Communications in Advanced Mathematical Sciences

VOLUME VII ISSUE I

ISSN 2651-4001

VOLUME 7 ISSUE 1
ISSN 2651-4001

March 2024
www.dergipark.org.tr/tr/pub/cams

COMMUNICATIONS IN ADVANCED MATHEMATICAL SCIENCES

Contents

Research Article

Communications in Advanced Mathematical Sciences
Vol. 7, No. 1, 1-13, 2024
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1394777

Cobalancing Numbers: Another Way of Demonstrating Their Properties

Arzu Özkoç Öztürk[1]*, Volkan Külahlı[2]

Abstract
In this study, previously obtained cobalancing numbers are considered from a different perspective, and the properties of the numbers are re-examined. The main purpose is to change the recurrence relation of cobalancing numbers and calculate some relations and properties in a more diverse and easier way. The reason that led us to this method is that the recurrence relation of cobalancing numbers has a second-order but non-homogeneous difference equations. Thus, it will be much easier to find the Binet formula, generating function, sum formulas, and many other relations with a sequence that is homogeneous and has a third-degree recurrence relation. Also some identities that have not been found before in the sequence are also included in this study.

Keywords: Binet formulas, Cobalancing numbers, Third-order recurrence relations
2010 AMS: 11B37, 11B83, 11D04

[1] *Department of Mathematics, Düzce University, Düzce, Türkiye, arzuozkoc@duzce.edu.tr, ORCID: 0000-0002-2196-3725*
[2] *Department of Mathematics, Düzce University, Düzce, Türkiye, volkankulahli@hotmail.com, ORCID: 0009-0003-2464-4628*
***Corresponding author**
Received: 23 November 2023, **Accepted:** 9 January 2023, **Available online:** 5 February 2023
How to cite this article: A. Özkoç Öztürk, V. Külahlı, *Cobalancing numbers: Another way of demonstrating their properties*, Commun. Adv. Math. Sci., 7(1) (2024), 1-13.

1. Introduction

Number sequences have been studied and researched by hundreds of mathematicians for many years. While the authors in [1] work with a special equation, Diophantine equation,

$$1+2+3+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1.1}$$

on triangular numbers, obtained an interesting relation of the numbers n in the solutions (n,r), which they call balancing numbers, with square triangular numbers. The number r in (n,r) is called the balancer corresponding to n. In the following years, Behera and Panda continued their work rapidly and continued to find interesting features related to this new number sequence. Later on, the first article that bridges the gap between Fibonacci numbers and balancing numbers was made by Panda [2].

Behera and Panda [1] proved that the square of any balancing number is a triangular number. Subramaniam [3] is another mathematician who established a relationship between balancing numbers and triangular numbers. Panda and Ray [4], studied another Diophantine equation

$$1+2+3+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{1.2}$$

on triangular numbers and call n a cobalancing number and r the cobalancer corresponding to n. Cobalancing numbers relate to different triangular numbers that can be given as the product of two consecutive natural numbers or as the arithmetic mean of the squares of two approximately consecutive natural numbers [5]. Also, Liptai [5] mentioned his name in a study that produced important results regarding balancing numbers.

Then, another article that aroused interest in other new topics by giving the literature the relationship between balancing and cobalancing numbers was written by Panda [6].

There are many studies on the Fibonacci sequence, which is related to the golden ratio, and the Pell sequence, which is related to the silver ratio, and these articles contain a lot of information that paves the way for integer sequences. Behera and Panda [1], introduced the Diophantine equation (1.1), then they obtained the sequence $\{B_n\}_{n\in\mathbb{N}}$ of balancing numbers and give some interesting properties of this sequence. $\{B_n\}_{n\in\mathbb{N}}$ is defined by the following recurrence relation of the second order, given by

$$B_{n+1} = 6B_n - B_{n-1} \quad , n \geq 1 \tag{1.3}$$

with initial terms $B_0 = 0$ and $B_1 = 1$, where B_n denotes the $n-$th balancing number. Taking $a_1 = 1+\sqrt{2}$ and $a_2 = 1-\sqrt{2}$, the Binet formula for B_n can be written as,

$$B_n = \frac{a_1^{2n} - a_2^{2n}}{4\sqrt{2}}.$$

The (ordinary) generating function of a sequence $\{x_n\}_{n\in\mathbb{N}^+}$ of real or complex numbers is given by

$$f(s) = \sum_{n=1}^{\infty} x_n s^n = x_1 s^1 + x_2 s^2 + x_3 s^3 + \cdots \tag{1.4}$$

[7]. The generating function for the sequence of balancing numbers $\{B_n\}_{n\in\mathbb{N}}$,is

$$g(s) = \frac{s}{1-6s+s^2}.$$

On the other hand, following Panda and Ray [4] a positive integer n is a cobalancing number with cobalancer r, if it is the solution of the Diophantine equation (1.2). The sequence $\{b_n\}_{n\in\mathbb{N}}$ is defined by the following recurrence relation of second-order given by

$$b_{n+1} = 6b_n - b_{n-1} + 2 \quad , n \geq 2 \tag{1.5}$$

with initial terms $b_1 = 0$ and $b_2 = 2$, where b_n denotes the $n-$th cobalancing number. We will denote cobalancing numbers with $\{b_{(2),n}\}_{n\in\mathbb{N}}$ instead of $\{b_n\}_{n\in\mathbb{N}}$ to avoid confusion throughout the article. Because throughout the article, a sequence that has a third-order recurrence relation with the new sequence that will be found shortly will be discussed. Since a cobalancing number with a second-order recurrence relation is expressed here, this notation will be used. The Binet formula for cobalancing numbers is given

$$b_{(2),n} = \frac{a_1^{2n-1} - a_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}. \tag{1.6}$$

The generating function for the sequence of cobalancing numbers $\{b_{(2),n}\}_{n\in\mathbb{N}}$,is

$$g(s) = \frac{2s^2}{(1-s)(1-6s+s^2)}. \tag{1.7}$$

As can be seen, information was given about the sequences with quadratic recurrence relations. Note that the recurrence relation we wrote with (1.5) has a non-homogeneous structure. Naturally, it is quite difficult to work with this recurrence, and the results will be obtained more easily if it is transformed into a sequences with another third-order recurrence relation, which makes it easier to use without changing the sequence. So, let's get some ideas about sequence with third-order recurrence relations [8]. The best known of these property is the tribonacci number sequence. The tribonacci sequence is defined by for

$$T_{n+1} = T_n + T_{n-1} + T_{n-2}, n \geq 2$$

with initial conditions

$$T_0 = 0, T_1 = 1, T_2 = 1.$$

Tribonacci sequence is a well known generalization of the Fibonacci sequence. The roots of characteristic equation of Tribonacci numbers are $\alpha_1, \beta_1, \gamma_1$ for the $x^3 - x^2 - x - 1 = 0$. The Binet formula of Tribonacci sequence is given by

$$T_n = \frac{\alpha_1^{n+1}}{(\alpha_1 - \beta_1)(\alpha_1 - \gamma_1)} + \frac{\beta_1^{n+1}}{(\beta_1 - \alpha_1)(\beta_1 - \gamma_1)} + \frac{\alpha_1 \gamma_1^{n+1}}{(\gamma_1 - \alpha_1)(\gamma_1 - \beta_1)}$$

[9]. The tribonacci numbers can also be computed using the generating function

$$g(z) = \frac{z}{1 - z - z^2 - z^3}.$$

Now let's talk about a method used for the recurrence change we mentioned above. In the study [10], as a result of the process performed for the Leonardo sequence with a non-homogeneous second-order recurrence relation, a new sequence with a third-order recurrence relation is obtained. Let's obtain the third-order recurrence relation we target with a similar method.

Let's write $n = n-1$ and $n = n$ in the equation (1.5)

$$\begin{aligned} b_{(2),n} &= 6b_{(2),n-1} - b_{(2),n-2} + 2 \\ b_{(2),n+1} &= 6b_{(2),n} - b_{(2),n-1} + 2. \end{aligned}$$

Now we wrote last above and subtract it side by side then the third-order recurrence relation to be obtained is as follows

$$b_{(3),n+1} = 7b_{(3),n} - 7b_{(3),n-1} + b_{(3),n-2} \tag{1.8}$$

for the initial conditions $b_{(3),0} = 0, b_{(3),1} = 0$ and $b_{(3),2} = 2$. The roots of characteristic equation $x^3 - 7x^2 + 7x - 1 = 0$ of cobalancing numbers are

$$\alpha = 3 + 2\sqrt{2}, \beta = 3 - 2\sqrt{2} \text{ and } \gamma = 1. \tag{1.9}$$

In mathematics, a recurrence relation is an equation that defines a sequence recursively; each term of the sequence is defined as a function of the preceding terms [11].

2. Cobalancing Numbers and Some Properties

In this section, it is aimed to find identities regarding cobalancing numbers. The first of these is to obtain the generating function that was previously found for the sequence that has a second-order recurrence relation. The usual generating function $g(x)$ for the sequence (1.8) of real numbers is defined as:

$$g(x) = b_{(3),0} + b_{(3),1}x + b_{(3),2}x^2 + b_{(3),3}x^3 + \cdots + b_{(3),n}x^n + b_{(3),n+1}x^{n+1} + \cdots.$$

Now let's write the $g(x)$ generating function in a different way from (1.7).

Theorem 2.1. *Let $b_{(3),n}$ can be the cobalancing number. The generating function $g(x)$, can be written as follows:*

$$g(x) = \frac{2x^2}{1 - 7x + 7x^2 - x^3}.$$

Proof. The generating function of the sequence (1.8) is

$$g(x) = b_{(3),0} + b_{(3),1}x + b_{(3),2}x^2 + b_{(3),3}x^3 + \cdots + b_{(3),n}x^n + b_{(3),n+1}x^{n+1} + \cdots.$$

Let's multiply the function $g(x)$ by $7x$, $7x^2$ and x^3.

$$\begin{aligned} g(x) &= b_{(3),0} + b_{(3),1}x + b_{(3),2}x^2 + b_{(3),3}x^3 + \cdots + b_{(3),n}x^n + b_{(3),n+1}x^{n+1} + \cdots \\ 7x.g(x) &= 7b_{(3),0}x + 7b_{(3),1}x^2 + 7b_{(3),2}x^3 + 7b_{(3),3}x^4 + \cdots + 7b_{(3),n}x^{n+1} + \cdots \\ 7x^2.g(x) &= 7b_{(3),0}x^2 + 7b_{(3),1}x^3 + 7b_{(3),2}x^4 + \cdots + 7b_{(3),n-1}x^{n+1} + \cdots \\ x^3.g(x) &= b_{(3),0}x^3 + b_{(3),1}x^4 + b_{(3),2}x^5 + b_{(3),3}x^3 + \cdots + b_{(3),n-2}x^{n+1} + \cdots. \end{aligned}$$

Let's take the necessary actions in the equations we have obtained and let's get the (2.1) equality.

$$\begin{aligned} & g(x)-7xg(x)+7x^2g(x)-x^3g(x) && (2.1)\\ = \; & b_{(3),0}+b_{(3),1}x+b_{(3),2}x^2-7b_{(3),0}x-7b_{(3),1}x^2+(b_{(3),3}-7b_{(3),2}+7b_{(3),1}-b_{(3),0})x^2+\cdots && (2.2)\\ & +(b_{(3),n+1}-7b_{(3),n}+7b_{(3),n-1}-b_{(3),n-2})x^{n+1}. \end{aligned}$$

Let's take the left side of the equation into the $g(x)$ bracket and write $b_{(3),0}=0, b_{(3),1}=0$ and $b_{(3),2}=2$ in the equality. In the last case, we find the $g(x)$ function written below.

$$g(x)(1-7x+7x^2-x^3)=b_{(3),0}+b_{(3),1}x+b_{(3),2}x^2-7b_{(3),0}x-7b_{(3),1}x^2+7b_{(3),0}x^2$$

then

$$g(x) \quad = \quad \frac{2x^2}{1-7x+7x^2-x^3}.$$

□

If the generating function is expressed in terms of the roots of the characteristic equation in (1.9), the following result is obtained.

Corollary 2.2. *Let $b_{(3),n}$ can be the cobalancing number. The generating function $g(x)$, can be written as follows:*

$$g(x)=\frac{2x^2}{1-7x+7x^2-x^3}=\frac{-\frac{1}{2}}{1-x}+\frac{\frac{5-\alpha}{2(\beta-\alpha)}}{1-\alpha x}+\frac{\frac{5-\beta}{2(\alpha-\beta)}}{1-\beta x}.$$

Proof. We can write the $g(x)$ function as the sum of rational numbers as follows,

$$\frac{2x^2}{(1-x)(1-6x+x^2)}=\frac{P}{1-x}+\frac{Q}{1-\alpha x}+\frac{R}{1-\beta x}.$$

Let's equalize the denominators and find the values P, Q and R using polynomial equality,

$$2x^2=P(1-\alpha x)(1-\beta x)+Q(1-x)(1-\beta x)+R(1-x)(1-\alpha x)$$

Let's write $x=1$ in the equation we found. $P=-\frac{1}{2}$ is found. We can write the equality we have found in the following way:

$$2x^2=(P+Q\beta+R\alpha)x^2+(-6P-Q(\beta+1)-R(\alpha+1))x+P+Q+R.$$

If we use the equality we have ended up with, we reach the following equations.

$$\begin{aligned} P+Q\beta+R\alpha &= 2,\\ -6P-Q(\beta+1)-R(\alpha+1) &= 0,\\ P+Q+R &= 0. \end{aligned}$$

Let's write $P=-\frac{1}{2}$ and let's solve the following system of equations

$$\begin{aligned} Q\beta+R\alpha &= 2+\frac{1}{2},\\ Q+R &= \frac{1}{2}. \end{aligned}$$

Finally, $Q=\frac{5-\alpha}{2(\beta-\alpha)}$ and $R=\frac{5-\beta}{2(\alpha-\beta)}$ is found, and the function $g(x)$ can be written as follows,

$$g(x)=\frac{-\frac{1}{2}}{1-x}+\frac{\frac{5-\alpha}{2(\beta-\alpha)}}{1-\alpha x}+\frac{\frac{5-\beta}{2(\alpha-\beta)}}{1-\beta x}.$$

□

The result given below for [8] was found in another way. We will find the Binet formula for the sequence $\left\{b_{(3),n}\right\}_{n\in\mathbb{N}}$.

Theorem 2.3. *Let $b_{(3),n}$ can be the cobalancing number. The Binet formula for the cobalancing number is as follows:*

$$b_{(3),n} = \frac{\alpha^{n-1}(\alpha^2-2\alpha+5)+\beta^{n-1}(\beta^2-2\beta+5)}{64} - \frac{1}{2}.$$

Proof. Let's write the following equation for the cobalancing sequence,

$$\begin{aligned} x^3-7x^2+7x-1 &= 0, \\ (x-1)(x^2-6x+1) &= 0. \end{aligned}$$

Let the roots of the equation we have arranged be (1.9), also

$$\begin{aligned} \alpha+\beta+\gamma &= 7, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

Let's write $b_{(3),n} = A1^n + B\alpha^n + C\beta^n$ and let us obtain the following equations

$$\text{for } n = 0 \Rightarrow b_{(3),0} = A1^0 + B\alpha^0 + C\beta^0 \Rightarrow A+B+C = 0, \tag{2.3}$$

$$\text{for } n = 1 \Rightarrow b_{(3),1} = A1^1 + B\alpha^1 + C\beta^1 \Rightarrow A+B\alpha+C\beta = 0, \tag{2.4}$$

$$\text{for } n = 2 \Rightarrow b_{(3),0} = A1^2 + B\alpha^2 + C\beta^2 \Rightarrow A+B\alpha^2+C\beta^2 = 2\ . \tag{2.5}$$

Let's subtract (2.3) from (2.4), also subtract (2.3) from (2.5) and get the following system of equations

$$\begin{aligned} B(\alpha-1)+C(\beta-1) &= 0 \\ B(\alpha^2-1)+C(\beta^2-1) &= 2. \end{aligned} \tag{2.6}$$

Let's multiply the (2.6) by $-(\alpha+1)$ and add the equations side by side.

$$\begin{aligned} C(\beta^2-1)-C(\beta-1)(\alpha+1) &= 2 \\ C(\beta-1)(\beta+1-\alpha-1) &= 2 \end{aligned}$$

then we find

$$C = \frac{2}{5\beta+\alpha-2}. \tag{2.7}$$

In the above equations, we have reached the (2.7) value by typing $\beta^2 = 6\beta-1$ and $\alpha.\beta = 1$. If $C = \frac{2}{5\beta+\alpha-2}$ is written in (2.6), $B = \frac{2}{5\alpha+\beta-2}$ is found. Let's write $B = \frac{2}{5\alpha+\beta-2}$ and $C = \frac{2}{5\beta+\alpha-2}$ in the (2.3). So

$$\begin{aligned} A+B+C &= 0 \\ A+\frac{2}{5\alpha+\beta-2}+\frac{2}{5\beta+\alpha-2} &= 0 \end{aligned}$$

then

$$A = -2\left(\frac{1}{5\alpha+\beta-2}+\frac{1}{5\beta+\alpha-2}\right). \tag{2.8}$$

If $\alpha.\beta = 1$ and $\alpha+\beta = 6$ are written in the (2.8), $A = -\frac{1}{2}$ is found. Let's write the values we found in their places in the equality $b_{(3),n} = A1^n + B\alpha^n + C\beta^n$ then,

$$\begin{aligned} b_{(3),n} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^n+\frac{2}{5\beta+\alpha-2}\beta^n \\ &= -\frac{1}{2}+2\left(\frac{\alpha^n(5\beta+\alpha-2)+\beta^n(5\alpha+\beta-2)}{(5\alpha+\beta-2)(5\beta+\alpha-2)}\right). \end{aligned}$$

In the last case, the Binet formula is as follows:

$$b_{(3),n} = \frac{\alpha^{n-1}(\alpha^2-2\alpha+5)+\beta^{n-1}(\beta^2-2\beta+5)}{64} - \frac{1}{2}.$$

□

Catalan, Cassini and d'Ocagne identities are given in [12], now let's talk about their proof in a different way. It is the Catalan identity in an identity that we can express with cobalancing numbers. We will show the correctness of this identity in the following theorem.

Theorem 2.4. *Let $b_{(3),n}$ can be the cobalancing number. The Catalan identity for the cobalancing number is as follows:*

$$b_{(3),n+k}b_{(3),n-k}-b_{(3),n}^2=\frac{1}{32}(\alpha^k-\beta^k)^2-\frac{\alpha^{n-k}(\alpha^k-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-k}(\beta^k-1)^2}{5\beta+\alpha-2}.$$

Proof. Let's write $b_{(3),n+k},\ b_{(3),n-k},\ b_{(3),n}$ then

$$\begin{aligned}
b_{(3),n} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^n+\frac{2}{5\beta+\alpha-2}\beta^n,\\
b_{(3),n+k} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n+k}+\frac{2}{5\beta+\alpha-2}\beta^{n+k},\\
b_{(3),n-k} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n-k}+\frac{2}{5\beta+\alpha-2}\beta^{n-k}.
\end{aligned}$$

Now $B=\frac{2}{5\alpha+\beta-2}$ and $C=\frac{2}{5\beta+\alpha-2}$ also we have $\alpha+\beta=6$ and $\alpha.\beta=1$.

$$b_{(3),n+k}b_{(3),n-k} = \frac{1}{4}-\frac{1}{2}B\alpha^{n-k}-\frac{1}{2}C\beta^{n-k}-\frac{1}{2}B\alpha^{n+k}-BB\alpha^{2n}+BC\alpha^{n+k}\beta^{n-k}-\frac{1}{2}C\beta^{n+k}+BC\alpha^{n-k}\beta^{n+k}+CC\beta^{2n}$$

and

$$b_{(3),n}b_{(3),n} = \frac{1}{4}-\frac{1}{2}B\alpha^n-\frac{1}{2}C\beta^n-\frac{1}{2}B\alpha^n+BB\alpha^{2n}+BC\alpha^n\beta^n-\frac{1}{2}C\beta^n+BC\alpha^n\beta^n+CC\beta^{2n}.$$

Let's subtract the equations written above from side to side

$$\begin{aligned}
b_{(3),n+k}b_{(3),n-k}-b_{(3),n}^2 &= BC\left(\left(\frac{\alpha}{\beta}\right)^k-2+\left(\frac{\beta}{\alpha}\right)^k\right)-\frac{B}{2}(\alpha^{n-k}-2\alpha^n+\alpha^{n+k})-\frac{C}{2}(\beta^{n-k}-2\beta^n+\beta^{n+k})\\
&= BC\left(\frac{\alpha^k}{\beta^k}-2+\frac{\beta^k}{\alpha^k}\right)-\frac{B}{2}\alpha^n\left(\frac{\alpha^k}{1}-2+\frac{1}{\alpha^k}\right)-\frac{C}{2}\beta^n\left(\beta^k-2+\frac{1}{\beta^k}\right)\\
&= BC(\alpha^k-\beta^k)^2-\frac{1}{2}\frac{2}{5\alpha+\beta-2}\alpha^{n-k}(\alpha^k-1)^2-\frac{1}{2}\frac{2}{5\beta+\alpha-2}\beta^{n-k}(\beta^k-1)^2.
\end{aligned}$$

Let's show that $B.C=\frac{1}{32}$ for $\alpha=3+2\sqrt{2}$ and $\beta=3-2\sqrt{2}$, we have $\alpha+\beta=6$ and $\alpha.\beta=1$,

$$\begin{aligned}
B.C &= \frac{2}{5\alpha+\beta-2}\frac{2}{5\beta+\alpha-2}\\
&= \frac{4}{26\alpha\beta+5[(\alpha+\beta)^2-2\alpha\beta]-12(\alpha+\beta)+4}\\
&= \frac{1}{32}.
\end{aligned}$$

The final state of equality is as follows:

$$b_{(3),n+k}b_{(3),n-k}-b_{(3),n}^2=\frac{1}{32}(\alpha^k-\beta^k)^2-\frac{\alpha^{n-k}(\alpha^k-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-k}(\beta^k-1)^2}{5\beta+\alpha-2}.$$

□

One of the other identities for cobalancing numbers are

$$b_{(3),n-1}b_{(3),n+1}-b_{(3),n}^2=\frac{(\alpha-\beta)^2}{32}-\frac{\alpha^{n-1}(\alpha-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-1}(\beta-1)^2}{5\beta+\alpha-2}.$$

We will show the correctness of this identity in the following theorem. This identity is called the Cassini identity.

Theorem 2.5. *Let $b_{(3),n}$ can be the cobalancing number. The Cassini identity for the cobalancing number is as follows:*

$$b_{(3),n-1}b_{(3),n+1}-b_{(3),n}^2=\frac{(\alpha-\beta)^2}{32}-\frac{\alpha^{n-1}(\alpha-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-1}(\beta-1)^2}{5\beta+\alpha-2}.$$

Proof. Let's use the Binet formula for the cobalancing numbers and write the following equations

$$\begin{aligned} b_{(3),n-1} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n-1}+\frac{2}{5\beta+\alpha-2}\beta^{n-1}, \\ &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n+1}+\frac{2}{5\beta+\alpha-2}\beta^{n+1}. \end{aligned}$$

Let's write $B=\frac{2}{5\alpha+\beta-2}$ and $C=\frac{2}{5\beta+\alpha-2}$ in the above equations, let's edit the $b_{(3),n-1}b_{(3),n+1}-b_{(3),n}^2$ equation as follows:

$$b_{(3),n-1}b_{(3),n+1} = \frac{1}{4}-\frac{B}{2}\alpha^{n+1}-\frac{C}{2}\beta^{n+1}-\frac{B}{2}\alpha^{n-1}+B^2\alpha^{2n}+BC(\alpha\beta)^n\left(\frac{\beta}{\alpha}\right)-\frac{C}{2}\beta^{n-1}+BC(\alpha\beta)^n\left(\frac{\beta}{\alpha}\right)+C^2\beta^{2n},$$

then

$$b_{(3),n}b_{(3),n} = \frac{1}{4}-\frac{B}{2}\alpha^{n}-\frac{C}{2}\beta^{n}-\frac{B}{2}\alpha^{n}+B^2\alpha^{2n}+BC(\alpha\beta)^n-\frac{C}{2}\beta^{n}+BC(\alpha\beta)^n+C^2\beta^{2n}.$$

Let's subtract the equations written above from side to side and get the expression $b_{n-1}b_{n+1}-b_n^2$,

$$\begin{aligned} b_{(3),n-1}b_{(3),n+1}-b_{(3),n}^2 &= -\frac{B}{2}\alpha^{n+1}-\frac{C}{2}\beta^{n+1}-\frac{B}{2}\alpha^{n-1}+BC\left(\frac{\beta}{\alpha}\right)-\frac{C}{2}\beta^{n-1}+BC\left(\frac{\alpha}{\beta}\right)+B\alpha^n+C\beta^n-2BC \\ &= -\frac{B}{2}\alpha^n\left(\alpha+\frac{1}{\alpha}-2\right)-\frac{C}{2}\beta^n\left(\beta+\frac{1}{\beta}-2\right)+BC\left(\frac{\beta}{\alpha}-2+\frac{\alpha}{\beta}\right) \\ &= \frac{1}{32}(\alpha-\beta)^2-\frac{1}{2}\frac{2}{5\alpha+\beta-2}\alpha^{n-1}(\alpha-1)^2-\frac{1}{2}\frac{2}{5\beta+\alpha-2}\beta^{n-1}(\beta-1)^2. \end{aligned}$$

The final state of equality is as follows

$$b_{(3),n-1}b_{(3),n+1}-b_{(3),n}^2=\frac{(\alpha-\beta)^2}{32}-\frac{\alpha^{n-1}(\alpha-1)^2}{5\alpha+\beta-2}-\frac{\beta^{n-1}(\beta-1)^2}{5\beta+\alpha-2}.$$

□

There are many identities for cobalancing numbers. Another one is d'Ocagne's identity. This equality will be proved in the following theorem.

Theorem 2.6. *Let $b_{(3),n}$ can be the cobalancing number. The d'Ocagne's identity for the cobalancing number is as follows:*

$$b_{(3),m}b_{(3),n+1}-b_{(3),m+1}b_{(3),n}=\frac{(\alpha-1)(\alpha^m-\alpha^n)}{5\alpha+\beta-2}+\frac{(\beta-1)(\beta^m-\beta^n)}{5\beta+\alpha-2}+\frac{(\alpha-\beta)(\alpha^n\beta^m-\alpha^m\beta^n)}{32}.$$

Proof. Helping by (1.8) sequence

$$b_{(3),m}=-\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^m+\frac{2}{5\beta+\alpha-2}\beta^m$$

and

$$b_{(3),n+1}=-\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n+1}+\frac{2}{5\beta+\alpha-2}\beta^{n+1}$$

is written. Let's write $\frac{2}{5\alpha+\beta-2}=B$ and $\frac{2}{5\beta+\alpha-2}=C$ before multiplying the expressions $b_{(3),m}$ and $b_{(3),n+1}$.

$$b_{(3),m}b_{(3),n+1} = \frac{1}{4}-\frac{B}{2}\alpha^{n+1}-\frac{C}{2}\beta^{n+1}-\frac{B}{2}\alpha^{m}+B^2\alpha^{m+n+1}+BC\alpha^m\beta^{n+1}-\frac{C}{2}\beta^m+BC\alpha^{n+1}\beta^m+C^2\beta^{m+n+1}.$$

Also from the following equations

$$\begin{aligned} b_{(3),m+1} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{m+1}+\frac{2}{5\beta+\alpha-2}\beta^{m+1} \\ b_{(3),n} &= -\frac{1}{2}+\frac{2}{5\alpha+\beta-2}\alpha^{n}+\frac{2}{5\beta+\alpha-2}\beta^{n}, \end{aligned}$$

we get

$$\begin{aligned} b_{(3),m+1}b_{(3),n} &= \frac{1}{4}-\frac{B}{2}\alpha^{n}-\frac{C}{2}\beta^{n}-\frac{B}{2}\alpha^{m+1}+B^2\alpha^{m+n+1} \\ &\quad +BC\alpha^{m+1}\beta^{n}-\frac{C}{2}\beta^{m+1}+BC\alpha^{n}\beta^{m+1}+C^2\beta^{m+n+1}. \end{aligned}$$

Let's show that $BC=\frac{1}{32}$ before finding the $b_{(3),m}b_{(3),n+1}-b_{(3),m+1}b_{(3),n}$ difference. For $\alpha=3+2\sqrt{2}$ and $\beta=3-2\sqrt{2}$, we have $\alpha+\beta=6$ and $\alpha.\beta=1$, so

$$\begin{aligned} B.C &= \frac{2}{5\alpha+\beta-2}\frac{2}{5\beta+\alpha-2} \\ &= \frac{4}{26\alpha\beta+5[(\alpha+\beta)^2-2\alpha\beta]-12(\alpha+\beta)+4} \\ &= \frac{1}{32}. \end{aligned}$$

Then

$$\begin{aligned} b_{(3),m}b_{(3),n+1}-b_{(3),m+1}b_{(3),n} &= \frac{1}{4}-\frac{B}{2}\alpha^{n+1}-\frac{C}{2}\beta^{n+1}-\frac{B}{2}\alpha^{m}+C^2\alpha^{m+n+1}+BC\alpha^{m}\beta^{n+1}-\frac{C}{2}\beta^{m}+BC\alpha^{n+1}\beta^{m}+C^2\beta^{m+n+1} \\ &\quad -\frac{1}{4}+\frac{B}{2}\alpha^{n}+\frac{C}{2}\beta^{n}+\frac{B}{2}\alpha^{m+1}-B^2\alpha^{m+n+1}-BC\alpha^{m+1}\beta^{n}+\frac{C}{2}\beta^{m+1}-BC\alpha^{n}\beta^{m+1}-C^2\beta^{m+n+1} \\ &= \frac{B}{2}(\alpha-1)(\alpha^m-\alpha^n)+\frac{C}{2}(\beta-1)(\beta^m-\beta^n)+BC(\alpha-\beta)(\alpha^n\beta^m-\alpha^m\beta^n) \\ &= \frac{1}{2}\frac{2}{5\alpha+\beta-2}(\alpha-1)(\alpha^m-\alpha^n)+\frac{1}{2}\frac{2}{5\beta+\alpha-2}(\beta-1)(\beta^m-\beta^n) \\ &\quad +\frac{1}{32}(\alpha-\beta)(\alpha^n\beta^m-\alpha^m\beta^n). \end{aligned}$$

The final state of equality is as follows.

$$b_{(3),m}b_{(3),n+1}-b_{(3),m+1}b_{(3),n}=\frac{(\alpha-1)(\alpha^m-\alpha^n)}{5\alpha+\beta-2}+\frac{(\beta-1)(\beta^m-\beta^n)}{5\beta+\alpha-2}+\frac{(\alpha-\beta)(\alpha^n\beta^m-\alpha^m\beta^n)}{32}.$$

□

3. Sum of Cobalancing Numbers

By performing various operations, we can show the cobalancing numbers in the form of a different summation formulas. Sum formulas of cobalancing numbers with positive subscripts are obtained in the following three theorem. The results regarding the sum formulas can be obtained from the following theorems.

Theorem 3.1. *Let $b_{(3),n}$ can be the cobalancing number. The following difference equation is valid:*

$$(x^2+7)W_1-(1+7x^2)W_2-14x=x^n(b_{(3),n+3}-7b_{(3),n+2})+x^{n-1}(b_{(3),n+2}-7b_{(3),n+1})$$

where

$$W_1=\sum_{k=0}^{n-2}x^k.b_{(3),k+2} \text{ and } W_2=\sum_{k=0}^{n-2}x^k.b_{(3),k+3}.$$

Proof. Let's leave the term $b_{(3),n-2}$ alone in the (1.8) equation and add the following equations from side to side after writing them

$$\begin{aligned}
n &= 2 \Rightarrow x^0 b_{(3),0} = x^0 b_{(3),3} - x^0 7b_{(3),2} + x^0 7b_{(3),1} \\
n &= 3 \Rightarrow x^1 b_{(3),1} = x^1 b_{(3),4} - x^1 7b_{(3),3} + x^1 7b_{(3),2} \\
&\vdots \\
n &= n \Rightarrow x^{n-2} b_{(3),n-2} = x^{n-2} b_{(3),n+1} - x^{n-2} 7b_{(3),n} + x^{n-2} 7b_{(3),n-1} \\
n &= n+1 \Rightarrow x^{n-1} b_{(3),n-1} = x^{n-1} b_{(3),n+2} - x^{n-1} 7b_{(3),n+1} + x^{n-1} 7b_{(3),n} \\
n &= n+2 \Rightarrow x^{n} b_{(3),n} = x^{n} b_{(3),n+3} - x^{n} 7b_{(3),n+2} + x^{n} 7b_{(3),n+1}.
\end{aligned}$$

Let's add the equations from side to side

$$\begin{aligned}
& b_{(3),0} + xb_{(3),1} + (x^2+7)\sum_{k=0}^{n-2} x^k b_{(3),k+2} + x^{n-1} 7b_{(3),n+1} + x^n 7b_{(3),n+2} \\
= \; & 14x + (1+7x^2)\sum_{k=0}^{n-2} x^k b_{(3),k+3} + x^{n-1} b_{(3),n+2} + x^n b_{(3),n+3}.
\end{aligned}$$

If the equations $b_{(3),0} = 0, b_{(3),1} = 0, b_{(3),2} = 2$ are written in their places, the following equality is obtained

$$\begin{aligned}
& (x^2+7)\sum_{k=0}^{n-2} x^k .b_{(3),k+2} - (1+7x^2)\sum_{k=0}^{n-2} x^k .b_{(3),k+3} - 14x \\
= \; & x^n (b_{(3),n+3} - 7b_{(3),n+2}) + x^{n-1}(b_{(3),n+2} - 7b_{(3),n+1}).
\end{aligned}$$

□

Now let the result be given that the even and odd terms of sums are in the same difference equation.

