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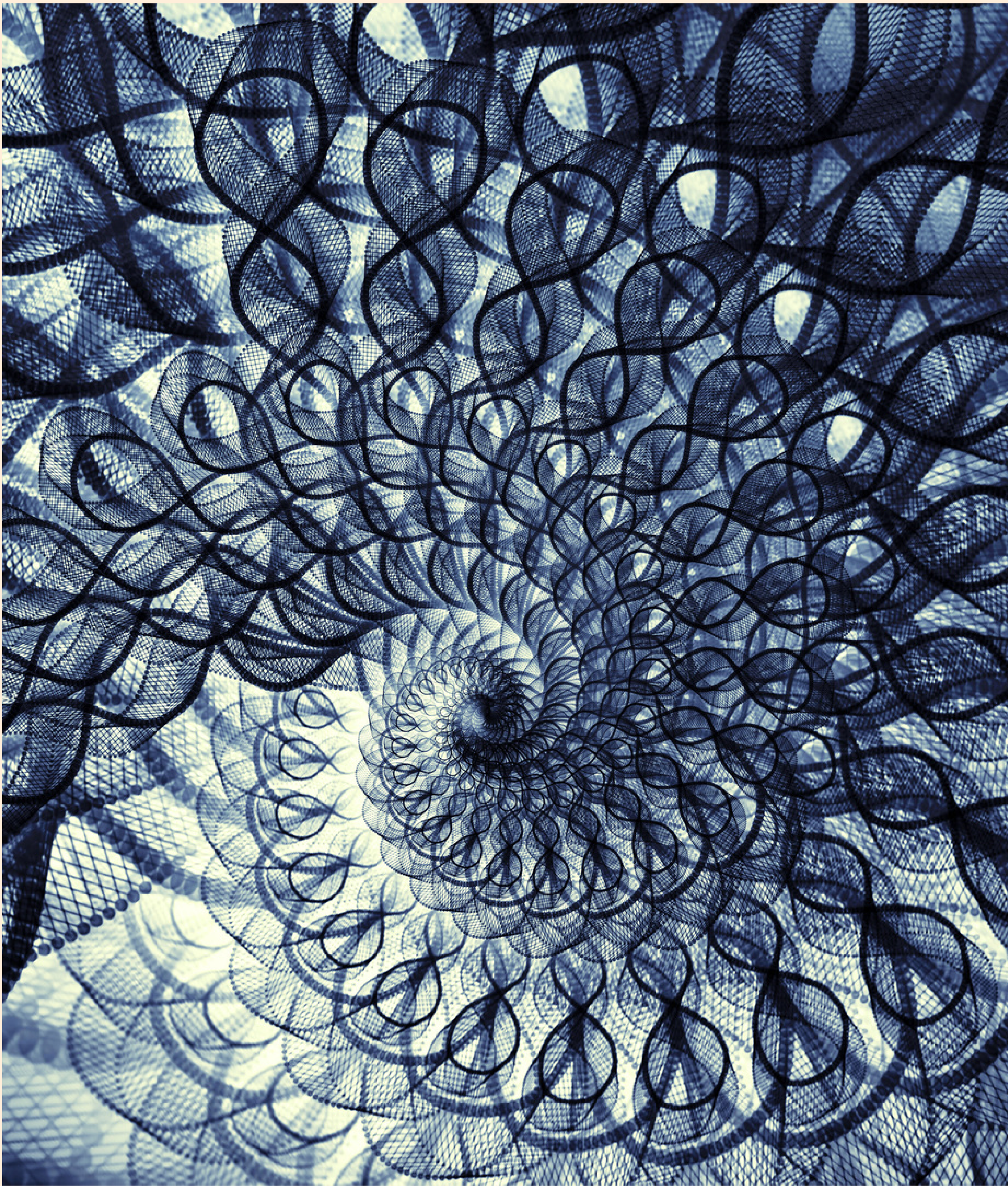

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Rough Convergence of Double Sequences in $n-$Normed Spaces

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Abstract
In this study, we introduced the concepts of rough convergence, rough Cauchy double sequence, and the set of rough limit points of a double sequence, as well as the rough convergence criteria associated with this set in n-normed spaces. Later, we proved that this set is both closed and convex. Finally, we presented the relationships between rough convergence and rough Cauchy double sequence in n-normed spaces.

Keywords: *Cauchy sequence, Double sequence, $n-$normed space, Rough convergence*

AMS Subject Classification (2020): *40A05; 40A35*

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1. Introduction

The concept of 2-normed spaces was initially introduced by Gähler [1, 2] in 1960. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [3] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Şahiner et al. [4] and Gürdal [5] studied $\mathcal{I}$-convergence in 2-normed spaces. Gürdal and Açık [6] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Also Çakallı and Ersan [7] studied new types of continuity in 2-normed spaces. Misiak [8] extended 2-normed spaces to $n-$normalized spaces. Since then, many researchers have studied this concept and obtained various results [9, 10]. Later, some studies on 2-normed spaces were transferred to $n-$normed spaces. For example, Reddy [11] investigated statistical convergence, the statistical Cauchy sequence and some properties of statistical convergence in $n-$normed spaces. Hazarika and Savaş [12] introduced the concept of λ-statistical convergence in $n-$normed spaces. They established some inclusion relations between the sets of statistically convergent and $\lambda-$statistically convergent sequences in [12]. Gürdal and Şahiner [13] studied ideal convergence in $n-$normed spaces and presented the main results.

In finite-dimensional normed spaces, Phu [14] was the first to present the concept of rough convergence. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in some normed linear space $(X, \|.\|)$ and r be a nonnegative real number, then $(x_i)_{i\in\mathbb{N}}$ is

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said to be $r-$convergent to x_*, denoted by $x_i \xrightarrow{r} x_*$, provided that

$$\forall\varepsilon > 0,\ \exists i_\varepsilon \in \mathbb{N}:\ i \geq i_\varepsilon \Rightarrow \|x_i - x_*\| < r + \varepsilon.$$

Also, the sequence (x_k) is said to be a rough Cauchy sequence satisfying

$$\forall\varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : k, m \geq K_\varepsilon \Rightarrow \|x_k - x_m\| < \rho + \varepsilon$$

for $\rho > 0$. ρ is roughness degree of (x_k). Shortly (x_k) is called a rough Cauchy sequence. ρ is also a Cauchy degree of (x_k). In [14], he showed that the set $\text{LIM}^r x$ is bounded, closed, and convex, and he introduced the notion of rough Cauchy sequence. He also investigated the relationships between rough convergence and other types of convergence, as well as the dependence of $\text{LIM}^r x$ on the roughness degree r. In another paper [15] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is $r-$continuous at every point $x \in X$ under the assumption $dimY < \infty$ and $r > 0$, where X and Y are normed spaces. In [16], he extended the results given in [14] to infinite-dimensional normed spaces. Aytar [17] studied rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [18] studied that the $r-$limit set of the sequence is equal to the intersection of these sets and that $r-$core of the sequence is equal to the union of these sets. In later times, Arslan and Dündar [19, 20] introduced the notions of rough convergence, rough Cauchy sequence, and the set of rough limit points of a sequence and obtained the rough convergence criteria associated with this set in 2-normed space first, then presented their work "On rough convergence in 2-normed spaces and some properties." They [21, 22] also examined rough statistical convergence and rough statistical cluster points in 2-normed spaces. Sunar and Arslan [23] introduced the concept of rough convergence in $n-$normed spaces by combining the concepts of rough convergence and $n-$normed spaces.

Pringsheim [24, 25] developed the idea of convergence for double sequences. He gave some examples of the convergence of double sequences with and without the usual convergence of rows and columns and defined the $P-$limit. $\mathbb{N}$ and $\mathbb{R}$ are used throughout the paper to denote the sets of all positive integers and all real numbers, respectively.

A double sequence $(x_{tk})_{t,k\in\mathbb{N}}$ in some linear space $(X, \|.\|)$ is said to converge to a point $L \in X$ in Pringsheim's sense, denoted by $(x_{tk}) \to L$, if for any $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L\| < \varepsilon \ \text{ for all } t, k \geq K_\varepsilon.$$

Further, a double sequence $(x_{tk})_{t,k\in\mathbb{N}}$ is said to be a Cauchy double sequence if for any $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - x_{mv}\| < \varepsilon \ \text{ for all } t, k, m, v \geq K_\varepsilon.$$

Contrary to the property of convergence in ordinary sequences, it is an important problem that convergent double sequences do not have to be bounded. Hardy [26] introduced the concept of regular convergence, which also needed the convergence of the rows and columns of a pair in addition to the Pringsheim convergence. Hence, this problem was eliminated. Later, many researchers used double sequences in their works in the area of summability theory. This work can be found in [27–32]. Malik and Maity [33] defined and examined rough convergence of double sequences, the set of $r-$limit points of double sequences and rough Cauchy double sequences. These concepts, given by Malik and Maity [33], are as follows:

Let (x_{tk}) be a double sequence in a normed space $(X, \|.\|)$ and r be a non-negative real number. (x_{tk}) is $r-$convergent to L in X, denoted by $x_{tk} \xrightarrow{\|.\|}_r L$ if

$$\forall\varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : t, k \geq K_\varepsilon \Rightarrow \|x_{tk} - L\| < r + \varepsilon.$$

A double sequence (x_{tk}) is called a rough Cauchy sequence with roughness degree ρ if for any $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - x_{mv}\| < \rho + \varepsilon, \ \text{ for all } t, k, m, v \geq K_\varepsilon.$$

Dündar and Çakan [34, 35] introduced the notions of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and studied the notions of rough convergence and the set of rough limit points of a double sequence. Also, Kişi and Dündar [36] presented the notion of rough $\mathcal{I}_2-$lacunary statistical limit set of a double sequence.

By combining the concepts of rough convergence, double sequences and $n-$normed spaces, we introduce the concept of rough convergence of double sequences in $n-$normed spaces. We obtain two convergence criteria associated with the set of rough limit points of a double sequence in $n-$normed spaces. Later, we prove that this set is both closed and convex. Finally, we investigate the relationships between a double sequence's cluster points and its rough limit points. The results and proof techniques presented in this paper are analogous to those presented in Phu's [14] paper. The concept of convergent double sequences given in our paper is used in the sense of Pringsheim. So a convergent double sequence may not be bounded. Namely, the actual origin of most of these results and proof techniques are the papers. The following theorems and results are extensions of the theorems and results in [14]. Currently, we recall the idea of $n-$normed spaces, some fundamental definitions, and notations.(See [8, 10, 11, 30, 33, 37]).

Definition 1.1. [37] Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ on X^n satisfying the following properties for all $y, z, x_1, x_2, \cdots, x_{n-1}, x_n \in X$

(i) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, x_2 \cdots, x_n$ are linearly dependent,

(ii) $\|x_1, x_2, \cdots, x_n\|$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$,

(iii) $\|x_1, x_2, \cdots, x_{n-1}, \alpha x_n\| = |\alpha| \|x_1, x_2, \cdots, x_{n-1}, x_n\|$ for all $\alpha \in \mathbb{R}$,

(iv) $\|x_1, x_2, \cdots, x_{n-1}, y + z\| \leq \|x_1, x_2, \cdots, x_{n-1}, y\| + \|x_1, x_2, \cdots, x_{n-1}, z\|$

is called an $n-$norm on X, and the pair $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is called an $n-$normed space.

An example of an $n-$normed space is $X = \mathbb{R}^n$ equipped with the followig Euclidean $n-$norm:

Example 1.1.

$$\|x_1, x_2 \cdots, x_{n-1}, x_n\|_E = \mid det(x_{ij}) \mid = abs \left(\begin{vmatrix} x_{11} \ldots & x_{1n} \\ x_{21} \ldots & x_{2n} \\ & \\ x_{n1} \ldots & x_{nn} \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$.

In this study, we suppose X to be an $n-$normed space having dimension d; where $2 \leq d < \infty$.

Definition 1.2. [37] A sequence (x_k) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be convergent to L in X if

$$\lim_{k \to \infty} \|x_k - L, z_2, \cdots, z_n\| = 0$$

for every $z_2, \cdots, z_n \in X$. In such a case, we write $\lim\limits_{k \to \infty} x_k = L$ and call L the limit of (x_k).

Example 1.2. [23] Let $x = (x_k) = (\frac{k}{k+1}, \frac{1}{k}, \ldots, \frac{1}{k})$, $L = (1, 0, \ldots, 0)$ and $z = (z_1, z_2, \ldots, z_n)$. It is clear that (x_k) is convergent to $L = (1, 0, \ldots, 0)$ in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$.

Definition 1.3. [37] A sequence (x_k) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be a Cauchy sequence in X if for every $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_k - x_m, z_2, z_3, \ldots, z_n\| < \varepsilon$$

for all $k, m \geq K_\varepsilon$ and every $z_2, z_3, \ldots, z_n \in X$.

Definition 1.4. [23] Let (x_k) be a sequence in $n-$normed linear space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ and r be a non-negative real number. (x_k) is said to be rough convergent (r-convergent) to L if

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : k \geq K_\varepsilon \Rightarrow \|x_k - L, z_2, \cdots, z_n\| < r + \varepsilon$$

for every $z_2, \cdots, z_n \in X$.

Definition 1.5. [23] Let (x_k) be a sequence in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$. (x_k) is said to be a rough Cauchy sequence satisfying

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : k, m \geq K_\varepsilon \Rightarrow \|x_k - x_m, z_2, \cdots, z_n\| < \rho + \varepsilon$$

for $\rho > 0$ and every $z_2, \cdots, z_n \in X$. ρ is roughness degree of (x_k).

Definition 1.6. (cf. [33]) A double sequence (x_{tk}) in $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be bounded if there exists a non-negative real number M such that $\|x_{tk}, z_2, \cdots, z_n\| < M$ for all $t, k \in \mathbb{N}$.

Definition 1.7. [30] A double sequence (x_{tk}) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be convergent to $L \in X$ if for each $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, \cdots, z_n\| < \varepsilon$$

for all $t, k \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$.

Definition 1.8. [30] A double sequence (x_{tk}) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be a Cauchy sequence if for each $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$

$$\|x_{tk} - x_{mv}, z_2, \cdots, z_n\| < \varepsilon$$

for all $t, k, m, v \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$.

2. Main results

We introduced the concepts of rough convergence, rough Cauchy double sequence and the set of rough limit points set of a double sequence in this work and we obtained the rough convergence criteria associated with this set in n-normed space. We later demonstrated that this set is both closed and convex. Finally, we investigated the relationships between rough convergence and rough Cauchy double sequence in n-normed spaces.

Definition 2.1. Let (x_{tk}) be a double sequence in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ and r be a non-negative real number. (x_{tk}) is said to be rough convergent ($r-$convergent) to L denoted by $x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L$ if

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : t, k \geq K_\varepsilon \Rightarrow \|x_{tk} - L, z_2 \cdots, z_n\| < r + \varepsilon \tag{2.1}$$

for every $z_2, \cdots, z_n \in X$.

If (2.1) holds, L is an $r-$limit point of (x_{tk}), which is usually no more unique (for $r > 0$). So, we have to consider the so-called $r-$limit set (or shortly $r-$limit) of (x_{tk}) defined by

$$\mathrm{LIM}_n^r x_{tk} := \{L \in X : x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L\}. \tag{2.2}$$

A double sequence (x_{tk}) is said to be $r-$convergent if $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$. In this case, r is called the convergence degree of the double sequence (x_{tk}). For $r = 0$ we have the classical convergence in $n-$normed space again. But our proper interest is the case $r > 0$. There are several reasons for this interest. For instance, since an originally convergent double sequence (y_{tk}) (with $y_{tk} \to L$) in $n-$normed space often cannot be determined (i.e., measured or calculated) exactly, one has to do with an approximated double sequence (x_{tk}) satisfying

$$\|x_{tk} - y_{tk}, z_2, \cdots, z_n\| \leq r$$

for all n and every $z_2, z_3, \ldots, z_n \in X$, where $r > 0$ is an upper bound of approximation error. Then, (x_{tk}) is no more convergent in the classical sense, but for every $z_2, \cdots z_n \in X$,

$$\|x_{tk} - L, z_2, \cdots, z_n\| \leq \|x_{tk} - y_{tk}, z_2, \cdots, z_n\| + \|y_{tk} - L, z_2, \cdots, z_n\| \leq r + \|y_{tk} - L, z_2, \cdots, z_n\|$$

implies that (x_{tk}) is $r-$convergent in the sense of (2.1).

