UJMA

Universal Journal of Mathematics and Applications

VOLUME VII
ISSUE I

ISSN 2619-9653

http://dergipark.gov.tr/ujma

VOLUME VII ISSUE I
ISSN 2619-9653

March 2024
http://dergipark.gov.tr/ujma

UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

Editor in Chief

Managing Editors

Editorial Board of Universal Journal of Mathematics and Applications

Statistics Editor

Technical Editor

Contents

Research Article

Universal Journal of Mathematics and Applications, 7 (1) (2024) 1-11
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1381621

Miao-Tam Equation and Ricci Solitons on Three-Dimensional Trans-Sasakian Generalized Sasakian Space-Forms

Uday Chand De[1], Tarak Mandal[2] and Avijit Sarkar[2*]

[1]*Department of Pure Mathematics, University of Calcutta, 35 Ballygunje Circular Road, Kolkata 700019, India*
[2]*Department of Mathematics, University of Kalyani, Kalyani 741235, West Bengal, India*

Article Info

Keywords: *Generalized Sasakian space-forms, Gradient Ricci solitons, Miao-Tam equation, Sasakian space-forms, Trans-Sasakian manifolds*
2010 AMS: *53C15, 53D25*
Received: *26 October 2023*
Accepted: *24 December 2023*
Available online: *9 January 2024*

Abstract

The aim of the present article is to characterize some properties of the Miao-Tam equation on three-dimensional generalized Sasakian space-forms with trans-Sasakian structures. It has been proved that in such space-forms if the Miao-Tam equation admits non-trivial solution, then the metric of the space form must be a gradient Ricci soliton. We have derived that a non-trivial solution of the Fischer-Marsden equation does not exist on the said space-forms. We have also investigated certain features of Ricci solitons and gradient Ricci solitons. At the end of the article, we construct an example to verify the obtained results

1. Introduction

Miao-Tam equation on f-cosymplectic manifolds was investigated by X. Chen [1]. He proved that under certain restrictions such a manifold is either locally the product of a Kähler manifold and an interval or a unit circle, or, the manifold is of constant scalar curvature. He also established that if the manifold is connected and satisfies the Miao-Tam equation, then the manifold is Einstein under certain conditions. Since an Einstein manifold or a manifold of constant curvature is model of some interesting physical systems, geometers are naturally motivated to find the conditions under which a manifold will be Einstein or, a manifold of constant scalar curvature. To this end we study Miao-Tam equation on generalized Sasakian space-forms with trans-Sasakian structure and established that if a generalized Sasakian space-form with trans-Sasakian structure admits a non-trivial solution of the Miao-Tam equation, then the scalar curvature is constant and the manifold is Einstein or the structure is β-Kenmotsu. Several researchers [2–10] have investigated the Miao-Tam equation for some classes of contact manifolds.
Let $(M^n, g), n>2$ be a compact orientable Riemannian manifold with a smooth boundary ∂M and $\lambda : M^n \to \mathbb{R}$ be a smooth function on the manifold. Then the Miao-Tam equation on M^n is given by

$$Hess\lambda = (\Delta\lambda)g + \lambda S + g, \tag{1.1}$$

on M and $\lambda = 0$ on ∂M, $Hess$, Δ being respectively the Hessian operator and Laplacian with respect to the metric g. S indicates the Ricci curvature and λ indicates the potential function. The metrics satisfying the equation (1.1) are known as Miao-Tam critical metrics [11].
A sub-class of the Miao-Tam equation is the Fischer-Marsden equation which is given by

$$Hess\lambda = (\Delta\lambda)g + \lambda S.$$

The Fischer-Marsden equation (FME, in short) was constructed by A.E. Fischer and J. Marsden in [12]. The authors [12] in their paper conjectured that a compact Riemannian manifold that admits a non-trivial solution of the FME is necessarily Einstein. This statement is known as Fischer-Marsden conjecture. Later Kobayashi [13] pointed out that the said conjecture is not true in general. They are valid only in some special cases. After that a huge number of works has been done to analyze Fischer-Marsden conjecture on Riemannian manifolds admitting several structures.
R. S. Hamilton [14] introduced the notion of the Ricci flow in 1988. On a Riemannian or semi-Riemannian manifold,

$$\frac{\partial g}{\partial t} + 2S = 0$$

***Email addresses and ORCID numbers*: uc_de@yahoo.com, 0000-0002-8990-4609 (U. C. De), mathtarak@gmail.com, 0000-0003-4808-8454 (T. Mandal), avjaj@yahoo.co.in, 0000-0002-7370-1698 (A. Sarkar)**
Cite as: U. C. De, T. Mandal, A. Sarkar, *Miao-Tam Equation and Ricci Solitons on Three-Dimensional Trans-Sasakian Generalized Sasakian Space-Forms*, Univers. J. Math. Appl., 7(1) (2024), 1-11.

denotes the Ricci flow equation. A self-similar solution of the above equation is called the Ricci soliton and the soliton equation is given by

$$\pounds_V g + 2S + 2\psi g = 0, \tag{1.2}$$

$\pounds$ denotes the Lie-derivative operator. Here V is called the potential vector field and ψ is the soliton constant. If the sign of ψ is positive then the soliton is known as expanding and for the cases where ψ is zero or negative, the soliton is steady or shrinking, respectively. For details about Ricci solitons see the articles [15–18]. If the potential vector field V is the gradient of a smooth function ζ, then it is called the gradient Ricci soliton. Thus the gradient Ricci soliton is given by

$$Hess(\zeta) + S + \psi g = 0, \tag{1.3}$$

here $Hess$ is the Hessian operator.
The theory of generalized Sasakian space-forms came into existence after the work of Alegre et al. [19]. A generalized Sasakian spce-form (GSSF, in short) is such a manifold whose Riemann curvature R is given by

$$R(V_1,V_2)V_3 = f_1 R_1(V_1,V_2)V_3 + f_2 R_2(V_1,V_2)V_3 + f_3 R_3(V_1,V_2)V_3, \tag{1.4}$$

f_1, f_2 and f_3 are smooth functions on M and

$$R_1(V_1,V_2)V_3 = g(V_2,V_3)V_1 - g(V_1,V_3)V_2,$$
$$R_2(V_1,V_2)V_3 = g(V_1,\phi V_3)\phi V_2 - g(V_2,\phi V_3)\phi V_1 + 2g(V_1,\phi V_2)\phi V_3,$$
$$R_3(V_1,V_2)V_3 = \eta(V_1)\eta(V_3)V_2 - \eta(V_2)\eta(V_3)V_1 + g(V_1,V_3)\eta(V_2)\xi - g(V_2,V_3)\eta(V_1)\xi.$$

Such a manifold admitting different almost contact structures like Sasakian, K-contact, trans-Sasakian, etc. was analyzed by Alegre and Carriazo. GSSF is now drawing attention of several geometers. In [20], it is proved that any GSSF with dimension greater than or equal to five must be Sasakian-space-form. It is also proved in the same article that a K-contact GSSF is a Sasakian manifold. For more details we cite the papers [21–25].
The present paper is organized as follows: After the introduction, we give some preliminaries in the Section 2. In Section 3, we have studied Miao-Tam equation on three dimensional GSSFs with trans-Sasakian structure. In the same section we have proved that if a non-trivial solution of the Miao-Tam equation exists then the metric must be a gradient Ricci soliton and non-existences of the non-trivial solution of the Fischer-Marsden equation is also deduced. In the next section, we have derived some new results of Ricci solitons and gradient Ricci solitons on the same space-forms. In the last section, we give an example to verify the deduced results.

2. Preliminaries

A smooth manifold M^{2n+1} is known as an almost contact manifold (ACM) if there exists a structure (ϕ,θ,η), where ϕ, θ and η are, respectively, a $(1,1)$-tensor field, a $(1,0)$ type vector field and a 1-form, such that

$$\phi^2 V_1 = -V_1 + \eta(V_1)\theta, \quad \eta(\theta) = 1, \quad \phi\theta = 0, \quad \eta.\phi = 0 \quad \text{rank}(\phi) = 2n,$$

for every vector field V_1 on M^{2n+1} [26, 27].
An ACM M^{2n+1} is called an almost contact metric manifold (ACMM) if it admits a Riemannian metric g such that

$$g(\phi V_1,\phi V_2) = g(V_1,V_2) - \eta(V_1)\eta(V_2), \tag{2.1}$$

for every vector fields V_1, V_2 on M^{2n+1}. Equation (2.1) gives

$$g(\phi V_1,V_2) = -g(V_1,\phi V_2).$$

An ACMM is called a contact metric manifold if there exists a 2-form Φ such that $d\eta = \Phi$, where $\Phi(V_1,V_2) = g(V_1,\phi V_2)$. An ACMM is called normal if Nijenhuis torsion tensor $[\phi,\phi](V_1,V_2) + 2d\eta(V_1,V_2)\theta$ vanishes, where $[\phi,\phi](V_1,V_2) = \phi^2[V_1,V_2] + [\phi V_1,\phi V_2] - \phi[\phi V_1,V_2] - \phi[V_1,\phi V_2]$. A normal contact metric manifold is called a Sasakian manifold. An ACMM is called a trans-Sasakian manifold [28] if there exist two smooth functions α and β such that

$$(\nabla_{V_1}\phi)V_2 = \alpha(g(V_1,V_2)\theta - \eta(V_2)V_1) + \beta(g(\phi V_1,V_2)\theta - \eta(V_2)\phi V_1), \tag{2.2}$$

for every vector fields V_1, V_2 on M^{2n+1}. Actually, trans-Sasakian manifolds are the generalizations of Sasakian manifolds and Kenmotsu manifolds, that means, if $\beta = 0$ (res. $\alpha = 0$) then the manifold reduces to α-Sasakian (res. β-Kenmotsu) manifold. For more details please follow the articles [29–33]. From equation (2.2), one can obtain

$$\nabla_{V_1}\theta = -\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta). \tag{2.3}$$

In view of (1.4), we have

$$S(V_2,V_3) = (2f_1 + 3f_2 - f_3)g(V_2,V_3) - (3f_2 + f_3)\eta(V_2)\eta(V_3), \tag{2.4}$$

which gives

$$QV_2 = (2f_1 + 3f_2 - f_3)V_2 - (3f_2 + f_3)\eta(V_2)\theta, \tag{2.5}$$

Q is the Ricci operator. Again, contracting V_2 in the foregoing equation, we get the scalar curvature as

$$r = 2(3f_1 + 3f_2 - 2f_3). \tag{2.6}$$

Lemma 2.1. *For a trans-Sasakian GSSF M, the following relation holds:*

$$f_1 - f_3 + \theta(\alpha) + \theta(\beta) - \alpha^2 + \beta^2 = 0. \tag{2.7}$$

Proof. According to the equations (2.2) and (2.3), we obtain

$$R(V_1,\theta)\theta = (\theta(\alpha) + \alpha\beta)\phi V_1 + (-\theta(\beta) - \beta^2 + \alpha^2 + \alpha\beta)(V_1 - \eta(V_1)\theta). \tag{2.8}$$

On the other hand, from equation (1.4), it can be easily seen that

$$R(V_1,\theta)\theta = (f_1 - f_3)(V_1 - \eta(V_1)\theta). \tag{2.9}$$

Comparing (2.8) and (2.9), we have

$$\theta(\alpha) + \alpha\beta = 0$$

and

$$-\theta(\beta) - \beta^2 + \alpha^2 + \alpha\beta = f_1 - f_3.$$

Combining the last two equations, we obtain the equation (2.7). □

Definition 2.2 ([34,35]). *A vector field V on a Riemannian manifold is called an infinitesimal contact transformation if*

$$\pounds_V \eta = \kappa\eta, \tag{2.10}$$

for some smooth function κ on the manifold. If $\kappa = 0$, then the vector field is called a strict infinitesimal contact transformation.

3. Miao-Tam Equation (MTE) on Trans-Sasakian Generalized Sasakian Space-forms

The prime aim of the present section is to study the Miao-Tam equation (MTE, in short) on three-dimensional trans-Sasakian GSSFs and make a bridge between MTE and Ricci solitons. Before going to main topic, we proof the following lemma.

Lemma 3.1. *Let M^3 be a trans-Sasakian GSSF of dimension three, then*

$$\begin{aligned}(\nabla_{V_1}Q)V_2 =& V_1(2f_1+3f_2-f_3)V_2 - V_1(3f_2+f_3)\eta(V_2)\theta - (3f_2+f_3)(-\alpha g(\phi V_1,V_2)\theta + \beta(g(V_1,V_2)\theta - \eta(V_1)\eta(V_2)\theta)) \\ &- (3f_2+f_3)(-\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta))\eta(V_2),\end{aligned} \tag{3.1}$$

$$\frac{1}{2}V_2(r) = V_2(2f_1+3f_2-f_3) - \theta(3f_2+f_3)\eta(V_2) - 2\beta(3f_2+f_3)\eta(V_2), \tag{3.2}$$

and

$$\theta(r) = 4(\theta(f_1 - f_3) - \beta(3f_2+f_3)), \tag{3.3}$$

for every vector fields V_1, V_2 on M^3.

Proof. Differentiating the equation (2.5) covariantly and using (2.3), one can obtain the equation (3.1). Contracting the equation (3.1) with respect to V_1, we obtain (3.2). Putting $V_2 = \xi$ in (3.2), we get the equation (3.3). □

Theorem 3.2. *If a three-dimensional trans-Sasakian GSSF admits non-trivial solution of the Miao-Tam equation then the scalar curvature is a constant.*

Proof. Let us suppose that the said space form admits non-trivial solution of the Miao-Tam equation. Then, from (1.1), we obtain

$$(\Delta\lambda)g(V_1,V_2) = (Hess\lambda)(V_1,V_2) - \lambda S(V_1,V_2) - g(V_1,V_2). \tag{3.4}$$

Let $\{u_1,u_2,\xi\}$ be an orthonormal set of tangent vector fields on M^3. Substituting $V_1 = V_2 = u_i$ in the previous equation and summing over i, we have

$$(\Delta\lambda) = -(3f_1+3f_2-2f_3)\lambda - \frac{3}{2}. \tag{3.5}$$

Using (3.5) in (3.4), we obtain

$$\nabla_{V_1}D\lambda = \lambda QV_1 - (3f_1+3f_2-2f_3)\lambda V_1 - \frac{1}{2}V_1. \tag{3.6}$$

The covariant derivative of the equation (3.6) in the direction of V_2 gives

$$\nabla_{V_2}\nabla_{V_1}D\lambda = V_2(\lambda)QV_1 + \lambda\nabla_{V_2}QV_1 - V_2(3f_1+3f_2-2f_3)\lambda V_1 - (3f_1+3f_2-2f_3)(V_2(\lambda)V_1 + \lambda\nabla_{V_2}V_1) - \frac{1}{2}\nabla_{V_2}V_1. \tag{3.7}$$

Interchanging V_1 and V_2 in (3.7), one can obtain

$$\nabla_{V_1}\nabla_{V_2}D\lambda = V_1(\lambda)QV_2 + \lambda\nabla_{V_1}QV_2 - V_1(3f_1+3f_2-2f_3)\lambda V_2 - (3f_1+3f_2-2f_3)(V_1(\lambda)V_2 + \lambda\nabla_{V_1}V_2) - \frac{1}{2}\nabla_{V_1}V_2. \tag{3.8}$$

Again, equation (3.6) gives

$$\nabla_{[V_1,V_2]}D\lambda = \lambda Q[V_1,V_2]-(3f_1+3f_2-2f_3)\lambda[V_1,V_2]-\frac{1}{2}[V_1,V_2]. \tag{3.9}$$

Using (3.7)-(3.9), we get the curvature tensor as

$$\begin{aligned}R(V_1,V_2)D\lambda =&V_1(\lambda)QV_2-V_2(\lambda)QV_1+\lambda((\nabla_{V_1}Q)V_2-(\nabla_{V_2}Q)V_1)-V_1(3f_1+3f_2-2f_3)\lambda V_2+V_2(3f_1+3f_2-2f_3)\lambda V_1\\&-(3f_1+3f_2-2f_3)(V_1(\lambda)V_2-V_2(\lambda)V_1).\end{aligned} \tag{3.10}$$

Contracting (3.10) along the vector field V_1, we obtain

$$\begin{aligned}S(V_2,D\lambda) =&(2f_1+3f_2-f_3)V_2(\lambda)-(3f_2+f_3)\theta(\lambda)\eta(V_2)+\lambda\{V_2(2f_1+3f_2-f_3)-\theta(3f_2+f_3)\eta(V_2)\\&-2\beta(3f_2+f_3)\eta(V_2)\}.\end{aligned} \tag{3.11}$$

According to (2.4), we find

$$S(V_2,D\lambda) = (2f_1+3f_2-f_3)V_2(\lambda)-(3f_2+f_3)\theta(\lambda)\eta(V_2). \tag{3.12}$$

Comparing (3.11) and (3.12), we get

$$V_2(2f_1+3f_2-f_3)-\theta(3f_2+f_3)\eta(V_2)-2\beta(3f_2+f_3)\eta(V_2)=0, \tag{3.13}$$

where we have used $\lambda\neq 0$. Substituting (3.13) in (3.2), we see that $V_2(r)=0$, that is, r is a constant.
This completes the proof. □

Theorem 3.3. *If a three-dimensional trans-Sasakian GSSF admits non-trivial solution of the Miao-Tam equation then either the structure is β-Kenmotsu or, the manifold is Einstein.*

Proof. Replacing V_1 by ξ and taking inner product with V_1 of the equation (3.10), we have

$$\begin{aligned}g(R(\theta,V_2)D\lambda,V_1) =&\theta(\lambda)\{-(f_1-f_3)g(V_1,V_2)-(3f_2+f_3)\eta(V_1)\eta(V_2)\}+(f_1+3f_3)V_2(\lambda)\eta(V_1)\\&+\lambda\{-\theta(f_1-f_3)g(V_1,V_2)-\theta(3f_2+f_3)\eta(V_1)\eta(V_2)+V_2(f_1+3f_2)\eta(V_1)\\&+(3f_2+f_3)(-\alpha g(V_1,\phi V_2)+\beta(g(V_1,V_2)-\eta(V_1)\eta(V_2)))\}.\end{aligned} \tag{3.14}$$

Putting $V_1=\xi$ in (1.4) and then taking inner product with $D\lambda$, one can obtain

$$g(R(\theta,V_2)V_1,D\lambda) = (f_1-f_3)(\theta(\lambda)g(V_1,V_2)-V_2(\lambda)\eta(V_1)). \tag{3.15}$$

Comparing (3.14) and (3.15), we find

$$\begin{aligned}&\theta(\lambda)\{-(f_1-f_3)g(V_1,V_2)-(3f_2+f_3)\eta(V_1)\eta(V_2)\}+(f_1+3f_2)V_2(\lambda)\eta(V_1)\\&+\lambda\left\{\begin{array}{l}-\theta(f_1-f_3)g(V_1,V_2)-\theta(3f_2+f_3)\eta(V_1)\eta(V_2)+V_2(f_1+3f_2)\eta(V_1)\\+(3f_2+f_3)(-\alpha g(V_1,\phi V_2)+\beta(g(V_1,V_2)-\eta(V_1)\eta(V_2)))\end{array}\right\}=(f_3-f_1)(\theta(\lambda)g(V_1,V_2)-V_2(\lambda)\eta(V_1)).\end{aligned} \tag{3.16}$$

Interchanging V_1 and V_2 in the foregoing equation, we find

$$\begin{aligned}&\theta(\lambda)\{-(f_1-f_3)g(V_1,V_2)-(3f_2+f_3)\eta(V_1)\eta(V_2)\}+(f_1+3f_3)V_1(\lambda)\eta(V_2)\\&+\lambda\left\{\begin{array}{l}-\theta(f_1-f_3)g(V_1,V_2)-\theta(3f_2+f_3)\eta(V_1)\eta(V_2)+V_1(f_1+3f_2)\eta(V_2)\\+(3f_2+f_3)(\alpha g(V_1,\phi V_2)+\beta(g(V_1,V_2)-\eta(V_1)\eta(V_2)))\end{array}\right\}=(f_3-f_1)(\theta(\lambda)g(V_1,V_2)-V_1(\lambda)\eta(V_2)).\end{aligned} \tag{3.17}$$

Subtracting (3.17) from (3.16), one can obtain

$$(3f_2+f_3)(V_2(\lambda)\eta(V_1)-V_1(\lambda)\eta(V_2))+\lambda\{V_2(f_1+3f_2)\eta(V_1)-V_1(f_1+3f_2)\eta(V_2)-2(3f_2+f_3)\alpha g(V_1,\phi V_2)\}=0.$$

Replacing V_1 and V_2 by ϕV_1 and ϕV_2, respectively, in the last equation, we obtain

$$(3f_2+f_3)\alpha g(V_1,\phi V_2)=0,$$

which implies that either $3f_2+f_3=0$ or, $\alpha=0$, i.e., the structure is β-Kenmotsu.
Let us now discuss the case when $3f_2+f_3=0$. Then from(2.6), we get $r=6(f_1-f_3)$. With the help of (2.4), (3.1), equation (3.16) can be written as

$$\theta(\lambda)(S(V_1,V_2)-(f_3-f_1)g(V_1,V_2))-3(f_1-f_3)V_2(\lambda)\eta(V_1)+\theta(f)g(V_1,V_2)-V_2(f)\eta(V_1)=0,$$

where $f=-\frac{r\lambda+1}{2}$ and

$$\nabla_{V_1}D\lambda = \lambda QV_1+fV_1. \tag{3.18}$$

As r is a constant, $2V_2(f)=-rV_2(\lambda)$ and so, $2\theta(f)=-r\theta(\lambda)$. Applying these relations in the above equation, we obtain

$$\theta(\lambda)\{S(V_1,V_2)-2(f_1-f_3)g(V_1,V_2)\}=0,$$

where we have used $r=6(f_1-f_3)$. From the foregoing equation we obtain either $\theta(\lambda)=0$ or, $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$. If we consider $\theta(\lambda)=0$, i.e., $g(\theta,D\lambda)=0$, then by covariant derivative

$$g(\nabla_{V_1}\theta,D\lambda)+g(\theta,\nabla_{V_1}D\lambda)=0.$$

Using (2.3) and (3.18) in the foregoing equation, we have

$$-\alpha\phi V_1(\lambda)+\beta V_1(\lambda)+\lambda S(V_1,\theta)+f\eta(V_1)=0, \tag{3.19}$$

where we have used $\theta(\lambda)=0$. Applying (1.5), $r=6(f_1-f_3)$ and $f=-\frac{r\lambda+1}{2}$ in (3.19), we obtain

$$-\alpha\phi V_1(\lambda)+\beta V_1(\lambda)-\{\lambda(f_1-f_3)+\frac{1}{2}\}\eta(V_1)=0. \tag{3.20}$$

Replacing V_1 by θ, equation (3.20) gives $\lambda(f_1-f_3)+\frac{1}{2}=0$, as $\theta(\lambda)=0$. Thus we find that $f=1$, a constant and hence λ is also a non-zero constant. Applying these data in (3.4), we see that $S(V_1,V_2)=-\frac{1}{\lambda}g(V_1,V_2)$, i.e, $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$, as $\lambda(f_1-f_3)+\frac{1}{2}=0$. Thus for every cases, the space-form obeys $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$. Hence the manifold is Einstein.
Thus the proof is completed. □

A consequence of the above theorem is

Corollary 3.4. *There does not exist a non-cosymplectic three-dimensional GSSF with β-Kenmotsu structure obeying non-trivial solution of the MTE, where β is a constant.*

Proof. Putting $V_1=V_2=u_i$ in (3.16), where $\{u_i\}$, $(i=1,2,3)$ being an orthonormal frame of the tangent space, and summing over i, we find

$$\theta(f_1-f_3)-\beta(3f_2+f_3)=0. \tag{3.21}$$

Comparing (3.3) and (3.21), we obtain $\theta(r)=0$. Using (2.7) in (3.21) and considering β as a constant, we find

$$\beta(3f_2+f_3)=0,$$

which gives $\beta=0$, as $3f_2+f_3\neq 0$. Hence the structure is cosymplectic. □

Corollary 3.5. *Let a trans-Sasakian GSSF be an Einstein manifold and the space form admit non-trivial solution of MTE. Then the metric is a gradient Ricci soliton.*

Proof. Using $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$ in (3.18), we see that

$$\nabla_{V_1}D\lambda=\{2(f_1-f_3)+1\}V_1.$$

The foregoing equation can be written as

$$Hess(\lambda)(V_1,V_2)+S(V_1,V_2)-\{2(f_1-f_3)(\lambda+1)+1\}g(V_1,V_2)=0,$$

which is the gradient Ricci soliton, where the soliton constant is $2(f_1-f_3)(\lambda+1)+1$. □

Theorem 3.6 ([36]). *If $\tilde{\lambda}$ is a solution of the Fischer-Marsden equation (FME, in short) on a three-dimensional trans-Sasakian GSSF, then the curvature tensor R is given by*

$$R(V_1,V_2)D\tilde{\lambda}=V_1(\tilde{\lambda})QV_2-V_2(\tilde{\lambda})QV_1+\tilde{\lambda}\{(\nabla_{V_1}Q)V_2-(\nabla_{V_2}Q)V_1\}+V_1(\tilde{f})V_2-V_2(\tilde{f})V_1, \tag{3.22}$$

for every vector fields V_1, V_2 on M and $\tilde{f}=-\frac{r\tilde{\lambda}}{2}$.
Moreover,

$$\nabla_{V_1}D\tilde{\lambda}=\tilde{\lambda}QV_1+\tilde{f}V_1. \tag{3.23}$$

Theorem 3.7. *In a three-dimensional trans-Sasakian GSSF, if the FME admits a solution then either the solution is trivial or, the scalar curvature is a constant.*

Proof. Using (2.4) in (3.22), one can obtain

$$\begin{aligned}R(V_1,V_2)D\tilde{\lambda}=&(2f_1+3f_2-f_3)V_1(\tilde{\lambda})V_2-(3f_2+f_3)V_1(\tilde{\lambda})\eta(V_2)\theta-(2f_1+3f_2-f_3)V_2(\tilde{\lambda})V_1+(3f_2+f_3)V_2(\tilde{\lambda})\eta(V_1)\theta\\&+\tilde{\lambda}\{(\nabla_{V_1}Q)V_2-(\nabla_{V_2}Q)V_1\}+V_1(\tilde{f})V_2-V_2(\tilde{f})V_1.\end{aligned} \tag{3.24}$$

Contracting (3.24) along V_1, we infer

$$S(V_2,D\tilde{\lambda})=(2f_1+3f_2-f_3)V_2(\tilde{\lambda})-(3f_2+f_3)\theta(\tilde{\lambda})\eta(V_2)+\frac{\tilde{\lambda}}{2}V_2(r), \tag{3.25}$$

where we have used $\tilde{f}=-\frac{r\tilde{\lambda}}{2}$. Comparing (3.25) with (3.12), we find that $\tilde{\lambda}V_2(r)=0$, which gives either $\tilde{\lambda}=0$, i.e., the solution is trivial or, $V_2(r)=0$, i.e., the scalar curvature is a constant.
This establishes the theorem. □

Theorem 3.8. *In a three-dimensional trans-Sasakian GSSF, if the FME admits a solution then either the structure is β-Kenmotsu or, the manifold is Einstein or, the solution is trivial.*

Proof. Taking inner product of (3.22) with θ, we find that

$$\begin{aligned} g(R(V_1,V_2)D\tilde{\lambda},\theta) =& 2(f_1-f_3)\{V_1(\tilde{\lambda})\eta(V_2)-V_2(\tilde{\lambda})\eta(V_1)\} \\ &+\tilde{\lambda}\{2V_1(f_1-f_3)\eta(V_2)-2V_2(f_1-f_3)\eta(V_1)+2(3f_2+f_3)\alpha g(\phi V_1,V_2)\} \\ &+V_1(\tilde{f})\eta(V_2)-V_2(\tilde{f})\eta(V_1). \end{aligned} \tag{3.26}$$

Replacing V_1 by ϕV_1 and V_2 by ϕV_2 in (3.26), one can obtain

$$g(R(\phi V_1,\phi V_2)D\tilde{\lambda},\theta) = -2\tilde{\lambda}(3f_2+f_3)\alpha g(V_1,\phi V_2). \tag{3.27}$$

Also, from(1.4), we have

$$g(R(\phi V_1,\phi V_2)D\tilde{\lambda},\theta) = 0. \tag{3.28}$$

Comparing (3.27) and (3.28), we obtain

$$\tilde{\lambda}(3f_2+f_3)\alpha g(V_1,\phi V_2) = 0.$$

Thus three possibility arise: (1) $\tilde{\lambda}=0$, (2) $(3f_2+f_3)=0$ and (3) $\alpha=0$.
Let us discuss the case when $(3f_2+f_3)=0$. Then, from (2.6), we find that $r=6(f_1-f_3)$. From (3.22), we get

$$g(R(\theta,V_2)D\tilde{\lambda},V_1) = \theta(\tilde{\lambda})S(V_1,V_2)-V_2(\tilde{\lambda})S(V_1,\theta)+\theta(\tilde{f})g(V_1,V_2)-V_2(\tilde{f})\eta(V_1). \tag{3.29}$$

Also, from (1.4), we infer

$$g(R(\theta,V_2)D\tilde{\lambda},V_1) = -(f_1-f_3)\{\theta(\tilde{f})g(V_1,V_2)-V_2(\tilde{f})\eta(V_1)\}. \tag{3.30}$$

Comparing (3.29) and (3.30) and using $r=6(f_1-f_3)$, $f=-\frac{r\tilde{\lambda}}{2}$ and the equation (2.4), one can obtain

$$\theta(\tilde{\lambda})(S(V_1,V_2)-2(f_1-f_3)g(V_1,V_2)) = 0,$$

which implies either $S(V_1,V_2)=2(f_1-f_3)g(V_1,V_2)$, i.e., the manifold is Einstein or, $\theta(\tilde{\lambda})=0$. Let us discuss the case when $(\theta\tilde{\lambda})=0$. Then we have $g(\theta,D\tilde{\lambda})=0$, which gives

$$g(\nabla_{V_2}\theta,D\tilde{\lambda})+g(\theta,\nabla_{V_2}D\tilde{\lambda}) = 0.$$

Applying (2.3), (2.4), (3.23) and $\tilde{f}=-\frac{r\tilde{\lambda}}{2}$ in the foregoing equation, we see that

$$-\alpha\phi V_2(\tilde{\lambda})+\beta V_2(\tilde{\lambda})-(f_1-f_3)\tilde{\lambda}\eta(V_2) = 0, \tag{3.31}$$

where we have used $\theta(\tilde{\lambda})=0$. Replacing V_2 by θ and taking $f_1\neq f_3$ in (3.31), we find that $\tilde{\lambda}=0$, i.e., the solution is trivial.
This ensures the validity of the theorem. □

4. Ricci Solitons on Three-Dimensional Generalized Sasakian Space-forms with Trans-Sasakian Structures

In the present section, we study Ricci solitons on three-dimensional generalized Sasakian space-forms with trans-Sasakian structure.

