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Pseudostarlikeness and Pseudoconvexity of Multiple Dirichlet Series

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Abstract

Let $p\in\mathbb{N}$, $s=(s_1,\dots,s_p)\in\mathbb{C}^p$, $h=(h_1,\dots,h_p)\in\mathbb{R}^p_+$, $(n)=(n_1,\dots,n_p)\in\mathbb{N}^p$ and the sequences $\lambda_{(n)}=(\lambda^{(1)}_{n_1},\dots,\lambda^{(p)}_{n_p})$ are such that $0<\lambda^{(j)}_1<\lambda^{(j)}_k<\lambda^{(j)}_{k+1}\uparrow+\infty$ as $k\to\infty$ for every $j=1,\dots,p$. For $a=(a_1,\dots,a_p)$ and $c=(c_1,\dots,c_p)$ let $(a,c)=a_1c_1+\ldots+a_pc_p$, and we say that $a>c$ if $a_j>c_j$ for all $1\le j\le p$. For a multiple Dirichlet series

$$F(s)=e^{(s,h)}+\sum_{\lambda_{(n)}>h} f_{(n)}\exp\{(\lambda_{(n)},s)\}$$

absolutely converges in $\Pi^p_0=\{s:\operatorname{Re}s<0\}$, concepts of pseudostarlikeness and pseudoconvexity are introduced and criteria for pseudostarlikeness and the pseudoconvexity are proved. Using the obtained results, we investigated neighborhoods of multiple Dirichlet series, Hadamard compositions, and properties of solutions of some differential equations.

1. Introduction

Let S be a class of functions $f(z)=z+\sum_{n=2}^{\infty}f_nz^n$ analytic univalent in $\mathbb{D}=\{z:|z|<1\}$. The function $f\in S$ is said to be starlike if $f(\mathbb{D})$ is a starlike domain concerning the origin. It is well-known [1] (see p. 202) that the condition

$$\operatorname{Re}\{zf'(z)/f(z)\}>0\ (z\in\mathbb{D})$$

is necessary and sufficient for the starlikeness of f. A. W. Goodman [2] (see also [3] p. 9) proved that if $\sum_{n=2}^{\infty}n|f_n|\le1$ then function $f\in S$ is starlike.

The concept of the starlikeness of function $f\in S$ got the series of generalizations. I. S. Jack [4] studied starlike functions of order $\alpha\in[0,1)$, i. e. such functions $f\in S$, for which

$$\operatorname{Re}\{zf'(z)/f(z)\}>\alpha\ (z\in\mathbb{D}).$$

It is proved [4], ([3], p. 13) that if $\sum_{n=2}^{\infty}(n-\alpha)|f_n|\le1-\alpha$ then function $f\in S$ is starlike function of order α. V. P. Gupta [5] introduced the concept of starlike function of order $\alpha\in[0,1)$ and type $\beta\in(0,1]$. A function $f\in S$ is so named for that

$$|zf'(z)/f(z)-1|<\beta|zf'(z)/f(z)+1-2\alpha|\quad\text{for all}\quad z\in\mathbb{D}.$$

It is proved [5] that if

$$\sum_{n=2}^{\infty}\{(1+\beta)n-\beta(2\alpha-1)-1)\}|f_n|\le2\beta(1-\alpha)$$

then function $f\in S$ is starlike function of the order α and the type β.

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For $f \in S$, following A. W. Goodman [6] and S. Ruscheweyh [7], its neighborhood is called a set

$$N_{\delta}(f) = \left\{ g(z) = z + \sum_{k=2}^{\infty} g_k z^k \in S : \sum_{k=2}^{\infty} k|g_k - f_k| \le \delta \right\}.$$

The neighborhoods of various classes of analytical in $\mathbb{D}$ functions were studied by many authors (we indicate here only in articles [8–14]). For power series

$$f_j(z) = \sum_{k=0}^{\infty} f_{k,j} z^k \quad (j = 1, 2)$$

the series

$$(f_1 * f_2)(z) = \sum_{k=0}^{\infty} f_{k,1} f_{k,2} z^k$$

is called the Hadamard composition (product) [15]. Obtained by J. Hadamard properties of this composition find the applications [16–18] in the theory of the analytic continuation of the functions represented by power series. Many authors (see for example [7, 19–22]) have studied Hadamard compositions of univalent, starlike, meromorphically starlike functions.
Let $h \ge 1$, $\Lambda = (\lambda_k)$ be an increasing to $+\infty$ sequence of positive numbers ($\lambda_1 > h$) and $S(\Lambda)$ be a class of Dirichlet series

$$F(s) = e^{sh} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\} \ (s = \sigma + it)$$

absolutely convergent in half-plane $\Pi_0 = \{s : \text{Re}\, s < 0\}$. It is known [24], ([3], p. 135) that each function $F \in S(\Lambda)$ is non-univalent in Π_0, but there exist conformal in Π_0 functions $F \in S(\Lambda)$, and if

$$\sum_{k=2}^{\infty} \lambda_k |f_k| \le h$$

then function F is conformal in Π_0. A conformal function F in Π_0 is said to be pseudostarlike if

$$\text{Re}\{F'(s)/F(s)\} > 0 \, (s \in \Pi_0).$$

In [24] (see also [3], p. 139) it is proved that if

$$\sum_{k=2}^{\infty} \lambda_k |f_k| \le h$$

then function F is pseudostarlike. A conformal function $F \in S(\Lambda)$ is said to be pseudostarlike of the order α if

$$\text{Re}\{F'(s)/F(s)\} > \alpha \in [0,\, 1) \quad \text{for all} \quad s \in \Pi_0.$$

Since the inequality $|w - h| < |w - (2\alpha - h)|$ holds if and only if $\text{Re}\, w > \alpha$, function $F \in S(\Lambda)$ is pseudostarlike of the order α if and only if

$$|F'(s)/F(s) - h| < |F'(s)/F(s) - (2\alpha - h)| fors \in \Pi_0.$$

Therefore, as in [25], we call a conformal function $F \in S(\Lambda)$ in Π_0 pseudostarlike of the order $\alpha \in [0,\, 1)$ and the type $\beta \in (0,\, 1]$ if

$$|F'(s)/F(s) - h| < \beta |F'(s)/F(s) - (2\alpha - h)| \quad \text{for} \quad s \in \Pi_0.$$

In [25], it is proved that if

$$\sum_{k=1}^{\infty} \{(1+\beta)\lambda_k - 2\beta\alpha - h(1-\beta)\}|f_k| \le 2\beta(h - \alpha)$$

then F is pseudostarlike of the order α and the type β. If in the definition of the pseudostarlikeness instead of F'/F to put F''/F' then we will get the definition of the pseudoconvexity.
S. M. Shah [26] indicated conditions on real parameters γ_0, γ_1, γ_2 of the differential equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (\gamma_0 z^2 + \gamma_1 z + \gamma_2) w = 0,$$

under which there exists an entire transcendental solution $f(z) = z + \sum_{n=2}^{\infty} f_n z^n$ such that f and all its derivatives are close-to-convex in $\mathbb{D}$. The convexity of solutions of the Shah equation has been studied in [27, 28]. Substituting $z = e^s$ we obtain the differential equation

$$\frac{d^2 w}{ds^2} + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2) w = 0$$

with $h = 1$. The pseudoconvexity and pseudostarlikeness of solutions of the last equation have been studied in [3] (see p. 147-153).
In the proposed article we will get similar results for multiple Dirichlet series. The theorems proved here complement the results of the papers [29–31].

2. Pseudostarlikeness and Pseudoconvexity

Let $p \in \mathbb{N}$, $h = (h_1, \dots, h_p) \in \mathbb{R}_+^p$, $(n) = (n_1, \dots, n_p) \in \mathbb{N}^p$ and the sequences $\lambda_{(n)} = \left(\lambda_{n_1}^{(1)}, \dots, \lambda_{n_p}^{(p)}\right)$ are such that

$$0 < \lambda_1^{(j)} < \lambda_k^{(j)} < \lambda_{k+1}^{(j)} \uparrow +\infty \quad \text{as} \quad k \to \infty \quad \text{for every} \quad j = 1, \dots, p.$$

We denote $\mathbf{H} = h_1 \cdot \ldots \cdot h_p$, $\Lambda_{(n)} = \left(\lambda_{n_1}^{(1)} \cdot \ldots \cdot \ldots \lambda_{n_p}^{(p)}\right)$. Also let $s = (s_1, \dots, s_p) \in \mathbb{C}^p$, $s_j = \sigma_j + it_j$, $\sigma = (\sigma_1, \dots, \sigma_p)$, and for $a = (a_1, \dots, a_p)$ and $c = (c_1, \dots, c_p)$ let $(a, c) = a_1 c_1 + \ldots + a_p c_p$. We say that $a > c$ if $a_j > c_j$ for all $1 \le j \le p$.
Suppose that the multiple Dirichlet series

$$F(s) = e^{(s,h)} + \sum_{\lambda_{(n)} > h} f_{(n)} \exp\{(\lambda_{(n)}, s)\} \tag{2.1}$$

absolutely converges in $\Pi_0^p = \{s : \operatorname{Re} s < 0\}$, where $\operatorname{Re} s < 0 \iff (\operatorname{Re} s_1 < 0, \dots, \operatorname{Re} s_p < 0)$.
For the definition of the pseudostarlikeness of the function (2.1) can be used either by one variable or in joint variables or in the direction. Here we will look at the pseudostarlikeness in joint variables.
We denote

$$F^{(p)}(s) = \frac{\partial^p F(s)}{\partial s_1 \cdot \ldots \cdot \partial s_p} \quad \text{and} \quad F^{(2p)}(s) = \frac{\partial^p F^{(p)}(s)}{\partial s_1 \cdot \ldots \cdot \partial s_p} = \frac{\partial^{2p} F(s)}{\partial^2 s_1 \cdot \ldots \cdot \partial^2 s_p}.$$

We say that function (2.1) is pseudostarlike of the order $\alpha \in [0, \mathbf{H})$ and the type $\beta > 0$ in joint variables if

$$\left| \frac{F^{(p)}(s)}{F(s)} - \mathbf{H} \right| < \beta \left| \frac{F^{(p)}(s)}{F(s)} - (2\alpha - \mathbf{H}) \right|, \quad s \in \Pi_0^p. \tag{2.2}$$

and function (2.1) is pseudoconvex of the order $\alpha \in [0, \mathbf{H})$ and the type $\beta > 0$ in joint variables if

$$\left| \frac{F^{(2p)}(s)}{F^{(p)}(s)} - \mathbf{H} \right| < \beta \left| \frac{F^{(2p)}(s)}{F^{(p)}(s)} - (2\alpha - \mathbf{H}) \right|, \quad s \in \Pi_0^p.$$

Theorem 2.1. *Let $\alpha \in [0, \mathbf{H})$ and $\beta > 0$. If*

$$\sum_{\lambda_{(n)} > h} \{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)\} |f_{(n)}| \le 2\beta(\mathbf{H} - \alpha) \tag{2.3}$$

then function (2.1) is pseudostarlike of the order α and the type β in joint variables. If

$$\sum_{\lambda_{(n)} > h} \Lambda_{(n)} \{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)\} |f_{(n)}| \le 2\mathbf{H}\beta(\mathbf{H} - \alpha) \tag{2.4}$$

then function (2.1) is pseudoconvex of the order α and the type β in joint variables.

Proof. Since

$$F^{(p)}(s) = \mathbf{H} e^{(s,h)} + \sum_{\lambda_{(n)} > h} \Lambda(n) f_{(n)} \exp\{(\lambda_{(n)}, s)\},$$

we have

$$\left| F^{(p)}(s) - \mathbf{H} F(s) \right| - \beta \left| F^{(p)}(s) - (2\alpha - \mathbf{H}) F(s) \right| = \left| \sum_{\lambda_{(n)} > h} (\Lambda(n) - \mathbf{H}) f_{(n)} \exp\{(\lambda_{(n)}, s)\} \right|$$
$$- \beta \left| 2(\mathbf{H} - \alpha) e^{(s,h)} + \sum_{\lambda_{(n)} > h} (\Lambda(n) + \mathbf{H} - 2\alpha) f_{(n)} \exp\{(\lambda_{(n)}, s)\} \right|.$$

Suppose that $\alpha < \mathbf{H}$. Since $-|a+b| \le -|a| + |b|$ and $\sigma < 0$, hence in view of (2.3) we get

$$\begin{aligned}
&\left| F^{(p)}(s) - \mathbf{H} F(s) \right| - \beta \left| F^{(p)}(s) - (2\alpha - \mathbf{H}) F(s) \right| \\
&\le \left| \sum_{\lambda_{(n)} > h} (\Lambda(n) - \mathbf{H}) f_{(n)} \exp\{(\lambda_{(n)}, s)\} \right| - \left| 2\beta(\mathbf{H} - \alpha) e^{(s,h)} \right| + \left| \beta \sum_{\lambda_{(n)} > h} (\Lambda_{(n)} + \mathbf{H} - 2\alpha) f_{(n)} \exp\{(\lambda_{(n)}, s)\} \right| \\
&\le \sum_{\lambda_{(n)} > h} (\Lambda(n) - \mathbf{H}) |f_{(n)}| \exp\{(\lambda_{(n)}, \sigma)\} - 2\beta(\mathbf{H} - \alpha) e^{(\sigma,h)} + \beta \sum_{\lambda_{(n)} > h} (\Lambda_{(n)} + \mathbf{H} - 2\alpha) |f_{(n)}| \exp\{(\lambda_{(n)}, \sigma)\} \\
&= e^{(\sigma,h)} \left(\sum_{\lambda_{(n)} > h} \{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)\} |f_{(n)}| \exp\{(\lambda_{(n)} - h, \sigma))\} - 2\beta(\mathbf{H} - \alpha) \right) \\
&< \sum_{\lambda_{(n)} > h} \{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)\} |f_{(n)}| - 2\beta(\mathbf{H} - \alpha) \le 0,
\end{aligned}$$

i. e.

$$\left|F^{(p)}(s)-\mathbf{H}F(s)\right|-\beta\left|F^{(p)}(s)-(2\alpha-\mathbf{H})F(s)\right|<0, \quad s\in\Pi_0^p. \tag{2.5}$$

Since conditions (2.2) and (2.5) are equivalent, function (2.1) is pseudostarlike of the order α and the type β in joint variables.
Since $F^{(2p)}(s)/F^{(p)}(s)=G^{(p)}(s)/G(s)$, where

$$G(s)=e^{sh}+\sum_{\lambda_{(n)}>h} g_{(n)}\exp\{(\lambda_{(n)},s)\}, \quad g_{(n)}=\frac{\Lambda_{(n)}}{\mathbf{H}}f_{(n)},$$

the function F is pseudoconvex of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ in joint variables if and only if the function G is pseudostarlike of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ in joint variables. Therefore, if (2.4) holds then the function F is pseudoconvex of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ in joint variables. The proof of Theorem 2.1 is complete. □

The following theorem complements the statement of Theorem 2.1.

Theorem 2.2. *Let $\alpha\in[0,\mathbf{H})$ and $\beta>0$. If function (2.1) is pseudostarlike of the order α and the type β in joint variables and all $f_{(m)}\le 0$ then (2.3) holds. If function (2.1) is pseudoconvex of the order α and the type β in joint variables and all $f_{(m)}\le 0$ then (2.4) holds.*

Proof. If function (2.1) is pseudostarlike of the order α and the type β in joint variables and $f_{(m)}=-|f_{(m)}|$ then in view of (2.5) as above we have for all $s\in\Pi_0^p$

$$\left|\frac{-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})|f_{(n)}|\exp\{(\lambda_{(n)},s)\}}{2(\mathbf{H}-\alpha)e^{(s,h)}-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)+\mathbf{H}-2\alpha)|f_{(n)}|\exp\{(\lambda_{(n)},s)\}}\right|=\left|\frac{F^{(p)}(s)-\mathbf{H}F(s)}{F^{(p)}(s)-(2\alpha-\mathbf{H})F(s)}\right|<\beta$$

and therefore,

$$\mathrm{Re}\,\frac{\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})|f_{(n)}|\exp\{(\lambda_{(n)},s)\}}{2(\mathbf{H}-\alpha)e^{(s,h)}-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)+\mathbf{H}-2\alpha)|f_{(n)}|\exp\{(\lambda_{(n)},s)\}}<\beta$$

whence for all $\sigma<0$ we obtain

$$\frac{\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})|f_{(n)}|\exp\{(\lambda_{(n)},\sigma)\}}{2(\mathbf{H}-\alpha)e^{(\sigma,h)}-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)+\mathbf{H}-2\alpha)|f_{(n)}|\exp\{(\lambda_{(n)},\sigma)\}}<\beta$$

Letting $\sigma\to 0$ from here we get

$$\frac{\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)-\mathbf{H})|f_{(n)}|}{2(\mathbf{H}-\alpha)-\sum\limits_{\lambda_{(n)}>h}(\Lambda(n)+\mathbf{H}-2\alpha),|f_{(n)}|}<\beta$$

whence (2.3) follows. The first part of Theorem 2.2 is proved. The second part is proved similarly. □

Theorems 2.1 and 2.2 imply the following statements.

Corollary 2.3. *In order that the function (2.1) is pseudostarlike of the order $\alpha\in[0,\mathbf{H})$ in joint variables, it is sufficient and in the case, when all $f_{(n)}\le 0$, it is necessary that*

$$\sum_{\lambda_{(n)}>h}\{\Lambda_{(n)}-\alpha)\}|f_{(n)}|\le\mathbf{H}-\alpha. \tag{2.6}$$

In order that the function (2.1) is pseudoconvex of the order $\alpha\in[0,\mathbf{H})$ in joint variables, it is sufficient and in the case, when all $f_{(n)}\le 0$, it is necessary that

$$\sum_{\lambda_{(n)}>h}\Lambda_{(n)}\{\Lambda_{(n)}-\alpha\}|f_{(n)}|\le\mathbf{H}(\mathbf{H}-\alpha).$$

Corollary 2.4. *In order that the function (2.1) is pseudostarlike in joint variables, it is sufficient and in the case, when all $f_{(n)}\le 0$, it is necessary that*

$$\sum_{\lambda_{(n)}>h}\Lambda_{(n)}|f_{(n)}|\le\mathbf{H}.$$

In order that the function (2.1) is pseudoconvex in joint variables, it is sufficient and in the case, when all $f_{(n)}\le 0$, it is necessary that $\sum\limits_{\lambda_{(n)}>h}(\Lambda_{(n)})^2|f_{(n)}|\le\mathbf{H}^2$.

3. Neighborhoods of Multiple Dirichlet Series

Here the class of series (2.1) absolutely convergent in Π_0^p we denote by D and we say that $F \in D^*$ if $F \in D$ and all $f_{(n)} \le 0$. By $PSD_{\alpha,\beta}$ we denote a class of pseudostarlike functions (2.1) of the order α and the type β in joint variables, and by $PCD_{\alpha,\beta}$ we denote a class of pseudoconvex functions (2.1) of the order α and the type β in joint variables.
For $j > 0$ and $\delta > 0$ we define the neighborhood of $F \in D$ in joint variables as follows

$$O_{j,\delta}(F) = \left\{ G(s) = e^{(s,h)} + \sum_{\lambda_{(n)}>h} g_{(n)} \exp\{(\lambda_{(n)}, s)\} \in D : \sum_{\lambda_{(n)}>h} \Lambda^j_{(n)} |g_{(n)} - f_{(n)}| \le \delta \right\}.$$

Similarly, for $F \in D^*$

$$O^*_{j,\delta}(F) = \left\{ G(s) = e^{(s,h)} + \sum_{\lambda_{(n)}>h} g_{(n)} \exp\{(\lambda_{(n)}, s)\} \in D^* : \sum_{\lambda_{(n)}>h} \Lambda^j_{(n)} |g_{(n)} - f_{(n)}| \le \delta \right\}.$$

Here we will establish a connection between classes $PSD_{\alpha,\beta}$, $PCD_{\alpha,\beta}$ and $O_{j,\delta}(F)$, $O^*_{j,\delta}(F)$. We need the following lemma.

Lemma 3.1. *Let $F \in D$. Then $G \in O_{2,\mathbf{H}\delta}(F)$ if and only if $\dfrac{G^{(p)}}{\mathbf{H}} \in O_{1,\delta}\left(\dfrac{F^{(p)}}{\mathbf{H}}\right)$.*

Indeed,

$$\frac{F^{(p)}}{\mathbf{H}} = e^{sh} + \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)}}{\mathbf{H}} f_{(n)} \exp\{(\lambda_{(n)}, s)\} \in D.$$

Therefore, $\dfrac{G^{(p)}}{\mathbf{H}} \in O_{1,\delta}\left(\dfrac{F^{(p)}}{\mathbf{H}}\right)$ if and only if $\sum\limits_{\lambda_{(n)}>h} \Lambda_{(n)} \left| \dfrac{\Lambda_{(n)}}{\mathbf{H}} f_{(n)} - \dfrac{\Lambda_{(n)}}{\mathbf{H}} g_{(n)} \right| \le \delta$, i. e. $G \in O_{2,\mathbf{H}\delta}(F)$.
At first, we consider the case when $F(s) = E(s) := e^{(h,s)}$ and we prove such theorem.

Theorem 3.2. *For the function $E(s) = e^{(h,s)}$ the following correlations are correct: $O_{1,\mathbf{H}}(E) \subset PSD_{0,1}$, $O^*_{1,\mathbf{H}}(E) = PSD_{0,1} \bigcap D^*$, $O_{2,\mathbf{H}^2}(E) \subset PCD_{0,1}$ and $O^*_{2,\mathbf{H}^2}(E) = PCD_{0,1} \bigcap D^*$.*

Proof. If $G \in O_{1,\mathbf{H}}(E)$ then $G \in D$ and $\sum\limits_{\lambda_{(n)}>h} \Lambda_{(n)} |g_{(n)}| \le \mathbf{H}$. Since

$$G^{(p)}(s) = \mathbf{H} e^{(s,h)} + \sum_{\lambda_{(n)}>h} \Lambda(n) g_{(n)} \exp\{(\lambda_{(n)}, s)\},$$

we have

$$\begin{aligned}
\left| \frac{G^{(p)}(s)}{\mathbf{H}} - G(s) \right| &= \left| \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)}}{\mathbf{H}} g_{(n)} \exp\{(\lambda_{(n)}, s)\} - \sum_{\lambda_{(n)}>h} g_{(n)} \exp\{(\lambda_{(n)}, s)\} \right| \\
&= \left| \sum_{\lambda_{(n)}>h} \left(\frac{\Lambda_{(n)}}{\mathbf{H}} - 1 \right) g_{(n)} \exp\{(\lambda_{(n)}, s)\} \right| \\
&\le \sum_{\lambda_{(n)}>h} \left(\frac{\Lambda_{(n)}}{\mathbf{H}} - 1 \right) |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} \\
&= \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)}}{\mathbf{H}} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} - \sum_{\lambda_{(n)}>h} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} \\
&\le \exp\{h, \sigma)\} \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)}}{\mathbf{H}} |g_{(n)}| - \sum_{\lambda_{(n)}>h} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} \\
&\le \exp\{h, \sigma)\} - \sum_{\lambda_{(n)}>h} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\}.
\end{aligned}$$

On the other hand,

$$|G(s)| = \left| e^{(h,s)} + \sum_{\lambda_{(n)}>h} |g_{(n)}| \exp\{\lambda_{(n)}, \sigma)\} \right| \ge e^{(h,\sigma)} - \sum_{\lambda_{(n)}>h} |g_{(n)}| e^{(\lambda_{(n)},\sigma)}$$

and thus, $\left| \dfrac{G^{(p)}(s)}{\mathbf{H}} - G(s) \right| \le |G(s)|$, i. e. $\left| \dfrac{G^{(p)}(s)}{\mathbf{H}G(s)} - 1 \right| \le 1$ for all $s \in \Pi_0^p$. From hence it follows that $\mathrm{Re}\left\{ \dfrac{G^{(p)}(s)}{\mathbf{H}G(s)} \right\} > 0$, i. e. $G \in PSD_{0,1}$ and $O_{1,\mathbf{H}}(E) \subset PSD_{0,1}$.

