667
1.05	0.1073	0.0572	0.0390
1.40	0.2967	0.1535	0.1035
1.75	0.3755	0.1984	0.1347
2.10	0.0963	0.0457	0.0298
2.45	0.6568	0.3473	0.2358
2.80	0.4962	0.2794	0.1939
3.15	0.6018	0.3206	0.2163
3.50	0.7171	0.5902	0.4409

Table 4: Table captions the different values of x.

We also set,

$$\mu_{n,k}(t)=\frac{t^2}{[(\gamma+1)nt+1]^2}$$

to use in Lemma 4.2 to investigate the moments of the operator $A_{n,\gamma}$.

Lemma 4.1. *Let $k,n\in\mathbb{N}$ and $\gamma\in(0,\infty)$, then we have:*

1. $\int_0^1\left(a_k\frac{k+t^\gamma}{n}\right)dt=\frac{k}{n}$
2. $\int_0^1\left(a_k\frac{k+t^\gamma}{n}\right)^2dt=\frac{k^2}{n^2}+\frac{\gamma^2k^2}{(2\gamma+1)n^2\left[(\gamma+1)k+1\right]^2}$

where $a_k=\frac{(\gamma+1)k}{[(\gamma+1)k+1]}$.

Lemma 4.2. *For each $n\in\mathbb{N}$, $\gamma\in(0,\infty)$ and $x\in[0,\infty)$ we obtain,*

1. $A_{n,\gamma}(1;x)=1$
2. $A_{n,\gamma}(t;x)=x$
3. $A_{n,\gamma}(t^2;x)=x^2+\frac{x}{n}+\frac{\gamma^2}{2\gamma+1}S_n(\mu_{n,k}(t);x)$.

where S_n is defined in (1.1).

Remark 4.3. *The operators $A_{n,\gamma}(f;x)$ in (4.1) reproduce linear polynomials, that is*

$$A_{n,\gamma}(ct+d;x)=cx+d,$$

where $c,d\in\mathbb{R}$.

Remark 4.4. *If $\gamma=1$ in (4.1), then $A_{n,\gamma}$ reduce to the operators in [32], which is introduced and studied by Bustamante.*

Theorem 4.5. *Let $A>0$, then for any $f\in C_B[0,A]$, and $\gamma\in(0,\infty)$, we obtain that $A_{n,\gamma}(f;x)$ are uniformly convergent to $f(x)$ on $[0,A]$.*

Proof. From Korovkin Theorem, it sufficies to demonstrate that $\lim_{n\to\infty}A_{n,\gamma}(t^i;x)=x^i$ where $i=0,1,2$. Evidently, as a consequence of Lemma 4.2, $\lim_{n\to\infty}A_{n,\gamma}(1;x)=1$ and $\lim_{n\to\infty}A_{n,\gamma}(t;x)=x$. Moreover, we can establish that

$$A_{n,\gamma}(t^2;x)=x^2+\frac{x}{n}+\frac{\gamma^2}{2\gamma+1}S_n(\mu_{n,k}(t);x).$$

Taking limit for both sides as $n\to\infty$, we get

$$\lim_{n\to\infty}A_{n,\gamma}(t^2;x)=\lim_{n\to\infty}x^2+\lim_{n\to\infty}\frac{x}{n}+\lim_{n\to\infty}\frac{\gamma^2}{2\gamma+1}S_n(\mu_{n,k}(t);x).$$

From the convergence of classical Szász–Mirakjan operators $\lim_{n\to\infty} S_n(\mu_{n,k}(t);x) = 0$ because of $\lim_{n\to\infty} \mu_{n,k}(x) = 0$. As a result,

$$\lim_{n\to\infty} A_{n,\gamma}(t^2;x) = x^2.$$

Hence, Korovkin theorem conditions are hold for $A_{n,\gamma}$. Then, the proof is completed. □

Note that, $A_{n,1}$ are the operators which is defined by Bustamante. In the following graph, we compare the error estimation results of $A_{n,1}$ and $A_{n,\gamma}$ for different values of γ, by using central moments. One can easily observe that, $A_{n,\gamma}$ have better error estimation when decreasing the value γ. Therefore, for any $0 \le \gamma \le 1$, the operators $A_{n,\gamma}$ have better error estimation than $A_{n,1}$.

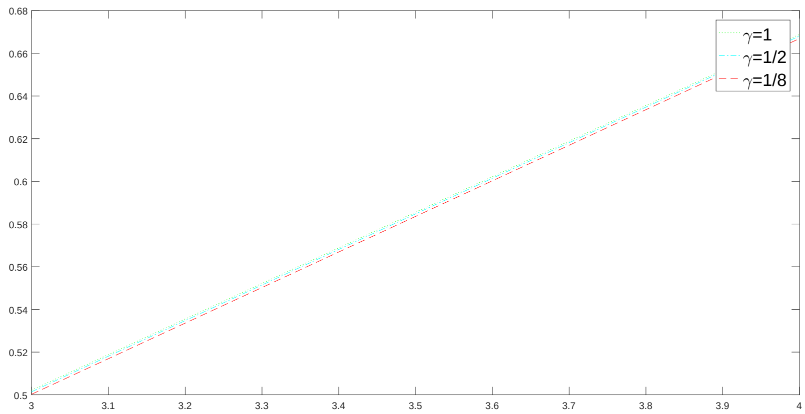

Figure 7: Error of approximation $A_{n,\gamma}((t-x)^2;x)$ for $\gamma = 1$, $\gamma = \frac{1}{2}$ and $\gamma = \frac{1}{8}$ when $n = 6$.

Now, we present graphical and numerical results to compare the convergence of the operators $A_{n,\gamma}(f;x)$ to $f(x)$, where

$$f(x) = \begin{cases} 1+x+x^2, & 0 \le x \le 2 \\ 7, & x > 2, \end{cases}$$

for different values of γ.

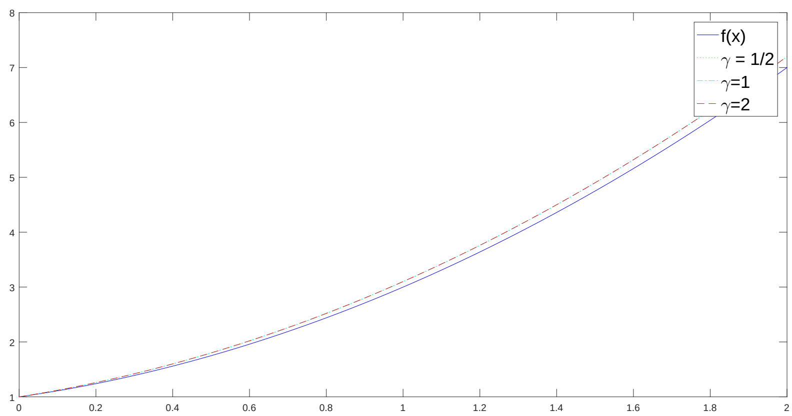

Figure 8: Approximation of $A_{n,\gamma}(f;x)$ to $f(x)$ for $\gamma = \frac{1}{2}$, $\gamma = 1$, and $\gamma = 2$ when $n = 10$.

As anticipated, Figure 8 and Table 5 illustrate that the approximation results of $A_{n,\gamma}(f;x)$ to a specific function $f(x)$ improve as the value of γ decreases. Now, in the upcoming figure, we compare the operators $A_{n,\gamma}(f;x)$ with the function $f(x)$, where

$$f(x) = \begin{cases} x(x-1)(x-\frac{1}{2}), & 0 \le x \le 1 \\ 0, & otherwise, \end{cases}$$

with different choices of n.

x	$\mid A_{10,\frac{1}{2}}(f;x)-f(x)\mid$	$\mid A_{10,1}(f;x)-f(x)\mid$	$\mid A_{10,2}(f;x)-f(x)\mid$
0.0	0.0000	0.0000	0.0000
0.2	0.0203	0.0205	0.0206
0.4	0.0404	0.0406	0.0407
0.6	0.0604	0.0607	0.0608
0.8	0.0805	0.0807	0.0808
1.0	0.1005	0.1007	0.1008
1.2	0.1205	0.1208	0.1208
1.4	0.1405	0.1408	0.1408
1.6	0.1605	0.1608	0.1609
1.8	0.1805	0.1808	0.1809
2.0	0.2005	0.2008	0.2009

Table 5: Table captions the different values of x.

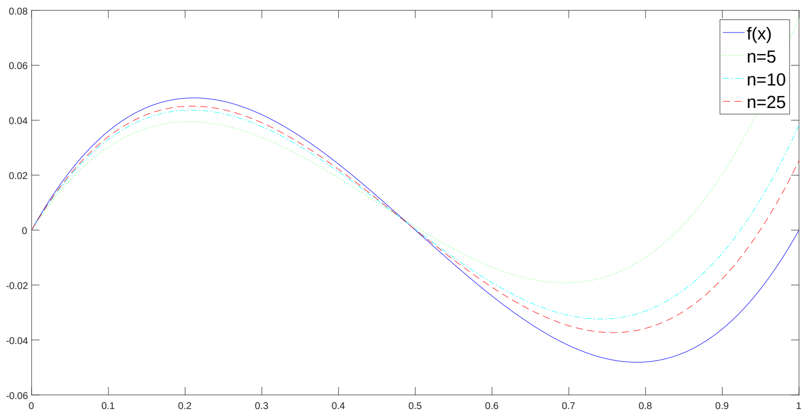

Figure 9: Approximation of $A_{n,\gamma}(f;x)$ to $f(x)$ for $n=5$, $n=10$, and $n=25$ when $\gamma=\frac{1}{3}$.

5. The bivariate case of $K_{n,\gamma}$

Favard [34] introduced and studied bivariate case of classical Szász–Mirakjan operators. Then, many researchers investigated these operators and their generalizations such as in [33] [35], [36], [37]. In this part, we define and investigate the bivariate case of $K_{n,\gamma}$. Consider, $C_B([0,\infty)\times[0,\infty))$ is the space of uniformly bounded and continuous bivariate functions on $[0,\infty)\times[0,\infty)$. We define the operators $K_{n_1,n_2}^{\gamma_1,\gamma_2}$ as

$$K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y) = \sum_{k_1=0}^{\infty} s_{n_1,k_1}(x)\sum_{k_2=0}^{\infty} s_{n_2,k_2}(y)\int_0^1\int_0^1 f\left(\frac{k_1+t_1^{\gamma}}{n_1},\frac{k_2+t_2^{\gamma}}{n_2}\right)dt_1dt_2, \tag{5.1}$$

where $(x,y)\in[0,\infty)\times[0,\infty)$ and $\gamma_1,\gamma_2\in(0,\infty)$ for an integrable functions $f:[0,\infty)\times[0,\infty)\to\mathbb{R}$.
Note that, the operators $K_{n_1,n_2}^{\gamma_1,\gamma_2}$ are linear and positive. Furthermore, choosing $\gamma_1=\gamma_2=1$ in (5.1), then we get the classical Bivariate Szász-Mirakjan Kantorovich operators.

Lemma 5.1. *For $(x,y)\in[0,\infty)\times[0,\infty)$ and $\gamma_1,\gamma_2\in(0,\infty)$, we have*

1. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(1;x,y)=1$
2. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_1;x,y)=x+\dfrac{1}{(\gamma_1+1)n_1}$
3. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_2;x,y)=y+\dfrac{1}{(\gamma_2+1)n_2}$
4. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_1^2;x,y)=x^2+\dfrac{(\gamma_1+3)x}{(\gamma_1+1)n_1}+\dfrac{1}{(2\gamma_1+1)n_1^2}$
5. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_2^2;x,y)=y^2+\dfrac{(\gamma_2+3)y}{(\gamma_2+1)n_2}+\dfrac{1}{(2\gamma_2+1)n_2^2}$.

Corollary 5.2. *From Lemma 5.1, we have the following central moments:*

1. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_1-x;x,y)=\dfrac{1}{(\gamma_1+1)n_1}=:\rho_{n_1,\gamma_1}(x)$
2. $K_{n_1,n_2}^{\gamma_1,\gamma_2}(t_2-y;x,y)=\dfrac{1}{(\gamma_2+1)n_2}=:\rho_{n_2,\gamma_2}(y)$
3. $K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_1-x)^2;x,y)=\dfrac{x}{n_1}+\dfrac{1}{(2\gamma_1+1)n_1^2}=:\varphi_{n_1,\gamma_1}(x)$
4. $K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_2-y)^2;x,y)=\dfrac{y}{n_2}+\dfrac{1}{(2\gamma_2+1)n_2^2}=:\varphi_{n_2,\gamma_2}(y).$

Theorem 5.3. *Let $A_1,A_2>0$, then for each $\gamma_1,\gamma_2\in(0,\infty)$ and $f\in C_B([0,A_1]\times[0,A_2])$, the operators $K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)$ uniformly convergent to f as $n_1,n_2\to\infty$ on $[0,A_1]\times[0,A_2]$.*

Proof. Volkov in [38] gives the conditions for the uniformly convergence of bivariate positive linear operators to continous functions. Using Lemma 5.1, one can easily see that the conditions for Volkov's theorem are hold. So, proof is completed. □

For any $f\in C([0,\infty)\times[0,\infty))$, the modulus of continuity for the bivariate case is

$$\omega(f;\delta_1,\delta_2)=\sup_{|t_1-\rho|\le\delta_1,|t_2-\sigma|\le\delta_2}\{|f(t_1,t_2)-f(\rho,\sigma)|:(t_1,t_2),(\rho,\sigma)\in[0,\infty)\times[0,\infty)\}$$

where $\delta_1,\delta_2\in\mathbb{R}^+$.

Moreover, the function $\omega(f;\delta_1,\delta_2)$ has the following inequality,

$$|f(t_1,t_2)-f(\rho,\sigma)|\le\left(1+\frac{|t_1-\rho|}{\delta_1}\right)\left(1+\frac{|t_2-\sigma|}{\delta_2}\right)\omega(f;\delta_1,\delta_2).$$

Theorem 5.4. *Assume that $f\in C([0,\infty)\times[0,\infty))$ and $\gamma_1,\gamma_2\in(0,\infty)$. Then for each $(x,y)\in[0,\infty)\times[0,\infty)$ we get,*

$$\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)-f(x,y)\right|\le4\omega\left(f;\sqrt{\varphi_{n_1,\gamma_1}(x)},\sqrt{\varphi_{n_2,\gamma_2}(y)}\right)$$

where $\varphi_{n_1,\gamma_1}(x)$ and $\varphi_{n_2,\gamma_2}(y)$ are given in Corollary 5.2.

