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PREFACE

Dear scientist,

I am happy to announce that Volume IX - Issue II of the Eastern Anatolian Journal of Science (EAJS) has been published. This issue is composed of 5 research articles that possess some of the leading and advanced techniques of natural and applied sciences. On behalf of the owner of EAJS, I would like to thank all authors, referees, our editorial board members and section editors that provide valuable contributions for the publication of the issue.

EAJS will publish original and high-quality articles covering a wide range of topics in scientific research, dedicated to promoting high standards and excellence in the creation and dissemination of scientific knowledge. EAJS published in English is open access journal and abstracting and indexing by various international index services.

Authors are solicited to contribute to the EAJS by submitting articles that illustrate research results, projects, surveying works and industrial experiences that describe significant advances in the following areas, but are not limited to:

- Biology
- Chemistry
- Engineering
- Mathematics
- Nanoscience and Nanotechnology
- Physics

Our previous issues have an attraction in terms of scientific quality and impact factor of articles by favorable feedbacks of readers. Our editorial team lend wings to be an internationally reputable and pioneer journal of science by their outstanding scientific personality. I am hoping to work effectively with our editorial team in the future.

I'd like to express my gratitude to all authors, members of editorial board and contributing reviewers. My sincere thanks go to Prof. Dr. Abdulhalik KARABULUT, the rector of Ağrı İbrahim Çeçen University, sets the goal of being also a top-ranking university in scientific sense, for supporting and motivating us in every respect. I express my gratitude to the members of technical staff of the journal for the design and proofreading of the articles. Last but not least, my special thanks go to the respectable businessman Mr. İbrahim ÇEÇEN who unsparingly supports our university financially and emotionally, to his team and to the director and staff of IC foundation.

I invite scientists from all branches of science to contribute our journal by sending papers for publication in EAJS.

Prof. Dr. İbrahim HAN

Editor-in-Chief

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The Effect of Grain Refiner and Mechanical Vibration on Feedability in Sand and Plaster Mold Casting of Etial 177 Aluminum Alloy

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Abstract

Aluminum alloys are widely used in industry due to their many engineering advantages. In aluminum alloys, the final product properties emerge during the solidifacation process. For this reasons, the goal in casting aluminum alloys is to obtain a fine-grained structure by using grain refining alloys such us Ti and B. There ara also alternative methods for achieving a fine grain structure. In this study, the effects of grain refiner addition and mechanical vibration on the feedability of Etial 177 aluminum alloy cast in sand and plaster molds were investigated. A model with different solidification times was designed and castings were made in molds prepared using this model. Liquid metal cleaning, sand and plaster casting, density measurement by Archimedes' principle, cross-section and surface examinations, and pore measurement techniques were used in the study. When the results were examined, it was determined that the feedability values changed depending on the solidification time and the amount of pores decreased in fine-grained structures. It was observed that mechanical vibration has a positive effect on the internal structure, and it was determined that the pore value decreased even more in castings with grain refiner addition compared to castings without addition.

Keywords: Etial 177, Sand mold, plaster mold, Feedability, Grain refinement, Mechanical vibration.

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1. *Introduction*

Aluminum, which is the second most abundant metal in the earth's crust, is widely used in today's industry with its advantageous properties. Aluminum and its alloys, which are the most preferred metal after steel, maintain their importance thanks to their mechanical properties and their usage areas in industry are increasing (Zeytin 2000). It is widely used in motor vehicles such as private vehicles, public transportation vehicles preferred aluminum alloys are also used in the aerospace industry. Aluminum alloys, whose strength and impact properties have been improved with new designs and additive components, have taken their place as an important material in the defense industry (Başer 2013). Aluminum alloys are widely used in the automotive and aerospace industries due to their high strength, easy castability and corrosion resistance. It is one of the most widely used non-ferrous metals in the world and is used in many sectors with energy recovery and environmentally friendly manufacturing (Tokatlı et al. 2022). It has long been known that grain refinement applications, which are widely used in aluminum alloys for development studies, have positive effects. The addition of Ti at some scale to the molten metal results in a significant reduction in the grain structure, which can make the alloy easier to cast. This process can be clearly seen in the grain structure on the etched surface of the samples without addition and with some Ti and B addition (Çolak and Kayıkçı 2009). With the addition of 0.15% Ti, $TiAl_3$ compounds are formed in the molten metal and the melting temperature of pure aluminum increases from 660°C to 665°C. In this way, it is observed that a heterogeneous fine-grained structure is spontaneously formed on the aluminum $TiAl_3$ compound without the need for any supercooling (ΔT) during the cooling period

(Sigworth 1984). In addition to grain refinement processes, modification studies are also carried out in foundries (Çolak, 2019). In experimental studies, the microstructure and mechanical properties of the liquid metal are affected by the alloying elements (Cu, Ni, Ti, Mn, etc.) added. Pressure, cooling rates, casting temperatures, vacuum treatment, mechanical vibration, etc. can be counted among these factors. These factors also affect the microstructure, good feedability and mechanical properties of the materials (Davis et al. 1990), (Elliott 1990).

As a result of the mechanical vibration process, which is accepted to increase the mechanical properties of aluminum as well as Ti and B elements, which are accepted as grain refiners, especially grain and phase sizes are reduced and the gases dissolved in the molten metal can be removed from the liquid metal by the effect of surface transport. The fluidity of the molten metal is therefore improved and the slowdown in heat transfer due to the segregation occurring at the mold and metal interface can be reduced (Hasırcı 2017). Mechanical vibration is one of the factors that should have a frequent continuity. When applied at high amplitude and wide time intervals, negative effects (turbulence formation, gas (air) cavity formation, mold distortion, etc.) can be seen. When vibration is applied under appropriate conditions, it is seen that results similar to grain refinement and modification processes can be obtained. This method is also an easy and low-cost process that can be used in all casting methods (Lin et al. 2000),(Minkoff, 1983).

It is known that the addition of grain refining mastic alloy and mechanical vibration in the casting of aluminum alloys will reduce the grain size and consequently increase the feedability and reduce the porosity, which is also known in the literature (Chen and Zhang 2010). One of the main purposes of grain refinement is to reduce the amount and size of porosity in the material (Sigworth and Kuhn, 2007). In the casting of aluminum alloys, the mechanical stirring process to ensure the homogeneity of the molten metal in the mechanical stirring method is mostly provided by a drill or impeller system (Fan 2002), (Figueredo 2001). Stirring is applied during the solidification period in order to form a non-dendritic microstructure. However, many problems have been encountered in the applications of mechanical mixing method. The electromagnetic mixing method was developed to solve the problems encountered in mechanical mixing. In this technique, rods with dendritic-free internal structure can be produced with electromagnetic stirring created in the casting line. It has been reported that the products manufactured by this method have a particle size between 30µm-100µm A study in which the effect of vibration during the solidification process of the A356 alloy was investigated and a comparison was made with the casting part solidified in a sand mold normally at different vibration intensities and without vibration was examined. As a result of the examination, it was seen that solidification under vibration had a positive effect on the properties of A356 aluminum casting alloy by reducing the grain size (Çolak and Balcı 2016). One of the conditions that have an effect on grain size is solidification time. Solidification time can be calculated by the Chvorinov formula, which ratios the volume (V) of the casting to the surface area (A). The equation obtained with this relation is $t=k(V/A)^2$. While t in the equation is the solidification time in minutes, k is the equation constant that varies depending on the casting alloy and mold material (Chvorinov 1940).

Under the basic basic integrity of the study is the issue of improving the properties of aluminum, and the concept of another branch of this improvement is the concept of feedability. In the production of Al alloys by casting, the timing of solidification and the design of the molds are very important. During the transition from liquid to solid state, Al alloys shrink in percentages ranging from 3.5% to 8.5% by volume, depending on their chemical composition. This volumetric shrinkage can be supported by feeders that can be placed in the required dimensions in places that seem appropriate in the design of the molds. As a result of the volumetric shrinkage not being adequately fed by the feeders, voids occur in the structure formed after solidification. Shrinkage, also known as shrinkage, which is one of the problems experienced in aluminum casting processes, can cause more than one casting product to be scrapped. In order to obtain robust parts, the amount of shrinkage in macro and micro dimensions must be reduced (Çolak and Arslan, 2018).

In addition to improving the properties of aluminum, mold materials are also available in the casting industry to support this improvement in a positive way. Economical and readily available sand molds are widely used for casting metals. Sand mold casting is a metal casting method with sand as the mold material. The mechanical properties of the parts manufactured in the casting process are proportional to the mold and mold materials in which the casting process is performed. Sand mold material, which is cost-effective and easily accessible as mold material, is the most widely used mold material. The molding steps are performed by compacting the sand material around a casting model and then removing the casting model from the mold. Approximately 60% of metal casting processes are performed by sand mold casting (Çalık et al. 2022). The biggest advantage of the plaster-mold method, which is another mold material, is known as surface smoothness. However, solidification times are prolonged due to low heat transfer. Cooling plates can be placed in areas where rapid cooling is required. However, the low gas permeability and fragility of the gypsum mold material are disadvantages (Altıparmak 2007).

In this study, castings will be made with Etial 177 alloy in sand and plaster molds prepared with a ladder-shaped model designed to produce different solidification times. The effects of castings without addition, grain refiner addition and mechanical vibration on feedability will be investigated. In addition, different solidification times will also occur depending on the geometry of the ladder model in the experiments, so that the effects of solidification time on feedability will be discussed.

2. Materials and Methods

In this study, Al-Si based Etial 177 alloy with a wide solidification range was cast to exhibit different solidification times according to changing casting conditions. For this purpose, the ladder model, whose dimensions and solid model image are given in Figure 1, including different section thicknesses specially designed for varying casting conditions, was used.

The dimensions of the rectangular prism shaped model given in Figure 1 are 20x20 mm lower step, 30x30 mm middle step and 50x50 mm upper step. Modulus calculations were made for each step of the model. As a result of the calculations, while the modulus value for the lowest part was 0.33 cm, the modulus in the thickest section was calculated as 0.83 cm. Thus, it was determined that each step has varying modulus values (Kayikçi 2008).

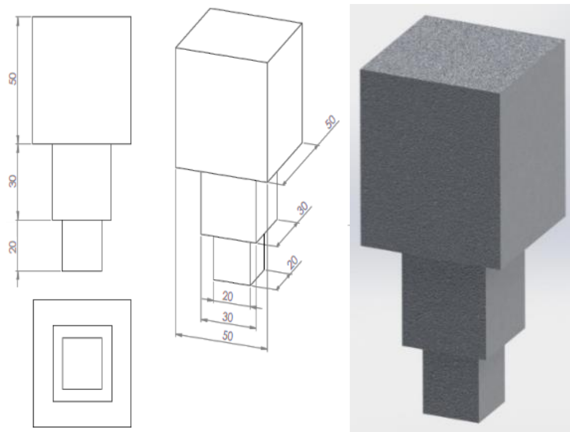

Figure 1. Model dimensions and solid model image.

Free model was used for the preparation of the molds. In the preparation of wet mold sand, 3-5% bentonite and 1-2% coal dust were added to dry silica sand with a grain size of 90-110 AFS and 5-7% water was added. After the necessary additions to the silica sand, mold material was obtained by homogeneous mixing. In the preparation of gypsum molds, the material called casting gypsum, which is sold commercially in the market and mainly used in the jewelry industry, was used. Gypsum mold slurry was obtained by mixing gypsum with water in the same ratio. By dissolving the gypsum material in water, the mold material was obtained in slurry consistency and poured into the degree in which the models were placed. Graphite was sprinkled as a mold release agent for easy removal of the model from the mold. Then, the molds of the models were formed with the prepared casting plaster. When the plaster molds reached a certain level of solidification and hardening, 2 hours after the gypsum slurry was poured into the degree, the models were removed. The plaster molds were then kept in the oven at 150°C for 24 hours. After firing, the molds were removed from the oven before casting, the runner connections were opened, the molds were made ready for casting and the stages were closed. Before casting, the mold was kept on the 80°C mold heater until casting to prevent moisture from the environment. Castings were made with alloys prepared in accordance with the experimental parameters.

Melting processes were carried out in the electric resistance furnace in our laboratories. SiC crucible with a capacity of 8 kg was used as a crucible in the furnace. In the study, castings were made in sand molds without addition, with grain refiner addition and under vibration in the same way. In addition, castings were made in gypsum molds in a total of 6 different changing parameters, including normal and under vibration. Al5Ti1B master alloy with 0.2% Ti was used for grain refiner addition. After the added alloy was melted in the furnace, nitrogen flushing was performed for 5 minutes with a graphite lance immersed in the crucible at 720°C to clean the liquid metal and then castings were made. For the casting experiments under mechanical vibration, the setup given in Figure 2 was prepared and the molds were solidified under vibration with the help of the setup. The vibration was turned on after filling the liquid metal into the mold and applied until the end of solidification. The sand mold was fixed to a plate to which an electric motor was connected with L-shaped profiles. The electric motor used is 130 W and 7000 rpm. In order for the electric motor to create vibration, vibration was provided with a weight attached to the rotor part of the motor.

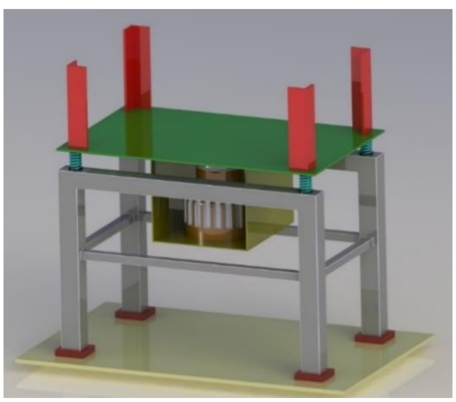

Figure 2. Illustration of the setup for solidification under mechanical vibration (Çolak and Balcı, 2016)

After cleaning, the RPT (Reduced Pressure Test) gas measurement test was applied to check the condition of the liquid metal. In this test method, the liquid metal is solidified under vacuum and then the sample is cut vertically in the middle and examined. The aim of the castings made within the scope of the study is to ensure that the gas level is at acceptable levels. Liquid metal cleaning was continued when necessary.

Following the casting tests, 2 ladder models were removed from each mold after solidification and separated from the runners. The demolded samples were first subjected to density measurements. Subsequently, one of the ladder model samples was cut vertically to check for depressions and the other one was cut horizontally for density measurement and pore determination from each step. The surfaces of the vertically cut specimens were sanded and transferred to the computer environment with the help of a scanner and the setup prepared for depression analysis in cross-section. Density measurements were made to determine the amount of pores in the inner cross-sections of the cast specimens under varying casting conditions and, accordingly, the feeding ability of the cast specimens removed from the mold. According to Archimedes' principle, each sample was immersed first in air and then in pure water and weighed in water. From the determined weights, the weight of the sample in air (m_h), the weight in water (m_s), the density of pure water at room temperature (d_s) and the density of the cast sample (d_n) were calculated according to the following formula (Taylor at al. 1999).

$$\mathbf{d_n = \frac{m_h}{m_h - m_s} x\, d_s} \qquad \text{(Equation 1)}$$

3. Experimental Results and Evaluation

3.1. Control of Chemical Composition Compliance

The chemical composition analysis results of the alloys used in casting experiments and the samples taken for the control of the amount of grain refiner addition are given in Table 1.

When the chemical composition values given in Table 1 are examined, it is determined that the alloys are within the standard composition range. In the experiments with grain refiner addition, Ti target ratios were found to be appropriate. It was targeted to add Ti at a rate of 0.2% for grain refinement, and since the critical threshold value for Ti addition is 0.15% Ti, it was determined that this ratio was achieved in all experiments.

Table 1. Chemical analysis result of alloys (% wt)

Alloy	Si	Fe	Cu	Mn	Mg	Zn	Ti	B	Sr	Al
Etial177	7.12	0.121	0.012	0.006	0.345	0.014	0.010	0.002	0.003	Rem.
Etial177+ TiB	7.01	0.122	0.008	0.007	0.364	0.011	0.192	0.011	0.004	Rem.

3.2. Liquid Metal Cleanliness Control Results

The formation of pores in the casting of aluminum alloys is due to various effects. These can occur due to shrinkage due to insufficient supply during solidification, due to gases dissolved in the melt, due to shrinkage and gas action. Pore formation in the samples included in the study will negatively affect the results. Due to this situation, in order to prevent pores due to dissolved gases, nitrogen cleaning was performed before liquid metal casting. Since the study compares pore formation based on underfeeding, possible gas-induced porosity will affect the results. For this reason, liquid metal cleaning process was carried out. After cleaning with nitrogen, the liquid metal was subjected to RPT test and the suitability of the cleaning was checked. The images of the specimens that were pressure tested after solidification are given in Figure 3. Figure 3 shows the cross-sectional surface results of the RPT sample taken after nitrogen purging for the alloy under all casting conditions. In all castings, it is understood that liquid metal cleaning process is suitable to prevent gas-induced porosity formation in the casting. Thus, it is thought that the possible porosities in the castings will not be gas- induced. Similar results were observed in the studies on liquid metal cleaning in the literature and the suitability of the liquid metal quality was confirmed (Dispinar and Campbell 2011), (Dışpınar and Campbell 2009).

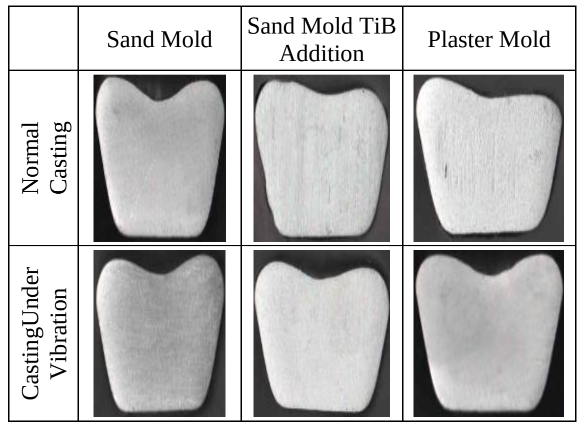

Figure 3. RPT specimen cross-sectional images obtained before casting tests.

3.3. Density Measurement and Pore Values

Table 2 gives the density and porosity results obtained after cutting the runners of the samples obtained from the casting experiments.

Table 2. Density measurement and pore values of cast samples as a whole.