From above it follows that $O^*_{1,\mathbf{H}}(E) \subset PSD_{0,1}$. On the other hand, if $G \in D$ and $G \in PSD_{0,1}$ then by Corollary 2 $\sum_{\lambda_{(n)}>h} \Lambda_{(n)}|g_{(n)}| \leq \mathbf{H}$, i. e. $G \in O^*_{1,\mathbf{H}}(E)$. Thus, $PSD_{0,1} \bigcap D \subset O^*_{1,\mathbf{H}}(E)$ and $PSD_{0,1} \bigcap D^* = O^*_{1,\mathbf{H}}(E)$.
Since $G \in PCD_{0,1}$ if and only if $G^{(p)}/\mathbf{H} \in PSD_{0,1}$, and by Lemma 3.1 $G \in O_{2,\mathbf{H}\delta}(E)$ if and only if $G^{(p)}/\mathbf{H} \in O_{1,\delta}(E/\mathbf{H}) = O_{1,\delta}(E)$, one can easily obtain the corresponding results for pseudoconvex functions.
For example, if $G \in O_{2,\mathbf{H}^2}(E)$ then $G^{(p)}/\mathbf{H} \in O_{1,\mathbf{H}}(E)$ and, thus, $G^{(p)}/\mathbf{H} \in PSD_{0,1}$ and $G^{(p)} \in PSD_{0,1}$. Therefore, $O_{2,\mathbf{H}^2}(E) \subset PCD_{0,1}$. The proof of Theorem 3.2 is completed. □

Now we investigate the neighborhoods of a pseudostarlike function of the order α. The following theorem is true.

Theorem 3.3. *Let* $0 \leq \alpha_1 < \alpha < \mathbf{H}$, $\Lambda = \min\{\Lambda_{(n)} : \lambda_{(n)} > h\}$, $\delta_1 = (\alpha - \alpha_1)\dfrac{\Lambda - \mathbf{H}}{\Lambda - \alpha}$, $\delta_2 = \Lambda\left(\dfrac{\mathbf{H}-\alpha}{\Lambda-\alpha} + \dfrac{\mathbf{H}-\alpha_1}{\Lambda-\alpha_1}\right)$ *and* $F \in D^* \bigcap PSD_{\alpha,1}$. *Then* $O^*_{1,\delta_1}(F) \subset PSD_{\alpha_1,1}$ *and* $D^* \bigcap PSD_{\alpha_1,1} \subset O^*_{1,\delta_2}(F)$, $O^*_{2,\mathbf{H}\delta_1}(F) \subset PSD_{\alpha_1,1}$ *and* $D^* \bigcap PSD_{\alpha_1,1} \subset O^*_{1,\mathbf{H}\delta_2}(F)$.

Proof. Let $G \in O^*_{1,\delta_1}(F)$. Since $F \in PSD_{\alpha,1}$, by Corollary 2.3 condition (2.6) holds. Therefore, for $\alpha_1 < \alpha$ we get

$$\begin{aligned}\sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|g_{(n)}| &\leq \sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|g_{(n)} - f_{(n)}| + \sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|f_{(n)}| \\ &= \sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|g_{(n)} - f_{(n)}| + \sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha\}|f_{(n)}| + (\alpha - \alpha_1)\sum_{\lambda_{(n)}>h} |f_{(n)}| \\ &\leq \delta_1 + \mathbf{H} - \alpha + (\alpha - \alpha_1)\sum_{\lambda_{(n)}>h} |f_{(n)}|.\end{aligned}$$

But in view of (2.6)

$$\sum_{\lambda_{(n)}>h} |f_{(n)}| \leq \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)} - \alpha}{\Lambda - \alpha}|f_{(n)}| \leq \frac{\mathbf{H}-\alpha}{\Lambda-\alpha}.$$

Therefore,

$$\sum_{\lambda_{(n)}>h} \{\Lambda_{(n)} - \alpha_1\}|g_{(n)}| \leq \delta_1 + \mathbf{H} - \alpha + (\alpha - \alpha_1)\frac{\mathbf{H}-\alpha}{\Lambda-\alpha} \leq \mathbf{H} - \alpha_1$$

i. e. by Corollary 2.3 the function $G \in PSD_{\alpha_1,1}$ and, thus, $O^*_{1,\delta_1}(F) \subset PSD_{\alpha_1,1}$.
Now suppose that $G \in D^*_0 \bigcap PSD_{\alpha_1,1}$. Then in view of (2.6) we have

$$\begin{aligned}\sum_{\lambda_{(n)}>h} \Lambda_{(n)}|g_{(n)} - f_{(n)}| &= \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)}}{\Lambda_{(n)} - \alpha_1}(\Lambda_{(n)} - \alpha_1)|g_{(n)} - f_{(n)}| \\ &\leq \frac{\Lambda}{\Lambda - \alpha_1}\sum_{\lambda_{(n)}>h} (\Lambda_{(n)} - \alpha_1)|g_{(n)} - f_{(n)}| \\ &\leq \frac{\Lambda}{\Lambda - \alpha_1}\left(\sum_{\lambda_{(n)}>h} (\Lambda_{(n)} - \alpha_1)|g_{(n)}| + \sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)} - \alpha_1}{\Lambda_{(n)} - \alpha}(\Lambda_{(n)} - \alpha)|f_{(n)}|\right) \\ &\leq \frac{\Lambda}{\Lambda - \alpha_1}\left(\mathbf{H} - \alpha_1 + \frac{\Lambda - \alpha_1}{\Lambda - \alpha}(\mathbf{H} - \alpha)\right) = \delta_2,\end{aligned}$$

i. e. $G \in O^*_{1,\delta_2}(F)$ and, thus, $D^* \bigcap PSD_{\alpha_1,1} \subset O^*_{1,\delta_2}(F)$.
Finally, since $F \in PCD_{0,1}$ if and only if $\dfrac{F^{(p)}}{\mathbf{H}} \in PSD_{0,1}$, and by Lemma 1 $G \in O_{2,\mathbf{H}\delta}(F)$ if and only if $\dfrac{G^{(p)}}{\mathbf{H}} \in O_{1,\delta}\left(\dfrac{F^{(p)}}{\mathbf{H}}\right)$, one can easily obtain the corresponding results for pseudoconvex functions. For example, if $G \in O_{2,\mathbf{H}\delta_1}(F)$ then $\dfrac{G^{(p)}}{\mathbf{H}} \in O_{1,\delta_1}\left(\dfrac{F^{(p)}}{\mathbf{H}}\right)$ and, thus, $\dfrac{G^{(p)}}{\mathbf{H}} \in PSD_{\alpha_1,1}$ and $G \in PCD_{\alpha_1,1}$. Therefore, $O_{2,\mathbf{H}\delta_1}(F) \subset PCD_{\alpha_1,1}$. The proof of Theorem 3.3 is completed. □

Finally, we consider the generalized case when the function F is a pseudostarlike in joint variables of the order α and the type β. The following theorem is true.

Theorem 3.4. *Let* $0 \leq \alpha < \mathbf{H}$, $0 < \beta < \beta_1 \leq 1$,

$$Q = \frac{(1+\beta_1)\Lambda - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)}{(1+\beta)\Lambda - 2\beta\alpha - \mathbf{H}(1-\beta)}, \quad \delta_1 = \frac{2(\mathbf{H}-\alpha)(\beta_1 - Q\beta)}{1+\beta_1}$$

$$\delta_2 = \frac{2\beta_1(\mathbf{H}-\alpha)\Lambda}{(1+\beta_1)\Lambda - \mathbf{H}(1-\beta_1) - 2\alpha\beta_1} + \frac{2\beta(\mathbf{H}-\alpha)\Lambda}{(1+\beta)\Lambda - \mathbf{H}(1-\beta) - 2\alpha\beta},$$

and

$$F \in D^* \bigcap PSD_{\alpha,\beta}.$$

Then, $O^*_{1,\delta_1}(F) \subset PSD_{\alpha,\beta_1}$, *and* $D^* \bigcap PSD_{\alpha,\beta_1} \subset O^*_{1,\delta_2}(F)$, $O^*_{2,\mathbf{H}\delta_1}(F) \subset PSD_{\alpha,\beta_1}$ *and* $D^* \bigcap PSD_{\alpha,\beta_1} \subset O^*_{2,\mathbf{H}\delta_2}(F)$.

Proof. At first we remark that in view of the conditions $0 \le \alpha < \mathbf{H}$ and $0 < \beta < \beta_1 \le 1$

$$\max_{\lambda_{(n)}>h} \frac{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)}{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)} = Q$$

and $\beta_1 - Q\beta > 0$. For $0 < \beta < \beta_1 \le 1$, we have

$$\begin{aligned}\sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta_1)\}|g_{(n)}| &\le \sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)\}|g_{(n)} - f_{(n)}| \\ &+ \sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)\}|f_{(n)}|\end{aligned} \tag{3.1}$$

If $G \in O^*_{1,\delta_1}(F)$, then

$$\sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)\}|g_{(n)} - f_{(n)}| \le (1+\beta_1) \sum_{\lambda_{(n)}>h} \Lambda_{(n)}|g_{(n)} - f_{(n)}| \le (1+\beta_1)\delta_1,$$

and, since $F \in D^* \bigcap PSD_{\alpha,\beta}$, by Theorem 2.1

$$\begin{aligned}\sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)\}|f_{(n)}| &= \sum_{\lambda_{(n)}>h} \frac{(1+\beta_1)\Lambda_{(n)} - 2\beta_1\alpha - \mathbf{H}(1-\beta_1)}{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)} \{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)\}|f_{(n)}| \\ &\le Q \sum_{\lambda_{(n)}>h} \{(1+\beta)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta)\}|f_{(n)}| \le 2Q\beta(\mathbf{H}-\alpha).\end{aligned}$$

Therefore, (3.1) implies

$$\sum_{\lambda_{(n)}>h} \{(1+\beta_1)\Lambda_{(n)} - 2\beta\alpha - \mathbf{H}(1-\beta_1)\}|g_{(n)}| \le (1+\beta_1)\delta_1 + 2Q\beta(\mathbf{H}-\alpha) = 2\beta_1(\mathbf{H}-\alpha),$$

i. e. by Theorem 2.2 $G \in PSD_{\alpha,\beta_1}$. Theorem 3.4 is proved. □

4. Hadamard Compositions of Multiple Dirichlet Series

For Dirichlet series $F_j(s) = e^{(s,h)} + \sum_{\lambda_{(n)}>h} f_{(n),j} \exp\{(\lambda_{(n)}, s)\}$ $(j = 1, 2)$ the Hadamard composition has the form

$$(F_1 * F_2(s) = e^{(s,h)} + \sum_{\lambda_{(n)}>h} f_{(n),1} f_{(n),2} \exp\{(\lambda_{(n)}, s)\}. \tag{4.1}$$

Theorem 2.1 and Corollary 2.3 imply the following statements.

Corollary 4.1. *If the functions $F_j \in D^*$ are pseudostarlike of the orders $\alpha_j \in [0, \mathbf{H})$ then Hadamard composition $F_1 * F_2$ is pseudostarlike of the order $\alpha = \max\{\alpha_1, \alpha_2\}$.*

If the functions $F_j \in D^*$ are pseudoconvex of the orders $\alpha_j \in [0, \mathbf{H})$ then Hadamard composition $F_1 * F_2$ is pseudoconvex of the order $\alpha = \max\{\alpha_1, \alpha_2\}$.
Indeed, since $F_j \in D^*$ that is $f_{(n),j} \le 0$ for all n and j, from (2.6) it follows that $|f_{(n),j}| \le (\mathbf{H} - \alpha_j)/(\Lambda_{(n)} - \alpha_j) < 1$ for $\lambda_{(n)} > h$ and therefore,

$$\sum_{\lambda_{(n)}>h} \frac{\Lambda_{(n)} - \alpha_1}{\mathbf{H} - \alpha_1} |f_{(n),1} f_{(n),2}| \le \sum_{k=1}^{\infty} \frac{\Lambda_{(n)} - \alpha_1}{\mathbf{H} - \alpha_1} |f_{(n),k}| \le 1$$

for each $k = 1$ and $k = 2$, i. e. the function $F_1 * F_2$ is pseudostarlike of the order α_1 and of the order α_2, and thus, $F_1 * F_2$ is pseudostarlike of the order $\alpha = \max\{\alpha_1, \alpha_2\}$.
The proof of the pseudoconvexity of $F_1 * F_2$ is similar.

Corollary 4.2. *If the functions $F_j \in D^*$ are pseudostarlike of the order $\alpha \in [0, \mathbf{H})$ and the type $\beta_j > 0$ then Hadamard composition $F_1 * F_2$ is pseudostarlike of the order α and the type $\beta = \min\{\beta_1, \beta_2\}$.*

If the functions $F_j \in D^*$ are pseudoconvex of the order $\alpha \in [0, \mathbf{H})$ and the type $\beta_j > 0$ then Hadamard composition $F_1 * F_2$ is pseudoconvex of the order α and the type $\beta = \min\{\beta_1, \beta_2\}$.
Indeed, from (2.3) it follows that

$$|f_{(n),j}| \le \frac{2\beta_j(\mathbf{H}-\alpha)}{(1+\beta_j)\Lambda_{(n)} - (1-\beta_j)\mathbf{H} - 2\beta_j\alpha} < 1$$

for $\lambda_{(n)} > h$ and therefore, as above we have

$$\sum_{\lambda_{(n)}>h} \frac{(1+\beta_j)\Lambda_{(n)} - (1-\beta_j)\mathbf{H} - 2\beta_j\alpha}{2\beta_j(\mathbf{H}-\alpha)} |f_{(n),1}||f_{(n),2}| \le 1, \quad j = 1, 2.$$

Hence it follows that $F_1 * F_2$ is pseudostarlike of the order α and the type β_j for each j and thus, $F_1 * F_2$ is pseudostarlike of the order α and the type $\beta = \min\{\beta_1, \beta_2\}$.
The proof of the pseudoconvexity of $F_1 * F_2$ is similar.

5. Differential Equation

Here we consider a differential equation

$$\frac{\partial^p w}{\partial s_1,\dots,\partial s_p}+(\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}+\gamma_2)w=0 \tag{5.1}$$

and we will find out at what conditions on the parameters γ_0, γ_1, γ_2 this equation has solution (2.1) pseudostarlike in joint variables of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$.
We will look for a solution to the equation in the form

$$F(s)=e^{(s,h)}+\sum_{n=1}^{\infty}f_{(\mathbf{n})}\exp\{((n+1)h,s)\}=e^{(s,h)}+\sum_{n=1}^{\infty}f_{(\mathbf{n})}\exp\{(n+1)(h,s)\}, \tag{5.2}$$

where $(\mathbf{n})=(n,\dots,n)$ (p times) and $\lambda_{(n)}=(n+1)h=((n+1)h_1,\dots,(n+1)h_p)$. Since

$$F^{(p)}(s)=\mathbf{H}e^{(s,h)}+\sum_{n=1}^{\infty}(n+1)^p\mathbf{H}f_{(\mathbf{n})}\exp\{(n+1)(h,s)\},$$

we have

$$\mathbf{H}e^{(s,h)}+\sum_{n=1}^{\infty}(n+1)^p\mathbf{H}f_{(\mathbf{n})}\exp\{(n+1)(h,s)\}+\gamma_0 e^{3(s,h)}+\gamma_0\sum_{n=1}^{\infty}f_{(\mathbf{n})}\exp\{(n+3)(h,s)\}+\gamma_1 e^{2(s,h)}$$
$$+\gamma_1\sum_{n=1}^{\infty}f_{(\mathbf{n})}\exp\{(n+2)(h,s)\}+\gamma_2 e^{(s,h)}+\gamma_2\sum_{n=1}^{\infty}f_{(\mathbf{n})}\exp\{(n+1)(h,s)\}\equiv 0,$$

i. e.

$$(\mathbf{H}+\gamma_2)e^{(s,h)}+\gamma_1 e^{2(s,h)}+\gamma_0 e^{3(s,h)}+(2^p\mathbf{H}+\gamma_2)f_{(\mathbf{1})}e^{2(s,h)}+(3^p\mathbf{H}+\gamma_2)f_{(\mathbf{2})}e^{3(s,h)}+\gamma_1 f_{(\mathbf{1})}e^{3(s,h)}+\sum_{n=3}^{\infty}\{((n+1)^p\mathbf{H}+\gamma_2)f_{(\mathbf{n})}$$
$$\geq\mathbf{3}+\gamma_1 f_{(\mathbf{n-1})}+\gamma_0 f_{(\mathbf{n-2})}\}\exp\{(n+1)(h,s)\}\equiv 0.$$

Hence it follows that

$$\mathbf{H}+\gamma_2=0,\quad (2^p\mathbf{H}+\gamma_2)f_{(\mathbf{1})}+\gamma_1=0,\quad (3^p\mathbf{H}+\gamma_2)f_{(\mathbf{2})}+\gamma_1 f_{(\mathbf{1})}+\gamma_0=0$$

and

$$((n+1)^p\mathbf{H}+\gamma_2)f_{(\mathbf{n})}+\gamma_1 f_{(\mathbf{n-1})}+\gamma_0 f_{(\mathbf{n-2})}\}\exp\{(n+1)(h,s),\quad n\geq 3.$$

Therefore, the following lemma is correct.

Lemma 5.1. *Function (5.2) satisfies differential equation (5.1) if and only if*

$$\gamma_2=-\mathbf{H},\quad f_{(\mathbf{1})}=-\frac{\gamma_1}{(2^p-1)\mathbf{H}},\quad f_{(\mathbf{2})}=-\frac{\gamma_1 f_{((\mathbf{1})}+\gamma_0}{(3^p-1)-1)\mathbf{H}}$$

and

$$f_{(\mathbf{n})}=-\frac{\gamma_1 f_{((\mathbf{n-1}))}+\gamma_0 f_{((\mathbf{n-2}))}}{((n+1)^p-1)\mathbf{H}}\quad (n\geq 3). \tag{5.3}$$

Using Lemma 5.1 now we prove the following theorem.

Theorem 5.2. *Let $\alpha\in[0,\mathbf{H})$, $\beta>0$, $\gamma_2=-\mathbf{H}$ and $|\gamma_0|+|\gamma_1|\leq\dfrac{2\beta}{1+\beta}(\mathbf{H}-\alpha)$. Then differential equation (5.1) has entire solution (5.2) pseudostarlike in joint variables of the order α and the type β.*

Proof. Recall that the function (2.1) is called pseudostarlike in joint variables of the order $\alpha\in[0,\mathbf{H})$ and type $\beta>0$ if

$$|F^{(p)}(s)/F(s)-\mathbf{H}|<\beta|F^{(p)}(s)/F(s)-(2\alpha-\mathbf{H})|.$$

Also, we remark that function F is a solution of differential equation (5.1) if and only if

$$F^{(p)}(s)+(\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}+\gamma_2)F(s)\equiv 0.$$

Hence it follows that F is pseudostarlike in joint variables of the order $\alpha\in[0,\mathbf{H})$ and the type $\beta>0$ if and only if

$$|-(\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}+\gamma_2)-\mathbf{H}|<\beta|-(\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}+\gamma_2)-(2\alpha-\mathbf{H})|,\quad s\in\Pi_0^p,$$

i. e.

$$|\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}|<\beta|\gamma_0 e^{2(s,h)}+\gamma_1 e^{(s,h)}-2(\mathbf{H}-\alpha)|,\quad s\in\Pi_0^p,$$

and thus,

$$\frac{1}{\beta} < \left| 1 - \frac{2(\mathbf{H}-\alpha)}{\gamma_0 e^{2(s,h)} + \gamma_1 e^{(s,h)}} \right|, \quad s \in \Pi_0^p.$$

Since $|w-1| \geq |w|-1$, this inequality holds if

$$\frac{2(\mathbf{H}-\alpha)}{|\gamma_0 e^{2(s,h)} + \gamma_1 e^{(s,h)}|} > 1 + \frac{1}{\beta},$$

i. e. if

$$|\gamma_0 e^{2(s,h)} + \gamma_1 e^{(s,h)}| < \frac{2\beta(\mathbf{H}-\alpha)}{1+\beta}, \quad s \in \Pi_0^p.$$

The last condition holds because

$$|\gamma_0 e^{2(s,h)} + \gamma_1 e^{(s,h)}| \leq |\gamma_0| e^{2(\sigma,h)} + |\gamma_1| e^{(\sigma,h)} < |\gamma_0| + |\gamma_1| \leq \frac{2\beta}{1+\beta}(\mathbf{H}-\alpha)$$

and thus, function (5.2) is pseudostarlike in joint variables of the order α and the type β.
Finally, since for every $\sigma \in \mathbb{R}^p$ there exits $n_0 = n_0(\sigma) \geq 1$ such that

$$\frac{(|\gamma_0| + |\gamma_1|)\exp\{2(h,\sigma)\}}{n^p \mathbf{H}} \leq \frac{1}{2},$$

in view of (5.3) we have

$$\begin{aligned}
\sum_{n=n_0}^{\infty} |f_{(\mathbf{n})}| \exp\{(n+1)(\sigma,h)\} &\leq \sum_{n=n_0}^{\infty} \frac{|\gamma_1| f_{(\mathbf{n-1})}|}{((n+1)^p-1)\mathbf{H}} e^{n(\sigma,h)} e^{(\sigma,h)} + \sum_{n=n_0}^{\infty} \frac{|\gamma_0| f_{(\mathbf{n-2})}|}{((n+1)^p-1)\mathbf{H}} e^{(n-1)(\sigma,h)} e^{2(\sigma,h)} \\
&= \sum_{n=n_0-1}^{\infty} \frac{|\gamma_1| f_{(\mathbf{n})}|}{((n+2)^p-1)\mathbf{H}} e^{(n+1)(\sigma,h)} e^{(\sigma,h)} + \sum_{n=n_0-2}^{\infty} \frac{|\gamma_0| f_{(\mathbf{n})}|}{((n+3)^p-1)\mathbf{H}} e^{(n+1)(\sigma,h)} e^{2(\sigma,h)} \\
&\leq \sum_{n=n_0}^{\infty} \frac{(|\gamma_1| + |\gamma_0|) e^{2(\sigma,h)}}{n^p \mathbf{H}} |f_{(\mathbf{n})}| e^{(n+1)(\sigma,h)} + \frac{|\gamma_1| f_{(\mathbf{n_0-1})}|}{((n_0+1)^p-1)\mathbf{H}} e^{(n_0+1)(\sigma,h)} \\
&\quad + \frac{|\gamma_0| f_{(\mathbf{n_0-2})}|}{((n_0+1)^p-1)\mathbf{H}} e^{(n_0+1)(\sigma,h)} + \frac{|\gamma_0| f_{(\mathbf{n_0-1})}|}{((n+2)^p-1)\mathbf{H}} e^{(n_0+2)(\sigma,h)} \\
&\leq \frac{1}{2} \sum_{n=n_0}^{\infty} |f_{(\mathbf{n})}| \exp\{(n+1)(\sigma,h)\} + \text{const},
\end{aligned}$$

i. e. Dirichlet series (5.2) is entire (absolutely convergent in $\mathbb{C}^p$). The proof of Theorem 5.2 is completed. □

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Some f-Divergence Measures Related to Jensen's One

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Abstract

In this paper, we introduce some f-divergence measures that are related to the Jensen's divergence introduced by Burbea and Rao in 1982. We establish their joint convexity and provide some inequalities between these measures and a combination of Csiszár's f-divergence, f-midpoint divergence and f-integral divergence measures.