Example 2.1. The double sequence $(x_{tk}) = ((-1)^{tk}, (-1)^{tk}, \ldots, (-1)^{tk})$ is not convergent in n-normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$, but it is rough convergent to $L = (0, 0, \ldots, 0)$ for every $z_2, \cdots, z_n \in X$. It is clear that

$$\mathrm{LIM}_n^r x_{tk} = \begin{cases} \emptyset, \text{ if } \quad r < 1 \\ [(-r, -r, \ldots, -r), (r, r, \ldots, r)], \text{ otherwise.} \end{cases}$$

Sometimes we are interested in the set of $r-$limit points lying in a given subset $D \subset X$, which is called $r-$limit in D and denoted by

$$\mathrm{LIM}_n^{D,r} x_{tk} := \{L \in D : x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L\}. \tag{2.3}$$

It is clear that

$$\mathrm{LIM}_n^{X,r} x_{tk} = \mathrm{LIM}_n^r x_{tk} \ \ and \ \ \mathrm{LIM}_n^{D,r} x_{tk} = D \cap \mathrm{LIM}_n^r x_{tk}.$$

First, let us transform some properties of classical convergence to rough convergence in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$. It is well known if a sequence converges then its limit is unique. This property is maintained for rough convergence with roughness degree $r > 0$, but only has the following analogy.

Theorem 2.1. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. We have $diam(\mathrm{LIM}_n^r x_{tk}) \leq 2r$. In general, $diam(\mathrm{LIM}_n^r x_{tk})$ has no smaller bound.*

Proof. We have to show that

$$diam(\mathrm{LIM}_n^r x_{tk}) = sup\left\{\|x_1 - x_2, z_2, \cdots, z_n\| : x_1, x_2 \in \mathrm{LIM}_n^r x_{tk} \leq 2r\right\}, \tag{2.4}$$

where $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is an $n-$normed space and for every $z_2, \cdots, z_n \in X$. Assume the contrary that

$$diam(\mathrm{LIM}_n^r x_{tk}) > 2r.$$

Then, there exist $x_1, x_2 \in \mathrm{LIM}_n^r x_{tk}$ satisfying

$$d := \|x_1 - x_2, z_2, \ldots, z_n\| > 2r$$

for every $z_2, z_3, \cdots, z_n \in X$. For an arbitrary $\varepsilon \in (0, \frac{d-2r}{2})$, it follows from (2.1) and (2.2) that there is a $K_\varepsilon \in \mathbb{N}$ such that for $t, k \geq K_\varepsilon$,

$$\|x_{tk} - x_1, z_2, \ldots, z_n\| < r + \varepsilon \ \ and \ \ \|x_{tk} - x_2, z_2, \ldots, z_n\| < r + \varepsilon$$

for every $z_2, z_3, \ldots, z_n \in X$. This implies

$$\begin{aligned} \|x_1 - x_2, z_2, \ldots, z_n\| &\leq \|x_{tk} - x_1, z_2, \ldots, z_n\| + \|x_{tk} - x_2, z_2, \ldots, z_n\| \\ &< 2(r + \varepsilon) \\ &< 2r + 2(\frac{d-2r}{2}) \\ &= d \end{aligned}$$

for every $z_2, z_3, \ldots, z_n \in X$, which conflicts with $d = \|x_1 - x_2, z_2, \ldots, z_n\|$. Hence, (2.4) must be true. Consider a convergent double sequence (x_{tk}) with $\lim\limits_{t,k\to\infty} x_{tk} = L$. Then, for

$$\overline{B}_r(L) := \{x_1 \in X : \|x_1 - L, z_2, z_3, \ldots, z_n\| \leq r\}$$

it follows from

$$\begin{aligned} \|x_{tk} - x_1, z_2, z_3, \ldots, z_n\| &\leq \|x_{tk} - L, z_2, z_3, \ldots, z_n\| + \|L - x_1, z_2, z_3, \ldots, z_n\| \\ &\leq \|x_{tk} - L, z_2, z_3, \ldots, z_n\| + r \end{aligned}$$

for every $z_2, z_3, \ldots, z_n \in X$ and for $x_1 \in \overline{B}_r(L)$, from (2.1) and (2.2) that

$$\mathrm{LIM}_n^r x_{tk} = \overline{B}_r(L).$$

Since $diam(\overline{B}_r(L)) = 2r$, this shows that in general the upper bound $2r$ of the diameter of an $r-$limit set cannot be decreased anymore. □

Obviously the uniqueness of limit (of classical convergence) can be regarded as a special case of latter property, because if $r = 0$ then $diam(\mathrm{LIM}_n^r x_{tk}) = 2r = 0$, that is, $\mathrm{LIM}_n^r x_{tk}$ is either empty or a singleton.

The following property shows an analogy between boundedness and rough convergence of a double sequence in $n-$normed space.

Theorem 2.2. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. If the double sequence (x_{tk}) is bounded then there exists an $r \geq 0$, such that $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$.*

$$\mathrm{LIM}_n^{(x_{t_v k_s}),r} x_{t_v k_s} \neq \emptyset.$$

Proof. For every $z_2, z_3, \ldots, z_n \in X$ if

$$s := sup\{\|x_{tk}, z_2, z_3, \ldots, z_n\| : t, k \in \mathbb{N}\} < \infty.$$

Then, $\mathrm{LIM}_n^s x_{tk}$ contains the origin of X. So, $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$. $\square$

The converse of the previous theorem might not hold true since a convergent double sequence is not always bounded. Let's now introduce the notion of loosely boundedness for $n-$normed spaces, which is analogous to [33].

Definition 2.2. A double sequence (x_{tk}) in X is said to be loosely bounded if there exist an $M \in \mathbb{R}^+$ and a $K \in \mathbb{N}$ such that $\|x_{tk}, z_2, z_3, \ldots, z_n\| < M$ for all $t, k \geq K$.

Every bounded double sequence is obviously loosely bounded, but the converse is not true.

Theorem 2.3. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. The double sequence (x_{tk}) is loosely bounded if and only if there exists an $r \geq 0$, such that $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$.*

Proof. Let (x_{tk}) be a loosely bounded double sequence. Then there exist an $M \in \mathbb{R}^+$ and a $K \in \mathbb{N}$ such that $\|x_{tk}, z_2, z_3, \ldots, z_n\| < M$ for all $t, k \geq K$. Then, $\mathrm{LIM}_n^M x_{tk}$ contains the origin of X. So, $\mathrm{LIM}_n^M x_{tk} \neq \emptyset$.

Conversely, let $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$ for some $r \geq 0$. Let $L \in \mathrm{LIM}_n^r x_{tk}$. We take $\varepsilon = 1$. Then there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, z_3, \ldots, z_n\| < r + 1 \text{ for all } t, k \geq K_\varepsilon.$$

So, (x_{tk}) is loosely bounded. $\square$

Now let $(t_i)_{i\in\mathbb{N}}$ and $(k_j)_{j\in\mathbb{N}}$ be two strictly increasing sequences of natural numbers. If $(x_{tk})_{t,k\in\mathbb{N}}$ is a double sequence in $(X, \|\bullet, \bullet, \ldots, \bullet\|)$, then we can define $(x_{t_i k_j})_{i,j\in\mathbb{N}}$ as a subsequence of $(x_{tk})_{t,k\in\mathbb{N}}$. (See, [33]).

Proposition 2.1. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. If $(x_{t_i k_j})$ is a subsequence of (x_{tk}) then,*

$$\mathrm{LIM}_n^r x_{tk} \subseteq \mathrm{LIM}_n^r x_{t_i k_j}$$

in n-normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$.

Proof. Let $L \in \mathrm{LIM}_n^r x_{tk}$. Then for any $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, z_3, \ldots, z_n\| < r + \varepsilon$$

for all $t, k \geq K_\varepsilon$ and every $z_2, z_3, \ldots, z_n \in X$. Since (t_i) and (k_j) are strictly increasing sequences, so there exists a $k_0 \in \mathbb{N}$ such that $t_{k_0} > K_\varepsilon$ and $k_{k_0} > K_\varepsilon$. Therefore, we get

$$\|x_{t_i k_j} - L, z_2, z_3, \ldots, z_n\| < r + \varepsilon$$

for all $t_i, k_j \geq K_\varepsilon$ and every $z_2, z_3, \ldots, z_n \in X$. So, $L \in \mathrm{LIM}_n^r x_{t_i k_j}$. $\square$

Theorem 2.4. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. For all $r \geq 0$, the $r-$limit set $\mathrm{LIM}_n^r x_{tk}$ of an arbitrary double sequence (x_{tk}) is closed.*

Proof. Let (y_{sv}) be an arbitrary double sequence in $\mathrm{LIM}_n^r x_{tk}$ which converges to some point L. For each $\varepsilon > 0$ and every $z_2, z_3, \ldots, z_n \in X$, by definition there exist $m_{\varepsilon/2}, k_{\varepsilon/2} \in \mathbb{N}$ such that

$$\|y_{m_{\varepsilon/2}} - L, z_2, z_3, \ldots, z_n\| < \frac{\varepsilon}{2} \text{ and } \|x_{tk} - y_{m_{\varepsilon/2}}, z_2, z_3, \ldots, z_n\| < r + \frac{\varepsilon}{2}$$

whenever $k \geq k_{\varepsilon/2}$. Consequently for every $z_2, z_3, \ldots, z_n \in X$,

$$\begin{aligned} \|x_{tk} - L, z_2, \ldots, z_n\| &\leq \|x_{tk} - y_{m_{\varepsilon/2}}, z_2, \ldots, z_n\| + \|y_{m_{\varepsilon/2}} - L, z_2, \ldots, z_n\| \\ &< r + \varepsilon \end{aligned}$$

for $k \geq k_{\varepsilon/2}$. That means $L \in \mathrm{LIM}_n^r x_{tk}$, too. Hence, $\mathrm{LIM}_n^r x_{tk}$ is closed. $\square$

Theorem 2.5. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. If*

$$y_0 \in \mathrm{LIM}_n^{r_0} x_{tk} \ \ and \ \ y_1 \in \mathrm{LIM}_n^{r_1} x_{tk},$$

then,

$$y_\alpha := (1-\alpha)y_0 + \alpha y_1 \in \mathrm{LIM}_n^{(1-\alpha)r_0+\alpha r_1} x_{tk}, \quad for \ \ \alpha \in [0,1].$$

Proof. By definition, for every $\varepsilon > 0$, $r_0, r_1 > 0$ and every $z_2, z_3, \ldots, z_n \in X$ there exists a $K_\varepsilon \in \mathbb{N}$ such that $t, k > K_\varepsilon$ implies

$$\|x_{tk} - y_o, z_2, \ldots, z_n\| < r_0 + \varepsilon \ \ and \ \ \|x_{tk} - y_1, z_2, \ldots, z_n\| < r_1 + \varepsilon,$$

which yields also, for every $z_2, z_3, \ldots, z_n \in X$,

$$\begin{aligned} \|x_{tk} - y_\alpha, z_2, z_3, \ldots, z_n\| &\leq (1-\alpha)\|x_{tk} - y_o, z_2, z_3, \ldots, z_n\| + \alpha\|x_{tk} - y_1, z_2, z_3, \ldots, z_n\| \\ &< (1-\alpha)(r_0+\varepsilon) + \alpha(r_1+\varepsilon) \\ &= (1-\alpha)r_0 + \alpha r_1 + \varepsilon. \end{aligned}$$

Hence, we have

$$y_\alpha \in \mathrm{LIM}_n^{(1-\alpha)r_0+\alpha r_1} x_{tk}.$$

□

Theorem 2.6. *Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an $n-$normed space and consider a double sequence $(x_{tk}) \in X$. $\mathrm{LIM}_n^r x_{tk}$ is convex.*

Proof. In particular, for $r = r_0 = r_1$, Theorem 2.5 yields immediately that $\mathrm{LIM}_n^r x_{tk}$ is convex. □

Theorem 2.7. *If $x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L_1$ and $y_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L_2$. Then,*

(i) $(x_{tk} + y_{tk}) \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r (L_1 + L_2)$ *and*

(ii) $\alpha(x_{tk}) \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r \alpha L_1$, $(\alpha \in \mathbb{R})$.

Proof. *(i)* By definition for every $z_2, z_3, \ldots, z_n \in X$,

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : t, k \geq K_\varepsilon \Rightarrow \|x_{tk} - L_1, z_2, z_3, \ldots, z_n\| < r_1 + \frac{\varepsilon}{2}$$

and

$$\forall \varepsilon > 0, \exists J_\varepsilon \in \mathbb{N} : t, k \geq J_\varepsilon \Rightarrow \|y_{tk} - L_2, z_2, z_3, \ldots, z_n\| < r_2 + \frac{\varepsilon}{2}.$$

Let $j = max\{K_\varepsilon, J_\varepsilon\}$ and $r_1 + r_2 = r$. For every $t, k > j$ and every $z_2, z_3, \ldots, z_n \in X$ we have

$$\begin{aligned} \|(x_{tk} + y_{tk}) - (L_1 + L_2), z_2, z_3, \ldots, z_n\| &= \|x_{tk} - L_1, z_2, z_3, \ldots, z_n\| + \|y_{tk} - L_2, z_2, z_3, \ldots, z_n\| \\ &< r_1 + \frac{\varepsilon}{2} + r_2 + \frac{\varepsilon}{2} \\ &= r + \varepsilon \end{aligned}$$

and so

$$(x_{tk} + y_{tk}) \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r (L_1 + L_2).$$

(ii) It is obvious for $\alpha = 0$. Let $\alpha \neq 0$. Since

$$x_{tk} \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r L_1$$

for every $\varepsilon > 0$ and every $z_2, z_3, \ldots, z_n \in X$, $\exists K_\varepsilon \in \mathbb{N}$ such that for every $t, k \geq K_\varepsilon$, we have

$$\|x_{tk} - L_1, z_2, z_3, \ldots, z_n\| < \frac{r + \varepsilon}{|\alpha|}.$$

According to this, for $\forall t,k \geq K_\varepsilon$ and every $z_2, z_3, \ldots, z_n \in X$, we can write

$$\begin{aligned} \|\alpha x_{tk} - \alpha L_1, z_2, z_3, \ldots, z_n\| &= |\alpha| \|x_{tk} - L_1, z_2, z_3, \ldots, z_n\| \\ &< |\alpha| \frac{r+\varepsilon}{|\alpha|} \\ &= r + \varepsilon. \end{aligned}$$

So,

$$(\alpha x_{tk}) \overset{\|\bullet,\bullet,\ldots,\bullet\|}{\longrightarrow}_r \alpha L_1.$$

□

Following, we give some relations between convergence and rough convergence of double sequences in $n-$normed space.