Lemma 3.2. *Let $b_{(3),n}$ can be the cobalancing number. The following difference equation is valid:*

$$(7+x)W_3 - (1+7x)W_4 = x^n (b_{(3),2n+1} + (7+x)b_{(3),2n})$$

where

$$W_3 = \sum_{k=1}^{n} x^k b_{(3),2k} \text{ and } W_4 = \sum_{k=1}^{n-1} x^k b_{(3),2k+1}.$$

Proof. Let's leave the term $7b_n$ alone in the (1.8) equation and add the following equations from side to side after writing them

$$7b_{(3),n} = b_{(3),n+1} + 7b_{(3),n-1} - b_{(3),n-2}.$$

$$\begin{aligned}
n &= 2 \Rightarrow 7x^1 b_{(3),2} = x^1 b_{(3),3} + 7x^1 b_{(3),1} - x^1 b_{(3),0} \\
n &= 4 \Rightarrow 7x^2 b_{(3),4} = x^2 b_{(3),5} + 7x^2 b_{(3),3} - x^2 b_{(3),2} \\
&\vdots \\
n &= 2n-2 \Rightarrow 7x^{n-1} b_{(3),2n-2} = x^{n-1} b_{(3),2n-1} + 7x^{n-1} b_{(3),2n-3} - x^{n-1} b_{(3),2n-4} \\
n &= 2n \Rightarrow 7x^{n} b_{(3),2n} = x^{n} b_{(3),2n+1} + 7x^{n} b_{(3),2n-1} - x^{n} b_{(3),2n-2}.
\end{aligned}$$

Let's add the equations from side to side, then

$$xb_{(3),0} + (7+x)\sum_{k=1}^{n-1} x^k .b_{(3),2k} = (1+7x)\sum_{k=1}^{n-1} x^k .b_{(3),2k+1}.$$

If the equations $b_0 = 0$, are written in their places, the following equality is obtained

$$x^n b_{(3),2n+1} = (7+x)\sum_{k=1}^{n-1} x^k b_{(3),2k} - (1+7x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1}.$$

Let's add and subtract the term $x^n b_{2n}$ in the sum symbol located on the right side of the equation

$$x^n b_{(3),2n+1} = (7+x)\sum_{k=1}^{n} x^k b_{(3),2k} - (1+7x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1} - (7+x)(x^n b_{(3),2n}).$$

In the last case, the equality is most regularly as follows:

$$(7+x)\sum_{k=1}^{n} x^k b_{(3),2k} - (1+7x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1} = x^n (b_{(3),2n+1} + (7+x) b_{(3),2n}).$$

□

Lemma 3.3. *Let $b_{(3),n}$ can be the cobalancing number. If $x \neq 0$, following equation is valid:*

$$(7+x)W_5 - (\frac{1}{x}+7)W_6 = x^n b_{(3),2n+2} - (\frac{1}{x}+7)2x$$

where

$$W_5 = \sum_{k=1}^{n-1} x^k b_{(3),2k+1} \text{ and } W_6 = \sum_{k=1}^{n} x^k b_{(3),2k}.$$

Proof. Let's leave the term $7b_{(3),n}$ alone in the (1.8) equation and add the following equations from side to side after writing them

$$7b_{(3),n} = b_{(3),n+1} + 7b_{(3),n-1} - b_{(3),n-2}.$$

$$\begin{aligned}
n &= 3 \Rightarrow 7x^1 b_{(3),3} = x^1 b_{(3),4} + 7x^1 b_{(3),2} - x^1 b_{(3),1}\\
n &= 5 \Rightarrow 7x^2 b_{(3),5} = x^2 b_{(3),6} + 7x^2 b_{(3),4} - x^2 b_{(3),3}\\
&\vdots\\
n &= 2n-1 \Rightarrow 7x^{n-1} b_{(3),2n-1} = x^{n-1} b_{(3),2n} + 7x^{n-1} b_{(3),2n-2} - x^{n-1} b_{(3),2n-3}\\
n &= 2n+1 \Rightarrow 7x^n b_{(3),2n+1} = x^n b_{(3),2n+2} + 7x^n b_{(3),2n} - x^n b_{(3),2n-1}.
\end{aligned}$$

Let's multiply and divide the first terms to the right of the equal sign by x

$$\begin{aligned}
n &= 3 \Rightarrow 7x^1 b_{(3),3} = \frac{x^2 b_4}{x} + 7x^1 b_{(3),2} - x^1 b_{(3),1}\\
n &= 5 \Rightarrow 7x^2 b_{(3),5} = \frac{x^3 b_{(3),6}}{x} + 7x^2 b_{(3),4} - x^2 b_{(3),3}\\
&\vdots\\
n &= 2n-1 \Rightarrow 7x^{n-1} b_{(3),2n-1} = \frac{x^n b_{(3),2n}}{x} + 7x^{n-1} b_{(3),2n-2} - x^{n-1} b_{(3),2n-3}\\
n &= 2n+1 \Rightarrow 7x^n b_{(3),2n+1} = x^n b_{(3),2n+2} + 7x^n b_{(3),2n} - x^n b_{(3),2n-1}.
\end{aligned}$$

Let's add the equations side by side

$$xb_{(3),1} + (7+x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1} = x^n b_{(3),2n+2} + (\frac{1}{x}+7)\sum_{k=2}^{n} x^k b_{(3),2k}.$$

In the last case, let's add the $xb_{(3),2}$ term to the sum term on the right side in the equality we obtained

$$xb_{(3),1} + (7+x)\sum_{k=1}^{n-1} x^k b_{(3),2k+1} \quad = \quad x^n b_{(3),2n+2} + (\frac{1}{x}+7)\sum_{k=1}^{n} x^k b_{(3),2k} - (\frac{1}{x}+7)xb_{(3),2}$$

then

$$(7+x)\sum_{k=1}^{n-1}x^k b_{(3),2k+1}-(\frac{1}{x}+7)\sum_{k=1}^{n}x^k b_{(3),2k} \quad = \quad x^n b_{(3),2n+2}-xb_{(3),1}-(\frac{1}{x}+7)xb_{(3),2}.$$

If it is written instead of $b_{(3),1}=0$ and $b_{(3),2}=2$, the most regular form of equality is as follows:

$$(7+x)\sum_{k=1}^{n-1}x^k b_{(3),2k+1}-(\frac{1}{x}+7)\sum_{k=1}^{n}x^k b_{(3),2k} \quad = \quad x^n b_{(3),2n+2}-14x-2.$$

□

The results to be found now will be found with the help of the theorems given above.

Theorem 3.4. *Let $b_{(3),n}$ can be the cobalancing number. The following equation is valid:*

$$\sum_{k=1}^{n}x^k b_{(3),2k}=\frac{\left\{\begin{array}{c} x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(x^2+7x) \\ +(x^n b_{(3),2n+2}-14x-2)(x+7x^2) \end{array}\right\}}{x^3-35x^2+35x-1} \tag{3.1}$$

and

$$\sum_{k=1}^{n-1}x^k b_{(3),2k+1}=\frac{\left\{\begin{array}{c} x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(1+7x) \\ +(x^n b_{(3),2n+2}-14x-2)(7x+x^2) \end{array}\right\}}{x^3-35x^2+35x-1}. \tag{3.2}$$

Proof. The following equations have been proved in the previous lemmas

$$(7+x)\sum_{k=1}^{n}x^k b_{(3),2k}-(1+7x)\sum_{k=1}^{n-1}x^k b_{(3),2k+1}) \quad = \quad x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})$$

and

$$(7+x)\sum_{k=1}^{n-1}x^k b_{(3),2k+1}-(\frac{1}{x}+7)\sum_{k=1}^{n}x^k b_{(3),2k} \quad = \quad x^n b_{(3),2n+2}-14x-2.$$

Let $A_1=\sum\limits_{k=1}^{n}x^k b_{(3),2k}$ and $B_1=\sum\limits_{k=1}^{n-1}x^k b_{(3),2k+1}$. Let's arrange the above equations in the following way and get a system of equations

$$(7+x)A_1-(1+7x)B_1 \quad = \quad x^n(b_{(3),2n+1}+(7+x)b_{(3),2n}) \tag{3.3}$$

$$-(\frac{1}{x}+7)A_1+(7+x)B_1 \quad = \quad x^n b_{(3),2n+2}-14x-2. \tag{3.4}$$

Let's multiply the first equation by $\frac{1}{x}+7$ and multiply the second equation by $7+x$, then add the equations side by side

$$\begin{aligned} (\frac{1}{x}+7)(7+x)A_1-(\frac{1}{x}+7)(1+7x)B_1 \quad &= \quad x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(\frac{1}{x}+7) \\ -(\frac{1}{x}+7)(7+x)A_1+(7+x)(7+x)B_1 \quad &= \quad (x^n b_{(3),2n+2}-14x-2)(7+x). \end{aligned}$$

If the equations are added side by side, the following equality is found

$$B_1=\frac{x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(1+7x)+(x^n b_{(3),2n+2}-14x-2)(7x+x^2)}{x^3-35x^2+35x-1}.$$

Let's multiply (3.3) by $(7+x)$ and multiply (3.4) by $(1+7x)$, then add the equations side by side

$$\begin{aligned} (7+x)(7+x)A_1-(7+x)(1+7x)B_1 \quad &= \quad x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(7+x) \\ -(\frac{1}{x}+7)(1+7x)A_1+(7+x)(1+7x)B_1 \quad &= \quad (x^n b_{(3),2n+2}-14x-2)(1+7x). \end{aligned}$$

When the above equations are added side by side, the equal of the expression A_1 will be as follows

$$A_1 = \frac{x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(x^2+7x)+(x^n b_{(3),2n+2}-14x-2)(x+7x^2)}{x^3-35x^2+35x-1}.$$

In the last case, the following equations are correct

$$\begin{aligned}
\sum_{k=1}^{n} x^k b_{(3),2k} &= \frac{\left\{\begin{array}{c} x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(x^2+7x) \\ +(x^n b_{(3),2n+2}-14x-2)(x+7x^2) \end{array}\right\}}{x^3-35x^2+35x-1} \\
\sum_{k=1}^{n-1} x^k b_{(3),2k+1} &= \frac{\left\{\begin{array}{c} x^n(b_{(3),2n+1}+(7+x)b_{(3),2n})(1+7x) \\ +(x^n b_{(3),2n+2}-14x-2)(7x+x^2) \end{array}\right\}}{x^3-35x^2+35x-1}.
\end{aligned}$$

□

The following theorem is special cases of the summation formulas that we have found.

Theorem 3.5. *Let $b_{(3),n}$ can be the cobalancing number. $n \geq 0$, we have the following sum formulas:*

1. $\sum_{k=1}^{n-1} (-1)^k b_{(3),2k+1} = \frac{(-1)^n}{12}(6b_{(3),2n}+b_{(3),2n+1}+b_{(3),2n+2})+1$
2. $\sum_{k=1}^{n} (-1)^k b_{(3),2k} = \frac{(-1)^n}{12}(6b_{(3),2n}+b_{(3),2n+1}-b_{(3),2n+2})-1$
3. $\sum_{k=0}^{n-2} (-1)^k b_{(3),k+2} = \frac{(-1)^n}{16}(b_{(3),n+3}-8b_{(3),n+2}+15b_{(3),n+1})+\frac{1}{8}$

Proof. 1) Let's write $x=-1$ in the (3.2) equation. Then

$$\begin{aligned}
\sum_{k=1}^{n-1} (-1)^k b_{(3),2k+1} &= \frac{(-1)^n(b_{(3),2n+1}+6b_{(3),2n})(-6)+((-1)^n b_{(3),2n+2}+14-2)(-6)}{-1-35-35-1} \\
&= \frac{(-1)^n(-6)(b_{(3),2n+1}+6b_{(3),2n})+(-1)^n(-6)b_{(3),2n+2}}{-72}+\frac{-72}{-72} \\
&= \frac{(-1)^n}{12}(6b_{(3),2n}+b_{(3),2n+1}+b_{(3),2n+2})+1
\end{aligned}$$

As in other cases, it is proven in a similar way. □

4. Conclusions

In this study, cobalancing numbers, which are an integer sequence with a non-homogeneous second-order recurrence relation, are transformed into a sequence with a homogeneous third-order recurrence relation, thus providing ease of operation. Some of the results found are the generating function, Binet formula, specially defined Catalan, Cassini and d'Ocagne identities and some sum formulas.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

References

[1] A. Behera, G. K. Panda, *On the square roots of triangular numbers*, Fibonacci Quart., **37** (2) (1999), 98-105.

[2] G. K. Panda, *Some fascinating properties of balancing numbers*, Congressus Numerantium, Proceedings of the Eleventh International Conference on Fibonacci Numbers and Their Applications, (Willian Webb, Ed.), 194, (2009), 185–189.

[3] K. B. Subramaniam, *A simple computation of square triangular numbers*, Int. J. Math. Educ. Sci. Technol., **23** (5) (1992), 790-793.

[4] G. K. Panda, P. K. Ray, *Cobalancing numbers and cobalancers*, Int. J. Math. Math. Sci., **2005** (8) (2005), 1189-1200.

[5] K. Liptai, *Fibonacci balancing numbers*, Fibonacci Quart., **42** (4) (2004), 330-340.

[6] G. K. Panda, *Sequence balancing and cobalancing numbers*, Fibonacci Quart., **45** (3) (2007), 265-271.

[7] J. J. Tattersall, *Elementary Number Theory in Nine Chapters*, Cambridge University Press, 1999.

[8] A.G. Shannon, A. F. Horadam, *Some properties of third-order recurrence relations*, Fibonacci Quart., **10**(2)(1972),135-146.

[9] H. Merzouk, A. Boussayoud, A. Abderrezzak, *Ordinary generating functions of binary products of third-order recurrence relations and 2-orthogonal polynomials*, Math. Slovaca, **72** (1) (2022), 11-34.

[10] P. Catarino, A. Borges, *On Leonardo numbers*, Acta Math. Univ. Comen., **89**(1)(2019), 75-86.

[11] S. Nanda, *Number Theory*, Allied Publishers, 1985.

[12] P. Catarino, H. Campos, P. Vasco, *On some identities for balancing and cobalancing numbers*, Ann. Math. Inform. **45** (2015), 11–24.

Communications in Advanced Mathematical Sciences
Vol. 7, No. 1, 14-26, 2024
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1395130

Partial Soft Derivative

Burak Arslan[1]*, Serdar Enginoğlu[2]

Abstract
The concept of soft derivative, introduced by Molodtsov in 1999, is one of the fundamental concepts in soft analysis. The handled paper defines partial soft derivative and studies some of its basic properties, such as the relation between partial soft derivative and boundedness, some basic partial soft derivative rules, e.g., sum rule, constant multiple rule, and difference rule, the relation between soft derivative and partial soft derivative, the relation between classical partial derivative and partial soft derivative, and the geometric interpretation of partial soft derivative. Moreover, it exemplifies the theoretical part of the study and provides figures for the geometric interpretation. Finally, this study discusses the need for further research.

Keywords: Partial soft derivative, Soft analysis, Soft derivative, Soft sets
2010 AMS: 26A24, 28A15

[1] *Department of Mathematics, Faculty of Science, Çanakkale Onsekiz Mart University, Çanakkale, Türkiye, tburakarslan@gmail.com, ORCID: 0000-0002-1724-8841*
[2] *Department of Mathematics, Faculty of Science, Çanakkale Onsekiz Mart University, Çanakkale, Türkiye, serdarenginoglu@gmail.com, ORCID: 0000-0002-7188-9893*
***Corresponding author**

Received: 23 November 2023, **Accepted:** 30 January 2024, **Available online:** 9 February 2024
How to cite this article: B. Arslan, S. Enginoğlu, *Partial soft derivative*, Commun. Adv. Math. Sci., 7(1) (2024) 14-26.

1. Introduction

Molodtsov [1] has introduced soft sets and discussed their relationships with several mathematical tools. Moreover, the author has investigated soft sets' applications to stability and regularization, game theory and operations research, and soft analysis. Molodtsov has studied soft limit, soft approximator (soft derivative), and upper and lower Riemann and Perron integrals in soft analysis. Afterward, the author has written a book entitled Soft Set Theory [2] that contains many topics related to soft sets. Then, Molodtsov et al. [3] have widely explored the basic concepts of soft analysis. Further, Molodtsov [4] has suggested higher-order soft derivative and higher-order almost soft derivative. Besides, the author [5, 6] has analyzed the basic concepts of rational analysis. Additionally, Acharjee and Molodtsov [7] have proposed soft rational line integral. However, since most of the aforesaid studies are in Russian, soft analysis studies have not become widespread.

On the other hand, despite the considerable developments in classical analysis, the fact that there are many types of uncertainty in real-life problems and that increasing the need for new mathematical tools makes soft analysis worth studying. Therefore, this paper focuses on the partial soft derivative, one of the essential concepts in soft analysis. Thus, this study aims to increase the widespread impact and make soft analysis studies more accessible. Moreover, the partial soft derivative will shed light on the concepts of higher-order partial soft derivative and soft gradients. Hence, this paper provides ideas concerning further studies to researchers. Section 2 of the present study provides some basic definitions and properties to be required in the next section. Section 3 defines partial soft derivative and studies some of its basic properties. The final section discusses the need for future studies.

2. Preliminaries

This section presents some of the basic definitions and properties to be needed for the following section. Across this paper, the notations $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^{\geq 0}$ represent the set of integer, real, positive real, and non-negative real numbers, respectively. Moreover, $\mathbb{R}^2 := \mathbb{R}\times\mathbb{R}$ and $P(U)$ denotes the set of all the classical subsets of U.

Definition 2.1. *[1, 2] Let U be a universal set, E be a parameter set, and $f : E \to P(U)$ be a function. Then, f is called a soft set parameterized via E over U (briefly over U).*

Example 2.2. *Let $f : \mathbb{Z} \to P(\mathbb{R})$ be a function defined by $f(x) = [x+2, x+4]$. Then, f is a soft set over $\mathbb{R}$.*

Definition 2.3. *[1, 2] Let M be a set called a model set, U be a universal set, E be a parameter set, and $f : M\times E \to P(U)$ be a function. Then, f is called a soft mapping parameterized via $M\times E$ over U (briefly over U).*

Definition 2.4. *[1, 2, 3] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_f(a) \neq \emptyset$, and $L \in \mathbb{R}$. Then, the real number L is called a (τ,ε)-soft derivative of f at the point a if $x \in \tau_f(a) \Rightarrow |f(x) - f(a) - L(x-a)| \leq \varepsilon(a)$. The set of all the (τ,ε)-soft derivatives of f at the point a is denoted by $D(f,a,\tau,\varepsilon)$. If $D(f,a,\tau,\varepsilon) = \emptyset$, then the (τ,ε)-soft derivative of f at the point a does not exist.*

Here, $\tau : \mathbb{R} \to P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ are two functions such that $\tau(a)$ is a set of points that are close to the point a but not equal to a. In addition, $\tau_f(a) := \tau(a)\cap \mathrm{Dom}(f)$, for all $a \in \mathbb{R}$, where $\mathrm{Dom}(f)$ stands for the domain set of f.

3. Partial Soft Derivative

This section defines the concept of partial soft derivative and studies some of its basic properties. Throughout this section, let $\tau,\lambda,\kappa : \mathbb{R}^2 \to P\left(\mathbb{R}^2\right)$, $\varepsilon,\alpha,\beta : \mathbb{R}^2 \to \mathbb{R}^{\geq 0}$, and $\delta : \mathbb{R}^2 \to \mathbb{R}^+$ be seven functions such that $\tau(a,b)$, $\lambda(a,b)$, and $\kappa(a,b)$ are sets of points that are close to the point (a,b) but not equal to (a,b). Besides, let $\tau_f(a,b) := \tau(a,b)\cap\mathrm{Dom}(f)$, for all $(a,b) \in \mathbb{R}^2$.

Definition 3.1. *The set of all the points belonging to $\tau(a,b)$ and the plane $y = b$ is defined by $\tau_x(a,b) := \tau(a,b)\cap(\mathbb{R}\times\{b\})$. Similarly, the set of all the points belonging to $\tau(a,b)$ and the plane $x = a$ is defined by $\tau_y(a,b) := \tau(a,b)\cap(\{a\}\times\mathbb{R})$. Therefore, the set of all the points belonging to $\tau(a,b)$ and whose first components are greater than a is defined by $\tau_x^+(a,b) := \tau(a,b)\cap((a,\infty)\times\mathbb{R})$ and the set of all the points belonging to $\tau(a,b)$ and whose first components are less than a is defined by $\tau_x^-(a,b) := \tau(a,b)\cap((-\infty,a)\times\mathbb{R})$. Similarly, the set of all the points belonging to $\tau(a,b)$ and whose second components are greater than b is defined by $\tau_y^+(a,b) := \tau(a,b)\cap(\mathbb{R}\times(b,\infty))$ and the set of all the points belonging to $\tau(a,b)$ and whose second components are less than b is defined by $\tau_y^-(a,b) := \tau(a,b)\cap(\mathbb{R}\times(-\infty,b))$.*

Moreover, if $\tau_x^-(a,b) = \emptyset$, for all $(a,b) \in \mathbb{R}^2$, then this mapping is called by τ_x-right mapping, and if $\tau_x^+(a,b) = \emptyset$, for all $(a,b) \in \mathbb{R}^2$, then this mapping is called by τ_x-left mapping. Similarly, if $\tau_y^-(a,b) = \emptyset$, for all $(a,b) \in \mathbb{R}^2$, then this mapping is called by τ_y-right mapping and if $\tau_y^+(a,b) = \emptyset$, for all $(a,b) \in \mathbb{R}^2$, then this mapping is called by τ_y-left mapping.

Furthermore, $\tau_\delta(a,b)$ is defined by

$$\tau_\delta(a,b) := \left\{(x,y) \in \mathbb{R}^2 \ : 0 < \sqrt{(x-a)^2+(y-b)^2} \leq \delta(a,b)\right\}$$

Thus, $\tau_{x\delta}^+(a,b) := \tau_\delta(a,b)\cap((a,\infty)\times\mathbb{R})$, $\tau_{x\delta}^-(a,b) := \tau_\delta(a,b)\cap((-\infty,a)\times\mathbb{R})$, $\tau_{y\delta}^+(a,b) := \tau_\delta(a,b)\cap(\mathbb{R}\times(b,\infty))$, and $\tau_{y\delta}^-(a,b) := \tau_\delta(a,b)\cap(\mathbb{R}\times(-\infty,b))$.

Note 3.2. *It must be noted that $\tau(a,b) = \tau_x^+(a,b)\cup\tau_x^-(a,b)$, $\tau(a,b) = \tau_y^+(a,b)\cup\tau_y^-(a,b)$, $\tau_\delta(a,b) = \tau_{x\delta}^+(a,b)\cup\tau_{x\delta}^-(a,b)$, and $\tau_\delta(a,b) = \tau_{y\delta}^+(a,b)\cup\tau_{y\delta}^-(a,b)$.*

Definition 3.3. *Let $A\times B \subseteq \mathbb{R}^2$, $f : A\times B \to \mathbb{R}$ be a function, $(a,b) \in A\times B$, $\tau_f(a,b) \neq \emptyset$, and $L \in \mathbb{R}$. Then, the real number L is called a partial (τ,ε)-soft derivative of f with respect to x at the point (a,b) if $(x,b) \in \tau_f(a,b) \Rightarrow |f(x,b) - f(a,b) - L(x-a)| \leq \varepsilon(a,b)$. The set of all the partial (τ,ε)-soft derivatives of f with respect to x at the point (a,b) is denoted by $D_x(f,(a,b),\tau,\varepsilon)$. If $D_x(f,(a,b),\tau,\varepsilon) = \emptyset$, then the partial (τ,ε)-soft derivative of f with respect to x at the point (a,b) does not exist.*

Definition 3.4. *Let $A\times B \subseteq \mathbb{R}^2$, $f : A\times B \to \mathbb{R}$ be a function, $(a,b) \in A\times B$, $\tau_f(a,b) \neq \emptyset$, and $L \in \mathbb{R}$. Then, the real number L is called a partial (τ,ε)-soft derivative of f with respect to y at the point (a,b) if $(a,y) \in \tau_f(a,b) \Rightarrow |f(a,y) - f(a,b) - L(y-b)| \leq \varepsilon(a,b)$. The set of all the partial (τ,ε)-soft derivatives of f with respect to y at the point (a,b) is denoted by $D_y(f,(a,b),\tau,\varepsilon)$. If $D_y(f,(a,b),\tau,\varepsilon) = \emptyset$, then the partial (τ,ε)-soft derivative of f with respect to y at the point (a,b) does not exist.*

Note 3.5. *Each of the concepts of partial (τ,ε)-soft derivative with respect to x and y is a soft mapping parameterized via $\Phi(A\times B,\mathbb{R})\times(A\times B)\times\Phi\left(\mathbb{R}^2,P\left(\mathbb{R}^2\right)\right)\times\Phi\left(\mathbb{R}^2,\mathbb{R}^{\geq 0}\right)$ over $\mathbb{R}$ such that $\emptyset \neq A\times B \subseteq \mathbb{R}^2$.*

Example 3.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x,y) = x^3 + 2y^2$ and $\varepsilon(-1,3) = 2$. Since*

$$\begin{aligned}(x,3) \in \tau_{\frac{1}{2}}(-1,3) \cap \mathbb{R}^2 &\Leftrightarrow 0 < \sqrt{(x+1)^2 + (3-3)^2} \leq \tfrac{1}{2} \\ &\Leftrightarrow 0 < |x+1| \leq \tfrac{1}{2}\end{aligned}$$

then, for all $(x,3) \in \tau_{\frac{1}{2}}(-1,3) \cap \mathbb{R}^2$,

$$\begin{aligned}|f(x,3) - f(-1,3) - L(x+1)| \leq 2 &\Leftrightarrow |x^3 + 1 - L(x+1)| \leq 2 \\ &\Leftrightarrow -x^3 - 1 - 2 \leq -L(x+1) \leq -x^3 - 1 + 2 \\ &\Leftrightarrow \tfrac{x^3+1}{x+1} - \tfrac{2}{|x+1|} \leq L \leq \tfrac{x^3+1}{x+1} + \tfrac{2}{|x+1|} \\ &\Leftrightarrow x^2 - x + 1 - \tfrac{2}{|x+1|} \leq L \leq x^2 - x + 1 + \tfrac{2}{|x+1|} \\ &\Leftrightarrow L \in \left[\tfrac{3}{4}, \tfrac{23}{4}\right]\end{aligned}$$

Therefore, $D_x\left(f, (-1,3), \tau_{\frac{1}{2}}, \varepsilon\right) = \left[\frac{3}{4}, \frac{23}{4}\right]$. Similarly, as

$$\begin{aligned}(-1,y) \in \tau_{\frac{1}{2}}(-1,3) \cap \mathbb{R}^2 &\Leftrightarrow 0 < \sqrt{(-1+1)^2 + (y-3)^2} \leq \tfrac{1}{2} \\ &\Leftrightarrow 0 < |y-3| \leq \tfrac{1}{2}\end{aligned}$$

then, for all $(-1,y) \in \tau_{\frac{1}{2}}(-1,3) \cap \mathbb{R}^2$,

$$\begin{aligned}|f(-1,y) - f(-1,3) - L(y-3)| \leq 2 &\Leftrightarrow |2y^2 - 18 - L(y-3)| \leq 2 \\ &\Leftrightarrow -2y^2 + 18 - 2 \leq -L(y-3) \leq -2y^2 + 18 + 2 \\ &\Leftrightarrow \tfrac{2y^2-18}{y-3} - \tfrac{2}{|y-3|} \leq L \leq \tfrac{2y^2-18}{y-3} + \tfrac{2}{|y-3|} \\ &\Leftrightarrow 2y + 6 - \tfrac{2}{|y-3|} \leq L \leq 2y + 6 + \tfrac{2}{|y-3|} \\ &\Leftrightarrow L \in [9, 15]\end{aligned}$$

Thus, $D_y\left(f, (-1,3), \tau_{\frac{1}{2}}, \varepsilon\right) = [9, 15]$.

Theorem 3.7. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\tau_f(a,b)$ be bounded. If $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$, then $z = f(x,b)$ is bounded on $\tau_f(a,b)$.*

Proof. Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, $\tau_f(a,b)$ be bounded, and $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$. Then, $\tau_f(a,b) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$\begin{aligned}(x,b) \in \tau_f(a,b) &\Rightarrow |f(x,b) - f(a,b) - L(x-a)| \leq \varepsilon(a,b) \\ &\Rightarrow f(a,b) + L(x-a) - \varepsilon(a,b) \leq f(x,b) \leq f(a,b) + L(x-a) + \varepsilon(a,b) \\ &\Rightarrow f(a,b) + \inf_{x \in \tau_f(a,b)}\{L(x-a)\} - \varepsilon(a,b) \leq f(x,b) \leq f(a,b) + \sup_{x \in \tau_f(a,b)}\{L(x-a)\} + \varepsilon(a,b)\end{aligned}$$

Since

$$f(a,b) + \inf_{x \in \tau_f(a,b)}\{L(x-a)\} - \varepsilon(a,b) \in \mathbb{R} \quad \text{and} \quad f(a,b) + \sup_{x \in \tau_f(a,b)}\{L(x-a)\} + \varepsilon(a,b) \in \mathbb{R}$$

then $z = f(x,b)$ is bounded on $\tau_f(a,b)$. □

Theorem 3.8. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\tau_f(a,b) \neq \emptyset$. If $z = f(x,b)$ is bounded on $\tau_f(a,b)$, then there exists a function $\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ such that $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$.*

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, $\tau_f(a,b)\neq\emptyset$, and $z=f(x,b)$ be bounded on $\tau_f(a,b)$. Then, there exists an $M\in\mathbb{R}$ such that $|f(x,b)|\leq M$, for all $(x,y)\in\tau_f(a,b)$. Then,

$$\begin{aligned}(x,b)\in\tau_f(a,b) &\Rightarrow |f(x,b)|\leq M\\ &\Rightarrow -M-f(a,b)\leq f(x,b)-f(a,b)\leq M-f(a,b)\\ &\Rightarrow |f(x,b)-f(a,b)-0(x-a)|\leq\max\{|M+f(a,b)|,|M-f(a,b)|\}\end{aligned}$$

Hence, for any function $\varepsilon:\mathbb{R}^2\to\mathbb{R}$ such that $\varepsilon(a,b)=\max\{|M+f(a,b)|,|M-f(a,b)|\}$, $0\in D_x(f,(a,b),\tau,\varepsilon)$. Consequently, $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$. □

Theorem 3.9. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $\tau_f(a,b)$ be bounded. If $D_y(f,(a,b),\tau,\varepsilon)\neq\emptyset$, then $z=f(a,y)$ is bounded on $\tau_f(a,b)$.*

Theorem 3.10. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $\tau_f(a,b)\neq\emptyset$. If $z=f(a,y)$ is bounded on $\tau_f(a,b)$, then there exists a function $\varepsilon:\mathbb{R}^2\to\mathbb{R}$ such that $D_y(f,(a,b),\tau,\varepsilon)\neq\emptyset$.*

The proofs are as in Theorems 3.7 and 3.8, respectively.

Theorem 3.11. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, and $(a,b)\in A\times B$. If $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$, then*

$$D_x(f,(a,b),\tau,\varepsilon)=\left[\sup_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\right),\inf_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|}\right)\right]$$

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$. Then, $\tau_f(a,b)\neq\emptyset$ and there exists an $L\in\mathbb{R}$ such that, for all $(x,b)\in\tau_f(a,b)$,

$$\begin{aligned}|f(x,b)-f(a,b)-L(x-a)|\leq\varepsilon(a,b) &\Rightarrow -(f(x,b)-f(a,b))-\varepsilon(a,b)\leq -L(x-a)\leq -(f(x,b)-f(a,b))+\varepsilon(a,b)\\ &\Rightarrow \begin{cases}\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{x-a}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{x-a}, & (x,b)\in\tau_x^+(a,b)\cap A\times B\\ \frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{x-a}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{x-a}, & (x,b)\in\tau_x^-(a,b)\cap A\times B\end{cases}\\ &\Rightarrow \begin{cases}\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|}, & (x,b)\in\tau_x^+(a,b)\cap A\times B\\ \frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|}, & (x,b)\in\tau_x^-(a,b)\cap A\times B\end{cases}\\ &\Rightarrow \frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|},\quad (x,b)\in\tau_f(a,b)\end{aligned}$$

Hence,

$$\sup_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)-f(a,b)}{x-a}-\tfrac{\varepsilon(a,b)}{|x-a|}\right)\leq L\quad\text{and}\quad L\leq\inf_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)-f(a,b)}{x-a}+\tfrac{\varepsilon(a,b)}{|x-a|}\right)$$

Consequently,

$$D_x(f,(a,b),\tau,\varepsilon)=\left[\sup_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\right),\inf_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|}\right)\right]$$

□

Theorem 3.12. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, and $(a,b)\in A\times B$. If $D_y(f,(a,b),\tau,\varepsilon)\neq\emptyset$, then*

$$D_y(f,(a,b),\tau,\varepsilon)=\left[\sup_{(a,y)\in\tau_f(a,b)}\left(\frac{f(a,y)-f(a,b)}{y-b}-\frac{\varepsilon(a,b)}{|y-b|}\right),\inf_{(a,y)\in\tau_f(a,b)}\left(\frac{f(a,y)-f(a,b)}{y-b}+\frac{\varepsilon(a,b)}{|y-b|}\right)\right]$$

The proof is as in Theorem 3.11.

Theorem 3.13. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $\beta(a,b)\leq\alpha(a,b)$. If $D_x(f,(a,b),\tau,\beta)\neq\emptyset$, then $D_x(f,(a,b),\tau,\alpha)\neq\emptyset$. Moreover, $D_x(f,(a,b),\tau,\beta)\subseteq D_x(f,(a,b),\tau,\alpha)$.*

Proof. Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, $\beta(a,b) \leq \alpha(a,b)$, and $D_x(f,(a,b),\tau,\beta) \neq \emptyset$. Then, $\tau_f(a,b) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$(x,b) \in \tau_f(a,b) \Rightarrow |f(x,b) - f(a,b) - L(x-a)| \leq \beta(a,b) \leq \alpha(a,b)$$

Therefore, $D_x(f,(a,b),\tau,\alpha) \neq \emptyset$. Moreover, since $\beta(a,b) \leq \alpha(a,b)$, then

$$\sup_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b)-f(a,b)}{x-a} - \frac{\alpha(a,b)}{|x-a|} \right) \leq \sup_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b)-f(a,b)}{x-a} - \frac{\beta(a,b)}{|x-a|} \right)$$

and

$$\inf_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b)-f(a,b)}{x-a} + \frac{\beta(a,b)}{|x-a|} \right) \leq \inf_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b)-f(a,b)}{x-a} + \frac{\alpha(a,b)}{|x-a|} \right)$$

Thus,

$$\begin{aligned} D_x(f,(a,b),\tau,\beta) &= \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\beta(a,b)}{|x-a|} \right), \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\beta(a,b)}{|x-a|} \right) \right] \\ &\subseteq \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\alpha(a,b)}{|x-a|} \right), \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\alpha(a,b)}{|x-a|} \right) \right] \\ &= D_x(f,(a,b),\tau,\alpha) \end{aligned}$$

□

Theorem 3.14. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\beta(a,b) \leq \alpha(a,b)$. If $D_y(f,(a,b),\tau,\beta) \neq \emptyset$, then $D_y(f,(a,b),\tau,\alpha) \neq \emptyset$. Moreover, $D_y(f,(a,b),\tau,\beta) \subseteq D_y(f,(a,b),\tau,\alpha)$.*

The proof is as in Theorem 3.13.