Casting Method	Sample Name	Weight in air (g)	Weight in water (g)	Experimental Density (g/cm^3)	Theoretical Density (g/cm^3)	Calculated Pore (%)
Sand Mold	Gravity Sand Casting	380.20	233.78	2.59	2.68	3.28
Sand Mold	Casting Under Vibration	384.26	238.12	2.62	2.68	2.06
Sand Mold	Sand Casting with TiB Addition	381.46	236.84	2.63	2.69	2.12
Sand Mold	Under Vibration with TiB Addition	378.24	237.12	2.68	2.69	0.54
Plaster Mold	Gravity Plaster Casting	370.07	224.37	2.53	2.68	5.40
Plaster Mold	Casting Under Vibration	386.45	238.42	2.61	2.69	3.12

When the values given in Table 2 are examined, it is observed that the density and calculated pore values of the cast samples vary depending on the casting conditions. While the pore value in the sample was 3.28% in the normal casting test in the sand mold, it was determined that the pore value was 2.06% in the casting samples obtained by solidifying the same alloy under vibration. It is understood that the vibration acting on the casting molds during solidification has a positive effect on the internal structure of the casting and therefore on its feeding. It was found that the pore values decreased even more in grain refiner added castings made in sand molds compared to similar castings. It was measured that the pore value decreased from 3.28% in normal casting without addition to 2.1% with the addition of grain fining and from 2.06% to 0.54% in casting without addition under vibration. It is thought that the increase in feedability and the decrease in the pore value in the cast structure in both grain refiner addition and vibration casting experiments are related to grain size. In the casting of aluminum alloys, grain size will decrease with the addition of grain refining mastic alloy and mechanical vibration, and accordingly, feedability will increase and porosity will decrease, which is also known in the literature (Chen et al. 2010).

In normal castings made in gypsum molds, the amount of pores was 5.4% in normal casting and 3.12% in castings solidified under vibration. As in the sand mold casting experiments, it was determined that the amount of pores decreased as a result of solidification under vibration in gypsum mold casting experiments. However, among all casting tests, the most pores were found in the gypsum mold casting specimens. This is thought to be related to the lower heat transfer coefficient of the gypsum mold compared to the sand mold and the later solidification in the mold. Modulus castings that solidify later are expected to form larger grain structures. The lowest porosity value was observed in the test results both with the addition of grain refiner and solidified under vibration. Therefore, it was observed that the most effective factor in the feeding of the castings and the resulting pore values within the scope of the experiments was the reduction in grain size. In addition, the cast samples were cut at each section level and density measurements were made. The lower part is coded as thin, the middle section as medium, and the uppermost region as thick, with density measurements conducted. The results obtained are given graphically in Figure 4.

As can be seen in the graph given in Figure 4, different densities and accordingly different values of pores appeared in the sections of the sample under all test conditions. Depending on the solidification time, the amount of pores in the lower parts with the thinnest cross-section was found to be low depending on the solidification time, while it was found to increase in the middle and upper parts. This is related to the solidification time, also known as the modulus criterion in castings (Kayıkçı and Akar 2007), (Şirin and Çolak 2009). Depending on the casting test conditions, mold material and section thickness, the parts that solidify earlier have a finer grained structure and it is observed from the related results that the amount of pores decreases and the feedability increases. For example, the pore value of the sample obtained under normal casting conditions in the sand mold was measured as 1.92% in the lowest small part, 2.38% in the middle part and 4.14% in the top part. As a result of the density measurements made on the whole sample for the relevant part, the pore value was found to be 3.28% as given in Table 2. When the sample obtained under the same conditions was cut horizontally, different pore values were found at each level. This difference reveals the difference in density due to the whole sample and solidification time.

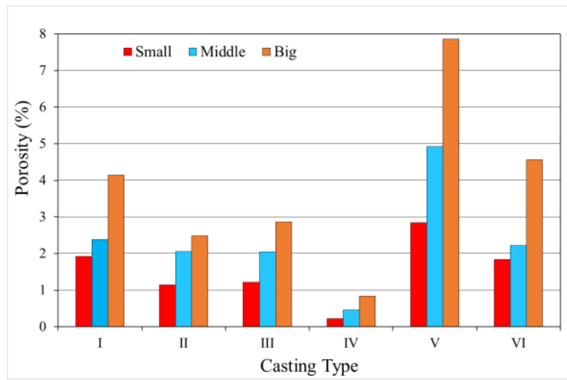

Figure 4. Pore values depending on casting sample section thicknesses. (I- Gravity Sand Casting, II- Casting Under Vibration, III- Sand Casting with TiB Addition, IV-Under Vibration with TiB Addition, V- Gravity Plaster Casting, VI- Casting Under Vibration)

3.4. Examination of the Cross-Sectional Surfaces of the Castings

In order to examine the pore condition and surface depression on the cross-sectional surfaces of the cast specimens, the specimens separated from the sprue connections were cut vertically in the middle and scanned images were obtained after sanding the surfaces. Figure 5 shows the scanned cross-sectional surface images of the specimens obtained from the experiments performed under normal casting conditions and Figure 6 shows the cross-sectional images of the cast specimens solidified under vibration.

Sand Mold	Sand Mold + TiB	Plaster Mold

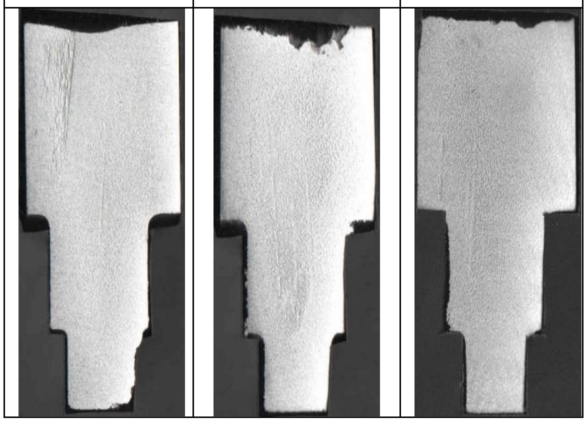

Figure 5. Cross sectional images of casting test speciment solidified under normal conditions.

Sand Mold	Sand Mold + TiB	Plaster Mold

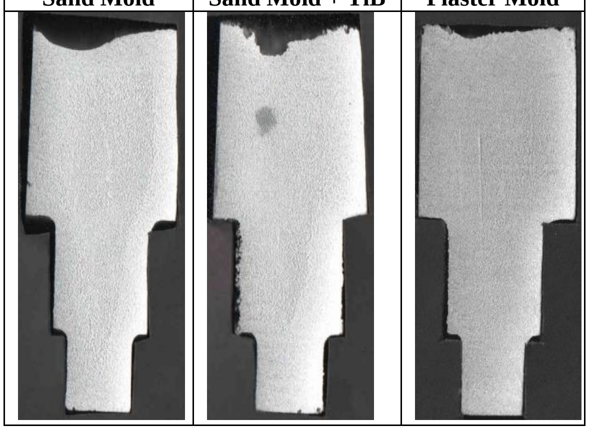

Figure 6. Cross sectional images of casting test specimen solidified under vibration.

When the cross-sectional surface scans of the cast samples given in Figure 5 and Figure 6 were examined, it was seen that the parts started to solidify from the bottom section, depending on the geometry of the cast samples. It is observed that the solidification direction continues and solidification occurs in the thickest section upper regions of the part. Therefore, shrinkage occurred on the casting surface under all casting conditions. Depending on the changing casting conditions, the shrinkage on the surface varies. This situation is thought to be related to the fact that the feed path remains open depending on the grain size, as expressed in the density measurements. When the existing literature on the subject is investigated, it is known that the addition of grain refiners increases the feedability in castings (Lee et al.1990), (Sabau and Viswanathan 2002). It is also known that mechanical vibration has a grain refining effect and reduces the grain size in castings (Kao and Chang 1996; (Sabau and Viswanathan 2002). In the examinations made according to the depressions on the casting surface, depending on the grain size, the shrinkage shape was observed to be conical downwards in some castings and wide in the castings with small grain sizes. Despite the pore values revealed according to the density measurement results in the cast samples, it is seen as if there is no shrinkage in the cross-sectional surface scanning picture. This situation shows that there is no macroporosity in the casting and possible shrinkages may occur as micropores. It is thought that this situation will be revealed in microstructure examinations in castings.

4. *General Results*

The results obtained from the experiments are listed below;

- It was determined that there were depressions on the surfaces of the RPT test samples and the liquid metal quality was appropriate in the cross-sectional surface examinations.
- It has been observed that the density and calculated porosity values in the casting samples vary depending on the casting conditions.
- It has been determined that the vibration affecting the casting molds during solidification has a positive effect on the internal structure and therefore the nutrition of the casting.
- It has been determined that the porosity values in castings made in sand molds with the addition of grain refiners decrease even more than in castings without additions.
- Among all casting tests, the highest porosity occurred in plaster mold casting samples. This is related to the fact that the heat conduction coefficient of the plaster mold is lower than the sand mold and the solidification in the mold is slower.
- It has been determined that the amount of pores in the thinnest-sectioned lower parts of the casting samples is low, depending on the solidification time, and increases in the middle and upper parts.
- It was observed that, depending on the geometry of the cast specimens, the parts started to solidify from the lower section and the solidification orientation continued and solidification took place in the upper regions of the part with the thickest cross-section.

5. *Acknowledgement*

We would like to thank Muhammet Emin KESKİN and Mehmet TOKATLI for their help during the casting experiments.

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Fekete-Sezegö problem for certain subclass of analytic and univalent functions associated with cosine and sine functions of complex order

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Abstract

The main focus of this paper is to give some coefficient estimates for the analytic and univalent functions on the open unit disk in the complex plane that are subordinated to cosine and sine functions of complex order. For the defined subclass of analytic and univalent functions $C(\tau)_{cos,sin}$, $\tau \in \mathbb{C} - \{0\}$ with the quantity

$$1 + \frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)} - 1\right]$$

subordinated to $\cos z + \sin z$ in the study, we obtain the coefficient estimates for the initial two coefficients and examine the Fekete-Szegö problem.

Keywords: Starlike function, convex function, cosine function, sine function, coefficient estimate, Fekete-Szegö problem, complex order.

1. *Introduction*

In this section, we give some basic information that we will use in proof of the main results and to discuss the studies known in the literature related to our subject.

Let $U = \{z \in \mathbb{C}: |z| < 1\}$ be open an unit disk in the complex plane $\mathbb{C}$ and $H(U)$ denote the class of all analytic functions in U.

By A, we will denote the class of the functions $f \in H(U)$ given by the following series expansion, which satisfying the conditions $f(0) = 0$ and $f'(0) - 1 = 0$

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$
$$= z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}. \quad (1.1)$$

As it is known that the subclass of univalent functions of A is denoted by S, in the literature. This class was first introduced by Köebe (Köebe 1909) and has become the core ingredient of advanced research in this field. Bieberbach (Bieberbach 1916) published a paper in which the famous coefficient hypothesis was proposed. This conjecture states that if $f \in S$ and has the series form (1.1), then $|a_n| \leq n$ for all $n \geq 2$. Many researchers worked hard to solve this problem. But for the first time this long-lasting conjecture was solved by De-Branges (De-Branges 1985) in 1985.

It is well-known that a univalent function $f \in S$ is called a convex function, if this function maps open unit disk U onto the convex shaped domain of the complex plane. The set of all convex functions in U, which satisfies the following condition is denoted by C

$$Re\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > 0, z \in U;$$

that is,

$$C = \left\{f \in S:\ Re\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > 0, z \in U\right\}.$$

Some of the important and well-investigated subclass of S include the class $C(\alpha)$ given below, which is called of the convex functions of order $\alpha(\alpha \in 0,1))$

$$C(\alpha) = \left\{f \in S:\ Re\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > \alpha, z \in U\right\}.$$

It is well-known that an analytical function ω satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ is called Schwartz function. Let $f, g \in H(U)$, then it is said that f is subordinate to g and denoted by $f \prec g$,

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if there exists a Schwartz function ω , such that $f(z) = g(\omega(z))$.

In 1992, Ma and Minda (Ma and Minda 1994) using subordination terminology presented the class $C(\varphi)$ as follows

$$C(\varphi) = \left\{ f \in S: \frac{\left(zf'(z)\right)'}{f'(z)} \prec \varphi(z), z \in U \right\},$$

$\beta \in [0,1]$,

where $\varphi(z)$ is a univalent function with $\varphi(0) = 1$, $\varphi'(0) > 0$ and the region $\varphi(U)$ is star-shaped relative to the point $\varphi(0) = 1$ and symmetric with respect to real axis. Such a function has a series expansion of the following form

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

$b_1 > 0$.

In the past few years, numerous subclasses of the collection S have been introduced as special choices of the class $C(\varphi)$ (see for example (Sokol 2011, Janowski 1970, Arif, et al 2019, Brannan 1969, Sokol and Stankiewcz et al. 2021, Sharma et al. 2016, Kumar and Arora 2020, Mendiratta et a.l 2015, Shi et al 2019, Bano and Raza 2020, Alotaibi et al. 2020, Ullah et al. 2021, Cho et al. 2019, Mustafa et al.2023a, Mustafa et al.2023b, Mustafa et al.2023c, Mustafa et al.2023d, Mustafa et al.2023e, Mustafa et al.2023f, Mustafa et al.2023g)).

Finding bounds for the function coefficients in a given collection is one of the most fundamental problems in geometric function theory, since it impacts geometric features.

The first order of Hankel determinant of the function $f \in S$ defined by

$$H_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and is known as the Fekete-Szegö functional. The functional $H_{2,1}(f, \mu) = a_3 - \mu a_2^2$ is known as the generalized Fekete-Szegö functional, where μ is a complex or real number (Duren 1983). Estimating the upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegö problem in the theory of analytic functions.

In this paper, we introduce a new subclass of analytic and univalent functions on the open unit disk in the complex plane that are subordinated to cosine and sine functions of complex order. For the defined here subclass $C(\tau)_{cos,sin}$, we obtain the coefficient estimates for the initial two coefficients and examine the Fekete-Szegö problem. A similar study for the class $S^{*(\tau)}_{cos,sin}$ was done in (Mustafa et al. 2023g).

2. *Materials and Methods*

By using the definition of subordination, we introduce a new subclass of analytic and univalent functions associated with cosine and sine functions of complex order.

Definition 2.1. For $\tau \in \mathbb{C} - \{0\}$ a function $f \in S$ is said to be in the class $C(\tau)_{cos,sin}$, if the following condition is satisfied

$$1 + \frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)} - 1\right] \prec \cos z + \sin z, z \in U;$$

that is,

$$C(\tau)_{cos,sin} = \left\{ f \in S \,\middle|\, 1 + \frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)} - 1\right] \prec \cos z + \sin z, z \in U \right\}$$

Remark 2.2. Taking $\tau = 1$ in the Definition 2.1, we have the class $C_{cos,sin}$given as follows

$$C_{cos,sin} = \left\{ f \in S: \frac{\left(zf'(z)\right)'}{f'(z)} \prec \cos z + \sinh z, z \in U \right\}.$$

Remark 2.3. In the case $\tau = 1$, we have the class $C_{cos,sin}$ which reviewed in (Mustafa et al. 2023d).

Let P be the class of analytic functions in U satisfied the conditions $p(0) = 1$ and $Re(p(z)) > 0$, $z \in U$, which from the subordination principle easily can written

$$P = \left\{ p \in A: p(z) \prec \frac{1+z}{1-z}, z \in U \right\},$$

where $p(z)$ has the series expansion of the form

$$\begin{aligned} p(z) &= 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} p_n z^n, z \in U. \end{aligned} \quad (1.2)$$

The class P defined above is known as the class Caratheodory functions (Miller 1975).

Now, let us present some necessary lemmas known in the literature for the proof of our main results.

Lemma 2.4 (Duren 1983). Let the function $p(z)$ belong in the class P. Then,

$|p_n| \le 2$ for each $n \in \mathbb{N}$ and $|p_n - \lambda p_k p_{n-k}| \le 2$ for $n, k \in \mathbb{N}, n > k$ and $\lambda \in [0,1]$.

The equalities hold for the function

$$p(z) = \frac{1+z}{1-z}.$$

Lemma 2.5 (Duren 1983). Let analytic function $p(z)$ be of the form (1.2), then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$
$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)y$$

for $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$.

3. *Main Results*

In this section, we give upper bound estimates for the initial two coefficients of the function belonging to the class $C(\tau)_{cos,sin}$ and examine the Fekete-Szegö problem for the class $C(\tau)_{cos,sin}$.
First of all, let us give the following theorem on coefficient estimates.

Theorem 3.1. Let the function $f \in A$ given by (1.1) belong to the class $C(\tau)_{cos,sin}$. Then,

$$|a_2| \leq \frac{|\tau|}{2} \text{ and } |a_3| \leq \frac{|\tau|}{6}\begin{cases} 1 & \text{if } |2\tau - 1| \leq 2, \\ \frac{|2\tau-1|}{2} & \text{if } |2\tau - 1| \geq 2. \end{cases}$$

Proof. Let $f \in C(\tau)_{cos,sin}$, $\tau \in \mathbb{C} - \{0\}$, then there exists a Schwartz function ω, such that

$$1 + \frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)} - 1\right] = cos\, \omega\,(z) + sin\, \omega\,(z).$$

We express the Schwartz function ω in terms of the Caratheodory function $p \in P$ as follows

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1z + p_2z^2 + \cdots .$$

It follows from that

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}p_1z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots \quad . \qquad (3.1)$$

From the series expansion (1.1) of the function $f(z)$, we can write

$$1 + \frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)} - 1\right] = 1 + \frac{1}{\tau}[2a_2z + (6a_3 - 4a_2^2)z^2 + \cdots]. \qquad (3.2)$$

Since

$$cos\, z + sin\, z = 1 + z - \frac{z^2}{2!} + \cdots \quad , (3.3)$$

using equality (3.1), we have

$$sin\, \omega\,(z) + cos\, \omega\,(z) = 1 + \frac{1}{2}p_1z + \frac{1}{2}\left(p_2 - \frac{3}{4}p_1^2\right)z^2 + \cdots \quad . \qquad (3.4)$$

By equalizing (3.2) and (3.4), then comparing the coefficients of the same degree terms on the right and left sides, we obtain the following equalities for two initial coefficients of the function $f(z)$

$$a_2 = \frac{\tau p_1}{4}, \qquad (3.5)$$

$$a_3 = \frac{\tau}{48}[4p_2 + (2\tau - 3)p_1^2]. \qquad (3.6)$$

Using Lemma 2.4, from the equalities (3.5) we can easily see that

$$|a_2| \leq \frac{|\tau|}{2}.$$

Using Lemma 2.5, from the equality of (3.6) we can write

$$a_3 = \frac{\tau}{48}[(2\tau - 1)p_1^2 + 2(4 - p_1^2)x],$$

for $x \in \mathbb{C}$ with $|x| \leq 1$. Applying triangle inequality, from this equality we obtain

$$|a_3| \leq \frac{|\tau|}{48}[|2\tau - 1|t^2 + 2(4 - t^2)\xi],$$

where $\xi = |x|$ and $t = |p_1|$.
If we maximize the function $\varphi: [0,1] \to \mathbb{R}$ defined as

$$\varphi(\xi) = |2\tau - 1|t^2 + 2(4 - t^2)\xi, \xi \in [0,1],$$

we write

$$|a_3| \leq \frac{|\tau|}{48}[(|2\tau - 1| - 2)t^2 + 8], t \in [0,2].$$

From this, immediately obtained the desired estimate for $|a_3|$.
Thus, the proof of theorem is completed.