Proof. Due to the linearity and positivity of $K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)$, we are able to write

$$\begin{aligned}\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)-f(x,y)\right| &\le K_{n_1,n_2}^{\gamma_1,\gamma_2}(|f(t_1,t_2)-f(x,y)|;x,y)\\ &\le \left(1+\frac{K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_1-x|;x,y)}{\delta_1}\right)\left(1+\frac{K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_2-y|;x,y)}{\delta_2}\right)\omega(f;\delta_1,\delta_2).\end{aligned}$$

Using Cauchy-Schwarz inequality,

$$K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_1-x|;x,y)\le\left[K_{n_1,n_2}^{\gamma_1,\gamma_2}\left((t_1-x)^2;x,y\right)\right]^{\frac{1}{2}}$$

and

$$K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_2-y|;x,y)\le\left[K_{n_1,n_2}^{\gamma_1,\gamma_2}\left((t_2-y)^2;x,y\right)\right]^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)-f(x,y)\right| &\le \left(1+\frac{\sqrt{K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_1-x)^2;x,y)}}{\delta_1}\right)\left(1+\frac{\sqrt{K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_2-y)^2;x,y)}}{\delta_2}\right)\omega(f;\delta_1,\delta_2)\\ &= \left(1+\frac{\sqrt{\varphi_{n_1,\gamma_1}(x)}}{\delta_1}\right)\left(1+\frac{\sqrt{\varphi_{n_2,\gamma_2}(y)}}{\delta_2}\right)\omega(f;\delta_1,\delta_2).\end{aligned}$$

Finally, by choosing $\delta_1=\sqrt{\varphi_{n_1,\gamma_1}(x)}$ and $\delta_2=\sqrt{\varphi_{n_2,\gamma_2}(y)}$, we get the desired result. □

For $0<a_1,a_2\le1$ the Lipschitz class $Lip_M(a_1,a_2)$ for bivariate case is defined as

$$Lip_M(a_1,a_2) := \left\{f(x,y)\in C([0,\infty)\times[0,\infty)):|f(t_1,t_2)-f(x,y)|\le M|t_1-x|^{a_1}|t_2-y|^{a_2}\right\}$$

where $M>0$ and $(x,y),(t_1,t_2)\in[0,\infty)\times[0,\infty)$.

Theorem 5.5. *If $f \in Lip_M(a_1,a_2)$, then we have,*

$$\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)-f(x,y)\right| \leq M\left[\varphi_{n_1,\gamma_1}(x)\right]^{\frac{a_1}{2}}\left[\varphi_{n_2,\gamma_2}(y)\right]^{\frac{a_2}{2}}$$

hold for all $(x,y)\in[0,\infty)\times[0,\infty)$, where $\varphi_{n_1,\gamma_1}(x)$ and $\varphi_{n_2,\gamma_2}(y)$ are given in Corollary 5.2.

Proof. Let $f \in Lip_M(a_1,a_2)$, then we have

$$\begin{aligned}\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)-f(x,y)\right| &\leq K_{n_1,n_2}^{\gamma_1,\gamma_2}(\left|f(t_1,t_2)-f(x,y)\right|;x,y)\\ &\leq MK_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_1-x|^{a_1}|t_2-y|^{a_2};x,y)\\ &= MK_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_1-x|^{a_1};x,y)K_{n_1,n_2}^{\gamma_1,\gamma_2}(|t_2-y|^{a_2};x,y).\end{aligned}$$

We apply Hölder's inequality with $p_1=\dfrac{2}{a_1}$, $q_1=\dfrac{2}{2-a_1}$ and $p_2=\dfrac{2}{a_2}$, $q_2=\dfrac{2}{2-a_2}$, to get

$$\begin{aligned}\left|K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)-f(x,y)\right| &\leq MK_{n_1,n_2}^{\gamma_1,\gamma_2}((t_1-x)^2;x,y)^{\frac{a_1}{2}}K_{n_1,n_2}^{\gamma_1,\gamma_2}((t_2-y)^2;x,y)^{\frac{a_2}{2}}\\ &= M[\varphi_{n_1,\gamma_1}(x)]^{\frac{a_1}{2}}[\varphi_{n_2,\gamma_2}(y)]^{\frac{a_2}{2}}.\end{aligned}$$

□

Now, we illustrate the approximation of $K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)$ to $f(x,y)$ for different choices of γ_1,γ_2 for, where

$$f(x,y) = \begin{cases} 4+(x-\frac{1}{2})(y-\frac{1}{2})(y-\frac{1}{4})(x-\frac{3}{4})(x-\frac{3}{2})(y-2), & 0\leq x,y\leq 2\\ 4, & x,y>2.\end{cases}$$

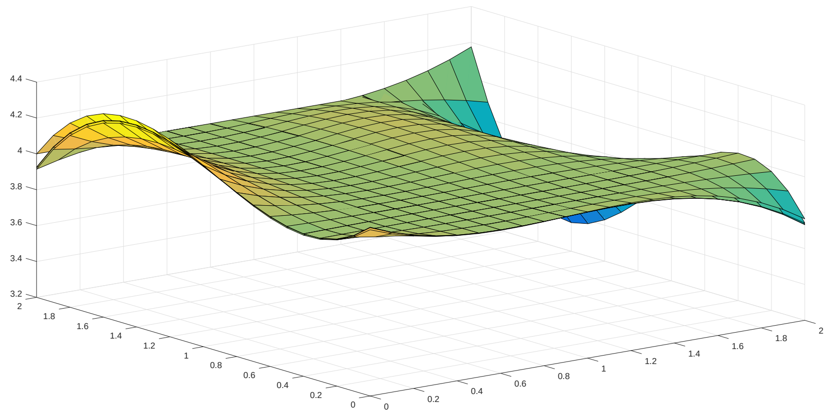

Figure 10: Approximation of $K_{n_1,n_2}^{\gamma_1,\gamma_2}(f;x,y)$ to $f(x,y)$ for $\gamma_1,\gamma_2=1$, $\gamma_1,\gamma_2=10$, $\gamma_1,\gamma_2=100$ when $n_1,n_2=50$.

6. Conclusion

We have introduced a novel generalization of the Szász-Mirakjan Kantorovich operators, denoted as $K_{n,\gamma}(f;x)$, defined by

$$K_{n,\gamma}(f;x) := \sum_{k=0}^{\infty} s_{n,k}(x)\int_0^1 f\left(\frac{k+t^{\gamma}}{n}\right)dt.$$

It's important to note that when $\gamma=1$, these operators reduce to the classical case.

The operators $K_{n,\gamma}$ has the following features:

- Uniformly convergent to any function $f\in C_B[0,A]$ on the interval $[0,A]$ for each $A,\gamma\in\mathbb{R}^+$.
- When γ is chosen to be greater than 1, these operators give improved error estimation compared to the classical case. Moreover, as the value of γ increases, the error estimation becomes smaller.
- Having shape preserving properties.

Furthermore, we introduced a new family of operators $A_{n,\gamma}(f;x)$. These operators reproduce linear(affine) polynomials. Note that, choosing $\gamma=1$, the new operators $A_{n,\gamma}(f;x)$ reduce to classical case which is introduced and studied by Bustamante in [32]. These operators have better error estimation than classical case if γ is choosen less than 1. Moreover, decreasing the value γ, the error is getting smaller. Finally, we defined the bivariate case of $K_{n,\gamma}(f;x)$, and investigated their approximation properties. As expected, increasing the value of γ_1 and γ_2, we got better error estimation than classical bivariate case.

Declarations

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Wijsman Deferred Invariant Statistical and Strong p-Deferred Invariant Equivalence of Order α

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Abstract

With this work, we present the asymptotical strongly p-deferred invariant and asymptotical deferred invariant statistical equivalence of order α $(0 < \alpha \leq 1)$ for sequences of sets in the Wijsman sense. Furthermore, we investigate the connections between these concepts and conduct their properties.

1. Introduction and backgrounds

One of the convergence concepts for sequences of sets (Ss) is convergence in the Wijsman sense (Ws) (see, [1, 2]). The statistical convergence in Ws was first introduced by Nuray and Rhoades [3]. Then, Ulusu and Nuray [4] studied the lacunary statistical convergence in Ws. Also, Pancaroğlu and Nuray [5] presented the invariant statistical convergence in Ws. Furthermore, Ulusu and Nuray [6] and Pancaroğlu et al. [7] introduced the asymptotical-asymptotical statistical equivalence and asymptotical invariant-asymptotical invariant statistical equivalence in Ws, respectively.
Agnew [8] first introduced the deferred Cesàro mean for real (complex) sequences. Subsequently, the deferred statistical convergence was studied by Küçükaslan and Yılmaztürk [9]. Then, Nuray [10] presented the deferred invariant and deferred invariant statistical convergence.
The deferred statistical convergence in Ws for Ss was introduced by Altınok et al. [11]. Also, Et and Yılmazer [12] studied on this concept. Then, Gülle [13] presented the deferred invariant statistical convergence of order α in Ws. Furthermore, Altınok et al. [14] and Et et al. [15] studied the asymptotical deferred statistical and asymptotical deferred statistical equivalence of order α in Ws, respectively.
In the metric space $(\mathcal{U}, d)$, the distance function $\rho(u, C) := \rho_u(C)$ is defined by

$$\rho_u(C) = \inf_{c \in C} d(u, c)$$

for each $u \in \mathcal{U}$ and non-empty $C \subseteq \mathcal{U}$.
For a function $f : \mathbb{N} \to 2^{\mathcal{U}}$ (power set) is defined by $f(j) = C_j \in 2^{\mathcal{U}}$ for each $j \in \mathbb{N}$ (the set of natural numbers), the sequence $\{C_j\} = \{C_1, C_2, \ldots\}$ is called sequence of sets.
Throughout the study, unless otherwise specified, $(\mathcal{U}, d)$ is regarded as a metric space and C, C_j, D_j, E_j, F_j as non-empty closed subsets of $\mathcal{U}$.
The Ss $\{C_j\}$ is called convergent in Ws to the set C if for each $u \in \mathcal{U}$

$$\lim_{j \to \infty} \rho_u(C_j) = \rho_u(C)$$

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and it is denoted in $C_j \xrightarrow{W} C$ format.
An invariant mean, also known as a σ-mean, is a continuous linear functional ψ in the bounded sequences space that adhere to the subsequent conditions:

(1) $\psi(x_t) \geq 0$ when the sequence (x_t) consists of non-negative elements for all t,
(2) $\psi(e) = 1$ for $e = (1,1,1,\ldots)$,
(3) $\psi(x_{\sigma(t)}) = \psi(x_t)$ for all the bounded sequences (x_t),

where σ is a mapping from the set of non-negative integers into itself.
The mappings σ are regarded as one-to-one and $\sigma^j(t) \neq t$ (jth iterate of σ) for all positive integers j. Therefore, ψ expands the limit functional on the convergent sequences space c such that $\psi(x_t) = \lim x_t$ for all $(x_t) \in c$.
The Ss $\{C_j\}$ is called;

(i) strongly invariant convergent in Ws to the set C if

$$\lim_{j\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left|\rho_u(C_{\sigma^j(t)}) - \rho_u(C)\right| = 0,$$

(ii) invariant statistically convergent in Ws to the set C if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{ j \leq n : |\rho_u(C_{\sigma^j(t)}) - \rho_u(C)| \geq \varepsilon \right\}\right| = 0$$

for each $u \in \mathfrak{U}$ and uniformly in t. These convergences are denoted in $C_j \xrightarrow{W[V_\sigma]} C$ and $C_j \xrightarrow{W(S_\sigma)} C$ formats, respectively.
For any non-empty closed subsets $C_j, D_j \in \mathfrak{U}$ such that $\rho_u(C_j) > 0$ and $\rho_u(D_j) > 0$ for each $u \in \mathfrak{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are called asymptotically equivalent to multiple η in Ws if for each $u \in \mathfrak{U}$

$$\lim_{j\to\infty} \frac{\rho(u, C_j)}{\rho(u, D_j)} = \eta$$

and it is denoted in $C_j \overset{W^\eta}{\sim} D_j$ format. These sequences are referred to as asymptotically equivalent in Ws when $\eta = 1$.
For any non-empty closed subsets $C_j, D_j \in \mathfrak{U}$ such that $\rho_u(C_j) > 0$ and $\rho_u(D_j) > 0$ for each $u \in \mathfrak{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are called;

(i) asymptotically strongly deferred Cesàro equivalent to multiple η in Ws if

$$\lim_{i\to\infty} \frac{1}{s(i) - r(i)} \sum_{j=r(i)+1}^{s(i)} \left| \frac{\rho(u, C_j)}{\rho(u, D_j)} - \eta \right| = 0,$$

(ii) asymptotically deferred statistical equivalent to multiple η in Ws if for every $\varepsilon > 0$

$$\lim_{i\to\infty} \frac{1}{s(i) - r(i)} \left| \left\{ r(i) < j \leq s(i) : \left| \frac{\rho(u, C_j)}{\rho(u, D_j)} - \eta \right| \geq \varepsilon \right\} \right| = 0$$

for each $u \in \mathfrak{U}$, where $(r(i))$ and $(s(i))$ are sequences of non-negative integers satisfying

$$r(i) < s(i) \quad \text{and} \quad \lim_{i\to\infty} s(i) = \infty. \tag{1.1}$$

These equivalences are denoted in $C_j \overset{W_d^\eta}{\sim} D_j$ and $C_j \overset{W_d^\eta(S)}{\sim} D_j$ formats, respectively.
Throughout the paper, unless otherwise specified, $(r(i))$ and $(s(i))$ is regarded as non-negative integer sequences satisfying (1.1).
An increasing sequence of integers $\theta = (k_i)$ is called a lacunary sequence when it satisfies two conditions: $k_0 = 0$ and $h_i = k_i - k_{i-1} \to \infty$ as $i \to \infty$.
For more study on the concepts of convergence, invariant summability, deferred mean and asymptotical equivalence for real or set sequences, we refer to [16, 17, 18, 19, 20, 21, 22].
From now on, for short, we will use the term $\rho_u\left(\frac{C_j}{D_j}\right)$ instead of the term $\frac{\rho(u, C_j)}{\rho(u, D_j)}$.

2. Main results

With this section, we present the asymptotical strongly p-deferred invariant and asymptotical deferred invariant statistical equivalence of order α $(0<\alpha\leq 1)$ in Ws for Ss. Furthermore, we investigate the connections between these concepts and conduct their properties.