Theorem 3.15. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\emptyset \neq \lambda_f(a,b) \subseteq \tau_f(a,b)$. If $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$, then $D_x(f,(a,b),\lambda,\varepsilon) \neq \emptyset$. Moreover, $D_x(f,(a,b),\tau,\varepsilon) \subseteq D_x(f,(a,b),\lambda,\varepsilon)$.*

Proof. Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, $\emptyset \neq \lambda_f(a,b) \subseteq \tau_f(a,b)$, and $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$. Then, there exists an $L \in \mathbb{R}$ such that

$$\begin{aligned} (x,b) \in \lambda_f(a,b) &\Rightarrow (x,b) \in \tau_f(a,b) \\ &\Rightarrow |f(x,b) - f(a,b) - L(x-a)| \leq \varepsilon(a,b) \end{aligned}$$

Therefore, $D_x(f,(a,b),\lambda,\varepsilon) \neq \emptyset$. Moreover, since $\emptyset \neq \lambda_f(a,b) \subseteq \tau_f(a,b)$, then

$$\sup_{(x,b)\in\lambda_f(a,b)} \left(\frac{f(x,b)-f(a,b)}{x-a} - \frac{\varepsilon(a,b)}{|x-a|} \right) \leq \sup_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b)-f(a,b)}{x-a} - \frac{\varepsilon(a,b)}{|x-a|} \right)$$

and

$$\inf_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b)-f(a,b)}{x-a} + \frac{\varepsilon(a,b)}{|x-a|} \right) \leq \inf_{(x,b)\in\lambda_f(a,b)} \left(\frac{f(x,b)-f(a,b)}{x-a} + \frac{\varepsilon(a,b)}{|x-a|} \right)$$

Thus,

$$\begin{aligned} D_x(f,(a,b),\tau,\varepsilon) &= \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|} \right), \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|} \right) \right] \\ &\subseteq \left[\sup_{(x,b)\in\lambda_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|} \right), \inf_{(x,b)\in\lambda_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|} \right) \right] \\ &= D_x(f,(a,b),\lambda,\varepsilon) \end{aligned}$$

□

Theorem 3.16. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\emptyset \neq \lambda_f(a,b) \subseteq \tau_f(a,b)$. If $D_y(f,(a,b),\tau,\varepsilon) \neq \emptyset$, then $D_y(f,(a,b),\lambda,\varepsilon) \neq \emptyset$. Moreover, $D_y(f,(a,b),\tau,\varepsilon) \subseteq D_y(f,(a,b),\lambda,\varepsilon)$.*

The proof is as in Theorem 3.15.

Theorem 3.17. *Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, and $(a,b)\in A\times B$. If $D_x(f,(a,b),\tau,\alpha)\neq\emptyset$ and $D_x(g,(a,b),\lambda,\beta)\neq\emptyset$, then $D_x(f+g,(a,b),\kappa,\varepsilon)\neq\emptyset$ such that $\emptyset\neq\kappa_{f+g}(a,b)\subseteq\tau_f(a,b)\cap\lambda_g(a,b)$ and $\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)$. Moreover,*

$$D_x(f,(a,b),\tau,\alpha)+D_x(g,(a,b),\lambda,\beta)\subseteq D_x(f+g,(a,b),\kappa,\varepsilon)$$

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, $(a,b)\in A\times B$, $D_x(f,(a,b),\tau,\alpha)\neq\emptyset$, and $D_x(g,(a,b),\lambda,\beta)\neq\emptyset$. Then, there exist $L_1,L_2\in\mathbb{R}$ such that

$$(x,b)\in\tau_f(a,b)\Rightarrow|f(x,b)-f(a,b)-L_1(x-a)|\leq\alpha(a,b)$$

and

$$(x,b)\in\lambda_g(a,b)\Rightarrow|g(x,b)-g(a,b)-L_2(x-a)|\leq\beta(a,b)$$

Therefore,

$$\begin{aligned}(x,b)\in\kappa_{f+g}(a,b) &\Rightarrow (x,b)\in\tau_f(a,b)\wedge(x,b)\in\lambda_g(a,b)\\ &\Rightarrow |f(x,b)-f(a,b)-L_1(x-a)|\leq\alpha(a,b)\wedge|g(x,b)-g(a,b)-L_2(x-a)|\leq\beta(a,b)\\ &\Rightarrow -\alpha(a,b)-\beta(a,b)\leq f(x,b)-f(a,b)-L_1(x-a)+g(x,b)-g(a,b)-L_2(x-a)\leq\alpha(a,b)+\beta(a,b)\\ &\Rightarrow |(f+g)(x,b)-(f+g)(a,b)-(L_1+L_2)(x-a)|\leq\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)\\ &\Rightarrow L_1+L_2\in D_x(f+g,(a,b),\kappa,\varepsilon)\\ &\Rightarrow D_x(f+g,(a,b),\kappa,\varepsilon)\neq\emptyset\end{aligned}$$

Moreover, for all $L\in D_x(f,(a,b),\tau,\alpha)+D_x(g,(a,b),\lambda,\beta)$, there exist $L_1\in D_x(f,(a,b),\tau,\alpha)$ and $L_2\in D_x(g,(a,b),\lambda,\beta)$ such that $L=L_1+L_2$. Then,

$$(x,b)\in\tau_f(a,b)\Rightarrow|f(x,b)-f(a,b)-L_1(x-a)|\leq\alpha(a,b)$$

and

$$(x,b)\in\lambda_g(a,b)\Rightarrow|g(x,b)-g(a,b)-L_2(x-a)|\leq\beta(a,b)$$

Hence,

$$(x,b)\in\kappa_{f+g}(a,b)\Rightarrow|(f+g)(x,b)-(f+g)(a,b)-(L_1+L_2)(x-a)|\leq\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)$$

Therefore, $L=L_1+L_2\in D_x(f+g,(a,b),\kappa,\varepsilon)$. Thus, $D_x(f,(a,b),\tau,\alpha)+D_x(g,(a,b),\lambda,\beta)\subseteq D_x(f+g,(a,b),\kappa,\varepsilon)$. □

Theorem 3.18. *Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, and $(a,b)\in A\times B$. If $D_y(f,(a,b),\tau,\alpha)\neq\emptyset$ and $D_y(g,(a,b),\lambda,\beta)\neq\emptyset$, then $D_y(f+g,(a,b),\kappa,\varepsilon)\neq\emptyset$ such that $\emptyset\neq\kappa_{f+g}(a,b)\subseteq\tau_f(a,b)\cap\lambda_g(a,b)$ and $\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)$. Moreover,*

$$D_y(f,(a,b),\tau,\alpha)+D_y(g,(a,b),\lambda,\beta)\subseteq D_y(f+g,(a,b),\kappa,\varepsilon)$$

The proof is as in Theorem 3.17.

Theorem 3.19. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $t\neq0$. Then, $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$ if and only if $D_x(tf,(a,b),\tau,|t|\varepsilon)\neq\emptyset$. Moreover,*

$$tD_x(f,(a,b),\tau,\varepsilon)=D_x(tf,(a,b),\tau,|t|\varepsilon)$$

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $t\neq0$.
($\Rightarrow$): Let $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$. Then, $\tau_f(a,b)\neq\emptyset$ and there exists an $L\in\mathbb{R}$ such that

$$\begin{aligned}(x,b)\in\tau_{tf}(a,b) &\Rightarrow (x,b)\in\tau_f(a,b)\\ &\Rightarrow |f(x,b)-f(a,b)-L(x-a)|\leq\varepsilon(a,b)\\ &\Rightarrow |t||f(x,b)-f(a,b)-L(x-a)|\leq|t|\varepsilon(a,b)\\ &\Rightarrow |tf(x,b)-tf(a,b)-tL(x-a)|\leq|t|\varepsilon(a,b)\\ &\Rightarrow |(tf)(x,b)-(tf)(a,b)-tL(x-a)|\leq|t|\varepsilon(a,b)\end{aligned}$$

Thus, $tL\in D_x(tf,(a,b),\tau,|t|\varepsilon)$. That is, $D_x(tf,(a,b),\tau,|t|\varepsilon)\neq\emptyset$.

($\Leftarrow$): Let $D_x(tf,(a,b),\tau,|t|\varepsilon) \neq \emptyset$. Then, $\tau_{tf}(a,b) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$\begin{aligned}(x,b) \in \tau_f(a,b) &\Rightarrow (x,b) \in \tau_{tf}(a,b)\\ &\Rightarrow |(tf)(x,b)-(tf)(a,b)-L(x-a)| \leq |t|\varepsilon(a,b)\\ &\Rightarrow |tf(x,b)-tf(a,b)-L(x-a)| \leq |t|\varepsilon(a,b)\\ &\Rightarrow |t|\left|f(x,b)-f(a,b)-\tfrac{L}{t}(x-a)\right| \leq |t|\varepsilon(a,b)\\ &\Rightarrow \left|f(x,b)-f(a,b)-\tfrac{L}{t}(x-a)\right| \leq \varepsilon(a,b)\end{aligned}$$

Thus, $\frac{L}{t} \in D_x(f,(a,b),\tau,\varepsilon)$. That is, $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$. Moreover, for all $L \in tD_x(f,(a,b),\tau,\varepsilon)$, there exists an $L^* \in D_x(f,(a,b),\tau,\varepsilon)$ such that $L = tL^*$. Since $L^* \in D_x(f,(a,b),\tau,\varepsilon)$, then $tL^* \in D_x(tf,(a,b),\tau,|t|\varepsilon)$ from the proof of the existence. That is, $L \in D_x(tf,(a,b),\tau,|t|\varepsilon)$. Hence,

$$tD_x(f,(a,b),\tau,\varepsilon) \subseteq D_x(tf,(a,b),\tau,|t|\varepsilon)$$

In addition, for all $L \in D_x(tf,(a,b),\tau,|t|\varepsilon)$, $\frac{L}{t} \in D_x(f,(a,b),\tau,\varepsilon)$ from the proof of the existence. Hence, $L = t\frac{L}{t} \in tD_x(f,(a,b),\tau,\varepsilon)$. Thus,

$$D_x(tf,(a,b),\tau,|t|\varepsilon) \subseteq tD_x(f,(a,b),\tau,\varepsilon)$$

Consequently, $tD_x(f,(a,b),\tau,\varepsilon) = D_x(tf,(a,b),\tau,|t|\varepsilon)$. □

Theorem 3.20. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $t \neq 0$. Then, $D_y(f,(a,b),\tau,\varepsilon) \neq \emptyset$ if and only if $D_y(tf,(a,b),\tau,|t|\varepsilon) \neq \emptyset$. Moreover,*

$$tD_y(f,(a,b),\tau,\varepsilon) = D_y(tf,(a,b),\tau,|t|\varepsilon)$$

The proof is as in Theorem 3.19.

Corollary 3.21. *Let $A \times B \subseteq \mathbb{R}^2$, $f,g : A \times B \to \mathbb{R}$ be two functions, and $(a,b) \in A \times B$. If $D_x(f,(a,b),\tau,\alpha) \neq \emptyset$ and $D_x(g,(a,b),\lambda,\beta) \neq \emptyset$, then $D_x(f-g,(a,b),\kappa,\varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{f-g}(a,b) \subseteq \tau_f(a,b) \cap \lambda_g(a,b)$ and $\alpha(a,b)+\beta(a,b) \leq \varepsilon(a,b)$. Moreover,*

$$D_x(f,(a,b),\tau,\alpha) - D_x(g,(a,b),\lambda,\beta) \subseteq D_x(f-g,(a,b),\kappa,\varepsilon)$$

Proof. Let $A,B \subseteq \mathbb{R}$, $f,g : A \times B \to \mathbb{R}$ be two functions, $(a,b) \in A \times B$, $D_x(f,(a,b),\tau,\alpha) \neq \emptyset$, and $D_x(g,(a,b),\lambda,\beta) \neq \emptyset$. From Theorem 3.19, for $t = -1$, $-D_x(g,(a,b),\lambda,\beta) = D_x(-g,(a,b),\lambda,\beta)$. Therefore, from Theorem 3.17, $D_x(f-g,(a,b),\kappa,\varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{f-g}(a,b) \subseteq \tau_f(a,b) \cap \lambda_g(a,b)$ and $\alpha(a,b)+\beta(a,b) \leq \varepsilon(a,b)$. Moreover,

$$D_x(f,(a,b),\tau,\alpha) - D_x(g,(a,b),\lambda,\beta) = D_x(f,(a,b),\tau,\alpha) + D_x(-g,(a,b),\lambda,\beta) \subseteq D_x(f+(-g),(a,b),\kappa,\varepsilon) = D_x(f-g,(a,b),\kappa,\varepsilon)$$

□

Corollary 3.22. *Let $A \times B \subseteq \mathbb{R}^2$, $f,g : A \times B \to \mathbb{R}$ be two functions, and $(a,b) \in A \times B$. If $D_y(f,(a,b),\tau,\alpha) \neq \emptyset$ and $D_y(g,(a,b),\lambda,\beta) \neq \emptyset$, then $D_y(f-g,(a,b),\kappa,\varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{f-g}(a,b) \subseteq \tau_f(a,b) \cap \lambda_g(a,b)$, and $\alpha(a,b)+\beta(a,b) \leq \varepsilon(a,b)$. Moreover,*

$$D_y(f,(a,b),\tau,\alpha) - D_y(g,(a,b),\lambda,\beta) \subseteq D_y(f-g,(a,b),\kappa,\varepsilon)$$

The proof is as in Corollary 3.21.

Theorem 3.23. *Let $A \times B \subseteq \mathbb{R}^2$, $f,g : A \times B \to \mathbb{R}$ be two functions, $(a,b) \in A \times B$, and $k,l \in \mathbb{R}$. If $g(x,y) = f(x,y)+kx+ly$, for all $(x,y) \in \tau_f(a,b) = \tau_g(a,b)$, and $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$, then $D_x(g,(a,b),\tau,\varepsilon) \neq \emptyset$. Moreover,*

$$D_x(g,(a,b),\tau,\varepsilon) = D_x(f,(a,b),\tau,\varepsilon)+k$$

Proof. Let $A \times B \subseteq \mathbb{R}^2$, $f,g : A \times B \to \mathbb{R}$ be two functions, $(a,b) \in A \times B$, $k,l \in \mathbb{R}$, $g(x,y) = f(x,y)+kx+ly$, for all $(x,y) \in \tau_f(a,b) = \tau_g(a,b)$, and $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$. Then, $\tau_f(a,b) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$(x,b) \in \tau_f(a,b) \Rightarrow |f(x,b)-f(a,b)-L(x-a)| \leq \varepsilon(a,b)$$

Therefore, for $L^* = k + L$ and for all $(x,b) \in \tau_g(a,b) = \tau_f(a,b)$,

$$\begin{aligned} |g(x,b) - g(a,b) - L^*(x-a)| &= |f(x,b) + kx + lb - f(a,b) - ka - lb - (k+L)(x-a)| \\ &= |f(x,y) - f(a,b) - L(x-a)| \\ &\leq \varepsilon(a) \end{aligned}$$

Thus, $L^* \in D_x(g,(a,b),\tau,\varepsilon)$. That is, $D_x(g,(a,b),\tau,\varepsilon) \neq \emptyset$. Moreover,

$$\begin{aligned} D_x(g,(a,b),\tau,\varepsilon) &= \left[\sup_{(x,b)\in\tau_g(a,b)}\left(\tfrac{g(x,b)-g(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|}\right), \inf_{(x,b)\in\tau_g(a,b)}\left(\tfrac{g(x,b)-g(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|}\right)\right] \\ &= \left[\sup_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)+kx+lb-f(a,b)-ka-lb}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|}\right), \inf_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)+kx+lb-f(a,b)-ka-lb}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|}\right)\right] \\ &= \left[\sup_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|} + k\right), \inf_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|} + k\right)\right] \\ &= \left[\sup_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|}\right) + k, \inf_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|}\right) + k\right] \\ &= \left[\sup_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|}\right), \inf_{(x,b)\in\tau_f(a,b)}\left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|}\right)\right] + k \\ &= D_x(f,(a,b),\tau,\varepsilon) + k \end{aligned}$$

□

Theorem 3.24. *Let $A \times B \subseteq \mathbb{R}^2$, $f,g : A \times B \to \mathbb{R}$ be two functions, $(a,b) \in A \times B$, and $k,l \in \mathbb{R}$. If $g(x,y) = f(x,y) + kx + ly$, for all $(x,y) \in \tau_f(a,b) = \tau_g(a,b)$, and $D_y(f,(a,b),\tau,\varepsilon) \neq \emptyset$, then $D_y(g,(a,b),\tau,\varepsilon) \neq \emptyset$. Moreover,*

$$D_y(g,(a,b),\tau,\varepsilon) = D_y(f,(a,b),\tau,\varepsilon) + l$$

The proof is as in Theorem 3.23

Example 3.25. *Let $f,g : \mathbb{R}^2 \to \mathbb{R}$, $\tau,\lambda,\kappa : \mathbb{R}^2 \to P\left(\mathbb{R}^2\right)$, and $\alpha,\beta,\varepsilon : \mathbb{R}^2 \to \mathbb{R}^{\geq 0}$ be seven functions defined by $f(x,y) = x^2 + 2y^2$, $g(x,y) = 2x + y$, $\tau(x,y) = \tau_1(x,y)$,*

$$\lambda(x,y) = \left\{(x_0,y_0) \in \mathbb{R}^2 \ : 0 < \sqrt{\frac{(x-x_0)^2}{9} + \frac{(y-y_0)^2}{16}} \leq 1\right\}$$

$\kappa(x,y) = \kappa_{\frac{1}{4}}(x,y)$, $\alpha(x,y) = |x| + |y|$, $\beta(x,y) = \max\{|x|,|y|\}$, and $\varepsilon(x,y) = 2(|x|+|y|)$, respectively. Here, for all $(x,y) \in \mathbb{R}^2$, $\tau(x,y) \subseteq \lambda(x,y)$, $\kappa(x,y) \subseteq \tau(x,y) \cap \lambda(x,y)$, $\beta(x,y) \leq \alpha(x,y)$, and $\alpha(x,y) + \beta(x,y) \leq \varepsilon(x,y)$. From Theorem 3.11, for $(2,-1) \in \mathbb{R}^2$,

$$D_x(f,(2,-1),\tau,\beta) = [3,5]$$

and

$$D_x(g,(2,-1),\lambda,\beta) = \left[\frac{4}{3},\frac{8}{3}\right]$$

From Theorem 3.13, since $\beta(2,-1) \leq \alpha(2,-1)$, $D_x(f,(2,-1),\tau,\alpha) \neq \emptyset$. Therefore, from Theorem 3.11,

$$D_x(f,(2,-1),\tau,\alpha) = [2,6]$$

and thus,

$$D_x(f,(2,-1),\tau,\beta) = [3,5] \subseteq [2,6] = D_x(f,(2,-1),\tau,\alpha)$$

From Theorem 3.15, as $\tau_f(2,-1) \subseteq \lambda_f(2,-1)$, $D_x(g,(2,-1),\tau,\beta) \neq \emptyset$. Hence, from Theorem 3.11,

$$D_x(g,(2,-1),\tau,\beta) = [0,4]$$

and thus,

$$D_x(g,(2,-1),\lambda,\beta)=\left[\frac{4}{3},\frac{8}{3}\right]\subseteq[0,4]=D_x(g,(2,-1),\tau,\beta)$$

Moreover, from Theorem 3.19, $D_x(2f,(2,-1),\tau,|2|\alpha)\neq\emptyset$ and $D_x(-g,(2,-1),\lambda,|-1|\beta)\neq\emptyset$. Thereby, from Theorem 3.11,

$$D_x(2f,(2,-1),\tau,|2|\alpha)=[4,12]$$

and

$$D_x(-g,(2,-1),\lambda,|-1|\beta)=\left[-\frac{8}{3},-\frac{4}{3}\right]$$

Therefore,

$$2D_x(f,(2,-1),\tau,\alpha)=2\,[2,6]=[4,12]=D_x(2f,(2,-1),\tau,|2|\alpha)$$

and

$$-D_x(g,(2,-1),\lambda,\beta)=-\left[\frac{4}{3},\frac{8}{3}\right]=\left[-\frac{8}{3},-\frac{4}{3}\right]=D_x(-g,(2,-1),\lambda,|-1|\beta)$$

From Theorem 3.17, because $\emptyset\neq\kappa_{f+g}(2,-1)\subseteq\tau_f(2,-1)\cap\lambda_g(2,-1)$ and $\alpha(2,-1)+\beta(2,-1)\leq\varepsilon(2,-1)$, then $D_x(f+g,(2,-1),\kappa,\varepsilon)\neq\emptyset$. Hereby, from Theorem 3.11,

$$D_x(f+g,(2,-1),\kappa,\varepsilon)=\left[-\frac{71}{4},\frac{119}{4}\right]$$

and thus,

$$\begin{aligned}D_x(f,(2,-1),\tau,\alpha)+D_x(g,(2,-1),\lambda,\beta) &= [2,6]+\left[\tfrac{4}{3},\tfrac{8}{3}\right]\\ &= \left[\tfrac{10}{3},\tfrac{26}{3}\right]\\ &\subseteq \left[-\tfrac{71}{4},\tfrac{119}{4}\right]\\ &= D_x(f+g,(2,-1),\kappa,\varepsilon)\end{aligned}$$

From Corollary 3.21, since $\emptyset\neq\kappa_{f-g}(2,-1)\subseteq\tau_f(2,-1)\cap\lambda_g(2,-1)$ and $\alpha(2,-1)+\beta(2,-1)\leq\varepsilon(2,-1)$, then $D_x(f-g,(2,-1),\kappa,\varepsilon)\neq\emptyset$. Herewith, from Theorem 3.11,

$$D_x(f-g,(2,-1),\kappa,\varepsilon)=\left[-\frac{87}{4},\frac{103}{4}\right]$$

and thus,

$$\begin{aligned}D_x(f,(2,-1),\tau,\alpha)-D_x(g,(2,-1),\lambda,\beta) &= [2,6]-\left[\tfrac{4}{3},\tfrac{8}{3}\right]\\ &= \left[-\tfrac{2}{3},\tfrac{14}{3}\right]\\ &\subseteq \left[-\tfrac{87}{4},\tfrac{103}{4}\right]\\ &= D_x(f-g,(2,-1),\kappa,\varepsilon)\end{aligned}$$

Besides, for the function $h:\mathbb{R}^2\to\mathbb{R}$ defined by $h(x,y)=f(x,y)+3x-5y$, from Theorem 3.23, $D_x(h,(2,-1),\tau,\alpha)\neq\emptyset$. Hence, from Theorem 3.11,

$$D_x(h,(2,-1),\tau,\alpha)=[5,9]$$

and thus,

$$D_x(f,(2,-1),\tau,\alpha)+3=[2,6]+3=[5,9]=D_x(h,(2,-1),\tau,\alpha)$$

Note 3.26. *For the functions f and τ in Example 3.25, $(a,b)=(2,-1)$, and $\varepsilon^*(2,-1)=\frac{3}{2}$, $D_x(f,(2,-1),\tau,\varepsilon^*)=\left[\frac{7}{2},\frac{9}{2}\right]$ and $D_y(f,(2,-1),\tau,\varepsilon^*)=\emptyset$. Similarly, for the function $h:\mathbb{R}^2\to\mathbb{R}$ defined by $h(x,y)=2x^2+y^2$, $D_x(h,(2,-1),\tau,\varepsilon^*)=\emptyset$ and $D_y(h,(2,-1),\tau,\varepsilon^*)=\left[-\frac{5}{2},-\frac{3}{2}\right]$. Hence, it is clear that the existence of partial soft derivative with respect to x does not require the existence of partial soft derivative with respect to y and vice versa.*

Note 3.27. *As in classical analysis, for a function with the variables x and y, if taking the partial soft derivative with respect to x, then y is fixed and vice versa. Thus, partial soft derivative turns into soft derivative. In other words, for a function $f:A\times B\to\mathbb{R}$ and $(a,b)\in A\times B$, if $L\in D_x(f,(a,b),\tau,\varepsilon)$, then $L\in D(g,a,\tau^*,\varepsilon^*)$ such that $g:A\to\mathbb{R}$, $\tau^*:\mathbb{R}\to P(\mathbb{R})$, and $\varepsilon^*:\mathbb{R}\to\mathbb{R}^{\geq 0}$ are three functions defined by $g(x)=f(x,b)$, $\tau^*(x)=\{x\in\mathbb{R}:(x,b)\in\tau(a,b)\}$, and $\varepsilon^*(x)=\varepsilon(x,b)$, for all $x\in A$, respectively. Similarly, for a function $f:A\times B\to\mathbb{R}$ and $(a,b)\in A\times B$, if $L\in D_y(f,(a,b),\tau,\varepsilon)$, then $L\in D(h,b,\tau^{**},\varepsilon^{**})$ such that $h:B\to\mathbb{R}$, $\tau^{**}:\mathbb{R}\to P(\mathbb{R})$, and $\varepsilon^{**}:\mathbb{R}\to\mathbb{R}^{\geq 0}$ are three functions defined by $h(y)=f(a,y)$, $\tau^{**}(y)=\{y\in\mathbb{R}:(a,y)\in\tau(a,b)\}$, and $\varepsilon^{**}(y)=\varepsilon(a,y)$, for all $y\in B$, respectively.*

In Theorems 3.28 and 3.29, the notations $(A\times B)^\circ$ and $(A\times B)'$ denote the set of all the interior and accumulation points of $A\times B$ according to the usual topology in $\mathbb{R}^2$, respectively.

Theorem 3.28. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, and $(a,b)\in(A\times B)^\circ\cap(A\times B)'$. If $f_x(a,b)\in\mathbb{R}$, then there exist τ and ε^* such that $D_x(f,(a,b),\tau,\varepsilon^*)\neq\emptyset$.*

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in(A\times B)^\circ\cap(A\times B)'$, and $f_x(a,b)\in\mathbb{R}$. Then, there exists an $L\in\mathbb{R}$ such that

$$f_x(a,b)=\lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a}=L$$

Therefore,

$$\forall\varepsilon>0,\exists\delta_\varepsilon>0\ni\left((x,b)\in B_0((a,b),\delta_\varepsilon)\cap A\times B\Rightarrow\left|\frac{f(x,b)-f(a,b)}{x-a}-L\right|\leq\varepsilon\right)$$

Thus,

$$\forall\varepsilon>0,\exists\delta_\varepsilon>0\ni\Big((x,b)\in B_0((a,b),\delta_\varepsilon)\cap A\times B\Rightarrow\big|f(x,b)-f(a,b)-L(x-a)\big|\leq\varepsilon|x-a|\leq\varepsilon\delta_\varepsilon\Big)$$

Hence, for an $\varepsilon>0$,

$$(x,b)\in\tau_f(a,b)\Rightarrow\big|f(x,b)-f(a,b)-L(x-a)\big|\leq\varepsilon^*(a,b)$$

such that $\tau(a,b)=B_0((a,b),\delta_\varepsilon)$ and $\varepsilon^*(a,b)=\varepsilon\delta_\varepsilon$. Thereby, $L\in D_x(f,(a,b),\tau,\varepsilon^*)$. Consequently, $D_x(f,(a,b),\tau,\varepsilon^*)\neq\emptyset$. □

Theorem 3.29. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, and $(a,b)\in(A\times B)^\circ\cap(A\times B)'$. If $f_y(a,b)\in\mathbb{R}$, then there exist τ and ε^* such that $D_y(f,(a,b),\tau,\varepsilon^*)\neq\emptyset$.*

The proof is as in Theorem 3.28.

Remark 3.30. *The geometric interpretation of the partial soft derivative of a function f with respect to x at a point (a,b) is the tangent of the slope angle of the bandwidth 2ε bounded by two linear functions $f(a,b)+L(x-a)+\varepsilon(a,b)$ and $f(a,b)+L(x-a)-\varepsilon(a,b)$ which contain the entire graph of $z=f(x,b)$ on the set $\tau_x(a,b)\cap\mathrm{Dom}(f)$. Similarly, the geometric interpretation of the partial soft derivative of a function f with respect to y at a point (a,b) is the tangent of the slope angle of the bandwidth 2ε bounded by two linear functions $f(a,b)+L(y-b)+\varepsilon(a,b)$ and $f(a,b)+L(y-b)-\varepsilon(a,b)$ which contain the entire graph of $z=f(a,y)$ on the set $\tau_y(a,b)\cap\mathrm{Dom}(f)$. For example, for the functions f, τ, and α and the point $(2,-1)\in\mathbb{R}^2$ in Example 3.25, $D_x(f,(2,-1),\tau,\alpha)=[2,6]$. Moreover, consider the following linear functions and ordered pairs:*

for $L=2\in[2,6]$,	$g_1(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=2x+5$	$A_1=(x,g_1(x))$
	$h_1(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=2x-1$	$B_1=(x,h_1(x))$
for $L=3\in[2,6]$,	$g_2(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=3x+3$	$A_2=(x,g_2(x))$
	$h_2(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=3x-3$	$B_2=(x,h_2(x))$
for $L=4\in[2,6]$,	$g_3(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=4x+1$	$A_3=(x,g_3(x))$
	$h_3(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=4x-5$	$B_3=(x,h_3(x))$
for $L=5\in[2,6]$,	$g_4(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=5x-1$	$A_4=(x,g_4(x))$
	$h_4(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=5x-7$	$B_4=(x,h_4(x))$
for $L=6\in[2,6]$,	$g_5(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=6x-3$	$A_5=(x,g_5(x))$
	$h_5(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=6x-9$	$B_5=(x,h_5(x))$

Then, it is clear that for all $i\in I_5=\{1,2,3,4,5\}$ and for all $(x,-1)\in\tau_x(2,-1)\cap\mathbb{R}^2$, $h_i(x)\leq f(x)\leq g_i(x)$ and the Euclidean distance of the ordered pairs $A_i=(x,g_i(x))$ and $B_i=(x,h_i(x))$ is 2α such that $|A_iB_i|=\sqrt{(x-x)^2+(g_i(x)-h_i(x))^2}=6=2\alpha$. Figures 3.1 and 3.2 show the graphs of the functions h_i, f, and g_i, for all $i\in I_5$, on the set $\tau_x(2,-1)\cap\mathbb{R}^2$ from different perspectives.

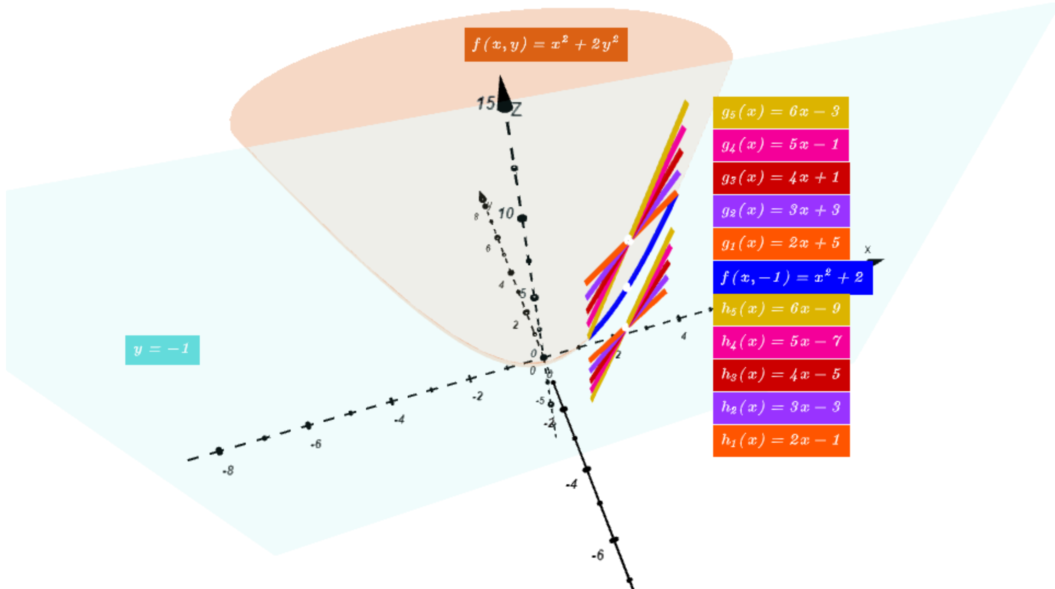

Figure 3.1. Graphs of the functions h_i, f, and g_i, for all $i \in I_5$, on the set $\tau_x(2,-1) \cap \mathbb{R}^2$

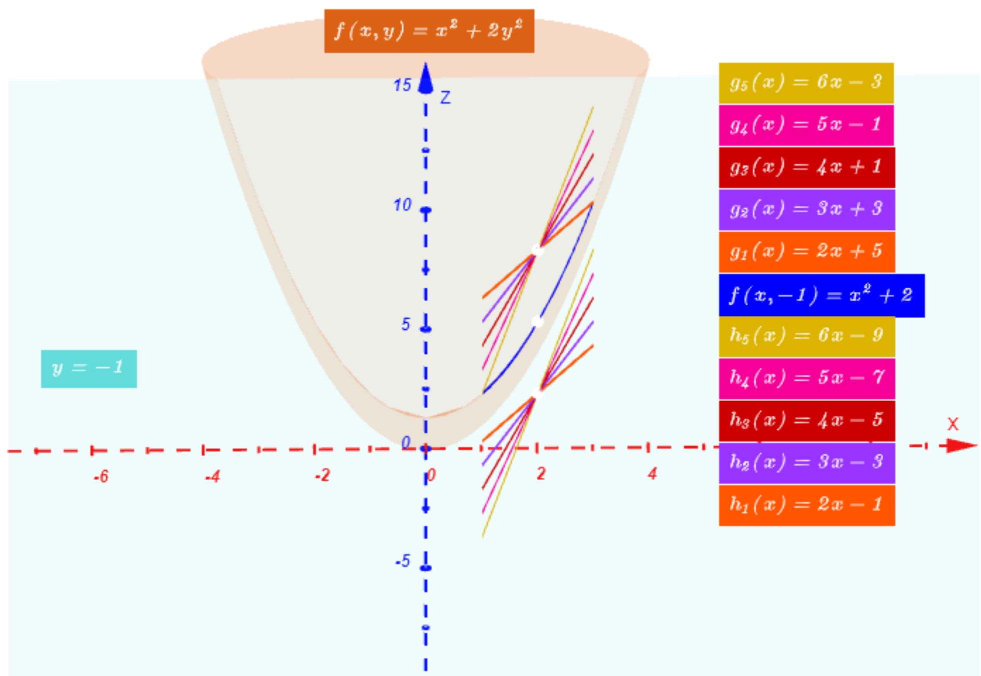

Figure 3.2. Graphs of the functions h_i, f, and g_i, for all $i \in I_5$, on the set $\tau_x(2,-1) \cap \mathbb{R}^2$ (another perspective)

Besides, for all $L \in D_x(f,(2,-1),\tau,\alpha) = [2,6]$, the pairs of all the linear functions h and g form two bundles of lines (see Figure 3.3).

