MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

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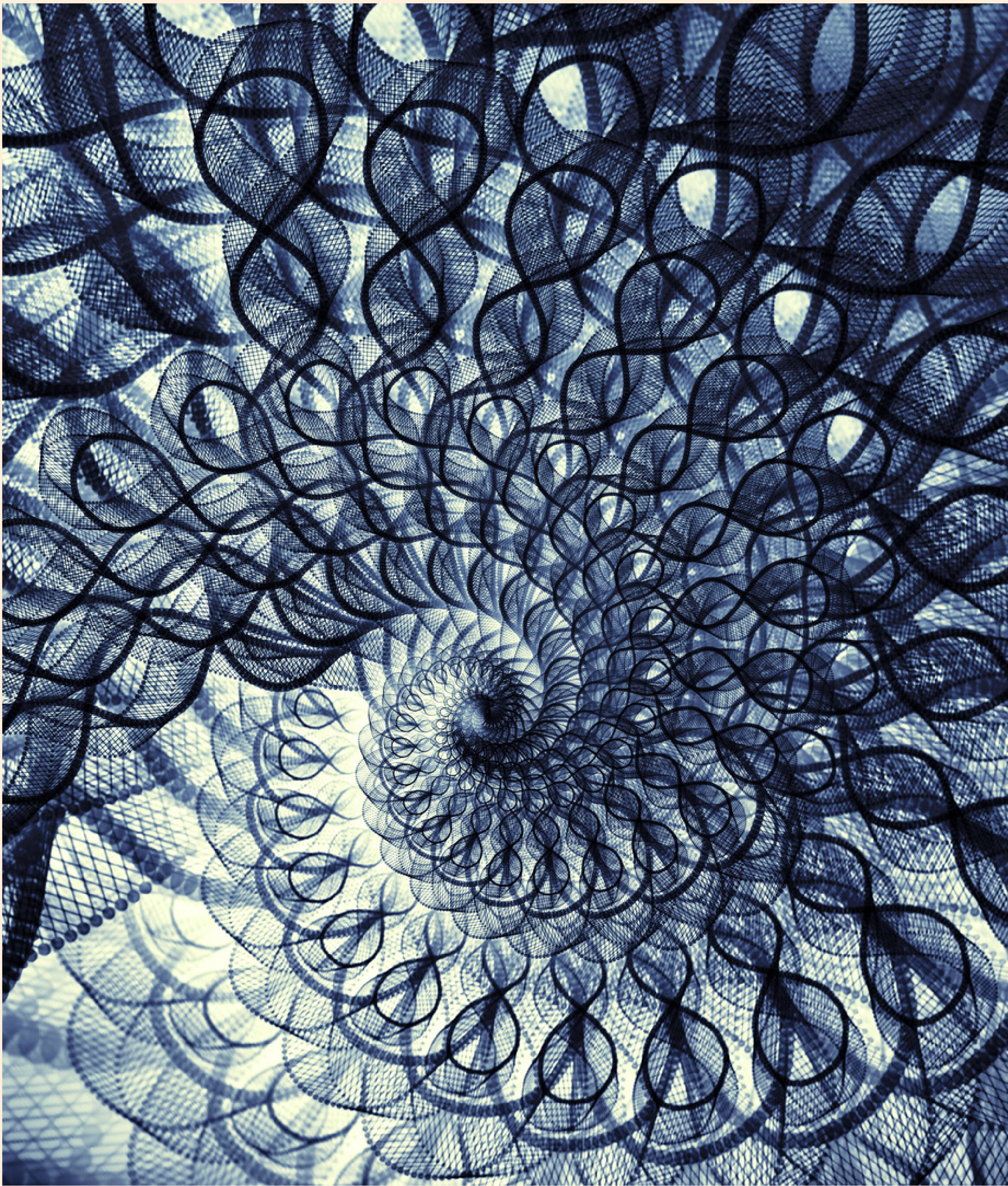

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Some New f-Divergence Measures and Their Basic Properties

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Abstract
In this paper, we introduce some new f-divergence measures that we call *t-asymmetric/symmetric divergence measure* and *integral divergence measure*, establish their joint convexity and provide some inequalities that connect these f-divergences to the classical one introduced by Csiszar in 1963. Applications for the *dichotomy class* of convex functions are provided as well.

Keywords: *f-divergence measures, Hellinger discrimination, HH f-divergence measures, Jeffrey's distance, Kullback-Leibler divergence, χ^2-divergence*

AMS Subject Classification (2020): *94A17; 26D1*

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1. Introduction

Let $(X,\mathcal{A})$ be a measurable space satisfying $|\mathcal{A}|>2$ and μ be a σ-finite measure on $(X,\mathcal{A})$. Let $\mathcal{P}$ be the set of all probability measures on $(X,\mathcal{A})$ which are absolutely continuous with respect to μ. For $P,Q\in\mathcal{P}$, let $p=\dfrac{dP}{d\mu}$ and $q=\dfrac{dQ}{d\mu}$ denote the *Radon-Nikodym* derivatives of P and Q with respect to μ.
Two probability measures $P,Q\in\mathcal{P}$ are said to be *orthogonal* and we denote this by $Q\perp P$ if

$$P(\{q=0\})=Q(\{p=0\})=1.$$

Let $f:[0,\infty)\to(-\infty,\infty]$ be a convex function that is continuous at 0, i.e., $f(0)=\lim_{u\downarrow 0}f(u)$.
In 1963, I. Csiszár [1] introduced the concept of f-divergence as follows.

Definition 1.1. Let $P,Q\in\mathcal{P}$. Then

$$I_f(Q,P)=\int_X p(x)\,f\left[\frac{q(x)}{p(x)}\right]d\mu(x), \tag{1.1}$$

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is called the f-divergence of the probability distributions Q and P.

Remark 1.1. Observe that, the integrand in the formula (1.1) is undefined when $p(x) = 0$. The way to overcome this problem is to postulate for f as above that

$$0f\left[\frac{q(x)}{0}\right] = q(x)\lim_{u\downarrow 0}\left[uf\left(\frac{1}{u}\right)\right], \; x \in X. \tag{1.2}$$

We now give some examples of f-divergences that are well-known and often used in the literature (see also [2]).

1.1 The class of χ^α-divergences

The f-divergences of this class, which is generated by the function χ^α, $\alpha \in [1,\infty)$, defined by

$$\chi^\alpha(u) = |u-1|^\alpha, \quad u \in [0,\infty)$$

have the form

$$I_f(Q,P) = \int_X p\left|\frac{q}{p}-1\right|^\alpha d\mu = \int_X p^{1-\alpha}|q-p|^\alpha d\mu. \tag{1.3}$$

From this class only the parameter $\alpha = 1$ provides a distance in the topological sense, namely the *total variation distance* $V(Q,P) = \int_X |q-p|\,d\mu$. The most prominent special case of this class is, however, *Karl Pearson's* χ^2-*divergence*

$$\chi^2(Q,P) = \int_X \frac{q^2}{p}d\mu - 1$$

that is obtained for $\alpha = 2$.

1.2 Dichotomy class

From this class, generated by the function $f_\alpha : [0,\infty) \to \mathbb{R}$

$$f_\alpha(u) = \begin{cases} u - 1 - \ln u & \text{for } \alpha = 0; \\ \dfrac{1}{\alpha(1-\alpha)}[\alpha u + 1 - \alpha - u^\alpha] & \text{for } \alpha \in \mathbb{R}\backslash\{0,1\}; \\ 1 - u + u\ln u & \text{for } \alpha = 1; \end{cases}$$

only the parameter $\alpha = \frac{1}{2}$ $\left(f_{\frac{1}{2}}(u) = 2(\sqrt{u}-1)^2\right)$ provides a distance, namely, the *Hellinger distance*

$$H(Q,P) = \left[\int_X (\sqrt{q}-\sqrt{p})^2 d\mu\right]^{\frac{1}{2}}.$$

Another important divergence is the *Kullback-Leibler divergence* obtained for $\alpha = 1$,

$$KL(Q,P) = \int_X q\ln\left(\frac{q}{p}\right)d\mu.$$

1.3 Matsushita's divergences

The elements of this class, which is generated by the function φ_α, $\alpha \in (0,1]$ given by

$$\varphi_\alpha(u) := |1-u^\alpha|^{\frac{1}{\alpha}}, \quad u \in [0,\infty),$$

are prototypes of metric divergences, providing the distances $[I_{\varphi_\alpha}(Q,P)]^\alpha$.

1.4 Puri-Vincze divergences

This class is generated by the functions Φ_α, $\alpha \in [1,\infty)$ given by

$$\Phi_\alpha(u) := \frac{|1-u|^\alpha}{(u+1)^{\alpha-1}}, \quad u \in [0,\infty).$$

It has been shown in [3] that this class provides the distances $[I_{\Phi_\alpha}(Q,P)]^{\frac{1}{\alpha}}$.

1.5 Divergences of Arimoto-type

This class is generated by the functions

$$\Psi_\alpha(u) := \begin{cases} \dfrac{\alpha}{\alpha-1}\left[(1+u^\alpha)^{\frac{1}{\alpha}} - 2^{\frac{1}{\alpha}-1}(1+u)\right] & \text{for } \alpha \in (0,\infty)\setminus\{1\}; \\ (1+u)\ln 2 + u\ln u - (1+u)\ln(1+u) & \text{for } \alpha = 1; \\ \dfrac{1}{2}|1-u| & \text{for } \alpha = \infty. \end{cases}$$

It has been shown in [4] that this class provides the distances $[I_{\Psi_\alpha}(Q,P)]^{\min\left(\alpha,\frac{1}{\alpha}\right)}$ for $\alpha \in (0,\infty)$ and $\frac{1}{2}V(Q,P)$ for $\alpha = \infty$.

For f continuous convex on $[0,\infty)$ we obtain the $*$-*conjugate* function of f by

$$f^*(u) = uf\left(\frac{1}{u}\right), \quad u \in (0,\infty)$$

and

$$f^*(0) = \lim_{u\downarrow 0} f^*(u).$$

It is also known that if f is continuous convex on $[0,\infty)$ then so is f^*.

The following two theorems contain the most basic properties of f-divergences. For their proofs we refer the reader to Chapter 1 of [5] (see also [2]).

Theorem 1.1 (Uniqueness and Symmetry Theorem). *Let f, f_1 be continuous convex on $[0,\infty)$. We have*

$$I_{f_1}(Q,P) = I_f(Q,P),$$

for all $P, Q \in \mathcal{P}$ if and only if there exists a constant $c \in \mathbb{R}$ such that

$$f_1(u) = f(u) + c(u-1),$$

for any $u \in [0,\infty)$.

Theorem 1.2 (Range of Values Theorem). *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous convex function on $[0,\infty)$.*

For any $P, Q \in \mathcal{P}$, we have the double inequality

$$f(1) \le I_f(Q,P) \le f(0) + f^*(0). \tag{1.4}$$

(i) If $P = Q$, then the equality holds in the first part of (1.4).

If f is strictly convex at 1, then the equality holds in the first part of (1.4) if and only if $P = Q$;

(ii) If $Q \perp P$, then the equality holds in the second part of (1.4).

If $f(0) + f^*(0) < \infty$, then equality holds in the second part of (1.4) if and only if $Q \perp P$.

The following result is a refinement of the second inequality in Theorem 1.2 (see [2, Theorem 3]).

Theorem 1.3. *Let f be a continuous convex function on $[0,\infty)$ with $f(1)=0$ (f is normalised) and $f(0)+f^*(0)<\infty$. Then*

$$0 \le I_f(Q,P) \le \frac{1}{2}\left[f(0)+f^*(0)\right] V(Q,P) \tag{1.5}$$

for any Q, $P \in \mathcal{P}$.

For other inequalities for f-divergence see [6], [7]-[21].

2. Some basic properties

Let f be a continuous convex function on $[0,\infty)$ with $f(1)=0$ and $t\in[0,1]$. We define the *t-asymmetric divergence measure* $A_{f,t}$ by

$$A_{f,t}(Q,P,W) := \int_X f\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right] w(x)\,d\mu(x) \tag{2.1}$$

and the *t-symmetric divergence measure* $S_{f,t}$ by

$$S_{f,t}(Q,P,W) := \frac{1}{2}\left[A_{f,t}(Q,P,W)+A_{f,1-t}(Q,P,W)\right] \tag{2.2}$$

for any Q, P, $W \in \mathcal{P}$.

For $t=\frac{1}{2}$ we consider the *mid-point divergence measure* M_f by

$$\begin{aligned} M_f(Q,P,W) &:= \int_X f\left[\frac{q(x)+p(x)}{2w(x)}\right] w(x)\,d\mu(x) \\ &= A_{f,1/2}(Q,P,W) = S_{f,1/2}(Q,P,W), \end{aligned}$$

for any Q, P, $W \in \mathcal{P}$.

We can also consider the *integral divergence measure*

$$\begin{aligned} A_f(Q,P,W) &:= \int_0^1 A_{f,t}(Q,P,W)\,dt = \int_0^1 S_{f,t}(Q,P,W) \\ &= \int_X \left(\int_0^1 f\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right] dt\right) w(x)\,d\mu(x). \end{aligned}$$

The following result contains some basic facts concerning the divergence measures above:

Theorem 2.1. *Let f be a continuous convex function on $[0,\infty)$ with $f(1)=0$. Then for all Q, P, $W\in\mathcal{P}$ and $t\in[0,1]$*

$$0 \le A_{f,t}(Q,P,W) \le (1-t)\,I_f(Q,W)+tI_f(P,W) \tag{2.3}$$

and the mapping

$$\mathcal{P}\times\mathcal{P} \ni (Q,P) \mapsto A_{f,t}(Q,P,W) \in [0,\infty) \tag{2.4}$$

is convex as a function of two variables.

We have the inequalities

$$0 \le M_f(Q,P,W) \le S_{f,t}(Q,P,W) \le \frac{1}{2}\left[I_f(Q,W)+I_f(P,W)\right] \tag{2.5}$$

for all Q, P, $W\in\mathcal{P}$ and the mapping

$$\mathcal{P}\times\mathcal{P} \ni (Q,P) \mapsto S_{f,t}(Q,P,W) \in [0,\infty) \tag{2.6}$$

is convex as a function of two variables.

Proof. Let $t \in [0,1]$ and $Q, P, W \in \mathcal{P}$. We use Jensen's integral inequality to get

$$\begin{aligned} A_{f,t}(Q,P,W) &= \int_X f\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right] w(x)\,d\mu(x) \\ &\geq f\left(\int_X \left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right] w(x)\,d\mu(x)\right) \\ &= f\left(\int_X [(1-t)\,q(x)+tp(x)]\,d\mu(x)\right) \\ &= f\left((1-t)\int_X q(x)\,d\mu(x) + t\int_X p(x)\,d\mu(x)\right) = f(1) = 0. \end{aligned}$$

By the convexity of f we also have

$$\begin{aligned} A_{f,t}(Q,P,W) &= \int_X f\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right] w(x)\,d\mu(x) \\ &\leq (1-t)\int_X f\left[\frac{q(x)}{w(x)}\right] w(x)\,d\mu(x) + t\int_X f\left[\frac{p(x)}{w(x)}\right] w(x)\,d\mu(x) \\ &= (1-t)\,I_f(Q,W) + tI_f(P,W) \end{aligned}$$

for $t \in [0,1]$ and $Q, P, W \in \mathcal{P}$, and the inequality (2.3) is proved.

Let $\alpha, \beta \geq 0$ and such that $\alpha+\beta = 1$. If $(Q_1,P_1), (Q_2,P_2) \in \mathcal{P}\times\mathcal{P}$, then

$$\begin{aligned} & A_{f,t}(\alpha(Q_1,P_1,W) + \beta(Q_2,P_2,W)) \\ = \;& A_{f,t}((\alpha Q_1+\beta Q_2, \alpha P_1+\beta P_2, W)) \\ = \;& \int_X f\left[\frac{(1-t)(\alpha Q_1+\beta Q_2) + t(\alpha P_1+\beta P_2)}{w(x)}\right] w(x)\,d\mu(x) \\ = \;& \int_X f\left[\frac{\alpha[(1-t)\,Q_1+tP_1] + \beta[(1-t)\,Q_2+tP_2]}{w(x)}\right] w(x)\,d\mu(x) \\ \leq \;& \alpha\int_X f\left[\frac{(1-t)\,Q_1+tP_1}{w(x)}\right] w(x)\,d\mu(x) + \beta\int_X f\left[\frac{(1-t)\,Q_2+tP_2}{w(x)}\right] w(x)\,d\mu(x) \\ = \;& \alpha A_{f,t}(Q_1,P_1,W) + \beta A_{f,t}(Q_2,P_2,W), \end{aligned}$$

which proves the joint convexity of the mapping defined in (2.4).

Using the convexity of f we have

$$f\left(\frac{1}{2}\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)} + \frac{(1-t)\,p(x)+tq(x)}{w(x)}\right]\right) \leq \frac{1}{2}\left\{f\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right] + f\left[\frac{(1-t)\,p(x)+tq(x)}{w(x)}\right]\right\},$$

namely

$$f\left(\frac{q(x)+p(x)}{2w(x)}\right) \leq \frac{1}{2}\left\{f\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right] + f\left[\frac{(1-t)\,p(x)+tq(x)}{w(x)}\right]\right\}, \tag{2.7}$$

for $x \in X$.

By multiplying (2.7) with $w(x)$ and integrating over $\mu(x)$ we get the second inequality inequality in (2.5).

We have, by (2.3) that

$$\begin{aligned} S_{f,t}(Q,P,W) &= \frac{1}{2}\left[A_{f,t}(Q,P,W) + A_{f,1-t}(Q,P,W)\right] \\ &\leq \frac{1}{2}\left[(1-t)\,I_f(Q,W) + tI_f(P,W) + tI_f(Q,W) + (1-tI)_f(P,W)\right] \\ &= \frac{1}{2}\left[I_f(Q,W) + I_f(P,W)\right], \end{aligned}$$

which proves the third inequality in (2.5).

The convexity of the mapping defined by (2.6) follows by the same property of the mapping defined by (2.4). $\square$

Corollary 2.1. *Let f be a continuous convex function on $[0,\infty)$ with $f(1)=0$. Then for all Q, P, $W\in\mathcal{P}$ we have the inequalities*

$$0\leq M_f(Q,P,W)\leq A_f(Q,P,W)\leq\frac{1}{2}\left[I_f(Q,W)+I_f(P,W)\right]. \tag{2.8}$$

The mapping

$$\mathcal{P}\times\mathcal{P}\ni(Q,P)\mapsto A_f(Q,P,W)\in[0,\infty) \tag{2.9}$$

is convex as a function of two variables.

Proof. The inequality (2.8) follows by integrating over t in the inequality (2.5). Since the mapping

$$\mathcal{P}\times\mathcal{P}\ni(Q,P)\mapsto S_{f,t}(Q,P,W)\in[0,\infty)$$

is convex as a function of two variables for all $t\in[0,1]$, then it remains convex if one takes the integral over $t\in[0,1]$. □

The following reverses of the Hermite-Hadamard inequality hold:

Lemma 2.1 (Dragomir, 2002 [9] and [10]). *Let $h:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned}0&\leq\frac{1}{8}\left[h_+\left(\frac{a+b}{2}\right)-h_-\left(\frac{a+b}{2}\right)\right](b-a)\\&\leq\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(x)\,dx\\&\leq\frac{1}{8}\left[h_-(b)-h_+(a)\right](b-a)\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}0&\leq\frac{1}{8}\left[h_+\left(\frac{a+b}{2}\right)-h_-\left(\frac{a+b}{2}\right)\right](b-a)\\&\leq\frac{1}{b-a}\int_a^b h(x)\,dx-h\left(\frac{a+b}{2}\right)\\&\leq\frac{1}{8}\left[h_-(b)-h_+(a)\right](b-a).\end{aligned} \tag{2.11}$$

The constant $\frac{1}{8}$ is best possible in all inequalities.

We have the reverse inequalities:

Theorem 2.2. *Let f be a differentiable convex function on $[0,\infty)$ with $f(1)=0$. Then for all Q, P, $W\in\mathcal{P}$ we have*

$$0\leq A_f(Q,P,W)-M_f(Q,P,W)\leq\frac{1}{8}\Delta_{f'}(Q,P,W) \tag{2.12}$$

and

$$0\leq\frac{1}{2}\left[I_f(Q,W)+I_f(P,W)\right]-A_f(Q,P,W)\leq\frac{1}{8}\Delta_{f'}(Q,P,W) \tag{2.13}$$

where

$$\Delta_{f'}(Q,P,W):=\int_X\left[f'\left(\frac{q(x)}{w(x)}\right)-f'\left(\frac{p(x)}{w(x)}\right)\right](q(x)-p(x))\,d\mu(x). \tag{2.14}$$

Proof. Let Q, P, $W\in\mathcal{P}$. By the inequality (2.11) we have

$$\begin{aligned}0&\leq\int_0^1 f\left[\frac{(1-t)q(x)+tp(x)}{w(x)}\right]dt-f\left(\frac{q(x)+p(x)}{2w(x)}\right)\\&\leq\frac{1}{8}\left[f'\left(\frac{q(x)}{w(x)}\right)-f'\left(\frac{p(x)}{w(x)}\right)\right]\left(\frac{q(x)}{w(x)}-\frac{p(x)}{w(x)}\right).\end{aligned}$$

If we multiply this inequality by $w(x) \geq 0$ and integrate on X we get (2.12).

From (2.10) we also have

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[f\left(\frac{q(x)}{w(x)}\right)+f\left(\frac{p(x)}{w(x)}\right)\right]-\int_0^1 f\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right]dt \\ &\leq \frac{1}{8}\left[f'\left(\frac{q(x)}{w(x)}\right)-f'\left(\frac{p(x)}{w(x)}\right)\right]\left(\frac{q(x)}{w(x)}-\frac{p(x)}{w(x)}\right). \end{aligned}$$

If we multiply this inequality by $w(x) \geq 0$ and integrate on X we get (2.12). □

Corollary 2.2. *Let f be a differentiable convex function on $[0,\infty)$ with $f(1)=0$ and $Q, P, W \in \mathcal{P}$. If there exists $0<r<1<R<\infty$ such that the following condition holds*

$$r \leq \frac{q(x)}{w(x)}, \frac{p(x)}{w(x)} \leq R \text{ for } \mu\text{-a.e. } x \in X, \tag{2.15}$$

then

$$0 \leq A_f(Q,P,W) - M_f(Q,P,W) \leq \frac{1}{8}\left[f'(R)-f'(r)\right] d_1(Q,P) \tag{2.16}$$

and

$$0 \leq \frac{1}{2}\left[I_f(Q,W)+I_f(P,W)\right] - A_f(Q,P,W) \leq \frac{1}{8}\left[f'(R)-f'(r)\right] d_1(Q,P) \tag{2.17}$$

where

$$d_1(Q,P) := \int_X |q(x)-p(x)|\, d\mu(x).$$

Proof. Since f' is increasing on $[r,R]$, then

$$|f'(t)-f'(s)| \leq f'(R)-f'(r)$$

for all $t, s \in [r,R]$.

Therefore

$$\begin{aligned} \Delta_{f'}(Q,P,W) &:= \int_X \left[f'\left(\frac{q(x)}{w(x)}\right)-f'\left(\frac{p(x)}{w(x)}\right)\right](q(x)-p(x))\,d\mu(x) \\ &\leq \int_X \left|f'\left(\frac{q(x)}{w(x)}\right)-f'\left(\frac{p(x)}{w(x)}\right)\right| |q(x)-p(x)|\,d\mu(x) \\ &\leq \left[f'(R)-f'(r)\right]\int_X |q(x)-p(x)|\,d\mu(x) \\ &= \left[f'(R)-f'(r)\right] d_1(Q,P), \end{aligned}$$

which proves the desired inequalities (2.16) and (2.17). □

Corollary 2.3. *Let f be a twice differentiable convex function on $[0,\infty)$ with $f(1)=0$ and $Q, P, W \in \mathcal{P}$. If there exists $0<r<1<R<\infty$ such that the condition (2.15) holds and*

$$\|f''\|_{[r,R],\infty} := \sup_{t\in[r,R]} |f''(t)| < \infty, \tag{2.18}$$

then

$$0 \leq A_f(Q,P,W) - M_f(Q,P,W) \leq \frac{1}{8}\|f''\|_{[r,R],\infty}\, d_{\chi^2}(Q,P,W) \tag{2.19}$$

and

$$0 \leq \frac{1}{2}\left[I_f(Q,W)+I_f(P,W)\right] - A_f(Q,P,W) \leq \frac{1}{8}\|f''\|_{[r,R],\infty}\, d_{\chi^2}(Q,P,W), \tag{2.20}$$

where

$$d_{\chi^2}(Q,P,W) := \int_X \frac{(q(x)-p(x))^2}{w(x)}\,d\mu(x). \tag{2.21}$$

Proof. We have

$$\begin{aligned}\Delta_{f'}(Q,P,W) &:= \int_X \left[f'\left(\frac{q(x)}{w(x)}\right) - f'\left(\frac{p(x)}{w(x)}\right)\right](q(x)-p(x))\,d\mu(x)\\ &\le \int_X \left|f'\left(\frac{q(x)}{w(x)}\right) - f'\left(\frac{p(x)}{w(x)}\right)\right| |q(x)-p(x)|\,d\mu(x)\\ &\le \|f''\|_{[r,R],\infty} \int_X \left|\frac{q(x)}{w(x)} - \frac{p(x)}{w(x)}\right| |q(x)-p(x)|\,d\mu(x)\\ &= \|f''\|_{[r,R],\infty} \int_X \frac{(q(x)-p(x))^2}{w(x)}\,d\mu(x),\end{aligned}$$

which proves the desired results (2.19) and (2.20). □

3. Further results

We have the following result for convex functions that is of interest in itself as well:

Lemma 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, $a, b \in \mathring{I}$, the interior of I, with $a < b$ and $\nu \in [0,1]$. Then*

$$\begin{aligned}&\nu(1-\nu)(b-a)\left[f'_+((1-\nu)a+\nu b) - f'_-((1-\nu)a+\nu b)\right]\\ &\le (1-\nu)f(a) + \nu f(b) - f((1-\nu)a+\nu b)\\ &\le \nu(1-\nu)(b-a)\left[f'_-(b) - f'_+(a)\right].\end{aligned} \tag{3.1}$$

In particular, we have

$$\begin{aligned}\frac{1}{4}(b-a)\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] &\le \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\\ &\le \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right].\end{aligned} \tag{3.2}$$

The constant $\frac{1}{4}$ is best possible in both inequalities from (3.2).

