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On Taxicab Circle Inverses of Lines

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Keywords:
Taxicab circle inversion, Taxicab inverses of lines, Taxicab plane.

Abstract — In this study, the inverses of lines with respect to the taxicab circle inversion are investigated. It is shown that the image of a line not passing through the inversion center is a closed curve consisting of two parabola arcs or a parabola arc and a line segment in taxicab plane. The properties of closed curves, which are taxicab circle inverses of lines are analytically determined according to vertical, horizontal, steep, gradual or separator line types. The distinctive properties of the taxicab circle inverses of lines are presented.

Subject Classification (2020): 51B20; 51F99; 51K99.

1. Introduction

The circle inversion is one of the most important and interesting geometric transformations. The inversion in a circle was introduced by Apollonius of Perga in his work "Plane Loci" and systematically studied by Steiner in 1830s. Since inversions have attracted attention of scientists from past to present, there are a lot of studies about them.

The circle inversions preserve angles and transform straight lines and circles into straight lines and/or circles. Many challenging problems in geometry become much more manageable when inversion is applied. Numerous scientists have studied and continue to study various aspects of this concept. Several generalizations of the inversion transformation have been introduced in the literature. In [7,9], the inversions with respect to the central conics were defined in Euclidean plane.

Non-Euclidean metric geometries have various applications in mathematics, physics, computer science, engineering and other fields, depending on the specific properties and distance functions they use. Among these geometries equipped with non-Euclidean metrics, taxicab geometry and maximum plane geometry have a rich literature [1-3,6,10,17-18,23]. The inversion with respect to taxicab circle has been defined and some properties such as cross ratio and harmonic conjugates have been given in [5]. Subsequently, the inversion in alpha plane [15], Chinese-Checker plane [21] and maximum plane [24] have been presented, and their corresponding features were examined. The circle inversion has been generalized to the spherical inversion in the three-dimensional taxicab space [20], Chinese-Checker space [19] and maximum space [8], utilizing a sphere. In [22], p-circle inversion which generalizes the

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classical inversion with respect to a circle (p = 2) and the taxicab inversion (p = 1), is defined, and new fractal patterns were obtained by applying this transformation to well-known fractals. A generalization of the alpha circle inversion fractal is also provided in [16].

In Euclidean geometry, the inverses of lines differ depending on whether they pass through the inversion center or not. In Euclidean circle inversion, inversion transforms the lines not passing through the inversion center into circles passing through the inversion center, circles passing through the inversion center into lines not passing through the center, and circles not passing through the center into circles not passing through the center. In some studies on circle inversion in non-Euclidean planes, the inverses of lines with this feature have been examined in the literature. However, it has been observed that this feature alone is not sufficient to classify the images of lines under the circle inversion in the taxicab plane and the maximum plane [4,11-17]. Therefore, in this study, it is aimed to determine the circle inversion of lines and their properties according to their types in the taxicab plane.

In this paper, the properties of images under the taxicab circle inversion have been analyzed analytically according to the types of lines. It is shown that the image of a line which does not pass through the center of the taxicab circle inversion is a closed curve different from a taxicab circle. The properties of these closed curves, which are taxicab circle inverses of lines, are determined according to vertical, horizontal, steep, gradual or separator line types. It is also demonstrated that the parallel line pencil forms a closed curve pencil passing through the inversion center under inversion with respect to the taxicab circle.

2. Preliminaries

We summarize below some definitions and theorems from the literature that are necessary for this study

The taxicab plane $\mathbb{R}^2_T$ is almost the same as the Euclidean plane $\mathbb{R}^2_T$. The points and the lines are the same, and the angles are measured in the same way. However, the distance function is different. In general, the taxicab distance between two points is measured as the sum of the change in horizontal and vertical directions between the two points, where Euclidean geometry is measured using the Pythagorean theorem.

Definition 2.1. Let P_1 and P_2 be two points whose coordinates are (x_1, y_1) and (x_2, y_2) in analytical plane, respectively. The taxicab distance between these points is $d_T(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$.

The isometry group of taxicab plane is the semi direct product of D(4) and T(2) where D(4) is the symmetry group of Euclidean square and T(2) is the group of all translations in the plane [23].

In [18], Krause classified lines depending on their slope as the following definition:

Definition 2.2 Let m be the slope of the line l in taxicab plane. The line l is called the steep line, the gradual line and the separator line in the cases of $|m|>1$, $|m|<1$ and $|m|=1$, respectively. In the special cases that the line l is parallel to x-axis or y-axis, l is named as the horizontal line or the vertical line, respectively [18].

Definition 2.3. The taxicab circle C_T with the center M and the radius r consists of the points X which satisfies the equation $d_T(M, X) = \text{r}$. The point M is called center of the taxicab circle, and r is called the length of the radius or simply the radius of the taxicab circle.

Every taxicab circle in the taxicab plane is an Euclidean square having sides with slopes ± 1. It is seen by definition 2.3 that the taxicab circle centered at the point $M = (m_1, m_2)$ with the radius r is the set $C_T = \{(x, y): |x_1 - m_1| + |y_1 - m_2| = \text{r}\}$. As particular case, the taxicab unit circle is the set $\{(x, y): |x| + |y| = 1\}$.

In the taxicab plane, the inversion with respect to the circle C_T with the center O and the radius r_T is denoted by $I_{(O,r)}$ and is defined as follows: For the collinear points O, the point P, and its image P' on the ray OP, $d_T(O,P).d_T(O,P') = r^2$, where $d_T(O,P)$ represents the taxicab distance between O and P, [5].

Clearly, if P' is the inverse point of P, then P is the inverse point of the P'. Note that if P is in the interior of C_T, P' is exterior to C_T; and vice-versa. So, the interior of C_T except for O is mapped to the exterior and the exterior to the interior. C_T itself is left by the inversion pointwise fixed. O has no image, and no point of the plane is mapped to O. However, we can add to the taxicab plane a single point at infinite O_∞, which is the inverse of the center O of taxicab inversion circle C_T. So, the taxicab circle inversion $I_{(O,r)}$ is one-to-one map of extended taxicab plane.

Now in the extended taxicab plane $\mathbb{R}_T^2 \cup \{O_\infty\}$, the definition of inversion with respect to a taxicab circle C_T can be given as follows:

Definition 2.4. The transformation

$$I_{(O,r)}: \mathbb{R}_T^2 \cup \{O_\infty\} \to \mathbb{R}_T^2 \cup \{O_\infty\}$$

$$P \to I_{(O,r)}(P) = P'$$

defined by the circle C_T is called the taxicab circle inversion. The circle C_T is known as taxicab inversion circle, O is called the center of the taxicab inversion, r is called the taxicab inversion radius, and P' is called the taxicab circle inverse of the point P, [5].

For any point P on the taxicab inversion circle, the taxicab circle inversion map has the property $I_{(O,r)}(P) = P$.

Theorem 2.5. The taxicab circle inversion maps the point inside of the taxicab inversion circle to the point outside of it, and vice versa, [5].

Teorem 2.6. If the points $P = (x,y)$ and $P' = (x', y')$ are a pair of the inverse points in the taxicab circle inversion with the center $O = (0,0)$ and radius r, the following equality exists between the coordinates of P and P'

$$(x',y') = \left(\frac{r^2x}{(|x|+|y|)^2}, \frac{r^2y}{(|x|+|y|)^2}\right),$$

[5].

Corollary 2.7. If the points $P = (x,y)$ and $P' = (x', y')$ are a pair of the inverse points in the taxicab circle inversion with the center $O = (a,b)$ and radius r, the following equality exists between the coordinates of P and P'

$$(x',y') = \left(\frac{r^2(x-a)}{(|x-a|+|y-b|)^2}, \frac{r^2(y-b)}{(|x-a|+|y-b|)^2}\right),$$

[5].

Since the translation transformation preserves the taxicab distance in the taxicab plane, the center of the taxicab inversion circle can be taken as the origin without loss of generality. Therefore, throughout this paper, the taxicab inversion center will be considered as the origin unless otherwise stated.

3. Images of the lines under the taxicab circle inversion

In the Euclidean circle, inversion transforms the lines not passing through the inversion center into circles passing through the inversion center, circles passing through the inversion center into lines not passing through the center and circles not passing through the center into circles not passing through the center. In taxicab plane and maximum plane, lines passing through the inversion center are invariant under the inversion transformation, but the images of the lines not passing through the center have different shapes [4, 5, 11, 24].

In this section, the images of lines not passing through the inversion center in the extended taxicab plane are analytically considered and their properties are presented depending on their positions in the plane.

Theorem 3.1. Lines not passing through the center of the taxicab inversion circle do not remain invariant under taxicab circle inversion.

Proof. Let $O=(0,0)$ be the center of the taxicab inversion circle C_T with radius *r*, and let l be a line defined by the equation $ax+by+c=0$, where at least one of a and b is non-zero, $c\neq 0, a,b,c\in\mathbb{R}$. The image of l under the taxicab inversion $I_{(\mathrm{O,r})}$ is given by the equation $ar^2x+br^2y+c(|x|+|y|)^2=0$. Since the coefficient c is not zero, the equation does not specify a line in the taxicab plane. Therefore, in the extented taxicab plane, lines that do not pass through the center of inversion do not remain invariant under the taxicab inversion transformation. This concludes the proof.

Theorem 3.2. The inverses of horizontal lines with respect to the taxicab circle are closed curves formed by the union of segments of two orthogonal parabolas passing through the inversion center."

Proof. Let $O=(0,0)$ be the center of the taxicab inversion circle with radius r and let l be a line defined by the equation $y=k, k\neq 0, k\in\mathbb{R}$. The inverse of the line l in C_T is the closed curve with equation $y=k(|x|+|y|)^2$. This means that the image is the closed curve consisting the union of two orthogonal parabola arcs passing through the origin and having the equations $y=k(x-y)^2$ and $y=k(-x+y)^2$. The axes of symmetry of the parabola segments forming this closed curve are $y=\frac{1}{4k}-|x|$ and directrices are $y=-\frac{1}{4k}+|x|$. So, the axes and directrices are perpendicular to each other since their slopes are 1 and -1. Hence, the inverses of the lines paralel to x-axis with respect to the taxicab circle are closed curves formed by the union of two parabola segments passing through the center of inversion and whose axes and directrices are perpendicular to each other.

In addition, when the slope of the symmetry axis is -1, the vertex and the focus of the parabola segment are obtained as $T_1=(\frac{3}{16k},\frac{1}{16k})$ and $O_1=(\frac{1}{8k},\frac{1}{8k})$, respectively. If the slope of the symmetry axis of the parabola is $+1$, the vertex and the focus of the parabola are obtained as $T_2=(-\frac{3}{16k},\frac{1}{16k})$ and $O_2=(-\frac{1}{8k},\frac{1}{8k})$, respectively.

Also, the following result are immediately obtained from the proof of Theorem 3.2.

Corollary 3.3. The taxicab inversion of a pencil of horizontal parallel lines not passing through the inversion center consists of a pencil of closed curves formed by the union of two parabola segments with symmetry axes are parallel to the separator lines. Also, each curve pencil in the taxicab inversion passes through the inversion center and is symmetric with respect to the perpendicular line passing through the inversion center.

Example 3.4. In Figure 1 (left), we show the taxicab inverse *I(l)* with the equation $y=k(|x|+|y|)^2$ in the taxicab unit circle centered at origin O of the line *l* with the equation $y=1$; in Figure 1 (right), we illustrate the taxicab inversion with respect to the taxicab unit circle centered at the origin O for a pencil of horizontal parallel lines that do not intersect the inversion center.

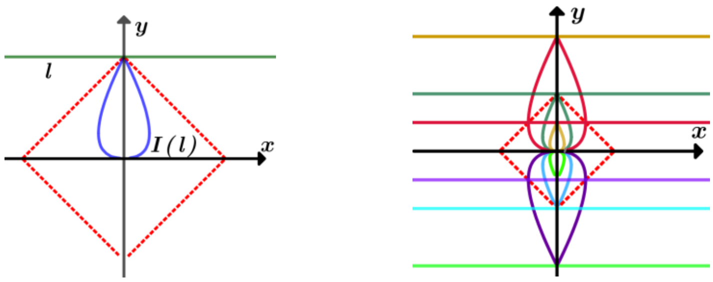

Figure 1. The taxicab circle inverses of parallel lines

Theorem 3.5. The symmetry axes and directrices of two parabola segments forming the inversion of a horizontal line with respect to a taxicab circle define a taxicab circle whose center is the center of inversion.

Proof. Inversion of the horizontal line with the equation $y = k,\ k \neq n, k \in \mathbb{R}$ with respect to a taxicab circle with center (m, n) and radius r is a closed curve with equation $(k-n)(|x-m|+|y-n|)^2 = r^2(y-n)$. This closed curve consists of the parabola segments with the symmetry axis $y = -x + m + n + \frac{r^2}{4(k-n)}$ and with the directrix $y = x + n - m - \frac{r^2}{4(k-n)}$ and the parabola segment with symmetry axis $y = x + n - m - \frac{r^2}{4(k-n)}$ and with the directrix $y = -x + n + m - \frac{r^2}{4(k-n)}$, respectively. The axes and directrices of these parabolas intersect at the points $\left(m, n + \frac{r^2}{4(k-n)}\right)$, $\left(m, n - \frac{r^2}{4(k-n)}\right)$, $\left(m + \frac{r^2}{4(k-n)}, n\right)$ and $\left(m - \frac{r^2}{4(k-n)}, n\right)$. Thus, the taxicab circle is formed, whose vertices are these points and whose edges are on the axes and directrices of the parabolas, with the equation $|x-m| + |y-n| = \frac{r^2}{4(k-n)}$. This completes the proof.

The reflection transformations with respect to the lines $y = x$ and $y = -x$ in the taxicab plane are isometries. Therefore, the theorems given for the taxicab inverses of horizontal lines can be given for taxicab inverses of vertical lines, too.

Theorem 3.6. The inverse of a vertical line with respect to the taxicab circle is closed curve formed by the union of two parabola arcs with axes and directrices perpendicular to each other and passing through inversion center.

Proof. Since the reflection transformation in the taxicab plane with respect to the line $y = x$ is an isometry, it can be easily proved by substituting the unknowns x and y in the proof of Theorem 3.2.

Corollary 3.7. The taxicab inversion of a pencil of vertical parallel lines with respect to the taxicab circle consists of a pencil of closed curves such that each closed curve in the pencil passes through the inversion center and is symmetric with respect to the horizontal line passing through the inversion center.

Proof. Since the inverse of each line in the vertical parallel line pencil with respect to the taxicab circle is closed curve formed by the union of two parabolas with axes and directrices perpendicular to each other and passing through the center of inversion, the proof is obvious.

Theorem 3.8. The axes and directrices of two parabola segments, which together compose the taxicab circle inverse of a vertical line with respect to a taxicab circle, determine a taxicab circle centered at the inversion center.

Proof. It can be easily proved by substituting the unknowns x and y in the proof of Theorem 3.4.

Theorem 3.9. The inverse of a separator line not passing through the center of the inversion circle is a closed figure consisting of a parabola arc with the vertex at the inversion center and a line segment.

Proof. Let $O=(0,0)$ be the center of the taxicab inversion circle C_T with radius r, and let l be a separator line. So, the line l can be defined by the equations $x+y+c=0$ or $x-y+c=0$, where $c\neq 0, c\in\mathbb{R}$. The image of the line l with equation $x+y+c=0$ under taxicab circular inversion is a closed curve with equation the equation $r^2x+r^2y+c(|x|+|y|)^2=0$. This equation gives a line segment with the equation $x+y+\frac{r^2}{c}=0$ parallel to the edge of the inversion circle when x and y coordinate values have the same sign, and a parabola segment with the equation $r^2x+r^2y+c(x-y)^2=0$ with the vertex at the origin and the symmetry axis the line $y=x$ when x and y have opposite signs. Similarly, the taxicab circular inverse of the line l with the equation x-y+c=0 is a closed curve with the equation $r^2x-r^2y+c(|x|+|y|)^2=0$. This equation represents a separator line segment with the equation $x-y+\frac{r^2}{c}=0$ when x and y coordinate values have opposite signs, and a parabola segment with the equation $r^2x-r^2y+c(x-y)^2=0$ with its vertex at the origin, and its symmetry axis is $y=x$ when x and y have the same sign. So, the proof is completed.

Theorem 3.10. The inverse of a gradual line or a step line in the taxicab plane not passing through the inversion center in taxicab circle is a closed curve consisting of two parabola arcs with axes perpendicular to each other and passing through the inversion center.

Proof. Suppose l be a gradual line not passing through origin in the taxicab plane. Then the equation of l is $y=mx+n$, where $m,n\in\mathbb{R}$ and $m\neq 0,\pm1,\infty$ and $n\neq 0$. The inverse of l in the taxicab circle centered at $O=(0,0)$ with the radius r has the equation $mr^2x-r^2y+n(|x|+|y|)^2=0$. This means that the image is the closed curve passing through the inversion center, formed by the union of two parabola arcs with equations $mr^2x-r^2y+n(x+y)^2=0$ for x and y coordinate values with the same sign, and $mr^2x-r^2y+n(x-y)^2=0$ for x and y coordinate values with the different signs. The symmetry axis of these two parabolas have the equations $x+y+\frac{(m-1)r^2}{4n}=0$ and $x-y+\frac{(m+1)r^2}{4n}=0$, respectively. Since the slopes of the symmetry axes of these parabolas are 1 and -1, they are perpendicular to each other. This completes the proof.

The vertices of these parabola arcs are $\left(\frac{(1-m)(3+m)}{16n(m+1)}r^2, \frac{(1-m)(1+3m)}{16n(m+1)}r^2\right)$ and $\left(\frac{(m+1)(3-m)}{(m-1)16n}r^2, \frac{(m+1)(3m-1)}{16n(m-1)}r^2\right)$, respectively.

Example 3.11. In Fig. 2 (left) we show the taxicab inversion in the taxicab unit circle of the separator line $x+y=2$; In Fig. 2 (right) we show the taxicab inverse with respect to the taxicab unit circle of the gradual line l with $y=0.5x-0.25$.

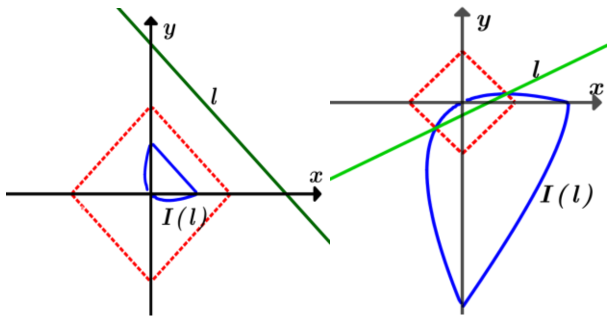

Figure 2. The taxicab circle inverses of separator line and gradual line

4. Conclusion

In In the present paper, we have explored the inverses of lines with respect to the taxicab circle inversion in the taxicab plane. We observed that the inverse of a line, different from the separator line and does not pass through the inversion center under the taxicab circle inversion, is the closed curve consisting of two parabola segments passing through the inversion center. On the other hand, the inverse of a separator line yields a closed curve comprising a line segment parallel to an edge of the inversion circle and a parabola segment. At the same time, it was seen that the axes and directrices of the parabola segments that form the inverse of a horizontal or vertical line determine a taxi circle whose center is the inversion center. It is also shown that the taxicab inversion of a pencil of parallel lines that do not pass through the center of inversion is a pencil of closed curves that are tangent at the center of inversion.
In conclusion, it is evident that taxicab circle inverses of lines in the analytic plane exhibit significantly different properties compared to Euclidean circle inverses.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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Trans-Sasakian Indefinite Finsler Manifolds

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Keywords:
Trans-Sasakian manifold, α −Sasakian manifold , β −Kenmotsu manifold , Pseudo-Finsler metric , Indefinite Finsler manifold

Abstract — In this paper we introduce some properties and results for trans-Sasakian structures on indefinite Finsler manifolds and give the examples of such manifolds.These structures are established on the $(M^0)^h$ and $(M^0)^v$ vector subbundles, where M is an $(2n+1)$ dimensional C^∞ manifold, $M^0 = (M^0)^h \oplus (M^0)^v$ is a non-empty open submanifold of TM. F^* is the fundamental Finsler function and $F^{2n+1} = (M, M^0, F^*)$ is an indefinite Finsler manifold. We use the Sasaki Finsler metric $G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*} \delta y^i \otimes \delta y^i$. Furthermore, we give some formulas for α −Sasakian and β −Kenmotsu Finsler manifolds with pseudo-Finsler metric. Finally, it is shown that the conformally flat trans-Sasakian indefinite Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are the η − Einstein manifolds if and only if $\alpha.\beta = 0$, where α, β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

Subject Classification (2020): 53B30, 53B35, 53B40

1. Introduction

Oubina introduced the idea of trans-Sasakian manifold of classification (α,β). Indefinite Sasakian manifold is a notable category of indefinite trans-Sasakian manifold for α=1, β=0. Also, indefinite cosymplectic manifold is the other category of indefinite trans-Sasakian manifold for α=0, β=0. Indefinite Kenmotsu manifold is given with α=0, β=1. M. D. Siddiqi, A. N. Siddiqui and O. Bahadır study the trans-Sasakian manifolds with a quarter-symmetric nonmetric connection [12]. R. Prasad, U. K. Gautam, J. Prakash and A. K. Rai study (ε)−Lorentzian trans-Sasakian manifolds [16].

The papers interested in contact structures with Riemannian metric or pseudo-Riemannian metric but in this paper, we are also related to the contact structures with pseudo-Finsler metric.