Definition 2.1. *For any non-empty closed subsets $C_j, D_j \in \mathfrak{U}$ such that $\rho_u(C_j)>0$ and $\rho_u(D_j)>0$ for each $u\in\mathfrak{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are said to be asymptotically strongly p-deferred invariant equivalent to multiple η of order α in Ws if for each $u\in\mathfrak{U}$*

$$\lim_{i\to\infty}\frac{1}{\big(s(i)-r(i)\big)^{\alpha}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^{p}=0$$

uniformly in t, where $0<p<\infty$ and $0<\alpha\leq 1$. For this case, the notation $C_j \overset{W_d^{\eta}[V_{\sigma}^{\alpha}]^p}{\sim} D_j$ is used, and these sequences are referred to as asymptotically strongly p-deferred invariant equivalent of order α in Ws when $\eta=1$.

Example 2.2. *Let us take $X=\mathbb{R}^2$ and the Ss $\{C_j\}$ and $\{D_j\}$ as follows:*

$$C_j:=\begin{cases}\{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+(x_2-1)^2=\frac{1}{j}\} & ;\ \text{if } j \text{ is a square integer}\\ \{(-1,0)\} & ;\ \text{if not}\end{cases}$$

and

$$D_j:=\begin{cases}\{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+(x_2+1)^2=\frac{1}{j}\} & ;\ \text{if } j \text{ is a square integer}\\ \{(-1,0)\} & ;\ \text{if not.}\end{cases}$$

Then, the Ss $\{C_j\}$ and $\{D_j\}$ are asymptotically strongly p-deferred invariant equivalent of order α $(0<\alpha\leq 1)$ in Ws.

Remark 2.3.

(i) For Ss, the asymptotical strongly p-deferred invariant equivalence of order α and asymptotical strongly p-invariant equivalence given in [7] coincide when $r(i)=0$, $s(i)=i$ and $\alpha=1$.
(ii) For Ss, the asymptotical strongly p-deferred invariant equivalence of order α and asymptotical strongly p-lacunary invariant equivalence given in [7] coincide when $r(i)=k_{i-1}$, $s(i)=k_i$ and $\alpha=1$.

Theorem 2.4. *Let $0<p<\infty$ and $0<\alpha\leq\beta\leq 1$. Then,*

$$C_j \overset{W_d^{\eta}[V_{\sigma}^{\alpha}]^p}{\sim} D_j \Rightarrow C_j \overset{W_d^{\eta}[V_{\sigma}^{\beta}]^p}{\sim} D_j.$$

Proof. Assume that $0<\alpha\leq\beta\leq 1$ and $C_j \overset{W_d^{\eta}[V_{\sigma}^{\alpha}]^p}{\sim} D_j$. For each $u\in\mathfrak{U}$, we can write

$$\frac{1}{\big(s(i)-r(i)\big)^{\beta}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^{p}\leq\frac{1}{\big(s(i)-r(i)\big)^{\alpha}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^{p}$$

for all t. Since the right side converges to 0 for $i\to\infty$ based on our assumption, we have $C_j \overset{W_d^{\eta}[V_{\sigma}^{\beta}]^p}{\sim} D_j$. □

The following corollary is obtained for $\beta=1$ in Theorem 2.4.

Corollary 2.5. *Let $0<p<\infty$ and $0<\alpha\leq 1$. If $C_j \overset{W_d^{\eta}[V_{\sigma}^{\alpha}]^p}{\sim} D_j$, then $C_j \overset{W_d^{\eta}[V_{\sigma}]^p}{\sim} D_j$ which this concept has not been studied yet.*

Theorem 2.6. *Let $0<p<q<\infty$ and $0<\alpha\leq 1$. Then,*

$$C_j \overset{W_d^{\eta}[V_{\sigma}^{\alpha}]^q}{\sim} D_j \Rightarrow C_j \overset{W_d^{\eta}[V_{\sigma}^{\alpha}]^p}{\sim} D_j.$$

Proof. Assume that $0<p<q<\infty$ and $C_j \overset{W_d^{\eta}[V_{\sigma}^{\alpha}]^q}{\sim} D_j$. By the Hölder inequality, for each $u\in\mathfrak{U}$, we can write

$$\frac{1}{\big(s(i)-r(i)\big)^{\alpha}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^{p}<\frac{1}{\big(s(i)-r(i)\big)^{\alpha}}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^{q}$$

for all t. Since the right side converges to 0 for $i\to\infty$ based on our assumption, we have $C_j \overset{W_d^{\eta}[V_{\sigma}^{\alpha}]^p}{\sim} D_j$. □

Definition 2.7. *For any non-empty closed subsets $C_j, D_j \in \mathfrak{U}$ such that $\rho_u(C_j) > 0$ and $\rho_u(D_j) > 0$ for each $u \in \mathfrak{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are said to be asymptotically deferred invariant statistical equivalent to multiple η of order α in Ws if for every $\varepsilon > 0$ and each $u \in \mathfrak{U}$*

$$\lim_{i\to\infty} \frac{1}{\big(s(i)-r(i)\big)^{\alpha}} \left| \left\{ r(i) < j \leq s(i) : \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right| \geq \varepsilon \right\} \right| = 0$$

uniformly in t, where $0 < \alpha \leq 1$. For this case, the notation $C_j \overset{W_d^{\eta}(S_\sigma^{\alpha})}{\sim} D_j$ is used, and these sequences are referred to as asymptotically deferred invariant statistical equivalent of order α in Ws when $\eta = 1$.

The set $\{W_d^{\eta}(S_\sigma^{\alpha})\}$ represents all Ss that asymptotically deferred invariant statistical equivalent of order α.

Example 2.8. *Let us take $X = \mathbb{R}^2$ and the Ss $\{C_j\}$ and $\{D_j\}$ as follows:*

$$C_j := \begin{cases} \left\{(x_1,x_2) \in \mathbb{R}^2 : (x_1 + j)^2 + x_2^2 = 1\right\} & ; \quad \textit{if } j \textit{ is a square integer} \\ \{(1,0)\} & ; \quad \textit{if not} \end{cases}$$

and

$$D_j := \begin{cases} \left\{(x_1,x_2) \in \mathbb{R}^2 : (x_1 - j)^2 + x_2^2 = 1\right\} & ; \quad \textit{if } j \textit{ is a square integer} \\ \{(1,0)\} & ; \quad \textit{if not.} \end{cases}$$

Then, the Ss $\{C_j\}$ and $\{D_j\}$ are asymptotically deferred invariant statistical equivalent order α $(0 < \alpha \leq 1)$ in Ws.

Remark 2.9.

(i) *For Ss, the asymptotical deferred invariant statistical equivalence of order α and asymptotical invariant statistical equivalence given in [7] coincide when $r(i) = 0$, $s(i) = i$ and $\alpha = 1$.*
(ii) *For Ss, the asymptotical deferred invariant statistical equivalence of order α and asymptotical lacunary invariant statistical equivalence given in [7] coincide when $r(i) = k_{i-1}$, $s(i) = k_i$ and $\alpha = 1$.*

Theorem 2.10. *Let $0 < \alpha \leq \beta \leq 1$. Then*

$$C_j \overset{W_d^{\eta}(S_\sigma^{\alpha})}{\sim} D_j \Rightarrow C_j \overset{W_d^{\eta}(S_\sigma^{\beta})}{\sim} D_j.$$

Proof. Assume that $0 < \alpha \leq \beta \leq 1$ and $C_j \overset{W_d^{\eta}(S_\sigma^{\alpha})}{\sim} D_j$. For every $\varepsilon > 0$ and each $u \in \mathfrak{U}$, we can write

$$\frac{1}{\big(s(i)-r(i)\big)^{\beta}} \left| \left\{ r(i) < j \leq s(i) : \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right| \geq \varepsilon \right\} \right| \leq \frac{1}{\big(s(i)-r(i)\big)^{\alpha}} \left| \left\{ r(i) < j \leq s(i) : \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right| \geq \varepsilon \right\} \right|$$

for all t. Since the right side converges to 0 for $i \to \infty$ based on our assumption, we have $C_j \overset{W_d^{\eta}(S_\sigma^{\beta})}{\sim} D_j$. □

The following corollary is obtained for $\beta = 1$ in Theorem 2.10.

Corollary 2.11. *Let $0 < \alpha \leq 1$. If $C_j \overset{W_d^{\eta}(S_\sigma^{\alpha})}{\sim} D_j$, then $C_j \overset{W_d^{\eta}(S_\sigma)}{\sim} D_j$ which this concept has not been studied yet.*

Theorem 2.12. *If the Ss $\{C_j\}$ and $\{D_j\}$ are asymptotically strongly p-deferred invariant equivalent to multiple η of order α in Ws, then the sequences are asymptotically deferred invariant statistical equivalent to multiple η of order α in Ws, where $0 < \alpha \leq 1$.*

Proof. Assume that $0 < \alpha \leq 1$ and $C_j \overset{W_d^{\eta}[V_\sigma^{\alpha}]^p}{\sim} D_j$. For every $\varepsilon > 0$ and each $u \in \mathfrak{U}$, we can write

$$\begin{aligned} \sum_{j=r(i)+1}^{s(i)} \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right|^p &\geq \sum_{\substack{j=r(i)+1 \\ \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right| \geq \varepsilon}}^{s(i)} \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right|^p \\ &\geq \varepsilon^p \left| \left\{ r(i) < j \leq s(i) : \left| \rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big) - \eta \right| \geq \varepsilon \right\} \right| \end{aligned}$$

and so,

$$\frac{1}{\varepsilon^p\left(s(i)-r(i)\right)^\alpha}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^p \geq \frac{1}{\left(s(i)-r(i)\right)^\alpha}\left|\left\{r(i)<j\leq s(i):\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|\geq\varepsilon\right\}\right|$$

for all t. Since the left side converges to 0 for $i\to\infty$ based on our assumption, we have $C_j \overset{W_d^\eta(S_\sigma^\alpha)}{\sim} D_j$. □

In the case of $\alpha=1$, the opposite of Theorem 2.12 is provided.

Theorem 2.13. *Let $\rho_u(C_j)\,\mathcal{O}\,\rho_u(D_j)$. If the Ss $\{C_j\}$ and $\{D_j\}$ are asymptotically deferred invariant statistical equivalent to multiple η in Ws, then the sequences are asymptotically strongly p-deferred invariant equivalent to multiple η in Ws.*

Proof. Suppose that $\rho_u(C_j)\,\mathcal{O}\,\rho_u(D_j)$ and $C_j \overset{W_d^\eta(S_\sigma)}{\sim} D_j$. Since $\rho_u(C_j)\,\mathcal{O}\,\rho_u(D_j)$, then there exists an $M>0$ such that

$$\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|\leq M$$

for all t and each $u\in\mathcal{U}$. For every $\varepsilon>0$, we can write

$$\begin{aligned}\frac{1}{s(i)-r(i)}\sum_{j=r(i)+1}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^p &= \frac{1}{s(i)-r(i)}\sum_{\substack{j=r(i)+1\\ \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\geq\varepsilon}}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^p\\ &\quad+\frac{1}{s(i)-r(i)}\sum_{\substack{j=r(i)+1\\ \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|<\varepsilon}}^{s(i)}\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|^p\\ &\leq \frac{M^p}{s(i)-r(i)}\left|\left\{r(i)<j\leq s(i):\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|\geq\varepsilon\right\}\right|+\varepsilon^p\end{aligned}$$

for all t. Since the left side converges to 0 for $i\to\infty$ based on our assumption, we have $C_j \overset{W_d^\eta[V_\sigma]^p}{\sim} D_j$. □

3. Auxiliary results

With this section, first of all, we define the asymptotical invariant statistical equivalence to multiple η of order α in Ws for Ss, then we examine the relationship between this concept and the asymptotical deferred invariant statistical equivalence to multiple η of order α.

Definition 3.1. *For any non-empty closed subsets $C_j,D_j\in\mathcal{U}$ such that $\rho_u(C_j)>0$ and $\rho_u(D_j)>0$ for each $u\in\mathcal{U}$, the Ss $\{C_j\}$ and $\{D_j\}$ are said to be asymptotically invariant statistical equivalent to multiple η of order α in Ws if for every $\varepsilon>0$ and each $u\in\mathcal{U}$*

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{j\leq n:\left|\rho_u\Big(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\Big)-\eta\right|\geq\varepsilon\right\}\right|=0$$

uniformly in t, where $0<\alpha\leq 1$. For this case, the notation $C_j \overset{W^\eta(S_\sigma^\alpha)}{\sim} D_j$ is used, and these sequences are referred to as asymptotically invariant statistical equivalent of order α in Ws when $\eta=1$.

The set $\{W^\eta(S_\sigma^\alpha)\}$ represents all Ss that asymptotically invariant statistical equivalent of order α.

Theorem 3.2. *If* $\left\{\frac{r(i)}{s(i)-r(i)}\right\}$ *is bounded, then* $\{W^{\eta}(S_{\sigma}^{\alpha})\} \subset \{W_d^{\eta}(S_{\sigma}^{\alpha})\}$, *where* $0<\alpha\le 1$.

Proof. Suppose that $0<\alpha\le 1$ and $C_j \overset{W^{\eta}(S_{\sigma}^{\alpha})}{\sim} D_j$. Then, for every $\varepsilon>0$ and each $u\in\mathcal{U}$, we have

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\left\{j\le n: \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\ge\varepsilon\right\}\right|=0$$

uniformly in t. Here using the well-known fact,

$$\lim_{i\to\infty}\frac{1}{(s(i))^{\alpha}}\left|\left\{j\le s(i): \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\ge\varepsilon\right\}\right|=0$$

is hold uniformly in t. Also, since

$$\left\{r(i)<j\le s(i): \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\ge\varepsilon\right\}\subset\left\{0<j\le s(i): \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\ge\varepsilon\right\},$$

we can write

$$\left|\left\{r(i)<j\le s(i): \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\ge\varepsilon\right\}\right|\le\left|\left\{0<j\le s(i): \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\ge\varepsilon\right\}\right|$$

for all t. Thus, the inequality is handled:

$$\frac{1}{(s(i)-r(i))^{\alpha}}\left|\left\{r(i)<j\le s(i): \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\ge\varepsilon\right\}\right|\le\left(1+\frac{r(i)}{s(i)-r(i)}\right)^{\alpha}\frac{1}{(s(i))^{\alpha}}\left|\left\{0<j\le s(i): \left|\rho_u\left(\frac{C_{\sigma^j(t)}}{D_{\sigma^j(t)}}\right)-\eta\right|\ge\varepsilon\right\}\right|.$$

If $\left\{\frac{r(i)}{s(i)-r(i)}\right\}$ is bounded in above inequality, then the desired result is obtained for $i\to\infty$. □

4. Conclusion

In this study, as a combination of asymptotical equivalence, deferred statistical convergence, invariant summability and order α, we defined new concepts for sequences of sets and obtained noteworthy results.

