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Preface

Seventh International Conference of Mathematical Sciences (ICMS 2023) Maltepe University, Istanbul-Turkey

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The International Conference of Mathematical Sciences 2023 (ICMS 2023) was scheduled to take place both face-to-face and online (Blackboard conference system) at Maltepe University, Istanbul, Turkey between July 05-09, 2023.

The aim of the conference is to bring together leading scientists of the International Mathematical Sciences community and attract researchers to present their original high quality research manuscripts.

Elenen special sessions were paralelly taken place after the plenary talks in the Seventh International Conference of Mathematical Sciences (ICMS 2023) both face-to-face and via Blackboard online conference system of Maltepe University, Istanbul-Turkey. High quality papers in this proceedings have been chosen from the presentations which had been included in the following sessions

SESSIONS

0) Title of Session: "Plenary" organized by Huseyin Cakalli,
1) Title of Special Session: “Topology” organized by Ljubisa D. R. KOCINAC, and Osman MUCUK,
2) Title of Special Session: “Analysis and Functional Analysis” organized by Hacer SENGUL KANDEMIR, and Nazlım Deniz ARAL
3) Title of Special Session: “Sequences, series, summability” organized by Ibrahim CANAK, and Sefa Anıl SEZER
4) Title of Special Session: “Fixed Point Theory” organized by Duran TURKOGLU, and Hakan SAHIN,
5) Title of Special Session: "Numerical Functional Analysis" organized by Allaberen ASHYRALYEV , and Charyyar ASHYRALYYEV
6) Title of Special Session: "Computer Science and Technology" organized by Sahin UYAVER, and Önder ŞAHİNASLAN

7) Title of Special Session: “Mathematical Methods in Physics” organized by Özay GÜRTUĞ, and Filiz CAGATAY UCGUN

8) Title of Special Session: “Applied Statistics” organized by Mujgan TEZ, and Kadri Ulas AKAY

9) Title of Special Session: "Differential Geometry" organized by Zerrin ŞENTÜRK

10) Title of Special Session: "Algebra" organized by Leyla BUGAY

11) Title of Special Session: "Fundamentals of Mathematics and Mathematical Logic" organized by Tahsin ÖNER and İbrahim ŞENTÜRK .

The selection of the papers included in this issue was based on an international peer review procedure by independently researchers, namely, each special session organizator acted as an area editor by assigning at least two referees.

We would like to acknowledge to thank:

- Special thanks to Hüseyin ŞİMŞEK (Founder of Maltepe University, Chairman of the Board of Trustees,Turkey), and Edibe SÖZEN, (Rector, Maltepe University, Turkey,
- The scientific Committee of ICMS 2023 for their valuable suggestions and reviews (See the conference details),
- Sessions organizers for their considerable work, as well as for their valuable suggestions on the further development of the conference.
- The distinguished plenary speakers for their difficult work and for their acceptance to give invited lectures on their respective fields of expertise,
- The Organizing Committee for their difficult task.

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ON THE BEHAVIORS OF SOLUTIONS IN LINEAR NONHOMOGENEOUS DELAY DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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Abstract. This paper deals with the behaviors of solutions for linear nonhomogeneous delay differential equations. In this study, a periodic solution, an asymptotic result and a useful exponential estimate of the solutions are established. Our results are obtained by the use of real roots of the corresponding characteristic equation.

1. Introduction and Preliminaries

The delay differential equation is considered as:

$$x'(t) = a(t)x(t) + \sum_{i\in I} b_i(t)x(t-\tau_i) + f(t), \quad t \geq 0, \tag{1.1}$$

$$x(t) = \phi(t)\ , \quad -\tau \leq t \leq 0. \tag{1.2}$$

where I is the initial segment of natural numbers, a and b_i for $i \in I$ the continuous real-valued functions on the interval $[0,\infty)$, f is a continuous real-valued function on the interval $[0,\infty)$, and τ_i for $i \in I$ positive real numbers with $\tau_{i_1} \neq \tau_{i_2}$ for $i_1, i_2 \in I$ such that $i_1 \neq i_2$. Suppose that the functions b_i for $i \in I$ are not identically zero on $[0,\infty)$ and also the coefficients a and b_i for $i \in I$ are the periodic functions with a common period $T > 0$ where $\tau_i = m_i T$ for positive integers m_i for $i \in I$. τ is positive number such that

$$\tau = \max_{i\in I} \tau_i.$$

ϕ is continuous real-valued given *the initial function* on the interval $[-\tau, 0]$.

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In the case where the function f is identically zero on the interval $[0,\infty)$, the delay differential equation (1.1) reduces to

$$x'(t) = a(t)x(t) + \sum_{i\in I} b_i(t)x(t-\tau_i), \quad t \ge 0. \tag{1.3}$$

As far as the applications' point of view is concerned, our literature review comprehensively offers the behaviors based on the solutions of delay differential equations [1-6]. As it concers the applications point view, first order linear delay differential equations appear as models in various problems in science and tecnology. For example, in [7], first order linear delay differential equations have been used for description of different economic processes. For the basic theory of delay differential equations with periodic coefficients, the reader is referred to the books by Farkas [8].

Our aim in this article is to obtain periodic solutions of the given equation, and to present some new results on asymptotic behavior for linear delay differential equations with periodic coefficients. Our results are motivated by those in two excellent papers by Philos [9] and Farkas [11]. The very recent results given by Philos [9] (and also [10]) for periodic first order linear (homogeneous) delay differential equations can be obtained from the results of the present paper. Also, the results given here contain essentially ones obtained by Farkas [11] for the particular case of first order linear nonhomogeneous one constant delay differential equations. Our results are derived by the use of a real root (with an appropriate property) of the corresponding (in a sense) characteristic equation. A combination of several methods [6, 9-11] are referred for the used techniques.

The function $x(t)$ is described as a *solution* of the initial value problem (1.1)-(1.3) on $[-\tau,\infty)$. This paper uses the notation

$$A = \frac{1}{T}\int_0^T a(t)dt, \quad \text{and} \quad B_i = \frac{1}{T}\int_0^T b_i(t)dt \quad \text{for} \quad i \in I.$$

Furthermore, we associate the following equation with the differential equation (1.3)

$$\lambda = A + \sum_{i\in I} B_i e^{-\lambda\tau_i}, \tag{1.4}$$

specified as the *characteristic equation* of (1.3). There were given sufficient conditions to obtain a unique real root of characteristic equation (1.4) in Philos [9].

In what follows, the T-periodic extensions are denoted by $\tilde{a}$ and $\tilde{b_i}$ for $i \in I$ for the coefficients a and b_i for $i \in I$ respectively on the interval $[-\tau,\infty)$. In order to construct a suitable mapping for the asymptotic criterion of the solutions, we should reach a finding as follows. Suppose that λ_0 is a real root of (1.4). We can now write

$$h_{\lambda_0}(t) = \tilde{a}(t) + \sum_{i\in I} \tilde{b_i}(t)e^{-\lambda_0\tau_i} \quad \text{for} \quad t \ge -\tau. \tag{1.5}$$

Next, we will establish some equalities needed below. For each index $i \in I$, we can use the assumption that the functions $\tilde{b_i}$ are T-periodic and that $\tau_i = m_iT$ to obtain for $t \geq 0$

$$\int_{t-\tau_i}^{t} \tilde{b}_i(u)du = \int_0^{\tau_i} b_i(u)du = \left[\frac{1}{\tau_i}\int_0^{\tau_i} b_i(u)du\right]\tau_i = \left[\frac{1}{T}\int_0^{T} b_i(u)du\right]\tau_i = B_i\tau_i. \tag{1.6}$$

In a similar manner, one can verify that

$$\int_{t-\tau_i}^{t} |\tilde{b}_i(u)|du = |B_i|\tau_i \quad \text{for every } t \geq 0 \quad \text{and all} \;\; i \in I. \tag{1.7}$$

Our aim in this paper is to study the periodic solutions of equation (1.1) when f is also T-periodic. We will show that, under certain conditions, equation (1.1) has periodic solutions. In the following discussion, without specific mention, we always assume that f is also T-periodic.

2. Periodic Solutions

In this section, we establish conditions under which equation (1.1) has a periodic solution. Consider, first, the homogeneous equation (1.3) and the equation without delay

$$x'(t) = a(t)x(t). \tag{2.1}$$

The general solution of (2.1) is

$$x(t) = c\; exp\left\{\int_0^t a(s)ds\right\}$$

where c is a constant. To find a solution of (1.3), we apply the variation of constants formula. Assume that

$$x(t) = C(t)\; exp\left\{\int_0^t \tilde{a}(s)ds\right\} \tag{2.2}$$

where

$$\tilde{a}(t) = \begin{cases} a(t)\,, & t \geq 0, \\ a(t+\tau)\,, & -\tau \leq t \leq 0, \end{cases}$$

is a solution of (1.3). Substituting this into (1.3) yields the condition

$$C'(t) = \sum_{i\in I} b_i(t)C(t-\tau_i)exp\left\{-\int_{-\tau_i}^{0} \tilde{a}(s)ds\right\} \tag{2.3}$$

for all $t \geq 0$ on $C(t)$. We define

$$g(t) = \sum_{i\in I} \tilde{b}_i(t),$$

where

$$\tilde{b}_i(t) = \begin{cases} b_i(t)\,, & t \geq 0, \\ b_i(t+\tau)\,, & -\tau \leq t \leq 0. \end{cases}$$

Assume that (2.3) has a solution of the form

$$C(t) = exp\left\{\mu\int_0^t g(s)ds\right\}. \tag{2.4}$$

Then, by using (2.4) in (2.3) for $t \geq 0$ we obtain

$$\mu\sum_{i\in I} b_i(t) = \sum_{i\in I} b_i(t)exp\left\{-\mu\int_{t-\tau_i}^{t} g(s)ds\right\} exp\left\{-\int_{-\tau_i}^{0} \tilde{a}(s)ds\right\}.$$

Since the functions $b_i(t)$ are T-periodic, from the last equation

$$\mu \sum_{i \in I} b_i(t) = \sum_{i \in I} b_i(t) exp \left\{ -\mu \int_{-\tau_i}^{0} g(s) ds \right\} exp \left\{ - \int_{-\tau_i}^{0} \tilde{a}(s) ds \right\}$$

or

$$\mu \sum_{i \in I} b_i(t) = \sum_{i \in I} b_i(t) exp \left\{ - \int_{0}^{\tau_i} (a(s) + \mu g(s))\, ds \right\}. \tag{2.5}$$

Next, for each index $i \in I$, we can use the assumption that the functions a and b_i are T-periodic and that $\tau_i = m_i T$ to obtain for $t \geq 0$

$$\begin{aligned} \int_{0}^{\tau_i} (a(s) + \mu g(s))\, ds &= \left[\frac{1}{\tau_i} \int_{0}^{\tau_i} (a(s) + \mu g(s))\, ds \right] \tau_i = \left[\frac{1}{T} \int_{0}^{T} (a(s) + \mu g(s))\, ds \right] \tau_i \\ &= \left\{ \left[\frac{1}{T} \int_{0}^{T} a(s) ds \right] + \mu \sum_{i \in I} \left[\frac{1}{T} \int_{0}^{T} b_i(s) \right] \right\} \tau_i \\ &= \left(A + \mu \sum_{i \in I} B_i \right) \tau_i. \end{aligned}$$

Thus, from (2.5) we get

$$\mu \sum_{i \in I} b_i(t) = \sum_{i \in I} b_i(t) exp \left\{ - \left(A + \mu \sum_{i \in I} B_i \right) \tau_i \right\}. \tag{2.6}$$

If we assume that $\sum_{i \in I} b_i(t) \neq 0$ for $t \geq -\tau$ and $A + \mu \sum_{i \in I} B_i = 0$ hold with $\mu = 1$, (2.6) establishes

$$C(t) = exp \left\{ \int_{0}^{t} \sum_{i \in I} \tilde{b}_i(s) ds \right\}$$

is a solution of (2.3). Hence, from (2.2)

$$x(t) = k\ exp \left\{ \int_{0}^{t} \left(\tilde{a}(s) + \sum_{i \in I} \tilde{b}_i(s) \right) ds \right\}, \tag{2.7}$$

where k is a constant, is a solution of equation (1.3). Also, since $A + \sum_{i \in I} B_i = 0$, it is easy to see that

$$\int_{0}^{\sigma} \left(a(s) + \sum_{i \in I} b_i(s) \right) ds = 0,$$

where $\sigma = \min_{i \in I} \tau_i$. Then, (2.7) is a σ-periodic solution of equation (1.3).