1. Introduction

Let $(X,\mathscr{A})$ be a measurable space satisfying $|\mathscr{A}|>2$ and μ be a σ-finite measure on $(X,\mathscr{A})$. Let $\mathscr{P}$ be the set of all probability measures on $(X,\mathscr{A})$ which are absolutely continuous with respect to μ. For $P,Q\in\mathscr{P}$, let $p=\frac{dP}{d\mu}$ and $q=\frac{dQ}{d\mu}$ denote the *Radon-Nikodym* derivatives of P and Q with respect to μ.
Two probability measures $P,Q\in\mathscr{P}$ are said to be *orthogonal* and we denote this by $Q\perp P$ if

$$P(\{q=0\})=Q(\{p=0\})=1.$$

Let $f:[0,\infty)\to(-\infty,\infty]$ be a convex function that is continuous at 0, i.e., $f(0)=\lim_{u\downarrow 0}f(u)$.
In 1963, I. Csiszár [1] introduced the concept of f-divergence as follows.

Definition 1.1. *Let $P,Q\in\mathscr{P}$. Then*

$$I_f(Q,P)=\int_X p(x)\,f\left[\frac{q(x)}{p(x)}\right]d\mu(x), \tag{1.1}$$

is called the f-divergence of the probability distributions Q and P.

Remark 1.2. *Observe that, the integrand in the formula (1.1) is undefined when $p(x)=0$. The way to overcome this problem is to postulate for f as above that*

$$0f\left[\frac{q(x)}{0}\right]=q(x)\lim_{u\downarrow 0}\left[uf\left(\frac{1}{u}\right)\right],\ x\in X. \tag{1.2}$$

We now give some examples of f-divergences that are well-known and often used in the literature (see also [2]).

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1.1. The class of χ^{α}-divergences

The f-divergences of this class, which is generated by the function χ^{α}, $\alpha \in [1,\infty)$, defined by

$$\chi^{\alpha}(u) = |u-1|^{\alpha}, \quad u \in [0,\infty)$$

have the form

$$I_f(Q,P) = \int_X p\left|\frac{q}{p}-1\right|^{\alpha} d\mu = \int_X p^{1-\alpha}|q-p|^{\alpha} d\mu. \tag{1.3}$$

From this class only the parameter $\alpha = 1$ provides a distance in the topological sense, namely the *total variation distance* $V(Q,P) = \int_X |q-p|\,d\mu$. The most prominent special case of this class is, however, *Karl Pearson's χ^2-divergence*

$$\chi^2(Q,P) = \int_X \frac{q^2}{p} d\mu - 1$$

that is obtained for $\alpha = 2$.

1.2. Dichotomy class

From this class, generated by the function $f_{\alpha} : [0,\infty) \to \mathbb{R}$

$$f_{\alpha}(u) = \begin{cases} u-1-\ln u & \text{for } \alpha = 0; \\ \frac{1}{\alpha(1-\alpha)}\left[\alpha u + 1 - \alpha - u^{\alpha}\right] & \text{for } \alpha \in \mathbb{R}\setminus\{0,1\}; \\ 1-u+u\ln u & \text{for } \alpha = 1; \end{cases}$$

only the parameter $\alpha = \frac{1}{2}$ $\left(f_{\frac{1}{2}}(u) = 2(\sqrt{u}-1)^2\right)$ provides a distance, namely, the *Hellinger distance*

$$H(Q,P) = \left[\int_X (\sqrt{q}-\sqrt{p})^2 d\mu\right]^{\frac{1}{2}}.$$

Another important divergence is the *Kullback-Leibler divergence* obtained for $\alpha = 1$,

$$KL(Q,P) = \int_X q\ln\left(\frac{q}{p}\right) d\mu.$$

1.3. Matsushita's divergences

The elements of this class, which is generated by the function φ_{α}, $\alpha \in (0,1]$ given by

$$\varphi_{\alpha}(u) := |1-u^{\alpha}|^{\frac{1}{\alpha}}, \quad u \in [0,\infty),$$

are prototypes of metric divergences, providing the distances $\left[I_{\varphi_{\alpha}}(Q,P)\right]^{\alpha}$.

1.4. Puri-Vincze divergences

This class is generated by the functions Φ_{α}, $\alpha \in [1,\infty)$ given by

$$\Phi_{\alpha}(u) := \frac{|1-u|^{\alpha}}{(u+1)^{\alpha-1}}, \quad u \in [0,\infty).$$

It has been shown in [3] that this class provides the distances $[I_{\Phi_{\alpha}}(Q,P)]^{\frac{1}{\alpha}}$.

1.5. Divergences of Arimoto-type

This class is generated by the functions

$$\Psi_{\alpha}(u) := \begin{cases} \frac{\alpha}{\alpha-1}\left[(1+u^{\alpha})^{\frac{1}{\alpha}} - 2^{\frac{1}{\alpha}-1}(1+u)\right] & \text{for } \alpha \in (0,\infty)\setminus\{1\}; \\ (1+u)\ln 2 + u\ln u - (1+u)\ln(1+u) & \text{for } \alpha = 1; \\ \frac{1}{2}|1-u| & \text{for } \alpha = \infty. \end{cases}$$

It has been shown in [4] that this class provides the distances $\left[I_{\Psi_{\alpha}}(Q,P)\right]^{\min\left(\alpha,\frac{1}{\alpha}\right)}$ for $\alpha \in (0,\infty)$ and $\frac{1}{2}V(Q,P)$ for $\alpha = \infty$.
For f continuous convex on $[0,\infty)$ we obtain the $*$-*conjugate* function of f by

$$f^*(u) = uf\left(\frac{1}{u}\right), \quad u \in (0,\infty)$$

and

$$f^{*}(0)=\lim_{u\downarrow 0}f^{*}(u).$$

It is also known that if f is continuous convex on $[0,\infty)$ then so is f^{*}.
The following two theorems contain the most basic properties of f-divergences. For their proofs we refer the reader to Chapter 1 of [5] (see also [2]).

Theorem 1.3 (Uniqueness and Symmetry Theorem)**.** *Let f, f_1 be continuous convex on $[0,\infty)$. We have*

$$I_{f_1}(Q,P)=I_f(Q,P),$$

for all P, $Q\in\mathscr{P}$ if and only if there exists a constant $c\in\mathbb{R}$ such that

$$f_1(u)=f(u)+c(u-1),$$

for any $u\in[0,\infty)$.

Theorem 1.4 (Range of Values Theorem)**.** *Let $f:[0,\infty)\to\mathbb{R}$ be a continuous convex function on $[0,\infty)$. For any $P,Q\in\mathscr{P}$, we have the double inequality*

$$f(1)\le I_f(Q,P)\le f(0)+f^{*}(0). \tag{1.4}$$

(i) If $P=Q$, then the equality holds in the first part of (1.4).

If f is strictly convex at 1, then the equality holds in the first part of (1.4) if and only if $P=Q$;

(ii) If $Q\perp P$, then the equality holds in the second part of (1.4).

If $f(0)+f^{*}(0)<\infty$, then equality holds in the second part of (1.4) if and only if $Q\perp P$.

The following result is a refinement of the second inequality in Theorem 1.4 (see [2, Theorem 3]).

Theorem 1.5. *Let f be a continuous convex function on $[0,\infty)$ with $f(1)=0$ (f is normalised) and $f(0)+f^{*}(0)<\infty$. Then*

$$0\le I_f(Q,P)\le\frac{1}{2}\left[f(0)+f^{*}(0)\right]V(Q,P) \tag{1.5}$$

for any Q, $P\in\mathscr{P}$.

For other inequalities for f-divergence see [6–20].

2. Some Preliminary Facts

For a function f defined on an interval I of the real line $\mathbb{R}$, by following the paper by Burbea & Rao [21], we consider the $\mathscr{J}$*-divergence* between the elements t, $s\in I$ given by

$$\mathscr{J}_f(t,s):=\frac{1}{2}\left[f(t)+f(s)\right]-f\left(\frac{t+s}{2}\right).$$

As important examples of such divergences, we can consider [21],

$$\mathscr{J}_\alpha(t,s):=\begin{cases}(\alpha-1)^{-1}\left[\frac{1}{2}(t^\alpha+s^\alpha)-\left(\frac{t+s}{2}\right)^\alpha\right], & \alpha\neq 1,\\[2ex] \left[t\ln(t)+s\ln(s)-(t+s)\ln\left(\frac{t+s}{2}\right)\right], & \alpha=1.\end{cases}$$

If f is convex on I, then $\mathscr{J}_f(t,s)\ge 0$ for all $(t,s)\in I\times I$.
The following result concerning the joint convexity of $\mathscr{J}_f$ also holds:

Theorem 2.1 (Burbea-Rao, 1982 [21])**.** *Let f be a C^2 function on an interval I. Then $\mathscr{J}_f$ is convex (concave) on $I\times I$, if and only if f is convex (concave) and $\frac{1}{f''}$ is concave (convex) on I.*

We define the *Hermite-Hadamard trapezoid* and *mid-point divergences*

$$\mathscr{T}_f(t,s):=\frac{1}{2}\left[f(t)+f(s)\right]-\int_0^1 f((1-\tau)t+\tau s)\,d\tau \tag{2.1}$$

and

$$\mathscr{M}_f(t,s):=\int_0^1 f((1-\tau)t+\tau s)\,d\tau-f\left(\frac{t+s}{2}\right) \tag{2.2}$$

for all $(t,s)\in I\times I$.
We observe that

$$\mathscr{J}_f(t,s)=\mathscr{T}_f(t,s)+\mathscr{M}_f(t,s) \tag{2.3}$$

for all $(t,s)\in I\times I$.

If f is convex on I, then by *Hermite-Hadamard inequalities*

$$\frac{f(a)+f(b)}{2} \geq \int_0^1 f((1-\tau)a+\tau b)\,d\tau \geq f\left(\frac{a+b}{2}\right)$$

for all $a,\, b \in I$, we have the following fundamental facts

$$\mathscr{T}_f(t,s) \geq 0 \text{ and } \mathscr{M}_f(t,s) \geq 0 \tag{2.4}$$

for all $(t,s) \in I \times I$.
Using *Bullen's inequality*, see for instance [22, p. 2],

$$\begin{aligned} 0 &\leq \int_0^1 f((1-\tau)a+\tau b)\,d\tau - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{f(a)+f(b)}{2} - \int_0^1 f((1-\tau)a+\tau b)\,d\tau \end{aligned}$$

we also have

$$0 \leq \mathscr{M}_f(t,s) \leq \mathscr{T}_f(t,s). \tag{2.5}$$

Let us recall the following special means:

a) The *arithmetic mean*

$$A(a,b) := \frac{a+b}{2}, \;\; a,b>0,$$

b) The *geometric mean*

$$G(a,b) := \sqrt{ab}; \;\; a,b \geq 0,$$

c) The *harmonic mean*

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}; \;\; a,b>0,$$

d) The *identric mean*

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad b \neq a \\ a & \text{if} \quad b=a \end{cases} ; \, a,b>0$$

e) The *logarithmic mean*

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if} \quad b \neq a \\ a & \text{if} \quad b=a \end{cases} ; \;\; a,b>0$$

f) The *p-logarithmic mean*

$$L_p(a,b) := \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if} \quad b \neq a, \;\; p \in \mathbb{R}\backslash\{-1,0\} \\ a & \text{if} \quad b=a \end{cases} ; \;\; a,b>0.$$

If we put $L_0(a,b) := I(a,b)$ and $L_{-1}(a,b) := L(a,b)$, then it is well known that the function $\mathbb{R} \ni p \mapsto L_p(a,b)$ is *monotonic increasing* on $\mathbb{R}$.
We observe that for $p \in \mathbb{R}\backslash\{-1,0\}$ we have

$$\int_0^1 [(1-\tau)a+\tau b]^p\,d\tau = L_p^p(a,b), \;\; \int_0^1 [(1-\tau)a+\tau b]^{-1}\,d\tau = L^{-1}(a,b)$$

and

$$\int_0^1 \ln[(1-\tau)a+\tau b]\,d\tau = \ln I(a,b).$$

Using these notations we can define the following divergences for $(t,s) \in I^n \times I^n$ where I is an interval of positive numbers:

$$\mathscr{T}_p(t,s) := A(t^p,s^p) - L_p^p(t,s)$$

and

$$\mathscr{M}_p(t,s) := L_p^p(t,s) - A^p(t,s)$$

for all $p \in \mathbb{R}\setminus\{-1,0\}$,

$$\mathscr{T}_{-1}(t,s) := H^{-1}(t,s) - L^{-1}(t,s)$$

and

$$\mathscr{M}_{-1}(t,s) := L^{-1}(t,s) - A^{-1}(t,s)$$

for $p = -1$ and

$$\mathscr{T}_0(t,s) := \ln\left(\frac{G(t,s)}{I(t,s)}\right)$$

and

$$\mathscr{M}_0(t,s) := \ln\left(\frac{I(t,s)}{A(t,s)}\right)$$

for $p = 0$.
Since the function $f(\tau) = \tau^p$, $\tau > 0$ is convex for $p \in (-\infty,0)\cup(1,\infty)$, then we have

$$\mathscr{T}_p(t,s),\ \mathscr{M}_p(t,s) \geq 0 \tag{2.6}$$

for all $(t,s) \in I\times I$.
For $p \in (0,1)$ the function $f(\tau) = \tau^p$, $\tau > 0$ and for $p = 0$, the function $f(\tau) = \ln\tau$ are concave, then we have for $p \in [0,1)$ that

$$\mathscr{T}_p(t,s),\ \mathscr{M}_p(t,s) \leq 0 \tag{2.7}$$

for all $(t,s) \in I\times I$.
Finally for $p = 1$ we have both $\mathscr{T}_1(t,s) = \mathscr{M}_1(t,s) = 0$ for all $(t,s) \in I\times I$.
We need the following convexity result that is a consequence of Burbea-Rao's theorem above:

Lemma 2.2. *Let f be a C^2 function on an interval I. Then $\mathscr{T}_f$ and $\mathscr{M}_f$ are convex (concave) on $I\times I$, if and only if f is convex (concave) and $\frac{1}{f''}$ is concave (convex) on I.*

Proof. If $\mathscr{T}_f$ and $\mathscr{M}_f$ are convex on $I\times I$ then the sum $\mathscr{T}_f + \mathscr{M}_f = \mathscr{J}_f$ is convex on $I\times I$, which, by Burbea-Rao theorem implies that f is convex and $\frac{1}{f''}$ is concave on I.
Now, if f is convex and $\frac{1}{f''}$ is concave on I, then by the same theorem we have that the function $\mathscr{J}_f : I\times I \to \mathbb{R}$

$$\mathscr{J}_f(t,s) := \frac{1}{2}[f(t)+f(s)] - f\left(\frac{t+s}{2}\right)$$

is convex.
Let t, s, u, $v \in I$. We define

$$\begin{aligned}
\varphi(\tau) &:= \mathscr{J}_f((1-\tau)(t,s)+\tau(u,v)) = \mathscr{J}_f(((1-\tau)t+\tau u,(1-\tau)s+\tau v)) \\
&= \frac{1}{2}[f((1-\tau)t+\tau u)+f((1-\tau)s+\tau v)] - f\left(\frac{(1-\tau)t+\tau u+(1-\tau)s+\tau v}{2}\right) \\
&= \frac{1}{2}[f((1-\tau)t+\tau u)+f((1-\tau)s+\tau v)] - f\left((1-\tau)\frac{t+s}{2}+\tau\frac{u+v}{2}\right)
\end{aligned}$$

for $\tau \in [0,1]$.
Let τ_1, $\tau_2 \in [0,1]$ and α, $\beta \geq 0$ with $\alpha+\beta = 1$. By the convexity of $\mathscr{J}_f$ we have

$$\begin{aligned}
&\varphi(\alpha\tau_1+\beta\tau_2) \\
&= \mathscr{J}_f((1-\alpha\tau_1-\beta\tau_2)(t,s)+(\alpha\tau_1+\beta\tau_2)(u,v)) \\
&= \mathscr{J}_f((\alpha+\beta-\alpha\tau_1-\beta\tau_2)(t,s)+(\alpha\tau_1+\beta\tau_2)(u,v)) \\
&= \mathscr{J}_f(\alpha(1-\tau_1)(t,s)+\beta(1-\tau_2)(t,s)+\alpha\tau_1(u,v)+\beta\tau_2(u,v)) \\
&= \mathscr{J}_f(\alpha[(1-\tau_1)(t,s)+\tau_1(u,v)]+\beta[(1-\tau_2)(t,s)+\tau_2(u,v)]) \\
&\leq \alpha\mathscr{J}_f((1-\tau_1)(t,s)+\tau_1(u,v))+\beta\mathscr{J}_f((1-\tau_2)(t,s)+\tau_2(u,v)) \\
&= \alpha\varphi(\tau_1)+\beta\varphi(\tau_2),
\end{aligned}$$

which proves that φ is convex on $[0,1]$ for all t, s, u, $v \in I$.
Applying the Hermite-Hadamard inequality for φ we get

$$\frac{1}{2}[\varphi(0)+\varphi(1)] \geq \int_0^1 \varphi(\tau)\,d\tau \tag{2.8}$$

and since

$$\varphi(0)=\frac{1}{2}[f(t)+f(s)]-f\left(\frac{t+s}{2}\right),$$

$$\varphi(1)=\frac{1}{2}[f(u)+f(v)]-f\left(\frac{u+v}{2}\right)$$

and

$$\int_0^1\varphi(\tau)d\tau=\frac{1}{2}\left[\int_0^1 f((1-\tau)t+\tau u)d\tau+\int_0^1 f((1-\tau)s+\tau v)d\tau\right]-\int_0^1 f\left((1-\tau)\frac{t+s}{2}+\tau\frac{u+v}{2}\right)d\tau,$$

hence by (2.8) we get

$$\frac{1}{2}\left\{\frac{1}{2}[f(t)+f(s)]-f\left(\frac{t+s}{2}\right)+\frac{1}{2}[f(u)+f(v)]-f\left(\frac{u+v}{2}\right)\right\}\geq\frac{1}{2}\left[\int_0^1 f((1-\tau)t+\tau u)d\tau+\int_0^1 f((1-\tau)s+\tau v)d\tau\right]$$
$$-\int_0^1 f\left((1-\tau)\frac{t+s}{2}+\tau\frac{u+v}{2}\right)d\tau.$$

Re-arranging this inequality, we get

$$\frac{1}{2}\left[\frac{f(t)+f(u)}{2}-\int_0^1 f((1-\tau)t+\tau u)d\tau\right]+\frac{1}{2}\left[\frac{f(s)+f(v)}{2}-\int_0^1 f((1-\tau)s+\tau v)d\tau\right]$$
$$\geq\frac{1}{2}\left[f\left(\frac{t+s}{2}\right)+f\left(\frac{u+v}{2}\right)-\int_0^1 f\left((1-\tau)\frac{t+s}{2}+\tau\frac{u+v}{2}\right)d\tau\right],$$

which is equivalent to

$$\frac{1}{2}\left[\mathscr{T}_f(t,u)+\mathscr{T}_f(s,v)\right]\geq\mathscr{T}_f\left(\frac{t+s}{2},\frac{u+v}{2}\right)=\mathscr{T}_f\left(\frac{1}{2}(t,u)+\frac{1}{2}(s,v)\right),$$

for all $(t,u),(s,v)\in I\times I$, which shows that $\mathscr{T}_f$ is Jensen's convex on $I\times I$. Since $\mathscr{T}_f$ is continuous on $I\times I$, hence $\mathscr{T}_f$ is convex in the usual sense on $I\times I$.
Now, if we use the second Hermite-Hadamard inequality for φ on $[0,1]$, we have

$$\int_0^1\varphi(\tau)d\tau\geq\varphi\left(\frac{1}{2}\right). \tag{2.9}$$

Since

$$\varphi\left(\frac{1}{2}\right)=\frac{1}{2}\left[f\left(\frac{t+u}{2}\right)+f\left(\frac{s+v}{2}\right)\right]-f\left(\frac{1}{2}\frac{t+s}{2}+\frac{1}{2}\frac{u+v}{2}\right)$$

hence by (2.9) we have

$$\frac{1}{2}\left[\int_0^1 f((1-\tau)t+\tau u)d\tau+\int_0^1 f((1-\tau)s+\tau v)d\tau\right]-\int_0^1 f\left((1-\tau)\frac{t+s}{2}+\tau\frac{u+v}{2}\right)d\tau$$
$$\geq\frac{1}{2}\left[f\left(\frac{t+u}{2}\right)+f\left(\frac{s+v}{2}\right)\right]-f\left(\frac{1}{2}\left(\frac{t+s}{2}+\frac{u+v}{2}\right)\right),$$

which is equivalent to

$$\frac{1}{2}\left[\int_0^1 f((1-\tau)t+\tau u)d\tau-f\left(\frac{t+u}{2}\right)\right]+\frac{1}{2}\left[\int_0^1 f((1-\tau)s+\tau v)d\tau-f\left(\frac{s+v}{2}\right)\right]$$
$$\geq\int_0^1 f\left((1-\tau)\frac{t+s}{2}+\tau\frac{u+v}{2}\right)d\tau-f\left(\frac{1}{2}\left(\frac{t+s}{2}+\frac{u+v}{2}\right)\right)$$

that can be written as

$$\frac{1}{2}\left[\mathscr{M}_f(t,u)+\mathscr{M}_f(s,v)\right]\geq\mathscr{M}_f\left(\frac{t+s}{2},\frac{u+v}{2}\right)=\mathscr{M}_f\left(\frac{1}{2}(t,u)+\frac{1}{2}(s,v)\right)$$

for all $(t,u),(s,v)\in I\times I$, which shows that $\mathscr{M}_f$ is Jensen's convex on $I\times I$. Since $\mathscr{M}_f$ is continuous on $I\times I$, hence $\mathscr{M}_f$ is convex in the usual sense on $I\times I$. □

The following reverses of the Hermite-Hadamard inequality hold:

Lemma 2.3 (Dragomir, 2002 [10] and [11]). *Let $h:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned} 0 &\le \frac{1}{8}\left[h_{+}\left(\frac{a+b}{2}\right)-h_{-}\left(\frac{a+b}{2}\right)\right](b-a) \\ &\le \frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(\tau)\,d\tau \\ &\le \frac{1}{8}\left[h_{-}(b)-h_{+}(a)\right](b-a) \end{aligned} \tag{2.10}$$

and

$$0 \le \frac{1}{8}\left[h_{+}\left(\frac{a+b}{2}\right)-h_{-}\left(\frac{a+b}{2}\right)\right](b-a) \le \frac{1}{b-a}\int_a^b h(\tau)\,d\tau - h\left(\frac{a+b}{2}\right) \le \frac{1}{8}\left[h_{-}(b)-h_{+}(a)\right](b-a). \tag{2.11}$$

The constant $\frac{1}{8}$ is best possible in all inequalities from (2.10) and (2.11).