Theorem 4.1. *In a three-dimensional trans-Sasakian GSSF obeying Ricci solitons, the potential vector field is an infinitesimal contact transformation.*

Proof. From (1.2), we have

$$(\pounds_V g)(V_1,V_2)+2S(V_1,V_2)+2\psi g(V_1,V_2) = 0.$$

Applying $V_2=\theta$ in the foregoing equation and using (2.4), we have

$$(\pounds_V g)(V_1,\theta) = -2(2(f_1-f_3)+\psi)\eta(V_1). \tag{4.1}$$

Again, changing V_1 by θ in (4.1), we get

$$\pounds_V\theta = (2(f_1-f_3)+\psi)\theta. \tag{4.2}$$

Applying Lie derivative of $\eta(V_1)=g(V_1,\theta)$ with respect to V and then using (4.1) and (4.2), we find that

$$(\pounds_V\eta)(V_1) = -(2(f_1-f_3)+\psi)\eta(V_1),$$

an infinitesimal contact transformation. □

From the above theorem, we prove the following:

Theorem 4.2. *In a three-dimensional trans-Sasakian GSSF obeying Ricci solitons, the soliton is shrinking, expanding or steady if $f_1 - f_3$ is positive, negative or zero, respectively.*

Proof. We have

$$(\pounds_V d\eta)(V_1,V_2) = (\pounds_V g)(V_1,\phi V_2) + g(V_1,(\pounds_V \phi)V_2).$$

Using (2.4) and (1.2) in the foregoing equation, we infer

$$\begin{aligned}(\pounds_V d\eta)(V_1,V_2) =& -2(2f_1+3f_2-f_3+\psi)g(V_1,\phi V_2)\\ &+g(V_1,(\pounds_V \phi)V_2).\end{aligned} \tag{4.3}$$

According to Theorem 4.1, V is an infinitesimal contact transformation. Also, since $\pounds$ and d commutes, equation (2.10) gives

$$\begin{aligned}(\pounds_V d\eta)(V_1,V_2) =&((d\kappa)\wedge\eta)(V_1,V_2)+\kappa g(V_1,\phi V_2)\\ =&\frac{1}{2}(V_1(\kappa)\eta(V_2)-V_2(\kappa)\eta(V_1))+\kappa g(V_1,\phi V_2).\end{aligned} \tag{4.4}$$

Comparing (4.3) and (4.4), we have

$$g(V_1,(\pounds_V \phi)V_2) = \frac{1}{2}(V_1(\kappa)\eta(V_2)-V_2(\kappa)\eta(V_1))+(2(2f_1+3f_2-f_3+\psi)+\kappa)g(V_1,\phi V_2),$$

which gives

$$(\pounds_V \phi)V_2 = \frac{1}{2}(\eta(V_2)D\kappa - V_2(\kappa)\theta)+(2(2f_1+3f_2-f_3+\psi)+\kappa)\phi V_2.$$

Changing V_2 by θ in the previous equation, we find

$$(\pounds_V \phi)\theta = \frac{1}{2}(D\kappa-\theta(\kappa)\theta). \tag{4.5}$$

But

$$(\pounds_V \phi)\theta = \pounds_V \phi\theta - \phi(\pounds_V \theta) = 0, \tag{4.6}$$

where we used (4.2) and $\phi\theta = 0$. Using (4.6) in (4.5), we obtain

$$D\kappa = \theta(\kappa)\theta,$$

which gives

$$d\kappa = \theta(\kappa)\eta. \tag{4.7}$$

By exterior derivative we find from (4.7) that

$$0 = d^2\kappa = d(\theta(\kappa))\wedge\eta + \theta(\kappa)d\eta.$$

Taking wedge product with η in the foregoing equation, we get

$$\theta(\kappa)\eta\wedge d\eta = 0.$$

As $\eta\wedge d\eta \neq 0$, the previous equation gives $\theta(\kappa) = 0$. Thus, from (4.7), we have $d\kappa = 0$, i.e., κ is a constant.
Due to Cartan's formula, for the closed volume form $\Omega(=\eta\wedge d\eta)$, we have

$$\pounds_V \Omega = (divV)\Omega, \tag{4.8}$$

where *div* is the divergence operator. Again, taking Lie derivative of the volume form $\Omega(=\eta\wedge d\eta)$ and using (4.4) and (4.8), we get

$$(divV)\Omega = 2\kappa\Omega,$$

which implies

$$divV = 2\kappa.$$

Integrating the above equation and using divergence theorem, we see that $\kappa = 0$. Thus V is the strict infinitesimal contact transformation and hence, we get $\psi = -2(f_1 - f_3)$.
This establishes the theorem. □

Theorem 4.3. *In a three dimensional trans-Sasakian GSSF obeying gradient Ricci solitons, either the structure is β-Kenmotsu or, the potential function is constant, i.e., the soliton is trivial.*

Proof. Let us suppose that a three dimensional trans-Sasakian generalized Sasakian space-form admit gradient Ricci solitons. Then, from (1.3), we can write

$$\nabla_{V_1} D\zeta = -QV_1 - \psi V_1. \tag{4.9}$$

Applying covariant derivative on (4.9), we get

$$\nabla_{V_2}\nabla_{V_1} D\zeta = -\nabla_{V_2} QV_1 - \psi \nabla_{V_2} V_1. \tag{4.10}$$

Interchanging V_1 and V_2 in the previous equation, we obtain

$$\nabla_{V_1}\nabla_{V_2} D\zeta = -\nabla_{V_1} QV_2 - \psi \nabla_{V_1} V_2. \tag{4.11}$$

Also, equation (4.9) gives

$$\nabla_{[V_1,V_2]} D\zeta = -Q[V_1,V_2] - \psi [V_1,V_2]. \tag{4.12}$$

Using (4.10)-(4.12), we get the curvature tensor as

$$\begin{aligned} R(V_1,V_2)D\zeta = -\{&V_1(2f_1+3f_2-f_3)V_2 - V_1(3f_2+f_3)\eta(V_2)\theta - V_2(2f_1+3f_2-f_3) + V_2(3f_2+f_3)\eta(V_1)\theta \\ &+2(3f_2+f_3)g(\phi V_1,V_2)\theta - (3f_2+f_3)(-\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta))\eta(V_2) \\ &+(3f_2+f_3)(-\alpha\phi V_2 + \beta(V_2 - \eta(V_2)\theta))\eta(V_1)\}. \end{aligned} \tag{4.13}$$

Replacing V_1 by θ in (4.13) and then taking inner product with V_1, we see that

$$\begin{aligned} g(R(\theta,V_2)D\zeta,V_1) = -\{&\theta(2f_1+3f_2-f_3)g(V_1,V_2) - \theta(3f_2+f_3)\eta(V_1)\eta(V_2) - 2V_2(f_1-f_3)\eta(V_1) \\ &+(3f_2+f_3)(-\alpha g(V_1,\phi V_2) + \beta(g(V_1,V_2) - \eta(V_1)\eta(V_2)))\}. \end{aligned} \tag{4.14}$$

Also, the equation (1.4) can be written as

$$g(R(\theta,V_2)D\zeta,V_1) = (f_1-f_3)\{V_2(\zeta)\eta(V_1) - \theta(\zeta)g(V_1,V_2)\}. \tag{4.15}$$

Comparing (4.14) and (4.15), we obtain

$$\begin{aligned} &\theta(2f_1+3f_2-f_3)g(V_1,V_2) - \theta(3f_2+f_3)\eta(V_1)\eta(V_2) - 2V_2(f_1-f_3)\eta(V_1) \\ &+(3f_2+f_3)(-\alpha g(V_1,\phi V_2) + \beta(g(V_1,V_2) - \eta(V_1)\eta(V_2))) + (f_1-f_3)\{V_2(\zeta)\eta(V_1) - \theta(\zeta)g(V_1,V_2)\} = 0. \end{aligned} \tag{4.16}$$

Interchanging V_1 and V_2 in (4.16), we have

$$\begin{aligned} &\theta(2f_1+3f_2-f_3)g(V_1,V_2) - \theta(3f_2+f_3)\eta(V_1)\eta(V_2) - 2V_1(f_1-f_3)\eta(V_2) \\ &+(3f_2+f_3)(\alpha g(V_1,\phi V_2) + \beta(g(V_1,V_2) - \eta(V_1)\eta(V_2))) + (f_1-f_3)\{V_1(\zeta)\eta(V_2) - \theta(\zeta)g(V_1,V_2)\} = 0. \end{aligned} \tag{4.17}$$

Subtracting (4.17) from (4.16), we see that

$$2V_1(f_1-f_3)\eta(V_2) - 2V_2(f_1-f_3)\eta(V_1) - 2(3f_2+f_3)\alpha g(V_1,\phi V_2) + (f_1-f_3)\{V_2(\zeta)\eta(V_1) - V_1(\zeta)\eta(V_2)\} = 0. \tag{4.18}$$

Replacing V_1 by ϕV_1 and V_2 by ϕV_2 in (4.18), we obtain

$$(3f_2+f_3)\alpha g(\phi V_1,V_2) = 0,$$

which indicates that either $\alpha = 0$, i.e., the structure is β-Kenmotsu or, $3f_2+f_3 = 0$. For the later case, with the help of (2.6) and (3.2), we get

$$V_1(f_1-f_3) = 0, \tag{4.19}$$

for every vector field V_1, i.e., f_1-f_3 is a constant. Thus, from (4.18), we obtain

$$(f_1-f_3)\{V_2(\zeta)\eta(V_1) - V_1(\zeta)\eta(V_2)\} = 0,$$

which gives either $f_1 = f_3$ or

$$V_2(\zeta)\eta(V_1) = V_1(\zeta)\eta(V_2). \tag{4.20}$$

Let us discuss the second possibility. Putting $V_2 = \theta$ in (4.20), we obtain

$$D\zeta = \theta(\zeta)\theta. \tag{4.21}$$

Taking covariant derivative of (4.21) with respect to V_1 and using (2.3), we obtain

$$\nabla_{V_1} D\zeta = V_1(\theta(\zeta))\theta + \theta(\zeta)(-\alpha\phi V_1 + \beta(V_1 - \eta(V_1)\theta)). \tag{4.22}$$

Comparing (4.22) with (4.9), we find that

$$V_1(\theta(\zeta))\eta(V_2) = -S(V_1,V_2) - \psi g(V_1,V_2) - \theta(\zeta)(-\alpha g(\phi V_1,V_2) + \beta(g(V_1,V_2) - \eta(V_1)\eta(V_2))).$$

Since $3f_2 + f_3 = 0$, using (2.4) in the above equation, we get

$$V_1(\theta(\zeta))\eta(V_2) = -\{2(f_1 - f_3) + \psi\}g(V_1, V_2) - \theta(\zeta)(-\alpha g(\phi V_1, V_2) + \beta(g(V_1, V_2) - \eta(V_1)\eta(V_2))). \tag{4.23}$$

Replacing V_2 by ϕV_2 in (4.23), we see that

$$\{2(f_1 - f_3) + \psi\}g(V_1, \phi V_2) + \theta(\zeta)(-\alpha(g(V_1, V_2) - \eta(V_1)\eta(V_2)) + \beta g(V_1, \phi V_2)) = 0.$$

Contracting the above equation and using $\mathrm{tr}\phi = 0$, we get

$$\alpha\theta(\zeta) = 0,$$

which gives $\theta(\zeta) = 0$, as we consider $\alpha \neq 0$. Thus, from (4.21), we see that $D\zeta = 0$, i.e., ζ is a constant.
Hence the proof is completed. □

From the equation(4.19), we can state the following corollary

Corollary 4.4. *If a three-dimensional trans-Sasakian GSSF admits gradient Ricci solitons, then either the structure is β-Kenmotsu or, $f_1 - f_3$ is a constant.*

5. Example

Let $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ be a three-dimensional manifold, where (x, y, z) are the standard co-ordinates in $\mathbb{R}^3$. We choose the basis vectors on M as

$$u_1 = e^{-2z}\frac{\partial}{\partial x}, \quad u_2 = e^{-2z}\frac{\partial}{\partial y}, \quad u_3 = \frac{\partial}{\partial z}.$$

Then we find by direct computation that

$$[u_1, u_2] = 0, \quad [u_1, u_3] = 2u_1, \quad [u_2, u_3] = 2u_2.$$

Let g be the metric tensor defined by

$$g(u_1, u_1) = 1, \quad g(u_2, u_2) = 1, \quad g(u_3, u_3) = 1, \quad g(u_1, u_2) = 0, \quad g(u_1, u_3) = 0, \quad g(u_2, u_3) = 0.$$

The 1-form η is given by $\eta(V_1) = g(V_1, u_3)$ for all V_1 on M. Let us define the $(1,1)$-tensor field ϕ as

$$\phi u_1 = -u_2, \quad \phi u_2 = u_1, \quad \phi u_3 = 0.$$

Then we see that

$$\eta(u_3) = 1, \quad \phi^2 V_1 = -V_1 + \eta(V_1)u_3, \quad g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2), \quad d\eta(V_1, V_2) = g(V_1, \phi V_2).$$

Thus the given manifold admits a contact metric structure (ϕ, u_3, η, g).Now, using Koszul's formula, we obtain

$$\nabla_{u_1}u_1 = -2u_3, \quad \nabla_{u_1}u_2 = 0, \quad \nabla_{u_1}u_3 = 2u_1, \quad \nabla_{u_2}u_1 = 0, \quad \nabla_{u_2}u_2 = -2u_3, \quad \nabla_{u_2}u_3 = 2u_2, \quad \nabla_{u_3}u_1 = 0,$$
$$\nabla_{u_3}u_2 = 0, \quad \nabla_{u_3}u_3 = 0.$$

Thus the given structure is a trans-Sasakian structure with $\alpha = 0$, $\beta = 2$. The components of the curvature tensor are given by

$$R(u_1, u_2)u_2 = -4u_1, \quad R(u_2, u_1)u_1 = -4u_2, \quad R(u_1, u_3)u_3 = -4u_1, \quad R(u_2, u_3)u_3 = -4u_2, \quad R(u_3, u_1)u_1 = -4u_3,$$
$$R(u_3, u_2)u_2 = -4u_3, \quad R(u_1, u_2)u_3 = 0, \quad R(u_1, u_3)u_2 = 0, \quad R(u_2, u_3)u_1 = 0.$$

From the above expressions, the given manifold is a generalized Sasakian space-form with $f_1 = \omega - 1$, $f_2 = -\frac{\omega+3}{3}$ and $f_3 = \omega + 3$, where ω is a smooth function on M.
The non-zero components of the Ricci tensor are given by

$$S(u_1, u_1) = -8, \quad S(u_2, u_2) = -8, \quad S(u_3, u_3) = -8.$$

Thus we see that $S(V_1, V_2) = -8g(V_1, V_2)$, for every vector fields V_1, V_2 on M. Hence the space-form is an Einstein manifold. The scalar curvature of the manifold is -24.
Let $\lambda = e^{-\frac{az}{2}} + b$, where a and b are scalars, so that, $e^{-\frac{az}{2}} = \lambda - b$. Now $D\lambda = -\frac{a}{2}e^{-\frac{az}{2}}u_3 = -\frac{a}{2}(\lambda - b)u_3$. Then

$$\nabla_{u_1}D\lambda = -a(\lambda - b)u_1, \quad \nabla_{u_2}D\lambda = -a(\lambda - b)u_2, \quad \nabla_{u_3}D\lambda = \frac{a^2}{4}(\lambda - b)u_3.$$

Thus $(\Delta_g\lambda) = (\frac{a^2}{4} - 2a)(\lambda - b)$. Now $-(\Delta_g\lambda)g(u_i, u_j) + g(\nabla_{u_i}D\lambda, u_j) - \lambda S(u_i, u_j) = g(u_i, u_j)$, $\quad i, j = 1, 2, 3$, gives the following two equations

$$(a - \frac{a^2}{4})(\lambda - b) + 8\lambda = 1$$

and

$$2a(\lambda-b)+8\lambda=1.$$

Comparing the above two equations, we see that $a=0$, $b=-\frac{7}{8}$ and $\lambda=\frac{1}{8}$ or $a=-4$, $b=\frac{1}{8}$ and $\lambda=e^{2z}+\frac{1}{8}$. Thus the non-trivial solution of the Miao-Tam equation exists on the given manifold. Since the manifold is Einstein and the structure is β-Kenmotsu (as $\alpha=0$), the Theorem 3.3 holds good.

Again, let $\tilde{\lambda}=e^{-\frac{az}{2}}+b$, where a and b are scalars, so that, $e^{-\frac{az}{2}}=\tilde{\lambda}-b$. Now $D\tilde{\lambda}=-\frac{a}{2}e^{-\frac{az}{2}}u_3=-\frac{a}{2}(\tilde{\lambda}-b)u_3$. Then

$$\nabla_{u_1}D\tilde{\lambda}=-a(\tilde{\lambda}-b)u_1,\quad \nabla_{u_2}D\tilde{\lambda}=-a(\tilde{\lambda}-b)u_2,\quad \nabla_{u_3}D\tilde{\lambda}=\frac{a^2}{4}(\tilde{\lambda}-b)u_3.$$

Thus $(\Delta_g\tilde{\lambda})=(\frac{a^2}{4}-2a)(\tilde{\lambda}-b)$. Now $-(\Delta_g\tilde{\lambda})g(u_i,u_j)+g(\nabla_{u_i}D\tilde{\lambda},u_j)-\tilde{\lambda}S(u_i,u_j)=0,\quad i,j=1,2,3$, gives the following two equations

$$(a-\frac{a^2}{4})(\lambda-b)+8\lambda=0$$

and

$$2a(\lambda-b)+8\lambda=0.$$

Solving the last two equations, we see that $\tilde{\lambda}=0$, i.e., the solution is trivial, which ensures the validity of the Theorem 3.8.

Let us consider the potential vector field $V=xe^{2z}u_1+ye^{2z}u_2+\frac{1}{2}(e^{2z}-1)u_3$. Then equation (1.2) is satisfied for that V with $\psi=8-e^{2z}$, i.e., the soliton is steady at $z=\frac{3}{2}\log 2$ and it is expanding or shrinking if z is less than or greater than $\frac{3}{2}\log 2$, respectively. Also $(\pounds_V\eta)(V_1)=e^{2z}\eta(V_1)$, for any vector field V_1 on M. Hence V is an infinitesimal contact transformation. In this way Theorem 4.1 is satisfied.

Next, we suppose that the potential vector field V is the gradient of a smooth function ζ, i.e., $V=D\zeta$. Then

$$D\zeta=e^{-2z}\frac{\partial\zeta}{\partial x}u_1+e^{-2z}\frac{\partial\zeta}{\partial y}u_2+\frac{\partial\zeta}{\partial z}u_3.$$

Therefore,

$$\nabla_{u_1}D\zeta=e^{-4z}\frac{\partial^2\zeta}{\partial x^2}u_1-2e^{-2z}\frac{\partial\zeta}{\partial x}u_3+e^{-4z}\frac{\partial^2\zeta}{\partial x\partial y}u_2+e^{-2z}\frac{\partial^2\zeta}{\partial x\partial z}u_3+2\frac{\partial\zeta}{\partial z}u_1,$$
$$\nabla_{u_2}D\zeta=e^{-4z}\frac{\partial^2\zeta}{\partial y^2}u_2-2e^{-2z}\frac{\partial\zeta}{\partial y}u_3+e^{-4z}\frac{\partial^2\zeta}{\partial y\partial x}u_1+e^{-2z}\frac{\partial^2\zeta}{\partial y\partial z}u_3+2\frac{\partial\zeta}{\partial z}u_2,$$
$$\nabla_{u_3}D\zeta=-2e^{-2z}\frac{\partial\zeta}{\partial x}u_1+e^{-2z}\frac{\partial^2\zeta}{\partial z\partial x}u_1-2e^{-2z}\frac{\partial\zeta}{\partial y}u_2+e^{-2z}\frac{\partial^2\zeta}{\partial z\partial y}u_2+\frac{\partial^2\zeta}{\partial z^2}u_3.$$

Thus the equation $\nabla_{V_1}D\zeta+QV_1+\psi V_1=0$ gives

$$e^{-4z}\frac{\partial^2\zeta}{\partial x^2}+2\frac{\partial\zeta}{\partial z}-8+\psi=0,$$
$$e^{-4z}\frac{\partial^2\zeta}{\partial y^2}+2\frac{\partial\zeta}{\partial z}-8+\psi=0,$$

and

$$\frac{\partial^2\zeta}{\partial z^2}-8+\psi=0.$$

The last three equations satisfy simultaneously only when ζ is a constant. Thus we see that the soliton is trivial, which verifies the Theorem 4.3.

Article Information

Acknowledgements: The authors are thankful to the referees for their valuable suggestions towards the improvement of the paper.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] X. Chen, *On almost f-cosymplectic manifolds satisfying the Miao-Tam equation*, J. Geom., **111** (2020), Article No 28.
[2] A. Barros, E. Ribeiro Jr., *Critical point equation on four-dimensional compact manifolds*, Math. Nachr., **287** (2014), 1618–1623.
[3] A. Besse, *Einstein Manifolds*, Springer-Verlag, New York, (2008).
[4] A. M. Blaga, *On harmonicity and Miao-Tam critical metrics in a perfect fluid spacetime*, Bol. Soc. Mat. Mexicana, **26** (2020), 1289–1299.
[5] A. Ghosh, D. S. Patra, *Certain almost Kenmotsu metrics satisfying the Miao-Tam equation*, arXiv:1701.04996v1[Math.DG], (2017).
[6] A. Ghosh, D. S. Patra, *The critical point equation and contact geometry*, arXiv:1711.05935v1[Math.DG], (2017).
[7] T. Mandal, *Miao-Tam equation on almost coKähler manifolds*, Commun. Korean Math. Soc., **37** (2022), 881–891.
[8] T. Mandal, *Miao-Tam equation on normal almost contact metric manifolds*, Differ. Geom.-Dyn. Syst., **23** (2021), 135–143.
[9] D. S. Patra, A. Ghosh, *Certain contact metrics satisfying the Miao-Tam critical condition*, Ann. Polon. Math., **116** (2016), 263–271.
[10] A. Sarkar, G. G. Biswas, *Critical point equation on K-paracontact manifolds*, Balkan J. Geom. Appl., **5** (2020), 117–126.
[11] P. Miao, L.-F. Tam, *On the volume functional of compact manifolds with boundary with constant scalar curvature*, Calc. Var. PDE., **36**(2009), 141–171.
[12] A. E. Fischer, J. Marsden, *Manifolds of Riemannian metrics with prescribed scalar curvature*, Bull. Am. Math. Soc., **80** (1974), 479–484.
[13] O. Kobayashi, *A differential equation arising from scalar curvature function*, J. Math. Soc. Jpn., **34** (1982), 665–675.
[14] R. S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math., **71** (1988), 237–261.
[15] B.-Y. Chen, *A survey on Ricci solitons on Riemannian submanifolds*, Contemp. Math., **674** (2016), 27–39.
[16] A. Sarkar, G. G. Biswas, *Ricci soliton on generalized Sasakian space forms with quasi-Sasakian metric*, Afr. Mat., **31** (2020), 455–463.
[17] Y. Wang, *Ricci solitons on 3-dimensional cosymplectic manifolds*, Math. Slovaca., **67** (2017), 979–984.
[18] Y. Wang, *Ricci solitons on almost coKähler manifolds*, Cand. Math. Bull., **62** (2019), 912–922.
[19] P. Alegre, D. E. Blair, A. Carriazo, *Generalized Sasakian-space-forms*, Israel J. Math., **141** (2004), 157–183.
[20] P. Alegre, A. Carriazo, *Structures on generalized Sasakian space forms*, Differential Geom. Appl., **26**(6) (2008), 656–666.
[21] P. Alegre, D. E. Blair, A. Carriazo, *Generalized Sasakian space-forms*, Israel J. Math., **141** (2004), 157–183.
[22] P. Alegre, A. Carriazo, C. Özgür, S. Sular, *New examples of Generalized Sasakian space-forms*, Proc. Est. Acad. Sci., **60** (2011), 251–257.
[23] U. C. De , A. Sarkar, *Some results on generalized Sasakian Space-forms*, Thai. J. Math., **8** (2010), 1–10.
[24] U. K. Kim, *Conformally flat generalized Sasakian space-forms and locally symmetric generalized Sasakian space-forms*, Note Mat., **26** (2006), 55–67.
[25] A. Sarkar, M. Sen, *Locally ϕ-symmetric generalized generalized Sasakian space-forms*, Ukr. Math. J., **65** (2014), 1588–1597.
[26] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., **19** (1995), 189–214.
[27] D. E. Blair, T. Koufogiorgos, R. Sharma, *A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$*, Kodai Math. J., **13**(1990), 391–401.
[28] J. A. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debreceen, **32** (1985), 187–193.
[29] U. C. De, M. M. Tripathi, *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, Kyungpook Math. J., **43** (2003), 247–255.
[30] S. Deshmukh, M. M. Tripathi, *A note on trans-Sasakian manifolds*, Math. Slovaca, **63** (2013), 1361–1370.
[31] Y. Wang, W. Wang, *A Remark on trans-Sasakian 3-manifolds*, Rev. de la Union Mat. Argentina, **60** (2019), 257–264.
[32] S. Deshmukh, F. Al-Solamy, *A note on compact trans-Sasakian manifolds*, Mediterr. J. Math. **13** (2016), 2099–2104.
[33] S. Deshmukh, *Trans-Sasakian manifolds homothetic to Sasakian manifolds*, Mediterr. J. Math. **13** (2016), 2951–2958.
[34] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, **203** (2010), Birkhäuser, New York.
[35] S. Tanno, *Some transformations on manifolds with almost contact and contact metric structure II*, Tohoku Math. J., **15** (1963), 322–331.
[36] D. S. Patra and A. Ghosh, *The Fischer-Marsden conjecture and contact geometry*, Period. Math. Hungar., **76**(2) (2017), 1–10.

Universal Journal of Mathematics and Applications, 7 (1) (2024) 12-29
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1379260

Measures of Distance and Entropy Based on the Fermatean Fuzzy-Type Soft Sets Approach

Murat Kirişci[1*]

[1]*Department of Biostatistics and Medical Informatics, Cerrahpaşa Medicine Faculty, Istanbul University-Cerrahpaşa, Istanbul, Türkiye*

Article Info

Keywords: *Decision-making, Entropy, Fermatean fuzzy soft set, Missile position estimation*
2010 AMS: *03E72, 94A17, 94D05 (Must be at least one and sequential)*
Received: *21 October 2023*
Accepted: *9 January 2024*
Available online: *15 January 2024*

Abstract

The definition of Fermatean fuzzy soft sets and some of its features are introduced in this study. A Fermatean fuzzy soft set is a parameterized family of Fermatean fuzzy sets and a generalization of intuitionistic and Pythagorean fuzzy soft sets. This paper presents a definition of the Fermatean fuzzy soft entropy. Also acquired are the formulae for standard distance measures such as Hamming and Euclidean distance. Other formulas have also been proposed for calculating the entropy and distance measurements of FFSSs. Even if the entropy and distance measures are defined for other set extensions, they cannot be applied directly to Fermatean fuzzy soft sets. It can be used to determine the uncertainty associated with a Fermatean fuzzy soft set, discover similarities between any two Fermatean fuzzy soft sets using the proposed distance measures, and compare it to other existing structures in the literature. Fermatean fuzzy soft set applications in decision-making and pattern recognition difficulties are also examined. Finally, comparison studies with other known equations are performed.

1. Introduction

1.1. Motivation

In generalized set theory, measures of entropy and distance are crucial notions. Distance is the difference between two patterns. The pattern could be a scalar number, vector, matrix, or other numeric data type. Distance metrics are effective for identifying parallels and differences in patterns. The dissimilarity, or distance, between two patterns, is zero if they are identical. As the difference between patterns develops, so does the dissimilarity or distance. Distance measuring between objects is essential in many fields, including information retrieval, data mining, and machine learning. The items under discussion are frequently made up of numerous components or groups of another object. A distance can be easily defined if the set or tuple is sorted and may be shown as a vector. However, it is expected to mistakenly believe that the $i-th$ index of a vector x equals the $i-th$ index of a vector y. One must rely on a less accurate distance measure if such a correlation does not exist. Entropy measures the degree of ambiguity. A system's entropy is directly proportional to its irregularity. If the entropy of each system is known, one can determine which is more stable. Entropy is the amount of practical work that can be produced from the heat energy emitted into the environment. You cannot work if the heat energy is equal inside and outside the engine. For example, suppose there is an energy difference between the engine and the external environment. The situation changes if it is cold outside and the engine's pistons are hot. The energy will flow from the hot to the cold, and at the same time, it will start the engine. Ludwig Boltzmann later generalized this definition: Accordingly, the particle configurations (low temperature) corresponding to the equal spread of energy in space are different from the combinations of energy concentration at a single point (high temperature). After all, space is more significant than a single point. If the energy is evenly distributed, the entropy always increases because we cannot do work. This is why heat flows from hot to cold. It is not impossible for the air in a room to spontaneously collect in one corner. However, this is such a low probability that you will not see it in your lifetime. Thus, Boltzmann showed that entropy is statistical.

Shannon [1] established the concept of information entropy. In information theory, entropy measures the uncertainty associated with a random variable. Entropy can be conceived of as a system's hidden information. Entropy, in more technical terms, is a measure of the

Email address and ORCID number: **mkirisci@hotmail.com, 0000-0003-4938-5207**
Cite as: M. Kirişci, ***Measures of Distance and Entropy Based on the Fermatean Fuzzy-Type Soft Sets Approach*****, Univers. J. Math. Appl., 7(1) (2024), 12-29.**

amount of information that may be acquired by measuring the system. In this context, very detailed information can be obtained if the air molecule particles collected in the corner of the room are measured. As a result, trillions of particles are measured at once. On the other hand, the air molecules surrounding the room provide more information. As for why the entropy is high, A particle gathered in a single corner of the room gives clear information about other particles. After all, they are all in the same place. However, if one of the air molecules emitted into the room is measured, not much information can be obtained about the location of the other molecules. According to Shannon entropy, the information does not disappear; in this case, it is hidden from view. The number of positional combinations of the air molecules occupying the room is higher than the number of molecules collected in one corner. In short, the more the different combinations of particles that make up a system look like the same thing at first glance (in this case, the air surrounding the room), the more information is hidden. Various entropy and distance measures are available in the literature, helpful in solving real-life problems. FS-type entropy and distance measures were later defined and studied in SS, IFSS, and PFSS. Fuzzy entropy is a term used to indicate the degree of uncertainty, and finding the entropy of a set is one of the essential applications of fuzzy set theory. It has yet to be suggested that these studies' definitions of entropy and distance measures be extended to include FFSSs. Filling this gap is the primary motivation of this study.

1.2. Literature

Multi-Criteria Decision Making (MCDM) is a collection of analytical approaches that evaluate the advantages and disadvantages of alternatives based on many criteria. MCDM methods are used to support the DM process and to select one or more alternatives from a set of alternatives with different characteristics according to conflicting criteria or to rank these alternatives. In other words, in MCDM methods, decision-makers rank the alternatives with different characteristics by evaluating them according to many criteria. MCDM is a set of methods frequently used in all areas of life and at all levels. There are many studies in the literature about MCDM in various fields [2]- [11].

Fuzzy sets(FS) and their expansions are a more effective tool for describing vague and imprecise information and explaining it in a way that is close to human thinking. Although the FSs that emerged with the membership function have made an innovative contribution to the solution of uncertainties, it is impossible to explain the problems and uncertainties in real life only through membership. Real life consists of degrees of non-membership and even hesitations as much as degrees of membership. This situation naturally leaves the solution to uncertainties incomplete. FS expansions proposed by many researchers, especially Atanassov [12], have been powerful tools to solve the problem. However, FSs must be more comprehensive in explaining uncertainties in real-life problems. Despite all of the possible responses, these theories have several drawbacks. These limitations include the inability to properly consider the parametrization tool and how to set the membership function for each unique item. Because of these restrictions, it is challenging for DMRs to make wise judgments throughout the analysis.

Since the formation of the membership function (MF) differs for each individual, the formation of more than one MF and its belonging to the set varies according to everyone. Thereupon, Molodtsov [13] initiated the SS theory. While the soft set (SS) theory deals with the set-valued function, FSs remove the uncertainty with the real-valued function. The problem of establishing an MF does not exist in the SS. So, the SS is much more useful. An SS is a parameterized family of sets extended into different hybrid structures, such as Fuzzy soft sets (FSS), intuitionistic fuzzy soft sets (IFSS), and Pythagorean fuzzy soft sets (PFSS). Since the Fermatean fuzzy set (FFS) can deal with vagueness or uncertainty, the parameterized family of FFSs, the Fermatean fuzzy soft sets (FFSS), also performs well. An FFS is obtained in the case of FFSSs, corresponding to each parameter. FFSs can manage several real-life situations where the intuitionistic fuzzy sets (IFS) and Pythagorean fuzzy sets (PFS) fail to explain. Suppose there is a case in which someone expresses his satisfaction to particular criteria as 0.6, and the degree of dissatisfaction is 0.7. Then, their sum exceeds one, but the square sum does not. So, FFSs can handle this. Thus, a FFSS is an effective parameterizing tool and an excellent medium to represent vagueness in many real-life situations. An SS is a parameterized family of sets that can be expanded into various hybrid structures such as FSSs, IFSSs, and PFSSs. The parameterized family of FFSs, the FFSS, also performs well because the FFS is highly competent in dealing with vagueness or uncertainty. In the case of FFSSs, an FFS is obtained for each parameter. FFSs can handle a variety of real-world circumstances that IFSs and PFSs cannot explain. Assume someone expresses his pleasure with specific criteria as 0.7 and his degree of discontent is 0.8. The amount then surpasses one, but not the square sum. As a result, FFSs can manage it. As a result, an FFSS is an effective parameterizing tool and an ideal medium for representing ambiguity in many real-world circumstances.

The concept of an FS proposed by Zadeh [14] was used to demonstrate the ambiguity and vagueness of a membership degree(MD). The IFS developed by Atanassov [12] can more fully explain evaluation information by linking an element's non-membership degree(ND) to an item. In light of the IFS mentioned above weakness, Yager [15] pioneered the PFS concept to increase the range of MD and ND so that $MD^2 + ND^2 \leq 1$. As a result, PFS provides additional evaluation opportunities for professionals to voice their opinions on numerous objectives. As the decision-making environment becomes more complex, it becomes increasingly challenging for professionals to provide more credible evaluation information. The notions of IFS and PFS have been supported to impact the vagueness and ambiguity created by the complicated subjectivity of human cognition. The FFS was the first to broaden the area of information statements by including the cubic sum of MD and ND. As a result, FFS is a more efficient and practical strategy than IFS and PFS for dealing with indeterminacy of choice difficulties. Due to its advantages in displaying ambiguous information and providing additional possibilities for professionals, academics have pushed to create many DM systems to handle genuine DM and evaluation problems.

Senepati and Yager [16] are the creators of the FFS. FFSs explain uncertainties better than IFSs and PFSs. Senapati and Yager [17] expanded on this article by examining a range of novel operations and arithmetic mean techniques over FFSs. They used the FF-weighted product model to handle MCDM problems. FFS-related novel aggregation operators have been defined, and [18] has investigated their properties. Many studies on FFS have appeared in the literature([17]- [31]).

The SS defines a distinct scenario to address ambiguity and vagueness [13]. A set of features produces a family of subsets regarded as approximation definitions of a notion (one for each property-defined viewpoint). Many academics with diverse interests were rapidly drawn to soft sets([26], [32]- [40]). With the advancement of communication and technology, many complex topics require more than one analytical instrument. In this respect, Maji et al. [35] demonstrated that FS and SS theories can coexist. Many articles ([41]- [46]) studied these models further. Researchers expanded on this sort of hybridization ([40], [47]- [49]). Of course, the motivations for studying these generalizations of IFS sets are reasonable.

In FS theory, which has gotten much attention in recent years, the distance measure is useful for representing the difference between two FSs. Many authors have proposed various distance measurements for IFSs and PFSs in IFS and PFS theories. Szmidt and Kacprzyk [50] defined and explained various distance measures for IFSs using the geometric method. Several forms of distance and similarity measurements for FS, IFS, and PFSs have been introduced since the evolution of FS theory. Several scholars have recently focused on distance or similarity measurements, which are significant mathematical instruments in DM and pattern recognition problems([51]- [58]). There are also new FFSS studies in the literature([19]- [21], [27], [59], [60]).

In the FSs theory, Zadeh [61] was the first to mention entropy as a measure of fuzziness or ambiguous information. De Luca and Termini [62] defined the entropy of FSs using Shannon's function and gave the hypotheses that the fuzzy entropy must follow. According to information entropy, the volume and quality of accessible information are the most significant factors of the accuracy and reliability of the decision to be made in a DM situation [63]. Kaufmann [64] showed how to determine an FS's entropy by measuring the distance between the FS and the nearest crisp set. In contrast, Yager [65] measured the distance between the FS and its complement.

Higashi and Klir [66] expanded Yager's [65] approach to a fairly general class of fuzzy complements. Using entropy in DM processes increases the method's usefulness in uncertain contexts because entropy is crucial for gauging uncertain information. Mohagheghi et al. [67] used the idea of entropy to weigh the importance of each criterion. So, the criteria' relevance was addressed directly by the DMs and indirectly by computing a weight based on the ideas obtained. Peng et al. [68] provide axiomatic definitions of PF-information metrics such as entropy, distance measure, inclusion measure, and similarity measure.

1.3. Necessity

"Keep the certain, avoid the uncertain" instructions are familiar to everyone. A person tends to choose what he knows, even if it is terrible, because that way, he feels more secure. This makes people distant and anxious about the "new". The goal is to live safely in a clear and specific world. The serenity of foretelling what might happen to one makes the story, which promises pain, very appealing. Unexpected situations, social protests, the virus, the evolution of education, and natural disasters convince us that uncertainty is life itself. In other words, uncertainty is an inevitable human reality. However, man is not faced with uncertainties only outside himself. One's expressions, expectations, what one does, and what one wants to do always contain uncertainties. Therefore, people have to make decisions based on these uncertainties at all times and everywhere, from the work they do during the day to the work they plan to do in the future.

The cognitive continuum, which leads to choosing a faith or plan of action from a range of potential possibilities, is known as decision-making. It might be logical or illogical. Making a choice is a method of deliberation based on the decision-maker's (DMR) values, preferences, and beliefs. Every decision-making(DM) continuum ends with a final alternative that may or may not be followed by action. Data are used in DM to minimize or completely remove ambiguity. Decisions are regularly taken while it is unknown whether they will be beneficial or harmful. Due to time constraints, a lack of knowledge, carelessness, and their ability to comprehend information, decision-makers often need more clarity and accurate information to address problems.

IFS and PFS-based techniques, by definition, cannot capture data in FFS format. It is also evident that IFS and PFS-based decision-making systems cannot withstand scenarios in which experts supply preference values in FFSs. Approaches based on FFS, such as the recommended technique, effectively obtain and analyze information to rate available options based on predefined criterion values. The FFS environment manages more information and covers a broader range of themes for dealing with uncertain information since it includes both IFSs and PFSs into a single platform. As a result, more information is needed in this collection.

An SS is a bag that contains an approximate representation of the objects. It is made up of two parts: a predicate and an approximate value set. It states the object-related information more accurately and precisely. Traditional mathematics machinery fails because the beginning description is approximate; however, the SS can manage several challenges in this respect. As a result, it is an effective instrument for dealing with ambiguous and perplexing knowledge during the DM process. Fuzzy-type soft sets (FSS, IFSS, PFSS, and others) outperform all other mathematical tools and produce significantly better outcomes, particularly in decision-making procedures. Existing SSs from the IFS and PFS are considered a subset of the proposed SSs. Furthermore, the proposed SSs can handle more data than the current ones. As a result, the presented method contains significantly more information than existing methods for dealing with data uncertainties in the IFS and PFS contexts.

Since the Fermatean fuzzy soft set is an excellent tool to handle more than IFSS and PFSS, obtaining the entropy is also relevant. It is discovered that entropy can be used in DM problems, and hence, DM problems handle certain broader domains. Distance measures can be used to solve problems such as DM, pattern identification, and machine learning. Hamming distance and Euclidean distance are the most commonly used distance measurements. The measure of distance is inversely proportional to the measure of similarity. As a result, determining the similarity between sets is beneficial. Senapati and Yager [16] provide an FFS distance measure and illustrate with a numerical example that the proposed distance measures are realistic and appropriate. This definition of FFSs is expanded to define the FFSS distance measure-both the proposed entropy and distance measurements aid in adequately understanding real-life events. Based on the inspiration of the soft set structure and the benefits of FFSs in dealing with uncertain and imprecise information, this work investigates the theory of FFSSs by establishing some new information measures called entropy and distance measures.

1.4. Contribution

The core contributions of the present work can be expressed as:

(i) The article sets up a Fermatean fuzzy-type soft set. Further, the main structures of FFSSs are investigated.
(ii) New measures of distance and entropy based on FFSSs are provided to measure uncertain information.
(iii) The theoretical background of new measures of distance and entropy has been given in detail.
(iv) DM algorithms related to new distance and entropy measures have been given.
(v) The suggested techniques were corroborated by numerical examples related to medical DM ve PR.
(vi) The new method is compared with PFS elements to see the advantages.

The following are the benefits of this work:

(i) It can be used to determine the level of uncertainty associated with an FFSS.
(ii) It can be used to identify the similarity between any two FFSSs using the provided distance measures.
(iii) It is comparable to other existing structures in the literature.

Entropy and distance metrics in another type of generalized structure will be compared in future work. In addition, topological, algebraic, and order theoretical structures for FFSSs can be introduced and examined.