Now, consider the original nonhomogeneous equation (1.1). The variation of constants formula is applied again. Assume that (1.1) has a solution of the form

$$x_p(t) = K(t)\ exp \left\{ \int_{0}^{t} \left(\tilde{a}(s) + \sum_{i \in I} \tilde{b}_i(s) \right) ds \right\}. \tag{2.8}$$

Using $A + \sum_{i \in I} B_i = 0$, substituting this into (1.1) yields the condition

$$K'(t) + \sum_{i \in I} b_i(t) \left(K(t) - K(t - \tau_i) \right) = f(t) exp \left\{ \int_{0}^{t} - \left(a(s) + \sum_{i \in I} b_i(s) \right) ds \right\}.$$

The equation (2.8) is a periodic solution of (1.1) if and only if $K(t)$ is periodic. But, this means that $K(t)-K(t-\tau_i)=0$, and so the differential equation for K is

$$K'(t)=f(t)exp\left\{\int_0^t-\left(a(s)+\sum_{i\in I}b_i(s)\right)ds\right\}.$$

It follows that

$$K(t)=\int_0^t f(u)exp\left\{\int_0^u-\left(a(s)+\sum_{i\in I}b_i(s)\right)ds\right\}du.$$

By noting that this function is the integral of a σ-periodic function, we see that it is a σ-periodic function if and only if

$$\int_0^\sigma f(u)exp\left\{\int_0^u-\left(a(s)+\sum_{i\in I}b_i(s)\right)ds\right\}du=0.$$

Substituting this into (2.8), we have the following result.

Theorem 2.1. *Assume that*

$$\sum_{i\in I}b_i(t)\neq 0 \quad for \quad t\geq -\tau,$$

$$A+\sum_{i\in I}B_i=0$$

where $A=\frac{1}{T}\int_0^T a(t)dt, \quad B_i=\frac{1}{T}\int_0^T b_i(t)dt$, *and suppose that*

$$\int_0^\sigma f(u)exp\left\{\int_0^u-\left(a(s)+\sum_{i\in I}b_i(s)\right)ds\right\}du=0,$$

where $\sigma=\min_{i\in I}\tau_i$. *Then, for each* $c\in\mathbb{R}$,

$$x(t)=c\;exp\left\{\int_0^t\left[a(s+\tau)+\sum_{i\in I}b_i(s+\tau)\right]ds\right\}+x_p(t)\quad for\quad t\geq-\tau,$$

where

$$\begin{aligned}x_p(t)=&exp\left\{\int_0^t\left[a(s+\tau)+\sum_{i\in I}b_i(s+\tau)\right]ds\right\}\\&\times\left\{\int_0^t f(u)exp\left[\int_0^u-\left(a(s+\tau)+\sum_{i\in I}b_i(s+\tau)\right)ds\right]du\right\}\end{aligned}$$

is a σ*-periodic solution of equation* (1.1).

Example 2.2. Consider

$$x'(t)=-2x(t)+(1-sint)x(t-2\pi)+(1+cost)x(t-4\pi)+sint-cost,\quad t\geq 0. \tag{2.9}$$

Since $A=\frac{1}{2\pi}\int_0^{2\pi}(-2)dt=-2$, $B_1=\frac{1}{2\pi}\int_0^{2\pi}(1-sint)dt=1$ and $B_2=\frac{1}{2\pi}\int_0^{2\pi}(1+cost)dt=1$, we have $A+B_1+B_2=0$. Also

$$\int_0^{2\pi}(sinu-cosu)exp\left\{\int_0^u(sins-coss)\,ds\right\}du=0.$$

Therefore, the conditions of Theorem 2.1 are satisfied. Then, for each $c \in \mathbb{R}$,

$$x(t) = c\, exp\left\{\int_0^t [coss - sins]\, ds\right\} + x_p(t) \quad \text{for} \quad t \geq -4\pi,$$

where

$$x_p(t) = exp\left\{\int_0^t [coss - sins]\, ds\right\}\left\{\int_0^t (sinu - cosu) exp\left[\int_0^u -(coss - sins)\, ds\right] du\right\}$$

or

$$x(t) = (c-1)\, exp\left\{sint + cost - 1\right\} + 1 \quad \text{for} \quad t \geq -4\pi$$

is 2π-periodic solution of equation (2.9).

3. An asymptotic result and estimation of solutions

We give a fundamental asymptotic criterion as a theorem to solve the problem (1.1)-(1.2).

Theorem 3.1. *Assume that λ_0 be a real root of the characteristic equation (1.4) and that the root λ_0 satisfies*

$$\mu(\lambda_0) = \sum_{i \in I} |B_i| \tau_i e^{-\lambda_0 \tau_i} + \int_0^\infty |f(u)| \exp\left[-\int_0^u h_{\lambda_0}(s) ds\right] du < 1, \tag{3.1}$$

where h_{λ_0} is defined as in (1.5). Then, for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution x of (1.1)-(1.2) satisfies

$$\lim_{t \to \infty} \left\{x(t) \exp\left[-\int_0^t h_{\lambda_0}(u) du\right]\right\} = \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \tag{3.2}$$

where

$$\begin{aligned} L(\lambda_0; \phi) =& \phi(0) + \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{-\tau_i}^0 \tilde{b}_i(s) \phi(s) \exp\left[-\int_0^s h_{\lambda_0}(u) du\right] ds \\ &+ \int_0^\infty f(u) \exp\left[-\int_0^u h_{\lambda_0}(s) ds\right] du \end{aligned} \tag{3.3}$$

and

$$\beta(\lambda_0) = \sum_{i \in I} B_i \tau_i e^{-\lambda_0 \tau_i}. \tag{3.4}$$

Note: It is guaranteed by the property (2.1) that $0 < 1 + \beta(\lambda_0) < 2$ and $\int_0^\infty f(u) \exp\left[-\int_0^u h_{\lambda_0}(s) ds\right]$ is finite.

Proof. By (3.1), we have $|\beta(\lambda_0)| \leq \mu(\lambda_0) < 1$. So, this yields that $0 < 1 + \beta(\lambda_0) < 2$ and
$-1 < \int_0^\infty f(u) \exp\left[-\int_0^u h_{\lambda_0}(s) ds\right] du < 1$.

Let us define

$$y(t) = x(t) \exp\left[-\int_0^t h_{\lambda_0}(u) du\right] \quad \text{for} \quad t \geq -\tau. \tag{3.5}$$

Then, we obtain for every $t \geq 0$

$$y'(t) = \big(a(t) - h_{\lambda_0}(t)\big)y(t) + \sum_{i \in I} b_i(t)e^{-\lambda_0 \tau_i} y(t - \tau_i) + f(t) \exp\Big[- \int_0^t h_{\lambda_0}(u)du \Big].$$

Thus, using (1.5), the fact that x satisfies (1.1) for all $t \geq 0$ is equivalent to

$$y'(t) = - \sum_{i \in I} b_i(t)e^{-\lambda_0 \tau_i} \big[y(t) - y(t - \tau_i)\big] + f(t) \exp\Big[- \int_0^t h_{\lambda_0}(u)du \Big]. \tag{3.6}$$

Furthermore, the initial condition (1.2) is equivalent to

$$y(t) = \phi(t) \exp\Big[- \int_0^t h_{\lambda_0}(u)du \Big], \quad t \in [-\tau, 0]. \tag{3.7}$$

When equation (3.6) is integrated from 0 to t, by taking into account the fact that the functions $\tilde{b}_i$ for each index $i \in I$ are T-periodic and that the delays τ_i, $i \in I$ are multiples of T, we can verify that (3.6) is equivalent to

$$y(t) = L(\lambda_0; \phi) - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t-\tau_i}^t \tilde{b}_i(s)y(s)ds - \int_t^\infty f(u) \exp\Big[- \int_0^u h_{\lambda_0}(s)ds \Big] du. \tag{3.8}$$

Now, for $t \geq -\tau$ we define

$$z(t) = y(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)}.$$

Hence, from the equation (3.8) it is reduced to the equation as below

$$z(t) = - \sum_{i \in I} e^{-\lambda_0 \tau_i} \int_{t-\tau_i}^t \tilde{b}_i(s)z(s)ds - \int_t^\infty f(u) \exp\Big[- \int_0^u h_{\lambda_0}(s)ds \Big] du \quad \text{for } t \geq 0. \tag{3.9}$$

Moreover, the initial condition (3.7) can be equivalently

$$z(t) = \phi(t) \exp\Big[- \int_0^t h_{\lambda_0}(u)du \Big] - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)}. \tag{3.10}$$

Using y and z, we should prove the equality (3.2), i.e.

$$\lim_{t \to \infty} z(t) = 0. \tag{3.11}$$

Put

$$W(\lambda_0; \phi) = \max\left\{ 1\,, \max_{t \in [-\tau, 0]} \left| \phi(t) \exp\Big[- \int_0^t h_{\lambda_0}(u)du \Big] - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \right| \right\}.$$

Thus, by (3.10) we obtain

$$|z(t)| \leq W(\lambda_0; \phi) \quad \text{for } -\tau \leq t \leq 0. \tag{3.12}$$

Now, the following inequality will be proved

$$|z(t)| \leq W(\lambda_0; \phi) \quad \text{for } t \geq -\tau. \tag{3.13}$$

To this end, let us consider an arbitrary number $\epsilon > 0$. We claim that

$$|z(t)| < W(\lambda_0; \phi) + \epsilon \quad \text{for } t \geq -\tau. \tag{3.14}$$

Otherwise, because of (3.13), there exists a point $t^* > 0$ such that

$$|z(t)| < W(\lambda_0;\phi) + \epsilon \quad \text{for } t \in [-\tau, t^*) \quad \text{and} \quad |z(t^*)| = W(\lambda_0;\phi) + \epsilon.$$

Then, by using (3.1) and (1.7), from (3.9) we obtain

$$\begin{aligned}
W(\lambda_0;\phi) + \epsilon &= |z(t^*)| \\
&= \left| -\sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t^*-\tau_i}^{t^*} \tilde{b}_i(s)z(s)ds - \int_t^\infty f(u)\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \right| \\
&\le \sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t^*-\tau_i}^{t^*} |\tilde{b}_i(s)||z(s)|ds + \int_t^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \\
&\le \left\{ \sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t^*-\tau_i}^{t^*} |\tilde{b}_i(s)|ds + \int_0^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \right\} \\
&\quad \times W(\lambda_0;\phi) + \epsilon \\
&\le \mu(\lambda_0)(W(\lambda_0;\phi) + \epsilon) < W(\lambda_0;\phi) + \epsilon.
\end{aligned}$$

This is a contradiction and so (3.14) holds true. Since (3.14) is satisfied for all $\epsilon > 0$, (3.13) is always fulfilled. Next, in view of (1.7), (3.1) and (3.13), from (3.9) we get for every $t \ge 0$

$$\begin{aligned}
|z(t)| &= \left| -\sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t-\tau_i}^{t} \tilde{b}_i(s)z(s)ds - \int_t^\infty f(u)\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \right| \\
&\le \sum_{i\in I} e^{-\lambda_0\tau_i} \int_{t-\tau_i}^{t} |\tilde{b}_i(s)||z(s)|ds + \int_t^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \\
&\le \left\{ \sum_{i\in I} e^{-\lambda_0\tau_i}|B_i|\tau_i + \int_0^\infty |f(u)|\exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \right\} W(\lambda_0;\phi) \\
&\le \mu(\lambda_0)W(\lambda_0;\phi).
\end{aligned}$$