After Finsler published his thesis about curves and surfaces, a lot of articles are dedicated to Finsler geometry, see references [4, 5, 10, 13, 14, 15] but the theory of indefinite Finsler manifold has been investigated by few researchers [1, 2, 7, 8, 9]. We also make reference to the reader to the recent monograph for detailed information in this field.

Hence, our aim is to present trans-Sasakian indefinite Finsler manifolds and to obtain the formulas for α −Sasakian and β −Kenmotsu indefinite Finsler manifolds. The paper is organized as follows: after

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introduction and background, we give some preliminaries about indefinite Finsler manifolds. Then, we deal with the trans-Sasakian indefinite Finsler manifolds, α −Sasakian and β −Kenmotsu indefinite Finsler manifolds. Finally, it is shown that the conformally flat trans-Sasakian indefinite Finsler manifolds $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are the $\eta -$ Einstein manifolds if and only if $\alpha.\beta = 0$, where α and β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

2. Preliminaries

2.1. Indefinite Finsler Manifolds

Let M be a real $(2n+1)-$ dimensional smooth manifold and TM be the tangent bundle of M. A coordinate system in M can be stated with $\{(U,\varphi): x^1, \dots, x^{2n+1}\}$, where U is an open subset of M; for any $x \in U, \varphi: U \to \mathbb{R}^{2n+1}$ is a diffeomorphism of U onto $\varphi(U)$, and $\varphi(x) = (x^1, \dots, x^{2n+1})$. On M, denote by π the canonical projection of TM and by T_xM the fibre, at $x \in M$, i.e., $T_xM = \pi^{-1}(x)$. Through the coordinate system $\left\{(U,\varphi): x^i\right\}$ in M, we can describe a new coordinate system $\{(U^*,\Phi); x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1}\}$ or shortly $\left\{(U^*,\Phi): x^i, y^i\right\}$ in TM, where $U^* = \pi^{-1}(U)$ and $\Phi: U^* \to \mathbb{R}^{4n+2}$ is a diffeomorphism of U^* on $\varphi(U) \times \mathbb{R}^{2n+1}$, and $\Phi(y_x) = (x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1})$ for any $x \in U$ and $y_x \in T_xM$. Let M^0 be a non-empty open submanifold of TM such that $\pi(M^0) = M$ and $\theta(M) \cap M^0 = \emptyset$, where θ is the zero section of TM. Assume that $M_x^0 = T_xM \cap M^0$ is a positive conic set, for any $k > 0$ and $y \in M_x^0$. we have $ky \in M_x^0$. Obviously, the largest M^0 holding the above circumstances is $TM \setminus \theta(M)$, ordinarily given with the description of a Finsler manifold.

We now consider a smooth function $F: M^0 \to (0,\infty)$ and take $F^* = F^2$. Then suppose that for any coordinate system $\left\{(U^0,\Phi^0);\ x^i, y^i\right\}$ in M^0, the following conditions are fulfilled:

(**$F1$**) F is positively homogenous of degree one regarding $(y^1, \dots, y^{2n+1})$, $i.e.$, we get, for all $k > 0$ and $(x,y) \in \Phi^0(U^0)$,

$$F(x^1, \dots, x^{2n+1}, ky^1, \dots, ky^{2n+1}) = k\, F(x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1})$$

($F2$) At any point $(x,y) \in \Phi^0(U^0)$,

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^*}{\partial y_i \partial y_j}(x,y), \quad i,j \in \{1,2,\dots,2n+1\}$$

are the components of a positive definite quadratic form on $\mathbb{R}^{2n+1}$.

We say that the triple $F^{2n+1} = (M, M^0, F)$ is a Finsler manifold, and F is the fundamental function of F^{2n+1}.

Certainly, condition $(F2)$ is not appropriate for some applications of Finsler geometry. To remove this inconvenience we consider a positive integer $0 < q < 2n+1$, and a smooth function $F^*: M^0 \to R$, where $M^0$is as above. Moreover, suppose that for any coordinate system $\left\{(U^0,\Phi^0);\ x^i, y^i\right\}$ in M^0, the following conditions are fulfilled:

($F1^*$) F^* is positively homogenous of degree two regarding $(y^1, \dots, y^{2n+1})$, we get, for all $k > 0$ and $(x,y) \in \Phi^0(U^0)$,

$$F^*(x^1, \dots, x^{2n+1}, ky^1, \dots, ky^{2n+1}) = k^2 F^*(x^1, \dots, x^{2n+1}, y^1, \dots, y^{2n+1})$$

($F2^*$) At all point $(x,y) \in \Phi^0(U^0)$,

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^*}{\partial y_i \partial y_j}(x,y), \quad i,j \in \{1,2,\dots,2n+1\}$$

are the components of a quadratic form on $\mathbb{R}^{2n+1}$ with $(2n+1)-q$ positive eigenvalues and q negative eigenvalues $(0<q<2n+1)$. In this state $F^{2n+1}=(M,M^0,F^*)$ is called indefinite Finsler manifolds with index q. Particularly, if chosing $q=1$, we get Lorentzian indefinite Finsler manifolds [2].

Consider the structure of $F^{2n+1}=(M,M^0,F^*)$ indefinite Finsler manifold with index q. Then the tangent mapping $\pi_*: TM^0 \to TM$ of the submersion $\pi: M^0 \to M$ and define the vector bundle $(TM^0)^V = ker\pi_*$. As locally, $\pi^i(x,y)=x^i$, we obtain $\pi_*^i\left(\frac{\partial}{\partial x^j}\right)=\delta_j^i$ *and* $\pi_*^i\left(\frac{\partial}{\partial y^j}\right)=0$, on the coordinate neighborhood $U^0 \subset M^0$. Thus, $\left\{\frac{\partial}{\partial y^i}\right\}$ is a basis of $\Gamma\left((TM^0)^V{}_{|_{U^0}}\right)$. We call $(TM^0)^V$ the vertical vector bundle of F^{2n+1}. Locally, on a coordinate neighborhood $U^0 \subset M^0$. we have $X^V = X^i(x,y)\frac{\partial}{\partial y^i}$, where X^i smooth functions on U^0. After we denote by $(T^*M^0)^V$ the dual vector bundle of $(TM^0)^V$. Thus a Finsler 1-form is smooth section of $(T^*M^0)^V$. Assume $\{\delta y^1,\dots,\delta y^{2n+1}\}$ is a dual basis to $\left\{\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^{2n+1}}\right\}$, i.e., $\delta y^i\left(\frac{\partial}{\partial y^j}\right)=\delta_j^i$. Then each for $w \in (T^*M^0)^V$, $w^V = w^i(x,y)\,\delta y^i$, where $w^i(x,y)=w\left(\frac{\partial}{\partial y^i}\right)$ [1, 2].

The complementary distribution $(TM^0)^H$ to $(TM^0)^V$ in TM^0 is said a horizontal distribution (non- linear connection) on M^0. Thus we can write

$$TM^0 = (TM^0)^H \oplus (TM^0)^V$$

The set of the local vector fields $\left\{\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^{2n+1}}\right\}$ is a basis in $\Gamma((TM^0)^H)$. Then

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N_i^j\frac{\partial}{\partial y^j}$$

Let X be a vector field on M^0. Then locally we get

$$X = X^i\frac{\delta}{\delta x^i}+\tilde{X}^i\frac{\partial}{\partial y^i}$$

Clearly, for $\tilde{X}^i(x,y)=0$, we obtain the subbundle of $(M^0)^h \subset M^0$ and for $X^i(x,y)=0$, we obtain the subbundle of $(M^0)^v \subset M^0$. Suppose $\{dx^1,\dots,dx^{2n+1}\}$ is a dual basis to $\left\{\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^{2n+1}}\right\}$, i.e., $dx^i\left(\frac{\delta}{\delta x^j}\right)=\delta_j^i$. Then each $w \in \Gamma(T^*M^0)^H$ is locally written as $w^H=\widetilde{w}_i(x,y)dx^i$, where $\widetilde{w}_i = w_i - N_i^j w_j$. Thus we can write

$$\delta y^i = dy^i + N_j^i(x,y)dx^j$$

Consider a w, 1-form, then

$$w=\widetilde{w}_i(x,y)dx^i+w_i(x,y)\ \delta y^i.$$

Also, $w^H(X^V)=0, w^V(X^H)=0$, where $w=w^H+w^V$ [2].

Definition 2.1. A Finsler connection is a linear connection $\nabla = F\Gamma$ with the property that the horizontal linear space $\left(T_{(x,y)}M^0\right)^H$, $(x,y)\in M^0$ of the distribution N is parallel with respect to ∇.

Similarly, a Finsler connection is called linear connection $\nabla = F\Gamma$ with the vertical linear space $\left(T_{(x,y)}M^0\right)^V$, $(x,y)\in M^0$ of the distribution N parallel relative to ∇.

Necessary and sufficient condition for linear connection ∇ on M^0 to be Finsler connection is

$$(\nabla_X^V Y^H)=0, (\nabla_X^H Y^V)=0$$

$$\nabla_X Y = \nabla_X^H Y^H + \nabla_X^V Y^V$$

for each $X, Y \in T_{(x,y)}M^0$.

$$\nabla_X w = \nabla_X^H w^H + \nabla_X^V w^V$$

for all $w \in T^*_{(x,y)}M^0$[15].

Let ∇ be a Finsler connection and the curvature of this connection is given with the below equation.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R^H(X^H, Y^H)Z^H + R^V(X^V, Y^V)Z^V$$

where $X, Y, Z \in T_{(x,y)}M^0$ [14].

Theorem 2.1. The curvature of a Finsler connection ∇ on $T_{(x,y)}M^0$ is totally stated with the following Finsler tensor fields equations:

$$R^H(X^H, Y^H)Z^H = \nabla_{X^H} \nabla_{Y^H} Z^H - \nabla_{Y^H} \nabla_{X^H} Z^H - \nabla_{[X^H, Y^H]} Z^H$$

$$R^V(X^V, Y^V)Z^V = \nabla_{X^V} \nabla_{Y^V} Z^V - \nabla_{Y^V} \nabla_{X^V} Z^V - \nabla_{[X^V, Y^V]} Z^V$$

[14].

2.2. Almost Contact Pseudo-Metric Finsler Structures

Consider tensor field ϕ, 1-form η and vector field ξ given as below:

$$\phi = \phi^H + \phi^V = \phi_i^j(x,y) \frac{\delta}{\delta x_i} \otimes dx^j + \widetilde{\phi_i^j}(x,y) \frac{\partial}{\partial y^i} \otimes \delta y^j \tag{2.1}$$

$$\eta = \eta^H + \eta^V = \eta_i(x,y) dx^i + \widetilde{\eta_i}(x,y) \delta y^i \tag{2.2}$$

$$\xi = \xi^H + \xi^V = \xi^i(x,y) \frac{\delta}{\delta x_i} + \widetilde{\xi^i}(x,y) \frac{\partial}{\partial y^i} \tag{2.3}$$

Then, we can write the following statements.

$$(\phi^H)^2 X^H = -X^H + \eta^H(X^H)\, \xi^H, (\phi^V)^2 X^V = -X^V + \eta^V(X^V)\, \xi^V \tag{2.4}$$

$$\eta^H(\xi^H) = \eta^V(\xi^V) = 1 \tag{2.5}$$

$$\phi^H(\xi^H) = \phi^V(\xi^V) = 0 \tag{2.6}$$

$$\eta^H \circ \phi^H = \eta^V \circ \phi^V = 0 \tag{2.7}$$

$$rank(\phi^H) = rank(\phi^V) = 2n \tag{2.8}$$

Thus, (ϕ^H, ξ^H, η^H) and (ϕ^V, ξ^V, η^V) are called the almost contact Finsler structures on vector bundles $(M^0)^h$ and $(M^0)^v$,respectively, where $M^0 = (M^0)^h \oplus (M^0)^v$. Also, we call that $\left((M^0)^h, \phi^H, \xi^H, \eta^H\right)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V)$ are almost contact Finsler manifolds [3].

Let $F^{2n+1} = (M, M^0, F^*)$ be an indefinite Finsler manifold. Then, we define

$$g^{F^*}: \Gamma(TM^0)^V \times \Gamma(TM^0)^V \to \mathfrak{F}(M^0),$$

$$g_{ij}^{F^*}(x,y) = g^{F^*}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)(x,y).$$

Obviously, g^{F^*} is a symmetric Finsler tensor field. g^{F^*} is called the pseudo-Finsler metric of F^{2n+1}. Thus, g^{F^*} is thought to be a pseudo-Riemannian metric on $(TM^0)^V$.

Similarly, we define the metric for horizontal distrubituon as following:

$$g^{F^*}: \Gamma(TM^0)^H \times \Gamma(TM^0)^H \to \mathfrak{F}(M^0),$$

$$g_{ij}^{F^*}(x,y) = g^{F^*}(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})(x,y)$$

[1, 2]. A Finsler vector can be described with below statements.

$$g^{F^*}(\mathcal{X},\mathcal{X}) = 0 \text{ and } \mathcal{X} \neq 0 \Rightarrow \text{light-like}$$

$$g^{F^*}(\mathcal{X},\mathcal{X}) > 0 \text{ or } \mathcal{X} = 0 \Rightarrow \text{space-like}$$

$$g^{F^*}(\mathcal{X},\mathcal{X}) < 0 \Rightarrow \text{time-like},$$

where $\mathcal{X} \in T_{(x,y)}M^0$, $(x,y) \in M^0$. The Finsler norm of $\mathcal{X}$ is a nonnegative number and $\|\mathcal{X}\|$ is described with following equation:

$$\|\mathcal{X}\| = \left|g^{F^*}(\mathcal{X},\mathcal{X})\right|^{1/2}.$$

If $g^{F^*}(\mathcal{X},\mathcal{X}) = 1$, $\mathcal{X}$ is called unit space-like Finsler vector or $g^{F^*}(\mathcal{X},\mathcal{X}) = -1$, $\mathcal{X}$ is called unit time-like Finsler vector. $g^{F^*}(\mathcal{X},\mathcal{X}) = \varepsilon$ and ε is said the signature of $\mathcal{X}$ when $\mathcal{X}$ is a unit Finsler vector.

Also,

$$G: \Gamma(TM^0) \times \Gamma(TM^0) \to \mathfrak{F}(M^0)$$

$$G(X,Y) = G^H(X,Y) + G^V(X,Y).$$

is defined. Obviously, G is a symmetric tensor field of type (0,2), non-degenerate and pseudo-Riemannian metric on M^0 with index $2q$. Then, G is called Sasaki Finsler metric on M^0. Then, G can be defined as below.

$$G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*}\, \delta y^i \otimes \delta y^i$$

[1, 2].

Definition 2.2. Suppose that (ϕ^H,ξ^H,η^H) and (ϕ^V,ξ^V,η^V) are almost contact structures on horizontal and vertical Finsler vector bundles $(M^0)^h$ and $(M^0)^v$. If the G^H and G^V satisfy the following conditions,

$$G^H(\phi X^H,\phi Y^H) = G^H(X^H,Y^H) - \varepsilon\eta^H(X^H)\eta^H(Y^H)$$

$$G^V(\phi X^V,\phi Y^V) = G^V(X^V,Y^V) - \varepsilon\eta^V(X^V)\eta^V(Y^V)$$

$$\eta^H(X^H) = \varepsilon\, G^H(X^H,\xi^H), \eta^V(X^V) = \varepsilon\, G^V(X^V,\xi^V)$$

where $\varepsilon = \pm 1$, then $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ are called almost contact pseudo-metric Finsler structures on $(M^0)^h$ and $(M^0)^v$.

Now, we define

$$\Omega\,(X\,,Y) = G(X\,,\phi Y), \qquad \Omega^H(X^H,Y^H) = G^H(X^H,\phi Y^H), \quad \Omega^V(X^V,Y^V) = G^V(X^V,\phi Y^V)$$

and call it the fundamental 2-form [4].

The fundamental 2-form, defined above, satisfies the following equations:

$$\Omega^H(\phi X^H,\, \phi Y^H) = \Omega^H(X^H,Y^H),\;\; \Omega^V(\phi X^V,\, \phi Y^V) = \Omega^V(X^V,Y^V)$$

$$\Omega^H(Y^H,X^H) = -\,\Omega^H(X^H,Y^H)\,,\;\; \Omega^V(Y^V,\, X^V) = -\Omega^V(X^V,Y^V)$$

Proposition 2.1. Let ∇ be a Finsler connection on M^0 and Ω be the fundamental 2-form which satisfies

$$d\eta^V(X^V,Y^V) = \Omega^V(X^V,Y^V), \qquad d\eta^H(X^H,Y^H) = \Omega^H(X^H,Y^H),$$

$$\Omega^H(X^H,Y^H) = (\nabla_X^H\eta^H)(Y^H) - (\nabla_Y^H\eta^H)(X^H) + \eta^H\big(T(X^H\,,Y^H)\big),$$

$$\Omega^V(X^V, Y^V) = (\nabla_X^V \eta^V)(Y^V) - (\nabla_Y^V \eta^V)(X^V) + \eta^V\big(T(X^V, Y^V)\big).$$

Then the almost contact pseudo-metric Finsler structure is called almost ε −Sasakian Finsler structure on M^0.

$(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ are called almost ε −Sasakian Finsler structures on $(M^0)^h$ and $(M^0)^v$, respectively [4].

Theorem 2.2. Let Ω be the fundamental 2-form and almost ε −Sasakian Finsler connection ∇ on M^0 is torsion free then

$$\Omega^H(X^H, Y^H) = (\nabla_X^H \eta^H)(Y^H) - (\nabla_Y^H \eta^H)(X^H)$$
$$\Omega^V(X^V, Y^V) = (\nabla_X^V \eta^V)(Y^V) - (\nabla_Y^V \eta^V)(X^V)$$

[4].

Definition 2.3. An almost ε −Sasakian Finsler structure on M^0 is said to be an ε −Sasakian Finsler structure if the 1-form η is a killing vector field, i.e.,

$$(\nabla_X^H \eta^H)(Y^H) + (\nabla_Y^H \eta^H)(X^H) = 0,\ (\nabla_X^V \eta^V)(Y^V) + (\nabla_Y^V \eta^V)(X^V) = 0$$
$$\Omega^H(X^H, Y^H) = 2(\nabla_X^H \eta^H)(Y^H),\ \Omega^V(X^V, Y^V) = 2(\nabla_X^V \eta^V)(Y^V)$$

[4].

3. Trans- Sasakian Indefinite Finsler Manifolds

We introduce trans-Sasakian indefinite Finsler manifolds in our main results. Also, we give the special case of these structures α −Sasakian and β −Kenmotsu indefinite Finsler manifolds.
The almost contact pseudo-metric Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ are said to be trans-Sasakian indefinite Finsler manifolds if and only if the following conditions are hold.

$$(\nabla_X^H \phi^H)Y^H = \frac{\alpha}{2}\{G^H(X^H, Y^H)\xi^H - \varepsilon\eta^H(Y^H)X^H\} + \frac{\beta}{2}\{\varepsilon G^H(\phi X^H, Y^H)\xi^H - \eta^H(Y^H)\phi X^H\} \tag{3.1}$$

$$(\nabla_X^V \phi^V)Y^V = \frac{\alpha}{2}\{G^V(X^V, Y^V)\xi^V - \varepsilon\eta^V(Y^V)X^V\} + \frac{\beta}{2}\{\varepsilon G^V(\phi X^V, Y^V)\xi^V - \eta^V(Y^V)\phi X^V\} \tag{3.2}$$

where α and β are smooth functions on $(M^0)^h$ and $(M^0)^v$ then we say such a structure the trans-Sasakian pseudo-metric Finsler structure of type (α, β). If α, β =constant, then the getting α, β =constant from (3.1) and (3.2) we get

$$(\nabla_X^H \xi^H) = -\varepsilon\frac{\alpha}{2}\ \phi X^H + \frac{\beta}{2}(X^H - \eta^H(X^H)\xi^H) \tag{3.3}$$

$$(\nabla_X^V \xi^V) = -\varepsilon\frac{\alpha}{2}\ \phi X^V + \frac{\beta}{2}(X^V - \eta^V(X^V)\xi^V) \tag{3.4}$$

$$(\nabla_X^H \eta^H)(Y^H) = \frac{\alpha}{2}G^H(X^H, \phi Y^H) + \varepsilon\frac{\beta}{2}G^H(\phi X^H, \phi Y^H) \tag{3.5}$$

$$(\nabla_X^V \eta^V)(Y^V) = \frac{\alpha}{2}G^V(X^V, \phi Y^V) + \varepsilon\frac{\beta}{2}G^V(\phi X^V, \phi Y^V) \tag{3.6}$$

Theorem 3.1. In the trans-Sasakian indefinite Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$ the following relations hold.