4. Conclusion

This study defined partial soft derivative and investigated some of its basic properties. This paper demonstrated that

- Every function with a partial soft derivative is bounded,
- Every bounded function has a partial soft derivative under certain conditions,
- A partial soft derivative of a function can be considered a soft derivative of the function (see Note 3.27), and

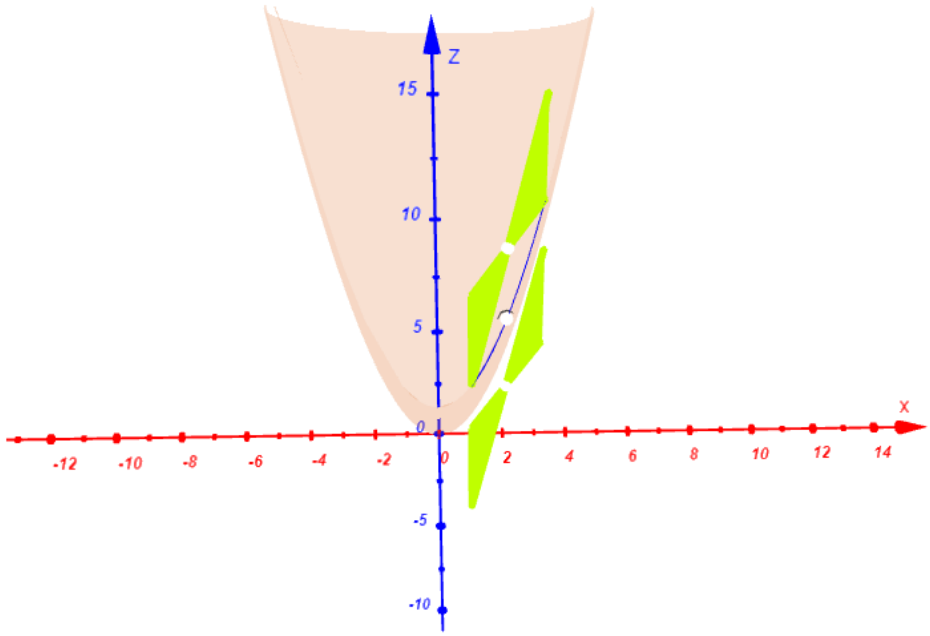

Figure 3.3. Bundles of lines formed by the pairs of all the linear functions h and g, for all $L \in [2,6]$

- Every function with a classical partial derivative has a partial soft derivative under certain conditions

and investigated algebraic properties and the geometric interpretation of partial soft derivative. Moreover, it clarified the theoretical section by examples and provided figures for the geometric interpretation. When the results herein are compared with those of in the classical analysis, the following comments can be briefly made:

- While the classical partial derivative of a function (if any) is equal to a real number, the partial soft derivative of a function (if any) is equal to a closed interval.
- While a bounded function does not always have a classical partial derivative, it has a partial soft derivative (see Theorems 3.8 and 3.10).
- While equality is valid for the sum rule in the partial derivative, inclusion is valid for the partial soft derivative. Similarly, while equality holds for the difference rule in the partial derivative, inclusion holds for the partial soft derivative.
- Geometrically, while a tangent line is obtained in the partial derivative, two bundles of lines are obtained in the partial soft derivative.

Partial soft derivative is a fundamental concept of soft analysis. Therefore, researchers can study this concept and its applications. Moreover, the concepts of higher-order partial soft derivative and soft gradient, associated with partial soft derivative, and the concept of directional soft derivative are also worth studying.

Article Information

Author Contributions: All the authors equally contributed to this work. This paper is a part of the first author's doctoral dissertation supervised by the second author. They all read and approved the final version of the paper.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: This study was supported by the Office of Scientific Research Projects Coordination at Çanakkale Onsekiz Mart University, Grant number: FDR-2022-4206.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

References

[1] D. A. Molodtsov, *Soft set theory – First results*, Comput. Math. Appl., **37**(4-5) (1999), 19–31.

[2] D. A. Molodtsov, *Soft Set Theory*, URSS, 2004. (In Russian)

[3] D. A. Molodtsov, A. A. Sokolov, D. V. Kovkov, *Basic foundations of soft analysis*, Nechetkie Sistemy i Myagkie Vychisleniya, **2**(1) (2007), 5–28. (In Russian)

[4] D. A. Molodtsov, *Higher order derivatives in soft analysis*, Nechetkie Sistemy i Myagkie Vychisleniya, **14**(1) (2019), 34–55. (In Russian)

[5] D. A. Molodtsov, *Principles of rational analysis – Continuity of functions*, Nechetkie Sistemy i Myagkie Vychisleniya, **14**(2) (2019), 126–141. (In Russian)

[6] D. A. Molodtsov, *Principles of rational analysis – Derivatives and integrals*, Nechetkie Sistemy i Myagkie Vychisleniya, **15**(1) (2020), 5–25. (In Russian)

[7] S. Acharjee, D. A. Molodtsov, *Soft rational line integral*, Vestnik Udmurtskogo Universiteta, Matematika, Mekhanika, Komp'yuternye Nauki, **31**(4) (2021), 578–596.

Communications in Advanced Mathematical Sciences
Vol. 7, No. 1, 27-41, 2024
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1414411

On Suzuki$-$Proinov Type Contractions in Modular $b-$Metric Spaces with an Application

Abdurrahman Büyükkaya[1], Mahpeyker Öztürk[2]*

Abstract
In this paper, by taking $\mathscr{C}_{\mathscr{A}}-$simulation function and Proinov type function into account, we set up a new contraction mapping called Suzuki$-$Proinov $\mathcal{Z}^{*}{}^{\mathcal{R}}_{\mathcal{E}^{*}}(\alpha)-$contraction, including both rational expressions that possess quadratic terms and $\mathcal{E}-$type contractions. Furthermore, we demonstrate a common fixed point theorem through the mappings endowed with triangular $\alpha-$admissibility in the setting of modular $b-$metric spaces. Besides that, we achieve some new outcomes that contribute to the current ones in the literature through the main theorem, and, as an application, we examine the existence of solutions to a class of functional equations emerging in dynamic programming.

Keywords: Common fixed point, Dynamic programming, Modular $b-$metric space, Proinov type mappings, Simulation functions
2010 AMS: 47H10,54H24

[1] *Department of Mathematics, Karadeniz Technical University, Trabzon, Türkiye, abdurrahman.giresun@hotmail.com, ORCID: 0000-0001-6197-8975*
[2] *Department of Mathematics, Sakarya University, Sakarya, Türkiye, mahpeykero@sakarya.edu.tr, ORCID: 0000-0003-2946-6114*
***Corresponding author**
Received: 3 January 2024, **Accepted:** 13 February 2024, **Available online:** 20 February 2024
How to cite this article: A. Büyükkaya, M. Öztürk, *On Suzuki$-$Proinov Type Contractions in Modular $b-$Metric Spaces with an Application*, Commun. Adv. Math. Sci., 7(1) (2024) 27-41.

1. Introduction and Preliminaries

The symbol $\mathbb{N}$ is used throughout the research to represent all positive natural numbers, whereas $\mathbb{R}^+$ represents the set of all non-negative real numbers.

Fixed point theory is a significant mathematical technique that finds applications in various scientific research areas. This theory has played a crucial role in creating several significant concepts and approaches and is an exciting area of ongoing study and advancement, which acts as an intermediary connecting topology and analysis and is commonly used in pure and applied mathematics. For the past several years, researchers in this field have been exploring potential applications of this field to a wide range of physically relevant engineering challenges. On the other hand, the metric fixed point theory is very attractive on account of the Banach Fixed Point Theorem or Banach Contraction Principle, which was conferred by S. Banach [1] in 1922. In this theorem, there is an answer about the existence and uniqueness of fixed point of contraction mappings in the setting of complete metric space. Further, many studies have been done to enhance this theorem's impressiveness, and it underwent several changes and generalizations as time progressed, see [2]-[5]. Simultaneously, in this direction, many authors try to obtain a more general metric space structure and diverse contractive conditions or both of them. Herewith, many new topological structures and contraction mappings have emerged. The notation of b-metric is one of the popular generalizations of the metric function, which was depicted by Bakhtin [6] and mainly, Czerwik [7, 8] in 1993 and 1998, as noted below.

Definition 1.1. [7] *A function $\rho : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ is a $b-$metric with $\tau \geq 1$ on a non−empty set $\mathcal{U}$ provided that the following axioms hold, for all $\lambda, \zeta, z \in \mathcal{U}$:*

(ρ_1) $\rho(\lambda,\zeta) = 0 \Leftrightarrow \lambda = \zeta$,

(ρ_2) $\rho(\lambda,\zeta) = \rho(\zeta,\lambda)$,

(ρ_3) $\rho(\lambda,\zeta) \leq \tau[\rho(\lambda,z)+\rho(z,\zeta)]$.

Thereupon, we say that the pair $(\mathcal{U},\rho)$ is a $b-$metric space, and, by choosing $\tau = 1$, $b-$metric is reduced to ordinary metric.

Also, except for the continuity, other topological features of $b-$metric can be defined as in metric ones. For continuity, the subsequent lemma can be a guide in $b-$metric.

Lemma 1.2. [9] *Let $(\mathcal{U},\rho)$ be a $b-$metric space with $\tau \geq 1$ and $\{\lambda_\mathfrak{y}\}$ and $\{\zeta_\mathfrak{y}\}$ be convergent to λ and ζ, respectively. Then*

$$\frac{1}{\tau^2}\rho(\lambda,\zeta) \leq \liminf_{\mathfrak{y}\to\infty}\rho(\lambda_\mathfrak{y},\zeta_\mathfrak{y}) \leq \limsup_{\mathfrak{y}\to\infty}\rho(\lambda_\mathfrak{y},\zeta_\mathfrak{y}) \leq \tau^2\rho(\lambda,\zeta).$$

Especially, if $\lambda = \zeta$, then $\lim_{\mathfrak{y}\to\infty}\rho(\lambda_\mathfrak{y},\zeta_\mathfrak{y}) = 0$. Also, for $z \in \mathcal{U}$, we have

$$\frac{1}{\tau}\rho(\lambda,z) \leq \liminf_{\mathfrak{y}\to\infty}\rho(\lambda_\mathfrak{y},z) \leq \limsup_{\mathfrak{y}\to\infty}\rho(\lambda_\mathfrak{y},z) \leq \tau\rho(\lambda,z).$$

On the other hand, in 2010, Chistyakov [10, 11] put forth a novel concept which is known as modular metric space.

Definition 1.3. [10, 11] *A function $\mu : (0,\infty)\times\mathcal{U}\times\mathcal{U} \to [0,\infty]$, defined by $\mu(\sigma,\lambda,\zeta) = \mu_\sigma(\lambda,\zeta)$, is called a modular metric on a non-void set $\mathcal{U}$ if it satisfies the below statements for all $\lambda,\zeta,z \in \mathcal{U}$:*

(μ_1) $\mu_\sigma(\lambda,\zeta) = 0$ *for all* $\sigma > 0 \Leftrightarrow \lambda = \zeta$,

(μ_2) $\mu_\sigma(\lambda,\zeta) = \mu_\sigma(\zeta,\lambda)$ *for all* $\sigma > 0$,

(μ_3) $\mu_{\sigma+\chi}(\lambda,\zeta) \leq \mu_\sigma(\lambda,z)+\mu_\chi(z,\zeta)$ *for all* $\sigma,\chi > 0$.

If instead of (μ_1), the condition

(μ_1') $\mu_\sigma(\lambda,\lambda) = 0$ for all $\sigma > 0$

is fulfilled, then μ is said to be a (metric) pseudomodular on $\mathcal{U}$.

By using the constant $\tau \geq 1$, the axiom (μ_3) is revised with the following one by M. E. Ege and C. Alaca [12], and in this case, the function μ is entitled as modular $b-$metric:

(μ_3') $\mu_{\sigma+\chi}(\lambda,\zeta) \leq \tau\left[\mu_\sigma(\lambda,z)+\mu_\chi(z,\zeta)\right]$ for all $\sigma,\chi > 0$.

Consequently, the pair $(\mathcal{U},\mu)$ is a modular $b-$metric space, which denotes $\mathcal{M}_b\mathcal{MS}$.

Note that the notation of modular $b-$metric and modular metric coincide when $\tau = 1$. Also, considering modular $b-$metric μ on $\mathcal{U}$, a modular set is specified by

$$\mathcal{U}_\mu = \left\{\zeta \in \mathcal{U} : \zeta \overset{\mu}{\sim} \lambda\right\},$$

where $\overset{\mu}{\sim}$ is a binary relation on $\mathcal{U}$ identified by $\lambda \sim \zeta \Leftrightarrow \lim_{\sigma\to\infty}\mu_\sigma(\lambda,\zeta) = 0$ for $\lambda,\zeta \in \mathcal{U}$. Moreover, the set

$$\mathcal{U}_\mu^* = \{\lambda \in \mathcal{U} : \exists\sigma = \sigma(\lambda) > 0 \text{ such that } \mu_\sigma(\lambda,\lambda_0) < \infty\}\ (\lambda_0 \in \mathcal{U})$$

is mentioned as $\mathcal{M}_b\mathcal{MS}$ (around λ_0).

Example 1.4. [12] *Consider the space*

$$\ell_p = \left\{ (\lambda_\mathfrak{y}) \subset \mathbb{R} : \sum_{j=1}^{\infty} |\lambda_\mathfrak{y}|^p < \infty \right\} \quad 0 < p < 1,$$

$\sigma \in (0,\infty)$ *and* $\mu_\sigma(\lambda,\zeta) = \frac{d(\lambda,\zeta)}{\sigma}$ *such that*

$$d(\lambda,\zeta) = \left(\sum_{j=1}^{\infty} |\lambda_\mathfrak{y} - \zeta_\mathfrak{y}|^p \right)^{\frac{1}{p}}, \quad \lambda = \lambda_\mathfrak{y},\, \zeta = \zeta_\mathfrak{y} \in \ell_p.$$

Eventually, one can conclude that $(\mathcal{U},\mu)$ *is an* $\mathcal{M}_\flat\mathcal{MS}$*.*

Example 1.5. [13] *Consider the equality* $\mu_\sigma(\lambda,\zeta) = (\omega_\sigma(\lambda,\zeta))^s$*, where* $(\mathcal{U},\omega)$ *is a modular metric space and* $s \geq 1$*. Thereupon, take into Jensen inequality account, together with the convexity of the function* $\mathcal{P}(\lambda) = \lambda^s$ *for* $\lambda \geq 0$*, we get*

$$(a+b)^s \leq 2^{s-1}(a^s + b^s)$$

for $a, b \in \mathbb{R}^+$*. Hence,* $(\mathcal{U},\mu)$ *is an* $\mathcal{M}_\flat\mathcal{MS}$ *with* $\tau = 2^{s-1}$*.*

Definition 1.6. [12] *Let* $\mathcal{U}_\mu^*$ *be an* $\mathcal{M}_\flat\mathcal{MS}$ *and* $\{\lambda_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}} \in \mathcal{U}_\mu^*$ *be a sequence.*

(c_1) *The sequence* $\{\lambda_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}}$ *is* $\mu-$*convergent to* $\lambda \in \mathcal{U}_\mu^* \Leftrightarrow \mu_\sigma(\lambda_\mathfrak{y},\lambda) \to 0$*, as* $\mathfrak{y} \to \infty$ *for all* $\sigma > 0$*.*

(c_2) *The* $\{\lambda_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}}$ *in* $\mathcal{U}_\mu^*$ *is a* $\mu-$*Cauchy sequence if* $\lim_{\mathfrak{y},m\to\infty} \mu_\sigma(\lambda_\mathfrak{y},\lambda_m) = 0$ *for all* $\sigma > 0$*.*

(c_3) *The space* $\mathcal{U}_\mu^*$ *is called* $\mu-$*complete provided that any* $\mu-$*Cauchy sequence in* $\mathcal{U}_\mu^*$ *is* $\mu-$*convergent to the point of* $\mathcal{U}_\mu^*$*.*

(c_4) $\mathcal{P} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ *is a* $\mu-$*continuous mapping if* $\mu_\sigma(\lambda_\mathfrak{y},\lambda) \to 0$*, provided to* $\mu_\sigma(\mathcal{P}\lambda_\mathfrak{y},\mathcal{P}\lambda) \to 0$ *as* $\mathfrak{y} \to \infty$*.*

Further, for more detail on modular $b-$metric, see [14]-[17].

As an auxiliary function, the class of simulation functions (briefly, $\mathcal{SF}$) was identified by Khojasteh et al. [18] in 2015, as noted below.

Definition 1.7. [18] *Let* $\Xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *be a mapping. If the axioms*

(Ξ_1) $\Xi(0,0) = 0$,

(Ξ_2) $\Xi(\ell,k) < k - \ell$ *for all* $\ell, k > 0$,

(Ξ_3) *if* $\{\ell_\mathfrak{y}\}$, $\{k_\mathfrak{y}\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{\mathfrak{y}\to\infty} \ell_\mathfrak{y} = \lim_{\mathfrak{y}\to\infty} k_\mathfrak{y} > 0$*, then* $\limsup_{\mathfrak{y}\to\infty} \Xi(\ell_\mathfrak{y},k_\mathfrak{y}) < 0$

are fulfilled, then, Ξ is an $\mathcal{SF}$, and $\mathcal{Z}$ represents the set of all $\mathcal{SF}$. Also, note that, from (Ξ_2), we have $\Xi(\ell,\ell) < 0$ for all $\ell > 0$.

Definition 1.8. [18] *A self-mapping* $\mathcal{P} : \mathcal{U} \to \mathcal{U}$ *on a metric space* $(\mathcal{U},d)$ *is called* $\mathcal{Z}$*-contraction with respect to* $\Xi \in \mathcal{Z}$ *provided that, for all* $\lambda,\zeta \in \mathcal{U}$*, the subsequent inequality hold:*

$$\Xi(d(\mathcal{P}\lambda,\mathcal{P}\zeta), d(\lambda,\zeta)) \geq 0.$$

Moreover, Banach contraction mapping can be expressed via $\mathcal{SF}$ $\Xi \in \mathcal{Z}$ for which $\Xi(\ell,k) = \gamma k - \ell$ for all $\ell, k \in [0,\infty)$ and $\gamma \in [0,1)$.

The following expression was used for the first time by Fulga and Proca [19] in 2017 and subsequently referred to as $\mathcal{E}-$contraction or $\mathcal{E}$ type contraction:

$$\mathcal{E}(\lambda,\zeta) = d(\lambda,\zeta) + |d(\lambda,\mathcal{P}\lambda) - d(\zeta,\mathcal{P}\zeta)|, \tag{1.1}$$

whenever $(\mathcal{U},d)$ is a complete metric space and $\lambda,\zeta \in \mathcal{U}$. Also, some studies involve such contraction; see [20]-[22]. One of them was presented by A. Fulga and E. Karapınar [23] via $\mathcal{SF}$ in 2018, as indicated below:

Theorem 1.9. [23] *Let $\mathcal{P}$ be a self-mapping on a complete metric space $(\mathcal{U},d)$. If there exists $\Xi \in \mathcal{Z}$ satisfying, for all $\lambda,\zeta \in \mathcal{U}$,*

$$\Xi(d(\mathcal{P}\lambda,\mathcal{P}\zeta),\mathcal{E}(\lambda,\zeta)) \geq 0,$$

where $\mathcal{E}(\lambda,\zeta)$ is defined as in (1.1), then $\mathcal{P}$ owns a fixed point.

In 2014, A.H. Ansari [24] proposed C−class functions as characterized in the subsequent definition.

Definition 1.10. [24] *A continuous function $\mathscr{A}:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is entitled C−class function if, for all $\ell,\mathcal{k}\in[0,\infty)$, the below statements hold:*

$(\mathscr{A}_1)$ *$\mathscr{A}(\ell,\mathcal{k}) \leq \ell$;*

$(\mathscr{A}_2)$ *$\mathscr{A}(\ell,\mathcal{k}) = \ell$ implies that either $\ell=0$ or $\mathcal{k}=0$.*

Let C−class functions symbolize as $\mathscr{C}$.

In 2018, Radenovic et al. [25] identified the idea of $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ by means of the C−class functions and $\mathcal{SF}$.

Definition 1.11. [25] *A mapping $\Omega:[0,\infty)^2\to\mathbb{R}$ is referred to as $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ if the conditions*

(Ω_1) *$\Omega(\ell,\mathcal{k}) \leq \mathscr{A}(\mathcal{k},\ell)$ for all $\ell,\mathcal{k}>0$, where $\mathscr{A}:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is a C−class functions,*

(Ω_2) *if $\{\ell_{\mathfrak{y}}\},\{\mathcal{k}_{\mathfrak{y}}\}\in(0,\infty)$ are sequences such that $\lim_{\mathfrak{y}\to\infty}\ell_{\mathfrak{y}} = \lim_{\mathfrak{y}\to\infty}\mathcal{k}_{\mathfrak{y}} > 0$ and $\mathcal{k}_{\mathfrak{y}}<\ell_{\mathfrak{y}}$, then $\limsup_{\mathfrak{y}\to\infty}\Omega(\ell_{\mathfrak{y}},\mathcal{k}_{\mathfrak{y}}) < \mathscr{C}_{\mathscr{A}}$*

are provided.

Presume that $\mathscr{Z}^*$ symbolizes the family of all $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$.

Definition 1.12. *[25] A mapping $\mathscr{A}:[0,\infty)\times[0,\infty)\to\mathbb{R}$ has the property $\mathscr{C}_{\mathscr{A}}$, if $\mathscr{C}_{\mathscr{A}}\geq 0$ exists such that*

(1) *$\mathscr{A}(\ell,\mathcal{k}) > \mathscr{C}_{\mathscr{A}}$ implies $\ell>\mathcal{k}$,*

(2) *$\mathscr{A}(\ell,\ell) \leq \mathscr{C}_{\mathscr{A}}$ for all $\ell\in[0,\infty)$.*

The following theorem has a new precondition added to a contractive mapping and was proved by Suzuki [26] in 2009. Herewith, many authors have mentioned this notation as a Suzuki-type contraction.

Theorem 1.13. [26] *Let $\mathcal{P}:\mathcal{U}\to\mathcal{U}$ be a self-mapping on a compact metric space $(\mathcal{U},d)$. If, for all distinc $\lambda,\zeta\in\mathcal{U}$, the statement*

$$\frac{1}{2}d(\lambda,\mathcal{P}\lambda) < d(\lambda,\zeta) \Rightarrow d(\mathcal{P}\lambda,\mathcal{P}\zeta) < d(\lambda,\zeta)$$

is hold, then, $\mathcal{P}$ owns a unique fixed point.

Very recently, Proinov [27] demonstrated a novel fixed point theorem by introducing some auxiliary functions, and subsequently, via this theorem, many significant results were obtained.

Definition 1.14. [27] *Let $\mathcal{P}:\mathcal{U}\to\mathcal{U}$ be a self mapping on a metric space $(\mathcal{U},d)$ and $\mathcal{F},\mathcal{Q}:(0,\infty)\to\mathbb{R}$ are two functions that provide the following features:*

(i) *$\mathcal{F}$ is non-decreasing,*

(ii) *$\mathcal{Q}(s) < \mathcal{F}(s)$ for all $s>0$,*

(iii) *$\limsup_{s\to s_0+}\mathcal{Q}(s) < \mathcal{F}(s_0+)$ for any $s_0>0$.*

If, for all $\lambda,\zeta\in\mathcal{U}$ and $d(\mathcal{P}\lambda,\mathcal{P}\zeta)>0$, the inequality

$$\mathcal{F}(d(\mathcal{P}\lambda,\mathcal{P}\zeta)) \leq \mathcal{Q}(d(\lambda,\zeta))$$

is fulfilled, then $\mathcal{P}$ is called Proinov type contraction.

Theorem 1.15. [27] *Let $\mathcal{P}:\mathcal{U}\to\mathcal{U}$ be a Proinov type contraction on a complete metric space $(\mathcal{U},m)$. Then, $\mathcal{P}$ admits a unique fixed point.*

Various fixed point results involving Proinov type contraction appear in the literature. Some examples are in [28]-[35].

In 2012, Samet et al. [36] introduced the class of $\alpha-$admissible mappings, and subsequently, many new notations appear via this mapping.

Definition 1.16. *Let $\mathcal{P},\mathcal{S}:\mathcal{U}\to\mathcal{U}$ be two mappings and $\alpha:\mathcal{U}\times\mathcal{U}\to\mathbb{R}$ be a function. Then, we have the following ideas.*

(α_1) [36] *If* $\alpha(\lambda,\zeta)\geq 1$ implies $\alpha(\mathcal{P}\lambda,\mathcal{P}\zeta)\geq 1$, *then $\mathcal{P}$ is $\alpha-$admissible,*

(α_2) [37] *if* $\alpha(\lambda,\mathcal{P}\lambda)\geq 1$ implies $\alpha\left(\mathcal{P}\lambda,\mathcal{P}^2\zeta\right)\geq 1$, *then, $\mathcal{P}$ is $\alpha-$orbital admissible,*

(α_3) [37] *together with (α_2), if* $\alpha(\lambda,\zeta)\geq 1$ and $\alpha(\zeta,\mathcal{P}\zeta)\geq 1$ imply $\alpha(\lambda,\mathcal{P}\zeta)\geq 1$, *then, $\mathcal{P}$ is triangular $\alpha-$orbital admissible,*

(α_4) [38] *together with (α_1), if* $\alpha(\lambda,z)\geq 1$ and $\alpha(z,\zeta)\geq 1$ imply $\alpha(\lambda,\zeta)\geq 1$, *then $\mathcal{P}$ is triangular $\alpha-$ admissible,*

(α_5) [39] *together with (α_4), if* $\alpha(\lambda,\zeta)\geq 1$ implies $\alpha(\mathcal{P}\lambda,\mathcal{S}\zeta)\geq 1$ and $\alpha(\mathcal{S}\mathcal{P}\lambda,\mathcal{P}\mathcal{S}\zeta)\geq 1$, *then, the pair $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible.*

Lemma 1.17. [37] *Let $\mathcal{P}:\mathcal{U}\to\mathcal{U}$ be a triangular $\alpha-$orbital admissible mapping. Assume that a $\lambda_0\in\mathcal{U}$ exists such that $\alpha(\lambda_0,\mathcal{P}\lambda_0)\geq 1$. Construct a sequence $\{\lambda_\mathfrak{y}\}$ by $\lambda_{\mathfrak{y}+1}=\mathcal{P}\lambda_\mathfrak{y}$. Then we have $\alpha(\lambda_\mathfrak{y},\lambda_m)\geq 1$ for all $\mathfrak{y},m\in\mathbb{N}$ with $\mathfrak{y}<m$.*

2. Main Results

Primarily, it is necessary to mention the below conditions to guarantee the existence and uniqueness of fixed points in $\mathcal{M}_\flat\mathcal{MS}$ owing to not having to be finite.

$(\mathfrak{C}_1)$ $\mu_\sigma(\lambda,\mathcal{P}\lambda)<\infty$ for all $\sigma>0$ and $\lambda\in\mathcal{U}^*_\mu$,

$(\mathfrak{C}_2)$ $\mu_\sigma(\lambda,\zeta)<\infty$ for all $\sigma>0$ and $\lambda,\zeta\in\mathcal{U}^*_\mu$.

Next, we establish a new contraction mapping by defining Suzuki−Proinov $\mathcal{Z}^{*\mathcal{R}}_{\mathcal{E}^*}(\alpha)-$contraction w.r.t Ω in the sense of $\mathcal{M}_\flat\mathcal{MS}$, as follows.

Definition 2.1. *Let $\mathcal{U}^*_\mu$ be an $\mathcal{M}_\flat\mathcal{MS}$ with constant $\tau\geq 1$ and let $\mathcal{P},\mathcal{S}:\mathcal{U}^*_\mu\to\mathcal{U}^*_\mu$ and $\alpha:\mathcal{U}^*_\mu\times\mathcal{U}^*_\mu\to\mathbb{R}$ be mappings. Then, we say that $\mathcal{P}$ and $\mathcal{S}$ are Suzuki−Proinov $\mathcal{Z}^{*\mathcal{R}}_{\mathcal{E}^*}(\alpha)-$contraction if there exists a $\mathscr{C}_\mathscr{A}-\mathcal{SF}$ $\Omega\in\mathscr{Z}^*$ such that*

$$\frac{1}{2\tau}\min\{\mu_\sigma(\lambda,\mathcal{P}\lambda),\mu_\sigma(\zeta,\mathcal{S}\zeta)\}\leq\mu_\sigma(\lambda,\zeta)$$

implies

$$\Omega\left(\alpha(\lambda,\zeta)\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta)^2\right),\mathcal{Q}(\mathcal{E}^*(\lambda,\zeta)\mathcal{R}(\lambda,\zeta))\right)\geq\mathscr{C}_\mathscr{A}, \tag{2.1}$$

where the functions $\mathcal{F},\mathcal{Q}:(0,\infty)\to\mathbb{R}$ are hold the below requirement:

(c_1) *$\mathcal{F}$ is lower semi-continuous and non-decreasing;*

(c_2) *$\mathcal{Q}(s)<\mathcal{F}(s)$ for all $s>0$;*

(c_3) *$\limsup_{s\to s_0+}\mathcal{Q}(s)<\mathcal{F}(s_0+)$ for any $s_0>0$,*

and also,

$$\mathcal{E}^*(\lambda,\zeta)=\mu_\sigma(\lambda,\zeta)+|\mu_\sigma(\lambda,\mathcal{P}\lambda)-\mu_\sigma(\zeta,\mathcal{S}\zeta)|$$

and

$$\mathcal{R}(\lambda,\zeta)=\frac{\mu_\sigma(\lambda,\mathcal{P}\lambda)\mu_\sigma(\lambda,\mathcal{S}\zeta)+[\mu_\sigma(\lambda,\zeta)]^2+\mu_\sigma(\lambda,\mathcal{P}\lambda)\mu_\sigma(\lambda,\zeta)}{\mu_\sigma(\lambda,\mathcal{P}\lambda)+\mu_\sigma(\lambda,\zeta)+\mu_\sigma(\lambda,\mathcal{S}\zeta)}$$

for all distinct $\lambda,\zeta\in\mathcal{U}^*_\mu$, $\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta)>0$ and for all $\sigma>0$.

Theorem 2.2. *Let $\mathcal{U}_\mu^*$ be a $\mu-$complete $\mathcal{M}_b\mathcal{MS}$ with constant $\tau \geq 1$ and $\mathcal{P}$ and $\mathcal{S}$ be a Suzuki−Proinov $\mathcal{Z}^{*\mathcal{R}}_{\mathcal{E}^*}(\alpha)-$contraction w.r.t. Ω. Assume that the following conditions hold:*

(i) the pair $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible,

(ii) there exists $\lambda_0 \in \mathcal{U}_\mu^*$ such that $\alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1$,

(iii) $\mathcal{P},\mathcal{S}$ are $\mu-$continuous,

(iv) there exists $\lambda,\zeta \in C_{Fix}(\mathcal{P},\mathcal{S})$, where $C_{Fix}(\mathcal{P},\mathcal{S})$ represents set of common fixed points of $\mathcal{P}$ and $\mathcal{S}$, such that $\alpha(\lambda,\zeta) \geq 1$.

In case of satisfying $(\mathfrak{C}_1)$, there there exists $\lambda^* \in \mathcal{U}_\mu^*$ such that $\lambda^* \in C_{Fix}(\mathcal{P},\mathcal{S})$. Also, additionally, if $(\mathfrak{C}_2)$ is hold, then $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$.