Proof. The case $\nu = 0$ or $\nu = 1$ reduces to equality in (3.1).

Since f is convex on I it follows that the function is differentiable on $\mathring{I}$ except a countably number of points, the lateral derivatives $f'_\pm$ exists in each point of $\mathring{I}$, they are increasing on $\mathring{I}$ and $f'_- \le f'_+$ on $\mathring{I}$.

For any $x, y \in \mathring{I}$ we have for the Lebesgue integral

$$f(x) = f(y) + \int_y^x f'(s)\,ds = f(y) + (x-y)\int_0^1 f'((1-t)y+tx)\,dt. \tag{3.3}$$

Assume that $a < b$ and $\nu \in (0,1)$. By (3.3) we have

$$\begin{aligned}&f((1-\nu)a+\nu b)\\ &= f(a) + \nu(b-a)\int_0^1 f'((1-t)a + t((1-\nu)a+\nu b))\,dt\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&f((1-\nu)a+\nu b)\\ &= f(b) - (1-\nu)(b-a)\int_0^1 f'((1-t)b + t((1-\nu)a+\nu b))\,dt.\end{aligned} \tag{3.5}$$

If we multiply (3.4) by $1-\nu$, (3.4) by ν and add the obtained equalities, then we get

$$\begin{aligned} f\left((1-\nu)a+\nu b\right) &= (1-\nu) f(a) + \nu f(b) \\ &\quad + (1-\nu)\nu(b-a)\int_0^1 f'\left((1-t)a + t\left((1-\nu)a+\nu b\right)\right) dt \\ &\quad - (1-\nu)\nu(b-a)\int_0^1 f'\left((1-t)b + t\left((1-\nu)a+\nu b\right)\right) dt, \end{aligned}$$

which is equivalent to

$$(1-\nu) f(a) + \nu f(b) - f\left((1-\nu)a+\nu b\right) = (1-\nu)\nu(b-a) \tag{3.6}$$

$$\times \int_0^1 \left[f'\left((1-t)b + t\left((1-\nu)a+\nu b\right)\right) - f'\left((1-t)a + t\left((1-\nu)a+\nu b\right)\right)\right] dt. \tag{3.7}$$

That is an equality of interest in itself.

Since $a<b$ and $\nu \in (0,1)$, then $(1-\nu)a+\nu b \in (a,b)$ and

$$(1-t)a + t\left((1-\nu)a+\nu b\right) \in \left[a, (1-\nu)a+\nu b\right]$$

while

$$(1-t)b + t\left((1-\nu)a+\nu b\right) \in \left[(1-\nu)a+\nu b, b\right]$$

for any $t \in [0,1]$.

By the monotonicity of the derivative we have

$$f'_+\left((1-\nu)a+\nu b\right) \le f'\left((1-t)b + t\left((1-\nu)a+\nu b\right)\right) \le f'_-(b) \tag{3.8}$$

and

$$f'_+(a) \le f'\left((1-t)a + t\left((1-\nu)a+\nu b\right)\right) \le f'_-\left((1-\nu)a+\nu b\right) \tag{3.9}$$

for any $t \in [0,1]$.

By integrating the inequalities (3.8) and (3.9) we get

$$f'_+\left((1-\nu)a+\nu b\right) \le \int_0^1 f'\left((1-t)b + t\left((1-\nu)a+\nu b\right)\right) dt \le f'_-(b)$$

and

$$f'_+(a) \le \int_0^1 f'\left((1-t)a + t\left((1-\nu)a+\nu b\right)\right) dt \le f'_-\left((1-\nu)a+\nu b\right),$$

which implies that

$$\begin{aligned} f'_+\left((1-\nu)a+\nu b\right) - f'_-\left((1-\nu)a+\nu b\right) &\le \int_0^1 f'\left((1-t)b + t\left((1-\nu)a+\nu b\right)\right) dt \\ - \int_0^1 f'\left((1-t)a + t\left((1-\nu)a+\nu b\right)\right) dt &\le f'_-(b) - f'_+(a). \end{aligned}$$

Making use of the equality (3.6) we the obtain the desired result (3.1).

If we consider the convex function $f:[a,b]\to\mathbb{R}$, $f(x) = \left|x - \frac{a+b}{2}\right|$, then we have $f'_+\left(\frac{a+b}{2}\right) = 1$, $f'_-\left(\frac{a+b}{2}\right) = -1$ and by replacing in (3.2) we get in all terms the same quantity $\frac{1}{2}(b-a)$ which show that the constant $\frac{1}{4}$ is best possible in both inequalities from (3.2). □

Corollary 3.1. *If the function $f: I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $a,b \in \mathring{I}$ and $\nu \in [0,1]$ we have*

$$\begin{aligned} 0 &\le (1-\nu) f(a) + \nu f(b) - f\left((1-\nu)a+\nu b\right) \\ &\le \nu(1-\nu)(b-a)\left[f'(b) - f'(a)\right]. \end{aligned} \tag{3.10}$$

Proof. If $a < b$, then the inequality (3.10) follows by (3.1). If $b < a$, then by (3.1) we get

$$\begin{aligned} 0 &\le (1-\nu) f(b) + \nu f(a) - f((1-\nu) b + \nu a) \\ &\le \nu (1-\nu)(b-a)[f'(b) - f'(a)] \end{aligned} \tag{3.11}$$

for any $\nu \in [0,1]$. If we replace ν by $1-\nu$ in (3.11), then we get (3.10). □

We can prove now the following reverse of the second inequality in (2.3) and the first inequality in (2.5).

Theorem 3.1. *Let f be a differentiable convex function on $[0,\infty)$ with $f(1) = 0$. Then for all $Q, P, W \in \mathcal{P}$ and $t \in [0,1]$ we have*

$$\begin{aligned} 0 &\le (1-t) I_f(Q,W) + t I_f(P,W) - A_{f,t}(Q,P,W) \\ &\le t(1-t)\Delta_{f'}(Q,P,W) \end{aligned} \tag{3.12}$$

and

$$0 \le S_{f,t}(Q,P,W) - M_f(Q,P,W) \le \frac{1}{2}\left(t - \frac{1}{2}\right)\Delta_{f',t}(Q,P,W), \tag{3.13}$$

where

$$\begin{aligned} \Delta_{f',t}(Q,P,W) &= \int_X (q(x) - p(x)) \\ \times \left[f'\left((1-t)\frac{p(x)}{w(x)} + t\frac{q(x)}{w(x)}\right) \right. &- \left. f'\left((1-t)\frac{q(x)}{w(x)} + t\frac{p(x)}{w(x)}\right)\right] d\mu(x). \end{aligned}$$

Proof. From the inequality (3.12) we get

$$\begin{aligned} 0 &\le (1-t) f\left(\frac{q(x)}{w(x)}\right) + t f\left(\frac{p(x)}{w(x)}\right) - f\left((1-t)\frac{q(x)}{w(x)} + t\frac{p(x)}{w(x)}\right) \\ &\le t(1-t)\left[f'\left(\frac{q(x)}{w(x)}\right) - f'\left(\frac{p(x)}{w(x)}\right)\right]\left(\frac{q(x)}{w(x)} - \frac{p(x)}{w(x)}\right). \end{aligned} \tag{3.14}$$

If we multiply this inequality by $w(x) \ge 0$ and integrate on X we get (3.12).

For any $x, y \in \mathring{I}$ we have

$$0 \le \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \le \frac{1}{4}(x-y)[f'(x) - f'(y)]. \tag{3.15}$$

If in this inequality we take $x = (1-t)a + tb$, $y = (1-t)b + ta$ with $a, b \in \mathring{I}$ and $t \in [0,1]$, then we get

$$\begin{aligned} 0 &\le \frac{f((1-t)a + tb) + f((1-t)b + ta)}{2} - f\left(\frac{a+b}{2}\right) \\ &\le \frac{1}{4}((1-t)a + tb - (1-t)b - ta) \\ &\times [f'((1-t)a + tb) - f'((1-t)b + ta)] \\ &= \frac{1}{2}\left(t - \frac{1}{2}\right)(b-a)[f'((1-t)a + tb) - f'((1-t)b + ta)]. \end{aligned} \tag{3.16}$$

From this inequality we have

$$\begin{aligned} 0 &\le \frac{1}{2}\left[f\left((1-t)\frac{q(x)}{w(x)} + t\frac{p(x)}{w(x)}\right) + f\left((1-t)\frac{p(x)}{w(x)} + t\frac{q(x)}{w(x)}\right)\right] \\ &- f\left(\frac{q(x) + p(x)}{2w(x)}\right) \\ &\le \frac{1}{2}\left(t - \frac{1}{2}\right)\left(\frac{q(x)}{w(x)} - \frac{p(x)}{w(x)}\right) \\ &\times \left[f'\left((1-t)\frac{p(x)}{w(x)} + t\frac{q(x)}{w(x)}\right) - f'\left((1-t)\frac{q(x)}{w(x)} + t\frac{p(x)}{w(x)}\right)\right]. \end{aligned}$$

If we multiply this inequality by $w(x) \ge 0$ and integrate on X we get (3.12). □

Corollary 3.2. *Let f be a differentiable convex function on $[0,\infty)$ with $f(1)=0$ and $Q,\ P,\ W\in\mathcal{P}$. If there exists $0<r<1<R<\infty$ such that the condition (2.15) holds, then*

$$\begin{aligned}0 &\le (1-t)\, I_f(Q,W) + tI_f(P,W) - A_{f,t}(Q,P,W) \\ &\le t(1-t)\left[f'(R)-f'(r)\right] d_1(Q,P)\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}0 &\le S_{f,t}(Q,P,W) - M_f(Q,P,W) \\ &\le \frac{1}{2}\left|t-\frac{1}{2}\right|\left[f'(R)-f'(r)\right] d_1(Q,P)\end{aligned} \tag{3.18}$$

Proof. The inequality (3.17) is obvious. For (3.18), we have

$$\begin{aligned}\frac{1}{2}\left(t-\frac{1}{2}\right)\Delta_{f',t}(Q,P,W) &= \frac{1}{2}\left|t-\frac{1}{2}\right|\left|\Delta_{f',t}(Q,P,W)\right| \\ &\le \frac{1}{2}\left|t-\frac{1}{2}\right|\int_X |q(x)-p(x)| \\ \times\left|f'\left((1-t)\frac{p(x)}{w(x)}+t\frac{q(x)}{w(x)}\right)-f'\left((1-t)\frac{q(x)}{w(x)}+t\frac{p(x)}{w(x)}\right)\right| d\mu(x) \\ &\le \frac{1}{2}\left[f'(R)-f'(r)\right]\left|t-\frac{1}{2}\right|\int_X |q(x)-p(x)|\, d\mu(x) \\ &= \frac{1}{2}\left|t-\frac{1}{2}\right|\left[f'(R)-f'(r)\right] d_1(Q,P).\end{aligned}$$

□

Corollary 3.3. *Let f be a twice differentiable convex function on $[0,\infty)$ with $f(1)=0$ and $Q,\ P,\ W\in\mathcal{P}$. If there exists $0<r<1<R<\infty$ such that the conditions (2.15) and (2.18) hold, then*

$$\begin{aligned}0 &\le (1-t)\, I_f(Q,W) + tI_f(P,W) - A_{f,t}(Q,P,W) \\ &\le t(1-t)\left\|f''\right\|_{[r,R],\infty} d_{\chi^2}(Q,P,W)\end{aligned} \tag{3.19}$$

and

$$0 \le S_{f,t}(Q,P,W) - M_f(Q,P,W) \le \left|t-\frac{1}{2}\right|^2 \left\|f''\right\|_{[r,R],\infty} d_{\chi^2}(Q,P,W). \tag{3.20}$$

Proof. We have

$$\begin{aligned}\frac{1}{2}\left(t-\frac{1}{2}\right)\Delta_{f',t}(Q,P,W) \quad &\le \quad \frac{1}{2}\left|t-\frac{1}{2}\right|\int_X |q(x)-p(x)| \\ &\times \quad \left|f'\left((1-t)\frac{p(x)}{w(x)}+t\frac{q(x)}{w(x)}\right)-f'\left((1-t)\frac{q(x)}{w(x)}+t\frac{p(x)}{w(x)}\right)\right| d\mu(x)\end{aligned}$$

$$\begin{aligned}&\le \frac{1}{2}\left|t-\frac{1}{2}\right|\left\|f''\right\|_{[r,R],\infty}\int_X |q(x)-p(x)| \\ &\times \left|(1-t)\frac{p(x)}{w(x)}+t\frac{q(x)}{w(x)}-(1-t)\frac{q(x)}{w(x)}-t\frac{p(x)}{w(x)}\right| d\mu(x) \\ &= \left|t-\frac{1}{2}\right|^2\left\|f''\right\|_{[r,R],\infty}\int_X |q(x)-p(x)|\,\frac{|q(x)-p(x)|}{w(x)}d\mu(x) \\ &= \left|t-\frac{1}{2}\right|^2\left\|f''\right\|_{[r,R],\infty} d_{\chi^2}(Q,P,W),\end{aligned}$$

which proves (3.20). □

4. Examples

Consider the *dichotomy class* generated by the function $f_\alpha : [0,\infty) \to \mathbb{R}$ that is given by

$$f_\alpha(u) = \begin{cases} u - 1 - \ln u & \text{for } \alpha = 0; \\ \dfrac{1}{\alpha(1-\alpha)}\left[\alpha u + 1 - \alpha - u^\alpha\right] & \text{for } \alpha \in \mathbb{R}\setminus\{0,1\}; \\ 1 - u + u\ln u & \text{for } \alpha = 1. \end{cases}$$

We have

$$\begin{aligned} A_{f_\alpha,t}(Q,P,W) &= \int_X f\left[\frac{(1-t)q(x)+tp(x)}{w(x)}\right] w(x)\,d\mu(x) \\ &= \begin{cases} -\int_X w(x)\ln\left[\dfrac{(1-t)q(x)+tp(x)}{w(x)}\right] d\mu(x) & \text{for } \alpha = 0; \\ \dfrac{1}{\alpha(1-\alpha)}\left[1 - \int_X \left[(1-t)q(x)+tp(x)\right]^\alpha w^{1-\alpha}(x)\,d\mu(x)\right] & \text{for } \alpha \in \mathbb{R}\setminus\{0,1\}; \\ \int_X \left[(1-t)q(x)+tp(x)\right]\ln\left[\dfrac{(1-t)q(x)+tp(x)}{w(x)}\right] d\mu(x) & \text{for } \alpha = 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} M_{f_\alpha}(Q,P,W) &= \int_X f\left[\frac{q(x)+p(x)}{2w(x)}\right] w(x)\,d\mu(x) \\ &= \begin{cases} -\int_X w(x)\ln\left[\dfrac{q(x)+p(x)}{2w(x)}\right] d\mu(x) & \text{for } \alpha = 0; \\ \dfrac{1}{\alpha(1-\alpha)}\left[1 - \int_X \left[\dfrac{q(x)+p(x)}{2}\right]^\alpha w^{1-\alpha}(x)\,d\mu(x)\right] & \text{for } \alpha \in \mathbb{R}\setminus\{0,1\}; \\ \int_X \left[\dfrac{q(x)+p(x)}{2}\right]\ln\left[\dfrac{q(x)+p(x)}{2w(x)}\right] d\mu(x) & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

Let us recall the following special means:

a) The *arithmetic mean*

$$A(a,b) := \frac{a+b}{2}, \quad a,b > 0,$$

b) The *geometric mean*

$$G(a,b) := \sqrt{ab}; \quad a,b \geq 0,$$

c) The *harmonic mean*

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}; \quad a,b > 0,$$

d) The *identric mean*

$$I(a,b) := \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } b \neq a \\ a & \text{if } b = a \end{cases}; \ a,b > 0$$

e) The *logarithmic mean*

$$L(a,b) := \begin{cases} \dfrac{b-a}{\ln b - \ln a} & \text{if} \quad b \neq a \\ a & \text{if} \quad b = a \end{cases} ; \quad a, b > 0$$

f) The *p-logarithmic mean*

$$L_p(a,b) := \begin{cases} \left(\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\dfrac{1}{p}} & \text{if} \quad b \neq a, \;\; p \in \mathbb{R} \backslash \{-1, 0\} \\ a & \text{if} \quad b = a \end{cases} ; \quad a, b > 0.$$

If we put $L_0(a,b) := I(a,b)$ and $L_{-1}(a,b) := L(a,b)$, then it is well known that the function $\mathbb{R} \ni p \mapsto L_p(a,b)$ is monotonic increasing on $\mathbb{R}$.

We observe that for $p \in \mathbb{R} \backslash \{-1, 0\}$ we have

$$\int_0^1 [(1-t)a + tb]^p \, dt = L_p^p(a,b), \quad \int_0^1 [(1-t)a + tb]^{-1} \, dt = L^{-1}(a,b)$$

and

$$\int_0^1 \ln [(1-t)a + tb] \, dt = \ln I(a,b).$$

We also have

$$\begin{aligned} &\int_0^1 [(1-t)a + tb] \ln [(1-t)a + tb] \, dt \\ &= \frac{1}{b-a} \int_a^b t \ln t dt = \frac{1}{2} \frac{1}{b-a} \int_a^b \ln t d\left(t^2\right) \\ &= \frac{1}{2} \frac{1}{b-a} \left[b^2 \ln b - a^2 \ln a - \frac{b^2 - a^2}{2} \right] \\ &= \frac{1}{2} \frac{1}{b-a} \left[\frac{b^2 \ln b^2 - a^2 \ln a^2}{2} - \frac{b^2 - a^2}{2} \right] \\ &= \frac{1}{2} \frac{1}{b-a} \frac{b^2 - a^2}{2} \left[\frac{b^2 \ln b^2 - a^2 \ln a^2}{b^2 - a^2} - 1 \right] \\ &= \frac{1}{4} (b+a) \ln I\left(a^2, b^2\right) = \frac{1}{2} A(a,b) \ln I\left(a^2, b^2\right). \end{aligned}$$

Therefore

$$\begin{aligned} A_{f_\alpha}(Q,P,W) &:= \int_0^1 A_{f_\alpha,t}(Q,P,W) \, dt \\ &= \int_X \left(\int_0^1 f\left[\frac{(1-t)\,q(x) + tp(x)}{w(x)} \right] dt \right) w(x) \, d\mu(x) \end{aligned}$$

$$
= \begin{cases} -\int_X \left(\int_0^1 \ln\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right]dt\right)w(x)\,d\mu(x) & \text{for } \alpha=0; \\ \frac{1}{\alpha(1-\alpha)}\left[1-\int_X\left(\int_0^1\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right]^{\alpha}dt\right)w(x)\,d\mu(x)\right] & \text{for } \alpha\in\mathbb{R}\backslash\{0,1\}; \\ \int_X\int_0^1\left(\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right]\ln\left[\frac{(1-t)\,q(x)+tp(x)}{w(x)}\right]dt\right)w(x)\,d\mu(x) & \text{for } \alpha=1 \end{cases}
$$

$$
= \begin{cases} -\int_X \ln I\left(\frac{q(x)}{w(x)},\frac{p(x)}{w(x)}\right)w(x)\,d\mu(x) & \text{for } \alpha=0; \\ \frac{1}{\alpha(1-\alpha)}\left[1-\int_X L_\alpha^\alpha\left(\frac{q(x)}{w(x)},\frac{p(x)}{w(x)}\right)w(x)\,d\mu(x)\right] & \text{for } \alpha\in\mathbb{R}\backslash\{0,1\}; \\ \frac{1}{2}\int_X A\left(\frac{q(x)}{w(x)},\frac{p(x)}{w(x)}\right)\ln I\left(\left(\frac{q(x)}{w(x)}\right)^2,\left(\frac{p(x)}{w(x)}\right)^2\right)w(x)\,d\mu(x) & \text{for } \alpha=1. \end{cases}
$$

According to Corollary 2.1 we have

$$
0\le M_{f_\alpha}(Q,P,W)\le A_{f_\alpha}(Q,P,W)\le\frac{1}{2}\left[I_{f_\alpha}(Q,W)+I_{f_\alpha}(P,W)\right] \tag{4.1}
$$

and the mapping

$$
\mathcal{P}\times\mathcal{P}\ni(Q,P)\mapsto A_{f_\alpha}(Q,P,W)\in[0,\infty) \tag{4.2}
$$

is convex.

Observe also that

$$
f_\alpha'(u)=\begin{cases} 1-\frac{1}{u} & \text{for } \alpha=0; \\ \frac{1}{1-\alpha}\left(1-u^{\alpha-1}\right) & \text{for } \alpha\in\mathbb{R}\backslash\{0,1\}; \\ \ln u & \text{for } \alpha=1, \end{cases}
$$

which implies that

$$
\begin{aligned}
\Delta_{f_\alpha'}(Q,P,W) &:= \int_X\left[f_\alpha'\left(\frac{q(x)}{w(x)}\right)-f_\alpha'\left(\frac{p(x)}{w(x)}\right)\right](q(x)-p(x))\,d\mu(x) \\
&= \begin{cases} \int_X\frac{(q(x)-p(x))^2}{p(x)\,q(x)}w(x)\,d\mu(x) & \text{for } \alpha=0; \\ \frac{1}{\alpha-1}\int_X\frac{q^{\alpha-1}(x)-p^{\alpha-1}(x)}{w^\alpha(x)}(q(x)-p(x))\,d\mu(x) & \text{for } \alpha\in\mathbb{R}\backslash\{0,1\}; \\ \int_X(q(x)-p(x))\ln\left(\frac{q(x)}{p(x)}\right)d\mu(x) & \text{for } \alpha=1. \end{cases}
\end{aligned}
$$

For all $Q,P,W\in\mathcal{P}$ we have by Theorem 2.2 that

$$
0\le A_{f_\alpha}(Q,P,W)-Mf_\alpha(Q,P,W)\le\frac{1}{8}\Delta_{f_\alpha'}(Q,P,W) \tag{4.3}
$$

and

$$
0\le\frac{1}{2}\left[I_{f_\alpha}(Q,W)+I_{f_\alpha}(P,W)\right]-A_{f_\alpha}(Q,P,W)\le\frac{1}{8}\Delta_{f_\alpha'}(Q,P,W). \tag{4.4}
$$

If there exists $0<r<1<R<\infty$ such that the following condition holds

$$
r\le\frac{q(x)}{w(x)},\frac{p(x)}{w(x)}\le R\text{ for }\mu\text{-a.e. }x\in X, \tag{(r,R)}
$$

then by Corollary 2.2

$$
\begin{aligned}
0 &\leq A_{f_\alpha}(Q,P,W) - M_{f_\alpha}(Q,P,W) && (4.5)\\
&\leq \frac{1}{8} d_1(Q,P) \begin{cases} \dfrac{R-r}{rR} & \text{for } \alpha = 0; \\[2ex] \dfrac{R^{\alpha-1} - r^{\alpha-1}}{\alpha - 1} & \text{for } \alpha \in \mathbb{R}\setminus\{0,1\}; \\[2ex] \ln\left(\dfrac{R}{r}\right) & \text{for } \alpha = 1 \end{cases} && (4.6)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{2}\left[I_f(Q,W) + I_f(P,W)\right] - A_f(Q,P,W) && (4.7)\\
&\leq \frac{1}{8} d_1(Q,P) \begin{cases} \dfrac{R-r}{rR} & \text{for } \alpha = 0; \\[2ex] \dfrac{R^{\alpha-1} - r^{\alpha-1}}{\alpha - 1} & \text{for } \alpha \in \mathbb{R}\setminus\{0,1\}; \\[2ex] \ln\left(\dfrac{R}{r}\right) & \text{for } \alpha = 1. \end{cases} && (4.8)
\end{aligned}
$$

Further, since

$$
f_\alpha''(u) = \begin{cases} \dfrac{1}{u^2} & \text{for } \alpha = 0; \\[2ex] u^{\alpha-2} & \text{for } \alpha \in \mathbb{R}\setminus\{0,1\}; \\[2ex] \dfrac{1}{u} & \text{for } \alpha = 1, \end{cases}
$$

hence by Corollary 2.3 we have

$$
\begin{aligned}
0 &\leq A_f(Q,P,W) - M_f(Q,P,W) && (4.9)\\
&\leq \frac{1}{8} d_{\chi^2}(Q,P,W) \begin{cases} \dfrac{1}{r^2} & \text{for } \alpha = 0; \\[2ex] R^{\alpha-2} & \text{for } \alpha \geq 2; \\[2ex] r^{\alpha-2} & \text{for } \alpha < 2,\ \alpha \in \mathbb{R}\setminus\{0,1\}; \\[2ex] \dfrac{1}{r} & \text{for } \alpha = 1, \end{cases} && (4.10)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{2}\left[I_f(Q,W) + I_f(P,W)\right] - A_f(Q,P,W) && (4.11)\\
&\leq \frac{1}{8} d_{\chi^2}(Q,P,W) \begin{cases} \dfrac{1}{r^2} & \text{for } \alpha = 0; \\[2ex] R^{\alpha-2} & \text{for } \alpha \geq 2; \\[2ex] r^{\alpha-2} & \text{for } \alpha < 2,\ \alpha \in \mathbb{R}\setminus\{0,1\}; \\[2ex] \dfrac{1}{r} & \text{for } \alpha = 1. \end{cases} && (4.12)
\end{aligned}
$$

The interested reader may apply the above general results for other particular divergences of interest generated by the convex functions provided in the introduction. We omit the details.