In the case $\tau \in \mathbb{R} - \{0\}$, Theorem 3.1 is given as follows.
Theorem 3.2. Let the function $f \in A$ given by (1.1) belong to the class $C(\tau)_{cos,sin}$,$\tau \in \mathbb{R} - \{0\}$. Then,

$$|a_2| \leq \begin{cases} \frac{-\tau}{2} & \text{if } \tau \in (-\infty, 0), \\ \frac{\tau}{2} & \text{if } \tau \in (0, +\infty). \end{cases}$$

and

$$|a_3| \leq \frac{\tau}{6}\begin{cases} \frac{2\tau-1}{2} & \text{if} \tau \in -\infty, -\frac{1}{2}, \\ -1 & \text{if } \tau \in -\frac{1}{2}, 0\Big), \\ 1 & \text{if } \tau \in 0, \frac{3}{2}, \\ \frac{2\tau-1}{2} & \text{if } \tau \in \frac{3}{2}, +\infty\Big). \end{cases}$$

Taking $\tau = 1$ in Theorem 3.1, we obtain the following estimates for the initial two coefficients for the function belonging to the class $C_{cos,sin}$.

Theorem 3.3. (Mustafa et al. 2023d) Let the function $f \in A$ given by (1.1) belong to the class $C_{cos,sin}$. Then,

$$|a_2| \leq \frac{1}{2} \text{ and } |a_3| \leq \frac{1}{6}.$$

Now, let us give the following theorem on the Fekete-Szegö inequality for the class $C(\tau)_{cos,sin}$.

Theorem 3.4. Let the function $f \in A$ given by (1.1) belong to the class $C(\tau)_{cos,sin}$, $\tau \in \mathbb{C} - \{0\}$ and $\mu \in \mathbb{C}$ or $\mu \in \mathbb{R}$. Then,

$$|a_3 - \mu a_2^2| \leq \frac{|\tau|}{12}\begin{cases} 2 & \text{if } |\tau(2-3\mu)-1| \leq 2, \\ |\tau(2-3\mu)-1| & \text{if } |\tau(2-3\mu)-1| \geq 2, \end{cases}$$

where

$$\gamma(\tau,\mu) = \frac{1}{12}|\tau(2-3\mu)-1|.$$

Proof. Let $f \in C(\tau)_{cos,sin}$,$\tau \in \mathbb{C} - \{0\}$ and $\mu \in \mathbb{C}$ or $\mu \in \mathbb{R}$. Then, from the equalities (3.5) and (3.6), we can write

$$a_3 - \mu a_2^2 = \frac{\tau}{48}\{4p_2 + [\tau(2-3\mu)-3]p_1^2\}.$$

Using Lemma 2.5, the last equality we can write as follows

$$a_3 - \mu a_2^2 = \frac{\tau}{48}\{[\tau(2-3\mu)-1]p_1^2 + 2(4-p_1^2)x\} \quad (3.7)$$

for $x \in \mathbb{C}$with $|x| \leq 1$. Applying triangle inequality to last equality, we have

$$|a_3 - \mu a_2^2| \leq \frac{|\tau|}{48}\{|\tau(2-3\mu)-1|t^2 + 2(4-p_1^2)\xi\}$$

with $t = |p_1| \in [0,2]$ and $\xi = |x|$.

By maximizing the function

$$\psi(\xi) = |\tau(2-3\mu)-1|t^2 + 2(4-p_1^2)\xi, \ \xi \in [0,1],$$

we obtain the following inequality

$$|a_3 - \mu a_2^2| \leq \frac{|\tau|}{48}\{[|\tau(2-3\mu)-1|-2]t^2 + 8\}, \quad t \in [0,2]. \quad (3.8)$$

From this, we obtain the desired result of theorem. Thus, the proof of theorem is completed.

In the cases $\tau \in \mathbb{R} - \{0\}$ and $\mu \in \mathbb{R}$, Theorem 3.4 is given as follows.

Theorem 3.5. Let the function $f \in A$ given by (1.1) belong to the class $C(\tau)_{cos,sin}$, $\tau \in \mathbb{R} - \{0\}$ and $\mu \in \mathbb{R}$. Then,

$$|a_3 - \mu a_2^2| \leq \frac{|\tau|}{6}\begin{cases} -6\gamma(\tau,\mu) & \text{if } \tau < 0 \text{ and } \mu \leq \mu_1(\tau), \\ 1 & \text{if } \tau < 0 \text{ and } \mu \in [\mu_1(\tau), \mu_2(\tau)], \\ 6\gamma(\tau,\mu) & \text{if } \tau < 0 \text{ and } \mu_2(\tau) \leq \mu, \\ 6\gamma(\tau,\mu) & \text{if } \tau > 0 \text{ and } \mu \leq \mu_2(\tau), \\ 1 & \text{if } \tau > 0 \text{ and } \mu \in [\mu_2(\tau), \mu_1(\tau)], \\ -6\gamma(\tau,\mu) & \text{if } \tau > 0 \text{ and } \mu_1(\tau) \leq \mu, \end{cases}$$

where

$$\gamma(\tau,\mu) = \frac{1}{12}|\tau(2-3\mu)-1|, \ \mu_1(\tau) = \frac{2\tau+1}{3\tau}, \ \mu_2(\tau) = \frac{2\tau-3}{3\tau}.$$

Taking $\tau = 1$ in Theorem 3.4, we obtain the following estimates for the Fekete-Szegö functional for the function belonging to the class $C_{cos,sin}$.

Theorem 3.6. (Mustafa et al. 2023d) Let the function $f \in A$ given by (1.1) belong to the class $C_{cos,sin}$ and $\mu \in \mathbb{C}$ or $\mu \in \mathbb{R}$. Then,

$$|a_3 - \mu a_2^2| \leq \frac{1}{12}\begin{cases} 2 & \text{if } |1-3\mu| \leq 2, \\ |1-3\mu| & \text{if } |1-3\mu| \geq 2. \end{cases}$$

Taking $\mu = 0$ and $\mu = 1$ in Theorem 3.4, we get the second result of Theorem 3.1 and the following result for the first order Hankel determinant, respectively.

Corollary 3.1. If the function $f \in A$ given by (1.1) belong to the class $C(\tau)_{cos,sin}$, then

$$|a_3 - a_2^2| \leq \frac{|\tau|}{12}\begin{cases} 2 & \text{if } |1+\tau| \leq 2, \\ |1+\tau| & \text{if } |1+\tau| \geq 2. \end{cases}$$

Setting $\tau \in \mathbb{R} - \{0\}$ in Theorem 3.4, we obtain the following result for the first order Hankel determinant.

Corollary 3.2. If the function $f \in A$ given by (1.1) belong to the class $C(\tau)_{cos,sin}$, then

$$|a_3 - a_2^2| \leq \frac{|\tau|}{12}\begin{cases}\tau + 1 \text{ if} & \tau \in -\infty, -3, \\ 2 \quad \text{if } \tau \in -3,0) \cup 0,1, \\ 2 \quad \text{if} & \tau \in 0,1, \\ \tau + 1 \text{ if} & \tau \in 1, +\infty).\end{cases}$$

Setting $\tau = 1$ in Corollary 3.2, we obtain the following result obtained in (Mustafa et al. 2023d).

Corollary 3.3. If the function $f \in A$ given by (1.1) belong to the class $C_{cos,sin}$, then,

$$|a_3 - a_2^2| \leq \frac{1}{6}.$$

4. Discussion

In the study, we defined a new subclass of analytic and univalent functions and give coefficient estimates for this class. Also, Fekete-Szegö problem is solved for defined class. In addition, the results obtained here found were compared with the results available in the literature.

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Effects of Exogen Lipoic Acid on the Mineral Compositions of Maize

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Abstract

This study, which does not aim to research the possible modulating role of exogenous lipoic acid supplementation on the inorganic composition of the root, coleoptile and endosperm of germinated maize, is the first one in the literature. Maize seeds were germinated with lipoic acid at concentrations of 10, 15, 25 and 30 μmol L^{-1} for 4 days as defined in the material method section for the study. Macro elements (magnesium (Mg), phosphorus (P), potassium (K) and calcium (Ca)), the concentrations of essential and nonessential micro elements (boron (B), manganese (Mn), iron (Fe), nickel (Ni), copper(Cu), zinc (Zn), beryllium (Be), aluminum (Al), selenium (Se) and molybdenum (Mo)) were analyzed using ICP-MS technique in root, coleoptile and endosperm of maize plant. When roots and coleoptiles treated with all lipoic acid applications compared with control groups, an important decrease was determined in endosperms (except for Mn) while a significant increase was recorded in Mg, P, K, Ca, B, Mn, Fe, Ni, Cu, Zn, Be, Se, Mo and Al contents (except for A1 in coleoptile groups). When taking account of the maximum changes in the concentrations of the analyzed elements, the best results were obtained in the application of 25 μmol L^{-1} lipoic acid. It can be said that the application of lipoic acid significantly affects the inorganic composition of the plant by increasing the inorganic element contents in maize through its transport from the endosperms to the roots and coleoptiles when all the results are considered together.

Keywords: Lipoic Acid, ICP-MS, Inorganic Element Contents, Maize, Germination.

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1. *Introduction*

Lipoic acid (LA; 6.8-thiotic acid or 1,2-dithiolane-3-pentanoic acid) is a sulphur-containing 8-carbon disulphide molecule which is in charge with energy production and metabolism, with important functions in pyruvate and glycine metabolism and it also plays a role as a cofactor of some enzymes (pyruvate dehydrogenase complex and glycine decarboxylase complex, and α-ketoglutarate dehydrogenase) (Gueguen et al., 2000; Packer and Cadenas, 2010). There are two forms of lipoic acid, LA and DHLA, and these forms can be easily transformed into one another by oxidation-reduction reactions (Karaca, 2008).

In addition to its role in energy metabolism mentioned above, in both LA and DHLA in-vivo and in-vitro conditions, it chelates metal ions by reducing the oxidized forms of vitamin C, glutathione and tocopherol molecules, detoxify free radicals, repairing oxidized proteins, as well as exhibit a unique antioxidant effect that can inhibit lipid peroxidation (Gorcek and Erdal, 2015; Sears, 2013; Turk et al., 2018). Due to the fact that the LA molecule has a dithiol ring, both oxidized and reduced forms of LA show strong redox activity, this situation makes it unique compared to other antioxidants (Navari-Izzo et al., 2002;Gorcek and Erdal, 2015; Yildiz et al., 2015; Turk et al., 2018). In addition, the only antioxidant property of LA and DHLA forms of lipoic acid, which can be easily soluble in both aqueous and lipid environments, facilitates its effect on both the cytoplasm and the cell membrane. Although numerous studies have been conducted on the effects of LA on the treatment of various metabolic pathways and diseases in animals, similar studies on the presence, effect, and function of LA in plants are still in its infancy. In the first studies, the biosynthetic pathway of LA was investigated in the plastid and mitochondria of Arabidopsis thaliana (Yasuno and Wada, 2002), changes in the endogenous content of LA and DHLA in wheat seedlings grown under copper stress and basil seeds exposed to salts stress were examined (Sgherri et al., 2002; Tarchoune et al., 2013). Another study noted that LA also had a significant effect on proteins associated with basic metabolic processes such as photosynthesis, energy metabolism, signal conduction, and stress defense (Yildiz et al., 2015). In other studies, in addition to the antioxidant properties of LA, it has also been proven to have the ability to chelate metal ions (Sgherri et al., 2002). Turk et al. (2018) investigated the antioxidant effects of lipoic acid in wheat seedlings germinated under lead stress. In addition to the effects of lipoic acid on the antioxidant system, the effect of LA on essential elements in wheat seedlings exposed to salt stress was determined in another study (Gorcek and Erdal, 2015). In a thesis study conducted with germinated maize seeds in recent years, LA's relationship with energy metabolism along with its antioxidant effect has been tried to be explained (Karayel, 2019). When the current literature is examined, the number of studies carried out, especially in the germination phase, which is the first and most important cycle of plants, is quite limited. However, there is no research in the current literature indicating the effect of lipoic acid on the content of inorganic elements of germinated seeds.

Nutrients are one of the important factors in the survival of plants like sufficient water, light, temperature, and air (Turk, 2019). They participate in many metabolic processes such as structural, electrochemical, and catalytic roles in biological organisms like plants. The lack of one or more of these essential elements can extremely affect the life cycle of plants negatively. Plants need 18 basic elements to maintain their growth and development properly. These basic elements are divided into two classes, including macro elements and microelements according to the amount needed by plants. While microelements (boron (B), manganese (Mn), iron (Fe), copper (Cu), zinc (Zn), nickel (Ni), and chlorine (Cl)) are vital for plants in very small quantities, excess amounts of them can cause toxic effects on plants. Plants need macroelements (magnesium (Mg), phosphorus (P), potassium (K), calcium (Ca), and sulfur (S)) at higher rates than microelements(Dumlupinar et al., 2007; Genisel et al., 2015; Turk, 2019). Minerals play vital roles in increasing plant tolerance, playing a signaling role and ensuring the stability of the cell membrane (Dumlupinar et al., 2007; Erdal and Dumlupinar, 2011). In addition, minerals take part in synthesis reactions such as photosynthesis and protein synthesis, and also function vitally as the building blocks of DNA, RNA, ATP, NADP, proteins, vitamins and enzymes. (Dumlupinar et al., 2007; Erdal and Dumlupinar, 2011).

When it is considered the significant effects of minerals and LA separately on plant metabolism as a whole in the present study, it was tried to clarify how maize had an effect on macro and microelement concentrations in root, coleoptile, and endosperm during the germination phase of exogenous LA application for the first time.

2. *Materials and Methods*

Experiment condition, Lipoic Acid applications, and growth analyses

In the present study, maize seed (*Zea mays* cv. Hido) was used. Before planting, the seeds were quickly washed with 96% alcohol and subjected to surface sterilization in 5% sodium hypochlorite for approximately 10 minutes. After the sterilization process, the seeds were thoroughly washed with distilled water. While the control group was shaken in pure water, the application groups were continuously shaken in LA solutions (10, 15, 25 and 30 μM L^{-1} lipoic acid) for 6 hours in room conditions. The solutions were prepared by dissolving L-(±)-α-Lipoic acid (Sigma Aldrich, ≥99.0% (titration) powder) in high purity water. After the swelling period, 13 ml of their own solution was added for each group and the 10-swollen maize seeds were placed on Petri dishes containing two layers of filter paper. The seeds were germinated at 25 °C for 4 days in dark conditions. The germinating seeds were harvested on day 4 the root and the cotyledon lengths of maize seedlings were measured by using the centimetric scale.

ICP-MS analysis and sample preparation procedure

Maize roots, coleoptiles and endosperms were oven-dried at 70 °C for 72 hours. The dried samples were then grounded with liquid nitrogen for microwave digestion procedure. Milestone Ethos up Microwave system was used for the digestion process. The specific procedure for maize was chosen and all samples were weighed in Teflon containers to a maximum of 0.5 g.

Next, nine milliliters of HNO_3 (65%) and one milliliter of H_2O_2 (30%) were added into the samples. Then all containers were sealed and placed in the microwave system and conducted digestion process at 210 °C for 35 minutes. After degradation, all samples were transferred to sterile vials and their volume was made up with 40 ml of ultrapure water. The samples were filtered through a 0.45 µm filter and diluted in 2% HNO_3 solution and loaded into the instrument. Concentrations of B, Mg, Al, P, K, Ca, Mn, Fe, Ni, Cu, Zn, Be, Se, and Mo were determined in inductively coupled plasma mass spectrophotometry (ICP / MS; Agilent 7800, Japan) (Turk, 2019). All standards and reagents were obtained from Merck and Agilent.

Before analysis, the quartz, glass materials and cone parts in the whole system were thoroughly cleaned with 5% HNO_3 solution. Before the analysis, the instrument was calibrated with Tune solution. Different seven calibration points were determined for elements at 0, 10, 25, 50, 100, 250 and 500 ppb concentrations. All of the standards were prepared in %2 HNO_3 solution.

Statistical analysis

The study was carried out with 3 independent samples and 3 parallel readings. The analysis of variance (ANOVA) was used to compare significant differences between samples. Statistical analyzes were performed using SPSS 20.0. Statistical significance was defined as $p < 0.05$ (Duncan's multiple range method).

3. Results and Discussion

Plants need essential elements to maintain their productivity and water-mineral balance alongside growth and development (Turk et al., 2018). Microelements are very important for plants although they are needed in very little quantity compared to macroelements. The quantitative decrease in the content of any microelements inhibits or delays the growth and development of the plant, no matter how small the amount needed. In the current study, we found that LA applications (especially 25 µM LA) significantly increased the concentrations of essential microelements (B, Mn, Fe, Ni, Cu and Zn) at the root and coleoptile of maize seeds. Compared to the control, it was observed that the concentration of Mn increased in endosperms while a decrease happened in the concentration of other microelements. It is known that Mn, Cu, Fe, Zn, and Ni activate many enzymes as cofactors. Zn catalyzes enzyme-substrate binding, participates in nucleic acid metabolism and protein synthesis, and also it may be necessary for chlorophyll synthesis in some plants (Erdal and Dumlupinar, 2011), while Fe is an essential element for chlorophyll production, protein biosynthesis, and starch content (Pazurkiewicz-Kocot et al., 2008). Cu plays a role as a redox cofactor in the basic processes of cellular metabolism such as respiration and photosynthesis, furthermore, like Ni, also participates in hormone signaling, lignin biosynthesis, and oxidative stress (Broadley et al., 2012; Vatansever et al., 2017). Moreover, while Mn and Ni act as cofactors in the activation of defense enzymes including SOD and CAT (Marschner, 1995; Wei Yang et al., 2008; Fabiano et al., 2015; Shahzad et al., 2018). B has a key role in cell division, elongation, and photosynthesis (Ahmad and Prasad, 2011).