Structure: We propose the notion of a FFSS. The essential characteristics of FFSS, such as FFS-subsets, "AND" and "OR" operators, union, intersection, and complement, are investigated in Section 3. Section 4 focused on the measures of entropy and distance concerning FFSS. Novel methods for DM and PR problems are devised, and concrete examples are provided in Section 5.

2. Preliminaries

Let $Z=\{z_1,z_2,\dots,z_n\}$ and $E=\{e_1,e_2,\dots,e_m\}$ be the universal and the parameter sets, respectively.

Definition 2.1.
- *An fuzzy set FS is defined as $F=\{(z,m_F(z)):z\in Z\}$, where $m_F(z):Z\to[0,1]$ is called MD [14].*
- *An IFS F is defined as $F=\{(z,m_F(z),n_F(z)):z\in Z\}$ such that $m_F(z)+n_F(z)\le 1$, where $m_F(z),n_F(z):Z\to[0,1]$ is called MD and ND, respectively [12].*
- *An PFS F is defined as $F=\{(z,m_F(z),n_F(z)):z\in Z\}$ such that $m_F^2(z)+n_F^2(z)\le 1$, where $m_F(z),n_F(z):Z\to[0,1]$ [15].*

Definition 2.2 ([16]). *The set $F=\{<k,m_F(k),n_F(k)>:k\in Z\}$ is called FFS, where $0\le m_F^3(k)+n_F^3(k)\le 1$ and $m_F,n_F:Z\to[0,1]$.*

The hesitancy degree (HD) of F is $h_F(k)=\sqrt[3]{1-(m_F^3(k)+n_F^3(k))}$.
For FFSs $F=(m_F,n_F)$, $F_1=(m_{F_1},n_{F_1})$ and $F_2=(m_{F_2},n_{F_2})$, [16]:

(i) $F_1\cap F_2=[\min(m_{F_1},m_{F_2}),\max(n_{F_1},n_{F_2})]$;
(ii) $F_1\cup F_2=[\max(m_{F_1},m_{F_2}),\min(n_{F_1},n_{F_2})]$;
(iii) $F^t=(n_F,m_F)$;
(iv) $F_1\boxplus F_2=\left(\sqrt[3]{m_{F_1}^3+m_{F_2}^3-m_{F_1}^3m_{F_2}^3},n_{F_1}n_{F_2}\right)$;
(v) $F_1\boxtimes F_2=\left(m_{F_1}m_{F_2},\sqrt[3]{n_{F_1}^3+n_{F_2}^3-n_{F_1}^3n_{F_2}^3}\right)$;
(vi) $\alpha F=\left(\sqrt[3]{1-(1-m_F^3)^\alpha},n_F^\alpha\right)$;
(vii) $F^\alpha=\left(m_{F_1}^3,\sqrt[3]{1-(1-n_F^3)^\alpha}\right)$.

The properties of complement of FFS [16]:

(i) $(F_1\cap F_2)^c=F_1^c\cup F_2^c$;
(ii) $(F_1\cup F_2)^c=F_1^c\cap F_2^c$;
(iii) $(F_1\boxplus F_2)^c=F_1^c\boxtimes F_2^c$;
(iv) $(F_1\boxtimes F_2)^c=F_1^c\boxplus F_2^c$;
(v) $\alpha(F)^c=(F^\alpha)^c$;
(vi) $(F^c)^\alpha=(\alpha F)^c$.

Proposition 2.3 ([16]). *Let three FSSs F,G,H. Then,*

(i) If $F\subseteq G$ and $G\subseteq H$, then $F\subseteq H$;
(ii) $(F^c)^c=F$;
(iii) The properties commutative, associative, and distributive are applied for the union and intersection;
(iv) Union and intersection provide DeMorgan's laws.

Definition 2.4. *The soft sets (SS) are a parameterized family of subsets of Z. That is, for the function $F:E\to SS(Z)$, (F,E) is denoted a SS, where $SS(Z)$ is a set of all subsets of Z.*

According to Definition 2.4, if $SS(Z)$ is selected as Z's F-, IF-, and PF-subsets, then (F,N) will be defined as fuzzy soft set (FSS) [35], intuitionistic fuzzy soft set (IFSS) [36] and Pythagorean fuzzy soft set (PFSS) [40], respectively. The definitions of FSS, IFSS, and PFSS are given as follows:

Definition 2.5.
- *The pair (F,E) is called FFS, if the function $F : E \to FS(Z)$ is a mapping from E into set of all fuzzy sets in Z, where if $FS(Z)$ is a set of all subsets of Z [35].*
- *The pair (F,E) is called IFFS, if the function $F : E \to IS(Z)$ is a mapping from E into set of all intuitionistic fuzzy power sets in Z, where if $IS(Z)$ is a set of all subsets of Z [36].*
- *The pair (F,E) is called PFFS, if the function $F : E \to PS(Z)$ is a mapping from E into set of all Pythagorean fuzzy sets in Z, where if $PS(Z)$ is a set of all subsets of Z [40].*

Definition 2.6. *Let d be a mapping $d : IFSS(Z) \times IFSS(Z) \to \mathbb{R}^+ \bigcup\{0\}$, where $\mathbb{R}^+ \bigcup\{0\}$ denotes the set of non-negative real numbers. For two $IFSS(Z)$ A,B, if $d(A,B)$ satisfies the following properties:*

- $d(A,B) \geq 0$;
- $d(A,B) = d(B,A)$;
- $d(A,B) = 0$ *if and only if* $A = B$;
- *For any* $C \in IFSS(Z)$, $d(A,B) + d(B,C) \geq d(A,C)$.

Then $d(A,B)$ is a distance measure between IFSSs A and B [69].

If the sets A and B in Definition 2.6 are taken as PFSSs and $d : PFSS(Z) \times PFSS(Z) \to \mathbb{R}^+ \bigcup\{0\}$, then the d transformation is called the "distance measure between PFSSs A and B" [32].

Definition 2.7. *A real function $T : IFFS(Z) \to \mathbb{R}^+$ is called a intuitionistic fuzzy soft entropy(IFSE) on $IFFS(Z)$ [69], if T has following properties;*

- $T(p) = 0$ *if and only if* $p \in S(Z)$.
- *Let* $p = (F,E) = [a_{ij}]_{m\times n}$, $T(p) = mn$ *if and only if* $m_{F(e)}(z) = 0 = n_{F(e)}(z)$, $\forall e \in E, \forall z \in Z$.
- $T(p) = T(p^c)$ $p \in IFSS(Z)$.
- *If* $p \preceq \bar{p}$, *then* $T(p) \geq T(\bar{p})$ *where* $(F,T) = p$ *and* $(G,T) = \bar{p}$.

If $IFSS(Z)$ in Definition 2.7 is taken as $PFSS(Z)$, then the T transformation is called a Pythagorean fuzzy soft entropy(PFSE) on $PFFS(Z)$ [32].

3. Fermatean Fuzzy Soft Sets

Definition 3.1. *For $M \subseteq E$, the FFSSs is defined as the pair (F,M) where $F : E \to FFS(Z)$ and $FFS(Z)$ is the set of all Fermatean fuzzy subsets of Z.*

For any parameter $e \in E$, $F(e)$ can be wirtten as a FFS such that

$$F(e) = \{(z, m_{F(e)}(z), n_{F(e)}(z)) : z \in Z\}$$

where $m_{F(e)}(z)$ and $n_{F(e)}(z)$ are the MD and ND with condition $m^3_{F(e)}(z) + n^3_{F(e)}(z) \leq 1$. Further, $h_{F(e)}(z) = \sqrt[3]{1 - (m_{F(e)}(z))^3 - (n_{F(e)}(z))^3}$.

Example 3.2. *The diseases set $Z = \{z_1, z_2, z_3\}$ and the symptoms set $M = \{e_1 = symptom1, e_2 = symptom2, e_3 = symptom3\}$. Hence*

$$F(e_1) = \{< z_1, 0.6, 0.9) >, < z_2, 0.8, 0.7) >, < z_3, 0.8, 0.9) >\}$$
$$F(e_2) = \{< z_1, 0.7, 0.9) >, < z_2, 0.9, 0.5) >, < z_3, 0.8, 0.8) >\}$$
$$F(e_3) = \{< z_1, 0.8, 0.7) >, < z_2, 0.8, 0.9) >, < z_3, 0.9, 0.6) >\}$$

and table representation as follows (Table 1):

	z_1	z_2	z_3
e_1	(0.6, 0.9)	(0.8, 0.7)	(0.8, 0.9)
e_2	(0.7, 0.9)	(0.9, 0.5)	(0.8, 0.8)
e_3	(0.8, 0.7)	(0.8, 0.9)	(0.9, 0.6)

Table 1: (F,M)

Definition 3.3. *Let $M,N \subset E$, and $(F,M),(G,N)$ be two FFSS (F,M) is called a FF soft subset of (G,N) (denoted by $(G,N)\hat{\subset}(F,M)$) if*

(i) $M \subseteq N$,
(ii) *For all* $z \in Z$, $e \in M$, $m_M(z) \geq m_N(z)$ *and* $n_M(z) \leq n_N(z)$.

Example 3.4. *Let $M = \{e_1 = symptom1\} \subset E$. Hence, we can written FFSS (G,N) as:*

$$G(e_1) = \{< z_1, 0.6, 0.8 >, < z_2, 0.6, 0.8) >, < z_3, 0.7, 0.9) >\}$$

This shows us that $(G,N)\hat{\subset}(F,M)$.

Definition 3.5. *Choose the two FFSS $(F,M),(F,N)$.*

(i) $(F,N)\hat{=}(F,M)$, *if* $(G,N)\hat{\subset}(F,M)$ *and* $(F,M)\hat{\subset}(G,N)$.
(ii) *The complement of (f,M) is identified $(F,M)^c$, where $F^c : M \to FFSS(Z)$ and $F^c(e) = (F(e))^c$ for every $e \in M$.*

Further, $((F,M)^c)^c = (F,M)$.

Example 3.6. *From Example 3.4. If*

$$G(e_1) = \{< z_1, 0.6, 0.8) >, < z_2, 0.5, 0.7) >, < z_3, 0.7, 0.6) >\},$$

then

$$G^c(e_1) = \{< z_1, 0.8, 0.6) >, < z_2, 0.7, 0.5) >, < z_3, 0.6, 0.7) >\}.$$

Definition 3.7. *Choose the two FFSS $(F,M),(G,N)$.*

- ***AND Operator:*** *(F,M) AND (G,N) is FFSS denoted by $(F,M)\wedge(G,N)$ is defined by $(F,M)\wedge(G,N) = (H, A\times B)$ where $H(\alpha,\beta) = F(\alpha)\cap G(\beta)$, $\forall\alpha,\beta\in A\times B$. That is, $H(\alpha,\beta)(z) = (z, \min\{m_{F(\alpha)(z)}, m_{G(\beta)(z)}\}, \max\{n_{F(\alpha)(z)}, n_{G(\beta)(z)}\})$, $\forall\alpha,\beta\in A\times B$ and $\forall z\in Z$.*
- ***OR Operator:*** *(F,M) OR (G,N) is FFSS denoted by $(F,M)\vee(G,N)$ is defined by $(F,M)\vee(G,N) = (H, A\times B)$ where $H(\alpha,\beta) = F(\alpha)\cup G(\beta)$, $\forall\alpha,\beta\in A\times B$. That is, $H(\alpha,\beta)(z) = (z, \max\{m_{F(\alpha)(z)}, m_{G(\beta)(z)}\}, \min\{n_{F(\alpha)(z)}, n_{G(\beta)(z)}\})$, $\forall\alpha,\beta\in A\times B$ and $\forall z\in Z$.*

Example 3.8. *Choose $N = \{e_1, e_2\}$. Then, FFSS (G,N) as:*

$$G(e_1) = \{< z_1, 0.6, 0.8) >, < z_2, 0.4, 0.8) >, < z_3, 0.8, 0.5) >\},$$
$$G(e_2) = \{< z_1, 0.6, 0.7) >, < z_2, 0.6, 0.4) >, < z_3, 0.9, 0.4) >\}.$$

It is seen that $(F,M)\hat{\subset}(G,N)$. AND and OR operations are shown by Tables 2 and 3.

	(e_1,e_1)	(e_1,e_2)	(e_2,e_1)	(e_2,e_2)	(e_3,e_1)	(e_3,e_2)
z_1	(0.6, 0.9)	(0.6, 0.9)	(0.6, 0.8)	(0.6, 0.7)	(0.6, 0.9)	(0.4, 0.9)
z_2	(0.4, 0.9)	(0.6, 0.9)	(0.4, 0.8)	(0.6, 0.5)	(0.4, 0.8)	(0.6, 0.8)
z_3	(0.8, 0.7)	(0.8, 0.7)	(0.8, 0.9)	(0.8, 0.9)	(0.8, 0.6)	(0.9, 0.6)

Table 2: $(F,M)\wedge(G,N)$

	(e_1,e_1)	(e_1,e_2)	(e_2,e_1)	(e_2,e_2)	(e_3,e_1)	(e_3,e_2)
z_1	(0.6, 0.8)	(0.6, 0.7)	(0.8, 0.7)	(0.8, 0.7)	(0.8, 0.8)	(0.8, 0.7)
z_2	(0.7, 0.8)	(0.7, 0.4)	(0.9, 0.5)	(0.9, 0.4)	(0.8, 0.8)	(0.8, 0.4)
z_3	(0.8, 0.5)	(0.9, 0.4)	(0.8, 0.5)	(0.9, 0.4)	(0.9, 0.5)	(0.9, 0.4)

Table 3: $(F,M)\vee(G,N)$

Theorem 3.9. *For two FFSSs (F,M) and (G,N),*

- *(i.) $((F,M)\wedge(G,N))^c = (F,M)^c\vee(G,N))^c$*
- *(ii.) $((F,M)\vee(G,N))^c = (F,M))^c\wedge(G,N))^c$.*

Proof. (i) First, take $(F,M)\wedge(G,N) = (H, M\times N)$, where $H(\alpha,\beta) = F(\alpha)\cap G(\beta)$, $\forall\alpha,\beta\in M\times N$. That is,

$$H(\alpha,\beta)) = (z, \min\{m_{F(\alpha)}(z), m_{G(\beta)}(z)\}, \max\{n_{F(\alpha)}(z), n_{G(\beta)}(z)\}), \quad \text{for all} \quad (\alpha,\beta)\in M\times N \quad \text{and} \quad z\in Z.$$

Second, $((F,M)\wedge(G,N))^c = (H, M\times N)^c = (H^c, M\times N)$. That is, $\forall\alpha,\beta\in M\times N$ and for all $z\in Z$,

$$H^c(\alpha,\beta)) = (z, \max\{n_{G(\alpha)}, n_{G(\beta)}\}, \min\{m_{F(\alpha)}, m_{F(\beta)}\}). \tag{3.1}$$

Let $(F,M)^c\vee(G,N)^c = (F^c,M)\vee(G^c,N) = (I,(M\times N))$, where $I(\alpha,\beta)) = F^c(\alpha)\cup G^c(\beta)$, $\forall(\alpha,\beta)\in M\times N$. So, for $z\in Z$, we get

$$\begin{aligned} I(\alpha,\beta) &= (z, \max\{m_{F^c(\alpha)}(z), m_{G^c(\beta)}(z)\}, \min\{n_{F^c(\alpha)}(z), n_{G^c(\beta)}(z)\}) \\ &= (z, \max\{n_{F(\alpha)}(z), n_{G(\beta)}(z)\}, \min\{m_{F(\alpha)}(z), m_{G(\beta)}(z)\}) \end{aligned} \tag{3.2}$$

We obtain $((F,M)\wedge(G,N))^c = (F,M)^c\vee(G,N)^c$, from (3.1) and (3.2).

(ii) can be proved similarly to (i).

□

Definition 3.10. *For two FFSSs (F,M) and (G,N), the union (H,P) of (F,M) and (G,N) $(F,M)\cup(G,N) = (H,P)$, is described as*

$$H(e) = \begin{cases} F(e) & , \quad e\in M/N \\ G(e) & , \quad e\in N/M \\ F(e)\cup G(e) & , \quad e\in M\cap N. \end{cases}$$

if $P = M\cup N$ and for all $e\in P$. So, for all $e\in M\cap N$, we have $F(e)\cup G(e) = (z, \max(m_{F(e)}(z), m_{G(e)}(z)), \min(n_{F(e)}(z), n_{G(e)}(z)) >: z\in Z$.

Theorem 3.11. *The union (H,P) is a FFSS.*

Proof. Using Definition 3.10, $\forall e \in P$ if $e \in M/N$ or $e \in N/M$, then $H(e) = F(e)$ or $H(e) = G(e)$. Therefore, $H(e)$ is FFSS.

If $e \in M \cap N$, for a fixed $z \in Z$, consider $m_{F(e)}(z) \leq m_{G(e)}(z)$, then,

$$\begin{aligned} m^3_{H(e)}(z) + n^3_{H(e)}(z) &= (m^3_{F(e)}(z) \vee m^3_{G(e)}(z)) + (n^3_{F(e)}(z) \wedge n^3_{G(e)}(z)) \\ &= m^3_{F(e)}(z) + (n^3_{F(e)}(z) \wedge n^3_{G(e)}(z)) \\ &\leq m^3_{G(e)}(z) + n^3_{G(e)}(z) \leq 1. \end{aligned}$$

Then, union (H,P) is a FFSS. □

Definition 3.12. *For two FFSSs (F,M) and (G,N), the union (H,P) of (F,M) and (G,N) $(F,M) \cup (G,N) = (H,P)$, is described as*

$$H(e) = \begin{cases} F(e) & , \quad e \in M/N \\ G(e) & , \quad e \in N/M \\ F(e) \cap G(e) & , \quad e \in M \cap N. \end{cases}$$

if $P = M \cup N$ and for all $e \in P$. So, for all $e \in M \cap N$, we have $F(e) \cap G(e) = (z, \min(m_{F(e)}(z), m_{G(e)}(z)), \max(n_{F(e)}(z), n_{G(e)}(z))) : z \in Z$.

Theorem 3.13. *The intersection (H,P) is a FFSS.*

Proof. Using Definition 3.12, $\forall e \in P$ if $e \in M/N$ or $e \in N/M$, then $H(e) = F(e)$ or $H(e) = G(e)$. Therefore, $H(e)$ is FFSS.

If $e \in M \cap N$, for a fixed $z \in Z$, consider $n_{F(e)}(z) \leq n_{G(e)}(z)$, then, we have,

$$\begin{aligned} m^3_{H(e)}(z) + n^3_{H(e)}(z) &= (m^3_{F(e)}(z) \wedge m^3_{G(e)}(z)) + (n^3_{F(e)}(z) \vee n^3_{G(e)}(z)) \\ &= (m^3_{F(e)}(z) \wedge m^3_{G(e)}(z)) + (n^3_{F(e)}(z) \wedge n^3_{G(e)}(z)) \\ &\leq m^3_{F(e)}(z) + n^3_{G(e)}(z) \leq 1. \end{aligned}$$

Hence, intersection (H,P) is a FFSS. □

Theorem 3.14. *Let (F,M), (G,N) and (H,P) be three FFSSs.*

(i) $(F,M) \cup (F,M) = (F,M)$
(ii) $(F,M) \cap (F,M) = (F,M)$
(iii) $(F,M) \cup (G\ N) = (G,N) \cup (F,M)$
(iv) $(F,M) \cap (G,N) = (G,N) \cap (F,M)$
(v) $((F,M) \cup (G,N)) \cup (H,P) = (F,M) \cup ((G,N) \cup (H,P))$
(vi) $((F,M) \cap (G,N)) \cap (H,P) = (F,M) \cap ((G,N) \cap (H,P))$.

Proof. The proof is obtained by Proposition (2.3), Definitions (3.10) and (3.12). □

Theorem 3.15. *Let (F,M) and (G,N) be two FFSSs.*

(i) $((F,M) \cap (G,N))^c = (F,M)^c \cup (G,N)^c$
(ii) $((F,M) \cup (G,N))^c = (F,M))^c \cap (G,N))^c$

Proof. If we take $P = M \cup N$ and $e \in P$, then $(F,M) \cap (G,N) = (H,P)$,

$$H(e) = \begin{cases} F(e) & , \quad e \in M/N \\ G(e) & , \quad e \in N/M \\ F(e) \cap G(e) & , \quad e \in M \cap N. \end{cases}$$

So, for all $e \in M \cap N$, we have $(F(e) \cap G(e) = (z, \min(m_{F(e)}(z), m_{G(e)}(z)), \max(n_{F(e)}(z), n_{G(e)}(z)) : z \in Z)$. So that, $((F,M) \cap (G,N))^t = (H,P))^c$ and $H^c(e) = (H(e))^c$. Then,

$$(H(e))^c = \begin{cases} (F(e))^c & , \quad e \in M/N \\ (G(e))^c & , \quad e \in N/M \\ (F(e) \cap G(e))^c & , \quad e \in M \cap N. \end{cases}$$

That is, $\forall e \in M \cap N$, we get

$$\begin{aligned} (F(e) \cap G(e))^c &= (z, \min(m_{F(e)}(z), m_{G(e)}(z)), \max(n_{F(e)}(z), n_{G(e)}(z)) : z \in Z >^c \\ &= (z, \max(n_{F^c(e)}(z), n_{G^c(e)}(z)), \min(m_{F^c(e)}(z), m_{G^c(e)}(z)) : z \in Z). \end{aligned}$$

Now, $(F,M))^c = (F^c,M)$ and $(G,N))^c = (G^c,N)$. So that $(F,M)^c \cup (G,N)^c = (F^c,M) \cup (G^c,N) = (H^c,P)$, where $P = M \cup N$ and

$$H^c(e) = \begin{cases} F^c(e) & , \quad e \in M/N \\ G^c(e) & , \quad e \in N/M \\ F(e)^c \cap G^c(e) & , \quad e \in M \cap N. \end{cases}$$

So, for all $e \in M \cap N$, $F^c(e) \cup G^c(e) = (k, \max(n_{F^c(e)}(k), n_{G^c(e)}(k)), \min(m_{F^c(e)}(k), m_{G^c(e)}(k)) : k \in Z)$. Hence, $((F,M) \cap (G,N))^c = (F,M)^c \cup (G,N)^c$. □

Example 3.16. *Take the diseases set* $Z = \{z_1, z_2, z_3, z_4\} = \{disease1, disease2, disease3, disease4\}$. *Select* $E = \{e_1, e_2, e_3, e_4, e_5\} = \{symptom1, symptom2, symptom3, symptom4, symptom5\}$ *as parameter set. Consider that* (F,M), $(F,M)^c$, (G,N), *and* (H,P) *are four FFSSs over* Z *given by* $M = \{e_1, e_2\}$, $N = \{e_1, e_2, e_4\}$ *and* $P = \{e_1, e_3, e_4\}$ *defined as follows (Tables (4)-(7)) :*

	e_1	e_2
z_1	(0.64, 0.88)	(0.81, 0.72)
z_2	(0.73, 0.79)	(0.94, 0.53)
z_3	(0.85, 0.59)	(0.92, 0.49)
z_4	(0.83, 0.67)	(0.67, 0.85)

Table 4: (F,M)

	e_1	e_2
z_1	(0.88, 0.64)	(0.72, 0.81)
z_2	(0.79, 0.73)	(0.53, 0.94)
z_3	(0.59, 0.85)	(0.49, 0.92)
z_4	(0.67, 0.83)	(0.85, 0.67)

Table 5: (F^c,M)

Table 6: (G,N)

	e_1	e_2	e_4
z_1	(0.82, 0.73)	(0.92, 0.57)	(0.85, 0.67)
z_2	(0.66, 0.78)	(0.75, 0.62)	(0.54, 0.91)
z_3	(0.84, 0.49)	(0.72, 0.39)	(0.71, 0.81)
z_4	(0.43, 0.87)	(0.67, 0.59)	(0.76, 0.37)

	e_1	e_3	e_4
z_1	(0.44, 0.95)	(0.57, 0.69)	(0.86, 0.59)
z_2	(0.56, 0.81)	(0.68, 0.69)	(0.79, 0.38)
z_3	(0.68, 0.56)	(0.92, 0.35)	(0.72, 0.65)
z_4	(0.63, 0.76)	(0.84, 0.37)	(0.95, 0.29)

Table 7: (H,P)

For the four FFSSs, the operations are in Tables 8 -11:

	e_1	e_2	e_3	e_4
z_1	(0.64, 0.88)	(0.81, 0.72)	(0.57, 0.69)	(0.86, 0.59)
z_2	(0.73, 0.81)	(0.94, 0.53)	(0.68, 0.69)	(0.79, 0.38)
z_3	(0.85, 0.56)	(0.92, 0.49)	(0.92, 0.35)	(0.72, 0.65)
z_4	(0.83, 0.67)	(0.67, 0.85)	(0.84, 0.37)	(0.95, 0.29)

Table 8: $(F,M) \cup (H,P)$

	e_1	e_2	e_3	e_4
z_1	(0.44, 0.95)	(0.81, 0.72)	(0.57, 0.69)	(0.86, 0.59)
z_2	(0.56, 0.81)	(0.94, 0.53)	(0.68, 0.69)	(0.79, 0.38)
z_3	(0.68, 0.59)	(0.92, 0.49)	(0.92, 0.35)	(0.72, 0.65)
z_4	(0.63, 0.76)	(0.67, 0.85)	(0.84, 0.37)	(0.95, 0.29)

Table 9: $(F,M) \cap (H,P)$

	(e_1,e_1)	(e_1,e_2)	(e_1,e_4)	(e_2,e_1)	(e_2,e_2)	(e_2,e_4)
z_1	(0.64, 0.88)	(0.64, 0.88)	(0.64, 0.88)	(0.81, 0.73)	(0.81, 0.72)	(0.81, 0.72)
z_2	(0.66, 0.79)	(0.73, 0.79)	(0.54, 0.91)	(0.66, 0.78)	(0.75, 0.62)	(0.54, 0.91)
z_3	(0.84, 0.59)	(0.72, 0.59)	(0.71, 0.81)	(0.84, 0.49)	(0.72, 0.49)	(0.71, 0.81)
z_4	(0.43, 0.87)	(0.67, 0.67)	(0.76, 0.67)	(0.43, 0.87)	(0.67, 0.85)	(0.67, 0.85)

Table 10: $(F,M)\wedge(G,N)$

	(e_1,e_1)	(e_1,e_2)	(e_1,e_4)	(e_2,e_1)	(e_2,e_2)	(e_2,e_4)
z_1	(0.82, 0.73)	(0.92, 0.57)	(0.85, 0.67)	(0.82, 0.72)	(0.92, 0.57)	(0.85, 0.67)
z_2	(0.73, 0.78)	(0.75, 0.62)	(0.73, 0.79)	(0.94, 0.53)	(0.94, 0.53)	(0.94, 0.53)
z_3	(0.85, 0.49)	(0.85, 0.39)	(0.85, 0.59)	(0.92, 0.49)	(0.92, 0.39)	(0.92, 0.49)
z_4	(0.83, 0.67)	(0.83, 0.59)	(0.83, 0.37)	(0.67, 0.85)	(0.67, 0.59)	(0.76, 0.37)

Table 11: $(F,M)\vee(G,N)$

4. Fermatean Fuzzy Measures

A crucial technique for quantifying uncertain information is entropy. One can quickly determine whether information is more stable if the entropy is lower because lower entropies also mean lower levels of uncertainty. Due to its greater generalization, the FFSS can represent information where other structures cannot. Therefore, introducing the measure of entropy is crucial in the current situation. The equations for entropy and distance measure for FFSSs are obtained and demonstrated with samples in this section by introducing various concepts and results.

4.1. Entropy meausre

Definition 4.1. *Take the two FFSSs (F,M) and (G,N). For all $z\in Z$ and $e\in E$, $m_{F(e)}(z)\leq m_{G(e)}(z)$ and $n_{F(e)}(z)\leq n_{G(e)}(z)$, $(F,M)\preceq(G,N)$ means that (F,M) is less than or equal to (G,N).*

The following definition is about a mapping that maps every FFSS to an FSS. It is also shown that the collection of images of FFSSs with $x\in[0,1]$ and with the relation $\subseteq$ is a totally ordered family of FSSs.

Definition 4.2. *For $x\in[0,1]$, the function $f_x: FFSS(Z)\to FSS(Z)$ is described as $f_x((F,E))=(F_xE)$, for each FFSS (F,E) with MV $m_{F(e)}$ and NV $n_{F(e)}$ and $F_x(e)=f_x(F_e)$ and,*

$$f_x(F_e)=(z,m^3_{F(\rho)}(z)+x.h^3_{F(e)}(z),1-m^3_{F(e)}(z)-x.h^3_{F(e)}(z):z\in Z). \tag{4.1}$$

As a result, every FFSS is given an FSS by the map f_x. A modification of [70] is the f_x. In contrast to the f_x described in [70], which is to assign an FFSS to an FS, the operator f_x is assigned to an FFSS to an FSS.

Example 4.3. *Take* $(F,E)=[a_{ij}]=\begin{pmatrix}(0.8,0.7) & (0.7,0.4)\\ (0.5,0.8 & (0.9,0.6)\end{pmatrix}$. *Choose $x=0.8$. Hence,*

$$F_x(e_1)=f_x[(k_1,0.8,0.7),(k_2,0.7,0.4)]=\{(k_1,0.628,0.372),(k_2,0.8174,0.1826)\}$$
$$F_x(e_2)=f_x[(k_1,0.5,0.8),(k_2,0.9,0.6)]=\{(k_1,0.4154,0.5846),(k_2,0.773,0.227)\}$$

Therefore, FSS is symbolized by the matrix $\begin{pmatrix}(0.628,0.372) & (0.8174,0.1826)\\ (0.4154,0.5846 & (0.773,0.227)\end{pmatrix}$

Theorem 4.4. *Let $\rho,\bar{\rho}\in FFSS(Z)$ and $x,y\in[0,1]$. Then,*

(i) If $x\leq y\Rightarrow f_x(\rho)\subset f_y(\rho)$.
(ii) If $\rho\subset\bar{\rho}\Rightarrow f_x(\rho)\subset f_x(\bar{\rho})$.
(iii) $f_x(f_y(\rho))=f_y(\rho)$
(iv) $(f_x(\rho^c))^c=f_{1-x}(\rho)$.

Proof. For $\forall e\in E$, take

$$\rho=(F,E),\quad F(e)=\{(z,m_{F(e)}(z),n_{F(e)}(z)):z\in Z\},$$
$$\bar{\rho}=(G,E),\quad G(e)=\{(z,m_{G(e)}(z),n_{G(e)}(z)):z\in Z\},$$

and $f_x(\rho)=(F_x,E)$, where

$$f_x(e)=(z,m^3_{F(e)}(z)+x.h^3_{F(e)}(z),1-m^3_{F(e)}(z)-x.h^3_{F(e)}(z):z\in Z).$$

(i) For $x \leq y$, for all $z \in Z$ and $e \in E$,

$$m^3_{F(e)}(z) + x.h^3_{F(e)}(z) \leq m^3_{F(e)}(z) + x.h^3_{F(e)}(z).$$

Thus,

$$m_{F_x(e)}(z) \leq m_{F_y(e)}(z)$$

for all $z \in Z$ and $e \in E$. Hence, $f_x(\rho) \subset f_y(\rho)$.

(ii) Take $\rho \subset \tilde{\rho}$. Therefore,

$$m_{F(e)}(z) \leq m_{G(e)(z)} \quad \textit{and} \quad n_{F(e)}(z) \geq n_{G(e)}(z), \quad \textit{for all} z \in Z, e \in E.$$

Then,

$$\begin{aligned} m_{F_x(e)}(z) &= m^3_{F(e)}(z) + x.h^3_{F(e)}(z) \\ &= m^3_{F(e)}(z) + x.(1 - m^3_{F(e)}(z) - n^3_{F(e)}(z)) \\ &= m^3_{F(e)}(z)(1-x) + x - x.n^3_{F(e)}(z) \\ &\leq m^3_{G(e)}(z)(1-x) + x - x.n^3_{G(e)}(z) \\ &= m^3_{G(e)}(z) + x.h^3_{G(e)}(z) = m_{G_x(e)}(z). \end{aligned}$$

Hence, $m_{F_x(e)}(z) \leq m_{G_x(e)}(z)$ and $f_x(\rho) \subset f_y(\rho)$.

(iii) Let $f_x(f_y(F,E)) = f_x(F_y,E) = ((F_y)_x, E)$ where $((F_y)_x(e) = f_x(F_y(e)) = f_x(f_y(F(e)))$ for all $e \in E$. It will be shown as $f_x(f_y(F(e))) = f_y(F(e))$. Since

$$((f_y)(F(e)) = \{(z, m^3_{F(e)}(z) + y.h^3_{F(e)}(z), 1 - m^3_{F(e)}(z) - y.h^3_{F(e)}(z) >: z \in Z\},$$

any $e \in E$, we get

$$\begin{aligned} f_x[f_y(F(e))] =& f_x(\{(z, m^3_{F(e)}(z) + y.h^3_{F(e)}(z), 1 - m^3_{F(e)}(z) - y.h^3_{F(e)}(z) >: z \in Z\}) \\ =& \{(z, (m^3_{F(e)}(z) + y.h^3_{F(e)}(z)) + xa.[1 - (m^3_{F(e)}(z) + ya.h^3_{F(e)}(z)) \\ & - (1 - m^3_{F(e)}(z) - y.\pi^3_{F(e)}(z))], 1 - [(m^3_{F(e)}(z) + y.\pi^3_{F(e)}(z)) \\ & + \alpha.(1 - (m^3_{F(e)}(z) + y.h^3_{F(e)}(z)) - (1 - m^3_{F(e)}(z) - y.h^3_{F(e)}(z)))]) : z \in Z\} \\ =& \{(z, m^3_{F(e)}(z) + y.h^3_{F(e)}(z), 1 - m^3_{F(e)}(z) - y.h^3_{F(e)}(z)) : z \in Z\} = f_y(F(e)) \end{aligned}$$

(iv) For all $e \in \neg E$, Take

$$\rho^c = (F,E)^c = (F^c, \neg E) = \{(z, n_{F(e)}(z), m_{F(e)}(z)) : z \in Z\}, \quad f_x(\rho^c) = f_x(F^c, \neg E) = ((F^c)_x, \neg E)$$

where

$$(F^c(x))(e) = \{(z, n^3_{F(\neg e)}(z) + x.h^3_{F(\neg e)}(z), 1 - n^3_{F(\neg e)}(z) - x.h^3_{F(\neg e)}(z) : z \in Z\}$$

$$(F^c_x)^c(e) = \{(z, 1 - n^3_{F(e)}(z) - x.h^3_{F(e)}(z)), n^3_{F(e)}(z) + x.h^3_{F(e)}(z) : z \in Z\}$$

for all $e \in E$.

$$\begin{aligned} f_{1-x}(\rho) &= f_{(1-x)}((F,E)) = f_{1-x}(\{(z, m_{F(e)}(z), n_{F(e)}(z)) : z \in Z\}) \\ &= (m^3_{F(e)}(z) + (1-x)h^3_{F(e)}(z), 1 - m^3_{F(e)}(z) - (1-x)h^3_{F(e)}(z), \\ & 1 - n^3_{F(e)}(z) - x.(1 - m^3_{F(e)}(z) - n^3_{F(e)}(z)), n^3_{F(e)}(z) + x.h^3_{F(e)}(z)) \\ &= F^3_{x^c}(e). \end{aligned}$$

Thus, $(f_x(\rho^c))^c = f_{1-x}(\rho)$.

□

Definition 4.5. *If the properties are satisfies, a real mapping $T : FFSS(Z) \to \mathbb{R}^+$ is called a FFSE on FFSS(Z):*

(i) $T(\rho) = 0 \Leftrightarrow \rho \in FSS(Z)$

(ii) Let $\rho = (F,T) = [a_{ij}]_{m\times n}$, for all $z \in Z$, $T(\rho) = mn \Leftrightarrow m_{F(e)}(z) = n_{F(e)}(z) = 0$, for all $e \in T$.

(iii) $T(\rho) = T(\rho^c)$, $\rho \in FFSS(Z)$.

(iv) For $(F,T) = \rho$ and $(G,T) = \tilde{\rho}$, if $\rho \preceq \tilde{\rho} \Rightarrow T(\rho) \geq T(\tilde{\rho})$.

From the definition, entropy is minimum(zero) when the FFSS degenerates into SS. The following theorem discusses the case when the entropy is maximum.