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On the Geometric and Physical Properties of Conformable Derivative

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Abstract

In this article, we explore the advantages geometric and physical implications of the conformable derivative. One of the key benefits of the conformable derivative is its ability to approximate the tangent at points where the classical tangent is not readily available. By employing conformable derivatives, alternative tangents can be created to overcome this limitation. Thanks to these alternative (conformable) tangents, physical interpretation can be made with alternative velocity vectors. Furthermore, the conformable derivative proves to be valuable in situations where the tangent plane cannot be defined. It enables the creation of alternative tangent planes, offering a solution in cases where the traditional approach falls short. Geometrically speaking, the conformable derivative carries significant meaning. It provides insights into the local behavior of a function and its relationship with nearby points. By understanding the conformable derivative, we gain a deeper understanding of how a function evolves and changes within its domain. A several examples are presented in the article to better understand the article and visualize the concepts discussed. These examples are accompanied by visual representations generated using the Mathematica program, aiding in a clearer understanding of the proposed ideas. By combining theoretical explanations, practical examples, and visualizations, this article aims to provide a comprehensive exploration of the advantages and geometric and physical implications of the conformable derivative.

Keywords: *Conformable derivative, Curvatures, Frenet frame, Surface*

AMS Subject Classification (2020): *53A04; 26A33*

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1. Introduction

Curves and surfaces hold great significance within the realm of differential geometry. When studying curves, one of the fundamental concepts is the tangent. The tangent vector of a curve plays a crucial role in analyzing the curve's behavior, such as exploring its Frenet frame or determining its curvatures. The construction of the Frenet

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frame heavily relies on the tangent vector of the curve. Similarly, the curvatures of the curve are computed based on its tangent. In the context of surfaces, tangent vectors and tangent planes assume a similar significance as the tangent vector does for curves. Tangent vectors and tangent planes are essential when investigating various surface concepts. These concepts include the surface's normal, first fundamental form, second fundamental form, and many others. Tangent vectors and tangent planes provide vital information for understanding the geometry and properties of surfaces. However, it becomes challenging to examine points where the concept of a tangent, either for curves or surfaces, does not exist. At such points, where the tangent is undefined, it becomes impossible to apply conventional methods that rely on tangent-based calculations and analyses. These points pose limitations in terms of exploring the local behavior and properties of curves and surfaces. Therefore, the existence and availability of tangent vectors and tangent planes are fundamental for the comprehensive study and analysis of curves and surfaces within the field of differential geometry. They serve as indispensable tools for understanding the geometry and various concepts associated with these mathematical objects.

The concept of the local conformable derivative and integral, initially introduced in 2014 by Khalil et al., has garnered significant attention from scientists and has been the subject of numerous publications. This novel definition incorporates a limit form similar to the classical derivative. Notably, the conformable derivative exhibits essential properties such as fractional linearity, the product rule, the quotient rule, Rolle's theorem, and the mean value theorem [1]. The subsequent development of this theory by Abdeljawad further enriched its applications. He introduced definitions for the left and right conformable derivatives, formulated higher-order conformable integral definitions for $\alpha > 1$, established conformable versions of the Gronwall inequality, chain rule, and partial integration formulas for congruent fractional derivatives. Additionally, power series expansions and Laplace transform techniques were extended to the conformable derivative framework [2]. The conformable derivative has found widespread use in various disciplines, with a particular emphasis on applied sciences [3–5] and physics [6–8]. Its application has proven to be valuable in solving problems and addressing phenomena in these fields. Researchers have leveraged the conformable derivative to gain deeper insights into complex systems, making it a powerful tool for analysis and modeling. The versatility and effectiveness of the conformable derivative have contributed to its growing popularity and adoption across different scientific domains. Its utilization in applied sciences and physics reflects its capability to capture the intricate dynamics and behaviors of real-world phenomena. As the research continues to advance, the conformable derivative is expected to continue playing a pivotal role in expanding our understanding of various disciplines.

The theory of curves can be described as the study of the motion of a point in a plane or space using the techniques of linear algebra and calculus. Considering the adventure of the literature in the last ten years, it is observed that fractional calculus is started to be used for curves and surfaces in differential geometry. T. Yajima and K. Kamasaki are made the first study on this subject by examining surfaces with fractional calculus [9]. Later, T. Yajima et al. are obtained Frenet formulas using fractional derivatives [10]. In another study, K.A. Lazopoulos and A.K. Lazopoulos are studied fractional differentiable manifolds [11]. M.E. Aydın et al. are studied plane curves in equiaffine geometry in fractional order [12]. U. Gozutok et al. are analyzed the basic concepts of curves and Frenet frame in fractional order with the help of conformable local fractional derivative [13]. On the other hand A. Has and B. Yılmaz are investigated some special curves and curve pairs in fractional order with the help of conformable Frenet frame [14, 15]. In addition, electromagnetic fields and magnetic curves are investigated under conformable derivative by A. Has and B. Yılmaz [16–18]. There are many more studies on this topic [19–21].

In this study, our objective is to present a geometric interpretation of conformable derivatives and highlight their advantages. We begin by delving into the concept of tangent vectors for curves. At points where the classical tangent does not exist, we introduce an alternative tangent that is defined using conformable derivatives. This allows us to establish a comprehensive understanding of the curve's behavior in those critical regions. Expanding upon this idea, we extend our investigation to surfaces. Similar to curves, we encounter points where the tangent vectors of the surface are undefined. To overcome this challenge, we employ conformable derivatives to generate alternative tangent vectors. These vectors collectively contribute to the formation of an alternative tangent plane that stretches across the surface at the respective point. Through this approach, we are able to explore the local behavior of surfaces in a way that would not be possible solely with classical tangents. Lastly, we aim to provide a geometric meaning to the conformable derivative. By studying its properties and implications within the context of curves and surfaces, we aim to shed light on its geometrical significance. This deeper understanding will allow us to grasp the underlying geometric principles that govern the behavior of functions and objects under conformable differentiation. Through this study, we strive to elucidate the geometric interpretation of conformable derivatives, showcase their advantages, and establish their relevance in various mathematical contexts.

2. Basics definitions and theorems in conformable calculus and conformable differential geometry

Given $s \mapsto x(s) \in \mathbb{E}^3$, $s \in I \subset \mathbb{R}$, the *conformable derivative* of x at s is defined by [1]

$$D_\alpha(x)(s) = \lim_{\varepsilon \to 0} \frac{x(s + \varepsilon s^{1-\alpha}) - x(s)}{\varepsilon}.$$

Let $Dx(s) = dx(s)/ds$. We then notice

$$D_\alpha x(s) = s^{1-\alpha} dx(s)/ds.$$

Denote by $D_\alpha x(s)$ the α-th order conformable derivative of $x(s)$ for each $s > 0$, $0 < \alpha < 1$.

It can be said that the conformable derivative provides some properties such as linearity, Leibniz rule and chain rule as in the classical derivative as follows

1. $D_\alpha(ax + by)(s) = aD_\alpha(x)(s) + bD_\alpha(y)(s),$ for all $a, b \in \mathbb{R}$,
2. $D_\alpha(s^p) = ps^{p-\alpha}$ for all $p \in \mathbb{R}$,
3. $D_\alpha(\lambda) = 0$, for all constant functions $x(s) = \lambda$,
4. $D_\alpha(xy)(s) = x(s)D_\alpha y(s) + y(s)D_\alpha x(s)$,
5. $D_\alpha(\frac{x}{y})(s) = \frac{x(s)D_\alpha y(s) - y(s)D_\alpha x(s)}{y^2(s)}$,
6. $D_\alpha(y \circ x)(s) = x(s)^{\alpha-1} D_\alpha x(s) D_\alpha y(x(s))$

where x, y be α-differentiable for each $s > 0$ and $0 < \alpha < 1$ [1].

The definition of the conformable integral is given as the inverse operator of the conformable derivative. The conformable integral of the function $x(s)$ is defined by [1]

$$I_\alpha^a f(t) = I_1^a(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx.$$

The effect of conformable analysis on vector-valued functions is investigated, and the limit and derivative of vector-valued functions also are investigated. In the following theorem, the conformable derivative of vector-valued functions is given.

Theorem 2.1. *Let x be a vector-valued function with n variables, and let x be a vector-valued function $x(s_1, ..., s_n) = (x_1(s_1, ..., s_n), ..., x_m(s_1, ..., s_n))$. So x is $\alpha-$differentiable at $t = (t_1, ..., t_n) \in \mathbb{R}$, for all $t_i > 0$ if and only if each x_i is, and [22]*

$$D_\alpha x(t) = (D_\alpha x_1(t), ..., D_\alpha x_m(t)).$$

Definition 2.1. Let $\mathbf{x} = \mathbf{x}(s)$ be a regular unit speed conformable curve in the Euclidean $3-$space where s measures its arc length. Also, let $\mathbf{t} = D^\alpha(\mathbf{x})(s)s^{\alpha-1}$ be its unit tangent vector, $\mathbf{n} = \frac{D^\alpha(\mathbf{t})(s)}{\|D^\alpha(\mathbf{t})(s)\|}$ be its principal normal vector and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ be its binormal vector. The triple $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the conformable Frenet frame of the curve x. Then the conformable Frenet formula of the curve is given by

$$\begin{pmatrix} D^\alpha(\mathbf{t})(s) \\ D^\alpha(\mathbf{n})(s) \\ D^\alpha(\mathbf{b})(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \tau_\alpha(s) \\ 0 & -\tau_\alpha(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} \tag{2.1}$$

where $\kappa_\alpha(s) = \|D^\alpha(\mathbf{t})(s)\|$ and $\tau_\alpha(s) = \langle D^\alpha(\mathbf{n})(s),\ \mathbf{b}\rangle$ are curvature and torsion of x, respectively (see details [13]).

Conclusion 1. *Let $\mathbf{x} = \mathbf{x}(s)$ be a regular unit speed conformable curve in the Euclidean $3-$space where s measures its arc length. The following relation exists between the curvature and torsion of the curve $\mathbf{x}$ and the conformable curvature and torsion [15]*

$$\kappa_\alpha = s^{1-\alpha}\kappa, \tag{2.2}$$
$$\tau_\alpha = s^{1-\alpha}\tau. \tag{2.3}$$

Conclusion 2. *Let $\mathbf{x} = \mathbf{x}(s)$ be a regular unit speed conformable curve where s measures its arc length. As can be seen from equation (2.1), when x is a unit speed curve, the conformable derivative has no effect on the Frenet vectors, so the Frenet vectors do not undergo any change. However, considering equations (2.2) and (2.3), the curvature and torsion of the curve $\mathbf{x}$ has changed under the conformable derivative [15].*

In this section, basic definitions and theorems of $\mathcal{C}_\alpha-$surfaces will be given. The concepts in this section are studied by A. Has and B. Yılmaz [23].

Definition 2.2. A subset $\mathcal{M} \subset \mathbb{R}^3$ is called a $\mathcal{C}_\alpha-$regular surface if for each point $p \in \mathcal{M}$, there exists a neighborhood V of $p \in \mathbb{R}^3$ and a map $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ of an open set $U \subset \mathbb{R}^2$ onto V intersection $\mathcal{M}$ such that
i. $\varphi : U \to V \cap \mathcal{M}$ is a homeomorphism,
ii. φ is conformable differentiable
iii. Each map $\varphi : U \to \mathcal{M}$ is a conformable regular patch.

Definition 2.3. Let $\mathcal{M}$ be the $\mathcal{C}_\alpha-$surface is given by the parameterization $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. In this case, the vectors $D_u^\alpha \varphi$ and $D_v^\alpha \varphi$ are the $\mathcal{C}_\alpha-$tangent vector of the $\mathcal{C}_\alpha-$surface. That is, they are $\mathcal{C}_\alpha-$tangent to the $\mathcal{C}_\alpha-$surface at point $P \in \mathcal{M}$. Thus, the $\mathcal{C}_\alpha-$plane span by the vectors $D_u^\alpha \varphi$ and $D_v^\alpha \varphi$ is called the $\mathcal{C}_\alpha-$tangent plane of the $\mathcal{C}_\alpha-$surface. Also, the space of $\mathcal{C}_\alpha-$tangent vectors we called $\mathcal{C}_\alpha-$tangent space and is denoted by $T_p^\alpha \mathcal{M}$.

3. Geometric meaning of the local conformable derivative

In this section, the geometric meaning of conformable derivative and its advantages over classical derivative will be explained.

3.1 Why local conformable derivative?

Differential geometry, a field that employs linear algebra and calculus techniques, focuses on the study of curves, surfaces, and high-dimensional manifolds. However, in the past decade, there has been a noticeable emergence of a new trend within differential geometry, where fractional calculus techniques are being utilized. This development has raised questions about the geometric properties associated with fractional derivative and integral operators. It is important to note that non-local fractional derivative operators such as Riemann-Liouville, Caputo, and Riesz do not adhere to the classical Leibniz and chain rules. These rules form the basis of the classical derivative used in differential geometry. Consequently, constructing a framework for differential geometry using non-local fractional derivatives poses significant challenges. Instead, a more advantageous approach is to employ local conformable derivative. Conformable local derivative operator satisfy the Leibniz and chain rules, making them suitable for constructing a differential geometry framework. By utilizing conformable local derivative, researchers can explore the geometric implications of fractional calculus techniques within the context of differential geometry. By embracing local conformable derivative operator, the field of differential geometry can navigate the complexities associated with non-local fractional derivatives and benefit from the inherent advantages provided by local derivative operators that adhere to the classical rules of calculus.

3.2 What is the advantage of the local conformable derivative?

Fractional derivatives, as is known, search for any fractional order derivatives of a function. Accordingly, at a point where there is no integer derivative of a function, its fractional derivative can be found. For example, consider the function $\mathbf{x}(u) = 2\sqrt{u}$. Here $\mathbf{x}'(0)$ does not exist. However, the result $D_{\frac{1}{2}}\mathbf{x}(0) = 1$ can be easily reached. As it can be easily seen from the example, the tangent of the curve $\mathbf{x}$ cannot be mentioned at the point where there is no classical derivative. The absence of the concept of tangent, which is the most basic concept of curves and surfaces, causes a bottleneck in this regard, in other words degeneration. At such points, which lead to degeneracy where there is no derivative, the tangent can be approached by choosing a fractional order derivative very close to first instead of the first-order derivative giving the tangent. Thus, necessary investigations can be made by making a fractional approximation to the tangent at a degenerate point where there is no tangent of the curve or the surface.

Case 1: Let x be a $\mathcal{C}_\alpha$(conformable)-differentiable curve. Let us choose a point u_0 where the traditional derivative of the curve $\mathbf{x}$ does not exist. At such a u_0 degenerate point, the tangent cannot be mentioned, because the derivative of the curve $\mathbf{x}$ does not exist. As it is known, when defining the Frenet frame at any point of the curve, the most important element is the tangent vector of the curve. In this case, the Frenet frame cannot be defined at the degenerate u_0 point. In such a case, with the help of conformable derivative, we can define a frame parallel to the Frenet frame by defining the classical tangent of the curve as an alternative to the $\mathcal{C}_\alpha-$tangent (see details [24]).

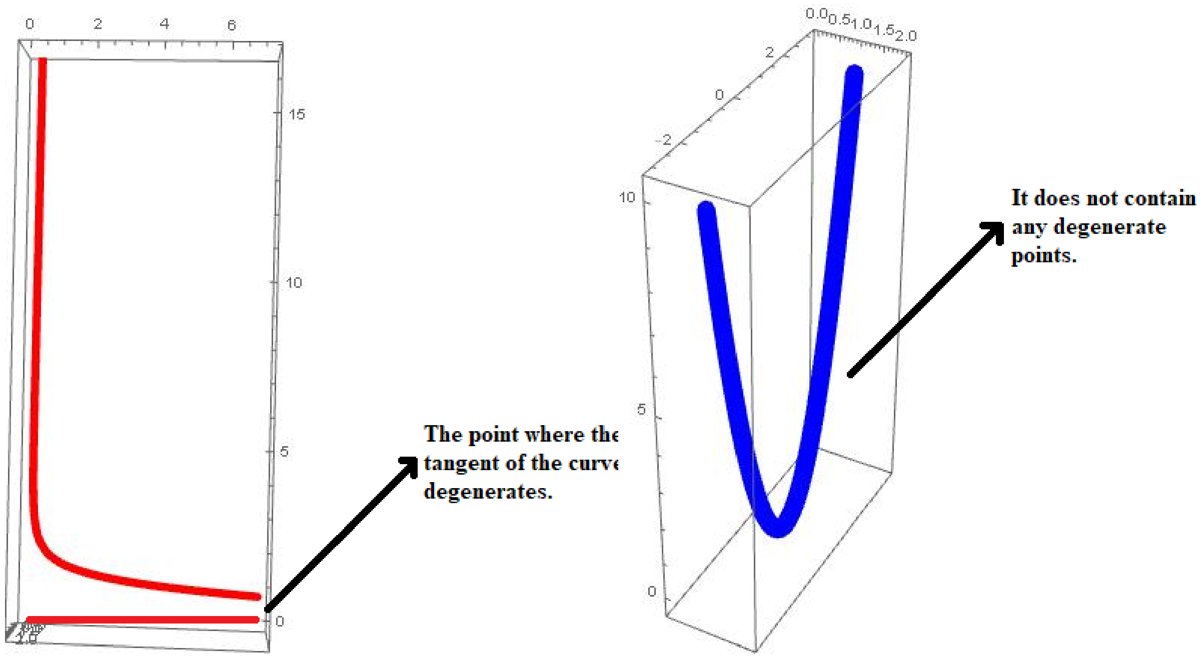
Example 3.1. Let $\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3$ be a $\mathcal{C}_\alpha-$curve in $\mathbb{R}^3$ parameterized by

$$\mathbf{x}(u) = \left(2u^{\frac{1}{2}},\ u^{\frac{3}{2}},\ u^{\frac{5}{2}}\right).$$

The tangent of the curve $\mathbf{x}$ obtained with the classical derivative and the tangent obtained with the conformable derivative for $\alpha = \frac{1}{2}$ and $-2 < u < 2$ are as follows, respectively.

$$\begin{aligned} T &= \frac{1}{\sqrt{\frac{1}{u} + \frac{9}{4}u + \frac{25}{4}u^2}} \left(u^{\frac{-1}{2}}, \frac{3}{2}u^{\frac{1}{2}}, \frac{5}{2}u^{\frac{3}{2}}\right), \\ T_{\frac{1}{2}} &= \frac{1}{\sqrt{1 + \frac{9}{4}u^2 + \frac{25}{4}u^3}} \left(1, \frac{3}{2}u, \frac{5}{2}u^2\right). \end{aligned}$$

where T and $T_{\frac{1}{2}}$ are the classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(u)$, respectively. It is clear that for the point $u_0 = 0$ the classical tangent of the $\mathbf{x}$ curve does not exist. In Fig. (1) we present the graph classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(u)$.



Classical tangent. $\mathcal{C}\alpha$-tangent.

Figure 1. Classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(s)$.

Case 2: Let $\mathcal{M}$ be the $\mathcal{C}_\alpha-$surface is given by the parameterization $\varphi(u, v)$. In order to define the tangent plane of the conformable surface M, the φ_u and φ_v partial derivatives that spanned the tangent plane are needed. In this case, the tangent plane of the surface cannot be mentioned at a point u_0 or v_0 where at least one of the derivatives of φ_u or φ_v does not exist. As it is well known, one of the most important concepts of the surface is the tangent plane. At the point where the tangent plane cannot be defined, it becomes impossible to work on the surface. Since the traditional derivative loses its function at such a degenerate u_0 or v_0 point, necessary studies can be done by obtaining the $\mathcal{C}_\alpha-$tangent plane with the help of $D_u^\alpha\varphi$ and $D_v^\alpha\varphi$ conformable partial derivatives (see details [23]).

Example 3.2. Let $\mathcal{M}$ be a $\mathcal{C}_\alpha-$surface in $\mathbb{R}^3$ parameterized by

$$\varphi(u, v) = \left(\int\int v^{\frac{1}{2}-\alpha}u^{\alpha-1}\cos u du dv, -\int\int v^{\frac{1}{2}-\alpha}u^{\alpha-1}\sin u du dv, \int v^{\frac{1}{2}-\alpha}dv\right).$$

In Fig. 2, the degenerate (there is no derivative) points the classical tangent plane for $v = 0$ and the $\mathcal{C}_\alpha-$-tangent plane for $v = 0$, $\alpha = \frac{1}{2}$ are given.

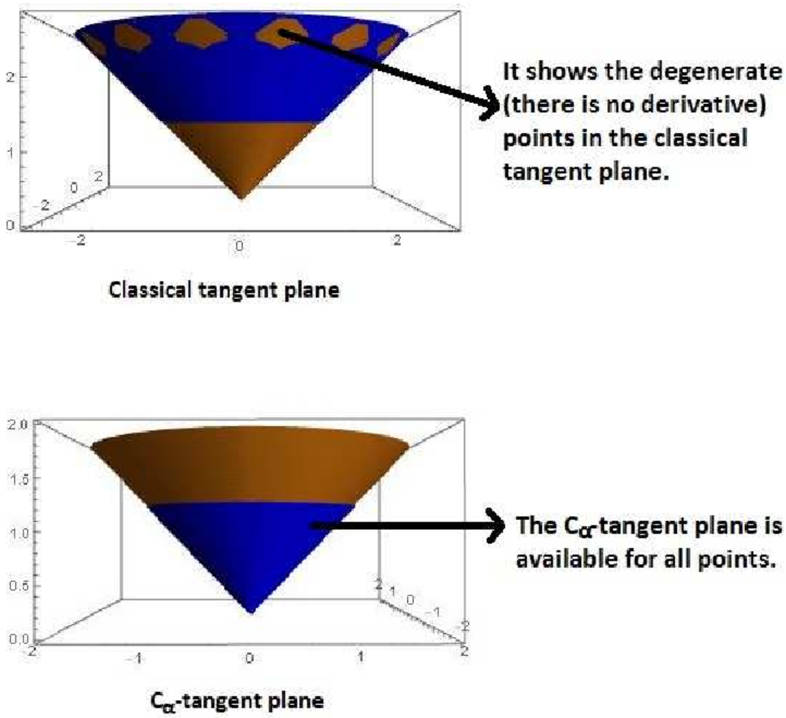


Figure 2. Yellow areas show the surface, and blue areas show tangent planes.

3.3 Geometric meaning of the conformable derivative

The geometric interpretation of the conformable derivative is based on the notion of fractal geometry. In fractal geometry, objects exhibit self-similarity at different scales. The conformable derivative captures this self-similar behavior of a function by considering its local fractional variations. Geometrically, it can be understood as analyzing the "zooming in" behavior of the function at that point, similar to the classical derivative capturing the local linear behavior. Overall, the geometric interpretation of the conformable derivative relates to the self-similarity and scaling properties of functions, enabling us to understand their behavior at different levels of detail and resolution. *More specifically, the conformable derivative can be explained as a measure of how much a straight line and plane bends to form a curve and a surface* (see details [24])

Example 3.3. Let consider the $s \mapsto \mathbf{x}(s) = (s, \int s^{1-\alpha} ds)$, $\mathcal{C}_\alpha-$line passing through the point $P = (0, 0)$ and whose direction is $v = (s^{1-\alpha}, s^{1-\alpha})$.
In Fig. (3) we present the graph of the conformable line for different α values.