Theorem 2.8. *Let $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ be an $n-$normed space and condiser a double sequence (x_{tk}) in X. If c is a cluster point of (x_{tk}), then $\|L - c, z_2, \cdots, z_n\| \leq r$ for every $L \in \mathrm{LIM}_n^r x_{tk}$.*

Proof. Let $L \in \mathrm{LIM}_n^r x_{tk}$. Assume the contrary that $d := \|L - c, z_2, \cdots, z_n\| > r$. Let $\varepsilon = \dfrac{d-r}{2}$. Since $L \in \mathrm{LIM}_n^r x_{tk}$, there exists a $K_\varepsilon \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, \cdots, z_n\| < r + \varepsilon$$

for all $t,k \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$. Then we write

$$\|L - c, z_2, \cdots, z_n\| \leq \|x_{tk} - L, z_2, \cdots, z_n\| + \|x_{tk} - c, z_2, \cdots, z_n\|$$

for all $t,k \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$. If we rewrite the inequality, we get

$$\begin{aligned} \|x_{tk} - c, z_2, \cdots, z_n\| &\geq \|L - c, z_2, \cdots, z_n\| - \|x_{tk} - L, z_2, \cdots, z_n\| \\ &> d - (r + \frac{d-r}{2}) \\ &= \varepsilon \end{aligned}$$

for all $t,k \geq K_\varepsilon$ and every $z_2, \cdots, z_n \in X$ which contradicts that c is a cluster point. So $\|L - c, z_2, \cdots, z_n\| \leq r$ for every $L \in \mathrm{LIM}_n^r x_{tk}$. □

Theorem 2.9. *Let $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ be an $n-$normed space and condiser a double sequence (x_{tk}) in X. Then (x_{tk}) converges to $L \in X$ if and only if $\mathrm{LIM}_n^r x_{tk} = \overline{B}_r(L)$.*

Proof. The first part of the proof is obtained directly from the second part of Theorem 2.1, that is, if (x_{tk}) converges to $L \in X$, then $\mathrm{LIM}_n^r x_{tk} = \overline{B}_r(L)$. Let us now show the second part of the theorem.

Conversely, let $\mathrm{LIM}_n^r x_{tk} = \overline{B}_r(L)$. Now let's show that (x_{tk}) converges to L, that is, for every $\alpha > 0$, there exists a $K_\alpha \in \mathbb{N}$ such that $\|x_{tk} - L, z_2, \cdots, z_n\| \leq \alpha$ for all $t,k \geq K_\alpha$ and every $z_2, \cdots, z_n \in X$. Now we can take a fixed $\alpha > 0$, such that $r + \varepsilon < \alpha$ for $r > 0$ and $\varepsilon > 0$. For $L \in \mathrm{LIM}_n^r x_{tk}$, there exists a $K_\alpha \in \mathbb{N}$ such that

$$\|x_{tk} - L, z_2, \cdots, z_n\| < r + \varepsilon < \alpha$$

for all $t,k \geq K_\alpha$ and every $z_2, \cdots, z_n \in X$. Therefore (x_{tk}) converges to $L \in X$. □

Definition 2.3. Let (x_{tk}) be a double sequence in $n-$normed space $(X, \|\bullet,\bullet,\ldots,\bullet\|)$. (x_{tk}) is said to be a rough Cauchy double sequence with roughness degree ρ, if

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N} : m,v,t,k \geq K_\varepsilon \Rightarrow \|x_{mv} - x_{tk}, z_2, z_3, \ldots, z_n\| < \rho + \varepsilon$$

is hold for $\rho > 0$, $L \in X$ and every $z_2, z_3, \ldots, z_n \in X$. ρ is also called a Cauchy degree of (x_{tk}).

Proposition 2.2. *(i) Monotonicity: Assume $\rho' > \rho$. If ρ is a Cauchy degree of a given double sequence (x_{tk}) in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$, so ρ' is a Cauchy degree of (x_{tk}).*

(ii) Boundedness: A double sequence (x_{tk}) is loosely bounded if and only if there exists a $\rho \geq 0$ such that (x_{tk}) is a $\rho-$Cauchy double sequence in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$.

Theorem 2.10. *If (x_{tk}) is rough convergent in $n-$normed space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$, i.e., $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$ if and only if (x_{tk}) is a ρ-Cauchy double sequence for every $\rho \geq 2r$. This bound for the Cauchy degree cannot be generally decreased.*

Proof. A Cauchy double sequence is loosely bounded. By Theorem 2.3, (x_{tk}) is rough convergent, that is, $\mathrm{LIM}_n^r x_{tk} \neq \emptyset$. So, it is sufficient to prove the first part of the theorem. Let L be any point in $\mathrm{LIM}_n^r x_{tk}$. Then, for all $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathbb{N}$ such that $m, v, t, k \geq K_\varepsilon$ implies

$$\|x_{mv} - L, z_2, z_3, \ldots, z_n\| \leq r + \frac{\varepsilon}{2} \text{ and } \|x_{tk} - L, z_2, z_3, \ldots, z_n\| \leq r + \frac{\varepsilon}{2}$$

for every $z_2, z_3, \ldots, z_n \in X$. Therefore, for $m, v, t, k \geq K_\varepsilon$, we have

$$\begin{aligned} \|x_{mv} - x_{tk}, z_2, z_3, \ldots, z_n\| &= \|x_{mv} - L + L - x_{tk}, z_2, z_3, \ldots, z_n\| \\ &\leq \|x_{mv} - L, z_2, z_3, \ldots, z_n\| + \|L - x_{tk}, z_2, z_3, \ldots, z_n\| \\ &\leq r + \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} \\ &= 2r + \varepsilon \end{aligned}$$

for every $z_2, z_3, \ldots, z_n \in X$. Hence, (x_{tk}) is a ρ-Cauchy double sequence for $\rho \geq 2r$. By Proposition 2.2, every $\rho \geq 2r$ is also a Cauchy degree of (x_{tk}). It is clear that this bound $2r$ can not be generally decreased, similar to Proposition 5.1 in [16]. □

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Some Fixed Point Results for α-Admissible Mappings on Quasi Metric Space Via θ-Contractions

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Abstract
By implying α-admissible mapping, this study expands and investigates generalized contraction mappings in quasi-metric spaces, aiming to establish the existence of fixed points. Moreover, we show that the main outcomes of the paper encompass several previously reported results in the literature.

Keywords: *Fixed point, Right K-Cauchy sequence, Quasi metric space, α admissible*

AMS Subject Classification (2020): *54H25; 47H10*

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1. Introduction and preliminaries

The Banach contraction principle, also known as the Banach fixed point theorem, is a fundamental result in mathematics, specifically in the field of functional analysis. It is named after the Polish mathematician Stefan Banach, who first stated and proved the theorem in 1922. The theorem provides conditions under which a mapping using a complete metric space to itself has a unique fixed point. A fixed point of a mapping is a point in the space that remains unchanged after applying the mapping. The proof of the Banach contraction principle typically involves constructing a sequence of iterates using the contraction property and showing that it converges to the fixed point. The completeness of the metric space is crucial for guaranteeing the convergence of the sequence.(see [1–5]). But owing to the strict conditions of the metric space and the specific properties imposed, the necessity to consider topological structures that have more flexible conditions than the metric space has emerged.Therefore, many generalizations of the Banach fixed point theorem have been obtained in this space by defining the quasi metric space. Furthermore, quasi-metric spaces are useful in numerous topics of mathematics, like optimization, functional analysis and computer science. They provided a more general framework for studying approachs related to distances and convergence, allowing for more flexible and adaptable notions of proximity. (see [6–11]). Now, review the definitions and notations related to quasi-metric space:

$\Lambda \neq \emptyset$ and ρ be a function $\rho : \Lambda \times \Lambda \to \mathbb{R}$ such that for each $\omega, \gamma, \eta \in \Lambda$:

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i) $\rho(\omega, \omega) = 0$ (Non-negativity),

ii) $\rho(\omega, \gamma) \leq \rho(\omega, \eta) + \rho(\eta, \gamma)$ (triangle inequality),

iii) $\rho(\omega, \gamma) = \rho(\gamma, \omega) = 0 \Rightarrow \omega = \gamma$ (asymmetry),

iv) $\rho(\omega, \gamma) = 0 \Rightarrow \omega = \gamma$.

If (i) and (ii) conditions are satisfied, then ρ it is called a quasi-pseudo metric(shortly q.-p.-m.), if (i), (ii) and (iii) conditions are satisfied, then ρ is called quasi metric(shortly q.-m.), in addition if a q.-m. ρ satisfies (iv), then ρ is called T_1-q.-m.. It is evident that

$$\begin{aligned} &\forall \text{ metric is a } T_1 \text{ quasi-metric,} \\ &\forall\, T_1 \text{ quasi-metric is a quasi-metric,} \\ &\forall \text{ quasi-metric is a quasi-pseudo metric.} \end{aligned}$$

Then, the pair (Λ, ρ) is also said to be a quasi pseudo metric space(shortly q.-p.-m. s.). Moreover, each q.-p. m. ρ on Λ generates a topology τ_ρ on Λ the family of open balls as a base defined as follows:

$$\{B_\rho(\omega, \varepsilon) : \omega \in \Lambda \text{ and } \varepsilon > 0\}$$

where $B_\rho(\omega_0, \varepsilon) = \{\gamma \in \Lambda : \rho(\omega_0, \gamma) < \varepsilon\}$.

If ρ is a q.-m. on Λ, then τ_ρ is a T_0 topology, and if ρ is a T_1-q.-m., then τ_ρ is a T_1 topology on Λ.

If ρ is a q.-m. and τ_ρ is T_1 topology, then ρ is T_1-q.-m.. In this case, the mappings, $\rho^{-1}, \rho^s, \rho_+ : \Lambda \times \Lambda \to [0, \infty)$ defines as

$$\begin{aligned} \rho^{-1}(\omega, \gamma) &= \rho(\gamma, \omega) \\ \rho^s(\omega, \gamma) &= \max\{\rho(\omega, \gamma), \rho^{-1}(\omega, \gamma)\} \\ \rho_+(\omega, \gamma) &= \rho(\omega, \gamma) + \rho^{-1}(\omega, \gamma) \end{aligned}$$

are also q.-p.-metrics on Λ. If ρ is a q.-m., then ρ^s and ρ_+ are (equivalent) metrics on Λ. To find the fixed point, the most important part is to use the completeness of the metric space. But since there is no symmetry conditions in a q.-m., there are many definitions of completeness in these spaces in the literature.(see [12–14])

Let (Λ, ρ) be a q.-m. and the convergence of a sequence $\{\omega_n\}$ to ω w. r. t.

$$\begin{aligned} \tau_\rho \text{ called } \rho - \text{convergence and is defined } \omega_n \overset{\rho}{\to} \omega \quad &\Leftrightarrow \quad \rho(\omega, \omega_n) \to 0, \\ \tau_{\rho^{-1}} \text{ called } \rho^{-1} - \text{convergence and is defined } \omega_n \overset{\rho^{-1}}{\to} \omega \quad &\Leftrightarrow \quad \rho(\omega_n, \omega) \to 0, \\ \tau_{\rho^s} \text{ called } \rho^s - \text{convergence and is defined } \omega_n \overset{\rho^s}{\to} \omega \quad &\Leftrightarrow \quad \rho(\omega_n, \omega) \to 0 \end{aligned}$$

for $\omega \in \Lambda$. A more detailed explanation of some essential metric properties can be found in [15]. Also, a sequence $\{\omega_n\}$ in Λ is called left(right) $K-$Cauchy if for every $\varepsilon > 0$,there exists $n_0 \in \mathbb{N}$ such that $\forall n, k, n \geq k \geq n_0 (k \geq n \geq n_0), \rho(\omega_k, \omega_n) < \varepsilon$. The left K-Cauchy property under ρ implies the right K-Cauchy property under ρ^{-1}. Assuming

$$\sum_{n=1}^{\infty} \rho(\omega_n, \omega_{n+1}) < \dot{\infty},$$

the sequence $\{\omega_n\}$ in the quasi-metric space (Λ, ρ) is left K-Cauchy.

In a metric space, every convergent sequence is indeed a Cauchy sequence, but since this may not hold true in q.-m., and so there have been several definitions of completeness. Let (Λ, ρ) be a q.-m.. Then (Λ, ρ) is said to be left(right) K (resp. $(M)(Smyth)$)- complete if every left(right) K -Cauchy sequence is ρ(resp. $(\rho^{-1})(\rho^s)$) -convergent.

Indeed, now explain the approach of α-admissibility as constructed by Samet et al. [16].

Let $\Lambda \neq \emptyset$, Υ be a self-mapping (a mapping from Λ to itself), and $\alpha : \Lambda \times \Lambda \to [0, \infty)$ be a function. In this context, Υ is said to be α-admissible if it satisfies the following condition:

$$\text{If } \alpha(\omega, \gamma) \geq 1, \;\; \text{then } \alpha(\Upsilon\omega, \Upsilon\gamma) \geq 1.$$

By introducing the approach of α-admissibility, Samet et al. [16] were able to establish some general fixed point results that encompassed many well-known theorems of complete metric spaces. These fixed point results provide a framework for studying the existence and properties of fixed points for self-mappings on a complete metric space, using the approach of α-admissibility.(see [17–23])

In addition to these, in the study conducted by Jleli and Samet in [24], they they led to the introduction of a new type of contractive mapping known as a θ-contraction. This θ-contraction serves as an attractive generalization within the field. To better understand this approach, let's review some notions and related results concerning θ-contraction.

The family of $\theta : (0, \infty) \to (1, \infty)$ functions that satisfy the following conditions can be denoted by the set Θ.

(θ_1) θ is nondecreasing;

(θ_2) Considering every sequence $\{\varkappa_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \varkappa_n = 0^+$ if only if $\lim_{n\to\infty} \theta(\varkappa_n) = 1$;

(θ_3) There exist $0 < p < 1$ and $\beta \in (0, \infty]$ such that $\lim_{\varkappa\to 0^+} \frac{\theta(\varkappa)-1}{\varkappa^p} = \beta$.

If we define $\theta(\varkappa) = e^{\sqrt{\varkappa}}$ for $\varkappa \leq 1$ and $\theta(\varkappa) = 9$ for $\varkappa > 1$, then $\theta \in \Theta$.

Let $\theta \in \Theta$ and (Λ, ρ) be a metric space. Then $\Upsilon : \Lambda \to \Lambda$ is said to be a θ-contraction if there exists $0 < \delta < 1$ such that

$$\theta(\rho(\Upsilon\omega, \Upsilon\gamma)) \leq [\theta(\rho(\omega, \gamma))]^{\delta} \tag{1.1}$$

for each $\omega, \gamma \in \Lambda$ with $\rho(\Upsilon\omega, \Upsilon\gamma) > 0$.

By choosing appropriate functions for θ, such as $\theta_1(\varkappa) = e^{\sqrt{\varkappa}}$ and $\theta_2(\varkappa) = e^{\sqrt{\varkappa e^{\varkappa}}}$, it is possible to obtain different types of nonequivalent contractions using (1.1).

Indeed, Jleli and Samet proved that every θ-contraction on a complete metric space possesses a unique fixed point. This result provides a valuable insight into the uniqueness and existence of fixed points for a wide range of contractive mappings. If you are interested in exploring more papers and literature related to θ-contractions, there are several resources available (see [25, 26]).

2. The results

Our basic results are based on a novel approach that we have developed.

Let (Λ, ρ) be a q.-m., $\Upsilon : \Lambda \to \Lambda$ be a given mapping and $\alpha : \Lambda \times \Lambda \to [0, \infty)$ be a function. We will consider the following set

$$\Upsilon_\alpha = \{(\omega, \gamma) \in \Lambda \times \Lambda : \alpha(\omega, \gamma) \geq 1 \text{ and } \rho(\Upsilon\omega, \Upsilon\gamma) > 0\}. \tag{2.1}$$

Let (Λ, ρ) be a q.-m. and $\Upsilon : \Lambda \to \Lambda$ be a mapping satisfying

$$\rho(\omega, \gamma) = 0 \Longrightarrow \rho(\Upsilon\omega, \Upsilon\gamma) = 0. \tag{2.2}$$

$\alpha : \Lambda \times \Lambda \to [0, \infty)$ and $\theta \in \Theta$ be two functions. Then we say that Υ is a generalized $(\alpha - \theta_\rho)$-contraction(shortly g. $(\alpha - \theta_\rho)$-c.) if there exists a constant $0 < \delta < 1$ such that

$$\theta(\rho(\Upsilon\omega, \Upsilon\gamma)) \leq [\theta(M(\omega, \gamma))]^{\delta}, \tag{2.3}$$

for each $\omega, \gamma \in \Upsilon_\alpha$, where

$$M(\omega, \gamma) = \max\left\{\rho(\omega, \gamma), \rho(\Upsilon\omega, \omega), \rho(\Upsilon\gamma, \gamma), \frac{1}{2}[\rho(\Upsilon\omega, \gamma) + \rho(\Upsilon\gamma, \omega)]\right\}.$$

Before presenting our main results, let us recall some important remarks:

- If (Λ, ρ) is a T_1-q.-m., then every mapping $\Upsilon : \Lambda \to \Lambda$ satisfies the condition (2.2).
- It is clear from (2.1), (2.2) and (2.3) that if Υ is an (α, θ_ρ)-contraction on a q.-m. (Λ, ρ), then

  $$\rho(\Upsilon\omega, \Upsilon\gamma) \leq \rho(\omega, \gamma),$$

  for each $\omega, \gamma \in \Lambda$ with $\alpha(\omega, \gamma) \geq 1$.