We also have:

Lemma 2.4. *Let f be a C^1 convex function on an interval I. If $\mathring{I}$ is the interior of I, then for all $(t,s)\in\mathring{I}\times\mathring{I}$ we have*

$$0 \le \mathscr{M}_f(t,s) \le \mathscr{T}_f(t,s) \le \frac{1}{8}\mathscr{C}_{f'}(t,s) \tag{2.12}$$

where

$$\mathscr{C}_{f'}(t,s) := \left[f'(t)-f'(s)\right](t-s). \tag{2.13}$$

Proof. Since for $b\neq a$

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \int_0^1 f((1-t)a+tb)\,dt,$$

then from (2.10) we get

$$\frac{f(t)+f(s)}{2}-\int_0^1 f((1-\tau)t+\tau s)\,dt \le \frac{1}{8}\left[f'(t)-f'(s)\right](t-s)$$

for all $(t,s)\in\mathring{I}\times\mathring{I}$. □

Remark 2.5. *If*

$$\gamma = \inf_{t\in\mathring{I}} f'(t) \text{ and } \Gamma = \sup_{t\in\mathring{I}} f'(t)$$

are finite, then

$$\mathscr{C}_{f'}(t,s) \le (\Gamma-\gamma)|t-s|$$

and by (2.12) we get the simpler upper bound

$$0 \le \mathscr{M}_f(t,s) \le \mathscr{T}_f(t,s) \le \frac{1}{8}(\Gamma-\gamma)|t-s|.$$

Moreover, if $t,\ s\in[a,b]\subset\mathring{I}$ and since f' is increasing on $\mathring{I}$, then we have the inequalities

$$0 \le \mathscr{M}_f(t,s) \le \mathscr{T}_f(t,s) \le \frac{1}{8}\left[f'(b)-f'(a)\right]|t-s|. \tag{2.14}$$

Since $\mathscr{J}_f(t,s)=\mathscr{T}_f(t,s)+\mathscr{M}_f(t,s)$, hence

$$0 \le \mathscr{J}_f(t,s) \le \frac{1}{4}\left[f'(b)-f'(a)\right]|t-s|.$$

Corollary 2.6. *With the assumptions of Lemma 2.4 and if the derivative f' is Lipschitzian with the constant $K>0$, namely*

$$\left|f'(t)-f'(s)\right| \le K|t-s| \text{ for all } t,\ s\in\mathring{I},$$

then we have the inequality

$$0 \le \mathscr{M}_f(t,s) \le \mathscr{T}_f(t,s) \le \frac{1}{8}K(t-s)^2. \tag{2.15}$$

3. Main Results

Let P, Q, $W \in \mathscr{P}$ and $f:(0,\infty)\to\mathbb{R}$. We define the following f-divergence

$$\begin{aligned}\mathscr{J}_f(P,Q,W) &:= \int_X w(x)\,\mathscr{J}_f\left(\frac{p(x)}{w(x)},\frac{q(x)}{w(x)}\right)d\mu(x)\\ &= \frac{1}{2}\left[\int_X w(x) f\left(\frac{p(x)}{w(x)}\right)d\mu(x)+\int_X w(x) f\left(\frac{q(x)}{w(x)}\right)d\mu(x)\right]-\int_X w(x) f\left(\frac{p(x)+q(x)}{2w(x)}\right).\end{aligned} \tag{3.1}$$

If we consider the *mid-point divergence measure* M_f defined by

$$M_f(Q,P,W) := \int_X f\left[\frac{q(x)+p(x)}{2w(x)}\right] w(x)\,d\mu(x)$$

for any Q, P, $W \in \mathscr{P}$, then from (3.1) we get

$$\mathscr{J}_f(P,Q,W) = \frac{1}{2}\left[I_f(P,W)+I_f(Q,W)\right]-M_f(Q,P,W). \tag{3.2}$$

We can also consider the *integral divergence measure*

$$A_f(Q,P,W) := \int_X\left(\int_0^1 f\left[\frac{(1-t)\,q(x)+t\,p(x)}{w(x)}\right]dt\right)w(x)\,d\mu(x).$$

We introduce the related f-divergences

$$\begin{aligned}\mathscr{T}_f(P,Q,W) &:= \int_X w(x)\,\mathscr{T}_f\left(\frac{p(x)}{w(x)},\frac{q(x)}{w(x)}\right)d\mu(x)\\ &= \frac{1}{2}\left[I_f(P,W)+I_f(Q,W)\right]-A_f(Q,P,W)\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\mathscr{M}_f(P,Q,W) &:= \int_X w(x)\,\mathscr{M}_f\left(\frac{p(x)}{w(x)},\frac{q(x)}{w(x)}\right)d\mu(x)\\ &= A_f(Q,P,W)-M_f(Q,P,W).\end{aligned} \tag{3.4}$$

We observe that

$$\mathscr{J}_f(P,Q,W) = \mathscr{T}_f(P,Q,W)+\mathscr{M}_f(P,Q,W).$$

If f is convex on $(0,\infty)$ then by the Hermite-Hadamard and Bullen's inequalities we have the positivity properties

$$0\le \mathscr{M}_f(P,Q,W)\le \mathscr{T}_f(P,Q,W)$$

and

$$0\le \mathscr{J}_f(P,Q,W)$$

for P, Q, $W \in \mathscr{P}$.
We have the following result:

Theorem 3.1. *Let f be a C^2 function on an interval $(0,\infty)$. If f is convex on $(0,\infty)$ and $\frac{1}{f''}$ is concave on $(0,\infty)$, then for all $W\in\mathscr{P}$, the mappings*

$$\mathscr{P}\times\mathscr{P}\ni(P,Q)\mapsto \mathscr{J}_f(P,Q,W),\ \mathscr{M}_f(P,Q,W),\ \mathscr{T}_f(P,Q,W)$$

are convex.

Proof. Let $(P_1,Q_1),(P_2,Q_2)\in\mathscr{P}\times\mathscr{P}$ and α, $\beta\ge 0$ with $\alpha+\beta=1$. We have

$$\begin{aligned}&\mathscr{J}_f(\alpha(P_1,Q_1,W)+\beta(P_2,Q_2,W)) = \mathscr{J}_f(\alpha P_1+\beta P_2,\alpha Q_1+\beta Q_2,W)\\ &= \int_X w(x)\,\mathscr{J}_f\left(\frac{\alpha p_1(x)+\beta p_2(x)}{w(x)},\frac{\alpha q_1(x)+\beta q_2(x)}{w(x)}\right)d\mu(x)\\ &= \int_X w(x)\,\mathscr{J}_f\left(\alpha\frac{p_1(x)}{w(x)}+\beta\frac{p_2(x)}{w(x)},\alpha\frac{q_1(x)}{w(x)}+\beta\frac{q_2(x)}{w(x)}\right)d\mu(x)\\ &= \int_X w(x)\,\mathscr{J}_f\left(\alpha\left(\frac{p_1(x)}{w(x)},\frac{q_1(x)}{w(x)}\right)+\beta\left(\frac{p_2(x)}{w(x)},\frac{q_2(x)}{w(x)}\right)\right)d\mu(x)\\ &=: \Psi\end{aligned}$$

Now, by the convexity of $\mathscr{J}_f$ on $I\times I$ proved in Theorem 2.1, we have that

$$\mathscr{J}_f\left(\alpha\left(\frac{p_1(x)}{w(x)},\frac{q_1(x)}{w(x)}\right)+\beta\left(\frac{p_2(x)}{w(x)},\frac{q_2(x)}{w(x)}\right)\right)\le\alpha\,\mathscr{J}_f\left(\left(\frac{p_1(x)}{w(x)},\frac{q_1(x)}{w(x)}\right)\right)+\beta\,\mathscr{J}_f\left(\left(\frac{p_2(x)}{w(x)},\frac{q_2(x)}{w(x)}\right)\right)$$

for $x \in X$. If we multiply by $w(x) \geq 0$ and integrate over $d\mu(x)$, then we get

$$\begin{aligned}\Psi &\leq \int_X w(x)\left[\alpha \mathscr{J}_f\left(\frac{p_1(x)}{w(x)}, \frac{q_1(x)}{w(x)}\right) + \beta \mathscr{J}_f\left(\frac{p_2(x)}{w(x)}, \frac{q_2(x)}{w(x)}\right)\right] d\mu(x) \\ &= \alpha \int_X w(x)\, \mathscr{J}_f\left(\frac{p_1(x)}{w(x)}, \frac{q_1(x)}{w(x)}\right) d\mu(x) + \beta \int_X w(x)\, \mathscr{J}_f\left(\frac{p_2(x)}{w(x)}, \frac{q_2(x)}{w(x)}\right) d\mu(x) \\ &= \alpha \mathscr{J}_f(P_1, Q_1, W) + \beta \mathscr{J}_f(P_2, Q_2, W),\end{aligned}$$

which proves the convexity of $\mathscr{P} \times \mathscr{P} \ni (P,Q) \mapsto \mathscr{J}_f(P,Q,W)$ for all $W \in \mathscr{P}$.
The convexity of the other two mappings follows in a similar way and we omit the details. □

Theorem 3.2. *Let f be a C^1 function on an interval $(0,\infty)$. If f is convex on $(0,\infty)$, then for all $W \in \mathscr{P}$*

$$0 \leq \mathscr{M}_f(P,Q,W) \leq \mathscr{T}_f(P,Q,W) \leq \frac{1}{8}\Delta_{f'}(Q,P,W) \tag{3.5}$$

where

$$\Delta_{f'}(Q,P,W) := \int_X \left[f'\left(\frac{q(x)}{w(x)}\right) - f'\left(\frac{p(x)}{w(x)}\right)\right](q(x) - p(x))\, d\mu(x). \tag{3.6}$$

Proof. From the inequality (2.12) we have

$$\frac{1}{2}\left[f\left(\frac{p(x)}{w(x)}\right) + f\left(\frac{q(x)}{w(x)}\right)\right] - \int_0^1 f\left((1-t)\frac{p(x)}{w(x)} + t\frac{q(x)}{w(x)}\right) dt \leq \frac{1}{8}\left(f'\left(\frac{p(x)}{w(x)}\right) - f'\left(\frac{q(x)}{w(x)}\right)\right)\left(\frac{p(x)}{w(x)} - \frac{q(x)}{w(x)}\right)$$

for all $x \in X$.
If we multiply by $w(x) > 0$ and integrate on X we get

$$\begin{aligned}\frac{1}{2}\left[I_f(P,W) + I_f(P,W)\right] - A_f(Q,P,W) &\leq \frac{1}{8}\int_X w(x)\left(f'\left(\frac{p(x)}{w(x)}\right) - f'\left(\frac{q(x)}{w(x)}\right)\right)\left(\frac{p(x)}{w(x)} - \frac{q(x)}{w(x)}\right) d\mu(x) \\ &= \frac{1}{8}\int_X \left(f'\left(\frac{p(x)}{w(x)}\right) - f'\left(\frac{q(x)}{w(x)}\right)\right)(p(x) - q(x))\, d\mu(x),\end{aligned}$$

which implies the desired inequality. □

Corollary 3.3. *With the assumptions of Theorem 3.2 and if f' is Lipschitzian with the constant $K > 0$, namely*

$$\left|f'(s) - f'(t)\right| \leq K|s-t| \text{ for all } t,\ s \in (0,\infty),$$

then

$$0 \leq \mathscr{M}_f(P,Q,W) \leq \mathscr{T}_f(P,Q,W) \leq \frac{1}{8}K d_{\chi^2}(Q,P,W), \tag{3.7}$$

where

$$d_{\chi^2}(Q,P,W) := \int_X \frac{(q(x) - p(x))^2}{w(x)} d\mu(x). \tag{3.8}$$

Remark 3.4. *If there exists $0 < r < 1 < R < \infty$ such that the following condition holds*

$$r \leq \frac{q(x)}{w(x)}, \frac{p(x)}{w(x)} \leq R \text{ for } \mu\text{-a.e. } x \in X, \tag{(r,R)}$$

then

$$0 \leq \mathscr{M}_f(P,Q,W) \leq \mathscr{T}_f(P,Q,W) \leq \frac{1}{8}\left[f'(R) - f'(r)\right] d_1(Q,P) \tag{3.9}$$

where

$$d_1(Q,P) := \int_X |q(x) - p(x)|\, d\mu(x).$$

Moreover, if f is twice differentiable and

$$\left\|f''\right\|_{[r,R],\infty} := \sup_{t\in[r,R]} \left|f''(t)\right| < \infty \tag{3.10}$$

then

$$0 \leq \mathscr{M}_f(P,Q,W) \leq \mathscr{T}_f(P,Q,W) \leq \frac{1}{8}\left\|f''\right\|_{[r,R],\infty} d_{\chi^2}(Q,P,W). \tag{3.11}$$

We also have:

Theorem 3.5. *Let f be a C^2 function on an interval $(0,\infty)$. If f is convex on $(0,\infty)$ and $\frac{1}{f''}$ is concave on $(0,\infty)$, then for all $W\in\mathscr{P}$,*

$$0\le \mathscr{I}_f(P,Q,W)\le \frac{1}{2}\left[\Psi_{f'}(P,Q,W)+\Psi_{f'}(Q,P,W)\right], \tag{3.12}$$

where

$$\Psi_{f'}(P,Q,W):=\int_X\left[f'\left(\frac{p(x)}{w(x)}\right)-f'\left(\frac{q(x)+p(x)}{2w(x)}\right)\right](p(x)-w(x))\,d\mu(x).$$

Proof. It is well known that if the function of two independent variables $F:D\subset\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is convex on the convex domain D and has partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ on D then for all $(t,s),(u,v)\in D$ we have the gradient inequalities

$$\frac{\partial F(t,s)}{\partial x}(t-u)+\frac{\partial F(t,s)}{\partial y}(s-v)\ge F(t,s)-F(u,v)\ge \frac{\partial F(u,v)}{\partial x}(t-u)+\frac{\partial F(u,v)}{\partial y}(s-v). \tag{3.13}$$

Now, if we take $F:(0,\infty)\times(0,\infty)\to\mathbb{R}$ given by

$$F(t,s)=\frac{1}{2}[f(t)+f(s)]-f\left(\frac{t+s}{2}\right)$$

and observe that

$$\frac{\partial F(t,s)}{\partial x}=\frac{1}{2}\left[f'(t)-f'\left(\frac{t+s}{2}\right)\right]$$

and

$$\frac{\partial F(t,s)}{\partial y}=\frac{1}{2}\left[f'(s)-f'\left(\frac{t+s}{2}\right)\right]$$

and since F is convex on $(0,\infty)\times(0,\infty)$, then by (3.13) we get

$$\begin{aligned}&\frac{1}{2}\left[f'(t)-f'\left(\frac{t+s}{2}\right)\right](t-u)+\frac{1}{2}\left[f'(s)-f'\left(\frac{t+s}{2}\right)\right](s-v)\\&\ge\frac{1}{2}[f(t)+f(s)]-f\left(\frac{t+s}{2}\right)-\frac{1}{2}[f(u)+f(v)]+f\left(\frac{u+v}{2}\right)\\&\ge\frac{1}{2}\left[f'(u)-f'\left(\frac{u+v}{2}\right)\right](t-u)+\frac{1}{2}\left[f'(v)-f'\left(\frac{u+v}{2}\right)\right](s-v).\end{aligned} \tag{3.14}$$

If we take $u=v=1$ in (3.14), then we have

$$\frac{1}{2}\left[f'(t)-f'\left(\frac{t+s}{2}\right)\right](t-1)+\frac{1}{2}\left[f'(s)-f'\left(\frac{t+s}{2}\right)\right](s-1)\ge\frac{1}{2}[f(t)+f(s)]-f\left(\frac{t+s}{2}\right)\ge 0 \tag{3.15}$$

for all $(t,s)\in(0,\infty)\times(0,\infty)$.
If we take $t=\frac{p(x)}{w(x)}$ and $s=\frac{q(x)}{w(x)}$ in (3.15) then we obtain

$$\begin{aligned}&\frac{1}{2}\left[f'\left(\frac{p(x)}{w(x)}\right)-f'\left(\frac{q(x)+p(x)}{2w(x)}\right)\right]\left(\frac{p(x)}{w(x)}-1\right)+\frac{1}{2}\left[f'\left(\frac{q(x)}{w(x)}\right)-f'\left(\frac{q(x)+p(x)}{2w(x)}\right)\right]\left(\frac{q(x)}{w(x)}-1\right)\\&\ge\frac{1}{2}\left[f\left(\frac{p(x)}{w(x)}\right)+f\left(\frac{q(x)}{w(x)}\right)\right]-f\left(\frac{q(x)+p(x)}{2w(x)}\right)\ge 0.\end{aligned}$$

By multiplying this inequality with $w(x)>0$ we get

$$\begin{aligned}0&\le\frac{1}{2}\left[w(x)f\left(\frac{p(x)}{w(x)}\right)+w(x)f\left(\frac{q(x)}{w(x)}\right)\right]-w(x)f\left(\frac{q(x)+p(x)}{2w(x)}\right)\\&\le\frac{1}{2}\left[f'\left(\frac{p(x)}{w(x)}\right)-f'\left(\frac{q(x)+p(x)}{2w(x)}\right)\right](p(x)-w(x))+\frac{1}{2}\left[f'\left(\frac{q(x)}{w(x)}\right)-f'\left(\frac{q(x)+p(x)}{2w(x)}\right)\right](q(x)-w(x))\end{aligned}$$

for all $x\in X$. □

Corollary 3.6. *With the assumptions of Theorem 3.2 and if f' is Lipschitzian with the constant $K>0$, then*

$$0\le\mathscr{I}_f(P,Q,W)\le\frac{1}{4}K\int_X|p(x)-q(x)|\left[\left|\frac{p(x)}{w(x)}-1\right|+\left|\frac{q(x)}{w(x)}-1\right|\right]d\mu(x). \tag{3.16}$$

Proof. We have that

$$\begin{aligned}\Psi_{f'}(P,Q,W) &\leq \int_X \left| f'\left(\frac{p(x)}{w(x)}\right) - f'\left(\frac{q(x)+p(x)}{2w(x)}\right)\right| |p(x)-w(x)|\, d\mu(x) \\ &\leq K \int_X \left|\frac{p(x)}{w(x)} - \frac{q(x)+p(x)}{2w(x)}\right| |p(x)-w(x)|\, d\mu(x) \\ &= K \int_X \left|\frac{p(x)-q(x)}{2w(x)}\right| |p(x)-w(x)|\, d\mu(x) \\ &= \frac{1}{2} K \int_X \frac{|p(x)-q(x)|\,|p(x)-w(x)|\, d\mu(x)}{w(x)} \\ &= \frac{1}{2} K \int_X |p(x)-q(x)| \left|\frac{p(x)}{w(x)} - 1\right| d\mu(x)\end{aligned}$$

and similarly

$$\Psi_{f'}(P,Q,W) \leq \frac{1}{2} K \int_X |p(x)-q(x)| \left|\frac{q(x)}{w(x)} - 1\right| d\mu(x).$$

Finally, by the use of (3.12) we get the desired result. □

Remark 3.7. *If there exist $0<r<1<R<\infty$ such that the following condition (r,R) holds and if f is twice differentiable and (3.10) is valid, then*

$$0 \leq \mathscr{J}_f(P,Q,W) \leq \frac{1}{4} \|f''\|_{[r,R],\infty} \times \int_X |p(x)-q(x)| \left[\left|\frac{p(x)}{w(x)} - 1\right| + \left|\frac{q(x)}{w(x)} - 1\right|\right] d\mu(x). \tag{3.17}$$

Since

$$\left|\frac{p(x)}{w(x)} - 1\right|, \left|\frac{q(x)}{w(x)} - 1\right| \leq \max\{R-1, 1-r\}$$

and

$$\left|\frac{p(x)}{w(x)} - \frac{q(x)}{w(x)}\right| \leq R - r,$$

hence by (3.17) we get the simpler bound

$$0 \leq \mathscr{J}_f(P,Q,W) \leq \frac{1}{2} \|f''\|_{[r,R],\infty} (R-r) \max\{R-1, 1-r\}. \tag{3.18}$$

We also have:

Theorem 3.8. *With the assumptions of Theorem 3.2 and if f' is Lipschitzian with the constant $K>0$, then*

$$0 \leq \mathscr{T}_f(P,Q,W) \leq \frac{1}{6} K \int_X |p(x)-q(x)| \left[\left|\frac{p(x)}{w(x)} - 1\right| + \left|\frac{q(x)}{w(x)} - 1\right|\right] d\mu(x). \tag{3.19}$$

Proof. Let $(x,y), (u,v) \in (0,\infty)\times(0,\infty)$. If we take $F:(0,\infty)\times(0,\infty)\to\mathbb{R}$ given by

$$F(t,s) = \frac{f(t)+f(s)}{2} - \int_0^1 f((1-\tau)t+\tau s)\, d\tau$$

then

$$\begin{aligned}\frac{\partial F(t,s)}{\partial x} &= \frac{1}{2} f'(t) - \int_0^1 (1-\tau) f'((1-\tau)t+\tau s)\, d\tau \\ &= \int_0^1 (1-\tau)\left[f'(t) - f'((1-\tau)t+\tau s)\right] d\tau\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F(t,s)}{\partial y} &= \frac{1}{2} f'(s) - \int_0^1 \tau f'((1-\tau)t+\tau s)\, d\tau \\ &= \int_0^1 \tau\left[f'(s) - f'((1-\tau)t+\tau s)\right] d\tau\end{aligned}$$

and since F is convex on $(0,\infty)\times(0,\infty)$, then by (3.1) we get

$$\begin{aligned}&(t-u)\int_0^1 (1-\tau)\left[f'(t) - f'((1-\tau)t+\tau s)\right] d\tau + (s-v)\int_0^1 \tau\left[f'(s) - f'((1-\tau)t+\tau s)\right] d\tau \\ &\geq \frac{f(t)+f(s)}{2} - \int_0^1 f((1-\tau)t+\tau s)\, d\tau - \frac{f(u)+f(v)}{2} + \int_0^1 f((1-\tau)u+\tau v)\, d\tau \\ &\geq (t-u)\int_0^1 (1-\tau)\left[f'(u) - f'((1-\tau)u+\tau v)\right] d\tau + (s-v)\int_0^1 \tau\left[f'(v) - f'((1-\tau)u+\tau v)\right] d\tau\end{aligned} \tag{3.20}$$

for all $(t,s),(u,v)\in(0,\infty)\times(0,\infty)$.
If we take $u=v=1$ in (3.20), then we have

$$\begin{aligned}&(t-1)\int_0^1(1-\tau)\left[f'(t)-f'((1-\tau)t+\tau s)\right]d\tau+(s-1)\int_0^1\tau\left[f'(s)-f'((1-\tau)t+\tau s)\right]d\tau\\&\geq\frac{f(t)+f(s)}{2}-\int_0^1 f((1-\tau)t+\tau s)\,d\tau\geq 0\end{aligned}\tag{3.21}$$

for all $(u,v)\in(0,\infty)\times(0,\infty)$.
If we take $t=\frac{p(x)}{w(x)}$ and $s=\frac{q(x)}{w(x)}$ in (3.21) then we get

$$\begin{aligned}&\left(\frac{p(x)}{w(x)}-1\right)\int_0^1(1-\tau)\left[f'\left(\frac{p(x)}{w(x)}\right)-f'\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)\right]d\tau\\&+\left(\frac{q(x)}{w(x)}-1\right)\int_0^1\tau\left[f'\left(\frac{q(x)}{w(x)}\right)-f'\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)\right]d\tau\\&\geq\frac{f\left(\frac{p(x)}{w(x)}\right)+f\left(\frac{q(x)}{w(x)}\right)}{2}-\int_0^1 f\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)d\tau\geq 0.\end{aligned}\tag{3.22}$$