Theorem 4.6. *FFSE ρ is maximum $\Leftrightarrow$ $\rho = (F,T) = [a_{ij}]_{m\times n} = [0]_{m\times n}$. So, $m_{F(e_j)}(z_i) = n_{F(e_j)}(z_i) = 0$, for all $e_j \in T$, $z_i \in Z$ where $i \in \{0,1,\cdots,m\}$ and $j \in \{0,1,\cdots,n\}$ and $\rho \in FFSS(Z)$.*

Proof. Take $\rho = (F,T) = [0]m \times n$. Choose $\bar{\rho} = (G,T)$ be any FFSS. Since $m_{F(e_j)}(z_i) \geq 0$ and $n_{F(e_j)}(z_i) \leq 0$ for all $e_j \in T, z_i \in Z$, where $i \in \{0,1,\cdots,m\}$ and $j \in \{0,1,\cdots,n\}$, by Definition 4.1, $\rho \preceq \bar{\rho}$. Therefore, from Definition 4.5 $T(\rho) \geq T(\bar{\rho})$ for all $\bar{\rho}$. Then, $T(\rho)$ is maximum.

Moreover, let $T(\rho)$ be the maximum. If we take $\rho = (F,T) \neq [0]_{m\times n}$, then there exist $e_j \in T$ and $z_i \in Z$ such that $m_{F(e_j)}(z_i) \neq 0$ or $n_{F(e_j)}(z_i) \neq 0$. Build the FFSS as $\bar{\rho} = (G,T)$ with $m_{G(e_j)}(z_i) = m_{F(e_j)}(z_i)/2$ and $n_{G(e_j)}(z_i) = n_{F(e_j)}(z_i)/2$ for all $e_j \in T$ and $z_i \in F$. Hence, using the Definition 4.1, $\rho \leq \bar{\rho}$. Thus $T(\bar{\rho}) \geq T(\rho)$ is obtained. This is a contradiction. Therefore, $\rho = [0]_{m\times n}$. □

The objective is to provide a statement enabling entropy generation for FFSSs. The method is the same as the one used to calculate concrete entropies for FFSSs: Let's build $\Lambda_{\mathscr{K}} : \mathscr{K} \to [0,1]$ using the set $\mathscr{K} = \{(u,v) \in [0,1]\times[0,1] : u^3+v^3 \leq 1\}$ given below, which meets the requirements listed below.

(i) $\Lambda_{\mathscr{K}}(u,v) = 1 \Leftrightarrow (u,v) = (0,1) \quad or \quad (u,v) = (1,0)$
(ii) $\Lambda_{\mathscr{K}}(u,v) = 0 \Leftrightarrow u = v = 0$
(iii) $\Lambda_{\mathscr{K}}(u,v) = \Lambda_{\mathscr{K}}(v,u)$
(iv) If $u \leq u'$ and $v \leq v'$ then $\Lambda_{\mathscr{K}}(u,v) \leq \Lambda_{\mathscr{K}}(u',v')$.

Theorem 4.7. *Let $T : FFSS(Z) \to \mathbb{R}^+$ and $\rho = (F,T) = [a_{ij}]_{m\times n} \in FFSS(Z)$. If $T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i), n_{F(e_j)}(z_i)))]$ where $\Lambda_{\mathscr{K}}$ satisfies the conditions (i)-(iv) of FFSE.*

Proof. $T(\rho) = 0 \Leftrightarrow T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i), n_{F(e_j)}(z_i)))] = 0 \Leftrightarrow \Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i), n_{F(e_j)}(z_i))) = 1, \forall e_j \in T \quad and z_i \in Z \Leftrightarrow m_{F(e_j)}(z_i) = 1, n_{F(e_j)}(z_i) = 0 \quad or \quad m_{F(e_j)}(z_i) = 0, n_{F(e_j)}(z_i) = 1 \Leftrightarrow \rho \quad is\ a\ SS$. Thus, T satisfies property (i) of Definition 4.5. $T(\rho) = mn \Leftrightarrow T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i), n_{F(e_j)}(z_i)))] = mn \Leftrightarrow (\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i), n_{F(e_j)}(z_i))) = 0, \forall e_j \in T \quad and z_i \in Z \Leftrightarrow m_{F(e_j)}(z_i) = 0 = n_{F(e_j)}(z_i)$ $\forall e_j \in N$ and $z_i \in Z$. Therefore, T satisfies property (ii) of Definition 4.5. For $F^c(e) = \{(z_i, n_{F(e_j)}(z_i), m_{F(e_j)}(z_i)) : z_i \in Z\}$, $\forall \neg e_j \in \neg T$, since $\rho = (F,T)^c = (F^c, \neg T)$, therefore,

$$
\begin{aligned}
T(\rho) &= \sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i), n_{F(e_j)}(z_i)))] \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{K}(n_{F(e_j)}(z_i), m_{F(e_j)}(z_i)))] = T(\rho^c)
\end{aligned}
$$

This property (iii) is provided for T. Let $\bar{\rho} = (G,T) = [b_{ij}]_{m\times n}$. If $\rho \leq \bar{\rho}$ then, $m_{F(e_j)}(z_i) \leq m_{G(e_j)}(z_i)$ and $n_{F(e_j)}(z_i) \leq n_{G(e_j)}(z_i)$ which implies $\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i), n_{F(e_j)}(z_i)) \leq \Lambda_{\mathscr{K}}(m_{G(e_j)}(k_i), n_{G(e_j)}(k_i))$. $T(\rho) \geq T(\bar{\rho})$ and so, property (iv) is provided for T. Therefore, T is a FFSE. □

Example 4.8. $T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}[1-(m_{F(e)}^4, n_{F(e)}^4)]$. *We must show that $T(\rho)$ is FFSE. To demonstrate this, it is necessary to prove that $m_{F(e)}^4 + n_{F(e)}^4$ meets the $\Lambda_{\mathscr{K}}$ requirements. $\Lambda_{\mathscr{K}} : \mathscr{K} = \{(m_{F(e)}, n_{F(e)}) \in [0,1]\times[0,1] : u^3+v^3 \leq 1\} \to [0,1]$, where $\Lambda_{\mathscr{K}}(u,v) = m_{F(e)}^4 + m_{F(e)}^4$. Further, $m_{F(e)}^4 + n_{F(e)}^4 = 1 \Leftrightarrow m_{F(e)} = 1, \quad n_{F(e)} = 0 \quad or \quad m_{F(e)} = 0, n_{F(e)} = 1$ in the domain $\mathscr{K}$.*

Definition 4.9. *Let $\Gamma, \Gamma' : [0,1] \to [0,1]$, if $u^3+v^3 \leq 1$, then $\Gamma(u^3) + \Gamma'(v^3) \leq 1$ with $u,v \in [0,1]$. Define the function $T_{\Gamma,\Gamma'}$ of the FFSS $\rho = (F,T) = [a_{ij}]_{m\times n}$ to $\mathbb{R}^+$ as,*

$$
T_{\Gamma,\Gamma'} = mn - \sum_{j=1}^{n}\sum_{i=1}^{m}\Gamma[m_{F(e_j)}(z_i)] + \Gamma'[n_{F(e_j)}(z_i)] \tag{4.2}
$$

Obviously $0 \leq T_{\Gamma,\Gamma'}(\rho) \leq mn$ and $\forall \rho = [a_{ij}]_{m\times n}$ belonging to FFSS(Z).

Theorem 4.10. *Let $\Gamma : [0,1] \to [0,1]$ provide the following items:*

(i) Γ is increasing
(ii) $\Gamma(u) = 0 \Leftrightarrow u = 0$
(iii) $\Gamma(u) + \Gamma(v) = 1 \Leftrightarrow (u,v) = (0,1) \quad or \quad (u,v) = (1,0)$.

Therefore, $\Gamma(u) + \Gamma(v)$ provides the properties (i)-(iv) of the $\Lambda_{\mathscr{K}}$.

Proof. The property (iii) of this theorem is identical to the condition (i) of $\Lambda_{\mathscr{K}}$ if $\Lambda_{\mathscr{K}}(u,v) = \Gamma(u) + \Gamma(v)$ is taken into account. $\Lambda_{\mathscr{K}}(u,v) = \Gamma(u) + \Gamma(v) = 0$ if and only if $\Gamma(u) = 0 = \Gamma(v)$ from property (ii), $u = v = 0$. As a result, the second condition of $\Lambda_{\mathscr{K}}$ is obtained. Additionally, as $\Gamma(u) + \Gamma(v) = \Gamma(v) + \Gamma(u)$, $\Lambda_{\mathscr{K}}(u,v) = \Lambda_{\mathscr{K}}(v,u)$. Γ is increasing, hence condition (iv) of $\Lambda_{\mathscr{K}}$ is obtained. Therefore, the $\Lambda_{\mathscr{K}}$ function's conditions (i) to (iv) are satisfied by $\Gamma(u) + \Gamma(v)$. □

Theorem 4.11. *Let $T : FFSS(Z) \to \mathbb{R}^+$, $\Gamma : [0,1] \to [0,1]$, and $\rho = (F,T) = [a_{ij}]_{m\times n} \in FFSS(Z)$. T is FFSE and a $T_{\Gamma,\Gamma}$ – function $\Leftrightarrow$ $T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)}) + \Gamma(n_{F(e_j)(z_i)})\right)$.*

Proof. Take $\Lambda : [0,1]\times[0,1] \to [0,1]$ with $\Lambda(u,v) = \Gamma(u)+\Gamma(v)$, and $\{(u,v) \in [0,1]\times[0,1] : u^3+v^3 \leq 1\}$. Restrict the $\Lambda_{\mathscr{K}}$ function from $\mathscr{K}$ to $[0,1]$. If $T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)$, then $T(\rho)$ is a FFSE. It is enough to prove that T is an $T_{\Gamma,\Gamma}$-function.

Let $\alpha,\beta \in [0,1]$ and $\alpha^3+\beta^3 \leq 1$ to prove $\Gamma(\alpha^3)+\Gamma(\beta^3) \leq 1$, construct the FFSS:

$$[a_{ij}]_{m\times n} = \begin{bmatrix} (\alpha^3,\beta^3) & (1,0) & \cdots & (1,0) \\ (\alpha^3,\beta^3) & (1,0) & \cdots & (1,0) \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha^3,\beta^3) & (1,0) & \cdots & (1,0) \end{bmatrix}$$

Thus,

$$\begin{aligned} T(p) &= \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right) \\ &= mn - m\left(\Gamma(\alpha^3)+\Gamma(\beta^3)\right) - m(n-1)(\Gamma(\alpha(1))+\Gamma(\beta(0))). \end{aligned}$$

Using Theorem 4.10, $\Gamma(\alpha(1))+\Gamma(\beta(0)) = 1$. Hence, $T(\rho) = mn - m\left(\Gamma(\alpha^3)+\Gamma(\beta^3)\right) - m(n-1) = m\left(\Gamma(\alpha^3)+\Gamma(\beta^3)\right)$. $T(\rho) \geq 0$ because T is entropy. $m\left(1-(\Gamma(\alpha^3)+\Gamma(\beta^3))\right) \geq 0$ which implies that $\Gamma(\alpha^3)+\Gamma(\beta^3) \leq 1$. Therefore, T is entropy and a $T_{\Gamma,\Gamma}$ function.

On the other hand, if T is an entropy and $T_{\Gamma,\Gamma}$ are functions, then T has the form $T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)$.

(i) Let $\alpha \leq \beta$, $\alpha,\beta \in [0,1]$ construct the following FFSSs:

$$\rho = (F,T) = [a_{ij}]_{m\times n} = \begin{bmatrix} (\alpha,0) & (0,0) & \cdots & (0,0) \\ (\alpha,0) & (0,0) & \cdots & (0,0) \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha,0) & (0,0) & \cdots & (0,0) \end{bmatrix}$$

and

$$\tilde{\rho} = (G,T) = [b_{ij}]_{m\times n} = \begin{bmatrix} (\beta,0) & (0,0) & \cdots & (0,0) \\ (\beta,0) & (0,0) & \cdots & (0,0) \\ \vdots & \vdots & \vdots & \vdots \\ (\beta,0) & (0,0) & \cdots & (0,0) \end{bmatrix}.$$

Thus,

$$\begin{aligned} T(\rho) &= \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right) \\ &= mn - m(\Gamma(\alpha)+\Gamma(0)) - m(n-1)2\Gamma(0), \end{aligned}$$

and

$$\begin{aligned} T(\tilde{\rho}) &= \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right) \\ &= mn - m(\Gamma(\Gamma(\beta)+\Gamma(0)) - m(n-1)2\Gamma(0). \end{aligned}$$

$T(\rho) \geq T(\tilde{\rho})$, because $\alpha \leq \beta$, $p \preceq \tilde{\rho}$. Therefore,

$$mn - m(\Gamma(\alpha)+\Gamma(0)) - m(n-1)2\Gamma(0) \geq mn - m(\Gamma(\Gamma(\beta)+\Gamma(0)) - m(n-1)2\Gamma(0)$$

implies $\Gamma(\alpha) \leq \Gamma(\beta)$. Therefore, Γ is increasing.

(ii) To prove $\Gamma(\alpha) = 0 \Leftrightarrow \alpha = 0$;
If $\alpha = 0$:

$$\rho = (F,T) = [a_{ij}]_{m\times n} = \begin{bmatrix} (0,0) & (0,0) & \cdots & (0,0) \\ (0,0) & (0,0) & \cdots & (0,0) \\ \vdots & \vdots & \vdots & \vdots \\ (0,0) & (0,0) & \cdots & (0,0) \end{bmatrix}.$$

Thus,

$$T(\rho) = mn - mm(\Gamma(0)+\Gamma(0))$$

hence $\Gamma(0)=0$ and so, $\Gamma(\alpha)=0$.
If $\Gamma(\alpha)=0$:

$$\rho=(F,T)=[a_{ij}]_{m\times n}=\begin{bmatrix}(\alpha,0) & (0,0) & \cdots & (0,0)\\ (\alpha,0) & (0,0) & \cdots & (0,0)\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha,0) & (0,0) & \cdots & (0,0)\end{bmatrix}.$$

Thus,

$$\begin{aligned}T(\rho)&=\sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)\\ &=mn-m(\Gamma(\Gamma(\alpha)+\Gamma(0))-m(n-1)2\Gamma(0).\end{aligned}$$

Since $\Gamma(0)=0$ in the preceding section and $\Gamma(\alpha)=0$, the conclusion is that $T(\rho)=mn$. Thus, α must be equal to 0.

(iii) To prove $\Gamma(\alpha)+\Gamma(\beta)=1\Leftrightarrow(\alpha,\beta)=(0,1)\quad or\quad(1,0)$;

If $(\alpha,\beta)=(0,1)$ or $(1,0)$:

$$\rho=(F,T)=[a_{ij}]_{m\times n}=\begin{bmatrix}(\alpha,\beta) & (\alpha,\beta) & \cdots & (\alpha,\beta)\\ (\alpha,\beta) & (\alpha,\beta) & \cdots & (\alpha,\beta)\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha,\beta) & (\alpha,\beta) & \cdots & (\alpha,\beta)\end{bmatrix}.$$

Then $\rho\in SS(Z)$. As a result, $T(\rho)=0$. Hence,

$$T(\rho)=\sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)=0.$$

Then, $\Gamma(\alpha)+\Gamma(\beta)=1$.

If $\Gamma(\alpha)+\Gamma(\beta)=1$:

$$\rho=([a_{ij}]_{m\times n}=\begin{bmatrix}(\alpha,\beta) & (1,0) & \cdots & (1,0)\\ (\alpha,\beta) & (1,0) & \cdots & (1,0)\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha,\beta) & (1,0) & \cdots & (1,0)\end{bmatrix}.$$

Thus

$$\begin{aligned}T(\rho)&=\sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)\\ &=mn-m(\Gamma(\alpha)+\Gamma(\beta))-m(n-1)(\Gamma(1)+\Gamma(0)).\end{aligned}$$

Given that $\Gamma(\alpha)+\varphi(\beta)=1$ and $\Gamma(0)+\Gamma(1)=1$ respectively, Then, $T(\rho)=0$. $\rho\in SS(Z)$, or $(\alpha,\beta)=(1,0)$ or $(0,1)$.

□

Let's note that: Let $\rho=(F,T)=[a_{ij}]_{m\times n}\in FFSS(Z)$, then entropy of ρ is,

$$T(\rho)=\sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m^{t}_{F(e_j)(z_i)})+\Gamma(n^{t}_{F(e_j)(z_i)})\right),\quad t=3,4,5,\ldots$$

4.2. Distance measure

Definition 4.12. *Let $\rho=(F,M)$ and $\tilde{\rho}=(G,N)$ be two FFSSs. Let U be a mapping given by $U:FFSS(Z)\times FFSS(Z)\to\mathbb{R}^{+}\cup\{0\}$ and $U(\rho,\tilde{\rho})$ satisfies the following axioms:*

(i) $0\leq U(\rho,\tilde{\rho})\leq 2^{1/2}$,
(ii) $U(\rho,\tilde{\rho})=U(\tilde{\rho},U(\rho,\tilde{\rho}))$,
(iii) $U(\rho,\tilde{\rho})=0\Leftrightarrow\rho=\tilde{\rho}$,
(iv) *For any* $\sigma=(H,P)\in FFSS(Z)$, $U(\rho,\tilde{\rho})+U(\tilde{\rho},\sigma)\geq U(\rho,\sigma)$.

Then $U(\rho,\tilde{\rho})$ is a distance measure between FFSSs ρ and $\tilde{\rho}$.

Definition 4.13. *Let $\rho_1 = (F,E), \rho_2 = (G,E)$ be two FFSSs over Z. Then normalized Euclidean distance between ρ_1, ρ_2 is defined as follows:*

$$U_E(\rho_1,\rho_2) = \left[\frac{1}{4mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left((m^3_{F(e_j)}(z_i) - m^3_{G(e_j)}(z_i))^2 + (n^3_{F(e_j)}(z_i) - n^3_{G(e_j)}(z_i))^2 + (h^3_{F(e_j)}(z_i) - h^3_{G(e_j)}(z_i))^2\right)\right]^{1/2}$$

Theorem 4.14. *Properties of Definition 4.12 are provided for normalized Euclidean distances of FFSSs.*

Theorem 4.15. *For three FFSSs $\rho_1 = (F,E), \rho_2 = (G,E), \rho_3 = (H,E)$ over Z, if $\rho_1 \leq \rho_2 \leq \rho_3$, then $U_E(\rho_1,\rho_2) \leq U_E\rho_1,\rho_3)$ and $U_E(\rho_2,\rho_3) \leq U_E(\rho_1,\rho_3)$.*

5. Applications

5.1. Entropy application

In this subsection, we will practice DM using entropy.

Algortihm:

Step 1: Input each of the FFSSs $\rho_1, \rho_2, \cdots \rho_k$

Step 2: Compute the entropy of each FFSS using the expression

$$T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}[1 - (m^3_{F(e_j)}(a_i) + n^3_{F(e_j)}(a_i))].$$

Step 3: Obtain ρ_r with the minimum of $T(\rho_i)$,

Step 4: The optimum result is to choose the ρ_r to get from Step 3.

Step 5: If more than one ideal solution is discovered, the user may select any of them.

Because the FFSS is an extension of existing sets such as the IFSS and PFSS, it is an excellent tool for representing information during decision-making. Consider a set of k options $V_1, V_2, ..., V_k$ examined by n experts $P_1, P_2, ..., P_n$. Each expert P_j evaluates the alternatives using the parameters $K = k_1, k_2, ..., k_m$ and assigns ratings to FFSNs. The challenge then seeks to select the best option among them. The provided methodology offers a method for solving the problem above using entropy measures.

Example 5.1. *Consider the selection of the car from a particular company. For it, a person wants to select a car from three different alternatives $V_1, V_2, \cdots, V_n$. To address it thoroughly and remove the hesitation between them, they hire three experts E_1, E_2, E_3 to evaluate each alternative under the three significant set of parameters K.*

Consider the purchase of an automobile from a specific firm. A customer wants to choose an automobile from three options: A, B, C. To address it adequately and remove any doubts, they appoint three experts, E_1, E_2, E_3, to analyze each possibility using the three critical sets of parameters $K = \{k_1, k_2, k_3\}$, where k_1 = expensive, k_2 = good engine capacity and k_3 = warranty.
Step 1: *Build $(F,A), (G,B), (H,C)$:*

$$F(k_1) = \{(E_1,(0.75,0.58)),(E_2,(0.98,0.15)),(E_3,(0.47,0.83))\}$$
$$F(k_2) = \{(E_1,(0.82,0.66)),(E_2,(0.59,0.51)),(E_3,(0.26,0.95))\}$$
$$F(k_3) = \{(E_1,(0.54,0.79)),(E_2,(0.73,0.55)),(E_3,(0.87,0.51))\}$$

$$G(k_1) = \{(E_1,(0.63,0.87)),(E_2,(0.80,0.72)),(E_3,(0.56,0.68))\}$$
$$G(k_2) = \{(E_1,(0.72,0.80)),(E_2,(0.51,0.92)),(E_3,(0.67,0.71))\}$$
$$G(k_3) = \{(E_1,(0.82,0.53)),(E_2,(0.88,0.45)),(E_3,(0.73,0.66))\}$$

$$H(k_1) = \{(E_1,(0.42,0.93)),(E_2,(0.56,0.70)),(E_3,(0.88,0.62))\}$$
$$H(k_2) = \{(E_1,(0.67,0.79)),(E_2,(0.77,0.64)),(E_3,(0.76,0.39))\}$$
$$H(k_3) = \{(E_1,(0.68,0.52)),(E_2,(0.91,0.36)),(E_3,(0.74,0.67))\}$$

Step 2: *Compute the FFSEs:*

$$T(F,A) = \sum_{j=1}^{n}\sum_{i=1}^{m}[1 - (m^4_{F(e_j)}(k_i) + n^4_{F(e_j)}(k_i))] = 4.60312528$$
$$T(G,B) = 3.96972281$$
$$T(H,C) = 4.17700393.$$

Step 3: *Find the FFSS with (G,B) as its entropy value, which is the smallest.*

Step 4: *The optimum decision is to choose* (G,B).

Step 5: *The B most likely has an infectious condition because there is only one best course of action.*

It is seen from these computed results that the best alternative for the given problem is B while the worst one is either C or A.

5.2. Distance measure application

PR is the process of identifying patterns in data and categorizing them so that there is a strong correlation between patterns belonging to the same category and a weak correlation between patterns belonging to different categories. FSs, SSs, FSSs, and other tools are helpful for modeling patterns. Due to the premise that similarity is a parallel concept to distance measurement, sets with reduced distances are presumed to be similar. FFSS can also illustrate patterns with a more precise representation of ambiguity.

Algorithm:

The supplied pattern is initially displayed in the feature space SSA as FFSSs $O_1, O_2, \cdots, O_k$. A single O_i, $i = 1,2,\cdots,k$ should be used to identify the pattern, also represented as FFSS B. The pattern O_i with the shortest distance to B is then found by calculating the distance between each O_i and B. This OI most closely resembles pattern B. The PR algorithm is provided below:
Step 1: Enter the patterns $O_1, O_2, \ldots O_k$

Step 2: Enter the expectedly recognizable pattern B.

Step 3: In Steps 1 and 2, determine the Euclidean distance between each set.

Step 4: The O_i with the smallest Euclidean distance will be chosen at the end.

The scenario and evaluation method of the example below are taken from reference [71].

Example 5.2. *We have proposed a MAGDM method based on the novel FFSS entropy measure. In this example, the method will be used in selecting a missile position. In making a battle plan, staff officers must select a place as a missile position. The following are the main attributes they took into account:* $S = \{k_1,k_2,k_3,k_4,k_5\}$, *as* k_1*-the operational intentions of superiors;* k_2*-the geological conditions of positions;* k_3*-the efficiency of firepower exertion;* k_4*-maneuverability;* k_4*-battlefield viability.*

After a thorough screening and comparison, three locations—$\{O_1,O_2,O_3\}$*—have been tentatively chosen as alternatives. Three experts are asked to rate the options using their FFNs based on gathered knowledge, facts, and experiences to help them make better decisions. Let* $E = \{e_1,e_2\}$ *be given such that the parameters for choosing the most suitable location according to these features are defined as* e_1*-appropriate and* e_2*-not appropriate. Let a P missile location be predetermined.*

$$O_1 = \left\{\begin{array}{l} e_1 = (k_1,(0.65,0.45)),(k_2,(0.52,0.54)),(k_3,(0.11,0.62)),(k_4,(0.35,0.72)),(k_5,(0.42,0.78)) \\ e_2 = (k_1,(0.92,0.11)),(k_2,(0.76,0.62)),(k_3,(0.94,0.10)),(k_4,(0.83,0.44)),(k_5,(0.69,0.58)) \end{array}\right\}$$
$$O_2 = \left\{\begin{array}{l} e_1 = (k_1,(0.86,0.14)),(k_2,(0.90,0.26)),(k_3,(0.73,0.52)),(k_4,(0.44,0.38)),(k_5,(0.68,0.60)) \\ e_2 = (k_1,(0.27,0.88)),(k_2,(0.16,0.82)),(k_3,(0.48,0.62)),(k_4,(0.65,0.54)),(k_5,(0.57,0.48)) \end{array}\right\}$$
$$O_3 = \left\{\begin{array}{l} e_1 = (k_1,(0.89,0.31)),(k_2,(0.87,0.35)),(k_3,(0.74,0.52)),(k_4,(0.78,0.25)),(k_5,(0.73,0.28)) \\ e_2 = (k_1,(0.14,0.43)),(k_2,(0.29,0.47)),(k_3,(0.12,0.57)),(k_4,(0.32,0.57)),(k_5,(0.40,0.70)) \end{array}\right\}$$

Build the FFSNs of pattern P

$$P = \left\{\begin{array}{l} e_1 = (k_1,(0.9,0.2)),(k_2,(0.8,0.3)),(k_3,(0.8,0.4)),(k_4,(0.7,0.5)),(k_5,(0.9,0.1)) \\ e_2 = (k_1,(0.1,0.9)),(k_2,(0.2,0.8)),(k_3,(0.3,0.8)),(k_4,(0.4,0.7)),(k_5,(0.2,0.9)) \end{array}\right\}.$$

The Euclidean distance values are:

- *For* e_1, $U_E(O_1,P) = 0.173$; $U_E(O_2,P) = 0.1$; $U_E(O_3,P) = 0.073$
- *For* e_2, $U_E(O_1,P) = 0.202$; $U_E(O_2,P) = 0.11$; $U_E(O_3,P) = 0.098$.

Between O_3 *and P, the Euclidean distance is the smallest. As a result, pattern* O_3 *resembles pattern P more. It can be concluded that the predetermined location P should be the location* O_3.

5.3. Comparison

For the PFSSs F_1, F_2, F_3 and the attribute set $A=\{k_1,k_2,k_3\}$, let the following values be given.

$$F_1=\left\{\begin{array}{l} e_1=(k_1,0.3,0.2),(k_2,0.6,0.0),(k_3,0.5,0.4), \\ e_2=(k_1,0.6,0.3),(k_2,0.7,0.2),(k_3,0.4,0.3), \\ e_3=(k_1,0.8,0.1),(k_2,0.8,0.1),(k_3,0.6,0.1) \end{array}\right\}$$
$$F_2=\left\{\begin{array}{l} e_1=(k_1,0.6,0.2),(k_2,0.8,0.1),(k_3,0.8,0.1), \\ e_2=(k_1,0.5,0.5),(k_2,0.7,0.2),(k_3,0.5,0.4), \\ e_3=(k_1,0.7,0.1),(k_2,0.6,0.3),(k_3,0.6,0.3) \end{array}\right\}$$
$$F_3=\left\{\begin{array}{l} e_1=(k_1,0.5,0.4),(k_2,0.4,0.1),(k_3,0.6,0.2), \\ e_2=(k_1,0.6,0.2),(k_2,0.7,0.1),(k_3,0.8,0.1), \\ e_3=(k_1,0.9,0.0),(k_2,0.5,0.1),(k_3,0.6,0.3) \end{array}\right\}$$

Let's compare the $I(w)$ given in Theorem 4 in the work of Jiang et al. [69], which offers the entropy measure related to IFSS, with the FFS entropy measure given in this article.

The IFSSVs are: $T_{IFSS}(F_1)=2.12, T_{IFSS}(F_2)=2.02, T_{IFSS}(F_3)=2.11$.

The Pythagorean fuzzy soft entropy(PFSE) described in [32] is utilized to compare the proposed entropy metric for FFSSs.
The PFSSVs are: $T_{PFSS}(F_1)=6.63, T_{PFSS}(F_2)=6.13, T_{PFSS}(F_3)=6.34$.

The FFSEVs were measured as $T_{FFSS}(F_1)=4.17, T_{FFSS}(F_2)=3.88, T_{FFSS}(F_3)=3.95$.

It can be seen that IFSE values are very close to each other. However, the result is that F_2 has the lowest entropy and F_1 has the highest entropy, which corresponds to T_{IFSS}, T_{PFSS} and T_{FFSS}. Therefore, the entropy equations that have been proposed are consistent (Table 12).

	T_{IFSS}	T_{PFSS}	T_{FFSS}
F_1	2.12	6.63	7.37
F_2	2.02	6.13	7.08
F_3	2.11	6.34	7.11

Table 12: Comparison of IFSE, PFSE and FFSE.

Now let's make a comparison of distance measures. Let's compare the normalized Euclidean distance based on IFFS given in Definition 8 in [69] and the normalized Euclidean distance based on PFFS given in Definition 3.6 in [32] with the normalized Euclidean distance based on FFSS proposed in this study:

Euclidean distance values for IFSS, PFSS, and FFSS are obtained as follows, respectively:

$$\left\{\begin{array}{llll} \text{For} \quad e_1, & U_{IFSS}(F_1,F_2)=0.281; & U_{IFSS}(F_2,F_3)=0.367; & U_{IFSS}(F_1,F_3)=0.302, \\ \text{For} \quad e_2, & U_{IFSS}(F_1,F_2)=0.274; & U_{IFSS}(F_2,F_3)=0.312; & U_{IFSS}(F_1,F_3)=0.291, \\ \text{For} \quad e_3, & U_{IFSS}(F_1,F_2)=0.241; & U_{IFSS}(F_2,F_3)=0.338; & U_{IFSS}(F_1,F_3)=0.278, \end{array}\right.$$

$$\left\{\begin{array}{llll} \text{For} \quad e_1, & U_{PFSS}(F_1,F_2)=0.277; & U_{PFSS}(F_2,F_3)=0.325; & U_{PFSS}(F_1,F_3)=0.318, \\ \text{For} \quad e_2, & U_{PFSS}(F_1,F_2)=0.253; & U_{PFSS}(F_2,F_3)=0.316; & U_{PFSS}(F_1,F_3)=0.286, \\ \text{For} \quad e_3, & U_{IFSS}(F_1,F_2)=0.266; & U_{IFSS}(F_2,F_3)=0.323; & U_{IFSS}(F_1,F_3)=0.303, \end{array}\right.$$

and

$$\left\{\begin{array}{llll} \text{For} \quad e_1, & U_{FFSS}(F_1,F_2)=0.225; & U_{FFSS}(F_2,F_3)=0.344; & U_{FFSS}(F_1,F_3)=0.327, \\ \text{For} \quad e_2, & U_{FFSS}(F_1,F_2)=0.212; & U_{FFSS}(F_2,F_3)=0.285; & U_{FFSS}(F_1,F_3)=0.266, \\ \text{For} \quad e_3, & U_{IFSS}(F_1,F_2)=0.198; & U_{IFSS}(F_2,F_3)=0.309; & U_{IFSS}(F_1,F_3)=0.275. \end{array}\right.$$

	U_{IFSS}			U_{PFSS}			U_{FFSS}		
	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
(F_1,F_2)	0.281	0.274	0.241	0.277	0.253	0.266	0.225	0.212	0.198
(F_2,F_3)	0.367	0.312	0.338	0.325	0.316	0.323	0.344	0.285	0.309
(F_1,F_3)	0.302	0.291	0.278	0.318	0.286	0.303	327	266	275

Table 13: Comparison of distance measures

Distance measures are given in Table 13. When these values for IFSS, PFSS, and FFSS are examined, it is seen that for each of them, the distance between F_1 and F_2 is the smallest, while the distance between F_2 and F_3 is the largest.

6. Conclusion

This study aims to define FFSSs and provide entropy and distance metrics. First, the FFSS idea is explained. Then, several FFSS activities and properties are covered. The idea of FSSs has been generalized in the form of FFSSs. Also introduced are the entropy and distance measures of FFSSs. It is simple to state that FFSS is more accurate and reasonable than current soft-set models. Then, DM issues and pattern identification on FFSS are suggested as applications. The recommended entropy was determined to be consistent when compared to FFSS entropy and Pythagorean fuzzy entropy.

This study still has several problems. The first distinction is between risk and uncertainty. The impacts of risk preference rather than uncertainty preference are the main focus of this study. One form of uncertainty avoidance is risk aversion. This is also important because it can be challenging to determine the exact probability of real-world problems. Aside from the benefits of the provided FFSS-based technique, its inability to generate a complete ranking of the available alternatives limits its usefulness in particular DM contexts. Furthermore, when the number of criteria and possibilities is enormous, constructing FFSSs can become difficult.

This study has some limitations while being objective and quantitative for DM problems. The arithmetic operations of FFSNs are more difficult to calculate than crisp or FNs; thus, computing solutions must be developed to lessen the effort of specialists.