$$R^H(X^H,Y^H)\xi^H = \frac{(\alpha^2-\beta^2)}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \eta^H(X^H)\phi Y^H\} \tag{3.7}$$

$$R^V(X^V,Y^V)\xi^V = \frac{(\alpha^2-\beta^2)}{4}\{\eta^V(Y^V)X^V - \eta^V(X^V)Y^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \eta^V(X^V)\phi Y^V\} \tag{3.8}$$

$$R^H(\xi^H,X^H)Y^H = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon G^H(X^H,Y^H)\xi^H - \eta^H(Y^H)X^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \varepsilon G^H(\phi X^H,Y^H)\xi^H\} \tag{3.9}$$

$$R^V(\xi^V,X^V)Y^V = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon G^V(X^V,Y^V)\xi^V - \eta^V(Y^V)X^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \varepsilon G^V(\phi X^V,Y^V)\xi^V\} \tag{3.10}$$

$$\eta^H(R^H(X^H,Y^H)Z^H) = \varepsilon\frac{(\alpha^2-\beta^2)}{4}\{G^H(Y^H,Z^H)\eta^H(X^H) - G^H(X^H,Z^H)\eta^H(Y^H)\} + \frac{\alpha\beta}{2}\{\eta^H(X^H)G(\phi Y^H,Z^H) - \eta^H(Y^H)G^H(\phi X^H,Z^H)\} \tag{3.11}$$

$$\eta^V(R^V(X^V,Y^V)Z^V) = \varepsilon\frac{(\alpha^2-\beta^2)}{4}\{G^V(Y^V,Z^V)\eta^V(X^V) - G^V(X^V,Z^V)\eta^V(Y^V)\} + \frac{\alpha\beta}{2}\{\eta^V(X^V)G(\phi Y^V,Z^V) - \eta^V(Y^V)G^V(\phi X^V,Z^V)\} \tag{3.12}$$

$$\eta^H(R^H(X^H,Y^H)\xi^H) = 0,\ \eta^V(R^V(X^V,Y^V)\xi^V) = 0 \tag{3.13}$$

$$S^H(X^H,\xi^H) = n\frac{(\alpha^2-\beta^2)}{2}\eta^H(X^H),\quad S^V(X^V,\xi^V) = n\frac{(\alpha^2-\beta^2)}{2}\eta^V(X^V) \tag{3.14}$$

$$S^H(\xi^H,\xi^H) = n\frac{(\alpha^2-\beta^2)}{2},\quad S^V(\xi^V,\xi^V) = n\frac{(\alpha^2-\beta^2)}{2} \tag{3.15}$$

$$QX^H = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}X^H,\ QX^V = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}X^V,\ Q\xi^H = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}\xi^H,\ Q\xi^V = \varepsilon\, n\frac{(\alpha^2-\beta^2)}{2}\xi^V \tag{3.16}$$

Proof:

$$R^H(X^H,Y^H)\xi^H = \nabla^H_{X^H}\nabla^H_{Y^H}\xi^H - \nabla^H_{Y^H}\nabla^H_{X^H}\xi^H - \nabla_{\nabla^H_{X^H}Y^H - \nabla^H_{Y^H}X^H}\xi^H$$

$$= \nabla^H_{X^H}\left\{-\varepsilon\frac{\alpha}{2}\phi Y^H + \frac{\beta}{2}(Y^H - \eta^H(Y^H)\xi^H)\right\} - \nabla^H_{Y^H}\left\{-\varepsilon\frac{\alpha}{2}\phi X^H + \frac{\beta}{2}(X^H - \eta^H(X^H)\xi^H)\right\}$$

$$-\nabla^H_{\nabla^H_{X^H}Y^H}\xi^H + \nabla^H_{\nabla^H_{Y^H}X^H}\xi^H$$

$$= \varepsilon\frac{\alpha}{2}\{(\nabla^H_Y\phi^H)X^H - (\nabla^H_X\phi^H)Y^H\} + \frac{\beta}{2}\{(\nabla^H_Y\eta^H)X^H\xi^H - (\nabla^H_X\eta^H)Y^H\xi^H + \eta^H(X^H)\nabla^H_Y\xi^H - \eta^H(Y^H)\nabla^H_X\xi^H\}$$

then we get following equation

$$= \varepsilon\frac{\alpha}{2}\Big\{-\varepsilon\frac{\alpha}{2}\eta^H(X^H)Y^H + \varepsilon\beta G^H(\phi Y^H, X^H) - \frac{\beta}{2}\eta^H(X^H)\phi Y^H + \varepsilon\frac{\alpha}{2}\eta^H(Y^H)X^H + \frac{\beta}{2}\eta^H(Y^H)\phi X^H\Big\} + \frac{\beta}{2}\Big\{\alpha G^H(Y^H, \phi X^H)\xi^H - \varepsilon\frac{\alpha}{2}\eta^H(X^H)\phi Y^H + \frac{\beta}{2}\eta^H(X^H)Y^H - \frac{\beta}{2}\eta^H(X^H)\eta^H(Y^H)\xi^H + \varepsilon\frac{\alpha}{2}\eta^H(Y^H)\phi X^H - \frac{\beta}{2}\eta^H(Y^H)X^H + \frac{\beta}{2}\eta^H(X^H)\eta^H(Y^H)\xi^H\Big\},$$

If we rearrange last equation, then we have the following one and the proof is completed

$$R(X^H, Y^H)\xi^H = \frac{(\alpha^2-\beta^2)}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \eta^H(X^H)\phi Y^H\}$$

By using similar processing steps, we can obtain the proof for vertical distribution.

Using the equations $G^H(R^H(X^H, Y^H)\xi^H, W^H) = G^H(R^H(\xi^H, W^H)\, X^H, Y^H)$ and $G^V(R^V(X^V, Y^V)\xi^V, W^V) = G^V(R^V(\xi^V, W^V)\, X^V, Y^V)$, we get

$$R^H(\xi^H, W^H)\, X^H = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon\, G^H(W^H, X^H)\xi^H - \eta^H(X^H)W^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(X^H)\phi W^H - \varepsilon G^V(\phi W^H, X^H)\xi^H\}.$$

and

$$R^V(\xi^V, W^V)\, X^V = \frac{(\alpha^2-\beta^2)}{4}\{\varepsilon\, G^V(W^V, X^V)\xi^V - \eta^V(X^V)W^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(X^V)\phi W^V - \varepsilon G^V(\phi W^V, X^V)\xi^V\}.$$

We have from equations (3.7) and (3.8), we get

$$\eta^H(R^H(X^H, Y^H)Z^H) = \varepsilon G(R^H(X^H, Y^H)Z^H, \xi^H) = -\varepsilon G(R^H(X^H, Y^H)\xi^H, Z^H)$$

$$= -\varepsilon G\left(\frac{(\alpha^2-\beta^2)}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(Y^H)\phi X^H - \eta^H(X^H)\phi Y^H\}, Z^H\right)$$

$$= \varepsilon\,\frac{(\alpha^2-\beta^2)}{4}\{G^H(Y^H, Z^H)\eta^H(X^H) - G^H(X^H, Z^H)\eta^H(Y^H)\}$$

$$+\frac{\alpha\beta}{2}\{\eta^H(X^H)G(\phi Y^H, Z^H) - \eta^H(Y^H)G^H(\phi X^H, Z^H)\}$$

and

$$\eta^V(R^V(X^V, Y^V)Z^V) = \varepsilon G(R^V(X^V, Y^V)Z^V, \xi^V) = -\varepsilon G(R^V(X^V, Y^V)\xi^V, Z^V)$$

$$= -\varepsilon G\left(\frac{(\alpha^2-\beta^2)}{4}\{\eta^V(Y^V)X^V - \eta^V(X^V)Y^V\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^V(Y^V)\phi X^V - \eta^V(X^V)\phi Y^V\}, Z^V\right)$$

$$= \varepsilon\,\frac{(\alpha^2-\beta^2)}{4}\{G^V(Y^V, Z^V)\eta^V(X^V) - G^V(X^V, Z^V)\eta^V(Y^V)\}$$

$$+\frac{\alpha\beta}{2}\{\eta^V(X^V)G(\phi Y^V, Z^V) - \eta^V(Y^V)G^V(\phi X^V, Z^V)\}.$$

Putting $Z^H = \xi^H$ and $Z^V = \xi^V$, we get $\eta^H(R^H(X^H, Y^H)\xi^H) = 0, \eta^V(R^V(X^V, Y^V)\xi^V)=0$.

For the trans-Sasakian indefinite Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$, the Ricci tensor S and scalar curvature r is defined by

$$S^H(X^H, Y^H) = \sum_{i=1}^{2n}\varepsilon_i\, G^H\left(R^H\left(E_i^H, X^H\right)Y^H, E_i^H\right) + \varepsilon\, G^H(R^H(\xi^H, X^H)Y^H, \xi^H),$$

$$r^H = \sum_{i=1}^{2n} S^H(E_i^H, E_i^H),\ r^V = \sum_{i=1}^{2n} S^V(E_i^V, E_i^V),$$

$$S^V(X^V, Y^V) = \sum_{i=1}^{2n}\varepsilon_i\, G^V\left(R^V\left(E_i^V, X^V\right)Y^V, E_i^V\right) + \varepsilon\, G^V(R^V(\xi^V, X^V)Y^V, \xi^V),$$

where $\{E_1^H, E_2^H, \dots, E_{2n}^H, \xi^H\}$ is orthonormal basis field in $(M^0)^h$and $G^H(E_i^H, E_i^H) = \varepsilon_i$

(similarly, $\{E_1^V, E_2^V, \dots, E_{2n}^V, \xi^V\}$ is orthonormal basis field in $(M^0)^v$and $G^v(E_i^v, E_i^v) = \varepsilon_i$).

Replacing Y^H by ξ^H, we get

$$S^H(X^H, \xi^H) = \sum_{i=1}^{2n} \varepsilon_i G^H(R^H(E_i^H, X^H)\xi^H, E_i^H) + \varepsilon G^H(R^H(\xi^H, X^H)\xi^H, \xi^H)$$

$$= \sum_{i=1}^{2n} \varepsilon_i G^H\left(\frac{(\alpha^2-\beta^2)}{4}\{\eta^H(X^H)E_i^H - \eta^H(E_i^H)X^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(X^H)\phi E_i^H - \eta^H(E_i^H)\phi X^H\}, E_i^H\right) +$$
$$\varepsilon G^H(\frac{(\alpha^2-\beta^2)}{4}\{\eta^H(X^H)\xi^H - \eta^H(\xi^H)X^H\} + \varepsilon\frac{\alpha\beta}{2}\{\eta^H(X^H)\phi\xi^H - \eta^H(\xi^H)\phi X^H\}, \xi^H),$$

where, since $G^H(R^H(\xi^H, X^H)\xi^H, \xi^H) = 0$ we get

$$S^H(X^H, \xi^H) = n\frac{(\alpha^2-\beta^2)}{2}\eta^H(X^H), \; S^H(\xi^H, \xi^H) = n\frac{(\alpha^2-\beta^2)}{2}.$$

The Ricci operatör Q given by

$$S^H(X^H, Y^H) = G^H(QX^H, Y^H) \text{ and } S^V(X^V, Y^V) = G^V(QX^V, Y^V).$$

By using $S^H(X^H, \xi^H) = G^H(QX^H, \xi^H)$ and $S^V(X^V, \xi^V) = G^V(QX^V, \xi^V)$, we obtain

$$QX^H = \varepsilon\frac{n(\alpha^2-\beta^2)}{2}(X^H), \; Q\xi^H = \varepsilon\frac{n(\alpha^2-\beta^2)}{2}\xi^H \text{ and } QX^V = \varepsilon\frac{n(\alpha^2-\beta^2)}{2}(X^V), \; Q\xi^H = \varepsilon\frac{n(\alpha^2-\beta^2)}{2}\xi^V.$$

Example 3.1. Consider the structure of $F^3 = (\mathbb{R}^3, (\mathbb{R}^3)^0, F^*)$ indefinite Finsler manifold. $(\mathbb{R}^3)^0 = \mathbb{R}^6 \setminus \{0\}$ is a real 6-dimensional C^∞ manifold and $T\mathbb{R}^3$ is the tangent bundle of $\mathbb{R}^3$. A coordinate system in $\mathbb{R}^3$can be stated with $\{(U, \varphi): x^1, x^2, x^3\}$, where U is an open subset of $\mathbb{R}^3$; for any $x \in U$, $\varphi: U \to \mathbb{R}^3$ is a diffeomorphism of U onto $\varphi(U)$, and $\varphi(x) = (x^1, x^2, x^3)$. On $\mathbb{R}^3$, denote by π the canonical projection of $T\mathbb{R}^3$ and by T_xM the fibre, at $x \in \mathbb{R}^3$, i.e., $T_x\mathbb{R}^3 = \pi^{-1}(x)$. Through the coordinate system $\{(U, \varphi): x^i\}$ in $\mathbb{R}^3$, we can describe a new coordinate system $\{(U^*, \Phi); x^1, x^2, x^3; y^1, y^2, y^3\}$ or shortly $\{(U^*, \Phi): x^i, y^i\}$ in $T\mathbb{R}^3$, where $U^* = \pi^{-1}(U)$ and $\Phi: U^* \to \mathbb{R}^6$ is a diffeomorphism of U^* on $\varphi(U) \times \mathbb{R}^3$, and $\Phi(y_x) = (x^1, x^2, x^3; y^1, y^2, y^3)$ for any $x \in U$ and $y_x \in T_x\mathbb{R}^3$. Let $(\mathbb{R}^3)^0$ be a non-empty open submanifold of $T\mathbb{R}^3$ such that $\pi((\mathbb{R}^3)^0) = \mathbb{R}^3$ and $\theta(\mathbb{R}^3) \cap (\mathbb{R}^3)^0 = \emptyset$, where θ is the zero section of $T\mathbb{R}^3$. Assume that $(\mathbb{R}^3)^0_x = T_x\mathbb{R}^3 \cap (\mathbb{R}^3)^0$ is a positive conic set, for any $k > 0$ and $y \in (\mathbb{R}^3)^0_x$. we have $ky \in (\mathbb{R}^3)^0_x$. Obviously, the largest $(\mathbb{R}^3)^0$ holding the above circumstances is $T\mathbb{R}^3 \setminus \theta(M)$, ordinarily given with the description of a Finsler manifold. The set of the local vector fields $\left\{\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \frac{\delta}{\delta x^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^H$ and $\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^V$. We get

$X^V = X_1^V(x,y)\frac{\partial}{\partial y^1} + X_2^V(x,y)\frac{\partial}{\partial y^2} + X_3^V(x,y)\frac{\partial}{\partial y^3}$, $X^H = X_1^H(x,y)\frac{\delta}{\delta x^1} + X_2^H(x,y)\frac{\delta}{\delta x^2} + X_3^H(x,y)\frac{\delta}{\delta x^3}$, for any $X^V \in (T(\mathbb{R}^3)^0)^V$and $X^H \in (T(\mathbb{R}^3)^0)^H$. Thus, for any $X \in T(\mathbb{R}^3)^0$, $X = X_i^H(x,y)\frac{\delta}{\delta x^i} + X_i^V(x,y)\frac{\partial}{\partial y^i}$ (i =1, 2, 3). Consider a η, 1-form, $\eta = \eta^H + \eta^V = \eta_i{}^H(x,y)dx^i + \eta_i{}^V(x,y)\delta y^i$ (i =1, 2, 3), $\eta^H \in (T^*(\mathbb{R}^3)^0)^H$ and $\eta^V \in (T^*(\mathbb{R}^3)^0)^V$.

G is a symmetric tensor field of type (0,2), non-degenerate and pseudo-Riemannian metric on $(\mathbb{R}^3)^0$. Then, G is called Sasaki Finsler metric on $(\mathbb{R}^3)^0$. Then, G can be defined as below:

$$G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*}\, \delta y^i \otimes \delta y^i \text{ (i=1, 2, 3).}$$

The vector fields

$$E_1^H = x_3 \frac{\delta}{\delta x^1} \ , \ E_2^H = x_3 \frac{\delta}{\delta x^2} \ , \ E_3^H = x_3 \frac{\delta}{\delta x^3} = \xi^H$$

are linear independent at every point of $((\mathbb{R}^3)^0)^h$. Let G be the Sasaki Finsler pseudo-metric given by

$$G^H(E_1^H, \xi^H) = G^H(E_1^H, E_2^H) = G^H(E_2^H, \xi^H) = 0$$

$$G^H(E_1^H, E_1^H) = G^H(E_2^H, E_2^H) = 1, G^H(\xi^H, \xi^H) = \varepsilon = -1.$$

Let η^H be the 1-form derscribed by

$$\eta^H(Z^H) = -G^H(Z^H, \ \xi^H) = -G^H(z_1 E_1^H + \ z_2 E_2^H + z_3 \, \xi^H, \ \xi^H) = z_3 \ , \ \forall \, Z^H \in (T(\mathbb{R}^3)^0)^H.$$

Consider ϕ^H the (1, 1) tensör field stated by

$$\phi^H(E_1^H) = - \ E_2^H \ , \phi^H(E_2^H) = E_1^H \ , \ \phi^H(\xi^H) = 0.$$

Then using the linearity of ϕ^H, we have

$$Z^H = z_1 E_1^H + \ z_2 E_2^H + z_3 \, \xi^H \ , W^H = w_1 E_1^H + \ w_2 E_2^H + w_3 \, \xi^H$$

$$\phi^H(Z^H) = \phi^H(z_1 E_1^H + \ z_2 E_2^H + z_3 \, \xi^H \ ) = z_1 \phi^H(E_1^H) + z_2 \phi^H(E_2^H) + z_3 \, \phi^H(\xi^H)$$

$$\phi^H(Z^H) = -z_1 E_2^H + \ z_2 E_1^H$$

$$\phi^H(W^H) = \phi^H(w E_1^H + \ w E_2^H + w_3 \, \xi^H \ ) = w_1 \phi^H(E_1^H) + w_2 \phi^H(E_2^H) + w_3 \, \phi^H(\xi^H)$$

$$\phi^H(W^H) = -w_1 E_2^H + \ w_2 E_1^H$$

$$(\phi^H)^2(Z^H) = -z_2 E_2^H - \ z_1 E_1^H = -Z + \eta^H(Z^H)\xi^H$$

Thus we get

$$G^H\big(\phi^H(Z^H), \phi^H(W^H)\big) = \ G^H(Z^H, \, W^H) + \eta^H(Z^H) \, \eta^H(W^H)$$

$\forall \, Z^H \in (T(\mathbb{R}^3)^0)^H$ and $\forall \, W^H \in (T(\mathbb{R}^3)^0)^H$. Thus the structure $(((\mathbb{R}^3)^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ define the almost contact pseudo-metric Finsler structure on $((\mathbb{R}^3)^0)^h$.

Let ∇ be the Levi-Civita connection with respect to pseudo-metric G^H. Then we have

$[E_1^H, E_2^H] = 0 \ , \ [E_1^H, \xi^H] = -E_1^H \ , \ [E_2^H, \xi^H] = - E_2^H.$

The connection ∇ of the pseudo-metric G^H is given by

$$2G^H\big(\nabla_{X^H} Y^H, Z^H\big) = X^H G^H(Y^H, Z^H) + Y^H G^H(X^H, Z^H) - Z^H G^H(X^H, Y^H)$$

$$-G^H(X^H, [Y^H, Z^H]) - G^H(Y^H, [X^H, Z^H]) + G^H(Z^H, [X^H, Y^H]) \tag{3.17}$$

Which is known as Koszul's formula. Using this formula, we have

$$2G^H\left(\nabla_{E_1^H} \xi^H, \, E_1^H\right) = -G^H(E_1^H, [\xi^H, \, E_1^H \,]) - G^H(\xi^H, [\, E_1^H, \, E_1^H]) + G^H(E_1^H \,, [\, E_1^H, \xi^H])$$

$$= 2G^H(-E_1^H, \, E_1^H).$$

Thus,

$$\nabla_{E_1^H} \xi^H = -E_1^H \ , \ \nabla_{\xi^H} E_1^H = 0.$$

Again by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_2^H} \xi^H, \, E_2^H\right) = -G^H(E_2^H, [\xi^H, \, E_2^H \,]) - G^H(\xi^H, [\, E_2^H, \, E_2^H]) + G^H(E_2^H \,, [\, E_2^H, \xi^H])$$

$$= 2G^H(- \, E_2^H, \, E_2^H).$$

Thus,

$$\nabla_{E_2^H}\xi^H = -E_2^H \quad , \nabla_{\xi^H}E_2^H = 0.$$

Also by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_1^H} E_2^H, \xi^H\right) = G^H(E_1^H, [\xi^H, E_2^H]) + G^H(\xi^H, [E_1^H, E_2^H]) - G^H(E_2^H, [E_1^H, \xi^H]) = 0.$$

Thus,

$$\nabla_{E_1^H} E_2^H = 0\,, \quad \nabla_{E_2^H} E_1^H = 0$$

Similarly we get

$$2G^H\left(\nabla_{E_1^H} E_1^H, \xi^H\right) = -G^H(E_1^H, [E_1^H, \xi^H]) + G^H(\xi^H, [E_1^H, E_1^H]) - G^H(E_1^H, [E_1^H, \xi^H])$$

$$= 2G^H(E_1^H, E_1^H) = -2G^H(\xi^H, \xi^H).$$

Thus,

$$\nabla_{E_1^H} E_1^H = -\xi^H.$$

(3.17) further yields

$$\nabla_{E_2^H} E_2^H = -\xi^H, \ \nabla_{\xi^H} E_1^H = 0, \ \nabla_{\xi^H} E_2^H = 0, \nabla_{E_2^H} E_1^H = 0.$$

If we use the equations we found

$$(\nabla_X^H\xi^H) = x_1\nabla_{E_1^H}\xi^H + x_2\nabla_{E_2^H}\xi^H = (-x_1)\, E_1^H - (x_2)E_2^H,$$

$\forall\, X^H \in (T(\mathbb{R}^3)^0)^H$.

The above equations tell us the almost contact pseudo-metric Finsler manifold $\left(((\mathbb{R}^3)^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ satisfy (3.3) for $\alpha = 0\,, \beta = -2\,,\ \varepsilon = -1$.