Since f' is Lipschitzian with the constant $K>0$, hence

$$\begin{aligned}0\leq&\frac{f\left(\frac{p(x)}{w(x)}\right)+f\left(\frac{q(x)}{w(x)}\right)}{2}-\int_0^1 f\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)d\tau\\\leq&\left|\frac{p(x)}{w(x)}-1\right|\int_0^1(1-\tau)\left|f'\left(\frac{p(x)}{w(x)}\right)-f'\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)\right|d\tau\\&+\left|\frac{q(x)}{w(x)}-1\right|\int_0^1\tau\left|f'\left(\frac{q(x)}{w(x)}\right)-f'\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)\right|d\tau\\\leq&K\left|\frac{p(x)}{w(x)}-1\right|\left|\frac{p(x)}{w(x)}-\frac{q(x)}{w(x)}\right|\int_0^1(1-\tau)\tau d\tau+K\left|\frac{q(x)}{w(x)}-1\right|\left|\frac{p(x)}{w(x)}-\frac{q(x)}{w(x)}\right|\int_0^1(1-\tau)\tau d\tau\\=&\frac{1}{6}K\left|\frac{p(x)}{w(x)}-\frac{q(x)}{w(x)}\right|\left[\left|\frac{p(x)}{w(x)}-1\right|+\left|\frac{q(x)}{w(x)}-1\right|\right].\end{aligned}$$

If we multiply this inequality by $w(x)>0$ and integrate, then we get the desired result (3.19). □

Corollary 3.9. *If there exist $0<r<1<R<\infty$ such that the condition (r,R) holds and if f is twice differentiable and (3.10) is valid, then*

$$0\leq\mathscr{T}_f(P,Q,W)\leq\frac{1}{3}\left\|f''\right\|_{[r,R],\infty}(R-r)\max\{R-1,1-r\}.\tag{3.23}$$

Finally, we also have:

Theorem 3.10. *With the assumptions of Theorem 3.2 and if f' is Lipschitzian with the constant $K>0$, then*

$$0\leq\mathscr{M}_f(P,Q,W)\leq\frac{1}{8}K\int_X|p(x)-q(x)|\left[\left|\frac{p(x)}{w(x)}-1\right|+\left|\frac{q(x)}{w(x)}-1\right|\right]d\mu(x).\tag{3.24}$$

Proof. Let $(t,s),(u,v)\in(0,\infty)\times(0,\infty)$. If we take $F:(0,\infty)\times(0,\infty)\to\mathbb{R}$ given by

$$F(t,s)=\int_0^1 f((1-\tau)t+\tau s)\,d\tau-f\left(\frac{t+s}{2}\right)$$

then

$$\begin{aligned}\frac{\partial F(t,s)}{\partial x}&=\int_0^1(1-\tau)f'((1-\tau)t+\tau s)\,d\tau-\frac{1}{2}f'\left(\frac{t+s}{2}\right)\\&=\int_0^1(1-\tau)\left[f'((1-\tau)t+\tau s)-f'\left(\frac{t+s}{2}\right)\right]d\tau,\end{aligned}$$

$$\begin{aligned}\frac{\partial F(t,s)}{\partial y}&=\int_0^1\tau f'((1-\tau)t+\tau s)\,d\tau-\frac{1}{2}f'\left(\frac{t+s}{2}\right)\\&=\int_0^1\tau\left[f'((1-\tau)t+\tau s)-f'\left(\frac{t+s}{2}\right)\right]d\tau\end{aligned}$$

and since F is convex on $(0,\infty)\times(0,\infty)$, then by (3.1) we get

$$\begin{aligned}&(t-u)\left[\int_0^1(1-\tau)\left[f'((1-\tau)t+\tau s)-f'\left(\frac{t+s}{2}\right)\right]d\tau\right]+(s-v)\left[\int_0^1\tau\left[f'((1-\tau)t+\tau s)-f'\left(\frac{t+s}{2}\right)\right]d\tau\right]\\&\geq\int_0^1 f((1-\tau)t+\tau s)\,d\tau-f\left(\frac{t+s}{2}\right)-\int_0^1 f((1-\tau)u+\tau v)\,d\tau+f\left(\frac{u+v}{2}\right)\\&\geq(t-u)\left[\int_0^1(1-\tau)\left[f'((1-\tau)u+\tau v)-f'\left(\frac{u+v}{2}\right)\right]d\tau\right]+(s-v)\int_0^1\tau\left[f'((1-\tau)u+\tau v)-f'\left(\frac{u+v}{2}\right)\right]d\tau.\end{aligned}\tag{3.25}$$

If we take $u=v=1$ in (3.25), then we have

$$\begin{aligned}&(t-1)\left[\int_0^1(1-\tau)\left[f'((1-\tau)t+\tau s)-f'\left(\frac{t+s}{2}\right)\right]d\tau\right]+(s-1)\left[\int_0^1\tau\left[f'((1-\tau)t+\tau s)-f'\left(\frac{t+s}{2}\right)\right]d\tau\right]\\&\geq\int_0^1 f((1-\tau)t+\tau s)\,d\tau-f\left(\frac{t+s}{2}\right)\geq 0\end{aligned}\tag{3.26}$$

for all $(t,s)\in(0,\infty)\times(0,\infty)$.
If we take $t=\frac{p(x)}{w(x)}$ and $s=\frac{q(x)}{w(x)}$ in (3.26) then we get

$$\begin{aligned}0\leq&\int_0^1 f\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)d\tau-f\left(\frac{p(x)+q(x)}{2w(x)}\right)\\\leq&\left(\frac{p(x)}{w(x)}-1\right)\times\left[\int_0^1(1-\tau)\left[f'\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)-f'\left(\frac{p(x)+q(x)}{2w(x)}\right)\right]d\tau\right]\\&+\left(\frac{q(x)}{w(x)}-1\right)\times\left[\int_0^1\tau\left[f'\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)-f'\left(\frac{p(x)+q(x)}{2w(x)}\right)\right]d\tau\right]\\\leq&\left|\frac{p(x)}{w(x)}-1\right|\times\left[\int_0^1(1-\tau)\left|f'\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)-f'\left(\frac{p(x)+q(x)}{2w(x)}\right)\right|d\tau\right]+\left|\frac{q(x)}{w(x)}-1\right|\\&\times\left[\int_0^1\tau\left|f'\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)-f'\left(\frac{p(x)+q(x)}{2w(x)}\right)\right|d\tau\right]\\\leq&K\left|\frac{p(x)}{w(x)}-1\right|\left|\frac{p(x)}{w(x)}-\frac{q(x)}{w(x)}\right|\int_0^1(1-\tau)\left|\tau-\frac{1}{2}\right|d\tau+K\left|\frac{q(x)}{w(x)}-1\right|\left|\frac{p(x)}{w(x)}-\frac{q(x)}{w(x)}\right|\int_0^1(1-\tau)\left|\tau-\frac{1}{2}\right|d\tau.\end{aligned}\tag{3.27}$$

Since

$$\int_0^1(1-\tau)\left|\tau-\frac{1}{2}\right|d\tau=\frac{1}{8},$$

hence

$$0\leq\int_0^1 f\left((1-\tau)\frac{p(x)}{w(x)}+\tau\frac{q(x)}{w(x)}\right)d\tau-f\left(\frac{p(x)+q(x)}{2w(x)}\right)\leq\frac{1}{8}K\left|\frac{p(x)}{w(x)}-\frac{q(x)}{w(x)}\right|\left[\left|\frac{p(x)}{w(x)}-1\right|+\left|\frac{q(x)}{w(x)}-1\right|\right]$$

for all $x\in X$.
If we multiply this inequality by $w(x)>0$ and integrate, then we get the desired result (3.19). □

Corollary 3.11. *If there exist $0<r<1<R<\infty$ such that the condition (r,R) holds and if f is twice differentiable and (3.10) is valid, then*

$$0\leq\mathscr{M}_f(P,Q,W)\leq\frac{1}{4}\left\|f''\right\|_{[r,R],\infty}(R-r)\max\{R-1,1-r\}.\tag{3.28}$$

4. Some Examples

The Dichotomy class of f-divergences are generated by the functions $f_\alpha:[0,\infty)\to\mathbb{R}$ defined as

$$f_\alpha(u)=\begin{cases}u-1-\ln u & \text{for } \alpha=0;\\ \frac{1}{\alpha(1-\alpha)}\left[\alpha u+1-\alpha-u^\alpha\right] & \text{for } \alpha\in\mathbb{R}\backslash\{0,1\};\\ 1-u+u\ln u & \text{for } \alpha=1.\end{cases}$$

Observe that

$$f''_\alpha(u)=\begin{cases}\frac{1}{u^2} & \text{for } \alpha=0;\\ u^{\alpha-2} & \text{for } \alpha\in\mathbb{R}\backslash\{0,1\};\\ \frac{1}{u} & \text{for } \alpha=1.\end{cases}$$

In this family of functions only the functions f_α with $\alpha\in[1,2)$ are both convex and with $\frac{1}{f''_\alpha}$ concave on $(0,\infty)$.
We have

$$I_{f_\alpha}(P,W)=\int_X w(x)f_\alpha\left(\frac{p(x)}{w(x)}\right)d\mu(x)=\begin{cases}\frac{1}{\alpha(\alpha-1)}\left[\int_X w^{1-\alpha}(x)p^\alpha(x)\,d\mu(x)-1\right], & \alpha\in(1,2),\\ \int_X p(x)\ln\left(\frac{p(x)}{w(x)}\right)d\mu(x), & \alpha=1,\end{cases}$$

and

$$M_{f_\alpha}(Q,P,W)=\int_X f\left[\frac{q(x)+p(x)}{2w(x)}\right]w(x)\,d\mu(x)=\begin{cases}\frac{1}{\alpha(\alpha-1)}\left[\int_X\left[\frac{q(x)+p(x)}{2}\right]^{\alpha}w^{1-\alpha}(x)\,d\mu(x)-1\right], & \alpha\in(1,2)\\ \int_X\left[\frac{q(x)+p(x)}{2}\right]\ln\left[\frac{q(x)+p(x)}{2w(x)}\right]d\mu(x), & \alpha=1.\end{cases}$$

We also have

$$\int_0^1[(1-t)a+tb]\ln[(1-t)a+tb]\,dt=\frac{1}{4}(b+a)\ln I\left(a^2,b^2\right)=\frac{1}{2}A(a,b)\ln I\left(a^2,b^2\right).$$

Therefore

$$\begin{aligned}A_{f_\alpha}(Q,P,W)&:=\int_X\left(\int_0^1 f\left[\frac{(1-t)q(x)+tp(x)}{w(x)}\right]dt\right)w(x)\,d\mu(x)\\&=\begin{cases}\frac{1}{\alpha(\alpha-1)}\left[\int_X L_\alpha^\alpha\left(\frac{q(x)}{w(x)},\frac{p(x)}{w(x)}\right)w(x)\,d\mu(x)-1\right], & \alpha\in(1,2)\\ \frac{1}{2}\int_X A\left(\frac{q(x)}{w(x)},\frac{p(x)}{w(x)}\right)\ln I\left(\left(\frac{q(x)}{w(x)}\right)^2,\left(\frac{p(x)}{w(x)}\right)^2\right)w(x)\,d\mu(x), & \alpha=1.\end{cases}\end{aligned}$$

We have

$$\mathscr{J}_{f_\alpha}(P,Q,W)=\frac{1}{2}\left[I_{f_\alpha}(P,W)+I_{f_\alpha}(Q,W)\right]-M_{f_\alpha}(Q,P,W),$$

$$\mathscr{T}_{f_\alpha}(P,Q,W)=\frac{1}{2}\left[I_{f_\alpha}(P,W)+I_{f_\alpha}(Q,W)\right]-A_{f_\alpha}(Q,P,W)$$

and

$$\mathscr{M}_{f_\alpha}(P,Q,W)=A_{f_\alpha}(Q,P,W)-M_{f_\alpha}(Q,P,W).$$

According to Theorem 3.1, for all $\alpha\in[1,2)$, the mappings

$$\mathscr{P}\times\mathscr{P}\ni(P,Q)\mapsto\mathscr{J}_{f_\alpha}(P,Q,W),\ \mathscr{M}_{f_\alpha}(P,Q,W),\ \mathscr{T}_{f_\alpha}(P,Q,W)$$

are *convex* for all $W\in\mathscr{P}$.
If $0<r<1<R$, then

$$\left\|f_\alpha''\right\|_{[r,R],\infty}=\sup_{t\in[r,R]}f_\alpha''(t)=\frac{1}{r^{2-\alpha}}\text{ for }\alpha\in[1,2).$$

If there exists $0<r<1<R<\infty$ such that the following condition holds

$$r\le\frac{q(x)}{w(x)},\frac{p(x)}{w(x)}\le R\text{ for }\mu\text{-a.e. }x\in X, \tag{(r,R)}$$

then by (3.18), (3.23) and (3.28) we get

$$0\le\mathscr{J}_{f_\alpha}(P,Q,W)\le\frac{1}{2}\left\|f''\right\|_{[r,R],\infty}(R-r)\max\{R-1,1-r\}, \tag{4.1}$$

$$0\le\mathscr{T}_{f_\alpha}(P,Q,W)\le\frac{1}{3}\frac{(R-r)}{r^{2-\alpha}}\max\{R-1,1-r\} \tag{4.2}$$

and

$$0\le\mathscr{M}_{f_\alpha}(P,Q,W)\le\frac{1}{4}\frac{(R-r)}{r^{2-\alpha}}\max\{R-1,1-r\}, \tag{4.3}$$

for all $\alpha\in[1,2)$ and $W\in\mathscr{P}$.
The interested reader may apply the above general results for other particular divergences of interest generated by the convex functions provided in the introduction. We omit the details.

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On Strongly Lacunary $\mathscr{I}_2^*$-Convergence and Strongly Lacunary $\mathscr{I}_2^*$-Cauchy Sequence

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Abstract

In the study conducted here, we have given some new concepts in summability theory. In this sense, firstly, using the lacunary sequence we have given the concept of strongly $\mathscr{I}_{\theta_2}^*$-convergence and we have examined the relations between $\mathscr{I}_{\theta_2}^*$-convergence and strongly $\mathscr{I}_{\theta_2}^*$-convergence and also between strongly $\mathscr{I}_{\theta_2}$-convergence and strongly $\mathscr{I}_{\theta_2}^*$-convergence. Also, using the lacunary sequence we have given the concept of strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence and examined the relations between strongly $\mathscr{I}_{\theta_2}$-Cauchy sequence and strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence.

1. Introduction and Definitions

In the mathematical literature, some types of convergence in summability theory and some applications and properties of these convergence types have been studied by many mathematicians. Especially recently, some types of convergence of double-indexed sequences have been frequently studied in summability theory. Also, the types of convergence defined in summability theory using the lacunary sequence have been studied by many authors. These studies were carried out by generalizing the theorems that give some similar properties in single-index sequences to double-index sequences. Classical convergence in real number sequences was generalized to the statistical convergence by Schoenberg [1] and Fast [2], independently. The ideal convergence, a generalization of statistical convergence that would later inspire many researchers, was first defined by Kostyrko et al. [3]. Nabiev [4] studied on the $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences with their characteristics. In the topology generated by n-normed spaces, the lacunary ideal convergence, lacunary ideal Cauchy sequence and their some important characteristics investigated by Yamancı and Gürdal [5]. The ideal lacunary convergence was introduced by Tripathy et al. [6]. In recent times, using the lacunary sequence, the $\mathscr{I}_\theta^*$-convergence, strongly $\mathscr{I}_\theta^*$-convergence, $\mathscr{I}_\theta^*$-Cauchy sequence and strongly $\mathscr{I}_\theta^*$-Cauchy sequence were introduced by Akın and Dündar [7,8]. The concept of ideal convergence and some of its important characteristics defined for single-index sequences have also been defined for double-index sequences in the linear metric space by many mathematicians [9–11] and many useful works have been done in this sense. In addition, the ideal convergence and strong ideal convergence and some of its characteristic properties using the lacunary sequence for single-index sequences were also introduced to the literature by Hazarika [12], Dündar et al. [13] and Akın and Dündar [14] for double sequences and double set sequences in metric spaces and normed spaces.
In recently, some convergence types such as classical convergence, statistical convergence and ideal convergence in some metric spaces and normed spaces were studied in summability theory by a lot of mathematicians. In the study conducted here, using the lacunary sequence, we have given the concept of strongly $\mathscr{I}_{\theta_2}^*$-convergence and we have investigated the relations between $\mathscr{I}_{\theta_2}^*$-convergence and strongly $\mathscr{I}_{\theta_2}^*$-convergence and also between strongly $\mathscr{I}_{\theta_2}$-convergence and strongly $\mathscr{I}_{\theta_2}^*$-convergence. Furthermore, using the lacunary sequence, we have given the concept of strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence and examined the relations between strongly $\mathscr{I}_{\theta_2}$-Cauchy sequence and strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence.
Some basic definitions, concepts and characteristics that will be used throughout the study and are available in the literature will now be noted (see [3–8, 10, 11, 14–21]).
Firstly, we want to give the ideas of ideal, filter and some properties about these ideas are used in our study.
For $\mathscr{I} \subseteq 2^{\mathbb{N}}$, if the following propositions

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(i) $\emptyset \in \mathscr{I}$,
(ii) If $T,U \in \mathscr{I}$, then $T \cup U \in \mathscr{I}$,
(iii) If $T \in \mathscr{I}$ and $U \subseteq T$, then $U \in \mathscr{I}$

are hold, then $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is named as an ideal.
If $\mathbb{N} \notin \mathscr{I}$, then $\mathscr{I}$ is named as a non-trivial ideal. Also, if $\{k\} \in \mathscr{I}$ for each $k \in \mathbb{N}$, then a non-trivial ideal is named as an admissible ideal.
For $\mathscr{F} \subseteq 2^{\mathbb{N}}$, if the following propositions

(i) $\emptyset \notin \mathscr{F}$,
(ii) If $T,U \in F$, then $T \cap U \in \mathscr{F}$,
(iii) If $T \in \mathscr{F}$ and $U \supseteq T$, then $U \in \mathscr{F}$

are hold, then $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is named as a filter.
For a non-trivial ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$, $\mathscr{F}(\mathscr{I}) = \{T \subset \mathbb{N} : T = \mathbb{N} \setminus U \text{ for } \exists U \in \mathscr{I}\}$ is named as the filter corresponding with $\mathscr{I}$.
Here, we want to give the ideas of lacunary sequence and some properties about lacunary sequence are used in our studypaper.
The increasing integer sequence $\theta = \{k_r\}$ is named as a lacunary sequence when it satisfies the propositions $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$, $(r \to \infty)$. During the study, $I_r = (k_{r-1}, k_r]$ and q_r represent the intervals determined by $\{k_r\}$ and the ratio $\frac{k_r}{k_{r-1}}$, respectively.
Then after this, we regard $\theta = \{k_r\}$ as a lacunary sequence and $\mathscr{I} \subseteq 2^{\mathbb{N}}$ as a non-trivial admissible ideal.
For a sequence $(x_k) \subset \mathbb{R}$, if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - \ell| = 0$$

is hold, then (x_k) is strongly lacunary convergent to $\ell \in \mathbb{R}$.
For a sequence $(x_k) \subset \mathbb{R}$, if

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |x_k - \ell| \geq \varepsilon \right\} \in \mathscr{I}, \text{ (for every } \varepsilon > 0)$$

is hold, then (x_k) is strongly lacunary $\mathscr{I}$-convergent to $\ell \in \mathbb{R}$ and denoted with $x_k \to \ell[\mathscr{I}_\theta]$.
For a sequence $(x_k) \subset \mathbb{R}$, if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |x_k - x_N| \geq \varepsilon \right\} \in \mathscr{I},$$

then (x_k) is strongly lacunary $\mathscr{I}$-Cauchy sequence.
For a sequence $(x_k) \subset \mathbb{R}$, iff there exists a set $G = \{g_1 < g_2 < \cdots < g_k < \cdots\} \subset \mathbb{N}$ such that for the set $G' = \{r \in \mathbb{N} : g_k \in I_r\} \in \mathscr{F}(\mathscr{I})$

$$\lim_{\substack{r\to\infty \\ (r\in G')}} \frac{1}{h_r} \sum_{k\in I_r} |x_{g_k} - \ell| = 0$$

is hold, then (x_k) is strongly lacunary $\mathscr{I}^*$-convergent to $\ell \in \mathbb{R}$ and denoted with $x_k \to \ell[\mathscr{I}_\theta^*]$.
For a sequence $(x_k) \subset \mathbb{R}$, iff there exists a set $G = \{g_1 < g_2 < \cdots < g_k < \cdots\} \subset \mathbb{N}$ such that for the set $G' = \{r \in \mathbb{N} : g_k \in I_r\} \in \mathscr{F}(\mathscr{I})$

$$\lim_{\substack{r\to\infty \\ (r\in G')}} \sum_{k,p\in I_r} |x_{g_k} - x_{g_p}| = 0$$

is hold, then (x_k) is strongly lacunary $\mathscr{I}^*$-Cauchy sequence.
For each $k \in \mathbb{N}$ and a non-trivial ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$, if $\{k\} \times \mathbb{N} \in \mathscr{I}_2$ and $\mathbb{N} \times \{k\} \in \mathscr{I}_2$, then $\mathscr{I}_2$ is named as strongly admissible ideal.
$\mathscr{I}_2^0 = \{T \subset \mathbb{N}^2 : (\exists m(T) \in \mathbb{N})(i, j \geq m(T) \Rightarrow (i,j) \notin T)\}$ ideal is a strongly admissible ideal. Furthermore, it is clearly that $\mathscr{I}_2$ is a strongly admissible iff $\mathscr{I}_2^0 \subset \mathscr{I}_2$.
It can be clearly seen that a strongly admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ is an admissible ideal.
The following set

$$\mathscr{F}(\mathscr{I}_2) = \{T \subset \mathbb{N}^2 : T = \mathbb{N}^2 \setminus U \text{ for } \exists U \in \mathscr{I}_2\}$$

is named a filter corresponding with $\mathscr{I}_2$.
For an admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$, if for every countable mutually disjoint set family $\{T_1, T_2, ...\} \in \mathscr{I}_2$, there exists a countable set family $\{U_1, U_2, ...\}$ such that $T_k \Delta U_k \in \mathscr{I}_2^0$, that is, $T_k \Delta U_k$ is involved in finite union of rows and columns in $\mathbb{N}^2$ for each $k \in \mathbb{N}$ and $U = \bigcup_{k=1}^{\infty} U_k \in \mathscr{I}_2$ ($U_k \in \mathscr{I}_2$ for each $k \in \mathbb{N}$), then $\mathscr{I}_2$ is named satisfying the property $(AP2)$.
If for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - \ell| < \varepsilon$, for all $k, j > n_\varepsilon$, then the double sequence $x = (x_{kj}) \subset \mathbb{R}$ is convergent to $\ell \in \mathbb{R}$ and denoted with $\lim_{k,j\to\infty} x_{kj} = \ell$ or $\lim_{k,j\to\infty} x_{kj} = \ell$.
Now, we want to give the idea of double lacunary sequence and some properties about it are used in our manuscript.
The double sequence $\theta_2 = \{(k_r, j_u)\}$ is named as a double lacunary sequence when two increasing integer sequences satisfy the propositions

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \text{ and } j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty,$$

for $r, u \to \infty$. We take the following screenings for double lacunary sequence:

$$k_{ru} = k_r j_u, \quad h_{ru} = h_r \bar{h}_u, \quad I_{ru} = \{(k,j) : k_{r-1} < k \leq k_r \quad and \quad j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \ \text{ and } \ q_u = \frac{j_u}{j_{u-1}}.$$

Then after this, we regard $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ as a strongly admissible ideal and $\theta_2 = \{(k_r, j_u)\}$ as a double lacunary sequence.
For a double sequence $(x_{kj}) \subset \mathbb{R}$, if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - \ell| = 0$$

is hold, then (x_{kj}) is strongly lacunary convergent to $\ell \in \mathbb{R}$.
For a double sequence $(x_{kj}) \subset \mathbb{R}$, if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| = 0$$

is hold, therefore (x_{kj}) is strongly lacunary Cauchy double sequence.
For a double sequence $(x_{kj}) \subset \mathbb{R}$, if for every $\varepsilon > 0$

$$\left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - \ell| \geq \varepsilon\right\} \in \mathscr{I}_2$$

is hold, then (x_{kj}) is strongly lacunary $\mathscr{I}_2$-convergent to $\ell \in \mathbb{R}$ and denoted with $x_{kj} \to \ell[\mathscr{I}_{\theta_2}]$.
For a double sequence $(x_{kj}) \subset \mathbb{R}$, if for every $\varepsilon > 0$, there exist $N = N(\varepsilon)$ and $S = S(\varepsilon)$ such that

$$\left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{NS}| \geq \varepsilon\right\} \in \mathscr{I}_2,$$

then (x_{kj}) is strongly lacunary $\mathscr{I}_2$-Cauchy double sequence.
Now, let's give a useful lemma that we will use in our work.