By utilizing the approach of g. $(\alpha - \theta_\rho)$-c., we will now present the following theorem.

Theorem 2.1. *Let (Λ, ρ) be a Hausdorff right K-complete T_1-q.-m., and let $\Upsilon : \Lambda \to \Lambda$ be a g. $(\alpha - \theta_\rho)$-c.. Presume that τ_ρ-continuous and Υ is α-admissible. If there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.*

Proof. Let $\omega_0 \in \Lambda$ be a such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$. Define a sequence $\{\omega_n\}$ in Λ by $\omega_{n+1} = \Upsilon\omega_n$ for each n in $\mathbb{N}$. Since Υ is α-admissible then $\alpha(\omega_{n+1}, \omega_n) \geq 1$ for each n in $\mathbb{N}$. If there exist $k \in \mathbb{N}$ with $\rho(\omega_n, \Upsilon\omega_n) = 0$ then $\omega_n = \Upsilon\omega_n$, ssince ρ is T_1 q.-m.. Hence, ω_k is a fixed point of Υ. Presume $\rho(\omega_n, \Upsilon\omega_n) > 0$ for each n in $\mathbb{N}$. In this case the pair (ω_{n+1}, ω_n) for each n in $\mathbb{N}$ belongs to Υ_α. Since Υ is g. $(\alpha - \theta_\rho)$-c. and (θ_1), we obtain

$$\begin{aligned} \theta(\rho(\omega_{n+1}, \omega_n)) &\leq [\theta(M(\omega_n, \omega_{n-1}))]^\delta \\ &= \left[\theta(\max\left\{ \begin{array}{c} \rho(\omega_n, \omega_{n-1}), \rho(\omega_{n+1}, \omega_n), \rho(\omega_n, \omega_{n-1}), \\ \frac{1}{2}\left[\rho(\omega_{n+1}, \omega_{n-1}) + \rho(\omega_n, \omega_n)\right] \end{array} \right\}\right]^\delta \\ &\leq [\theta(\max\{\rho(\omega_{n+1}, \omega_n), \rho(\omega_n, \omega_{n-1})\}]^\delta . \end{aligned} \tag{2.4}$$

If $\max\{\rho(\omega_{n+1}, \omega_n), \rho(\omega_n, \omega_{n-1})\} = \rho(\omega_{n+1}, \omega_n)$, using (2.4), we get

$$\theta(\rho(\omega_{n+1}, \omega_n)) \leq [\theta(\rho(\omega_{n+1}, \omega_n)]^\delta < \theta(\rho(\omega_{n+1}, \omega_n)),$$

which is a contradiction. Thus, $\max\{\rho(\omega_{n+1}, \omega_n), \rho(\omega_n, \omega_{n-1})\} = \rho(\omega_n, \omega_{n-1})$, and then we obtain

$$\theta(\rho(\omega_{n+1}, \omega_n)) \leq [\theta(\rho(\omega_n, \omega_{n-1}))]^\delta , \tag{2.5}$$

for each n in $\mathbb{N}$. Denote $f_n = \rho(\omega_{n+1}, \omega_n)$ for n in $\mathbb{N}$. Then $f_n > 0$ for each n in $\mathbb{N}$ and repeating this process with using (2.5), we have

$$\theta(f_n) \leq [\theta(f_0)]^{\delta^n} ,$$

i.e.

$$1 < \theta(f_n) \leq [\theta(f_0)]^{\delta^{n-1}} \tag{2.6}$$

for each n in $\mathbb{N}$. When taking the limit as $n \to \infty$ in (2.6), we obtain

$$\lim_{n\to\infty} \theta(f_n) = 1. \tag{2.7}$$

Using (θ_2), we can deduce that $\lim_{n\to\infty} f_n = 0^+$, thus using (θ_3), there exist $p \in (0,1)$ and $\beta \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(f_n) - 1}{(f_n)^p} = \beta.$$

Presume that $\beta < \infty$. In this case, let $F = \frac{\beta}{2} > 0$. Using the definition of the limit, there exists n_0 in $\mathbb{N}$ such that, for each $n_0 \leq n$,

$$\left| \frac{\theta(f_n) - 1}{(f_n)^p} - \beta \right| \leq F.$$

This implies that, for each $n_0 \leq n$,

$$\frac{\theta(f_n) - 1}{(f_n)^p} \geq \beta - F = F.$$

Then, for each $n_0 \leq n$,

$$n(f_n)^p \leq Dn\left[\theta(f_n) - 1\right],$$

where $D = 1/F$.

Presume now that $\beta = \infty$. Let $F > 0$ be an arbitrary positive number. Using the definition of the limit, there exists n_0 in $\mathbb{N}$ such that, for each $n_0 \leq n$,

$$\frac{\theta(f_n) - 1}{(f_n)^p} \geq F.$$

This implies that, for each $n_0 \leq n$,

$$n\left[f_n\right]^p \leq Dn\left[\theta(f_n) - 1\right],$$

where $D = 1/F$.

Thus, in all cases, there exist $D > 0$ and n_0 in $\mathbb{N}$ such that

$$n\left[f_n\right]^p \leq Dn\left[\theta(f_n) - 1\right],$$

for each $n_0 \leq n$. Using (2.6), we obtain

$$n\,[f_n]^p \leq Dn\left[[\theta(f_0)]^{\delta^{n-1}} - 1\right],$$

for each $n_0 \leq n$. Letting $n \to \infty$ from the given inequality, we have

$$\lim_{n\to\infty} n\,[f_n]^p = 0.$$

Thus, there exists n_1 in $\mathbb{N}$ such that $n\,[f_n]^p \leq 1$ for each $n \geq n_1$, so we have, for each $n \geq n_1$,

$$f_n \leq \frac{1}{n^{1/p}}. \tag{2.8}$$

In order to show that $\{\omega_n\}$ is a left K-Cauchy sequence, consider m, n in $\mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for ρ and using (2.8), we have

$$\begin{aligned}
\rho(\omega_m, \omega_n) &\leq \rho(\omega_m, \omega_{m-1}) + \rho(\omega_{m-1}, \omega_{m-2}) + \cdots + \rho(\omega_{n+1}, \omega_n) \\
&= f_{m-1} + f_m + \cdots + f_n \\
&= \sum_{i=n}^{m-1} f_i \leq \sum_{i=n}^{\infty} f_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/p}}.
\end{aligned}$$

By the convergence of the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{1/p}}$, we get $\rho(\omega_m, \omega_n) \to 0$ as $n \to \infty$. This yields that $\{\omega_n\}$ is a right K-Cauchy sequence in the q.-m. (Λ, ρ). Since (Λ, ρ) is a right K-complete, there exists $\eta \in \Lambda$ such that the sequence $\{\omega_n\}$ is ρ-converges to $\eta \in \Lambda$; that is, $\rho(\eta, \omega_n) \to 0$ as $n \to \infty$. Since Υ is τ_ρ-continuous then $\rho(\Upsilon\eta, \Upsilon\omega_n) = \rho(\Upsilon\eta, \omega_{n+1}) \to 0$ as $n \to \infty$. Since Λ is Hausdorff, we get $\Upsilon\eta = \eta$. □

In Theorem 2.1, if we consider the approach of $\tau_{\rho^{-1}}$-continuity, we can derive the following theorem.

Theorem 2.2. *Let (Λ, ρ) be a right M-complete T_1-q.-m. such that $(\Lambda, \tau_{\rho^{-1}})$ is Hausdorff and $\Upsilon : \Lambda \to \Lambda$ be a g. $(\alpha - \theta_\rho)$-c.. Presume that Υ is $\tau_{\rho^{-1}}$-continuous and α-admissible. If there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.*

Proof. Similar to the proof of Theorem 2.1, we can take iterative sequence $\{\omega_n\}$ right K-Cauchy. Since (Λ, ρ) right M-complete, there exists $\eta \in \Lambda$ such that $\{\omega_n\}$ is ρ^{-1}-converges to η, that is, $\rho(\omega_n, \eta) \to 0$ as $n \to \infty$. Using $\tau_{\rho^{-1}}$-continuity of Υ, we get $\rho(\Upsilon\omega_n, \Upsilon\eta) = \rho(\omega_{n+1}, \Upsilon\eta) \to 0$ as $n \to \infty$. Since $(\Lambda, \tau_{\rho^{-1}})$ is Hausdorff, we get $\eta = \Upsilon\eta$. □

Theorem 2.3. *Let (Λ, ρ) be a right Smyth complete T_1 q.-m. and $\Upsilon : \Lambda \to \Lambda$ be a g. $(\alpha - \theta_\rho)$-c.. Presume that Υ is τ_ρ or $\tau_{\rho^{-1}}$-continuous and α-admissible. If there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.*

Proof. Similar to the proof of Theorem 2.1, we can take iterative sequence $\{\omega_n\}$ right K-Cauchy. Since (Λ, ρ) is right Smyth complete, there exists $\eta \in \Lambda$ such that $\{\omega_n\}$ is ρ^s-converges to $\eta \in \Lambda$; that is, $\rho^s(\omega_n, \eta) \to 0$ as $n \to \infty$. If Υ is τ_ρ-continuous, then

$$\rho(\Upsilon\eta, \Upsilon\omega_n) = \rho(\Upsilon\eta, \omega_{n+1}) \to 0 \text{ as } n \to \infty.$$

Therefore we get,

$$\rho(\Upsilon\eta, \eta) \leq \rho(\Upsilon\eta, \omega_{n+1}) + \rho(\omega_{n+1}, \eta) \to 0 \text{ as } n \to \infty.$$

If Υ is $\tau_{\rho^{-1}}$-continuous, then

$$\rho(\Upsilon\omega_n, \Upsilon\eta) = \rho(\omega_{n+1}, \Upsilon\eta) \to 0 \text{ as } n \to \infty.$$

Therefore we have,

$$\rho(\eta, \Upsilon\eta) \leq \rho(\eta, \omega_{n+1}) + \rho(\Upsilon\omega_{n+1}, \Upsilon\eta) \to 0 \text{ as } n \to \infty.$$

Since Υ is T_1-q.-m., we obtain $\Upsilon\eta = \eta$. □

Based on Theorem 2.1, we can derive the following corollaries.

Corollary 2.1. *Let (Λ, ρ) be a Hausdorff right K-complete T_1-q.-m. and $\Upsilon : \Lambda \to \Lambda$ be given a mapping that satisfies*

$$\theta(\rho(\Upsilon\omega, \Upsilon\gamma)) \leq \left[\theta(t_1\rho(\omega,\gamma) + t_2\rho(\Upsilon\omega,\omega) + t_3\rho(\Upsilon\gamma,\gamma), t_4\left[\rho(\Upsilon\omega,\gamma) + \rho(\Upsilon\gamma,\omega)\right])\right]^{\delta}, \tag{2.9}$$

for each $\omega, \gamma \in \Lambda$, where $0 < \delta < 1$, $t_1, t_2, t_3, t_4 \geq 0$, and $t_1 + t_2 + t_3 + 2t_4 < 1$. Presume that Υ is τ_ρ-continuous and α-admissible . If there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.

Proof. for each $\omega, \gamma \in \Lambda$, we have

$$\begin{aligned} & t_1\rho(\omega,\gamma) + t_2\rho(\Upsilon\omega,\omega) + t_3\rho(\Upsilon\gamma,\gamma), t_4\left[\rho(\Upsilon\omega,\gamma) + \rho(\Upsilon\gamma,\omega)\right] \\ \leq \quad & (t_1 + t_2 + t_3 + 2t_4)\max\left\{\rho(\omega,\gamma), \rho(\Upsilon\omega,\omega), \rho(\Upsilon\gamma,\gamma), \frac{1}{2}\left[\rho(\Upsilon\omega,\gamma) + \rho(\Upsilon\gamma,\omega)\right]\right\} \\ \leq \quad & M(\omega,\gamma). \end{aligned}$$

Then using (θ_1) we see that (2.3) is a consequence of (2.9). Therefore, the proof is concluded. □

Corollary 2.2. *Let (Λ, ρ) be a Hausdorff right K-complete T_1-q.-m. and $\Upsilon : \Lambda \to \Lambda$ be given a mapping that satisfies*

$$\rho(\Upsilon\omega, \Upsilon\gamma) \leq t_1\rho(\omega,\gamma) + t_2\rho(\Upsilon\omega,\omega) + t_3\rho(\Upsilon\gamma,\gamma),$$

for each $\omega, \gamma \in \Lambda$, where $t_1 + t_2 + t_3 \geq 0$ and $t_1 + t_2 + t_3 < 1$. Presume that Υ is τ_ρ-continuous or α-admissible. If there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.

Proof. If $\theta(\varkappa) = e^{\sqrt{\varkappa}}$ and $\delta = \sqrt{t_1 + t_2 + t_3}$, since $\rho(\Upsilon\omega, \Upsilon\gamma) \leq (t_1 + t_2 + t_3)M(\omega,\gamma)$, using Theorem 2.1, then the proof is concluded. □

Corollary 2.3. *Let (Λ, ρ) be a Hausdorff right K-complete T_1-q.-m. and $\Upsilon : \Lambda \to \Lambda$ be given a mapping that satisfies*

$$\rho(\Upsilon\omega, \Upsilon\gamma) \leq L\max\{\rho(\Upsilon\omega,\omega), \rho(\Upsilon\gamma,\gamma)\}$$

for each $\omega, \gamma \in \Lambda$, where $L \in [0, 1)$. Presume that Υ is τ_ρ-continuous or α-admissible. If there exists $\omega_0 \in \Lambda$ such that $\alpha(\Upsilon\omega_0, \omega_0) \geq 1$, then Υ has a fixed point in Λ.

Proof. If $\theta(\varkappa) = e^{\sqrt{\varkappa}}$ and $\delta = \sqrt{L}$, since $\rho(\Upsilon\omega, \Upsilon\gamma) \leq \lambda M(\omega,\gamma)$, using Theorem 2.1, then the proof is concluded. □

Remark 2.1. By considering the notion of left completeness in the sense of K, M and $Smyth$, we can extend similar fixed point results to the setting of q.- m. spaces.

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Strongly Lacunary $\mathcal{I}^*$-Convergence and Strongly Lacunary $\mathcal{I}^*$-Cauchy Sequence

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Abstract
In this paper, we defined the concepts of lacunary $\mathcal{I}^*$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. We investigated the relations between strongly lacunary $\mathcal{I}$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. Also, we defined the concept of strongly lacunary $\mathcal{I}^*$-Cauchy sequence and investigated the relations between strongly lacunary $\mathcal{I}$-Cauchy sequence and strongly lacunary $\mathcal{I}^*$-Cauchy sequence.