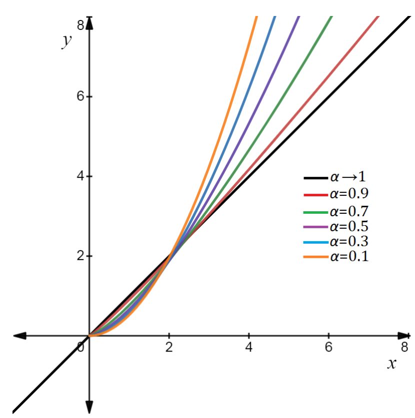


Figure 3. Transformation from line to curve.

Remark 3.1. In differential geometry, a classical line bends depending on the α values with the effect of conformable derivative. Thus, with specially selected α values, how much the line deviates from the plane can be measured.

Example 3.4. Let X be a representation point of the $\mathcal{C}_\alpha-$plane that contains the point $P = (0, 0, 0)$ and whose normal is $v = (2^{1-\alpha}, -3^{1-\alpha}, 0)$. If X representative point is chosen as follows

$$\begin{aligned}
\mathbf{x}_1(s) &= \int x^{1-\alpha} dx, \\
\mathbf{x}_2(s) &= \int y^{1-\alpha} dy, \\
\mathbf{x}_3(s) &= 0
\end{aligned}$$

we get the conformable plane. In Fig. (4) we present the graph of the conformable plane for different α values.

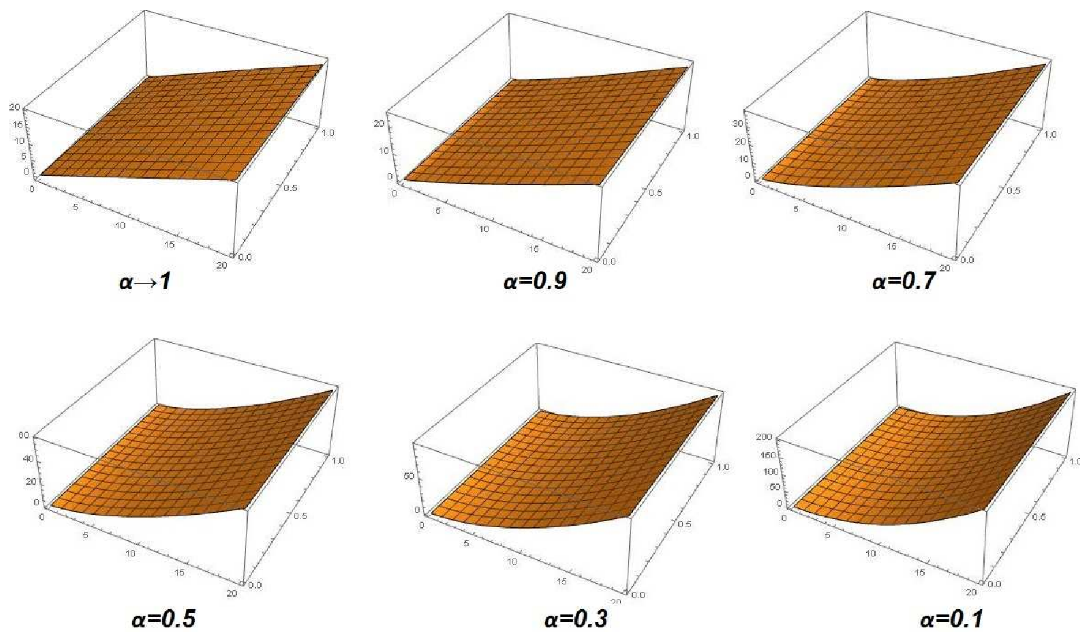


Figure 4. Transformation from plane to surface.

Remark 3.2. In differential geometry, a classical plane bends and transforms into a surface, depending on the α values, with the effect of the concerted derivative. Thus, with specially selected α values, the measure of separation of a surface from the plane can be obtained.

3.4 Physical meaning of the conformable derivative

The velocity vector plays a crucial role in describing the motion of an object over time. It represents how the object's position changes as time progresses. The concept of velocity is defined as the ratio of the change in position of an object to the change in time, and its direction indicates the object's direction of motion.

The relationship between the tangent vector and the velocity vector can be explained as follows: When an object is in motion, its velocity vector is aligned with the tangent direction of the path along which the object moves. In other words, the velocity vector is parallel to the slope (direction) of the object's path and, therefore, points in the direction of the tangent vector. If the object moves in a straight line, the velocity vector and the tangent vector align in the same direction. However, when the object follows a curvilinear path, the velocity vector constantly adjusts in parallel with the slope of the path, hence always directed towards the tangent vector.

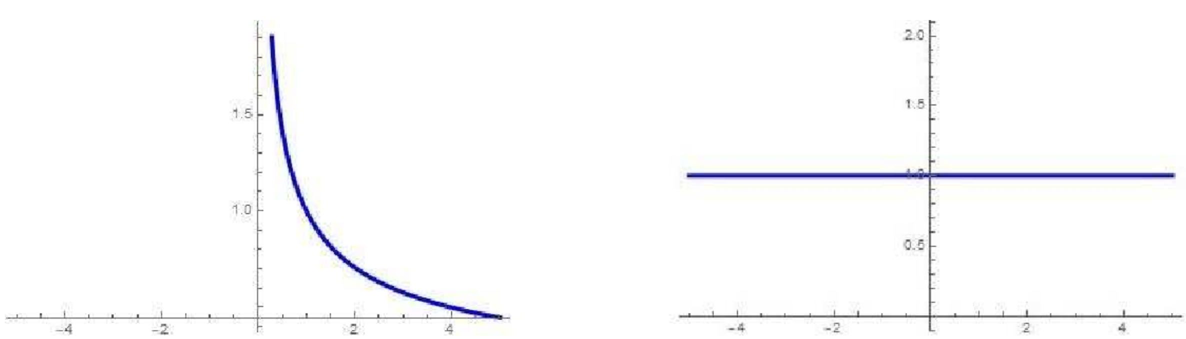
In certain instances, the tangent vector may be undefined at certain points along the path. Consequently, at these points, the velocity vector will also be undefined. This situation arises when the classical derivative is not defined. At such points, discussing a physical interpretation becomes challenging since the velocity vector's meaning is lost.

Nevertheless, the conformable derivative comes to the rescue by addressing this undefinedness and allowing for the notion of "conformable velocity" to be introduced. As a result, the conformable derivative provides an advantage in terms of physical interpretation, enabling us to understand the object's behavior even at points where the classical derivative fails.

In conclusion, the velocity vector is vital in characterizing object motion, and it aligns with the tangent vector of the object's path during movement. In cases where the tangent vector is undefined, the velocity vector also becomes undefined, hindering a physical interpretation. However, the conformable derivative offers a solution, eliminating this undefinedness and facilitating a meaningful interpretation through the concept of conformable velocity.

Example 3.5. In Subsection 3.2, an example is presented where the velocity vector is not defined at the point $u_0 = 0$, making it difficult to establish a physical interpretation. However, a solution is found using the conformable derivative, allowing the creation of an alternative (conformable) velocity vector at the point $u_0 = 0$ for $\alpha = \frac{1}{2}$ and enabling a meaningful physical interpretation. The conformable derivative is a mathematical tool used to define derivatives of non-integer order. In this context, it helps overcome the limitation of traditional derivatives, which are not defined for non-integer values. By introducing the conformable derivative with $\alpha = \frac{1}{2}$, it becomes possible to extend the concept of the velocity vector to points like $u_0 = 0$, where traditional derivatives fail. With the introduction of the conformable derivative and considering $\alpha = \frac{1}{2}$, a new velocity vector can be constructed at the point $u_0 = 0$. This new velocity vector provides valuable insights into the physical interpretation of the system, even at previously undefined points. It allows us to understand the behavior of the system at $u_0 = 0$ in a way that was not possible before, offering new perspectives and understanding of the underlying dynamics. In summary, the use of the conformable derivative with $\alpha = \frac{1}{2}$ provides a powerful mathematical tool that enables the definition of the velocity vector at points where it was previously undefined. This breakthrough allows for a more comprehensive and meaningful physical interpretation of the system, enriching our understanding of its behavior and characteristics.

In Fig. 5, the classical velocity vector of the $\mathbf{x}(u) = 2\sqrt{u}$ equation, where $u_0 = 0$, and the compatible velocity vector for $\alpha = \frac{1}{2}$ are illustrated, respectively. The Fig. 5 demonstrates an intriguing contrast between the two velocity vectors. When considering the classical velocity vector, where $u_0 = 0$, it is evident that there is a limitation in terms of physical interpretation. This is because, at this particular point, the classical derivative fails to provide meaningful information about the body's motion. The classical derivative, which relies on integer values for differentiation, encounters issues when dealing with non-integer values like $u_0 = 0$. As a result, it becomes impossible to establish a clear physical interpretation for the body's velocity at this specific point. However, the situation takes a different turn with the introduction of the compatible velocity vector for $\alpha = \frac{1}{2}$. With the help of the conformable derivative, which extends the concept of differentiation to non-integer values, the compatible velocity vector becomes accessible and well-defined along the entire real axis. This remarkable advantage of the conformable derivative enables a continuous interpretation of the body's velocity, even at points where the classical derivative fails. As a consequence, the compatible velocity vector not only provides a consistent interpretation throughout the entire real axis but also removes the ambiguity associated with the classical velocity vector when $u_0 = 0$. The conformable derivative allows us to overcome the limitations of traditional derivatives, granting a more comprehensive understanding of the body's motion and behavior.



Classical velocity vector. **Conformable velocity vector.**

Figure 5. Classical and conformable velocity vectors, respectively.

Conclusion

As is known, the relationship between the conformable derivative and the classical derivative is given by $D_\alpha x(s) = s^{1-\alpha} dx(s)/ds$. When examining the effect of the conformable derivative on lines and planes, the term $s^{1-\alpha}$ causes the line to bend, transforming it into a curve, and the plane to bend, transforming it into a surface. We can refer to the expression $s^{1-\alpha}$ as the bending measure, as it quantifies the amount of bending based on the specific values assigned to α. Geometrically, the conformable derivative can be interpreted as a measure of bending. It captures the degree to which the curve or surface deviates from its original straight form. The bending measure provides valuable insight into the geometric properties and behavior of the objects under consideration. Furthermore, as observed in *Case 1* and *Case 2*, the conformable derivative demonstrates significant advantages in the realm of differential geometry. It enables a deeper understanding and analysis of the geometrical aspects associated with curves, surfaces, and their transformations. The conformable derivative opens up new avenues and perspectives for exploring the intricate connections between bending, geometry, and the underlying mathematical principles. Through these observations, it becomes apparent that the conformable derivative plays a crucial role in differential geometry, offering powerful tools for investigating and comprehending the bending phenomena exhibited by curves and surfaces.

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Asymptotic Stability of Neutral Differential Systems with Variable Delay and Nonlinear Perturbations

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Abstract
In this paper, the problem of asymptotic stability of a kind of nonlinear perturbed neutral differential system with variable delay is discussed. The Lyapunov-Krasovskiĭ functional constructed, is used to obtain conditions for asymptotic stability of the nonlinear perturbed neutral differential system in terms of linear matrix inequality (LMI). The two new results (delay-independent and delay-dependent criteria) include and extend the existing results in the literature. Finally, an example of delay-dependent criteria is supplied and the simulation result is shown to justify the effectiveness and reliability of the used techniques.

Keywords: *Asymptotic stability, Lyapunov-Krasovskiĭ functional, Neutral delay differential system, Perturbation analysis*

AMS Subject Classification (2020): *34K05; 34K20; 34K27; 34K40; 37C75*

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1. Introduction

Qualitative behaviour of solutions of differential equations with or without delay and/or randomness of various orders have received appreciable attention in recent years, see for instance the papers in [1–9]. These significant improvements in the study of differential equations are not unconnected to umpteen areas of applications in electrical networks containing lossless transmission lines [10–12], stability properties of electrical power systems, and macroeconomic models, the motion of nuclear reactors, feedback control loops involving sensors in integrated communication and control systems, energy or signal transmission, see [13–18]. Accordingly, researchers have developed efficient and effective techniques such as the Lyapunov direct method, the technique of characteristic equation, the fixed point principle, the state trajectory method, and so on, to discuss systems of first-, second-, third-, and higher-order differential equations.

In their contributions, Hale *et al.* [19], Hale and Verdny, Lunel [20], Li [21], Slemrod and Infante [22] to mention but a few, have developed delay-independent criteria for the asymptotic stability of neutral delay differential

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systems. In addition, sufficient delay-dependent conditions were presented by Brayton and Willoughby [23] and Khusainov and Yun'kova [24] for the asymptotic stability of neural delay differential system and systems with nonlinear perturbations respectively. The obtained sufficient conditions are presented in terms of either matrix norm or matrix measure operations which by Park and Won [25], are conservative. Other relevant papers on chaotic control and hyperchaotic systems with time delay include Feng *et al.* [26], Onasanya *et al.* [27], stability tests and solution estimates for non-linear differential equations Tunç [28], among others.

In this investigation, we consider the problem of asymptotic stability of a neutral differential system with a variable delay. Lyapunov-Krasovskiĭ functionals are developed to derive sufficient conditions for the asymptotic stability of the system under investigation in terms of LMI. Two new results (i.e., sufficient criteria for delay-independent and delay-dependent) which generalize that of Park and Won are presented. The motivation for this paper comes from the work of Park and Won, where sufficient conditions for the stability of perturbed constant delay differential systems are discussed in terms of LMI. If $\tau(t) = h$, $h > 0$ is a constant delay, then the nonlinear perturb neutral variable delay differential systems considered, the Lyapunov-Krasovskiĭ functional employed and the two new results coincide with that of Park and Won.

2. Stability results

Consider a perturbed neutral variable delay differential system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau(t)) + C\dot{x}(t-\tau(t)) + Q(x(t), x(t-\tau(t)), \dot{x}(t-\tau(t))) \tag{2.1}$$

with the initial condition $x(t) = \phi(t)$, where $x(t) \in \mathbb{R}^n$ is the state vector, A, B and $C \in \mathbb{R}^{n\times n}$ are constant matrices, $Q \in C(\mathbb{R}^{3n}, \mathbb{R}^n)$, the positive constants α_1, α_2 exist and satisfying the following inequalities $\tau(t) \le \alpha_1$ and its derivative $\dot{\tau}(t) \le \alpha_2$, $(0 < \alpha_2 < 1)$, $\phi(t)$ is the continuously differentiable function on $[-\alpha_1, 0]$ and for all $t \in [-\alpha_1, 0]$, $Q(t) = Q(x(t), x(t-\tau(t)), \dot{x}(t-\tau(t)))$ a nonlinear perturbation that satisfies the following estimate

$$\|Q(t)\| \le \lambda_1\|x(t)\| + \lambda_2\|x(t-\tau(t))\| + \lambda_3\|\dot{x}(t-\tau(t))\| \tag{2.2}$$

where λ_i, $(i = 1, 2, 3)$ is a positive constant.

Lemma 2.1. *(Khargonekar et al. [29])*
Let D and E be real matrices of appropriate dimensions. Then, for any scalar $\epsilon > 0$

$$DE + E^TD^T < \epsilon DD^T + \epsilon^{-1}E^TE.$$

The main tool employed in this investigation is the functional $V = V(x)$ defined as

$$V = x^T(t)Px(t) + \sum_{i=1}^{2} W_i, \quad (i = 1, 2) \tag{2.3}$$

where

$$W_1 := \int_{t-\tau(t)}^{t} \dot{x}^T(\theta)\dot{x}(\theta)d\theta \text{ and } W_2 := \int_{t-\tau(t)}^{t} x^T(\theta)Rx(\theta)d\theta.$$

Next, we shall state and prove the first stability result of this paper as follows.

Theorem 2.1. *In addition to the basic assumptions on functions τ and Q, suppose that ε_i $(i = 1, 2, 3, 4)$ are positive constants, P and R are $n\times n$ symmetric positive definite matrices satisfying the following LMI*

$$M_1 = \begin{bmatrix} A_1 & A^TB + PB & A^TC + PC \\ B^TA + B^TP & A_2 & B^TC \\ C^TA + C^TP & C^TB & A_3 \end{bmatrix} < 0$$

where

$$A_1 := D_1 + \varepsilon_1^{-1}PP + \varepsilon_2^{-1}A^TA; \;\; A_2 := D_2 + \varepsilon_3^{-1}B^TB;$$

$$A_3 := D_3 + \varepsilon_4^{-1}C^TC; \;\; D_1 := A^TP + PA + A^TA + R + 3\big(1 + \sum_{i=1}^{4}\varepsilon_i\big)\lambda_1^2 I;$$

$$D_2 := B^TB - (1-\alpha_2)R + 3\big(1 + \sum_{i=1}^{4}\varepsilon_i\big)\lambda_2^2 I; \;\; \text{and} \;\; D_3 := C^TC - (1-\alpha_2)I + 3\big(1 + \sum_{i=1}^{4}\varepsilon_i\big)\lambda_3^2 I.$$

Then the solution of system (2.1) is asymptotically stable.

Proof. Let $x = x(t),\ \tau = \tau(t)$ and $x_\tau = x(t - \tau(t))$, the derivative of V with respect to the independent variable t along the solution path of (2.1) is

$$\dot{V}_{(2.1)} = x^T(PA + A^TP)x + 2x_\tau^T B^T P x + 2\dot{x}_\tau^T C^T P x + 2Q^T P x + \sum_{i=1}^{2} \dot{W}_i, \tag{2.4}$$

where

$$\begin{aligned}\dot{W}_1 &= x^T A^T A x + x_\tau^T B^T B x_\tau + \dot{x}_\tau^T C^T C \dot{x}_\tau + Q^T Q + 2x^T A^T B x_\tau + 2x^T A^T C \dot{x}_\tau \\ &+ 2x_\tau^T B^T C \dot{x}_\tau + 2Q^T A x + 2Q^T B x_\tau + 2Q^T C \dot{x}_\tau - (1 - \dot{\tau})\dot{x}_\tau^T \dot{x}_\tau\end{aligned} \tag{2.5}$$

and

$$\dot{W}_2 = x^T R x - (1 - \dot{\tau}) x_\tau^T R x_\tau. \tag{2.6}$$

Re-arranging $2Q^T Px, 2Q^T Ax, 2Q^T Bx_\tau$ and $2Q^T C\dot{x}_\tau$ using Lemma 2.1, we obtain

$$\begin{aligned}2Q^T P x &\le \varepsilon_1 Q^T Q + \varepsilon_1^{-1} x^T P P x,\\ 2Q^T A x &\le \varepsilon_2 Q^T Q + \varepsilon_2^{-1} x^T A^T A x,\\ 2Q^T B x_\tau &\le \varepsilon_3 Q^T Q + \varepsilon_3^{-1} x_\tau^T B^T B x_\tau,\\ 2Q^T C \dot{x}_\tau &\le \varepsilon_4 Q^T Q + \varepsilon_4^{-1} \dot{x}_\tau^T C^T C \dot{x}_\tau.\end{aligned} \tag{2.7}$$

In view of inequality (2.2) and the fact that $2ab\|x\|\|y\| \le a^2\|x\|^2 + b^2\|y\|^2$, it follows that

$$Q^T Q \le 3(\lambda_1^2 x^T x + \lambda_2^2 x_\tau^T x_\tau + \lambda_3^2 \dot{x}_\tau^T \dot{x}_\tau). \tag{2.8}$$

From inequalities (2.7) and (2.8) we find

$$\big(1 + \sum_{i=1}^{4} \varepsilon_i\big) Q^T Q \le 3\big(1 + \sum_{i=1}^{4} \varepsilon_i\big)(\lambda_1^2 x^T x + \lambda_2^2 x_\tau^T x_\tau + \lambda_3^2 \dot{x}_\tau^T \dot{x}_\tau). \tag{2.9}$$

Engaging (2.5), (2.6), estimates (2.7), and (2.9) in (2.4), we have

$$\begin{aligned}\dot{V}_{(2.1)} \le & x^T\big[A^T P + PA + \varepsilon_1^{-1} PP + (1 + \varepsilon_2^{-1}) A^T A + R + 3\big(1 + \sum_{i=1}^{4} \varepsilon_i\big)\lambda_1^2 I\big]x\\ &+ x_\tau^T\big[(1 + \varepsilon_3^{-1}) B^T B + 3\big(1 + \sum_{i=1}^{4} \varepsilon_i\big)\lambda_2^2 I - (1 - \alpha_2) R\big]x_\tau\\ &+ \dot{x}_\tau^T\big[(1 + \varepsilon_4^{-1}) C^T C + 3\big(1 + \sum_{i=1}^{4} \varepsilon_i\big)\lambda_3^2 I - (1 - \alpha_2) I\big]\dot{x}_\tau\\ &+ 2x^T(A^T B + PB)x_\tau + 2x^T(A^T C + PC)\dot{x}_\tau + 2x_\tau^T B^T C \dot{x}_\tau.\end{aligned} \tag{2.10}$$

Inequality (2.10) can be recast as

$$\dot{V}_{(2.1)} \le \begin{bmatrix} x \\ x_\tau \\ \dot{x}_\tau \end{bmatrix}^T M_1 \begin{bmatrix} x \\ x_\tau \\ \dot{x}_\tau \end{bmatrix}$$

where

$$M_1 = \begin{bmatrix} A_1 & PB + A^T B & PC + A^T C \\ B^T P + B^T A & A_2 & B^T C \\ C^T P + C^T A & C^T B & A_3 \end{bmatrix}.$$

Therefore, $\dot{V}_{(2.1)}$ is negative definite if the matrix M_1 is negative definite. This completes the proof. □

Remark 2.1. We have the following observations:

(i) If $\tau(t) = h$ for some constant $h > 0$, then Theorem 2.1 coincide with Theorem 1 in [25]. Thus system (2.1) and the Lyapunov-Krasovskiĭ functional defined by (2.3) is an extension of the one used in [25];

(ii) If the perturbed function $Q(t) = 0$ in (2.1) then the sufficient condition for stability of the trivial solution is

$$M_2 = \begin{bmatrix} A_4 & A^TB + PB & A^TC + PC \\ B^TA + B^TP & A_5 & B^TC \\ C^TA + C^TP & C^TB & A_6 \end{bmatrix} < 0 \tag{2.11}$$

where

$$A_4 := A^TP + PA + A^TA + R, \quad A_5 := B^TB - (1-\alpha_2)R, \text{ and } A_6 := C^TC - (1-\alpha_2)I.$$

Thus if $\tau(t) = h$ inequality (2.11) coincide with inequality (17) in [25].

Next, we shall discuss the delay-dependent stability criteria for system (2.1). Let $x(t)$ be continuously differentiable on $[-2\alpha_1, -\alpha_1]$, then system (2.1) can be represented as

$$\dot{x}(t) = A_0x(t) - B\sum_{i=1}^{3}\eta_i - BCx_\tau(t) + BCx_{2\tau}(t) + C\dot{x}_t(t) + Q(t), \tag{2.12}$$

where

$$A_0 := A + B,\ \eta_1 := \int_{t-\tau}^{t} Ax(\theta)d\theta,\ \eta_2 := \int_{t-\tau}^{t} Bx_\tau(\theta)d\theta, \text{ and } \eta_3 := \int_{t-\tau}^{t} Q(\theta)d\theta.$$

A continuously differentiable functional $W = W(x)$ used in this case is defined by

$$W = x^T(t)Px(t) + \sum_{i=1}^{4} W_i, \tag{2.13}$$

where

$$W_1 := \int_{t-\tau}^{t} \dot{x}^T(\theta)\dot{x}(\theta)d\theta,\ W_2 := \int_{t-\tau}^{t} x^T(\theta)R_1x(\theta)d\theta,\ W_3 := \int_{t-2\tau}^{t} x^T(\theta)R_2x(\theta)d\theta,$$
$$W_4 = \int_{-\tau}^{0}\int_{t+\mu}^{t} \left[\|Ax(\theta)\|^2 + \|Bx_\tau(\theta)\|^2 + \|Q(\theta)\|^2\right]d\theta d\mu,$$

R_1 and R_2 are positive semi-definite symmetric matrices to be determined later.

Remark 2.2. If $\tau = h$ ($h > 0$ a constant) then the functional (2.13) specialized to that of Park and Won [25].

The stability result for the delay-dependent (2.12) is as follows.