There are many studies in the literature showing that microelements affect growth, plant yield, and macroelement balance. For example; B can affect the absorption of N, P, and K, and while a lack of B can change the optimal balance of these three macroelements (Turk et al., 2016) a high concentration of Mn accumulation can prevent the reception and displacement of other essential elements such as Ca, Mg, Fe, and P due to their similarities in ionic radii (Marschner, 2012; Millaleo et al., 2013). In the current study, significant increases were found in B, Cu, Zn, Fe, Mn, and Ni concentrations of 107.07%, 68.71%, 159.80, 102.02%, 78.78%, and 157.33 respectively, compared with control groups in roots treated with 25 µM LA (Table 1a-1b). In coleoptiles, it was determined that the contents B, Cu, Zn, Fe, Mn, and Ni increased significantly in 25 µM LA groups according to control, 56.49%, 74.57%, 95.92%, 266.76%, 66.11%, and 8772, 27% respectively (Table 1a-1b). Unlike root and coleoptile results, 96.63%, 73.69%, 55.27%, 32.69%, and 76.82% reductions were found in B, Cu, Zn, Fe and Ni concentrations respectively, compared to control in endosperms treated with 25 µM LA (Table 1a-1b). Unlike other elements, an increase of Mn content of 77.46% was recorded in endosperms applied 25 μM LA compared to the control (Table 1a).

It can be expressed that these increases, which occur by transporting from endosperms to the root and coleoptile, contribute to plant growth with cell elongation. These results are similar to increases in root and coleoptile lengths of LA application groups (Figure 1). In the 25 μM LA application, an increase of 30.44% was recorded in root length and 28.21% in coleoptile length compared to control groups (Figure 1). The decrease in Cu content in endosperms can be said to have a positive effect on plant growth, taking into account root and coleoptile length values, as well as being effective in all cellular metabolic processes by moving from here to the root and coleoptiles. The increase of mineral Zn in roots and coleoptiles and the decrease in endosperms support that Zn is transported from endosperms to roots and coleoptiles as in the Cu element. The increase in the determined mineral Fe is very remarkable in terms of the inclusion of various enzymes, proteins and chlorophyll, which are as important for the growth and development of the plant as well as for its survival. This increase is very important in terms of cofactoring the enzymes that play a vital role in the plant, positively affecting lipid and carbohydrate metabolism and triggering plant growth. Besides, it is possible to mention that increases in Fe, Ca, Mg and P contents with Mn (to be mentioned later) are not high in the groups that applied LA and this accumulation is at the demanded level for the plant, since extreme Mn accumulation can prevent the removal and displacement of essential elements such as Ca, Mg, Fe, and P because of similarities in ionic radii (Marschner, 2012; Millaleo et al., 2013).

When the vital importance of the Ni element mentioned above is taken into account, the increase in Ni content is quite important for germinated maize. The increase in this element in all LA groups can contribute to plant growth by positively affecting many vital processes from photosynthesis to stress, endurance and defense, from nitrogen metabolism to plant growth. In this study, it can be stated that essential microelements are transported from endosperms to roots and coleoptiles, and LA contributes to supporting plant growth by increasing microelements content, especially in roots and coleoptiles. It can be said that lipoic acid can achieve this in line with the faster growth of plants by affecting the intake, distribution, and relocation of inorganic elements

Non-essential elements have also important roles in plant metabolism. Beryllium (Be) is not a major nutrient for higher plants but a low amount of Be can stimulate the growth of certain plant species and also activate certain enzymes (Sajwan et al., 2003). It has also been reported that it is possible to interfere with plant nutrition in line with the inhibition of certain enzymes of Be as well as an antagonism pathway between Be and Ca, Mg, or phosphate (Williams and Le Riche, 1968). In our study, while the Be content decreased by 59.44% compared to the control in endosperms treated with 25μM LA, it increased by 70.56% in the root and 82.65% in coleoptile (Table 2). Selenium (Se), one of the trace elements, has a toxic effect on plants at high concentrations whereas at low concentrations it has positive effects on plants (Calabrese and Baldwin, 2003). Se is similar to sulfur which is a chemically important plant macronutrient (Cruz-Jímenez et al., 2005). A high rate of Se content suppresses protein synthesis, nucleic acid synthesis, growth and organogenesis (Terry et al., 2000). Se, with affecting the distribution of metals, sometimes also increases the excretion of toxic elements due to its antioxidant properties (Pazurkiewicz-Kocot et al., 2003, 2008). In this study, while an increase in Se content in the root by 38.17% and 163.67% in coleoptile in 25μM LA application compared with the control, a rate of 40.21% reduction was recorded in endosperm tissues (Table 2). Se accumulation in the root and coleoptile of the plant with LA application may stem from the antioxidant properties of this mineral and its capacity to protect the plant against possible oxidative damage. Aluminum (Al) is the third most abundant element in the earth's crust. In this research, Al content decreased in coleoptile and endosperms but increased in the roots. For increased Al in the roots, we can say that it can stimulate nutrient absorption by promoting root growth in plants. A high rate of Al concentration causes rapid inhibition of plant root growth under normal conditions (Horst et al., 1992; Ryan et al., 1995) damaging the root system, restricting nutrient and water intake (Kochian et al., 2004). In this study, it was determined that in the 25μM LA application, while Al content in coleoptile and endosperm tissues decreased by 59.17% and 61.43% respectively compared to the control, an increase of 11.30% in the root (Table 2). The increase in the root length and Al content (Figure 1) shows that Al concentration is not toxic to germinating maize seeds. Because the primary purpose of Al toxicity is the root tips in the distal part of the transition zone between the elongation area and cell division, and in the main binding regions where the root meristem is localized in the cell walls (Sun et al., 2020). In some studies with the tea plant, the increase in root growth and rooting in the presence of Al has a quality that supports our finding (Fung et al., 2008; Mukhopadyay et al., 2012; Sun et al., 2020). Molybdenum (Mo) is a necessary micronutrient for the growth of many living organisms (Sauer and Frebort, 2003; Kaiser et al., 2005). The main physiological function of molybdenum in higher plants is that an essential component of mononuclear Mo enzymes, which play a vital role in many metabolic processes such as C-, N- and S-cycles in plants (Stallmeyer et al., 1999; Mendel and HaÈnsch, 2002). In the current study, it was determined that the Mo content decreased by 63.60% in endosperm compared to the control in 25µM LA application but increased by 214.90% and 283.03% in root and coleoptile respectively (Table 2). It can be stated that it has a positive effect on plant growth and development by participating in the structure of enzymes affecting important metabolic processes thanks to the transportation of Mo content from endosperms to roots and cotyledons.

Potassium (K) which is known as the macro element and has essential importance is a cation abundant in plants and it has vital roles in a wide range of metabolic pathways, including the transport of sugar and nutrients, photosynthesis, protein synthesis, enzyme activation, product quality, osmotic potential and regulation of stoma movement (Erdal and Dumlupinar, 2011; Genisel et al., 2012; Turk and Erdal, 2015; Turk, 2019). In a study, it was noted that LA application provided a significant increase in the K content of wheat seedlings grown under normal and salty conditions compared to the control group (Gorcek and Erdal, 2015). In this study, again, it was also noted that lipoic acid had a significant effect on photosynthetic activity by improving the leaf surface area, chlorophyll content and Rubisco activity. In the current study, with 25µM LA application compared to the control, the K content increased significantly by 69.95% in the root and 56.71% in coleoptile while the endosperm decreased by 31.20% (Table 3). It can be said that increased K content may cause an increase in growth parameters by stimulating biosynthesis reactions and the transportation of related products. Another important macro element, phosphorus (P), has many important roles in plant metabolism (Turk et al., 2016). Since P takes part in the structure of NADP and ATP, it plays very important roles in metabolic events such as photosynthesis, respiration and fatty acid biosynthesis (Dumlupinar et al., 2007; Erdal and Dumlupinar, 2011). P is also a component of DNA and RNA (Taiz and Zeiger, 2003). In the current study, it was recorded a decrease of 25,09% in P content in endosperms applied 25µM LA compared to the control while a 72.09% increase in roots and 61.68% in coleoptiles (Table 3). It can be said that the P content which is vitally important during germination and necessary for energy metabolism can have an influence upon energy and other metabolic pathways, moving from the endosperms of LA applied groups to their roots and cotyledons. In a thesis study on wheat seedlings exposed to salt stress, significant increases in P content with the LA application were observed compared to the control group (Gorcek, 2013). In another thesis study with maize, significant increases were recorded in the gene activities of the enzymes (citrate synthase, cytochrome oxidase, pyruvate dehydrogenase, ATP synthase) which participate in the energy metabolism in LA-applied roots and coleoptiles compared to the control (Karayel, 2019). Based on these results, it can be said that LA application can stimulate mitochondrial respiration and other pathways by increasing the P content at the roots and coleoptiles. Similar to P, endosperms treated with 25 µM LA showed a decrease in calcium (Ca) and magnesium (Mg) contents by 63.07% and 17.23% respectively. Increases of 145.44% and 68.54% were recorded in the root for Ca and Mg, and 30.13% and 65.40% for Ca and Mg in coleoptile respectively (Table 3). Ca is a component of the cell wall and ensures root growth even though the effects of elements on plant metabolism are different from each other. Ca also acts as a secondary messenger against both environmental and hormonal stimuli in addition to its use in the synthesis of middle lamella in newly divided cells in the plant (Sanders et al., 1999). Ca also affects many metabolic processes by binding with the calmodulin present in the cytosol. While Ca is essential for plant cell membranes to maintain normal function, Mg participates in many physiological and biochemical reactions involved in plant growth and development (Turk and Erdal, 2015; Turk, 2019). Mg, which participates in the structure of chlorophyll, has an important effect on chloroplast enzymes (Mengel et al., 2001). It was recorded that in a study with wheat seedlings grown under normal and salty conditions, LA application increases Ca and Mg and also ensures important increases in total chlorophyll content (Gorcek and Erdal, 2015). In the current study, the decrease in the minerals Ca and Mg in the endosperms and the increase in the roots and coleoptiles indicate that they are transported in these elements as in others. Increases in Ca content support the increase in root and coleoptile length (Figure 1). Considering this situation, it can be expressed that Ca is transported to be used in many processes such as root and coleoptile formation, cell permeability, intracellular communication and osmotic potential. The increase in Mg content in root and coleoptile may suggest that it is carried for use specifically in DNA and RNA synthesis and respiration, as well as it can act as a cofactor for chlorophyll formation and chloroplast enzymes.

4. *Conclusion*

Considering all these results, LA applications stimulated nonessential element contents as well as the positive changes it created in basic macro and microelement concentrations, and the best results were obtained in the application of 25µM LA. It can be stated that the positive effect of lipoic acid on macro and microelements regulates plant metabolism by affecting biochemical processes, thus contributing to the growth and development of the plant. Lipoic acid indicated this effect by transporting macro and microelements from endosperm to root, especially to coleoptiles. This finding will provide a very important contribution in order to illuminate the other metabolic processes lipoic acid connected in addition to understanding its mechanism on plant metabolism at the macro and microelement level.

5. *Acknowledgement*

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Declaration of competing interest

The author declares no conflict of interest.

CRediT authorship contribution statement

Hulya Turk: Conceptualization, Methodology, Formal analysis, Investigation, Resources, Writing - Original Draft, Writing - Review & Editing, Visualization.

Table 1a. Effects of lipoic acid applications on essential microelement concentrations in 4-day-old maize endosperm, root and coleoptiles

Essential Micro Elements (mg. kg^{-1})									
	Boron			Copper			Zinc		
	E	R	C	E	R	C	E	R	C
C	33.87^{a}	21.60^{e}	26.12^{e}	0.74^{b}	6.96^{e}	4.45^{e}	65.13^{a}	65.27^{e}	64.63^{c}
10 LA	17.55^{b}	25.36^{d}	31.40^{c}	0.59^{c}	7.19^{d}	5.14^{d}	53.67^{b}	139.88^{c}	79.11^{c}
15 LA	7.75^{c}	34.09^{b}	35.33^{b}	0.27^{d}	7.41^{c}	5.98^{b}	37.80^{c}	148.05^{b}	108.36^{ab}
25 LA	1.48^{e}	44.72^{a}	40.87^{a}	0.19^{e}	11.74^{a}	7.76^{a}	29.14^{d}	169.56^{a}	126.63^{a}
30 LA	6.83^{d}	28.10^{c}	30.24^{d}	1.13^{a}	8.29^{b}	5.72^{c}	56.90^{b}	130.08^{d}	84.68^{bc}
Different letters in the same group indicate statistically significant differences ($p<0.05$)									

(C: Control, 10 LA: 10 µmol L^{-1} Lipoic acid, 15 LA: 15 µmol L^{-1} Lipoic acid, 25 LA: 25 µmol L^{-1} Lipoic acid, 30 LA: 30 µmol L^{-1} Lipoic acid and E: Endosperm, R: Root and C: Coleoptile)

Table 1b. Effects of lipoic acid applications on essential microelement concentrations in 4-day-old maize endosperm, root and coleoptiles

Essential Micro Elements (mg. kg^{-1})									
	Iron			Manganese			Nickel		
	E	R	C	E	R	C	E	R	C
C	39.52^{a}	54.38^{e}	55.57^{e}	8.46^{d}	8.75^{e}	13.72^{d}	0.51^{a}	1.71^{d}	0.13^{e}
10 LA	35.01^{b}	68.75^{d}	73.19^{c}	9.71^{c}	9.50^{d}	15.48^{c}	0.47^{b}	1.85^{c}	1.95^{c}
15 LA	30.04^{d}	75.79^{c}	85.40^{b}	12.00^{b}	10.36^{c}	18.44^{b}	0.43^{c}	2.81^{b}	2.99^{b}
25 LA	26.60^{e}	109.85^{a}	205.47^{a}	15.02^{a}	15.65^{a}	22.80^{a}	0.12^{d}	4.40^{a}	11.50^{a}
30 LA	34.50^{c}	77.84^{b}	61.71^{d}	9.86^{c}	11.02^{b}	12.67^{e}	0.42^{c}	1.26^{e}	0.90^{d}
Different letters in the same group indicate statistically significant differences ($p<0.05$)									

(C: Control, 10 LA: 10 µmol L^{-1} Lipoic acid, 15 LA: 15 µmol L^{-1} Lipoic acid, 25 LA: 25 µmol L^{-1} Lipoic acid, 30 LA: 30 µmol L^{-1} Lipoic acid and E: Endosperm, R: Root and C: Coleoptile)

Table 2. Effects of lipoic acid applications on non-essential microelement concentrations in 4-day-old maize endosperm, root and coleoptiles

Non-Essential Micro Elements (mg. kg^{-1})												
	Beryllium			Aluminum			Selenium			Molybdenum		
	E	R	C	E	R	C	E	R	C	E	R	C
C	0.11^{a}	0.06^{c}	0.07^{cd}	14.28^{a}	7.85^{d}	9.13^{a}	0.54^{a}	0.72^{e}	0.51^{e}	2.02^{a}	0.81^{e}	0.83^{e}
10 LA	0.07^{c}	0.08^{ab}	0.08^{c}	12.74^{b}	12.52^{c}	5.02^{c}	0.47^{b}	0.83^{d}	0.58^{d}	1.34^{b}	1.09^{d}	1.33^{d}
15 LA	0.06^{c}	0.09^{a}	0.11^{b}	9.92^{c}	14.53^{b}	4.23^{d}	0.39^{c}	0.85^{c}	0.89^{c}	0.99^{d}	1.27^{c}	1.75^{b}
25 LA	0.04^{d}	0.10^{a}	0.13^{a}	5.83^{e}	17.15^{a}	3.52^{e}	0.32^{d}	1.00^{a}	1.35^{a}	0.73^{e}	2.57^{a}	3.18^{a}
30 LA	0.08^{b}	0.06^{bc}	0.07^{d}	9.43^{d}	16.79^{a}	6.38^{b}	0.43^{b}	0.90^{b}	0.99^{b}	1.17^{c}	1.38^{b}	1.50^{c}
Different letters in the same group indicate statistically significant differences ($p<0.05$)												

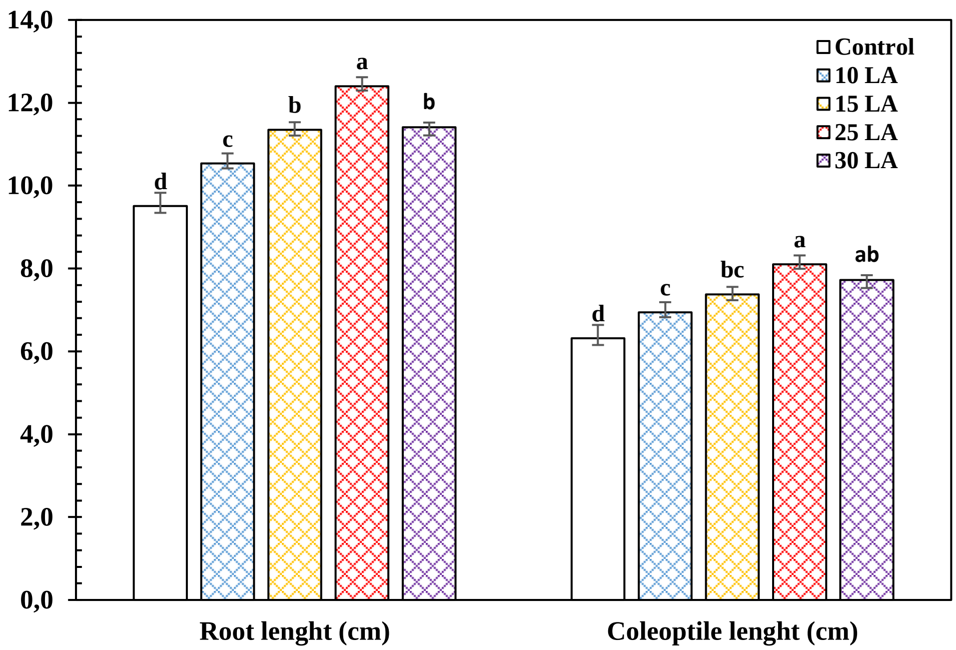
(C: Control, 10 LA: 10 µmol L^{-1} Lipoic acid, 15 LA: 15 µmol L^{-1} Lipoic acid, 25 LA: 25 µmol L^{-1} Lipoic acid, 30 LA: 30 µmol L^{-1} Lipoic acid and E: Endosperm, R: Root and C: Coleoptile)

Table 3. Effects of lipoic acid applications on macro element concentrations in 4-day-old maize endosperm, root and coleoptiles

Macro Elements (mg. kg^{-1})						
	Magnesium			Phosphorus		
	E	R	C	E	R	C
C	1775.38^a	1299.32^e	1672.38^e	4233.60^a	6927.87^e	8076.71^e
10 LA	1657.67^b	1419.22^d	1772.53^d	3856.51^b	7198.39^d	8716.00^d
15 LA	1594.22^c	1469.41^c	1807.46^c	3635.23^c	7567.66^c	9504.55^c
25 LA	1469.46^e	2191.29^a	2766.10^a	3171.29^e	11921.94^a	13074.92^a
30 LA	1503.12^d	1623.00^b	1853.46^b	3364.39^d	8219.89^b	10195.38^b
	Potassium			Calcium		
	E	R	C	E	R	C
C	1869.88^a	17164.46^e	18121.44^e	84.53^a	133.54^e	73.82^e
10 LA	1626.48^b	19474.60^d	19183.46^d	54.48^b	147.99^d	78.22^d
15 LA	1444.30^d	20906.10^c	22367.04^c	50.64^c	259.40^b	81.02^b
25 LA	1286.47^e	29170.74^a	28398.61^a	31.22^e	331.78^a	96.06^a
30 LA	1601.20^c	21301.02^b	25112.39^b	37.02^d	199.38^c	79.82^c
Different letters in the same group indicate statistically significant differences ($p<0.05$)						

(C: Control, 10 LA: 10 µmol L^{-1} Lipoic acid, 15 LA: 15 µmol L^{-1} Lipoic acid, 25 LA: 25 µmol L^{-1} Lipoic acid, 30 LA: 30 µmol L^{-1} Lipoic acid and E: Endosperm, R: Root and C: Coleoptile)

Figure 1. Effects of exogenous lipoic acid supplementations on root and coleoptile lengths in 4-day-old maize seeds



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A Large Class of Closed, Bounded and Convex Subsets in Köthe-Toeplitz Duals of Certain Generalized Difference Sequence Spaces with Fixed Point Property

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Abstract

In the present study, we consider the Köthe-Toeplitz duals for the 2nd order and 3rd order types difference sequence space generalizations by Et and Esi studied in 2000. We work on Goebel and Kuczumow analogy for those spaces to obtain large classes of closed, bounded and convex subsets satisfying the fixed point property. In the study, we also study some other Banach spaces in connection with the Köthe-Toeplitz duals for the 2nd order and 3rd order generalized difference sequence spaces.