In other words, we have

$$|z(t)| \le \mu(\lambda_0)W(\lambda_0;\phi) \quad \text{for } \ t \ge 0. \tag{3.15}$$

By (3.1), (3.13) and (3.15), using an easy induction, that z satisfies

$$|z(t)| \le [\mu(\lambda_0)]^n\, W(\lambda_0;\phi) \quad \text{for } \ t \ge n\tau - \tau \ \ (n = 0, 1, \cdots). \tag{3.16}$$

Due to (2.1) , we get $\lim_{n\to\infty} [\mu(\lambda_0)]^n = 0$. Thus, from (3.16) we get

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \left\{ x(t)\exp\left[-\int_0^t h_{\lambda_0}(u)du\right] - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \right\} = 0$$

i.e. (3.2) satisfies. Theorem 3.1 has been already proven. □

Corollary 3.2. *Assume that*

$$a(t) + \sum_{i\in I} b_i(t) = 0 \quad \textit{for} \quad t \in [0,\infty) \tag{3.17}$$

and

$$\sum_{i\in I} |B_i|\tau_i + \int_0^\infty |f(u)|du < 1. \tag{3.18}$$

Thus, the solution x of (1.1)-(1.2) satisfies for any $\phi \in ([-\tau, 0], \mathbb{R})$,

$$\lim_{t\to\infty} x(t) = \frac{\phi(0) + \sum_{i\in I} \int_{-\tau_i}^{0} \tilde{b}_i(s)\phi(s)ds + \int_0^\infty f(u)du}{1 + \sum_{i\in I} B_i\tau_i}.$$

Note: It is guaranteed by (3.18) that $2 > 1 + \sum_{i\in I} B_i\tau_i > 0$.

Proof. It immediately follows from (3.17) that $A+\sum_{i\in I} B_i = 0$ and hence $\lambda_0 = 0$ is a real root of (1.4). By using again (3.18), we see that, for $\lambda_0 = 0$, we have $h_{\lambda_0} = 0$ on the interval $[-\tau, \infty)$. Moreover, (3.18) facilitates the verification of which the root $\lambda_0 = 0$ of (1.4) has the property (2.1). Therefore this can be applied Theorem 3.1. □

Theorem 3.3. *Let λ_0 be a real root of the characteristic equation (1.4) with the property (3.1), and let $h_{\lambda_0}(t)$ and $\beta(\lambda_0)$ are specified by (1.5) and (3.4), respectively. Set*

$$N(\lambda_0) = \frac{(1 + \mu(\lambda_0))^2}{1 + \beta(\lambda_0)} + \mu(\lambda_0). \tag{3.19}$$

Then, for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution x of (1.1)-(1.2) satisfies

$$|x(t)| \le N(\lambda_0)R(\lambda_0; \phi) \exp\left[\int_0^t h_{\lambda_0}(u)du\right], \quad \textit{for all} \quad t \ge 0, \tag{3.20}$$

where

$$R(\lambda_0; \phi) = \max\left\{1\ ,\ \max_{-\tau\le t\le 0} |\phi(t)|\ ,\ \max_{-\tau\le t\le 0}\left[|\phi(t)| \exp\left[-\int_0^t h_{\lambda_0}(u)du\right]\right]\right\}. \tag{3.21}$$

Note: It is guaranteed by the property (2.1) that $0 < 1 + \beta(\lambda_0) < 2$.

Proof. Suppose that x is the solution of (1.1)-(1.2) and y, z are defined as above, i.e. for $t \ge -\tau$

$$y(t) = x(t) \exp\left[-\int_0^t h_{\lambda_0}(u)du\right] \quad \text{and} \quad z(t) = y(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)},$$

where $L(\lambda_0; \phi)$ is defined as in (3.3). Therefore, we specify $W(\lambda_0; \phi)$ as in the proof of Theorem 3.1. Hence, as in Theorem 3.1, it can be also proved that z satisfies inequality (3.15), and thus for $t \ge 0$ we get

$$|y(t)| \le \mu(\lambda_0)W(\lambda_0; \phi) + \frac{|L(\lambda_0; \phi)|}{1 + \beta(\lambda_0)}. \tag{3.22}$$

Using (3.1) and (3.21), from (3.3) we obtain

$$
\begin{aligned}
|L(\lambda_0;\phi)| &\le |\phi(0)| + \sum_{i\in I} e^{-\lambda_0\tau_i}\int_{-\tau_i}^{0} |\tilde{b}_i(s)||\phi(s)| \exp\left[-\int_0^s h_{\lambda_0}(u)du\right]ds \\
&\quad + \int_0^\infty |f(u)| \exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du \\
&\le \left(1 + \sum_{i\in I} e^{-\lambda_0\tau_i}\int_{-\tau_i}^{0} |\tilde{b}_i(s)|ds + \int_0^\infty |f(u)| \exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du\right) \\
&\quad \times R(\lambda_0;\phi) \\
&= \left(1 + \sum_{i\in I} |B(i)|\tau_i e^{-\lambda_0\tau_i} + \int_0^\infty |f(u)| \exp\left[-\int_0^u h_{\lambda_0}(s)ds\right]du\right) \\
&\quad \times R(\lambda_0;\phi) \\
" &= (1+\mu(\lambda_0))R(\lambda_0;\phi).
\end{aligned}
$$

Furthermore, using the definition of $W(\lambda_0;\phi)$ we have

$$
\begin{aligned}
W(\lambda_0;\phi) &\le \max\left\{1\ ,\ R(\lambda_0;\phi) + \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)}\right\} = R(\lambda_0;\phi) + \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} \\
&\le R(\lambda_0;\phi) + \frac{(1+\mu(\lambda_0))R(\lambda_0;\phi)}{1+\beta(\lambda_0)} = \left(1 + \frac{(1+\mu(\lambda_0))}{1+\beta(\lambda_0)}\right) R(\lambda_0;\phi).
\end{aligned}
$$

So, using (3.19) and (3.21), by (3.22) we reach for $t \ge 0$

$$
\begin{aligned}
|y(t)| &\le \mu(\lambda_0)\left(1 + \frac{(1+\mu(\lambda_0))}{1+\beta(\lambda_0)}\right) R(\lambda_0;\phi) + \frac{(1+\mu(\lambda_0))R(\lambda_0;\phi)}{1+\beta(\lambda_0)} \\
&= \left\{\mu(\lambda_0)\left(1 + \frac{(1+\mu(\lambda_0))}{1+\beta(\lambda_0)}\right) + \frac{(1+\mu(\lambda_0))}{1+\beta(\lambda_0)}\right\} R(\lambda_0;\phi) \\
&= N(\lambda_0)R(\lambda_0;\phi).
\end{aligned}
$$

Last of all, using the definition of y we get

$$
|x(t)| \le N(\lambda_0)R(\lambda_0;\phi)\exp\left[\int_0^t h_{\lambda_0}(u)du\right], \quad \text{for all} \quad t \ge 0.
$$

Therefore, this completes the proof of the theorem.

□

Example 3.4. In the following example, we will apply Theorem 3.1 and Theorem 3.3. For simplicity of example we consider the problem as follows:

$$
x'(t) = \left(\frac{1}{3} + \sin 2\pi t\right) x(t) - \left(\frac{1}{3} + \sin 2\pi t\right) x(t-1) - \frac{e^{-t}}{3}, \quad t \ge 0, \tag{3.23}
$$

$$
x(t) = 1\ , \quad -1 \le t \le 0 \tag{3.24}
$$

where $\frac{1}{3} + \sin 2\pi t$ and $-\frac{1}{3} - \sin 2\pi t$ with period $T = 1$. The characteristic equation of the homogeneous equation of (3.23) is from (1.4)

$$
\lambda = \frac{1}{3} - \frac{1}{3}e^{-\lambda}. \tag{3.25}
$$

We have $\lambda_1 \approx -1.9$ and $\lambda_2 = 0$ are real roots of characteristic equation (3.25). Let $\lambda_0 \approx -1.9$. Then, the first term in (3.1) $\frac{e^{1.9}}{3} \approx 2.23$. Therefore, Theorem 3.1 and

Theorem 3.3 cannot be applied to equation (3.23). But, let $\lambda_0 = 0$. We check the condition for Theorem 3.1 as follows: Since $h_{\lambda_0}(t) = 0$, from (3.1) we obtained easily

$$\mu(\lambda_0) = \mu(0) = \frac{1}{3} + \int_0^\infty \frac{e^{-u}}{3} du = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < 1.$$

Therefore, (3.1) is satisfied. Then, from (3.2) and (3.20), the solution x of (3.23) and (3.24) satisfies

$$\lim_{t\to\infty} x(t) = \frac{1 - \int_{-1}^{0} \left(\frac{1}{3} + \sin 2\pi s\right) ds - \int_0^\infty \frac{e^{-t}}{3} du}{1 - \frac{1}{3}} = \frac{3}{2}$$

and

$$|x(t)| \le \left(\frac{(1+2/3)^2}{1-\frac{1}{3}} + \frac{2}{3} \right) = \frac{29}{6}, \quad \text{for all} \quad t \ge 0.$$

4. The Special Case of Linear Nonhomogeneous Delay Differential Equations with Constant Coefficients

In this section, we will consider the special case of first order linear nonhomogeneous delay differential equations with constant coefficients and constant delays. The linear autonomous delay differential equation is a special version of the delay differential equation (1.1)

$$x'(t) = ax(t) + \sum_{i\in I} b_i x(t-\tau_i) + f(t), \quad t \ge 0, \tag{4.1}$$

where a, b_i for $i \in I$ are the real constants, and τ_i for $i \in I$ the positive real numbers with $\tau_{i_1} \neq \tau_{i_2}$ for i_1, i_2 with $i_1 \neq i_2$ and f is a continuous real-valued function on the interval $[0, \infty)$. Let τ be defined by $\tau = \max_{i\in I} \tau_i$. and the initial function be given as in (1.2). The characteristic equation of the homogeneous equation of (4.1) is

$$\lambda = a + \sum_{i\in I} b_i e^{-\lambda\tau_i}. \tag{4.2}$$

There were given sufficient conditions to obtain a unique real root of characteristic equation (4.2) in Philos [2, Chapter 5]. The constant coefficients a and b_i of (4.1) can be considered as T -periodic functions, for each real number $T > 0$. Moreover, as it concerns the autonomous delay differential equation (4.1), the hypothesis that there exists positive integers m_i for $i \in I$ such that $\tau_i = m_i T$ holds by itself. After these observations, it is not difficult to apply the main results of this paper, i.e., Theorem 3.1, Corollary 3.2 and Theorem 3.3, to the special case of the autonomous linear nonhomogeneous delay differential equation (4.1). Because of equation (4.1) is a constant coefficient equation, we needn't to prove below Theorem 4.1 and Theorem 4.3.

Theorem 4.1. *Suppose that λ_0 be a real root of (4.2) with*

$$\mu(\lambda_0) = \sum_{i\in I} |b_i| \tau_i e^{-\lambda_0 \tau_i} + \int_0^\infty |f(u)|\, e^{-\lambda_0 u} du < 1. \tag{4.3}$$

Thus the solution x of the system (4.1) *and* (1.2) *satisfies*

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right] = \frac{L(\lambda_0;\phi)}{1+\sum_{i\in I} b_i\tau_i e^{-\lambda_0\tau_i}},$$

where

$$L(\lambda_0;\phi) = \phi(0) + \sum_{i\in I} b_i e^{-\lambda_0\tau_i}\int_{-\tau_i}^{0}\phi(s)e^{-\lambda_0 s}ds + \int_0^\infty f(u)e^{-\lambda_0 u}du.$$

Note: It is guaranteed by the property (4.3) that $0 < 1+\sum_{i\in I} b_i\tau_i e^{-\lambda_0\tau_i} < 2$.