Declarations

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The Qualitative Analysis of Some Difference Equations Using Homogeneous Functions

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Abstract

This article deals with the qualitative analysis of a general class of difference equations. That is, we examine the periodicity nature and the stability character of some non-linear second-order difference equations. Homogeneous functions are used while examining the character of the solutions of introduced difference equations. Moreover, a new technique available in the literature is used to examine the periodic solutions of these equations.

1. Introduction

Although it is known that the theory of difference equations emerged with the rabbit problem introduced by the famous Italian mathematician Fibonacci in 1202 has been a field of study that has been of interest to many scientists, especially in the last 30 years (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). Difference equations are an important field of study in many applied sciences, including mathematics, physics, chemistry, statistics, sociology, psychology, and engineering. Different mathematical models are needed to examine situations related to different living conditions, such as the climate crisis, the arms race, plant populations, animal populations, human populations, birth and death rates, migration rates, the spread of diseases. Here, difference equations come into play, and ecological, biological, economic, statistical, sociological and psychological mathematical models that can be used in different fields of science are created (see [10, 11, 12, 13, 14, 15, 16, 17]). In this context, the examination of difference equations (because it models various systems) is of great importance in that it is applicable not only in mathematics but also in different branches.

In recent years, many studies have been done on difference equations in mathematics, sub-branches of mathematics and other sciences (see [18, 19, 20]). Any quantitative and qualitative research, especially in the field of difference equations, is very important. Detailed qualitative studies in this field are invaluable when considering any result obtained by examining the global behavior, asymptotic behavior, boundedness nature and the stability character of solutions of difference equations. However, considering difference equation theory, it should be noted that there are not many general theorems and techniques that study difference equation classes. The structure of higher-order non-linear difference equation classes is quite complex and challenging. For this reason, although there are many articles and books on linear difference equations, there are not many sources on higher-order non-linear difference equations. On account of this, it is very important to examine various difference equations that will both contribute to the literature and expand and improve the difference equation theory.

In [21], Elsayed introduced a new method for the prime period two solutions and the prime period three solutions of the rational difference equation

$$\omega_{n+1} = \mu + \phi \frac{\omega_n}{\omega_{n-1}} + \gamma \frac{\omega_{n-1}}{\omega_n}, \quad n = 0, 1, \ldots$$

where the parameters μ, ϕ, γ and initial values ω_{-1}, ω_0 are positive real numbers. Besides, the global convergence and the boundedness nature were investigated.

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In [22], Moaaz *et al.* examined the asymptotic behavior, that is, the stability, the oscillation and the periodicity character of solutions of a general class of difference equations

$$z_{n+1}=g(z_n,z_{n-1}),\quad n=0,1,\ldots$$

where the initial conditions z_{-1},z_0 are real numbers and g is a continuous homogeneous function with degree zero.
In [23], Moaaz investigated the asymptotic behavior of solutions of the following general class of difference equations

$$\omega_{n+1}=g(\omega_{n-l},\omega_{n-k})$$

where l, k are positive integers, the initial conditions $\omega_{-\mu},\omega_{-\mu+1},\ldots,\omega_0$ are real numbers for $\mu=\max\{l,k\}$ and g is a continuous homogeneous real function of degree γ. Namely, the periodic solutions, the global attractiveness and the stability have been examined.
In [24], Stevic has shown that the claim given in Theorem 3.3 in [23] is not true. Essentially, he has improved and expanded global attractiveness results.
In [25], Abdelrahman *et al.* investigated the local stability, the periodicity and the boundedness character of solutions of a new class of the difference equations

$$\omega_{n+1}=\zeta\omega_{n-l}+\varphi\omega_{n-k}+g(\omega_{n-l},\omega_{n-k}),\quad n=0,1,\ldots \tag{1.1}$$

where l,k are non-negative integers, the parameters ζ,φ are non-negative real numbers and the initial values $\omega_{-s},\omega_{-s+1},...,\omega_0$ are positive real numbers for $s=\max\{l,k\}$ and $g:(0,\infty)^2\to(0,\infty)$ is a continuous homogeneous function with degree zero.
In [26], Abdelrahman investigated the dynamical behavior of solutions of a general class of difference equations

$$x_{m+1}=g(x_m,x_{m-1},\ldots,x_{m-k}),\quad m=0,1,\ldots$$

where $g:(0,\infty)^{k+1}\to(0,\infty)$ is a continuously homogeneous function of degree zero and k is a positive integer. That is, the stability, the periodicity and the oscillatory have been examined.
In [27], Moaaz *et al.* examined the existence and non-existence of periodic solutions of some non-linear difference equations. Especially, they studied the existence of periodic solutions of the difference equation

$$\omega_{n+1}=\gamma\omega_{n-1}F(\omega_n,\omega_{n-1})$$

where the parameter γ is positive real number, the initial values ω_{-1},ω_0 are positive real numbers and F is a homothetic function, namely there exists a strictly increasing function $F_1:\mathbb{R}\to\mathbb{R}$ and $F_2:\mathbb{R}^2\to\mathbb{R}$ are homogenous function with degree ρ, such that $F=F_1(F_2)$ and also studied the following second-order difference equation

$$\omega_{n+1}=\mu+\eta\frac{\omega_{n-1}^{\rho}}{h(\omega_n,\omega_{n-1})}$$

where ρ is a positive real number, the parameters μ,η are arbitrary real numbers, the initial values ω_{-1}, ω_0 arbitrary real numbers and h is a continuous homogeneous function with degree ρ. Finally, they obtained the periodicity results of the closed-form difference equations

$$\omega_{n+1}=\zeta(\omega_n,\omega_{n-1})$$

and

$$\omega_{n+1}=\zeta(\omega_n,\omega_{n-2})$$

where $\zeta\in C\left((0,\infty)^2,(0,\infty)\right)$ and the initial values $\omega_{-2},\omega_{-1},\omega_0$ are positive arbitrary real numbers.
In [28], Gümüş and Eğilmez investigated the global behavior of solutions, that is, the prime period two solutions, the prime period three solutions and the stability character of a new general class of the second-order difference equation

$$\delta_{m+1}=\omega+\zeta\frac{f(\delta_m,\delta_{m-1})}{\delta_{m-1}^{\beta}},\quad m=0,1,...$$

where the parameters $\omega,\zeta\in\mathbb{R}$, the initial conditions $\delta_{-1},\delta_0\in\mathbb{R}$ and $f:(0,\infty)^2\to(0,\infty)$ is a continuous homogeneous function with degree β.
This paper aims to investigate the global dynamics of solutions for a new general class of the second-order difference equations

$$\omega_{m+1}=\sigma+\zeta\frac{g(\omega_m,\omega_{m-1})}{\omega_m^{\gamma}},\quad m=0,1,\ldots \tag{1.2}$$

$$\omega_{m+1}=\sigma+\zeta\frac{\omega_m^{\gamma}}{g(\omega_m,\omega_{m-1})},\quad m=0,1,\ldots \tag{1.3}$$

where the parameters σ,ζ are arbitrary real numbers, the initial conditions ω_{-1},ω_0 are arbitrary real numbers and $g:(0,\infty)^2\to(0,\infty)$ is a continuous homogeneous function with degree γ. In other words, the prime period two solutions, the prime period three solutions and the stability character are discussed in detail. Also, periodic solutions are studied using a new technique. In addition, stability analysis of the equilibrium point is performed and new sufficient conditions for stability character are specified.
In the following, we will give a very useful theorem to examine the stability character of the solutions of difference equations, which we will benefit from in this paper.

Theorem 1.1. *[19] (Clark Theorem) Assume that $a_0,a_1\in\mathbb{R}$ and $k\in\{0,1,\ldots\}$. Then, the difference equation*

$$\gamma_{m+1}+a_0\gamma_m+a_1\gamma_{m-k}=0,\quad m=0,1,\ldots.$$

is the asymptotic stability if

$$|a_0|+|a_1|<1.$$

2. The behavior of solutions of the difference equation $\omega_{m+1}=\sigma+\zeta\frac{g(\omega_m,\omega_{m-1})}{\omega_m^{\gamma}}$

This section is devoted to investigating the dynamical behavior of solutions, that is, the two periodic solutions, the three periodic solutions and the local stability of second-order rational difference equation (1.2).
Here, we can easily find the positive equilibrium point of Eq.(1.2) as

$$\bar{\omega}=\sigma+\zeta g(1,1).$$

Now, let's define the function $f:(0,\infty)^2\to(0,\infty)$ by

$$f(u,v)=\sigma+\zeta\frac{g(u,v)}{u^{\gamma}}.$$

Hence, we get the partial derivatives of the function f

$$\frac{\partial f}{\partial u}(u,v)=\zeta\frac{ug_u(u,v)-\gamma g(u,v)}{u^{\gamma+1}}$$

and

$$\frac{\partial f}{\partial v}(u,v)=\zeta\frac{g_v(u,v)}{u^{\gamma}}.$$

In the next theorem, the locally asymptotic stability of Eq.(1.2) will be examined.

Theorem 2.1. *The equilibrium point of Eq.(1.2) $\bar{\omega}=\sigma+\zeta g(1,1)$ is locally asymptotically stable if*

$$|g_u(1,1)-\gamma g(1,1)|+|g_v(1,1)|<\left|\frac{\sigma+\zeta g(1,1)}{\zeta}\right|. \tag{2.1}$$

Proof. By using the Euler's Homogeneous Function Theorem, we obtain that

$$\begin{aligned} f_u(\bar{\omega},\bar{\omega}) &= \zeta\frac{\bar{\omega}g_u(\bar{\omega},\bar{\omega})-\gamma g(\bar{\omega},\bar{\omega})}{\bar{\omega}^{\gamma+1}} \\ &= \zeta\frac{\bar{\omega}^{\gamma}g_u(1,1)-\gamma\bar{\omega}^{\gamma}g(1,1)}{\bar{\omega}^{\gamma+1}} \\ &= \zeta\frac{g_u(1,1)-\gamma g(1,1)}{\bar{\omega}}, \end{aligned}$$

and

$$\begin{aligned} f_v(\bar{\omega},\bar{\omega}) &= \zeta\frac{g_v(\bar{\omega},\bar{\omega})}{\bar{\omega}^{\gamma}} \\ &= \zeta\frac{\bar{\omega}^{\gamma-1}g_v(1,1)}{\bar{\omega}^{\gamma}} \\ &= \zeta\frac{g_v(1,1)}{\bar{\omega}}. \end{aligned}$$

Now, by applying Clark Theorem, we find

$$\left|\zeta\frac{g_u(1,1)-\gamma g(1,1)}{\bar{\omega}}\right|+\left|\zeta\frac{g_v(1,1)}{\bar{\omega}}\right|<1.$$

Since $\bar{\omega}=\sigma+\zeta g(1,1)$, we find

$$\left|\zeta\frac{g_u(1,1)-\gamma g(1,1)}{(\sigma+\zeta g(1,1))}\right|+\left|\zeta\frac{g_v(1,1)}{(\sigma+\zeta g(1,1))}\right|<1,$$

and so

$$|g_u(1,1)-\gamma g(1,1)|+|g_v(1,1)|<\left|\frac{\sigma+\zeta g(1,1)}{\zeta}\right|.$$

The proof is completed. □

In the next theorem, the two periodic solutions of Eq.(1.2) will be examined.

Theorem 2.2. *Eq.(1.2) has the prime period two solution*

$$\ldots,\phi,\vartheta,\phi,\vartheta,\ldots.$$

if and only if

$$\sigma=\zeta\frac{\Omega g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)} \tag{2.2}$$

where $\Omega=\frac{\phi}{\vartheta}$, $\Omega\in\mathbb{R}-\{0,\pm1\}$.

Proof. Suppose that Eq.(1.2) has a prime period two solution in the following form

$$\ldots,\phi,\vartheta,\phi,\vartheta,\ldots.$$

Let's define $\omega_{n-(2s+1)}=\phi$ and $\omega_{n-2s}=\vartheta$ for $s=0,1,2,\ldots$. From Eq.(1.2), we obtain

$$\phi=\sigma+\zeta\frac{g(\vartheta,\phi)}{\vartheta^{\gamma}},$$

and

$$\vartheta=\sigma+\zeta\frac{g(\phi,\vartheta)}{\phi^{\gamma}}.$$

Since g is a continuous homogeneous function of degree γ, we obtain

$$\phi=\sigma+\zeta\frac{\phi^{\gamma}g\left(\frac{\vartheta}{\phi},1\right)}{\vartheta^{\gamma}}\Rightarrow\phi=\sigma+\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right), \tag{2.3}$$

and

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}g\left(1,\frac{\vartheta}{\phi}\right)}{\phi^{\gamma}}\Rightarrow\vartheta=\sigma+\zeta g\left(1,\frac{1}{\Omega}\right). \tag{2.4}$$

By using the fact $\phi-\Omega\vartheta=0$, we find

$$0=\phi-\Omega\vartheta=\sigma+\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)-\Omega\left(\sigma+\zeta g\left(1,\frac{1}{\Omega}\right)\right),$$

and so

$$\sigma(1-\Omega)=\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right).$$

Therefore, we get

$$\sigma=\zeta\frac{\Omega g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}.$$

Thus, from (2.3) and (2.4) respectively, we find

$$\begin{aligned} \phi &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right) \\ &= \zeta\frac{\Omega g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma+1}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \vartheta &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta g\left(1,\frac{1}{\Omega}\right) \\ &= \zeta\frac{g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}. \end{aligned} \tag{2.6}$$

Secondly, assume (2.2) holds. Let's choose the initial conditions

$$\omega_{-1}=\phi \text{ and } \omega_0=\vartheta,$$

where ϕ,ϑ are defined as (2.3) and (2.4), respectively. Hence, we obtain that

$$\begin{aligned} \omega_1 &= \sigma+\zeta\frac{g(\omega_0,\omega_{-1})}{\omega_0^{\gamma}} \\ &= \sigma+\zeta\frac{g(\vartheta,\phi)}{\vartheta^{\gamma}} \\ &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta\frac{\phi^{\gamma}g\left(\frac{\vartheta}{\phi},1\right)}{\vartheta^{\gamma}} \\ &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right) \\ &= \zeta\frac{\Omega g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma+1}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}=\phi, \end{aligned}$$

and

$$\begin{aligned} \omega_2 &= \sigma+\zeta\frac{g(\omega_1,\omega_0)}{\omega_1^{\gamma}} \\ &= \sigma+\zeta\frac{g(\phi,\vartheta)}{\phi^{\gamma}} \\ &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta\frac{\phi^{\gamma}g\left(1,\frac{\vartheta}{\phi}\right)}{\phi^{\gamma}} \\ &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta g\left(1,\frac{1}{\Omega}\right) \\ &= \zeta\frac{g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}=\vartheta. \end{aligned}$$

Then, by induction, we can obtain that for all $n\geq 0$

$$\omega_{2n-1}=\phi \text{ and } \omega_{2n}=\vartheta.$$

Hence, Eq.(1.2) has a prime period two solution. The proof is completed. □

In the following theorem, the prime period three solution of Eq.(1.2) will be investigated.