Keywords: Fixed point property, nonexpansive mapping, Köthe-Toeplitz dual.

1. Introduction

When a Banach space satisfies the condition that every invariant nonexpansive mappings defined on any closed, bounded and convex (cbc) nonemtpy subset has a fixed point, then it is said that the space has the fixed point property for nonexpansive mappings. We need to note that distances between images of distant points under nonexpansive mapping cannot exceed the distances between the points taken. Researchers have considered categorizing Banach spaces with this property.

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Firstly, in (Browder 1965) it is found that Hilbert spaces have the property and the result was generalized in (Kirk 1965) to reflexive Banach spaces with normal structure.

Then, researchers have especially investigated nonreflexive classical Banach spaces and wondered if they can be renormable and falls in the same category with their equivalent norm while they fail to be members of the category with their usual norm but they were able to detect some nonreflexive Banach spaces which have equivalent norms and they become to have the fixed point property with those renormings. The first example was given by Lin (2008) for ℓ^1. Then even it has been asked if the same could have been done for c_0, but the answer still remains open. Since the researchers have considered trying to obtain the analogous results for well-known other classical nonreflexive Banach spaces, another experiment was done for Lebesgue integrable functions space $L_1[0,1]$ by Hernandes Lineares and Maria (2012) but they were able to obtain the positive answer when they restricted the nonexpansive mappings by assuming they were affine as well. One can say that there is no doubt most tries have been inspired by the ideas of the study (Goebel and Kuczumow 1979) where Goebel and Kuczumow proved that while ℓ^1 fails the fixed point property since one can easily find a cbc nonweakly compact subset there and a fixed point free invariant nonexpansive map, it is possible to find a very large class subsets in target such that invariant nonexpansive mappings defined on the members of the class have fixed points. In fact, it is easy to notice the traces of those ideas in (Lin 2008) work. Even Goebel and Kuczumow's work has inspired many other researchers to investigate if there exist more example of nonreflexive Banach spaces with large classes satisfying fixed point property. For example, Kaczor and Prus (2004) wanted to generalize Goebel and Kuczumow's findings by investigating if the same could be done for asymptotically nonexpansive mappings. Then, as their result, they proved that under affinity condition, asymptotically nonexpansive invariant mappings defined on a large class of cbc subsets in ℓ^1 can have fixed points. Moreover, in (Everest 2013) Kaczor and Prus's results were extended by having been found larger classes satisfying the fixed point property for affine asymptotically nonexpansive mappings. Thus, affinity condition become an easiness tool for their works. In fact, as an another well-known nonreflexive Banach space, Lebesgue space $L_1[0,1]$ was studied in (Hernandes Lineares and Japón 2012) and in their study they obtained an analogous result to (Lin 2008) as they showed that $L_1[0,1]$ can be renormed to have the fixed point property for affine nonexpansive mappings.

In this study we will investigate some Banach spaces analogous to ℓ^1. We aim to discuss the analogous results for Köthe-Toeplitz duals of certain generalized difference sequence spaces studied by Et and Esi (2000). We show that there exists a very large class of cbc subsets in those spaces with fixed point property for nonexpansive mappings. Thus, first we will recall the definition of Cesàro sequence spaces introduced by Shiue (1970) and next we will give Kızmaz's construction in (Kızmaz 1981) for difference sequence spaces since the dual space we work on is obtained from the generalizations of Kızmaz's idea which are derived differently by many researchers such as (Çolak 1989), (Et 1996), (Et and Çolak 1995), (Et and Esi 2000), (NgPeng-Nung and and LeePeng-Yee 1978), (Orhan 1983), and (Tripathy et. al. 2005). But we need to note that Et and Esi's work (Et and Esi 2000) and the further study by Et and Çolak (1995) used the new type of difference sequence definition from Çolak's work (Çolak 1989).

2. *Materials and Methods*

First we recall that (Shiue 1970) introduced the Cesàro sequence spaces written as

$$\mathrm{ces}_p = \left\{ (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

such that $\ell^p \subset \mathrm{ces}_p$ and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

such that $\ell^\infty \subset ces_\infty$ where $1 \leq p < \infty$. Their topological properties have been investigated and it has been seen that for $1 < p < \infty$, ces_p is a seperable reflexive Banach space. Furthermore, many researchers such as (Cui 1999), (Cui, Hudzik, and Li 2000) and (Cui, Meng, and Pluciennik 2000) were able to prove that for $1 < p < \infty$, Cesàro sequence space ces_p has the fixed point property.

Easiest way to show that was due to both reflexivity by the fact the space has normal structure when $1 < p < \infty$ (using the fact via (Kirk 1965)) and the space having the weak fixed point property because of its Garcia-Falset coefficient is less than 2 (see for example (Falset 1997)). A good reference about fixed point theory results for Cesàro sequence spaces can be a survey in (Chen et. al. 2001).

After the introduction of Cesàro sequence spaces, Kızmaz (1981), denoting by $\ell^\infty(\triangle)$, $\mathrm{c}(\triangle)$ and $\mathrm{c}_0(\triangle)$, introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences, respectively. Here $\triangle$ represented the difference operator applied to the sequence $x = (x_n)_n$ with the rule given by $\triangle x = (x_k - x_{k+1})_k$. Kızmaz studied then Köthe-Toeplitz Duals and topological properties for them.

As earlier it was stated, Çolak was one of the researchers generalizing Kızmaz's (1981) ideas. In his work, Çolak (1989) obtained the generalized version of the difference sequence space in the following way by picking an arbitrary sequence of nonzero complex values $v = (v_n)_n$. The new difference operator is denoted by $\triangle_v$ and the difference sequence of a sequence $x = (x_n)_n$ is written as $\triangle_v x = (v_k x_k - v_{k+1} x_{k+1})_k$. Then, in their study, Et and Esi (2000) defined a generalized difference sequence space as below.

$$\triangle_v (\ell^\infty) = \{ x = (x_n)_n \subset \mathbb{R} \,|\, \triangle_v x \in \ell^\infty \},$$
$$\triangle_v (\mathrm{c}) = \{ x = (x_n)_n \subset \mathbb{R} \,|\, \triangle_v x \in \mathrm{c} \},$$
$$\triangle_v (\mathrm{c}_0) = \{ x = (x_n)_n \subset \mathbb{R} \,|\, \triangle_v x \in \mathrm{c}_0 \}.$$

Then, they also defined m^{th} order generalized type difference sequence for any $m \in \mathbb{N}$ given by

$$\triangle_v^0 x = (v_k x_k)_k,$$

$\triangle_v^m x = (\triangle_v^m x_k)_k = (\triangle_v^{m-1} x_k - \triangle_v^{m-1} x_{k+1})_k$ with $\triangle_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ for each $k \in \mathbb{N}$.

In fact, Et and Esi (2000) further generalized the above difference sequence spaces and Bektaş, Et and

Çolak (2004) not only found the Köthe-Toeplitz duals for them but also obtained the duals for the generalized types of Et and Esi's. In this study, we will consider the 2nd order and 3rd order types which have the following norms respectively:

$$\|x\|_v^{(2)} = |v_1x_1| + |v_2x_2| + \|\triangle_v^m x\|_\infty,$$

$$\|x\|_v^{(3)} = |v_1x_1| + |v_2x_2| + |v_3x_3| + \|\triangle_v^m x\|_\infty.$$

Then the corresponding Köthe-Toeplitz duals were obtained as in (Bektaş, Et and Çolak 2004) and (Et and Esi 2000) such that they are written as below:

$$U_1^2 \coloneqq \{a = (a_n)_n \subset \mathbb{R} | (n^2 {v_n}^{-1} a_n)_n \in \ell^1\}$$

$$= \left\{a = (a_n)_n \subset \mathbb{R}:\ \|a\|^{(2)} = \sum_{k=1}^{\infty} \frac{k^2|a_k|}{|v_k|} < \infty\right\}$$

and

$$U_1^3 \coloneqq \{a = (a_n)_n \subset \mathbb{R} | (n^3 {v_n}^{-1} a_n)_n \in \ell^1\}$$

$$= \left\{a = (a_n)_n \subset \mathbb{R}:\ \|a\|^{(3)} = \sum_{k=1}^{\infty} \frac{k^3|a_k|}{|v_k|} < \infty\right\}.$$

Note that $U_1^m \subset \ell^1$ if $k^m|{v_k}^{-1}| > 1$ for each $k \in \mathbb{N}$ and $\ell^1 \subset U_1^m$ if $k^m|{v_k}^{-1}| < 1$ for each $k \in \mathbb{N}$ and $m = 2, 3$.

In this study, we will also condiser two more Banach spaces which are closely related to the above ones. We will denote them by W_1^2 and W_1^3 and their definitions are as follow:

$$W_1^2 \coloneqq \left\{a = (a_n)_n \subset \mathbb{R} \middle| \left(\frac{a_n}{n^2 v_n}\right)_n \in \ell^1\right\}$$

$$= \left\{a = (a_k)_k \subset \mathbb{R}:\ \|a\|_{(2)} = \sum_{k=1}^{\infty} \frac{|a_k|}{k^2|v_k|} < \infty\right\}$$

and

$$W_1^3 \coloneqq \left\{a = (a_n)_n \subset \mathbb{R} \middle| \left(\frac{a_n}{n^3 v_n}\right)_n \in \ell^1\right\}$$

$$= \left\{a = (a_k)_k \subset \mathbb{R}:\ \|a\|_{(3)} = \sum_{k=1}^{\infty} \frac{|a_k|}{k^3|v_k|} < \infty\right\}.$$

Note that $W_1^m \subset \ell^1$ if $k^m|v_k| < 1$ for each $k \in \mathbb{N}$ and $\ell^1 \subset W_1^m$ if $k^m|v_k| > 1$ for each $k \in \mathbb{N}$ and $m = 2, 3$.

These Banach spaces in connection with the above Köthe-Toeplitz duals are types of degenerate Lorentz-Marcinkiewicz spaces. The reader is recommended to see for example (Lindenstrauss and Tzafriri 1977) about them.

We will need the below well-known preliminaries before giving our main results. (Goebel and Kirk 1990) may be suggested as a good reference for these fundamentals.

Definition 2.1. *Consider that $(X, \|\cdot\|)$ is a Banach space and let C be a non-empty cbc subset. Let* $: C \to C$ *be a mapping. We say that*

1. *T is an affine mapping if for every $t \in [0,1]$ and $a, b \in C$, $T\big((1-t)a + tb\big) = (1-t)T(a) + t\,T(b)$.*
2. *T is a nonexpansive mapping if for every $a, b \in C$, $\| T(a) - T(b) \| \leq \| a - b \|$.*

Then, we will easily obtain an anologous key lemma from the below lemma in the work (Goebel and Kuczumow 1979).

Lemma 2.2. *Let $\{u_n\}$ be a sequence in ℓ^1 converging to u in weak-star topology, then for every $w \in \ell^1$,*

$$r(w) = r(u) + \|w - u\|_1$$

where

$$r(w) = \limsup_n \|u_n - w\|_1.$$

Note that our scalar field in this study will be real numbers although (Çolak 1989) considers complex values of $v = (v_n)_n$ while introducing his structer of the difference sequence which is taken as the fundamental concept in this study.

3. *Results*

In this section, we will present our results. As earlier it has been mentioned in the first section, we investigate Goebel and Kuczumow's analogy for the spaces U_1^2, U_1^3, W_1^2 and W_1^3. We aim to show that there are large classes of cbc subsets in these spaces such that every nonexpansive invariant mapping defined on the subsets in the classes taken has a fixed point. Recall that the invariant mappings have the same domain and range.

Firstly, due to isometric isomorphism, using Lemma 2.2, we will provide the straight analogous result as a lemma below which will be a key step as in the works such as (Goebel and Kuczumow 1979) and (Everest 2013) and in fact the methods in the study (Everest 2013) will be our lead in this work.

Lemma 3.1. *Let $\{u_n\}$ be a sequence in a Banach space Z which is a member of the spaces U_1^2, U_1^3, W_1^2 or W_1^3* such that $\|.\|$ denotes the norm for each space and assume $\{u_n\}$ *converges to u in weak-star topology, then for every $w \in Z$,*

$$r(w) = r(u) + \|w - u\|$$

where $r(w) = \limsup_n \|u_n - w\|.$

Then we prove the following theorems as our main results.

Theorem 3.2. *Fix* $b \in (0,1)$. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence defined by* $f_1 := b\, v_1\, e_1$, $f_2 := \frac{b\, v_2\, e_2}{2^2}$, *and* $f_n := \frac{v_n}{n^2} e_n$ *for all integers* $n \geq 3$ *where the sequence* $(e_n)_{n\in\mathbb{N}}$ *is the canonical basis of both* c_0 *and* ℓ^1. *Then, consider the cbc subset* $E^{(2)} = {E_b}^{(2)}$ *of* U_1^2 *by*

$$E^{(2)} := \left\{ \sum_{n=1}^{\infty} t_n f_n : \forall n \in \mathbb{N}, t_n \geq 0 \; and \; \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

Then, $E^{(2)}$ *has the fixed point property for* $\|\,.\,\|^{(2)}$*-nonexpansive mappings.*

Proof. Fix $b \in (0,1)$. Let $T: E^{(2)} \to E^{(2)}$ be a nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E^{(2)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|^{(2)} \underset{n}{\to} 0$. Due to isometric isomorphism U_1^2 shares common geometric properties with ℓ^1 and so both U_1^2 and its predual have same fixed point theory facts to ℓ^1 and c_0, respectly. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in U_1^2, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E^{(2)}$ by

$$C^{(2)} := \overline{E^{(2)}}^{w^*}$$
$$= \left\{ \sum_{n=1}^{\infty} t_n \; f_n : each\; t_n \geq 0 \; and \; \sum_{n=1}^{\infty} t_n \leq 1 \right\}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u \in C^{(2)}$ of $u^{(n)}$. Then, by Lemma 3.1, we have a function $r: U_1^2 \to [0, \infty)$ defined by

$$r(w) = \limsup_n \left\|u^{(n)} - w\right\|^{(2)}, \qquad \forall w \in U_1^2$$

such that *for every* $w \in U_1^2$,

$$r(w) = r(u) + \|u - w\|^{(2)}.$$

Case 1: $u \in E^{(2)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|^{(2)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\|Tu - u^{(n)}\right\|^{(2)} \\ &\leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\leq \limsup_n \left\|u - u^{(n)}\right\|^{(2)} + 0 \\ &= r(u). \end{aligned} \qquad (3.2.1)$$

Thus, $r(Tu) = r(u) + \|Tu - u\|^{(2)} \leq r(u)$ and so $\|Tu - u\|^{(2)} = 0$. Therefore, $Tu = u$.

Case 2: $u \in C^{(2)} \backslash E^{(2)}$.

Then, we may find scalars satisfying $\sum_{n=1}^{\infty} \delta_n < 1 \; and \; \forall n \in \mathbb{N}, \delta_n \geq 0$ such that $u = \sum_{n=1}^{\infty} \delta_n f_n$. Then, let $\gamma := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\alpha \in \left[\frac{-\delta_1}{\gamma}, \frac{\delta_2}{\gamma} + 1\right]$ define

$$h_\alpha := (\delta_1 + \alpha\gamma) f_1 + (\delta_2 + (1-\alpha)\gamma) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\begin{aligned} \|h_\alpha - u\|^{(2)} &= \left\|\alpha b \gamma v_1 e_1 + (1-\alpha)\gamma \frac{b\, v_2 e_2}{2^2}\right\|^{(2)} \\ &= \mathrm{b}|\alpha|\gamma + b|1-\alpha|\gamma. \end{aligned}$$

$\|h_\alpha - u\|^{(2)}$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b\gamma$.

Now fix $w \in E^{(2)}$. Then, we may find scalars t_n satisfying $\forall n \in \mathbb{N}, t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1$ such that $w = \sum_{n=1}^{\infty} t_n f_n$.