Application of the Theorem 4.1 with $\lambda_0 = 0$ leads to the following corollary.

Corollary 4.2. *Assume that*

$$a + \sum_{i\in I} b_i = 0 \quad \text{and} \quad \sum_{i\in I}|b_i|\tau_i + \int_0^\infty f(u)du < 1. \tag{4.4}$$

The solution x of the system (4.1) *and* (1.3) *satisfies*

$$\lim_{t\to\infty} x(t) = \frac{\phi(0)+\sum_{i\in I} b_i\int_{-\tau_i}^{0}\phi(s)ds + \int_0^\infty f(u)du}{1+\sum_{i\in I} b_i\tau_i}.$$

Theorem 4.3. *Assume that Theorem 4.1 is satisfied and Let λ_0 be a real root of* (4.2) *satisfying* (4.3) *and set*

$$R(\lambda_0;\phi) = \max\left\{1\ ,\ \max_{-\tau\le t\le 0}|\phi(t)|\ ,\ \max_{-\tau\le t\le 0}\left[e^{-\lambda_0 t}|\phi(t)|\right]\right\}.$$

Thus the solution x of the system (4.1) *and* (1.3) *satisfies*

$$|x(t)| \le N(\lambda_0)R(\lambda_0;\phi)e^{\lambda_0 t} \quad \text{for} \quad t\ge 0,$$

where

$$N(\lambda_0) = \frac{(1+\mu(\lambda_0))^2}{1+\sum_{i\in I} b_i\tau_i e^{-\lambda_0\tau_i}} + \mu(\lambda_0).$$

5. Conclusions

In this study, firstly, we have obtained sufficient conditions for (1.1) to have periodic solutions. Later, we have proved that there is a basic asymptotic criterion for the solutions of the initial value problem (1.1)-(1.2). Finally, using this asymptotic criterion, we obtained a useful exponential boundary for solutions of (1.1)-(1.2). These results were obtained using a suitable real root for the characteristic equation. Namely that, this real root played an important role in establishing the results of the article. We have also presented the application in the special case of constant coefficients of the results obtained. We also gave two examples.

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AN ALTERNATIVE DISCRETE ANALOGUE OF THE HALF-LOGISTIC DISTRIBUTION

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Abstract. A discrete version of the continuous half-logistic distribution is introduced, which is based on the minimization of the Cramér distance between the corresponding continuous and step-wise cumulative distribution functions. The expression of the probability mass function is derived in analytic form and some properties of the distribution are discussed, as well as sample estimation. A comparison is also made with a discrete version already proposed in the literature, which is based on a different rationale. An application to real data is finally presented.

1. Introduction

The half-logistic distribution is a random distribution supported on $\mathbb{R}^+$ obtained by folding the logistic distribution about the origin [1]. Thus, if Y is a random variable (rv) following the logistic distribution with parameter $\theta > 0$, with cumulative distribution function (cdf) $F_Y(y) = \frac{1}{1+e^{-\theta y}}$ and probability density function (pdf) $f_Y(y) = \frac{\theta e^{-\theta y}}{(1+e^{-\theta y})^2}$, the rv $X = |Y|$ follows the the half-logistic distribution with the same parameter θ; its pdf is

$$f(x) = \frac{2\theta e^{-\theta x}}{(1+e^{-\theta x})^2}, \quad x \in \mathbb{R}^+, \theta \in \mathbb{R}^+; \tag{1.1}$$

its cdf is

$$F(x) = \frac{2}{1+e^{-\theta x}} - 1 = \frac{2e^{\theta x}}{e^{\theta x}+1} - 1 = \frac{e^{\theta x}-1}{e^{\theta x}+1} = 1 - \frac{2}{1+e^{\theta x}}, \quad x \in \mathbb{R}^+. \tag{1.2}$$

The expectation is $\mu = \log 4/\theta$. [2] introduced a discrete analogue of the half-logistic distribution, defined through (1.1) or (1.2), by imposing the matching of

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the survival function (sf) $P(X \geq x)$ at each integer value of the support, i.e., defining the probability mass function (pmf) as $p(x) = F(x+1) - F(x)$. The pmf of the discrete analogue of the half-logistic distribution has thus the following expression:

$$p_i = p(i) = 2\left[1 + e^{-\theta(i+1)}\right]^{-1} - 2\left[1 + e^{-\theta i}\right]^{-1}, i = 0, 1, 2, \ldots \tag{1.3}$$

In this paper, we introduce and discuss an alternative discrete version of the continuous half-logistic distribution by following a different approach, based on the minimization of a discrepancy measure between the continuous cdf of the parent distribution and the step-wise cdf of the discrete counterpart [3]. The distance chosen is the Cramér distance, defined as

$$d(F, G) = \int_{\mathbb{R}} |F(x) - G(x)|^2 \mathrm{d}x, \tag{1.4}$$

where F and G are the continuous and step-wise cdf of the continuous random distribution and of its discrete version, respectively. The paper is structured as follows: In the next section, we provide the general solution to the problem stated above and then derive the "optimal" discrete counterpart of the half-logistic distribution, by providing the analytic expression of its pmf and some properties. The third section is devoted to sample estimation and discusses the maximum likelihood method, the method of moment and the method of proportion. The fourth and final section presents an application to a real dataset, on which the proposed discrete distribution is fitted.

2. Definition of an alternative discrete version of the half-logistic distribution

If G is a stepwise cdf, supported on the non-negative integers $i \in \{0, 1, 2, \ldots\}$, which can be seen as a discrete version of a continuous cdf F, supported on the positive half-line, letting $Q_i = G(i)$, the Cramér distance (1.4) can be rewritten as

$$d(F, G) = \sum_{i=0}^{\infty} |F(x) - Q_i|^2.$$

By minimizing the function above with respect to the Q_i's, we obtain the "optimal" values as $Q_i = \int_i^{i+1} F(x)\mathrm{d}x$ [3]. The optimal discrete analogue of the half-logistic distribution has then cumulative probabilities given by

$$Q_i = \int_i^{i+1} \left(1 - \frac{2}{1 + e^{\theta x}}\right) \mathrm{d}x = 1 - 2 + \left[\frac{2\log(1 + e^{\theta x})}{\theta}\right]_i^{i+1} = \frac{2}{\theta} \log \frac{1 + e^{\theta(i+1)}}{1 + e^{\theta i}} - 1,$$

for $i = 0, 1, 2, \ldots$, so that the probabilities are

$$\begin{cases} p_0 = Q_0 = \dfrac{2}{\theta} \log \dfrac{1 + e^{\theta}}{2} - 1 \\ p_i = Q_i - Q_{i-1} = \dfrac{2}{\theta} \log \dfrac{(1 + e^{\theta(i+1)})(1 + e^{\theta(i-1)})}{(1 + e^{\theta i})^2}, \quad i = 1, 2, \ldots. \end{cases} \tag{2.1}$$

It can be proved that $p_0 < p_1$ if θ is smaller than $\theta^* = 2.12255$. Conversely, for any $\theta > \theta^*$, $p_0 > p_1$, whereas if $\theta = \theta^*$, it follows that $p_0 = p_1$. It can be also proved that $p_i > p_{i+1}$ for any $i \geq 1$. As a direct consequence of the two results above, we have that the proposed alternative discrete counterpart of the half-logistic distribution is unimodal with mode equal to 1 if $\theta < \theta^*$, with mode

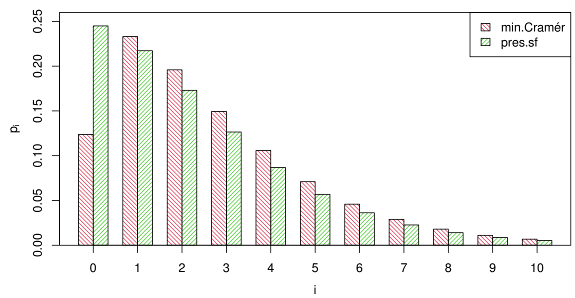

FIGURE 1. Pmf of the proposed discrete counterpart, based on Eq. (2.1), and of the discrete counterpart proposed by [2], Eq. (1.3), based on the preservation of the sf; $\theta = 1/2$.

equal to 0 if $\theta > \theta^*$; it is bimodal with modes at 0 and 1 if $\theta = \theta^*$. This is a very relevant difference with respect to the model of Eq. (1.3), which is unimodal with mode at 0. Figure 1 displays, for the integers 0 to 10, the probabilities for the two models when $\theta = 1/2$. It can be easily shown that the expectation of the alternative discrete half-logistic coincides with that of the parent continuous distribution. This is a general property holding for the discrete counterparts of positive rvs obtained by minimizing the Cramér distance (1.4). In fact, denoting the continuous rv and its optimal counterpart by X and $\tilde{X}$, respectively, and recalling an alternative formulation of the expected value for non-negative rvs, one shows that

$$\mathbb{E}(\tilde{X}) = \sum_{i=0}^{\infty}(1 - Q_i) = \sum_{i=0}^{\infty}\left(1 - \int_i^{i+1} F(x)\mathrm{d}x\right) = \int_0^{\infty}(1 - F(x))\mathrm{d}x = \mathbb{E}(X).$$

Determining if and when the proposed discretization is "better", according to some appropriate criterion, than the usual one, based on the matching of the sf, can the the object of further study.

3. Parameter estimation

Given an iid sample $(x_1, x_2, \ldots, x_n)$ which we assume to come from the alternative discrete half-logistic distribution (2.1), the unknown parameter θ can be estimated by resorting to one of the following methods.

3.1. **Maximum likelihood method.** The maximum likelihood estimate $\hat{\theta}_{ML}$ of θ is the value maximizing the log-likelihood function $\ell(\theta; x_1, \ldots, x_n) = \sum_{i=1}^{n} \log p_{x_i}(\theta)$. Due to the complicated expression of the pmf, it is not possible to derive a closed-form expression of $\hat{\theta}_{ML}$, but any standard optimization routine can be used in order to obtain it numerically.

3.2. **Method of moment.** By equating this expectation of the proposed model to the sample mean $\bar{x} = \sum_{i=1}^{n} x_i/n$, one derives the moment estimate as $\hat{\theta}_M = \log 4/\bar{x}$.

3.3. **Method of proportion.** This method, suitable for discrete distributions, consists in considering an assigned support value and determining the value of the parameter for which the probability of that value equals the corresponding relative sample frequency. One can consider matching the probability of 0, available from (2.1), and the corresponding relative sample frequency of zeros, $\hat{p}_0$. After

Table 1. Distribution of number of outbreaks of strikes, from [4]

count	observed frequency	theoretical frequency
0	46	51.39
1	76	69.69
2	24	25.61
3	9	7.01
(≥)4	1	2.30
total	156	156

simple algebraic steps, one obtains the following equation in $\omega = e^\theta$, $2\omega^{(1+\hat{p}_0)/2} - \omega - 1 = 0$, which yields a unique root $\hat{\omega}_P$ and the corresponding estimate $\hat{\theta}_P = \log \hat{\omega}_P$.

4. A real data example

We consider the dataset presented in [4] and reported in Table 1. Fitting the data through the alternative half-logistic might be plausible, since the mode is 1. The MLE of θ is 1.4238 and using this estimate we reconstruct the theoretical frequencies, which are displayed in the last column of Table 1. Pooling the last two counts (3 and 4), we calculate the usual chi-square statistic, $X^2 = \sum_{i=0}^{3}(n_i - n_i^*)^2/\hat{n}_i$, where n_i and n_i^* are the observed and theoretical frequencies of the count i; its value is 1.2896 and the approximate p-value of the chi-square test is 0.5247, thus indicating a more than satisfactory fit of the model. The maximum value of the log-likelihood function is -188.104; the AIC value is 378.208. All these results, if compared to those of the statistical models analyzed in [5], highlight that the alternative discrete half-logistic distribution has a superior goodness-of-fit.