With the help of the above results it can be verified that

$$R^H(E_1^H, E_2^H)\, E_2^H = E_1^H, \qquad R^H(\xi^H, E_2^H)\, E_2^H = \xi^H, \qquad R^H(E_1^H, \xi^H)\, \xi^H = -E_1^H$$

$$R^H(E_2^H, \xi^H)\, \xi^H = -E_2^H, \qquad R^H(E_2^H, E_1^H)\, E_1^H = E_2^H, \qquad R^H(\xi^H, E_1^H)\, E_1^H = \xi^H$$

$$S^H(\xi^H, \xi^H) = G^H(R^H(E_1^H, \xi^H)\, \xi^H, E_1^H) + G^H(R^H(E_2^H, \xi^H)\, \xi^H, E_2^H) = G^H(-E_1^H, E_1^H) + G^H(-E_2^H, E_2^H)$$

$$S^H(\xi^H, \xi^H) = n\frac{(\alpha^2-\beta^2)}{2} = -2$$

Example 3.2. Consider the structure of $F^3 = (\mathbb{R}^3, (\mathbb{R}^3)^0, F^*)$ indefinite Finsler manifold. $(\mathbb{R}^3)^0 = \mathbb{R}^6 \setminus \{0\}$ is a real 6-dimensional C^∞ manifold and $T\mathbb{R}^3$ is the tangent bundle of $\mathbb{R}^3$. A coordinate system in $\mathbb{R}^3$ can be stated with $\{(U,\varphi): x^1, x^2, x^3\}$, where U is an open subset of $\mathbb{R}^3$; for any $x \in U$, $\varphi: U \to \mathbb{R}^3$ is a diffeomorphism of U onto $\varphi(U)$, and $\varphi(x) = (x^1, x^2, x^3)$. On $\mathbb{R}^3$, denote by π the canonical projection of $T\mathbb{R}^3$ and by T_xM the fibre, at $x \in \mathbb{R}^3$, i.e., $T_x\mathbb{R}^3 = \pi^{-1}(x)$. Through the coordinate system $\{(U,\varphi): x^i\}$ in $\mathbb{R}^3$, we can describe a new coordinate system $\{(U^*,\Phi); x^1, x^2, x^3; y^1, y^2, y^3\}$ or shortly $\{(U^*,\Phi): x^i, y^i\}$ in $T\mathbb{R}^3$, where $U^* = \pi^{-1}(U)$ and $\Phi: U^* \to \mathbb{R}^6$ is a diffeomorphism of U^* on $\varphi(U) \times \mathbb{R}^3$, and $\Phi(y_x) = (x^1, x^2, x^3; y^1, y^2, y^3)$ for any $x \in U$ and $y_x \in T_x\mathbb{R}^3$. Let $(\mathbb{R}^3)^0$ be a non-empty open submanifold of $T\mathbb{R}^3$ such that $\pi((\mathbb{R}^3)^0) = \mathbb{R}^3$ and $\theta(\mathbb{R}^3) \cap (\mathbb{R}^3)^0 = \emptyset$, where θ is the zero section of $T\mathbb{R}^3$. Assume that $(\mathbb{R}^3)^0_x = T_x\mathbb{R}^3 \cap (\mathbb{R}^3)^0$ is a positive conic set, for any $k > 0$ and $y \in (\mathbb{R}^3)^0_x$. we have $ky \in (\mathbb{R}^3)^0_x$. Obviously, the largest $(\mathbb{R}^3)^0$ holding the above circumstances is $T\mathbb{R}^3 \setminus \theta(M)$, ordinarily given with the description of a Finsler manifold. The set of the local vector fields $\left\{\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \frac{\delta}{\delta x^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^H$ and $\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}\right\}$ is a basis in $(T(\mathbb{R}^3)^0)^V$. We get

$X^V = X_1^V(x,y)\frac{\partial}{\partial y^1} + X_2^V(x,y)\frac{\partial}{\partial y^2} + X_3^V(x,y)\frac{\partial}{\partial y^3}$, $X^H = X_1^H(x,y)\frac{\delta}{\delta x^1} + X_2^H(x,y)\frac{\delta}{\delta x^2} + X_3^H(x,y)\frac{\delta}{\delta x^3}$, for any $X^V \in (T(\mathbb{R}^3)^0)^V$ and $X^H \in (T(\mathbb{R}^3)^0)^H$. Thus, for any $X \in T(\mathbb{R}^3)^0$, $X = X_i^H(x,y)\frac{\delta}{\delta x^i} + X_i^V(x,y)\frac{\partial}{\partial y^i}$ (i=1, 2, 3). Consider a η, 1-form, $\eta = \eta^H + \eta^V = {\eta_i}^H(x,y)dx^i + {\eta_i}^V(x,y)\delta y^i$ (i=1, 2, 3), $\eta^H \in (T^*(\mathbb{R}^3)^0)^H$ and $\eta^V \in (T^*(\mathbb{R}^3)^0)^V$.

G is a symmetric tensor field of type (0,2), non-degenerate and pseudo-Riemannian metric on $(\mathbb{R}^3)^0$. Then, G is called Sasaki Finsler metric on $(\mathbb{R}^3)^0$. Then, G can be defined as below:

$$G = G^H + G^V = g_{ij}^{F^*} dx^i \otimes dx^j + g_{ij}^{F^*} \delta y^i \otimes \delta y^i \text{ (i=1, 2, 3).}$$

The vector fields

$$E_1^H = \frac{x_1}{x_3}\frac{\delta}{\delta x^1} \ , \ E_2^H = \frac{x_2}{x_3}\frac{\delta}{\delta x^2} \ , \ E_3^H = \frac{\delta}{\delta x^3} = \xi^H$$

are linear independent at every point of $((\mathbb{R}^3)^0)^h$. Let G^H be the Sasaki Finsler pseudo-metric of index 2 given by

$$G^H(E_1^H,\xi^H) = G^H(E_1^H,E_2^H) = G^H(E_2^H,\xi^H) = 0$$

$$G^H(E_1^H,E_1^H) = G^H(E_2^H,E_2^H) = -1, G^H(\xi^H,\xi^H) = \varepsilon = 1.$$

Let η^H be the 1-form derscribed by

$$\eta^H(Z^H) = G^H(Z^H,\ \xi^H) = G^H(z_1E_1^H + z_2E_2^H + z_3\,\xi^H,\ \xi^H) = z_3 ,\ \forall\, Z^H \in (T(\mathbb{R}^3)^0)^H.$$

Consider ϕ^H the (1, 1) tensör field stated by

$$\phi^H(E_1^H) = E_2^H \ , \phi^H(E_2^H) = -E_1^H \ , \ \phi^H(\xi^H) = 0.$$

Then using the linearity of ϕ^H, we have

$$Z^H = z_1E_1^H + z_2E_2^H + z_3\,\xi^H \ , W^H = w_1E_1^H + w_2E_2^H + w_3\,\xi^H$$

$$\phi^H(Z^H) = \phi^H(z_1E_1^H + z_2E_2^H + z_3\,\xi^H) = z_1\phi^H(E_1^H) + z_2\phi^H(E_2^H) + z_3\,\phi^H(\xi^H)$$

$$\phi^H(Z^H) = z_1E_2^H - z_2E_1^H$$

$$\phi^H(W^H) = w_1\phi^H(E_1^H) + w_2\phi^H(E_2^H) + w_3\,\phi^H(\xi^H) = w_1E_2^H - w_2E_1^H$$

$$(\phi^H)^2(Z^H) = -z_2E_2^H - z_1E_1^H = -Z + \eta^H(Z^H)\xi^H$$

Thus we get

$$G^H\big(\phi^H(Z^H),\phi^H(W^H)\big) = G^H(Z^H,\ W^H) - \eta^H(Z^H)\,\eta^H(W^H)$$

$\forall\, Z^H \in (T(\mathbb{R}^3)^0)^H$ and $\forall\, W^H \in (T(\mathbb{R}^3)^0)^H$. Thus the structure $(((\mathbb{R}^3)^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ define the almost contact pseudo-metric Finsler structure on $((\mathbb{R}^3)^0)^h$.

Let ∇ be the Levi-Civita connection with respect to pseudo-metric G^H. Then we have

$$[E_1^H,E_2^H] = 0 \ , \ [E_1^H,\ \xi^H] = \frac{1}{x_3}E_1^H \ , \ [E_2^H,\ \xi^H] = \frac{1}{x_3}E_2^H.$$

The connection ∇ of the pseudo-metric G^H is given by

$$2G^H\big(\nabla_{X^H}Y^H,Z^H\big) = X^HG^H(Y^H,Z^H) + Y^HG^H(X^H,Z^H) - Z^HG^H(X^H,Y^H) - G^H(X^H,[Y^H,Z^H])$$
$$-g(Y^H,[X^H,Z^H]) + G^H(Z^H,[X^H,Y^H])$$

Which is known as Koszul's formula. Using this formula, we have

$$2G^H\left(\nabla_{E_1^H}\xi^H,\ E_1^H\right) = -G^H(E_1^H,[\xi^H,\ E_1^H\ ]\ ) - G^H(\xi^H,[\ E_1^H,\ E_1^H]) + G^H(E_1^H\ ,[\ E_1^H,\ \xi^H])$$

$$= 2G^H(\tfrac{1}{x_3}E_1^H,\ E_1^H).$$

Thus,

$$\nabla_{E_1^H}\xi^H = \tfrac{1}{x_3}E_1^H\ ,\ \nabla_{\xi^H}E_1^H = 0.$$

Again by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_2^H}\xi^H,\ E_2^H\right) = -G^H(E_2^H,[\xi^H,\ E_2^H\ ]\ ) - G^H(\xi^H,[\ E_2^H,\ E_2^H]) + G^H(E_2^H\ ,[\ E_2^H,\xi^H])$$

$$= 2G^H(\tfrac{1}{x_3}\ E_2^H,\ E_2^H).$$

Thus,

$$\nabla_{E_2^H}\xi^H = \tfrac{1}{x_3}\ E_2^H \quad , \quad \nabla_{\xi^H}E_2^H = 0.$$

Also by using Koszul's formula we obtain

$$2G^H\left(\nabla_{E_1^H}\ E_2^H,\ \xi^H\right) = G^H(E_1^H,[\xi^H,\ E_2^H\ ]\ ) + (\xi^H,[\ E_1^H,\ E_2^H]) - G^H(E_2^H\ ,[\ E_1^H,\xi^H]) = 0.$$

Thus,

$$\nabla_{E_1^H}\ E_2^H = 0\ , \quad \nabla_{E_2^H}\ E_1^H = 0$$

Similarly we get

$$2G^H\left(\nabla_{E_1^H}\ E_1^H,\ \xi^H\right) = -\ G^H(E_1^H,[E_1^H,\xi^H\ ]\ ) + (\xi^H,[\ E_1^H,\ E_1^H]) - G^H(E_1^H\ ,[\ E_1^H,\xi^H])$$

$$= -2G^H\left(\tfrac{1}{x_3}E_1^H\ ,\ E_1^H\right) = = \tfrac{2}{x_3} = 2G^H\left(\tfrac{1}{x_3}\xi^H\ ,\ \xi^H\right).$$

Thus,

$$\nabla_{E_1^H}\ E_1^H = \frac{1}{x_3}\ \xi^H\ .$$

If we use the equations we found

$$(\nabla_X^H\xi^H) = x_1\nabla_{E_1^H}\xi^H + x_2\nabla_{E_2^H}\xi^H = x_1\ \tfrac{1}{x_3}\ E_1^H + x_2\ \tfrac{1}{x_3}E_2^H,$$

$\forall\ X^H \in (T(\ \mathbb{R}^3)^0)^H$.

The above equations tell us the almost contact pseudo-metric Finsler manifold $\left((\mathbb{R}^3)^0\right)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ satisfy (3.3) for $\alpha = 0\ ,\ \beta = \frac{2}{x_3},\ \varepsilon =\ 1.$

3.1. α −Sasakian Indefinite Finsler Manifolds

$F^{2n+1} = (M, M^0, F^*)$ be an indefinite Finsler manifold. The almost contact pseudo-metric Finsler structures $(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ on $(M^0)^h$ and $(M^0)^v$ are the α −Sasakian pseudo-metric Finsler structures if and only if

$$(\nabla_X^H\phi^H)Y^H = \tfrac{\alpha}{2}\{G^H(X^H,Y^H)\xi^H - \varepsilon\eta^H(Y^H)X^H\} \tag{3.18}$$

$$(\nabla_X^V\phi^V)Y^V = \tfrac{\alpha}{2}\{G^V(X^V,Y^V)\xi^V - \varepsilon\eta^V(Y^V)X^V\} \tag{3.19}$$

and

$$(\nabla^H_X \xi^H) = -\varepsilon \frac{\alpha}{2}\ \phi X^H, \qquad (\nabla^V_X \xi^V) = -\varepsilon \frac{\alpha}{2}\ \phi X^V.$$

Moreover, from (3.18) and (3.19) we obtain

$$(\nabla^H_X \eta^H)(Y^H) = \frac{\alpha}{2}\Omega^H(X^H, Y^H) = \frac{\alpha}{2}G^H(X^H, \phi Y^H)$$

$$(\nabla^V_X \eta^V)(Y^V) = \frac{\alpha}{2}\Omega^V(X^V, Y^V) = \frac{\alpha}{2}G^V(X^V, \phi Y^V)$$

Thus, these structures are the $\alpha-$Sasakian pseudo-metric structures in the $\alpha-$Sasakian indefinite Finsler manifolds $((M^0)^h, \phi^H, \xi^H, \eta^H, G^H)$ and $((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$. Also, the following relations hold.

$$R^H(X^H, Y^H)\xi^H = \frac{\alpha^2}{4}\{\eta^H(Y^H)X^H - \eta^H(X^H)Y^H\}$$

$$R^V(X^V, Y^V)\xi^V = \frac{\alpha^2}{4}\{\eta^V(Y^V)X^V - \eta^V(X^V)Y^V\}$$

$$\eta^H(R^H(X^H, Y^H)Z^H) = \varepsilon\frac{\alpha^2}{4}\{G^H(Y^H, Z^H)\eta^H(X^H) - G^H(X^H, Z^H)\eta^H(Y^H)\}$$

$$\eta^V(R^V(X^V, Y^V)Z^V) = \varepsilon\frac{\alpha^2}{4}\{G^V(Y^V, Z^V)\eta^V(X^V) - G^V(X^V, Z^V)\eta^V(Y^V)\}$$

$$(\nabla^H_Z R^H)(X^H, Y^H)\xi^H = \varepsilon\frac{\alpha^2}{8}\{G^H(Y^H, Z^H)X^H - G^H(X^H, Z^H)Y^H\} - \frac{1}{2}\ R^H(X^H, Y^H)Z^H$$

$$(\nabla^V_Z R^V)(X^V, Y^V)\xi^V = \varepsilon\frac{\alpha^2}{8}\{G^V(Y^V, Z^V)X^V - G^V(X^V, Z^V)Y^V\} - \frac{1}{2}\ R^V(X^V, Y^V)Z^V$$

$$R^H(X^H, Y^H)Z^H = \varepsilon\frac{\alpha^2}{4}\{G^H(Y^H, Z^H)X^H - G^H(X^H, Z^H)Y^H\}$$

$$R^V(X^V, Y^V)Z^V = \varepsilon\frac{\alpha^2}{4}\{G^V(Y^V, Z^V)X^V - G^V(X^V, Z^V)Y^V\}$$

$$R^H(X^H, \xi^H)Y^H = \frac{\alpha^2}{4}\{\eta^H(Y^H)X^H - \varepsilon G^H(X^H, Y^H)\xi^H\}$$

$$R^V(X^V, \xi^V)Y^V = \frac{\alpha^2}{4}\{\eta^V(Y^V)X^V - \varepsilon G^V(X^V, Y^V)\xi^V\}$$

$$R^H(\xi^H, X^H)Y^H = \frac{\alpha^2}{4}\{\varepsilon G^H(X^H, Y^H)\xi^H - \eta^H(Y^H)X^H\}$$

$$R^V(\xi^V, X^V)Y^V = \frac{\alpha^2}{4}\{\varepsilon G^V(X^V, Y^V)\xi^V - \eta^V(Y^V)X^V\}$$

$$S^H(\xi^H, \xi^H) = \begin{cases} \alpha^2\left(\frac{2n-q}{4}\right), \xi^H \ is\ a\ space-like\ vector \\ \alpha^2\left(\frac{2n-q+1}{4}\right), \xi^H \ is\ a\ time-like\ vector \end{cases}$$

$$S^V(\xi^V,\xi^V)=\begin{cases}\alpha^2\left(\frac{2n-q}{4}\right), \xi^V \textit{ is a space}-\textit{like vector}\\ \alpha^2\left(\frac{2n-q+1}{4}\right), \xi^V \textit{ is a time}-\textit{like vector}\end{cases}$$

$$S^H(X^H,\xi^H)=\begin{cases}\alpha^2\left(\frac{2n-q}{4}\right)\eta^H(X^H), \xi^H \textit{ is a space}-\textit{like vector}\\ \alpha^2\left(\frac{2n-q+1}{4}\right)\eta^H(X^H), \xi^H \textit{ is a time}-\textit{like vector}\end{cases}$$

$$S^V(X^V,\xi^V)=\begin{cases}\alpha^2\left(\frac{2n-q}{4}\right)\eta^V(X^V), \xi^V \textit{ is a space}-\textit{like vector}\\ \alpha^2\left(\frac{2n-q+1}{4}\right)\eta^V(X^V), \xi^V \textit{ is a time}-\textit{like vector}\end{cases}$$

If ξ^H and ξ^V are the space-like vectors, then we get

$$S^H(\phi X^H,\phi Y^H)=S^H(X^H,Y^H)+\alpha^2\left(\frac{q-2n}{4}\right)\eta^H(X^H)\eta^H(Y^H)$$

$$S^V(\phi X^V,\phi Y^V)=S^V(X^V,Y^V)+\alpha^2\left(\frac{q-2n}{4}\right)\eta^V(X^V)\eta^V(Y^V).$$

If ξ^H and ξ^V are the time-like vectors, then we get

$$S^H(\phi X^H,\phi Y^H)=S^H(X^H,Y^H)+\alpha^2\left(\frac{q-2n-1}{4}\right)\eta^H(X^H)\eta^H(Y^H)$$

$$S^V(\phi X^V,\phi Y^V)=S^V(X^V,Y^V)+\alpha^2\left(\frac{q-2n-1}{4}\right)\eta^V(X^V)\eta^V(Y^V).$$

3.2. $\boldsymbol{\beta}$ −Kenmotsu Indefinite Finsler Manifolds

Let $F^{2n+1}=(M,M^0,F^*)$ be an indefinite Finsler manifold with the warped product space $M^{2n+1}=\mathbb{R}\times_f N^{2n}$. We suppose that $(N^0)^{2n}=TN^{2n}\setminus\theta$ is a Kahlerian manifold and $f(t)=ce^{\beta\frac{t}{2}}$. For the almost Kenmotsu pseudo-metric Finsler structures $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ on $(M^0)^h$ and $(M^0)^v$ resp., 1-forms η^H and η^V and 2-forms Ω^H and Ω^V satisfy the below conditions.

$$d\eta^H=d\eta^V=0,\qquad d\eta^H=\beta\eta^H\wedge\Omega^H,\qquad d\eta^V=\beta\eta^V\wedge\Omega^V$$

where β being a non-zero real constant.

The almost contact pseudo-metric Finsler structures $(\phi^H,\xi^H,\eta^H,G^H)$ and $(\phi^V,\xi^V,\eta^V,G^V)$ on $(M^0)^h$ and $(M^0)^v$ resp., are the β-Kenmotsu pseudo-metric Finsler structures if and only if

$$(\nabla_X^H\phi)Y^H=\tfrac{\beta}{2}\{\varepsilon G^H(\phi X^H,Y^H)\xi^H-\eta^H(Y^H)\phi X^H\} \tag{3.20}$$

$$(\nabla_X^V\phi)Y^V=\tfrac{\beta}{2}\{\varepsilon G^V(\phi X^V,Y^V)\xi^V-\eta^V(Y^V)\phi X^V\} \tag{3.21}$$

and

$$(\nabla_X^H\xi^H)=\frac{\beta}{2}(X^H-\eta^H(X^H)\xi^H)=-\frac{\beta}{2}\phi^2X^H$$

$$(\nabla_X^V\xi^V)=\frac{\beta}{2}(X^V-\eta^V(X^V)\xi^V)=-\frac{\beta}{2}\phi^2X^V.$$

Moreover from (3.20) and (3.21) we obtain

$$(\nabla_X^H \eta^H)(Y^H) = \frac{\beta}{2} G^H(\phi X^H, \phi Y^H) = \frac{\beta}{2} \Omega^H(\phi X^H, Y^H)$$

$$(\nabla_X^V \eta^V)(Y^V) = \frac{\beta}{2} G^V(\phi X^V, \phi Y^V) = \frac{\beta}{2} \Omega^V(\phi X^V, Y^V).$$

Thus, these structures are the β −Kenmotsu pseudo-metric Finsler structures.