Lemma 1.1 ([10]). *Let $\mathscr{F}(\mathscr{I}_2)$ be a filter corresponding with a strongly admissible ideal $\mathscr{I}_2$ with $(AP2)$. Thus, there exists a set $T \subset \mathbb{N}^2$ such that $T \in \mathscr{F}(\mathscr{I}_2)$ and the set $T \backslash T_k$ is finite for all k, where $\{T_k\}_{k=1}^{\infty}$ is a countable collection of subsets of $\mathbb{N}^2$ such that $T_k \in F(\mathscr{I}_2)$ for each k.*

2. Main Results

In the original part of our work, using the lacunary sequence, we will define for double-indexed sequences the definitions and concepts available in the literature for single-indexed sequences. For double sequences, we first defined lacunary $\mathscr{I}_2^*$-convergence and strongly lacunary $\mathscr{I}_2^*$-convergence with theorems examining the relationship between these new convergence types.

Definition 2.1. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$ iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} = \ell$$

and so we can write $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$.

Definition 2.2. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}_2^*$-convergent to $\ell \in \mathbb{R}$ iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - \ell| = 0$$

and so we can write $x_{kj} \to \ell[\mathscr{I}_{\theta_2}^*]$.

Theorem 2.3. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly $\mathscr{I}_{\theta_2}^*$-convergent to $\ell \in \mathbb{R}$, then it is $\mathscr{I}_{\theta_2}^*$-convergent to same point.*

Proof. As per our assumption, let $x_{kj} \to \ell[\mathscr{I}_{\theta_2}^*]$. Therefore, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., U = \mathbb{N}^2 \backslash G' \in \mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r,u > r_0$ we have

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - \ell| < \varepsilon, \ ((r,u) \in G').$$

Therefore, for every $\varepsilon > 0$ and all $r,u > r_0$ we have

$$\left|\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{kj} - \ell\right| \leq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - \ell| < \varepsilon, \ ((r,u) \in G')$$

and so $x_{kj} \to \ell(\mathscr{I}_{\theta_2}^*)$. □

Theorem 2.4. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly $\mathscr{I}^*_{\theta_2}$-convergent to $\ell \in \mathbb{R}$, then this sequence is strongly $\mathscr{I}_{\theta_2}$-convergent to same point.*

Proof. As per our assumption, let $x_{kj} \to \ell[\mathscr{I}^*_{\theta_2}]$. Therefore, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., U = \mathbb{N}^2 \backslash G' \in \mathscr{I}_2)$ there is a $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r,u > r_0$ we have

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| < \varepsilon, \ ((r,u) \in G').$$

Then,

$$A(\varepsilon) = \left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| \geq \varepsilon\right\} \subset U \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\})\right)\right].$$

Since $\mathscr{I}_2$ is a strongly admissible ideal, we have

$$U \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\})\right)\right] \in \mathscr{I}_2.$$

Thus $A(\varepsilon) \in \mathscr{I}_2$ and $x_{kj} \to \ell[\mathscr{I}_{\theta_2}]$. □

Theorem 2.5. *Let a strongly admissible ideal $\mathscr{I}_2$ with $(AP2)$. If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly $\mathscr{I}_{\theta_2}$-convergent to $\ell \in \mathbb{R}$, then this sequence is strongly $\mathscr{I}^*_{\theta_2}$-convergent to same point.*

Proof. As per our assumption, let $x_{kj} \to \ell[\mathscr{I}_{\theta_2}]$. Then, for every $\varepsilon > 0$,

$$\mathscr{M}(\varepsilon) = \left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| \geq \varepsilon\right\} \in \mathscr{I}_2.$$

Let us take

$$\mathscr{M}_1 = \left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| \geq 1\right\} \text{ and}$$

$$\mathscr{M}_\beta = \left\{(r,u) \in \mathbb{N}^2 : \frac{1}{\beta} \leq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| < \frac{1}{\beta - 1}\right\},$$

for a natural number $\beta \geq 2$. Obviously, $\mathscr{M}_\alpha \cap \mathscr{M}_\gamma = \emptyset$ for $\alpha \neq \gamma$ and $\mathscr{M}_\alpha \in \mathscr{I}_2$ for each $\alpha \in \mathbb{N}$. Also, by $(AP2)$, there is a sequence $\{\mathscr{V}_\beta\}_{\beta\in\mathbb{N}}$ such that $\mathscr{M}_\alpha \Delta \mathscr{V}_\alpha$ is involved in finite union of rows and columns in $\mathbb{N}^2$ for each $\alpha \in \mathbb{N}$ and

$$\mathscr{V} = \bigcup_{\alpha=1}^{\infty} \mathscr{V}_\alpha \in \mathscr{I}_2.$$

We prove that

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| = 0,$$

for $G' = \mathbb{N}^2 \setminus \mathscr{V} \in \mathscr{F}(\mathscr{I}_2)$. For $\delta > 0$ select $q \in \mathbb{N}$ with the inequality $\frac{1}{q} < \delta$. Therefore,

$$\left\{(r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| \geq \delta\right\} \subset \bigcup_{j=1}^{q-1} \mathscr{M}_j.$$

Since $\mathscr{M}_\alpha \Delta \mathscr{V}_\alpha$ is a finite set for $\alpha \in \{1,2,\cdots,q-1\}$, there exists $r_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q-1} \mathscr{M}_j\right) \cap \left\{(r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq r_0\right\} = \left(\bigcup_{j=1}^{q-1} \mathscr{V}_j\right) \cap \left\{(r,u) \in \mathbb{N}^2 : r \geq r_0 \wedge u \geq r_0\right\}.$$

If $r \geq r_0$ and $(r,u) \notin V$, then

$$(r,u) \notin \bigcup_{j=1}^{q-1} \mathscr{V}_j \text{ and so } (r,u) \notin \bigcup_{j=1}^{q-1} \mathscr{M}_j.$$

We have

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| < \frac{1}{q} < \delta.$$

And so from this inequality, we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left|x_{kj} - \ell\right| = 0.$$

Therefore, we have $x_{kj} \to \ell[\mathscr{I}^*_{\theta_2}]$. □

Now, for double sequences, we have defined $\mathscr{I}^*_{\theta_2}$-Cauchy and strongly $\mathscr{I}^*_{\theta_2}$-Cauchy sequence with theorems examining the relationships between $\mathscr{I}^*_{\theta_2}$-Cauchy sequence and strongly $\mathscr{I}^*_{\theta_2}$-Cauchy sequence, and also between strongly $\mathscr{I}^*_{\theta_2}$-Cauchy sequence and strongly $\mathscr{I}_{\theta_2}$-Cauchy sequence.

Definition 2.6. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is lacunary $\mathscr{I}^*_2$-Cauchy sequence iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) = 0.$$

Definition 2.7. *A double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence iff there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$, we have*

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| = 0.$$

Theorem 2.8. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence, then (x_{kj}) is lacunary $\mathscr{I}^*_2$-Cauchy sequence.*

Proof. As per our assumption, let (x_{kj}) is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence. Thus, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., H = \mathbb{N}^2 \setminus G' \in \mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in \mathbb{N}$, we have

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| < \varepsilon, \ \ ((r,u) \in G')$$

for all $r,u > r_0$. Then, we have

$$\left| \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} (x_{kj} - x_{st}) \right| \leq \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| < \varepsilon, \ \ ((r,u) \in G')$$

for every $\varepsilon > 0$ and all $r,u > r_0$ and so (x_{kj}) is lacunary $\mathscr{I}^*_2$-Cauchy sequence. □

Theorem 2.9. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence, then (x_{kj}) is strongly lacunary $\mathscr{I}_2$-Cauchy sequence.*

Proof. As per our assumption, let (x_{kj}) is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence. Then, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., U = \mathbb{N}^2 \backslash G' \in \mathscr{I}_2)$ and for every $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in \mathbb{N}$, we have

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| < \varepsilon, \ \ ((r,u) \in G')$$

for all $r,u > r_0$. Let $N = N(\varepsilon) \in I_{r_0+1,u_0+1}$ and $S = S(\varepsilon) \in I_{r_0+1,u_0+1}$ Then, for every $\varepsilon > 0$ and all $r,u > r_0 = r_0(\varepsilon)$

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{NS}| < \varepsilon, \ \ ((r,u) \in G').$$

Now, let $U = \mathbb{N}^2 \setminus G'$. It is clear that $U \in \mathscr{I}_2$. Then,

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{NS}| \geq \varepsilon \right\} \subset U \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right].$$

Since $\mathscr{I}_2$ is a strongly admissible ideal, we have

$$U \cup \left[G' \cap \left((\{1,2,\cdots,r_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,r_0\}) \right) \right] \in \mathscr{I}_2$$

and so $A(\varepsilon) \in \mathscr{I}_2$. Hence, (x_{kj}) is strongly lacunary $\mathscr{I}_2$-Cauchy sequence. □

Theorem 2.10. *Let a strongly admissible ideal $\mathscr{I}_2$ with $(AP2)$. If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly lacunary $\mathscr{I}_2$-Cauchy sequence, then (x_{kj}) is strongly lacunary $\mathscr{I}^*_2$-Cauchy sequence.*

Proof. As per our assumption, let (x_{kj}) is strongly lacunary $\mathscr{I}_2$-Cauchy sequence. Then, for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $S = S(\varepsilon)$ such that

$$A(\varepsilon) = \left\{ (r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{NS}| \geq \varepsilon \right\} \in \mathscr{I}_2.$$

Let us take

$$T_j = \left\{ (r,u) \in \mathbb{N}^2 : \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{m_j n_j}| \geq \frac{1}{j} \right\}, \ j = 1,2,\ldots,$$

where $m_j = P\left(\frac{1}{j}\right)$ and $n_j = S\left(\frac{1}{j}\right)$. It is clear that $T_j \in \mathscr{F}(\mathscr{I}_2)$ for $j = 1, 2, \cdots$. Since $\mathscr{I}_2$ has the $(AP2)$ property, then by Lemma 1.1, there exists a set $T \subset \mathbb{N}^2$ such that $T \in \mathscr{F}(\mathscr{I}_2)$ and $T \setminus T_j$ is finite for all j. Now, we demonstrate that

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in T)}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| = 0.$$

To show this, let $\varepsilon > 0$ and a natural number m be $m > \frac{2}{\varepsilon}$. If $(r,u) \in T$, then $T \setminus T_m$ is a finite set, so there exists $r_0 = r_0(m)$ such that $(r,u) \in T_m$ for all $r,u > r_0(m)$. Therefore, for all $r,u > r_0(m)$

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{s_m t_m}| < \frac{1}{m} \text{ and } \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} |x_{st} - x_{s_m t_m}| < \frac{1}{m}.$$

Hence, for all $r,u > r_0(m)$ it follows that

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| \leq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - x_{s_m t_m}| + \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} |x_{st} - x_{s_m t_m}| < \frac{1}{m} + \frac{1}{m} < \varepsilon.$$

Therefore, for any $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that for $r,u > r_0(\varepsilon)$ and $(r,u) \in T \in \mathscr{F}(\mathscr{I}_2)$

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| < \varepsilon.$$

This demonstrates that (x_{kj}) is strongly lacunary $\mathscr{I}_2^*$-Cauchy sequence. □

Theorem 2.11. *If a double sequence $(x_{kj}) \subset \mathbb{R}$ is strongly $\mathscr{I}_{\theta_2}^*$-convergent to ℓ, then (x_{kj}) is strongly $\mathscr{I}_{\theta_2}$-Cauchy double sequence.*

Proof. As per our assumption, let $x_{kj} \to \ell[\mathscr{I}_{\theta_2}^*]$. Therefore, there exists a set $G = \{(k,j) \in \mathbb{N}^2\}$ such that for the set $G' = \{(r,u) \in \mathbb{N}^2 : (k,j) \in I_{ru}\} \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., U = \mathbb{N}^2 \backslash G' \in \mathscr{I}_2)$ there is a $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r,u > r_0$ we have

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - \ell| < \frac{\varepsilon}{2}, \ ((r,u) \in G').$$

Since

$$\frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| \leq \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{kj} - \ell| + \frac{1}{h_{ru}} \sum_{(s,t)\in I_{ru}} |x_{st} - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ ((r,u) \in G')$$

for all $r,u > r_0$ and so we have

$$\lim_{\substack{r,u\to\infty \\ ((r,u)\in G')}} \frac{1}{h_{ru}} \sum_{(k,j),(s,t)\in I_{ru}} |x_{kj} - x_{st}| = 0.$$

That is, (x_{kj}) is a strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence. Thus,(x_{kj}) is a strongly $\mathscr{I}_{\theta_2}$-Cauchy sequence by Theorem 2.9. □

3. Conclusion

Using the lacunary sequence, for double sequences, we have first defined lacunary $\mathscr{I}_2^*$-convergence and strongly lacunary $\mathscr{I}_2^*$-convergence with theorems examining the relationship between these new convergence types. Furthermore, we have defined $\mathscr{I}_{\theta_2}^*$-Cauchy and strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence with theorems examining the relationships between $\mathscr{I}_{\theta_2}^*$-Cauchy sequence and strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence, and also between strongly $\mathscr{I}_{\theta_2}^*$-Cauchy sequence and strongly $\mathscr{I}_{\theta_2}$-Cauchy sequence. In the future, these studies are also debatable in terms of regularly convergence for double sequences.

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The New Class $L_{p,\Phi}$ of s-Type Operators

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Abstract

In this study, the class of s-type $\ell_p(\Phi)$ operators is introduced and it is shown that $L_{p,\Phi}$ is a quasi-Banach operator ideal. Also, some other classes are defined by using approximation, Gelfand, Kolmogorov, Weyl, Chang, and Hilbert number sequences. Then, some properties are examined.

1. Introduction

In this study, all natural numbers are symbolized by $\mathbb{N}$ and all non-negative real numbers are symbolized by $\mathbb{R}^+$. If the dimension of the range space of a bounded linear operator is finite, it is called a finite rank operator [1].
Throughout this study, E and F represent real or complex Banach spaces. The space of all bounded linear operators from E to F is denoted by $\mathscr{B}(E,F)$ and the space of all bounded linear operators from an arbitrary Banach space to another arbitrary Banach space is denoted by $\mathscr{B}$. The operator ideal theory is a very important field in functional analysis. The theory of normed operator ideals first appeared in the 1950s in [2]. In functional analysis, most of the operator ideals are constructed via different scalar sequence spaces. s-number sequence is one of the most important example of this. For more information about operator ideals and $s-$numbers, we refer to [3–8]. The definition of s-numbers goes back to E. Schmidt [9] who used this concept in the theory of non-selfadjoint integral equations. In Banach spaces, there are many different possibilities of defining some equivalents for s-numbers, namely Kolmogorov numbers, Gelfand numbers, approximation numbers, etc. In the following years, Pietsch developed the concept of s-number sequence to collect all s-numbers in a single definition [10–12].
A map

$$S : K \to (s_r(K))$$

which assigns a non-negative scalar sequence to each operator, is called an s-number sequence if for all Banach spaces E, F, E_0, and F_0 the following conditions are satisfied:

(*i*) $\|K\| = s_1(K) \geq s_2(K) \geq \ldots \geq 0$, for every $K \in \mathscr{B}(E,F)$,
(*ii*) $s_{p+r-1}(L+K) \leq s_p(L) + s_r(K)$ for every L, $K \in \mathscr{B}(E,F)$ and p, $r \in \mathbb{N}$,
(*iii*) $s_r(MLK) \leq \|M\| s_r(L) \|K\|$ for some $M \in \mathscr{B}(F,F_0)$, $L \in \mathscr{B}(E,F)$ and $K \in \mathscr{B}(E_0,E)$, where E_0, F_0 are arbitrary Banach spaces,
(*iv*) If $rank(K) \leq r$, then $s_r(K) = 0$,
(*v*) $s_r(I_r) = 1$, where I_r is the identity map of r-dimensional Hilbert space l_2^r to itself [13].

$s_r(K)$ represents the r-th $s-$number of the operator K.
Pietsch defined approximation numbers, which are frequently used examples of s-number sequence, $a_r(K)$, the r-th approximation number of a bounded linear operator as

$$a_r(K) = \inf\{\, \|K-A\| : A \in \mathscr{B}(E,F),\, rank(A) < r\,\},$$

where $K \in \mathscr{B}(E,F)$ and $r \in \mathbb{N}$ [10]. Let $K \in \mathscr{B}(E,F)$ and $r \in \mathbb{N}$. *Gel′fand* numbers $(c_r(K))$, *Kolmogorov* numbers $(d_r(K))$, *Weyl* numbers $(x_r(K))$, *Chang* numbers $(y_r(K))$, *Hilbert* numbers $(h_r(K))$, are some other examples of s-number sequences. For more information about these sequences, we refer to [1].

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Some necessary properties of $s-$number sequences are given in the sequel.
Let $\mathscr{J} \in \mathscr{B}(F,F_0)$ be a metric injection. If the sequence $s=(s_r)$ satisfies $s_r(K)=s_r(\mathscr{J}K)$ for all $K\in\mathscr{B}(E,F)$ the sequence of s-number is named injective [14, p.90].

Proposition 1.1. *[14, p.90-94] The number sequences $(c_r(K))$ and $(x_r(K))$ are injective.*

Let $\mathscr{S}\in\mathscr{B}(E_0,E)$ be a metric surjection. If the sequence $s=(s_r)$ satisfies $s_r(K)=s_r(K\mathscr{S})$ for all $K\in\mathscr{B}(E,F)$ the s-number sequence is named surjective [14, p.95].

Proposition 1.2. *[14, p.95] The number sequences $(d_r(K))$ and $(y_r(K))$ are surjective.*

Proposition 1.3. *[14, p.115] Let $K\in\mathscr{B}(E,F)$. Then, the following inequalities hold:*

(i) $h_r(K)\le x_r(K)\le c_r(K)\le a_r(K)$,
(ii) $h_r(K)\le y_r(K)\le d_r(K)\le a_r(K)$.

Lemma 1.4. *[11] Let $S,K\in\mathscr{B}(E,F)$, then $|s_r(K)-s_r(S)|\le\|K-S\|$ for $r=1,2,\ldots$.*

Let the space of all real-valued sequences be denoted by ω. Then, a sequence space is any vector subspace of ω.
Maddox defined the linear space $l(p)$ as follows in [15]:

$$l(p)=\left\{x\in\omega:\sum_{n=1}^{\infty}|x_n|^{p_n}<\infty\right\},$$

where(p_n) is a bounded sequence of strictly positive real numbers.
If an operator $K\in\mathscr{B}(E,F)$ satisfies $\sum_{n=1}^{\infty}(a_n(K))^p<\infty$ for $0<p<\infty$, K is defined as an l_p- type operator in [10] by Pietsch. Afterward *ces-p* type operators which is a new class obtained via Cesaro sequence space are introduced by Constantin [16]. Later on, Tita in [17], proved that the class of l_p type operators and *ces-p* type operators coincide.
In this paper φ denotes the Euler function. For every $u\in\mathbb{N}$ with $u\ge 1$, $\varphi(u)$ is the number of positive integers less than u which are coprime with u and $\varphi(1)=1$. If $p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ is the prime factorization of a natural number $u>1$ then

$$\varphi(u)=u(1-\frac{1}{p_1})(1-\frac{1}{p_2})...(1-\frac{1}{p_r}).$$

Also, the equality

$$u=\sum_{t|u}\varphi(t)$$

holds for every $u\in\mathbb{N}$ and $\varphi(u_1u_2)=\varphi(u_1)\varphi(u_2)$, where $u_1,u_2\in\mathbb{N}$ are coprime [18].
In [19] the sequence space $\ell_p(\Phi)$ is defined as:

$$\ell_p(\Phi)=\left\{x=(x_n)\in\omega:\sum_n\left|\frac{1}{n}\sum_{t|n}\varphi(t)x_t\right|^p<\infty\right\}\quad(1\le p<\infty).$$

Let E^*, the dual of E, be the set of continuous linear functionals on E. The map $x^{'}\otimes y:E\to F$ is defined as

$$\left(x^{'}\otimes y\right)(x)=x^{'}(x)y$$

where $x\in E$, $x^{'}\in E^*$ and $y\in F$.
A subcollectionϑ of $\mathscr{B}$ is said to be an operator ideal if for each component $\vartheta(E,F)=\vartheta\cap\mathscr{B}(E,F)$ the following conditions are hold:

(*i*) if $x^{'}\in E^*$, $y\in F$, then $x^{'}\otimes y\in\vartheta(E,F)$,
(*ii*) if $L,K\in\vartheta(E,F)$, then $L+K\in\vartheta(E,F)$,
(*iii*) if $L\in\vartheta(E,F)$, $K\in\mathscr{B}(E_0,E)$ and $M\in\mathscr{B}(F,F_0)$, then $MLK\in\vartheta(E_0,F_0)$ [12].