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ABOUT VARIATIES OF G-SEQUENTAILLY METHODS, G-HULLS AND G-CLOSURES

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Abstract. In the first countable spaces many topological concepts such as open and closed subsets; and continuous functions are defined for convergent sequences. The concept of limit defines a function from the set of all convergence sequences in X to X itself if X is a Hausdorff space. This is extended not only to topological spaces but also to sets. More specifically a *G-method* is defined to be a function defined on a subset of all sequences We say that a sequence $\mathbf{x} = (x_n)$ *G-convergences* to a if $G(\mathbf{x}) = a$. Then many topological objects such as open and closed subsets and many others including these sets have been extended in terms of G-convergence. G-continuity, G-compactness and G-connectedness have been studied by several authors ([1], [2], [3], [4]). On the other hand we know that in a topological space X, a sequence (x_n) converges to a point $a \in X$ if any open neighbourhood of a includes all terms except finite number. Similarly we define a sequence (x_n) to be G-sequentially converging to a if any G-open neighbourhood of a includes almost all terms. In this work provided some examples we indicate that G-convergence and G-sequentially convergence are different. We will prove that G-closed and G-sequentially closed subsets and therefore many others are different.ed.

1. Introduction

Useful tools for defining topological concepts in sequential terms are the convergences of the sequences.

Some authors explored A-continuity for methods of almost convergence and for related approaches, including Savaş and Das [5], Borsik and Salat [6].

The effects of substituting G-methods defined on a subspace of the real sequences for sequential convergence were examined by Connor and Grosse-Erdmann [7]. In order to apply this idea to topological groups, Çakallı extending this concept to

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topological groups, defined G- continuity in [1] (see also [8] for various additional forms of continuities). In [9] Mucuk and Şahan introduced the concepts of G-open sets and G-neighbourhoods in topological groups and looked into additional G-continuity features. Recently, Lin and Liu in [10] proposed the ideas of G-methods, G-submethods and G-topologies for arbitrary sets as well as topological spaces, and they also looked into the operations involving G-hulls, G-closures, G-kernels and G-interiors.

Yongxing and Fucai [11] expanded on several findings and discussed some G-connectedness, G-hull, and G-kernel properties. In [12] Brown and Mucuk studied the covering of disconnected topological groups. In their article [13] L. Liu and Z. Ping proposed the idea of the product G-method on sets, which results in a G-generalized topology. They also talked about the G-connectedness of the Cartesian product. We studied G-connectedness and G-sequential methods for product spaces in the works [14] and [15]. Authors explore the concepts of countably G- compact and sequentially GO-compact spaces in article [16]. The first countable spaces are sequential topological spaces and can be completely characterized by convergent sequences. A subset A of sequential space X is said to be closed, whenever any convergence sequence $\mathbf{x} = (x_n)$ in A has sequential limit in the same subset A. Open subsets in sequential spaces can be also defined in terms of sequences. Subset A is open if and only if any sequence converging to a point $a \in A$ is almost in A.

In [17] some counter examples of convergent G-methods are given; and G-open, G-closed subsets for these G-convergent methods are characterised. The main object of this paper is to define G-methods as G-sequential convergence and then to characterize a variety of G-open, G-closed subsets associated with these G-methods.

2. G-sequential convergence

Throughout the text, the letter X designates a topological space unless otherwise stated. The boldface letters x, y, z,... stand for the sequences of terms $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$,, whereas $s(X)$ and $c(X)$ stand for the sequences of all terms and the sequence of all convergent sequences of points in X, respectively. We define a G-method of sequential convergence for X as a map defined on a subset $c_G(X)$ of $s(X)$ into X. When for $\mathbf{x} \in c_G(X)$ and $G(\mathbf{x}) = \ell$, a sequence $\mathbf{x} = (x_n)$ is said to be *G-convergent* to ℓ. In particular, the G-method with $G = \lim$ is the lim function defined on $c(X)$. When a sequence $\mathbf{x}$ is G-convergent to ℓ, then any subsequence of $\mathbf{x}$ is likewise G-convergent to the same point ℓ, is referred to as the *preservation of the G-convergence of subsequences*. A sequence $\mathbf{x}$ is described as *regular* whenever any convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. We remind that in a topological space X, a sequence $\mathbf{x} = (x_n)$ has limit a if and only if every open neighbourhood of a includes almost all terms of $\mathbf{x} = (x_n)$. Parallel to this, we can define a variety of G-convergence as follows:

For a set X, we say that a sequence $\mathbf{x} = (x_n)$ in X is *G-sequentially convergent* to a point $a \in X$, if every G-open neighbourhood U of a includes almost all terms of the sequence. Note that we here use additionally the word "sequentially" to distinguish from G-convergence. The notion of G-sequentially convergence defined in this manner enables us to obtain a variety of G-open, G-closed subsets and some others. We keep to use the word "sequentially" additionally for these varieties of the notions.

***G-hull* and *G-closed* subsets** : The point $\ell \in X$ is said to be in the *G-hull* of A if the subset A has a sequence $\mathbf{x} = (x_n)$ with $G(\mathbf{x}) = \ell$. A is said to be *G-closed* if $[A]^G \subseteq A$, which denotes the G-hull of A. A is G-closed if and only if $[A]^G = A$ since for a regular method G, one has $A \subseteq [A]^G$. Here it should be noted that $\emptyset$ is G-closed since $[\emptyset]^G = \emptyset$ and X is G-closed since $[X]^G \subseteq X$; and $[X]^G = X$ if G is regular. As seen in Example 2.1, even for a regular G-method, G-closure $[A]^G$ is not necessarily a G-closed subset. A subset A with $[A]^G = \emptyset$ is G-closed. The union of G-closed subsets of X is not always G-closed, but the intersection of G-closed subsets is also G-closed. The *G-closure* of A is defined to be the intersection of all G-closed subsets containing A, and denoted by $\overline{A}^G$ which is a G-closed subset. By the fact that $[A]^G \subseteq [K]^G \subseteq K$ whenever $A \subseteq K$ and K is a G-closed subset, we can deduce that $[A]^G \subseteq \overline{A}^G$.

A subset $A \subseteq X$ is called *G-open* if $X \setminus A$ is G-closed.

X and $\emptyset$ are G-open since they are both G-closed. Eventually the union of G-open subsets of X is G-open and the intersection of G-open subsets is not necessarily G-open. A subset $A \subseteq X$ is a *G-neighborhood* of a if there exists a G-open subset U of X such that $a \in U \subseteq A$. The union of G-open subsets of A is called *G-interior* of A and denoted by A^{0G} which is the largest G-open subset of A [9]. A is G-open if only if $A = A^{0G}$.

***G-sequentially hull* and *G-sequentially closed* subsets** : We say a point $l \in X$ is in the *G-sequentially hull* of a subset A if there exists a sequence $\mathbf{x} = (x_n)$ of the terms in A which G-sequentially converges to l and write $[A]_G$ for the set of G-sequentially hull points of A. Since for $a \in A$, the constant sequence $(x_n) = (a, a, \dots)$ is G-sequentially convergent to a we conclude that $A \subseteq [A]_G$. A is G-sequentially closed if $[A]_G \subseteq A$. Note that $[X]_G = X$ and $[\emptyset]_G = \emptyset$; and therefore $\emptyset$ and X are G-sequentially closed.

The *G-sequentially closure* of A, denoted by $\overline{A}_G$, is the intersection of all G-sequentially closed subsets containing A, which is also a G-sequentially closed subset. If $A \subseteq K$ and K is a G-sequentially closed subset, then $[A]_G \subseteq [K]_G \subseteq K$ and therefore $[A]_G \subseteq \overline{A}_G$.

We remark that a point a in a first countable space X is an interior point of the subset A if any sequence $\mathbf{x} = (x_n)$ converging to a is almost in A. Hence we can extend this notion to a G-method as follows: A point a is said to be a *G-sequentially interior point* of A and write $a \in A^0_G$ whenever any sequence $\mathbf{x} = (x_n)$ with G-sequentially convergence to a is almost in A or equivalently there is no any sequence $\mathbf{x} = (x_n)$ in $X \setminus A$ with G-sequentially convergence to a. By the fact that the constant sequence $(x_n) = (a, a, \dots)$ is G-sequentially convergent to a, one can see that $A^0_G \subseteq A$ and therefore A is G-sequentially open when $A \subseteq A^0_G$.

We can state following theorems to support the idea of G-sequential convergence.

Theorem 2.1. [3] *Let X be a set with a G-method. A subset A is G-sequentially open if and only if $X \setminus A$ is G-sequentially closed.*

Equivalently we can state the following theorem.

Theorem 2.2. *A is G-sequentially closed if and only if $X \setminus A$ is G-sequentially open.*

In the following examples, we shall define two G-methods and the compare G-convergence and G-sequential convergence together with associated features of G-sequentially closed subsets.

Example 2.1. Let G be a convergent method on $\mathbb{R}$ defined by $G(\mathbf{x}) = \lim \frac{x_n + x_{n+1}}{2}$ for some sequences $\mathbf{x} = (x_n)$. We can check the following properties for G-convergence.

(i) **G-closed and G-open subsets**. Since A is regular we have $A \subseteq \overline{A}^G$. Hence a subset A is G-closed if and only if $A = \overline{A}^G$. For the subset $A = \{0, 1\}$ one has $\overline{A}^G = \{0, \frac{1}{2}, 1\}$. Here note that since the sequence $(x_n) = (0, 1, 0, 1....)$ is G-converging to $1/2$ one has $1/2 \in \overline{A}^G$. Hence A is not G-closed.

If $A = \{x\}$ and $y \in \overline{A}^G$, then there exists a sequence $\mathbf{x} = (x_n)$ in A with $G(\mathbf{x}) = y$. But $\mathbf{x} = (x_n) = (x, x, \dots)$ and since G is regular $G(\mathbf{x}) = \lim(\mathbf{x}) = x$ and therefore $y = x$. Hence $A = \{x\}$ is G-closed and therefore G-open subsets are the complements $\mathbb{R} \setminus \{x\}$ for $x \in \mathbb{R}$.

(ii) **G-convergence and G-sequential convergence** This method is G converging for some sequences but it is not G-sequentially converging to any point. In below we give different types of examples for the G-sequentially convergence of sequences.

(a) For example the sequence $\mathbf{x} = (x_n) = (1, 3, 1, 3, \dots)$ is G-convergent to 2 but not G-sequentially converging to any point, because for any point $x \in \mathbb{R}$, we can choose a G-open neighbourhood $\mathbb{R} \setminus \{a\}$ of x, which does not include almost all terms of $\mathbf{x} = (x_n)$.

(b) For a constant $a \in X$ consider the sequence $\mathbf{x} = (x_n)$ defined by

$$x_n = \begin{cases} n, & \text{if } n \text{ is odd} \\ a, & \text{if } n \text{ is even} \end{cases}$$

Then $\mathbf{x} = (x_n)$ is not G-convergent to any point but G-sequentially convergent to the point a because any G-open neighbourhood $\mathbb{R} \setminus \{x\}$ of a includes almost all terms of the sequence. For any point x, which is different from a, the subset $\mathbb{R} \setminus \{a\}$ is a G-open neighbourhood of x but it does not include almost all terms and therefore $\mathbf{x} = (x_n)$ does not G-sequentially convergent to x.

(c) The sequence $\mathbf{x} = (x_n) = (\frac{1}{n})$ is G-convergent to 0 G-sequentially convergent to all points x's, because any G-open neighbourhood $\mathbb{R} \setminus \{a\}$ of x includes almost all terms of $\mathbf{x}$.