Theorem 2.3. *Eq.(1.2) has the prime period three solution* $\{\omega_n\}_{n=-1}^{\infty}$ *where*

$$\omega_n=\begin{cases}\phi, & \text{for } n=3z-1\\ \vartheta, & \text{for } n=3z\\ \nu, & \text{for } n=3z+1\end{cases}, \quad z=0,1,\ldots$$

if and only if

$$\begin{aligned} \eta\left(\sigma+\frac{\zeta}{\psi^{\gamma}}g(\psi,\eta)\right) &= \sigma+\zeta g(1,\psi) \\ \psi\left(\sigma+\frac{\zeta}{\psi^{\gamma}}g(\psi,\eta)\right) &= \sigma+\frac{\zeta}{\eta^{\gamma}}g(1,\psi) \end{aligned} \tag{2.7}$$

where $\eta=\frac{\vartheta}{\phi}$ *and* $\psi=\frac{\nu}{\phi}, \eta,\psi\in\mathbb{R}-\{0,\mp 1\}$.

Proof. Suppose that Eq.(1.2) has a prime period three solution in the following form

$$\ldots,\phi,\vartheta,\nu,\phi,\vartheta,\nu,\ldots.$$

From Eq.(1.2), we obtain that

$$\phi=\sigma+\zeta\frac{g(\nu,\vartheta)}{\nu^{\gamma}},$$

$$\vartheta=\sigma+\zeta\frac{g(\phi,\nu)}{\phi^{\gamma}},$$

and

$$\nu=\sigma+\zeta\frac{g(\vartheta,\phi)}{\vartheta^{\gamma}}.$$

By using the homogeneous function definition, we can find the equalities

$$\phi=\sigma+\zeta\frac{\phi^{\gamma}g(\psi,\eta)}{\nu^{\gamma}}\Rightarrow\phi=\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}$$

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}g(1,\psi)}{\phi^{\gamma}}\Rightarrow\vartheta=\sigma+\zeta g(1,\psi)$$

and

$$\nu=\sigma+\zeta\frac{\phi^{\gamma}g(\eta,1)}{\vartheta^{\gamma}}\Rightarrow\nu=\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}.$$

Therefore, we can easily see that

$$\eta=\frac{\vartheta}{\phi}=\frac{\sigma+\zeta g(1,\psi)}{\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}}$$

and

$$\psi=\frac{\nu}{\phi}=\frac{\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}}{\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}}.$$

Thus, we can rewrite the equalities

$$\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)=\sigma+\zeta g(1,\psi),$$

$$\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)=\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}.$$

Secondly, assume (2.7) holds. Let's choose the initial conditions for all $\eta,\psi\in\mathbb{R}-\{0,\mp 1\}$

$$\omega_{-1}=\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}$$

and

$$\omega_0=\sigma+\zeta g(1,\psi).$$

Thus, we obtain that

$$\begin{aligned}
\omega_1 &= \sigma+\zeta\frac{g(\omega_0,\omega_{-1})}{\omega_0^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\sigma+\zeta g(1,\psi),\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)}{(\sigma+\zeta g(1,\psi))^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right),\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)}{\left(\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)^{\gamma}} \\
&= \sigma+\zeta\frac{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}g(\eta,1)}{\left(\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)^{\gamma}} \\
&= \sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}=\nu,
\end{aligned}$$

$$\begin{aligned}
\omega_2 &= \sigma+\zeta\frac{g(\omega_1,\omega_0)}{\omega_1^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}},\sigma+\zeta g(1,\psi)\right)}{\left(\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}\right)^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right),\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)}{\left(\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)^{\gamma}} \\
&= \sigma+\zeta\frac{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}g(\psi,\eta)}{\left(\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)^{\gamma}} \\
&= \sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}=\phi,
\end{aligned}$$

and

$$\begin{aligned}
\omega_3 &= \sigma+\zeta\frac{g(\omega_2,\omega_1)}{\omega_2^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}},\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}\right)}{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}},\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)}{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}} \\
&= \sigma+\zeta\frac{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}g(1,\psi)}{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}} \\
&= \sigma+\zeta g(1,\psi)=\vartheta.
\end{aligned}$$

Then, by induction, we can obtain that for all $n\geq 0$.

$$\omega_{3n-1}=\phi,\ \omega_{3n}=\vartheta \text{ and } \omega_{3n+1}=\nu.$$

Hence, Eq.(1.2) has a prime period three solution. The proof is completed. □

3. The behavior of solutions of the difference equation $\omega_{m+1}=\sigma+\zeta\frac{\omega_m^{\gamma}}{g(\omega_m,\omega_{m-1})}$

This section is devoted to examining the asymptotic behavior of the solutions of non-linear rational difference equation (1.3). Here, we can easily obtain the positive equilibrium point of Eq.(1.3) as

$$\bar{\omega}=\sigma+\frac{\zeta}{g(1,1)}.$$

Now, let's define the function $z:(0,\infty)^2 \to (0,\infty)$ as

$$z(u,v)=\sigma+\zeta\frac{u^\gamma}{g(u,v)}.$$

Therefore, we find

$$\frac{\partial z}{\partial u}(u,v)=\zeta\frac{\gamma u^{\gamma-1}g(u,v)-g_u(u,v)u^\gamma}{(g(u,v))^2}$$

and

$$\frac{\partial z}{\partial v}(u,v)=-\zeta\frac{g_v(u,v)u^\gamma}{(g(u,v))^2}.$$

In the next theorem, the locally asymptotic stability for Eq.(1.3) will be examined.

Theorem 3.1. *The equilibrium point of Eq.(1.3) $\bar{\omega}=\sigma+\frac{\zeta}{g(1,1)}$ is locally asymptotically stable if*

$$|\gamma g(1,1)-g_u(1,1)|+|g_v(1,1)|<\left|\frac{\left(\sigma+\frac{\zeta}{g(1,1)}\right)g^2(1,1)}{\zeta}\right|.$$

Proof. Since g is a homogeneous function with degree γ, the partial derivatives are of degree $\gamma-1$. Thus, we obtain that

$$\begin{aligned} z_u(\bar{\omega},\bar{\omega}) &= \zeta\frac{\gamma\bar{\omega}^{\gamma-1}g(\bar{\omega},\bar{\omega})-g_u(\bar{\omega},\bar{\omega})\bar{\omega}^\gamma}{(g(\bar{\omega},\bar{\omega}))^2} \\ &= \zeta\frac{\gamma\bar{\omega}^{2\gamma-1}g(1,1)-g_u(1,1)\bar{\omega}^{2\gamma-1}}{(\bar{\omega}^\gamma g(1,1))^2} \\ &= \zeta\frac{\gamma g(1,1)-g_u(1,1)}{\bar{\omega}g^2(1,1)}, \end{aligned}$$

and

$$\begin{aligned} z_v(\bar{\omega},\bar{\omega}) &= -\zeta\frac{g_v(\bar{\omega},\bar{\omega})\bar{\omega}^\gamma}{(g(\bar{\omega},\bar{\omega}))^2} \\ &= -\zeta\frac{g_v(1,1)\bar{\omega}^{2\gamma-1}}{\bar{\omega}^{2\gamma}g^2(1,1)} \\ &= -\zeta\frac{g_v(1,1)}{\bar{\omega}g^2(1,1)}. \end{aligned}$$

Now, by using Clark Theorem, we obtain

$$\left|\zeta\frac{\gamma g(1,1)-g_u(1,1)}{\bar{\omega}g^2(1,1)}\right|+\left|\zeta\frac{g_v(1,1)}{\bar{\omega}g^2(1,1)}\right|<1.$$

Since the equilibrium point $\bar{\omega}=\sigma+\zeta\frac{1}{g(1,1)}$, we find

$$\left|\zeta\frac{\gamma g(1,1)-g_u(1,1)}{\left(\sigma+\frac{\zeta}{g(1,1)}\right)g^2(1,1)}\right|+\left|\zeta\frac{g_v(1,1)}{\left(\sigma+\frac{\zeta}{g(1,1)}\right)g^2(1,1)}\right|<1,$$

and so,

$$|\gamma g(1,1)-g_u(1,1)|+|g_v(1,1)|<\left|\frac{\left(\sigma+\zeta\frac{1}{g(1,1)}\right)g^2(1,1)}{\zeta}\right|.$$

This completes the proof. □

In the next theorem, the prime period two solutions of Eq.(1.3) will be investigated.

Theorem 3.2. *Eq.(1.3) has the prime period two solution*

$$\dots,\phi,\vartheta,\phi,\vartheta,\dots$$

if and only if

$$\sigma=\frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right) \tag{3.1}$$

where $\Omega=\frac{\phi}{\vartheta}$, $\Omega\in\mathbb{R}-\{0,\pm1\}$.

Proof. Suppose that Eq.(1.3) has a prime period two solution in the following form

$$\dots,\phi,\vartheta,\phi,\vartheta,\dots.$$

Let's define $\omega_{n-(2s+1)}=\phi$ and $\omega_{n-2s}=\vartheta$ for $s=0,1,2,\dots$. From Eq.(1.3), we find

$$\phi=\sigma+\zeta\frac{\vartheta^{\gamma}}{g(\vartheta,\phi)},$$

and

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}}{g(\phi,\vartheta)}.$$

From the definition of the homogeneous function, we can easily obtain that

$$\phi=\sigma+\zeta\frac{\vartheta^{\gamma}}{\phi^{\gamma}g\left(\frac{\vartheta}{\phi},1\right)}\Rightarrow\phi=\sigma+\frac{\zeta}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)} \tag{3.2}$$

and

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}}{\phi^{\gamma}g\left(1,\frac{\vartheta}{\phi}\right)}\Rightarrow\vartheta=\sigma+\frac{\zeta}{g\left(1,\frac{1}{\Omega}\right)}. \tag{3.3}$$

Now, by using the fact $\phi-\Omega\vartheta=0$, we find

$$0=\phi-\Omega\vartheta=\sigma+\frac{\zeta}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}-\Omega\left(\sigma+\frac{\zeta}{g\left(1,\frac{1}{\Omega}\right)}\right)$$

and so,

$$\sigma(1-\Omega)=\zeta\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right).$$

Hence, we find

$$\sigma=\frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right).$$

Then, from Eq.(3.2) and (3.3), we obtain

$$\begin{aligned}\phi &= \sigma+\frac{\zeta}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)} \qquad (3.4)\\ &= \frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)+\frac{\zeta}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\\ &= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}+\frac{(1-\Omega)}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)\\ &= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{\Omega}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right),\end{aligned}$$

and

$$\begin{aligned}\vartheta &= \sigma+\zeta\frac{1}{g\left(1,\frac{1}{\Omega}\right)} \\ &= \frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)+\zeta\frac{1}{g\left(1,\frac{1}{\Omega}\right)} \\ &= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}+\frac{(1-\Omega)}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}\right) \\ &= \zeta\left(\frac{1}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right).\end{aligned} \tag{3.5}$$

On the other hand, suppose (3.1) holds. Let's choose the initial conditions

$$\omega_{-1}=\phi \text{ and } \omega_0=\vartheta,$$

where ϕ,ϑ are defined as (3.2) and (3.3), respectively. Therefore, we find

$$\begin{aligned}\omega_1 &= \sigma+\zeta\frac{\omega_0^{\gamma}}{g(\omega_0,\omega_{-1})} \\ &= \sigma+\zeta\frac{\vartheta^{\gamma}}{g(\vartheta,\phi)} \\ &= \frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)+\zeta\frac{1}{\Omega^{\gamma}g\left(\frac{\vartheta}{\phi},1\right)} \\ &= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}+\frac{(1-\Omega)}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right) \\ &= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{\Omega}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)=\phi\end{aligned}$$

and

$$\begin{aligned}\omega_2 &= \sigma+\zeta\frac{\omega_1^{\gamma}}{g(\omega_1,\omega_0)} \\ &= \sigma+\zeta\frac{\phi^{\gamma}}{g(\phi,\vartheta)} \\ &= \frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)+\zeta\frac{1}{g\left(1,\frac{1}{\Omega}\right)} \\ &= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}+\frac{(1-\Omega)}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}\right) \\ &= \zeta\left(\frac{1}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)=\vartheta\end{aligned}$$

Then, by induction, we can obtain that for all $n\geq 0$

$$\omega_{2n-1}=\phi \text{ and } \omega_{2n}=\vartheta.$$

Hence, Eq.(1.3) has a prime period two solution. The proof is completed. □

In the following theorem, the three periodic solutions of Eq.(1.3) will be studied.

Theorem 3.3. *Eq.(1.3) has a prime period three solution $\{\omega_n\}_{n=-1}^{\infty}$ where*

$$\omega_n=\begin{cases}\phi, & \textit{for } n=3z-1\\ \vartheta, & \textit{for } n=3z\\ \nu, & \textit{for } n=3z+1\end{cases}, \quad z=0,1,\ldots$$

if and only if

$$\begin{aligned} \eta\left(\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}\right) &= \sigma+\zeta\frac{1}{g(1,\psi)} \\ \psi\left(\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}\right) &= \sigma+\zeta\frac{\eta^{\gamma}}{g(\eta,1)} \end{aligned} \tag{3.6}$$

where $\eta=\frac{\vartheta}{\phi}$ *and* $\psi=\frac{\nu}{\phi},\eta,\psi\in\mathbb{R}-\{0,\pm1\}$.