Proof. Let $\lambda_0 \in \mathcal{U}_\mu^*$ be a specified point such that $\alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1$. Construct an iterative sequence $\{\lambda_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}}$ in $\mathcal{U}_\mu{}^*$ such that

$$\lambda_{2\mathfrak{y}+1} = \mathcal{P}\lambda_{2\mathfrak{y}} \quad \text{and} \quad \lambda_{2\mathfrak{y}+2} = \mathcal{S}\lambda_{2\mathfrak{y}+1}, \quad \text{for all } \mathfrak{y} \in \mathbb{N}.$$

On the other hand, regarding that $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible, we derive

$$\alpha(\lambda_0,\lambda_1) = \alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1 \ \Rightarrow \begin{cases} \alpha(\mathcal{P}\lambda_0,\mathcal{S}\lambda_1) = \alpha(\lambda_1,\lambda_2) \geq 1 \\ \text{and} \\ \alpha(\mathcal{S}\mathcal{P}\lambda_0,\mathcal{P}\mathcal{S}\lambda_1) = \alpha(\mathcal{S}\lambda_1,\mathcal{P}\lambda_2) = \alpha(\lambda_2,\lambda_3) \geq 1. \end{cases}$$

Likewise, we get

$$\alpha(\lambda_2,\lambda_3) \geq 1 \ \Rightarrow \begin{cases} \alpha(\mathcal{P}\lambda_2,\mathcal{S}\lambda_3) = \alpha(\lambda_3,\lambda_4) \geq 1 \\ \text{and} \\ \alpha(\mathcal{S}\mathcal{P}\lambda_2,\mathcal{P}\mathcal{S}\lambda_3) = \alpha(\mathcal{S}\lambda_3,\mathcal{P}\lambda_4) = \alpha(\lambda_2,\lambda_3) \geq 1. \end{cases}$$

Thereby, recursively, we conclude that

$$\alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) \geq 1. \tag{2.2}$$

Also, if there is some $\mathfrak{y}_0 \in \mathbb{N}$ such that $\lambda_{\mathfrak{y}_0} = \lambda_{\mathfrak{y}_0+1}$, then $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\mathfrak{y}_0\}$. Thereupon, we presume that $\lambda_k \neq \lambda_{k+1}$ for all $k \in \mathbb{N}$, which indicates that $\mu_\sigma(\lambda_k,\lambda_{k+1}) > 0$ for all $\sigma > 0$. Next, we assume that $k = 2\mathfrak{y}$ for some $\mathfrak{y} \in \mathbb{N}$. Because

$$\begin{aligned}\tfrac{1}{2\tau}\min\{\mu_\sigma(\lambda_{2\mathfrak{y}},\mathcal{P}\lambda_{2\mathfrak{y}}),\mu_\sigma(\lambda_{2\mathfrak{y}+1},\mathcal{S}\lambda_{2\mathfrak{y}+1})\} &= \tfrac{1}{2\tau}\min\{\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}),\mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})\} \\ &\leq \mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}),\end{aligned}$$

from (2.1) and (Θ_1), we have

$$\begin{aligned}\mathscr{C}_\mathscr{A} &\leq \Omega\left(\alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda_{2\mathfrak{y}},\mathcal{S}\lambda_{2\mathfrak{y}+1})^2\right),\mathcal{Q}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}))\right) \\ &= \Omega\left(\alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{F}\left(\tau^6\mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})^2\right),\mathcal{Q}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}))\right) \\ &< \mathscr{A}\left(\mathcal{Q}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})),\alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{F}\left(\tau^6\mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})^2\right)\right),\end{aligned}$$

and by (c_2), (2.2) and the properties $\mathscr{C}_\mathscr{A}$, we yield

$$\begin{aligned}\mathcal{F}\left(\tau^6\mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})^2\right) \leq \alpha(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{F}\left(\tau^6\mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})^2\right) &< \mathcal{Q}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})) \\ &< \mathcal{F}(\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\,\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) &= \mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) + |\mu_\sigma(\lambda_{2\mathfrak{y}},\mathcal{P}\lambda_{2\mathfrak{y}}) - \mu_\sigma(\lambda_{2\mathfrak{y}+1},\mathcal{S}\lambda_{2\mathfrak{y}+1})| \\ &= \mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) + |\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) - \mu_\sigma(\lambda_{2\mathfrak{y}+1},\lambda_{2\mathfrak{y}+2})|\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}) &= \frac{\mu_\sigma(\lambda_{2\mathfrak{y}},\mathcal{P}\lambda_{2\mathfrak{y}})\mu_\sigma(\lambda_{2\mathfrak{y}},\mathcal{S}\lambda_{2\mathfrak{y}+1})+\left[\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\right]^2+\mu_\sigma(\lambda_{2\mathfrak{y}},\mathcal{P}\lambda_{2\mathfrak{y}})\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})}{\mu_\sigma(\lambda_{2\mathfrak{y}},\mathcal{P}\lambda_{2\mathfrak{y}})+\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})+\mu_\sigma(\lambda_{2\mathfrak{y}},\mathcal{S}\lambda_{2\mathfrak{y}+1})}\\
&= \frac{\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+2})+\left[\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\right]^2+\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})}{\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})+\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})+\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+2})}\\
&= \frac{\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\left[\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+2})+\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})+\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})\right]}{\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})+\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})+\mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+2})}\\
&= \mu_\sigma(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1}).\end{aligned}$$

Denote $\mu_\sigma(\lambda_\mathfrak{y},\lambda_{\mathfrak{y}+1})$ by $\kappa_\mathfrak{y}$. Now, if $\max\{\kappa_{2\mathfrak{y}},\kappa_{2\mathfrak{y}+1}\}=\kappa_{2\mathfrak{y}+1}$, then, we get $\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})=\kappa_{2\mathfrak{y}+1}$ and $\mathcal{R}(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})=\kappa_{2\mathfrak{y}}$. Thereupon, (2.3) turns into

$$\mathcal{F}\left(\kappa^2_{2\mathfrak{y}+1}\right)\le\mathcal{F}\left(\tau^6\kappa^2_{2\mathfrak{y}+1}\right)<\mathcal{Q}(\kappa_{2\mathfrak{y}+1}.\kappa_{2\mathfrak{y}})<\mathcal{F}(\kappa_{2\mathfrak{y}+1}.\kappa_{2\mathfrak{y}}),$$

such that, by utilizing the function $\mathcal{F}$'s characteristics, we conclude that $\kappa_{2\mathfrak{y}+1}<\kappa_{2\mathfrak{y}}$. However, this contradicts our assumptions. Thereby, we achieve $\max\{\kappa_{2\mathfrak{y}},\kappa_{2\mathfrak{y}+1}\}=\kappa_{2\mathfrak{y}}$, which implies that $\mathcal{E}^*(\lambda_{2\mathfrak{y}},\lambda_{2\mathfrak{y}+1})=2\kappa_{2\mathfrak{y}}-\kappa_{2\mathfrak{y}+1}$. Then, (2.3) becomes

$$\mathcal{F}\left(\kappa^2_{2\mathfrak{y}+1}\right)\le\mathcal{F}\left(\tau^6\kappa^2_{2\mathfrak{y}+1}\right)<\mathcal{Q}((2\kappa_{2\mathfrak{y}}-\kappa_{2\mathfrak{y}+1}.)\kappa_{2\mathfrak{y}})<\mathcal{F}((2\kappa_{2\mathfrak{y}}-\kappa_{2\mathfrak{y}+1}.)\kappa_{2\mathfrak{y}}), \tag{2.4}$$

by (c_1), we obtain that

$$\begin{aligned}\kappa^2_{2\mathfrak{y}+1}<(2\kappa_{2\mathfrak{y}}-\kappa_{2\mathfrak{y}+1}.)\kappa_{2\mathfrak{y}} &\Leftrightarrow \kappa^2_{2\mathfrak{y}+1}<2\kappa^2_{2\mathfrak{y}}-\kappa_{2\mathfrak{y}}\kappa_{2\mathfrak{y}+1}<2\kappa^2_{2\mathfrak{y}}-\kappa^2_{2\mathfrak{y}+1}\\
&\Leftrightarrow 2\kappa^2_{2\mathfrak{y}+1}<2\kappa^2_{2\mathfrak{y}}\\
&\Leftrightarrow \kappa_{2\mathfrak{y}+1}<\kappa_{2\mathfrak{y}}.\end{aligned}$$

Likewise, one concludes that $\kappa_{2\mathfrak{y}}<\kappa_{2\mathfrak{y}-1}$. So, we say that that $\{\kappa_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}}=\{\mu_\sigma(\lambda_\mathfrak{y},\lambda_{\mathfrak{y}+1})\}_{\mathfrak{y}\in\mathbb{N}}$ is a non-increasing sequence of non-negative real numbers. Also, a similar consequence can be obtained when k is an odd number. Then, there exists $p\ge 0$ such that $\lim_{\mathfrak{y}\to\infty}\kappa_\mathfrak{y}=p$. Assume on the contrary, we aim to show that $p>0$. Then, by (2.4), we have

$$\mathcal{F}\left(p^2\right)\le\lim_{\mathfrak{y}\to\infty}\mathcal{F}\left(\kappa^2_{2\mathfrak{y}+1}\right)<\lim_{\mathfrak{y}\to\infty}\mathcal{Q}((2\kappa_{2\mathfrak{y}}-\kappa_{2\mathfrak{y}+1}.)\kappa_{2\mathfrak{y}})<\lim_{\mathfrak{y}\to\infty}\mathcal{F}((2\kappa_{2\mathfrak{y}}-\kappa_{2\mathfrak{y}+1}.)\kappa_{2\mathfrak{y}})=\mathcal{F}\left(p^2\right),$$

which emerges a contradiction, which means that, for all $\sigma>0$,

$$\lim_{\mathfrak{y}\to\infty}\mu_\sigma(\lambda_\mathfrak{y},\lambda_{\mathfrak{y}+1})=0. \tag{2.5}$$

Now, it is required to indicate $\{\lambda_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}}$ is a $\mu-$Cauchy sequence. Rather, presume that $\{\lambda_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}}$ is not a $\mu-$Cauchy sequence. Then, for at least a $\varepsilon>0$ and $\mathfrak{y}_\hbar>m_\hbar>\hbar$ whenever $\hbar\in\mathbb{N}\cup\{0\}$ and let $\mathfrak{y}_\hbar$ be the smallest index such that the following expressions are provided:

$$\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar}\right)\ge\varepsilon \quad\text{and}\quad \mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar-2}\right)<\varepsilon,\quad\text{for all }\sigma>0. \tag{2.6}$$

By using (2.5), (2.6) and (μ_3'), we yield

$$\begin{aligned}\varepsilon\le\mu_{4\sigma}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar}\right)&\le\tau\mu_{2\sigma}\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)+\tau^2\mu_\sigma\left(\lambda_{2m_\hbar+1},\lambda_{2\mathfrak{y}_\hbar+2}\right)\\
&+\tau^3\mu_{\sigma/2}\left(\lambda_{2\mathfrak{y}_\hbar+2},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\tau^3\mu_{\sigma/2}\left(\lambda_{2\mathfrak{y}_\hbar+1},\lambda_{2\mathfrak{y}_\hbar}\right)\end{aligned}$$

such that

$$\limsup_{\hbar\to\infty}\mu_\sigma\left(\lambda_{2m_\hbar+1},\lambda_{2\mathfrak{y}_\hbar+2}\right)\ge\frac{\varepsilon}{\tau^2}. \tag{2.7}$$

Also, we get

$$\begin{aligned}\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right) &\leq \tau\mu_{\sigma/2}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar-2}\right)+\tau^2\mu_{\sigma/4}\left(\lambda_{2\mathfrak{y}_\hbar-2},\lambda_{2\mathfrak{y}_\hbar-1}\right)\\ &\quad+\tau^3\mu_{\sigma/8}\left(\lambda_{2\mathfrak{y}_\hbar-1},\lambda_{2\mathfrak{y}_\hbar}\right)+\tau^3\mu_{\sigma/8}\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right).\end{aligned} \tag{2.8}$$

Thereby, by taking the limit superior in (2.8) and using (2.5), we obtain that

$$\limsup_{\hbar\to\infty}\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\leq\tau\varepsilon. \tag{2.9}$$

Also, from the (2.5) and (2.6), we achieve that

$$\begin{aligned}\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+2}\right) &\leq \tau\mu_{\sigma/2}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar-2}\right)+\tau^2\mu_{\sigma/4}\left(\lambda_{2\mathfrak{y}_\hbar-2},\lambda_{2\mathfrak{y}_\hbar-1}\right)\\ &\quad+\tau^3\mu_{\sigma/8}\left(\lambda_{2\mathfrak{y}_\hbar-1},\lambda_{2\mathfrak{y}_\hbar}\right)+\tau^4\mu_{\sigma/16}\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\tau^4\mu_{\sigma/16}\left(\lambda_{2\mathfrak{y}_\hbar+1},\lambda_{2\mathfrak{y}_\hbar+2}\right)\end{aligned}$$

and letting $\hbar\to\infty$, we attain

$$\limsup_{\hbar\to\infty}\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+2}\right)\leq\tau\varepsilon. \tag{2.10}$$

Furthermore, if $\mathfrak{y}_\hbar>m_\hbar>\hbar$ for sufficiently large $\hbar\in\mathbb{N}$, we assert

$$\frac{1}{2\tau}\min\left\{\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\mathcal{P}\lambda_{2\mathfrak{y}_\hbar}\right),\mu_\sigma\left(\lambda_{2m_\hbar-1},\mathcal{S}\lambda_{2m_\hbar-1}\right)\right\}\leq\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right). \tag{2.11}$$

Given the fact that, $\mathfrak{y}_\hbar>m_\hbar$ and $\{\mu_\sigma(\lambda_\mathfrak{y},\lambda_{\mathfrak{y}+1})\}_{\mathfrak{y}\geq1}$ is non-decreasing, we acquire

$$\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\mathcal{P}\lambda_{2\mathfrak{y}_\hbar}\right)=\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\leq\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)\leq\mu_\sigma\left(\lambda_{2m_\hbar-1},\lambda_{2m_\hbar}\right)=\mu_\sigma\left(\lambda_{2m_\hbar-1},\mathcal{S}\lambda_{2m_\hbar-1}\right).$$

Hence,

$$\frac{1}{2\tau}\min\left\{\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\mathcal{P}\lambda_{2\mathfrak{y}_\hbar}\right),\mu_\sigma\left(\lambda_{2m_\hbar-1},\mathcal{S}\lambda_{2m_\hbar-1}\right)\right\}=\frac{1}{2\tau}\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\mathcal{P}\lambda_{2\mathfrak{y}_\hbar}\right)=\frac{1}{2\tau}\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right).$$

According to (2.5), there exists $\hbar_1\in\mathbb{N}$ such that for any $\hbar>\hbar_1$,

$$\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)<\frac{\varepsilon}{2\tau}.$$

Also, there exists $\hbar_2\in\mathbb{N}$ such that for any $\hbar>\hbar_2$,

$$\mu_\sigma\left(\lambda_{2m_\hbar-1},\lambda_{2m_\hbar}\right)<\frac{\varepsilon}{2\tau}.$$

Hence, for any $\hbar>\max\{\hbar_1,\hbar_2\}$ and $\mathfrak{y}_\hbar>m_\hbar>\hbar$, we get

$$\varepsilon\leq\mu_{2\sigma}\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar}\right)\leq\tau\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right)+\tau\mu_\sigma\left(\lambda_{2m_\hbar-1},\lambda_{2m_\hbar}\right)\leq\tau\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right)+\tau\frac{\varepsilon}{2\tau}.$$

So, one can conclude that

$$\frac{\varepsilon}{2\tau}\leq\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right).$$

Thus, we deduce that for any $\hbar>\max\{\hbar_1,\hbar_2\}$ and $\mathfrak{y}_\hbar>m_\hbar>\hbar$,

$$\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)<\frac{\varepsilon}{2\tau}\leq\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar},\lambda_{2m_\hbar-1}\right)$$

which implies that (2.11) is hold. Also, by using that $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible pair, we can write $\alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\geq1$. Therefore, from (2.1), we conclude that

$$\begin{aligned}\mathscr{C}_\mathscr{A} &\leq\Omega\left(\alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right),\mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right)\right)\\ &=\Omega\left(\alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right),\mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right)\right)\\ &<\mathscr{A}\left(\mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right),\alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right)\right),\end{aligned}$$

and by the properties of $\mathscr{C}_{\mathscr{A}}$ and (c_2), we deduce that

$$\begin{aligned} \mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right) &\leq \alpha\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right) \\ &< \mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right) \\ &< \mathcal{F}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right) &= \mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\left|\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{P}\lambda_{2m_\hbar}\right)-\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar+1},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)\right| \\ &= \mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\left|\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)-\mu_\sigma\left(\lambda_{2\mathfrak{y}_\hbar+1},\lambda_{2\mathfrak{y}_\hbar+2}\right)\right| \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right) &= \frac{\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{P}\lambda_{2m_\hbar}\right)\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)+\left[\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right]^2+\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{P}\lambda_{2m_\hbar}\right)\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)}{\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{P}\lambda_{2m_\hbar}\right)+\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\mu_\sigma\left(\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)} \\ &= \frac{\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+2}\right)+\left[\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right]^2+\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)}{\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2m_\hbar+1}\right)+\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)+\mu_\sigma\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+2}\right)}. \end{aligned} \tag{2.14}$$

Next, letting $\hbar\to\infty$ in (2.12), (2.13) and (2.14), and also, by using (2.5), (2.7), (2.9) and (2.10), we acquire that

$$\begin{aligned} \mathcal{F}\left(\tau^2\varepsilon^2\right) = \mathcal{F}\left(\tau^6\left(\frac{\varepsilon}{\tau^2}\right)^2\right) &\leq \limsup_{\mathfrak{y}\to\infty}\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda_{2m_\hbar},\mathcal{S}\lambda_{2\mathfrak{y}_\hbar+1}\right)^2\right) \\ &< \limsup_{\mathfrak{y}\to\infty}\mathcal{Q}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right) \\ &< \limsup_{\mathfrak{y}\to\infty}\mathcal{F}\left(\mathcal{E}^*\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\mathcal{R}\left(\lambda_{2m_\hbar},\lambda_{2\mathfrak{y}_\hbar+1}\right)\right) \\ &\leq \mathcal{F}\left(\tau\varepsilon.\frac{(\tau\varepsilon)^2}{\tau\varepsilon+\tau\varepsilon}\right) = \mathcal{F}\left(\frac{\tau^2\varepsilon^2}{2}\right). \end{aligned}$$

This causes a contradictory, that is, $\{\lambda_\mathfrak{y}\}_{\mathfrak{y}\in\mathbb{N}}$ is a $\mu-$Cauchy sequence on a $\mu-$complete $\mathcal{M}_b\mathcal{MS}$. Thereby, a point λ^* exists in $\mathcal{U}^*_\mu$ such that

$$\lim_{\mathfrak{y}\to\infty}\lambda_\mathfrak{y} = \lambda^*. \tag{2.15}$$

Considering the continuity of the mappings and (2.15), we get

$$\begin{aligned} \mathcal{P}\lambda^* &= \mathcal{P}\left(\lim_{\mathfrak{y}\to\infty}\lambda_{2\mathfrak{y}}\right) = \lim_{\mathfrak{y}\to\infty}\mathcal{P}\lambda_{2\mathfrak{y}} = \lim_{\mathfrak{y}\to\infty}\lambda_{2\mathfrak{y}+1} = \lambda^* \\ &= \lim_{\mathfrak{y}\to\infty}\lambda_{2\mathfrak{y}+2} = \lim_{\mathfrak{y}\to\infty}\mathcal{S}\lambda_{2\mathfrak{y}+1} \\ &= \mathcal{S}\left(\lim_{\mathfrak{y}\to\infty}\lambda_{2\mathfrak{y}+1}\right) = \mathcal{S}\lambda^*. \end{aligned}$$

Thereupon, we conclude that λ^* is a common fixed point of $\mathcal{P}$ and $\mathcal{S}$. Finally, we prove that the point λ^* is unique. For this, there is $\hat{\lambda}$, which is another common fixed point, such that $\lambda^*\neq\hat{\lambda}$. So, from the condition (iv), we deduce that $\alpha\left(\lambda^*,\hat{\lambda}\right)\geq 1$. Hence, since

$$0 = \frac{1}{2\tau}\min\left\{\mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right),\mu_\sigma\left(\hat{\lambda},\mathcal{S}\hat{\lambda}\right)\right\} \leq \mu_\sigma\left(\lambda^*,\hat{\lambda}\right),$$

by using (2.1) and (Θ_1), we gain

$$\begin{aligned}\mathscr{C}_{\mathscr{A}} &\leq \Omega\left(\alpha\left(\lambda^*,\hat{\lambda}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\mathcal{P}\lambda^*,\mathcal{S}\hat{\lambda}\right)^2\right),\mathcal{Q}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right)\right)\\ &= \Omega\left(\alpha\left(\lambda^*,\hat{\lambda}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right),\mathcal{Q}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right)\right)\\ &< \mathscr{A}\left(\mathcal{Q}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right),\alpha\left(\lambda^*,\hat{\lambda}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right)\right)\end{aligned}$$

and by Definition 1.12 and (c_2), we get

$$\begin{aligned}\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right) \leq \alpha\left(\lambda^*,\hat{\lambda}\right)\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right) &< \mathcal{Q}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right)\\ &< \mathcal{F}\left(\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right)\mathcal{R}\left(\lambda^*,\hat{\lambda}\right)\right),\end{aligned} \tag{2.16}$$

where

$$\mathcal{E}^*\left(\lambda^*,\hat{\lambda}\right) = \mu_\sigma\left(\lambda^*,\hat{\lambda}\right) + \left|\mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right) - \mu_\sigma\left(\hat{\lambda},\mathcal{S}\hat{\lambda}\right)\right| = \mu_\sigma\left(\lambda^*,\hat{\lambda}\right)$$

and

$$\mathcal{R}\left(\lambda^*,\hat{\lambda}\right) = \frac{\mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right)\mu_\sigma\left(\lambda^*,\mathcal{S}\hat{\lambda}\right) + \left[\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)\right]^2 + \mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right)\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)}{\mu_\sigma\left(\lambda^*,\mathcal{P}\lambda^*\right) + \mu_\sigma\left(\lambda^*,\hat{\lambda}\right) + \mu_\sigma\left(\lambda^*,\mathcal{S}\hat{\lambda}\right)} = \frac{\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)}{2}.$$

Consequently, considering the above equalities, the inequality (2.16) turns into

$$\mathcal{F}\left(\tau^6\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2\right) < \mathcal{F}\left(\mu_\sigma\left(\lambda^*,\hat{\lambda}\right).\frac{\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)}{2}\right) = \mathcal{F}\left(\frac{\mu_\sigma\left(\lambda^*,\hat{\lambda}\right)^2}{2}\right),$$

which causes a contradiction. In turn, we achieve that $\lambda^* = \hat{\lambda}$, which means that $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$. This ends the proof. □

3. Consequences

In this part of the study, we discuss some of the implications of the fundamental observation. Primarily, if the restriction

$$\frac{1}{2\tau}\min\{\mu_\sigma(\lambda,\mathcal{P}\lambda),\mu_\sigma(\zeta,\mathcal{S}\zeta)\} \leq \mu_\sigma(\lambda,\zeta)$$

is ignored, Theorem 2.2 yields the subsequent consequence.

Corollary 3.1. *Let $\mathcal{U}_\mu^*$ be a $\mu-$complete $\mathcal{M}_b\mathcal{MS}$ with $\tau \geq 1$, $\alpha : \mathcal{U}_\mu^* \times \mathcal{U}_\mu^* \to \mathbb{R}$ be a function and $\mathcal{P},\mathcal{S} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ be two self-mappings. Assume that the following assertions are true:*

(i) there exists $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ such that

$$\Omega\left(\alpha(\lambda,\zeta)\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta)^2\right),\mathcal{Q}(\mathcal{E}^*(\lambda,\zeta)\mathcal{R}(\lambda,\zeta))\right) \geq \mathscr{C}_{\mathscr{A}},$$

where $\mathcal{F}$, $\mathcal{Q}$, $\mathcal{E}(\lambda,\zeta)$ and $\mathcal{R}(\lambda,\zeta)$ are defined as in Definition 2.1 for all distinct $\lambda,\zeta \in \mathcal{U}_\mu^*$, $\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta) > 0$ and for all $\sigma > 0$,

(ii) the pair $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible and there exists $\lambda_0 \in \mathcal{U}_\mu^*$ such that $\alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1$,

(iii) $\mathcal{P},\mathcal{S}$ are $\mu-$continuous,

(iv) there exist $\lambda,\zeta \in C_{Fix}(\mathcal{P},\mathcal{S})$ such that $\alpha(\lambda,\zeta) \geq 1$.

Under the conditions $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$, $\lambda^* \in \mathcal{U}_\mu^*$ exists such that $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$.

Moreover, take into $\alpha(\lambda,\zeta) = 1$ account in Corollary 3.1, the next result is determined.

Corollary 3.2. *Let $\mathcal{U}_\mu^*$ be a $\mu-$complete $\mathcal{M}_\flat\mathcal{MS}$ with $\tau \geq 1$ and $\mathcal{P},\mathcal{S} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ be two self-mappings there exists $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ such that*

$$\Omega\left(\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta)^2\right), \mathcal{Q}\left(\mathcal{E}^*(\lambda,\zeta)\,\mathcal{R}(\lambda,\zeta)\right)\right) \geq \mathscr{C}_{\mathscr{A}},$$

where $\mathcal{F}, \mathcal{Q}$, $\mathcal{E}(\lambda,\zeta)$ and $\mathcal{R}(\lambda,\zeta)$ are defined as in Definition 2.1 for all distinct $\lambda,\zeta \in \mathcal{U}_\mu^*$, $\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta) > 0$ and for all $\sigma > 0$. Thereupon, together with $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$, we conclude that $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$.

Corollary 3.3. *Let $\mathcal{U}_\mu^*$ be a $\mu-$complete $\mathcal{M}_\flat\mathcal{MS}$ with a constant $\tau \geq 1$, $\alpha : \mathcal{U}_\mu^* \times \mathcal{U}_\mu^* \to \mathbb{R}$ be a function and $\mathcal{P} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ be a self-mapping. Assume that the below requirements are met:*

(i) there exists $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ such that

$$\frac{1}{2\tau}\mu_\sigma(\lambda,\mathcal{P}\lambda) \leq \mu_\sigma(\lambda,\zeta)$$

implies

$$\Omega\left(\alpha(\lambda,\zeta)\,\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda,\mathcal{P}\zeta)^2\right), \mathcal{Q}\left(\mathcal{E}^*(\lambda,\zeta)\,\mathcal{R}(\lambda,\zeta)\right)\right) \geq \mathscr{C}_{\mathscr{A}},$$

where the functions $\mathcal{F}, \mathcal{Q}$ are as indicated in Definition 2.1 and also, $\mathcal{E}(\lambda,\zeta)$ as in (1.1) and

$$\mathcal{R}(\lambda,\zeta) = \frac{\mu_\sigma(\lambda,\mathcal{P}\lambda)\mu_\sigma(\lambda,\mathcal{P}\zeta) + [\mu_\sigma(\lambda,\zeta)]^2 + \mu_\sigma(\lambda,\mathcal{P}\lambda)\mu_\sigma(\lambda,\zeta)}{\mu_\sigma(\lambda,\mathcal{P}\lambda) + \mu_\sigma(\lambda,\zeta) + \mu_\sigma(\lambda,\mathcal{P}\zeta)}$$

for all distinct $\lambda,\zeta \in \mathcal{U}_\mu^*$, $\mu_\sigma(\mathcal{P}\lambda,\mathcal{P}\zeta) > 0$ and for all $\sigma > 0$,

(ii) $\mathcal{P}$ is a triangular $\alpha-$orbital admissible mapping and there exists $\lambda_0 \in \mathcal{U}_\mu^*$ such that $\alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1$,

(iii) $\mathcal{P}$ is $\mu-$continuous,

(iv) there exist $\lambda,\zeta \in Fix(\mathcal{P})$ such that $\alpha(\lambda,\zeta) \geq 1$.

So, under the conditions $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$, $\mathcal{P}$ has a unique fixed point.

Proof. Letting $\mathcal{P} = \mathcal{S}$ in Theorem 2.2, and by Lemma 1.17, we achieve the desired results. □

Corollary 3.4. *Let $\mathcal{U}_\mu^*$ be a $\mu-$complete $\mathcal{M}_\flat\mathcal{MS}$ with a constant $\tau \geq 1$, $\alpha : \mathcal{U}_\mu^* \times \mathcal{U}_\mu^* \to \mathbb{R}$ be a function and $\mathcal{P},\mathcal{S} : \mathcal{U}_\mu^* \to \mathcal{U}_\mu^*$ be two self-mappings. Assume that the following assertions are true:*

(i) there exists $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ such that

$$\frac{1}{2\tau}\min\{\mu_\sigma(\lambda,\mathcal{P}\lambda), \mu_\sigma(\zeta,\mathcal{S}\zeta)\} \leq \mu_\sigma(\lambda,\zeta)$$

implies

$$\alpha(\lambda,\zeta)\,\mathcal{F}\left(\tau^6\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta)^2\right) \leq \mathcal{Q}\left(\mathcal{E}^*(\lambda,\zeta)\,\mathcal{R}(\lambda,\zeta)\right),$$

where $\mathcal{F}, \mathcal{Q}$, $\mathcal{E}(\lambda,\zeta)$ and $\mathcal{R}(\lambda,\zeta)$ are defined as in Definition 2.1 for all distinct $\lambda,\zeta \in \mathcal{U}_\mu^*$, $\mu_\sigma(\mathcal{P}\lambda,\mathcal{S}\zeta) > 0$ and for all $\sigma > 0$;

(ii) the pair $(\mathcal{P},\mathcal{S})$ is triangular $\alpha-$admissible and there exists $\lambda_0 \in \mathcal{U}_\mu^*$ such that $\alpha(\lambda_0,\mathcal{P}\lambda_0) \geq 1$,

(iii) $\mathcal{P},\mathcal{S}$ are $\mu-$continuous,

(iv) there exists $\lambda,\zeta \in C_{Fix}(\mathcal{P},\mathcal{S})$ such that $\alpha(\lambda,\zeta) \geq 1$.

Thereupon, $C_{Fix}(\mathcal{P},\mathcal{S}) = \{\lambda^*\}$ provided that $(\mathfrak{C}_1)$ and $(\mathfrak{C}_2)$ are met.

Proof. Letting $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$ with the properties $\mathscr{C}_{\mathscr{A}}$ in Definition 1.12. □

Remark 3.5. *Note that all of the results can be again evaluated with respect to $\Xi \in \mathcal{Z}$ in place of $\mathscr{C}_{\mathscr{A}}-\mathcal{SF}$ $\Omega \in \mathscr{Z}^*$. Besides, as in Corollary 3.3, different results can be obtained when $\mathcal{P} = \mathcal{S}$.*

4. An Application to Dynamic Programming

We assume that Λ and Φ are Banach spaces, $\Sigma \subseteq \Lambda$ and $\Upsilon \subseteq \Phi$ such that Σ and Υ are state space and decision space, respectively. Consider the system of functional equations:

$$q(\lambda) = \max_{\zeta \in \Upsilon} \{f(\lambda,\zeta) + G(\lambda,\zeta,q(\xi(\lambda,\zeta)))\}, \ \lambda \in \Sigma$$

where $f : \Sigma \times \Upsilon \to \mathbb{R}$ and $G : \Sigma \times \Upsilon \times \mathbb{R} \to \mathbb{R}$ are bounded, $\xi : \Sigma \times \Upsilon \to \Sigma$. Let $\mathcal{U}_\mu = B(\Sigma)$ denotes the space of all bounded real-valued functions on Σ. Consider the metric defined by

$$\mu_\sigma(\varsigma,\varpi) = \frac{1}{\sigma} \max_{\lambda \in \Sigma} |\varsigma(\lambda) - \varpi(\lambda)|^2, \text{ for all } \varsigma,\varpi \in \Lambda \text{ and } \sigma > 0.$$

Then, $\mathcal{U}_\mu$ is a $\mu-$complete $\mathcal{M}_b\mathcal{MS}$ with $\tau = 2$. Moreover, let $\mathcal{P} : \mathcal{U}_\mu \to \mathcal{U}_\mu$ be given by

$$\mathcal{P}\varsigma(\lambda) = \sup_{\zeta \in \Upsilon} \{f(\lambda,\zeta) + G(\lambda,\zeta,\varsigma(\xi(\lambda,\zeta)))\}, \tag{4.1}$$

where $\lambda \in \Sigma$ and $\varsigma \in \mathcal{U}_\mu$. If the functions f and G are bounded, then Λ and Φ are well-defined.

Theorem 4.1. *Let $\mathcal{P} : \mathcal{U}_\mu \to \mathcal{U}_\mu$ be an operator defined by (4.1) and suppose that the following conditions are hold:*

(i) f and G are bounded;

(ii) for $\forall \varsigma,\varpi \in \mathcal{U}_\mu$, $\forall \lambda \in \Sigma$, $\forall \zeta \in \Upsilon$, there exists $\delta \in (0,1)$ such that

$$|G(\lambda,\zeta,\varsigma(\lambda)) - G(\lambda,\zeta,\varpi(\lambda))| < \delta^{1/4} |\varsigma(\lambda) - \varpi(\lambda)|.$$

Then, the function equation (4.1) has a bounded solution; that is, $\mathcal{P}$ has a fixed point.

Proof. Let $\varepsilon \in \mathbb{R}^+$ be arbitrary, $\lambda \in \Sigma$ and $\varsigma \in \mathcal{U}_\mu$. Assume that $\mathcal{P}\varsigma \neq \varsigma$. Then, $\zeta_1, \zeta_2 \in \Upsilon$ exist such that

$$\mathcal{P}\varsigma(\lambda) < f(\lambda,\zeta_1) + G(\lambda,\zeta_1,\varsigma(\xi(\lambda,\zeta_1))) + \varepsilon, \tag{4.2}$$

$$\varpi(\lambda) < f(\lambda,\zeta_2) + G(\lambda,\zeta_2,\varpi(\xi(\lambda,\zeta_1))) + \varepsilon, \tag{4.3}$$

$$\mathcal{P}\varsigma(\lambda) \geq f(\lambda,\zeta_2) + G(\lambda,\zeta_2,\varsigma(\xi(\lambda,\zeta_2))), \tag{4.4}$$

$$\varpi(\lambda) \geq f(\lambda,\zeta_1) + G(\lambda,\zeta_1,\varpi(\xi(\lambda,\zeta_1))). \tag{4.5}$$

Then, from (4.2) and (4.5), we yield that

$$\begin{aligned}
\mathcal{P}\varsigma(\lambda) - \varpi(\lambda) &< G(\lambda,\zeta_1,\varsigma(\xi(\lambda,\zeta_1))) - G(\lambda,\zeta_1,\varpi(\xi(\lambda,\zeta_1))) + \varepsilon \\
&\leq |G(\lambda,\zeta_1,\varsigma(\xi(\lambda,\zeta_1))) - G(\lambda,\zeta_1,\varpi(\xi(\lambda,\zeta_1)))| + \varepsilon \\
&< \delta^{1/4} |\varsigma(\lambda) - \varpi(\lambda)| + \varepsilon.
\end{aligned}$$

Likewise, from (4.3) and (4.4), we get

$$\begin{aligned}
\varpi(\lambda) - \mathcal{P}\varsigma(\lambda) &< G(\lambda,\zeta_2,\varpi(\xi(\lambda,\zeta_2))) - G(\lambda,\zeta_2,\varsigma(\xi(\lambda,\zeta_2))) + \varepsilon \\
&\leq |G(\lambda,\zeta_2,\varpi(\xi(\lambda,\zeta_2))) - G(\lambda,\zeta_2,\varsigma(\xi(\lambda,\zeta_2)))| + \varepsilon \\
&< \delta^{1/4} |\varsigma(\lambda) - \varpi(\lambda)| + \varepsilon.
\end{aligned}$$

Hence, by considering the above inequalities, we conclude that

$$|\mathcal{P}\varsigma(\lambda)-\varpi(\lambda)|<\delta^{1/4}|\varsigma(\lambda)-\varpi(\lambda)|+\varepsilon,$$

and, for an arbitrary ε

$$|\mathcal{P}\varsigma(\lambda)-\varpi(\lambda)|\leq\delta^{1/4}|\varsigma(\lambda)-\varpi(\lambda)|.$$

So, we have

$$\mu_\sigma(\mathcal{P}\varsigma(\lambda),\varpi(\lambda))=\frac{1}{\sigma}|\mathcal{P}\varsigma(\lambda)-\varpi(\lambda)|^2\leq\frac{1}{\sigma}\delta^{1/2}|\varsigma(\lambda)-\varpi(\lambda)|^2=\delta^{1/2}\mu_\sigma(\varsigma(\lambda),\varpi(\lambda)).\tag{4.6}$$

Now, in Theorem 2.2, we take $\Omega(\ell,\mathcal{k})=\gamma\mathcal{k}-\ell$ with $\gamma\in(0,1)$, $\mathscr{C}_{\mathscr{A}}=0$ and $\mathscr{A}(\ell,\mathcal{k})=\ell-\mathcal{k}$, and also, $\alpha(\lambda,\zeta)=1$, $\mathcal{F}(s)=s$, $\mathcal{Q}(s)=\frac{s}{2}$ and lastly $\mathcal{S}=I$, which means that

$$\mathcal{E}^*(\varsigma(\lambda),\varpi(\lambda))=\mu_\sigma(\varsigma(\lambda),\varpi(\lambda))+\mu_\sigma(\varsigma(\lambda),\mathcal{P}\varsigma(\lambda))$$

and

$$\mathcal{R}(\varsigma(\lambda),\varpi(\lambda))=\frac{\mu_\sigma(\varsigma(\lambda),\varpi(\lambda))[2\mu_\sigma(\varsigma(\lambda),\mathcal{P}\varsigma(\lambda))+\mu_\sigma(\varsigma(\lambda),\varpi(\lambda))]}{\mu_\sigma(\varsigma(\lambda),\mathcal{P}\varsigma(\lambda))+2\mu_\sigma(\varsigma(\lambda),\varpi(\lambda))}.$$

Thereby, by a simple calculation, Theorem 2.2 turns into

$$\begin{aligned}\mu_\sigma(\mathcal{P}\varsigma(\lambda),\varpi(\lambda))^2&\leq\tfrac{\gamma}{128}\mathcal{E}^*(\varsigma(\lambda),\varpi(\lambda))\mathcal{R}(\varsigma(\lambda),\varpi(\lambda))\\&\leq\tfrac{\gamma}{128}[\mu_\sigma(\varsigma(\lambda),\varpi(\lambda))+\mu_\sigma(\varsigma(\lambda),\mathcal{P}\varsigma(\lambda))\mu_\sigma(\varsigma(\lambda),\varpi(\lambda))].\end{aligned}\tag{4.7}$$

Consequently, from the inequality (4.6), we deduce that

$$\begin{aligned}\mu_\sigma(\mathcal{P}\varsigma(\lambda),\varpi(\lambda))^2&\leq\delta\mu_\sigma(\varsigma(\lambda),\varpi(\lambda))^2\\&\leq\delta[\mu_\sigma(\varsigma(\lambda),\varpi(\lambda))+\mu_\sigma(\varsigma(\lambda),\mathcal{P}\varsigma(\lambda))\mu_\sigma(\varsigma(\lambda),\varpi(\lambda))],\end{aligned}$$

which means that, by taking $\delta=\frac{\gamma}{128}\in(0,1)$, (4.7) is satisfied, that is, all the conditions of Theorem 2.2 are met. Thus, we gain that $\mathcal{P}$ has a fixed point, i.e., the functional equation (4.1) has a bounded solution.