Then,

$$\begin{aligned} \|\mathrm{w} - u\|^{(2)} &= \left\|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(2)} \\ &= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + b \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\quad + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\geq \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \delta_k\right| + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \delta_k\right| + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}|1 - (1-\gamma)| + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k|. \end{aligned}$$

Hence,

$$\|\mathrm{w} - u\|^{(2)} \geq b\gamma + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k| \geq b\gamma$$

and the equality is obtained if and only if $(1 - b) \sum_{k=3}^{\infty} |t_k - \delta_k| = 0$; that is, we have $\|\mathrm{w} - u\|^{(2)} =$

$b\gamma$ if and only if $t_k = \delta_k$ for every $k \geq 3$; or say, $\|\mathrm{w} - u\|^{(2)} = b\gamma$ if and only if $\mathrm{w} = h_\alpha$ for some $\alpha \in [0,1]$.

Then, there exists a continuous function $\rho: [0,1] \to E^{(2)}$ defined by $\rho(\alpha) = h_\alpha$ and $\Lambda \coloneqq \rho([0,1])$ is a compact convex subset and so $\|\mathrm{w} - u\|^{(2)}$ achieves its minimum value at $\mathrm{w} = h_\alpha$ and for any $h_\alpha \in \Lambda$, we get

$$\begin{aligned} r(h_\alpha) &= r(u) + \|h_\alpha - u\|^{(2)} \\ &\leq r(u) + \|Th_\alpha - u\|^{(2)} \\ &= r(Th_\alpha) = \limsup_n \left\|Th_\alpha - u^{(n)}\right\|^{(2)} \end{aligned}$$

then same as the inequality (3.2.1), we get

$$\begin{aligned} r(h_\alpha) &\leq \limsup_n \left\|Th_\alpha - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|^{(2)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|^{(2)} + 0 \\ &= r(h_\alpha). \end{aligned}$$

Hence, $r(h_\alpha) \leq r(Th_\alpha) \leq r(h_\alpha)$ and so $r(Th_\alpha) = r(h_\alpha)$.

Therefore,

$r(u) + \|Th_\alpha - u\|^{(2)} = r(u) + \|h_\alpha - u\|^{(2)}$.

Thus, $\|Th_\alpha - u\|^{(2)} = \|h_\alpha - u\|^{(2)}$ and so $Th_\alpha \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's Fixed Point Theorem (Schauder 1930) easily we get the result T has a fixed point since T is continuous; thus, h_α is the unique minimizer of $\|\mathrm{w} - u\|^{(2)} : w \in E^{(2)}$ and $Th_\alpha = h_\alpha$.

Therefore, $E^{(2)}$ has the fixed point property for nonexpansive mappings.

Theorem 3.3. *Fix $b \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := b\, v_1\, e_1$, $f_2 := \frac{b\, v_2\, e_2}{2^3}$, and $f_n := \frac{v_n}{n^3} e_n$ for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E^{(3)} = {E_b}^{(3)}$ of U_1^3 by*

$$E^{(3)} := \left\{ \sum_{n=1}^{\infty} t_n f_n : \forall n \in \mathbb{N}, t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

Then, $E^{(3)}$ has the fixed point property for $\|\,.\,\|^{(3)}$-nonexpansive mappings.

Proof. Fix $b \in (0,1)$. Let $T: E^{(3)} \to E^{(3)}$ be a nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E^{(3)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|^{(3)} \underset{n}{\to} 0$. Due to isometric isomorphism U_1^3 shares common geometric properties with ℓ^1 and so both U_1^3 and its predual have same fixed point theory facts to ℓ^1 and c_0, respectly. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in U_1^3, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E^{(3)}$ by

$$\begin{aligned} C^{(3)} &:= \overline{E^{(3)}}^{w^*} \\ &= \left\{ \sum_{n=1}^{\infty} t_n\ f_n : each\ t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n \leq 1 \right\} \end{aligned}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u \in C^{(3)}$ of $u^{(n)}$. Then, by Lemma 3.1, we have a function $r: U_1^3 \to [0, \infty)$ defined by

$$r(w) = \limsup_n \left\|u^{(n)} - w\right\|^{(3)},\ \forall w \in U_1^3$$

such that *for every* $w \in U_1^3$,

$$r(w) = r(u) + \|u - w\|^{(3)}.$$

Case 1: $u \in E^{(3)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|^{(3)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\|Tu - u^{(n)}\right\|^{(3)} \\ &\leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|^{(3)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(3)} \\ &\leq \limsup_n \left\|u - u^{(n)}\right\|^{(3)} + 0 \\ &= r(u). \end{aligned} \tag{3.3.1}$$

Thus, $r(Tu) = r(u) + \|Tu - u\|^{(3)} \leq r(u)$ and so $\|Tu - u\|^{(3)} = 0$. Therefore, $Tu = u$.

Case 2: $u \in C^{(3)} \backslash E^{(3)}$.

Then, we may find scalars satisfying $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\forall n \in \mathbb{N}, \delta_n \geq 0$ such that $u = \sum_{n=1}^{\infty} \delta_n f_n$. Then, let $\gamma := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\alpha \in \left[\frac{-\delta_1}{\gamma}, \frac{\delta_2}{\gamma} + 1\right]$ define

$$h_\alpha := (\delta_1 + \alpha\gamma) f_1 + (\delta_2 + (1 - \alpha)\gamma) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\|h_\alpha - u\|^{(3)} = \left\|\alpha b\gamma v_1 e_1 + (1-\alpha)\gamma \frac{b\, v_2 e_2}{2^3}\right\|^{(3)}$$
$$= \mathrm{b}|\alpha|\gamma + b|1-\alpha|\gamma.$$

$\|h_\alpha - u\|^{(3)}$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b\gamma$.

Now fix $w \in E^{(3)}$. Then, we may find scalars t_n satisfying $\forall n \in \mathbb{N}, t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1$ such that $w = \sum_{n=1}^{\infty} t_n f_n$.

Then,

$$\|\mathrm{w} - u\|^{(3)} = \left\|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(3)}$$
$$= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + \sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + b\sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$+(1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$\geq \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$= \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$= \mathrm{b}|1-(1-\gamma)| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|.$$

Hence,

$$\|\mathrm{w} - u\|^{(3)} \geq b\gamma + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \geq b\gamma$$

and the equality is obtained if and only if $(1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| = 0$; that is, we have $\|\mathrm{w} - u\|^{(2)} = b\gamma$ if and only if $t_k = \delta_k$ for every $k \geq 3$; or say, $\|\mathrm{w} - u\|^{(3)} = b\gamma$ if and only if $\mathrm{w} = h_\alpha$ for some $\alpha \in [0,1]$.

Then, there exists a continuous function $\rho: [0,1] \to E^{(3)}$ defined by $\rho(\alpha) = h_\alpha$ and $\Lambda \coloneqq \rho([0,1])$ is a compact convex subset and so $\|\mathrm{w} - u\|^{(3)}$ achieves its minimum value at $\mathrm{w} = h_\alpha$ and for any $\mathrm{h}_\alpha \in \Lambda$, we get

$$r(h_\alpha) = r(u) + \|h_\alpha - u\|^{(3)}$$
$$\leq r(u) + \|Th_\alpha - u\|^{(3)}$$
$$= r(Th_\alpha)$$
$$= \limsup_n \left\|Th_\alpha - u^{(n)}\right\|^{(3)}$$

then same as the inequality (3.3.1), we get

$$r(h_\alpha) \leq \limsup_n \left\|Th_\alpha - T\left(u^{(n)}\right)\right\|^{(3)}$$
$$+\limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(3)}$$
$$\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|^{(3)}$$
$$+\limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(3)}$$
$$\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|^{(3)} + 0$$
$$= r(h_\alpha).$$

Hence, $r(h_\alpha) \leq r(Th_\alpha) \leq r(h_\alpha)$ and so $r(Th_\alpha) = r(h_\alpha)$.

Therefore,

$$r(u) + \|Th_\alpha - u\|^{(3)} = r(u) + \|h_\alpha - u\|^{(3)}.$$

Thus, $\|Th_\alpha - u\|^{(3)} = \|h_\alpha - u\|^{(3)}$ and so $Th_\alpha \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's Fixed Point Theorem (Schauder 1930) easily we get the result T has a fixed point since T is continuous; thus, h_α is the unique minimizer of $\|\mathrm{w} - u\|^{(3)} : w \in E^{(3)}$ and $Th_\alpha = h_\alpha$.

Therefore, $E^{(3)}$ has the fixed point property for nonexpansive mappings.

Theorem 3.4. *Fix $b \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := b\, v_1\, e_1$, $f_2 := 2^2 b\, v_2\, e_2$, and $f_n := n^2 v_n e_n$ for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E_{(2)} = E_{b(2)}$ of W_1^2 by*

$$E_{(2)} := \left\{\sum_{n=1}^{\infty} t_n f_n : \forall n \in \mathbb{N}, t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1\right\}.$$

Then, $E_{(2)}$ has the fixed point property for $\|\,.\,\|_{(2)}$-nonexpansive mappings.

Proof. Fix $b \in (0,1)$. Let $T: E_{(2)} \to E_{(2)}$ be a nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E_{(2)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|_{(2)} \underset{n}{\to} 0$. Due to isometric isomorphism W_1^2 shares common geometric properties with ℓ^1 and so both W_1^2 and its predual have same fixed point theory facts to ℓ^1 and c_0, respectly. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in W_1^2, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E_{(2)}$ by

$$C_{(2)} := \overline{E_{(2)}}^{w^*}$$
$$= \left\{ \sum_{n=1}^{\infty} t_n \ f_n : each\ t_n \geq 0\ and\ \sum_{n=1}^{\infty} t_n \leq 1 \right\}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u \in C_{(2)}$ of $u^{(n)}$. Then, by Lemma 3.1, we have a function $r: W_1^2 \to [0, \infty)$ defined by

$$r(w) = \limsup_n \left\| u^{(n)} - w \right\|_{(2)}, \qquad \forall w \in W_1^2$$

such that *for every* $w \in W_1^2$,

$$r(w) = r(u) + \|u - w\|_{(2)}.$$

Case 1: $u \in E_{(2)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{(2)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\| Tu - u^{(n)} \right\|_{(2)} \\ &\leq \limsup_n \left\| Tu - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\quad + \limsup_n \left\| u^{(n)} - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\leq \limsup_n \left\| u - u^{(n)} \right\|_{(2)} + 0 \\ &= r(u). \end{aligned} \qquad (3.4.1)$$

Thus, $r(Tu) = r(u) + \|Tu - u\|_{(2)} \leq r(u)$ and so $\|Tu - u\|_{(2)} = 0$. Therefore, $Tu = u$.

Case 2: $u \in C^{(2)} \backslash E_{(2)}$.

Then, we may find scalars satisfying $\sum_{n=1}^{\infty} \delta_n < 1\ and\ \forall n \in \mathbb{N}, \delta_n \geq 0$ such that $u = \sum_{n=1}^{\infty} \delta_n f_n$. Then, let $\gamma := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\alpha \in \left[\frac{-\delta_1}{\gamma}, \frac{\delta_2}{\gamma} + 1\right]$ define

$$h_\alpha := (\delta_1 + \alpha\gamma) f_1 + (\delta_2 + (1 - \alpha)\gamma) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\begin{aligned} \|h_\alpha - u\|_{(2)} &= \|\alpha b \gamma v_1 e_1 + (1 - \alpha)\gamma 2^2 b\ v_2 e_2\|_{(2)} \\ &= \mathrm{b}|\alpha|\gamma + b|1 - \alpha|\gamma. \end{aligned}$$

$\|h_\alpha - u\|_{(2)}$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b\gamma$.

Now fix $w \in E_{(2)}$. Then, we may find scalars t_n satisfying $\forall n \in \mathbb{N}, t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1$ such that $w = \sum_{n=1}^{\infty} t_n f_n$.

Then,

$$\|\mathrm{w} - u\|_{(2)} = \left\| \sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \delta_k f_k \right\|_{(2)}$$
$$= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + \sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + b \sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$+ (1 - b) \sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$\geq \mathrm{b} \left| \sum_{k=1}^{\infty} t_k - \delta_k \right| + (1 - b) \sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$= \mathrm{b} \left| \sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \delta_k \right| + (1 - b) \sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$= \mathrm{b}|1 - (1 - \gamma)| + (1 - b) \sum_{k=3}^{\infty} |t_k - \delta_k|.$$

Hence,

$$\|\mathrm{w} - u\|_{(2)} \geq b\gamma + (1 - b) \sum_{k=3}^{\infty} |t_k - \delta_k| \geq b\gamma$$

and the equality is obtained if and only if $(1 - b) \sum_{k=3}^{\infty} |t_k - \delta_k| = 0$; that is, we have $\|\mathrm{w} - u\|_{(2)} = b\gamma$ if and only if $t_k = \delta_k$ for every $k \geq 3$; or say, $\|\mathrm{w} - u\|_{(2)} = b\gamma$ if and only if $\mathrm{w} = h_\alpha$ for some $\alpha \in [0,1]$.

Then, there exists a continuous function $\rho: [0,1] \to \mathrm{E}_{(2)}$ defined by $\rho(\alpha) = h_\alpha$ and $\Lambda := \rho([0,1])$ is a compact convex subset and so $\|w - u\|_{(2)}$ achieves its minimum value at $\mathrm{w} = h_\alpha$ and for any $\mathrm{h}_\alpha \in \Lambda$, we get

$$\begin{aligned} r(h_\alpha) &= r(u) + \|h_\alpha - u\|_{(2)} \\ &\leq r(u) + \|Th_\alpha - u\|_{(2)} \\ &= r(Th_\alpha) \\ &= \limsup_n \left\| Th_\alpha - u^{(n)} \right\|_{(2)} \end{aligned}$$

then same as the inequality (3.4.1), we get

$$\begin{aligned} r(h_\alpha) &\leq \limsup_n \left\| Th_\alpha - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\quad + \limsup_n \left\| u^{(n)} - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\leq \limsup_n \left\| h_\alpha - u^{(n)} \right\|_{(2)} \\ &\quad + \limsup_n \left\| u^{(n)} - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\leq \limsup_n \left\| h_\alpha - u^{(n)} \right\|_{(2)} + 0 \\ &= r(h_\alpha). \end{aligned}$$

Hence, $r(h_\alpha) \leq r(Th_\alpha) \leq r(h_\alpha)$ and so $r(Th_\alpha) = r(h_\alpha)$.

Therefore,

$$r(u) + \|Th_\alpha - u\|_{(2)} = r(u) + \|h_\alpha - u\|_{(2)}.$$

Thus, $\|Th_\alpha - u\|_{(2)} = \|h_\alpha - u\|_{(2)}$ and so $Th_\alpha \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's Fixed Point Theorem (Schauder 1930) easily we get the result T has a fixed point since T is continuous; thus, h_α is the unique minimizer of $\|w-u\|_{(2)} : w \in E_{(2)}$ and $Th_\alpha = h_\alpha$.

Therefore, $E_{(2)}$ has the fixed point property for nonexpansive mappings.

Theorem 3.5. *Fix $b \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := b\, v_1\, e_1$, $f_2 := 2^3 b\, v_2\, e_2$, and $f_n := n^3 v_n e_n$ for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E_{(3)} = E_{b(3)}$ of W_1^3 by*

$$E_{(3)} := \left\{\sum_{n=1}^{\infty} t_n f_n : \forall n \in \mathbb{N}, t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1\right\}.$$

Then, $E_{(3)}$ has the fixed point property for $\|.\|_{(3)}$-nonexpansive mappings.

Proof. Fix $b \in (0,1)$. Let $T: E_{(3)} \to E_{(3)}$ be a nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E_{(3)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|_{(3)} \underset{n}{\to} 0$. Due to isometric isomorphism W_1^2 shares common geometric properties with ℓ^1 and so both W_1^3 and its predual have same fixed point theory facts to ℓ^1 and c_0, respectly. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in W_1^3, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E_{(3)}$ by

$$C_{(3)} := \overline{E_{(3)}}^{w^*} = \left\{\sum_{n=1}^{\infty} t_n\ f_n : \text{each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n \leq 1\right\}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u \in C_{(3)}$ of $u^{(n)}$. Then, by Lemma 3.1, we have a function $r: W_1^3 \to [0,\infty)$ defined by

$$r(w) = \limsup_n \left\|u^{(n)} - w\right\|_{(3)}, \qquad \forall w \in W_1^3$$

such that *for every* $w \in W_1^3$,

$$r(w) = r(u) + \|u - w\|_{(3)}.$$

Case 1: $u \in E_{(3)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{(3)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\|Tu - u^{(n)}\right\|_{(3)} \\ &\leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\leq \limsup_n \left\|u - u^{(n)}\right\|_{(3)} + 0 \\ &= r(u). \end{aligned} \tag{3.5.1}$$

Thus, $r(Tu) = r(u) + \|Tu - u\|_{(3)} \leq r(u)$ and so $\|Tu - u\|_{(3)} = 0$. Therefore, $Tu = u$.

Case 2: $u \in C^{(3)} \backslash E_{(3)}$.

Then, we may find scalars satisfying $\sum_{n=1}^{\infty} \delta_n < 1$ and $\forall n \in \mathbb{N}, \delta_n \geq 0$ such that $u = \sum_{n=1}^{\infty} \delta_n f_n$. Then, let $\gamma := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\alpha \in \left[\frac{-\delta_1}{\gamma}, \frac{\delta_2}{\gamma} + 1\right]$ define

$$h_\alpha := (\delta_1 + \alpha\gamma) f_1 + (\delta_2 + (1-\alpha)\gamma) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\begin{aligned} \|h_\alpha - u\|_{(3)} &= \|\alpha b \gamma v_1 e_1 + (1-\alpha)\gamma 2^3 b\, v_2\, e_2\|_{(3)} \\ &= b|\alpha|\gamma + b|1-\alpha|\gamma. \end{aligned}$$

$\|h_\alpha - u\|_{(3)}$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b\gamma$.

Now fix $w \in E_{(3)}$. Then, we may find scalars t_n satisfying $\forall n \in \mathbb{N}, t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1$ such that $w = \sum_{n=1}^{\infty} t_n f_n$.

Then,

$$\begin{aligned} \|w - u\|_{(3)} &= \left\|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|_{(3)} \\ &= b|t_1 - \delta_1| + b|t_2 - \delta_2| + \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= b|t_1 - \delta_1| + b|t_2 - \delta_2| + b\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\quad + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\geq b\left|\sum_{k=1}^{\infty} t_k - \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= b\left|\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= b|1 - (1-\gamma)| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|. \end{aligned}$$

Hence,

$$\|w-u\|_{(3)} \geq b\gamma + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \geq b\gamma$$

and the equality is obtained if and only if $(1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| = 0$; that is, we have $\|w-u\|_{(3)} = b\gamma$ if and only if $t_k = \delta_k$ for every $k \geq 3$; or say, $\|w-u\|_{(3)} = b\gamma$ if and only if $w = h_\alpha$ for some $\alpha \in [0,1]$.

