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RESEARCH ARTICLE

A Trapezoid Type Tensorial Norm Inequality for Continuous Functions of Selfadjoint Operators in Hilbert Spaces

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ABSTRACT
Generalized trapezoid and trapezoid rules play an important role in approximating the Lebesgue integral in the case of scalar-valued functions defined on a finite interval. Motivated by this reason, in this paper we provided several norm error bounds in approximation the integral of continuous function of the convex combination of some tensorial products in terms of the corresponding tensorial generalized and trapezoid rules. The case of continuously differentiable functions is analysed in detail in the case when the derivative is bounded on a finite interval. Related results for the case when the absolute value of the derivative is convex is also provided. The important particular case for the operator exponential function is also considered and the corresponding norm inequalities revealed.

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1. INTRODUCTION

Assume that the function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$, then we have the *generalized trapezoid inequality*, see for instance Cerone, P., Dragomir, S. S. (2000)

$$\left| \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

For $x = \frac{a+b}{2}$ we get the *trapezoid inequality*

$$\left| \frac{f(b) + f(a)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{4} \|f'\|_\infty (b-a),$$

with $\frac{1}{4}$ as best possible constant.

In order to extend this result for tensorial products of selfadjoint operators and norms, we need the following preparations.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; where $E_i(\cdot)$ is the spectral measure of A_i for $i = 1, ..., k$; by following Araki, H.,

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Hansen, F. (2000), we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.2}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction Araki, H., Hansen, F. (2000) extends the definition of Korányi Korányi, A. (1961) for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is known that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq)\, f(s)\, f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then (Furuta, T., Mićić Hot, J., Pečarić, J., Seo, Y. 2005, p. 173)

$$f(A \otimes B) \geq (\leq)\, f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.3}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.4}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A, B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of # and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B)\,\#\,(B \otimes A).$$

In 2007, S. Wada Wada, S. (2007) obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A\#B) \otimes (A\#B) &\leq \frac{1}{2}\left[(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)\right] \\ &\leq \frac{1}{2}(A \otimes B + B \otimes A) \end{aligned} \tag{1.5}$$

for $A, B > 0$ and $\alpha \in [0, 1]$. For other similar results, see Ando, T. (1979), Aujila, J. S., Vasudeva, H. L. (1995) and Ebadian, A., Nikoufar, I., Gordji, M. E. (2011)-Kitamura, K., Seo, Y. (1998).

Motivated by the above results, if f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t \in I} |f'(t)| < \infty$ and A, B are selfadjoint operators with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then

$$\begin{aligned} &\left\|(1-\lambda) f(A) \otimes 1 + \lambda 1 \otimes f(B) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du\right\| \\ &\leq \|f'\|_{I,\infty} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \|1 \otimes B - A \otimes 1\| \end{aligned}$$

for $\lambda \in [0, 1]$. In particular, we have the trapezoid inequality

$$\begin{aligned} &\left\|\frac{f(A) \otimes 1 + 1 \otimes f(B)}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du\right\| \\ &\leq \frac{1}{4} \|f'\|_{I,\infty} \|1 \otimes B - A \otimes 1\|. \end{aligned}$$

2. MAIN RESULTS

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D) \tag{2.1}$$

that holds for any $A, B, C, D \in B(H)$, the Banach algebra of all bounded linear operators on H.

If we take $C = A$ and $D = B$, then we get

$$A^2 \otimes B^2 = (A \otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n \otimes B^n = (A \otimes B)^n \text{ for natural } n \geq 0. \tag{2.2}$$

In particular

$$A^n \otimes 1 = (A \otimes 1)^n \text{ and } 1 \otimes B^n = (1 \otimes B)^n \tag{2.3}$$

for all $n \geq 0$.