Proof. Suppose that Eq.(1.3) has a prime period three solution in the following form

$$\ldots,\phi,\vartheta,\nu,\phi,\vartheta,\nu,\ldots$$

From Eq.(1.3), we obtain that

$$\phi=\sigma+\zeta\frac{\nu^{\gamma}}{g(\nu,\vartheta)},$$

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}}{g(\phi,\nu)}$$

and

$$\nu=\sigma+\zeta\frac{\vartheta^{\gamma}}{g(\vartheta,\phi)}.$$

Since g is a homogeneous function with degree γ, we obtain the equalities

$$\phi=\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)},$$

$$\vartheta=\sigma+\zeta\frac{1}{g(1,\psi)}$$

and

$$\nu=\sigma+\zeta\frac{\eta^{\gamma}}{g(\eta,1)}.$$

Hence, we find

$$\eta=\frac{\vartheta}{\phi}=\frac{\sigma+\zeta\frac{1}{g(1,\psi)}}{\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}}$$

and

$$\psi=\frac{\nu}{\phi}=\frac{\sigma+\zeta\frac{\eta^{\gamma}}{g(\eta,1)}}{\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}}.$$

Thus, we obtain that

$$\eta\left(\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}\right)=\sigma+\zeta\frac{1}{g(1,\psi)},$$

$$\psi\left(\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}\right)=\sigma+\zeta\frac{\eta^{\gamma}}{g(\eta,1)}.$$

Now, assume (3.6) holds. Let's choose the initial values for all $\eta,\psi\in\mathbb{R}-\{0,\pm1\}$

$$\omega_{-1}=\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}$$

and

$$\omega_0 = \sigma + \frac{\zeta}{g(1,\psi)}.$$

Therefore, we obtain

$$\begin{aligned}
\omega_1 &= \sigma + \zeta \frac{\omega_0^\gamma}{g(\omega_0, \omega_{-1})} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{1}{g(1,\psi)}\right)^\gamma}{g\left(\sigma + \zeta \frac{1}{g(1,\psi)}, \sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)} \\
&= \sigma + \zeta \frac{\left(\eta \left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)^\gamma}{g\left(\left(\eta \left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right), \sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)} \\
&= \sigma + \zeta \frac{\left(\eta \left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)^\gamma}{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma g(\eta, 1)} \\
&= \sigma + \zeta \frac{\eta^\gamma}{g(\eta, 1)} = \nu,
\end{aligned}$$

$$\begin{aligned}
\omega_2 &= \sigma + \zeta \frac{\omega_1^\gamma}{g(\omega_1, \omega_0)} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{\eta^\gamma}{g(\eta,1)}\right)^\gamma}{g\left(\sigma + \zeta \frac{\eta^\gamma}{g(\eta,1)}, \sigma + \zeta \frac{1}{g(1,\psi)}\right)} \\
&= \sigma + \zeta \frac{\left(\psi \left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)^\gamma}{g\left(\psi \left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right), \eta \left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)} \\
&= \sigma + \zeta \frac{\left(\psi \left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)^\gamma}{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma g(\psi, \eta)} \\
&= \sigma + \zeta \frac{\psi^\gamma}{g(\psi, \eta)} = \phi
\end{aligned}$$

and

$$\begin{aligned}
\omega_3 &= \sigma + \zeta \frac{\omega_2^\gamma}{g(\omega_2, \omega_1)} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma}{g\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}, \sigma + \zeta \frac{\eta^\gamma}{g(\eta,1)}\right)} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma}{g\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}, \psi \left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma}{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma g(1, \psi)} \\
&= \sigma + \zeta \frac{1}{g(1, \psi)} = \vartheta.
\end{aligned}$$

Then, by induction, we obtain for all $n \geq 0$

$$\omega_{3n-1} = \phi, \ \omega_{3n} = \vartheta \text{ and } \omega_{3n+1} = \nu.$$

Hence, Eq.(1.3) has a prime period three solution. The proof is completed. □

4. Conclusions and suggestions

In this article, we have considered the detailed qualitative behavior of a general class of difference equations, which can be seen as an extension of [22, 23, 24, 25, 26, 27]. The qualitative behavior of the solutions of the introduced non-linear difference equations has been examined. In other words, the two periodic solutions, the three periodic solutions and the stability character of difference equations have been discussed. Qualitative research of mathematical models created using difference equations has an important place in mathematics, sub-branches of mathematics and other applied sciences. Here, the two periodic solutions of Eq.(1.2) and Eq.(1.3) in Theorem 3.2 and Theorem 2.2 and the three periodic solutions in Theorem 3.3 and Theorem 2.3 have been examined in detail. In these theorems, using the new technique, the periodicity character of Eq.(1.2) and Eq.(1.3) have been determined and necessary and sufficient conditions have been created for the existence of periodic solutions. In addition, the equilibrium points of Eq.(1.2) and Eq.(1.3) have been investigated and sufficient conditions have been obtained for the local asymptotic stability of these equilibrium points.
It can be suggested to those who do research in this field that research can be done in the equations established with the help of homogeneous functions. Difference equations created with these functions are very convenient and useful for researching general classes of difference equations.
In our future studies, we will aim to investigate some general classes of difference equations formed by homogeneous functions of different degrees.

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Solvability of a Second-Order Rational System of Difference Equations

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Abstract

In this paper, we represent the admissible solutions of the system of second-order rational difference equations given below in terms of Lucas and Fibonacci sequences:

$$x_{n+1}=\frac{L_{m+2}+L_{m+1}y_{n-1}}{L_{m+3}+L_{m+2}y_{n-1}},\quad y_{n+1}=\frac{L_{m+2}+L_{m+1}z_{n-1}}{L_{m+3}+L_{m+2}z_{n-1}},$$
$$z_{n+1}=\frac{L_{m+2}+L_{m+1}w_{n-1}}{L_{m+3}+L_{m+2}w_{n-1}},\quad w_{n+1}=\frac{L_{m+2}+L_{m+1}x_{n-1}}{L_{m+3}+L_{m+2}x_{n-1}}.$$

where $n\in\mathbb{N}_0$, $\{L_m\}_{m=-\infty}^{+\infty}$ is Lucas sequence and the initial conditions x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0, w_{-1}, w_0 are arbitrary real numbers such that $v_{-i}\neq-\frac{L_{m+3}}{L_{m+2}}$, where $v_{-i}=x_{-i},y_{-i},z_{-i},w_{-i}$, $i=0,1$ and $m\in\mathbb{Z}$.

1. Introduction and preliminaries

Recently, there has been a growing interest in the study of finding closed-form solutions of difference equations and systems of difference equations. Some of the forms of solutions of these equations are representable via well-known integer sequences such as Fibonacci numbers [1, 2], Horadam numbers [3], Lucas numbers [4, 5], and Padovan numbers [6]. For more on Fibonacci and Lucas numbers, one can see [7, 8], for more on difference equations and systems of difference equations solvable in closed form, one can see [9]-[24].

The Lucas sequence has the same recursive relationship as the Fibonacci sequence, where each term is the sum of the two previous terms, and defined as follows:

$$L_{n+1}=L_n+L_{n-1},\quad n\geq 1, \tag{1.1}$$

but with different initial values , $L_0=2$, $L_1=1$. The solution of Equation (1.1) is given by the formula

$$L_n=\alpha^n+\beta^n,$$

where

$$\alpha=\frac{1+\sqrt{5}}{2}\quad \textit{and}\quad \beta=\frac{1-\sqrt{5}}{2}.$$

The formula of terms with negative indices in the Lucas sequence is

$$L_{-n}=(-1)^nL_n.$$

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In [25], the authors represented the general solution of the following difference equation

$$x_{n+1} = \frac{1}{1+x_n}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

in terms of the initial value x_0 and Fibonacci sequence. It was proved by induction that, every well-defined solution of equation (1.2) can be written in the following form

$$x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0}, \quad n \in \mathbb{N}_0,$$

where $\{F_n\}_{n=0}^{\infty}$ is Fibonacci sequence. They also proved that, every well-defined solution of the equation

$$x_{n+1} = \frac{1}{-1+x_n}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

can be written in the following form

$$x_n = \frac{F_{-n} + F_{-(n-1)}x_0}{F_{-(n+1)} + F_{-n}x_0}, \quad n \in \mathbb{N}_0,$$

where the terms of the Fibonacci sequence with negative indices are calculated by the formula

$$F_{-n} = F_{-n+2} - F_{-n+1}, \quad n \in \mathbb{N}_0,$$

where $F_0 = 0$ and $F_1 = 1$.

Khelifa et al. [5] give some theoretical explanations related to the representation of the general solution to the system of three higher-order rational difference equations

$$x_{n+1} = \frac{1+2y_{n-k}}{3+y_{n-k}}, \quad y_{n+1} = \frac{1+2z_{n-k}}{3+z_{n-k}}, \quad z_{n+1} = \frac{1+2x_{n-k}}{3+x_{n-k}},$$

where $n,k \in \mathbb{N}_0$, giving its solution in terms of Fibonacci and Lucas sequences.

Recently in Khelifa et al. [4], the following higher-order rational difference equations

$$x_{n+1}^{(1)} = \frac{1+2x_{n-k}^{(2)}}{3+x_{n-k}^{(2)}}, \; x_{n+1}^{(2)} = \frac{1+2x_{n-k}^{(3)}}{3+x_{n-k}^{(3)}}, \cdots, \; x_{n+1}^{(2p+1)} = \frac{1+2x_{n-k}^{(1)}}{3+x_{n-k}^{(1)}},$$

in terms of Fibonacci and Lucas sequences, where the initial values $x_{-k}^{(i)}$, $x_{-k+1}^{(i)}, \cdots, x_{-1}^{(i)}$ and $x_0^{(i)}$, $i = 1,2,\cdots,2p+1$ are real numbers such that, the denominator does not equal zero in each equation. Some theoretical explanations related to the representation of the general solution are also given.

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \; n = 0,1,\cdots. \tag{1.4}$$

The Good set to Equation (1.4) is the set of all initial points $(x_0, x_{-1}, ..., x_{-k})$ for which the corresponding solution $\{x_n\}_{n=-k}^{\infty}$ is well-defined or admissible solution.

Here, we list a set of identities concerning the Fibonacci and Lucas sequences that may be used in the paper [7, 8].

For $s, m, r, \theta \in \mathbb{N}$, we have

1. $F_m = F_{s+1}F_{m-s} + F_sF_{m-(s+1)}$,
2. $L_m = F_{s+1}L_{m-s} + F_sL_{m-(s+1)}$,
3. $F_sL_{m+3} + F_{s-1}L_{m+2} = L_{s+m+2}$,
4. $L_rL_{(\theta-1)r+1} + L_{r-1}L_{(\theta-1)r} = 5F_{\theta r}$,
5. $L_{r+1}L_{(\theta-1)r} + L_rL_{(\theta-1)r-1} = 5F_{\theta r}$,
6. $L_{r+1}L_{(\theta-1)r+1} + L_rL_{(\theta-1)r} = 5F_{\theta r+1}$,
7. $L_rL_{(\theta-1)r} + L_{r-1}L_{(\theta-1)r-1} = 5F_{\theta r-1}$,
8. $L_{\theta(m+2)-1} + L_{\theta(m+2)+1} = 5F_{\theta(m+2)}$.

Now, consider the system of second-order rational difference equations

$$\begin{aligned} x_{n+1} &= \frac{L_{m+2} + L_{m+1}y_{n-1}}{L_{m+3} + L_{m+2}y_{n-1}}, \quad y_{n+1} = \frac{L_{m+2} + L_{m+1}z_{n-1}}{L_{m+3} + L_{m+2}z_{n-1}}, \\ z_{n+1} &= \frac{L_{m+2} + L_{m+1}w_{n-1}}{L_{m+3} + L_{m+2}w_{n-1}}, \quad w_{n+1} = \frac{L_{m+2} + L_{m+1}x_{n-1}}{L_{m+3} + L_{m+2}x_{n-1}}, \end{aligned} \tag{1.5}$$

where $\{L_m\}_{m=-\infty}^{+\infty}$ is Lucas sequence and the initial conditions x_{-1}, x_0, y_{-1}, y_0, z_{-1},z_0 and w_{-1},w_0 are arbitrary real numbers such that $v_{-i} \neq -\frac{L_{m+3}}{L_{m+2}}$, where $v_{-i} = x_{-i}, y_{-i}, z_{-i}, w_{-i}$, $i = 0, 1$ and $m \in \mathbb{Z}$.

In this paper, we shall represent the admissible solutions of the system (1.5) in terms of Fibonacci and Lucas sequences.

2. Solvability of system (1.5)

In this section, we investigate the solvability of the system (1.5).

From (1.5), we can write for $t = 0, 1$

$$x_{2(n+1)-t} = \frac{L_{m+2} + L_{m+1}y_{2n-t}}{L_{m+3} + L_{m+2}y_{2n-t}}, \quad y_{2(n+1)-t} = \frac{L_{m+2} + L_{m+1}z_{2n-t}}{L_{m+3} + L_{m+2}z_{2n-t}},$$
$$z_{2(n+1)-t} = \frac{L_{m+2} + L_{m+1}w_{2n-t}}{L_{m+3} + L_{m+2}w_{2n-t}}. \quad w_{2(n+1)-t} = \frac{L_{m+2} + L_{m+1}x_{2n-t}}{L_{m+3} + L_{m+2}x_{2n-t}}.$$

Let

$$x'_n = x_{2n-t},\ y'_n = y_{2n-t},\ z'_n = z_{2n-t},\ w'_n = w_{2n-t}, \tag{2.1}$$

where $t = 0, 1$.

Then, the system (1.5) becomes

$$\begin{aligned} x'_{n+1} &= \frac{L_{m+2} + L_{m+1}y'_n}{L_{m+3} + L_{m+2}y'_n}, \quad y'_{n+1} = \frac{L_{m+2} + L_{m+1}z'_n}{L_{m+3} + L_{m+2}z'_n}, \\ z'_{n+1} &= \frac{L_{m+2} + L_{m+1}w'_n}{L_{m+3} + L_{m+2}w'_n}, \quad w'_{n+1} = \frac{L_{m+2} + L_{m+1}x'_n}{L_{m+3} + L_{m+2}x'_n}. \end{aligned} \tag{2.2}$$

If we use the second recurrence relation in (2.2) in the first, we obtain

$$x'_{n+1} = \frac{F_{2m+4} + F_{2m+3}z'_{n-1}}{F_{2m+5} + F_{2m+4}z'_{n-1}}, \quad n \geq 1.$$

The substitution of z'_{n-1} into x'_{n+1}, leads to

$$x'_{n+1} = \frac{L_{3m+6} + L_{3m+5}w'_{n-2}}{L_{3m+7} + L_{3m+6}w'_{n-2}}, \quad n \geq 2.$$

Finally, after substituting with w'_{n-2} into x'_{n+1}, we get

$$x'_{n+1} = \frac{F_{4m+8} + F_{4m+7}x'_{n-3}}{F_{4m+9} + F_{4m+8}x'_{n-3}}, \quad n \geq 3.$$

Therefore, the system (2.2) can be written in the following form:

$$x'_{n+1} = \frac{F_{4m+8} + F_{4m+7}x'_{n-3}}{F_{4m+9} + F_{4m+8}x'_{n-3}}, \quad n \geq 3. \tag{2.3}$$

Let us introduce the notation

$$x_n^{\prime(j)} = x'_{4n+j}, \quad n \in \mathbb{N}_0, \tag{2.4}$$

where $j \in \{0, 1, 2, 3\}$.