Keywords: *Ideal, Lacunary sequence, $\mathcal{I}$-Convergence, $\mathcal{I}$-Cauchy Sequence*

AMS Subject Classification (2020): *40A05; 40A35*

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1. Introduction and definitions

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [1] and Schoenberg [2]. The concept of $\mathcal{I}$-convergence in a metric space, which is a generalized from of statistical convergence, was introduced by Kostyrko et al. [3]. Later it was further studied many others. Nabiev et al. [4] studied on $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence with some properties. Recently, Das et al. [5] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal. Also, Yamancı and Gürdal [6] introduced the notions lacunary $\mathcal{I}$-convergence and lacunary $\mathcal{I}$-Cauchy in the topology induced by random n-normed spaces and prove some important results. Debnath [7] studied the notion of lacunary ideal convergence in intuitionistic fuzzy normed linear spaces as a variant of the notion of ideal convergence. Tripathy et al. [8] introduced the concept of lacunary $\mathcal{I}$-convergent sequences. A lot of development have been made about the statistical convergence and ideal convergence defined in different setups [9–11].

In this paper, we defined the concepts of lacunary $\mathcal{I}^*$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. We investigated the relations between strongly lacunary $\mathcal{I}$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. Also, we defined the concept of strongly lacunary $\mathcal{I}^*$-Cauchy sequence and investigated the relations between strongly

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lacunary $\mathcal{I}$-Cauchy sequence and strongly lacunary $\mathcal{I}^*$-Cauchy sequence.

Now, we recall some basic concepts and definitions (see [3, 4, 6–8, 12–21]).
A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathcal{I}$,
(ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$,
(iii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.
A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter if and only if
(i) $\emptyset \notin \mathcal{F}$,
(ii) If $A, B \in F$, then $A \cap B \in \mathcal{F}$,
(iii) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.
$\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$, then the set

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter in $\mathbb{N}$, called the filter associated with $\mathcal{I}$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \cdots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}.$$

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-Cauchy sequence if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\} \in \mathcal{I}.$$

A sequence (x_n) is said to be $\mathcal{I}^*$-convergent to L if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{k \to \infty} x_{m_k} = L.$$

A sequence (x_n) is said to be $\mathcal{I}^*$-Cauchy sequence if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that the subsequence $x_M = (x_{m_k})$ is an ordinary Cauchy sequence, that is,

$$\lim_{k,p \to \infty} |x_{m_k} - x_{m_p}| = 0.$$

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty$$

as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by

$$I_r = (k_{r-1}, k_r]$$

and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Throughout the paper, we take $\theta = \{k_r\}$ be a lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal.
A sequence (x_n) of elements of $\mathbb{R}$ is said to be strongly lacunary convergent to $L \in \mathbb{R}$ if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| = 0.$$

A sequence (x_n) is said to be a strongly lacunary $\mathcal{I}$-convergent to L, if for every $\varepsilon > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $x_n \to L[\mathcal{I}_\theta]$.

A sequence (x_n) is said to be a strongly lacunary $\mathcal{I}$-Cauchy if for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - x_N| \geq \varepsilon \right\} \in \mathcal{I}.$$

Lemma 1.1. *[4] Let $\{P_i\}_1^\infty$ be a countable collection of subsets of $\mathbb{N}$ such that $P_i \in F(\mathcal{I})$ for each i, where $F(\mathcal{I})$ is a filter associate with an admissible ideal $\mathcal{I}$ with property (AP). Then there exists a set $P \subset \mathbb{N}$ such that $P \in F(\mathcal{I})$ and the set $P \backslash P_i$ is finite for all i.*

2. Main results

In this section, firstly, we gave the concepts of lacunary $\mathcal{I}^*$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. We investigated the relations between strongly lacunary $\mathcal{I}$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. Then after, we gave the concept of strongly lacunary $\mathcal{I}^*$-Cauchy sequence and investigated the relations between strongly lacunary $\mathcal{I}$-Cauchy sequence and strongly lacunary $\mathcal{I}^*$-Cauchy sequence.

Definition 2.1. [12]. A sequence (x_n) is said to be lacunary $\mathcal{I}^*$-convergent to L if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{k \in I_r} x_{m_k} = L.$$

In this case, we write $x_n \to L(\mathcal{I}_\theta^*)$.

Definition 2.2. A sequence (x_n) is said to be strongly lacunary $\mathcal{I}^*$-convergent to L if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| = 0.$$

In this case, we write $x_n \to L[\mathcal{I}_\theta^*]$.

Theorem 2.1. *If a sequence (x_n) is strongly lacunary $\mathcal{I}^*$-convergent to L, then it is lacunary $\mathcal{I}^*$-convergent to L.*

Proof. Let $x_n \to L[\mathcal{I}_\theta^*]$. Then, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ $(i.e. H = \mathbb{N} \backslash M' \in \mathcal{I})$ and for every $\varepsilon > 0$ there is a $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r > r_0$ we have

$$\frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| < \varepsilon, \ (r \in M').$$

Then, we have

$$\begin{aligned} \left| \frac{1}{h_r} \sum_{k \in I_r} x_{m_k} - L \right| &\leq \frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| \\ &< \varepsilon, \ (r \in M') \end{aligned}$$

for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$ and so $x_n \to L(\mathcal{I}_\theta^*)$. □

Theorem 2.2. *If a sequence (x_n) is strongly lacunary $\mathcal{I}^*$-convergent to L, then it is strongly lacunary $\mathcal{I}$-convergent to L.*

Proof. Let $x_n \to L[\mathcal{I}_\theta^*]$. Then, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ $(i.e. H = \mathbb{N} \backslash M' \in \mathcal{I})$ and for every $\varepsilon > 0$ there is a $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r > r_0$ we have

$$\frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| < \varepsilon, \ (r \in M').$$

Then,

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| \geq \varepsilon \right\} \subset H \cup \{1, 2, \cdots, r_0\}.$$

Since $\mathcal{I}$ is an admissible ideal, we have

$$H \cup \{1, 2, \cdots, r_0\} \in \mathcal{I}$$

and so $A(\varepsilon) \in \mathcal{I}$. Hence, $x_n \to L[\mathcal{I}_\theta]$. □

Theorem 2.3. *Let $\mathcal{I}$ be a admissible ideal with property (AP). If (x_n) is strongly lacunary $\mathcal{I}$-convergent to L, then it is strongly lacunary $\mathcal{I}^*$-convergent to L.*

Proof. Assume that $x_n \to L[\mathcal{I}_\theta]$. Then, for every $\varepsilon > 0$,

$$T(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| \geq \varepsilon \right\} \in \mathcal{I}.$$

Put

$$T_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| \geq 1 \right\} \text{ and } T_p = \left\{ r \in \mathbb{N} : \frac{1}{p} \leq \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| < \frac{1}{p-1} \right\},$$

for $p \geq 2$ and $p \in \mathbb{N}$. It is clear that $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathcal{I}$ for each $i \in \mathbb{N}$. By property (AP) there is a sequence $\{V_p\}_{p \in \mathbb{N}}$ such that $T_j \Delta V_j$ is a finite set for each $j \in \mathbb{N}$ and

$$V = \bigcup_{j=1}^{\infty} V_j \in \mathcal{I}.$$

We prove that,

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| = 0,$$

for $M' = \mathbb{N} \backslash V \in \mathcal{F}(\mathcal{I})$. Let $\delta > 0$ be given. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Then,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| \geq \delta \right\} \subset \bigcup_{j=1}^{q-1} T_j.$$

Since $T_j \Delta V_j$ is a finite set for $j \in \{1, 2, \cdots, q-1\}$, there exists $r_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q-1} T_j \right) \cap \{ r \in \mathbb{N} : r \geq r_0 \} = \left(\bigcup_{j=1}^{q-1} V_j \right) \cap \{ r \in \mathbb{N} : r \geq r_0 \}.$$

If $r \geq r_0$ and $r \notin V$, then

$$r \notin \bigcup_{j=1}^{q-1} V_j \text{ and so } r \notin \bigcup_{j=1}^{q-1} T_j.$$

We have

$$\frac{1}{h_r} \sum_{n \in I_r} |x_n - L| < \frac{1}{q} < \delta.$$

This implies that

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| = 0.$$

Hence, we have $x_n \to L[\mathcal{I}_\theta^*]$. This completes the proof. □

Definition 2.3. [12]. A sequence (x_n) is said to be lacunary $\mathcal{I}^*$-Cauchy sequence if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{k,p \in I_r} (x_{m_k} - x_{m_p}) = 0.$$

Definition 2.4. A sequence (x_n) is said to be strongly lacunary $\mathcal{I}^*$-Cauchy sequence if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r\to\infty \\ (r\in M')}} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| = 0.$$

Theorem 2.4. *If the sequence (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence, then (x_n) is lacunary $\mathcal{I}^*$-Cauchy sequence.*

Proof. Suppose that (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence. Then, for every $\varepsilon > 0$, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$

$$\frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| < \varepsilon, \ \ (r \in M')$$

for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$. Then, we have

$$\begin{aligned} \left| \frac{1}{h_r} \sum_{k,p\in I_r} (x_{m_k} - x_{m_p}) \right| &\leq \frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| \\ &< \varepsilon, \ \ (r \in M') \end{aligned}$$

for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$ and so (x_n) is lacunary $\mathcal{I}^*$-Cauchy sequence. □

Theorem 2.5. *If the sequence (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence, then (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence.*

Proof. Suppose that (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence. Then, for every $\varepsilon > 0$, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$

$$\frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| < \varepsilon, \ \ (r \in M')$$

for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$. Let $N = N(\varepsilon) \in I_{r_0+1}$. Then, for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$

$$\frac{1}{h_r} \sum_{k\in I_r} |x_{m_k} - x_N| < \varepsilon, \ \ (r \in M').$$

Now, let $H = \mathbb{N} \setminus M'$. It is clear that $H \in \mathcal{I}$. Then,

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in I_r} |x_n - x_N| \geq \varepsilon \right\} \subset H \cup \{1, 2, \cdots, r_0\}.$$

Since $\mathcal{I}$ is an admissible ideal, we have

$$H \cup \{1, 2, \cdots, r_0\} \in \mathcal{I}$$

and so $A(\varepsilon) \in \mathcal{I}$. Hence, (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence. □

Theorem 2.6. *If $\mathcal{I}$ admissible ideal with property (AP). The sequence (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence, then (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence.*

Proof. Assume that (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence. Then, for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in I_r} |x_n - x_N| \geq \varepsilon \right\} \in \mathcal{I}.$$

Let

$$P_i = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in I_r} |x_n - x_{m_i}| \geq \frac{1}{i} \right\}, \ i = 1, 2, \ldots,$$

where $m_i = N\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I})$ for $i = 1, 2, \cdots$. Since $\mathcal{I}$ has the (AP) property, then by Lemma 1.1 there exists a set $P \subset \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I})$ and $P \setminus P_i$ is finite for all i. Now, we show that

$$\lim_{\substack{r\to\infty \\ (r\in P)}} \frac{1}{h_r} \sum_{n,m\in I_r} |x_n - x_m| = 0.$$

To prove this let $\varepsilon > 0$, $j \in \mathbb{N}$ such that $j > \frac{2}{\varepsilon}$. If $r \in P$ then $P \setminus P_j$ is a finite set, so there exists $r_0 = r_0(j)$ such that $r \in P_j$ for all $r > r_0(j)$. Therefore, for all $r > r_0(j)$

$$\frac{1}{h_r} \sum_{n\in I_r} |x_n - x_{m_j}| < \frac{1}{j} \text{ and } \frac{1}{h_r} \sum_{m\in I_r} |x_m - x_{m_j}| < \frac{1}{j}.$$

Hence, for all $r > r_0(j)$ it follows that

$$\begin{aligned} \frac{1}{h_r} \sum_{n,m\in I_r} |x_n - x_m| &\leq \frac{1}{h_r} \sum_{n\in I_r} |x_n - x_{m_j}| + \frac{1}{h_r} \sum_{m\in I_r} |x_m - x_{m_j}| \\ &< \frac{1}{j} + \frac{1}{j} < \varepsilon. \end{aligned}$$

Thus, for any $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that for all $r > r_0(\varepsilon)$ and $r \in P \in \mathcal{F}(\mathcal{I})$

$$\frac{1}{h_r} \sum_{n,m\in I_r} |x_n - x_m| < \varepsilon.$$

This shows that the sequence (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence. □

Theorem 2.7. *If a sequence (x_n) is strongly lacunary $\mathcal{I}^*$-convergent to L, then (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence.*

Proof. Let $x_n \to L[\mathcal{I}_\theta^*]$. Then, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r\to\infty \\ (r\in M')}} \frac{1}{h_r} \sum_{k\in I_r} |x_{m_k} - L| = 0.$$

It shows that there exists $r_0 = r_0(\varepsilon)$ such that

$$\frac{1}{h_r} \sum_{k\in I_r} |x_{m_k} - L| < \frac{\varepsilon}{2},\ (r \in M')$$

for every $\varepsilon > 0$ and all $r > r_0$. Since

$$\begin{aligned} \frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| &\leq \frac{1}{h_r} \sum_{k\in I_r} |x_{m_k} - L| + \frac{1}{h_r} \sum_{p\in I_r} |x_{m_p} - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ (r \in M') \end{aligned}$$

for all $r > r_0$, so we have

$$\lim_{\substack{r\to\infty \\ (r\in M')}} \frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| = 0$$

i.e., (x_n) is a strongly lacunary $\mathcal{I}^*$-Cauchy sequence. Then, by Theorem 2.5 (x_n) is a strongly lacunary $\mathcal{I}$-Cauchy sequence. □

Conclusions and future work

We investigated the concepts of strongly lacunary $\mathcal{I}^*$-convergence and strongly lacunary $\mathcal{I}^*$-Cauchy sequence. These concepts can also be studied for the double sequence in the future.

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The Farey Sum of Pythagorean and Eisenstein Triples

Mircea Crasmareanu*

Abstract
A composition law, inspired by the Farey addition, is introduced on the set of Pythagorean triples. We study some of its properties as well as two symmetric matrices naturally associated to a given Pythagorean triple. Several examples are discussed, some of them involving the degenerated Pythagorean triple $(1, 0, 1)$. The case of Eisenstein triples is also presented.

Keywords: *Farey sum, Pythagorean (Eisenstein) triple, Symmetric matrix*

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1. The Farey composition law on Pythagorean triples

Fix the set $\mathbb{N}^2(<) := \{(p, q) \in \mathbb{N}^* \times \mathbb{N}^*; p < q\}$ and the map:

$$P : \mathbb{N}^2(<) \to (\mathbb{N}^*)^3, \quad P(p, q) := (q^2 - p^2, 2pq, p^2 + q^2).$$

It is well-known that P provides a parametrization (up to a strictly positive multiplicative factor) of the set of Pythagorean triples $PT := \{(a, b, c) \in (\mathbb{N}^*)^3; 2|b, a^2 + b^2 = c^2\}$. If, in addition $\gcd(p, q) = 1$ with $2 \nmid (q - p)$ then (a, b, c) is a primitive (i.e. $\gcd(a, b) = 1$) Pythagorean triple.