Article Information

Acknowledgements: The author would like to express his sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: The article has a single author. The author has read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] C. E. Shannon, *A mathematical theory of communication*, Bell Syst. Tech. J., **27** (1948), 379–423 623–656.
[2] M. Akram, M. Ashraf, *Multi-criteria group decision-making based on spherical fuzzy rough numbers*, Granul. Comput. **8** (2023), 1267–1298.
[3] M. Akram, S. Zahid, *Group decision-making method with a Pythagorean fuzzy rough number for the evaluation of best design concept*, Granul. Comput. **8** (2023), 1121–1148.
[4] M. Akram, S. Zahid, M. Deveci, *Enhanced CRITIC-REGIME method for decision making based on Pythagorean fuzzy rough number*, Expert Systems with Applications, **238** (2023), 122014.
[5] M. Akram, F. Ilyas, M. Deveci, *Interval rough integrated SWARA-ELECTRE model: An application to machine tool remanufacturing*, Expert Systems with Applications, **238** (2023), 122067.
[6] S. Broumi, A. Bakali, M. Talea, F. Smarandache, F. Karaaslan, *Interval valued neutrosophic soft graphs*, New Trends in Neutrosophic Theory and Applications, **2** (2018), 218–251.
[7] M. Palanikumar, A. Iampan, S. Broumi, L. J. Manavalan, K. Sundareswari, *Multi-criteria group decision-making method in Pythagorean interval-valued neutrosophic fuzzy soft using VIKOR approach*, Int. J. Neutrosophic Sci. **22** (2023), 104–113.
[8] M. Palanikumar, A. Iampan, S. Broumi, G.Balaj, *Generalization of Neutrosophic interval-valued soft sets with different aggregating operators using multi-criteria group decision-making*, Int. J. Neutrosophic Sci., **22** (2023), 114–123.
[9] S. Priyadarsini, A. V. Singh, S. Broumi, *Review of Generalized Neutrosophic soft set in solving multiple expert decision making problems*, Int. J. Neutrosophic Sci., **19** (2022), 48–59.
[10] K. Zahid, M. Akram, *Multi-criteria group decision-making for energy production from municipal solid waste in Iran based on spherical fuzzy sets*, Granul. Comput. **8** (2023), 1299–1323.
[11] M. Akram, H. S. Nawaz, M. Deveci, *Attribute reduction, and information granulation in Pythagorean fuzzy formal contexts*, Expert Syst. Appl., **222** (2023), 119794.
[12] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst. **20** (1986), 87–96.
[13] D. Molodtsov, *Soft set theory first results*, Comp. Math. Appl., **7**(1) (2019), 91.
[14] L. A. Zadeh, *Fuzzy sets*, Inf. Comp., **8** (1965), 338–353.
[15] R. R. Yager, *Pythagorean fuzzy subsets*. In: Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada 57—61, (2013).
[16] T. Senapati, R. R. Yager, *Fermatean fuzzy sets*, J. Ambient Intell. Humaniz. Comput., **11** (2020), 663–674.
[17] T. Senapati, R. R. Yager, *Some new operations over Fermatean fuzzy numbers and application of Fermatean fuzzy WPM in multiple criteria decision making*, Informatica, **30** (2019), 391–412.
[18] T. Senapati, R. R. Yager, *Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria DM methods*, Eng. Appl. Artif. Intell., **85** (2019), 112–121.
[19] M. Akram, G. Ali, J. C. R. Alcantud, A. Riaz, *Group decision-making with Fermatean fuzzy soft expert knowledge*, Artif. Intell. Rev., **55** (2022), 5349–5389.
[20] M. Akram, U. Amjad, J. C. R. Alcantud, G. Santos-Garcis, *Complex fermatean fuzzy N-soft sets: a new hybrid model with applications*, J. Ambient Intell. Humaniz. Comput., **14** (2023), 8765–8798.
[21] M. Kirişci, *New cosine similarity and distance measures for Fermatean fuzzy sets and TOPSIS approach*, Knowl. Inf. Syst., **65** (2023), 855–868.
[22] M. Kirişci, I. Demir, N. Şimşek, *Fermatean fuzzy ELECTRE multi-criteria group decision-making and most suitable biomedical material selection*, Artif. Intell. Med., **127** (2022), 102278.
[23] M. Kirişci, *Data analysis for lung cancer: Fermatean hesitant fuzzy sets approach*, Math. Models Comput. Simul., **30** (2022), 701–710.
[24] M. Kirişci, *Fermatean Hesitant Fuzzy Sets for Multiple Criteria Decision-Making with Applications*, Fuzzy Information and Engineering, **15**(2), (2023), 100–127. doi: 10.26599/FIE.2023.9270011

[25] M. Kirişci, *Data analysis for panoramic X-ray selection: Fermatean fuzzy type correlation coefficients approach*, Engineering Applications of Artificial Intelligence, **126**, (2023), 106824. doi:10.1016/j.engappai.2023.106824
[26] G. Shahzadi, A. Akram, *Hypergraphs Based on Pythagorean Fuzzy Soft Model*. Math. Comput. Appl., **24**, (2019), 100. doi:10.3390/mca24040100
[27] G. Shahzadi, A. Akram, *Group decision-making for the selection of an antivirus mask under fermatean fuzzy soft information*. Journal of Intelligent & Fuzzy Systems, **40**, (2021), 1401–1416.
[28] P.A. Ejegwa, P. Muhiuddin, E.A. Algehyne, J.M. Agbetayo, D. Al-Kadi, *An enhanced Fermatean fuzzy composition relation based on a maximum-average approach and its application in diagnostic analysis*. Journal of Mathematics, (2022), Article ID 1786221, 12 pages.
[29] P.A. Ejegwa, I. C. Onyeke, *Fermatean fuzzy similarity measure algorithm and its application in students' admission process*. International Journal of Fuzzy Computation and Modelling, **4**(1), (2022), 34–50.
[30] A. Mehmood, W. Ullah, S. Broumi, M. I. Khan, H. Qureshi, M. I. Abbas, H.Kalsoom, F. Nadeem, *Neutrosophic Soft Structures*, Neutrosophic Sets and Systems, **33**, (2020), 23–58.
[31] I. C. Onyeke, P.A. Ejegwa, *Modified Senapati and Yager's Fermatean Fuzzy Distance and Its Application in Students' Course Placement in Tertiary Institution*, In: Sahoo, L., Senapati, T., Yager, R.R. (eds) Real Life Applications of Multiple Criteria Decision Making Techniques in Fuzzy Domain; Studies in Fuzziness and Soft Computing, vol **420**, 237–253, Springer.
[32] T. M. Athira, S. J. John, H. Garg, *Entropy and distance measures of Pythagorean fuzzy soft sets and their applications*, J. Intell. Fuzzy Syst., **37**(3) (2019), 4071–4084.
[33] T. M. Athira, S. J. John, H. Garg, *A novel entropy measure of Pythagorean fuzzy soft sets*, AIMS Mathematics, **5**(2) (2020), 1050–1061.
[34] A. Guleria, R. K. Bajaj, *On Pythagorean fuzzy soft matrices, operations and their applications in decision making and medical diagnosis*, Soft Comput. **23** (2019), 7889–7900.
[35] P. Maji, A. Biswas, A. Roy, *Fuzzy soft sets*, J. Fuzzy Math., **9**(3) (2001), 589–602.
[36] P. Maji, A. Biswas, A. Roy, *Intuitionistic Fuzzy soft sets*, J. Fuzzy Math., **9**(3) (2001), 677–692.
[37] P. Maji, A. Biswas, A. Roy, *Soft set theory*, Comp. Math Appl., **45**(4-5) (2003), 555–562.
[38] P. Majumdar, S. Samanta, *Similarity measure of soft sets*, New Math. Nat. Comput., **4**(1) (2008), 1–12.
[39] K. Naeem, M. Riaz, X. Peng, D. Afzal, *Pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR, and aggregation operators*, J. Intell. Fuzzy Syst., **37**(5) (2019), 6937–6957.
[40] X. Peng, Y. Yang, J. Song, Y. Jiang, *Pythagorean fuzzy soft set and its application*, Computer Engineering, **41**(7) (2015), 224–229.
[41] J. C. R. Alcantud, G. Varela, B. Santos-Buitrago, G. Santos-Garcia, M. F. Jimenez, *Analysis of survival for lung cancer resection cases with fuzzy and soft set theory in surgical decision-making*, Plos One, **14** (2019), e0218283.
[42] M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Shabir, *On some new operations in soft set theory*, Comput. Math. Appl. **57** (2009), 1547–1553.
[43] N. Çağman, S. Enginoğlu, F. Citak, *Fuzzy soft set theory and its application*, Iran. J. Fuzzy Syst., **8** (2011), 137–147.
[44] M. Kirişci, *A case study for medical decision making with the fuzzy soft sets*, Afr. Mat., **31** (2020), 557–564.
[45] A. R. Roy, P. K. Maji, *A fuzzy soft set theoretic approach to decision-making problems*, J. Comput. Appl. Math., **203** (2007), 412–418.
[46] S. Saleh, R. Abu-Gdairi, T. M. Al-Shami, Mohammed S. Abdo, *On categorical property of fuzzy soft topological spaces*, Appl. Math. Inform. Sci., **16** (2022), 635–641.
[47] M. T. Hamida, M. Riaz, D. Afzal, *Novel MCGDM with q-rung orthopair fuzzy soft sets and TOPSIS approach under q-Rung orthopair fuzzy soft topology*, J. Intell. Fuzzy Syst., **39** (2020), 3853–3871.
[48] A. Sivadas, S. J. John, *Fermatean fuzzy soft sets and its applications*, In: A. Awasthi, S. J.John, S. Panda (eds), CSMCS 2020. Communications in Computer and Information Science, Springer, Singapore, **1345** (2021).
[49] Y. J. Xu, Y. K. Sun, D. F. Li, *Intuitionistic fuzzy soft set*, 2nd International Workshop on Intelligent Systems and Applications, (2010), 1–4.
[50] E. Szmidt, J. Kacprzyk, *Distances between intuitionistic fuzzy sets*, Fuzzy Sets Syst. **114** (2000), 505–518.
[51] P. A. Ejegwa, E. S. Modom, *Diagnosis of viral hepatitis using new distance measure of intuitionistic fuzzy sets*, Int. J. Fuzzy Math. Arch. **8** (1) (2015), 1–7.
[52] P. A. Ejegwa, *Distance and similarity measures for Pythagorean fuzzy sets*, Granul. Comput. **5** (2020), 225–238.
[53] P. A. Ejegwa, *Modified Zhang and Xu's distance measure for Pythagorean fuzzy sets and its application to pattern recognition problems*, Neural. Comput. Appl., **32** (2020), 10199–10208.
[54] A. H. Ganie, S. Singh, M. M. Khalaf, M. M. A. Al-Shamiri, *On some measures of similarity and entropy for Pythagorean fuzzy sets with their applications*, Comp. Appl. Math., **41** (2022), 420.
[55] X. L. Zhang, Z. S. Xu, *Extension of TOPSIS to multi-criteria decision making with Pythagorean fuzzy sets*, Int. J. Intell. Syst. **29** (2014), 1061—1078.
[56] G. Wei, R. Lin, H. Wang, *Distance and similarity measures for hesitant interval-valued fuzzy sets*, J. Intell. Fuzzy Syst., **27**(1) (2014), 19–36.
[57] Y. Song, X. Wang, H. Zhang, *A distance measure between intuitionistic fuzzy belief functions*, Knowl. Based Syst., **86** (2015), 288–298.
[58] G. Wei, *Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making*, Informatica, **28** (2017), 547–564.
[59] R. A. Alahmadi, A. H. Ganie, Y. Al-Qudah, M. M. Khalaf, A. H. Ganie, *Multi-attribute decision-making based on novel Fermatean fuzzy similarity measure and entropy measure*. Granul. Comput., **8** (2023), 1385–1405.
[60] G. Ali, M. N. Ansari, *Multiattribute decision-making under Fermatean fuzzy bipolar soft framework*, Granul. Comput., **7** (2022), 337–352.
[61] L. A. Zadeh, *Fuzzy sets and systems*, In Proceedings of the Symposium on Systems Theory, (1965), 29–37.
[62] A. De Luca, S. Termini, *A definition of a nonprobabilistic entropy in the setting of fuzzy set theory*, In: Readings in Fuzzy Sets for Intelligent Systems, Elsevier, (1993), 197–202.
[63] J. Wu, J. Sun, L. Liang, Y. Zha. *Determination of weights for ultimate cross efficiency using Shannon entropy*, Expert Syst. Appl., **38** (2011), 5162–5165.
[64] A. Kaufmann, *Introduction to the Theory of Fuzzy Sets:* Fundamental Theoretical Elements, Vol. 1, Academic Press, New York, 1975.
[65] R. R. Yager, *On the measure of fuzziness and negation, Part 1: Membership in the unit interval*, Int. J. Gener. Syst., **5** (1979), 221–229.
[66] M. Higashi, G. Klir, *On measures of fuzziness and fuzzy complements*, Int. J. General Syst., **8** (1982), 169–180.
[67] V. Mohagheghi, S. M. Mousavi, B. Vahdani, *Enhancing DM flexibility by introducing a new last aggregation evaluating approach based on multi-criteria group decision making and Pythagorean fuzzy sets*, Appl. Soft Comput., **61** (2017), 527–535.
[68] X. Peng, H. Yuan, Y. Yang, *Pythagorean fuzzy information measures and their applications*, Int. J. Intell., **32** (2017), 991–1029.
[69] Y. Jiang, Y. Tang, H. Liu, Z. Chen, *Entropy on intuitionistic fuzzy soft sets and on interval-valued fuzzy soft sets*, Inf. Sci., **240** (2013), 95–114.
[70] P. Burillo, H. Bustince, *Entropy on intuitionistic fuzzy sets and interval-valued fuzzy sets*, Fuzzy Sets Syst, **78**(3) (1996), 305–316.
[71] Q. Han, L. Weimin, S. Yafei, T. Zhang, R. Wang, *A new method for MAGDM based on improved TOPSIS and a novel Pythagorean fuzzy soft entropy*, Symmetry **11** (2019), 905.

Universal Journal of Mathematics and Applications, 7 (1) (2024) 30-37
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1397051

New Perspectives on Fractional Milne-Type Inequalities: Insights from Twice-Differentiable Functions

Henok Desalegn Desta[1], Hüseyin Budak[2*] and Hasan Kara[2]

[1]*Department of Mathematics, Haramaya University, Ethiopia*
[2]*Department of Mathematics, Faculty of Science and Arts, Duzce University, Türkiye*
[*]*Corresponding author*

Article Info

Keywords: *Convex function, Fractional integrals, Milne type inequalities, Twice differentiable*
2010 AMS: *26D10, 26D07, 26D15*
Received: *28 November 2023*
Accepted: *16 January 2024*
Available online: *16 January 2024*

Abstract

This paper delves into an inquiry that centers on the exploration of fractional adaptations of Milne-type inequalities by employing the framework of twice-differentiable convex mappings. Leveraging the fundamental tenets of convexity, Hölder's inequality, and the power-mean inequality, a series of novel inequalities are deduced. These newly acquired inequalities are fortified through insightful illustrative examples, bolstered by rigorous proofs. Furthermore, to lend visual validation, graphical representations are meticulously crafted for the showcased examples.

1. Introduction

Convex functions and the inequalities that describe their properties are fundamental concepts in mathematical optimization and economic theory, among other fields. These functions are characterized by the shape of their graph, which curves such that any line segment between two points on the graph does not fall below it. This curvature leads to interesting and useful properties, particularly in the realm of optimization where they guarantee local minima are also global minima. Inequalities related to convex functions, such as Jensen's inequality, play a crucial role in various analytical and theoretical proofs. They are also instrumental in establishing conditions for optimality and convergence in more complex scenarios. Understanding these functions and their associated inequalities provides a solid foundation for delving into more advanced topics in mathematics and economics. Now, let's define the basic notion of a convex function to further grasp the essence of these intriguing concepts.

Definition 1.1 ([1]). *Let I be convex set on $\mathbb{R}$. The function $f : I \to \mathbb{R}$ is called convex on I, if it satisfies the following inequality:*

$$f(tx+(1-t)\,y) \le t f(x) + (1-t) f(y) \tag{1.1}$$

for all $(x,y) \in I$ and $t \in [0,1]$. The mapping f is a concave on I if the inequality (1.1) holds in reversed direction for all $t \in [0,1]$ and $x,y \in I$.

Within the domain of mathematical analysis, inequalities serve as pivotal tools for examining the intricate nuances of numerical relationships. These inequalities provide a framework for exploring the dynamic interplay between quantities, shedding light on the disparities that permeate mathematical landscapes. As researchers endeavor to unveil the mysteries of mathematical systems, inequalities offer a lens through which the fundamental variations between values can be rigorously scrutinized. Their presence underscores the recognition that the mathematical continuum is far from a monolithic entity; rather, it is a rich tapestry of gradations and magnitudes. By meticulously navigating the terrain of inequalities, scholars gain insights into the profound structure and underlying principles governing mathematical phenomena.
In recent times, the attention of researchers has turned significantly toward diverse classes of integral inequalities, including types like Trapezoid, Midpoint, and Simpson inequalities. Numerous scholars have made substantial contributions to extending and generalizing these fundamental inequalities. For instance, noteworthy progress has been achieved by Dragomir and Agarwal in investigating error estimates for the trapezoidal formula, as highlighted in [2]. The variations of the trapezoid formula's boundedness were explored by Dragomir in [3]. Additionally, Sarikaya and Aktan, in their work [4], delved into novel inequalities of both the Simpson and Trapezoid types, focusing particularly on functions characterized by a convex absolute value of the second derivative. Fractional trapezoid-type inequalities found their

Email addresses and ORCID numbers: **henokddesta@gmail.com, 0000-0003-0395-4857 (H. D. Desta), hsyn.budak@gmail.com, 0000-0001-8843-955X (H. Budak), hasan64kara@gmail.com, 0000-0002-2075-944X (H. Kara)**
Cite as: H. D. Desta, H. Budak, H. Kara, *New Perspectives on Fractional Milne-Type Inequalities: Insights from Twice-Differentiable Functions*, Univers. J. Math. Appl., 7(1) (2024), 30-37.

exploration in [5, 6]. Kırmacı, in [7] introduced midpoint-type inequalities tailored for differentiable convex functions, while Sarıkaya and colleagues derived an array of fresh inequalities suitable for twice differentiable functions as expounded in [8]. The fractional counterparts of these findings are also comprehensively discussed in [9, 10]. Moreover, a series of mathematical luminaries have established results applicable to twice differentiable convex functions, exemplified by works such as [11–13].

Explorations within the realm of numerical integration and the establishment of error bounds have assumed a pivotal position within the tapestry of mathematical literature. Furthermore, scholars have meticulously examined the error bounds of functions that exhibit diverse levels of differentiability from once to multiple times. The spectrum of mathematical inequalities, including those of Simpson, Newton, and Milne types, emerges with distinct purposes and applications. These inequalities find their roles reverberating across various domains of mathematics and numerical analysis, facilitating meticulous scrutiny and enhancement of the efficiency and accuracy of computational techniques. Remarkably, this journey doesn't halt here a multitude of researchers have ventured into uncharted territories, harnessing the potency of fractional calculus to derive novel bounds, expanding the frontiers of understanding and application in this intricate mathematical terrain.

The Milne-type inequality delves into the realm of mathematical analysis within the context of a function's behavior over a closed interval. This inequality offers insights into the intricate connections between a function's values at the endpoints of the interval, its integral over the interval, and the fourth derivative of the function. It forms a crucial bridge between differentiability and integration, highlighting the delicate interplay between these mathematical concepts. The essence of this inequality lies in its ability to encapsulate the behavior of a function in terms of its derivatives and integrals, offering a powerful tool for understanding and quantifying the relationships within the mathematical landscape.

The Milne-type inequality, an essential mathematical inequality within the realm of integral estimation, draws its nomenclature from the distinguished British mathematician Edward Arthur Milne, who bestowed this inequality upon the mathematical community during the early 20th century. The roots of this inequality trace back to the strategic interplay between integral values and specific points, enabling an upper boundary to be established.

Significant strides in Milne-type inequality research have been witnessed, Budak et al. [14] elegantly derived Milne-type inequalities for differentiable convex functions through the application of Riemann-Liouville fractional integrals. Inspired by their contributions to literature, our current study embarks on a journey of exploration, aiming to unveil novel inequalities by harnessing the potential of Riemann-Liouville integrals to characterize twice-differentiable functions.

Our endeavor begins with a thorough review of the established definitions underpinning the Milne-type inequality and the Riemann-Liouville integral. These definitions, widely recognized and foundational within the literature, lay the groundwork for our research.

Furthermore, delving into the context of Newton-Cotes formulas reveals intriguing parallels between Milne's formula, an open-type variant, and the Simpson's formula, representing the closed-type counterpart. These similarities emerge as a result of both formulas adhering to identical conditions. Consider a function $f:[a,b]\to\mathbb{R}$, which boasts four times continuous differentiability over the open interval (a,b). Here, the expression $\left\|f^{(4)}\right\|_\infty=\sup\limits_{\upsilon\in(a,b)}\left|f^{(4)}(\upsilon)\right|<\infty$, signifies the supremum of the absolute values of the fourth derivative, symbolizing the upper echelon of its variations. Under these stipulated conditions, our pursuit culminates in the emergence of the Milne-type inequality:

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(\upsilon)d\upsilon\right|\leq\frac{7(b-a)^4}{23040}\left\|f^{(4)}\right\|_\infty$$

This compelling result, as unveiled in [15], substantiates the elegant interplay between mathematical constructs in the exploration of Milne-type inequalities. For more studies on Milne type inequalities, you can refer to [16–18].

Definition 1.2 ([19])**.** *Let us consider a function f belonging to the space $L_1[a,b]$. Within this context, we introduce the Riemann-Liouville fractional integrals, denoted as $\mathscr{J}^{\alpha}_{(a)^+}f$ and $\mathscr{J}^{\alpha}_{(b)^-}f$, where $\alpha>0$, by invoking the following equalities:*

$$\mathscr{J}^{\alpha}_{(a)^+}f(\upsilon)=\frac{1}{\Gamma(\alpha)}\int_a^{\upsilon}(\upsilon-t)^{\alpha-1}f(t)dt,\quad \upsilon>a,$$

which represents the definitive expression of the left-sided Riemann-Liouville fractional integral of function f with order α at the point a, and

$$\mathscr{J}^{\alpha}_{(b)^-}f(\upsilon)=\frac{1}{\Gamma(\alpha)}\int_{\upsilon}^{b}(t-\upsilon)^{\alpha-1}f(t)dt,\quad \upsilon<b,$$

depicting the clear-cut equation that defines the right-sided Riemann-Liouville fractional integral of function f with order α at the point b. It is pertinent to note that $\Gamma(\alpha)$ represents the Gamma function, and $\mathscr{J}^{0}_{(a)^+}f(\upsilon)=\mathscr{J}^{0}_{(b)^-}f(\upsilon)=f(\upsilon)$ in accordance with the defined context.

For an in-depth exploration into the intricacies of Riemann-Liouville fractional integrals, we refer the interested reader to [19–21]. Armed with this foundational understanding from the existing literature, we embark on the journey to unveil our novel contributions in the subsequent sections.

2. Main Results

In this study, we will initially derive an equation for twice differentiable functions. By taking the absolute value of this equation and employing convexity, we will establish an inequality. Furthermore, leveraging Hölder's and the power mean inequalities, we will deduce novel inequalities.

Lemma 2.1. *Consider a mapping $f:[a,b]\to\mathbb{R}$ that is twice differentiable on the interval (a,b) and satisfies $f''\in L_1([a,b])$. Under these conditions, the subsequent lemma establishes the following equality:*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\\&=\frac{(b-a)^2}{8(\alpha+1)}\int_0^1\left(t^{\alpha+1}-\frac{4}{3}(\alpha+1)t\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right]dt.\end{aligned}$$

Proof. Applying the method of integration by parts, we are able to derive the subsequent expression:

$$\begin{aligned}
I_1 &= \int_0^1 \left(t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right) f''\left(\frac{2-t}{2}a + \frac{t}{2}b\right) dt \\
&= \frac{2}{b-a}\left[\left(t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right) f'\left((1-t)a + t\frac{a+b}{2}\right)\right]\Bigg|_0^1 - \frac{2(\alpha+1)}{b-a}\int_0^1 \left(t^{\alpha} - \frac{4}{3}\right) f'\left((1-t)a + t\frac{a+b}{2}\right) dt \\
&= \frac{2}{b-a}\left[\left(1 - \frac{4}{3}(\alpha+1)\right) f'\left(\frac{a+b}{2}\right)\right] - \frac{2(\alpha+1)}{b-a}\left\{\int_0^1 \left(t^{\alpha} - \frac{4}{3}\right) f'\left((1-t)a + t\frac{a+b}{2}\right) dt\right\} \\
&= \frac{2}{b-a}\left[\left(1 - \frac{4}{3}(\alpha+1)\right) f'\left(\frac{a+b}{2}\right)\right] - \frac{2(\alpha+1)}{b-a}\left\{-\frac{2}{3(b-a)} f\left(\frac{a+b}{2}\right) + \frac{8}{3(b-a)} f(a) - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a)\right\} \\
&= \frac{2}{b-a}\left[\left(1 - \frac{4}{3}(\alpha+1)\right) f'\left(\frac{a+b}{2}\right)\right] + \frac{4(\alpha+1)}{3(b-a)^2} f\left(\frac{a+b}{2}\right) - \frac{16(\alpha+1)}{3(b-a)^2} f(a) + \frac{2^{\alpha+2}\Gamma(\alpha+1)(\alpha+1)}{(b-a)^{\alpha+2}} \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a).
\end{aligned}$$

Likewise, we acquire

$$\begin{aligned}
I_2 &= \int_0^1 \left(t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right) f''\left(\frac{2-t}{2}b + \frac{t}{2}a\right) dt \\
&= -\frac{2}{b-a}\left[\left(1 - \frac{4}{3}(\alpha+1)\right) f'\left(\frac{a+b}{2}\right)\right] + \frac{4(\alpha+1)}{3(b-a)^2} f\left(\frac{a+b}{2}\right) - \frac{16(\alpha+1)}{3(b-a)^2} f(b) + \frac{2^{\alpha+2}\Gamma(\alpha+1)(\alpha+1)}{(b-a)^{\alpha+2}} \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b).
\end{aligned}$$

Subsequently, we can observe the following computation:

$$\frac{(b-a)^2}{8(\alpha+1)}[I_1 + I_2] = \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - \frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right].$$

Hence, this concludes the proof. □

Theorem 2.2. *Assuming $f : [a,b] \to \mathbb{R}$ is a function with twice differentiable function on the open interval (a,b), and $f'' \in L_1([a,b])$, with $|f''|$ exhibiting convexity across $[a,b]$, the subsequent inequality is valid:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - \frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right]\right| \\
&\leq \frac{(b-a)^2\,(2(\alpha+1)(\alpha+2) - 3)}{24(\alpha+1)(\alpha+2)}\left[|f''(a)| + |f''(b)|\right].
\end{aligned} \tag{2.1}$$

Proof. Upon applying the absolute value to Lemma 2.1 and leveraging the convex property of $|f''|$, we deduce the following result:

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + \mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - \frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right]\right| \\
&= \frac{(b-a)^2}{8(\alpha+1)}\left|\int_0^1 \left(t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right)\left[f''\left(\frac{2-t}{2}a + \frac{t}{2}b\right) + f''\left(\frac{2-t}{2}b + \frac{t}{2}a\right)\right] dt\right| \\
&\leq \frac{(b-a)^2}{8(\alpha+1)}\int_0^1 \left|t^{\alpha+1} - \frac{4}{3}(\alpha+1)t\right|\left[\frac{2-t}{2}|f''(a)| + \frac{t}{2}|f''(b)| + \frac{2-t}{2}|f''(b)| + \frac{t}{2}|f''(a)|\right] dt \\
&= \frac{(b-a)^2}{8(\alpha+1)}\left[|f''(a)| + |f''(b)|\right]\int_0^1 \frac{4}{3}(\alpha+1)t - t^{\alpha+1} dt \\
&= \frac{(b-a)^2\,(2(\alpha+1)(\alpha+2) - 3)}{24(\alpha+1)(\alpha+2)}\left[|f''(a)| + |f''(b)|\right].
\end{aligned}$$

The necessary inequality (2.1) is established. □

Corollary 2.3. *The choice $\alpha = 1$ in Theorem 2.2 yields the following result:*

$$\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2}{16}\left[|f''(a)| + |f''(b)|\right].$$

Example 2.4. *Consider the interval $[a,b] = [0,1]$, and let's define the function $f : [0,1] \to \mathbb{R}$ as $f(t) = \frac{t^4}{12}$, so that $f''(t) = t^2$ and $|f''|$ is convex over the interval $[0,1]$. Given these conditions,*

$$\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] = \frac{31}{576}$$

Using the definition of the Riemann-Liouville fractional integral, we attain

$$\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) = \mathscr{J}^{\alpha}_{\left(\frac{1}{2}\right)^-} f(0) = \frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{2}} t^{\alpha-1}\frac{t^4}{12} dt = \frac{1}{12\Gamma(\alpha)(\alpha+4)2^{\alpha+4}}$$

and

$$\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)=\mathscr{J}^{\alpha}_{\left(\frac{1}{2}\right)^+}f(1)=\frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}(1-t)^{\alpha-1}\frac{t^4}{12}dt$$
$$=\frac{\alpha^4-50\alpha^3+83\alpha^2+262\alpha+384}{12\alpha\Gamma(\alpha)(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)2^{\alpha+4}}.$$

Hence, we possess

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]$$
$$=2^{\alpha-1}\Gamma(\alpha+1)\left[\frac{1}{12\Gamma(\alpha)(\alpha+4)2^{\alpha+4}}+\frac{\alpha^4-50\alpha^3+83\alpha^2+262\alpha+384}{12\alpha\Gamma(\alpha)(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)2^{\alpha+4}}\right]$$
$$=\frac{2\alpha^4-44\alpha^3+94\alpha^2+268\alpha+384}{384(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}.$$

Consequently, the left-hand side of inequality (2.1) simplified to

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right|$$
$$=\left|\frac{2\alpha^4-44\alpha^3+94\alpha^2+268\alpha+384}{384(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}-\frac{31}{576}\right|=:LHS. \tag{2.2}$$

In a similar manner, the right-hand side of inequality (2.1) was brought down to

$$\frac{(b-a)^2\left(2(\alpha+1)(\alpha+2)-3\right)}{24(\alpha+1)(\alpha+2)}\left[|f''(a)|+|f''(b)|\right]=\frac{2(\alpha+1)(\alpha+2)-3}{24(\alpha+1)(\alpha+2)}=:RHS.$$

The outcomes from Example 2.4 are illustrated in Figure 2.1.

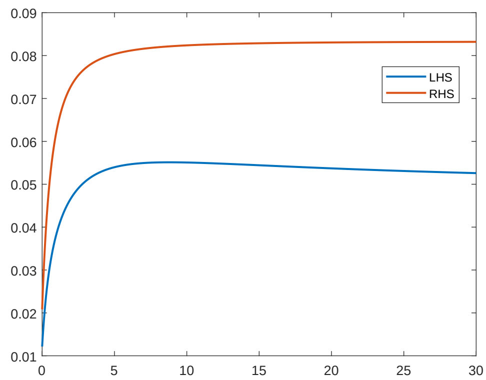

Figure 2.1: Graph of Example 2.4.

Theorem 2.5. *Consider a function $f:[a,b]\to\mathbb{R}$, twice differentiable on the interval (a,b), with $f''\in L_1([a,b])$. Additionally, let $|f''|^q$ exhibit convexity on $[a,b]$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. As a result of these conditions, the following inequality is established.*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right|$$
$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\times\left[\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{4}\right)^{\frac{1}{q}}\right]. \tag{2.3}$$

Proof. By considering the absolute value in Lemma 2.1, we find that

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{J}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right|$$
$$=\frac{(b-a)^2}{8(\alpha+1)}\left|\int_0^1\left(t^{\alpha+1}-\frac{4}{3}(\alpha+1)t\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right]dt\right| \tag{2.4}$$
$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left[\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt+\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right|dt\right].$$

Exploiting the convex nature of $|f''|^q$ and employing the Holder inequality, leads us to the conclusion that

$$\begin{aligned}
&\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\\
&\le\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q\right)^{\frac{1}{q}}\\
&\le\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{2-t}{2}|f''(a)|^q+\frac{t}{2}|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\\
&=\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left[\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right]^{\frac{1}{q}}.
\end{aligned}\tag{2.5}$$

Similarly,

$$\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right|dt\le\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\left[\frac{3|f''(b)|^q+|f''(a)|^q}{4}\right]^{\frac{1}{q}}.\tag{2.6}$$

Through the incorporation of inequalities (2.5) and (2.6) into (2.4) we arrive at inequality (2.3), thus finalizing the proof. □

Corollary 2.6. *When $\alpha=1$, based on Theorem 2.5, we get*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\
&\le\frac{(b-a)^2}{16}\left(\int_0^1\left(\frac{8}{3}t-t^2\right)^p\right)^{\frac{1}{p}}\times\left[\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Example 2.7. *Considering $[a,b]=[0,1]$, let's define the function $f:[0,1]\to\mathbb{R}$ as $f(t)=\frac{t^4}{12}$, satisfying $f''(t)=t^2$, and ensuring $|f''|$ exhibits convexity over $[0,1]$, with $p=q=2$. The left-hand side of the inequality (2.3) resembles the equation presented in (2.2), while the right-hand side of (2.3) simplifies to*

$$\begin{aligned}
&\frac{(b-a)^2}{8(\alpha+1)}\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}\times\left[\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(b)|^q+|f''(a)|^q}{4}\right)^{\frac{1}{q}}\right]\\
&=\frac{1}{8(\alpha+1)}\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)^2 dt\right)^{\frac{1}{2}}\times\left[\left(\frac{3|f''(0)|^2+|f''(1)|^2}{4}\right)^{\frac{1}{2}}+\left(\frac{3|f''(1)|^2+|f''(0)|^2}{4}\right)^{\frac{1}{2}}\right]\\
&=\frac{\sqrt{3}+1}{16(\alpha+1)}\left[\frac{16(\alpha+1)^2(2\alpha+3)(\alpha+3)-72(2\alpha+3)(\alpha+1)+27(\alpha+3)}{27(2\alpha+3)(\alpha+3)}\right]^{\frac{1}{2}}=:RHS.
\end{aligned}$$

The findings from Example 2.7 are visually presented in Figure 2.2.

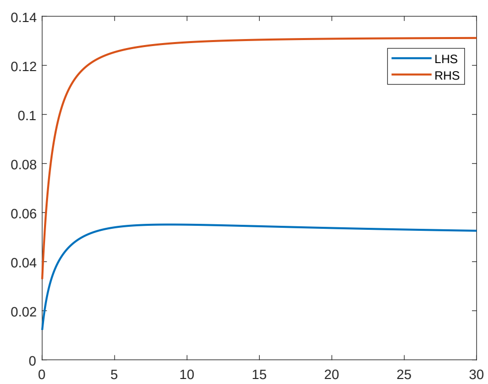

Figure 2.2: Graph of Example 2.7.

Theorem 2.8. *Consider a function $f:[a,b]\to\mathbb{R}$ that is twice differentiable over the interval (a,b), with $f''\in L_1([a,b])$ and $|f''|^q$, where $q\geq 1$, demonstrating convexity across $[a,b]$. As a result of these conditions, the subsequent inequality is satisfied.*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right|\\
&\leq \frac{(b-a)^2}{8(\alpha+1)}\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(a)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(b)|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(b)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(a)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{2.7}$$

Proof. Through the process of taking the absolute value within Lemma 2.1, we arrive at

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]\right|\\
&=\frac{(b-a)^2}{8(\alpha+1)}\left|\int_0^1\left(t^{\alpha+1}-\frac{4}{3}(\alpha+1)t\right)\left[f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right]dt\right|\\
&\leq\frac{(b-a)^2}{8(\alpha+1)}\left[\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt+\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right|dt\right].
\end{aligned}\tag{2.8}$$

By exploiting the power-mean inequality in conjunction with the convex property of $|f''|^q$, we establish

$$\begin{aligned}
&\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\\
&\leq\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left(\frac{2-t}{2}|f''(a)|^q+\frac{t}{2}|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\\
&=\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}}\\
&\times\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(a)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(b)|^q\right).
\end{aligned}\tag{2.9}$$

Similarly,

$$\begin{aligned}
&\int_0^1\left(\frac{4}{3}(\alpha+1)t-t^{\alpha+1}\right)\left|f''\left(\frac{2-t}{2}b+\frac{t}{2}a\right)\right|dt\\
&\leq\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}}\\
&\times\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(b)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(a)|^q\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.10}$$

Upon substituting (2.9) and (2.10) into (2.8), we arrive at the intended inequality denoted as (2.7). □

Corollary 2.9. *If we choose $\alpha=1$ in Theorem 2.8, then we acquire*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\
&\leq\frac{(b-a)^2}{16}\left[\left(\frac{147|f''(a)|^q+69|f''(b)|^q}{216}\right)^{\frac{1}{q}}+\left(\frac{147|f''(b)|^q+69|f''(a)|^q}{216}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Example 2.10. *Let's take the interval $[a,b]=[0,1]$ and define the function $f:[0,1]\to\mathbb{R}$ as $f:[0,1]\to\mathbb{R}$, $f(t)=\frac{t^4}{12}$, yielding $f''(t)=t^2$. Notably, $|f''|$ exhibits convex behavior across $[0,1]$ and q is assigned the value of 2. Drawing a parallel, the left-hand side of inequality (2.7)* *shares a resemblance with equality (2.2), while the right-hand side of (2.7) simplifies to*

$$\frac{(b-a)^2}{8(\alpha+1)}\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(a)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)|f''(b)|^q+\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)|f''(a)|^q\right)^{\frac{1}{q}}\right]$$
$$=\frac{1}{8(\alpha+1)}\left(\frac{2(\alpha+1)(\alpha+2)-3}{3(\alpha+2)}\right)^{\frac{1}{2}}$$
$$\times\left[\left(\frac{4(\alpha+1)(\alpha+3)-9}{18(\alpha+3)}\right)^{\frac{1}{2}}+\left(\frac{8(\alpha+1)(\alpha+2)(\alpha+3)-18(\alpha+3)+9(\alpha+2)}{18(\alpha+2)(\alpha+3)}\right)^{\frac{1}{2}}\right]=:RHS.$$

The findings extracted from Example 2.10 have been depicted in Figure 2.3.

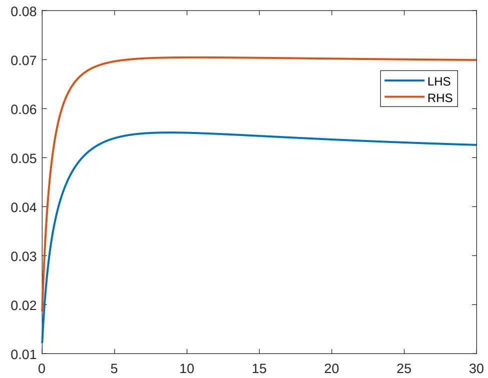

Figure 2.3: Graph of Example 2.10.