We also observe that, by (2.1), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B. \tag{2.4}$$

Moreover, for two natural numbers m, n we have

$$(A \otimes 1)^m (1 \otimes B)^n = (1 \otimes B)^n (A \otimes 1)^m = A^m \otimes B^n. \tag{2.5}$$

We have the following representation results for continuous functions:

Lemma 2.1. *Assume A and B are selfadjoint operators with $\operatorname{Sp}(A) \subset I$ and $\operatorname{Sp}(B) \subset J$. Let f, h be continuous on I, g, k continuous on J and φ continuous on an interval K that contains the sum of the intervals $h(I) + k(J)$, then*

$$\begin{aligned} &(f(A) \otimes 1 + 1 \otimes g(B)) \varphi (h(A) \otimes 1 + 1 \otimes k(B)) \\ &= \int_I \int_J (f(t) + g(s)) \varphi (h(t) + k(s)) \, dE_t \otimes dF_s, \end{aligned} \tag{2.6}$$

where A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_J s dF(s). \tag{2.7}$$

Proof. By Stone-Weierstrass, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^n$ with n any natural number.

For natural number $n \geq 1$ we have

$$\begin{aligned} \mathcal{K} &:= \int_I \int_J (f(t) + g(s)) (h(t) + k(s))^n \, dE_t \otimes dF_s \\ &= \int_I \int_J (f(t) + g(s)) \sum_{m=0}^{n} C_n^m [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \\ &= \sum_{m=0}^{n} C_n^m \int_I \int_J (f(t) + g(s)) [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \\ &= \sum_{m=0}^{n} C_n^m \left[\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \right. \\ &\left. + \int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} \, dE_t \otimes dF_s \right]. \end{aligned} \tag{2.8}$$

Observe that

$$\begin{aligned} &\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \\ &= f(A) [h(A)]^m \otimes [k(B)]^{n-m} = (f(A) \otimes 1) ([h(A)]^m \otimes [k(B)]^{n-m}) \\ &= (f(A) \otimes 1) ([h(A)]^m \otimes 1) (1 \otimes [k(B)]^{n-m}) \\ &= (f(A) \otimes 1) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m} \end{aligned}$$

and

$$\int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} dE_t \otimes dF_s$$
$$= [h(A)]^m \otimes (g(B) [k(B)]^{n-m}) = (1 \otimes g(B)) ([h(A)]^m \otimes [k(B)]^{n-m})$$
$$= (1 \otimes g(B)) ([h(A)]^m \otimes 1) (1 \otimes [k(B)]^{n-m})$$
$$= (1 \otimes g(B)) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m},$$

with $h(A) \otimes 1$ and $1 \otimes k(B)$ commutative.

Therefore

$$\mathcal{K} = (f(A) \otimes 1 + 1 \otimes g(B)) \sum_{m=0}^{n} C_n^m (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m}$$
$$= (f(A) \otimes 1 + 1 \otimes g(B)) (h(A) \otimes 1 + 1 \otimes k(B))^n,$$

for which the commutativity of $h(A) \otimes 1$ and $1 \otimes k(B)$ has been employed.

We have the following representation result:

Theorem 2.2. *Assume that f is continuously differentiable on I, A and B are selfadjoint operators with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, *then*

$$(1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \tag{2.9}$$
$$= (1 \otimes B - A \otimes 1) \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B) du$$

for all $\lambda \in [0,1]$.

In particular, we have the trapezoid identity

$$\frac{1 \otimes f(B) + f(A) \otimes 1}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \tag{2.10}$$
$$= (1 \otimes B - A \otimes 1) \int_0^1 \left(u - \frac{1}{2}\right) f'((1-u) A \otimes 1 + u 1 \otimes B) du.$$

Proof. Integrating by parts in the Lebesgue integral, we have

$$\int_a^b (t-x) f'(t) dt = (t-x) f(t)|_a^b - \int_a^b f(t) dt \tag{2.11}$$
$$= (b-x) f(b) + (x-a) f(a) - \int_a^b f(t) dt$$

for $a \le x \le b$ and f absolutely continuous on $[a,b]$.