Theorem 2.2. *Suppose there exist positive constants* ξ_i $(i = 1, 2, \cdots, 12)$, P, R_1 *and* R_2 *are* $n \times n$ *symmetric, positive definite matrices which satisfy the following LMI:*

$$M_3 = \begin{bmatrix} A_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_8 \end{bmatrix} < 0$$

with $21B^TB = (1-\alpha_2)I$ *where*

$$A_7 := A_0^TP + PA_0 + \gamma_1A_0^TA_0 + \gamma_2PP + \alpha_1A^TA + [(1-2\alpha_2)^{-1}\gamma_3 + (1-\alpha_2)^{-1}\gamma_4]C^TB^TBC$$
$$+ (1-\alpha_2)^{-1}\left[\alpha_1B^TB + (\xi_7^{-1} + \xi_8^{-1})C^TC\right] + \left[\lambda_1^2 + (1-\alpha_2)\lambda_2^2\right]\gamma_3I,$$
$$A_8 := \gamma_6C^TC + \left[\lambda_3^2\gamma_3 - (1-\alpha_2)\right]I,\ \gamma_1 := 1 + 3\alpha_1 + \xi_3^{-1} + \xi_6 + \xi_7 + \xi_{10},$$
$$\gamma_2 := 3(\alpha_1 + \xi_1^{-1}) + \xi_5 + \xi_8 + \xi_9,\ \ \gamma_3 := 3\big(1 + 4\alpha_1 + \xi_3 + \sum_{i=1}^{4}\xi_i\big),$$
$$\gamma_4 := 1 + 3\alpha_1 + \xi_3^{-1} + \xi_5^{-1} + \xi_6^{-1} + \xi_{11} + \xi_{12},\ \ \gamma_5 := 1 + 3\alpha_1 + \xi_3^{-1} + \xi_9^{-1} + \xi_{10}^{-1} + \xi_{11}^{-1} + \xi_{12}, \textit{and}$$
$$\gamma_6 := 1 + 3\alpha_1 + \xi_4^{-1} + 2\xi_{12}^{-1}.$$

Then the solution of (2.12) *is asymptotically stable.*

Proof. The derivative of the functional (2.13) along solution of (2.12) is given by

$$\begin{aligned}\dot W_{(2.12)} &= x^T(A_0^TP + PA_0)x - 2\sum_{i=1}^{3}\eta_i^TB^TPx - 2x_\tau^TC^TB^TPx + 2x_{2\tau}^TC^TB^TPx \\ &\quad + 2\dot x_\tau^TC^TPx + 2Q^TPx + \sum_{i=1}^{4}\dot W_i,\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}\dot W_1 :=\ & x^TA_0^TA_0x + x_\tau^TC^TB^TBCx_\tau + x_{2\tau}^TC^TB^TBCx_{2\tau} + \dot x_\tau^TC^TC\dot x_\tau - 2x^TA_0^TBCx_\tau \\ & + 2x^TA_0^TBCx_{2\tau} + 2x^TA_0^TC\dot x_\tau - 2x_\tau^TC^TB^TBCx_{2\tau} - 2x_\tau^TC^TB^TC\dot x_\tau + 2x_{2\tau}^TC^TB^TC\dot x_\tau \\ & + \sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i - 2x^TA_0^TB\sum_{i=1}^{3}\eta_i + 2\sum_{i=1}^{3}\eta_i^TB^TBCx_\tau - 2\sum_{i=1}^{3}\eta_i^TB^TBCx_{2\tau} \\ & - 2\sum_{i=1}^{3}\eta_i^TB^TC\dot x_\tau - 2\sum_{i=1}^{3}\eta_i^TB^TQ + 2x^TA_0^TQ - 2x_\tau^TC^TB^TQ + 2x_{2\tau}^TC^TB^TQ + 2\dot x_\tau^TC^TQ \\ & + Q^TQ - (1-\dot\tau)\dot x_\tau^T\dot x_\tau,\end{aligned} \tag{2.15}$$

$$\dot W_2 := x^TR_1x - (1-\dot\tau)x_\tau^TR_1x_\tau, \tag{2.16}$$

$$\dot W_3 := x^TR_2x - (1-2\dot\tau)x_{2\tau}^TR_2x_{2\tau}, \text{ and} \tag{2.17}$$

$$\dot W_4 := \tau\big[\|Ax\|^2 + \|Bx_\tau\|^2 + \|Q(t)\|^2\big] - (1-\dot\tau)\int_{t-\tau}^{t}\big[\|Ax(\theta)\|^2 + \|Bx_\tau(\theta)\|^2 + \|Q(\theta)\|^2\big]d\theta. \tag{2.18}$$

By Lemma 2.1, the following inequalities are fulfilled:

$$\begin{aligned}-2\sum_{i=1}^{3}\eta_i^TB^TPx &\le 3\tau x^TPPx + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i; \\ -2\sum_{i=1}^{3}\eta_i^TB^TA_0x &\le 3\tau x^TA_0^TA_0x + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i; \\ 2\sum_{i=1}^{3}\eta_i^TB^TBCx_\tau &\le 3\tau x_\tau^TC^TB^TBCx_\tau + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i; \\ -2\sum_{i=1}^{3}\eta_i^TB^TBCx_{2\tau} &\le 3\tau x_{2\tau}^TC^TB^TBCx_{2\tau} + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i; \\ -2\sum_{i=1}^{3}\eta_i^TB^TC\dot x_\tau &\le 3\tau\dot x_\tau^TC^TC\dot x_\tau + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i;\end{aligned} \tag{2.19}$$

$$\begin{aligned}-2\sum_{i=1}^{3}\eta_i^TB^TQ &\le 3\tau Q^TQ + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i; \\ 2Q^TPx &\le \xi_1Q^TQ + \xi_1^{-1}x^TPPx; \\ 2Q^TA_0x &\le \xi_2Q^TQ + \xi_2^{-1}x^TA_0^TA_0x; \\ -2Q^TBCx_\tau &\le \xi_3Q^TQ + \xi_3^{-1}x_\tau^TC^TB^TBCx_\tau; \\ 2Q^TBCx_{2\tau} &\le \xi_3Q^TQ + \xi_3^{-1}x_{2\tau}^TC^TB^TBCx_{2\tau}; \\ 2Q^TC\dot x_\tau &\le \xi_4Q^TQ + \xi_4^{-1}\dot x_\tau^TC^TC\dot x_\tau.\end{aligned} \tag{2.20}$$

First collate the like-terms in Q^TQ from (2.15) and (2.20) using estimate (2.2) we find

$$\big(1 + 3\tau + \xi_3 + \sum_{i=1}^{3}\xi_i\big)Q^TQ \le 3\big(1 + 3\tau + \xi_3 + \sum_{i=1}^{3}\xi_i\big)\big(\lambda_1^2x^Tx + \lambda_2^2x_\tau^Tx_\tau + \lambda_3^2\dot x_\tau^T\dot x_\tau\big), \tag{2.21}$$

also terms involving η_i from (2.15) and (2.19) are

$$\sum_{i=1}^{3}\eta_i^T B^T B\sum_{i=1}^{3}\eta_i + 18\tau^{-1}\sum_{i=1}^{3}\eta_i^T B^T B\sum_{i=1}^{3}\eta_i \leq 21B^T B\int_{t-\tau}^{t}\big[\|Ax(\theta)\|^2 + \|Bx_\tau(\theta)\|^2 + \|Q(\theta)\|^2\big]d\theta, \tag{2.22}$$

and the first term of (2.18) is

$$\tau\big[\|Ax\|^2 + \|Bx_\tau\|^2 + \|Q(t)\|^2\big] \leq \tau\big[x^T(A^T A + 3\lambda_1^2 I)x + x_\tau^T(B^T B + 3\lambda_2^2 I)x_\tau + 3\lambda_3^2\dot{x}_\tau^T\dot{x}_\tau\big]. \tag{2.23}$$

Next, engaging (2.15)-(2.17), second term of (2.18), and inequalities (2.19)-(2.23) in (2.14), noting that $\tau \leq \alpha_1$, $\dot{\tau} \leq \alpha_2$, and $21B^T B = (1-\alpha_2)I$, we obtain

$$\begin{aligned}
\dot{W}_{(2.12)} &= x^T\big[A_0^T P + PA_0 + (1+3\alpha_1+\xi_2^{-1})A_0^T A_0 + (3\alpha_1+\xi_1^{-1})PP + \alpha_1 A^T A + R_1 + R_2\\
&+ 3(1+4\alpha_1+\xi_3+\sum_{i=1}^{4}\xi_i)\lambda_1^2 I\big]x + x_\tau^T\big[(1+3\alpha_1+\xi_3^{-1})C^T B^T BC + \alpha_1 B^T B - (1-\alpha_2)R_1\\
&+ 3(1+4\alpha_1+\xi_3+\sum_{i=1}^{4}\xi_i)\lambda_2^2 I\big]x_\tau + x_{2\tau}^T\big[(1+3\alpha_1+\xi_2^{-1})C^T B^T BC - (1-2\alpha_2)R_2\big]x_{2\tau}\\
&+ \dot{x}_\tau^T\big[(1+3\alpha_1+\xi_4^{-1})C^T C + 3(1+4\alpha_1+\xi_3+\sum_{i=1}^{4}\xi_i)\lambda_3^2 I - (1-\alpha_2)\big]\dot{x}_\tau\\
&- 2x^T(PBC + A_0^T BC)x_\tau + 2x^T(A_0^T C + PC)\dot{x}_\tau + 2x^T(PBC + A_0^T BC)x_{2\tau}\\
&- 2x_\tau^T C^T B^T BCx_{2\tau} - 2x_\tau^T C^T B^T C\dot{x}_\tau + 2x_{2\tau}^T C^T B^T C\dot{x}_\tau.
\end{aligned} \tag{2.24}$$

Rearranging the mixed terms in (2.24) using Lemma 2.1, the following inequalities hold:

$$\begin{aligned}
-2x^T PBCx_\tau &\leq \xi_5 x^T PPx + \xi_5^{-1}x_\tau^T C^T B^T BCx_\tau,\\
-2x^T A_0^T BCx_\tau &\leq \xi_6 x^T A_0^T A_0 x + \xi_6^{-1}x_\tau^T C^T B^T BCx_\tau,\\
2x^T A_0^T C\dot{x}_\tau &\leq \xi_7 x^T A_0^T A_0 x + \xi_7^{-1}x_\tau^T C^T Cx_\tau,\\
2x^T PC\dot{x}_\tau &\leq \xi_8 x^T PPx + \xi_8^{-1}x_\tau^T C^T Cx_\tau,\\
-2x^T PBCx_{2\tau} &\leq \xi_9 x^T PPx + \xi_9^{-1}x_{2\tau}^T C^T B^T BCx_{2\tau},\\
-2x^T A_0^T BCx_{2\tau} &\leq \xi_{10} x^T A_0^T A_0 x + \xi_{10}^{-1}x_{2\tau}^T C^T B^T BCx_{2\tau},\\
-2x_\tau^T C^T B^T BCx_{2\tau} &\leq \xi_{11} x_\tau^T C^T B^T BCx_\tau + \xi_{11}^{-1}x_{2\tau}^T C^T B^T BCx_{2\tau},\\
-2x_\tau^T C^T B^T C\dot{x}_\tau &\leq \xi_{12} x_\tau^T C^T B^T BCx_\tau + \xi_{12}^{-1}\dot{x}_\tau^T C^T C\dot{x}_\tau,\\
2x_{2\tau}^T C^T B^T C\dot{x}_\tau &\leq \xi_{12} x_{2\tau}^T C^T B^T BCx_{2\tau} + \xi_{12}^{-1}\dot{x}_\tau^T C^T C\dot{x}_\tau.
\end{aligned} \tag{2.25}$$

Employing inequalities (2.25) in (2.24), we obtain

$$\begin{aligned}
\dot{W}_{(2.12)} &\leq x^T\big[A_0^T P + PA_0 + \gamma_1 A_0^T A_0 + \gamma_2 PP + \alpha_1 A^T A + \lambda_1^2\gamma_3 I + R_1 + R_2\big]x\\
&+ x_\tau^T\big[\gamma_4 C^T B^T BC + \alpha_1 B^T B + (\xi_7^{-1}+\xi_8^{-1})C^T C + \gamma_3\lambda_2^2 I - (1-\alpha_2)R_1\big]x_\tau\\
&+ x_{2\tau}^T\big[\gamma_5 C^T B^T BC - (1-2\alpha_2)R_2\big]x_{2\tau} + \dot{x}_\tau^T\big[\gamma_6 C^T C + \gamma_3\lambda_3^2 I - (1-\alpha_2)\big]\dot{x}_\tau.
\end{aligned} \tag{2.26}$$

Choose $R_1 := (1-\alpha_2)^{-1}\big[\gamma_4 C^T B^T BC + \alpha_1 B^T B + (\xi_7^{-1}+\xi_8^{-1})C^T C + \gamma_3\lambda_2^2 I\big]$ and $R_2 := (1-2\alpha_2)^{-1}\gamma_5 C^T B^T BC$ inequality (2.26) yields

$$\dot{W}_{(2.12)} \leq \begin{bmatrix} x\\ x_\tau\\ x_{2\tau}\\ \dot{x}_\tau\end{bmatrix}^T M_3 \begin{bmatrix} x\\ x_\tau\\ x_{2\tau}\\ \dot{x}_\tau\end{bmatrix}$$

where

$$M_3 := \begin{bmatrix} A_7 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & A_8\end{bmatrix},$$

$$A_7 := A_0^T P + PA_0 + \gamma_1 A_0^T A_0 + \gamma_2 PP + \alpha_1 A^T A + \left[(1-2\alpha_2)^{-1}\gamma_3 + (1-\alpha_2)^{-1}\gamma_4\right] C^T B^T BC$$
$$+ (1-\alpha_2)^{-1}\left[\alpha_1 B^T B + (\xi_7^{-1} + \xi_8^{-1}) C^T C\right] + \left[\lambda_1^2 + (1-\alpha_2)^{-1}\gamma_3 I\right] \text{ and}$$
$$A_8 := \gamma_6 C^T C + \left[\lambda_3^2 \gamma_3 - (1-\alpha_2)\right] I.$$

The function $\dot{W}_{(2.12)}$ is negative definite if M_3 is negative definite, thus the solution of system (2.12) is asymptotically stable. □

3. Examples

Example 3.1. Consider the following neutral delay differential system

$$\dot{x} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} x + \begin{bmatrix} 2 & 10 \\ 10 & 3 \end{bmatrix} x_\tau + \begin{bmatrix} 6 & 7 \\ 7 & 8 \end{bmatrix} \dot{x}_\tau + Q(t). \tag{3.1}$$

Equations (2.1) and (3.1) established that

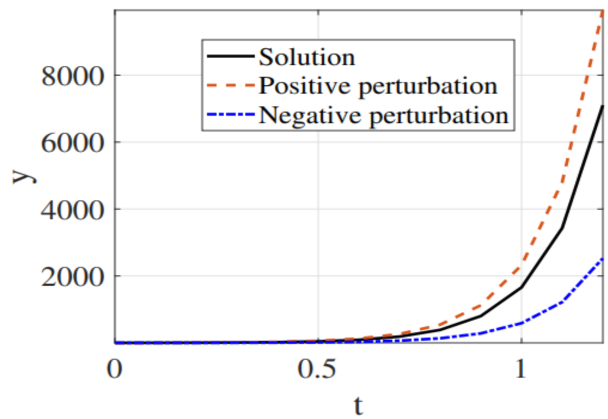

Figure 1. Path of solution of (3.1) and its perturbations in the neighbourhood $(-1, 1)$.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 10 \\ 10 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 7 \\ 7 & 8 \end{bmatrix}$$

and since β_i $(i = 1, 2, 3)$ is positive, the nonlinear perturbation Q is estimated to be

$$\|Q(t)\| \leq 0.1\|x\| + 0.01\|x_\tau\| + 0.001\|\dot{x}_\tau\|$$

so that $\beta_1 = 0.1$, $\beta_2 = 0.01$ and $\beta_3 = 0.001$. Moreover, since $0 < \alpha_2 < 1$ it follows that for any $\alpha_2 \in [0.001,\ 0.9]$ with $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.11$, $\varepsilon_3 = 0.111$ and $\varepsilon_4 = 0.1111$ the matrix $M_2 < 0$ if matrices

$$P := \begin{bmatrix} 2 & 0.1 \\ 0.1 & 3 \end{bmatrix} \text{ and } R := \begin{bmatrix} 4 & 0.01 \\ 0.01 & 2 \end{bmatrix}.$$

If $\alpha_2 = 0$, the case discussed in [25] is verified, i.e., $M_2 < 0$, thus the solution of system (3.1) is stable.
In addition, the exact solution of (3.1) using Matlab software is shown in Figure 1 in the neighbourhood $(-1, 1)$ with $0 < \alpha_2 < 1$, i.e., $\alpha_2 \in [0.001, 09]$, hence the solutions of (3.1) is not only stable but asymptotically stable.

4. Conclusion

In this paper, we have investigated the asymptotic stability of neutral differential systems with variable delay and nonlinear perturbations. We have established sufficient conditions for the asymptotic stability of the systems using Lyapunov-Krasovskiĭ functionals technique. The results obtained in this paper provide important insights into the stability properties of neutral differential systems with variable delay and nonlinear perturbations. Our findings contribute to the existing body of knowledge on stability analysis of time-delay systems and have potential applications in various engineering and scientific fields. Further research can be conducted to extend the results to more general classes of systems and to explore practical implementation of the stability conditions derived in this paper.

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Some Parseval-Goldstein Type Theorems For Generalized Integral Transforms

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Abstract

In this work, we establish some Parseval-Goldstein type identities and relations that include various new generalized integral transforms such as $\mathcal{L}_{\alpha,\mu}$-transform and generalized Stieltjes transform. In addition, we evaluated improper integrals of some fundamental and special functions using our results.

Keywords: *Generalized Laplace transform, Generalized Stieltjes transform, Laplace transform, Parseval-Goldstein theorem*

AMS Subject Classification (2020): *44A05; 44A10; 44A15; 44A20*

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1. Introduction, definitions and preliminaries

The theory of special functions and integral transforms constitute an important part of research subjects in mathematics, physics and engineering. Generally, an integral transform is defined by

$$T(y) = \mathcal{T}\{f(t); y\} = \int_a^b K(y,t) f(t) dt \tag{1.1}$$

where the function $f(t)$ defined in $a \le t \le b$, $K(y,t)$ is called the kernel of transform, and y is called the transform variable [1]. In the literature, some famous integral transforms are Laplace, Fourier and Stieltjes transforms. Many researchers have defined new integral transforms in the form of (1.1) by choosing different kernels and boundaries. In particular, the kernels of the transforms can be selected from special functions as well as elementary functions. The reader may refer to [1].

The Stieltjes transform of a function is obtained by applying the Laplace transform of the function twice. These kinds of relations, where consecutive integral transforms are applied, are referred to as Parseval-Goldstein type relations or theorems. Thus, the image of a function under an unknown new integral transform can be obtained through the successive applications of known integral transforms. As a result, these relations shed light on the calculation of many generalized integrals that have not yet been evaluated.

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In 1989, Yürekli [2] proved a Parseval-Goldstien type theorem which gives the relationship between Laplace and Stieltjes transforms and many results arising from this theorem. In 1992, he did a similar study for the generalized Stieltjes transform [3]. Later, many authors examined similar relationships between different integral transforms based on Parseval-Goldstein type theorems.[2–7].

Albayrak [8] considers a different generalization of Laplace transform over the set of functions

$$A=\left\{f(t)|\exists K,M,a\in\mathbb{R},\left|t^{\alpha-\mu}f\left(t\right)\right|\leq Ke^{at^{\mu}}\text{ for all }t\geq M,K>0\right\},$$

which is defined by

$$F\left(y\right)=\mathcal{L}_{\alpha,\mu}\left\{f\left(t\right);y\right\}=\int_{0}^{\infty}t^{\alpha-1}e^{-y^{\mu}t^{\mu}}f\left(t\right)dt, \tag{1.2}$$

and the inverse of $\mathcal{L}_{\alpha,\mu}-$transform is defined by

$$f(t)=\mathcal{L}_{\alpha,\mu}^{-1}\left\{F\left(y\right);t\right\}=\frac{\mu t^{\mu-\alpha}}{2\pi i}\int_{C}e^{yt^{\mu}}F\left(y^{1/\mu}\right)dy,$$

where $\alpha,y\in\mathbb{C}$, $\mu\in\mathbb{R}$, $\operatorname{Re}\alpha>\mu>0$, $\operatorname{Re}y>0.$ A generalization of the harmonic oscillator in non-resisting and resisting medium problems, initial-boundary value problems and integral equations are solved via this integral transform. Furthermore, the alternative solution of well-known series entitled as Basel problem is obtained in a similar way. The reader may refer to [8] for detailed information.

In this study, Parseval-Goldstein type theorem involving $\mathcal{L}_{\alpha,\mu}-$transform will be proved. Later, some generalized integrals will be evaulated as applications of these theorems.

With special choices of α and μ, $\mathcal{L}_{\alpha,\mu}-$transform can be reduced to some classical integral transforms, such as $\mathcal{L}_{1,1}\left\{f\left(t\right);y\right\}=\mathcal{L}\left\{f\left(t\right);y\right\}$ Laplace transform [1], $\mathcal{L}_{2,2}\left\{f\left(t\right);y\right\}=\mathcal{L}_{2}\left\{f\left(t\right);y\right\}$ $\mathcal{L}_2-$transform which was introduced by Yürekli and Sadek [9], $\mathcal{L}_{\alpha,1}\left\{f\left(t\right);y\right\}=\mathcal{L}_{\alpha}\left\{f\left(t\right);y\right\}$ another generalized Laplace transform which is defined by Karataş et al [6, 7] and $\mathcal{L}_{\mu\omega,\mu}\left\{f\left(t\right);y\right\}=\frac{1}{\omega y^{\mu\omega-1}}\mathfrak{B}_{\omega,\mu}\left\{f\left(t\right);y\right\}$ Borel-Džrbashjan transform [10, 11]. If we make a change of variable $t=u^{\frac{1}{\mu}}$ in the right-hand side of (1.2), we get the following relationship between the Laplace transform and the $\mathcal{L}_{\alpha,\mu}-$transform

$$\mathcal{L}_{\alpha,\mu}\left\{f\left(t\right);y\right\}=\frac{1}{\mu}\mathcal{L}\left\{t^{\frac{\alpha}{\mu}-1}f\left(t^{\frac{1}{\mu}}\right);y^{\mu}\right\}. \tag{1.3}$$

In the literature, some generalizations of the Stieltjes transform have been examined by many authors and their applications have been included. We will also describe a new generalized Stieltjes transform obtained by applying $\mathcal{L}_{\alpha,\mu}$-transform sequentially. In addition, under appropriate conditions of convergence, we will introduce some new generalized integral transforms with the help of $\mathcal{L}_{\alpha,\mu}$ or some integral transforms.

The generalized Stieltjes-type transform of $f\left(x\right)$, is defined by

$$S_{\alpha,\mu,\rho}(y)=\mathcal{S}_{\alpha,\mu,\rho}\left\{f\left(t\right);y\right\}=\int_{0}^{\infty}\frac{t^{\alpha-1}}{\left(y^{\mu}+t^{\mu}\right)^{\rho}}f\left(t\right)dt \tag{1.4}$$

where $\operatorname{Re}\alpha>0$, $\operatorname{Re}\mu>0$, $\operatorname{Re}\rho>0$ and the inverse of generalized Stieltjes-type transform is defined by

$$f(t)=\mathcal{S}_{\alpha,\mu,\rho}^{-1}\left\{S_{\alpha,\mu,\rho}(y);t\right\}=\frac{\mu t^{\mu-\alpha}(\rho-1)}{2\pi i}\int_{C}\left(t^{\mu}+y\right)^{\rho-2}S_{\alpha,\mu,\rho}\left(y^{1/\mu}\right)dy,$$

where $\operatorname{Re}\alpha>0$, $\operatorname{Re}\mu>0$, $\operatorname{Re}\rho>1$. With special choices of α,μ and ρ, $\mathcal{S}_{\alpha,\mu,\rho}-$transform can be reduced to some classical integral transforms, such as $\mathcal{S}_{1,1,1}\left\{f\left(t\right);y\right\}=\mathcal{S}\left\{f\left(t\right);y\right\}$ Stieltjes transform [1, 12], $\mathcal{S}_{2,2,1}\left\{f\left(t\right);y\right\}=\mathcal{P}\left\{f\left(t\right);y\right\}$ Widder-Potential transform [13], $\mathcal{S}_{1,2,1/2}\left\{f\left(t\right);y\right\}=\mathcal{G}\left\{f\left(t\right);y\right\}$, Glasser transform [14], $\mathcal{S}_{1,1,\rho}\left\{f\left(t\right);y\right\}=\mathcal{S}_{\rho}\left\{f\left(t\right);y\right\}$ generalized Stieltjes transform [12], $\mathcal{S}_{2,2,\rho}\left\{f\left(t\right);y\right\}=\mathcal{P}_{\rho,2}\left\{f\left(t\right);y\right\}$ generalized Widder-Potential transform [15].

If we make a change of variable $t=u^{\frac{1}{\mu}}$ in the right-hand side of (1.4), we have the following relationship between the generalized Stieltjes transform and the generalized Stieltjes-type transform

$$\mathcal{S}_{\alpha,\mu,\rho}\left\{f\left(t\right);y\right\}=\frac{1}{\mu}\mathcal{S}_{\rho}\left\{t^{\frac{\alpha}{\mu}-1}f\left(t^{\frac{1}{\mu}}\right);y^{\mu}\right\}.$$

Beside the generalized Stieltjes type integral transform, some generalized integral transforms that will shed light on the study will be defined as follows under appropriate convergence conditions.