3. Conclusion

In this article, Milne-type inequalities have been derived using Fractional Integrals. The obtained inequalities are exemplified, and the accuracy of these examples is validated through graphical representations. Future researchers could explore novel inequalities for different fractional integrals. Furthermore, the current study focused on functions that are twice differentiable. By considering a broader scope of differentiable functions, new inequalities could potentially be discovered. By employing various types of inequalities in Functional Analysis and Numerical Analysis, novel results could be obtained using the methodologies presented in this paper.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
[2] S. S. Dragomir, R. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., **11**(5) (1998), 91–95.
[3] S. S. Dragomir, *On trapezoid quadrature formula and applications*, Kragujevac J. Math., **23** (2001), 25–36.

[4] M. Z. Sarıkaya, N. Aktan, *On the generalization of some integral inequalities and their applications*, Math. Comput. Model., **54**(9-10) (2011), 2175–2182.
[5] M. Z. Sarıkaya, H. Budak, *Some Hermite-Hadamard type integral inequalities for twice differentiable mappings via fractional integrals*, Facta Univ. Ser. Math. Inform., **29**(4) (2014), 371–384.
[6] M. Z. Sarıkaya, E. Set, H. Yaldiz, N. Basak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model., **57**(9-10) (2013), 2403–2407
[7] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput., **147**(5) (2004), 137–146.
[8] M. Z. Sarikaya, A. Saglam, H. Yıldırım, *New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex*, Int. J. Open Probl. Comput. Sci. Math., **5**(3) (2012), 1–14.
[9] M. Iqbal, M. I. Bhatti, K. Nazeer, *Generalization of inequalities analogous to Hermite–Hadamard inequality via fractional integrals*, Bull. Korean Math. Soc., **52**(3) (2015), 707–716.
[10] M.Z. Sarikaya, H. Yildirim, *On Hermite–Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Math. Notes, **17**(2)(2016), 1049–1059.
[11] S. Hussain, S. Qaisar, *More results on Simpson's type inequality through convexity for twice differentiable continuous mappings*, SpringerPlus, 5(1) (2016), 1–9.
[12] J. Nasir, S. Qaisar, S. I. Butt, A. Qayyum, *Some Ostrowski type inequalities for mappings whose second derivatives are preinvex function via fractional integral operator*, AIMS Mathematics, **7**(3) (2022), 3303–3320.
[13] M. Z. Sarikaya, E. Set, M. E. Özdemir, *On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex*, J. Appl. Math. Stat. Inform., **9**(1) (2013), 37–45.
[14] H. Budak, P. Kösem, H. Kara, *On new Milne-type inequalities for fractional integrals*, J. Inequal. Appl., (2023), Art. 10.
[15] A. D. Booth, *Numerical methods*, 3rd Ed., Butterworths, California, 1966.
[16] H. Budak, A. A. Hyder, *Enhanced bounds for Riemann-Liouville fractional integrals: Novel variations of Milne inequalities*, AIMS Mathematics, **8**(12) (2023), 30760–30776.
[17] P. Bosch, J. M. Rodríguez, J. M Sigarreta, *On new Milne-type inequalities and applications*, J. Inequal. Appl., (2023), Art. 3.
[18] B. Meftah, A. Lakhdari, W. Saleh, A. Kiliçman, *Some new fractal Milne-type integral inequalities via generalized convexity with applications*, Fractal Fract., **7**(2) (2023), Art. 166.
[19] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations, North-Holland Mathematics Studies*, 204, Elsevier Sci. B. V., Amsterdam, 2006.
[20] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Wien: Springer-Verlag, 1997, 223–276.
[21] S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

Universal Journal of Mathematics and Applications, 7 (1) (2024) 38-45
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1390222

Laguerre Collocation Approach of Caputo Fractional Fredholm-Volterra Integro-Differential Equations

Dilek Varol[1] and Ayşegül Daşcıoğlu[1*]

[1] *Department of Mathematics, Faculty of Science, Pamukkale University, Denizli, Türkiye*
[*] *Corresponding author*

Article Info

Keywords: *Caputo fractional derivatives, Fredholm-Volterra integro-differential equations, Laguerre polynomials*
2010 AMS: *26A33, 33C45, 34A08, 34K37, 45J05, 65D15, 90C32*
Received: *13 November 2023*
Accepted: *6 February 2024*
Available online: *15 February 2024*

Abstract

This paper discusses the linear fractional Fredholm-Volterra integro-differential equations (IDEs) considered in the Caputo sense. For this purpose, Laguerre polynomials have been used to construct an approximation method to obtain the solutions of the linear fractional Fredholm-Volterra IDEs. By this approximation method, the IDE has been transformed into a linear algebraic equation system using appropriate collocation points. In addition, a novel and exact matrix expression for the Caputo fractional derivatives of Laguerre polynomials and an associated explicit matrix formulation has been established for the first time in the literature. Furthermore, a comparison between the results of the proposed method and those of methods in the literature has been provided by implementing the method in numerous examples.

1. Introduction

The integro-differential equations (IDEs) of the fractional order are used by mathematicians and other scientists to model different physical and biological processes just as the heat conduction problem, radiative equilibrium, fracture mechanics, elasticity, signal processing, control and robotics, population dynamics, and health issues [1]- [12]. Hence, solving these types of equations and investigating the exact and approximate solutions has gained importance in recent years. When these investigations are reviewed it can be obviously seen that the methods handled to solve the fractional Fredholm-Volterra integro-differential equations (FVIDEs) are presented as reliable modified Laplace Adomian decomposition method [13], generalized hat functions [14], Nyström and Newton-Kantorovitch [15], Chebyshev wavelet [16]- [18], wavelet-based methods [19], Chebyshev Neural Network [20], Taylor expansion [21], sinccollocation [22], Legendre wavelet [23], Lucas wavelets with Legendre–Gauss quadrature [24], Bessel polynomials [25], fractional differential transform [26], Bernstein polynomials [27], Genocchi polynomials [28], spectral Jacobi-collocation [29], Block pulse functions [30], fractional-order Bernoulli functions [31], hybrid functions [32], Bernoulli wavelets [33], hybrid orthonormal Bernstein and block-pulse functions wavelet method [34].

Additionally, Laguerre polynomials have been used to solve the IDEs of integer order. Obviously, these integer-order equations can be specialized as 2-evolution equation [35], Altarelli-Parisi equation [36], Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equation [37], linear Fredholm IDE [38], [39], Volterra IDE of pantograph-type [40], delay partial functional differential equation [41], Volterra partial IDE of parabolic-type [42], [43], and nonlinear partial IDE [44]. In other respects, Laguerre polynomials have been applied to attain the solutions of the fractional IDE of the Fredholm type [45].

Moreover, in our research articles, approximation methods based on Laguerre polynomials have been developed. Daşcıoğlu et al. [46] have used a collocation method based upon the Laguerre polynomials to attain the solutions of the linear fractional FVIDEs in conformable sense. The method described in [46] is an improvement of the method that used for the solutions of the linear fractional IDEs of the Fredholm type in the Caputo sense [47] and Caputo fractional linear IDEs of the Volterra type [48].

However, for the linear fractional IDEs of the Fredholm-Volterra type in the Caputo sense with mixed conditions there is no method in the sense of Laguerre polynomials. In this work, a method based on these polynomials is proposed to obtain the solutions of the fractional

***Email addresses and ORCID numbers*:** **dvarol@pau.edu.tr, 0000-0002-7750-6910 (D. Varol) aakyuz@pau.edu.tr, 0000-0001-8931-6930 (A. Daşcıoğlu)**
Cite as: D. Varol, A. Daşcıoğlu, *Laguerre Collocation Approach of Caputo Fractional Fredholm-Volterra Integro-Differential Equations*, Univers. J. Math. Appl., 7(1) (2024), 38-45.

linear IDE of the Fredholm-Volterra type in the following general form:

$$\sum_{i=1}^{m} p_i(x)D^{\alpha_i}y(x) + \sum_{i=1}^{l} q_i(x)y^i(x) = g(x) + \lambda_1 \int_a^b F(x,t)y(t)\,dt + \lambda_2 \int_a^x V(x,t)y(t)\,dt, \qquad a \le x \le b, \tag{1.1}$$

with the conditions

$$\sum_{k=0}^{v-1} B_{jk}y^{(k)}(\beta_{jk}) = \mu_j, \qquad v_i - 1 < \alpha_i < v_i, \qquad j = 0,1,...,v-1, \tag{1.2}$$

where $m,l \in \mathbb{N}$, $v_i \in \mathbb{Z}^+$; $\mu_j, \beta_{jk}, B_{jk}, \lambda_1, \lambda_2 \in \mathbb{R}$, $v = max\left(\left(\underbrace{max}_{0\le i\le m} v_i\right), l\right)$. Here $p_i(x), q_i(x), F(x,t), V(x,t)$, and $g(x)$ are known functions, $y(x)$ is the unknown function that has to be determined, $y^i(x)$, shows the ordinary derivatives of the unknown function $y(x)$, $D^{\alpha_i}y(x)$ stands for the Caputo fractional derivative of $y(x)$ whose definition has been given below:

Definition 1.1. *[49] The Caputo fractional differentiation operator D^α of order α is defined as:*

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-n)^{\alpha+1-n}} dt, \qquad \alpha > 0,$$

where $-1 < \alpha < n, n \in \mathbb{Z}^+$ and Γ is the well-known Gamma function.

The main purpose of this work is to obtain an approximate solution of given problem (1.1)-(1.2) in the form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n L_n(x), \tag{1.3}$$

where N is any taken positive integer such that $N \ge v$, the unknown coefficients a_n's must be discovered,and $L_n(x)$ stand for the Laguerre polynomials of the order n stated by Bell [50] as:

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!(k!)^2} x^k.$$

The rest of the paper is arranged as follows: In section 2, the fundamental matrix relations for each term in fractional IDE (1.1) are constituted. In section 3, a functional collocation method based on the Laguerre polynomials is introduced. In section 4, numerical examples are resolved, their results are presented, and these solutions are compared with the existing results in the literature to affirm the precision and effectiveness of the proposed method. The last section of the paper presents the conclusions.

2. Elementary Matrix Formulas

In this section, we attempt to transform Eq. (1.1) by formulating the matrix forms of the unknown function and its fractional derivatives in the Caputo sense.

First, we can formulate the approximate solution (1.3) as the product of $\mathbf{L}(x)$ which can be called as the Laguerre matrix and the coefficient matrix $\mathbf{A}$ by

$$y_N(x) = \mathbf{L}(x)\mathbf{A}, \tag{2.1}$$

where the matrices are given as

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T \text{ and } \mathbf{L}(x) = \begin{bmatrix} L_0(x) & L_1(x) & \cdots & L_N(x) \end{bmatrix}.$$

Then, the following theorem has been given which demonstrates the connection between the Laguerre polynomials and the fractional derivative of Laguerre polynomials in the Caputo sense, which has been given and proved in our previous paper:

Theorem 2.1. *[46] Let $L_n(x)$ be the Laguerre polynomial of order n, then the Caputo fractional derivative of $L_n(x)$ in terms of Laguerre polynomials is found as follows:*

$$D^\alpha L_n(x) = 0, n < \lceil \alpha \rceil,$$

and otherwise

$$D^\alpha L_n(x) = x^{1-\alpha} \sum_{k=\lceil \alpha \rceil}^{n} \sum_{r=0}^{k-1} (-1)^{r+k} \frac{(k-1)!}{\Gamma(k+1-\alpha)} \binom{n}{k} \binom{k-1}{r} L_r(x),$$

where $\lceil \alpha \rceil$ indicates the smallest integer greater than or equal to α which is known as the ceiling function.

Secondly, the matrix relations of the differential side of the Eq. (1.1) are formulated. The relation between the Laguerre matrix $\mathbf{L}(x)$ and its integer order derivatives of the Laguerre matrix $\mathbf{L}(x)$ will be used in the form given in Eq. (2.2) which can be seen in Ref. [40] to present the matrix relation for the derivatives of the integer order of the unknown function $y(x)$,

$$\mathbf{L}^{(i)}(x) = \mathbf{L}(x)\mathbf{M}^i, \qquad i = 0,1,\ldots,N, \tag{2.2}$$

where the matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & -1 & & -1 \\ 0 & 0 & -1 & \cdots & -1 \\ 0 & 0 & 0 & & -1 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & & 0 \end{bmatrix}.$$

Therefore, the derivatives of integer order of the unknown function $y(x)$ in Eq. (1.1) can be represented as below by using Eq. (2.2),

$$y^{(i)}(x) \cong \mathbf{L}(x)\mathbf{M}^i\mathbf{A}. \tag{2.3}$$

Theorem 2.2. *Let $\boldsymbol{L}(x)$ be the Laguerre matrix defined in (2.1) and $D^{\alpha}\boldsymbol{L}(x)$ be the Caputo fractional derivative of $\boldsymbol{L}(x)$ of the α-th order, then the Caputo fractional derivative of Laguerre matrix is given as*

$$D^{\alpha}\boldsymbol{L}(x) = x^{1-\alpha}\boldsymbol{L}(x)\boldsymbol{S}_{\alpha}, \tag{2.4}$$

where $\boldsymbol{S}_{\alpha}$ is an $(N+1)$ dimensional square matrix specified as

$$\boldsymbol{S}_{\alpha} = \begin{bmatrix} 0 & \binom{0}{0}S_{1,1} & \binom{0}{0}S_{1,2}+\binom{1}{0}S_{2,2} & \cdots & \sum_{k=1}^{N}\binom{k-1}{0}S_{k,N} \\ 0 & 0 & -\binom{1}{1}S_{2,2} & \cdots & -\sum_{k=2}^{N}\binom{k-1}{1}S_{k,N} \\ 0 & 0 & 0 & \cdots & \sum_{k=3}^{N}\binom{k-1}{2}S_{k,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^{N+1}S_{N,N} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

or

$$\boldsymbol{S}_{\alpha} = \left[(-1)^i\textstyle\sum_{k=i+1}^{j}\binom{k-1}{i}S_{k,j}\right], \qquad i,j = 0,1,\ldots,N.$$

Here, the $S_{k,j}$ terms in the entries of the matrix S_{α} are defined as

$$\boldsymbol{S}_{k,j} = \begin{cases} (-1)^k\frac{(k-1)!}{\Gamma(k+1-\alpha)}\binom{j}{k}, & \textit{if} \quad \lceil\alpha\rceil \le k \le j \\ 0, & \textit{otherwise} \end{cases}.$$

Proof. First, the Caputo fractional derivative of $\mathbf{L}(x)$ which is denoted by $D^{\alpha}\mathbf{L}(x)$ has been defined by

$$D^{\alpha}\mathbf{L}(x) = \begin{bmatrix} D^{\alpha}L_0(x) & D^{\alpha}L_1(x) & \cdots & D^{\alpha}L_N(x) \end{bmatrix}.$$

By using Theorem 1 above, for $j < \lceil\alpha\rceil, D^{\alpha}L_j(x) = 0$, and for $j \ge \lceil\alpha\rceil, k = 1,2,\ldots,j$

$$D^{\alpha}L_j(x) = x^{1-\alpha}\sum_{k=\lceil\alpha\rceil}^{j}\sum_{r=0}^{k-1}(-1)^{r+k}\frac{(k-1)!}{\Gamma(k+1-\alpha)}\binom{j}{k}\binom{k-1}{r}L_r(x).$$

At this point, since the term $S_{k,j}, k = 1,2,\ldots,j$ is defined as follows:

$$S_{k,j} = \begin{cases} (-1)^k\frac{(k-1)!}{\Gamma(k+1-\alpha)}\binom{j}{k}, & \lceil\alpha\rceil \le k \le j \\ 0, & \text{otherwise} \end{cases}.$$

$D^{\alpha}L_0(x) = 0$ and for $j = 1,2,\ldots,N$

$$D^{\alpha}L_j(x) = x^{1-\alpha}\sum_{k=1}^{j}\sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}S_{k,j}L_r(x).$$

Here, for $j = 0, D^{\alpha}L_0(x) = 0$ and for $j \in \{1,\ldots,N\}$

$$D^{\alpha}L_j(x) = x^{1-\alpha}\sum_{k=1}^{j}\sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}S_{k,j}L_r(x)$$

$$= x^{1-\alpha}\left\{\sum_{k=1}^{j}\binom{k-1}{0}S_{k,j}L_0(x) - \sum_{k=2}^{j}\binom{k-1}{1}S_{k,j}L_1(x) - \cdots(-1)^{j-1}\binom{j-1}{j-1}S_{j,j}L_{j-1}(x)\right\}$$

Therefore, all the entries in the 0-th column and all the entries in the N-th row of $D^{\alpha}\mathbf{L}(x)$ is zero, and otherwise, the i,j-th element of the matrix $D^{\alpha}\mathbf{L}(x)$ is given as

$$x^{1-\alpha}\sum_{k=i+1}^{j}(-1)^{i}\binom{k-1}{i}S_{k,j}L_i(x).$$

Thus, the relation between $D^{\alpha}\mathbf{L}(x)$ and $\mathbf{L}(x)$ as expressed in Eq. (2.4) has been obtained. □

This relation proves the theorem.

Then, using the result of Theorem 2 and using relations (2.1) and (2.4), the Caputo fractional derivative of the unknown function $y(x)$ which is the differential part of Eq. (1.1) can be represented by

$$D^{\alpha}y(x)\cong D^{\alpha}\mathbf{L}(x)\mathbf{A}=x^{1-\alpha}\mathbf{L}(x)\mathbf{S}_{\alpha}\mathbf{A}. \tag{2.5}$$

Now, finally, the corresponding matrix formula for mixed conditions (1.1) could be given in the form

$$\sum_{k=0}^{v-1}B_{jk}\mathbf{L}(\beta_{jk})\mathbf{M}^{k}\mathbf{A}=\mu, \qquad j=0,1,\ldots,v-1. \tag{2.6}$$

by using Eq. (2.3).

Finally, when the matrix in the summation in the left-hand side of Eq. (2.6) is called as $\mathbf{U}_j$ that is an $1\times(N+1)$ vector matrix, Eq. (2.6) transforms into

$$\mathbf{U}_j\mathbf{A}=\mu_j, \qquad j=0,1,\ldots,v-1.$$

3. Solution Method

In this part of the paper, we maintain the approximate solution method which can be specified as a collocation method, because we use the collocation points at the end to solve the matrix equation. In other words, we determine the unknown coefficients a_i's in Eq. (1.3) to obtain the solution of Equations (1.1)-(1.2) using a collocation method.

Theorem 3.1. *Suppose that the fractional FVIDE defined by Eq. (1.1) is given. Utilizing the collocation points $x_s>0$ and $x_s\in[a,b]$, this IDE can be abbreviated as the following matrix equation:*

$$\left\{\sum_{i=0}^{m}\boldsymbol{P}_i\boldsymbol{X}_{\alpha_i}\boldsymbol{L}\boldsymbol{S}_{\alpha_i}+\sum_{i=0}^{l}\boldsymbol{Q}_i\boldsymbol{L}\boldsymbol{M}^{i}-\lambda_1\boldsymbol{F}-\lambda_2\boldsymbol{V}\right\}\boldsymbol{A}=\boldsymbol{G}.$$

Here, the matrices $\boldsymbol{M}$ and $\boldsymbol{S}_{\alpha_i}$ are in forms as in Eq. (2.2) and (2.4), respectively. In addition, $\boldsymbol{G}=[g(x_s)]$ is an $(N+1)\times 1$ dimensional matrix; $\boldsymbol{X}_{\alpha_i}=diag[x_s^{1-\alpha_i}]$, $\boldsymbol{P}_i=diag[p_i(x_s)]$, $\boldsymbol{Q}_i=diag[q_i(x_s)]$, $\boldsymbol{L}=[\boldsymbol{L}(x_s)]$, $\boldsymbol{F}=[\boldsymbol{f}(x_s)]$, and $\boldsymbol{V}=[\boldsymbol{v}(x_s)]$ are $(N+1)\times(N+1)$ dimensional square matrices. Moreover, $\boldsymbol{L}(x)$ corresponds for the Laguerre matrix, as described in Eq. (2.1), $\boldsymbol{f}(x_s)$ and $\boldsymbol{v}(x_s)$ represent the given integrals; $\boldsymbol{f}(x_s)=\int_a^b F(x_s,t)\boldsymbol{L}(t)dt$ and $\boldsymbol{v}(x_s)=\int_a^{x_s}V(x_s,t)\boldsymbol{L}(t)dt$

Proof. Firstly, substituting matrix relations (2.1), (2.3) and (2.5) into the Eq. (1.1), the following matrix equation has been obtained

$$\sum_{i=0}^{m}p_i(x)x^{1-\alpha_i}\mathbf{L}(x)\mathbf{S}_{\alpha_i}\mathbf{A}+\sum_{i=0}^{l}q_i(x)\mathbf{L}(x)\mathbf{M}^{i}\mathbf{A}=g(x)+\lambda_1\int_a^b F(x,t)\mathbf{L}(t)\mathbf{A}dt+\lambda_2\int_0^x V(x,t)\mathbf{L}(t)\mathbf{A}dt. \tag{3.1}$$

By substituting the non-negative collocation points $x_s(s=0,1,\ldots,N)$ into Eq. (3.1), the following system of linear matrix equations has been gained

$$\sum_{i=0}^{m}p_i(x_s){x_s}^{1-\alpha_i}\mathbf{L}(x_s)\mathbf{S}_{\alpha_i}\mathbf{A}+\sum_{i=0}^{l}q_i(x_s)\mathbf{L}(x_s)\mathbf{M}^{i}\mathbf{A}=g(x_s)+\lambda_1\mathbf{f}(x_s)\mathbf{A}+\lambda_2\mathbf{v}(x_s)\mathbf{A}, \tag{3.2}$$

where $\mathbf{f}(x_s)=\int_a^b F(x_s,t)\mathbf{L}(t)\,dt$ and $\mathbf{v}(x_s)=\int_a^{x_s}V(x_s,t)\mathbf{L}(t)\,dt$.

The system given by Eq. (3.2) can be written in the compact forms in the form

$$\left\{\sum_{i=0}^{m}\mathbf{P}_i\mathbf{X}_{\alpha_i}\mathbf{L}\mathbf{S}_{\alpha_i}+\sum_{i=0}^{l}\mathbf{Q}_i\mathbf{L}\mathbf{M}^{i}-\lambda_1\mathbf{F}-\lambda_2\mathbf{V}\right\}\mathbf{A}=\mathbf{G}, \tag{3.3}$$

where the matrices mentioned above are given as follows:

$$\mathbf{X}_{\alpha_i}=\begin{bmatrix}{x_0}^{1-\alpha_i} & 0 & & 0\\ 0 & {x_1}^{1-\alpha_i} & \cdots & 0\\ & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & {x_N}^{1-\alpha_i}\end{bmatrix},\mathbf{L}=\begin{bmatrix}\mathbf{L}(x_0)\\ \mathbf{L}(x_1)\\ \vdots\\ \mathbf{L}(x_N)\end{bmatrix},\mathbf{P}_i=\begin{bmatrix}p_i(x_0) & 0 & & 0\\ 0 & p_i(x_0) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & p_i(x_N)\end{bmatrix},$$

$$\mathbf{Q}_i = \begin{bmatrix} q_i(x_0) & 0 & & 0 \\ 0 & q_i(x_0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_i(x_N) \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \mathbf{f}(x_0) \\ \mathbf{f}(x_1) \\ \vdots \\ \mathbf{f}(x_N) \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \mathbf{v}(x_0) \\ \mathbf{v}(x_1) \\ \vdots \\ \mathbf{v}(x_N) \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \mathbf{g}(x_0) \\ \mathbf{g}(x_1) \\ \vdots \\ \mathbf{g}(x_N) \end{bmatrix}.$$

□

For simplicity, symbolizing the expression in the parenthesis of Eq. (3.3) by $\mathbf{W}$, the fundamental matrix equation associated with Eq. (1.1) can be abbreviated to $\mathbf{WA} = \mathbf{G}$. Apparently, this equation substitutes for a $(N+1)$ dimensional linear algebraic equations system with the unknown coefficients a_i's for $i = 0, 1, \ldots, N$ hich we can call as Laguerre coefficients.

Consequently, to find the solution of Eq. (1.1) with given conditions (1.2), the n rows of the obtained augmented matrix $[\mathbf{W}; \mathbf{G}]$ are stacked or replaced by the n rows of the augmented matrix $[\mathbf{U}_j; \mu_j]$. Therefore, because the unknown Laguerre coefficients are discovered by resolving this system, we obtain the solution of Eq. (1.1) under Conditions (1.2).

4. Numerical Examples

In this section, four examples have been tried to solve by the proposed method. All the numerical calculations were executed with the aid of Mathcad 15.

Example 4.1. *Consider the given fractional Fredholm IDE*

$$y''(x) + D^{\frac{1}{2}} y(x) + y(x) = \frac{9}{4} - \frac{1}{3}x - \frac{2}{\Gamma(\frac{5}{2})} x^{\frac{3}{2}} + x^2 + \int_0^1 (x-t) y(t) dt$$

with the conditions $y(0) = y'(0) = 0$*. This problem has the exact solution* $y(x) = x^2$*.*

Implementing the methodology explained in Section 3, the expected fundamental matrix equation of the given problem and its conditions can be presented as

$$\left\{ \mathbf{X}_{\frac{1}{2}} \mathbf{L} \mathbf{S}_{\frac{1}{2}} + \mathbf{L} + \mathbf{L}\mathbf{M}^2 - \mathbf{V} \right\} \mathbf{A} = \mathbf{G}$$

and

$$\mathbf{U}_0 \mathbf{A} = \mathbf{L}(0) \mathbf{A} = 0, \quad \mathbf{U}_1 \mathbf{A} = \mathbf{L}(0) \mathbf{M} \mathbf{A} = 0.$$

Here, the collocation points for $N = 2$ such as $x_0 = 0.25$, $x_1 = 0.75$, $x_2 = 1$ were used. Then the matrices mentioned above are

$$\mathbf{X}_{\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & \frac{3}{4} & \frac{17}{32} \\ 1 & \frac{1}{4} & \frac{-7}{32} \\ 1 & 0 & \frac{-1}{2} \end{bmatrix}, \quad \mathbf{S}_{\frac{1}{2}} = \begin{bmatrix} 0 & \frac{-2}{\sqrt{\Pi}} & \frac{-8}{3\sqrt{\Pi}} \\ 0 & 0 & \frac{-4}{3\sqrt{\Pi}} \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} \frac{-1}{4} & \frac{-1}{24} & \frac{1}{12} \\ \frac{1}{4} & \frac{5}{24} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{5}{24} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \frac{1}{3\sqrt{\Pi}} + \frac{107}{48} \\ \frac{\sqrt{3}}{\sqrt{\Pi}} + \frac{41}{16} \\ \frac{8}{3\sqrt{\Pi}} + \frac{35}{12} \end{bmatrix}, \quad \mathbf{U}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \mathbf{U}_1 = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix}$$

By solving this system, we obtain $a_0 = 2$, $a_1 = -4$, $a_2 = 2$. In the final step, we substitute these coefficients into the approximate Eq. (1.3) and obtain the exact solution. This problem was solved by Ordokhani et al. [25] by using the Bessel collocation method. They found an approximate solution with absolute maximum errors 3.70×10^{-3} for $N = 2$, 3.28×10^{-4} for $N = 4$ and 8.58×10^{-5} for $N = 6$. We found the exact solution for $N = 2$ with symbolic evaluation in Mathcad 15 using the proposed method. Clearly, the proposed method is more accurate than the other method.

Example 4.2. *Let us consider the fractional FVIDE having the exact solution* $y(x) = x^2 + x^3$*,*

$$D^{1.7} y(x) = g(x) + \int_0^x (x-t) y(t) dt + \int_0^1 (x+t) y(t) dt$$

with the given initial conditions $y(0) = y'(0) = 0$ *where*

$$g(x) = \frac{6}{\Gamma(2.3)} x^{1.3} + \frac{2}{\Gamma(1.3)} x^{0.3} - \frac{x^5}{20} - \frac{x^4}{12} - \frac{7x}{12} - \frac{9}{20}.$$

Implementing the methodology explained in Section 3, the expected fundamental matrix equation of the given problem and its conditions can be presented as

$$\{\mathbf{X}_{1.7}\mathbf{LS}_{1.7}-\mathbf{F}-\mathbf{V}\}\mathbf{A}=\mathbf{G}$$

and

$$\mathbf{U}_0\mathbf{A}=0, \qquad \mathbf{U}_1\mathbf{A}=0.$$

Here, we use the collocation points for $N=3$ such as $x_0=0.25$, $x_1=0.5$, $x_2=0.75$, $x_3=1$. We obtain the Laguerre coefficients as $a_0=8$, $a_1=-22$, $a_2=20$, $a_3=-6$ by solving this system. In the final step, we substitute these coefficients into the approximate Eq. (1.3), then we obtain the exact solution.

The approximate solutions to this problem using the Legendre wavelet method were given by Meng et al. [23]. Therefore, the maximum absolute errors of their method were calculated as 5.3×10^{-2} for 16 terms, 2.7×10^{-2} for 32 terms, 1.2×10^{-2} for 64 terms and 9.0×10^{-4} for 128 terms. In addition, Genocchi polynomials were used by Loh et al. [28] to obtain the numerical solution of the above problem with the maximum absolute error 7.0×10^{-2} for $N=8$. Since we obtain the exact solution for $N=3$,the proposed method is faster, more efficient, and more accurate compared than the other methods.

Example 4.3. *Consider the given fractional FVIDE with the exact solution $y(x)=x^{\frac{7}{2}}$ which is nonpolynomial:*

$$D^{2.3}y(x)=g(x)+\frac{1}{4}\int_0^x (x-t)y(t)dt+\frac{1}{2}\int_0^1 xty(t)dt$$

with following three conditions $y(0)=y'(0)=y''(0)=0$ where the non-homogenous function given as $g(x)=\frac{\Gamma(4.5)}{\Gamma(2.2)}x^{1.2}-\frac{x^{5.5}}{99}-\frac{x}{11}$.

Implementing the methodology explained in Section 3, the expected fundamental matrix equation of the given fractional equation and its conditions can be presented as

$$\left\{\mathbf{X}_{2.3}\mathbf{LS}_{2.3}-\frac{1}{2}\mathbf{F}-\frac{1}{4}\mathbf{V}\right\}\mathbf{A}=\mathbf{G}$$

and

$$\mathbf{U}_0\mathbf{A}=0, \qquad \mathbf{U}_1\mathbf{A}=0, \qquad \mathbf{U}_2\mathbf{A}=0$$

This problem was solved using the collocation points with the formula $x_s=\left[1-\cos\left(\frac{(s+1)\Pi}{N+1}\right)\right]/2$ and the numerical results are given in Table 1 for $N=8$ and $N=9$. Besides, the illustration of the results for $N=9$ is given in Figure 4.1.

	LWM	ADM	FBF	GHF	GP	Present	method
x	k=2,M=5	n=5	m=8	n=32	N=9	N=8	N=9
$\frac{1}{8}$	6.6×10^{-6}	1.0	6.9×10^{-7}	4.2×10^{-6}	1.5×10^{-4}	1.3×10^{-10}	1.6×10^{-8}
$\frac{2}{8}$	4.5×10^{-5}	4.2	3.5×10^{-7}	5.6×10^{-5}	6.3×10^{-4}	9.7×10^{-10}	6.3×10^{-9}
$\frac{3}{8}$	3.1×10^{-5}	9.2	2.4×10^{-7}	6.2×10^{-5}	1.3×10^{-3}	7.0×10^{-9}	4.0×10^{-9}
$\frac{4}{8}$	7.4×10^{-5}	4.2	2.3×10^{-7}	6.9×10^{-5}	2.0×10^{-3}	3.3×10^{-8}	9.1×10^{-11}
$\frac{5}{8}$	2.4×10^{-4}	8.1	8.3×10^{-7}	3.2×10^{-4}	2.8×10^{-3}	1.0×10^{-7}	3.9×10^{-8}
$\frac{6}{8}$	3.8×10^{-4}	2.3	2.3×10^{-7}	4.5×10^{-4}	3.7×10^{-3}	2.5×10^{-7}	1.2×10^{-7}
$\frac{7}{8}$	6.0×10^{-4}	8.1	4.6×10^{-7}	6.2×10^{-4}	4.6×10^{-3}	5.0×10^{-7}	2.4×10^{-7}

Table 1: Comparison of absolute maximum errors of Example 4.3.

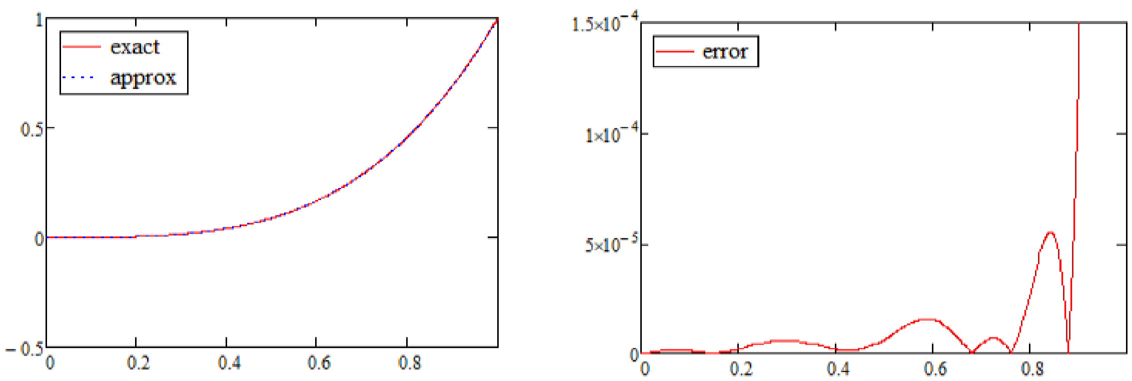

Figure 4.1: Graphical analysis of Example 4.3 for N=9

The results of the Legendre wavelet method (LWM) and the Adomian decomposition method(ADM) were provided by Meng et al. [23]. In addition, the fractional order Bernoulli functions (FBF) were used by Rahimkhani et al. [31], Genocchi polynomials (GP) wwere used by Loh et al. [28] and generalized hat functions (GHF) were used by Li [14] to obtain the approximate solution of this problem. The numerical results are presented in Table 1. It is obviously seen from the table that the proposed method is more effective and more accurate than the other methods compared.

Example 4.4. *Let us consider the following fractional IDE*

$$y''(x)+\frac{1}{x}D^{\frac{1}{2}}y(x)+\frac{1}{x^2}y(x)=g(x)+\int_0^1\cos(x-t)y(t)dt+\int_0^x\sin(x-t)y(t)dt$$

with the boundary conditions $y(0)=y(1)=0$. *The exact solution of this problem is* $y(x)=x^2-x^3$.

This problem was also solved by sinc-collocation method proposed by Alkan et al. [22]. They found an approximate solution with the maximum absolute errors 4.6×10^{-2} for $N=4$, 2.7×10^{-2} for $N=8$, 1.8×10^{-3} for $N=16$, 2.6×10^{-5} for $N=32$ and 3.9×10^{-7} for $N=64$. However, we found the exact solution using the proposed method with $N=3$. Therefore, it is evident that the proposed method is more efficient than the other methods.