Using this notation, Equation (2.3) can be written as

$$x_{n+1}^{\prime(j)} = \frac{F_{4m+8} + F_{4m+7}x_n^{\prime(j)}}{F_{4m+9} + F_{4m+8}x_n^{\prime(j)}}, \quad j \in \{0, 1, 2, 3\} \text{ and } n \geq 3. \tag{2.5}$$

Now consider the equation

$$u_{n+1} = \frac{F_{4m+8} + F_{4m+7}u_n}{F_{4m+9} + F_{4m+8}u_n}, \quad n \geq 3. \tag{2.6}$$

The solution of Equation (2.6) is (can be found in [20])

$$u_n=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}u_0}{F_{4n(m+2)+1}+F_{4n(m+2)}u_0},\ n\in\mathbb{N}_0,$$

where $(F_m)_{n=-\infty}^{+\infty}$ is Fibonacci sequence.

Then the solution of Equation (2.5) is given by

$$x_n'^{(j)}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x_0'^{(j)}}{F_{4n(m+2)+1}+F_{4n(m+2)}x_0'^{(j)}},\quad j\in\{0,1,2,3\}\text{ and }n\in\mathbb{N}_0.$$

Therefore, the solution of Equation (2.5) can be written as

$$x'_{4n+j}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_j}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_j},\quad j\in\{0,1,2,3\}\text{ and }n\in\mathbb{N}_0.$$

Theorem 2.1. *Let $(x'_n,y'_n,z'_n,w'_n)_{n\geq 0}$ be an admissible solution of the system (2.2). Then we get*

$$\begin{aligned}
x'_{4n}&=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_0}, & z'_{4n}&=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}z'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}z'_0},\\
x'_{4n+1}&=\frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}y'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}y'_0}, & z'_{4n+1}&=\frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}w'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}w'_0},\\
x'_{4n+2}&=\frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}z'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}z'_0}, & z'_{4n+2}&=\frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}x'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}x'_0},\\
x'_{4n+3}&=\frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}w'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}w'_0}, & z'_{4n+3}&=\frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}y'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}y'_0},\\
y'_{4n}&=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}y'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}y'_0}, & w'_{4n}&=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}w'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}w'_0},\\
y'_{4n+1}&=\frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}z'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}z'_0}, & w'_{4n+1}&=\frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}x'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}x'_0},\\
y'_{4n+2}&=\frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}w'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}w'_0}, & w'_{4n+2}&=\frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}y'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}y'_0},\\
y'_{4n+3}&=\frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}x'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}x'_0}, & w'_{4n+3}&=\frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}z'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}z'_0},
\end{aligned}\tag{2.7}$$

where $n\in\mathbb{N}_0$, $(L_m)_{m=-\infty}^{+\infty}$ is Lucas sequence and $(F_m)_{m=-\infty}^{+\infty}$ is Fibonacci sequence.

Proof. Let $(x'_n,y'_n,z'_n,w'_n)_{n\geq 0}$ be a solution to system (2.2). Then, $(x'_n)_{n\geq 0}$ is a solution to Equation (2.5) and so

$$x'_{4n+j}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_j}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_j},$$

where $m\in\mathbb{Z}$, $j\in\{0,1,2,3\}$. For $j=0$, we have

$$x'_{4n}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_0}.$$

We also have

$$x'_{4n+1}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_1}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_1},$$

where $x'_1=\dfrac{L_{m+2}+L_{m+1}y'_0}{L_{m+3}+L_{m+2}y'_0}$.

Using identity (2), we get

$$x'_{4n+1}=\frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}y'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}y'_0}.$$

Similarly,

$$x'_{4n+2}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_2}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_2},$$

where $x'_2=\dfrac{F_{2m+4}+F_{2m+3}z'_0}{F_{2m+5}+F_{2m+4}z'_0}$.

Using identity (1), we get

$$x'_{4n+2}=\frac{F_{4n(m+2)+2m+4}+F_{4n(m+2)+2m+3}z'_0}{F_{4n(m+2)+2m+5}+F_{4n(m+2)+2m+4}z'_0}.$$

Finally, for $j=3$, we have

$$x'_{4n+3}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_3}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_3},$$

where $x'_1=\dfrac{L_{3m+6}+L_{3m+5}w'_0}{L_{3m+7}+L_{3m+6}w'_0}$.

Again using identity (2), we get

$$x'_{4n+3}=\frac{L_{4n(m+2)+3m+6}+L_{4n(m+2)+3m+5}w'_0}{L_{4n(m+2)+3m+7}+L_{4n(m+2)+3m+6}w'_0}.$$

Then

$$\begin{aligned}
x'_{4n} &= \frac{F_{4n(m+2)}+F_{4n(m+2)-1}x'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}x'_0},\\
x'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}y'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}y'_0},\\
x'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}z'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}z'_0},\\
x'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}w'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}w'_0}.
\end{aligned}$$

In the same way, after some calculations and using the fact that

$$y'_n=\frac{L_{m+2}+L_{m+1}z'_{n-1}}{L_{m+3}+L_{m+2}z'_{n-1}},\quad z'_n=\frac{L_{m+2}+L_{m+1}w'_{n-1}}{L_{m+3}+L_{m+2}w'_{n-1}},w'_n=\frac{L_{m+2}+L_{m+1}x'_{n-1}}{L_{m+3}+L_{m+2}x'_{n-1}}.$$

we find

$$\begin{aligned}
y'_{4n} &= \frac{F_{4n(m+2)}+F_{4n(m+2)-1}y'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}y'_0}, & z'_{4n} &= \frac{F_{4n(m+2)}+F_{4n(m+2)-1}z'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}z'_0},\\
y'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}z'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}z'_0}, & z'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}w'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}w'_0},\\
y'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}w'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}w'_0}, & z'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}x'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}x'_0},\\
y'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}x'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}x'_0}, & z'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}y'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}y'_0},
\end{aligned}$$

$$\begin{aligned}
w'_{4n} &= \frac{F_{4n(m+2)}+F_{4n(m+2)-1}w'_0}{F_{4n(m+2)+1}+F_{4n(m+2)}w'_0},\\
w'_{4n+1} &= \frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}x'_0}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}x'_0},\\
w'_{4n+2} &= \frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}y'_0}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}y'_0},\\
w'_{4n+3} &= \frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}z'_0}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}z'_0}.
\end{aligned}$$

□

The following theorem is our main result that shows the solvability of the system (1.5).

Theorem 2.2. *Let* $\{x_n, y_n, z_n, w_n\}_{n\geq -1}$ *be an admissible solution of system (1.5). Then for* $n \in \mathbb{N}$*, we get*

$$x_{8n-1} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x_{-1}}{F_{4n(m+2)+1} + F_{4n(m+2)}x_{-1}}, \quad x_{8n+3} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}z_{-1}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}z_{-1}},$$
$$x_{8n} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}x_{0}}{F_{4n(m+2)+1} + F_{4n(m+2)}x_{0}}, \quad x_{8n+4} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}z_{0}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}z_{0}},$$
$$x_{8n+1} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}y_{-1}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}y_{-1}}, \quad x_{8n+5} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}w_{-1}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}w_{-1}},$$
$$x_{8n+2} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}y_{0}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}y_{0}}, \quad x_{8n+6} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}w_{0}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}w_{0}},$$

$$y_{8n-1} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}y_{-1}}{F_{4n(m+2)+1} + F_{4n(m+2)}y_{-1}}, \quad y_{8n+3} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}w_{-1}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}w_{-1}},$$
$$y_{8n} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}y_{0}}{F_{4n(m+2)+1} + F_{4n(m+2)}y_{0}}, \quad y_{8n+4} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}w_{0}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}w_{0}},$$
$$y_{8n+1} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}z_{-1}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}z_{-1}}, \quad y_{8n+5} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}x_{-1}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}x_{-1}},$$
$$y_{8n+2} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}z_{0}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}z_{0}}, \quad y_{8n+6} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}x_{0}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}x_{0}},$$

$$z_{8n-1} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}z_{-1}}{F_{4n(m+2)+1} + F_{4n(m+2)}z_{-1}}, \quad z_{8n+3} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}x_{-1}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}x_{-1}},$$
$$z_{8n} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}z_{0}}{F_{4n(m+2)+1} + F_{4n(m+2)}z_{0}}, \quad z_{8n+4} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}x_{0}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}x_{0}},$$
$$z_{8n+1} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}w_{-1}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}w_{-1}}, \quad z_{8n+5} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}y_{-1}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}y_{-1}},$$
$$z_{8n+2} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}w_{0}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}w_{0}}, \quad z_{8n+6} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}y_{0}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}y_{0}},$$

and

$$w_{8n-1} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}w_{-1}}{F_{4n(m+2)+1} + F_{4n(m+2)}w_{-1}}, \quad w_{8n+3} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}y_{-1}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}y_{-1}},$$
$$w_{8n} = \frac{F_{4n(m+2)} + F_{4n(m+2)-1}w_{0}}{F_{4n(m+2)+1} + F_{4n(m+2)}w_{0}}, \quad w_{8n+4} = \frac{F_{4n(m+2)+(2m+4)} + F_{4n(m+2)+(2m+3)}y_{0}}{F_{4n(m+2)+(2m+5)} + F_{4n(m+2)+(2m+4)}y_{0}},$$
$$w_{8n+1} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}x_{-1}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}x_{-1}}, \quad w_{8n+5} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}z_{-1}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}z_{-1}},$$
$$w_{8n+2} = \frac{L_{4n(m+2)+(m+2)} + L_{4n(m+2)+(m+1)}x_{0}}{L_{4n(m+2)+(m+3)} + L_{4n(m+2)+(m+2)}x_{0}}, \quad w_{8n+6} = \frac{L_{4n(m+2)+(3m+6)} + L_{4n(m+2)+(3m+5)}z_{0}}{L_{4n(m+2)+(3m+7)} + L_{4n(m+2)+(3m+6)}z_{0}},$$

where $(L_m)_{m=-\infty}^{+\infty}$ *is the Lucas sequence,* $(F_m)_{m=-\infty}^{+\infty}$ *is the Fibonacci sequence.*

Proof. We have

$$x'_n = x_{2n-t},\ y'_n = y_{2n-t},\ z'_n = z_{2n-t},\ w'_n = w_{2n-t},\ t = 0, 1.$$

Then for $t = 0, 1$, we have

$$x'_{4n} = x_{8n-t},\ y'_{4n} = y_{8n-t},\ z'_{4n} = z_{8n-t},\ w'_{4n} = w_{8n-t},$$

and

$$x'_0 = x_{-t},\ y'_0 = y_{-t},\ z'_0 = z_{-t},\ w'_0 = w_{-t}.$$

Using Theorem (2.1), we can write for $t=0,1$

$$x_{8n-t}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}x_{-t}}{F_{4n(m+2)+1}+F_{4n(m+2)}x_{-t}},$$
$$y_{8n-t}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}y_{-t}}{F_{4n(m+2)+1}+F_{4n(m+2)}y_{-t}},$$
$$z_{8n-t}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}z_{-t}}{F_{4n(m+2)+1}+F_{4n(m+2)}z_{-t}},$$
$$w_{8n-t}=\frac{F_{4n(m+2)}+F_{4n(m+2)-1}w_{-t}}{F_{4n(m+2)+1}+F_{4n(m+2)}w_{-t}}.$$

Also, for $t=0,1$, we have

$$x'_{4n+1}=x_{8n+2-t},\ y'_{4n+1}=y_{8n+2-t},\ z'_{4n+1}=z_{8n+2-t},\ w'_{4n+1}=w_{8n+2-t}.$$

Using Theorem (2.1), we get for $t=0,1$

$$x_{8n+2-t}=\frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}y_{-t}}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}y_{-t}},$$
$$y_{8n+2-t}=\frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}z_{-t}}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}z_{-t}},$$
$$z_{8n+2-t}=\frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}w_{-t}}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}w_{-t}},$$
$$w_{8n+2-t}=\frac{L_{4n(m+2)+(m+2)}+L_{4n(m+2)+(m+1)}x_{-t}}{L_{4n(m+2)+(m+3)}+L_{4n(m+2)+(m+2)}x_{-t}}.$$

In the same way, we get for $t=0,1$

$$x_{8n+4-t}=\frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}z_{-t}}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}z_{-t}},$$
$$y_{8n+4-t}=\frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}w_{-t}}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}w_{-t}},$$
$$z_{8n+4-t}=\frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}x_{-t}}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}x_{-t}},$$
$$w_{8n+4-t}=\frac{F_{4n(m+2)+(2m+4)}+F_{4n(m+2)+(2m+3)}y_{-t}}{F_{4n(m+2)+(2m+5)}+F_{4n(m+2)+(2m+4)}y_{-t}},$$

and

$$x_{8n+6-t}=\frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}w_{-t}}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}w_{-t}},$$
$$y_{8n+6-t}=\frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}x_{-t}}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}x_{-t}},$$
$$z_{8n+6-t}=\frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}y_{-t}}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}y_{-t}},$$
$$w_{8n+6-t}=\frac{L_{4n(m+2)+(3m+6)}+L_{4n(m+2)+(3m+5)}z_{-t}}{L_{4n(m+2)+(3m+7)}+L_{4n(m+2)+(3m+6)}z_{-t}}.$$

This completes the proof. □

3. Special cases

We end this paper by illustrating the cases $m=-1$ and $m=0$ in system (1.5).