Then, there exists a continuous function $\rho: [0,1] \to E_{(3)}$ defined by $\rho(\alpha) = h_\alpha$ and $\Lambda \coloneqq \rho([0,1])$ is a compact convex subset and so $\|w-u\|_{(3)}$ achieves its minimum value at $w = h_\alpha$ and for any $h_\alpha \in \Lambda$, we get

$$\begin{aligned} r(h_\alpha) &= r(u) + \|h_\alpha - u\|_{(3)} \\ &\leq r(u) + \|Th_\alpha - u\|_{(3)} \\ &= r(Th_\alpha) \\ &= \limsup_n \left\|Th_\alpha - u^{(n)}\right\|_{(3)} \end{aligned}$$

then same as the inequality (3.5.1), we get

$$\begin{aligned} r(h_\alpha) &\leq \limsup_n \left\|Th_\alpha - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|_{(3)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|_{(3)} + 0 \\ &= r(h_\alpha). \end{aligned}$$

Hence, $r(h_\alpha) \leq r(Th_\alpha) \leq r(h_\alpha)$ and so $r(Th_\alpha) = r(h_\alpha)$. Therefore, $r(u) + \|Th_\alpha - u\|_{(3)} = r(u) + \|h_\alpha - u\|_{(3)}$. Thus, $\|Th_\alpha - u\|_{(3)} = \|h_\alpha - u\|_{(3)}$ and so $Th_\alpha \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's Fixed Point Theorem (Schauder 1930) easily we get the result T has a fixed point since T is continuous; thus, h_α is the unique minimizer of $\|w-u\|_{(3)} : w \in E_{(3)}$ and $Th_\alpha = h_\alpha$.

Therefore, $E_{(3)}$ has the fixed point property for nonexpansive mappings.

4. *Discussion*

In this study, we have considered the Köthe-Toeplitz duals for the 2nd order and 3rd order types difference sequence space generalizations by Et and Esi (2000). We have studied Goebel and Kuczumow (1979) analogy for those spaces and showed that there exist large classes of cbc subsets in those Köthe-Toeplitz duals with fixed point property for nonexpansive mappings. We have also another study in preparation to get larger classes for mth order types. Furthermore, the first author has another study in preparation to investigate Kaczor and Prus (2004) analogy for the 2nd order and 3rd order types difference sequence space generalizations by Et and Esi (2000), which is to look for large classes of cbc subsets satisfying the fixed point property for asymptotically nonexpansive mappings. These spaces we have studied are analogous Banach spaces to ℓ^1. There are many Banach spaces analogous to ℓ^1 and Goebel-Kuczumow analogy or Kaczor-Prus analogy might be investigated by researchers.

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Integral Inequalities of Steffensen Type for Some Different Classes of Functions

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Abstract

In this paper, we have obtained some new integral inequalities of Steffensen type for P-function and η-convex function with the help of identities proved by Mitrinovic et al..

Keywords: Convex function, η-convex function, P-function, Steffensen inequality, Hermite-Hadamard inequality.

1. Introduction

Many function classes have been defined in the historical process of mathematics and one of these function classes is the class of convex functions. This function class has offered new application areas to mathematicians. The definition of this function class, which allows many new results to be obtained with the studies carried out on it and therefore attracts the attention of mathematicians, is as follows.

Definition 1.1 *The function* $f:[a,b]\subseteq \mathbb{R}\to\mathbb{R}$, *is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y) \quad (1.1)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

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One of the most important results obtained for convex functions is the inequality given below which is known as Hermite Hadamard inequality in the literature.

Assume that $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex function defined on the interval I of $\mathbb{R}$ where $a<b$. The following statement holds

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \quad (1.2)$$

Both inequalities hold in the reversed direction if f is concave.

One of the functions defined in the class of convex functions is the η-convex function. In (Gordji et al. 2015), Gordji et al. introduced the idea of η-convex functions as generalization of ordinary convex functions and gave the following definition for η-convexity of functions.

Definition 1.2 *A function* $f:[a,b]\to\mathbb{R}$ *is said to be η-convex (or convex with respect to η) if the inequality*

$$f(\lambda x+(1-\lambda)y)\le f(y)+\lambda\eta(f(x),f(y))$$

holds for all $x,y\in[a,b]$, $\lambda\in[0,1]$ and η is defined by $\eta: f([a,b])\times f([a,b])\to\mathbb{R}$.

In the above definition if we set $\eta(x,y)=x-y$, then we can directly obtain the classical definition of a convex function. To see more results and details on η-convex functions see (Delavar and Dragomir 2017, Gordji et al. 2015, Gordji et al. 2016).

Another function defined in the different class of functions is the P function and its definition is given below.

Definition 1.3 (Dragomir et al. 1995) *A function $f: I \subset \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and for $a, b \in I$ and $\lambda \in [0,1]$ satisfies the following inequality,*

$$f(\lambda a + (1-\lambda)b) \le f(a) + f(b).$$

The Hermite-Hadamard inequality obtained for P functions is as follows.

Theorem 1.1 (Dragomir et al. 1995, Dragomir and Pearce 200) *Let $I = [a,b]$. Let P(I) be class of P functions defined on I and $f \in P(I)$ be integrable function, then the following inequality of Hermite-Hadamard type holds*

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_a^b f(x)dx \le 2\big(f(a)+f(b)\big). \tag{1.3}$$

In order to study certain inequalities between mean values, Steffensen (Steffensen 1918) has proved to following inequality (see also (Mitrinovic 1993, p.311)):

Theorem 1.2 *Let f and g be two integrable functions defined on (a,b). f is decreasing and for each $t \in (a,b)$, $0 \le g(t) \le 1$. Then, the following inequality*

$$\int_{b-\lambda}^b f(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt \tag{1.4}$$

holds, where $\lambda = \int_a^b g(t)dt$.

Some minor generalization of Steffensen's inequality in the (1.4) was considered by Hayashi (Hayashi 1919), using the substituting $g(t)/A$ for $g(t)$, where A is positive constant. For other result involving Steffensen's type inequality, see (Hayashi 1919, Mitrinovic et al. 1993).

In (Mitrinovic et al. 1993), Mitrinovic et al. proved the following equality:

Lemma 1.1 *Let $f, g: [a,b] \subset \mathbb{R}^+ \to \mathbb{R}$ be integrable such that $0 \le g(t) \le 1$, for all $t \in [a,b]$ and $\int_a^b g(t) f'(t)dt$ exists. Then we have the following representation*

$$\int_a^{a+\lambda} f(x)dx - \int_a^b f(x)g(x)dx \tag{1.5}$$

$$= -\int_a^{a+\lambda}\left(\int_a^x (1-g(t))dt\right) f'(x)dx - \int_{a+\lambda}^b \left(\int_x^b g(t)dt\right) f'(x)dx,$$

and

$$\int_a^b f(x)g(x)dx - \int_{b-\lambda}^b f(x)dx \tag{1.6}$$

$$= -\int_a^{b-\lambda}\left(\int_a^x g(t)dt\right) f'(x)dx - \int_{b-\lambda}^b \left(\int_x^b (1-g(t))dt\right) f'(x)dx,$$

where $\lambda := \int_a^b g(t)dt$.

In (Alomari et al. 2017), using this identity, Alomari introduced some new Steffensen type inequalities for s-convex functions.

The aim of this paper is to obtain new integral inequalities of Steffensen type for η-convex functions and P function.

2. *Steffensen's type inequalities for η-convex function*

Theorem 2.1 *Let $f, g: [a,b] \subset \mathbb{R}^+ \to \mathbb{R}$ be integrable such that $0 \le g(t) \le 1$, for all $t \in [a,b]$ such that $\int_a^b g(t) f'(t)dt$ exists. If f is absolutely continuous on $[a,b]$ such that $|f'|$ is $\eta-$convex function on $[a,b]$, then we have*

$$\left|\int_a^{a+\lambda} f(x)dx - \int_a^b f(x)g(x)dx\right|$$

$$\le \frac{\lambda^2}{6}\left[3|f'(a)| + 2\eta(|f'(a+\lambda)|, |f'(a)|)\right]$$

$$+\frac{(b-a-\lambda)^2}{6}\left[3|f'(a+\lambda)| + \eta(|f'(b)|, |f'(a+\lambda)|)\right] \tag{2.1}$$

and

$$\left|\int_a^b f(x)g(x)dx - \int_{b-\lambda}^b f(x)dx\right|$$

$$\le \frac{(b-a-\lambda)^2}{6}\left[3|f'(a)| + \frac{2(b-a-\lambda)\eta(|f'(a+\lambda)|, |f'(a)|)}{\lambda}\right]$$

$$+\frac{\lambda^2}{6}\left[3|f'(a+\lambda)| + \frac{(3b-3a-5\lambda)\eta(|f'(b)|, |f'(a+\lambda)|)}{b-a-\lambda}\right] \tag{2.2}$$

where $\lambda := \int_a^b g(t)dt$.

Proof. Using Lemma 1.1 and since $|f'|$ is η-convex function, we have

$$\left|\int_a^{a+\lambda} f(x)dx - \int_a^b f(x)g(x)dx\right|$$

$$\leq \left|\int_a^{a+\lambda}\left(\int_a^x (1-g(t))dt\right)f'(x)dx\right| + \left|\int_{a+\lambda}^b\left(\int_x^b g(t)dt\right)f'(x)dx\right|$$

$$\leq \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right| |f'(x)|dx + \int_{a+\lambda}^b\left|\int_x^b g(t)dt\right| |f'(x)|dx$$

$$= \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\left|f'\left(\frac{x-a}{\lambda}(a+\lambda)+\frac{a+\lambda-x}{\lambda}a\right)\right|dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\left|f'\left(\frac{x-a-\lambda}{b-a-\lambda}b+\frac{b-x}{b-a-\lambda}(a+\lambda)\right)\right|dx$$

$$\leq \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\left[|f'(a)|+\frac{x-a}{\lambda}\eta\left(|f'(a+\lambda)|,|f'(a)|\right)\right]dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\left[|f'(a+\lambda)|+\frac{x-a-\lambda}{b-a-\lambda}\eta\left(|f'(b)|,|f'(a+\lambda)|\right)\right]dx$$

$$= \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right| |f'(a)|dx$$

$$+\int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\frac{x-a}{\lambda}\eta\left(|f'(a+\lambda)|,|f'(a)|\right)dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right| |f'(a+\lambda)|dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\frac{x-a-\lambda}{b-a-\lambda}\eta\left(|f'(b)|,|f'(a+\lambda)|\right)dx$$

$$\leq |f'(a)|\int_a^{a+\lambda}(x-a)dx+\frac{\eta(|f'(a+\lambda)|,|f'(a)|)}{\lambda}\int_a^{a+\lambda}(x-a)^2dx$$

$$+|f'(a+\lambda)|\int_{a+\lambda}^b(b-x)dx+\frac{\eta(|f'(b)|,|f'(a+\lambda)|)}{b-a-\lambda}\int_{a+\lambda}^b(b-x)(x-a-\lambda)dx$$

$$=\frac{\lambda^2}{2}|f'(a)|+\frac{\lambda^3}{3}\frac{\eta(|f'(a+\lambda)|,|f'(a)|)}{\lambda}$$

$$+\frac{(b-a-\lambda)^2}{2}|f'(a+\lambda)|+\frac{(b-a-\lambda)^3}{6}\frac{\eta(|f'(b)|,|f'(a+\lambda)|)}{b-a-\lambda}$$

$$=\frac{\lambda^2}{6}[3|f'(a)|+2\eta(|f'(a+\lambda)|,|f'(a)|)]$$

$$+\frac{(b-a-\lambda)^2}{6}[3|f'(a+\lambda)|+\eta(|f'(b)|,|f'(a+\lambda)|)]$$

and so we proved inequality (2.1). Then similarly

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^b f(x)dx\right|$$

$$\leq\left|\int_a^{b-\lambda}\left(\int_a^x g(t)dt\right)f'(x)dx\right|+\left|\int_{b-\lambda}^b\left(\int_x^b(1-g(t))dt\right)f'(x)dx\right|$$

$$\leq\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right| |f'(x)|dx+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right| |f'(x)|dx$$

$$=\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\left|f'\left(\frac{x-a}{\lambda}(a+\lambda)+\frac{a+\lambda-x}{\lambda}a\right)\right|dx$$

$$+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right|\left|f'\left(\frac{x-a-\lambda}{b-a-\lambda}b+\frac{b-x}{b-a-\lambda}(a+\lambda)\right)\right|dx$$

$$\leq\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\left[|f'(a)|+\frac{x-a}{\lambda}\eta(|f'(a+\lambda)|,|f'(a)|)\right]dx$$

$$+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right|\left[|f'(a+\lambda)|+\frac{x-a-\lambda}{b-a-\lambda}\eta(|f'(b)|,|f'(a+\lambda)|)\right]dx$$

$$=|f'(a)|\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|dx+\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\frac{x-a}{\lambda}\eta(|f'(a+\lambda)|,|f'(a)|)dx$$

$$+|f'(a+\lambda)|\int_{b-\lambda}^b\left|\int_x^b g(t)dt\right|dx$$

$$+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right|\frac{x-a-\lambda}{b-a-\lambda}\eta(|f'(b)|,|f'(a+\lambda)|)dx$$

$$\leq |f'(a)|\int_a^{b-\lambda}(x-a)dx+\frac{\eta(|f'(a+\lambda)|,|f'(a)|)}{b-a-\lambda}\int_a^{b-\lambda}(x-a)^2dx$$

$$+|f'(a+\lambda)|\int_{b-\lambda}^{b}(b-x)dx+\frac{\eta(|f'(b)|,|f'(a+\lambda)|)}{b-a-\lambda}\int_{b-\lambda}^{b}(b-x)(x-a-\lambda)dx$$

$$=\frac{(b-a-\lambda)^2}{2}|f'(a)|+\frac{(b-a-\lambda)^3}{3}\frac{\eta(|f'(a+\lambda)|,|f'(a)|)}{\lambda}$$

$$+|f'(a+\lambda)|\frac{\lambda^2}{2}+\frac{\lambda^2(3b-3a-5\lambda)}{6}\frac{\eta(|f'(b)|,|f'(a+\lambda)|)}{b-a-\lambda}$$

and so we proved inequality (2.2).

Theorem 2.2 *Let* $f,g:[a,b]\subset \mathbb{R}^+\to\mathbb{R}$ *be integrable such that* $0\leq g(t)\leq 1$, *for all* $t\in[a,b]$ *such that* $\int_a^b g(t)f'(t)dt$ *exists. If* f *is absolutely continuous on* $[a,b]$ *such that* $|f'|$ *is* $\eta-$*convex function on* $[a,b]$, *then we have*

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\leq\int_{a+\lambda}^{b}g(t)dt\left[\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|}{2}\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\right]$$

$$\leq(b-a-\lambda)\left[\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|}{2}\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\right] \quad (2.3)$$

and

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^{b}f(x)dx\right|$$

$$\leq\int_{b-\lambda}^{b}g(t)dt\left[(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|}{2}\right)\right.$$

$$\left.+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\right]$$

$$\leq(b-a-\lambda)\left[(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|}{2}\right)\right.$$

$$\left.+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\right]$$

where $\lambda:=\int_a^b g(t)dt$.

Proof. From Lemma 1.1, we have

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|\leq\sup_{x\in[a,a+\lambda]}\left[\int_a^x(1-g(t))dt\right]\int_a^{a+\lambda}|f'(x)|dx$$

$$+\sup_{x\in[a+\lambda,b]}\left[\int_x^b g(t)dt\right]\int_{a+\lambda}^{b}|f'(x)|dx.$$

Since $|f'|$ is $\eta-$convex on $[a,b]$, we get

$$\int_a^{a+\lambda}|f'(x)|dx\leq\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right),$$

and

$$\int_{a+\lambda}^{b}|f'(x)|dx\leq(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right).$$

Therefore, we have

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\leq\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right)\left[\int_a^{a+\lambda}(1-g(t))dt\right]$$

$$+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\left[\int_{a+\lambda}^{b}g(t)dt\right]$$

$$\leq\max\left\{\int_a^{a+\lambda}(1-g(t))dt,\int_{a+\lambda}^{b}g(t)dt\right\}\left[\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\right]$$

$$=\int_{a+\lambda}^{b}g(t)dt\left[\lambda\left(|f'(a+\lambda)|+\frac{\eta(|f'(a)|,|f'(a+\lambda)|)}{2}\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(b)|+\frac{\eta(|f'(a+\lambda)|,|f'(b)|)}{2}\right)\right]$$

which proves the first inequality (2.3). The second inequality in (2.3) follows directly, since $0\le g(t)\le 1$ for all $t\in[a,b]$, then

$$0\le\int_{a+\lambda}^{b}g(t)dt\le b-a-\lambda.$$

Similarly

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^{b}f(x)dx\right|\le\sup_{x\in[a,b-\lambda]}\left[\int_a^x g(t)dt\right]\int_a^{b-\lambda}|f'(x)|dx$$
$$+\sup_{x\in[b-\lambda,b]}\left[\int_x^b(1-g(t))dt\right]\int_{b-\lambda}^{b}|f'(x)|dx.$$

Since $|f'|$ is $\eta-$convex on $[a,b]$, we get

$$\int_a^{b-\lambda}|f'(x)|dx\le(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right),$$

and

$$\int_{b-\lambda}^{b}|f'(x)|dx\le\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right).$$

Therefore, we have

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^{b}f(x)dx\right|$$
$$\le(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right)\left[\int_a^{b-\lambda}g(t)dt\right]$$
$$+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\left[\int_{b-\lambda}^{b}(1-g(t))dt\right]$$
$$\le\max\left\{\int_a^{b-\lambda}g(t)dt,\int_{b-\lambda}^{b}(1-g(t))dt\right\}\left[(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right)\right.$$
$$\left.+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\right]$$
$$=\int_{b-\lambda}^{b}g(t)dt\left[(b-a-\lambda)\left(|f'(b-\lambda)|+\frac{\eta(|f'(a)|,|f'(b-\lambda)|)}{2}\right)\right.$$
$$\left.+\lambda\left(|f'(b)|+\frac{\eta(|f'(b-\lambda)|,|f'(b)|)}{2}\right)\right]$$

which proves the first inequality (2.4). The second inequality in (2.4) follows directly, since $0\le g(t)\le 1$ for all $t\in[a,b]$, then

$$0\le\int_a^{b-\lambda}g(t)dt\le b-a-\lambda.$$

Theorem 2.3 *Let* $f,g:[a,b]\subset\mathbb{R}^+\to\mathbb{R}$ *be integrable such that* $0\le g(t)\le 1$*, for all* $t\in[a,b]$ *such that* $\int_a^b g(t)f'(t)dt$ *exists. If* f *is absolutely continuous on* $[a,b]$ *such that* $|f'|$ *is* $\eta-$*convex function on* $[a,b]$ *and* $q>1$*, then we have*

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|$$
$$\le\frac{\lambda^2}{(p+1)^{\frac{1}{p}}}\left[\left(|f'(a+\lambda)|^q+\frac{\eta(|f'(a)|^q,|f'(a+\lambda)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-a-\lambda)^2}{(p+1)^{\frac{1}{p}}}\left(|f'(b)|^q+\frac{\eta(|f'(a+\lambda)|^q,|f'(b)|^q)}{2}\right)^{\frac{1}{q}}\right]\tag{2.5}$$

and

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^{b}f(x)dx\right|$$
$$\le\frac{(b-a-\lambda)^2}{(p+1)^{\frac{1}{p}}}\left[\left(|f'(b-\lambda)|^q+\frac{\eta(|f'(a)|^q,|f'(b-\lambda)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{\lambda^2}{(p+1)^{\frac{1}{p}}}\left(|f'(b)|^q+\frac{\eta(|f'(b-\lambda)|^q,|f'(b)|^q)}{2}\right)^{\frac{1}{q}}\right]\tag{2.6}$$

where $\lambda:=\int_a^b g(t)dt$.