The aim of this short note is to study the transport of a natural sum from $\mathbb{N}^2(<)$ to PT. Namely, defining $(p, q) \oplus (p', q') := (p + p', q + q')$ it follows the pair $(PT, \oplus)$ with:

$$(a, b, c,) \oplus (a', b', c') := (a'', b'', c'') = ((q + q')^2 - (p + p')^2, 2(p + p')(q + q'), (p + p')^2 + (q + q')^2).$$

More precisely, we have:

$$a'' := a + a' + 2(qq' - pp'), \quad b'' := b + b' + 2(pq' + qp'), \quad c'' := c + c' + 2(qq' + pp'). \tag{1.1}$$

Remark 1.1. If the initial pair (p, q) from $\mathbb{N}^2(<)$ is considered as the ratio $\frac{p}{q} \in (0, 1)$ then the sum:

$$\frac{p}{q} \oplus \frac{p'}{q'} := \frac{p + p'}{q + q'}$$

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is called *the mediant* in [1] due to the double inequality:

$$\frac{p}{q} < \frac{p+p'}{q+q'} < \frac{p'}{q'}.$$

But we prefer to use the name of *Farey sum* after [2, p. 209] although, obviously, the initial sum on $\mathbb{N}^2(<)$ is the restriction of the additive law of the real 2-dimensional linear space $\mathbb{R}^2$; another source for the applications of the Farey sequences in hyperbolic dynamics is [3]. Our choice for this name is also inspired by the very nice picture of page 23 from the book [4] illustrating a relationship between the circular Farey diagram and the Pythagorean triples. We point out that a group structure on the subset of primitive Pythagorean triples is considered in [5]. □

Properties 1.1. 1) The composition law $\oplus$ on PT is commutative but without a neutral element.
2) The *height* of the Pythagorean triple $pt := (a, b, c)$ is $h(pt) := c - b = (q-p)^2$. For our triple of Pythagorean triples it follows:

$$h((pt)'') = h((pt)') + h(pt) - 2(q-p)(q'-p') < h((pt)') + h(pt).$$

3) The usual CBS inequality provides an upper bound for the resulting Pythagorean triple in terms of the given $(a, b, c,), (a', b', c') \in PT$:

$$a'' < a + a' + 2\sqrt{cc'}, \quad b'' < b + b' + 2\sqrt{cc'}, \quad \sqrt{c''} \leq \sqrt{c} + \sqrt{c'} \tag{1.2}$$

with equality in the last relation if and only if $c = c'$ which, in turn, yields $c'' = 4c = 4c'$ as consequence of the relation:

$$(a, b, c) \oplus (a, b, c) = 4(a, b, c).$$

□

Example 1.1. 1) Since $(1, 3) \oplus (2, 3) = (3, 6)$ we have $2(4, 3, 5) \oplus (5, 12, 13) = 9(3, 4, 5)$.
2) The sum $(1, 2) \oplus (1, 3) = (2, 5)$ gives $(3, 4, 5) \oplus 2(4, 3, 5) = (21, 20, 29)$.
3) The restriction of the complex multiplication to the unit circle S^1 gives a group multiplication on the set of *all* Pythagorean triples:

$$(a, b, c) \odot (a', b', c') = (aa' - bb', ab' + a'b, cc'), \quad (a, b, c) \odot (a, b, c) = (a^2 - b^2, 2ab, c^2 = a^2 + b^2)$$

having as neutral element the degenerate Pythagorean triple $(1, 0, 1)$ which can be considered as the image through the map P of the pair $(\tilde{p}, \tilde{q}) = (0, 1)$. For our sum we have:

$$(a, b, c) \oplus (1, 0, 1) = (a + 2q + 1, b + 2p, c + 2q + 1), \quad (3, 4, 5) \oplus (1, 0, 1) = 2(4, 3, 5).$$

4) Fix $k \in \mathbb{N}^*$ and a triangle Δ. Then we call Δ as being a *k-triangle* if its area $\mathcal{A}$ is k times its semi-parameter $s = \frac{1}{2}(a + b + c)$. Let us find the k-rectangular triangles for a prime number k. From $ab = k(a + b + c)$ it results:

$$p(q-p) = k$$

with only two solutions:

$$\begin{cases} (p = 1, q = k+1), & (pt)_1 = (k(k+2), 2(k+1), k^2 + 2k + 2), \quad \mathcal{A}_1 = k(k+1)(k+2), \\ (p = k, q = k+1), & (pt)_2 = (2k+1, 2k(k+1), 2k^2 + 2k + 1), \quad \mathcal{A}_2 = k(k+1)(2k+1) \end{cases}$$

Hence, their Farey sum is:

$$(pt)_1 \oplus (pt)_2 = (k+1)^2(3, 4, 5).$$

Also concerning the area there exist pairs of Pythagorean triples sharing it; for example the area $\mathcal{A} = 210$ is provided by:

$$(p_1 = 2, q_1 = 5) \quad (pt)_1 = (21, 20, 29), \quad (p_2 = 1, q_2 = 6), \quad (pt)_2 = (35, 12, 37)$$

and their Farey sum is:

$$(p = 3, q = 11), \quad pt = 2(56, 33, 65).$$

5) Let $(F_n)_{n\in\mathbb{N}}$ be the Fibonacci sequence and let $p = p_n := F_{n+1} < q = q_n := F_{n+2}$. It results the *$n$-Fibonacci-Pythagorean triple* $(Fpt)_n = (a_n, b_n, c_n)$:

$$a_n = F_n F_{n+3}, \quad b_n = 2F_{n+1}F_{n+2}, \quad c_n = F^2_{n+1} + F^2_{n+2}$$

for which we have the Farey sum of Fibonacci type:

$$(Fpt)_n \oplus (Fpt)_{n+1} = (Fpt)_{n+2}.$$

6) Fix c a hypotenuse which as natural number has only two representations as sum of different squares; for example $65 = 1^2 + 8^2 = 4^2 + 7^2$ or $145 = 1^2 + 12^2 = 8^2 + 9^2$. Then we call the corresponding Pythagorean triples (a_1, b_1, c), (a_2, b_2, c) as being $hypotenuse - related$ and we can perform their Farey sum. For our examples above we have:

$$\begin{cases} (p_1 = 1, q_1 = 8) \oplus (p_2 = 4, q_2 = 7) = (p = 5, q = 15), (63, 16, 65) \oplus (33, 56, 65) = 50(4, 3, 5), \\ (p_1 = 1, q_1 = 12) \oplus (p_2 = 8, q_2 = 9) = (p = 9, q = 21), (143, 24, 145) \oplus (17, 144, 145) = 18(20, 21, 29). \end{cases}$$

The class of these c is provided by the expression $c = p_1^{a_1} p_2^{a_2}$ with $p_1 < p_2$ prime numbers of the form $4k + 1$; recall also that any prime number of the form $4k + 1$ is a sum of two squares. Related to this discussion we recall that a positive integer k is a sum of two triangular numbers:

$$k = \frac{u(u+1)}{2} + \frac{v(v+1)}{2} \tag{1.3}$$

if and only if $4k + 1$ is a sum of squares; namely (1.3) implies $4k + 1 = (v - u)^2 + (u + v + 1)^2$. Hence this k with $u < v$ provides the Pythagorean triple:

$$(p = v - u < q = u + v + 1), \quad a = (2u + 1)(2v + 1), \quad b = 2(v - u)(u + v + 1), \quad c = 4k + 1. \tag{1.4}$$

As example, $c = 65$ is provided by $k = 16$ which is generated by two triangular numbers:

$$u_1 = 3 < v_2 = 4, \quad (a_1, b_1, c) = (63, 16, 65), \quad u_2 = 1 < v_2 = 5, \quad (a_2, b_2, c) = (33, 56, 65).$$

7) Fix $2N$ an even number and ask the given triangle has the perimeter $2s = 2N$. It follows the quadratic Diophantine equation:

$$q(p + q) = N$$

which for some value of N has only two solutions; namely $N \in \{120, 180, 240, 252, 336, ...\}$. Then we call the corresponding Pythagorean triples (a_1, b_1, c), (a_2, b_2, c) as being $perimeter - related$ and we can perform their Farey sum. For the example of $N = 120$ we have $(p_1 = 2 < q_1 = 10)$ and $(p_1 = 7 < p_2 = 8)$ and then:

$$2^3(12, 5, 13) \oplus (15, 112, 113) = 3^4(3, 4, 5).$$

Returning to the last inequality (1.2) the right-hand-side of it can be interpreted in terms of a quasi-arithmetic mean. Fix an open real interval I and $M : I \times I \to I$ a *mean* i.e. for any pair $(x, y) \in I \times I$ we have the double inequality:

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}.$$

Recall also that M is called *quasi-arithmetic* if there exists a continuous and strictly monotonic function $f : I \to \mathbb{R}$ such that:

$$M(x, y) = M_f(x, y) := f^{-1}\left(\frac{f(x) + f(y)}{2}\right).$$

Hence, with $I = \mathbb{R}^*_+ := (0, +\infty)$ the last inequality (1.2) reads:

$$c'' \leq 4M_{\sqrt{\cdot}}(c, c').$$

2. Two symmetric matrices associated to a given Pythagorean triple

In the following we provide a matrix formalism associated to a given Pythagorean triple. Namely, the relations (1.1) can be put into the form:

$$\begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} + 2\Gamma \cdot \begin{pmatrix} p' \\ q' \end{pmatrix}, \quad \Gamma := \begin{pmatrix} -p & q \\ q & p \\ p & q \end{pmatrix} \in M_{3,2}(\mathbb{Z}^*).$$

The matrix Γ and its transpose Γ^t provides two new matrices.

I) a symmetric 2×2 one:

$$A := \Gamma^t \cdot \Gamma = \begin{pmatrix} 2p^2+q^2 & pq \\ pq & p^2+2q^2 \end{pmatrix} = \begin{pmatrix} c+p^2 & \frac{b}{2} \\ \frac{b}{2} & c+q^2 \end{pmatrix} \in Sym(2, \mathbb{N}^*), \det A = 2c^2 > 0.$$

Allowing the pair (p,q) to be a point in the Euclidean plane $\mathbb{R}^2$ then the map P is a regular parametrization from $\mathbb{R}^2 \setminus \{(0,0)\}$ of the cone $C : x^2+y^2-z^2=0$ (which can be called *the Pythagorean cone*) and hence A is exactly the first fundamental form of this quadric in $\mathbb{R}^3$. Its coefficients of the fundamental forms are in the Gauss notation:

$$\begin{cases} E = 4(2p^2+q^2), \quad F = 4pq(=2b), \quad G = 4(p^2+2q^2) \\ L = \frac{2\sqrt{2}p^2}{p^2+q^2} \leq 2\sqrt{2}, \quad M = \frac{\sqrt{2}pq}{p^2+q^2} \left(= \frac{\sqrt{2}}{2}\sin B < \frac{\sqrt{2}}{2}\right), \quad N = \frac{2\sqrt{2}q^2}{p^2+q^2} \leq 2\sqrt{2}. \end{cases}$$

Returning to the matrix A, recall after [6] that any symmetric 2×2 matrix has two Hermitian parameters, one real being half of its trace, and one complex, called *Hopf invariant*, which for our A is:

$$H(A) = \frac{p^2-q^2}{2} - (pq)i = \frac{1}{2}(p-iq)^2 \in \mathbb{C}^*.$$

Let us remark that if p and q share the same parity (which means that (a,b,c) is not a primitive Pythagorean triple since 2 divides also a) then $H(A)$ is a Gaussian integer. Recall also that a proof of the fact that the map P is a parametrization of the set PT is based exactly on the complex number $(p+iq)^2 = 2\overline{H(A)}$ since $c = |2H(A)|$. The eigenvalues and associated eigenvectors of the matrix A are:

$$\lambda_1 = c < \lambda_2 = 2c, \quad \bar{v}_1 = (-q,p) = -q+ip = i \cdot \sqrt{2\overline{H(A)}}, \quad \bar{v}_2 = (p,q) = p+iq = \sqrt{2\overline{H(A)}}.$$

So, the invertible matrix making A a diagonal one is:

$$\begin{cases} S = \begin{pmatrix} -q & p \\ p & q \end{pmatrix} \in GL(2,\mathbb{Z}) \cap Sym(2), \quad S^{-1} = \frac{1}{c}S \in GL(2,\mathbb{Q}) \cap Sym(2), \\ S^{-1} \cdot A \cdot S = diag(c, 2c), \quad H(S) = -q-pi, \quad \det S = -c < 0. \end{cases}$$

For example:

$$A(1,0,1) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad S(1,0,1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall that a matrix $U \in GL(n,\mathbb{R})$ can be consider as corresponding to a mathematical game $G(U)$ of two persons, both having n strategies; then *the value* of this game is ([7, p. 449]) $v(G(U)) = \frac{1}{s(U^{-1})}$ where $s(U^{-1})$ means the sum of all elements of U^{-1}. For the matrix A the value of its corresponding game is:

$$v(G(A)) = \frac{2c^2}{3c-b} < c, \quad v(G(p=1,q=2)) = \frac{50}{11}.$$

□

II) a symmetric 3×3 one:

$$\begin{cases} B := \Gamma \cdot \Gamma^t = \begin{pmatrix} c & 0 & a \\ 0 & c & b \\ a & b & c \end{pmatrix} \in Sym(3, \mathbb{N}^*), \\ \frac{1}{c}B = \begin{pmatrix} 1 & 0 & \sin(\angle A) \\ 0 & 1 & \sin(\angle B) = \cos(\angle A) \\ \sin(\angle A) & \sin(\angle B) = \cos(\angle A) & 1 \end{pmatrix} \in Sym(3) = Sym(3,\mathbb{R}). \end{cases} \tag{2.1}$$

Again, its eigenvalues and associated eigenvectors are:

$$\lambda_1 = 0 < \lambda_2 = c < \lambda_3 = 2c, \quad \bar{v}_1 = (-a,-b,c), \quad \bar{v}_2 = (-b,a,0), \quad \bar{v}_3 = (a,b,c).$$

Hence, the invertible matrix making the matrix B a diagonal one is:

$$\begin{cases} S = \begin{pmatrix} -a & -b & a \\ -b & a & b \\ c & 0 & c \end{pmatrix} \in GL(3,\mathbb{Z}) \quad \det S = -2c^3 < 0, \\ S^{-1} = \frac{1}{2c^2}\begin{pmatrix} -a & -b & c \\ -2b & 2a & 0 \\ a & b & c \end{pmatrix} \in GL(3,\mathbb{Q}), \quad S^{-1}\cdot B\cdot S = diag(0,c,2c). \end{cases}$$

Recall also that a matrix from $Sym(3)$ represents geometrically a conic, see for example [8]. The conic associated to the matrix B reduces to the double point $\left(-\frac{a}{c}, -\frac{b}{c}\right) \in S^1$. For example:

$$B(1,0,1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad S(1,0,1) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let us remark that the second matrix from the relation (2.1) yields the function:

$$f : \left(0, \frac{\pi}{2}\right) \to Sym(3) \setminus GL(3,\mathbb{R}), \quad f(t) := \begin{pmatrix} 1 & 0 & \sin t \\ 0 & 1 & \cos t \\ \sin t & \cos t & 1 \end{pmatrix}$$

as restriction to (the first quadrant of) the unit circle S^1 of the map $F : \mathbb{R}^2 \to Sym(3)$:

$$\begin{cases} F(x,y) := \begin{pmatrix} x^2+y^2 & 0 & y \\ 0 & x^2+y^2 & x \\ y & x & x^2+y^2 \end{pmatrix}, \quad \det F(x,y) = (x^2+y^2)^2(x^2+y^2-1), \\ F|_{\mathbb{C}^*} : (x,y) = r(\cos\varphi, \sin\varphi), F(r,\varphi) := r\begin{pmatrix} r & 0 & \sin\varphi \\ 0 & r & \cos\varphi \\ \sin\varphi & \cos\varphi & r \end{pmatrix}, \det F(r,\varphi) = r^4(r^2-1). \end{cases}$$

The matrix $S \in GL(3,\mathbb{R})$ making diagonal the symmetric matrix $f(t)$ is:

$$\begin{cases} S(t) := \frac{1}{2}\begin{pmatrix} -\sin t & -\cos t & 1 \\ -\sin 2t & 2\sin^2 t & 0 \\ \sin t & \cos t & 1 \end{pmatrix}, \quad S^{-1}(t) := \begin{pmatrix} -\sin t & -\frac{\cos t}{\sin t} & \sin t \\ -\cos t & 1 & \cos t \\ 1 & 0 & 1 \end{pmatrix}, \\ S(t)\cdot f(t)\cdot S^{-1}(t) = diag(0,1,2) \end{cases}$$

while the matrix $S \in GL(3,\mathbb{R})$ making diagonal the symmetric matrix $F|_{\mathbb{C}^*}$ is:

$$\begin{cases} S(x,y) := \frac{1}{2r^2}\begin{pmatrix} -yr & -xr & r^2 \\ -2xy & 2y^2 & 0 \\ yr & xr & r^2 \end{pmatrix}, \quad S^{-1}(x,y) := \begin{pmatrix} -\frac{y}{r} & -\frac{x}{y} & \frac{y}{r} \\ -\frac{x}{r} & 1 & \frac{x}{r} \\ 1 & 0 & 1 \end{pmatrix}, \\ S(t)\cdot F(r,\varphi)\cdot S^{-1}(t) = diag(r^2-r, r^2, r^2+r). \end{cases}$$

From a differentiable point of view F is an immersion of $\mathbb{R}^2$ into $\mathbb{R}^6 = Sym(3)$ since the rank of the Jacobian matrix of F is 2. With the notation $u = x^2+y^2$ the equation $\det F = 1$, i.e. $F(x,y) \in SL(3,\mathbb{R})$, means the cubic equation:

$$u^3 - u^2 - 1 = \left(u - \frac{1}{3}\right)^3 - \frac{1}{3}\left(u - \frac{1}{3}\right) - \frac{29}{27} = 0$$

which admits only one real (and positive) solution $u_1 \simeq 1.4656$. Naturally, we can associate the cubic (in fact elliptic) plane curve:

$$\mathcal{C} : v^2 = u^3 - u^2 - 1$$

whose details can be found on: https://www.lmfdb.org/EllipticCurve/Q/496/e/1. □

Returning to the case of 2×2 matrices let us remark that the first part of relations (1.4) gives an affine map:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} p \\ q \end{pmatrix} := C\cdot\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C := \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues of C and the square matrix S making C a diagonal one are:

$$\lambda_1 = -\sqrt{2} < \lambda_2 = \sqrt{2}, \quad S = \begin{pmatrix} -1-\sqrt{2} & \sqrt{2}-1 \\ 1 & 1 \end{pmatrix}, \quad S^{-1} = \frac{1}{4}\begin{pmatrix} -\sqrt{2} & 2-\sqrt{2} \\ \sqrt{2} & 2+\sqrt{2} \end{pmatrix}$$

with $S^{-1}CS = diag(-\sqrt{2}, \sqrt{2})$.