5. Conclusion

In this paper, Laguerre polynomials were applied to construct a numerical approximation method to obtain the solutions of the fractional linear IDEs of the Fredholm-Volterra type. Using this approximation method a great variety of differential and integral (or both) equations has been covered since the equation in (1) has been presented in a general manner including not only the fractional IDEs of the Fredholm-Volterra type but also the fractional IDEs of the Fredholm or Volterra type and the fractional differential equations. Specifically, the given general fractional IDE of the Fredholm-Volterra type is converted into the fractional IDE of the Volterra type for $\lambda_1=0$, $\lambda_2\neq0$; the fractional IDE of the Fredholm type for $\lambda_1\neq0$, $\lambda_2=0$ and the fractional differential equation for both $\lambda_1=\lambda_2=0$. For this reason, the relation for the matrix of the Caputo fractional derivative of the Laguerre polynomials and the related exact matrix relation have been obtained for the first time in the fractional calculus literature. Utilizing suitable collocation points and the obtained matrix relations, the fractional IDE was transformed into an algebraic equations system. This method is more efficient, faster, and easier to apply than the other methods in the literature.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] M. Yi, J. Huang, *CAS wavelet method for solving the fractional integro-differential equation with a weakly singular kernel*, Int. J. Comput. Math., **92**(8) (2015), 1715-1728.
[2] B. Q. Tang, X. F. Li, *Solution of a class of Volterra integral equations with singular and weakly singular kernels*, Appl. Math. Comput., **199** (2008), 406413 .
[3] P. K. Kythe, P. Puri, *Computational Method for Linear Integral Equations*, Birkhauser, Boston, 2002.
[4] V. V. Zozulya, P. I. Gonzalez-Chi, *Weakly singular, singular and hypersingular integrals in 3-D elasticity and fracture mechanics*, J. Chin. Inst. Eng., **22** (2002), 763775 .
[5] C. Li, Y. Wang, *Numerical algorithm based on Adomian decomposition for fractional differential equations* Comput. Math. with Appl., **57**(10) (2009), 1672-1681 .
[6] B. Ghanbari, S. Djilali, *Mathematical and numerical analysis of a three-species predator-prey model with herd behavior and time fractional-order derivative* Math. Method Appl. Sci., **43**(4) (2020), 1736-1752 .
[7] P. Veeresha, D. G. Prakasha, S. Kumar, *A fractional model for propagation of classical optical solitons by using nonsingular derivative*, Math. Method Appl. Sci., **2020** (2020), 1–15. https://doi.org/10.1002/mma.6335
[8] S. Kumar, A. Kumar, B. Samet, J. F. Gómez-Aguilar, M. S. Osman, *A chaos study of tumor and effector cells in fractional tumor-immune model for cancer treatment*, Chaos Solitons Fractals **141** (2020), 110321.
[9] S. Kumar, A. Kumar, B. Samet, H. Dutta, *A study on fractional host–parasitoid population dynamical model to describe insect species*, Numer. Methods Partial Differ. Equ., **37**(2) (2021), 1673-1692.
[10] S. Kumar, S. Ghosh, B. Samet, E. F. D. Goufo, *An analysis for heat equations arises in diffusion process using new Yang-Abdel-Aty-Cattani fractional operator*, Math. Method Appl. Sci., **43**(9) (2020), 6062-6080.
[11] S. Kumar, R. Kumar, R. P. Agarwal, B. Samet, *A study of fractional Lotka-Volterra population model using Haar wavelet and Adams-Bashforth-Moulton methods*, Math. Method Appl. Sci., **43**(8), (2020) 5564-5578.
[12] B. Ghanbari, S. Kumar, R. Kumar, *A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative*, Chaos Solitons Fractals, **133** (2020), 109619.
[13] A. A. Hamoud, K. H. Hussain, K. P. Ghadle, *The reliable modified Laplace Adomian decomposition method to solve fractional Volterra-Fredholm integro differential equations*, Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications Algorithms, **26** (2019), 171-184.
[14] B. Li, *Numerical solution of fractional Fredholm-Volterra integro-differential equations by means of generalized hat functions method*, CMES Comput. Model. Eng. Sci., **99**(2) (2014), 105-122.
[15] D. Nazari Susahab, M. Jahanshahi, *Numerical solution of nonlinear fractional Volterra-Fredholm integro-differential equations with mixed boundary conditions*, Int. J. Ind. Math., **7**(1) (2015), 00563.
[16] S. T. Mohyud-Din, H. Khan, M. Arif, M. Rafiq, *Chebyshev wavelet method to nonlinear fractional Volterra–Fredholm integro-differential equations with mixed boundary conditions*, Adv. Mech. Eng., **9**(3) (2017), 1-8.

[17] A. Setia, Y. Liu, A. S. Vatsala, *Numerical solution of Fredholm-Volterra fractional integro-differential equations with nonlocal boundary conditions*, J. Fract. Calc. Appl., **5**(2) (2014), 155-165.
[18] Y. Wang, L. Zhu, *SCW method for solving the fractional integro-differential equations with a weakly singular kernel*, Appl. Math. Comput., **275** (2016), 72-80.
[19] F. Mohammadi, A. Ciancio, *Wavelet-based numerical method for solving fractional integro-differential equation with a weakly singular kernel*, Wavelets Linear Algebr., **4**(1) (2017), 53-73.
[20] S. S. Chaharborj, S. S. Chaharborj, Y. Mahmoudi, *Study of fractional order integrodifferential equations by using Chebyshev neural network*, J. Math. Stat., **13**(1) (2017), 1-13.
[21] L. Huang, X. F. Li, Y. Zhao, X. Y. Duan, *Approximate solution of fractional integro-differential equations by Taylor expansion method*, Comput. Math. Appl., **62** (2011), 1127–1134.
[22] S. Alkan, V. F. Hatipoğlu, *Approximate solutions of Volterra-Fredholm integro-differential equations of fractional order*, Tbil. Math. J., **10**(2) (2017), 1-13.
[23] Z. Meng, L. Wang, H. Li, W. Zhang, *Legendre wavelets method for solving fractional integro-differential equations*, Int. J. Comput. Math., **92**(6) (2015), 1275-1291.
[24] H. Dehestani, Y. Ordokhani, M. Razzaghi, *Combination of Lucas wavelets with Legendre–Gauss quadrature for fractional Fredholm–Volterra integro-differential equations*, J. Comput. Appl. Math., **382** (2021), 113070.
[25] Y.Ordokhani, H. Dehestani, *Numerical solution of linear Fredholm-Volterra integro-differential equations of fractional order*, World J. Model. Simul., **12**(3) (2016), 204-216.
[26] D. Nazari, S. Shahmorad, *Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions*, J. Comput. Appl. Math., **234** (2010), 883-891.
[27] M. Jani, D. Bhatta, S. Javadi, *Numerical solution of fractional integro-differential equations with nonlocal conditions*, Appl. Appl. Math., **12**(1) (2017), 98 – 111.
[28] J. R. Loh, C. Phang, A. Isah, *New operational matrix via Genocchi polynomials for solving Fredholm-Volterra fractional integro-differential equations*, Adv. Math. Phys., **2017** (2017), 3821870.
[29] Y. Yang, Y. Chen, Y. Huang, *Spectral-collocation method for fractional Fredholm integro-differential equations*, J. Korean Math. Soc., **51**(1) (2014), 203-224.
[30] F. Mohammadi, *Fractional integro-differential equation with a weakly singular kernel by using block pulse functions*, U.P.B. Sci. Bull. Series A., **79**(1) (2017).
[31] P. Rahimkhani, Y. Ordokhani, E. Babolian, *Fractional-order Bernoulli functions and their applications in solving fractional Fredholem–Volterra integro-differential equations*, Appl. Numer. Math., **122** (2017), 66–81.
[32] H. Dehestani, Y. Ordokhani, M. Razzaghi, *Hybrid functions for numerical solution of fractional Fredholm-Volterra functional integro-differential equations with proportional delays*, Int. J. Numer. Model. El., **32**(5) (2019), e2606.
[33] E. Keshavarz, Y. Ordokhani, M. Razzaghi, *Numerical solution of nonlinear mixed Fredholm-Volterra integro-differential equations of fractional order by Bernoulli wavelets*, Comput. Methods Differ. Equ., **7**(2) (2019), 163-176.
[34] M. R. Ali, A. R. Hadhoud, H. M. Srivastava, *Solution of fractional Volterra–Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method*, Adv. Differ. Equ., **2019**(1) (2019), 115.
[35] S. Kumano, T. H. Nagai, *Comparison of numerical solutions for Q2 evolution equations*, J. Comput. Phys., **201**(2) (2004), 651-664.
[36] R. Kobayashi, M. Konuma, S. Kumano, *FORTRAN program for a numerical solution of the nonsinglet Altarelli-Parisi equation*, Comput. Phys. Commun., **86** (1995), 264-278.
[37] L. Schoeffel, *An elegant and fast method to solve QCD evolution equations. Application to the determination of the gluon content of the Pomeron*, Nucl. Instrum. Meth. A., **423** (1999), 439-445.
[38] N. Baykus Savasaneril, M. Sezer, *Laguerre polynomial solution of high-order linear Fredholm integro-differential equations*, New Trends in Math. Sci., **4**(2) (2016), 273-284.
[39] B. Gürbüz, M. Sezer, C. Güler, *Laguerre collocation method for solving Fredholm integro-differential equations with functional arguments*, J. Appl. Math., (2014) 682398, 1-12.
[40] S. Yuzbası, *Laguerre approach for solving pantograph-type Volterra integro-differential equations*, Appl. Math. Comput., **232** (2014), 1183–1199.
[41] B. Gürbüz, M. Sezer, *Laguerre polynomial solutions of a class of delay partial functional differential equations*, Acta Phys. Polon. A., **132**(3) (2017c), 558-560.
[42] K. A. Al-Zubaidy, *A Numerical solution of parabolic-type Volterra partial integro-differential equations by Laguerre collocation method*, Int. J. Sci. Technol., **8**(4) (2013), 51-55.
[43] B. Gürbüz, M. Sezer, *A numerical solution of parabolic-type Volterra partial integro-differential equations by Laguerre collocation method*, Int. J. Appl. Phys. Math., **7**(1) (2017a), 49-58.
[44] B. Gürbüz, M. Sezer, *A new computational method based on Laguerre polynomials for solving certain nonlinear partial integro differential equations*, Acta Phys. Polon. A., **132**(3) (2017b), 561-563.
[45] A. M. S. Mahdy, R. T. Shwayyea, *Numerical solution of fractional integro-differential equations by least squares method and shifted Laguerre polynomials pseudo-spectral method*, IJSER, **7**(4) (2016), 1589-1596.
[46] A. Daşcıoğlu, D. Varol, *Laguerre polynomial solutions of linear fractional integro-differential equations*, Math. Sci., **15** (2021), 47-54. https://doi.org/10.1007/s40096-020-00369-y
[47] A. Daşcıoğlu, D. Varol Bayram, *Solving fractional Fredholm integro-differential equations by Laguerre polynomials*, Sains Malaysiana, **48**(1) (2019), 251–257.
[48] D. Varol Bayram, A. Daşcıoğlu, *A method for fractional Volterra integro-differential equations by Laguerre polynomials*, Adv. Differ. Equ., **2018** (2018), 466.
[49] I. Podlubny, *Fractional Differential Equations*, Academic Press, USA, 1999.
[50] W. W. Bell, *Special Functions for Scientists and Engineers*, D. Van Nostrand Company, London, 1968.

Universal Journal of Mathematics and Applications, 7 (1) (2024) 46-58
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1387528

On the Solvability of Iterative Systems of Fractional-Order Differential Equations with Parameterized Integral Boundary Conditions

Boddu Muralee Bala Krushna[1] and Mahammad Khuddush[2*]

[1]*Department of Mathematics, MVGR College of Engineering, Vizianagaram, 535 005, India*
[2]*Department of Mathematics, Chegg India Pvt. Ltd., Visakhapatnam, 530002, India*
[*]*Corresponding author*

Article Info

Keywords: *Boundary value problem, Fixed-point theorems, Fractional derivative, Kernel, Positive solution*
2010 AMS: *26A33, 34A08, 47H10*
Received: *7 November 2023*
Accepted: *5 March 2024*
Available online: *10 March 2024*

Abstract

The aim of this paper is to determine the eigenvalue intervals of $\mu_{\mathtt{k}},\ 1 \leq \mathtt{k} \leq \mathtt{n}$ for which an iterative systems of a class of fractional-order differential equations with parameterized integral boundary conditions (BCs) has at least one positive solution by means of standard fixed point theorem of cone type. To the best of our knowledge, this will be the first time that we attempt to reach such findings for the topic at hand in the literature. The obtained results in the paper are illustrated with an example for their feasibility.

1. Introduction

There is a strong impetus for the study of nonlinear fractional systems, and significant research efforts have been made undertaken lately for these systems with the aim of implementing findings on the existence of positive solutions in related fields. At this point, differential calculus expanded its scope to include the dynamics of the complex real world, and new theories began to be put into effect and assessed on real data [1]. A variety of materials and processes with characteristics of heredity and memory can be accurately described by the nonlocal nature of fractional calculus [2, 3]. There are numerous applications in a variety of scientific disciplines, including biomathematics [4], viscoelasticity [5], non-Newtonian fluid mechanics [6], and characterization of anomalous diffusion [7].
Progressively, distinctive scientific advances and tools are created specifically for fractional differential equations (FDEqs). Due to this, a significant amount of scientists concentrate on boundary value problems (BVPs) for FDEqs involving various derivatives, such as Riemann–Liouville or Caputo, as well as some novel derivatives, including conformable fractional derivatives [8]. The literature on FDEqs of the conformable type is not enriched yet. The conformable fractional derivative was first proposed in 2014. The conformable derivative can be utilized for modeling many physical problems as DEqs with conformable fractional derivatives are easier to solve numerically in comparison to those with Riemann–Liouville or Caputo fractional derivatives. A new concept, known as the conformable fractional derivative, has recently [9, 10] been defined. Indeed, several researchers have previously applied conformable fractional derivatives to a wide range of domains, and numerous replicating methodologies have been established, see [11]. In different industries, such as telecommunication equipment, synthetic chemicals, automobiles, and pharmaceuticals, BVPs are frequently used. In these processes, positive solutions seem to be beneficial. In these contexts, the existence of positive solutions is often advantageous. For instance, in [12], the authors established the existence of multiple positive solutions for a coupled system of Riemann–Liouville FBVPs by means of an Avery generalization of the Leggett–Williams FPT. Subsequently, in [13], the same authors determined the eigenvalue intervals of the parameters leading to a positive solution for an iterative system of nonlinear Sturm–Liouville FBVPs by utilizing the Guo–Krasnosel'skii FPT on a cone. Additionally, in [14], the authors examined p-Laplacian fractional higher-order BVPs, establishing criteria for determining parameter values ensuring at least one positive solution. Furthermore, they derived sufficient conditions for the existence of an even number of positive solutions for FBVPs using an Avery–Henderson functional FPT. Moreover, in [15], the authors established the existence of at least three positive solutions to a system of FBVPs by employing a five-functionals FPT. Lastly, in [16], the authors investigated the eigenvalue intervals of parameters guaranteeing at least one positive solution for an iterative system of four-point FBVPs under suitable conditions.

***Email addresses and ORCID numbers*: muraleebalu@yahoo.com, 0000-0001-7426-7451 (B. M. B. Krushna), khuddush89@gmail.com, 0000-0002-1236-8334 (M. Khuddush)**
Cite as: B. M. B. Krushna, M. Khuddush, *On the Solvability of Iterative Systems of Fractional-Order Differential Equations with Parameterized Integral Boundary Conditions*, Univers. J. Math. Appl., 7(1) (2024), 46-58.

Recently, Zhou et al. [17] the existence, uniqueness, and multiplicity of findings associated with positive solutions to various types of conformable FBVPs. By using conventional fixed point theorems (FPTs) in conjunction with the theory of the cosine family of linear operators, Bouaouid [18] showed the existence and continuous dependence of mild solutions for a class of conformable FDEqs with nonlocal initial conditions. In their study of conformable stochastic functional DEqs of the neutral type, Xiao et al. [19] examined the existence and stability outcomes. A mild solution to a conformable FBVP was introduced by Jaiswal et al. [20] and the existence, uniqueness of solutions to the considered problem employing the contraction principle have been proven.

Conformable FDEqs with integral BCs provide a more flexible framework for modeling complex systems that exhibit non-local or memory-dependent behavior. Many real-world processes, such as heat conduction in non-homogeneous materials or transport phenomena in porous media, can be better described using fractional calculus. Gokdogan et al. demonstrated the uniqueness of solutions for sequential linear conformable FDEqs in [21]. Khuddush et al. [22] obtained the existence of positive solutions for an iterative system of conformable fractional dynamic BVPs on time scales by an application of FPT on a Banach space. Zhong and Wang in [23], where they studied the existence of positive solutions to the FBVP

$$\mathtt{D}^{\mathtt{q}}\mathtt{u}(\mathtt{z})+\mathfrak{f}(\mathtt{z},\mathtt{u}(\mathtt{z}))=0, \quad \mathtt{z}\in(0,1),$$
$$\mathtt{u}(0)=0, \ \ \mathtt{u}(1)=\lambda\int_0^1 \mathtt{u}(\mathtt{z})d\mathtt{z},$$

where $\mathtt{q}\in(1,2]$, λ is a constant and $\mathtt{D}^{\mathtt{q}}$ is the conformable derivative. By utilizing the solution-tube approach and Schauder's FPT, Bendouma et al. [24] investigated the existence of solutions to systems of conformable FDEqs concerning periodic conditions.

In [25], Haddouchi used the Kernel characteristics along with the FPT in a cone to investigate the existence of positive solutions to conformable FBVPs

$$\mathtt{D}^{\mathtt{q}}\mathtt{u}(\mathtt{z})+\mathfrak{f}(\mathtt{z},\mathtt{u}(\mathtt{z}))=0, \quad \mathtt{z}\in(0,1),$$
$$\mathtt{u}(0)=0, \ \ \mathtt{u}(1)=\lambda\int_0^\eta \mathtt{u}(\mathtt{z})d\mathtt{z},$$

where $\mathtt{q}\in(1,2]$, $\eta\in(0,1]$, λ is a constant and $\mathtt{D}^{\mathtt{q}}$ is the conformable derivative.

Through the use of various FPTs found in the literature, numerous authors have explored the existence of positive solutions to a variety BVPs for ordinary, FDEqs during the past few years. Motivated and inspired by above highly decorated topics, by employing the Guo–Krasnosel'skii FPT of cone compression and expansion of norm kind (see [26, 27]) to the considered problem. More explicitly, we construct the Kernel for the associated linear FBVP, and estimate the bounds of this Kernel in more detail since they are essential for finding suitable fixed points for the newly indicated operator on a cone in a Banach space. Furthermore, it was explained how to utilize the fixed point technique and the bootstrapping argument to establish the existence of positive solutions to the iterative system. To the best of our knowledge, in this work, we attempt for the first time to determine the eigenvalue intervals of parameters that have positive solutions for the following iterative systems of conformable FDEqs

$$\left.\begin{aligned}\mathtt{D}^{\mathtt{q}}\mathtt{u}_{\mathtt{k}}(\mathtt{z})+\mu_{\mathtt{k}}\mathtt{p}_{\mathtt{k}}(\mathtt{z})\mathfrak{g}_{\mathtt{k}}(\mathtt{u}_{\mathtt{k}+1}(\mathtt{z}))=0\\ \mathtt{u}_{\mathtt{n}+1}(\mathtt{z})=\mathtt{u}_1(\mathtt{z}), \quad \mathtt{z}\in(0,1),\end{aligned}\right\} \tag{1.1}$$

with parameterized integral BCs

$$\left.\begin{aligned}\mathtt{u}_{\mathtt{k}}(0)=0, \quad \mathtt{u}_{\mathtt{k}}(1)=\vartheta\int_0^\xi \mathtt{u}_{\mathtt{k}}(\mathtt{z})d\mathtt{z},\\ \text{for } 1\le\mathtt{k}\le\mathtt{n},\end{aligned}\right\} \tag{1.2}$$

where $\mathtt{q}\in(1,2]$, $\xi\in(0,1]$, $\vartheta\in\mathbb{R}^+$ is constant and $\mathtt{D}^{\mathtt{q}}$ is the conformable fractional derivative. Iterative FDEqs have a variety of applications, which makes studying them preferable to non-iterative DEqs. For instance, IFDEqs are the most suitable for studying problems associated with infectious models and the kinetics of particles that are charged with delayed contact and can't be employed to study such problems via ordinary non-iterative DEqs. Iterative DEqs model dynamic systems where a variable's rate of change depends not only on its current value but also on its past values. These equations capture the influence of a system's history on its current state, often in a nonlinear fashion. They find applications across various fields, including modeling object motion, fluid dynamics, disease spread, chemical reactions, population growth, control systems, electrical circuits, and economic systems. The equation (1.1) relates a diffusion phenomena with source or reaction term. For example, in thermal conduction, it can be understood as a one dimensional heat conduction equation modeling steady states of a heating rod of length $\mathtt{c}$ with the controller at $\mathtt{r}=\mathtt{c}$, while the left end is held at 0°C and $\mathtt{h}$ is function of source distribution temperature over time delays in thermal conduction [28, 29]. The main advantage of studying IFDEqs over non-iterative DEqs exist in its various applications. For example, the problems related to infectious models and the motion of charge particles with retarded interaction are best described using IFDEqs and cannot be studied by general non DEqs.

We provide varied conditions for the functions $\mathfrak{g}_1,\mathfrak{g}_2,\cdots,\mathfrak{g}_{\mathtt{n}}$ and the intervals of $\mu_1,\mu_2,\cdots,\mu_{\mathtt{n}}$ ensuring that positive solutions to the iterative system of FBVP (1.1)–(1.2). A positive solution of the problem (1.1)–(1.2), we mean $\big(\mathtt{u}_1(\mathtt{z}),\mathtt{u}_2(\mathtt{z}),\cdots,\mathtt{u}_{\mathtt{n}}(\mathtt{z})\big)\in\big(\mathtt{C}^2[0,1]\big)^{\mathtt{n}}$ satisfying (1.1) and (1.2) with $\mathtt{u}_{\mathtt{k}}(\mathtt{z})>0, \mathtt{k}=1,2,\cdots,\mathtt{n}, \forall\, \mathtt{z}\in(0,1]$.

Throughout the article, we propose the following hypotheses:

(H_1) $\Delta=2-\vartheta\xi^2>0$.

(H_2) $\mathtt{p}_{\mathtt{k}}:[0,1]\to\mathbb{R}^+$ is continuous and $\mathtt{p}_{\mathtt{k}}$ does not vanish identically on any closed subinterval of $[0,1]$, for $\mathtt{k}=\overline{1,\mathtt{n}}$.

(H_3) $\mathfrak{g}_{\mathtt{k}}:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, for $\mathtt{k}=\overline{1,\mathtt{n}}$.

(H_4) each of $\mathfrak{g}_{\mathtt{k}0}=\lim\limits_{\mathtt{x}\to0^+}\dfrac{\mathfrak{g}_{\mathtt{k}}(\mathtt{x})}{\mathtt{x}}$ and $\mathfrak{g}_{\mathtt{k}\infty}=\lim\limits_{\mathtt{x}\to\infty}\dfrac{\mathfrak{g}_{\mathtt{k}}(\mathtt{x})}{\mathtt{x}}$, for $1\le\mathtt{k}\le\mathtt{n}$, exists as positive real numbers.

The paper is arranged as follows: The preliminary results presented in Sect. 2 serve as foundations for the subsequent sections that follow. This covers the solution to the corresponding linear problem, an investigation of the characteristics of Kernels, and other pertinent information. The key existence theorems for the problem (1.1)–(1.2) are the focus of Sect. 3. In Sect. 4, an example is coined in support of validity of the findings concerning the earlier sections.

2. Preliminaries, Kernel and Bounds

In order to move on to the key results in the subsequent sections, the necessary results are provided here.

Definition 2.1. *[8] The conformable derivative of* $\mathtt{h}:[0,\infty)\to\mathbb{R}$ *is defined as*

$$\mathtt{D}_0^{\zeta}\mathtt{h}(\mathbf{r})=\lim_{\varepsilon\to 0}\left[\frac{\mathtt{h}(\mathbf{r}+\varepsilon\mathbf{r}^{1-\zeta})-\mathtt{h}(\mathbf{r})}{\varepsilon}\right],\quad \mathbf{r}>0,\quad \zeta\in(0,1],$$

and

$$\mathtt{D}_0^{\zeta}\mathtt{h}(0)=\lim_{\mathbf{r}\to 0^+}\mathtt{D}_0^{\zeta}\mathtt{h}(\mathbf{r}).$$

If $\mathtt{h}$ *is differentiable then* $\mathtt{D}_0^{\zeta}\mathtt{h}(\mathbf{r})=\mathbf{r}^{1-\zeta}\mathtt{h}'(\mathbf{r})$.

Definition 2.2. *[8] The conformable fractional integral of a function of order* ζ *is defined for* $\mathtt{h}:[0,\infty)\to\mathbb{R}$ *as*

$$\mathtt{I}_0^{\zeta}\mathtt{h}(\mathbf{r})=\int_0^{\mathbf{r}}\mathfrak{s}^{\zeta-1}\mathtt{h}(\mathfrak{s})d\mathfrak{s},\quad \mathfrak{s}>0,\ \zeta\in(0,1].$$

Lemma 2.3. *[30] Let* $\zeta\in(0,1]$ *and* $\mathtt{h}:(0,\infty)\to\mathbb{R}$ *be differentiable. Then*

$$\mathtt{I}_0^{\zeta}\mathtt{D}_0^{\zeta}\mathtt{h}(\mathbf{r})=\mathtt{h}(\mathbf{r})-\mathtt{h}(0),\quad \forall\,\mathbf{r}>0.$$

Lemma 2.4. *Suppose* $(\mathtt{H}_1)$ *holds, let* $\mathtt{h}(\mathtt{z})\in\mathtt{C}([0,1],\mathbb{R})$. *Then* $\mathtt{u}_1(\mathtt{z})\in\mathtt{C}([0,1],\mathbb{R})$ *is a solution of the FBVP*

$$^{\mathtt{H}}\mathtt{D}_{1^+}^{\mathtt{q}}\mathtt{u}_1(\mathtt{z})+\mathtt{h}(\mathtt{z})=0,\quad \mathtt{z}\in(0,1),\tag{2.1}$$

$$\mathtt{u}_1(0)=0,\quad \mathtt{u}_1(1)=\vartheta\int_0^{\xi}\mathtt{u}_1(\mathtt{z})d\mathtt{z},\tag{2.2}$$

has a unique solution

$$\mathtt{u}_1(\mathtt{z})=\int_0^1\aleph(\mathtt{z},\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y},$$

where

$$\aleph(\mathtt{z},\mathtt{y})=\aleph_1(\mathtt{z},\mathtt{y})+\frac{\vartheta\mathtt{z}}{\Delta}\aleph_2(\xi,\mathtt{y}),\tag{2.3}$$

$$\aleph_1(\mathtt{z},\mathtt{y})=\begin{cases}(1-\mathtt{z})\mathtt{y}^{\mathtt{q}-1}, & 0\le\mathtt{y}\le\mathtt{z}\le 1,\\ \mathtt{z}(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}, & 0\le\mathtt{z}\le\mathtt{y}\le 1,\end{cases}$$

$$\aleph_2(\mathtt{z},\mathtt{y})=\begin{cases}(2\mathtt{z}-\mathtt{z}^2-\mathtt{y})\mathtt{y}^{\mathtt{q}-1}, & \mathtt{y}\le\mathtt{z},\\ \mathtt{z}^2(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}, & \mathtt{z}\le\mathtt{y}.\end{cases}$$

Proof. Let $\mathtt{u}_1(\mathtt{z})\in\mathtt{C}^2[0,1]$ be a solution of FBVP (2.1)-(2.2) and is uniquely expressed as

$$\mathtt{u}_1(\mathtt{z})=\sum_{\mathtt{k}=1}^{2}\mathtt{c}_\mathtt{k}\mathtt{z}^{2-\mathtt{k}}-\int_1^{\mathtt{z}}(\mathtt{z}-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}\mathtt{h}(\mathtt{y})d\mathtt{y}.$$

By the condition (2.2), we get $\mathtt{c}_2=0$ and $\mathtt{c}_1=\mathtt{I}^{\mathtt{q}}\mathtt{h}(1)+\mathtt{u}_1(1)$. Hence the unique solution of FBVP (2.1)-(2.2) is

$$\begin{aligned}\mathtt{u}_1(\mathtt{z})&=\left\{\begin{array}{l}\displaystyle\int_0^{\mathtt{z}}(1-\mathtt{z})\mathtt{y}^{\mathtt{q}-1}\mathtt{h}(\mathtt{y})d\mathtt{y}+\int_{\mathtt{z}}^1\mathtt{z}(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}\mathtt{h}(\mathtt{y})d\mathtt{y}+\\ \displaystyle\frac{\vartheta\mathtt{z}}{\Delta}\int_0^{\xi}\mathtt{y}^{\mathtt{q}-2}\left[\xi^2(1-\mathtt{y})-(\xi-\mathtt{y})^2\right]\mathtt{h}(\mathtt{y})d\mathtt{y}+\\ \displaystyle\frac{\vartheta\mathtt{z}}{\Delta}\int_{\xi}^1\xi^2(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}\mathtt{h}(\mathtt{y})d\mathtt{y}\end{array}\right.\\ &=\left\{\begin{array}{l}\displaystyle\int_0^1\aleph_1(\mathtt{z},\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y}+\frac{\vartheta\mathtt{z}}{\Delta}\int_0^{\xi}\mathtt{y}^{\mathtt{q}-1}(2\xi-\xi^2-\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y}+\\ \displaystyle\frac{\vartheta\mathtt{z}}{\Delta}\int_{\xi}^1\xi^2(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}\mathtt{h}(\mathtt{y})d\mathtt{y}\end{array}\right.\\ &=\int_0^1\aleph_1(\mathtt{z},\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y}+\frac{\vartheta\mathtt{z}}{\Delta}\int_0^1\aleph_2(\xi,\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y}\\ &=\int_0^1\aleph(\mathtt{z},\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y},\end{aligned}$$

where $\aleph(\mathtt{z},\mathtt{y})$ is given in (2.3). The proof is completed. □

Lemma 2.5. *The Kernel* $\aleph(z,y)$ *given in* (2.3) *is nonnegative, for all* $z,y\in[0,1]$.

Proof. The Kernel $\aleph(z,y)$ is given in (2.3). Let $0\le z\le y\le 1$. Then:

$$\aleph_1(z,y)=z(1-y)y^{q-2}\ge 0.$$

Let $0\le y\le z\le 1$. Then:

$$\aleph_1(z,y)=(1-z)y^{q-1}\ge 0.$$

On the other hand, let $0\le\xi\le y\le 1$. Then:

$$\aleph_2(\xi,y)=\xi^2(1-y)y^{q-2}\ge 0.$$

Let $0\le y\le\xi\le 1$. Then:

$$\aleph_2(\xi,y)=(2\xi-\xi^2-y)y^{q-1}\ge 0.$$

Hence $\aleph(z,y)\ge 0$. □

Lemma 2.6. *Let* $\sigma\in\left(0,\frac{1}{2}\right)$. *The Kernel* $\aleph_1(z,y)$ *has the properties:*

(1) $\aleph_1(z,y)\le\aleph_1(y,y),\ \forall\, z,y\in(0,1]$,

(2) $\aleph_1(z,y)\ge z(1-z)\aleph_1(y,y),\forall\, z,y\in(0,1]$,

(3) $\aleph_1(z,y)\ge\sigma^2\aleph_1(y,y),\forall\, z\in[\sigma,1-\sigma],y\in(0,1]$.

Proof. We prove (1). Let $0\le z\le y\le 1$. Then:

$$\begin{aligned}\aleph_1(z,y)&=z(1-y)y^{q-2}\\&\le(1-y)y^{q-1}\\&=\aleph_1(y,y).\end{aligned}$$

Let $0\le y\le z\le 1$. Then:

$$\begin{aligned}\aleph_1(z,y)&=(1-z)y^{q-1}\\&\le(1-y)y^{q-1}\\&=\aleph_1(y,y).\end{aligned}$$

Hence the inequality (1). We establish the inequality (2). Let $0\le z\le y\le 1$. Then:

$$\begin{aligned}\aleph_1(z,y)&=z(1-y)y^{q-2}\\&\ge(1-z)y^{q-1}\\&\ge z(1-z)\aleph_1(y,y).\end{aligned}$$

Let $0\le y\le z\le 1$. Then:

$$\begin{aligned}\aleph_1(z,y)&=(1-z)y^{q-1}\\&\ge(1-z)(1-y)y^{q-1}\\&\ge z(1-z)\aleph_1(y,y).\end{aligned}$$

Hence the inequality (2). On the other hand, if $\sigma\in\left(0,\dfrac{1}{2}\right)$, then $\aleph_1(z,y)$ satisfies

$$\aleph_1(z,y)\ge\sigma^2\aleph_1(y,y),\forall\, z\in[\sigma,1-\sigma],y\in(0,1].$$

□

Lemma 2.7. *Let* $\sigma\in\left(0,\frac{1}{2}\right)$. *The Kernels* $\aleph_1(z,y)$ *and* $\aleph_2(z,y)$ *have the properties:*

(1) $\aleph_2(z,y)\le\aleph_1(y,y),\ \forall\, z,y\in(0,1]$,

(2) $\aleph_2(z,y)\ge\theta(z)\aleph_1(y,y),\forall\, z,y\in(0,1]$,

$$\textit{where }\theta(z)=\min\left\{z^2,z(1-z)\right\}=\begin{cases}z^2, & z\le\frac{1}{2},\\ z(1-z), & z>\frac{1}{2},\end{cases}$$

(3) $\aleph_2(z,y)\ge\sigma^2\aleph_1(y,y),\forall\, z\in[\sigma,1-\sigma],y\in(0,1]$.