Proof. From Lemma 1.1, $\eta-$convexity of $|f'|^q$ and using the Hölder inequality for $q>1$, and $p=\dfrac{q}{q-1}$, we obtain

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|$$
$$\le\int_a^{a+\lambda}\left|\int_a^x(1-g(t))dt\right||f'(x)|dx+\int_{a+\lambda}^{b}\left|\int_x^b g(t)dt\right||f'(x)|dx$$
$$\le\left(\int_a^{a+\lambda}\left|\int_a^x(1-g(t))dt\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^{a+\lambda}|f'(x)|^q dx\right)^{\frac{1}{q}}$$

$$+\left(\int_{a+\lambda}^{b}\left|\int_{x}^{b}g(t)dt\right|^{p}dx\right)^{\frac{1}{p}}\left(\int_{a+\lambda}^{b}|f^{'}(x)|^{q}dx\right)^{\frac{1}{q}}:=M \tag{2.7}$$

where p is the conjugate of q.

Since $|f^{'}|^{q}$ is $\eta-$convex on $[a,b]$, we have

$$\int_{a}^{a+\lambda}|f^{'}(x)|^{q}dx\le\lambda\left(|f^{'}(a+\lambda)|^{q}+\frac{\eta(|f^{'}(a)|^{q},|f^{'}(a+\lambda)|^{q})}{2}\right),$$

and

$$\int_{a+\lambda}^{b}|f^{'}(x)|^{q}dx\le(b-a-\lambda)\left(|f^{'}(b)|^{q}+\frac{\eta(|f^{'}(a+\lambda)|^{q},|f^{'}(b)|^{q})}{2}\right)$$

which gives by (2.7)

$$M\le\left(\int_{a}^{a+\lambda}(x-a)^{p}dx\right)^{\frac{1}{p}}\lambda^{\frac{1}{q}}\left[|f^{'}(a+\lambda)|^{q}+\frac{\eta(|f^{'}(a)|^{q},|f^{'}(a+\lambda)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$+\left(\int_{a+\lambda}^{b}(b-x)^{p}dx\right)^{\frac{1}{p}}(b-a-\lambda)^{\frac{1}{q}}\left[|f^{'}(b)|^{q}+\frac{\eta(|f^{'}(a+\lambda)|^{q},|f^{'}(b)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$=\frac{\lambda^{2}}{(p+1)^{\frac{1}{p}}}\left[|f^{'}(a+\lambda)|^{q}+\frac{\eta(|f^{'}(a)|^{q},|f^{'}(a+\lambda)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$+\frac{(b-a-\lambda)^{2}}{(p+1)^{\frac{1}{p}}}\left[|f^{'}(b)|^{q}+\frac{\eta(|f^{'}(a+\lambda)|^{q},|f^{'}(b)|^{q})}{2}\right]^{\frac{1}{q}}$$

giving the inequality (2.5). Similarly

$$\left|\int_{a}^{b}f(x)g(x)dx-\int_{b-\lambda}^{b}f(x)dx\right| \tag{2.8}$$

$$\le\int_{a}^{b-\lambda}\left|\int_{a}^{x}g(t)dt\right||f^{'}(x)|dx+\int_{b-\lambda}^{b}\left|\int_{x}^{b}(1-g(t))dt\right||f^{'}(x)|dx$$

$$\le\left(\int_{a}^{b-\lambda}\left|\int_{a}^{x}g(t)dt\right|^{p}dx\right)^{\frac{1}{p}}\left(\int_{a}^{b-\lambda}|f^{'}(x)|^{q}dx\right)^{\frac{1}{q}}$$

$$+\left(\int_{b-\lambda}^{b}\left|\int_{x}^{b}(1-g(t))dt\right|^{p}dx\right)^{\frac{1}{p}}\left(\int_{b-\lambda}^{b}|f^{'}(x)|^{q}dx\right)^{\frac{1}{q}}:=N$$

where p is the conjugate of q.

Since $|f^{'}|^{q}$ is $\eta-$convex on $[a,b]$, we have

$$\int_{a}^{b-\lambda}|f^{'}(x)|^{q}dx\le(b-a-\lambda)\left(|f^{'}(b-\lambda)|^{q}+\frac{\eta(|f^{'}(a)|^{q},|f^{'}(b-\lambda)|^{q})}{2}\right),$$

and

$$\int_{b-\lambda}^{b}|f^{'}(x)|^{q}dx\le\lambda\left(|f^{'}(b)|^{q}+\frac{\eta(|f^{'}(b-\lambda)|^{q},|f^{'}(b)|^{q})}{2}\right)$$

which gives by (2.8)

$$N\le\left(\int_{a}^{b-\lambda}(x-a)^{p}dx\right)^{\frac{1}{p}}(b-a-\lambda)^{\frac{1}{q}}\left[|f^{'}(b-\lambda)|^{q}+\frac{\eta(|f^{'}(a)|^{q},|f^{'}(b-\lambda)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$+\left(\int_{b-\lambda}^{b}(b-x)^{p}dx\right)^{\frac{1}{p}}\lambda^{\frac{1}{q}}\left[|f^{'}(b)|^{q}+\frac{\eta(|f^{'}(b-\lambda)|^{q},|f^{'}(b)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$=\frac{(b-a-\lambda)^{2}}{(p+1)^{\frac{1}{p}}}\left[|f^{'}(b-\lambda)|^{q}+\frac{\eta(|f^{'}(a)|^{q},|f^{'}(b-\lambda)|^{q})}{2}\right]^{\frac{1}{q}}$$

$$+\frac{\lambda^{2}}{(p+1)^{\frac{1}{p}}}\left[|f^{'}(b)|^{q}+\frac{\eta(|f^{'}(b-\lambda)|^{q},|f^{'}(b)|^{q})}{2}\right]^{\frac{1}{q}}$$

giving the inequality (2.6).

3. *Steffensen's type inequalities for P- function*

Theorem 3.1 *Let* $f,g:[a,b]\subset\mathbb{R}^{+}\to\mathbb{R}$ *be integrable such that* $0\le g(t)\le1$*, for all* $t\in[a,b]$ *such that* $\int_{a}^{b}g(t)f^{'}(t)dt$ *exists. If* f *is absolutely continuous on* $[a,b]$ *such that* $|f^{'}|$ *is* P *function on* $[a,b]$*, then we have*

$$\left|\int_{a}^{a+\lambda}f(x)dx-\int_{a}^{b}f(x)g(x)dx\right| \tag{3.1}$$

$$\le\frac{\lambda^{2}}{2}\left[|f^{'}(a)|+|f^{'}(a+\lambda)|\right]+\frac{(b-a-\lambda)^{2}}{2}\left[|f^{'}(b)|+|f^{'}(a+\lambda)|\right]$$

and

$$\left|\int_{a}^{b}f(x)g(x)dx-\int_{b-\lambda}^{b}f(x)dx\right| \tag{3.2}$$

$$\le\frac{(b-a-\lambda)^{2}}{2}\left[|f^{'}(a)|+|f^{'}(a+\lambda)|\right]+\frac{\lambda^{2}}{2}\left[|f^{'}(b)|+|f^{'}(a+\lambda)|\right]$$

where $\lambda:=\int_{a}^{b}g(t)dt$.

Proof. Using Lemma 1.1 and since $|f^{'}|$ is P function, we have

$$\left|\int_{a}^{a+\lambda}f(x)dx-\int_{a}^{b}f(x)g(x)dx\right|$$

$$\le\left|\int_{a}^{a+\lambda}\left(\int_{a}^{x}(1-g(t))dt\right)f^{'}(x)dx\right|+\left|\int_{a+\lambda}^{b}\left(\int_{x}^{b}g(t)dt\right)f^{'}(x)dx\right|$$

$$\le\int_{a}^{a+\lambda}\left|\int_{a}^{x}(1-g(t))dt\right||f^{'}(x)|dx+\int_{a+\lambda}^{b}\left|\int_{x}^{b}g(t)dt\right||f^{'}(x)|dx$$

$$= \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\left|f'\left(\frac{x-a}{\lambda}(a+\lambda)+\frac{a+\lambda-x}{\lambda}a\right)\right|dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\left|f'\left(\frac{x-a-\lambda}{b-a-\lambda}b+\frac{b-x}{b-a-\lambda}(a+\lambda)\right)\right|dx$$

$$\le \int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right|\left[|f'(a+\lambda)|+|f'(a)|\right]dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right|\left[|f'(b)|+|f'(a+\lambda)|\right]dx$$

$$=\int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right||f'(a+\lambda)|dx+\int_a^{a+\lambda}\left|\int_a^x (1-g(t))dt\right||f'(a)|dx$$

$$+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right||f'(b)|dx+\int_{a+\lambda}^b\left|\int_x^b g(t)dt\right||f'(a+\lambda)|dx$$

$$\le \int_a^{a+\lambda}(x-a)|f'(a+\lambda)|dx+\int_a^{a+\lambda}(x-a)|f'(a)|dx$$

$$+\int_{a+\lambda}^b(b-x)|f'(b)|dx+\int_{a+\lambda}^b(b-x)|f'(a+\lambda)|dx$$

$$=|f'(a+\lambda)|\int_a^{a+\lambda}(x-a)dx+|f'(a)|\int_a^{a+\lambda}(x-a)dx$$

$$+|f'(b)|\int_{a+\lambda}^b(b-x)dx+|f'(a+\lambda)|\int_{a+\lambda}^b(b-x)dx$$

$$=\frac{\lambda^2}{2}[|f'(a+\lambda)|+|f'(a)|]+\frac{(b-a-\lambda)^2}{2}[|f'(b)|+|f'(a+\lambda)|]$$

and so we proved inequality (3.1). Then similarly

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^b f(x)dx\right|$$

$$\le\left|\int_a^{b-\lambda}\left(\int_a^x g(t)dt\right)f'(x)dx\right|+\left|\int_{b-\lambda}^b\left(\int_x^b(1-g(t))dt\right)f'(x)dx\right|$$

$$\le\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right||f'(x)|dx+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right||f'(x)|dx$$

$$=\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\left|f'\left(\frac{x-a}{\lambda}(a+\lambda)+\frac{a+\lambda-x}{\lambda}a\right)\right|dx$$

$$+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right|\left|f'\left(\frac{x-a-\lambda}{b-a-\lambda}b+\frac{b-x}{b-a-\lambda}(a+\lambda)\right)\right|dx$$

$$\le\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|\left[|f'(a+\lambda)|+|f'(a)|\right]dx$$

$$+\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right|\left[|f'(b)|+|f'(a+\lambda)|\right]dx$$

$$=|f'(a+\lambda)|\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|dx+|f'(a)|\int_a^{b-\lambda}\left|\int_a^x g(t)dt\right|dx$$

$$+|f'(b)|\int_{b-\lambda}^b\left|\int_x^b g(t)dt\right|dx+|f'(a+\lambda)|\int_{b-\lambda}^b\left|\int_x^b(1-g(t))dt\right|dx$$

$$\le|f'(a+\lambda)|\int_a^{b-\lambda}(x-a)dx+|f'(a)|\int_a^{b-\lambda}(x-a)dx$$

$$+|f'(b)|\int_{b-\lambda}^b(b-x)dx+|f'(a+\lambda)|\int_{b-\lambda}^b(b-x)dx$$

$$=\frac{(b-a-\lambda)^2}{2}[|f'(a+\lambda)|+|f'(a)|]+\frac{\lambda^2}{2}[|f'(b)|+|f'(a+\lambda)|]$$

and so we proved inequality (3.2).

Theorem 3.2 *Let* $f,g:[a,b]\subset \mathbb{R}^+\to\mathbb{R}$ *be integrable such that* $0\le g(t)\le 1$*, for all* $t\in[a,b]$ *such that* $\int_a^b g(t)f'(t)dt$ *exists. If* f *is absolutely continuous on* $[a,b]$ *such that* $|f'|$ *is* P *function on* $[a,b]$*, then the following inequalities holds*

$$\left|\int_a^{a+\lambda}f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\le\left(\int_{a+\lambda}^b g(t)dt\right)[\lambda(|f'(a)|+|f'(a+\lambda)|)+(b-a-\lambda)(|f'(a+\lambda)+|f'(b)|)]$$

$$\le(b-a-\lambda)[\lambda(|f'(a)|+|f'(a+\lambda)|)+(b-a-\lambda)(|f'(a+\lambda)+|f'(b)|)]$$

and

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^b f(x)dx\right|$$

$$\le\left(\int_a^{b-\lambda}g(t)dt\right)[(b-a-\lambda)(|f'(b-\lambda)|+|f'(a)|)+\lambda(|f'(b)|+|f'(b-\lambda)|)]$$

$$\le\lambda[(b-a-\lambda)(|f'(b-\lambda)|+|f'(a)|)+\lambda(|f'(b)|+|f'(b-\lambda)|)]$$

where $\lambda:=\int_a^b g(t)dt$.

Proof. From Lemma 1.1, we can write

$$\left|\int_a^{a+\lambda} f(x)dx-\int_a^b f(x)g(x)dx\right| \le \sup_{x\in[a,a+\lambda]}\left[\int_a^x (1-g(t))dt\right]\int_a^{a+\lambda} |f'(x)|\,dx$$

$$+\sup_{x\in[a+\lambda,b]}\left[\int_x^b g(t)dt\right]\int_{a+\lambda}^b |f'(x)|\,dx.$$

Since $|f'|$ is P function on $[a,b]$ and using (1.3), we have

$$\int_a^{a+\lambda} |f'(x)|\,dx \le \lambda\left(|f'(a)|+|f'(a+\lambda)|\right),$$

and

$$\int_{a+\lambda}^b |f'(x)|\,dx \le (b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right).$$

Therefore, we have

$$\left|\int_a^{a+\lambda} f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\le \lambda\left(|f'(a)|+|f'(a+\lambda)|\right)\left[\int_a^{a+\lambda}(1-g(t))dt\right]$$

$$+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\left[\int_{a+\lambda}^b g(t)dt\right]$$

$$\le \max\left\{\int_a^{a+\lambda}(1-g(t))dt,\int_{a+\lambda}^b g(t)dt\right\}\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)\right.$$

$$\left.+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right]$$

$$=\left(\int_{a+\lambda}^b g(t)dt\right)\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right].$$

Since $0\le g(t)\le 1$ for all $t\in[a,b]$, we can write

$$0\le \int_{a+\lambda}^b g(t)dt \le b-a-\lambda.$$

So, we obtained

$$\left|\int_a^{a+\lambda} f(x)dx-\int_a^b f(x)g(x)dx\right|$$

$$\le\left(\int_{a+\lambda}^b g(t)dt\right)\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right]$$

$$\le (b-a-\lambda)\left[\lambda\left(|f'(a)|+|f'(a+\lambda)|\right)+(b-a-\lambda)\left(|f'(a+\lambda)+|f'(b)|\right)\right].$$

Similarly

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^b f(x)dx\right| \le \sup_{x\in[a,b-\lambda]}\left[\int_a^x g(t)dt\right]\int_a^{b-\lambda} |f'(x)|\,dx$$

$$+\sup_{x\in[b-\lambda,b]}\left[\int_x^b (1-g(t))dt\right]\int_{b-\lambda}^b |f'(x)|\,dx.$$

Since $|f'|$ is P function on $[a,b]$ and using (1.3), we get

$$\int_a^{b-\lambda} |f'(x)|\,dx \le (b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right),$$

and

$$\int_{b-\lambda}^b |f'(x)|\,dx \le \lambda\left(|f'(b)|+|f'(b-\lambda)|\right).$$

Therefore, we have

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^b f(x)dx\right|$$

$$\le (b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)\left[\int_a^{b-\lambda} g(t)dt\right]$$

$$+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\left[\int_{b-\lambda}^b(1-g(t))dt\right]$$

$$\le \max\left\{\int_a^{b-\lambda} g(t)dt,\int_{b-\lambda}^b (1-g(t))dt\right\}\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)\right.$$

$$\left.+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

$$=\left(\int_{b-\lambda}^b (1-g(t))dt\right)\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

Since $0\le g(t)\le 1$ for all $t\in[a,b]$, we can write

$$0\le \int_{b-\lambda}^b (1-g(t))dt \le \lambda.$$

So, we obtained

$$\left|\int_a^b f(x)g(x)dx-\int_{b-\lambda}^b f(x)dx\right|$$

$$\le\left(\int_a^{b-\lambda} g(t)dt\right)\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

$$\le \lambda\left[(b-a-\lambda)\left(|f'(b-\lambda)|+|f'(a)|\right)+\lambda\left(|f'(b)|+|f'(b-\lambda)|\right)\right]$$

and the proof is completed.

4. Conclusion

In the present paper, we prove some Steffensen's type inequalities by utilizing P-function and η-convex function. The new bounds can establish by used different classes of convex functions instead of this convex functions by researches interested in the subject.

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