(iii) **G-sequentially closed and G-sequentially open subsets**

We can now characterize G-sequentially closure and hence G-sequentially closed subsets. Consider the following cases.

(a) If A is an infinite set, then we have a sequence $\mathbf{x} = (x_n) = (x_1, x_2, \dots)$ in A with different terms and $\mathbf{x} = (x_n)$ is G-sequentially convergent to every point $x \in \mathbb{R}$, since each G-open neighbourhood $\mathbb{R} \setminus \{a\}$ of x includes almost all terms of $\mathbf{x}$. Hence all points of $\mathbb{R}$ are in the G-sequentially hull of A and therefore $[A]_G = \mathbb{R}$.

(b) Let A be a finite set and $x \notin A$. If $\mathbf{x} = (x_n)$ is a sequence of the terms of A, then (x_n) is in the form $(x_n) = (.., x_{n_0}, \dots, x_{n_0}, ...)$ and G-open neighbourhood $\mathbb{R} \setminus \{x_{n_0}\}$ of x does not include almost all the terms. Hence $x \notin [A]_G$ for all $x \notin A$ and therefore $[A]_G = A$. We can write the generalization

$$[A]_G = \begin{cases} \mathbb{R}, & A \text{ is infinite} \\ A, & A \text{ is finite} \end{cases}$$

Hence we can conclude that finite subsets are G-sequentially closed, cofinite subsets are G-sequentially open and

$$A_G^0 = \begin{cases} A, & \text{if } A \text{ is cofinite} \\ \emptyset, & \text{otherwise} \end{cases}$$

Example 2.2. Let $c \in X$ be a constant element and G a method on the set X defined by $G(\mathbf{x}) = c$ for any sequence $\mathbf{x} = (x_n)$. Then we check the following.

(i) **G-closed and G-open subsets**. One can check that $[A]^G \subseteq A$ if and only if $c \in A$. Hence A is G-closed if and only if $c \in A$ If $(a_n) \subseteq A$ and $G(a_n) = c \in A$, then $[A]^G \subseteq A$. Thus $[A]^G = \{c\}$. and therefore we can state G-closed and G-open subsets as follows

$$\begin{cases} A \text{ is G-closed}, & \text{if } c \in A \text{ or } A = \emptyset \\ A \text{ is G-open}, & \text{if } c \notin A \text{ or } A = X \end{cases}$$

(ii) **G- convergence and G-sequential convergence**. For an $a \in X$ with $a \neq c$, the sequence $\mathbf{x} = (x_n)$ is G-sequentially convergent to a if and only if the terms of $\mathbf{x} = (x_n)$ is almost a since by (i) $\{a\}$ is a G-open neighbourhood of $a \in X$. Moreover by (i) the only G-open neighbourhood of c is $\mathbb{R}$ and therefore any sequence is also G-sequentially converging to c.

(iii) **G-sequentially closed and G-sequentially open subsets**.

Let $a \neq c$. Then by (i) $\{a\}$ is a G-open neighbourhood of a. Hence a sequence $\mathbf{x} = (x_n)$ in A is G-sequentially convergent to a if and only if the terms are almost a, i.e., $(x_n) = (a_1, a_2, ..., a_{n_0}, a, a, \dots)$. Hence $[A]_G \subseteq A$ and therefore all subsets are G- sequentially closed and also G-sequentially open.

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A NOTE ON STONE SPACES

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Abstract. The goal of this paper is to characterize each of compact, totally disconnected, Stone relation spaces, and Stone reflexive spaces as well as examine the relationships between them. Finally, we investigate some properties of them and compare our results.

1. Introduction

If a topological space X is Hausdorff, totally disconnected, and compact, then X is called a Stone space [14]. Stone spaces are used in algebra, topology, functional analysis, the representation theory of rings, algebraic geometry, and mathematical logic [13, 14, 16, 17].
Categorical setting of compact Hausdorff spaces are studied by several authors[5, 9, 12, 15].

The notion of closedness which is being used in defining the Hausdorffness, openness, compactness, total disconnectedness was introduced in [3].

The category **Rel** of relation spaces where objects are sets with a binary relation and where morphisms $f : (A_1, R) \to (B_1, S)$ are functions with $f(a)Sf(b)$ if aRb for all $a, b \in A_1$ [10].
The category **RRel** of reflexive relation spaces is the full subcategory of **Rel** and they are topological categories [10].

Let $B \neq \emptyset$ and let $B^2 \bigvee_\Delta B^2$ be taking two distinct copies of B^2 identified along Δ.
The map $S : B^2 \vee_\Delta B^2 \to B^2$ is given by $S(a,b)_1 = (a,b,b)$ and $S(a,b)_2 = (a,a,b)$ and the map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(a,b)_1 = (a,b,a)$ and $A(a,b)_2 = (a,a,b)$.
The map $\nabla : B^2 \bigvee_\Delta B^2 \to B^2$ is given by $\bigtriangledown((a,b)_j) = (a,b)$ for $j = 1,2$ [3].

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Let $X \in Ob(\mathbf{E})$ with $U(X) = B$, where $\mathbf{E}$ is a set based topological category.
Let S_W (resp. A_W) be the initial lift of the U-source S (resp. A) : $B^2 \bigvee_\Delta B^2 \to U(X^3) = B^3$.

Definition 1.1. *(cf. [3, 4]).*
(1) *If the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $A : B^2 \vee_\Delta B^2 \to U(X^3)$ is discrete, then X is said to be a $\overline{T_0}$ object, where D is the discrete functor.*
(2) *If the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $id : B^2 \vee_\Delta B^2 \to U(B^2 \vee_\Delta B^2)'$ is discrete, then X is said to be a T_0' object.*
(3) *If $S_W = A_W$, then X is said to be a $Pre\overline{T}_2$ object.*
(4) *If X is $Pre\overline{T}_2$ and T_0' (resp. $\overline{T}_0$), then X is said to be a KT_2 (resp. $\overline{T}_2$) object.*

Let $\bigvee_x^\infty B$ be taking countably many disjoint copies of B and identifying them at the point $x \in B$. The map $A_x^\infty : \bigvee_x^\infty B \to B^\infty$ (resp. $\bigtriangledown_x^\infty : \bigvee_x^\infty B \longrightarrow B$) is given by $A_x^\infty(a_i) = (x, ..., x, a, x, x, ...)$ (resp. $\bigtriangledown_x^\infty(a_i) = a$ for all $i \in I$), where a_i is in the i-th component of $\bigvee_x^\infty B$ and B^∞ is the countable product of B [3].

Definition 1.2. *(cf. [3, 5]).*
(1) *If the initial lift of the U-source $\bigtriangledown_x^\infty : \vee_x^\infty B \to UD(B)$ and $A_x^\infty : \vee_x^\infty B \to U(X^\infty)$ is discrete, then $\{x\}$ is said to be closed.*
(2) *If $\{*\}$, the image of N, is closed in X/N or $N = \emptyset$, then N is said to be closed, where X/N is the final lift of the epi U-sink $Q : U(X) \to B/N = (B \backslash N) \cup \{*\}$, identifying N with a point *.*
(3) *If N^C, the complement of N, is closed, then N is said to be open.*
(4) *If the projection map $\pi_2 : X \times Z \longrightarrow Z$ is closed for each object Z in $\mathbf{E}$, then X is said to be a compact object.*

In **Top** (the category of topological spaces and continuous functions), $\overline{T}_0$ and T_0' (resp. $\overline{T}_2$ and KT_2) reduce to T_0 (resp. T_2) axiom [3]. Also, compactness (resp. openness and closedness) coincides with the usual compactness (resp. openness and closedness) [5].

Theorem 1.1. ***(1)*** *Every subset of a relation space is closed.*
(2) *Every relation space is compact.*

Proof. **(1)** Let (B, R)be a relation space and $N \subset B$. If $N = \emptyset$, then by Definition 1.2, N is closed. If $N = \{x\}$ for some $x \in B$, then let R_1 be the initial structure on $\vee_x^\infty B$ induced by $\bigtriangledown_x^\infty : \vee_x^\infty B \to (B, \emptyset)$ and $A_x^\infty : \vee_x^\infty B \to (B^\infty, R^\infty)$, where $\emptyset$ is the discrete relation on B and R^∞ is the product relation on B^∞. Since $\bigtriangledown_x^\infty : \vee_x^\infty B \to (B, \emptyset)$ is a relation preserving map and $(B, \emptyset)$ is discrete, we have $R_1 = \emptyset$ and so, $\{x\}$ is closed in (B, R).
If N has cardinality at least 2, then $\{*\}$ is closed in B/N and by Definition 1.2, N is closed.
(2) follows from Part (1) and Definition 1.2. □

Theorem 1.2. *A reflexive space (B, R) is compact iff for every $x \in A$ there exist $a, b \in B$ with xRa and bRx.*

Proof. It is proved in [8]. □

2. Stone Spaces

We introduce two new Stone objects in a topological category and find relationships between them. Moreover, we characterize each of Stone relation spaces and

Stone reflexive spaces and compare our results.
Let $X \in Ob(\mathbf{E})$ and $N \subset X$. Recall, in [7], that the quasi-component closure $Q_X(N)$ of N is the intersection of all open and closed subsets of X containing N.

Definition 2.1. ***(1)*** *If every quasi-component of X contains only one point, then X is said to be totally disconnected.*
(2) *If X is KT_2 (resp. $\overline{T}_2$), compact, and totally disconnected, then X is called a TKT_2 (resp. $T\overline{T}_2$) object.*
An object satisfying the condition (2) will be called a Stone object.

In **Top**, the notion of total disconnectedness coincide with the usual total disconnectedness [2, 7, 11]. Moreover, TKT_2 and $T\overline{T}_2$ Stone spaces reduce to the usual Stone spaces [14].

Theorem 2.1. *Every $T\overline{T}_2$ Stone object is TKT_2.*

Proof. Let $X \in Ob(\mathbf{E})$, where $U : \mathbf{E} \to \mathbf{Set}$ is topological.
If X is a $T\overline{T}_2$ Stone object, then, X is $\overline{T}_2$ and by Definition 1.1, X is $Pre\overline{T_2}$ and $\overline{T}_0$. Since X is $\overline{T}_0$, by Theorem 2.7 of [4], X is T_0' and so, X is KT_2. Hence, X is TKT_2. □

Theorem 2.2. ***(1)*** *Every relation space is totally disconnected.*
(2) *For a relation space (B, R), the following are equivalent:*
(i) *(B, R) is $T\overline{T}_2$.*
(ii) *(B, R) is TKT_2.*
(iii) *For each $x, y \in B$ there exists $z \in B$ with xRz and yRz, then for any $w \in B$, xRw iff yRw.*

Proof. **(1)** Since by Theorem 1.3, $Q(s) = \{s\}$ for all $s \in B$, then (B, R) is totally disconnected.
(2) By Theorem 1.3 and Part (1), a relation space (B, R) is compact and totally disconnected. By Theorem 3.5 of [8], we get the result. □

Theorem 2.3. *A reflexive space(B, R) is TKT_2 iff it is $T\overline{T}_2$.*

Proof. By Theorems 3.2 and 5.2 of [7], a reflexive space (B, R) is $\overline{T}_2$ iff it is KT_2 and totally disconnected, and by Definition 2.1, one has the result. □

Let $\mathbf{TKT_2Rel}$ and $\mathbf{T\overline{T}_2Rel}$ be the full subcategory of **Rel** whose objects are the TKT_2 or $T\overline{T}_2$ Stone relation spaces.

Theorem 2.4. *The categories $\mathbf{TKT_2Rel}$ and $\mathbf{T\overline{T}_2Rel}$ are isomorphic topological categories.*

Proof. By Theorem 2.2 and Theorem 3.4 of [6], one has the result. □

Recall, in [8], that if X is KT_2 (resp. $\overline{T}_2$), compact, and extremally disconnected, then X is called a EKT_2 (resp. $E\overline{T}_2$) Stone object.