□

Article Information

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed, and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism was detected.

References

[1] S. Banach, *Sur les operations dans les emsembles abstraits et leurs applications aux equations integrales*, Fund. Math., **1** (1922) 133-181.

[2] M. Younis, A. A. N. Abdou, *Novel fuzzy contractions and applications to engineering science*, Fractal Fract., **8** (2024), 28.

[3] M. Younis, D. Singh, A. A. N. Abdou, *A fixed point approach for tuning circuit problem in b-dislocated metric spaces*, Math. Methods Appl. Sci., **45** (2022), 2234–2253.

[4] M. Younis, D. Bahuguna, *A unique approach to graph-based metric spaces with an application to rocket ascension*, Comput. Appl. Math., **42** (2023), 44.

[5] M. Younis, H. Ahmad, L. Chen, M. Han, *Computation and convergence of fixed points in graphical spaces with an application to elastic beam deformations*, J. Geom. Phys. **192** (2023), 104955.

[6] I. A. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Funct. Anal. Unianowsk Gos. Ped. Inst., **30** (1989) 26–37.

[7] S. Czerwik, *Contraction mappings in $b-$metric spaces*, Acta. Math. Inform. Univ. Ostrav., **1**(1) (1993), 5-11.

[8] S. Czerwik, *Nonlinear set-valued contraction mappings in $b-$metric spaces*, Atti Semin. Mat. Fis. Univ.Modena, **46** (1998), 263-276.

[9] A. Aghajani, M. Abbas, J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca, **64**(4) (2014), 941–960.

[10] V. V. Chistyakov,*Modular metric spaces, I: Basic concepts*, Nonlinear Anal., **72** (2010), 1-14.

[11] V. V. Chistyakov, *Modular metric spaces, II: Application to superposition operators*, Nonlinear Anal., **72** (2010), 15-30.

[12] M. E. Ege, C. Alaca, *Some results for modular $b-$metric spaces and an application to a system of linear equations*, Azerbaijan J. Math. **8**(1) (2018), 3-14.

[13] V. Parvaneh, N. Hussain, M. Khorshidi, N. Mlaiki, H. Aydi, *Fixed point results for generalized $F-$contractions in modular $b-$metric spaces with applications*, Mathematics, **7**(10) (2019), 1-16.

[14] M. Öztürk, A. Büyükkaya, *Fixed point results for Suzuki-type $\Sigma-$contractions via simulation functions in modular $b-$metric spaces*, Math Meth Appl Sci., **45** (2022), 12167-12183. https://doi.org/10.1002/mma.7634

[15] A. Büyükkaya, M. Öztürk, *Some fixed point results for Sehgal-Proinov type contractions in modular $b-$metric spaces*, Analele Stiint. Ale Univ. Ovidius Constanta Ser. Mat., **31(3)** (2023), 61-85.

[16] A. Büyükkaya, A. Fulga, M. Öztürk, *On Generalized Suzuki-Proinov type $(\alpha, \mathscr{Z}_E^*)-$contractions in modular $b-$metric spaces*, Filomat, **37**(4) (2023), 1207–1222.

[17] M. Öztürk, F. Golkarmanesh, A. Büyükkaya, V. Parvaneh, *Generalized almost simulative $\hat{Z}^{\Theta}_{\Psi^*}-$contraction mappings in modular $b-$metric spaces*, J. Math. Ext., **17**(2) (2023), 1-37.

[18] F. Khojasteh, S. Shukla, S. Radenovic, *A new approach to the study of fixed point theorems for simulation functions*, Filomat, **29** (2015), 1189-1194.

[19] A. Fulga, A. M. Proca, *Fixed point for Geraghty ϑ_E-contractions*, J. Nonlinear Sci. Appl., **10** (2017), 5125-5131.

[20] A. Fulga, A. M. Proca, *A new Generalization of Wardowski fixed point theorem in complete metric spaces*, Adv. Theory Nonlinear Anal. Appl., **1** (2017), 57-63.

[21] A. Fulga, E. Karapınar, *Some results on $S-$contractions of Type E*, Mathematics, **195** (6) (2018), 1-9.

[22] B. Alqahtani, A. Fulga, E. Karapınar, *A short note on the common fixed points of the Geraghty contraction of type $E_{S,T}$*, Demonstr. Math., **51** (2018), 233-240.

[23] A. Fulga, E. Karapınar, *Some results on $S-$contractions of Type $\mathcal{E}$*, Mathematics, **195**(6) (2018), 1-9.

[24] A. H. Ansari, *Note on "$\varphi-\psi-$contractive type mappings and related fixed point"*, In The 2qd Regional Conference on Mathematics and Applications, Payame Noor University, (2014), 377-380.

[25] S. Radenovic, S. Chandok, *Simulation type functions and coincidence point results*, Filomat, **32** (2018), 141–147.

[26] T. Suzuki, *A new type of fixed point theorem in metric spaces*, Nonlinear Anal., **71** (2009), 5313–5317.

[27] P. Proinov, *Fixed point theorems for generalized contractive mappings in metric spaces*, J. Fixed Point Theory Appl., **22** (2020), 21.

[28] E. Karapınar, A. Fulga, *A Fixed point theorem for Proinov mappings with a contractive iterate*, Appl. Math. J. Chinese Univ. **38** (2023), 403–412.

[29] E. Karapınar, A. Fulga, *Discussions on Proinov$-\mathbf{C}_b-$contraction mapping on $b-$metric space*, J. Funct. Spaces, **ID:1411808** (2023). https://doi.org/10.1155/2023/1411808

[30] E. Karapınar, M. De La Sen, A. Fulga, *A note on the Gornicki-Proinov type contraction*, J. Funct. Spaces **2021** (2021). https://doi.org/10.1155/2021/6686644

[31] E. Karapınar, A. Fulga, S.S. Yesilkaya, *Fixed points of Proinov type multivalued mappings on quasi metric spaces*, J. Funct. Spaces, **ID:7197541** (2022). https://doi.org/10.1155/2022/7197541

[32] E. Karapınar, J. Martinez-Moreno, N. Shahzad, A.F. Roldan Lopez de Hierro, *Extended Proinov $\mathfrak{X}-$contractions in metric spaces and fuzzy metric spaces satisfying the property $\mathcal{NC}$ by avoiding the monotone condition*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **116**(4) (2022), 1-28.

[33] A. F. Roldan Lopez de Hierro, A. Fulga, E. Karapınar, N. Shahzad, *Proinov type fixed point results in non-Archimedean fuzzy metric spaces*, Mathematics, **9**(14) (2021), 1594.

[34] M. Zhou, X. Liu, N. Saleem, A. Fulga, N. Özgür, *A new study on the fixed point sets of Proinov-type contractions via rational forms*, Symmetry, **14**(1) (2022), 93.

[35] M. Zhou, N. Saleem, X. Liu, A. Fulga, A. F. Roldán López de Hierro, *A new approach to Proinov-type fixed point results in Non-archimedean fuzzy metric spaces*, Mathematics, **9**(23) (2021), 3001.

[36] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha-\psi-$contractive mappings*, Nonlinear Anal., **75** (2012), 2154-2165

[37] O. Popescu, *Some new fixed point theorems for $\alpha-$Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl., **2014** (2014), 190.

[38] E. Karapınar, P. Kumam, P. Salimi, *On $\alpha-\varrho-$Meir-Keeler contractive mappings*, Fixed Point Theory Appl., **2013** (2013), 1-12.

[39] H. Qawaqneh, M. S. M. Noorani, W. Shatanawi, H. Alsamir, *Common fixed points for pairs of triangular $\alpha-$admissible mappings*, J. Nonlinear Sci. Appl., **10** (2017), 6192–6204.

Communications in Advanced Mathematical Sciences
Vol. 7, No. 1, 42-55, 2024
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1414791

Mersenne Matrix Operator and Its Application in $p-$Summable Sequence Space

Serkan Demiriz[1]*, Sezer Erdem[2]

Abstract
In this study, it is introduced the regular Mersenne matrix operator which is obtained by using Mersenne numbers and examined sequence spaces described as the domain of this matrix in the space of p-summable sequences for $1 \leq p \leq \infty$. After that, it investigated some properties and inclusion relations, established the Schauder basis, and stated $\alpha-$, $\beta-$, and $\gamma-$duals of the aforementioned spaces. Additionally, it is characterized by the matrix classes from newly described spaces to classical sequence spaces. Finally, we studied the compactness of matrix operators on related sequence spaces.

Keywords: Compact operators, Duals, Matrix transformations, Mersenne numbers, Sequence spaces
2010 AMS: 11B83, 46A45, 46B45, 47B07, 47B37

[1] *Department of Mathematics, Tokat Gaziosmanpaşa University, Tokat, Türkiye, serkandemiriz@gmail.com, ORCID: 0000-0002-4662-6020*
[2] *Department of Basic Engineering Sciences, Malatya Turgut Özal University, Malatya, Türkiye, sezererdem8344@gmail.com, ORCID: 0000-0001-9420-8264*
***Corresponding author**
Received: 4 January 2024, **Accepted:** 13 February 2024, **Available online:** 27 February 2024
How to cite this article: S. Demiriz, S. Erdem, *Mersenne Matrix Operator and Its Application in $p-$Summable Sequence Space*, Commun. Adv. Math. Sci., 7(1) (2024), 42-55.

1. Introduction

Mersenne numbers, named after the French theologian, philosopher, mathematician, music theorist and priest Marin Mersenne, who is known as the father of acoustics, in the first half of the 17th century, have an important place in number theory and computer science. rth Mersenne number m_r is stated by $m_r = 2^r - 1$ with $r \in \mathbb{N}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$ and this is called as the Binet formula of the Mersenne sequence.

The Mersenne numbers m_r can be described by the recurrence relations

$$m_{r+2} = 3m_{r+1} - 2m_r \quad \text{and} \quad \sum_{s=1}^{r} m_s = 2m_r - r.$$

The first 10 terms of the Mersenne sequence are as follows:

$$1, 3, 7, 15, 31, 63, 127, 255, 511, 1023 \ldots.$$

There are prime and non-prime Mersenne numbers, and studies on Mersenne primes have held an important place in the fields of number theory and computer science until today. It is known that if m_r is prime, then r must be a prime, but the its reverse is not true.

Now, we may give basic information about sequence spaces and summability theory. ω represents all real or complex sequence's space and each $\Gamma \subset \omega$ named as sequence space. The spaces ℓ_∞, c, c_0 and ℓ_p $(1 \leq p < \infty)$ express the set of all bounded, convergent, null and convergent p-absolutely summable sequences' well known spaces, respectively. The spaces mentioned above are Banach spaces with $\|u\|_{\ell_\infty} = \|u\|_c = \|u\|_{c_0} = \sup_{r\in\mathbb{N}} |u_r|$ and $\|u\|_{\ell_p} = (\sum_r |u_r|^p)^{\frac{1}{p}}$, where $\sum_r |u_r| = \sum_{r=1}^{\infty} |u_r|$. Moreover, every finite sequences' space is represented by Ω and by cs, cs_0 and bs, we mean the spaces of all convergent, null and bounded series, respectively.

Banach spaces in which all coordinate functionals t_s described with $t_s(u) = u_s$ are continuous are called BK-spaces. Additionally, metric vector spaces in which all coordinate functionals are continuous are called FK-spaces.

Let $e^{(1)} = (1,0,0,\ldots)$, $e^{(2)} = (0,1,0,\ldots)$, $e^{(3)} = (0,0,1,0,\ldots)$,.... If each $u = (u_r) \in \Gamma \subset \omega$ can be expressed uniquely as $u = \sum_r u_r e^r$, in that case, it is said that the BK-space Γ holds the AK-property. The spaces ℓ_p $(1 \le p < \infty)$ and c_0 hold AK-property however the spaces c and ℓ_∞ do not hold.

For an infinite matrix $B = (b_{rs})$ with real entries, B_r represent the rth row for each $r \in \mathbb{N}$. The B-transform of $u = (u_s) \in \omega$ is described by $(Bu)_r = \sum_s b_{rs} u_s$ provided that the series is convergent for each $r \in \mathbb{N}$. If $Bu \in \Psi$, in that case it is said that B is a matrix transformation from Γ to Ψ for all $u \in \Gamma$. The class of every matrices transform Γ to Ψ is represented by $(\Gamma : \Psi)$. Matrix domain of B in Γ is described as

$$\Gamma_B = \{u \in \omega : Bu \in \Gamma\}. \tag{1.1}$$

If Γ and Ψ are two sequence spaces, then the multiplier set $D(\Gamma : \Psi)$ is described as

$$D(\Gamma : \Psi) = \left\{ x = (x_r) \in \omega : xu = (x_r u_r) \in \Psi \quad \text{for all} \quad (u_r) \in \Gamma \right\}.$$

In that case, α-, β- and γ-duals of Γ are described as $\Gamma^\alpha = D(\Gamma : \ell_1)$, $\Gamma^\beta = D(\Gamma : cs)$ and $\Gamma^\gamma = D(\Gamma : bs)$.

Sequences, their spaces and matrix domains have been seen as interesting topics in mathematics by the authors, and in recent years, many studies have been done in this area. Researchers who want to get more detailed information about summability theory, infinite matrices, sequences and their spaces, matrix domains and other related subjects can benefit from the studies [1]-[10] and textbooks [11]-[13].

Special integer sequences have been used extensively in sequence space studies in recent years. In this context, the first study done is the study with a tag [14] made by Başarır and Kara. After this study, some special integer sequences such as Lucas, Padovan, Pell, Leanardo, Catalan, Bell, Schröder and Motzkin were used to define new sequence spaces in summability theory. Researchers who want to get more detailed information about literature can benefit from the studies [15]-[25].

In parallel with the studies mentioned above, this article aims to construct a novel regular matrix operator μ obtained by the aid of Mersenne sequence and examine sequence spaces described as the domain of μ in ℓ_p $(1 \le p \le \infty)$. It is investigated algebraic and topological properties, established Schauder basis and stated $\alpha-$, $\beta-$ and $\gamma-$duals of the aforementioned spaces and additionally, it is featured the matrix classes from new sequence spaces to the classical sequence spaces. At the end, it is studied the compactness of matrix operators on related sequence spaces.

2. Mersenne Matrix Operator and Mersenne Sequence Spaces

It is described the Mersenne matrix operator generated with the help of the Mersenne numbers and it is observed that this aforementioned matrix is regular. After that, we introduced the normed spaces $\ell_p(\mu)$ and $\ell_\infty(\mu)$ and shown that these are complete and linearly isomorphic to ℓ_p and ℓ_∞, respectively, for $1 \le p < \infty$. Then, it is shown that except for the case $p = 2$, $\ell_p(\mu)$ is not a Hilbert space, it is established Schauder basis and to determine the location of the newly defined spaces among the other spaces, it is given the inclusion relations at the end.

Now, we construct the Mersenne matrix operator $\mu = (\mu_{rs})$ with the help of Mersenne numbers as follows:

$$\mu_{rs} := \begin{cases} \dfrac{m_s}{2m_r - r} & , \quad \text{if } 1 \le s \le r, \\ 0 & , \quad \text{if } s > r, \end{cases}$$

for all $r, s \in \mathbb{N}$. The Mersenne matrix μ can be expressed more clearly in the following form:

$$\mu := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & \cdots \\ \frac{1}{11} & \frac{3}{11} & \frac{7}{11} & 0 & 0 & \cdots \\ \frac{1}{26} & \frac{3}{26} & \frac{7}{26} & \frac{15}{26} & 0 & \cdots \\ \frac{1}{57} & \frac{3}{57} & \frac{7}{57} & \frac{15}{57} & \frac{31}{57} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

From its definition, we can understand that μ is a triangle. Moreover, μ-transform of a sequence $u = (u_s)$ is stated as

$$v_r := (\mu u)_r = \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \quad (r \in \mathbb{N}). \tag{2.1}$$

It is known that, an infinite matrix is named as regular if it maps any convergent sequence into a convergent sequence with the same limit.

Lemma 2.1. *An infinite matrix B is regular if and only if the following conditions hold:*

(i) $\sup_{r\in\mathbb{N}} \sum_s |b_{rs}| < \infty$,

(ii) $\lim_{r\to\infty} \sum_s b_{rs} = 1$,

(iii) $\lim_{r\to\infty} b_{rs} = 0$.

Theorem 2.2. *The Mersenne matrix μ is regular.*

Proof. From the equality

$$\sum_s |\mu_{rs}| = \sum_s \mu_{rs} = \sum_{s=1}^{r} \frac{m_s}{2m_r - r} = 1,$$

it is easily seen that the conditions (**i**) and (**ii**) hold. It is reached the validity of the condition (**iii**) from the equality

$$\begin{aligned} \lim_{r\to\infty} \mu_{rs} = \lim_{r\to\infty} \frac{m_s}{2m_r - r} &= m_s . \lim_{r\to\infty} \frac{1}{2m_r - r} \\ &= m_s . \lim_{r\to\infty} \frac{1}{2^{r+1} - r - 2} = 0. \end{aligned}$$

□

Now, let us introduce the sets $\ell_p(\mu)$ and $\ell_\infty(\mu)$ of all Mersenne p-absolutely convergent and Mersenne bounded sequences by

$$\ell_p(\mu) = \left\{ u = (u_s) \in \omega : \sum_{r=1}^{\infty} \left| \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \right|^p < \infty \right\} \quad (1 \le p < \infty)$$

and

$$\ell_\infty(\mu) = \left\{ u = (u_s) \in \omega : \sup_{r\in\mathbb{N}} \left| \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \right| < \infty \right\}.$$

In that case, the sets $\ell_p(\mu)$ can be rewritten as $\ell_p(\mu) = (\ell_p)_\mu$ for $1 \le p \le \infty$ with the notation (1.1). If $\Gamma \subset \omega$ is normed, in that case $\Gamma(\mu)$ is called as a Mersenne sequence space.

Unless otherwise stated in the following parts of the study, $1 \le p < \infty$ will be assumed.

Wilansky [26] proved that, if B is triangle and Γ is BK-space, in that case the domain Γ_B is BK-space too, with $\|u\|_{\Gamma_B} = \|Bu\|_\Gamma$. Therefore, we are ready to give the theorem without proof regarding the BK-spaceness of the sets we just defined.

Theorem 2.3. *$\ell_p(\mu)$ and $\ell_\infty(\mu)$ are BK-spaces with*

$$\|u\|_{\ell_p(\mu)} = \left(\sum_{r=1}^{\infty} \left| \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \right|^p \right)^{\frac{1}{p}}$$

and

$$\|u\|_{\ell_\infty(\mu)} = \sup_{r\in\mathbb{N}} \left| \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \right|,$$

respectively.

Theorem 2.4. *$\ell_p(\mu)$ and $\ell_\infty(\mu)$ are linearly isomorphic to the spaces ℓ_p and ℓ_∞, respectively.*

Proof. Since, it can be shown similarly for the other spaces, the theorem will be proven only for the spaces $\ell_\infty(\mu)$ and ℓ_∞.

For the proof, it must be shown that there is a norm-preserving bijection between the aforementioned spaces. The linearity of the function described for this purpose as $\mathscr{A} : \ell_\infty(\mu) \to \ell_\infty$, $\mathscr{A}(u) = \mu u$ can be seen immediately. Besides this, from the proposition $\mathscr{A}(u) = 0 \Rightarrow u = 0$, $\mathscr{A}$ is decided to be an injection.

By taking into account the sequences $\nu = (\nu_s) \in \ell_\infty$ and $u = (u_s) \in \omega$ whose terms are

$$u_s = \sum_{i=s-1}^{s} (-1)^{s-i} \frac{2m_i - i}{m_s} \nu_i$$

with $u_1 = \nu_1$ for all $s \geq 2$, we reach the surjectivity of $\mathscr{A}$ from the expression

$$\begin{aligned} (\mu u)_r &= \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s u_s \\ &= \frac{1}{2m_r - r} \sum_{s=1}^{r} m_s \sum_{i=s-1}^{s} (-1)^{s-i} \frac{2m_i - i}{m_s} \nu_i \\ &= \nu_r. \end{aligned}$$

Additionally, since the relation $\|u\|_{\ell_\infty(\mu)} = \|\mu u\|_{\ell_\infty}$ holds, then $\mathscr{A}$ keeps the norm. □

Theorem 2.5. *Except for the case $p = 2$, $\ell_p(\mu)$ is not a Hilbert space.*

Proof. If we consider that $x = (1, 1, -\frac{4}{7}, 0, 0, \ldots)$ and $y = (1, -\frac{5}{3}, \frac{4}{7}, 0, 0, \ldots)$, in that case it is obtain that $\mu x = (1, 1, 0, 0, \ldots)$ and $\mu y = (1, -1, 0, 0, \ldots)$ and

$$\|x+y\|^2_{\ell_p(\mu)} + \|x-y\|^2_{\ell_p(\mu)} = 8 \neq 2^{2+\frac{2}{p}} = 2\left(\|x\|^2_{\ell_p(\mu)} + \|y\|^2_{\ell_p(\mu)}\right).$$

Hence, the norm associated with the space $\ell_p(\mu)$ for $p \neq 2$ doesn't hold the parallelogram equality, which is desired result. □

Consider the normed sequence space $(\Gamma, \|.\|)$ and $(\eta_r) \in \Gamma$. In that case, (η_r) is Schauder basis for Γ if for any $u \in \Gamma$, there is a unique scalars' sequence (σ_r) as

$$\left\| u - \sum_{s=1}^{r} \sigma_s \eta_s \right\| \longrightarrow 0$$

as $r \to \infty$ and it is written as $u = \sum_s \sigma_s \eta_s$.

Now, it will be given the result that determines the Schauder basis of $\ell_p(\mu)$. It is concluded that the inverse image of the basis $(e^{(r)})_{r \in \mathbb{N}}$ of ℓ_p composes the basis of $\ell_p(\mu)$ because the function $\mathscr{A} : \ell_p(\mu) \to \ell_p$ described above is an isomorphism. In this way, we can present the following theorem about the Schauder basis without proof.

Theorem 2.6. *Let us consider the sequences $\sigma_s = (\mu u)_s$ and $\eta^{(s)} = \left(\eta_r^{(s)}\right) \in \ell_p(\mu)$ described as*

$$\eta_r^{(s)} := \begin{cases} (-1)^{r-s} \dfrac{2m_s - s}{m_r} & , \quad \textit{if } r-1 \leq s \leq r, \\ 0 & , \quad \textit{otherwise}. \end{cases}$$

In that case; the set $\eta^{(s)}$ is a basis for the space $\ell_p(\mu)$ and the unique representation of any $u \in \ell_p(\mu)$ is stated as $u = \sum_s \sigma_s \eta^{(s)}$ for $1 \leq p < \infty$.

Theorem 2.7. *The inclusion $\ell_p(\mu) \subset \ell_{\tilde{p}}(\mu)$ strictly holds for $1 \leq p < \tilde{p} < \infty$.*

Proof. Consider the sequence $u = (u_s) \in \ell_p(\mu)$ such that $\mu u \in \ell_p$. Furthermore, it is known that $\ell_p \subset \ell_{\tilde{p}}$ for $1 \leq p < \tilde{p} < \infty$ and thus $\mu u \in \ell_{\tilde{p}}$. Consequently, we can write $u = (u_s) \in \ell_{\tilde{p}}(\mu)$.

The strictness of inclusion can be easily seen when $\tilde{\nu} = \mu \tilde{u} \in \ell_{\tilde{p}} \setminus \ell_p$ is taken. □

Theorem 2.8. *The inclusion $\ell_\infty \subset \ell_\infty(\mu)$ holds.*

Proof. By taking a sequence $u=(u_s)\in\ell_\infty$, from the inequality

$$\begin{aligned}\|u\|_{\ell_\infty(\mu)} &= \sup_{r\in\mathbb{N}}\left|\frac{1}{2m_r-r}\sum_{s=1}^{r}m_su_s\right|\\ &\leq \|u\|_\infty\sup_{r\in\mathbb{N}}\left|\frac{1}{2m_r-r}\sum_{s=1}^{r}m_s\right|\\ &= \|u\|_\infty<\infty,\end{aligned}$$

it is reached that $u\in\ell_\infty(\mu)$, which is desired result. □

Theorem 2.9. *The inclusion $\ell_p\subset\ell_p(\mu)$ holds.*

Proof. By taking a sequence $u=(u_s)\in\ell_p$ for $1<p<\infty$, from the inequality

$$\begin{aligned}\sum_{r=1}^{\infty}|(\mu u)_r|^p &\leq \sum_{r=1}^{\infty}\left(\sum_{s=1}^{r}\frac{m_s}{2m_r-r}|u_s|\right)^p\\ &\leq \sum_{r=1}^{\infty}\left(\sum_{s=1}^{r}\frac{m_s}{2m_r-r}|u_s|^p\right)\left(\sum_{s=1}^{r}\frac{m_s}{2m_r-r}\right)^{p-1}\\ &= \sum_{r=1}^{\infty}\left(\sum_{s=1}^{r}\frac{m_s}{2m_r-r}|u_s|^p\right)\\ &= \sum_{s=1}^{\infty}|u_s|^p\left(\sum_{r=s}^{\infty}\frac{m_s}{2m_r-r}\right),\end{aligned}$$

we reach that $\|u\|^p_{\ell_p(\mu)}\leq N.\|u\|^p_{\ell_p}$ for $N=\sup_{s\in\mathbb{N}}\left\{\sum_{r=s}^{\infty}\frac{m_s}{2m_r-r}\right\}$. This implies that $u\in\ell_p(\mu)$ and $\ell_p\subset\ell_p(\mu)$. It can be shown that $\ell_1\subset\ell_1(\mu)$ similarly. □

3. Dual Spaces

It will be calculated duals of the spaces $\ell_p(\mu)$ in the current part. Since, the following results related the duals can be seen similar to the case $1<p\leq\infty$, the proofs of results involving the case $p=1$ will be omitted. In the rest of the paper, unless otherwise stated, $q=\frac{p}{p-1}$ will be assumed and $\mathscr{F}$ will represented the family of all finite subsets of $\mathbb{N}$.

For the determination of duals, it may be given the following lemmas collected from the study [27] to characterize some classical matrix classes:

Lemma 3.1. *For $1<p\leq\infty$, $B=(b_{rs})\in(\ell_p:\ell_1)$ if and only if*

$$\sup_{E\in\mathscr{F}}\sum_{s=1}^{\infty}\left|\sum_{r\in E}b_{rs}\right|^q<\infty.$$

Lemma 3.2. *For $1<p<\infty$, $B=(b_{rs})\in(\ell_p:c)$ if and only if*

$$\lim_{r\to\infty}b_{rs}\text{ exists for all }s\in\mathbb{N},\tag{3.1}$$

$$\sup_{r\in\mathbb{N}}\sum_{s=1}^{\infty}|b_{rs}|^q<\infty.\tag{3.2}$$

Lemma 3.3. *$B=(b_{rs})\in(\ell_\infty:c)$ if and only if the conditions (3.1),*

$$\sup_{r\in\mathbb{N}}\sum_{s=1}^{\infty}|b_{rs}|<\infty,$$
$$\lim_{r\to\infty}\sum_{s=1}^{\infty}\left|b_{rs}-\lim_{r\to\infty}b_{rs}\right|=0$$

hold.

Lemma 3.4. *$B=(b_{rs})\in(\ell_p:\ell_\infty)$ if and only if (3.2) holds for $1<p\leq\infty$.*

Theorem 3.5. *Let us consider the set ϖ_1 and the infinite matrix $G=(g_{rs})$ described by*

$$\varpi_1=\left\{\tau=(\tau_s)\in\omega:\sup_{E\in\mathscr{F}}\sum_{s=1}^{\infty}\left|\sum_{r\in E}g_{rs}\right|^q<\infty\right\}$$

and

$$g_{rs}:=\begin{cases}(-1)^{r-s}\dfrac{2m_s-s}{m_r}\tau_r & , \quad \text{if } r-1\leq s\leq r,\\ \\ 0 & , \quad \text{otherwise}.\end{cases}$$

In that case; $[\ell_p(\mu)]^\alpha=\varpi_1$ for $1<p\leq\infty$.

Proof. By using the equality (2.1), we obtain that

$$\begin{aligned}\tau_r u_r &= \tau_r\left(\sum_{s=r-1}^{r}(-1)^{r-s}\frac{2m_s-s}{m_r}v_s\right)\\ &= \sum_{s=r-1}^{r}\left((-1)^{r-s}\frac{2m_s-s}{m_r}\tau_r\right)v_s=(Gv)_r\end{aligned}\tag{3.3}$$

for all $r\in\mathbb{N}$. Hence, it is obtained by the relation (3.3) that $\tau u=(\tau_r u_r)\in\ell_1$ when $u\in\ell_p(\mu)$ if and only if $Gv\in\ell_1$ when $v\in\ell_p$. In that case, it is reached the biconditional statement $\tau\in[\ell_p(\mu)]^\alpha$ if and only if $G\in(\ell_p:\ell_1)$. By taking into consideration the condition of Lemma 3.1 with together $G=(g_{rs})$ in place of $B=(b_{rs})$, it is seen that $[\ell_p(\mu)]^\alpha=\varpi_1$ for $1<p\leq\infty$, which is desired result. □

Theorem 3.6. *Let us consider the sets $\varpi_2^{(q)}$, ϖ_3 and ϖ_4 by*

$$\begin{aligned}\varpi_2^{(q)} &= \left\{\tau=(\tau_s)\in\omega:\sum_{s=1}^{\infty}\left|(2m_s-s)\left(\frac{\tau_s}{m_s}-\frac{\tau_{s+1}}{m_{s+1}}\right)\right|^q<\infty\right\},\\ \varpi_3 &= \left\{\tau=(\tau_s)\in\omega:\sup_{r\in\mathbb{N}}\left|\frac{2m_r-r}{m_r}\tau_r\right|<\infty\right\},\\ \varpi_4 &= \left\{\tau=(\tau_s)\in\omega:\lim_{r\to\infty}\frac{2m_r-r}{m_r}\tau_r=0\right\}.\end{aligned}$$

In that case; $[\ell_p(\mu)]^\beta=\varpi_2^{(q)}\cap\varpi_3$ for $1<p<\infty$ and $[\ell_\infty(\mu)]^\beta=\varpi_2^{(1)}\cap\varpi_4$.

Proof. Let us choose two sequences $\tau=(\tau_s)\in\omega$ and $u\in\ell_p(\mu)$ such that $v\in\ell_p$ with the relation (2.1). Then, we reach that

$$\begin{aligned}\psi_r=\sum_{s=1}^{r}\tau_s u_s &= \sum_{s=1}^{r}\tau_s\left(\sum_{i=s-1}^{s}(-1)^{s-i}\frac{2m_i-i}{m_s}v_i\right)\\ &= \sum_{s=1}^{r-1}(2m_s-s)\left(\frac{\tau_s}{m_s}-\frac{\tau_{s+1}}{m_{s+1}}\right)v_s+\frac{2m_r-r}{m_r}\tau_r v_r\\ &= (Ov)_r\end{aligned}\tag{3.4}$$

where the matrix $O=(o_{rs})$ is described as

$$o_{rs}:=\begin{cases}(2m_s-s)\left(\frac{\tau_s}{m_s}-\frac{\tau_{s+1}}{m_{s+1}}\right) & , \quad 1\leq s<r,\\ \frac{2m_r-r}{m_r}\tau_r & , \quad s=r,\\ 0 & , \quad \text{otherwise.}\end{cases} \tag{3.5}$$

It can be checked that

$$\lim_{r\to\infty} o_{rs}=(2m_s-s)\left(\frac{\tau_s}{m_s}-\frac{\tau_{s+1}}{m_{s+1}}\right). \tag{3.6}$$

In that case, from the relation (3.4), it is infered that $\tau u\in cs$ whenever $u=(u_s)\in\ell_p(\mu)$ if and only if $\psi=(\psi_r)\in c$ when $v\in\ell_p$. Thus, $\tau\in[\ell_p(\mu)]^\beta$ if and only if $O\in(\ell_p:c)$ for $1<p<\infty$. Hence, in view of (3.4), (3.6) and the conditions of Lemma 3.2, it is reached that

$$\sum_{s=1}^{\infty}\left|(2m_s-s)\left(\frac{\tau_s}{m_s}-\frac{\tau_{s+1}}{m_{s+1}}\right)\right|^q<\infty \quad \text{and} \quad \sup_{r\in\mathbb{N}}\left|\frac{2m_r-r}{m_r}\tau_r\right|<\infty$$

which is desired result.

It can be shown similarly for the case $p=\infty$ by the aid of Lemma 3.3 and the relations (3.4) and (3.6). □

Theorem 3.7. *For* $1<p\leq\infty$, $[\ell_p(\mu)]^\gamma=\varpi_2^{(q)}\cap\varpi_4$.

Proof. It can be obtained with similar approach in the proof of the Theorem 3.6 by considering with together the Lemma 3.4 with the matrix $O=(o_{rs})$ described by (3.5). □

4. Matrix Transformations

Current part aims to present the matrix classes $(\ell_p(\mu):\Psi)$, where $\Psi\in(\ell_\infty,c,c_0)$ and $1\leq p\leq\infty$. For brevity, we take

$$\phi_{rs}=(2m_s-s)\left(\frac{b_{rs}}{m_s}-\frac{b_{r,s+1}}{m_{s+1}}\right) \tag{4.1}$$

in the rest for infinite matrices $\Phi=(\phi_{rs})$ and $B=(b_{rs})$ and $r,s\in\mathbb{N}$.