In the β −Kenmotsu indefinite Finsler manifolds $\left((M^0)^h, \phi^H, \xi^H, \eta^H, G^H\right)$ and$((M^0)^v, \phi^V, \xi^V, \eta^V, G^V)$, the following relations hold.

$$R^H(X^H, Y^H)\xi^H = \frac{\beta^2}{4}\{\eta^H(X^H)Y^H - \eta^H(Y^H)X^H\}$$

$$R^V(X^V, Y^V)\xi^V = \frac{\beta^2}{4}\{\eta^V(X^V)Y^V - \eta^V(Y^V)X^V\}$$

$$\eta^H(R^H(X^H, Y^H)Z^H) = \varepsilon\frac{\beta^2}{4}\{G^H(X^H, Z^H)\eta^H(Y^H) - G^H(Y^H, Z^H)\eta^H(X^H)\}$$

$$\eta^V(R(X^V, Y^V)Z^V) = \varepsilon\frac{\beta^2}{4}\{G^V(Y^V, Z^V)\eta^V(Y^V) - G^V(Y^V, Z^V)\eta^V(X^V)\}$$

$$(\nabla_Z^H R^H)(X^H, Y^H)\xi^H = \varepsilon\frac{\beta^2}{8}\{G^H(X^H, Z^H)Y^H - G^H(Y^H, Z^H)X^H\} - \frac{1}{2}R^H(X^H, Y^H)Z^H$$

$$(\nabla_Z^V R^V)(X^V, Y^V)\xi^V = \varepsilon\frac{\beta^2}{8}\{G^V(X^V, Z^V)Y^V - G^V(Y^V, Z^V)X^V\} - \frac{1}{2}R^V(X^V, Y^V)Z^V$$

$$R^H(X^H, Y^H)Z^H = -\varepsilon\frac{\beta^2}{4}\{G^H(Y^H, Z^H)X^H - G^H(X^H, Z^H)Y^H\}$$

$$R^V(X^V, Y^V)Z^V = -\varepsilon\frac{\beta^2}{4}\{G^V(Y^V, Z^V)X^V - G^V(X^V, Z^V)Y^V\}$$

$$R^H(\xi^H, X^H)Y^H = \varepsilon\frac{\beta^2}{4}\{-G^H(X^H, Y^H)\xi^H + \varepsilon\eta^H(Y^H)X^H\}$$

$$R^V(\xi^V, X^V)Y^V = \varepsilon\frac{\beta^2}{4}\{-G^V(X^V, Y^V)\xi^V + \varepsilon\eta^V(Y^V)X^V\}$$

$$S^H(\xi^H, \xi^H) = \begin{cases} \beta^2\left(\dfrac{q-2n}{4}\right), \xi^H \textit{ is a space} - \textit{like vector} \\ \beta^2\left(\dfrac{q-2n-1}{4}\right), \xi^H \textit{ is a time} - \textit{like vector} \end{cases}$$

$$S^V(\xi^V, \xi^V) = \begin{cases} \beta^2\left(\dfrac{q-2n}{4}\right), \xi^V \textit{ is a space} - \textit{like vector} \\ \beta^2\left(\dfrac{q-2n-1}{4}\right), \xi^V \textit{ is a time} - \textit{like vector} \end{cases}$$

$$S^H(X^H, \xi^H) = \begin{cases} \beta^2\left(\dfrac{q-2n}{4}\right)\eta^H(X^H), \xi^H \textit{ is a space} - \textit{like vector} \\ \beta^2\left(\dfrac{q-2n-1}{4}\right)\eta^H(X^H), \xi^H \textit{ is a time} - \textit{like vector} \end{cases}$$

$$S^V(X^V,\xi^V) = \begin{cases} \beta^2\left(\frac{q-2n}{4}\right)\eta^V(X^V), \xi^V \text{ is a space} - \text{like vector} \\ \beta^2\left(\frac{q-2n-1}{4}\right)\eta^V(X^V), \xi^V \text{ is a time} - \text{like vector} \end{cases}$$

$$S^H(\phi X^H,\phi Y^H) = S^H(X^H,Y^H) + \beta^2\left(\frac{2n-q}{4}\right)\eta^H(X^H)\eta^H(Y^H)$$

$$S^V(\phi X^V,\phi Y^V) = S^V(X^V,Y^V) + \beta^2\left(\frac{2n-q}{4}\right)\eta^V(X^V)\eta^V(Y^V).$$

4. Conformally Flat Trans-Sasakian Indefinite Finsler Manifolds

We consider conformally flat trans-Sasakian indefinite Finsler manifolds $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$. The conformal curvature tensor field C is given by

$$C^H(X^H,\ Y^H)Z^H = R^H(X^H,\ Y^H)Z^H - \frac{1}{(2n-1)}[S^H(Y^H,\ Z^H)X^H - S^H(X^H,\ Z^H)Y^H + G^H(Y^H,Z^H)QX^H - G^H(X^H,\ Z^H)QY^H] + \frac{r}{2n(2n-1)}[G^H(Y^H,Z^H)X^H - G^H(X^H,Z^H)Y^H] \quad (4.1)$$

and

$$C^V(X^V,\ Y^V)Z^V = R^V(X^V,\ Y^V)Z^V - \frac{1}{(2n-1)}[S^V(Y^V,\ Z^V)X^V - S^V(X^V,\ Z^V)Y^V + G^V(Y^V,\ Z^V)QX^V - G^V(X^V,\ Z^V)QY^V] + \frac{r}{2n(2n-1)}[G^V(Y^V,Z^V)X^V - G^V(X^V,\ Z^V)Y^V] \quad (4.2),$$

where R^H, S^H, Q^H and r are the curvature tensor, the Ricci tensor, the Ricci operatör and the scalar curvature tensor of the $(M^0)^h$, respectively. (R^V, S^V, Q^V and r are the curvature tensor, the Ricci tensor, the Ricci operatör and the scalar curvature tensor of the $(M^0)^v$). If the trans-Sasakian indefinite Finsler manifolds $\left((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\right)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are conformally flat, i. e.

$C^H = 0$ and $C^V = 0$, then from (4.1) and (4.2), we have

$$R^H(X^H,\ Y^H)Z^H = \frac{1}{(2n-1)}[S^H(Y^H,\ Z^H)X^H - S^H(X^H,\ Z^H)Y^H + G^H(Y^H,Z^H)QX^H - G^H(X^H,\ Z^H)QY^H] - \frac{r}{2n(2n-1)}[G^H(Y^H,Z^H)X^H - G^H(X^H,Z^H)Y^H]$$

$$R^V(X^V,\ Y^V)Z^V = \frac{1}{(2n-1)}[S^V(Y^V,\ Z^V)X^V - S^V(X^V,\ Z^V)Y^V + G^V(Y^V,\ Z^V)QX^V - G^V(X^V,\ Z^V)QY^V] - \frac{r}{2n(2n-1)}[G^V(Y^V,Z^V)X^V - G^V(X^V,\ Z^V)Y^V]$$

Now, taking scalar product on both side of above equations with W^H and W^V, we have

$$G^H(R^H(X^H,\ Y^H)Z^H,\ W^H) = G^H(\frac{1}{(2n-1)}[S^H(Y^H,\ Z^H)X^H - S^H(X^H,\ Z^H)Y^H + G^H(Y^H,\ Z^H)QX^H - G^H(X^H,\ Z^H)QY^H] + \frac{r}{2n(2n-1)}[G^H(Y^H,\ Z^H)X^H - G^H(X^H,\ Z^H)Y^H],W^H)$$

and

$$G^V(R^V(X^V,\ Y^V)Z^V,\ W^V) = G^V(\frac{1}{(2n-1)}[S^V(Y^V,\ Z^V)X^V - S^V(X^V,\ Z^V)Y^V + G^V(Y^V,\ Z^V)QX^V - G^V(X^V,\ Z^V)QY^V] + \frac{r}{2n(2n-1)}[G^V(Y^V,\ Z^V)X^V - G^V(X^V,\ Z^V)Y^V],W^V)$$

$$G^H(R^H(X^H,\ Y^H)Z^H,\ W^H) = \frac{1}{(2n-1)}[S^H(Y^H,\ Z^H)G^H(X^H,\ W^H) - S^H(X^H,\ Z^H)G^H(Y^H,\ W^H) + G^H(Y^H,\ Z^H)G^H(QX^H,\ W^H) - G^H(X^H,\ Z^H)G^H(QY^H,\ W^H)]$$

$$+\frac{r}{2n(2n-1)}[G^H(Y^H,\ Z^H)G^H(X^H,\ W^H)-G^H(X^H,\ Z^H)G^H(Y^H,\ W^H)]\,,$$

on putting $W^H=\xi^H$ we get

$$G^H(R^H(X^H,\ Y^H)Z^H,\ \xi^H)=\frac{1}{(2n-1)}[S^H(Y^H,\ Z^H)\varepsilon\,\eta^H(X^H)-\ S^H(X^H,\ Z^H)\,\varepsilon\,\eta^H(Y^H)+G^H(Y^H,$$
$$Z^H)\,S^H(X^H,\xi^H)-G^H(X^H,\ Z^H)\,S^H(Y^H,\xi^H)]-\varepsilon\,\frac{r}{2n(2n-1)}[G^H(Y^H,\ Z^H)\,\eta^H(X^H)-G^H(X^H,\ Z^H)\,\eta^H(Y^H)]\,.$$

Replacing Y^H by ξ^H in equation (3.11) we have

$$G^H(R^H(X^H,\xi^H)Z^H,\xi^H)=\varepsilon\,\eta^H(R^H(X^H,\xi^H)Z^H)=\frac{(\alpha^2-\beta^2)}{4}\{\varepsilon\,\eta^H(X^H)\eta^H(Z^H)-G^H(X^H,Z^H)\}$$
$$-\varepsilon\frac{\alpha\beta}{2}\{G^H(\phi X^H,Z^H)\}=\frac{1}{(2n-1)}[\varepsilon\,S^H(\xi^H,\ Z^H)\,\eta^H(X^H)-\ \varepsilon\,S^H(X^H,\ Z^H)\ +\varepsilon\,\eta^H(Z^H)S^H(X^H,\xi^H)-$$
$$G^H(X^H,\ Z^H)\,S^H(\xi^H,\ \xi^H)]-\frac{r}{2n(2n-1)}[\,\eta^H(Z^H)\,\eta^H(X^H)-\varepsilon\,G^H(X^H,\ Z^H)]\,.$$

by using equations (3.14), (3.15) and (3.16)

$$S^H(X^H,\ Z^H)=\left[\frac{r}{2n}+\frac{(\alpha^2-\beta^2)}{4}\big((2n-1)-\varepsilon(2n)\big)\right]G^H(X^H,Z^H)$$
$$+[\frac{-\varepsilon\,r}{2n}+\frac{(\alpha^2-\beta^2)}{4}(4n-\varepsilon\,(2n-1)]\,\eta^H(Z^H)\,\eta^H(X^H)+\varepsilon\frac{\alpha\beta}{2}(2n-1)\{G^H(\phi X^H,Z^H)\}$$

and

$$S^V(X^V,\ Z^V)=\left[\frac{r}{2n}+\frac{(\alpha^2-\beta^2)}{4}\big((2n-1)-\varepsilon(2n)\big)\right]G^V(X^V,Z^V)$$
$$+[\frac{-\varepsilon\,r}{2n}+\frac{(\alpha^2-\beta^2)}{4}(4n-\varepsilon\,(2n-1)]\,\eta^V(Z^V)\,\eta^V(X^V)+\varepsilon\frac{\alpha\beta}{2}(2n-1)\{G^V(\phi X^V,Z^V)\}$$

Hence we have the following theorem

Theorem 4.1. The conformally flat trans-Sasakian indefinite Finsler manifolds $\big((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\big)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are the $\eta-$ Einstein manifolds if and only if $\alpha.\beta=0$, where α,β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

Corollary 4.1. The conformally flat α -Sasakian indefinite Finsler manifolds $\big((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\big)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are the $\eta-$ Einstein manifolds.

Corollary 4.2. The conformally flat β -Kenmotsu indefinite Finsler manifolds $\big((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\big)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are the $\eta-$ Einstein manifolds.

5. Conclusion

In this article, we study indefinite trans-Sasakian structures on indefinite Finsler manifolds by using pseudo-Finsler metric. Also, α −Sasakian and β −Kenmotsu indefinite Finsler manifolds are presented. The conformally flat trans-Sasakian indefinite Finsler manifolds $\big((M^0)^h,\phi^H,\xi^H,\eta^H,G^H\big)$ and $((M^0)^v,\phi^V,\xi^V,\eta^V,G^V)$ are the $\eta-$ Einstein manifolds if and only if $\alpha.\beta=0$, where α,β are constant functions defined on $(M^0)^h$ and $(M^0)^v$.

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Application of Bipolar Near Soft Sets

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Keywords
Soft sets,
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Abstract — The bipolar soft set is supplied with two soft sets, one positive and the other negative. Whichever feature is stronger can be selected to find the object we want. In this paper, the notion of bipolar near soft set, which near set features are added to a bipolar soft set, and its fundamental properties are introduced. In this new set, its features can be restricted and the basic properties and topology of the set can be examined accordingly. With the soft set close to bipolar, it will be easier for us to decide to find the most suitable object in the set of objects. This new idea is illustrated with real-life examples. With the help of the bipolar near soft set, we make it easy to choose the one closest to the criteria we want in decision making. Among the many given objects, we can find the one with the properties we want by using the ones with similar properties.

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1. Introduction

The models used for each uncertainty problem are different from each other. For this, different set concepts have been created. With the help of objects and features on these objects, Pawlak [17] first presented the concept of rough set and then Peters [18, 19] presented the concept of near set, in which he examined sets close to each other with these features. Another set, the soft set, was created by Molodtsov [14] and has been studied by many people both in practice and in theory [1–3, 5–7, 12, 13, 15]. Feng and Li [9], on the other hand, established a new concept by integrating the concepts of soft set and near set. Similarly, Tasbozan et al. [22] combined the concepts of near and soft set. These concepts have been developed and produced in the topology [23, 24].

The idea of bipolar soft set was presented by Shabir and Naz [20] and later this definition was used by many researchers in applications. Karaaslan and Karatas [10] created the idea of bipolar soft cluster and used it in applications. Mahmood [11] gave the bipolar soft set approach and its application. The notion of bipolar soft set, which is a set in which human decisions are made with two types of notice, positive and negative, was defined [4, 8, 10, 11, 16, 21]. Parameters with positive or negative properties give us information about objects. In some uncertainty problems, a decision making approach should be established in order to make the most accurate object selection under these conditions with the parameters determined by the decision maker. The construction of all these mathematical models is up to the decision maker. By restricting this

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information to the selected parameters, we have obtained the concept of bipolar near soft set in order to distinguish the ones with similar properties more quickly. In the application with bipolar near soft sets, practicality can be provided in decision making so that we can find the object we will choose. Therefore, it can be applied to multi-criteria decision making problems. Today, bipolar theory is used in the evaluation system to understand people's positive or negative opinions about objects. In this way, organizations can track how much their products are liked or help buyers find the products closest to their needs.

In this study, the necessary definitions were given in the first part, and in the other part, we reached the concept of bipolar near soft sets, in which we added set characteristics near to bipolar sets. It is exemplified how this concept can be applied in an environment of uncertainty. In order to find the one with the features we want among many objects, we were restricted to the features desired by the decision maker, and with the choices we made, we were able to see the objects with similar features more clearly. This has provided us with the practice of choosing the most suitable products for us that we need.

2. Preliminary

Let $\mathscr{O}$ be an objects set, $\mathscr{F}$ be a set of parameters that define properties on objects and $\mathscr{P}(\mathscr{O})$ is the set of all subsets of $\mathscr{O}$.

Definition 2.1. [3] Let $B \subseteq \mathscr{F}$ and $F : B \to \mathscr{P}(\mathscr{O})$, then (F,B) is a soft set(SS) over $\mathscr{O}$.

Definition 2.2. [22] Let $NAS = (\mathscr{O}, \mathscr{F}, \sim_{Br}, N_r, \nu_{N_r})$ be a nearness approximation space, B be a non-empty subsets of $\mathscr{F}$ and (F,B) be a SS over $\mathscr{O}$. Then

$$N_r*((F,B)) = (N_r*(F(k) = \cup\{x \in \mathscr{O} : [x]_{Br} \subseteq F(k)\}, B))$$

and

$$N_r^*((F,B)) = (N_r^*(F(k) = \cup\{x \in \mathscr{O} : [x]_{Br} \cap F(k) \neq \emptyset\}, B))$$

are lower and upper near approximation operators where $[x]_{Br}$ be equivalence classes denoted by the subscript r for the cardinality of the restricted subset B_r. The SS $N_r((F,B))$ with $Bnd_{N_r(B)}((F,B)) \geqslant 0$ called a near soft set(NSS) where

$$Bnd_{N_r(B)}((F,B)) = N_r^*((F,B)) \setminus N_r*((F,B)).$$

Definition 2.3. [21] Let $F : B \to P(\mathscr{O})$ and $G : \neg B \to P(\mathscr{O})$ be a mappings which $F(k) \cap G(\neg k) = \emptyset$, $\forall k \in B$. (F,G,B) is called a bipolar soft set (BSS) over $\mathscr{O}$.

3. Bipolar Near Soft Set

In this section, by introducing the bipolar set, we have reached the concept of bipolar near soft sets, to which we add near set properties. How this concept can be applied in an environment of uncertainty is discussed with assumptions about the values of the data on the example. Thus, in order to find the one with the features we want among many objects, we can select objects with similar features by limiting them to the features the decision maker wants.

Definition 3.1. Let $\sigma = (F,B)$, $N_r(\sigma)$ be a NSS and (F,G,B) be a BSS over $\mathscr{O}$. $F : B \to P(\mathscr{O})$ and $G : \neg B \to P(\mathscr{O})$ are mappings which $F(k) \cap G(\neg k) = \emptyset$, $\forall k \in B$. Then the triplet $N(F,G,B)$ is called a bipolar near soft set over $\mathscr{O}$ ($BNSS$).

Definition 3.2. Let $N(F_s, G_s, A)$ and $N(F_1, G_1, B)$ be $BNSS$ over $\mathscr{O}$, if

1. $A \subseteq B$,
2. $F_s(k) \subseteq F_1(k)$ and $G_1(\neg k) \subseteq G_s(\neg k)$, $\forall k \in A$,
3. For $N_*(\sigma) = N_*(F_s(k), A)$ of a set (F_s, A) and $N_*(\mu) = N_*(F_1(k), B)$ of a set (F_1, B), $N_*(\sigma) \subseteq N_*(\mu)$,

then $N(F_s, G_s, A)$ is a bipolar near soft subset $BNSs$ of $N(F_1, G_1, B)$ and denoted by $N(F_s, G_s, A) \subseteq N(F_1, G_1, B)$.

Definition 3.3. If $N(F_s, G_s, A)$ is a $BNSs$ of $N(F_1, G_1, B)$ and $N(F_1, G_1, B)$ is a $BNSs$ of $N(F_s, G_s, A)$, then $N(F_s, G_s, A)$ and $N(F_1, G_1, B)$ are equal $BNSS$ over $\mathscr{O}$.

Definition 3.4. Let F^c and G^c be mappings where $F^c(k) = G(\neg k)$ and $G^c(\neg k) = F(k)$, $\forall k \in A$. $N(F, G, A)^c = N(F^c, G^c, A)$ is a complement of a $BNSS$.

Definition 3.5. If $\Phi(k) = \emptyset$ and $N(\mathscr{O}(\neg k)) = \mathscr{O}$, for all $k \in A$, then $N(\Phi, \mathscr{O}, A)$ is a null $BNSS$ over $\mathscr{O}$.

Definition 3.6. If $N(\mathscr{O}(k)) = \mathscr{O}$ and $N(\Phi(\neg k)) = \emptyset$, for all $k \in A$, then $N(\mathscr{O}, \Phi, A)$ is an absolute $BNSS$ over $\mathscr{O}$.

Definition 3.7. Let $N(F, G, A)$ and $N(F_1, G_1, B)$ be two $BNSS$ over $\mathscr{O}$. The intersection of $N(F, G, A)$ and $N(F_1, G_1, B)$, denoted by $N(H, I, C) = N(F, G, A) \cap N(F_1, G_1, B)$, $\forall k \in C = A \cap B$ where $H = F \cap F_1$ and $I = G \cap G_1$, the union of $N(F, G, A)$ and $N(F_1, G_1, B)$ where denoted by $N(H, I, C)$, $\forall k \in C = A \cup B$ where $H = F \cup F_1$ and $I = G \cup G_1$.

Example 3.8. Let $\mathscr{O} = \{y_1, y_2, y_3, y_4, y_5\}$ be a five person and $B = \{k_1, k_2\} \subseteq \mathscr{F} = \{k_1, k_2, k_3, k_4\}$ be a set of parameters, where k_1, k_2, k_3, k_4 stand for tall, strong, well dressed and intelligent, respectively. Sample values of the $k_i, i = 1, 2, 3, 4$ functions are shown

$$\begin{aligned}
[y_1]_{k_1} &= \{y_1, y_4\}, [y_2]_{k_1} = \{y_2, y_3, y_5\}, \\
[y_1]_{k_2} &= \{y_1, y_4\}, [y_2]_{k_2} = \{y_2, y_3\}, [y_5]_{k_2} = \{y_5\}, \\
[y_1]_{k_1,k_2} &= \{y_1, y_4\}, \\
[y_2]_{k_1,k_2} &= \{y_2, y_3\}, \\
[y_5]_{k_1,k_2} &= \{y_5\}.
\end{aligned}$$

Let $B = \{k_1, k_2\}$ and (F, B) be a SS defined by $(F, B) = ((k_1, \{y_1, y_4\}), (k_2, \{y_3, y_5\}))$ is a NSS with $r = 1$ and $r = 2$. We get

$$N_*((F, B)) = (F_*(k_2), B) = \{(k_2, \{y_5\})\}, for k_2 \in B$$

and

$$N^*((F, B)) = \{(k_2, \{y_2, y_3, y_5\})\}, for k_1, k_2 \in B.$$

Hence, $Bnd_N(\sigma) \geq 0$, then (F, B) is a NSS.

Let $F : B \to P(\mathscr{O})$ and $G : -B \to P(\mathscr{O})$ be mappings given as follows:

$$\begin{aligned}
F(k_1) &= \{y_1, y_4\}, G(\neg k_1) = \{y_3\}, \\
F(k_2) &= \{y_3, y_5\}, G(\neg k_2) = \emptyset.
\end{aligned}$$

Then

$$N(F, G, B) = \{(k_1, \{y_1, y_4\}, \{y_3\}), (k_2, \{y_3, y_5\}, \emptyset)\}$$

is a $BNSS$.