Let ϑ be an operator ideal and $\rho:\vartheta\to\mathbb{R}^+$ be a function on ϑ. Then, if the following conditions are hold:

(*i*) if $x^{'}\in E^*$, $y\in F$, then $\rho\left(x^{'}\otimes y\right)=\left\|x^{'}\right\|\|y\|$;
(*ii*) if $\exists\mathscr{C}\ge 1$ such that $\rho(L+K)\le\mathscr{C}[\rho(L)+\rho(K)]$;
(*iii*) if $L\in\vartheta(E,F)$, $K\in\mathscr{B}(E_0,E)$ and $M\in\mathscr{B}(F,F_0)$, then $\rho(MLK)\le\|M\|\rho(L)\|K\|$,

ρ is called a quasi-norm on the operator ideal ϑ [12].
For special case $\mathscr{C}=1$, ρ is a norm on the operator ideal ϑ.
If ρ is a quasi-norm on an operator ideal ϑ, it is denoted by $[\vartheta,\rho]$. Also, if every component $\vartheta(E,F)$ is complete with the quasi-norm ρ, $[\vartheta,\rho]$ is called a quasi-Banach operator ideal.
Let $[\vartheta,\rho]$ be a quasi-normed operator ideal and $\mathscr{J}\in\mathscr{B}(F,F_0)$ be a metric injection. If for every operator $K\in\mathscr{B}(E,F)$ and $\mathscr{J}K\in\vartheta(E,F_0)$ we have $K\in\vartheta(E,F)$ and $\rho(\mathscr{J}K)=\rho(K)$, $[\vartheta,\rho]$ is called an injective quasi-normed operator ideal. Furthermore, let $[\vartheta,\rho]$ be a quasi-normed operator ideal and $\mathscr{S}\in\mathscr{B}(E_0,E)$ be a metric surjection. If for every operator $K\in\mathscr{B}(E,F)$ and $K\mathscr{S}\in\vartheta(E_0,F)$ we have $K\in\vartheta(E,F)$ and $\rho(K\mathscr{S})=\rho(K)$, $[\vartheta,\rho]$ is called a surjective quasi-normed operator ideal [12].
Let K^* be the dual of K. An $s-$number sequence is called symmetric and completely symmetric if for all $K\in\mathscr{B}$, $s_r(K)\ge s_r(K^*)$ and $s_r(K)=s_r(K^*)$, respectively [12].
The dual of an operator ideal ϑ is denoted by ϑ^* and it is defined as

$$\vartheta^*(E,F)=\left\{K\in\mathscr{B}(E,F):K'\in\vartheta(F^*,E^*)\right\}$$

[12].
An operator ideal ϑ is called symmetric if $\vartheta\subset\vartheta^*$ and is called completely symmetric if $\vartheta=\vartheta^*$ [12].

2. Main Results

An operator $K\in\mathscr{B}(E,F)$ is in the class of $s-$type $\ell_p(\Phi)$ if

$$\sum_n\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p<\infty\quad(1\le p<\infty).$$

The class of all $s-$type $\ell_p(\Phi)$ operators is denoted by $L_{p,\Phi}(E,F)$.

Theorem 2.1. *The class $L_{p,\Phi}$ is a quasi-normed operator ideal by*

$$\|K\|_{p,\Phi}=\frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}},\quad(1<p<\infty).$$

Proof. In this proof we show that the class $L_{p,\Phi}$ satisfies the conditions of an operator ideal and $\|K\|_{p,\Phi}$ satisfies the conditions for a quasi-norm. Let $x^{'}\in E^*$ and $y\in F$. Then,

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(x^{'}\otimes y)\right)^p&=\left(\sum_{t|1}\varphi(t)s_t(x^{'}\otimes y)\right)^p+\left(\frac{1}{2}\sum_{t|2}\varphi(t)s_t(x^{'}\otimes y)\right)^p+\left(\frac{1}{3}\sum_{t|3}\varphi(t)s_t(x^{'}\otimes y)\right)^p+\dots\\&=\left(\varphi(1)s_1(x^{'}\otimes y)\right)^p+\left(\frac{1}{2}\varphi(1)s_1(x^{'}\otimes y)\right)^p+\left(\frac{1}{3}\varphi(1)s_1(x^{'}\otimes y)\right)^p+\dots\\&=\left(s_1(x^{'}\otimes y)\right)^p\left(1+(\frac{1}{2})^p+(\frac{1}{3})^p+\dots\right)\\&<\infty.\end{aligned}$$

Since the operator $x^{'}\otimes y$ has rank one, $s_n\left(x^{'}\otimes y\right)=0$ for $n\ge 2$. Therefore, $x^{'}\otimes y\in L_{p,\Phi}(E,F)$.
And also,

$$\begin{aligned}\left\|x^{'}\otimes y\right\|_{p,\Phi}&=\frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(x^{'}\otimes y)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}\\&=\frac{\left[\left(s_1(x^{'}\otimes y)\right)^p\left(1+(\frac{1}{2})^p+(\frac{1}{3})^p+\dots\right)\right]^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}\\&=s_1(x^{'}\otimes y)=\|x^{'}\otimes y\|=\|x^{'}\|\|y\|.\end{aligned}$$

Hence $\left\|x^{'}\otimes y\right\|_{p,\Phi}=\|x^{'}\|\|y\|$.
Let $L,K\in L_{p,\Phi}(E,F)$. Then,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(L)\right)^p<\infty,\quad\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p<\infty.$$

To show that $L+K\in L_{p,\Phi}(E,F)$, let begin with

$$\begin{aligned}\sum_{t|n}\varphi(t)s_t(L+K)&\le\sum_{t|n}\varphi(2t-1)s_{2t-1}(L+K)+\sum_{t|n}\varphi(2t)s_{2t}(L+K)\\&\le\sum_{t|n}\left(\varphi(2t-1)+\varphi(2t)\right)s_{2t-1}(L+K)\\&\le\sum_{t|n}\mathscr{C}\varphi(t)s_{2t-1}(L+K)\\&\le\mathscr{C}\left(\sum_{t|n}\varphi(t)s_t(L)+\sum_{t|n}\varphi(t)s_t(K)\right)\end{aligned}$$

since $\exists\mathscr{C}\ge 1$ which satisfies $\varphi(2t-1)+\varphi(2t)\le\mathscr{C}\varphi(t)$.

By using Minkowski inequality, we get;

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(L+K)\right)^p\right)^{\frac{1}{p}} \leq C\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(L)+\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p\right)^{\frac{1}{p}}$$
$$\leq C\left[\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(L)\right)^p\right)^{\frac{1}{p}}+\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p\right)^{\frac{1}{p}}\right]$$
$$< \infty.$$

Hence, $L+K \in L_{p,\Phi}(E,F)$. Additionally,

$$\|L+K\|_{p,\Phi} = \frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(L+K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}$$
$$\leq C\left[\frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(L)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}+\frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}\right]$$
$$= C\left[\|L\|_{p,\Phi}+\|K\|_{p,\Phi}\right].$$

Let $M \in \mathscr{B}(F,F_0)$, $S \in L_{p,\Phi}(E,F)$ and $K \in \mathscr{B}(E_0,E)$. Then,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(MLK)\right)^p \leq \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)\,\|R\|\,s_t(L)\,\|K\|\right)^p$$
$$\leq \|R\|^p\,\|T\|^p\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(L)\right)^p < \infty.$$

So $MLK \in L_{p,\Phi}(E_0,F_0)$. Furthermore,

$$\|MLK\|_{p,\Phi} = \frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(MLK)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}$$
$$\leq \|R\|\,\|K\|\,\frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(L)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}} = \|R\|\,\|\,\|L\|_{p,\Phi}\,\|K\|\,.$$

Therefore, $L_{p,\Phi}(E,F)$ is an operator ideal, and $\|K\|_{p,\Phi}$ is a quasi-norm on this operator ideal. □

Theorem 2.2. *Let* $1 < p < \infty$. $\left[L_{p,\Phi}(E,F), \|K\|_{p,\Phi}\right]$ *be a quasi-Banach operator ideal.*

Proof. Let E,F be any two Banach spaces and $1 \leq p < \infty$. The following inequality holds

$$\|K\|_{p,\Phi} = \left[\frac{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(K)\right)^p}{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}} \geq \|K\| = s_1(K)$$

for $K \in L_{p,\Phi}(E,F)$.

Let (K_m) be a Cauchy sequence in $L_{p,\Phi}(E,F)$. Then, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|K_m - K_l\|_{p,\Phi} < \varepsilon \tag{2.1}$$

for all $m,l \geq n_0$. It follows that

$$\|K_m - K_l\| \leq \|K_m - K_l\|_{p,\Phi} < \varepsilon.$$

Then, (K_m) is a Cauchy sequence in $\mathscr{B}(E,F)$. $\mathscr{B}(E,F)$ is a Banach space since F is a Banach space. Therefore, $\|K_m - K\| \to 0$ as $m \to \infty$ for $K \in \mathscr{B}(E,F)$. Now we show that $\|K_m - K\|_{p,\Phi} \to 0$ as $m \to \infty$ for $K \in L_{p,\Phi}(E,F)$.
The operators $K_l - K_m$, $K - K_m$ are in the class $\mathscr{B}(E,F)$ for $K_m, K_l, K \in \mathscr{B}(E,F)$. Then,

$$|s_n(K_l - K_m) - s_n(K - K_m)| \le \|K_l - K_m - (K - K_m)\| = \|K_l - K\|.$$

Since $K_l \to T$ as $l \to \infty$ that is $\|K_l - K\| < \varepsilon$ we obtain

$$s_n(K_l - K_m) \to s_n(K - K_m) \text{ as } l \to \infty. \tag{2.2}$$

It follows from (2.1) that the statement

$$\|K_m - K_l\|_{p,\Phi} = \left[\frac{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(K_m - K_l)\right)^p}{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}} < \varepsilon$$

valid for all $m, l \ge n_0$. From (2.2) the following inequality is obtained.

$$\left[\frac{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(K_m - K)\right)^p}{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}} < \varepsilon, \qquad \text{as } l \to \infty.$$

Hence we have

$$\|K_m - K\|_{p,\Phi} < \varepsilon, \text{ for all } m \ge n_0.$$

Finally, we show that $K \in L_{p,\Phi}(E,F)$.

$$\begin{aligned}\sum_{t|n}\varphi(t)s_t(K) &\le \sum_{t|n}\varphi(2t-1)s_{2t-1}(K) + \sum_{t|n}\varphi(2t)s_{2t}(K)\\ &\le \sum_{t|n}\big(\varphi(2t-1)+\varphi(2t)\big)s_{2t-1}(K - K_m + K_m)\\ &\le \sum_{t|n}\mathscr{C}\varphi(t)s_{2t-1}(K - K_m + K_m)\\ &\le \mathscr{C}\left(\sum_{t|n}\varphi(t)s_t(K - K_m) + \sum_{t|n}\varphi(t)s_t(K_m)\right).\end{aligned}$$

By using Minkowski inequality; since $K_m \in L_{p,\Phi}(E,F)$ for all m and $\|K_m - K\|_{p,\Phi} \to 0$ as $m \to \infty$, we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p\right)^{\frac{1}{p}} &\le \mathscr{C}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_k(K - K_m) + \frac{1}{n}\sum_{t|n}\varphi(t)s_k(K_m)\right)^p\right)^{\frac{1}{p}}\\ &\le \mathscr{C}\left[\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_k(K - K_m)\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_k(K_m)\right)^p\right)^{\frac{1}{p}}\right]\\ &< \infty\end{aligned}$$

which means $K \in L_{p,\Phi}(E,F)$. □

Definition 2.3. *Let $\mu = (\mu_i(K))$ be one of the sequences $s = (s_n(K))$, $c = (c_n(K))$, $d = (d_n(K))$, $x = (x_n(K))$, $y = (y_n(K))$ and $h = (h_n(K))$. Then, the space $L_{p,\Phi}^{(\mu)}$ generated via $\mu = (\mu_i(K))$ is defined as*

$$L_{p,\Phi}^{(\mu)}(E,F) = \left\{K \in \mathscr{B}(E,F) : \sum_n\left(\frac{1}{n}\sum_{t|n}\varphi(t)\mu_t(K)\right)^p < \infty, (1 < p < \infty)\right\}.$$

The corresponding norm $\|K\|_{p,\Phi}^{(\mu)}$ for each class is defined as

$$\|K\|_{p,\Phi}^{(\mu)} = \frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)\mu_t(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}.$$

Proposition 2.4. *The inclusion $L^{(a)}_{p,\Phi} \subseteq L^{(a)}_{q,\Phi}$ holds for $1 < p \leq q < \infty$.*

Proof. Since $l_p \subseteq l_q$ for $1 < p \leq q < \infty$ we have $L^{(a)}_{p,\Phi} \subseteq L^{(a)}_{q,\Phi}$. □

Theorem 2.5. *Let $1 < p < \infty$. The quasi-Banach operator ideal $\left[L^{(s)}_{p,\Phi}, \|T\|^{(s)}_{p,\Phi}\right]$ is injective if the sequence $s_r(K)$ is injective.*

Proof. Let $1 < p < \infty$ and $K \in \mathscr{B}(E,F)$ and $\mathscr{J} \in \mathscr{B}(F,F_0)$ be any metric injection. Suppose that $\mathscr{J}K \in L^{(s)}_{p,\Phi}(E,F_0)$. Then,

$$\sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(\mathscr{J}K)\right)^p$$

Since $s = (s_r)$ is injective, we have

$$s_r(K) = s_r(\mathscr{J}K) \text{ for all } K \in \mathscr{B}(E,F), r = 1,2,\ldots. \tag{2.3}$$

Hence, we get

$$\sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p = \sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(\mathscr{J}K)\right)^p < \infty.$$

Thus $K \in L^{(s)}_{p,\Phi}(E,F)$ and we have from (2.3)

$$\|\mathscr{J}K\|_{p,\Phi} = \left[\frac{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(\mathscr{J}T)\right)^p}{\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}}$$

$$= \left[\frac{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p}{\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}} = \|K\|^{(s)}_{p,\Phi}.$$

So, the operator ideal $\left[L^{(s)}_{p,\Phi}, \|K\|^{(s)}_{p,\Phi}\right]$ is injective. □

Corollary 2.6. *It is known that $(c_r(K))$ and $(x_r(K))$ are injective, therefore the quasi-Banach operator ideals $\left[L^{(c)}_{p,\Phi}, \|K\|^{(c)}_{p,\Phi}\right]$ and $\left[L^{(x)}_{p,\Phi}, \|K\|^{(x)}_{p,\Phi}\right]$ are injective [14, p.90-94].*

Theorem 2.7. *Let $1 < p < \infty$. The quasi-Banach operator ideal $\left[L^{(s)}_{p,\Phi}, \|K\|^{(s)}_{p,\Phi}\right]$ is surjective if the sequence $(s_r(K))$ is surjective.*

Proof. Let $1 < p < \infty$ and $K \in \mathscr{B}(E,F)$ and $\mathscr{S} \in \mathscr{B}(E_0,E)$ be any metric surjection. Suppose that $K\mathscr{S} \in L^{(s)}_{p,\Phi}(E_0,F)$. Then,

$$\sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K\mathscr{S})\right)^p < \infty.$$

Since $s = (s_r)$ is surjective, we have

$$s_r(K) = s_r(K\mathscr{S}) \text{ for all } K \in \mathscr{B}(E,F), r = 1,2,\ldots. \tag{2.4}$$

Hence, we get

$$\sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p = \sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K\mathscr{S})\right)^p < \infty.$$

Thus, $K \in L^{(s)}_{p,\Phi}(E,F)$ and we have from (2.4)

$$\|K\mathscr{S}\|^{(s)}_{p,\Phi} = \left[\frac{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K\mathscr{S})\right)^p}{\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}}$$

$$= \left[\frac{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p}{\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}} = \|K\|^{(s)}_{p,\Phi}.$$

Hence, the operator ideal $\left[L^{(s)}_{p,\Phi}, \|K\|^{(s)}_{p,\Phi}\right]$ is surjective. □

Corollary 2.8. *It is known that* $(d_r(K))$ *and* $(y_r(K))$ *are surjective, therefore, quasi-Banach operator ideals* $\left[L^{(d)}_{p,\Phi}, \|K\|^{(d)}_{p,\Phi}\right]$ *and* $\left[L^{(y)}_{p,\Phi}, \|K\|^{(y)}_{p,\Phi}\right]$ *are surjective [14, p.95].*

Theorem 2.9. *The inclusion relations in the sequel hold for* $1 < p < \infty$ *:*

(i) $L^{(a)}_{p,\Phi} \subseteq L^{(c)}_{p,\Phi} \subseteq L^{(x)}_{p,\Phi} \subseteq L^{(h)}_{p,\Phi}$,

(ii) $L^{(a)}_{p,\Phi} \subseteq L^{(d)}_{p,\Phi} \subseteq L^{(y)}_{p,\Phi} \subseteq L^{(h)}_{p,\Phi}$.

Proof. Let $K \in L^{(a)}_{p,\Phi}$. Then,

$$\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) s_t(K)\right)^p < \infty$$

where $1 < p < \infty$. And from Proposition 1.3, we have;

$$\begin{aligned}
\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) h_k(K)\right)^p &\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) x_k(K)\right)^p \\
&\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) c_k(K)\right)^p \\
&\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K)\right)^p \\
&< \infty
\end{aligned}$$

and

$$\begin{aligned}
\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) h_k(K)\right)^p &\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) y_k(K)\right)^p \\
&\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) d_k(K)\right)^p \\
&\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K)\right)^p \\
&< \infty.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.10. *For* $1 < p < \infty$, $L^{(a)}_{p,\Phi}$ *is a symmetric operator ideal, and* $L^{(h)}_{p,\Phi}$ *is a completely symmetric operator ideal.*

Proof. Let $1 < p < \infty$.
Firstly, we show that $L^{(a)}_{p,\Phi}$ is symmetric in other words $L^{(a)}_{p,\Phi} \subseteq \left(L^{(a)}_{p,\Phi}\right)^*$ holds. Let $K \in L^{(a)}_{p,\Phi}$. Then,

$$\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K)\right)^p < \infty.$$

It follows from [12, p.152] $a_r(K^*) \leq a_r(K)$ for $K \in \mathscr{B}$. Hence, we get

$$\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K^*)\right)^p \leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K)\right)^p < \infty.$$

Therefore, $K \in \left(L^{(a)}_{p,\Phi}\right)^*$. Thus, $L^{(a)}_{p,\Phi}$ is symmetric.
Let show that the equation $L^{(h)}_{p,\Phi} = \left(L^{(h)}_{p,\Phi}\right)^*$ is satisfied. It follows from [14, p.97] that $h_r(K^*) = h_r(K)$ for $K \in \mathscr{B}$. Then, we can write

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j\in E_k} v_j h_j(K^*)\right)^p = \sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j\in E_k} v_j h_j(K)\right)^p \text{ and}$$

$$\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) h_k(K^*)\right)^p = \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) h_k(K)\right)^p.$$

Hence, $L^{(h)}_{p,\Phi}$ is completely symmetric. □

Theorem 2.11. *Let* $1 < p < \infty$. $L^{(c)}_{p,\Phi} = \left(L^{(d)}_{p,\Phi}\right)^*$ *and* $L^{(d)}_{p,\Phi} \subseteq \left(L^{(c)}_{p,\Phi}\right)^*$ *holds. Also, for any compact operators* $L^{(d)}_{p,\Phi} = \left(L^{(c)}_{p,\Phi}\right)^*$ *holds.*

Proof. Let $1 < p < \infty$. For $T \in \mathscr{B}$ it is known from [14] that $c_r(K) = d_r(K^*)$ and $c_r(T^*) \leq d_r(K)$. Also, when K is a compact operator, the equality $c_r(K^*) = d_r(K)$ holds. So the proof is complete. □

Theorem 2.12. $L^{(x)}_{p,\Phi} = \left(L^{(y)}_{p,\Phi}\right)^*$ *and* $L^{(y)}_{p,\Phi} = \left(L^{(x)}_{p,\Phi}\right)^*$ *hold.*

Proof. Let $1 < p < \infty$. For $K \in \mathscr{B}$ we have from [14] that $x_n(K) = y_n(K^*)$ and $y_n(K) = x_n(K^*)$. Thus the proof is clear. □

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Lifts of Hypersurfaces on a Sasakian Manifold with a Quartersymmetric Semimetric Connection (QSSC) to Its Tangent Bundle

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Abstract

The aim of the present paper is to introduce a Sasakian manifold immersed with a quarter-symmetric semimetric connection to a tangent bundle. Some basic results are given on a Riemannian connection and a QSSC to the tangent bundle on a Sasakian manifold. The geometrical properties of a Sasakian manifold to its tangent bundle are also discussed.

1. Introduction

A quartersymmetric linear connection with an affine connection ∇ in differentiable manifolds was defined and studied by Golab [1]. Let $\tilde{T}_0$ be a torsion tensor defied as

$$\tilde{T}_0(X_0,Y_0) = u(Y_0)\phi X_0 - u(X_0)\phi Y_0, \tag{1.1}$$

where $u \in \Im_0^1(M), \phi \in \Im_1^1(M)$, then ∇ is known as a quarter symmetric connection.
Several authors made precious contributions to a QSSC including ([2], [3]). Dida et. al. ([4], [5]) studied the geometry of II order tangent bundle and Ricci soliton on the tangent bundle with semisymmetric metric connection. Golden Riemannian structure on tangent bundles were studied and some basic results was proved on it by Peyghan et. al. [6]. Recently, Altunbas et. al. [7] introduced and obtained fundamental results on Ricci soliton on tangent bundles by applying complete lifts. Some theorems on a Lorentzian para-Sasakian manifold with a quartersymmetric metric connection on tangent bundles are determined [8].
In this study, we apply the complete and vertical lifts on tensor fields and connections. The development of the theory of hypersurfaces prolonged to tangent bundle with respect to complete lifts of metric tensor of a Riemannian manifold is attributed to Tani [9]. In 2022, Khan [10] studied submanifolds of a Riemannian manifold endowed with a new type of semi-symmetric non-metric connection in the tangent bundle. Different geometers have studied and defined different types of connections and structures which can be seen in ([11]- [17]).
Lifts of hypersurface from a Sasakian manifold to its tangent bundle connected to a QSSC are examined in the proposed work. Key findings include the following:

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- We proved the induced connection on a Sasakian manifold with QSSC concerning the unit normal is also a QSSC.
- We determined the formula for $\bar{\nabla}^C$ and $\hat{\nabla}^C$ with a QSSC on TM.
- We developed a relation between a QSSC $\mathring{\nabla}^C$ with respect to Riemannian connection ∇^C in $(TS, \tilde{g})$.
- We proved some theorems on geometrical properties with respect to $\mathring{\nabla}^C$ and ∇^C.