Case $m=-1$ When $m=-1$ in system (1.5), we obtain the system of difference equations

$$x_{n+1}=\frac{1+2y_{n-1}}{3+y_{n-1}},\quad y_{n+1}=\frac{1+2z_{n-1}}{3+z_{n-1}},z_{n+1}=\frac{1+2w_{n-1}}{3+w_{n-1}},\quad w_{n+1}=\frac{1+2x_{n-1}}{3+x_{n-1}},\quad n\in\mathbb{N}_0. \tag{3.1}$$

For system (3.1), by applying Theorem (2.2) we get

$$x_{8n-1}=\frac{F_{4n}+F_{4n-1}x_{-1}}{F_{4n+1}+F_{4n}x_{-1}},\qquad x_{8n+3}=\frac{F_{4n+2}+F_{4n+1}z_{-1}}{F_{4n+3}+F_{4n+2}z_{-1}},$$
$$x_{8n}=\frac{F_{4n}+F_{4n-1}x_{0}}{F_{4n+1}+F_{4n}x_{0}},\qquad x_{8n+4}=\frac{F_{4n+2}+F_{4n+1}z_{0}}{F_{4n+3}+F_{4n+2}z_{0}},$$
$$x_{8n+1}=\frac{L_{4n+1}+L_{4n}y_{-1}}{L_{4n+2}+L_{4n+1}y_{-1}},\qquad x_{8n+5}=\frac{L_{4n+3}+L_{4n+2}w_{-1}}{L_{4n+4}+L_{4n+3}w_{-1}},$$
$$x_{8n+2}=\frac{L_{4n+1}+L_{4n}y_{0}}{L_{4n+2}+L_{4n+1}y_{0}},\qquad x_{8n+6}=\frac{L_{4n+3}+L_{4n+2}w_{0}}{L_{4n+4}+L_{4n+3}w_{0}},$$

$$y_{8n-1}=\frac{F_{4n}+F_{4n-1}y_{-1}}{F_{4n+1}+F_{4n}y_{-1}},\qquad y_{8n+3}=\frac{F_{4n+2}+F_{4n+1}w_{-1}}{F_{4n+3}+F_{4n+2}w_{-1}},$$
$$y_{8n}=\frac{F_{4n}+F_{4n-1}y_{0}}{F_{4n+1}+F_{4n}y_{0}},\qquad y_{8n+4}=\frac{F_{4n+2}+F_{4n+1}w_{0}}{F_{4n+3}+F_{4n+2}w_{0}},$$
$$y_{8n+1}=\frac{L_{4n+1}+L_{4n}z_{-1}}{L_{4n+2}+L_{4n+1}z_{-1}},\qquad y_{8n+5}=\frac{L_{4n+3}+L_{4n+2}x_{-1}}{L_{4n+4}+L_{4n+3}x_{-1}},$$
$$y_{8n+2}=\frac{L_{4n+1}+L_{4n}z_{0}}{L_{4n+2}+L_{4n+1}z_{0}},\qquad y_{8n+6}=\frac{L_{4n+3}+L_{4n+2}x_{0}}{L_{4n+4}+L_{4n+3}x_{0}},$$

$$z_{8n-1}=\frac{F_{4n}+F_{4n-1}z_{-1}}{F_{4n+1}+F_{4n}z_{-1}},\qquad z_{8n+3}=\frac{F_{4n+2}+F_{4n+1}x_{-1}}{F_{4n+3}+F_{4n+2}x_{-1}},$$
$$z_{8n}=\frac{F_{4n}+F_{4n-1}z_{0}}{F_{4n+1}+F_{4n}z_{0}},\qquad z_{8n+4}=\frac{F_{4n+2}+F_{4n+1}x_{0}}{F_{4n+3}+F_{4n+2}x_{0}},$$
$$z_{8n+1}=\frac{L_{4n+1}+L_{4n}w_{-1}}{L_{4n+2}+L_{4n+1}w_{-1}},\qquad z_{8n+5}=\frac{L_{4n+3}+L_{4n+2}y_{-1}}{L_{4n+4}+L_{4n+3}y_{-1}},$$
$$z_{8n+2}=\frac{L_{4n+1}+L_{4n}w_{0}}{L_{4n+2}+L_{4n+1}w_{0}},\qquad z_{8n+6}=\frac{L_{4n++3}+L_{4n+2}y_{0}}{L_{4n+4}+L_{4n+3}y_{0}},$$

$$w_{8n-1}=\frac{F_{4n}+F_{4n-1}w_{-1}}{F_{4n+1}+F_{4n}w_{-1}},\qquad w_{8n+3}=\frac{F_{4n+2}+F_{4n+1}y_{-1}}{F_{4n+3}+F_{4n+2}y_{-1}},$$
$$w_{8n}=\frac{F_{4n}+F_{4n-1}w_{0}}{F_{4n+1}+F_{4n}w_{0}},\qquad w_{8n+4}=\frac{F_{4n+2}+F_{4n+1}y_{0}}{F_{4n+3}+F_{4n+2}y_{0}},$$
$$w_{8n+1}=\frac{L_{4n+1}+L_{4n}x_{-1}}{L_{4n+2}+L_{4n+1}x_{-1}},\qquad w_{8n+5}=\frac{L_{4n+3}+L_{4n+2}z_{-1}}{L_{4n+4}+L_{4n+3}z_{-1}},$$
$$w_{8n+2}=\frac{L_{4n+1}+L_{4n}x_{0}}{L_{4n+2}+L_{4n+1}x_{0}},\qquad w_{8n+6}=\frac{L_{4n+3}+L_{4n+2}z_{0}}{L_{4n+4}+L_{4n+3}z_{0}},$$

Case $m=0$ When $m=0$ in system (1.5), we obtain the system of difference equations

$$x_{n+1}=\frac{3+y_{n-1}}{4+3y_{n-1}},\quad y_{n+1}=\frac{3+z_{n-1}}{4+3z_{n-1}},z_{n+1}=\frac{3+w_{n-1}}{4+3w_{n-1}},\quad w_{n+1}=\frac{3+x_{n-1}}{4+3x_{n-1}},\quad n\in\mathbb{N}_0. \tag{3.2}$$

For system (3.2), applying Theorem (2.2) we get

$$x_{8n-1}=\frac{F_{8n}+F_{8n-1}x_{-1}}{F_{8n+1}+F_{8n}x_{-1}},\qquad x_{8n+3}=\frac{F_{8n+4}+F_{8n+3}z_{-1}}{F_{8n+5}+F_{8n+4}z_{-1}},$$
$$x_{8n}=\frac{F_{8n}+F_{8n-1}x_{0}}{F_{8n+1}+F_{8n}x_{0}},\qquad x_{8n+4}=\frac{F_{8n+4}+F_{8n+3}z_{0}}{F_{8n+5}+F_{8n+4}z_{0}},$$
$$x_{8n+1}=\frac{L_{8n+2}+L_{8n+1}y_{-1}}{L_{8n+3}+L_{8n+2}y_{-1}},\qquad x_{8n+5}=\frac{L_{8n+6}+L_{8n+5}w_{-1}}{L_{8n+7}+L_{8n+6}w_{-1}},$$
$$x_{8n+2}=\frac{L_{8n+2}+L_{8n+1}y_{0}}{L_{8n+3}+L_{8n+2}y_{0}},\qquad x_{8n+6}=\frac{L_{8n+6}+L_{8n+5}w_{0}}{L_{8n+7}+L_{8n+6}w_{0}},$$

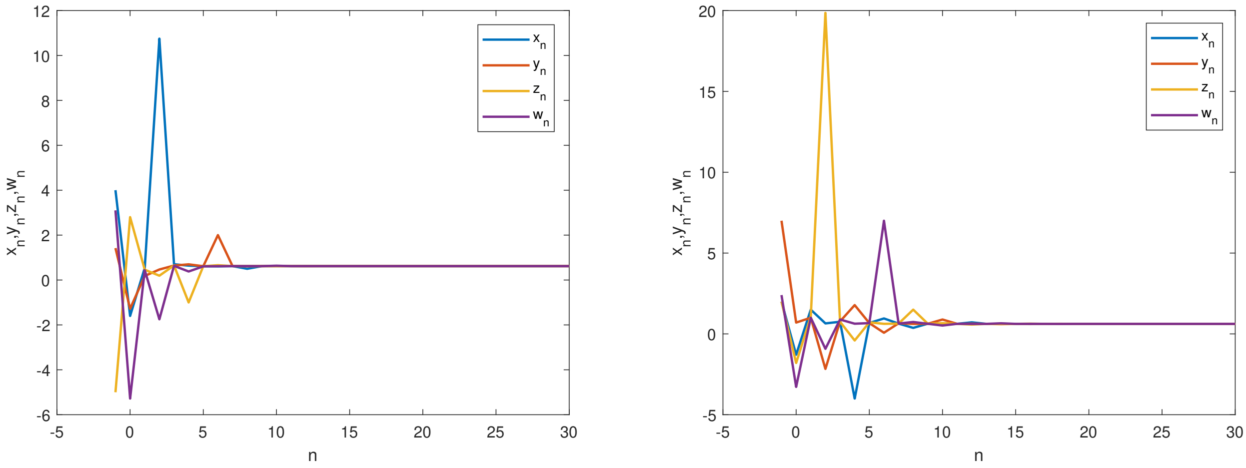

Figure 1: System (3.1) (left) and System (3.2) (right).

$$y_{8n-1}=\frac{F_{8n}+F_{8n-1}y_{-1}}{F_{8n+1}+F_{8n}y_{-1}},\qquad y_{8n+3}=\frac{F_{8n+4}+F_{8n+3}w_{-1}}{F_{8n+5}+F_{8n+4}w_{-1}},$$
$$y_{8n}=\frac{F_{8n}+F_{8n-1}y_{0}}{F_{8n+1}+F_{8n}y_{0}},\qquad y_{8n+4}=\frac{F_{8n+4}+F_{8n+3}w_{0}}{F_{8n+5}+F_{8n+4}w_{0}},$$
$$y_{8n+1}=\frac{L_{8n+2}+L_{8n+1}z_{-1}}{L_{8n+3}+L_{8n+2}z_{-1}},\qquad y_{8n+5}=\frac{L_{8n+6}+L_{8n+5}x_{-1}}{L_{8n+7}+L_{8n+6}x_{-1}},$$
$$y_{8n+2}=\frac{L_{8n+2}+L_{8n+1}z_{0}}{L_{8n+3}+L_{8n+2}z_{0}},\qquad y_{8n+6}=\frac{L_{8n+6}+L_{8n+5}x_{0}}{L_{8n+7}+L_{8n+6}x_{0}},$$

$$z_{8n-1}=\frac{F_{8n}+F_{8n-1}z_{-1}}{F_{8n+1}+F_{8n}z_{-1}},\qquad z_{8n+3}=\frac{F_{8n+4}+F_{8n+3}x_{-1}}{F_{8n+5}+F_{8n+4}x_{-1}},$$
$$z_{8n}=\frac{F_{8n}+F_{8n-1}z_{0}}{F_{8n+1}+F_{8n}z_{0}},\qquad z_{8n+4}=\frac{F_{8n+4}+F_{8n+3}x_{0}}{F_{8n+5}+F_{8n+4}x_{0}},$$
$$z_{8n+1}=\frac{L_{8n+2}+L_{8n+1}w_{-1}}{L_{8n+3}+L_{8n+2}w_{-1}},\qquad z_{8n+5}=\frac{L_{8n+6}+L_{8n+5}y_{-1}}{L_{8n+7}+L_{8n+6}y_{-1}},$$
$$z_{8n+2}=\frac{L_{8n+2}+L_{8n+1}w_{0}}{L_{8n+3}+L_{8n+2}w_{0}},\qquad z_{8n+6}=\frac{L_{8n+6}+L_{8n+5}y_{0}}{L_{8n+7}+L_{8n+6}y_{0}},$$

$$w_{8n-1}=\frac{F_{8n}+F_{8n-1}w_{-1}}{F_{8n+1}+F_{8n}w_{-1}},\qquad w_{8n+3}=\frac{F_{8n+4}+F_{8n+3}y_{-1}}{F_{8n+5}+F_{8n+4}y_{-1}},$$
$$w_{8n}=\frac{F_{8n}+F_{8n-1}w_{0}}{F_{8n+1}+F_{8n}w_{0}},\qquad w_{8n+4}=\frac{F_{8n+4}+F_{8n+3}y_{0}}{F_{8n+5}+F_{8n+4}y_{0}},$$
$$w_{8n+1}=\frac{L_{8n+2}+L_{8n+1}x_{-1}}{L_{8n+3}+L_{8n+2}x_{-1}},\qquad w_{8n+5}=\frac{L_{8n+6}+L_{8n+5}z_{-1}}{L_{8n+7}+L_{8n+6}z_{-1}},$$
$$w_{8n+2}=\frac{L_{8n+2}+L_{8n+1}x_{0}}{L_{8n+3}+L_{8n+2}x_{0}},\qquad w_{8n+6}=\frac{L_{8n+6}+L_{8n+5}z_{0}}{L_{8n+7}+L_{8n+6}z_{0}},$$

Example 3.1. *Fig.1. (left) represents system (3.1) with* $x_{-1}=2$, $x_0=-1.29$, $y_{-1}=7$, $y_0=0.7$, $z_{-1}=2$, $z_0=-1.8$, $w_{-1}=2.4$, $w_0=-3.28$

Example 3.2. *Fig.1. (right) represents system (3.2) with* $x_{-1}=4$, $x_0=-1.6$, $y_{-1}=1.42$, $y_0=-1.28$, $z_{-1}=-5$, $z_0=2.8$, $w_{-1}=3.1$, $w_0=-5.28$.

4. Conclusion

In this paper, we showed that the system of difference equations

$$x_{n+1} = \frac{L_{m+2} + L_{m+1}y_{n-1}}{L_{m+3} + L_{m+2}y_{n-1}}, \quad y_{n+1} = \frac{L_{m+2} + L_{m+1}z_{n-1}}{L_{m+3} + L_{m+2}z_{n-1}},$$

$$z_{n+1} = \frac{L_{m+2} + L_{m+1}w_{n-1}}{L_{m+3} + L_{m+2}w_{n-1}}, \quad w_{n+1} = \frac{L_{m+2} + L_{m+1}x_{n-1}}{L_{m+3} + L_{m+2}x_{n-1}}.$$

where the coefficients are the well-known Lucas numbers is solvable in closed form.

In fact, its solution is represented in terms of Lucas and Fibonacci numbers.

We also provided two illustrative examples for the case $m = -1$ and $m = 0$.

We conjecture that, the results in this paper can be satisfied to a more general case of the aforementioned system.

Declarations

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