First, let's give definitions of integral transforms that we want to generalize. Fourier sine and Fourier cosine integral transforms [1], respectively, are defined by

$$\mathcal{F}_s\left\{f(t);y\right\}=\int_0^\infty \sin(yt)\,f(t)\,dt,$$
$$\mathcal{F}_c\left\{f(t);y\right\}=\int_0^\infty \cos(yt)\,f(t)\,dt.$$

Now, we will define a generalized form of these integral transforms under appropriate convergence conditions. Generalized Fourier sine and cosine integral transforms are defined as follow

$$F_{s,\alpha,\mu}(y)=\mathcal{F}_{s,\alpha,\mu}\left\{f(t);y\right\}=\int_0^\infty t^{\alpha-1}\sin\left(y^\mu t^\mu\right)f(t)\,dt, \tag{1.5}$$
$$F_{c,\alpha,\mu}(y)=\mathcal{F}_{c,\alpha,\mu}\left\{f(t);y\right\}=\int_0^\infty t^{\alpha-1}\cos\left(y^\mu t^\mu\right)f(t)\,dt, \tag{1.6}$$

where $y^\mu>0$, $f(t)$ is piecewise continuous and $t^{\alpha-1}f(t)$ is absolutely integrable over $[0,\infty)$. The inverse of $\mathcal{F}_{s,\alpha,\mu}-$transform and $\mathcal{F}_{c,\alpha,\mu}-$transform are defined by

$$f(t)=\mathcal{F}^{-1}_{s,\alpha,\mu}\left\{F_{s,\alpha,\mu}(y);t\right\}=\frac{2\mu t^{\mu-\alpha}}{\pi}\int_0^\infty \sin\left(t^\mu y\right)F_{s,\alpha,\mu}(y^{1/\mu})dy,$$
$$f(t)=\mathcal{F}^{-1}_{c,\alpha,\mu}\left\{F_{c,\alpha,\mu}(y^{1/\mu});t\right\}=\frac{2\mu t^{\mu-\alpha}}{\pi}\int_0^\infty \cos\left(t^\mu y\right)F_{c,\alpha,\mu}(y^{1/\mu})dy,$$

where $F_{s,\alpha,\mu}(y^{1/\mu})$ and $F_{c,\alpha,\mu}(y^{1/\mu})$ are piecewise continuous and absolutely integrable over $[0,\infty)$. Now, let's give special functions that will be used throughout the study [16].

The Gamma function is defined by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\quad \mathrm{Re}(z)>0.$$

Basic properties of Gamma function are given in [16]. Pochammer symbol is defined by the following relation,

$$(\alpha)_n=\begin{cases}\alpha(\alpha+1)\dots(\alpha+n-1), & n=1,2,3\dots\\ 1, & n=0\end{cases}$$

where $\alpha\in\mathbb{R}$. The relationship between the Pochammer symbol and the gamma function is given by

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},\quad \alpha\neq 0,1,2,\dots$$

The generalized hypergeometric series is defined as

$${}_rF_s\left[\begin{array}{c}\alpha_1,\alpha_2,\dots,\alpha_r\\ \beta_1,\beta_2,\dots,\beta_s\end{array}\mid z\right]=\sum_{n=0}^\infty\frac{(\alpha_1)_n(\alpha_2)_n\dots(\alpha_r)_n}{(\beta_1)_n(\beta_2)_n\dots(\beta_s)_n}\frac{z^n}{n!},$$

where $r,s\in\mathbb{Z}^+\cup\{0\}$ and $\alpha_i,\beta_j\neq 0,-1,-2,\dots(1\le i\le r,1\le j\le s)$. The reader may refer to [16] for detailed information about the convergence conditions of this series. The Laplace transform of a generalized hypergeometric function ${}_rF_s$ in [17, p.219,Entry(17)] as follows:

$$\int_0^\infty e^{-yt}t^{v-1}{}_rF_s\left[\begin{array}{c}\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{array}\mid at\right]dt=\frac{\Gamma(v)}{y^v}{}_{r+1}F_s\left[\begin{array}{c}v,\alpha_1,\cdots,\alpha_r\\ \beta_1,\cdots,\beta_s\end{array}\mid \frac{a}{y}\right] \tag{1.7}$$

provided if $r<s,\mathrm{Re}(v)>0,\mathrm{Re}(y)>0$ and a is arbitrary or if $r=s>0,\mathrm{Re}(v)>0$ and $\mathrm{Re}(y)>\mathrm{Re}(a)$.

The confluent hypergeometric function is defined [16] as follows:

$$
{}_1\Phi_1(a;c;x) = M(a;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!}
$$

where $|x| < \infty$; $c \neq 0, -1, -2, \ldots$. The confluent hypergeometric function of second kind is defined by

$$
U(a;c;x) = \frac{\pi}{\sin(\pi c)} \left[\frac{M(a;c;x)}{\Gamma(1+a-c)\Gamma(c)} - \frac{x^{1-c} M(1+a-c; 2-c; x)}{\Gamma(a)\Gamma(2-c)} \right].
$$

The integral representation of $U(a;c;x)$ is given by

$$
U(a;c;x) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-xt} t^{a-1} (1+t)^{c-a-1} dt
$$

where $a > 0, c > 0, c \neq 1, 2, \ldots$. In [18], Ferreira and Salinas defined the incomplete generalized gamma function by using the confluent hypergeometric function of the second kind as follows:

$$
{}_{\lambda}\gamma_{\omega}\left(p, \delta; a; c; \nu\right) = \int_0^{\omega} x^{\lambda-1} e^{-px^{\delta}} U(a;c;\nu x^{\delta}) dx
$$

where $x > 0$, $\delta > 0$, $p > 0$, a and c are arbitrary constants. Motivated by this definition, we define the following integral transform

$$
{}_{\lambda}\gamma_{\infty}\left(p, \delta; a; c; \nu; f(x)\right) = \int_0^{\infty} x^{\lambda-1} e^{-px^{\delta}} U(a;c;\nu x^{\delta}) f(x) dx.
$$

In [18], Ferreira and Salinas evaluated the following integral,

$$
\int_0^{\infty} x^{\lambda-1} e^{-px^{\delta}} U\left(a;c;vx^{\delta}\right) dx = \frac{\pi}{\delta \sin(\pi c) p^{\frac{\lambda}{\delta}}} \left[\frac{A(p,\delta,\lambda,a,c,v) - B(p,\delta,\lambda,a,c,v)}{p^{1-c}\Gamma(1+a-c)\Gamma(c)\Gamma(a)\Gamma(2-c)} \right] \tag{1.8}
$$

where

$$
A(p,\delta,\lambda,a,c,v) = p^{1-c}\Gamma(a)\Gamma\left(\frac{\lambda}{\delta}\right)\Gamma(2-c)\,{}_2F_1\left(a, \frac{\lambda}{\delta}; c; \frac{v}{p}\right),
$$
$$
B(p,\delta,\lambda,a,c,v) = v^{1-c}\Gamma(1+a-c)\Gamma(c)\Gamma\left(\frac{\lambda}{\delta} - c + 1\right){}_2F_1\left(1+a-c, \frac{\lambda}{\delta} - c + 1; 2-c; \frac{v}{p}\right)
$$

where $\lambda, \delta, p > 0$, λ, v, p are constants such as $0 < v < p, c < 1, c \notin \mathbb{Z}$ and $a, 1+a-c \notin \mathbb{Z}^-$. But, using the relation

$$
\begin{aligned}
{}_2F_1(a,b;c;z) = {} & \frac{\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(1-c)\Gamma(a+b-c+1)}\,{}_2F_1(a,b;a+b-c+1;1-z) \\
& - \frac{\Gamma(a-c+1)\Gamma(b-c+1)\Gamma(c-1)}{\Gamma(a)\Gamma(b)\Gamma(1-c)} z^{1-c}\,{}_2F_1(a-c+1, b-c+1; 2-c; z)
\end{aligned}
$$

of the generalized hypergeometric function, we can write it as

$$
\int_0^{\infty} x^{\lambda-1} e^{-px^{\delta}} U\left(a;c;vx^{\delta}\right) dx = \frac{1}{\delta p^{\frac{\lambda}{\delta}}} \frac{\Gamma\left(\frac{\lambda}{\delta}\right)\Gamma\left(\frac{\lambda}{\delta} - c + 1\right)}{\Gamma\left(a + \frac{\lambda}{\delta} - c + 1\right)}\,{}_2F_1\left(a, \frac{\lambda}{\delta}; a + \frac{\lambda}{\delta} - c + 1; 1 - \frac{v}{p}\right). \tag{1.9}
$$

2. Parseval-Goldstein type theorems

In this section, we will prove some identities and Parseval-Goldstein type theorems.

The following lemma shows that the generalized Stieltjes transform can be obtained by applying $\mathcal{L}_{\alpha,\mu}$-transform and $\mathcal{L}_{\delta,\mu}$-transform consecutively.

Lemma 2.1. *Let $F(y)=\mathcal{L}_{\delta,\mu}\{f(t);y\}$. If $x,y,\alpha,\delta\in\mathbb{C}$, $\mu\in\mathbb{R}$ and $f,F\in A$, then the following identity*

$$\mathcal{L}_{\alpha,\mu}\{\mathcal{L}_{\delta,\mu}\{f(t);x\};y\}=\frac{1}{\mu}\Gamma\left(\frac{\alpha}{\mu}\right)\mathcal{S}_{\delta,\mu,\frac{\alpha}{\mu}}\{f(t);y\} \tag{2.1}$$

holds true for $\operatorname{Re}\alpha>\mu>0$, $\operatorname{Re}\delta>\mu>0$, $\operatorname{Re}y>0$, $\operatorname{Re}x>0$, $\operatorname{Re}\left(\frac{\alpha}{\mu}\right)>0$ provided that the integrals involved converge absolutely.

Proof. Using the definition of (1.2), changing the order of integration, which is permissible by absolute convergence of the integrals involved, we get

$$\mathcal{L}_{\alpha,\mu}\{\mathcal{L}_{\delta,\mu}\{f(t);x\};y\}=\int_0^\infty t^{\delta-1}f(t)\mathcal{L}_{\alpha,\mu}\{1;\sqrt[\mu]{t^\mu+y^\mu}\}dt$$

and using the relation (1.3) and the formula

$$\mathcal{L}_{\alpha,\mu}\{1;\sqrt[\mu]{t^\mu+y^\mu}\}=\Gamma\left(\frac{\alpha}{\mu}\right)\frac{1}{\mu}\frac{1}{(t^\mu+y^\mu)^{\frac{\alpha}{\mu}}},$$

we arrive at (2.1). □

The following is a Parseval-Goldstein type theorem for $\mathcal{L}_{\alpha,\mu}$-transform and generalized Stieltjes transform.

Theorem 2.1. *If $f,g\in A$, $\alpha,\delta\in\mathbb{C}$, $\mu,y\in\mathbb{R}$ and then the following identities*

$$\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\{f(t);y\}\mathcal{L}_{\delta,\mu}\{g(x);y\}dy=\frac{1}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\int_0^\infty t^{\alpha-1}f(t)\mathcal{S}_{\delta,\mu,\frac{\lambda}{\mu}}\{g(x);t\}dt, \tag{2.2}$$

$$\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\{f(t);y\}\mathcal{L}_{\delta,\mu}\{g(x);y\}dy=\frac{1}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\int_0^\infty x^{\delta-1}g(x)\mathcal{S}_{\alpha,\mu,\frac{\lambda}{\mu}}\{f(t);x\}dx, \tag{2.3}$$

hold true for $\operatorname{Re}\alpha>\mu>0$, $\operatorname{Re}\delta>\mu>0$, $y>0$, $\operatorname{Re}\left(\frac{\lambda}{\mu}\right)>0$ provided that the integrals involved converge absolutely.

Proof. Using the definition (1.2) and changing the order of integration, we have

$$\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\{f(t);y\}\mathcal{L}_{\delta,\mu}\{g(x);y\}dy=\int_0^\infty t^{\alpha-1}f(t)\mathcal{L}_{\lambda,\mu}\{\mathcal{L}_{\delta,\mu}\{g(x);y\};t\}dt.$$

Using the identity (2.1) of Lemma 2.1, we arrive at (2.2). Proof of (2.3) is similar. □

As a result of Theorem 2.1, the following relation can be obtained from the equivalence of relations (2.2) and (2.3).

$$\int_0^\infty t^{\alpha-1}f(t)\mathcal{S}_{\delta,\mu,\frac{\lambda}{\mu}}\{g(x);t\}dt=\int_0^\infty x^{\delta-1}g(x)\mathcal{S}_{\alpha,\mu,\frac{\lambda}{\mu}}\{f(t);x\}dx.$$

The following lemma shows that the generalized Stieltjes transform can be obtained by applying $\mathcal{L}_{\alpha,\mu}$-transform and $\mathcal{F}_{s,\delta,\mu}$-transform consecutively or $\mathcal{L}_{\alpha,\mu}$-transform and $\mathcal{F}_{c,\delta,\mu}$-transform consecutively in both order.

Lemma 2.2. *Let $F(x)=\mathcal{L}_{\alpha,\mu}\{f(t);x\}$, $F_s(x)=\mathcal{F}_{s,\delta,\mu}\{f(t);x\}$ and $F_c(x)=\mathcal{F}_{c,\delta,\mu}\{f(t);x\}$. If $f,F_s,F_c\in A$, $\alpha,\delta\in\mathbb{C}$, and $x,y,\mu\in\mathbb{R}$, then the following identities*

$$\mathcal{L}_{\alpha,\mu}\{\mathcal{F}_{s,\delta,\mu}\{f(t);x\};y\}=\frac{1}{\mu}\Gamma\left(\frac{\alpha}{\mu}\right)\mathcal{S}_{\delta,2\mu,\frac{\alpha}{2\mu}}\left\{\sin\left[\frac{\alpha}{\mu}\arctan\left(\frac{t^\mu}{y^\mu}\right)\right]f(t);y\right\}, \tag{2.4}$$

$$\mathcal{F}_{s,\delta,\mu}\{\mathcal{L}_{\alpha,\mu}\{f(t);x\};y\}=\frac{1}{\mu}\Gamma\left(\frac{\delta}{\mu}\right)\mathcal{S}_{\alpha,2\mu,\frac{\delta}{2\mu}}\left\{\sin\left[\frac{\delta}{\mu}\arctan\left(\frac{y^\mu}{t^\mu}\right)\right]f(t);y\right\}, \tag{2.5}$$

$$\mathcal{L}_{\alpha,\mu}\{\mathcal{F}_{c,\delta,\mu}\{f(t);x\};y\}=\frac{1}{\mu}\Gamma\left(\frac{\alpha}{\mu}\right)\mathcal{S}_{\delta,2\mu,\frac{\alpha}{2\mu}}\left\{\cos\left[\frac{\alpha}{\mu}\arctan\left(\frac{t^\mu}{y^\mu}\right)\right]f(t);y\right\}, \tag{2.6}$$

$$\mathcal{F}_{c,\delta,\mu}\{\mathcal{L}_{\alpha,\mu}\{f(t);x\};y\}=\frac{1}{\mu}\Gamma\left(\frac{\delta}{\mu}\right)\mathcal{S}_{\alpha,2\mu,\frac{\delta}{2\mu}}\left\{\cos\left[\frac{\delta}{\mu}\arctan\left(\frac{y^\mu}{t^\mu}\right)\right]f(t);y\right\}, \tag{2.7}$$

hold true for $\operatorname{Re}\alpha>\mu>0$, $x^\mu>0$, $y^\mu>0$, $\operatorname{Re}\left(\frac{\alpha}{2\mu}\right)>0$, $\operatorname{Re}\left(\frac{\delta}{2\mu}\right)>0$, $f(t)$ and $F(x)$ are piecewise continuous, $t^{\delta-1}f(t)$ and $x^{\delta-1}F(x)$ are absolutely integrable over $[0,\infty)$ provided that the integrals involved converge absolutely.

Proof. Using the definitions of (1.2) and (1.5), changing the order of integration, which is permissible by absolute convergence of the integrals involved, we have

$$\mathcal{L}_{\alpha,\mu}\left\{\mathcal{F}_{s,\delta,\mu}\left\{f\left(t\right);x\right\};y\right\}=\int_0^\infty t^{\delta-1}f\left(t\right)\mathcal{L}_{\alpha,\mu}\left\{\sin\left(x^\mu t^\mu\right);y\right\}dt.$$

Using the relation (1.3), the known formula [17, p.152, Entry(15)]

$$\mathcal{L}\left\{t^{\nu-1}\sin\left(at\right);y\right\}=\frac{\Gamma\left(a\right)}{\left(a^2+y^2\right)^{\nu/2}}\sin\left[\nu\arctan\left(\frac{a}{y}\right)\right]$$

where $\mathrm{Re}(\nu)>-1$, $\mathrm{Re}(y)>|\,\mathrm{Im}(a)|$ and definition of (1.4), we arrive at (2.4). Similarly, using the definitions of (1.2) and (1.6) changing the order of integration, which is permissible by absolute convergence of the integrals involved, we have

$$\mathcal{L}_{\alpha,\mu}\left\{\mathcal{F}_{c,\delta,\mu}\left\{f\left(t\right);x\right\};y\right\}=\int_0^\infty t^{\delta-1}f\left(t\right)\mathcal{L}_{\alpha,\mu}\left\{\cos\left(x^\mu t^\mu\right);y\right\}dt.$$

Using the relation (1.3), the known formula [17, p.157, Entry(58)]

$$\mathcal{L}\left\{t^{\nu-1}\cos\left(at\right);y\right\}=\frac{\Gamma\left(a\right)}{\left(a^2+y^2\right)^{\nu/2}}\cos\left[\nu\arctan\left(\frac{a}{y}\right)\right]$$

where $\mathrm{Re}(\nu)>0$, $\mathrm{Re}(y)>|\,\mathrm{Im}(a)|$ and definition of (1.4), we arrive at (2.6). Proof of (2.5) and (2.7) are similar and can be made using the same definitions, relations and formulas. □

The following is a Parseval-Goldstein type theorem for $\mathcal{L}_{\alpha,\mu}$-transform, generalized Fourier cosine and sine transforms and generalized Stieltjes transform.

Theorem 2.2. *If $f\in A$, $g(x)$ is piecewise continuous and $t^{\delta-1}f(t)$ is absolutely integrable over $[0,\infty)$, $\alpha\in\mathbb{C}$, $\mu,y\in\mathbb{R}$, then the following identities*

$$\begin{aligned}&\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\left\{f\left(t\right);y\right\}\mathcal{F}_{s,\delta,\mu}\left\{g\left(x\right);y\right\}dy\\&\quad=\frac{1}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\int_0^\infty t^{\alpha-1}f\left(t\right)\mathcal{S}_{\delta,2\mu,\frac{\lambda}{2\mu}}\left\{\sin\left[\frac{\lambda}{\mu}\arctan\left(\frac{x^\mu}{t^\mu}\right)\right]g\left(x\right);t\right\}dt,\end{aligned}\tag{2.8}$$

$$\begin{aligned}&\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\left\{f\left(t\right);y\right\}\mathcal{F}_{s,\delta,\mu}\left\{g\left(x\right);y\right\}dy\\&\quad=\frac{1}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\int_0^\infty x^{\delta-1}g\left(x\right)\mathcal{S}_{\delta,2\mu,\frac{\lambda}{2\mu}}\left\{\sin\left[\frac{\lambda}{\mu}\arctan\left(\frac{x^\mu}{t^\mu}\right)\right]f\left(t\right);x\right\}dx,\end{aligned}\tag{2.9}$$

$$\begin{aligned}&\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\left\{f\left(t\right);y\right\}\mathcal{F}_{c,\delta,\mu}\left\{g\left(x\right);y\right\}dy\\&\quad=\frac{1}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\int_0^\infty t^{\alpha-1}f\left(t\right)\mathcal{S}_{\delta,2\mu,\frac{\lambda}{2\mu}}\left\{\cos\left[\frac{\lambda}{\mu}\arctan\left(\frac{x^\mu}{t^\mu}\right)\right]g\left(x\right);t\right\}dt,\end{aligned}\tag{2.10}$$

$$\begin{aligned}&\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\left\{f\left(t\right);y\right\}\mathcal{F}_{c,\delta,\mu}\left\{g\left(x\right);y\right\}dy\\&\quad=\frac{1}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\int_0^\infty x^{\delta-1}g\left(x\right)\mathcal{S}_{\delta,2\mu,\frac{\lambda}{2\mu}}\left\{\cos\left[\frac{\lambda}{\mu}\arctan\left(\frac{x^\mu}{t^\mu}\right)\right]f\left(t\right);x\right\}dx,\end{aligned}\tag{2.11}$$

hold true for $\mathrm{Re}\,\alpha>\mu>0$, $y^\mu>0$, $\mathrm{Re}\left(\frac{\lambda}{2\mu}\right)>0$ provided that the integrals involved converge absolutely.

Proof. Using the definition (1.2) and changing the order of integration, we have

$$\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\left\{f\left(t\right);y\right\}\mathcal{F}_{s,\delta,\mu}\left\{g\left(x\right);y\right\}dy=\int_0^\infty t^{\alpha-1}f\left(t\right)\mathcal{L}_{\lambda,\mu}\left\{\mathcal{F}_{s,\delta,\mu}\left\{g\left(x\right);y\right\};t\right\}dt.$$

Using the identity (2.4) of Lemma (2.2), we arrive at (2.8). Proof of (2.9) is similar and can be made using the definition (1.5) and identity (2.5) of Lemma (2.2). Using the definition (1.2), changing the order of integration and using the identity (2.6) of Lemma (2.2), we arrive at (2.10). Proof of (2.11) is similar and can be made using the definition (1.6) and identity (2.7) of Lemma (2.2). □

The following lemma shows that the improper integral involving the confluent hypergeometric function of second kind can be obtained by applying $\mathcal{L}_{\alpha,\mu}$-transform and generalized Stieltjes integral transform consecutively in both order.

Lemma 2.3. *Let $S(x) = \mathcal{S}_{\delta,\mu,\rho}\{f(t);x\}$. If $\alpha \in \mathbb{C}$, $x, y, \mu \in \mathbb{R}$ and $f, S \in A$, then the following identities*

$$\mathcal{L}_{\alpha,\mu}\{\mathcal{S}_{\delta,\mu,\rho}\{f(t);x\};y\} = \frac{\Gamma(\frac{\alpha}{\mu})}{\mu}\int_0^\infty t^{\delta+\alpha-\mu\rho-1}U\left(\frac{\alpha}{\mu};1+\frac{\alpha}{\mu}-\rho;t^\mu y^\mu\right)f(t)dt, \tag{2.12}$$

$$\mathcal{S}_{\delta,\mu,\rho}\{\mathcal{L}_{\alpha,\mu}\{f(t);x\};y\} = \frac{y^{\delta-\mu\rho}}{\mu}\Gamma\left(\frac{\delta}{\mu}\right)\int_0^\infty t^{\alpha-1}U\left(\frac{\delta}{\mu};1+\frac{\delta}{\mu}-\rho;t^\mu y^\mu\right)f(t)dt \tag{2.13}$$

hold true for $\operatorname{Re}\alpha > \mu > 0$, $\operatorname{Re}\delta > 0$, $x > 0$, $y > 0$, $\operatorname{Re}\left(\frac{\alpha}{\mu}\right) > 0$, $\operatorname{Re}\left(\frac{\delta}{\mu}\right) > 0$, $\operatorname{Re}\left(1+\frac{\alpha}{\mu}\right) > \operatorname{Re}\rho >$, $\operatorname{Re}\left(1+\frac{\delta}{\mu}\right) > \operatorname{Re}\rho > 0$ *provided that the integrals involved converge absolutely.*

Proof. Using the definitions of (1.2) and (1.4), changing the order of integration, which is permissible by absolute convergence of the integrals involved, we have

$$\mathcal{L}_{\alpha,\mu}\{\mathcal{S}_{\delta,\mu,\rho}\{f(t);x\};y\} = \int_0^\infty t^{\delta-1}f(t)\left[\int_0^\infty \frac{x^{\alpha-1}e^{-x^\mu y^\mu}}{(t^\mu+x^\mu)^\rho}dx\right]dt.$$

Now, making the change of variable $x = tu^{\frac{1}{\mu}}$ in the inner integral, we get

$$\mathcal{L}_{\alpha,\mu}\{\mathcal{S}_{\delta,\mu,\rho}\{f(t);x\};y\} = \frac{1}{\mu}\int_0^\infty t^{\delta+\alpha-\mu\rho-1}f(t)\left[\int_0^\infty \frac{u^{\frac{\alpha}{\mu}-1}}{(1+u)^\rho}e^{-t^\mu y^\mu u}du\right]dt.$$

Using the integral representation of the confluent hypergeometric function $U(a,b,z)$, we arrive at (2.12). Proof of (2.13) is similar and can be made using the same definitions and formulas. □

The following is a Parseval-Goldstein type theorem for $\mathcal{L}_{\alpha,\mu}$-transform, generalized Stieltjes transform and ${}_\lambda\gamma_\infty-$transform.