2. Preliminaries

Let M (dim= n) be a Riemannian manifold. For $\phi \in \Im^1_1(M), \eta_0 \in \Im^0_1(M), \xi \in \Im^1_0(M)$ fulfilling

$$\phi^2 X = -X_0 + \eta_0(X_0)\xi, \tag{2.1}$$

M is called an almost contact manifold [18] and the structure (ϕ, ξ, η_0) is called an almost contact structure on M. In addition, there exists a metric tensor g satisfying

$$\begin{aligned} g(\phi X_0, \phi Y_0) =& g(X_0, Y_0) - \eta_0(X_0)\eta_0(Y_0), \\ g(X_0, \xi) =& \eta_0(X_0), \end{aligned}$$

then M is called an almost contact metric manifold [19].
The vector field ξ is said to be a Killing vector field if it generates a group of isometries or equivalently if $g(\nabla_{X_0}\xi, Y_0) + g(\nabla_{Y_0}\xi, X_0) = 0$.
If ξ is a Killing vector field then the contact metric manifold (ϕ, ξ, η_0) is called a K-contact structure, and such a manifold is called a K-contact manifold [19]. A K-contact Riemannian manifold (M, g) is called a Sasakian manifold ([2], [20]) if $\forall X_0, Y_0 \in \Im^1_0(M)$, we have

$$(\nabla_{X_0}\phi)(Y_0) = g(Y_0, X_0)\xi - \eta_0(Y_0)X_0. \tag{2.2}$$

Besides the relations (2.1) and (2.2), the following relations also hold in a Sasakian manifold

$$\phi\xi = 0, \ \ \eta_0(\xi) = 1, \ \ \nabla_{X_0}\xi = -\phi X_0, \ \ g(\phi X_0, Y_0) + g(X_0, \phi Y_0) = 0,$$

$\forall X_0, Y_0 \in \Im^1_0(M)$.
The torsion tensor $\hat{T}_0$ with the Levi-Civita connection ∇ and the linear connection $\hat{\nabla}$ is defined as

$$\hat{T}_0(\hat{X}_0, \hat{Y}_0) = \hat{\nabla}_{\hat{X}_0}\hat{Y}_0 - \hat{\nabla}_{\hat{Y}_0}\hat{X}_0 - [\hat{X}_0, \hat{Y}_0], \tag{2.3}$$

$\forall \hat{X}_0, \hat{Y}_0 \in \Im^1_0(M)$.
A QSSC $\overline{\nabla}$ in $(M, \hat{g})$ is defined as [21]

$$\overline{\nabla}_{\hat{X}_0}\hat{Y}_0 = \hat{\nabla}_{\hat{X}_0}\hat{Y}_0 - \hat{\eta}(\hat{X}_0)\hat{\phi}\hat{Y}_0 + \hat{g}(\hat{\phi}\hat{X}_0, \hat{Y}_0), \tag{2.4}$$

which satisfies

$$(\overline{\nabla}_{\hat{X}_0}\hat{g}(\hat{X}_0, \hat{Y}_0) = 2\hat{\eta}(\hat{X}_0)\hat{g}(\hat{Y}_0, \hat{Z}_0) - \hat{\eta}(\hat{Y}_0)\hat{g}(\hat{\phi}\hat{X}_0, \hat{Z}_0) + \hat{\eta}(\hat{Z}_0)\hat{g}(\hat{\phi}\hat{X}_0, \hat{Y}_0), \tag{2.5}$$

$\forall \hat{X}_0, \hat{Y}_0 \in \Im^1_0(M)$, where $\hat{\nabla}$ is a Riemannian connection in $(M, \hat{g})$ and $\hat{P} \in \Im^1_0(M)$ given by $\hat{g}(\hat{P}, \hat{X}_0) = \hat{\eta}(\hat{X}_0)$.

Let TM be the tangent bundle of M. Superscripts C and V denote the complete and vertical lifts of the tensor fields. The following characteristics of these lifts ([10, 22, 23]):

$$\begin{aligned} [\hat{X}_0{}^C, \hat{Y}_0{}^C] =& [\hat{X}_0, \hat{Y}_0]^C \ ; \ \hat{\phi}^C(\hat{X}_0{}^C) = (\hat{\phi}(\hat{X}_0))^C, \\ \hat{\phi}^V(\hat{X}_0{}^C) =& \hat{\phi}^C(\hat{X}_0{}^V) = (\hat{\phi}(\hat{X}_0))^V \ ; \ \hat{\phi}^V(\hat{X}_0{}^V) = 0, \\ \hat{\eta}_0{}^V(\hat{X}_0{}^C) =& (\hat{\eta}_0(\hat{X}_0))^V \ ; \ \hat{\eta}_0{}^C(\hat{X}_0{}^C) = (\hat{\eta}_0(\hat{X}_0))^C, \\ \hat{g}^C(\hat{X}_0{}^V, \hat{Y}_0{}^C) =& \hat{g}^C(\hat{X}_0{}^C, \hat{Y}_0{}^V) = (\hat{g}(\hat{X}_0, \hat{Y}_0))^V, \\ \hat{g}^C(\hat{X}_0{}^V, \hat{Y}_0{}^C) =& (\hat{g}(\hat{X}_0, \hat{Y}_0))^C, \\ \hat{\nabla}^C(\hat{X}_0{}^C, \hat{Y}_0{}^C) =& (\hat{\nabla}(\hat{X}_0, \hat{Y}_0))^C, \\ \hat{\nabla}^C(\hat{X}_0{}^C, \hat{Y}_0{}^V) =& (\hat{\nabla}(\hat{X}_0, \hat{Y}_0))^V, \end{aligned} \tag{2.6}$$

$\forall X_0, Y_0 \in \Im^1_0(M), \eta \in \Im^0_1(M), \phi \in \Im^1_1(M)$.

Let S (dim$=n-1$) be a manifold such that a mapping $B:S\to M$. The tangent map of B represented by $\tilde{B}:T(TS)\to T(TM)$, where $\tilde{B}:TS\to TM$. The hypersurface S is a Riemannian manifold and g is induce metric on S such that

$$g(X_0,Y_0)=\hat{g}(BX_0,BY_0),$$

and

$$\hat{\nabla}_{BX_0}BY_0=B(\nabla_{X_0}Y_0)+h(X_0,Y_0)N, \tag{2.7}$$

$\forall X_0,Y_0\in\Im_0^1(M)$, where $\hat{\nabla}$ is induced connection, N is the unit normal vector field and h is the second fundamental tensor field on (S,g) ([9, 24]). The relation

$$h(X_0,Y_0)=g(HX_0,Y_0), H\in\Im_1^1(S).$$

Definition 2.1 (i) If $h=0$ then S is said to be totally geodesic with respect to ∇.
(ii) If h is proportional to g then S is said to be totally umbilical with respect to ∇ [25].

3. Lifts of a QSSC to the Tangent Bundle on a Sasakian Manifold

Let $\mathring{\nabla}$ is a QSSC induced on the hypersurface S from $\bar{\nabla}$, fulfills

$$\bar{\nabla}_{BX_0}BY_0=B(\mathring{\nabla}_{X_0}Y_0)+h(X_0,Y_0)N, \tag{3.1}$$

$\forall X_0,Y_0\in\Im_0^1(S), m\in\Im_0^2(S)$.
Putting $M=H-\lambda I$, we get the relation

$$m(X_0,Y_0)=g(MX,Y_0),$$

$\forall I\in\Im_1^1(S)$.
Definition 3.1 (i) If $m=0$ then S is said to be totally geodesic with respect to $\mathring{\nabla}$.
(ii) If $m\propto g$ then S is said to be totally umbilical with respect to $\mathring{\nabla}$.
In view of (2.4), we infer

$$\bar{\nabla}_{BX}BY=\hat{\nabla}_{BX}BY-\hat{\eta}_0(BX)B\phi Y_0+\hat{g}(B\phi X_0,BY)\hat{\xi}, \tag{3.2}$$

$\forall X_0,Y_0\in\Im_0^1(S)$.
Using equations (2.7), (3.1) and (3.2), we obtain

$$B(\mathring{\nabla}_{X_0}Y_0)+m(X_0,Y_0)N \quad=\quad B(\nabla_{X_0}Y_0)+h(X_0,Y_0)N-\hat{\eta}_0(BX_0)B\phi Y_0+\hat{g}(B\phi X_0,BY_0)(B\xi+\lambda N),$$

Put $\hat{\xi}=B\xi+\lambda N$, where λ is a function, $\xi\in\Im_0^1(S)$ and $\eta_0\in\Im_1^0(S)$ determined by $\eta_0(X_0)=\hat{\eta}_0(BX_0)$ ([19, 27, 28]).
Comparing the tangential and normal parts from both sides, we infer

$$\begin{aligned}\mathring{\nabla}_{X_0}Y_0 =&\nabla_XY-\eta_0(X_0)\phi Y_0+g(\phi X_0,Y_0)\xi,\\ m(X_0,Y_0) =&h(X_0,Y_0)+\lambda g(\phi X_0,Y_0).\end{aligned}$$

Hence, we state the following:

Theorem 3.1. *The connection induced on a Riemannian manifold's hypersurfaces with a QSSC on a Sasakian manifold with respect to the unit normal is also a QSSC.*

Let $\hat{g}$ be an element of M and the complete lift $\hat{g}^C$ be the element of TM. The induced metric on TS from $\hat{g}^C$ by $\hat{g}$. Then

$$\tilde{g}(X_0^C,Y_0^C)=\hat{g}^C(\tilde{B}X_0^C,\tilde{B}Y_0^C), \forall X_0,Y_0\in\Im_1^0(S).$$

Let Riemannian connection $\hat{\nabla}$ be an element of $(M,\hat{g})$, then $\hat{\nabla}^C$ will be an element of $(TM,\hat{g}^C)$. Let ∇ be an induced connection in (S,g), then ∇^C is an element of $(TM,\tilde{g})$.
We shall first state known results ([6, 29])

Theorem 3.2. *If $\hat{T}_0$ is torsion tensor of $\hat{\nabla}$ in $(M,\hat{g})$, then ${\hat{T}_0}^C$ is torsion tensor of $\hat{\nabla}^C$ in $(TM,\hat{g}^C)$.*

Theorem 3.3. $\forall X_0,Y_0\in\Im_0^1(S)$

$$\begin{aligned}{\hat{\eta}_0}^V(\tilde{B}X_0^C) =&{\hat{\eta}_0}^V(\tilde{B}X_0^{\bar{C}})=\#({\hat{\eta}_0}^V(\tilde{B}X_0^C)=\#(\hat{\eta}_0(\tilde{B}X_0)^V=(\hat{\eta}_0(\tilde{B}X_0)^{\bar{V}},\\ {\hat{\eta}_0}^C(\tilde{B}X_0^C) =&{\hat{\eta}_0}^C(\tilde{B}X_0^{\bar{C}})=\#({\hat{\eta}_0}^V(\tilde{B}X_0^)=\#(\hat{\eta}_0(\tilde{B}X_0)^C=(\hat{\eta}_0(\tilde{B}X_0)^{\bar{C}},\end{aligned}$$

where # represents an operation of restriction and $\bar{C}$ and $\bar{V}$ represent complete and vertical lifts operatons on $\pi_M^{-1}(\tau(S))$. Applying the complete lifts on (2.4) and with the help of (2.6), we infer

$$\begin{aligned}
(\bar{\nabla}_{BX}BY)^{\bar{C}} =& (\hat{\nabla}_{BX}BY)^{\bar{C}} - (\hat{\eta}_0(BX)B\phi Y_0)^{\bar{C}} + (\hat{g}(B\phi X_0, BY)\hat{\xi})^{\bar{C}}\\
(\bar{\nabla}_{BX}BY)^{\bar{C}} =& (\hat{\nabla}_{BX}BY)^{\bar{C}} - (\hat{\eta}_0(BX))^{\bar{C}}(B\phi Y_0)^{\bar{C}} + \hat{\eta}_0(BX))^{\bar{V}}(B\phi Y_0)^{\bar{V}})\\
&+ (\hat{g}(B\phi X_0, BY))^{\bar{C}}\hat{\xi})^{\bar{V}} + (\hat{g}(B\phi X_0, BY))^{\bar{V}}\hat{\xi})^{\bar{C}}\\
\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C =& (\hat{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C - \hat{\eta}_0{}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta}_0{}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C)\\
&+ (\hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}(\phi X_0)^C\hat{\xi})^{\bar{V}} + (\hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}(\phi X_0)^C\hat{\xi})^{\bar{C}}.
\end{aligned}$$

We have

$$\begin{aligned}
\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C - \bar{\nabla}^C_{\tilde{B}Y_0^C}\tilde{B}X_0^C - [X_0^C, Y_0^C] =& - \hat{\eta}_0{}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta}_0{}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C) + \hat{\eta}_0{}^C(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^V)\\
&+ \hat{\eta}_0{}^V(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^C).
\end{aligned}$$

From equation (2.3) and Theorem 3.2, we get

$$\bar{T}^C(BX^C, BY^C) = \hat{\eta}_0{}^C(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^V) + \hat{\eta}_0{}^V(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^C) - \hat{\eta}_0{}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta}_0{}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C). \quad (3.3)$$

Now,

$$\begin{aligned}
\hat{g}^C(\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C, \tilde{B}Z_0^C) + \hat{g}^C(\tilde{B}Y_0^C, \bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Z_0^C) =& \hat{g}^C(\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C - \hat{\eta}_0{}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta}_0{}^C(\tilde{B}Y_0^C)(\tilde{B}(\phi X_0)^V)\\
&+ \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)\hat{\xi}^{\bar{V}} + \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C)\hat{\xi}^{\bar{C}}, \tilde{B}Z_0^C)\\
&+ \hat{g}^C(\tilde{B}Y_0, \hat{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Z_0^C - \hat{\eta}_0{}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Z_0)^V)\\
&- \hat{\eta}_0{}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Z_0)^C) + \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Z_0^C)\hat{\xi}^{\bar{V}}\\
&+ \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Z_0^C)\hat{\xi}^{\bar{C}})\\
=& \hat{g}^C(\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C, \tilde{B}Z_0^C) + \hat{g}^C(\tilde{B}Y_0^C, \bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Z_0^C)\\
&+ (\hat{\eta}_0(\tilde{B}Z_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^C\\
&+ (\hat{\eta}_0(\tilde{B}Z_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^V + (\hat{\eta}_0(\tilde{B}Y_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^C\\
&+ (\hat{\eta}_0(\tilde{B}Y_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^V\\
=& (\tilde{B}X_0^C)\hat{g}^C(\tilde{B}Y_0^C, \tilde{B}Z_0^C) + (\hat{\eta}_0(\tilde{B}Z_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^C\\
&+ (\hat{\eta}_0(\tilde{B}Z_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^V + (\hat{\eta}_0(\tilde{B}Y_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^C\\
&+ (\hat{\eta}_0(\tilde{B}Y_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^V.
\end{aligned}$$

On solving, we get

$$\begin{aligned}
\hat{g}^C(\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C, \tilde{B}Z_0^C) =& (\hat{\eta}_0(\tilde{B}Z_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^C\\
+& (\hat{\eta}_0(\tilde{B}Z_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Y_0))^V\\
+& (\hat{\eta}_0(\tilde{B}Y_0))^V(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^C\\
+& (\hat{\eta}_0(\tilde{B}Y_0))^C(\hat{g}(\tilde{B}\phi X_0, \tilde{B}Z_0))^V. \quad (3.4)
\end{aligned}$$

Hence, we state the following:

Theorem 3.4. *Let $\bar{\nabla}$ be a QSSC with respect to $\hat{\nabla}$ in $(M, \hat{g})$ that fulfills equations (2.4) and (2.5). Then the QSSC $\bar{\nabla}^C$ on a Sasakian manifold with respect to $\hat{\nabla}$ in $(TM, \hat{g}^C)$ is represented by (3.4).*

Now, applying the complete lifts on (2.4) and with the help of (2.6), we infer

$$\begin{aligned}
(\bar{\nabla}_{BX}BY)^{\bar{C}} =& (\hat{\nabla}_{BX}BY)^{\bar{C}} - (\hat{\eta}_0(BX)(B\phi Y_0))^{\bar{C}} + (\hat{g}(B\phi X_0, BY)\hat{\xi})^{\bar{C}},\\
(\bar{\nabla}_{BX}BY)^{\bar{C}} =& (\hat{\nabla}_{BX}BY)^{\bar{C}} - (\hat{\eta}_0(BX)^{\bar{C}}(B\phi Y_0)^{\bar{V}} + \hat{\eta}_0(BX)^{\bar{V}}(B\phi Y_0)^{\bar{C}}) + (\hat{g}(B\phi X_0, BY))^{\bar{C}}\hat{\xi})^{\bar{V}} + (\hat{g}(B\phi X_0, BY))^{\bar{V}}\hat{\xi})^{\bar{C}},\\
\bar{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C =& \hat{\nabla}^C_{\tilde{B}X_0^C}\tilde{B}Y_0^C - \hat{\eta}_0{}^C(\tilde{B}X_0^C)(\tilde{B}\phi Y_0^V) + \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C))\hat{\xi})^{\bar{V}} + \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C))\hat{\xi})^{\bar{C}},
\end{aligned}$$

for arbitrary vector fields X_0 and Y_0 in S. Hence, from the equation (2.7) and the equation (3.1), we get

$$\begin{aligned}
(B(\mathring{\nabla}_X Y) + m(X_0, Y_0)N)^{\bar{C}} =& (B(\nabla_X Y) + h(X_0, Y_0)N)^C - \hat{\eta}_0{}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta}_0{}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C)\\
&+ \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C))(\tilde{B}\hat{\xi})^V + \lambda^V N^{\bar{V}}) + \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C))(\tilde{B}\hat{\xi})^C + \lambda^C N^{\bar{C}} + \lambda^C N^V),
\end{aligned}$$

$$\tilde{B}(\mathring{\nabla}_X Y)^C = \tilde{B}(\nabla_X Y)^C - \hat{\eta}_0{}^C(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^V) - \hat{\eta}_0{}^V(\tilde{B}X_0^C)(\tilde{B}(\phi Y_0)^C) + \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C))(\tilde{B}\hat{\xi})^V + \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C))(\tilde{B}\hat{\xi})^C. \tag{3.5}$$

$$m^V(X_0^C, Y_0^C)N^{\bar{C}} + m^C(X_0^C, Y_0^C)N^{\bar{V}} = h^V(X_0^C, Y_0^C)N^{\bar{C}} + h^C(X_0^C, Y_0^C)N^{\bar{V}}) + \lambda^V \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)N^{\bar{V}}) + \lambda^c \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)N^{\bar{C}}) + \lambda^C \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C)N^{\bar{V}}). \tag{3.6}$$

$$(\mathring{\nabla}_X Y)^C = (\nabla_X Y)^C - \eta_0^C(X_0^C)(\phi Y_0)^V - \eta_0^V(X_0^C)(\phi Y_0)^C + \tilde{g}((\phi X_0)^C, Y_0^C)\xi^V + \tilde{g}((\phi X_0)^V, Y_0^C)\xi^C, \mathring{\nabla}^C_{X_0^C} Y_0^C \tag{3.7}$$

$$= \nabla^C_{X_0^C} Y_0^C - \eta_0^C(X_0^C)(\phi Y_0)^V - \eta_0^V(X_0^C)(\phi Y_0)^C + \tilde{g}((\phi X_0)^C, Y_0^C)\xi^V + \tilde{g}((\phi X_0)^V, Y_0^C)\xi^C \tag{3.8}$$

We have

$$\mathring{\nabla}^C_{X_0^C} Y_0^C - \mathring{\nabla}^C_{Y_0^C} X_0^C - [X_0^C, Y_0^C] = \eta_0^C(Y_0^C)(\phi X_0)^V + \eta_0^V(Y_0^C)(\phi X_0)^C - \eta_0^C(X_0^C)(\phi Y_0)^V - \eta_0^V(X_0^C)(\phi Y_0)^C$$

Similarly,

$$\tilde{g}(\mathring{\nabla}^C_{X_0^C} Y_0^C, Z_0^C) = X_0^C(\tilde{g}(Y_0^C, Z_0^C)) + (\eta_0(Z_0))^V \tilde{g}((\phi X_0)^C, Y_0^C) + (\eta_0(Z_0))^C \tilde{g}((\phi X_0)^V, Y_0^C) + (\eta_0(Y_0))^V \tilde{g}((\phi X_0)^C, Z_0^C) + (\eta_0(Y_0))^C \tilde{g}((\phi X_0)^C, Z_0^V),$$

$$\mathring{\nabla}^C_{X_0^C}\tilde{g})(Y_0^C, Z_0^C) = (\eta_0(Z_0))^V \tilde{g}((\phi X_0)^C, Y_0^C) + (\eta_0(Z_0))^C \tilde{g}((\phi X_0)^V, Y_0^C) + (\eta_0(Y_0))^V \tilde{g}((\phi X_0)^C, Z_0^C) + (\eta_0(Y_0))^C \tilde{g}((\phi X_0)^V, Z_0^C). \tag{3.9}$$

Hence, we state the following:

Theorem 3.5. *Let $\mathring{\nabla}$ be a QSSC with respect to ∇ in (S, g). Then the QSSC $\mathring{\nabla}^C$ on a Sasakian manifold with respect to ∇^C in $(TS, \tilde{g})$ is represented by (3.9).*

The QSSC $\mathring{\nabla}^C$ on $(TS, \tilde{g})$ is defined as

$$\mathring{\nabla}^C_{X_0^C} Y_0^C = \nabla^C_{X_0^C} Y_0^C - \eta_0^C(X_0^C)(\phi Y_0)^V - \eta_0^V(X_0^C)(\phi Y_0)^C + \tilde{g}((\phi X_0)^C, Y_0^C)\xi^V + \tilde{g}((\phi X_0)^V, Y_0^C)\xi^C.$$

On applying the complete lifts of (3.1), we get

$$\bar{\nabla}^C_{\tilde{B}X_0^C} \tilde{B}Y_0^C = \tilde{B}(\mathring{\nabla}^C_{X_0^C} Y_0^C) + m^V(X_0^C, Y_0^C)N^{\bar{C}} + m^C(X_0^C, Y_0^C)N^{\bar{V}}.$$

From the equation (3.6), we acquire

$$m^V(X_0^C, Y_0^C) = h^V(X_0^C, Y_0^C) + \lambda^C \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C)$$
$$m^C(X_0^C, Y_0^C) = h^V(X_0^C, Y_0^C) + \lambda^V \hat{g}^C(\tilde{B}(\phi X_0)^C, \tilde{B}Y_0^C) + \lambda^C \hat{g}^C(\tilde{B}(\phi X_0)^V, \tilde{B}Y_0^C)N^{\bar{V}}).$$

Thus, TS is totally umbilical iff

$$m^V(\tilde{X}_0, \tilde{Y}_0) = \delta \tilde{g}(\tilde{X}_0, \tilde{Y}_0),$$
$$m^C(\tilde{X}_0, \tilde{Y}_0) = \mu \tilde{g}(\tilde{X}_0, \tilde{Y}_0),$$

$\forall X_0, Y_0 \in \Im_0^1(S)$, where δ and μ are differentiable functions. If $\delta = \mu = 0$, then TS is totally geodesic.
Hence, we state the following:

Theorem 3.6. *TS is totally umbilical corresponding to the QSSC $\mathring{\nabla}^C$ on a Sasakian manifold iff it is totally umbilical or totally geodesic with respect to ∇^C.*

4. Conclusion

We introduced and studied a Sasakian manifold immersed with a QSSM connection to the tangent bundle and some fundamental results are obtained of it. Certain theorems on geometrical properties like totally umbilical, totally geodesic on a Sasakian manifold on the tangent bundle are proved.

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