We finish this section by introducing a composition law on $\mathbb{R}^*_+ = (0, +\infty)$, inspired by the equality case discussed in the Property 1.2.3):

$$x \oplus_F y := (\sqrt{x} + \sqrt{y})^2.$$

Apart from commutativity and $x \oplus_F x = 4x$ we note the property $\cos^2 t \oplus_F \sin^2 t = 1 + \sin 2t$.

3. The Farey sum of a class of Eisenstein triples

For the sake of completeness we present now the case of Eisenstein triples. Recall that an Eisenstein triangle has an angle of $60°$. By supposing this angle to be $\angle C$ it results:

$$a^2 - ab + b^2 = c^2$$

and hence, a *Eisenstein triple* is a triple of positive integers satisfying this Diophantine equation; then $\min\{a, b\} \leq c \leq \max\{a, b\}$. We point out that recently, the Eisenstein triples are used in [9] to characterize the bijective digitized rotations on the hexagonal grid. Contrary to the Pythagorean case we have only a *partial* parametrization:

$$a = a(p, q) := q^2 - p^2, \quad b = b(p, q) := 2pq - p^2, \quad c = c(p, q) := p^2 + q^2 - pq = (q - p)^2 + pq \tag{3.1}$$

and the limit case $p = q$ gives the degenerate Eisenstein triple $p^2(0, 1, 1)$. Then we can define a Farey sum on the class of (3.1) (p, q)-*Eisenstein triples*:

$$(a, b, c,) \oplus (a', b', c') := (a'', b'', c'') =$$

$$= ((q + q')^2 - (p + p')^2, 2(p + p')(q + q') - (p + p')^2, (p + p')^2 + (q + q')^2 - (p + p')(q + q')). \tag{3.2}$$

Example 3.1. 1) Since $(p = 1, q = 2)$ yields the equilateral triangle $3(1, 1, 1)$ and $(p = 1, q = 3)$ gives the Eisenstein triple $(8, 5, 7)$ we have:

$$(3, 3, 3) \oplus (8, 5, 7) = (21, 16, 19), \quad (p'' = 2, q'' = 5).$$

2) Again $(a, b, c) \oplus (a, b, c) = 4(a, b, c)$ and $(a, b, c) \oplus (0, 1, 1) = (a + 2(q - p), b + 2q + 1, c + p + q + 1)$ with the example $3(1, 1, 1) \oplus (0, 1, 1) = (5, 8, 7)$.

The matrix expression of the Farey sum (3.2) is:

$$\begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} + \Gamma \cdot \begin{pmatrix} p' \\ q' \end{pmatrix}, \quad \Gamma := \begin{pmatrix} -2p & 2q \\ 2(q-p) & 2p \\ 2p - q & 2q - p \end{pmatrix} \in M_{3,2}(\mathbb{Z}).$$

The associated symmetric matrices are:

I)

$$A := \Gamma^t \cdot \Gamma = \begin{pmatrix} 12(p^2 - pq) + 5q^2 & -6p^2 + 5pq - 2q^2 \\ -6p^2 + 5pq - 2q^2 & 5p^2 + 4(2q^2 - pq) \end{pmatrix} \in Sym(2, \mathbb{Z})$$

with:

$$TrA = 17p^2 - 16pq + 13q^2, \quad \det A = 12(2p^4 - 4p^3q + 10p^2q^2 - 8pq^3 + 3q^4).$$

II)

$$B := \Gamma \cdot \Gamma^t = \begin{pmatrix} 4(p^2 + q^2) & 4p^2 & -4p^2 + 6pq - 2q^2 \\ 4p^2 & 4(2p^2 - 2pq + q^2) & -6p^2 + 10pq - 2q^2 \\ -4p^2 + 6pq - 2q^2 & -6p^2 + 10pq - 2q^2 & 5p^2 - 8pq + 5q^2 \end{pmatrix} \in Sym(3, \mathbb{Z})$$

with:

$$TrB = 17p^2 - 16pq + 13q^2, \quad \det B = 48q(2p^5 - 6p^4q + 10p^3q^2 - 7p^2q^3 + q^5).$$

To the equilateral triangle $3(1,1,1)$ corresponds the matrices:

I)

$$A(p=1,q=2)=\begin{pmatrix} 8 & -4 \\ -4 & 29 \end{pmatrix}\in Sym(2,\mathbb{Z}),\quad \lambda_1=\frac{37-\sqrt{505}}{2}<\lambda_2=\frac{37+\sqrt{505}}{2}$$

with $TrA=37$, $\det A=6^3$, Hopf invariant $H(A)=-\frac{21}{2}+4i$ and:

$$S=\frac{1}{8}\begin{pmatrix} 21+\sqrt{505} & 21-\sqrt{505} \\ 8 & 8 \end{pmatrix}\in GL(2,\mathbb{R}),\quad S^{-1}=\frac{1}{2\sqrt{505}}\begin{pmatrix} 8 & \sqrt{505}-21 \\ -8 & \sqrt{505}+21 \end{pmatrix}.$$

II)

$$\begin{cases} B(p=1,q=2)=\begin{pmatrix} 20 & 4 & 0 \\ 4 & 8 & 6 \\ 0 & 6 & 9 \end{pmatrix}\in Sym(3,\mathbb{Z}), \\ \lambda_1\simeq 1.98<\lambda_2\simeq 13.51<\lambda_3\simeq 21.50, \quad \det B=576=24^2. \end{cases}$$

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The Source of Γ-Primeness on Γ-Rings

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Abstract

The source of the primeness texture is a skeleton that generalizes traditional prime rings. In this context, our primary aim in this study is to describe the source of Γ-primeness in Γ-rings not included in the literature. This work's purpose is to generalize the concept of the source of primeness to a Γ-ring. In this study, the characteristics provided by the defined concept are also discussed, and the results achieved are exemplified.

Keywords: *Prime* Γ*-ideal, Source of* Γ*-primeness,* Γ*-ring*

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1. Introduction

The structure of the Γ-ring was first proposed by Nobusawa in 1964 as a generalization of the ring [1]. The author determined the notion of the Γ-ring under certain conditions and obtained some significant results. Afterward, Barnes [2], inspired by Nabusawa, introduced and analyzed some concepts for Γ-rings. Many studies have extended important results on the structure of rings to Γ-rings [3–8].

Prime and semiprime ideals contribute extremely to important results in ring theory. Some properties of prime and semiprime ideals are studied in ring theory and generalized to Γ-rings. Recently, Aydın et al. [9] and Camcı [11] suggested the concept of the source of semiprimeness for a ring and described three ring types that were not previously included in the literature. Next, Arslan and Düzkaya [10] generalized the set of the source of semiprimeness defined for a ring to the Γ-ring and inquired about the properties of the set. Motivated by the set of the source of semiprimeness, Yeşil and Camcı [12] characterized the concept of the source of primeness for a ring. The authors regarded the relation between a ring's idempotent, nilpotent, and zero divisor elements and the set of the source of primeness and described new ring types.

This study set one's sights on generalizing the set of the source of primeness of a ring to the Γ-ring. Moreover, in this paper, the characteristics of the concept of the source of Γ-primeness of a Γ-ring and the different results created by idempotent, strongly nilpotent, nilpotent, and zero divisor elements in the set of the source of Γ-primeness are mentioned. The relationship between the source of semiprimeness and the source of Γ-primeness in the Γ-ring was also observed.

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2. Preliminaries

In this section, basic definitions previously lay one's laid in the literature are presented [1, 2, 8, 10, 13–16].

Definition 2.1. Let R and Γ be two additive abelian groups. If there exists a mapping $(a,\gamma,b) \to a\gamma b$ of $R \times \Gamma \times R \to R$ satisfies the following conditions:

1. $a\gamma b \in R$
2. $(a+b)\gamma c = a\gamma c + b\gamma c$, $a\gamma(b+c) = a\gamma b + a\gamma c$, and $a(\beta+\gamma)b = a\beta b + a\gamma b$
3. $(a\gamma b)\beta c = a\gamma(b\beta c) = a\gamma b\beta c$

for all $a, b, c \in R$ and $\beta, \gamma \in \Gamma$, then R is called a Γ-ring.

Definition 2.2. Let A be an additive subgroup of a Γ-ring R. If $a\gamma b \in A$, for all $a, b \in A$ and $\gamma \in \Gamma$, then A is called a Γ-subring of R.

Equivalently; if $A\Gamma A \subseteq A$, then A is called a Γ-subring of R.

Definition 2.3. Let A be an additive subgroup of a Γ-ring R. If $r\gamma a \in A$ (left ideal), $a\gamma r \in A$(right ideal), for all $r \in R$, $\gamma \in \Gamma$, and $a \in A$, then A is called a Γ-ideal of R.

Equivalently; if $A\Gamma R \subseteq A$ and $R\Gamma A \subseteq A$, then A is called a Γ-ideal of R.

Definition 2.4. Let P be a proper Γ-ideal of R. If $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$, for Γ-ideals A and B of R, then P is called a prime Γ-ideal of R.

Definition 2.5. For $a, b \in R$, if $a\Gamma R\Gamma b = (0)$ implies that $a = 0$ or $b = 0$, then R is called a prime Γ-ring.

Definition 2.6. Let R be a Γ-ring and $e \in R$. If $\gamma \in \Gamma$ exists such that $e\gamma e = e$, then the element $e \in R$ is called an idempotent element.

Definition 2.7. Let R be a Γ-ring. R is called a Boolean Γ-ring if $m\gamma m = m$, for all $m \in R$ and $\gamma \in \Gamma$.

Definition 2.8. An element x of a Γ-ring R is called nilpotent element if for some $\gamma \in \Gamma$, there exists a positive integer n such that $(x\gamma)^n x = 0$.

Definition 2.9. An element x of a Γ-ring R is called strongly nilpotent if there exist a positive integer n such that $(x\Gamma)^n x = 0$.

Definition 2.10. If there exist $1 \in R$ and $\gamma \in \Gamma$ such that $1\gamma r = r\gamma 1 = r$, for all $r \in R$, then R is called a Γ-ring with unit.

Definition 2.11. An element $0 \neq a \in R$ is called a zero divisor if there exists $b \neq 0$ such that $a\gamma b = b\gamma a = 0$.

Definition 2.12. Let R and S be Γ_1-ring and Γ_2-ring respectively. An ordered (ϕ, ψ) is called a Γ-homomorphism if the following conditions are satisfied:

1. $\phi : R \to S$ is a group homomorphism
2. $\psi : \Gamma_1 \to \Gamma_2$ is a group homomorphism
3. $\phi(x\gamma y) = \phi(x)\psi(\gamma)\phi(y)$

for all $x, y \in R$ and $\gamma \in \Gamma$.

Remark 2.1. Let R and S be Γ_1-ring and Γ_2-ring respectively. The product $R \times S$ is a $\Gamma_1 \times \Gamma_2$-ring with the following operation:

$$(a,b) + (c,d) = (a+c, b+d)$$
$$(\alpha,\delta) + (\beta,\gamma) = (\alpha+\beta, \delta+\gamma)$$
$$(a,b)(\beta,\gamma)(c,d) = (a\beta c, b\gamma d)$$

for all $(a,b), (c,d) \in R \times S$ and $(\beta,\gamma), (\alpha,\delta) \in \Gamma_1 \times \Gamma_2$.

Definition 2.13. Let A be a subset of a Γ-ring R. The source of semiprimeness of A is defined as $S_R(A) = \{b \in R : b\Gamma A\Gamma b = (0)\}$. When $A = R$, S_R is adopting instead of $S_R(R)$.

3. Results

In this section, the concept of the source of Γ-primeness is characterized for the Γ-ring. To understand the concept better, the basic characteristics of the set are first inspected. Furthermore, the relationship between a ring with unit, zero divisor, idempotent, and nilpotent elements, and the set of the source of Γ-primeness is discoursed.

Definition 3.1. Let A be a non-empty subset of the Γ-ring R and $a \in R$. The set described as

$$\{b \in R : a\Gamma A\Gamma b = (0)\}$$

is denoted by $S^a_{R_\Gamma}(A)$. The intersection of sets $S^a_{R_\Gamma}(A)$ is demonstrated by $P_{R_\Gamma}(A)$, and $P_{R_\Gamma}(A)$ is called the source of Γ-primeness of A in R. When $A = R$, the $S^a_{R_\Gamma}$ notation will be operated instead of $S^a_{R_\Gamma}(R)$. Therefore, the source of Γ-primeness of the R is

$$P_{R_\Gamma} = \bigcap_{a\in R} S^a_{R_\Gamma}$$

The primary and necessary features are stated below to comprehend the concept of Γ-primeness's source.

1. Let R be a Γ-ring. $P_{R_\Gamma}(A) = \bigcap_{a\in R} S^a_{R_\Gamma}(A) \neq \emptyset$ because of $a\Gamma A\Gamma 0 = (0)$, for all $a \in R$.
2. $S^0_{R_\Gamma}(A) = R$.
3. Let A be a Γ-subring of R. If $x \in S^a_{A_\Gamma}$, then $x \in A$ and $a\Gamma A\Gamma x = (0)$. Since $A \subseteq R$, $x \in S^a_{R_\Gamma}(A)$. Therefore, $S^a_{A_\Gamma} \subseteq S^a_{R_\Gamma}(A)$.