Theorem 2.3. *If $\alpha, \delta, \rho \in \mathbb{C}$, $\mu \in \mathbb{R}$ and $f \in A$, then the following identities*

$$\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\{f(t);y\}\mathcal{S}_{\delta,\mu,\rho}\{g(x);y\}dy$$
$$= \frac{\Gamma(\frac{\lambda}{\mu})}{\mu}\int_0^\infty t^{\alpha-1}f(t)\ {}_{\delta+\lambda-\mu\rho}\gamma_\infty\left(0;\mu;\frac{\lambda}{\mu};1+\frac{\lambda}{\mu}-\rho;t^\mu;g(x)\right)dt, \tag{2.14}$$

$$\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\{f(t);y\}\mathcal{S}_{\delta,\mu,\rho}\{g(x);y\}dy$$
$$= \frac{\Gamma(\frac{\lambda}{\mu})}{\mu}\int_0^\infty x^{\delta+\lambda-\mu\rho-1}g(x)\ {}_\alpha\gamma_\infty\left(0;\mu;\frac{\lambda}{\mu};1+\frac{\lambda}{\mu}-\rho;x^\mu;f(t)\right)dx, \tag{2.15}$$

hold true for $\operatorname{Re}\alpha > \mu > 0$, $\operatorname{Re}\delta > 0$, $y > 0$, $\operatorname{Re}\left(\frac{\alpha}{\mu}\right) > 0$, $\operatorname{Re}\left(\frac{\delta}{\mu}\right) > 0$, $\operatorname{Re}\left(1+\frac{\alpha}{\mu}\right) > \operatorname{Re}\rho >$, $\operatorname{Re}\left(1+\frac{\delta}{\mu}\right) > \operatorname{Re}\rho > 0$ *provided that the integrals involved converge absolutely.*

Proof. Using the definition (1.2) and changing the order of integration, we have

$$\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\{f(t);y\}\mathcal{S}_{\delta,\mu,\rho}\{g(x);y\}dy = \int_0^\infty t^{\alpha-1}f(t)\mathcal{L}_{\lambda,\mu}\{\mathcal{S}_{\delta,\mu,\rho}\{g(x);y\};t\}dt.$$

Using the identity (2.12) of Lemma 2.3, we arrive at (2.14). Proof of (2.15) is similar and can be using the definition (1.4) and identity (2.13) of Lemma 2.3. □

3. Applications

We know that [8]

$$\mathcal{L}_{\alpha,\mu}\left\{x^{\lambda-1};y\right\}=\frac{1}{\mu}\Gamma\left(\frac{\alpha+\lambda-1}{\mu}\right)\frac{1}{y^{\alpha+\lambda-1}} \tag{3.1}$$

where $\operatorname{Re} y>0$ and $\operatorname{Re}\left(\frac{\alpha+\lambda-1}{\mu}\right)>-1$.

In this section we give some applications of above lemmas and theorems.

Example 3.1. We show that

$$\mathcal{S}_{\delta,\mu,\frac{\alpha}{\mu}}\left\{t^{\lambda-1};y\right\}=\frac{1}{\mu y^{\alpha-\delta-\lambda+1}}B\left(\frac{\delta+\lambda-1}{\mu},\frac{\alpha-\delta-\lambda+1}{\mu}\right), \tag{3.2}$$

$$\mathcal{S}_{\delta,\mu,\frac{\alpha}{\mu}}\left\{e^{-a^{\mu}t^{\mu}};y\right\}=\frac{a^{\alpha-\delta}}{\mu}\Gamma\left(\frac{\delta}{\mu}\right)U\left(\frac{\alpha}{\mu};1+\frac{\alpha}{\mu}-\frac{\delta}{\mu};a^{\mu}y^{\mu}\right) \tag{3.3}$$

where $\operatorname{Re}\left(\frac{\delta+\lambda-1}{\mu}\right)>0$, $\operatorname{Re}\left(\frac{\alpha-\delta-\lambda+1}{\mu}\right)>0$, $\operatorname{Re}\left(\frac{\delta}{\mu}\right)>0$, $\operatorname{Re}\left(\frac{\alpha}{\mu}\right)>0$, $\operatorname{Re}\left(\frac{\alpha}{\mu}-\frac{\delta}{\mu}\right)>-1$ and U is a second kind of confluent hypergeometric function.

Setting $f(t)=t^{\lambda-1}$ in (2.1), we have

$$\mathcal{S}_{\delta,\mu,\frac{\alpha}{\mu}}\left\{t^{\lambda-1};y\right\}=\mu\left\{\Gamma\left(\frac{\alpha}{\mu}\right)\right\}^{-1}\mathcal{L}_{\alpha,\mu}\left\{\mathcal{L}_{\delta,\mu}\left\{t^{\lambda-1};x\right\};y\right\}.$$

Using the formula (3.1) successively, we obtain the formula (3.2). Setting $f(t)=e^{-a^{\mu}t^{\mu}}$ in (2.1), we have

$$\mathcal{S}_{\delta,\mu,\frac{\alpha}{\mu}}\left\{e^{-a^{\mu}t^{\mu}};y\right\}=\mu\left\{\Gamma\left(\frac{\alpha}{\mu}\right)\right\}^{-1}\mathcal{L}_{\alpha,\mu}\left\{\mathcal{L}_{\delta,\mu}\left\{e^{-a^{\mu}t^{\mu}};x\right\};y\right\}.$$

Using the definition (1.2) and the formula (3.1) for $\lambda=1$, we have

$$\begin{aligned}\mathcal{S}_{\delta,\mu,\frac{\alpha}{\mu}}\left\{e^{-a^{\mu}t^{\mu}};y\right\}&=\mu\left\{\Gamma\left(\frac{\alpha}{\mu}\right)\right\}^{-1}\mathcal{L}_{\alpha,\mu}\left\{\mathcal{L}_{\delta,\mu}\left\{1;\sqrt[\mu]{a^{\mu}+x^{\mu}}\right\};y\right\}\\&=\left\{\Gamma\left(\frac{\alpha}{\mu}\right)\right\}^{-1}\Gamma\left(\frac{\delta}{\mu}\right)\mathcal{L}_{\alpha,\mu}\left\{\frac{1}{(a^{\mu}+x^{\mu})^{\frac{\delta}{\mu}}};y\right\}\\&=\left\{\Gamma\left(\frac{\alpha}{\mu}\right)\right\}^{-1}\Gamma\left(\frac{\delta}{\mu}\right)\int_0^{\infty}\frac{x^{\alpha-1}e^{-y^{\mu}x^{\mu}}}{(a^{\mu}+x^{\mu})^{\frac{\delta}{\mu}}}dx.\end{aligned}$$

Now, making the change of variable $x=au^{\frac{1}{\mu}}$, we get

$$\mathcal{S}_{\delta,\mu,\frac{\alpha}{\mu}}\left\{e^{-a^{\mu}t^{\mu}};y\right\}=\frac{a^{\alpha-\delta}}{\mu}\int_0^{\infty}u^{\frac{\alpha}{\mu}-1}(1+u)^{-\frac{\delta}{\mu}}e^{-y^{\mu}a^{\mu}u}dx.$$

Using the integral representation of the confluent hypergeometric function $U(a,b,z)$, we arrive at (3.3).

Example 3.2. We show that

$$\int_0^{\infty}\frac{y^{\lambda-1}}{(a^{\mu}+y^{\mu})^{\frac{\alpha}{\mu}}(b^{\mu}+y^{\mu})^{\frac{\delta}{\mu}}}dy=\frac{b^{\lambda-\delta}}{\mu a^{\alpha}}\frac{\Gamma\left(\frac{\lambda}{\mu}\right)\Gamma\left(\frac{\alpha-\lambda+\delta}{\mu}\right)}{\Gamma\left(\frac{\alpha+\delta}{\mu}\right)}{}_2F_1\left(\frac{\alpha}{\mu},\frac{\lambda}{\mu};\frac{\alpha+\delta}{\mu};1-\frac{b^{\mu}}{a^{\mu}}\right). \tag{3.4}$$

where $\operatorname{Re}\left(\frac{\lambda}{\mu}\right)>0$, $\operatorname{Re}\left(\frac{\alpha}{\mu}\right)>0$, $\operatorname{Re}\left(\frac{\alpha+\lambda}{\mu}\right)>\operatorname{Re}\left(\frac{\lambda}{\mu}\right)>0$.

If we set $f(t)=e^{-a^\mu t^\mu}$ and $g(x)=e^{-b^\mu x^\mu}$ in (2.2) and use the definition (1.2) and the formula (3.3), we have

$$\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\left\{e^{-a^\mu t^\mu};y\right\}\mathcal{L}_{\delta,\mu}\left\{e^{-b^\mu x^\mu};y\right\}dy=\frac{1}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\int_0^\infty t^{\alpha-1}e^{-a^\mu t^\mu}\mathcal{S}_{\delta,\mu,\frac{\lambda}{\mu}}\left\{e^{-b^\mu x^\mu};t\right\}dt$$

and

$$\begin{aligned}&\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\left\{1;\sqrt[\mu]{a^\mu+y^\mu}\right\}\mathcal{L}_{\delta,\mu}\left\{1;\sqrt[\mu]{b^\mu+y^\mu}\right\}dy\\&=\frac{b^{\lambda-\delta}}{\mu^2}\Gamma\left(\frac{\lambda}{\mu}\right)\Gamma\left(\frac{\delta}{\mu}\right)\int_0^\infty t^{\alpha-1}e^{-a^\mu t^\mu}U\left(\frac{\lambda}{\mu};1+\frac{\lambda-\delta}{\mu};b^\mu t^\mu\right)dt.\end{aligned}$$

Using the formulas (3.1) for $\lambda=1$ and (1.9), we arrive at (3.4).

Example 3.3. We show that

$$\mathcal{S}_{\delta,2\mu,\frac{\alpha}{2\mu}}\left\{t^{-\nu}\sin\left[\frac{\alpha}{\mu}\arctan\left(\frac{t^\mu}{y^\mu}\right)\right];y\right\}=\frac{1}{y^{\alpha+\nu-\delta}\mu}B\left(\frac{\alpha-\delta+\nu}{\mu},\frac{\delta-\nu}{\mu}\right)\sin\left[\frac{\pi}{2}\left(\frac{\delta-\nu}{\mu}\right)\right]\tag{3.5}$$

where $0<\operatorname{Re}\left(\frac{\delta-\nu}{\mu}\right)<2$.

If we choose $f(t)=t^{-\nu}$ in (2.4), we have

$$\mathcal{L}_{\alpha,\mu}\left\{\mathcal{F}_{s,\delta,\mu}\left\{t^{-\nu};x\right\};y\right\}=\frac{1}{\mu}\Gamma\left(\frac{\alpha}{\mu}\right)\mathcal{S}_{\delta,2\mu,\frac{\alpha}{2\mu}}\left\{\sin\left[\frac{\alpha}{\mu}\arctan\left(\frac{t^\mu}{y^\mu}\right)\right]t^{-\nu};y\right\}.\tag{3.6}$$

Firstly, let's find the inner transform on the left side of the identity

$$\mathcal{F}_{s,\delta,\mu}\left\{t^{-\nu};x\right\}=\int_0^\infty t^{\delta-\nu-1}\sin\left(x^\mu t^\mu\right)dt.$$

Making the change of variable $x=u^{\frac{1}{\mu}}$ and using the formula [17, p.68, (1)], we get

$$\mathcal{F}_{s,\delta,\mu}\left\{t^{-\nu};x\right\}=\frac{x^{\nu-\delta}}{\mu}\Gamma\left(\frac{\delta-\nu}{\mu}\right)\sin\left[\frac{\pi}{2}\left(\frac{\delta-\nu}{\mu}\right)\right].\tag{3.7}$$

Finally, setting the result (3.7) in (3.6) and using the formula (3.1), we arrive at (3.5).

Example 3.4. We show that

$$\mathcal{S}_{\lambda-\delta,2\mu,\frac{\alpha+\beta}{2\mu}}\left\{P^{-\nu}_{\frac{\alpha+\beta}{\mu}-1}\left[\frac{y^\mu}{\sqrt{y^{2\mu}+a^{2\mu}}}\right];a\right\}=\Gamma\left(\frac{\lambda-\delta}{\mu}\right)\frac{2^{\frac{\alpha+\beta+\delta-\lambda}{\mu}-1}a^{\lambda-\alpha-\beta-\delta-3\mu}\Gamma\left(\frac{\alpha+\beta+\delta-\lambda}{\mu}+\frac{\nu}{2}\right)}{\mu\Gamma\left(\frac{\alpha+\beta}{\mu}+\nu\right)\Gamma\left(\frac{\nu-1}{2}-\frac{\alpha+\beta+\delta-\lambda}{\mu}\right)\cos\left(\frac{\pi\delta}{2\mu}\right)}.\tag{3.8}$$

where $\operatorname{Re}\left(\frac{\alpha+\beta+\delta-\lambda}{\mu}+\frac{\nu}{2}\right)>0$, $\operatorname{Re}\left(\frac{\alpha+\beta}{\mu}+\nu\right)>0$ and $\operatorname{Re}\left(\frac{\nu-1}{2}-\frac{\alpha+\beta+\delta-\lambda}{\mu}\right)>0$.

Setting $f(t)=t^\beta\mathcal{J}_\nu\left(a^\mu t^\mu\right)$ and $g(x)=1$ in (2.8), we obtain

$$\begin{aligned}&\int_0^\infty y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\left\{t^\beta\mathcal{J}_\nu\left(a^\mu t^\mu\right);y\right\}\mathcal{F}_{s,\delta,\mu}\left\{1;y\right\}dy\\&=\frac{1}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\int_0^\infty t^{\alpha+\beta-1}\mathcal{J}_\nu\left(a^\mu t^\mu\right)\mathcal{S}_{\delta,2\mu,\frac{\lambda}{2\mu}}\left\{\sin\left[\frac{\lambda}{\mu}\arctan\left(\frac{x^\mu}{t^\mu}\right)\right];t\right\}dt.\end{aligned}\tag{3.9}$$

To start with, let's find the first transform on the left side of the identity

$$\mathcal{L}_{\alpha,\mu}\left\{t^\beta\mathcal{J}_\nu\left(a^\mu t^\mu\right);y\right\}=\frac{1}{\mu}\int_0^\infty t^{\alpha+\beta-1}e^{t^\mu y^\mu}\mathcal{J}_\nu\left(a^\mu t^\mu\right)dt.$$

Making the change of variable $x = u^{\frac{1}{\mu}}$ and use the formula [12, p.29, (6)], we get

$$\mathcal{L}_{\alpha,\mu}\left\{t^{\beta}\mathcal{J}_{\nu}\left(a^{\mu}t^{\mu}\right);y\right\} = \frac{1}{\mu}\Gamma\left(\frac{\alpha+\beta}{\mu}+\nu\right)\frac{1}{\left(y^{2\mu}+a^{2\mu}\right)^{\frac{\alpha+\beta}{2\mu}}}P^{-\nu}_{\frac{\alpha+\beta}{\mu}-1}\left[\frac{y^{\mu}}{\sqrt{y^{2\mu}+a^{2\mu}}}\right]. \tag{3.10}$$

Secondly, using the formula (3.7) for $v = 0$, we get

$$\mathcal{F}_{s,\delta,\mu}\left\{1;y\right\} = \frac{1}{\mu}y^{-\delta}\sin\left(\frac{\pi\delta}{\mu}\right)\Gamma\left(\frac{\delta}{\mu}\right), \tag{3.11}$$

and using the formula (3.5), for $v = 0$, we obtain

$$\mathcal{S}_{\delta,2\mu,\frac{\lambda}{2\mu}}\left\{\sin\left[\frac{\lambda}{\mu}\arctan\left(\frac{x^{\mu}}{t^{\mu}}\right)\right];t\right\} = \frac{1}{t^{\lambda-\delta}\mu}B\left(\frac{\lambda-\delta}{\mu},\frac{\delta}{\mu}\right)\sin\left(\frac{\pi\delta}{2\mu}\right). \tag{3.12}$$

Now, setting the results (3.10), (3.11) and (3.12) in (3.9), we obtain

$$\begin{aligned}&\Gamma\left(\frac{\alpha+\beta}{\mu}+\nu\right)\Gamma\left(\frac{\delta}{\mu}\right)\cos\left(\frac{\pi\delta}{2\mu}\right)\mathcal{S}_{\lambda-\delta,2\mu,\frac{\alpha+\beta}{2\mu}}\left\{P^{-\nu}_{\frac{\alpha+\beta}{\mu}-1}\left[\frac{y^{\mu}}{\sqrt{y^{2\mu}+a^{2\mu}}}\right];a\right\}\\&=\frac{1}{2}\Gamma\left(\frac{\lambda}{\mu}\right)B\left(\frac{\lambda-\delta}{\mu},\frac{\delta}{\mu}\right)\int_0^{\infty}t^{\alpha+\beta+\delta-\lambda-1}\mathcal{J}_{\nu}\left(a^{\mu}t^{\mu}\right)dt.\end{aligned}$$

Finally, making the change of variable $x = u^{\frac{1}{\mu}}$ on the right side of the equation and using the formula [12, p.22, (7)], we arrive at (3.8).

Example 3.5. We show that

$$\begin{aligned}&\mathcal{S}_{\lambda+\delta-\mu\rho,\mu,\frac{\alpha}{\mu}}\left\{U\left(\frac{\delta}{\mu};1+\frac{\delta}{\mu}-\rho;b^{\mu}y^{\mu}\right);a\right\} = \frac{1}{a^{\alpha+\lambda-\mu\rho}}\frac{\Gamma\left(\rho-\frac{\lambda}{\mu}\right)}{\Gamma(\rho)\Gamma\left(\frac{\alpha}{\mu}\right)\Gamma\left(\frac{\delta}{\mu}\right)\Gamma\left(1+\rho-\frac{\lambda}{\mu}\right)}\\&\times\left\{\frac{a^{\mu\rho-\lambda}}{b^{\delta}\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\Gamma\left(\frac{\delta}{\mu}\right)\Gamma\left(\frac{\alpha}{\mu}+\frac{\lambda}{\mu}-\rho\right)\Gamma\left(1+\rho-\frac{\lambda}{\mu}\right){}_3F_1\left(\frac{\delta}{\mu},\frac{\lambda}{\mu},\frac{\delta}{\mu}+\frac{\lambda}{\mu}-\rho;1+\frac{\lambda}{\mu}-\rho;\frac{b^{\mu}}{a^{\mu}}\right)\right.\\&\left.+\frac{1}{b^{\rho-\frac{\lambda}{\mu}+\frac{\alpha}{\mu}}\mu}\Gamma(\rho)\Gamma\left(\frac{\alpha}{\mu}\right)\Gamma\left(\frac{\lambda}{\mu}+1-\rho\right)\Gamma\left(\frac{\alpha}{\mu}-\frac{\lambda}{\mu}+\rho\right){}_3F_1\left(\frac{\alpha}{\mu}-\frac{\lambda}{\mu}+\rho,\rho,\frac{\alpha}{\mu};1+\rho-\frac{\lambda}{\mu};\frac{b^{\mu}}{a^{\mu}}\right)\right\}.\end{aligned} \tag{3.13}$$

where $\operatorname{Re}\frac{\alpha}{\mu} > 0$, $\operatorname{Re}\frac{\delta}{\mu} > 0$, $\operatorname{Re}\frac{\lambda}{\mu} > 0$ and $\operatorname{Re}\left(\frac{\alpha+\lambda}{\mu}\right) > \operatorname{Re}\rho > \operatorname{Re}\frac{\lambda}{\mu} - 1$.

Setting $f(t) = e^{-a^{\mu}t^{\mu}}$ and $g(x) = e^{-b^{\mu}x^{\mu}}$ in (2.15), we obtain

$$\begin{aligned}&\int_0^{\infty}y^{\lambda-1}\mathcal{L}_{\alpha,\mu}\left\{e^{-a^{\mu}t^{\mu}};y\right\}\mathcal{S}_{\delta,\mu,\rho}\left\{e^{-b^{\mu}t^{\mu}};y\right\}dy\\&=\frac{\Gamma(\frac{\lambda}{\mu})}{\mu}\int_0^{\infty}x^{\delta-1}e^{-b^{\mu}x^{\mu}}\;{}_{\alpha+\lambda-\mu\rho}\gamma_{\infty}\left(0;\mu;\frac{\lambda}{\mu};1+\frac{\lambda}{\mu}-\rho;x^{\mu};e^{-a^{\mu}t^{\mu}}\right)dx.\end{aligned}$$

On the other hand, using the formulas (3.3), (1.8), (1.7) and

$$\mathcal{L}_{\alpha,\mu}\left\{e^{-a^{\mu}t^{\mu}};y\right\} = \mathcal{L}_{\alpha,\mu}\left\{1;\sqrt[\mu]{y^{\mu}+a^{\mu}}\right\} = \Gamma\left(\frac{\alpha}{\mu}\right)\frac{1}{\mu}\frac{1}{\left(a^{\mu}+y^{\mu}\right)^{\frac{\alpha}{\mu}}}$$

and the definition (1.4), we arrive at (3.13).

4. Conclusion

In this work, we establish Parseval-Goldstein type relations and identities that include various integral transforms such as $\mathcal{L}_{\alpha,\mu}$-transform and generalized Stieltjes transform. Thus, using these results, we show how simple it can be to evaulate integral transforms of some elementary and special functions. It is possible to obtain all the results and applications in [2, 3, 9] when $\alpha = \mu = \delta = \lambda = \rho = 1$ is chosen in all lemmas, theorems and applications in Sections 2 and 3.

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Certain Results for Invariant Submanifolds of an Almost α-Cosymplectic (k, μ, ν)-Space

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Abstract
In this paper we present invariant submanifolds of an almost α-cosymplectic (k, μ, ν)-space. Then, we gave some results for an invariant submanifold of an almost α-cosymplectic (k, μ, ν)-space to be totally geodesic. As a result, we have discovered some interesting conclusions about invariant submanifolds of an almost cosymplectic (k, μ, ν)-space.

Keywords: *α-cosymplectic (k, μ, ν)-space, W_1^*-curvature tensor, W_7-curvature tensor*

AMS Subject Classification (2020): *53C15; 53C25; 53D25*

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1. Introduction

T. Koufogiorgos and C. Tsichlias found a new class of 3-dimensional contact metric manifolds that k and μ are non-constant smooth functions. They generalized $(k, \mu)-$contact metric manifolds on non-Sasakian manifolds for $n > 1$, where the functions k, μ are constants [1].

S. I. Goldberg and K. Yano obtained integrability conditions for almost cosymplectic structures on almost contact manifolds. The simplest examples of almost cosymplectic manifolds are these structures of almost Kaehler manifolds, the real $\mathbb{R}$ line and the circle S^1. Besides, they studied an almost cosymplectic manifold is cosymplectic only in the case it is locally flat [2].

İ. Küpeli Erken researched almost $\alpha-$cosymplectic manifolds. They studied, respectively, projectively flat, conformally flat and concircularly flat almost $\alpha-$cosymplectic manifolds (with the $\eta-$parallel tensor field ϕh). They devoted to properties of almost with the $\eta-$parallel tensor field ϕh [3].

For an almost contact metric structure to be almost cosymplectic, Z. Olszak provided a few necessary requirements. They established the absence of virtually cosymplectic manifolds in dimensions bigger than three with non-zero constant sectional curvature. Fortunately, such locally flat manifolds with zero sectional curvature do exist and were cosymplectic. Additionally, they looked at several constraints on virtually cosymplectic manifolds that had conformally flat surfaces or constant ϕ-sectional curvature [4].

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In 2022, M. Atçeken studied the invariant submanifolds of an almost α-cosymplectic (k,μ,ν)-space that satisfying certain geometric requirements so that $Q(\sigma,R)=0$,
$Q(S,\sigma)=0$, $Q(S,\widetilde{\nabla}\sigma)=0$, $Q(S,\widetilde{R}\cdot\sigma)=0$, $Q(g,C\cdot R)=0$ and $Q(S,C\cdot\sigma)=0$. He showed that under certain circumstances, these conditions are identical to totally geodesic [5]. Additionally, some geometers have worked on the almost Kenmotsu (k,μ,ν)-space [6–8].

Our article's focus is on invariant submanifolds of an almost α-cosymplectic (k,μ,ν)-space, which is inspired by the works mentioned studies. In addition, we research several conditions for an α-cosymplectic (k,μ,ν)-space's invariant submanifold to be totally geodesic. Then, some classifications and characterizations have been developed.