Definition 3.9. Let (F,B) be a NSS over $\mathcal{O}$, $u \in \mathcal{O}$. Then $N(F_u;G_u;B)$ denotes the bipolar near soft set over $\mathcal{O}$ and $(u_k, u'_{(-k)}, B)$ called a bipolar near soft point, defined by $F_u(k) = \{u\}$ and $F_u(k') = \emptyset$ for all $k' \in B - \{k\}$ and $G_u(-k) = \mathcal{O} - \{u\} = u'$, for each $k \in B$.

Definition 3.10. Let $\mu = N(F,G,B)$ be a $BNSS$ over $\mathcal{O}$ and τ be the collection of $BNSs$ of $\mathcal{O}$. If the following are provided

i) $(\emptyset, G, B), (\mathcal{O}, G, B) \in \tau$,

ii) $N(F_1, G_1, B), N(F_2, G_2, B) \in \tau$ then $N(F_1, G_1, B) \cap N(F_2, G_2, B) \in \tau$,

iii) $N(F_i, G_i, B), \forall k \in B$ then $\bigcup_i N(F_i, G_i, B) \in \tau$,

then $N(\mathcal{O}, \tau, B, -B)$ is a bipolar near soft topological space($BNSTS$).

Definition 3.11. Let $N(\mathcal{O}, \tau, B, -B)$ be called a $BNSTS$ over $\mathcal{O}$. Then the collection $\tau_k = \{F(k) : N(F,G,B) \in \tau\}$ for each $k \in B$ defines a topology on $\mathcal{O}$.

Definition 3.12. Let $N(\mathcal{O}, \tau, B, -B)$ be a $BNSTS$ over $\mathcal{O}$ and $N(F,G,B)$ be a $BNSS$ over $\mathcal{O}$. Then $N(F,G,B)$ is said to be bipolar near soft closed ($BNSC$) if and only if $N(F,G,B)^c$ in τ. Then $(\mathcal{O}, \tau, B, -B)$ is a $BNSTS$ over $\mathcal{O}$ and the members of are bipolar near soft open ($BNSO$) sets in $\mathcal{O}$.

Definition 3.13. Let $N(F,G,B)$ be a $BNSS$ over $\mathcal{O}$ and $Y \neq \emptyset \subseteq \mathcal{O}$. Then the $BNSS$ of $N(F,G,B)$ over Y is defined as follows: ${}^{Y}F(k) = Y \cap F(k)$ and ${}^{Y}G(-k) = Y \cap G(-k)$; for each $k \in B$ and denoted by $N({}^{Y}F, {}^{Y}G, B)$.

Definition 3.14. Let $N(\mathcal{O}, \tau, B, -B)$ be a $BNSTS$ over $\mathcal{O}$ and $Y \neq \emptyset \subseteq \mathcal{O}$. Then $\tau_Y = \{N({}^{Y}F, {}^{Y}G, B) : N(F,G,B) \in \tau\}$ is a $BNST$on Y.

Definition 3.15. Let $N(\mathcal{O}, \tau_s, B, -B)$ be a $BNSTS$ over $\mathcal{O}$. Then the collection consisting of $BNSS$, $N(F,G,B)$ such that $(F,B) \in \tau$, $G(-k) = F'(k) = \mathcal{O} \backslash F(k)\ \forall -k \in -B$, defined a $BNST$ over $\mathcal{O}$.

Example 3.16. Let $\mathcal{O}$, B be sets, $F : B \to P(\mathcal{O})$ and $G : -B \to P(\mathcal{O})$ be two maps as in Example 16. Then

$$\begin{aligned} N(F_1, G_1, B) &= \{(k_1, \{y_1, y_4\}, \{y_1\}), (k_2, \{y_3, y_5\}, \emptyset)\}, \\ N(F_2, G_2, B) &= \{(k_1, \{y_1, y_4, y_2\}, \{y_1\}), (k_2, \{y_5\}, \emptyset)\}, \\ N(F_3, G_3, B) &= \{(k_1, \{y_1, y_4\}, \{y_1\}), (k_2, \{y_5\}, \emptyset)\} \end{aligned}$$

are $BNSS$. Also, we obtained

$$\tau = \{N(F_1, G_1, B), N(F_2, G_2, B), N(F_3, G_3, B), (\emptyset, G, B), (\mathcal{O}, G, B)\}.$$

Then, $N(\mathcal{O}, \tau, B, -B)$ is a $BNSTS$.

Definition 3.17. Let $Y \neq \emptyset$ and $Y \subseteq \mathcal{O}$, then the whole $BNSS$, $N(Y,G,B)$ over $\mathcal{O}$ for which $Y(k) = Y$, for all $k \in B$.

Definition 3.18. Let $N(F,G,B)$ be a $BNSS$ over $\mathcal{O}$, $Y \neq \emptyset$ and $Y \subseteq \mathcal{O}$. Then the $BNSsS$ of $N(F,G,B)$ over Y is defined as follows:

$${}^{Y}F(k) = Y \cap F(k), \forall k \in B$$

and denoted by $N({}^{Y}F, G, B)$.

Definition 3.19. Let $N(K_1, P_1, B)$ and $N(K_2, P_2, D)$ be two $BNSS$ over $\mathscr{O}_1$ and $\mathscr{O}_2$, respectively. The cartesian product $N(K_1, P_1, B) \times N(K_2, P_2, D)$ is defined by $(K_1 \times K_2)_{(B\times C)}$ where $(K_1 \times K_2)_{(B\times D)}(k, m) = K_1(k) \times K_2(m)$, $\forall(k, m) \in B \times D$. According to this definition, the soft set $N(K_1, P_1, B) \times N(K_2, P_2, D)$ is a $BNSS$ over $\mathscr{O}_1 \times \mathscr{O}_2$ and its parameter universe is $B \times D$.

Definition 3.20. Let $N(\mathscr{O}, \tau, B, -B)$ be a $BNSTS$ over $\mathscr{O}$, then the members of τ are said to be bipolar near soft open($BNSO$) sets in $\mathscr{O}$.

Definition 3.21. Let $N(\mathscr{O}, \tau, B, -B)$ be a $BNSTS$ and $N(F, G, B)$ be a $BNSS$ over $\mathscr{O}$. Then the BNS closure $N(F, G, B)^-$ is the intersection of all $BNSC$ sets of (F, G, B) is the smallest $BNSC$ over $\mathscr{O}$ and the BNS interior $N(F, G, B)^\circ$ is the combination of all $BNSO$ sets of $N(F, G, B)$ is the biggest $BNSO$ set over $\mathscr{O}$.

3.1. Application of Bipolar Near Soft Sets

In this part, we will use the notion of bipolar near soft sets to make the best choice available to us. In order to do this, we will follow some steps. Let us now consider this with an example.

Example 3.22. Assume that a house selling firm has a set of houses $\mathscr{O}$ with a set of parameters $\mathscr{F}$. Let $\mathscr{O} = \{y_1, y_2, y_3, y_4, ..., y_{12}\}$ be a set of twelve house and $B = \{k_5, k_7\} \subseteq \mathscr{F} = \{k_1, k_2, k_3, ..., k_7\}$ be a set of seven parameters, where k_i, $i = (1, 2, 3, 4, 5, 6, 7)$ stand for "expensive,""cheap," "modern," "earthquake resistant" "good location," "multi-storey,"and "quality material,"respectively. We should noted that -k_1 does not denote "cheap" and $\neg k_2$ does not denote "expensive." Now, assume that a house selling firm categorises these houses with interest to the set of parameters using a concept of a $BNSS$, $N(F, G, B)$ as follows:

$$\begin{aligned}
F(k_1) &= \{y_i : i = 1,2,5,7,9\}, G(-k_1) = \{y_i : i = 3,4,10\}, \\
F(k_2) &= \{y_i : i = 3,5,8,11,12\}, G(-k_2) = \{y_i : i = 1,2,9,10\}, \\
F(k_3) &= \{y_i : i = 1,7,8,9,12\}, G(-k_3) = \{y_i : i = 2,5,10\}, \\
F(k_4) &= \{y_i : i = 1,5,8,9,11,12\}, G(-k_4) = \{y_i : i = 2,3,4\}, \\
F(k_5) &= \{y_i : i = 1,2,7,8,9,10,11,12\}, G(-k_5) = \{y_i : i = 4,5\}, \\
F(k_6) &= \{y_i : i = 2,10\}, G(-k_6) = \{y_i : i = 7,9,11\}, \\
F(k_7) &= \{y_i : i = 8,12\}, G(-k_7) = \{y_i : i = 6,7,11\}.
\end{aligned}$$

Now, suppose that we want to select a house with respect to $B = \{k_5, k_7\} \subseteq \mathscr{F}$. We will construct table respect with $F : B \to P(\mathscr{O})$ and $G : -B \to P(\mathscr{O})$.

Table 1

	k_1	$-k_1$	k_2	$-k_2$	k_3	$-k_3$	k_4	$-k_4$	k_5	$-k_5$	k_6	$-k_6$	k_7	$-k_7$
y_1	1	0	0	−1	1	0	1	0	1	0	0	0	0	0
y_2	1	0	0	−1	0	−1	0	−1	1	0	1	0	0	0
y_3	0	−1	1	0	0	0	0	−1	0	0	0	0	0	0
y_4	0	−1	0	0	0	0	0	−1	0	−1	0	0	0	0
y_5	1	0	1	0	0	−1	1	0	0	−1	0	0	0	0
y_6	0	0	0	0	0	0	0	0	0	0	0	0	0	−1
y_7	1	0	0	0	1	0	0	0	1	0	0	−1	0	−1
y_8	0	0	1	0	1	0	1	0	1	0	0	0	1	0
y_9	1	0	0	−1	1	0	1	0	1	0	0	−1	0	0
y_{10}	0	−1	0	−1	0	−1	0	0	1	0	1	0	0	0
y_{11}	0	0	1	0	0	0	1	0	1	0	0	−1	0	−1
y_{12}	0	0	1	0	1	0	1	0	1	0	0	0	1	0

We determine the value of $(y_n, F(k_i))$ and $(y_n, G(-k_i))$ by the following two roles and construct table:

$$(y_n, F(k_i)) = \begin{cases} 1, & y_n \in F(k_i) \\ 0, & y_n \notin F(k_i) \end{cases} .$$

If we combine it using $F(k_i) \cap G(\neg k_i) = \emptyset$ for each $k_i \in \mathscr{F}$, we get the following table:

$$(y_n, (F(k_i), G(-k_i))) = \begin{cases} 1, & y_n \in F(k_i) \\ -1, & y_n \in G(-k_i) \\ 0, & y_n \notin F(k_i) \cup G(-k_i) \end{cases} .$$

Table 2

	$(k_1,-k_1)$	$(k_2,-k_2)$	$(k_3,-k_3)$	$(k_4,-k_4)$	$(k_5,-k_5)$	$(k_6,-k_6)$	$(k_7,-k_7)$	Sum
y_1	1	−1	1	1	1	0	0	3
y_2	1	−1	−1	−1	1	1	0	0
y_3	−1	1	0	−1	0	0	0	−1
y_4	−1	0	0	−1	−1	0	0	−3
y_5	1	1	−1	1	−1	0	0	1
y_6	0	0	0	0	0	0	−1	−1
y_7	1	0	1	0	1	−1	−1	1
y_8	0	1	1	1	1	0	1	5
y_9	1	−1	1	1	1	−1	0	2
y_{10}	−1	−1	−1	0	1	1	0	−1
y_{11}	0	1	0	1	1	−1	−1	1
y_{12}	0	1	1	1	1	0	1	5

Enumerate the values by the rule

$$Sumn = \sum_{i=1}^{7} (y_n, (F(k_i), G(\neg k_i))).$$

Find the verdict, denoted by d, for which $d = \{maxSumn : n = 1, 2, ..., s\}$, where $s = |y|$. Then, d is the suitable select house. If d has more than one value, any of them can be selected.

Let $\sigma = (F, B)$, $B = \{k_5, k_7\}$ be a SS defined by

$N(F, G, B) = ((k_5, \{y_1, y_2, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}\}, \{y_4, y_5\}), (k_7, \{y_8, y_{12}\}, \{y_6, y_7, y_{11}\}))$ is a $BNSS$ with $r = 2$. From the table, we obtained

$$\begin{aligned}
[y_1]_{k_5} &= \{y_1, y_2, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}\}, [y_3]_{k_5} = \{y_3, y_4, y_5, y_6\}, \\
[y_8]_{k_7} &= \{y_8, y_{12}\}, [y_1]_{k_7} = \{y_1, y_{2,} y_3, y_4, y_5, y_6, y_7, y_9, y_{10}, y_{11}\}, \\
[y_8]_{k_5, k_7} &= \{y_8, y_{12}\}, \\
[y_1]_{k_5, k_7}, &= \{y_1, y_2, y_7, y_9, y_{10}, y_{11}\}, \\
[y_3]_{k_5, k_7} &= \{y_3, y_4, y_5, y_6\}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
N_*(\sigma) &= N_*(F(k), B) \\
&= (N_* F(k), B) \\
&= (F_*(k), B) \\
&= (F_*(k_7), B) \\
&= \{(k_7, \{y_8, y_{12}\})\}, \quad for \quad k_7 \in B
\end{aligned}$$

and

$$\begin{aligned}
N^*(\sigma) &= (F^*(k), B) \\
&= \{((k_5, \{y_1, y_2, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}\}), (k_7, \{y_8, y_{12}\}))\} \quad for \quad k_5, k_7 \in B.
\end{aligned}$$

Then, $N(F, G, B)$ is a $BNSS$.

Table 3

	$(k_5,-k_5)$	$(k_7,-k_7)$	Sum
y_1	1	0	1
y_2	1	0	1
y_3	0	0	0
y_4	-1	0	-1
y_5	-1	0	-1
y_6	0	-1	-1
y_7	1	-1	0
y_8	1	1	2
y_9	1	0	1
y_{10}	1	0	1
y_{11}	1	-1	0
y_{12}	1	1	2

From the table, we obtained

$$Sumn = \sum_{i=1}^{7} (y_n, (F(k_i), G(\neg k_i))) = 2.$$

Now, one can note from table that houses y_8 and y_{12} are the optimal houses. Therefore, any of them can be chosen by us to get the house, we want. Accordingly, we find the most suitable house or houses according to the k_5 and k_7 features.

4. Conclusions

In order to find the one with the properties we want among the many objects given in this study, we reached the concept of near soft sets, which we obtained that have properties near to each other, with a bipolar approach. The concept of bipolar near soft set enabled us to see more clearly what we want, namely the practice of choosing the best products. It reduced the features so we could choose what we needed. We aim to obtain similar examples according to the definitions we will find in future studies.

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Bimultipliers of R-algebroids

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Keywords
R-Algebroid,
Bimultiplier.

Abstract — Group action is determined by the automorphism group and algebra action is defined by the multiplication algebra. In the study we generalize the multiplication algebra by defining multipliers of an R-algebroid M. Firstly, the set of bimultipliers on an R-algebroid is introduced, it is denoted by $Bim(M)$, then it is proved that this set is an R-algebroid, it is called multiplication R-algebroid. Using this $Bim(M)$, for an R-algebroid morphism $A \longrightarrow Bim(M)$ it is shown that this morphism gives an R-algebroid action. Then we examine some of the properties associated with this action.

Subject Classification (2020): 18E05, 18A40, 18G30.

1. Introduction

In the realm of group theory, the interplay between groups and their actions on one another is a subject of profound importance. Central to this discourse is the notion that the action of a group on another group is intricately determined by the automorphism group. This relationship is encapsulated in the form of a homomorphism, mapping the acting group to the automorphism group of the target group. Moreover, any extension of groups also finds its roots in such homomorphisms, further underscoring their significance in understanding the dynamics between groups.

Extending beyond the confines of group theory, similar principles resonate in the domain of algebra, where the action of an algebra on another is closely intertwined with the concept of multiplication algebras. The seminal work of Maclane [1] lays the foundation for this concept, elucidating its pivotal role in algebraic structures. Building upon this framework, Ege and Arvasi [2] introduce actor crossed modules of commutative algebras, leveraging multiplication algebras to generalize aspects from commutative algebras to crossed modules [13], [14].

Within the realm of R-algebroids, a branch of algebraic structures, significant attention has been directed towards their study, notably by Mitchell [3], [4], [5] and Amgott [6]. Mitchell's categorical definition of R-algebroids and Mosa's introduction of crossed modules of R-algeb-
roids serve as pivotal contributions to this field. Notably, the equivalence between crossed modules of R-algebroids and special double algebroids with connections, established by Mosa [7], further enriches our

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understanding of these structures. Subsequent investigations by Akca and Avcioglu [8], [9], [10], [11], [12] delve deeper into crossed modules of R-algebroids, unraveling intricate connections and properties. By means of algebra action, the 2-crossed module structure is defined [15] and the equivalence of 2-crossed modules to simplicial algebras is shown [16]. There are also studies [17], [18], [19], [20], [21] on 2-crossed modules.

In this paper, we embark on a journey to explore the multifaceted landscape of R-algebroids, with a specific focus on their actions and associated properties. Our endeavor begins with the introduction of the set denoted Bim(M), comprising multipliers of an R-algebroid M. Through a rigorous exposition, we establish that this set itself forms an R-algebroid, aptly termed the multiplication R-algebroid, by defining suitable operations. Leveraging this newfound structure, we define an R-algebroid morphism from an arbitrary algebra to Bim(M), thereby elucidating the mechanism through which actions manifest. Finally, we undertake a comprehensive examination of the properties entailed by this action, shedding light on its intricacies and implications.

Throughout our discourse, we maintain R as a fixed commutative ring, anchoring our investigations within a well-defined mathematical framework. As we delve deeper into the intricacies of R-algebroids and their actions, we aim to uncover novel insights and forge connections that resonate across various mathematical domains.

Throughout this paper R will be a fixed commutative ring.

1.1. Preliminaries

Most of the following data can be found in [3–5].

Definition 1.1. An R-category is defined as a category in which each homset possesses an R-module structure, and the composition is R-bilinear. Consequently, a category earns the designation of an R-category only when it satisfies these conditions.

Specifically, a small R-category, termed as an R-algebroid, delineates a more specialized class within this framework. This classification is attributed to a category where homsets exhibit an R-module structure, composition is R-bilinear, and additionally, the category is small in size.

Definition 1.2. An R-linear functor, denoted as an R-functor, denotes a functorial mapping between two R-categories, preserving the R-module structures inherent in their homsets. This functor encapsulates the essence of R-linearity within the categorical framework.

Moreover, within the realm of R-algebroids, an R-functor between two such structures assumes the appellation of an R-algebroid morphism. This morphism elucidates the preservation of the algebraic structure, including R-linearity and compositionality, between the respective R-algebroids.

Definition 1.3. Let A be a pre-R-algebroid, and consider the family $I = \{I(x;y) \subseteq A(x;y) : x,y \in A_0\}$ of R-submodules. If $ab, ba' \in I$ for all $b \in I$, $a, a' \in A$ with $ta = sb$, $tb = sa'$, then I is denoted as a two-sided ideal of A.

Definition 1.4. Let A and N be two pre-R-algebroids sharing the same object set A_0. Consider a family of maps defined for all $x, y, z \in A_0$ as follows:

$$\begin{array}{ccc} N(x,y) \times A(y,z) & \longrightarrow & N(x,z) \\ (n,a) & \mapsto & n^a \end{array}$$

is called a right action of A on N if the conditions

$$\begin{array}{llll} 1. & n^{a_1+a_2} = n^{a_1} + n^{a_2} & 4. & (n'n) = n'n^a \\ 2. & (n_1+n_2)^a = n_1^a + n_2^a & 5. & r \centerdot n^a = (r \centerdot n)^a = n^{r \centerdot a} \\ 3. & (n^a)^{a'} = n^{aa'} & & \end{array}$$

and the condition $n^{1_{tn}} = n$, whenever 1_{tn} exists, are satisfied for all $r \in R,\ a, a', a_1, a_2 \in A,\ n, n', n_1, n_2 \in N$ with compatible sources and targets.

In a similar vein, a left action of A on N is established, albeit with a distinction in the side of application. Additionally, if A exhibits both a right and a left action on N, and if the actions conform to the condition $({}^a n)^{a'} = {}^a(n^{a'})$ for all $n \in N,\ a,\ a' \in A$ with $ta = sn$ and $tn = sa'$, where t denotes the target map and s denotes the source map, then A is termed to possess an associative action on N, or to act associatively on N.

Corollary 1.5. Given two pre-R-algebroids A and N with the same object set

i. if A has a left action on N then ${}^{0_{A(x,sn)}} n = 0_{A(x,tn)}$ and ${}^{-a}n = {}^a(-n) = -{}^a n,$

ii. if A has a right action on N then $n^{0_{A(tn,y)}} = 0_{A(sn,y)}$ and $n^{-a'} = (-n)^{a'} = -n^{a'}$

for all $n \in N, a, a' \in A, x, y \in A_0$ with $ta = sn, tn = sa'$.

Definition 1.6. Let M is an R-Algebroid, for all $m, m', m'' \in M$, with $t(m) = s(m')$ and $t(m'') = s(m)$

$$Ann_M M = \{m \in M : mm' = 0, m''m = 0, m', m'' \in M\}$$

is called Annihilator of M R-Algebroid.