Consider that u and v with the relation (2.1). In that case, it is reached that

$$\sum_{s=1}^{n}b_{rs}u_s=\sum_{s=1}^{n-1}\phi_{rs}v_s+\frac{2m_n-n}{m_n}b_{rn}v_n. \tag{4.2}$$

Now, it may be given the following conditions to characterize new matrix classes:

$$\left(\frac{2m_s-s}{m_s}b_{rs}\right)_{s=1}^{\infty}\in\ell_\infty \text{ for all } r\in\mathbb{N}, \tag{4.3}$$

$$\sup_{r,s\in\mathbb{N}}|\phi_{rs}|<\infty, \tag{4.4}$$

$$\sup_{r\in\mathbb{N}}\sum_{s=1}^{\infty}|\phi_{rs}|^q<\infty, \tag{4.5}$$

$$\left(\frac{2m_s-s}{m_s}b_{rs}\right)_{s=1}^{\infty}\in c_0 \text{ for all } r\in\mathbb{N}, \tag{4.6}$$

$$\lim_{r\to\infty}\phi_{rs} \text{ exists for all } s\in\mathbb{N}, \tag{4.7}$$

$$\lim_{r\to\infty}\sum_{s=1}^{\infty}|\phi_{rs}-\rho_s|=0 \text{ for all } s\in\mathbb{N} \text{ and } (\rho_s)\in\omega, \tag{4.8}$$

$$\lim_{r\to\infty}|\phi_{rs}|=0 \text{ for all } s\in\mathbb{N}. \tag{4.9}$$

In that case; from the conditions of the matrix classes in [27] with together Theorem 3.6 and the relation (4.2), it may be given the following results:

Theorem 4.1. *The following statements hold:*

(i) $B=(b_{rs})\in(\ell_1(\mu):\ell_\infty)$ *if and only if (4.3) and (4.4) hold.*

(ii) $B=(b_{rs})\in(\ell_1(\mu):c)$ *if and only if (4.3), (4.4) and (4.7) hold.*

(iii) $B=(b_{rs})\in(\ell_1(\mu):c_0)$ *if and only if (4.3), (4.4) and (4.9) hold.*

Theorem 4.2. *For $1<p<\infty$, the following statements hold:*

(i) $B=(b_{rs})\in(\ell_p(\mu):\ell_\infty)$ *if and only if (4.3) and (4.5) hold.*

(ii) $B=(b_{rs})\in(\ell_p(\mu):c)$ *if and only if (4.3), (4.5) and (4.7) hold.*

(iii) $B=(b_{rs})\in(\ell_p(\mu):c_0)$ *if and only if (4.3), (4.5) and (4.9) hold.*

Theorem 4.3. *The following statements hold:*

(i) $B=(b_{rs})\in(\ell_\infty(\mu):\ell_\infty)$ *if and only if (4.5) and (4.6) hold with $q=1$.*

(ii) $B=(b_{rs})\in(\ell_\infty(\mu):c)$ *if and only if (4.5), (4.6), (4.7) and (4.8) hold with $q=1$.*

(iii) $B=(b_{rs})\in(\ell_\infty(\mu):c_0)$ *if and only if (4.6) and (4.8) hold for $\rho_s=0$ and $s\in\mathbb{N}$.*

5. Compactness by Hausdorff Measure of Non-compactness

This part aims to acquire the necessary and sufficient conditions for an operator to be compact from $\ell_p(\mu)$ to the space Ψ, where $1\le p\le\infty$ and $\Psi\in\{c_0,c,\ell_\infty,\ell_1,cs_0,cs,bs\}$.

For a normed space Γ, $\mathscr{D}_\Gamma$ represents the unit sphere in Γ. It is used the notation

$$\|u\|_\Gamma^\diamond=\sup_{x\in\mathscr{D}_\Gamma}\left|\sum_s u_s x_s\right|$$

for a BK-space $\Gamma\supset\Omega$ and $u=(u_s)\in\omega$, where Ω represents all finite sequences's space and it is assumed that the series above is exists and then it is reached that $u\in\Gamma^\beta$.

Lemma 5.1. *[28] The following statements hold:*

(i) $\ell_\infty^\beta=c^\beta=c_0^\beta=\ell_1$ *and* $\|u\|_\Gamma^\diamond=\|u\|_{\ell_1}$ *for all $u\in\ell_1$ and $\Gamma\in\{\ell_\infty,c,c_0\}$.*

(ii) $\ell_1^\beta=\ell_\infty$ *and* $\|u\|_{\ell_1}^\diamond=\|u\|_{\ell_\infty}$ *for all $u\in\ell_\infty$.*

(iii) $\ell_p^\beta=\ell_q$ *and* $\|u\|_{\ell_p}^\diamond=\|u\|_{\ell_q}$ *for all $u\in\ell_q$.*

The set $\mathfrak{B}(\Gamma:\Psi)$ represents all bounded (continuous) linear operators' set from Γ to Ψ.

Lemma 5.2. *[28] Let Γ and Ψ are the BK-spaces. In that case, for all $B\in(\Gamma:\Psi)$, there exists a linear operator $\mathscr{L}_B\in\mathfrak{B}(\Gamma:\Psi)$ as $\mathscr{L}_B(u)=Bu$ for every $u\in\Gamma$.*

Lemma 5.3. *[28] Consider that $\Gamma\supset\Omega$ is a BK-space. If $B\in(\Gamma:\Psi)$, in that case $\|\mathscr{L}_B\|=\|B\|_{(\Gamma:\Psi)}=\sup_{r\in\mathbb{N}}\|B_r\|_\Gamma^\diamond<\infty$.*

Let us consider a metric space Γ and $A\subset\Gamma$ is bounded. The Hausdorff measure of non-compactness of A is represented with $\chi(A)$ and it is described by

$$\chi(A)=\inf\left\{\varepsilon>0:A\subset\cup_{j=1}^r A(u_j,n_j),u_j\in\Gamma,n_j<\varepsilon,r\in\mathbb{N}\right\},$$

where $A(u_j,n_j)$ is the open ball centred at u_j and radius n_j for each $j=1,2,\ldots,r$. Researchers who want to get more detailed information about Hausdorff measure of non-compactness can benefit from [28] and its references.

Theorem 5.4. *[29] Let $A\subset\ell_p$ is bounded and the operator $\lambda_n:\ell_p\longrightarrow\ell_p$ described as $\lambda_n(u)=(u_1,u_2,u_3,\ldots,u_n,0,0,\ldots)$ for every $u=(u_s)\in\ell_p$, $1\le p<\infty$ and each $n\in\mathbb{N}$. In that case, for the identity operator I on ℓ_p, it is reached that*

$$\chi(A)=\lim_{n\to\infty}\left(\sup_{u\in A}\|(I-\lambda_n)(u)\|_{\ell_p}\right).$$

For the Banach spaces Γ and Ψ, a linear operator $\mathscr{L}:\Gamma\to\Psi$ is named as compact operator if domain of $\mathscr{L}$ is whole of Γ and $\mathscr{L}(A)$ is totally bounded set in Ψ for all $u=(u_s)\in\ell_\infty\cap\Gamma$. Equivalently, $\mathscr{L}$ is compact if $(\mathscr{L}(u))$ has a convergent subsequence in Ψ for all $u=(u_s)\in\ell_\infty\cap\Gamma$.

Let $\|\mathscr{L}\|\chi$ represents Hausdorff measure of non-compactness of $\mathscr{L}$ and it is described by $\|\mathscr{L}\|\chi=\chi(\mathscr{L}(\mathscr{D}_\Gamma))$. The notions Hausdorff measure of non-compactness and compact operators have a distinct relationship of "$\mathscr{L}$ is compact if and only if $\|\mathscr{L}\|\chi=0$".

Readers can use the studies [30, 31, 32, 33, 34, 35] for sequence space studies where Hausdorff measure of non-compactness is used to determine compact operators between BK-spaces.

Lemma 5.5. *[30] Let $\Gamma\supset\Omega$ is BK-space. In that case:*

(i) *If $B\in(\Gamma:c_0)$, then $\|\mathscr{L}_B\|_\chi=\limsup_r\|B_r\|_\Gamma^\diamond$ and $\mathscr{L}_B$ is compact if and only if $\lim_r\|B_r\|_\Gamma^\diamond=0$.*

(ii) *If Γ has AK property or $\Gamma=\ell_\infty$ and $B\in(\Gamma:c)$, then*

$$\frac{1}{2}\limsup_r\|B_r-\kappa\|_\Gamma^\diamond\le\|\mathscr{L}_B\|_\chi\le\limsup_r\|B_r-\kappa\|_\Gamma^\diamond$$

and $\mathscr{L}_B$ is compact if

$$\lim_r\|B_r-\kappa\|_\Gamma^\diamond=0$$

where $\kappa=(\kappa_s)$ and $\kappa_s=\lim_r b_{rs}$.

(iii) *If $B\in(\Gamma:\ell_\infty)$, then $0\le\|\mathscr{L}_B\|_\chi\le\limsup_r\|B_r\|_\Gamma^\diamond$ and $\mathscr{L}_B$ is compact if $\lim_r\|B_r\|_\Gamma^\diamond=0$.*

(iv) *If $B\in(\Gamma:\ell_1)$, then*

$$\lim_j\left(\sup_{E\in\mathscr{F}_j}\left\|\sum_{r\in E}B_r\right\|_\Gamma^\diamond\right)\le\|\mathscr{L}_\mathscr{B}\|_\chi\le 4.\lim_j\left(\sup_{E\in\mathscr{F}_j}\left\|\sum_{r\in E}B_r\right\|_\Gamma^\diamond\right)$$

and $\mathscr{L}_\mathscr{B}$ is compact if and only if $\lim_j\left(\sup_{E\in\mathscr{F}_j}\|\sum_{r\in E}B_r\|_\Gamma^\diamond\right)=0$, where $\mathscr{F}$ represents the family of all finite subsets of $\mathbb{N}$ and $\mathscr{F}_j$ is the subcollection of $\mathscr{F}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than j.

In the sequel of the study, it is used the matrices $\Phi=(\phi_{rs})$ and $B=(b_{rs})$ connected with the relation (4.1).

Lemma 5.6. *Let $\Psi\subset\omega$. If $B\in(\ell_p(\mu):\Psi)$, then $\Phi\in(\ell_p:\Psi)$ and $Bu=\Phi v$ hold for all $u\in\ell_p(\mu)$ and $1\le p\le\infty$.*

Theorem 5.7. *Let $1<p<\infty$. In that case:*

(i) *If $B\in(\ell_p(\mu):c_0)$, then $\|\mathscr{L}_B\|_\chi=\limsup_r(\sum_s|\phi_{rs}|^q)^{\frac{1}{q}}$ and $\mathscr{L}_B$ is compact if and only if $\lim_r(\sum_s|\phi_{rs}|^q)^{\frac{1}{q}}=0$.*

(ii) *If $B\in(\ell_p(\mu):c)$, then*

$$\frac{1}{2}\limsup_r\left(\sum_s|\phi_{rs}-a_s|^q\right)^{\frac{1}{q}}\le\|\mathscr{L}_B\|_\chi\le\limsup_r\left(\sum_s|\phi_{rs}-a_s|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if and only if $\lim_r(\sum_s|\phi_{rs}-a_s|^q)^{\frac{1}{q}}=0$, where $a_s=\lim_r\phi_{rs}$.

(iii) *If $B\in(\ell_p(\mu):\ell_\infty)$, then $0\le\|\mathscr{L}_B\|_\chi\le\limsup_r(\sum_s|\phi_{rs}|^q)^{\frac{1}{q}}$ and $\mathscr{L}_B$ is compact if $\lim_r(\sum_s|\phi_{rs}|^q)^{\frac{1}{q}}=0$.*

(iv) *If $B\in(\ell_p(\mu):\ell_1)$, then $\lim_j\|B\|_{(\ell_p(\mu):\ell_1)}^{(j)}\le\|\mathscr{L}_B\|_\chi\le 4.\lim_j\|B\|_{(\ell_p(\mu):\ell_1)}^{(j)}$ and $\mathscr{L}_B$ is compact if and only if $\lim_j\|B\|_{(\ell_p(\mu):\ell_1)}^{(j)}=0$, where $\|B\|_{(\ell_p(\mu):\ell_1)}^{(j)}=\sup_{E\in\mathscr{F}_j}(\sum_s|\sum_{r\in E}\phi_{rs}|^q)^{\frac{1}{q}}$.*

Proof. **(i)** Let $B \in (\ell_p(\mu) : c_0)$. It is seen that

$$\|B_r\|^{\diamond}_{\ell_p(\mu)} = \|\Phi_r\|^{\diamond}_{\ell_p} = \|\Phi_r\|_{\ell_q} = \left(\sum_s |\phi_{rs}|^q\right)^{\frac{1}{q}}.$$

Thus, in view of Lemma 5.5-(i), it is reached that

$$\|\mathscr{L}_B\|_\chi = \limsup_r \|B_r\|^{\diamond}_{\ell_p(\mu)} = \limsup_r \left(\sum_s |\phi_{rs}|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if $\lim_r (\sum_s |\phi_{rs}|^q)^{\frac{1}{q}}$.

(ii) Let $B \in (\ell_p(\mu) : c)$. In that case, $\Phi \in (\ell_p : c)$ by Lemma 5.6. From Lemma 5.1-(iii) it is concluded that

$$\|\Phi_r - a\|^{\diamond}_{\ell_p} = \|\Phi_r - a\|_{\ell_q} = \left(\sum_s |\phi_{rs} - a_s|^q\right)^{\frac{1}{q}}. \tag{5.1}$$

By the aid of the Lemma 5.5-(ii), it is reached that

$$\frac{1}{2}\limsup_r \|\Phi_r - a\|^{\diamond}_{\ell_p} \leq \|\mathscr{L}_B\|_\chi \leq \limsup_r \|\Phi_r - a\|^{\diamond}_{\ell_p}. \tag{5.2}$$

Then, considering (5.1) and (5.2) together, it is obtained that

$$\frac{1}{2}\limsup_r \left(\sum_s |\phi_{rs} - a_s|^q\right)^{\frac{1}{q}} \leq \|\mathscr{L}_B\|_\chi \leq \limsup_r \left(\sum_s |\phi_{rs} - a_s|^q\right)^{\frac{1}{q}}.$$

Moreover, it is seen by Lemma 5.5-(ii) that $\mathscr{L}_B$ is compact if and only if

$$\lim_r \left(\sum_s |\phi_{rs} - a_s|^q\right)^{\frac{1}{q}} = 0.$$

(iii) This proof can be made analogous to that of (i) and (ii) considering Lemma 5.5-(iii).

(iv) It is reached that

$$\|\sum_{r\in E} B_r\|^{\diamond}_{\ell_p(\mu)} = \|\sum_{r\in E} \Phi_r\|^{\diamond}_{\ell_p} = \|\sum_{r\in E} \Phi_r\|^{\diamond}_{\ell_q} = \left(\sum_s \left|\sum_{r\in E} \phi_{rs}\right|^q\right)^{\frac{1}{q}}.$$

Let $B \in (\ell_p(\mu) : \ell_1)$, then by Lemma 5.6, $\Phi \in (\ell_p : \ell_1)$ holds. In that case, by taking account the Lemma 5.5-(iv), it is concluded that

$$\lim_j \left(\sup_{E\in\mathscr{F}_j} \sum_s \left|\sum_{r\in E} \phi_{rs}\right|^q\right)^{\frac{1}{q}} \leq \|\mathscr{L}_B\|_\chi \leq 4. \lim_j \left(\sup_{E\in\mathscr{F}_j} \sum_s \left|\sum_{r\in E} \phi_{rs}\right|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if and only if

$$\lim_j \left(\sup_{E\in\mathscr{F}_j} \sum_s \left|\sum_{r\in E} \phi_{rs}\right|^q\right)^{\frac{1}{q}} = 0.$$

□

Lemma 5.8. *[30] Let $\Gamma \supset \Omega$ is BK-space and*

$$\|B\|^{[r]}_{(\Gamma:bs)} = \left\|\sum_{n=1}^{r} B_n\right\|^{\diamond}_{\Gamma}.$$

In that case:

(i) *If $B \in (\Gamma : cs_0)$, then $\|\mathscr{L}_B\|_\chi = \limsup_r \|B\|^{[r]}_{(\Gamma:bs)}$ and $\mathscr{L}_B$ is compact if and only if $\lim_r \|B\|^{[r]}_{(\Gamma:bs)} = 0$.*

(ii) *If Γ has AK and $B \in (\Gamma : cs)$, in that case*

$$\frac{1}{2}\limsup_r \left\|\sum_{n=1}^{r} B_n - \xi\right\|_{\Gamma}^{\diamond} \leq \|\mathscr{L}_B\|_{\chi} \leq \limsup_r \left\|\sum_{n=1}^{r} B_n - \xi\right\|_{\Gamma}^{\diamond}$$

and $\mathscr{L}_B$ is compact if and only if $\lim_r \|\sum_{n=1}^{r} B_n - \xi\|_{\Gamma}^{\diamond} = 0$, where $\xi = \xi_s$ with $\xi_s = \lim_{r\to\infty}\sum_{n=1}^{r} b_{ns}$ for each $s \in \mathbb{N}$.

(iii) *If $B \in (\Gamma : bs)$, then $0 \leq \|\mathscr{L}_B\|_{\chi} \leq \limsup_r \|B\|_{(\Gamma:bs)}^{[r]}$ and $\mathscr{L}_B$ is compact if $\lim_r \|B\|_{(\Gamma:bs)}^{[r]} = 0$.*

Theorem 5.9. *Let $1 < p < \infty$. In that case:*

(i) *If $B \in (\ell_p(\mu) : cs_0)$, then $\|\mathscr{L}_B\|_{\chi} = \limsup_r (\sum_s |\sum_{n=1}^{r} \phi_{rs}|^q)^{\frac{1}{q}}$ and $\mathscr{L}_B$ is compact if and only if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{rs}|^q)^{\frac{1}{q}} = 0$.*

(ii) *If $B \in (\ell_p(\mu) : cs)$, then*

$$\frac{1}{2}\limsup_r \left(\sum_s \left|\sum_{n=1}^{r} \phi_{ns} - \tilde{a}_s\right|^q\right)^{\frac{1}{q}} \leq \|\mathscr{L}_B\|_{\chi} \leq \limsup_r \left(\sum_s \left|\sum_{n=1}^{r} \phi_{ns} - \tilde{a}_s\right|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if and only if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{ns} - \tilde{a}_s|^q)^{\frac{1}{q}} = 0$, where $\tilde{a} = (\tilde{a}_s)$ and $\tilde{a}_s = \lim_r \sum_{n=1}^{r} \phi_{ns}$.

(iii) *If $B \in (\ell_p(\mu) : bs)$, then*

$$0 \leq \|\mathscr{L}_B\|_{\chi} \leq \limsup_r \left(\sum_s \left|\sum_{n=1}^{r} \phi_{ns}\right|^q\right)^{\frac{1}{q}}$$

and $\mathscr{L}_B$ is compact if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{ns}|^q)^{\frac{1}{q}} = 0$.

Proof. **(i)** It is clear that

$$\left\|\sum_{n=1}^{r} B_n\right\|_{\ell_p(\mu)}^{\diamond} = \left\|\sum_{n=1}^{r} \Phi_n\right\|_{\ell_p}^{\diamond} = \left\|\sum_{n=1}^{r} \phi_{ns}\right\|_{\ell_q}^{\diamond} = \left(\sum_s \left|\sum_{n=1}^{r} \phi_{ns}\right|^q\right)^{\frac{1}{q}}.$$

Hence, by using Lemma 5.8-(i), it is obtained that $\|\mathscr{L}_B\|_{\chi} = \limsup_r (\sum_s |\sum_{n=1}^{r} \phi_{ns}|^q)^{\frac{1}{q}}$ and $\mathscr{L}_B$ is compact if and only if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{ns}|^q)^{\frac{1}{q}}$.
(ii) We have

$$\left\|\sum_{n=1}^{r} \Phi_n - \tilde{a}\right\|_{\ell_p}^{\diamond} = \left\|\sum_{n=1}^{r} \Phi_n - \tilde{a}\right\|_{\ell_q} = \left(\sum_s \left|\sum_{n=1}^{r} \phi_{ns} - \tilde{a}\right|^q\right)^{\frac{1}{q}}. \tag{5.3}$$

If $B \in (\ell_p(\mu) : cs)$, in that case by Lemma 5.6, it is reached that $\Phi \in (\ell_p : cs)$. In that case, by the aid of the Lemma 5.8-(b), it is deduced that

$$\frac{1}{2}\limsup_r \left\|\sum_{n=1}^{r} \Phi_n - \tilde{a}\right\|_{\ell_p}^{\diamond} \leq \|\mathscr{L}_B\|_{\chi} \leq \limsup_r \left\|\sum_{n=1}^{r} \Phi_n - \tilde{a}\right\|_{\ell_p}^{\diamond},$$

which on using (5.3) gives us

$$\frac{1}{2}\limsup_r \left(\sum_s \left|\sum_{n=1}^{r} \phi_{ns} - \tilde{a}\right|^q\right)^{\frac{1}{q}} \leq \|\mathscr{L}_B\|_{\chi} \leq \limsup_r \left(\sum_s \left|\sum_{n=1}^{r} \phi_{ns} - \tilde{a}\right|^q\right)^{\frac{1}{q}}$$

and also, $\mathscr{L}_B$ is compact if and only if $\lim_r (\sum_s |\sum_{n=1}^{r} \phi_{ns} - \tilde{a}_s|^q)^{\frac{1}{q}} = 0$.
(iii) It can be done similarly to the proof of the first part, considering Lemma 5.8-(iii). □

Theorem 5.10. **(i)** *If $B \in (\ell_\infty(\mu) : c_0)$, in that case $\|\mathscr{L}_B\|_{\chi} = \limsup_r \sum_s |\phi_{rs}|$ and $\mathscr{L}_B$ is compact if $\lim_r \sum_s |\phi_{rs}| = 0$.*

(ii) *If* $B \in (\ell_\infty(\mu) : c)$, *then*

$$\frac{1}{2}\limsup_{r}\left(\sum_{s}|\phi_{rs}-a_s|\right) \leq \|\mathscr{L}_B\|_\chi \leq \limsup_{r}\left(\sum_{s}|\phi_{rs}-a_s|\right)$$

and $\mathscr{L}_B$ *is compact if and only if* $\lim_r\left(\sum_s|\phi_{rs}-a_s|\right)=0$.

(iii) *If* $B \in (\ell_\infty(\mu) : \ell_\infty)$, *in that case* $0 \leq \|\mathscr{L}_B\|_\chi \leq \limsup_r \sum_s |\phi_{rs}|$ *and* $\mathscr{L}_B$ *is compact if* $\lim_r \sum_s |\phi_{rs}| = 0$.

(iv) *If* $B \in (\ell_\infty(\mu) : \ell_1)$, *then* $\lim_j \|B\|^{(j)}_{(\ell_\infty(\mu):\ell_1)} \leq \|\mathscr{L}_B\|_\chi \leq 4.\lim_j \|B\|^{(j)}_{(\ell_\infty(\mu):\ell_1)}$ *and* $\mathscr{L}_B$ *is compact if and only if* $\lim_j \|B\|^{(j)}_{(\ell_\infty(\mu):\ell_1)} = 0$, *where* $\|B\|^{(j)}_{(\ell_\infty(\mu):\ell_1)} = \sup_{E\in\mathscr{F}_j}\left(\sum_s\left|\sum_{r\in E}\phi_{rs}\right|\right)$ *for all* $j \in \mathbb{N}$.

Proof. It can be obtained in a similar way to the proof of Theorem 5.7. So, it is omitted. □

Theorem 5.11. **(i)** *If* $B \in (\ell_1(\mu) : c_0)$, *in that case* $\|\mathscr{L}_B\|_\chi = \limsup_r\left(\sup_s|\phi_{rs}|\right)$ *and* $\mathscr{L}_B$ *is compact if and only if* $\lim_r\left(\sup_s|\phi_{rs}|\right) = 0$.

(ii) *If* $B \in (\ell_1(\mu) : c)$, *in that case*

$$\frac{1}{2}\limsup_{r}\left(\sup_{s}|\phi_{rs}-a_s|\right) \leq \|\mathscr{L}_B\|_\chi \leq \limsup_{r}\left(\sup_{s}|\phi_{rs}-a_s|\right)$$

and $\mathscr{L}_B$ *is compact if and only if* $\lim_r\left(\sup_s|\phi_{rs}-a_s|\right)=0$.

(iii) *If* $B \in (\ell_1(\mu) : \ell_\infty)$, *in that case* $0 \leq \|\mathscr{L}_B\|_\chi \leq \limsup_r\left(\sup_s|\phi_{rs}|\right)$ *and* $\mathscr{L}_B$ *is compact if* $\lim_r\left(\sup_s|\phi_{rs}|\right) = 0$.

Proof. It can be acquired in a analogous procedure of Theorem 5.7. Thence, it is omitted, too. □

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

References

[1] B. Altay, F. Başar, *Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space*, J. Math. Anal. Appl., **336** (2007), 632-645.

[2] T. Yaying, B. Hazarika, M. İlkhan, M.Mursaleen, *Poisson like matrix operator and its application in $p-$summable space*, Math. Slovaca, **71**(5) (2021), 1189-1210.

[3] T. Yaying, *On Λ-Fibonacci difference sequence spaces of fractional order*, Dera Natung Government College Research Journal, **6**(1) (2021), 92-102. https://doi.org/10.56405/dngcrj.2021.06.01.10

[4] M. Karakaş, M.C. Dağlı, *A new paranormed sequence space defined by regular Bell matrix*, Dera Natung Government College Research Journal, **8**(1) (2023), 30-45. https://doi.org/10.56405/dngcrj.2023.08.01.03

[5] O. Tuğ, E. Malkowsky, B. Hazarika, T. Yaying, *On the new generalized Hahn sequence space h_d^p*, Abstr. Appl. Anal., **2022**, Article ID 6832559, 11 pages, 2022. https://doi.org/10.1155/2022/6832559

[6] M. Candan, E. E. Kara, *A study on topological and geometrical characteristics of a new Banach sequence spaces*, Gulf. J. Math., **3**(4) (2015), 67-84.

[7] M. Candan, *Some Characteristics of matrix operators on generalized Fibonacci weighted difference sequence space*, Symmetry, **14**(7), 2022.

[8] S. Erdem, S. Demiriz, *On the new generalized block difference sequence space*, Appl. Appl. Math.(AAM), Special Issue 5 (2019), 68-83.

[9] H. B. Ellidokuzoğlu, S. Demiriz, *On some generalized q-difference sequence spaces*, AIMS Mathematics, **8**(8) (2023), 18607-18617.

[10] M. İlkhan, P. Z. Alp, E. E. Kara, *On the spaces of linear operators acting between asymmetric cone normed spaces*, Mediterr. J. Math., **15**, 136 (2018).

[11] F. Başar, *Summability Theory and Its Applications*, Istanbul, 2012.

[12] J. Boos, *Classical and Modern Methods in Summability*, Oxford Science Publications, Oxford University Press, 2000.

[13] M. Mursaleen, F. Başar, *Sequence Spaces: Topic in Modern Summability*, CRC Press, Taylor Frencis Group, Series:Mathematics and its applications, Boca Raton, London, New York, 2020.

[14] E. E. Kara, M. Başarır, *An application of Fibonacci numbers into infinite Toeplitz matrices*, Casp. J. Math. Sci., **1**(1) (2012), 43-47.

[15] M. C. Dağlı, *A novel conservative matrix arising from Schröder numbers and its properties*, Linear and Multilinear Algebra, **71**(8) (2023), 1338-1351. DOI: 10.1080/03081087.2022.2061401.

[16] M. C. Dağlı, *Matrix mappings and compact operators for Schröder sequence spaces*, Turkish J. Math., **46** (2022), 2304-2320.

[17] M. Karakaş, M. C. Dağlı, *Some topologic and geometric properties of new Catalan sequence spaces*, Adv. Oper. Theory, **8**(14) (2023), 15 pages.

[18] E. E. Kara, *Some topological and geometrical properties of new Banach sequence spaces*, J. Inequal. Appl., **2013**, 2013:38.

[19] M. Karakaş, H. Karabudak, *An application on the Lucas numbers and infinite Toeplitz matrices*, Cumhuriyet Sci. J., **38**(3) (2017), 557-562.

[20] M. Karakaş, A. M. Karakaş, *New Banach sequence spaces that is defined by the aid of Lucas numbers*, Iğdır Univ. J. Inst. Sci. Tech., **7**(4) (2017), 103-111.

[21] T. Yaying, B. Hazarika, S. A. Mohiuddine, *Domain of Padovan $q-$difference matrix in sequence spaces ℓ_p and ℓ_∞*, Filomat, **36**(3) (2022), 905-919.

[22] T. Yaying, B. Hazarika, O. M. Kalthum S. K. Mohamed, Awad A. Bakery, *On new Banach sequence spaces involving Leonardo numbers and the associated mapping ideal*, J. Funct. Spaces, **2022**, Article ID: 8269000, 21 Pages.

[23] M. İlkhan, *A new conservative matrix derived by Catalan numbers and its matrix domain in the spaces c and c_0*, Linear and Multilinear Algebra, https://doi.org/10.1080/03081087.2019.1635071

[24] M.İlkhan, E. E. Kara, *Matrix transformations and compact operators on Catalan sequence spaces*, J. Math. Anal. Appl., **498** (2021), 124925.

[25] M. Karakaş, *On the sequence spaces involving Bell numbers*, Linear and Multilinear Algebra, https://doi.org/10.1080/03081087.2022.2098225

[26] A. Wilansky, *Summability Through Functional Analysis*. Amsterdam-New York-Oxford: North- Holland Mathematics Studies 85; 1984.

[27] M. Stieglitz, H. Tietz, *Matrix transformationen von folgenraumen eine ergebnisbersicht*, Math Z., **154** (1977), 1-16.

[28] E. Malkowsky, V. Rakocevic, *An introduction into the theory of sequence spaces and measure of noncompactness*, Zbornik Radova, Matematicki Inst SANU, Belgrad, **9**(17) (2000), 143-234.

[29] V. Rakocevic, *Measures of noncompactness and some applications*, Filomat, **12** (1998), 87-120.

[30] M. Mursaleen, AK. Noman, *Compactness by the Hausdorff measure of noncompactness*, Nonlinear Anal., **73**(8) (2010), 2541-2557.

[31] M. Mursaleen, AK. Noman, *Applications of the Hausdorffmeasure of noncompactness in some sequence spaces of weighted means*, Comput Math Appl. 2010;**60**(5) (2010), 1245-1258.

[32] F. Başar, E. Malkowsky, *The characterization of compact operators on spaces of strongly summable and bounded sequences*, Appl Math Comput., **217** (2011), 5199-5207.

[33] E.E. Kara, M. Başarır, *On compact operators and some Euler $B^{(m)}$ difference sequence spaces and compact operators*, J. Math. Anal. Appl., **379** (2011), 499-511.

[34] M. Başarır, E. E. Kara, *On some difference sequence spaces of weighted means and compact operators*, Ann. Funct. Anal.,**2** (2011), 114-129.

[35] M. Başarır, E. E. Kara, *On the B-difference sequence space derived by generalized weighted mean and compact operators*, J. Math. Anal. Appl., **391** (2012), 67-81.

Communications in Advanced Mathematical Sciences
Vol. 7, No. 1, 56-70, 2024
Research Article
e-ISSN: 2651-4001
DOI:10.33434/cams.1362711

Refinements and Reverses of Tensorial and Hadamard Product Inequalities for Selfadjoint Operators in Hilbert Spaces Related to Young's Result

Silvestru Sever Dragomir[1,2]

Abstract
Let H be a Hilbert space. In this paper we show among others that, if the selfadjoint operators A and B satisfy the condition $0 < m \leq A,\ B \leq M$, for some constants $m,\ M$, then

$$\begin{aligned}0 &\leq \frac{m}{M^2}\nu(1-\nu)\left(\frac{A^2\otimes 1+1\otimes B^2}{2}-A\otimes B\right)\\ &\leq (1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\\ &\leq \frac{M}{m^2}\nu(1-\nu)\left(\frac{A^2\otimes 1+1\otimes B^2}{2}-A\otimes B\right)\end{aligned}$$

for all $\nu\in[0,1]$. We also have the inequalities for Hadamard product

$$\begin{aligned}0 &\leq \frac{m}{M^2}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1-A\circ B\right)\\ &\leq [(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\\ &\leq \frac{M}{m^2}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1-A\circ B\right)\end{aligned}$$

for all $\nu\in[0,1]$.

Keywords: Tensorial product, Hadamard product, Selfadjoint operators, Convex functions
2010 AMS: Primary 47A63, Secondary 47A99

[1] *Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.*
[2] *DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa, ORCID: 0000-0003-2902-6805*
sever.dragomir@vu.edu.au, http://rgmia.org/dragomir

Received: 19-09-2023, **Accepted:** 19-02-2024, **Available online:** 04-03-2024
How to cite this article: S. S. Dragomir, *Refinements and Reverses of Tensorial and Hadamard Product Inequalities for Selfadjoint Operators in Hilbert Spaces Related to Young's Result*, Commun. Adv. Math. Sci., 7(1) (2024) 56-70.

1. Introduction

The famous *Young inequality* for scalars says that if $a,\ b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called ν*-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [1]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty)\\ \\ 1 \text{ if } h=1.\end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.3}$$

where $a,\ b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (1.3) is due to Tominaga [2] while the first one is due to Furuichi [3].

Kittaneh and Manasrah [4, 5] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{1.4}$$

where $a,\ b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We also consider the *Kantorovich's ratio* defined by

$$K(h):=\frac{(h+1)^{2}}{4h},\ h>0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.6}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [6] while the second by Liao et al. [7].

In [6] the authors also showed that $K^{r}(h)\geq S(h^{r})$ for $h>0$ and $r\in\left[0,\frac{1}{2}\right]$ implying that the lower bound in (1.6) is better than the lower bound from (1.3).

In the recent paper [8] we obtained the following reverses of Young's inequality as well:

$$0\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\nu(1-\nu)(a-b)(\ln a-\ln b) \tag{1.7}$$

and

$$1\leq\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.8}$$

where $a,b>0$, $\nu\in[0,1]$.

In [9], we obtained the following Young related inequalities:

Theorem 1.1. *For any $a,b>0$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\min\{a,b\}&\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\&\leq\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\max\{a,b\}\end{aligned} \tag{1.9}$$

and

$$\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\right]\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\right].\tag{1.10}$$

For an equivalent form and a different approach in proving the results (1.9) and (1.10) see [10].