2. Preliminaries

An almost contact manifold is of 1-form η satisfying on M^{2n+1}, an odd-dimensional manifold, a field ϕ of endomorphisms of the tangent spaces, a characteristic or Reeb vector field, and a vector field ξ

$$\phi^2=-I+\eta\otimes\xi,\ \ \eta(\xi)=1, \tag{2.1}$$

in which $I:TM^{2n+1}\to TM^{2n+1}$ denotes an identity mapping. Because of (2.1), it follows

$$\eta\circ\phi=0,\ \phi\xi=0,\ rank(\phi)\text{=}2n. \tag{2.2}$$

An almost contact manifold $M^{2n+1}(\phi,\xi,\eta)$ is said to be normal if the tensor field $N=[\phi,\phi]+2d\eta\otimes\xi=0$, where $[\phi,\phi]$ denote the Nijenhuis tensor field of ϕ. Any almost contact manifold $M^{2n+1}(\phi,\xi,\eta)$ is known to have a Riemannian metric like that

$$g(\phi\omega_1,\phi\omega_2)=g(\omega_1,\omega_2)-\eta(\omega_1)\eta(\omega_2), \tag{2.3}$$

for all vector fields $\omega_1,\omega_2\in\Gamma(TM)$ [9]. A metric of this type, g is known as an equipped metric, and the structure (ϕ,η,ξ,g) and manifold $M^{2n+1}(\phi,\eta,\xi,g)$, associated with it, are known as an almost contact metric manifolds and denoted by as $M^{2n+1}(\phi,\eta,\xi,g)$. It is defined for $M^{2n+1}(\phi,\eta,\xi,g)$ to have a 2-form Φ. It is known as the fundamental form of $M^{2n+1}(\phi,\eta,\xi,g)$ when $\Phi(\omega_1,\omega_2)=g(\phi\omega_1,\omega_2)$. An almost contact metric manifold is referred to as a cosymplectic manifold if η and Φ are closed, that is, $d\eta=d\Phi=0$ [10].
The definition of an almost α-cosymplectic manifold for every real number α is [11]

$$d\eta=0,\ d\Phi=2\alpha\eta\wedge\Phi. \tag{2.4}$$

An $\alpha-$cosymplectic refers to a normal almost $\alpha-$cosymplectic manifold [12]. It is well known that the following equality holds for the tensor h on the contact metric manifold $M^{2n+1}(\phi,\eta,\xi,g)$, described by $2h=L_\xi\phi$,

$$\widetilde{\nabla}_{\omega_1}\xi=-\phi\omega_1-\phi h\omega_1,\ h\phi+\phi h=0,\ trh=tr\phi h=0,\ h\xi=0, \tag{2.5}$$

where,$\widetilde{\nabla}$ is the Levi-Civita connection on M^{2n+1} [6].

The following presented the notation of the $(k,\mu,\nu)-$contact metric manifold, which expands above generalized (k,μ)-spaces:

$$R(\omega_1,\omega_2)\xi=\eta(\omega_2)\left[kI+\mu h+\nu\phi h\right]\omega_1+\eta(\omega_1)\left[kI+\mu h+\nu\phi h\right]\omega_2, \tag{2.6}$$

where R is the Riemannian curvature tensor of M^{2n+1} and certain smooth functions k, μ and ν on M^{2n+1}, ω_1,ω_2 are vector fields [13].

Lemma 2.1. *Given $M^{2n+1}(\phi,\eta,\xi,g)$ is an almost $\alpha-$cosymplectic $(k,\mu,\nu)-$space, so*

$$h^2=(k+\alpha^2)\phi^2, \tag{2.7}$$

$$\xi(k)=2(k+\alpha^2)(\nu-2\alpha), \tag{2.8}$$

$$\begin{aligned}R(\xi,\omega_1)\omega_2 \;=\;& k[g(\omega_1,\omega_2)\xi-\eta(\omega_2)\omega_1]+\mu[g(h\omega_1,\omega_2)\xi-\eta(\omega_2)h\omega_1]\\ &+\nu[g(\phi h\omega_1,\omega_2)\xi-\eta(\omega_2)\phi h\omega_1],\end{aligned} \tag{2.9}$$

$$(\widetilde{\nabla}_{\omega_1}\phi)\omega_2 = g(\alpha\phi\omega_1 + hu_1, \omega_2)\xi - \eta(\omega_2)(\alpha\phi\omega_1 + h\omega_1), \tag{2.10}$$

$$\widetilde{\nabla}_{\omega_1}\xi = -\alpha\phi^2\omega_1 - \phi h\omega_1, \tag{2.11}$$

for any vector fields ω_1, ω_2 on M^{2n+1} [9].

Let M be an immersed submanifold of $\widetilde{M}^{2n+1}$, which is an almost $\alpha-$cosymplectic (k,μ,ν)-space. We denote the tangent and normal subspaces of M in $\widetilde{M}$ by $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively, the Gauss and Weingarten formulas are provided, respectively, by

$$\widetilde{\nabla}_{\omega_1}\omega_2 = \nabla_{\omega_1}\omega_2 + \sigma(\omega_1, \omega_2), \tag{2.12}$$

and

$$\widetilde{\nabla}_{\omega_1}\omega_5 = -A_{\omega_5}\omega_1 + \nabla^{\perp}_{\omega_1}\omega_5 \tag{2.13}$$

for all $\omega_1, \omega_2 \in \Gamma(TM)$ and $\omega_5 \in \Gamma(T^{\perp}M)$, σ and A are referred to as the second fundamental form and shape operators of M, respectively, ∇ and $\nabla^{\perp}$ are the induced connections on M and $\Gamma(T^{\perp}M)$. $\Gamma(TM)$ stands for the set of differentiable vector fields on M. They are associated by

$$g(A_{\omega_5}\omega_1, \omega_2) = g(\sigma(\omega_1, \omega_2), \omega_5). \tag{2.14}$$

The second fundamental form σ is first covariant derivative is given by

$$(\widetilde{\nabla}_{\omega_1}\sigma)(\omega_2, \omega_3) = \nabla^{\perp}_{\omega_1}\sigma(\omega_2, \omega_3) - \sigma(\nabla_{\omega_1}\omega_2, \omega_3) - \sigma(\omega_2, \nabla_{\omega_1}\omega_3), \tag{2.15}$$

for all $\omega_1, \omega_2, \omega_3 \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, the second fundamental form is parallel.

By R, we denote the Riemannian curvature tensor of submanifold, then we have the Gauss formulae.

$$\begin{aligned} \widetilde{R}(\omega_1, \omega_2)\omega_3 &= R(\omega_1, \omega_2)\omega_3 + A_{\sigma(\omega_1,\omega_3)}\omega_2 - A_{\sigma(\omega_2,\omega_3)}\omega_1 + (\widetilde{\nabla}_{\omega_1}\sigma)(\omega_2, \omega_3) \\ &\quad -(\widetilde{\nabla}_{\omega_2}\sigma)(\omega_1, \omega_3), \end{aligned} \tag{2.16}$$

for all $\omega_1, \omega_2, \omega_3 \in \Gamma(TM)$.

$\widetilde{R} \cdot \sigma$ is given by

$$\begin{aligned} (\widetilde{R}(\omega_1, \omega_2) \cdot \sigma)(\omega_4, \omega_5) &= R^{\perp}(\omega_1, \omega_2)\sigma(\omega_4, \omega_5) - \sigma(R(\omega_1, \omega_2)\omega_4, \omega_5) \\ &\quad -\sigma(\omega_4, R(\omega_1, \omega_2)\omega_5), \end{aligned} \tag{2.17}$$

where

$$R^{\perp}(\omega_1, \omega_2) = [\nabla^{\perp}_{\omega_1}, \nabla^{\perp}_{\omega_2}] - \nabla^{\perp}_{[\omega_1,\omega_2]},$$

denote the normal bundle's Riemannian curvature tensor.

For the Riemannian manifold (M^{2n+1}, g), the W_1^* curvature tensor is determined by

$$W_1^*(\omega_1, \omega_2)\omega_3 = R(\omega_1, \omega_2)\omega_3 - \frac{1}{2n}[S(\omega_2, \omega_3)\omega_1 - S(\omega_1, \omega_3)\omega_2], \tag{2.18}$$

for all $\omega_1, \omega_2, \omega_3 \in \Gamma(TM)$ [14].

Similarly, the tensor $W_1^* \cdot \sigma$ is defined by

$$\begin{aligned} (W_1^*(\omega_1, \omega_2) \cdot \sigma)(\omega_4, \omega_5) &= R^{\perp}(\omega_1, \omega_2)\sigma(\omega_4, \omega_5) - \sigma(W_1^*(\omega_1, \omega_2)\omega_4, \omega_5) \\ &\quad -\sigma(\omega_4, W_1^*(\omega_1, \omega_2)\omega_5), \end{aligned} \tag{2.19}$$

for all $\omega_1, \omega_2, \omega_4, \omega_5 \in \Gamma(TM)$.

Furthermore, the W_7-curvature tensor for Riemannian manifold (M^{2n+1}, g) is given by

$$W_7(\omega_1, \omega_2)\omega_3 = R(\omega_1, \omega_2)\omega_3 - \frac{1}{2n}[S(\omega_2, \omega_3)\omega_1 - g(\omega_2, \omega_3)Q\omega_1] \tag{2.20}$$

for all $\omega_1, \omega_2, \omega_3 \in \Gamma(TM)$ [15].

On a semi-Riemannian manifold (M, g), for a $(0, k)-$type tensor field $(0, k)$-type tensor field T and $(0, 2)$-type tensor field A, $(0, k+2)$-type tensor field Tachibana $Q(A, T)$ is defined as

$$\begin{aligned} Q(A,T)(\omega_{11},\omega_{12},...,\omega_{1k};\omega_1,\omega_2) &= -T((\omega_1 \wedge_A \omega_2)\omega_{11},\omega_{12},...,\omega_{1k}) \\ &- T(\omega_{11},(\omega_1 \wedge_A \omega_2)\omega_{13},...,\omega_{1k}) \\ &. \\ &. \\ &. \\ &- T(\omega_{11},\omega_{12},...,(\omega_1 \wedge_A \omega_2)\omega_{1k}), \end{aligned} \tag{2.21}$$

for all $\omega_{11}, \omega_{12}, ..., \omega_{1k}, \omega_1, \omega_2 \in \chi(M)$, where

$$(\omega_1 \wedge_A \omega_2)\omega_3 = A(\omega_2, \omega_3)\omega_1 - A(\omega_1, \omega_3)\omega_2. \tag{2.22}$$

3. Invariant submanifolds of an almost $\alpha-$cosymplectic (k, μ, ν)-space

Now, let M be an immersed submanifold of $\widetilde{M}^{2n+1}$ and M be an almost $\alpha-\text{cosymplectic}$ $(k, \mu, \nu)-$space. If $\phi(T_{\omega_{11}}M) \subseteq T_{\omega_{11}}M$, for each point at $\omega_{11} \in M$, then M is said to be an invariant submanifold of $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$ with respect to ϕ. Following, it will be clear that a submanifold that is invariant with respect to ϕ is also invariant with respect to h.

Proposition 3.1. *ξ is tangent to M, let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. Hence, the following equalities hold on M ;*

$$\begin{aligned} R(\omega_1,\omega_2)\xi &= k[\eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2] + \mu[\eta(\omega_2)h\omega_1 - \eta(\omega_1)h\omega_2] \\ &\quad +\nu[\eta(\omega_2)\phi h\omega_1 - \eta(\omega_1)\phi h\omega_2] \end{aligned} \tag{3.1}$$

$$(\nabla_{\omega_1}\phi)\omega_2 = g(\alpha\phi\omega_1 + h\omega_1, \omega_2)\xi - \eta(\omega_2)(\alpha\phi\omega_1 + h\omega_1) \tag{3.2}$$

$$\nabla_{\omega_1}\xi = -\alpha\phi^2\omega_1 - \phi h\omega_1 \tag{3.3}$$

$$\phi\sigma(\omega_1, \omega_2) = \sigma(\phi\omega_1, \omega_2) = \sigma(\omega_1, \phi\omega_2), \quad \sigma(\omega_1, \xi) = 0, \tag{3.4}$$

where ∇, σ and R stand for M's Levi-Civita connection, shape operator and the Riemannian curvature tensor on M, respectively.

Proof. As the proof is a consequence of straightforward, we omit it. □

We shall assume for the remainder of this work that M is an invariant submanifold of an $\alpha-$cosymplectic (k, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. From (2.5), we have in this instance

$$\phi h\omega_1 = -h\phi\omega_1, \tag{3.5}$$

for all $\omega_1 \in \Gamma(TM)$, in other words M is also invariant with respect to the tensor field h.

Theorem 3.1. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k, μ, ν)-space $\widetilde{M}^{2n+1}(\phi, \xi, \eta, g)$. Then $Q(g, W_1^* \cdot \sigma) = 0$ if and only if M is either totally geodesic or $\mu^2 + \nu^2 = 0$.*

Proof. We suppose that $Q(g, W_1^* \cdot \sigma) = 0$. This means that

$$(W_1^*(\omega_1, \omega_2) \cdot \sigma)((\omega_3 \wedge_g \omega_6)\omega_4, \omega_5) + (W_1^*(\omega_1, \omega_2) \cdot \sigma)(\omega_4, (\omega_3 \wedge_g \omega_6)\omega_5) = 0,$$

for all $\omega_1, \omega_2, \omega_4, \omega_5, \omega_3, \omega_6 \in \Gamma(TM)$, which implies that

$$\begin{aligned} &(W_1^*(\omega_1, \omega_2) \cdot \sigma) + (g(\omega_4, \omega_6)\omega_3 - g(\omega_3, \omega_4)\omega_6, \omega_5) + (W_1^*(\omega_1, \omega_2) \cdot \sigma) \\ &+(\omega_4, g(\omega_5, \omega_6)\omega_3 - g(\omega_3, \omega_5)\omega_6) = 0. \end{aligned} \tag{3.6}$$

In (3.6), putting $\omega_2=\omega_4=\omega_3=\omega_5=\xi$ and using (2.18), (2.19),(3.1), we observe

$$\begin{aligned} & (W_1^*(\omega_1,\xi)\cdot\sigma)(\eta(\omega_6)\xi-\omega_6,\xi)=(W_1^*(\omega_1,\xi)\cdot\sigma)(\eta(\omega_6)\xi,\xi) \\ & -(W_1^*(\omega_1,\xi)\cdot\sigma)(\omega_6,\xi) \\ = & R^{\perp}(\omega_1,\xi)\sigma(\eta(\omega_6)\xi,\xi)-\sigma(\eta(\omega_6)W_1^*(\omega_1,\xi)\xi,\xi) \\ & -\sigma(\eta(\omega_6)\xi,W_1^*(\omega_1,\xi)\xi)-R^{\perp}(\omega_1,\xi)\sigma(\omega_6,\xi) \\ & +\sigma(W_1^*(\omega_1,\xi)\omega_6,\xi)+\sigma(\omega_6,W_1^*(\omega_1,\xi)\xi)=0. \end{aligned} \tag{3.7}$$

In view of (2.6) and (2.16), non-zero components of (3.7) vectors give us

$$\sigma(W_1^*(\omega_1,\xi)\xi,\omega_6)=\sigma(\omega_6,\mu h\omega_1+\nu\phi h\omega_1)=0. \tag{3.8}$$

Also taking $\phi\omega_1$ instead of ω_1 in (3.8) and by virtue of lemma 2.1 and proposition 1, we have

$$-\mu\sigma(h\omega_1,\omega_6)+\nu\sigma(h\omega_1,\omega_6)=0. \tag{3.9}$$

Equations (3.8) and (3.9) implies that

$$\mu^2+\nu^2=0 \text{ or } \sigma=0.$$

This proves our assertion. □

Theorem 3.2. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S,W_1^*\cdot\sigma)=0$ if and only if M is either totally geodesic or $2nk(\mu^2+\nu^2)=0$.*

Proof. We believe that $Q(S,W_1^*\cdot\sigma)=0$, which follows that

$$Q(S,W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_5;\omega_3,\omega_6)=0,$$

for all $\omega_1,\omega_2,\omega_4,\omega_5,\omega_3,\omega_6\in\Gamma(TM)$, by virtue of (2.19) and (2.21), we obtain

$$\begin{aligned} & S(\omega_3,\omega_4)(W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_6,\omega_5)-S(\omega_6,\omega_4)(W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_3,\omega_5) \\ & +S(\omega_3,\omega_5)(W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_6) \\ & -S(\omega_6,\omega_5)(W_1^*(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_3)=0. \end{aligned} \tag{3.10}$$

Expanding (3.10) and putting $\omega_2=\omega_4=\omega_3=\omega_5=\xi$, non-zero components is

$$2nk\sigma(\omega_6,W_1^*(\omega_1,\xi)\xi)=0. \tag{3.11}$$

As a result, by combining the previous equation and applying (2.20), we reach

$$2nk\mu\sigma(\omega_6,\mu h\omega_1)+2nk\nu\sigma(\omega_6,\phi h\omega_1)=0. \tag{3.12}$$

On the other hand, substituting $\phi\omega_1$ for ω_1 in (3.12) and taking into account (2.7) and (3.4), we conclude that $2nk\left[(\mu^2+\nu^2)\right]\sigma(h\omega_1,\omega_6)=0$,which follows that, $2nk(\mu^2+\nu^2)=0$ or $\sigma=0$.
Thus proof is completed. □

Theorem 3.3. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(g,W_7\cdot\sigma)=0$ if and only if M is either totally geodesic or $\left[k^2+(k+\alpha^2)(\mu^2+\nu^2)\right]=0$.*

Proof. We suppose that $Q(g,W_7\cdot\sigma)=0$. This means that

$$(W_7(\omega_1,\omega_2)\cdot\sigma)((\omega_3\wedge_g\omega_6)\omega_4,\omega_5)+(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_4,(\omega_3\wedge_g\omega_6)\omega_5)=0,$$

for all $\omega_1,\omega_2,\omega_4,\omega_5,\omega_3,\omega_6\in\Gamma(TM)$, which implies that

$$\begin{aligned} & (W_7(\omega_1,\omega_2)\cdot\sigma)+(g(\omega_4,\omega_6)\omega_3-g(\omega_3,\omega_4)\omega_6,\omega_5)+(W_7(\omega_1,\omega_2)\cdot\sigma) \\ & +(\omega_4,g(\omega_5,\omega_6)\omega_3-g(\omega_3,\omega_5)\omega_6)=0. \end{aligned} \tag{3.13}$$

In (3.13), putting $\omega_2 = \omega_4 = \omega_3 = \omega_5 = \xi$ and by using (2.6), (2.20), we observe

$$\begin{aligned} & (W_7(\omega_1,\xi)\cdot\sigma)(\eta(\omega_6)\xi-\omega_6,\xi) = (W_7(\omega_1,\xi)\cdot\sigma)(\eta(\omega_6)\xi,\xi) \\ & -(W_7(\omega_1,\xi)\cdot\sigma)(\omega_6,\xi) \\ = \; & R^{\perp}(\omega_1,\xi)\sigma(\eta(\omega_6)\xi,\xi)-\sigma(\eta(\omega_6)W_7(\omega_1,\xi)\xi,\xi) \\ & -\sigma(\eta(\omega_6)\xi,W_7(\omega_1,\xi)\xi)-R^{\perp}(\omega_1,\xi)\sigma(\omega_6,\xi) \\ & +\sigma(W_7(\omega_1,\xi)\omega_6,\xi)+\sigma(\omega_6,W_7(\omega_1,\xi)\xi)=0. \end{aligned} \tag{3.14}$$

In view of (2.17) and (2.20), non-zero components of (3.14) give us

$$\sigma(W_7(\omega_1,\xi)\xi,\omega_6)=\sigma(\omega_6,k\omega_1+\mu h\omega_1+\nu\phi h\omega_1)=0. \tag{3.15}$$

Substituting $\phi\omega_1$ for ω_1 in (3.15) and considering the equations (2.1) and (2.7), then we get

$$k\sigma(\phi\omega_6,\omega_1)-\mu\sigma(\omega_6,\phi h\omega_1)+\nu\sigma(\omega_6,h\omega_1)=0. \tag{3.16}$$

From (3.15) and (3.16), we conclude that

$$\left[k^2+(k+\alpha^2)(\mu^2+\nu^2)\right]\sigma(\omega_6,h\omega_1)=0$$

So, the proof is finished. □

Theorem 3.4. *Let M be an invariant submanifold of an almost $\alpha-$cosymplectic (k,μ,ν)-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then $Q(S,W_7\cdot\sigma)=0$ if and only if M is either totally geodesic or $2nk\left[k^2+(k+\alpha^2)(\mu^2+\nu^2)\right]=0$.*

Proof. Let us assume that $Q(S,W_7\cdot\sigma)=0$. It follows that

$$Q(S,W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_5;\omega_3,\omega_6)=0,$$

for all $\omega_1,\omega_2,\omega_4,\omega_5,\omega_3,\omega_6\in\Gamma(TM)$, by virtue of (2.17) and (2.20), we deduce that

$$\begin{aligned} & S(\omega_3,\omega_4)(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_6,\omega_5)-S(\omega_6,\omega_4)(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_3,\omega_5) \\ & +S(\omega_3,\omega_5)(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_6)-S(\omega_6,\omega_5)(W_7(\omega_1,\omega_2)\cdot\sigma)(\omega_4,\omega_3)=0. \end{aligned} \tag{3.17}$$

By setting $\omega_2=\omega_4=\omega_3=\omega_5=\xi$ in the last equation and it non-zero components is

$$2nk\sigma(\omega_6,W_7(\omega_1,\xi)\xi)=0. \tag{3.18}$$

On the other hand (3.18) can be written as follows:

$$2nk\left[k\sigma(\omega_6,\omega_1)+\mu\sigma(\omega_6,h\omega_1)+\nu\sigma(\omega_6,\phi h\omega_1)\right]=0. \tag{3.19}$$

In the same way, by using (3.15) and (3.16), we get
$2nk\left[k^2+(k+\alpha^2)(\mu^2+\nu^2)\right]\sigma(h\omega_1,\omega_6)=0$, this means that,
$2nk\left[k^2+(k+\alpha^2)(\mu^2+\nu^2)\right]=0$ or $\sigma=0$.
This proves our assertion. □

Example 3.1. Let $M=\{(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5)\in\mathbb{R}^5,\omega_5\neq\pm1,0\}$ and we take

$$\begin{aligned} e_1 &= (\omega_5+1)\frac{\partial}{\partial\omega_1}, \quad e_2=\frac{1}{\omega_5-1}\frac{\partial}{\partial\omega_2}, \quad e_3=\frac{1}{2}(\omega_5+1)^2\frac{\partial}{\partial\omega_3}, \\ e_4 &= \frac{5}{\omega_5-1}\frac{\partial}{\partial\omega_4}, \qquad e_5=(\omega_5-1)\frac{\partial}{\partial\omega_5} \end{aligned}$$

are linearly independent vector fields on M. We also definite $(1,1)-$type tensor field ϕ by $\phi e_1=e_2$, $\phi e_2=-e_1$, $\phi e_3=e_4$, $\phi e_4=-e_3$ and $\phi e_5=0$.

Furthermore, the Riemannian metric tensor g is given by

$$g(e_i,e_j)=\{1,\ i=j;\quad 0,\ i\neq j\}.$$

By direct computations, we can easily to see that

$$\phi^2\omega_1 = -\omega_1 + \eta(\omega_1)\xi, \;\; \eta(\omega_1) = g(\omega_1, \xi)$$

and

$$g(\phi\omega_1, \phi\omega_2) = g(\omega_1, \omega_2) - \eta(\omega_1)\eta(\omega_2).$$

Thus $M^5(\phi, \xi, \eta, g)$ is a 5-dimensional almost contact metric manifold. From the Lie-operatory, we have the non-zero components

$$\begin{aligned} [e_1, e_5] &= -(\omega_5 - 1)e_1, \;\; [e_2, e_5] = (\omega_5 + 1)e_2, \quad [e_3, e_5] = -(\omega_5 - 1)e_3, \\ [e_4, e_5] &= (\omega_5 + 1)e_4. \end{aligned}$$

Furthermore, by ∇, we denote the Levi-Civita connection on M, by using Koszul's formula, we can reach at the non-zero components

$$\begin{aligned} \nabla_{e_1} e_5 &= -(\omega_5 - 1)e_1, \quad \nabla_{e_2} e_5 = (\omega_5 + 1)e_2, \quad \nabla_{e_3} e_5 = -(\omega_5 - 1)e_3, \\ \nabla_{e_4} e_5 &= (\omega_5 + 1)e_4. \end{aligned}$$

Comparing the above relations with

$$\nabla_{\omega_1} e_5 = \omega_1 - \eta(\omega_1)e_5 - \phi h\omega_1,$$

we can observe

$$he_1 = -\omega_5 e_2, \quad he_2 = -\omega_5 e_1, \quad he_3 = -\omega_5 e_4, \quad he_4 = -\omega_5 e_3 \text{ and } \;\; he_5 = 0.$$

By direct calculations, we get

$$\begin{aligned} R(e_1, e_5)e_5 &= ke_1 + \mu he_1 + \nu\phi he_1 = 2(\omega_5 - 1)e_1, \\ R(e_2, e_5)e_5 &= ke_2 + \mu he_2 + \nu\phi he_2 = -2w_5(\omega_5 + 1)e_2, \\ R(e_3, e_5)e_5 &= ke_3 + \mu he_3 + \nu\phi he_3 = 2(\omega_5 + 1)e_3, \end{aligned}$$

and

$$R(e_4, e_5)e_5 = ke_4 + \mu he_4 + \nu\phi he_4 = -2w_5(\omega_5 + 1)e_4,$$

which imply that $k = -(\omega_5 + 1)$, $\mu = 0$ and $\nu = 2 - \frac{1}{\omega_5} + \omega_5$.

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