Proof. Let $0 \leq z \leq y \leq 1$. Then:

$$\begin{aligned}\aleph_2(z,y) &= z^2(1-y)y^{q-2}\\ &\leq z(1-y)y^{q-2}\\ &\leq (1-y)y^{q-1}\\ &= \aleph_1(y,y).\end{aligned}$$

Let $0 \leq y \leq z \leq 1$. Then:

$$\begin{aligned}\aleph_2(z,y) &= (2z-z^2-y)y^{q-1}\\ &\leq \left[(1-y)-(1-z)^2\right]y^{q-1}\\ &\leq (1-y)y^{q-1}\\ &= \aleph_1(y,y).\end{aligned}$$

Hence the inequality (1). Let $0 \leq z \leq y \leq 1$. Then:

$$\begin{aligned}\aleph_2(z,y) &= z^2(1-y)y^{q-2}\\ &\geq z^2y(1-y)y^{q-2}\\ &= z^2(1-y)y^{q-1}\\ &= z^2\aleph_1(y,y).\end{aligned}$$

Let $0 \leq y \leq z \leq 1$. Then:

$$\begin{aligned}\aleph_2(z,y) &= (2z-z^2-y)y^{q-1}\\ &= \left[z(1-z)+(z-y)\right]y^{q-1}\\ &\geq z(1-z)y^{q-1}\\ &\geq z(1-z)(1-y)y^{q-1}\\ &\geq z(1-z)\aleph_1(y,y).\end{aligned}$$

Therefore $\aleph_2(z,y) \geq \theta(z)\aleph_1(y,y), \forall\, z,y \in (0,1]$, where

$$\theta(z) = \min\left\{z^2, z(1-z)\right\} = \begin{cases} z^2, & z \leq \frac{1}{2},\\ z(1-z), & z > \frac{1}{2}.\end{cases}$$

Hence the inequality (2). On the other hand, if $\sigma \in \left(0,\frac{1}{2}\right)$, then it follows immediately from (2):

$$\aleph_2(z,y) \geq \sigma^2\aleph_1(y,y), \forall\, z \in [\sigma, 1-\sigma], y \in (0,1].$$

□

3. Existence of Positive Solutions

An n-tuple $\Big(u_1(z),u_2(z),\cdots,u_n(z)\Big)$ is a solution of the FBVP (1.1)-(1.2) if and only if $u_k(z) \in C^2[0,1],\ k=1,2,\cdots,n$ satisfies:

$$u_1(z) = \begin{cases} \mu_1\int_0^1\left[\aleph_1(z,y_1)+\dfrac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\right]p_1(y_1)\\ g_1\Bigg(\mu_2\int_0^1\left[\aleph_1(y_1,y_2)+\dfrac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right]p_2(y_2)\cdots\\ g_{n-1}\Bigg(\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\dfrac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\\ p_n(y_n)g_n\Big(u_1(y_n)\Big)dy_n\Bigg)\cdots dy_2\Bigg)dy_1,\end{cases}$$

and

$$\begin{cases} u_2(z) = \mu_2\int_0^1\left[\aleph_1(z,y)+\dfrac{\vartheta z}{\Delta}\aleph_2(\xi,y)\right]p_2(y)g_2\big(u_3(y)\big)dy,\\ u_3(z) = \mu_3\int_0^1\left[\aleph_1(z,y)+\dfrac{\vartheta z}{\Delta}\aleph_2(\xi,y)\right]p_3(y)g_3\big(u_4(y)\big)dy,\\ \cdots\\ u_n(z) = \mu_n\int_0^1\left[\aleph_1(z,y)+\dfrac{\vartheta z}{\Delta}\aleph_2(\xi,y)\right]p_n(y)g_n\big(u_{n+1}(y)\big)dy,\end{cases}$$

where $u_{n+1}(z) = u_1(z), \quad 0 < z < 1$. By a positive solution of the FBVP (1.1)-(1.2), we mean $\big(u_1(z), u_2(z), \cdots, u_n(z)\big) \in \Big(C^2[0,1]\Big)^n$ which satisfying the FDEq (1.1) and BCs (1.2) with $u_k(z) > 0, k = \overline{1,n}\ \forall\, z \in [0,1]$.
Let $B = \big\{x : x \in C[0,1]\big\}$ be the Banach space endowed with the norm

$$\|x\| = \max_{z\in[0,1]} |x(z)|$$

and $P \subset B$ be a cone defined as

$$P = \left\{ x \in B \ : \ x(z) \geq 0 \text{ on } [0,1] \text{ and } \min_{z\in\left[\sigma,1-\sigma\right]} x(z) \geq \sigma^2 \|x\| \right\},$$

where $\sigma \in \left(0, \frac{1}{2}\right)$. Construct an integral operator $T : P \to B$, for $u_1 \in P$, as

$$Tu_1(z) = \begin{cases} \mu_1 \displaystyle\int_0^1 \Big[\aleph_1(z,y_1) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\Big] p_1(y_1) g_1 \Bigg(\mu_2 \int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] \\ p_2(y_2)\cdots g_{n-1}\Bigg(\mu_n \displaystyle\int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\Big] p_n(y_n) g_n\Big(u_1(y_n)\Big)dy_n\Bigg)\cdots dy_2\Bigg)dy_1. \end{cases}$$

Notice from (H_1) and Lemma 2.5 that, for $u_1 \in P$, $Tu_1(z) \geq 0$ on $[0,1]$. In addition, we have

$$Tu_1(z) \leq \begin{cases} \mu_1 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\Big] p_1(y_1) \\ g_1\Bigg(\mu_2 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] p_2(y_2)\cdots \\ g_{n-1}\Bigg(\mu_n \displaystyle\int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\Big] \\ p_n(y_n) g_n\Big(u_1(y_n)\Big)dy_n\Bigg)\cdots dy_2\Bigg)dy_1 \end{cases}$$

so that

$$\left.\begin{aligned} \|Tu_1\| \leq \mu_1 \int_0^1 \Big[\aleph_1(y_1,y_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\Big] p_1(y_1) \\ g_1\Bigg(\mu_2 \int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] \\ p_2(y_2)\cdots g_{n-1}\Bigg(\mu_n \int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta} \\ \aleph_2(\xi,y_n)\Big] p_n(y_n) g_n\Big(u_1(y_n)\Big)dy_n\Bigg)\cdots dy_2\Bigg)dy_1. \end{aligned}\right\} \tag{3.1}$$

If $u_1 \in P$, from Lemmas 2.6, 2.7 and (3.1), we deduce that

$$\begin{aligned} \min_{z\in\left[\sigma,1-\sigma\right]} Tu_1(z) &= \begin{cases} \displaystyle\min_{z\in\left[\sigma,1-\sigma\right]} \mu_1 \int_0^1 \Big[\aleph_1(z,y_1) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\Big] \\ p_1(y_1) g_1\Bigg(\mu_2 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] \\ p_2(y_2)\cdots g_{n-1}\Bigg(\mu_n \displaystyle\int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta} \\ \aleph_2(\xi,y_n)\Big] p_n(y_n) g_n\Big(u_1(y_n)\Big)dy_n\Bigg)\cdots dy_2\Bigg)dy_1 \end{cases} \\ &\geq \begin{cases} \mu_1\sigma^2 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\Big] \\ p_1(y_1) g_1\Bigg(\mu_2 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] \\ p_2(y_2)\cdots g_{n-1}\Bigg(\mu_n \displaystyle\int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta} \\ \aleph_2(\xi,y_n)\Big] p_n(y_n) g_n\Big(u_1(y_n)\Big)dy_n\Bigg)\cdots dy_2\Bigg)dy_1 \end{cases} \\ &\geq \sigma^2 \|Tu_1\|. \end{aligned}$$

Therefore $\displaystyle\min_{z\in\left[\sigma,1-\sigma\right]} Tu_1(z) \geq \sigma^2\|Tu_1\|$. Hence $Tu_1 \in P$ and so $T : P \to P$. An application of the Arzela–Ascoli Theorem indicates that the operator T remains completely continuous.

3.1. Notations

We introduce:

$$\sigma_1 = \max\left\{\begin{array}{c} \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_1(y)dy\mathfrak{g}_{1\infty}\right]^{-1}, \\ \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_2(y)dy\mathfrak{g}_{2\infty}\right]^{-1}, \\ \cdots \\ \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_n(y)dy\mathfrak{g}_{n\infty}\right]^{-1} \end{array}\right\},$$

$$\sigma_2 = \min\left\{\begin{array}{c} \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_1(y)dy\mathfrak{g}_{10}\right]^{-1}, \\ \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_2(y)dy\mathfrak{g}_{20}\right]^{-1}, \\ \cdots \\ \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_n(y)dy\mathfrak{g}_{n0}\right]^{-1} \end{array}\right\}.$$

Theorem 3.1. *Suppose* (H_1)-(H_4) *hold. Then for each* $\mu_k, k=\overline{1,n}$ *satisfying*

$$\sigma_1 < \mu_k < \sigma_2, \quad k=\overline{1,n}, \tag{3.2}$$

there exists an n*-tuple* $(u_1,u_2,\cdots,u_n)$ *satisfying the FBVP* (1.1)-(1.2) *s.t.* $u_k(z)>0$, $k=\overline{1,n}$ *on* $(0,1)$.

Proof. Let μ_k, $k=\overline{1,n}$ be found as in (3.2). Now let $\varepsilon>0$ be chosen s.t.

$$\max\left\{\begin{array}{c} \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_1(y)dy(\mathfrak{g}_{1\infty}-\varepsilon)\right]^{-1}, \\ \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_2(y)dy(\mathfrak{g}_{2\infty}-\varepsilon)\right]^{-1}, \\ \cdots \\ \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_n(y)dy(\mathfrak{g}_{n\infty}-\varepsilon)\right]^{-1} \end{array}\right\} \le \min\left\{\begin{array}{c}\mu_1,\\ \mu_2,\\ \cdot\\ \cdot\\ \cdot\\ \mu_n\end{array}\right\}$$

and

$$\max\left\{\begin{array}{c}\mu_1,\\ \mu_2,\\ \cdot\\ \cdot\\ \cdot\\ \mu_n\end{array}\right\} \le \min\left\{\begin{array}{c} \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_1(y)dy(\mathfrak{g}_{10}+\varepsilon)\right]^{-1}, \\ \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_2(y)dy(\mathfrak{g}_{20}+\varepsilon)\right]^{-1}, \\ \cdots \\ \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_n(y)dy(\mathfrak{g}_{n0}+\varepsilon)\right]^{-1} \end{array}\right\}.$$

Furthermore, according to $\mathfrak{g}_{k0}$, $k=\overline{1,n}$, there exists an $N_1>0$ s.t., for each $1\le k\le n$, $\mathfrak{g}_k(x)\le(\mathfrak{g}_{k0}+\varepsilon)x$, $\quad 1<x\le N_1$.
Let $u_1\in P$ with $\|u_1\|=N_1$. By Lemmas 2.6, 2.7 and the choice of ε, for $0\le y_{n-1}\le 1$,

$$\begin{aligned} &\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)\mathfrak{g}_n(u_1(y_n))dy_n \\ &\le \mu_n\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)(\mathfrak{g}_{n0}+\varepsilon)u_1(y_n)dy_n \\ &\le \mu_n\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)dy_n(\mathfrak{g}_{n0}+\varepsilon)\|u_1\| \\ &\le \|u_1\| = N_1. \end{aligned}$$

It follows from Lemmas 2.6, 2.7 in the same way, for $0 \le y_{n-2} \le 1$,

$$\left.\begin{array}{l} \mu_{n-1}\displaystyle\int_0^1 \left[\aleph_1(y_{n-2},y_{n-1}) + \frac{\vartheta y_{n-2}}{\Delta}\aleph_2(\xi,y_{n-1})\right] p_{n-1}(y_{n-1}) \\ \qquad g_{n-1}\left(\mu_n \displaystyle\int_0^1 \left[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\right. \\ \qquad\qquad \left. p_n(y_n) g_n(u_1(y_n)) dy_n \right) dy_{n-1} \end{array}\right\}$$
$$\le \left\{ \begin{array}{l} \mu_{n-1}\displaystyle\int_0^1 \left[\aleph_1(y,y_{n-1}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_{n-1})\right] \\ p_{n-1}(y_{n-1}) dy_{n-1} \left(g_{\overline{n-1}0} + \varepsilon\right) \|u_1\| \end{array}\right.$$
$$\le \|u_1\| = N_1.$$

Proceeding with the bootstrapping assertion, for $0 \le z \le 1$,

$$\left.\begin{array}{l} \mu_1 \displaystyle\int_0^1 \left[\aleph_1(z,y_1) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\right] p_1(y_1) \\ \quad g_1\left(\mu_2 \displaystyle\int_0^1 \left[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right] p_2(y_2) \cdots \right. \\ \qquad g_{n-1}\left(\mu_n \displaystyle\int_0^1 \left[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\right. \\ \qquad\qquad \left.\left. p_n(y_n) g_n(u_1(y_n)) dy_n \right) \cdots dy_2 \right) dy_1 \end{array}\right\} \le N_1,$$

so that, for $0 \le z \le 1$, $Tu_1(z) \le N_1$. Hence $\|Tu_1\| \le N_1 = \|u_1\|$. If we set $E_1 = \left\{x \in B : \|x\| < N_1\right\}$, then

$$\|Tu_1\| \le \|u_1\|, \text{ for } u_1 \in P \cap \partial E_1. \tag{3.3}$$

Additionally, according to $g_{k\infty}$, $k = \overline{1,n}$, there exists $\overline{N}_2 > 0$ s.t., for each $1 \le k \le n$, $g_k(x) \ge (g_{k\infty} - \varepsilon)x$, $x \ge \overline{N}_2$. Choose $N_2 = \max\left\{2N_1, \frac{\overline{N}_2}{\sigma^2}\right\}$. Let $u_1 \in P$ and $\|u_1\| = N_2$. Then

$$\min_{z\in[\sigma,1-\sigma]} u_1(z) \ge \sigma^2 \|u_1\| \ge \overline{N}_2.$$

Based on Lemmas 2.6, 2.7 and choice of ε, for $0 \le y_{n-1} \le 1$, we have:

$$\begin{aligned} &\mu_n \int_0^1 \left[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right] p_n(y_n) g_n(u_1(y_n)) dy_n \\ &\ge \sigma^2 \mu_n \int_\sigma^{1-\sigma} \left[\aleph_1(y,y_n) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right] p_n(y_n)(g_{n\infty} - \varepsilon) u_1(y_n) dy_n \\ &\ge \sigma^2 \mu_n \int_\sigma^{1-\sigma} \left[\aleph_1(y,y_n) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right] p_n(y_n) dy_n (g_{n\infty} - \varepsilon) \|u_1\| \\ &\ge \|u_1\| = N_2. \end{aligned}$$

It stems in the same way from Lemmas 2.6, 2.7 and choice of ε, for $0 \le y_{n-2} \le 1$:

$$\left.\begin{array}{l} \mu_{n-1}\displaystyle\int_0^1 \left[\aleph_1(y_{n-2},y_{n-1}) + \frac{\vartheta y_{n-2}}{\Delta}\aleph_2(\xi,y_{n-1})\right] p_{n-1}(y_{n-1}) \\ \qquad g_{n-1}\left(\mu_n \displaystyle\int_0^1 \left[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\right. \\ \qquad\qquad \left. p_n(y_n) g_n(u_1(y_n)) dy_n \right) dy_{n-1} \end{array}\right\}$$
$$\ge \left\{ \begin{array}{l} \sigma^2\mu_{n-1}\displaystyle\int_\sigma^{1-\sigma} \left[\aleph_1(y,y_{n-1}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_{n-1})\right] \\ p_{n-1}(y_{n-1}) dy_{n-1} \left(g_{\overline{n-1}\infty} - \varepsilon\right) \|u_1\| \end{array}\right.$$
$$\ge \|u_1\| = N_2.$$

By bootstrapping argument, we discover:

$$\left.\begin{array}{l} \mu_1 \displaystyle\int_0^1 \left[\aleph_1(z,y_1) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\right] p_1(y_1) \\ \quad g_1\left(\mu_2 \displaystyle\int_0^1 \left[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right] p_2(y_2) \cdots \right. \\ \qquad g_{n-1}\left(\mu_n \displaystyle\int_0^1 \left[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\right. \\ \qquad\qquad \left.\left. p_n(y_n) g_n(u_1(y_n)) dy_n \right) \cdots dy_2 \right) dy_1 \end{array}\right\} \ge N_2,$$

so that $\mathtt{Tu}_1(\mathtt{z}) \geq \mathtt{N}_2 = \|\mathtt{u}_1\|$. Hence $\|\mathtt{Tu}_1\| \geq \|\mathtt{u}_1\|$. So if we set $\mathtt{E}_2 = \{\mathtt{x} \in \mathtt{B} : \|\mathtt{x}\| < \mathtt{N}_2\}$, then

$$\|\mathtt{Tu}_1\| \geq \|\mathtt{u}_1\|, \text{ for } \mathtt{u}_1 \in \mathtt{P} \cap \partial \mathtt{E}_2. \tag{3.4}$$

By utilizing (3.3), (3.4) and Guo–Krasnosel'skii FPT (see [26, 27]), we discover that $\mathtt{T}$ has a fixed point $\mathtt{u}_1 \in \mathtt{P} \cap (\bar{\mathtt{E}}_2 \backslash \mathtt{E}_1)$. Setting $\mathtt{u}_1 = \mathtt{u}_{n+1}$ yields a positive solution $(\mathtt{u}_1, \mathtt{u}_2, \cdots, \mathtt{u}_n)$ of the FBVP (1.1)–(1.2) iteratively indicated by:

$$\mathtt{u}_k(\mathtt{z}) = \mu_k \int_0^1 \left[\aleph_1(\mathtt{z},\mathtt{y}) + \frac{\vartheta \mathtt{z}}{\Delta} \aleph_2(\xi,\mathtt{y}) \right] \mathtt{p}_k(\mathtt{y}) \mathfrak{g}_k(\mathtt{u}_{k+1}(\mathtt{y}))d\mathtt{y}, \quad \mathtt{k} = \mathtt{n}, \mathtt{n}-1, \cdots, 1.$$

□

3.2. Notations

We introduce:

$$\sigma_3 = \max \left\{ \begin{array}{c} \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_1(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{10}\right]^{-1}, \\ \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_2(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{20}\right]^{-1}, \\ \cdots \\ \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_n(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{n0}\right]^{-1} \end{array} \right\} \text{ and}$$

$$\sigma_4 = \min \left\{ \begin{array}{c} \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_1(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{1\infty}\right]^{-1}, \\ \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_2(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{2\infty}\right]^{-1}, \\ \cdots \\ \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_n(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{n\infty}\right]^{-1} \end{array} \right\}.$$

Theorem 3.2. *Suppose* (H_1)-(H_4) *hold, then for each* $\mu_k, \mathtt{k} = \overline{1,\mathtt{n}}$ *satisfying*

$$\sigma_3 < \mu_k < \sigma_4, \ \mathtt{k} = \overline{1,\mathtt{n}}, \tag{3.5}$$

there exists an $\mathtt{n}$*-tuple* $(\mathtt{u}_1, \mathtt{u}_2, \cdots, \mathtt{u}_n)$ *satisfying the FBVP* (1.1)-(1.2) *s.t.* $\mathtt{u}_k(\mathtt{z}) > 0$, $\mathtt{k} = \overline{1,\mathtt{n}}$ *on* $(0,1)$.

Proof. Let μ_k, $\mathtt{k} = \overline{1,\mathtt{n}}$ be provided as in (3.5). Now let $\varepsilon > 0$ be chosen s.t.

$$\max \left\{ \begin{array}{c} \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_1(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{10}-\varepsilon)\right]^{-1}, \\ \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_2(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{20}-\varepsilon)\right]^{-1}, \\ \cdots \\ \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_n(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{n0}-\varepsilon)\right]^{-1} \end{array} \right\} \leq \min \left\{ \begin{array}{c} \mu_1, \\ \mu_2, \\ \cdot \\ \cdot \\ \cdot \\ \mu_n \end{array} \right\}$$

and

$$\max \left\{ \begin{array}{c} \mu_1, \\ \mu_2, \\ \cdot \\ \cdot \\ \cdot \\ \mu_n \end{array} \right\} \leq \min \left\{ \begin{array}{c} \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_1(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{1\infty}+\varepsilon)\right]^{-1}, \\ \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_2(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{2\infty}+\varepsilon)\right]^{-1}, \\ \cdots \\ \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_n(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{n\infty}+\varepsilon)\right]^{-1} \end{array} \right\}.$$

Based on the rules of $\mathfrak{g}_{k0}$, $1 \leq \mathtt{k} \leq \mathtt{n}$ there exists $\bar{\mathtt{N}}_3 > 0$ s.t., for each $1 \leq \mathtt{k} \leq \mathtt{n}$,

$$\mathfrak{g}_k(\mathtt{x}) \geq (\mathfrak{g}_{k0} - \varepsilon)\mathtt{x}, \ 1 < \mathtt{x} \leq \bar{\mathtt{N}}_3.$$

According to the definitions of $\mathfrak{g}_{k0}$, it follows that $\mathfrak{g}_{k0}(1)=0,\ 1\le k\le n$ and so there exist $1<\Theta_n<\Theta_{n-1}<\cdots<\Theta_2<\overline{N}_3$ s.t.

$$\left.\begin{aligned}
\mu_k\mathfrak{g}_k(z) &\le \frac{\Theta_{k-1}}{\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_k(y)dy},\ z\in\left[1,\Theta_k\right],\\
&\qquad 3\le k\le n,\ \text{and}\\
\mu_2\mathfrak{g}_2(z) &\le \frac{\overline{N}_3}{\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_2(y)dy},\ z\in\left[1,\Theta_2\right].
\end{aligned}\right\}$$

Let $u_1\in P$ with $\|u_1\|=\Theta_n$. Then:

$$\begin{aligned}
&\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)\mathfrak{g}_n\big(u_1(y_n)\big)dy_n\\
&\quad\le \mu_n\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)\mathfrak{g}_n\big(u_1(y_n)\big)dy_n\\
&\quad\le \frac{\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)\Theta_{n-1}dy_n}{\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)dy_n}\\
&\quad\le \Theta_{n-1}.
\end{aligned}$$

Utilizing this bootstrapping technique, it implies that

$$\left.\begin{aligned}
&\mu_2\int_0^1\left[\aleph_1(y_1,y_2)+\frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right]p_2(y_2)\\
&\quad\mathfrak{g}_2\bigg(\mu_3\int_0^1\left[\aleph_1(y_2,y_3)+\frac{\vartheta y_2}{\Delta}\aleph_2(\xi,y_3)\right]p_3(y_3)\cdots\\
&\quad\quad\mathfrak{g}_{n-1}\bigg(\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\\
&\quad\quad\quad p_n(y_n)\mathfrak{g}_n\big(u_1(y_n)\big)dy_n\bigg)\cdots dy_2\bigg)dy_1
\end{aligned}\right\}\le\overline{N}_3.$$

Then

$$\begin{aligned}
Tu_1(z) &= \left\{\begin{aligned}
&\mu_1\int_0^1\left[\aleph_1(z,y_1)+\frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\right]p_1(y_1)\\
&\quad\mathfrak{g}_1\bigg(\mu_2\int_0^1\left[\aleph_1(y_1,y_2)+\frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right]p_2(y_2)\cdots\\
&\quad\quad\mathfrak{g}_{n-1}\bigg(\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\\
&\quad\quad\quad p_n(y_n)\mathfrak{g}_n\big(u_1(y_n)\big)dy_n\bigg)\cdots dy_2\bigg)dy_1
\end{aligned}\right.\\
&\ge \sigma^2\mu_1\int_\sigma^{1-\sigma}\left[\aleph_1(y,y_1)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\right]p_1(y_1)\big(\mathfrak{g}_{10}-\varepsilon\big)\|u_1\|dy_1\\
&\ge \|u_1\|.
\end{aligned}$$

So $\|Tu_1\|\ge\|u_1\|$. If we set $E_1=\left\{x\in B:\|x\|<\Theta_n\right\}$, then

$$\|Tu_1\|\ge\|u_1\|,\ \text{for } u_1\in P\cap\partial E_1. \tag{3.6}$$

It follows that $\mathfrak{g}_k$, $1\le k\le n$ is unbounded at ∞. Since each $\mathfrak{g}_{k\infty}$ is considered to be a positive real number. For each $1\le k\le n$, set

$$\mathfrak{g}_k^*(x)=\sup_{y\in[1,x]}\mathfrak{g}_k(y).$$

Based on the definition of $\mathfrak{g}_{k\infty}$, $1\le k\le n$, there exists $\overline{N}_4$ s.t., for each $1\le k\le n$,

$$\mathfrak{g}_k^*(x)\le\big(\mathfrak{g}_{k\infty}+\varepsilon\big)x,\ x\ge\overline{N}_4.$$

It follows that there exists $N_4=\max\left\{2\overline{N}_3,\overline{N}_4\right\}$ s.t., for each $1\le k\le n$,

$$\mathfrak{g}_k^*(x)\le\mathfrak{g}_k^*\big(N_4\big),\ 1<x\le N_4.$$

Choose $\mathtt{u}_1 \in \mathtt{P}$ with $\|\mathtt{u}_1\| = \mathtt{N}_4$. Then, by using bootstrapping argument, we have:

$$\mathtt{Tu}_1(\mathtt{z}) = \begin{cases} \mu_1 \int_0^1 \left[\aleph_1(\mathtt{z},\mathtt{y}_1) + \frac{\vartheta \mathtt{z}}{\Delta}\aleph_2(\xi,\mathtt{y}_1)\right]\mathtt{p}_1(\mathtt{y}_1) \\ \quad \mathfrak{g}_1\Bigg(\mu_2 \int_0^1 \left[\aleph_1(\mathtt{y}_1,\mathtt{y}_2) + \frac{\vartheta \mathtt{y}_1}{\Delta}\aleph_2(\xi,\mathtt{y}_2)\right]\mathtt{p}_2(\mathtt{y}_2)\cdots \\ \quad\quad \mathfrak{g}_{\mathtt{n}-1}\Bigg(\mu_\mathtt{n} \int_0^1 \left[\aleph_1(\mathtt{y}_{\mathtt{n}-1},\mathtt{y}_\mathtt{n}) + \frac{\vartheta \mathtt{y}_{\mathtt{n}-1}}{\Delta}\aleph_2(\xi,\mathtt{y}_\mathtt{n})\right] \\ \quad\quad\quad \mathtt{p}_\mathtt{n}(\mathtt{y}_\mathtt{n})\mathfrak{g}_\mathtt{n}(\mathtt{u}_1(\mathtt{y}_\mathtt{n}))d\mathtt{y}_\mathtt{n}\Bigg)\cdots d\mathtt{y}_2\Bigg)d\mathtt{y}_1 \end{cases}$$

$$\le \begin{cases} \mu_1 \int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y}_1)\right]\mathtt{p}_1(\mathtt{y}_1) \\ \quad \mathfrak{g}_1^*\Bigg(\mu_2 \int_0^1 \left[\aleph_1(\mathtt{y}_1,\mathtt{y}_2) + \frac{\vartheta \mathtt{y}_1}{\Delta}\aleph_2(\xi,\mathtt{y}_2)\right]\mathtt{p}_2(\mathtt{y}_2)\cdots \\ \quad\quad \mathfrak{g}_{\mathtt{n}-1}\Bigg(\mu_\mathtt{n} \int_0^1 \left[\aleph_1(\mathtt{y}_{\mathtt{n}-1},\mathtt{y}_\mathtt{n}) + \frac{\vartheta \mathtt{y}_{\mathtt{n}-1}}{\Delta}\aleph_2(\xi,\mathtt{y}_\mathtt{n})\right] \\ \quad\quad\quad \mathtt{p}_\mathtt{n}(\mathtt{y}_\mathtt{n})\mathfrak{g}_\mathtt{n}(\mathtt{u}_1(\mathtt{y}_\mathtt{n}))d\mathtt{y}_\mathtt{n}\Bigg)\cdots d\mathtt{y}_2\Bigg)d\mathtt{y}_1 \end{cases}$$

$$\le \mu_1 \int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y}_1)\right]\mathtt{p}_1(\mathtt{y}_1)\mathfrak{g}_1^*(\mathtt{N}_4)d\mathtt{y}_1$$

$$\le \mu_1 \int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y}_1)\right]\mathtt{p}_1(\mathtt{y}_1)d\mathtt{y}_1(\mathfrak{g}_{1\infty}+\varepsilon)\mathtt{N}_4$$

$$\le \mathtt{N}_4 = \|\mathtt{u}_1\|.$$

Thus $\|\mathtt{Tu}_1\| \le \|\mathtt{u}_1\|$. So, if we let $\mathtt{E}_2 = \left\{\mathtt{x} \in \mathtt{B} : \|\mathtt{x}\| < \mathtt{N}_4\right\}$, then

$$\|\mathtt{Tu}_1\| \le \|\mathtt{u}_1\|, \text{ for } \mathtt{u}_1 \in \mathtt{P} \cap \partial \mathtt{E}_2. \tag{3.7}$$

By utilizing (3.6), (3.7) and Guo–Krasnosel'skii FPT (see [26, 27]), we get that $\mathtt{T}$ has a fixed point $\mathtt{u}_1 \in \mathtt{P} \cap (\overline{\mathtt{E}}_2 \backslash \mathtt{E}_1)$, which in turn with $\mathtt{u}_1 = \mathtt{u}_{\mathtt{n}+1}$ yields an $\mathtt{n}$-tuple $(\mathtt{u}_1, \mathtt{u}_2, \cdots, \mathtt{u}_\mathtt{n})$ satisfying the FBVP (1.1)-(1.2) for the chosen values of $\mu_\mathtt{k}$, $\mathtt{k} = \overline{1,\mathtt{n}}$. □

4. Application

Let $\mathtt{n} = 2, \mathtt{p}_1(\mathtt{z}) = \mathtt{z}+1, \mathtt{p}_2(\mathtt{z}) = \mathtt{z}+2,\ \xi = \frac{1}{2}, \nu = 4, \sigma = \frac{1}{4},\ \mathtt{g}_1(\mathtt{u}) = \mathtt{u}\left(1 - \frac{19}{20}e^{-\mathtt{u}}\right),\ \ \mathtt{g}_2(\mathtt{u}) = \mathtt{u} - \frac{39}{40}\sin\mathtt{u}$. Then $\Delta = 1,\ \mathtt{g}_{10} = \frac{1}{20},\ \mathtt{g}_{20} = \frac{1}{40},\ \mathtt{g}_{1\infty} = \mathtt{g}_{2\infty} = 1$.

$$\begin{aligned} \sigma_1 &= \max_{1\le i\le 2}\left\{\left[\sigma^2 \int_\sigma^{1-\sigma}\left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right]\mathtt{p}_i(\mathtt{y})d\mathtt{y}\mathfrak{g}_{i\infty}\right]^{-1}\right\} \\ &= \max\left\{22.67613805, 13.41689359\right\} \\ &= 22.67613805. \end{aligned}$$

$$\begin{aligned} \sigma_2 &= \min_{1\le i\le 2}\left\{\left[\int_0^1\left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right]\mathtt{p}_i(\mathtt{y})d\mathtt{y}\mathfrak{g}_{i0}\right]^{-1}\right\} \\ &= \min\left\{28.34517257, 33.54223396\right\} \\ &= 28.34517257. \end{aligned}$$

Theorem 3.1's requirements are all met. Therefore by Theorem 3.1 the following BVP

$$\left.\begin{aligned} \mathtt{D}^{1.5}\mathtt{u}_1(\mathtt{z}) + \mu_1(\mathtt{z}+1)\mathtt{u}_2(\mathtt{z})\left(1 - \frac{19}{20}e^{\mathtt{u}_2(\mathtt{z})}\right) = 0 \quad & \mathtt{z} \in (0,1), \\ \mathtt{D}^{1.5}\mathtt{u}_2(\mathtt{z}) + \mu_2(\mathtt{z}+2)\left(\mathtt{u}_1(\mathtt{z}) - \frac{39}{40}\sin\mathtt{u}_1(\mathtt{z})\right) = 0, \quad & \mathtt{z} \in (0,1), \end{aligned}\right\}$$

$$\mathtt{u}_\mathtt{k}(0) = 0, \quad \mathtt{u}_\mathtt{k}(1) = 4\int_0^{1/2}\mathtt{u}_\mathtt{k}(\mathtt{z})d\mathtt{z}, \qquad \text{for } \mathtt{k} = 1,2,$$

has a positive solution if $22.67613805 < \mu_\mathtt{k} < 28.34517257$ for $\mathtt{k} = 1,2$.

5. Conclusion

In conclusion, this paper effectively fulfills its objective of identifying the eigenvalue intervals of $\mu_{\mathtt{k}}$, $1 \leq \mathtt{k} \leq \mathtt{n}$, for which an iterative system of a class of fractional-order DEqs with parameterized integral BCs possesses at least one positive solution. This is accomplished through the utilization of the standard fixed-point theorem of cone type. The significance of this work lies in its novelty; the authors assert that it represents the inaugural endeavor in the literature to derive such insights for this specific domain.

6. Comparison

In comparison to existing approaches, our study explores the eigenvalue intervals of $\mu_{\mathtt{k}}$, $1 \leq \mathtt{k} \leq \mathtt{n}$ for a class of FDEqs with parameterized integral BCs. By employing standard FPT of cone and combining an incomplete $\aleph$-function with a broad category of polynomials, the researchers devised generalized fractional calculus formulations [31]. Additionally, they utilized the natural transform method along with graph-based approaches to represent solutions for the M-Sturm-Liouville problem [32]. Moreover, the MDLTM was applied to provide analytic solutions for the fractional pseudo hyperbolic telegraph equation [33]. Notably, the existence and uniqueness of the model underlying the Caputo-Fabrizio-fractal-fractional derivative were demonstrated using FPTs [34]. Furthermore, the fundamental properties of a new integral transformation were elucidated, and its application to elementary functions was discussed in [35].

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] H. G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Q. Chen, *A new collection of real world applications of fractional calculus in science and engineering*, Commun. Nonlinear Sci. Numer. Simul., **64** (2018), 213–231.
[2] A. A. Kilbas, H. M. Srivasthava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.
[3] I. Podulbny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[4] E. F. D. Goufo, *A biomathematical view on the fractional dynamics of cellulose degradation*, Fract. Calc. Appl. Anal., **18**(3) (2015), 554–564.
[5] H. H. Sherief, M. A. el-Hagary, *Fractional order theory of thermo-viscoelasticity and application*, Mech. Time-Depend Mater., **24** (2020), 179–195.
[6] G. Alotta, E. Bologna, G. Failla, M. Zingales, *A fractional approach to Non-Newtonian blood rheology in capillary vessels*, J. Peridyn Nonlocal Model, **1** (2019), 88–96.
[7] H. Sun, W. Chen, C. Li, Y. Q. Chen, *Fractional differential models for anomalous diffusion*, Physica A Stat. Mech. Appl., **389** (2010), 2719–2724.
[8] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70.
[9] M. A. Hammad, R. Khalil, *Abel's formula and Wronskian for conformable fractional differential equations*, Int. J. Diff. Equ. Appl., **13** (2014), 177–183.
[10] U. Katugampola, *A new fractional derivative with classical properties*, J. American Math. Soc., arXiv:1410.6535v2.
[11] D. R. Anderson, R. I. Avery, *Fractional order boundary value problem with Sturm–Liouville boundary conditions*, Electron. J. Differential Equations, **2015** (2015), 1–10.
[12] K. R. Prasad, B. M. B. Krushna, *Multiple positive solutions for a coupled system of Riemann–Liouville fractional order two-point boundary value problems*, Nonlinear Stud., **20** (2013), 501–511.
[13] K. R. Prasad, B. M. B. Krushna, *Eigenvalues for iterative systems of Sturm–Liouville fractional order two-point boundary value problems*, Fract. Calc. Appl. Anal., **17** (2014), 638–653.
[14] K. R. Prasad, B. M. B. Krushna, *Solvability of p-Laplacian fractional higher order two-point boundary value problems*, Commun. Appl. Anal., **19** (2015), 659–678.
[15] K. R. Prasad, B. M. B. Krushna, V. V. R. R. B. Raju, Y. Narasimhulu, *Existence of positive solutions for systems of fractional order boundary value problems with Riemann–Liouville derivative*, Nonlinear Stud., **24** (2017), 619–629.
[16] K. R. Prasad, B. M. B. Krushna, L.T. Wesen, *Existence results for positive solutions to iterative systems of four-point fractional-order boundary value problems in a Banach space*, Asian-Eur. J. Math., **13** (2020), 1–17.
[17] B. Zhou, L. Zhang, *Uniqueness and iterative schemes of positive solutions for conformable fractional differential equations via sum-type operator method*, J. Math. Res. Appl., **42**(4) (2022), 349–362.
[18] M. Bouaouid, *Mild solutions of a class of conformable fractional differential equations with nonlocal conditions*, Acta Math. Appl. Sin., Engl. Ser., **39**(2) (2023), 249–261.
[19] G. Xiao, J. Wang, D. O'Regan, *Existence and stability of solutions to neutral conformable stochastic functional differential equations*, Qual. Theory Dyn. Syst., **21**(1) (2022), 1–22.
[20] A. Jaiswal, D. Bahuguna, *Semilinear conformable fractional differential equations in Banach spaces*, Differ. Equ. Dyn. Syst., **27** (2019), 313–325.
[21] A. Gokdogan, E. Unal, E. Celik, *Existence and uniqueness theorems for sequential linear conformable fractional differential equations*, Miskolc Math. Notes, **17** (2016), 267–279.
[22] M. Khuddush, K. R. Prasad, *Infinitely many positive solutions for an iterative system of conformable fractional order dynamic boundary value problems on time scales*, Turk. J. Math., **46**(2) (2022), 338-359. https://doi.org/10.3906/mat-2103-117,
[23] W. Zhong, L. Wang, *Positive solutions of conformable fractional differential equations with integral boundary conditions*, Bound. Value Probl., **2018** (2018), 1–12.
[24] B. Bendouma, A. Cabada, A. Hammoudi, *Existence results for systems of conformable fractional differential equations*, Arch. Math., **55**(2) (2019), 69–82.

[25] F. Haddouchi, *Existence of positive solutions for a class of conformable fractional differential equations with parameterized integral boundary conditions*, Kyungpook Math. J., **61** (2021), 139–153.
[26] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, 1988.
[27] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.
[28] J. Cannon, *The solution of the heat equation subject to the specification of energy*, Quart. Appl. Math., **21** (1963), 155–160.
[29] R. Chegis, *Numerical solution of a heat conduction problem with an integral boundary condition*, Litovsk. Mat. Sb., **24** (1984), 209–215.
[30] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66.
[31] S. Bhatter, S. D. Purohit, K. S. Nisar, S. R. Munjam, *Some fractional calculus findings associated with the product of incomplete $\aleph$-function and Srivastava polynomials*, Int. J. Math. Comput. Eng., **2**(1) (2024), 97–116.
[32] E. Bas, M. Karaoglan, *Representation of the solution of the M-Sturm–Liouville problem with natural transform*, Int. J. Math. Comput. Eng., **1**(2) (2023), 243–252.
[33] S. T. Abdulazeez, M. Modanli, *Analytic solution of fractional order Pseudo-Hyperbolic Telegraph equation using modified double Laplace transform method*, Int. J. Math. Comput. Eng., **1**(1) (2023), 105–114.
[34] R. Singh, J. Mishra, V. K. Gupta, *Dynamical analysis of a Tumor Growth model under the effect of fractal fractional Caputo-Fabrizio derivative*, Int. J. Math. Comput. Eng., **1**(1) (2023), 115–126.
[35] E. Ata, I. O. Kiymaz, *New generalized Mellin transform and applications to partial and fractional differential equations*, Int. J. Math. Comput. Eng., **1**(1) (2023), 45–66.