Definition 1.7. [7] For R-algebroids A and M sharing the same object sets and with A exhibiting an associative action on M, an R-algebroid morphism $\eta : M \to A$ is termed a crossed module of R-algebroids if it satisfies the following conditions:

$$\begin{array}{ll} CM1. & \eta({}^a m) = a\eta(m) \\ & \eta(m^{a'}) = \eta(m)a' \\ CM2. & m^{\eta(m')} = mm' = {}^{\eta(m)} m' \end{array}$$

2. Bimultipliers of an R-algebroid

In this section, we commence our exploration by defining the bimultipliers of an R-algebroid M. Subsequently, we embark on a rigorous proof, establishing that the set of bimultipliers of M indeed forms an R-algebroid, which we aptly term the multiplication R-algebroid. This designation arises from the inherent structure and operations defined on this set, which align with the fundamental principles of R-algebroids.

Definition 2.1. Let M is an R-Algebroid and $f, g : M \to M$ be an R-Linear mappings with identity on object set satisfying the following equations for $m, m' \in M$ with $t(m) = s(m')$,

$$\begin{aligned} f(mm') &= mf(m') \\ g(mm') &= g(m)m' \\ f(m)m' &= mg(m') \end{aligned}$$

The pair (f, g) is called bimultipliers of M. Set of all bimultipliers of M are denoted by $Bim(M)$.

Theorem 2.2. Let $Bim(M)$ be a set of bimultipliers of M. $Bim(M)$ is an R-Algebroid with single object and with the following operations,

$$\begin{aligned} (f,g)+(f',g') &= (f+f', g+g') \\ (f,g)\circ(f',g') &= (f'\circ f, g\circ g') \\ r\cdot(f,g) &= (r\cdot f, r\cdot g) \end{aligned}$$

Proof.

$$\begin{aligned} r\cdot((f,g)+(f',g')) &= r\cdot(f+f', g+g') \\ &= (r\cdot f + r\cdot f', r\cdot g + r\cdot g') \\ &= r\cdot(f,g) + r\cdot(f',g') \end{aligned}$$

$$\begin{aligned} (r_1+r_2)\cdot(f,g) &= ((r_1+r_2)\cdot f, (r_1+r_2)\cdot g) \\ &= (r_1\cdot f + r_2\cdot f, r_1\cdot g + r_2\cdot g) \\ &= (r_1\cdot f, r_1\cdot g) + (r_2\cdot f, r_2\cdot g) \\ &= r_1\cdot(f,g) + r_2\cdot(f,g) \end{aligned}$$

$$\begin{aligned} (r_1r_2)\cdot(f,g) &= (r_1r_2\cdot f, r_1r_2\cdot g) \\ &= r_1(r_2\cdot f, r_2\cdot g) \\ &= r_1\cdot(r_2\cdot(f,g)) \end{aligned}$$

$$\begin{aligned} r\cdot(f,g)\circ(f',g') &= (r\cdot f, r\cdot g)\circ(f',g') \\ &= ((r\cdot f')\circ f, (r\cdot g)\circ g') \\ &= (r\cdot(f'\circ f), r\cdot(g\circ g')) \\ &= r\cdot(f'\circ f, g\circ g') \\ &= r\cdot((f,g)\circ(f',g')) \end{aligned}$$

$$\begin{aligned} (f,g)\circ r\cdot(f',g') &= (f,g)\circ(r\cdot f', r\cdot g') \\ &= ((r\cdot f')\circ f, g\circ(r\cdot g')) \\ &= (r\cdot(f'\circ f), r\cdot(g\circ g')) \\ &= r\cdot(f'\circ f, g\circ g') \\ &= r\cdot((f,g)\circ(f'\circ g')) \end{aligned}$$

In the realm of group theory, the characterization of an action is facilitated by the automorphism group. Specifically, for any group A, its action on itself is delineated by a homomorphism $A \to \mathrm{Aut}(A)$. However, in certain algebraic contexts, the mere structure of automorphisms proves insufficient to define an action. Unlike groups, the set of automorphisms of an algebra typically does not form an algebra itself.

In the study conducted by Arvasi and Ege [2], attention is directed towards the case of commutative algebras, where the limitations of the automorphism structure are explored. Furthermore, MacLane [1] delves into the realm of associative algebras, introducing the notion of the bimultiplication algebra $Bim(M)$ associated with an associative algebra M. This concept serves as an alternative to the automorphism group, effectively fulfilling the role of providing an action within the associative algebraic framework.

Definition 2.3. Let A and M be R-Algebroids with same object we define the set

$$M \,{}_t\overset{a}{\times}_s\, M = \{(m, m') \in M \times M : t(m) = s(a), t(a) = s(m')\}$$

for an $a \in A$.

Theorem 2.4. Let A and M be R-Algebroids with same object set and $Ann(M) = 0$ or $M^2 = M$. For the maps

$$\begin{aligned} f_a : \quad & M \to M \\ & m \mapsto f_a(m) = m^a \end{aligned}$$

and

$$\begin{aligned} g_a : \quad & M \to M \\ & m' \mapsto g_a(m') = {}^a m' \end{aligned}$$

for an $a \in A$ with $(m, m') \in M \,{}_t\overset{a}{\times}_s\, M$, let $(f_a, g_a) \in Bim(M)$. Then the R-Algebroid morphism

$$\begin{aligned} \phi : \quad & A \to Bim(M) \\ & a \mapsto \phi(a) = \phi_a = (f_a, g_a) \end{aligned}$$

gives an R-Algebroid action of A on M.

Proof.

(i) Since ϕ is an R-algebroid homomorphism, then

$$r \cdot \phi(a) = \phi(r \cdot a) \Rightarrow r \cdot \phi(a) = \phi(r \cdot a)$$

for $a \in A$ and

$$\begin{aligned} r \cdot \phi_a(m, m') &= r \cdot (f_a, g_a)(m, m') \\ &= r \cdot (f_a(m), g_a(m')) \\ &= (r \cdot f_a(m), r \cdot g_a(m')) \end{aligned}$$

$$\begin{aligned} \phi_{r \cdot a}(m, m') &= (f_{r \cdot a}, g_{r \cdot a})(m, m') \\ &= (f_{r \cdot a}(m), g_{r \cdot a}(m')) \end{aligned}$$

for $(m, m') \in M \,{}_t\overset{a}{\times}_s\, M$. Therefore we get

$$f_{r \cdot a}(m) = r \cdot f_a(m) \Rightarrow m^{r \cdot a} = r \cdot m^a$$

$$g_{r \cdot a}(m') = r \cdot g_a(m') \Rightarrow m'^{(r \cdot a)} = r \cdot (m'^a) = {}^{r \cdot a} m' = r \cdot {}^a m'.$$

(ii) Since ϕ is an R-Algebroid homomorphism, then

$$\phi(a_1+a_2)=\phi(a_1)+\phi(a_2)\Rightarrow \phi_{a_1+a_2}=\phi_{a_1}+\phi_{a_2}$$

for $a_1,a_2\in A$ with $s(a_1)=s(a_2), t(a_1)=t(a_2)$ and

$$\phi_{a_1+a_2}(m,m')=(f_{(a_1+a_2)},g_{(a_1+a_2)})(m,m')$$

$$\begin{aligned}\phi_{a_1}(m,m')+\phi_{a_2}(m,m') &= (f_{a_1}(m),g_{a_1}(m'))+(f_{a_2}(m),g_{a_2}(m'))\\ &= (f_{a_1}(m)+f_{a_2}(m),g_{a_1}(m')+g_{a_2}(m'))\end{aligned}$$

for $(m,m')\in M\ {}_t\overset{a}{\times}{}_s\ M$.

Therefore we get

$$f_{a_1+a_2}(m)=f_{a_1}(m)+f_{a_2}(m)\Rightarrow m^{a_1+a_2}=m^{a_1}+m^{a_2}$$

$$g_{a_1+a_2}(m')=g_{a_1}(m')+g_{a_2}(m')\Rightarrow {}^{a_1+a_2}m'={}^{a_1}m'+{}^{a_2}m'$$

.

(iii) Since $\phi_a=(f_a,g_a)\in Bim(M)$ for $a\in A$, then,

$$\phi_a((m_1,m_1')+(m_2,m_2'))=\phi_a(m_1,m_1')+\phi_a(m_2,m_2')$$

and

$$\begin{aligned}\phi_a((m_1,m_1')+(m_2,m_2')) &= \phi_a(m_1+m_2,m_1'+m_2')\\ &= (f_a(m_1+m_2),g_a(m_1'+m_2'))\\ &= ((m_1+m_2)^a,{}^a(m_1'+m_2')),\end{aligned}$$

$$\begin{aligned}\phi_a(m_1,m_1')+\phi_a(m_2,m_2') &= (f_a(m_1),g_a(m_1'))+(f_a(m_2),g_a(m_2'))\\ &= (m_1^a,{}^a m_1')+(m_2^a,{}^a m_2')\\ &= (m_1^a+m_2^a,{}^a m_1'+{}^a m_2')\end{aligned}$$

for $(m_1,m_1'),(m_2,m_2')\in M\ {}_t\overset{a}{\times}{}_s\ M$,$(s(m_1)=s(m_2))$ and $(t(m_1')=t(m_2'))$ therefore we get

$$(m_1+m_2)^a=m_1^a+m_2^a$$

and

$${}^a(m_1'+m_2')={}^a m_1'+{}^a m_2'.$$

(iv) Since $\phi_a=(f_a,g_a)\in Bim(M)$ for $a\in A$, then

$$\begin{aligned}\phi_a(m_1m_2,m_1'm_2') &= (f_a,g_a)(m_1m_2,m_1'm_2')\\ &= (f_a(m_1m_2),g_a(m_1'm_2'))\\ &= ((m_1m_2)^a,{}^a(m_1'm_2'))\end{aligned}$$

and

$$\begin{aligned}(f_a(m_1m_2),g_a(m_1'm_2')) &= (m_1f_a(m_2),g_a(m_1')m_2')\\ &= (m_1(m_2^a),(({}^a m_1')m_2'))\end{aligned}$$

for $(m_1 m_2, m_1' m_2') \in M \ {}_t\overset{a}{\times}_s M$ and $t(m_1) = s(m_2)$, $t(m_1') = s(m_2')$ therefore we get

$$m_1 m_2^a = m_1 (m_2)^a$$

and

$$^a m_1' m_2' = (^a m_1' m_2').$$

(v) Since ϕ is an R-Algebroid homomorphism, then

$$\phi_{aa'} = \phi_a \circ \phi_{a'}$$

$$\phi_{aa'} = (f_{aa'}, g_{aa'})$$

$$\begin{aligned} \phi_a \circ \phi_{a'} &= (f_a, g_a) \circ (f_{a'}, g_{a'}) \\ &= (f_{a'} \circ f_a, g_a \circ g_{a'}) \end{aligned}$$

for $a, a' \in A$ with $t(a) = s(a')$ and

$$\begin{aligned} \phi_{aa'}(m, m') &= (f_{aa'}, g_{aa'})(m, m') \\ &= (f_{aa'}(m), g_{aa'}(m')) \\ &= (m^{aa'}, {}^{aa'} m') \end{aligned}$$

$$\begin{aligned} (\phi_a \circ \phi_{a'})(m, m') &= (f_{a'} \circ f_a, g_a \circ g_{a'})(m, m') \\ &= ((f_{a'} \circ f_a)(m), (g_a \circ g_{a'})(m')) \\ &= (f_{a'}(f_a(m)), g_a(g_{a'}(m'))) \\ &= (f_{a'}(m^a), g_a({}^{a'} m')) \\ &= ((m^a)^{a'}, {}^a({}^{a'} m')) \end{aligned}$$

for $(m, m') \in M \ {}_t\overset{aa'}{\times}_s M$, therefore we get $m^{aa'} = (m^a)^{a'}$ and ${}^{aa'} m' = {}^a ({}^{a'} m')$.

Thus, $\phi : A \to Bim(M)$ R-Algebroid morphism induces an R-Algebroid action of A on M.

Definition 2.5. Let A be an R-Algebroid. For an R-Algebroid morphism

$$\begin{aligned} \phi : A &\to Bim(A) \\ a &\mapsto \phi(a) = (f_a, g_a) \end{aligned}$$

the pair $(f_a, g_a)(a', a'') = (f_a(a'), g_a(a'')) = (a'a, aa'')$ is called inner bimultipliers of A for $(a', a'') \in A \ {}_t\overset{a}{\times}_s A$. Set of all bimultipliers of A are denoted by $I(A)$ and $I(A) = Im(\phi)$.

Theorem 2.6. Let M be an R-Algebroid. The kernel of homomorphism

$$\begin{aligned} \phi : \quad & M \to Bim(M) \\ & m \mapsto \phi(m) = (f_m, g_m) \end{aligned}$$

is Annihilator of M.

Proof.

The annihilator of M is

$$Ann_M(M) = \{m \in M : f_m(m') = m'm = 0, g_m(m'') = m''m = 0, m', m'' \in M\}.$$

$$\begin{aligned} f_{m_1 m_2}(m') &= m'(m_1 m_2) \\ &= (m' m_1) m_2 \\ &= f_{m_2}(m' m_1) \\ &= f_{m_2}(f_{m_1}(m')) \\ &= (f_{m_2} \circ f_{m_1})(m') \end{aligned}$$

$$\begin{aligned} g_{m_1 m_2}(m'') &= (m_1 m_2)(m'') \\ &= m_1(m_2 m'') \\ &= g_{m_1}(m_2 m'') \\ &= g_{m_1}(g_{m_2}(m'')) \\ &= (g_{m_1} \circ g_{m_2})(m'') \end{aligned}$$

and

$$\begin{aligned} \phi_{m_1 m_2} &= (f_{m_1 m_2}, g_{m_1 m_2}) \\ &= (f_{m_2} \circ f_{m_1}, g_{m_1} \circ g_{m_2}) \\ &= (f_{m_1}, g_{m_1}) \circ (f_{m_2}, g_{m_2}) \\ &= (\phi_{m_1} \phi_{m_2}) \end{aligned}$$

for $(m', m'') \in M \ {}_t\overset{m_1 m_2}{\times}{}_s \ M$. Also

$$m \in Ker\phi \quad \Leftrightarrow \phi_m = (f_m, g_m) = (\mathbf{0}, \mathbf{0})$$

and

$$f_m(m') = \mathbf{0}, g_m(m'') = \mathbf{0} \Leftrightarrow m'm = 0, mm'' = 0 \Leftrightarrow m \in Ann_M(M)$$

for $(m', m'') \in M \ {}_t\overset{m}{\times}{}_s \ M$. Thus $Ker\phi = Ann_M(M)$.

Theorem 2.7. Let $I(M)$ be image of $\phi : M \to Bim(M)$ algebroid homomorphism. $I(M)$ is ideal of $Bim(M)$.

Proof.
For $(f_m, g_m) \in I(M)$ and $(f', g') \in Bim(M)$ and $(m', m'') \in M \ {}_t\overset{m}{\times}{}_s \ M$.

$$\begin{aligned} I(M) \times Bim(M) &\to I(M) \\ ((f_m, g_m), (f', g')) &\mapsto ((f_m, g_m) \circ (f', g')) = ((f' \circ f_m), (g_m \circ g')) \end{aligned}$$

$$\begin{aligned} f' f_m(m') &= f'(m'm) \\ &= m' f'(m) \\ &= f_{f'(m)}(m') \end{aligned}$$

$$\begin{aligned} g_m g'(m'') &= m g'(m'') \\ &= f_{g'(m'')}(m) \end{aligned}$$

and

$$\begin{aligned} Bim(M)\times I(M) &\to I(M)\\ ((f',g'),(f_m,g_m)) &\mapsto ((f',g')\circ(f_m,g_m)) = ((f_m\circ f'),(g'\circ g_m)) \end{aligned}$$

$$\begin{aligned} f_m f'(m') &= f'(m')m\\ &= g_{f'(m')}(m) \end{aligned}$$

$$\begin{aligned} g'g_m(m'') &= g'(mm'')\\ &= g'(m)m''\\ &= g_{g'(m)}(m'') \end{aligned}$$

Thus $I(M)$ is ideal of $Bim(M)$.

Definition 2.8. Let $I(M)$ be ideal of $Bim(M)$ algebroid,

$$O(M) = Bim(M)/I(M)$$

division algebroid is called the outer multiplication of M algebroid and denoted by $O(M)$.

Theorem 2.9. Let M be an R-Algebroid such that $Ann(M) = 0$ or $M^2 = M$ and

$$\begin{aligned} \eta : M &\to Bim(M)\\ m &\mapsto \eta(m) = (f_m, g_m) \end{aligned}$$

be an R-Algebroid morphism with the pair $(f_m, g_m)(m', m'') = (f_m(m'), g_m(m'')) = (m'm, mm'')$ for $(m', m'') \in M\ {}_t\overset{m}{\times}_s\ M$. Then $(M, Bim(M), \eta)$ is a crossed module.

Proof.

$Bim(M)$ acts on M by

$$\begin{aligned} Bim(M)\times M &\to M\\ ((f',g'),m') &\mapsto (f',g')\cdot m' = g'(m') \end{aligned}$$

and

$$\begin{aligned} M\times Bim(M) &\to M\\ (m'',(f',g')) &\mapsto m''\cdot(f',g') = f'(m'') \end{aligned}$$

for $(m', m'') \in M\ {}_t\overset{m}{\times}_s\ M$ and

$$\begin{aligned} f'_m : M &\to M\\ m' &\mapsto f'_m(m') = m'm \end{aligned}$$

and

$$\begin{aligned} g'_m : M &\to M\\ m'' &\mapsto g'_m(m'') = mm'' \end{aligned}$$

such that

$$\begin{aligned} \eta : M &\to Bim(M)\\ m &\mapsto \eta(m) = (f'_m, g'_m). \end{aligned}$$

CM1.

$$
\begin{aligned}
\eta((f',g')\cdot m)(m',m'') &= \eta(g'(m))(m',m'') \\
&= (f'_{g'(m)}, g'_{g'(m)})(m',m'') \\
&= (m'g'(m), g'(m)m'') \\
&= (f'(m')m, g'(mm'')) \\
&= (f'_m(f'(m')), g'(g'_m(m''))) \\
&= (f'_m f', g'g'_m)(m',m'') \\
&= (f',g')\circ(f'_m,g'_m)(m',m'')
\end{aligned}
$$

then

$$
\begin{aligned}
\eta((f',g')\cdot m) &= (f',g')\circ(f'_m,g'_m) \\
&= (f',g')\circ\eta(m)
\end{aligned}
$$

$$
\begin{aligned}
\eta(m\cdot(f',g'))(m',m'') &= \eta(f'(m))(m',m'') \\
&= (f'_{f'(m)}, g'_{f'(m)})(m',m'') \\
&= (m'f'(m), f'(m)m'') \\
&= (f'(m'm), mg'(m'')) \\
&= (f'(f'_m(m')), g'_m(g'(m''))) \\
&= (f'f'_m, g'_m g')(m',m'') \\
&= (f'_m,g'_m)\circ(f',g')(m',m'')
\end{aligned}
$$

then

$$
\begin{aligned}
\eta(m\cdot(f',g')) &= (f'_m,g'_m)\circ(f',g') \\
&= \eta(m)\circ(f',g')
\end{aligned}
$$

CM2.

$$
\begin{aligned}
\eta(m')\cdot m &= (f'_{m'}, g'_{m'}) \\
&= g'_{m'}(m) \\
&= m'm
\end{aligned}
$$

$$
\begin{aligned}
m'\cdot\eta(m) &= m'\cdot(f'_m,g'_m) \\
&= f'_m(m') \\
&= m'm
\end{aligned}
$$

Thus $(M, Bim(M), \eta)$ is a crossed module.

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Formulas for Bernoulli and Euler Numbers and Polynomials with the aid of Applications of Operators and Volkenborn Integral

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Keywords
Bernoulli numbers and polynomials, Euler numbers and polynomials, Fubini polynomials, Array polynomials, Generating function, Operators, p-adic integral, Special functions, Stirling numbers

Abstract — We study on applications of operators and the (p-adic) Volkenborn integral in order to investigate fundamental properties of the special numbers and polynomials. The aim of this article is to derive new formulas for these numbers and polynomials and finite sums by using operators and the Volkenborn integral. These formulas are related to the Stirling numbers, array polynomials, the Fubini-type polynomials and numbers, and also the Bernoulli and Euler numbers and polynomials. Moreover, in the light of our new formulas, we set new special number families with their generating functions, and give very important footnotes about their definitions and properties.

Subject Classification (2020): 12D10, 11B68, 11S80, 26C05, 26C10.

1. Introduction

Recently, examining the properties of polynomials with operator theory and deriving special numbers with the help of operators are among the trendy topics in mathematics. Because special numbers and polynomials are among the basic tools that can be easily applied in

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mathematical modeling problems used in problem solving. Especially the special numbers and polynomials have also been used in almost all areas of mathematics, and in all applied sciences (*cf.* [1]-[40]). Investigating formulas and finite sums for certain family of polynomials and numbers using operators and Volkenborn integral methods also form the basis of the motivation of this article.

We use the following basic standard notations and definitions:

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

$\mathbb{C}$ denotes a set of complex numbers,

$$0^n = \begin{cases} 1, & (n = 0) \\ 0, & (n \in \mathbb{N}) \end{cases}$$

and

$$\binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{v} = \frac{\lambda(\lambda-1)\cdots(\lambda-v+1)}{v!} = \frac{(\lambda)^{(v)}}{v!},$$

where $v \in \mathbb{N}$, $\lambda \in \mathbb{C}$ (see [1]-[40]).

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.1}$$

where $|t| < 2\pi$ and when $x = 0$, we have $B_n := B_n(0)$ denotes the Bernoulli numbers (see [1]-[40]).