The second inequalities in (1.9) and (1.10) are better than the corresponding results obtained by Furuichi and Minculete in [11] where instead of constant $\frac{1}{2}$ they had the constant 1. Let $I_1,\dots,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times\dots\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,\dots,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,\dots,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,\dots,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,\dots,k$; by following [12], we define

$$f(A_1,\dots,A_k):=\int_{I_1}\dots\int_{I_k}f(\lambda_1,\dots,\lambda_k)\,dE_1(\lambda_1)\otimes\dots\otimes dE_k(\lambda_k)\tag{1.11}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes\dots\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [12] extends the definition of Korányi [13] for functions of two variables and have the property that

$$f(A_1,\dots,A_k)=f_1(A_1)\otimes\dots\otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,\dots,t_k)=f_1(t_1)\dots f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st)\ge(\le)f(s)f(t)\ \text{ for all } s,t\in[0,\infty)$$

and if f is continuous on $[0,\infty)$, then [14, p. 173]

$$f(A\otimes B)\ge(\le)f(A)\otimes f(B)\ \text{ for all } A,\ B\ge 0.\tag{1.12}$$

This follows by observing that, if

$$A=\int_{[0,\infty)}t\,dE(t)\ \text{ and } B=\int_{[0,\infty)}s\,dF(s)$$

are the spectral resolutions of A and B, then

$$f(A\otimes B)=\int_{[0,\infty)}\int_{[0,\infty)}f(st)\,dE(t)\otimes dF(s)\tag{1.13}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B>0$

$$A\#_t B:=A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2},$$

where $t\in[0,1]$ and

$$A\# B:=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B=B\#A\ \text{ and }\ (A\#B)\otimes(B\#A)=(A\otimes B)\#(B\otimes A).$$

In 2007, Wada [15] obtained the following *Callebaut type inequalities* for tensorial product

$$(A\#B)\otimes(A\#B) \leq \frac{1}{2}\left[(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\right] \leq \frac{1}{2}(A\otimes B+B\otimes A) \tag{1.14}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\left\langle (A\circ B)e_j, e_j\right\rangle = \left\langle Ae_j,e_j\right\rangle\left\langle Be_j,e_j\right\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [16], we have the representation

$$A\circ B=\mathscr{U}^*(A\otimes B)\mathscr{U} \tag{1.15}$$

where $\mathscr{U}: H\to H\otimes H$ is the isometry defined by $\mathscr{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [14, p. 173]

$$f(A\circ B)\geq(\leq)f(A)\circ f(B) \text{ for all } A,\ B\geq 0. \tag{1.16}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\geq 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq 1 \text{ for } A>0. \tag{1.17}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [17] showed that

$$A\circ B\leq\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2} \text{ for } A,\ B\geq 0$$

and Aujla and Vasudeva [18] gave an alternative upper bound

$$A\circ B\leq\left(A^2\circ B^2\right)^{1/2} \text{ for } A,\ B\geq 0.$$

It has been shown in [19] that $\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}$ and $\left(A^2\circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by these results, in this paper we provide among others some upper and lower bounds for the Young differences

$$(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}$$

and

$$[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}$$

for $\nu\in[0,1]$ and $A,\ B>0$.

2. Main Results

The first main result is as follows:

Theorem 2.1. *Assume that the selfadjoint operators A and B satisfy the condition $0<m\leq A,\ B\leq M$, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}m\nu(1-\nu)\left[\left(\ln^2 A\right)\otimes 1+1\otimes\left(\ln^2 B\right)-2\ln A\otimes\ln B\right] \\ &\leq (1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu} \\ &\leq \frac{1}{2}M\nu(1-\nu)\left[\left(\ln^2 A\right)\otimes 1+1\otimes\left(\ln^2 B\right)-2\ln A\otimes\ln B\right] \\ &\leq \frac{1}{2}\nu(1-\nu)M(\ln M-\ln m)^2 \end{aligned} \tag{2.1}$$

for all $\nu \in [0,1]$.

In particular,

$$
\begin{aligned}
0 &\le \frac{1}{8} m\left[\left(\ln^2 A\right)\otimes 1 + 1\otimes\left(\ln^2 B\right) - 2\ln A\otimes \ln B\right] \\
&\le \frac{A\otimes 1 + 1\otimes B}{2} - A^{1/2}\otimes B^{1/2} \\
&\le \frac{1}{8} M\left[\left(\ln^2 A\right)\otimes 1 + 1\otimes\left(\ln^2 B\right) - 2\ln A\otimes \ln B\right] \\
&\le \frac{1}{8} M\left(\ln M - \ln m\right)^2 .
\end{aligned} \tag{2.2}
$$

Proof. If $t,\, s \in [m,M] \subset (0,\infty)$, then by (1.9) we get

$$
\begin{aligned}
0 &\le \frac{1}{2} m\nu\left(1-\nu\right)\left(\ln t - \ln s\right)^2 \le \left(1-\nu\right)t + \nu s - t^{1-\nu}s^{\nu} \\
&\le \frac{1}{2} M\nu\left(1-\nu\right)\left(\ln t - \ln s\right)^2 \\
&\le \frac{1}{2} M\nu\left(1-\nu\right)\left(\ln M - \ln m\right)^2 .
\end{aligned} \tag{2.3}
$$

If

$$
A = \int_m^M t dE\left(t\right) \text{ and } B = \int_m^M s dF\left(s\right)
$$

are the spectral resolutions of A and B, then by taking in (2.3) the double integral $\int_m^M \int_m^M$ over $dE\left(t\right)\otimes dF\left(s\right)$, we get

$$
\begin{aligned}
0 &\le \frac{1}{2} m\nu\left(1-\nu\right)\int_m^M \int_m^M \left(\ln t - \ln s\right)^2 dE\left(t\right)\otimes dF\left(s\right) \\
&\le \int_m^M \int_m^M \left[\left(1-\nu\right)t + \nu s - t^{1-\nu}s^{\nu}\right] dE\left(t\right)\otimes dF\left(s\right) \\
&\le \frac{1}{2} M\nu\left(1-\nu\right)\int_m^M \int_m^M \left(\ln t - \ln s\right)^2 dE\left(t\right)\otimes dF\left(s\right) \\
&\le \frac{1}{8} M\left(\ln M - \ln m\right)^2 \int_m^M \int_m^M dE\left(t\right)\otimes dF\left(s\right).
\end{aligned} \tag{2.4}
$$

Now, observe that, by (1.11)

$$
\begin{aligned}
\int_m^M \int_m^M \left(\ln t - \ln s\right)^2 dE\left(t\right)\otimes dF\left(s\right) &= \int_m^M \int_m^M \left(\ln^2 t - 2\ln t \ln s + \ln^2 s\right) dE\left(t\right)\otimes dF\left(s\right) \\
&= \int_m^M \int_m^M \ln^2 t dE\left(t\right)\otimes dF\left(s\right) + \int_m^M \int_m^M \ln^2 s dE\left(t\right)\otimes dF\left(s\right) \\
&\quad - 2\int_m^M \int_m^M \ln t \ln s dE\left(t\right)\otimes dF\left(s\right) \\
&= \left(\ln^2 A\right)\otimes 1 + 1\otimes\left(\ln^2 B\right) - 2\ln A\otimes \ln B,
\end{aligned}
$$

$$
\begin{aligned}
\int_m^M \int_m^M \left[\left(1-\nu\right)t + \nu s - t^{1-\nu}s^{\nu}\right] dE\left(t\right)\otimes dF\left(s\right) &= \left(1-\nu\right)\int_m^M \int_m^M t dE\left(t\right)\otimes dF\left(s\right) + \nu \int_m^M \int_m^M s dE\left(t\right)\otimes dF\left(s\right) \\
&\quad - \int_m^M \int_m^M t^{1-\nu}s^{\nu} dE\left(t\right)\otimes dF\left(s\right) \\
&= \left(1-\nu\right)A\otimes 1 + \nu 1\otimes B - A^{1-\nu}\otimes B^{\nu}
\end{aligned}
$$

and

$$
\int_m^M \int_m^M dE\left(t\right)\otimes dF\left(s\right) = 1\otimes 1 = 1.
$$

By employing (2.4), we then get the desired result (2.1). $\square$

Corollary 2.2. *With the assumptions of Theorem 2.1,*

$$\begin{aligned}0 &\leq \frac{1}{2}m\nu(1-\nu)\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right]\\ &\leq [(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\\ &\leq \frac{1}{2}M\nu(1-\nu)\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right]\\ &\leq \frac{1}{2}\nu(1-\nu)M(\ln M-\ln m)^2\end{aligned} \tag{2.5}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0 &\leq \frac{1}{8}m\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right]\\ &\leq \frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2}\\ &\leq \frac{1}{8}M\left[\left(\ln^2 A+\ln^2 B\right)\circ 1-2\ln A\circ\ln B\right]\\ &\leq \frac{1}{8}M(\ln M-\ln m)^2.\end{aligned} \tag{2.6}$$

Proof. The proof follows from Theorem 2.1 by taking to the left $\mathscr{U}^*$, to the right $\mathscr{U}$, where $\mathscr{U}: H\to H\otimes H$ is the isometry defined by $\mathscr{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$ and utilizing the representation (1.15). □

Remark 2.3. *If we take $B=A$ in Corollary 2.2, then we get*

$$\begin{aligned}0 &\leq m\nu(1-\nu)\left[\left(\ln^2 A\right)\circ 1-\ln A\circ\ln A\right]\leq A\circ 1-A^{1-\nu}\circ A^{\nu}\\ &\leq M\nu(1-\nu)\left[\left(\ln^2 A\right)\circ 1-\ln A\circ\ln A\right]\\ &\leq \frac{1}{2}\nu(1-\nu)M(\ln M-\ln m)^2\end{aligned} \tag{2.7}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0 &\leq \frac{1}{4}m\left[\left(\ln^2 A\right)\circ 1-\ln A\circ\ln A\right]\leq A\circ 1-A^{1/2}\circ A^{1/2}\\ &\leq \frac{1}{4}M\left[\left(\ln^2 A\right)\circ 1-\ln A\circ\ln A\right]\leq \frac{1}{8}M(\ln M-\ln m)^2.\end{aligned} \tag{2.8}$$

Theorem 2.4. *With the assumptions of Theorem 2.1, we have*

$$\begin{aligned}0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\\ &\leq (1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\\ &\leq \frac{M}{2m^2}\nu(1-\nu)\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2\end{aligned} \tag{2.9}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0 &\leq \frac{m}{8M^2}\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\\ &\leq \frac{A\otimes 1+1\otimes B}{2}-A^{1/2}\otimes B^{1/2}\\ &\leq \frac{M}{8m^2}\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\leq \frac{M}{8m^2}(M-m)^2.\end{aligned} \tag{2.10}$$

Proof. We observe that

$$0<\frac{1}{\max\{a,b\}}\leq\frac{\ln a-\ln b}{a-b}\leq\frac{1}{\min\{a,b\}},$$

which implies that

$$0<\frac{1}{\max^2\{a,b\}}\leq\left(\frac{\ln a-\ln b}{a-b}\right)^2\leq\frac{1}{\min^2\{a,b\}}$$

for all $a,b>0$.

By making use of (1.9), we derive

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)(b-a)^2\frac{\min\{a,b\}}{\max^2\{a,b\}}\\&\leq\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\&\leq\frac{1}{2}\nu(1-\nu)(b-a)^2\frac{\max\{a,b\}}{\min^2\{a,b\}}.\end{aligned}\tag{2.11}$$

If $t,\,s\in[m,M]\subset(0,\infty)$, then by (2.11) we get

$$\begin{aligned}0&\leq\frac{m}{2M^2}\nu(1-\nu)(t-s)^2\leq(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\\&\leq\frac{M}{2m^2}\nu(1-\nu)(t-s)^2.\end{aligned}\tag{2.12}$$

If

$$A=\int_m^M t\,dE(t)\text{ and }B=\int_m^M s\,dF(s)$$

are the spectral resolutions of A and B, then by taking in (2.12) the double integral $\int_m^M\int_m^M$ over $dE(t)\otimes dF(s)$, we get

$$\begin{aligned}0&\leq\frac{m}{2M^2}\nu(1-\nu)\int_m^M\int_m^M(t-s)^2E(t)\otimes dF(s)\\&\leq\int_m^M\int_m^M\left[(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\right]E(t)\otimes dF(s)\\&\leq\frac{M}{2m^2}\nu(1-\nu)\int_m^M\int_m^M(t-s)^2E(t)\otimes dF(s).\end{aligned}\tag{2.13}$$

Since, by (1.11)

$$\begin{aligned}\int_m^M\int_m^M(t-s)^2E(t)\otimes dF(s)&=\int_m^M\int_m^M\left(t^2-2ts+s^2\right)E(t)\otimes dF(s)\\&=\int_m^M\int_m^M t^2E(t)\otimes dF(s)+\int_m^M\int_m^M s^2E(t)\otimes dF(s)-\int_m^M\int_m^M 2tsE(t)\otimes dF(s)\\&=A^2\otimes1+1\otimes B^2-2A\otimes B,\end{aligned}$$

then by (2.13) we derive the first part of (2.9).

The last part follows by the fact that

$$(t-s)^2\leq(M-m)^2$$

for all $t,\,s\in[m,M]$. □

Corollary 2.5. *With the assumptions of Theorem 2.1, we have the following inequalities for the Hadamard product*

$$\begin{aligned}
0 &\le \frac{m}{M^2}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \\
&\le [(1-\nu)A+\nu B]\circ 1 - A^{1-\nu}\circ B^{\nu} \\
&\le \frac{M}{m^2}\nu(1-\nu)\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \le \frac{M}{2m^2}\nu(1-\nu)(M-m)^2
\end{aligned} \tag{2.14}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}
0 &\le \frac{m}{4M^2}\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \le \frac{A+B}{2}\circ 1 - A^{1/2}\circ B^{1/2} \\
&\le \frac{M}{4m^2}\left(\frac{A^2+B^2}{2}\circ 1 - A\circ B\right) \le \frac{M}{8m^2}(M-m)^2.
\end{aligned} \tag{2.15}$$

The proof of this corollary is similar to the one of Corollary 2.2 by utilizing Theorem 2.4 and we omit the details.

Remark 2.6. *If we take $B=A$ in Corollary 2.5, then we get*

$$\begin{aligned}
0 &\le \frac{m}{M^2}\nu(1-\nu)\left(A^2\circ 1 - A\circ A\right) \le A - A^{1-\nu}\circ A^{\nu} \\
&\le \frac{M}{m^2}\nu(1-\nu)\left(A^2\circ 1 - A\circ A\right) \le \frac{M}{2m^2}\nu(1-\nu)(M-m)^2
\end{aligned} \tag{2.16}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}
0 &\le \frac{m}{4M^2}\left(A^2\circ 1 - A\circ A\right) \le A\circ 1 - A^{1/2}\circ A^{1/2} \\
&\le \frac{M}{4m^2}\left(A^2\circ 1 - A\circ A\right) \le \frac{M}{8m^2}(M-m)^2.
\end{aligned} \tag{2.17}$$

Further, we also have:

Theorem 2.7. *Assume that the selfadjoint operators A and B satisfy the condition $0<A,\, B\le M$, then*

$$\begin{aligned}
0 &\le (1-\nu)A\otimes 1 + \nu 1\otimes B - A^{1-\nu}\otimes B^{\nu} \\
&\le M\nu(1-\nu)\left(\frac{A^{-1}\otimes B + A\otimes B^{-1}}{2} - 1\right)
\end{aligned} \tag{2.18}$$

for all $\nu\in[0,1]$.

In particular,

$$0 \le \frac{A\otimes 1 + 1\otimes B}{2} - A^{1/2}\otimes B^{1/2} \le \frac{1}{4}M\left(\frac{A^{-1}\otimes B + A\otimes B^{-1}}{2} - 1\right). \tag{2.19}$$

Proof. Recall that if $a,\, b>0$ and

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } a\ne b, \\ \\ b \text{ if } a=b \end{cases}$$

is the *logarithmic mean* and $G(a,b) := \sqrt{ab}$ is the *geometric mean*, then $L(a,b)\ge G(a,b)$ for all $a,\, b>0$.

Then from (1.9) we have for $a\ne b$ that

$$\begin{aligned}
(1-\nu)a + \nu b - a^{1-\nu}b^{\nu} &\le \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\} \\
&= \frac{1}{2}\nu(1-\nu)(b-a)^2\left(\frac{\ln a - \ln b}{b-a}\right)^2 \max\{a,b\} \\
&\le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{ab}\max\{a,b\} \\
&= \frac{1}{2}\nu(1-\nu)\left(\frac{b}{a}+\frac{a}{b}-2\right)\max\{a,b\},
\end{aligned}$$

which implies that

$$(1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)\left(\frac{b}{a}+\frac{a}{b}-2\right)\max\{a,b\} \tag{2.20}$$

for all $a,\, b>0$.

If $t,\, s\in[m,M]\subset(0,\infty)$, then by (2.20) we get

$$\begin{aligned}(1-\nu)t+\nu s-t^{1-\nu}s^{\nu} &\le \frac{1}{2}\nu(1-\nu)\left(\frac{s}{t}+\frac{t}{s}-2\right)\max\{t,s\} \\ &\le \frac{1}{2}M\nu(1-\nu)\left(\frac{s}{t}+\frac{t}{s}-2\right).\end{aligned} \tag{2.21}$$

By taking in (2.21) the double integral $\int_m^M\int_m^M$ over $dE(t)\otimes dF(s)$, we get

$$\int_m^M\int_m^M\left[(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\right]dE(t)\otimes dF(s) \le \frac{1}{2}M\nu(1-\nu)\int_m^M\int_m^M\left(\frac{s}{t}+\frac{t}{s}-2\right)dE(t)\otimes dF(s). \tag{2.22}$$

Since

$$\begin{aligned}\int_m^M\int_m^M\left(\frac{s}{t}+\frac{t}{s}-2\right)dE(t)\otimes dF(s) =& \int_m^M\int_m^M t^{-1}sE(t)\otimes dF(s)+\int_m^M\int_m^M ts^{-1}dE(t)\otimes dF(s) \\ &-\int_m^M\int_m^M dE(t)\otimes dF(s) \\ =& A^{-1}\otimes B+A\otimes B^{-1}-2,\end{aligned}$$

hence by (2.22) we derive (2.18). □

Corollary 2.8. *With the assumptions of Theorem 2.7, we have the inequalities for the Hadamard product*

$$\begin{aligned}0 &\le [(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu} \\ &\le M\nu(1-\nu)\left(\frac{A^{-1}\circ B+A\circ B^{-1}}{2}-1\right)\end{aligned} \tag{2.23}$$

for all $\nu\in[0,1]$.

In particular,

$$0\le \frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2}\le \frac{1}{4}M\left(\frac{A^{-1}\circ B+A\circ B^{-1}}{2}-1\right). \tag{2.24}$$

The proof of this corollary is similar to the one of Corollary 2.2 by utilizing Theorem 2.7.

We observe that, if we take $B=A$ in Corollary 2.8, then we get

$$0\le A\circ 1-A^{1-\nu}\circ A^{\nu}\le M\nu(1-\nu)\left(A^{-1}\circ A-1\right) \tag{2.25}$$

for all $\nu\in[0,1]$.

In particular,

$$0\le A\circ 1-A^{1/2}\circ A^{1/2}\le \frac{1}{8}M\left(A^{-1}\circ A-1\right). \tag{2.26}$$

We also have the following multiplicative results:

Theorem 2.9. *Assume that the selfadjoint operators A and B satisfy the condition* $0<m\le A,\, B\le M$, *then*

$$\begin{aligned}A^{1-\nu}\otimes B^{\nu} &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right]A^{1-\nu}\otimes B^{\nu} \\ &\le (1-\nu)A\otimes 1+\nu 1\otimes B \\ &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right]A^{1-\nu}\otimes B^{\nu}\end{aligned} \tag{2.27}$$

for all $\nu \in [0,1]$.

In particular,

$$A^{1-\nu} \otimes B^{\nu} \leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] A^{1/2} \otimes B^{1/2} \tag{2.28}$$
$$\leq \frac{A \otimes 1 + 1 \otimes B}{2}$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] A^{1/2} \otimes B^{1/2}.$$

Proof. Since

$$\frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{\max\{a,b\} - \min\{a,b\}}{\max\{a,b\}}\right)^2 = \left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2$$

and

$$\frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{\max\{a,b\} - \min\{a,b\}}{\min\{a,b\}}\right)^2 = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2,$$

hence by (1.10) we derive

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \leq \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^{\nu}} \tag{2.29}$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right].$$

If $t,\ s \in [m,M] \subset (0,\infty)$, then by (2.29) we get

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] t^{1-\nu}s^{\nu} \leq (1-\nu)t + \nu s \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] t^{1-\nu}s^{\nu}. \tag{2.30}$$

Now, if we take the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$ in (2.30), we derive the desired result (2.27). □

Corollary 2.10. *With the assumptions of Theorem 2.9, we have the inequalities for Hadamard product*

$$A^{1-\nu} \circ B^{\nu} \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] A^{1-\nu} \circ B^{\nu} \tag{2.31}$$
$$\leq (1-\nu)A + \nu B$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] A^{1-\nu} \circ B^{\nu}$$

for all $\nu \in [0,1]$.

In particular,

$$A^{1/2} \circ B^{1/2} \leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] A^{1/2} \circ B^{1/2} \tag{2.32}$$
$$\leq \frac{A+B}{2} \circ 1$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] A^{1/2} \circ B^{1/2}.$$

The proof of this corollary is similar to the one of Corollary 2.2 by utilizing Theorem 2.9.

If we take $B = A$ in Corollary 2.10, then we get the following inequalities for one operator A satisfying the condition $0 < m \leq A \leq M$,

$$
\begin{aligned}
A^{1-\nu} \circ A^{\nu} &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right] A^{1-\nu} \circ A^{\nu} \\
&\leq A \circ 1 \\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right] A^{1-\nu} \circ A^{\nu}
\end{aligned} \tag{2.33}
$$

for all $\nu \in [0,1]$.

In particular,

$$
\begin{aligned}
A^{1/2} \circ A^{1/2} &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] A^{1/2} \circ A^{1/2} \\
&\leq A \circ 1 \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] A^{1/2} \circ A^{1/2}.
\end{aligned} \tag{2.34}
$$

3. Inequalities for Sums

We also have the following inequalities for sums of operators:

Proposition 3.1. *Assume that $0 < m \leq A_i,\, B_j \leq M$ and $p_i,\, q_j \geq 0$ for $i \in \{1,...,n\}$, $j \in \{1,...,k\}$, and put $P_n := \sum_{i=1}^n p_i$, $Q_k := \sum_{j=1}^k q_j$. Then*

$$
\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left[Q_k\left(\sum_{i=1}^n p_i A_i^2\right)\otimes 1 + P_n 1\otimes\left(\sum_{j=1}^k q_j B_j^2\right) - 2\left(\sum_{i=1}^n p_i A_i\right)\otimes\left(\sum_{j=1}^k q_j B_j\right)\right] \\
&\leq (1-\nu)Q_k\left(\sum_{i=1}^n p_i A_i\right)\otimes 1 + \nu P_n 1\otimes\left(\sum_{j=1}^k q_j B_j\right) - \left(\sum_{i=1}^n p_i A_i^{1-\nu}\right)\otimes\left(\sum_{j=1}^k q_j B_j^{\nu}\right) \\
&\leq \frac{M}{2m^2}\nu(1-\nu)\left[Q_k\left(\sum_{i=1}^n p_i A_i^2\right)\otimes 1 + P_n 1\otimes\left(\sum_{j=1}^k q_j B_j^2\right) - 2\left(\sum_{i=1}^n p_i A_i\right)\otimes\left(\sum_{j=1}^k q_j B_j\right)\right] \\
&\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 P_n Q_k
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
0 &\leq (1-\nu)Q_k\left(\sum_{i=1}^n p_i A_i\right)\otimes 1 + \nu P_n 1\otimes\left(\sum_{j=1}^k q_j B_j\right) - \left(\sum_{i=1}^n p_i A_i^{1-\nu}\right)\otimes\left(\sum_{j=1}^k q_j B_j^{\nu}\right) \\
&\leq M\nu(1-\nu)\times\left[\frac{\left(\sum_{i=1}^n p_i A^{-1}\right)\otimes\left(\sum_{j=1}^k q_j B\right) + \left(\sum_{i=1}^n p_i A\right)\otimes\left(\sum_{j=1}^k q_j B^{-1}\right)}{2} - P_n Q_k\right].
\end{aligned} \tag{3.2}
$$

Proof. From (2.9) we get

$$
\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left(A_i^2\otimes 1 + 1\otimes B_j^2 - 2A_i\otimes B_j\right) \\
&\leq (1-\nu)A_i\otimes 1 + \nu 1\otimes B_j - A_i^{1-\nu}\otimes B_j^{\nu} \\
&\leq \frac{M}{2m^2}\nu(1-\nu)\left(A_i^2\otimes 1 + 1\otimes B_j^2 - 2A_i\otimes B_j\right) \\
&\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2
\end{aligned}
$$

for all for $i \in \{1,...,n\}$, $j \in \{1,...,k\}$ and $\nu \in [0,1]$.

If we multiply by $p_i q_j \geq 0$ and sum, then we get

$$
\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left[(1-\nu)A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu}\right] \\
&\leq \frac{M}{2m^2}\nu(1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \\
&\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i.
\end{aligned} \tag{3.3}
$$

Observe that

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left(A_i^2 \otimes 1 + 1 \otimes B_j^2 - 2A_i \otimes B_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i^2 \otimes 1 + \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i 1 \otimes B_j^2 - 2\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i \otimes B_j \\
&= Q_k \left(\sum_{i=1}^{n} p_i A_i^2\right) \otimes 1 + P_n 1 \otimes \left(\sum_{j=1}^{k} q_j B_j^2\right) - 2\left(\sum_{i=1}^{n} p_i A_i\right) \otimes \left(\sum_{j=1}^{k} q_j B_j\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i \left[(1-\nu)A_i \otimes 1 + \nu 1 \otimes B_j - A_i^{1-\nu} \otimes B_j^{\nu}\right] &= (1-\nu)\sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i \otimes 1 + \nu \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i 1 \otimes B_j \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{k} q_j p_i A_i^{1-\nu} \otimes B_j^{\nu} \\
&= (1-\nu) Q_k \left(\sum_{i=1}^{n} p_i A_i\right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^{k} q_j B_j\right) \\
&\quad - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \otimes \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right).
\end{aligned}
$$

By (3.3) we then get the desired result (3.1).

The inequality (3.2) follows in a similar way from (2.18). □

Corollary 3.2. *With the assumptions of Proposition 3.1, we have the Hadamard product inequalities*

$$
\begin{aligned}
0 &\leq \frac{m}{2M^2}\nu(1-\nu)\left[\left(Q_k \left(\sum_{i=1}^{n} p_i A_i^2\right) + P_n \left(\sum_{j=1}^{k} q_j B_j^2\right)\right) \circ 1 \; - 2\left(\sum_{i=1}^{n} p_i A_i\right) \circ \left(\sum_{j=1}^{k} q_j B_j\right)\right] \\
&\leq \left[(1-\nu) Q_k \left(\sum_{i=1}^{n} p_i A_i\right) + \nu P_n \left(\sum_{j=1}^{k} q_j B_j\right)\right] \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right) \\
&\leq \frac{M}{2m^2}\nu(1-\nu)\left[\left(Q_k \left(\sum_{i=1}^{n} p_i A_i^2\right) + P_n \left(\sum_{j=1}^{k} q_j B_j^2\right)\right) \circ 1 \; - 2\left(\sum_{i=1}^{n} p_i A_i\right) \circ \left(\sum_{j=1}^{k} q_j B_j\right)\right] \\
&\leq \frac{M}{2m^2}\nu(1-\nu)(M-m)^2 P_n Q_k
\end{aligned} \tag{3.4}
$$

and

$$0 \le \left[(1-\nu) Q_k \left(\sum_{i=1}^{n} p_i A_i \right) + \nu P_n \left(\sum_{j=1}^{k} q_j B_j \right) \right] \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu} \right) \tag{3.5}$$
$$\le M\nu(1-\nu) \times \left[\frac{\left(\sum_{i=1}^{n} p_i A^{-1}\right) \circ \left(\sum_{j=1}^{k} q_j B\right) + \left(\sum_{i=1}^{n} p_i A\right) \circ \left(\sum_{j=1}^{k} q_j B^{-1}\right)}{2} - P_n Q_k \right].$$

If we take $k = n$, $p_i = q_i$ and $B_i = A_i$, then we get the simpler inequalities

$$0 \le \frac{m}{M^2} \nu(1-\nu) \times \left[P_n \left(\sum_{i=1}^{n} p_i A_i^2 \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i \right) \circ \left(\sum_{i=1}^{n} p_i A_i \right) \right] \tag{3.6}$$
$$\le P_n \left(\sum_{i=1}^{n} p_i A_i \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu} \right)$$
$$\le \frac{M}{2m^2} \nu(1-\nu) \times \left[P_n \left(\sum_{i=1}^{n} p_i A_i^2 \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i \right) \circ \left(\sum_{i=1}^{n} p_i A_i \right) \right]$$
$$\le \frac{M}{2m^2} \nu(1-\nu)(M-m)^2 P_n^2$$

and

$$0 \le P_n \left(\sum_{i=1}^{n} p_i A_i \right) \circ 1 - \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu} \right) \tag{3.7}$$
$$\le M\nu(1-\nu) \left[\left(\sum_{i=1}^{n} p_i A^{-1} \right) \circ \left(\sum_{i=1}^{n} p_i A \right) - P_n^2 \right],$$

for all $\nu \in [0,1]$, provided that $0 < m \le A_i \le M$ and $p_i \ge 0$ for $i \in \{1, ..., n\}$.

We also have the multiplicative inequalities:

Proposition 3.3. *With the assumptions of Proposition 3.3,*

$$\left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \otimes \left(\sum_{j=1}^{k} q_j B_j^{\nu} \right) \le \exp\left[\frac{1}{2} \nu(1-\nu) \left(\frac{M-m}{M} \right)^2 \right] \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \otimes \left(\sum_{j=1}^{k} q_j B_j^{\nu} \right) \tag{3.8}$$
$$\le (1-\nu) Q_k \left(\sum_{i=1}^{n} p_i A_i \right) \otimes 1 + \nu P_n 1 \otimes \left(\sum_{j=1}^{k} q_j B_j \right)$$
$$\le \exp\left[\frac{1}{2} \nu(1-\nu) \left(\frac{M-m}{m} \right)^2 \right] \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \otimes \left(\sum_{j=1}^{k} q_j B_j^{\nu} \right)$$

and

$$\left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu} \right) \le \exp\left[\frac{1}{2} \nu(1-\nu) \left(\frac{M-m}{M} \right)^2 \right] \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu} \right) \tag{3.9}$$
$$\le (1-\nu) Q_k \left(\sum_{i=1}^{n} p_i A_i \right) \circ 1 + \nu P_n 1 \circ \left(\sum_{j=1}^{k} q_j B_j \right)$$
$$\le \exp\left[\frac{1}{2} \nu(1-\nu) \left(\frac{M-m}{m} \right)^2 \right] \left(\sum_{i=1}^{n} p_i A_i^{1-\nu} \right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu} \right),$$

for all $\nu \in [0,1]$.

If we take $k = n$, $p_i = q_i$ and $B_i = A_i$ in (3.9), then we get the simpler inequalities

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right) &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right]\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{j=1}^{k} q_j B_j^{\nu}\right) \\ &\leq P_n\left(\sum_{i=1}^{n} p_i A_i\right) \circ 1 \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right]\left(\sum_{i=1}^{n} p_i A_i^{1-\nu}\right) \circ \left(\sum_{i=1}^{n} p_i A_i^{\nu}\right),\end{aligned} \tag{3.10}$$

for all $\nu \in [0,1]$, provided that $0 < m \leq A_i \leq M$ and $p_i \geq 0$ for $i \in \{1,...,n\}$.

4. Conclusion

In this paper, by utilizing some recent refinements and reverses of scalar Young's inequality, we provided some upper and lower bounds for the Young differences

$$(1-\nu)A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}$$

and

$$[(1-\nu)A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu}$$

for $\nu \in [0,1]$ and $A, B > 0$. The case of weighted sums for sequences of operators were also investigated.

Article Information

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: The article has a single author. The author has read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

References

[1] W. Specht, *Zer Theorie der elementaren Mittel*, Math. Z., **74** (1960), 91–98.

[2] M. Tominaga, *Specht's ratio in the Young inequality*, Sci. Math. Japon., **55** (2002), 583–588.

[3] S. Furuichi, *Refined Young inequalities with Specht's ratio*, Journal of the Egyptian Mathematical Society, **20**(2012), 46–49.

[4] F. Kittaneh, Y. Manasrah, *Improved Young and Heinz inequalities for matrix*, J. Math. Anal. Appl., **361** (2010), 262–269.

[5] F. Kittaneh, Y. Manasrah, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra., **59** (2011), 1031–1037.

[6] G. Zuo, G. Shi, M. Fujii, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., **5** (2011), 551–556.

[7] W. Liao, J. Wu, J. Zhao, *New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant*, Taiwanese J. Math., **19**(2) (2015), 467–479.

[8] S. S. Dragomir, *A note on Young's inequality*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, **111**(2) (2017), 349–354. Preprint RGMIA Res. Rep. Coll., **18** (2015), Art. 126. [`http://rgmia.org/papers/v18/v18a126.pdf`].

[9] S. S. Dragomir, *A note on new refinements and reverses of Young's inequality*, Transyl. J. Math. Mec. **8**(1) (2016), 45–49. Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. [`https://rgmia.org/papers/v18/v18a131.pdf`].

[10] H. Alzer, C. M. da Fonseca, A. Kovačec, *Young-type inequalities and their matrix analogues*, Linear and Multilinear Algebra, **63**(3) (2015), 622–635.

[11] S. Furuichi, N. Minculete, *Alternative reverse inequalities for Young's inequality*, J. Math Inequal., **5**(4) (2011), 595–600.

[12] H. Araki, F. Hansen, *Jensen's operator inequality for functions of several variables*, Proc. Amer. Math. Soc., **128**(7) (2000), 2075–2084.

[13] A. Korányi, *On some classes of analytic functions of several variables*, Trans. Amer. Math. Soc., **101** (1961), 520–554.

[14] T. Furuta, J. Mićić Hot, J. Pečarić, Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[15] S. Wada, *On some refinement of the Cauchy-Schwarz inequality*, Lin. Alg. & Appl., **420** (2007), 433–440.

[16] J. I. Fujii, *The Marcus-Khan theorem for Hilbert space operators*, Math. Jpn., **41**(1995), 531–535.

[17] T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Lin. Alg. & Appl., **26** (1979), 203–241.

[18] J. S. Aujila, H. L. Vasudeva, *Inequalities involving Hadamard product and operator means*, Math. Japon., **42** (1995), 265–272.

[19] K. Kitamura, Y. Seo, *Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities*, Scient. Math., **1**(2) (1998), 237–241.