We can infer the foolowing results:
(1) In **Rel**, by Theorem 2.2 and Theorem 4.5 of [8], all TKT_2, EKT_2, $E\overline{T}_2$, and $T\overline{T}_2$ Stone relation spaces are equivalent and by Theorem 2.4, the subcategories $\mathbf{TKT_2Rel}$, $\mathbf{T\overline{T}_2Rel}$, $\mathbf{EKT_2Rel}$, and $\mathbf{E\overline{T}_2Rel}$ have all limits and colimits. By Theorem 4.5 of [8] and Theorems 1.3 and 2.2, a relation space is totally disconnected

iff it is extremally disconnected.
(2) In **RRel**, by Theorems 3.2 and 5.2 of [7] and Theorem 4.6 of [8], $\overline{T}_2$ implies each of KT_2, extremally disconnected, and totally disconnected. The indiscrete reflexive space $(\{m,n\},\{m,n\}^2)$ is KT_2 and extremally disconnected but it is neither $\overline{T}_2$ nor totally disconnected. $(\{m,n\},\{(m,m),(n,n),(n,m)\})$ is totally disconnected but it is neither KT_2 nor $\overline{T}_2$. By Theorem 4.6 of [8] and Theorem 2.3, $TKT_2 = T\overline{T}_2 = E\overline{T}_2 \Rightarrow EKT_2$ but $(\{m,n\},\{m,n\}^2)$ is EKT_2 but it is neither TKT_2 nor $T\overline{T}_2$ nor a $E\overline{T}_2$ Stone reflexive space.
(3) In arbitrary topological category, by Theorem 2.1, every $T\overline{T}_2$ Stone object is TKT_2.

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DECENTRALIZED MACHINE LEARNING APPROACH ON ICU ADMISSION PREDICTION FOR ENHANCED PATIENT CARE USING COVID-19 DATA

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Abstract. The Intensive Care Unit (ICU) represents a constrained healthcare resource, involving invasive procedures and high costs, with significant psychological effects on patients and their families. The traditional approach to ICU admissions relies on observable behavioral indicators like breathing patterns and consciousness levels, which may lead to delayed critical care due to deteriorating conditions. Therefore, in the ever-evolving healthcare landscape, predicting whether patients will require admission to the ICU plays a pivotal role in optimizing resource allocation, improving patient outcomes, and reducing healthcare costs. Essentially, in the context of the post-COVID-19 pandemic, aside from many other diseases, this prediction not only forecasts the likelihood of ICU admission but also identifies patients at an earlier stage, allowing for timely interventions that can potentially mitigate the need for ICU care, thereby improving overall patient outcomes and healthcare resource utilization. However, this task usually requires a lot of diverse data from different healthcare institutions for a good predictive model, leading to concerns regarding sensitive data privacy. This paper aims to build a decentralized model using deep learning techniques while maintaining data privacy among different institutions to address these challenges.

1. Introduction

The COVID-19 pandemic confronted health systems worldwide with an unprecedented challenge. According to the World Health Organization (WHO), approximately 14.9 million deaths were associated with this novel coronavirus during 2020 and 2021 [1]. Surging cases overwhelmed hospitals and depleted essential resources globally, especially in intensive care units (ICUs) where shortages of beds, equipment, and staff severely constrained life-saving care [2].

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The ICU is a crucial but limited healthcare resource [3]. Especially under the context of the COVID-19 era, a large number of cases have particularly stressed ICU settings with an increased need for ICU beds [4]. As cases skyrocketed in pandemic hotspots from Wuhan, Italy, to New York, ICUs were immediately overloaded with exceeding capacity [5] [6]. This emergency of ICU and other medical resources scarcity extremely affected patient outcomes and mortality throughout the pandemic [7]. Additionally, medical treatment in the ICU has the disadvantages of possible invasive procedures [8], high cost, and significant psychological effects on both patients, their families and the medical institution [9], compared to the equivalent but earlier treatment outside the ICU. Moreover, the traditional approach to determining if someone should be admitted to the ICU primarily depends on observable indicators such as the patient's breathing pattern, consciousness, and medical instability, which means decisions for sending some patients without notable into ICU are made at relatively later points waiting until the patient's health condition has already deteriorated [10]. This decision-making strategy could potentially result in delayed medical treatment, thus leading to a poor survival rate and long-term effects on the patient's physical condition[10].

To summarize, the challenges and disadvantage of critical care is: limited resources, possible late admission decisions, and significant burden on different aspects of different groups. To solve the root cause, Machine Learning (ML) methods have been proven as a robust tool to reduce the necessity for patients to be sent to ICU by facilitating earlier clinical decision-making and critical care intervention which in turn helps with better ICU resource allocation [11] [12] [13]. A more robust model can help with making more accurate and reliable decisions for an earlier intervention, which will lead to a positive change in survival rate, long-term effects, and readmission rates among patients carrying a wide range of diseases [10] [14] [15]. However, creating a robust model that can produce reliable information also necessitates access to a wide range of diverse data from different institutions [16], which challenges data privacy and integrity significantly [17]. We recognize the importance of data privacy and the distributed nature of healthcare data, which provides significant challenges to this traditional centralized approach. On the one hand, healthcare data contains highly sensitive and private information, requiring high privacy protection measures. However, healthcare data also distributed across various countries and institutions, prevents data accessibility and holds back the development of accurate predictive models. To alleviate the need for data transfer between institutions, which is a primary concern of data privacy, this paper aims to deploy a predictive model using decentralized deep learning architecture that enables model transfer among different institutions to maintain data privacy, which is commonly known as Federated Learning (FL) [18].

FL has emerged as a promising approach for training machine learning models in the biomedical field, specifically in healthcare, to address the challenges of data privacy and data accessibility. By enabling collaborative model training without the need for centralizing patient data, FL allows healthcare institutions to collectively leverage their datasets while preserving data privacy [16]. For instance, in the pioneering publication on FL in the medical domain, Sheller et al. [19] have successfully applied FL on studying brain tumor. The results showed that the deep learning model trained using FL could reach 99% of the performance of the same model trained with the traditional data-sharing method, highlighting the potential of this technique in maintaining data privacy while effectively utilizing distributed healthcare data. Overall, this study aims to explore the potential of FL techniques on deep learning models in improving the accuracy of ICU admission prediction models and addressing the challenges posed by healthcare data privacy.

2. Methods

2.1. **Data Overview.** The original data is provided by the Mexican Government [20]. We translated the attributes and chose 21 medical-related features from the data set for this research (Table 1). Irrelevant features, like registration ID, migration status, and whether the patient speak an indigenous language or not, were dropped. The data set is being updated regularly. As of the day the research began, 1,048,575 records were collected.

Table 1. Table includes the features included in the dataset, 20 features and 1 target column.

Name	Type	Description
USMR	Categorical	medical units of the first, second or third level
Medical Unit	Categorical	type of institution that provided the care
Sex	Categorical	biological gender 1 for female and 2 for male
Patient Type	Categorical	type of care. (1 = returned; 2 = hospitalization)
Date Died	Date	the date of death
Intubed	Categorical	whether the patient was connected to the ventilator
Pneumonia	Categorical	air sacs inflammation in the past
Age	Discrete	years of age
Pregnant	Categorical	whether the patient is pregnant or not.
Diabetes	Categorical	whether the patient has diabetes or not
COPD	Categorical	Chronic obstructive pulmonary disease
Asthma	Categorical	whether the patient has asthma or not
INMSUPR	Categorical	whether the patient is immunosuppressed or not.
Hypertension	Categorical	whether the patient has hypertension or not
Other Disease	Categorical	whether the patient has other disease or not
Cardiovascular	Categorical	heart or blood vessels related disease
Obesity	Categorical	whether the patient is obese or not
Renal Chronic	Categorical	chronic renal disease
Tobacco	Categorical	whether the patient is a tobacco user
Classification Final	Discrete	covid test findings. $1 \sim 3$=COVID; ≥ 4=negative
ICU	Categorical	admitted to an Intensive Care Unit

2.2. **Data Preprocessing.** For this research, the categorical feature ICU was used as the target attribute for prediction. Besides AGE and DATE DIED, all other categorical features implied if a record has the diseases or not. Among them, CLASSIFICATION FINAL indicated whether a patient tested positive for COVID-19 or not. The original data used $1 \sim 3$ for positive, ≥ 4 for negative, and we converted this attribute into binary. DATE DIED indicates when the patient deceased. A empty value in this column indicated that the person survived. The DATE DIED column was transformed into a binary attribute. Eventually, this column was dropped and converted into records in ICU column. So, the final dataset, after dropping all records with a null value (about 1% of the dataset), includes 189112 records that are hospitalized. Within those records, Column ICU has 75011 (about 39.7% of the entire dataset) entries indicating this patient will need critical medical care and should consider early intervention. And, within all those 75011 records, 16397 records were originally included in the ICU column before the data processing. The rest 58641 records came from the hospitalized records that died without being sent into the ICU. Those records was originally from DATE DIED column (Figure 1). Because a patient died under hospitalized status but not in ICU indicates that they were supposed to received early intervention medical care for a potential better outcome, we combined the ICU and DATE DIED columns.

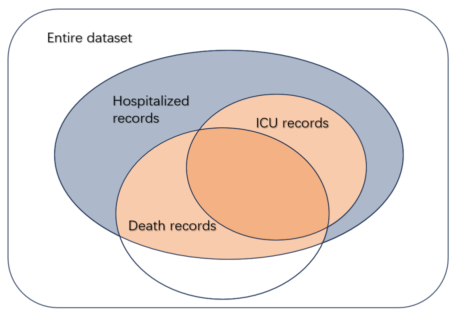

FIGURE 1. A Venn Diagram indicates the composition of the target column. The grey area is the records hospitalized. The orange area is the final target group, which consists of ICU records and records that died in the hospital but not in the ICU.

2.3. **Baseline Training.** To create a baseline understanding of our dataset and test data cleaning, a series of traditional machine learning techniques were performed on all of our datasets. Models like Decision Trees (DT) [21], Random Forests (RF) [22], Bayesian Classifiers (BC) [23], SVM [24], deep learning models like Convolutional Neural Networks (CNN) [25], and Recurrent Neural Networks (RNN) [26] were used. They mainly served as the comparison group that trained on the global dataset without considering data privacy.

2.4. **Federated Learning.** Federated learning (FL) is a decentralized machine learning approach that allows multiple devices or nodes to collaboratively update a shared model and hold local data samples [18] . In this research, the goal is to develop a federated learning architecture that can retain moderate accuracy, recall, and precision while not sharing the information between edges.

A global model was distributed physically to different medical institutions (Figure 2). After the local model was trained on each dataset. The updates from different medical institutions were sent back to make the update. The following Algorithm 1 explains the federated learning pipeline in more detail.

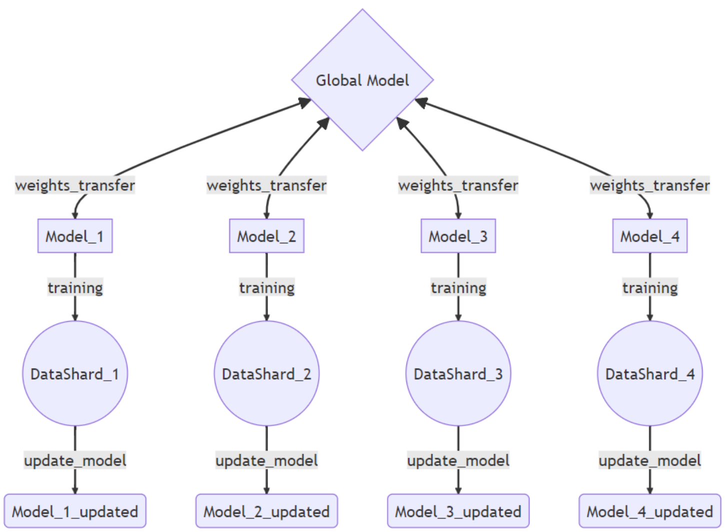

FIGURE 2. a round of Federated Learning pipeline with 4 datasets. On the top is the Global Model, the first level (top down) is the weight transfer back and forth between the global model and the models on different edges. The second level (Model to Datashard) indicates the training processes between locals models with local data. The third level (Datashard to Model updated) indicates that the updated weights of each local model are collected and ready to compile into one global model update.