The Euler numbers are defined by

$$h(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

where $|t| < \pi$ (see [1]-[40]).

The Euler polynomials are defined by

$$g(t, x) = h(t) e^{tx} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.2}$$

which satisfies $E_n := E_n(0)$ (see [1]-[40]).

The Stirling numbers of the second kind $S_2(n, k)$ are defined by means of the following generating function:

$$F_s(t,k) = \frac{\left(e^t-1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}, \tag{1.3}$$

which satisfies

$$S_2(n,k) = 0$$

if $n < k$ or $k < 0$ and $k \in \mathbb{N}_0$ (see [1]-[40]).

By combining (1.2) and (1.3), assuming $\left|e^t - 1\right| < 1$, we reach the following functional equation:

$$g(t,x) = h(t)\sum_{m=0}^{\infty}\binom{x}{m} m!F_s(t,m) \tag{1.4}$$

and

$$e^{t(x+v)} = \sum_{m=0}^{\infty}\binom{x+v}{m} m!F_s(t,m). \tag{1.5}$$

By using Eq. (1.4), we obtain

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v}E_{n-v}\sum_{m=0}^{v}\binom{x}{m} m!S_2(v,m)\frac{t^n}{n!}.$$

By equalizing the coefficients of $\frac{t^n}{n!}$ found on both sides of the previous equation, we reach the proof of the following theorem:

Theorem 1.1. Let $n \in \mathbb{N}_0$. Then we have

$$E_n(x) = \sum_{v=0}^{n}\binom{n}{v}E_{n-v}\sum_{m=0}^{v}\binom{x}{m} m!S_2(v,m).$$

By using Eq. (1.5), we obtain

$$\sum_{n=0}^{\infty}(x+v)^n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{x+v}{m} m!S_2(n,m)\frac{t^n}{n!}.$$

By equalizing the coefficients of $\frac{t^n}{n!}$ found on both sides of the previous equation, we reach the proof of the following theorem:

Theorem 1.2. Let $n, v \in \mathbb{N}_0$. Then we have

$$(x+v)^n = \sum_{m=0}^{n}\binom{x+v}{m} m!S_2(n,m). \tag{1.6}$$

When $v = 0$, Eq. (1.6) reduces to

$$x^n = \sum_{m=0}^{n} \binom{x}{m} m! S_2(n,m) \tag{1.7}$$

(*cf.* [6, 27, 28, 39]).

The λ-array polynomials $S_k^n(x;\lambda)$ are defined by means of the following generating function:

$$\frac{1}{k!} e^{tx} \left(\lambda e^t - 1\right)^k = \sum_{n=0}^{\infty} S_k^n(x;\lambda) \frac{t^n}{n!} \tag{1.8}$$

(see [1, 28]).

Substituting $\lambda = 1$ into (1.8), we have

$$S_k^n(x) := S_k^n(x;1) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(x+j\right)^n \tag{1.9}$$

with

$$S_0^0(x) = S_n^n(x) = 1, S_0^n(x) = x^n.$$

If $k > n$, then

$$S_k^n(x) = 0$$

(see [1, 3, 28]; and also the references cited therein).

The Fubini-type numbers and polynomials of order k are defined, respectively, by

$$\left(\frac{2}{2-e^t}\right)^k = \sum_{n=0}^{\infty} a_n^{(k)} \frac{t^n}{n!} \tag{1.10}$$

and

$$\left(\frac{2}{2-e^t}\right)^k e^{xt} = \sum_{n=0}^{\infty} a_n^{(k)}(x) \frac{t^n}{n!} \tag{1.11}$$

which satisfies $a_n^{(k)} := a_n^{(k)}(0)$ (see [9]; and also [8, 10, 12, 13, 36]).

When $k = 1$ in (1.10), we have

$$a_n := a_n^{(1)}$$

and

$$a_n = 2\sum_{j=0}^{n} \binom{n}{j} w_g(j) w_g(n-j),$$

where $w_g(n)$ denote the Fubini numbers which are defined by

$$\frac{1}{2-e^t}=\sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!} \tag{1.12}$$

(see [4]; and also [8–10, 12, 13, 36]).

Using (1.3) and (1.12), we have the following well-known relation [4]:

$$w_g(n)=\sum_{j=0}^{n} j!S_2(n,j).$$

From (1.11) and (1.3), Kilar and Simsek [9] gave the following formula:

$$x^n=2^{-k}\sum_{r=0}^{n}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\binom{n}{r}j!S_2(r,j)a_{n-r}^{(k)}(x). \tag{1.13}$$

1.1. The operators $\mathscr{O}_\lambda\left[f;a,b\right]$ and $T_\lambda\left[f;a,b\right]$

Let

$$E^a\left[f\right](x)=f(x+a),$$

(see [1, 18, 23, 37]). We [30] gave the following operator $\mathscr{O}_\lambda\left[f;a,b\right]$ for real parameters λ, a and b:

$$\mathscr{O}_\lambda\left[f;a,b\right](x)=\lambda E^a\left[f\right](x)+E^b\left[f\right](x), \tag{1.14}$$

where $x\in\mathbb{R}$ and

$$T_\lambda\left[f;a,b\right](x)=\frac{\mathscr{O}_\lambda\left[f;a,b\right](x)}{a+b+1}. \tag{1.15}$$

We [30] showed that

$$\begin{aligned}
\frac{1}{2}T_1\left[f;0,0\right](x) &= I\left[f\right](x), \text{ (Identity Operator)}\\
-2T_{-1}\left[f;1,0\right](x) &= \Delta\left[f\right](x), \text{ (Forward Difference Operator)}\\
I\left[f\right](x)+\frac{1}{2}T_1\left[f;-1,-1\right](x) &= \nabla\left[f\right](x), \text{ (Backward Difference Operator)}\\
T_1\left[f;1,0\right](x) &= M\left[f\right](x), \text{ (Means Operator)}\\
-T_{-1}\left[f;\frac{1}{2},-\frac{1}{2}\right](x) &= \delta\left[f\right](x), \text{ (Central Difference Operator)}\\
\frac{1}{2}T_1\left[f;\frac{1}{2},-\frac{1}{2}\right](x) &= \mu\left[f\right](x), \text{ (Averaging Difference Operator)}
\end{aligned}$$

and also

$$\begin{aligned} -(2a+b+1)T_{-1}\left[f;a+b,a\right](x) &= \Delta_b E^a\left[f\right](x), (a \neq b, \text{ Gould Operator}) \\ -2T_{-\lambda}\left[f;1,0\right](x) &= \Delta_\lambda\left[f\right](x). \end{aligned}$$

For details about the above operators and their applications, see [30] and also [38].

We [32] modified the operators $\mathcal{O}_\lambda\left[f;a,b\right]$ and $T_\lambda\left[f;a,b\right]$ as follows:

$$\mathbb{Y}_{\lambda,\beta}\left[f;a,b\right](x) = \lambda E^a\left[f\right](x) + \beta E^b\left[f\right](x) \tag{1.16}$$

and

$$\mathbb{Y}_{\lambda,\beta}\left[f;a,b\right](x) = \beta\mathcal{O}_{\frac{\lambda}{\beta}}\left[f;a,b\right](x) = \beta(a+b+1)\, T_{\frac{\lambda}{\beta}}\left[f;a,b\right](x),$$

where λ and β are complex or real parameters, a and b are real parameters.

We [32] showed that

$$\mathbb{Y}_{-\lambda,1}\left[f;1,0\right](x) = -\Delta_\lambda\left[f\right](x)$$

(see also [1]),

$$E^a\left[f\right](x) = \mathbb{Y}_{1,0}\left[f;a,0\right](x)$$

and

$$\Delta_a\left[f\right](x) = \mathbb{Y}_{1,-1}\left[f;a,0\right](x),$$

where Δ_a denotes the forward difference operator,

$$\nabla_{-b}\left[f\right] = \mathbb{Y}_{1,-1}\left[f;0,-b\right],$$

which yields

$$\mathbb{Y}_{1,-1}\left[f;b,0\right]\mathbb{Y}_{1,0}\left[f;-b,0\right] = \mathbb{Y}_{1,-1}\left[f;b,0\right]\mathbb{Y}_{1,0}\left[f;0,-b\right],$$

where ∇_{-b} denotes the backward difference operator.

$$\delta_a\left[f\right] = \mathbb{Y}_{1,-1}\left[f;\frac{a}{2},-\frac{a}{2}\right],$$

where δ_a denotes the central difference operator. The Gould operator

$$G_{a,b}\left[f\right] = \mathbb{Y}_{1,0}\left[f;a+b,0\right] - \mathbb{Y}_{1,0}\left[f;a,0\right],$$

where $a \neq b$. Let $k \in \mathbb{N}$. With the aid of (1.14), we [32] also showed that

$$\mathbb{Y}_{\lambda,\beta}^{k}\left[f;a,b\right](x)=\sum_{j=0}^{k}\binom{k}{j}\lambda^{k-j}\beta^{j}f(x+jb+(k-j)a). \tag{1.17}$$

Putting $b=0$ and $\beta=-1$ in (1.17), we have

$$\mathbb{Y}_{\lambda,-1}^{k}\left[f;1,0\right](x)=\sum_{j=0}^{k}\binom{k}{j}\lambda^{k-j}(-1)^{j}f(x+(k-j)a)=\Delta_{\lambda}^{k}\left[f\right](x)$$

(see [1, p. 155, Eq. (29)], [32]).

Putting $b=0$ and $\beta=-1$ in the above equation, we have

$$\begin{aligned}\mathbb{Y}_{\lambda,-1}^{k}\left[x^{n};1,0\right](x) &= \Delta_{\lambda}^{k}\left[x^{n}\right](x)\\ &= S_{k}^{n}(x,\lambda)\end{aligned}$$

(*cf.* [1, p. 155], [32]).

Therefore,

$$S_{k}^{n}(x)=\frac{1}{k!}\Delta^{k}\left[x^{n}\right]$$

(*cf.* [1, p. 155], [3], [32]).

The results of this article are briefly summarized for the reader as follows, section by section.

In Section 2, some basic properties of the Euler polynomials are given with the help of operators. We also give formulas for the Fubini-type polynomials, the Stirling numbers of the second kind and the Euler polynomials.

In Section 3, we derive some formulas, identities and finite sums for the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the array polynomials, and the Stirling numbers of the second kind with the aid of operators and Volkenborn integrals.

In Section 4, we give a conclusion section.

2. Formulas for Euler Polynomials in terms of Operators

The purpose of this section is to study the Euler polynomials with the help of operators and to provide an introductory discussion of some of their properties and applications. Here, we note that the operators $T_{\lambda}\left[f;a,b\right]$ and derivative operator D action the variable x (see [22, p.

406]). Using the averaging operator

$$M[f] = T_1[f;1,0] = \frac{E+I}{2}[f],$$

we have

$$T_1[E_n(x);1,0](x) = x^n, \tag{2.1}$$

which satisfies

$$\frac{E_n(x+1)+E_n(x)}{2} = x^n$$

and

$$E_n(x) \quad = \quad T_1^{-1}[x^n;1,0](x).$$

Thus, we see that

$$E_n(x) = \sum_{j=0}^{\infty} \left(T_{-1}[x^n;1,0](x)\right)^j.$$

For $j > n$,

$$\Delta^j\{x^n\} = 0,$$

the Euler polynomials are given by

$$E_n(x) = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \Delta^j\{x^n\} \tag{2.2}$$

(see [22, p. 406]).

Applying derivative operator D to the equation (2.1) yields

$$D[M[E_n(x)]] = D\{x^n\}.$$

Therefore

$$D\left\{\frac{E_n(x+1)+E_n(x)}{2}\right\} = nx^{n-1}.$$

Combining the above equation with the following derivative formula for the Euler polynomials, which are members of Appell polynomials,

$$D\{E_n(x)\} = nE_{n-1}(x),$$

we get

$$\frac{E_{n-1}(x+1)+E_{n-1}(x)}{2}=x^{n-1}.$$

Thus we get

$$D\{M[E_n(x)]\}=M[E_{n-1}(x)].$$

From the above equation, we get

$$M^{-1}[D\{M[E_n(x)]\}]=E_{n-1}(x).$$

Hence

$$D\{E_n(x)\}=\frac{M^{-1}[D\{M[E_n(x)]\}]}{n},$$

and

$$D^k\{E_n(x)\}=\begin{cases}(n)^{(k)}E_{n-k}(x), & 1\le k<n\\ k!, & n=k\\ 0, & n<k\end{cases}$$

(see [22, p. 406]).

Combining (1.13) with (2.2), we have the following result:

Corollary 2.1. Let $k,n\in\mathbb{N}_0$. Then we have

$$T_1[E_n(x);1,0](x)=2^{-k}\sum_{r=0}^{n}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\binom{n}{r}j!S_2(r,j)a_{n-r}^{(k)}(x).$$

or, equivalently,

$$E_n(x+1)+E_n(x)=2^{-k+1}\sum_{r=0}^{n}\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\binom{n}{r}j!S_2(r,j)a_{n-r}^{(k)}(x).$$

3. Formulas for the Bernoulli and Euler Numbers and Polynomials with the aid of Operators and Volkenborn Integrals

The purpose of this section is to derive formulas, finite sums and relations involving the Bernoulli and Euler numbers and polynomials, and the Stirling numbers using operators and applications of the Volkenborn integral.

Before giving the essential formulas of this section, the following some properties of the Volkenborn integral are given with a very brief introduction.

Let $\mathbb{Z}_p$ be a set of p-adic integers. Let $f:\mathbb{Z}_p \to \mathbb{C}_p$, where $\mathbb{C}_p$ is a field of p-adic completion of algebraic closure of set of p-adic rational numbers. f is called a uniformly differential function at a point $a \in \mathbb{Z}_p$ if f satisfies the following conditions:

If the difference quotients $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$ such that

$$\Phi_f(x,y) = \frac{f(x)-f(y)}{x-y}$$

have a limit $f'(z)$ as $(x,y) \to (0,0)$ (with $x \neq y$). A set of uniformly differential functions is briefly indicated by $f \in UD(\mathbb{Z}_p)$ or $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$.

The Volkenborn integral of the uniformly differential function f is given as follows:

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{3.1}$$

where $\mu_1(x)$ denote the Haar distribution, given by

$$\mu_1(x) = \frac{1}{p^N}$$

(see [7, 15, 17, 21, 25, 31, 34, 40]).

Let $n \in \mathbb{N}_0$. Some examples for p-adic integrals are given as follows:

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x) \tag{3.2}$$

and

$$B_n(y) = \int_{\mathbb{Z}_p} \left(x+y\right)^n d\mu_1(x), \tag{3.3}$$

where B_n and $B_n(y)$ denote the Bernoulli numbers and the Bernoulli polynomials, respectively (see [7, 15, 16, 21, 25, 31, 34]).

By applying the Volkenborn integral to the Eq. (2.2), we obtain

$$\int_{\mathbb{Z}_p} E_n(x) d\mu_1(x) = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \int_{\mathbb{Z}_p} \Delta^j \left\{x^n\right\} d\mu_1(x).$$

Combining the above equation with the following well-known formulas

$$\Delta = E - I$$

and

$$\Delta^j = \sum_{v=0}^{j} (-1)^v \binom{j}{v} E^v,$$

we get

$$\int_{\mathbb{Z}_p} E_n(x) d\mu_1(x) = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^v \binom{j}{v} \int_{\mathbb{Z}_p} (x+v)^n d\mu_1(x).$$

Combining the above equation with (3.3) yields the following theorem:

Theorem 3.1. Let $n \in \mathbb{N}_0$. Then we have

$$\int_{\mathbb{Z}_p} E_n(x) d\mu_1(x) = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^v \binom{j}{v} B_n(v). \tag{3.4}$$

By combining (3.4) with the following known formula:

$$E_n(x) = \sum_{v=0}^{n} \binom{n}{v} x^{n-v} E_v,$$

we also get

$$\sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^v \binom{j}{v} B_n(v) = \sum_{v=0}^{n} \binom{n}{v} E_v \int_{\mathbb{Z}_p} x^{n-v} d\mu_1(x).$$

Combining the above equation with (3.2), we arrive at the following theorem:

Theorem 3.2. Let $n \in \mathbb{N}_0$. Then we have

$$(B+E)^n = \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^v \binom{j}{v} B_n(v),$$

where

$$(B+E)^n = \sum_{v=0}^{n} \binom{n}{v} E_v B_{n-v}$$

and after applying binomial expansion, each index of B^n and E^n are to be replaced by the corresponding suffix: B_n and E_n, respectively.

By applying the Volkenborn integral to the Eq. (1.6), we get

$$\int_{\mathbb{Z}_p} (x+v)^n \, d\mu_1(x) = \sum_{m=0}^{n} m! S_2(n,m) \int_{\mathbb{Z}_p} \binom{x+v}{m} d\mu_1(x).$$

Combining the left-hand side of the above equation with (3.3), we obtain

$$B_n(v) = \sum_{m=0}^{n} m! S_2(n,m) \int_{\mathbb{Z}_p} \binom{x+v}{m} d\mu_1(x).$$

Combining the right-hand side of the above equation with the following formula

$$\int_{\mathbb{Z}_p} \binom{x+v}{m} d\mu_1(x) = \sum_{k=0}^{m} (-1)^k \binom{v}{m-k} \frac{1}{k+1}$$

(see [34, p. 21]), we arrive at the following theorem:

Theorem 3.3. Let $n, v \in \mathbb{N}_0$. Then we have

$$B_n(v) = \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^k \binom{v}{m-k} \frac{m! S_2(n,m)}{k+1}. \tag{3.5}$$

Combining (3.4) with (3.5), we also arrive at the following theorem:

Theorem 3.4. Let $n \in \mathbb{N}_0$. Then we have

$$\begin{aligned} \int_{\mathbb{Z}_p} E_n(x) d\mu_1(x) &= \sum_{j=0}^{n} \frac{(-1)^j}{2^j} \sum_{v=0}^{j} (-1)^v \binom{j}{v} \\ &\quad \times \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^k \binom{v}{m-k} \frac{m! S_2(n,m)}{k+1}. \end{aligned}$$

By using (1.9) and (1.8), we have (*cf.* [28]):

$$\begin{aligned} S_k^n(x) &= \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n \\ &= \sum_{j=0}^{n} \binom{n}{j} S_2(j,k) x^{n-j}. \end{aligned}$$

By applying the Volkenborn integral to the above equation, we get

$$\begin{aligned} & \sum_{j=0}^{n} \binom{n}{j} S_2(j,k) \int_{\mathbb{Z}_p} x^{n-j} d\mu_1(x) \\ = & \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \int_{\mathbb{Z}_p} (x+j)^n d\mu_1(x). \end{aligned}$$

Combining the above equation with (3.3), we obtain the following theorem:

Theorem 3.5. Let $n, k \in \mathbb{N}_0$. Then we have

$$\sum_{j=0}^{n} \binom{n}{j} S_2(j,k) B_{n-j} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} B_n(j). \tag{3.6}$$

Here we note that using (3.6), we set the following sequences of numbers:

$$Y_{10}(n,k) = \sum_{j=0}^{n} \binom{n}{j} S_2(j,k) B_{n-j} \tag{3.7}$$

and

$$Y_{11}(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} B_n(j). \tag{3.8}$$

Thus, generating function for the numbers $Y_{10}(n,k)$ is defined by

$$F(t) = \sum_{n=0}^{\infty} Y_{10}(n,k) \frac{t^n}{n!} \tag{3.9}$$

and generating function for the numbers $Y_{11}(n,k)$ is defined by

$$G(t) = \sum_{n=0}^{\infty} Y_{11}(n,k) \frac{t^n}{n!}. \tag{3.10}$$

Examination of the fundamental properties of the functions $F(t)$ and $G(t)$ is left to the reader. With the help of these functions, interesting and applicable results can be derived by examining the fundamental properties of the numbers $Y_{10}(n,k)$ and $Y_{11}(n,k)$.

Let us end our article with guiding tips by giving the reader a brief introduction about the functions $F(t)$ and $G(t)$.

Observe that

$$F(t) = \frac{t(e^t - 1)^{k-1}}{k!}, \tag{3.11}$$

where k is a positive integer. By using the above function, we get

$$\sum_{n=0}^{\infty} Y_{10}(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty} S_2(n,k-1)\frac{t^{n+1}}{kn!} \tag{3.12}$$

By equalizing the coefficients of $\frac{t^n}{n!}$ found on both sides of the previous equation, we reach the proof of the following theorem:

Theorem 3.6. Let $n,k \in \mathbb{N}$. Then we have

$$Y_{10}(n,k) = \frac{n}{k} S_2(n-1,k-1). \tag{3.13}$$

Thus, by combining (3.6) and (3.7) with (3.13), we also have the following result:

Theorem 3.7. Let $n,k \in \mathbb{N}$. Then we have

$$S_2(n-1,k-1) = \frac{k}{n}\sum_{j=0}^{n}\binom{n}{j} S_2(j,k)B_{n-j}. \tag{3.14}$$

With the help of similar operations and methods above, new and applicable formulas can be achieved by performing the function $G(t)$ and the numbers $Y_{11}(n,k)$.

4. Conclusions

We gave generating functions for the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Fubini-type polynomials, and the Stirling numbers. We also gave some properties of the operator. Some properties of the Euler polynomials were examined with the aid of operators.

By using operators and the Volkenborn integrals, we derived some formulas, identities and finite sums involving the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Fubini numbers and polynomials, the array polynomials, and Stirling numbers.

With the help of Theorem 3.5, we set new special number families with their generating functions, and gave very important footnotes about their definitions and properties.

We think that these formulas will have the potential to be used in mathematics, mathematical physics, and engineering.

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