Remark 3.1. Let $K = \{b \in R : a\Gamma A\Gamma b = (0), \forall a \in R\}$, for a non-empty subset A of a Γ-ring R. If $x \in P_{R_\Gamma}(A)$, then $a\Gamma A\Gamma x = (0)$, for all $a \in R$. Hence, $P_{R_\Gamma}(A) \subseteq K$. Similarly, $K \subseteq P_{R_\Gamma}(A)$. In line with this explanation, the source of Γ-primeness of A in R is expressed as

$$P_{R_\Gamma}(A) = \{b \in R : R\Gamma A\Gamma b = (0)\}$$

Proposition 3.1. *Let A and B be two non-empty subsets of a Γ-ring R. Then,*

$$P_{(R\times R)_\Gamma}(A \times B) = P_{R_\Gamma}(A) \times P_{R_\Gamma}(B)$$

Proof. Let $(x, y) \in P_{(R\times R)_\Gamma}(A \times B)$. Then, $(R \times R)(\Gamma \times \Gamma)(A \times B)(\Gamma \times \Gamma)(x, y) = (0, 0)$. Thus, $(R\Gamma A\Gamma x, R\Gamma B\Gamma y) = (0, 0)$. From here, $R\Gamma A\Gamma x = (0)$ and $R\Gamma B\Gamma y = (0)$. This means that $x \in P_{R_\Gamma}(A)$ and $y \in P_{R_\Gamma}(B)$. Hence, $(x, y) \in P_{R_\Gamma}(A) \times P_{R_\Gamma}(B)$. The converse is similar. Therefore, $P_{(R\times R)_\Gamma}(A \times B) = P_{R_\Gamma}(A) \times P_{R_\Gamma}(B)$. □

Example 3.1. Let $R = \mathbf{Z_4}$ and $S = \mathbf{Z_6}$ be $\mathbf{Z_4}$-ring and $\mathbf{Z_6}$-ring respectively, and $A = \{\overline{0}, \overline{2}\} \subseteq R$ and $B = \{\overline{0}, \overline{3}\} \subseteq S$. Then, $R \times S$ is a $\mathbf{Z_4} \times \mathbf{Z_6}$-ring and $A \times B \subseteq R \times S$.

$$P_{(R\times S)_{\mathbf{Z_4}\times\mathbf{Z_6}}}(A \times B) = \{(\overline{c}, \overline{d}) \in R \times S : (R \times S)(\mathbf{Z_4} \times \mathbf{Z_6})(A \times B)(\mathbf{Z_4} \times \mathbf{Z_6})(\overline{c}, \overline{d}) = (\overline{0}, \overline{0})\}.$$

$$\begin{aligned} (\overline{c}, \overline{d}) \in P_{(R\times S)_{\mathbf{Z_4}\times\mathbf{Z_6}}}(A \times B) \quad &\Rightarrow \quad (R \times S)(\mathbf{Z_4} \times \mathbf{Z_6})(A \times B)(\mathbf{Z_4} \times \mathbf{Z_6})(\overline{c}, \overline{d}) = (\overline{0}, \overline{0}) \\ &\Rightarrow \quad (R\mathbf{Z_4}A\mathbf{Z_4}\overline{c}, S\mathbf{Z_6}B\mathbf{Z_6}\overline{d}) = (\overline{0}, \overline{0}) \\ &\Rightarrow \quad R\mathbf{Z_4}A\mathbf{Z_4}\overline{c} = (\overline{0}) \text{ and } S\mathbf{Z_6}B\mathbf{Z_6}\overline{d} = (\overline{0}) \\ &\Rightarrow \quad \overline{c} \in \{\overline{0}, \overline{2}\} \text{ and } \overline{d} \in \{\overline{0}, \overline{2}\} \end{aligned}$$

Therefore,

$$P_{(R\times S)_{\mathbf{Z_4}\times\mathbf{Z_6}}}(A \times B) = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{2}), (\overline{2}, \overline{0}), (\overline{2}, \overline{2})\}$$

Let $(a, b) \in P_{R_{\mathbf{Z_4}}}(A) \times P_{S_{\mathbf{Z_6}}}(B)$. Then, $R\mathbf{Z_4}A\mathbf{Z_4}\overline{a} = (\overline{0})$ and $S\mathbf{Z_6}B\mathbf{Z_6}\overline{b} = (\overline{0})$. From here, $\overline{a} \in \{\overline{0}, \overline{2}\}$ and $\overline{b} \in \{\overline{0}, \overline{2}\}$. Thus,

$$P_{R_{\mathbf{Z_4}}}(A) \times P_{S_{\mathbf{Z_6}}}(B) = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{2}), (\overline{2}, \overline{0}), (\overline{2}, \overline{2})\}$$

Proposition 3.2. *Let R be a Γ-ring with unit. Then, $P_{R_\Gamma} \subseteq \{x \in R : x\Gamma x = (0)\}$.*

Proof. Let $M = \{x \in R : x\Gamma x = (0)\}$. If $x \in P_{R_\Gamma}$, then $R\Gamma R\Gamma x = (0)$. Since R is a Γ-ring with unit, $(0) = x\Gamma 1\Gamma x = x\Gamma x$. Hence, $x \in M$. □

Proposition 3.3. *Let A and B be two non-empty subsets of a Γ-ring R. Then, the following holds.*

1. *If $A \subseteq B$, then $P_{R_\Gamma}(B) \subseteq P_{R_\Gamma}(A)$. In particular, $P_{R_\Gamma} \subseteq P_{R_\Gamma}(A)$ is provided.*
2. *If A is a Γ-subring of R, then $A \cap P_{R_\Gamma}(A) \subseteq P_{A_\Gamma}$.*

Proof.
1. Let $A \subseteq B$. If $x \in P_{R_\Gamma}(B)$, then $R\Gamma B\Gamma x = (0)$. Since $A \subseteq B$, $R\Gamma A\Gamma x = (0)$. Therefore, $x \in P_{R_\Gamma}(A)$.
2. Let $x \in A \cap P_R^\Gamma(A)$. Since $x \in A$ and $R\Gamma A\Gamma x = (0)$, $x \in P_{A_\Gamma}$.

□

Proposition 3.4. *Let A be a nonempty subset of a Γ-ring R. Then, $P_{R_\Gamma}(A) \subset S_R(A)$.*

Proof. If $x \in P_{R_\Gamma}(A)$, then $R\Gamma A\Gamma x = (0)$. Thus, $x\Gamma A\Gamma x = (0)$. Hence, $x \in S_R(A)$. □

Lemma 3.1. *Let R be a Γ-ring and $\emptyset \neq I \subseteq R$. Then,*

1. *$S^a_{R_\Gamma}(I)$ is a right Γ-ideal of R.*
2. *If I is a right Γ-ideal, then $S^a_{R_\Gamma}(I)$ is a Γ-ideal of R. In addition, $S^a_{R_\Gamma}$ is a Γ-ideal of R.*

Proof.
1. If $x, y \in S^a_{R_\Gamma}(I)$, then $a\Gamma I\Gamma x = (0)$ and $a\Gamma I\Gamma y = (0)$, for all $a \in R$. Thus, $x\Gamma R \subseteq S^a_{R_\Gamma}(I)$ and $x - y \in S^a_{R_\Gamma}(I)$ because of
$$a\Gamma I\Gamma(x - y) = a\Gamma I\Gamma x - a\Gamma I\Gamma y = (0)$$
and
$$a\Gamma I\Gamma(x\Gamma R) = (a\Gamma I\Gamma x)\Gamma R = 0\Gamma R = (0)$$
Accordingly, $S^a_{R_\Gamma}(I)$ is a right Γ-ideal of R.
2. From 3.1, $S^a_{R_\Gamma}(I)$ is a right Γ-ideal of R. In addition
$$a\Gamma I\Gamma(R\Gamma x) = (a\Gamma I\Gamma R)\Gamma x \subseteq a\Gamma I\Gamma x = (0).$$
Thus, $R\Gamma x \subseteq S^a_{R_\Gamma}(I)$. Consequently, $S^a_{R_\Gamma}(I)$ is a Γ-ideal of R. Moreover, since R is its ideal, $S^a_{R_\Gamma}$ is a Γ-ideal of R.

□

Theorem 3.1. *Let R be a Γ-ring and $\emptyset \neq I \subseteq R$. Then,*

1. *$P_{R_\Gamma}(I)$ is a right Γ-ideal of R.*
2. *If I is a right Γ-ideal of Γ-ring R, then $P_{R_\Gamma}(I)$ is a Γ-ideal of R. Specially, P_{R_Γ} is a Γ-ideal of R.*

Proof.
1. If $x, y \in P_{R_\Gamma}(I) = \bigcap_{a\in R} S^a_{R_\Gamma}(I)$, then $x, y \in S^a_{R_\Gamma}(I)$, for all $a \in R$. From Lemma 3.1, $S^a_{R_\Gamma}(I)$ is a right Γ-ideal of R. As a result, $x\Gamma R \subseteq \bigcap_{a\in R} S^a_{R_\Gamma}(I) = P_{R_\Gamma}(I)$ and $x - y \in S^a_{R_\Gamma}(I) = P_{R_\Gamma}(I)$, for all $x \in S^a_{R_\Gamma}(I)$. Therefore, $P_{R_\Gamma}(I)$ is a right Γ-ideal of R.
2. From 1, $P_{R_\Gamma}(I)$ is a right Γ-ideal of R. Moreover, if I is a right Γ-ideal, then by Lemma 3.1, $S^a_{R_\Gamma}(I)$ is a Γ-ideal of R. Thence, $R\Gamma x \subseteq \bigcap_{a\in R} S^a_{R_\Gamma}(I) = P_R^\Gamma(I)$, for all $x \in S^a_{R_\Gamma}(I)$. Therefore, $P_R^\Gamma(I)$ is Γ-ideal of R. Furthermore, since R is its ideal, P_R^Γ is a Γ-ideal of R.

□

Example 3.2. Let $R = M_{2\times 2}(\mathbb{R}) = \left\{\begin{pmatrix} a & x \\ b & y \end{pmatrix} : a, b, x, y \in \mathbb{R}\right\}$ and $\Gamma = M_{2\times 2}(\mathbb{Z}) = \left\{\begin{pmatrix} k & 0 \\ 0 & h \end{pmatrix} : k, h \in \mathbb{R}\right\}$. Then, R is a Γ-ring according to the addition and multiplication operations in matrices. Let $I = \left\{\begin{pmatrix} 0 & t \\ 0 & t \end{pmatrix} : t \in \mathbb{R}\right\}$. Here, I is a subset of R but is not a right or left Γ-ideal. Hence, when the set $P_{R_\Gamma}(I)$ is observed, it is concluded that $P_{R_\Gamma}(I) = \left\{\begin{pmatrix} e & f \\ 0 & 0 \end{pmatrix} : e, f \in \mathbb{R}\right\}$. Evidently, $P_{R_\Gamma}(I)$ is a right Γ- ideal but not a left Γ-ideal of R.

Theorem 3.2. *Let R be a Γ-ring. The following are provided.*

1. *If R is a prime Γ-ring, then $P_{R_\Gamma} = \{0\}$.*
2. *The source of Γ-primeness P_{R_Γ} is contained by every prime Γ-ideal of the R.*

Proof. 1. Let R be a prime Γ-ring and $x \in P_{R_\Gamma} = \bigcap_{a\in R} S^a_{R_\Gamma}$. Then, $b\Gamma R\Gamma x = (0)$, for all $0 \neq b \in R$. From the hypothesis, $x = 0$. Therefore, $P_{R_\Gamma} = \{0\}$.

2. Let I be a prime Γ-ideal of R and $x \in P_{R_\Gamma}$. Then, $R\Gamma R\Gamma x = (0) \subseteq I$. Since I is a prime Γ-ideal, $R \subseteq I$ or $x \in I$. Accordingly, $P_{R_\Gamma} \subseteq I$.

□

Example 3.3. Let $R = M_{1\times 2}(\mathbb{R}) = \left\{\begin{pmatrix} a & a \end{pmatrix} : a \in \mathbb{R}\right\}$ and $\Gamma = M_{2\times 1}(\mathbb{Z}) = \left\{\begin{pmatrix} k \\ 0 \end{pmatrix} : k \in \mathbb{Z}\right\}$. Then, R is a Γ-ring. It is straightforward to verify that R is a prime Γ-ring. Further, it can be examined that $P_{R_\Gamma} = \left\{\begin{pmatrix} 0 & 0 \end{pmatrix}\right\}$.

The following example can be donated to signalize that the reverse does not work.

Example 3.4. Let $R = \mathbf{Z_4}$ and $\Gamma = \mathbb{Z}$. Then, R is a Γ-ring. Precisely, $P_{R_\Gamma} = \{0\}$. However, since $\overline{x}\Gamma R\Gamma\overline{y} = \overline{0}$, for $\overline{x} = \overline{y} = \overline{2}$, R is not a prime Γ-ring.

Proposition 3.5. *Let R be a Γ-ring. The followings are satisfied.*

1. *If R is a Boolean Γ-ring, then $P_{R_\Gamma} = \{0\}$.*
2. *If $a \in P_{R_\Gamma}$, then a is a zero divisor element of R.*
3. *If R is a Γ-ring with unit, then $P_{R_\Gamma} = \{0\}$.*

Proof. 1. If $x \in P_{R_\Gamma}$, then $R\Gamma R\Gamma x = (0)$. Thus, $(0) = x\Gamma x\Gamma x = x$. Hence, $P_{R_\Gamma} = \{0\}$.

2. If $0 \neq a \in P_{R_\Gamma}$, then $R\Gamma R\Gamma a = (0)$. Thus, $a\Gamma a\Gamma a = (0)$. If it is stated that this equality is $a\Gamma(a\Gamma a) = (0)$ or $(a\Gamma a)\Gamma a = (0)$, then $a\Gamma a = (0)$ or $a\Gamma a \neq (0)$ since $a \neq 0$. If $a\Gamma a = (0)$, then a is a zero divisor element. If $a\Gamma a \neq (0)$, a is a zero divisor element because of $a\Gamma(a\Gamma a) = (0)$.

3. If $a \in P_{R_\Gamma}$, then $R\Gamma R\Gamma a = (0)$. Thus, $(0) = 1\Gamma 1\Gamma a = a$. Consequently, $P_{R_\Gamma} = \{0\}$.

□

As a result of the above proposition, the following corollary is acquired.

Corollary 3.1. *Let R be a Γ-ring. Then,*

1. *There is no idempotent element other than zero in P_{R_Γ}.*
2. *If $x \in P_{R_\Gamma}$, then x is a strongly nilpotent element of R.*
3. *Every element in P_{R_Γ} is a nilpotent element.*

Proof. 1. Let $x \in P_{R_\Gamma}$ be an idempotent element. Then, $R\Gamma R\Gamma x = (0)$. Thus, $x\Gamma x\Gamma x = 0$. Since x is an idempotent element, $x = 0$.

2. If $x \in P_{R_\Gamma}$, then $R\Gamma R\Gamma x = (0)$. Therefore, $(0) = x\Gamma x\Gamma x = (x\Gamma)^2 x$.

3. Since every strongly nilpotent element is nilpotent, every element of P_{R_Γ} is a nilpotent.

□

Theorem 3.3. *Let R and S be Γ_1-ring and Γ_2-ring, respectively. If ordered pair (f, ψ) is a Γ-homomorphism, then $f(P_{R_\Gamma}) \subseteq P_{f(R)_\Gamma}$. If f is an injective, then $f(P_{R_\Gamma}) = P_{f(R)_\Gamma}$.*

Proof. Since (f, ψ) is a Γ-ring homomorphism, $f(R)$ is a $\psi(\Gamma_1)$-ring with multiplication

$$f(a)\psi(\gamma)f(b) = f(a\gamma b).$$

Let $x \in f(P_{R_\Gamma})$. Then, there exists $a \in P_{R_\Gamma}$ such that $x = f(a)$. Since $a \in P_{R_\Gamma}$, $R\Gamma R\Gamma a = (0)$. From here,

$$(0) = f(R\Gamma R\Gamma a) = f(R)\psi(\Gamma)f(R)\psi(\Gamma)f(a)$$

Thence, $x = f(a) \in P_{f(R)_\Gamma}$.

Let f be an injective function and $a \in P_{f(R)_\Gamma}$. Then, $f(R)\psi(\Gamma)f(R)\psi(\Gamma)a = (0)$. Since

$$f(R\Gamma R\Gamma x) = f(R)\psi(\Gamma)f(R)\psi(\Gamma)a = (0)$$

$R\Gamma R\Gamma x = (0)$ is obtained. This means $x \in P_{R_\Gamma}$. Accordingly, $a = f(x) \in f(P_{R_\Gamma})$. □

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