Algorithm 1 Basic Federated Learning Architecture

$W_{0,0} \sim F_w$
$\alpha_1, \alpha_2, \alpha_3, \alpha_4$
$N = 100$
while $n < N$ **do**
  //initialize models on edges base on the global model
  $W_{n,1}, W_{n,2}, W_{n,3}, W_{n,4} = W_{n,0}$

  //Training each model on their dataset for one epoch
  $W'_{n,1}, W'_{n,2}, W'_{n,3}, W'_{n,4} \sim W_{n,1}, W_{n,2}, W_{n,3}, W_{n,4}$

  //update the global model weight based on the weighted average
  $W_{n+1,0} = \alpha_1 W'_{n,1} + \alpha_2 W'_{n,2} + \alpha_3 W'_{n,3} + \alpha_4 W'_{n,4}$

end while

The training ran 100 rounds in total. The local model trained on each edge for 1 epoch, which refers to local training. The updates of the global model were calculated based on the weighted average of the models from different shards. During the training, global accuracy and loss were monitored.

3. Results

3.1. **Baseline Training's performance.** Accuracy, recall, and precision are monitored for baseline training. Accuracy is the overall accuracy. Due to the nature of binary classification, the recall and the precision are measured on the class indicating the need for critical care and early intervention. Overall accuracy ranges from 70% (DT) to 76% (DNN). Precision ranges from 65 % (DT) to 78% (RNN). Recall ranges from 38% (SVM), to 59% (DNN) (Table 2).

Table 2. Baseline Training models' prediction performance, need add precision also

	Decision Tree	Random Forest	SVM	Bayesian Classifier	DNN	CNN	RNN
Acc	70.82%	72.68%	73.49%	76.30%	76.39%	76.25%	76.26%
Precision	65.74%	67.95%	88.22%	81.16%	76.14%	77.56%	78.25%
Recall	55.34%	59.0%	38.35%	52.50%	59.02%	56.53%	55.68%

3.2. **Federated Learning's performance.**

3.2.1. *Accuracy.* Accuracy serves as a general measurement of the architectures predicting power.

$$Accuracy = \frac{TP + TN}{TP + TN + FP + FN}$$

Deep learning models like CNN, RNN, and DNN are used separately as the base models for FL. The accuracy ranges from 76.18% to 76.28% on the global dataset and performs equivalently well on each data shard (Table. 3).

Table 3. Federated Learning model Accuracy

Base Models	Global	Shard1	Shard2	Shard3	Shard4
DNN	76.22%	76.14%	76.61%	76.59%	75.97%
CNN	76.18%	76.14%	76.40%	76.28%	75.90%
RNN	76.28%	76.17%	76.57%	76.40%	76.00%

3.2.2. *Precision.* Precision is measured by True Positive rate over True Positive rate and False Positive rate. A way to interpret this is how many correct predictions the model made about a class are correct among all predictions made for this class.

$$Precision = \frac{TP}{TP + FP}$$

The precision of three FL architectures ranges from 74.75% by the CNN-based model to 75.20% by the RNN-based model (Table. 3).

Table 4. Federated Learning architecture precision

Base Models	Global	Shard1	Shard2	Shard3	Shard4
DNN	74.83%	74.59%	74.54%	75.43%	74.76%
CNN	74.75%	74.82%	74.20%	75.13%	74.83%
RNN	75.20%	75.05%	74.70%	75.70%	75.34%

3.2.3. *Recall.* Recall is measured by the True Positive record number divided by the sum of the True Positive record number and the False Negative record number. Recall can be interpreted as among all records that need early intervention, how many of them are successfully detected.

$$Recall = \frac{TP}{TP + FN}$$

The recall rate for FL with DL models ranges from 60.07% by the RNN-based model, to 60.84% by the DNN-based model (Table. 5).

Table 5. Federated Learning architecture recall

Base Models	Global	Shard1	Shard2	Shard3	Shard4
DNN	60.84%	60.81%	60.9%	61.05%	60.61%
CNN	60.40%	60.44%	60.61%	60.34%	60.20%
RNN	60.07%	60.17%	60.43%	59.90%	59.78%

4. Discussion

Our FL architecture with Deep Learning models reached 99.8% accuracy of the baseline modeling, where data privacy is not well preserved (Figure 3). FL with DNN models achieved 76.3% accuracy, which surpass all machine learning models and some deep learning models that are trained on congregated dataset.

In the research, the focus is on the prediction of records that actually need early medical care to prevent ICU entrance. So, the higher the measurement of the precision, the more records that are predicted as needing ICU-level treatment are correct. The precision of our FL architecture reached 85% (75.2% by FL with RNN models compared to 88.22% by Bayesian Classifier) of maximum precision from baseline training (Figure 4). However, the recall of Bayesian Classifier is considered extremely low, only 38.35%. Therefore, Bayesian Classifier should be considered as an outlier and not considered for the comparison. Then, the precision of the highest FL architecture reached 92.7% of the highest precision by the SVM from baseline training.

Recall plays a vital role in real-world applications, too. Based on the focus of this research, the measuring for recall indicates how many actual records that need ICU entrance have been successfully detected. The recall of our FL architecture outperforms all the other models. The lowest recall by the CNN-based model in FL architecture obtained a recall of 60.4%, compared to the highest recall by DNN

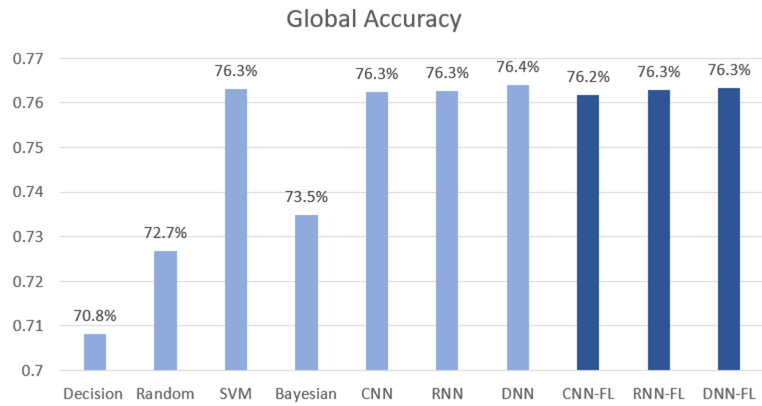

FIGURE 3. Global Accuracy for the FL. X-axis indicating with model is trained and tested, the y-axis indicating the accuracy. Dark blue indicating the federated learning design, and light blue indicating the baseline traditional machine learning models' results.

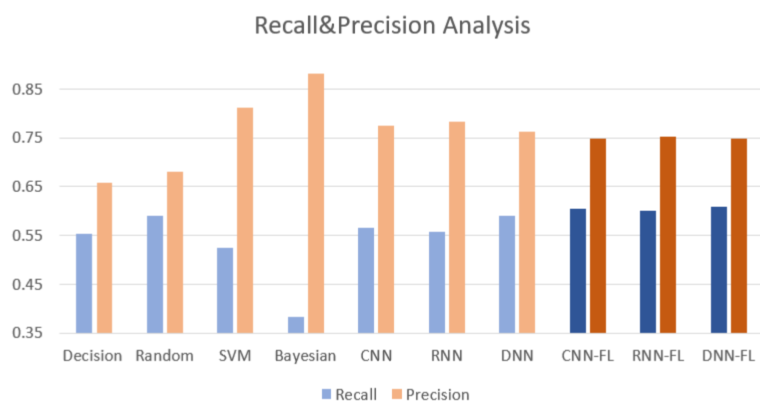

FIGURE 4. Global Recall and Precision Analysis. Each pair of columns consists of Recall (left, blue) and Precision (right, organce). The darker pairs indicating the federated learning models and its corresponding precision and recall. X-axis is the models used, y-axis is the percentage for recall and precision.

alone with a recall of 59.02% in baseline training. This indicates our FL architecture can successfully predict more records that need earlier ICU-level treatment. The result also implies FL architecture can improve the recall of target attributes in general.

Considering all three valuation factors, the FL architecture has proven to be a robust tool for better overall accuracy and recall, and tantamount precision than traditional machine learning. Most importantly, FL reaches the predicting power under the circumstance of keeping the privacy of data under no risk of leaking or retrieving. The scalability of the architecture from a data perspective, the main concern about ML in the medical field [17], can be improved.

5. Limitations & Future Work

Developing robust, trustworthy AI tools that preserve fairness is critically important, especially in high-stakes applications like healthcare. Achieving trustworthiness encompasses attributes like explainability, fairness, privacy preservation, and robustness. However, the overall prediction performance of health AI models is often prioritized over potential biases they may have [27]. In FL, there are potential risks of under-representing minority groups, if the contribution to the global model from different edges is guided by the training size, which is statistical heterogeneity [28]. In the future, we would like to explore the potential of leveraging fairness through multiple methods, like local debiasing [29] and fairer aggregation strategies[28].

Although the FL system aims to address privacy concerns by keeping patients' private data in local storage during training, potential security issues persist, particularly in the transmission of gradients and partial parameters, leading to indirect privacy leakage [30]. Three main attack categories in FL are identified: Data poisoning attacks, involving the embedding of tainted data to compromise data integrity [31]; Model poisoning, which manipulate machine learning models to produce incorrect results [31]; and Inferring attacks, focused on detecting privacy records or restoring training data [32]. Existing defense methodologies have some potential in more research, and the need for stronger protection measures, such as anomaly detection and data encryption, is emphasized to mitigate these attacks in the federated setting [31][32]. Future work on this should explore and develop more robust protection methods.

There is rich literature discussing whether FL overfits or underfits under different data quality, parameters' sizes, and extents of the local updates [33] [34] [35]. Evaluation of overfitting and underfitting usually requires a validation dataset during the training phase. However, traditionally collecting a validation dataset violates the main data privacy protection schema provided by FL. A representative and effective validation set needs to combine a certain amount of data from each dataset on the edge, but a congregated dataset is what FL trying to avoid due to privacy concerns. Moreover, validation and testing datasets are usually not directly accessible to the FL server [36], and the global model is tested on selected clients or data shards separately. In the future, our team will investigate more about the necessity of evaluating the global model of FL and its corresponding metrics.

An ICU decision will potentially put pressure on both medical institutions as well as the patients themselves [9], both mentally and physically. Traditional Machine Learning, even Federated Learning, produces a one-number confident prediction that might worsen this situation. Ethically, using probability estimation instead of one-point prediction could be a challenging but effective improvement to this situation. Plus, due to the nature of most probability predictions that produce a distribution of predictions as the output, incorporating differential privacy can add an extra layer of data protection as well as help balance the trade-off between privacy and accuracy [37]. Inspired by this idea, we would like to further investigate the feasibility of incorporating probability estimation and differential privacy in FL architectures.

6. Conclusion

In conclusion, Federated Learning demonstrated to be an effective tool to help clinical decision-making without losing data privacy. Particularly, our design of FL outperformed other traditional machine learning and deep learning techniques on the ICU admission data set. This design and architecture imply that, with the help of FL, medical institutions can potentially make more effective decisions regarding early interventions on patients to improve the treatment outcome, critical medical resource allocation, and alleviation of avoidable burdens on both sides. Besides that, this paper also tried to raise public's awareness of data privacy and ethics to encourage us to rethink our machine learning pipeline when building models for supporting clinical decision-making.

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