If we take $x = (1-\lambda) a + \lambda b$, $\lambda \in [0,1]$ and change the variable $t = (1-u) a + ub$, then $dt = (b-a) du$ and by (2.11) we derive

$$(1-\lambda)(b-a) f(b) + \lambda (b-a) f(a) - (b-a) \int_0^1 f((1-u) a + ub) du$$
$$= (b-a)^2 \int_0^1 (u-\lambda) f'((1-u) a + ub) du,$$

namely

$$(1-\lambda) f(b) + \lambda f(a) - \int_0^1 f((1-u) a + ub) du \tag{2.12}$$
$$= (b-a) \int_0^1 (u-\lambda) f'((1-u) a + ub) du,$$

for all $a, b \in I$ and $\lambda \in [0,1]$.

Assume that A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s).$$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$ in (2.12) written for $b = s$, $a = t$, then we get

$$\int_I \int_I \left[(1-\lambda) f(s) + \lambda f(t) - \int_0^1 f((1-u)t + us)\, du \right] dE_t \otimes dF_s \tag{2.13}$$
$$= \int_I \int_I \left[(s-t) \int_0^1 (u-\lambda) f'((1-u)t + us) \right] dE_t \otimes dF_s.$$

By utilizing Fubini's theorem and Lemma 2.1 we derive

$$\int_I \int_I \left[(1-\lambda) f(s) + \lambda f(t) - \int_0^1 f((1-u)t + us)\, du \right] dE_t \otimes dF_s \tag{2.14}$$
$$= (1-\lambda) \int_I \int_I f(s)\, dE_t \otimes dF_s + \lambda \int_I \int_I f(t)\, dE_t \otimes dF_s$$
$$- \int_0^1 \left(\int_I \int_I (f((1-u)t + us))\, dE_t \otimes dF_s \right) du$$
$$= (1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du$$

and

$$\int_I \int_I \left[(s-t) \int_0^1 (u-\lambda) f'((1-u)t + us)\, du \right] dE_t \otimes dF_s \tag{2.15}$$
$$= \int_0^1 (u-\lambda) \left[\int_I \int_I (s-t) f'((1-u)t + us)\, dE_t \otimes dF_s \right] du$$
$$= \int_0^1 (u-\lambda)(1 \otimes B - A \otimes 1) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du$$
$$= (1 \otimes B - A \otimes 1) \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du.$$

Therefore, by (2.13)-(2.15) we get the desired identity (2.9).

We have the following generalized trapezoid inequality:

Theorem 2.3. *Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t \in I} |f'(t)| < \infty$ and A, B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then*

$$\left\| (1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\| \tag{2.16}$$
$$\leq \|1 \otimes B - A \otimes 1\| \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|f'\|_{I,\infty}$$

for all $\lambda \in [0,1]$.

In particular, we have the trapezoid inequality

$$\left\| \frac{1 \otimes f(B) + f(A) \otimes 1}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\| \tag{2.17}$$
$$\leq \frac{1}{4} \|f'\|_{I,\infty} \|1 \otimes B - A \otimes 1\|.$$

Proof. If we take the norm in the identity (2.9) and use the properties of the integral, then we get

$$\left\| (1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\| \tag{2.18}$$
$$= \left\| (1 \otimes B - A \otimes 1) \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\|$$
$$\leq \|1 \otimes B - A \otimes 1\| \left\| \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\|$$
$$\leq \|1 \otimes B - A \otimes 1\| \int_0^1 |u-\lambda| \, \|f'((1-u) A \otimes 1 + u 1 \otimes B)\|\, du$$

for all $\lambda \in [0,1]$.

Observe that, by Lemma 2.1

$$|f'((1-u)A\otimes 1+u1\otimes B)|=\int_I\int_I|f'((1-u)t+us)|\,dE_t\otimes dF_s$$

for $u\in[0,1]$.

Note that

$$|f'((1-u)t+us)|\le\|f'\|_{I,\infty}$$

for $u\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}&|f'((1-u)A\otimes 1+u1\otimes B)| \\ &=\int_I\int_I|f'((1-u)t+us)|\,dE_t\otimes dF_s\le\|f'\|_{I,\infty}\int_I\int_I dE_t\otimes dF_s=\|f'\|_{I,\infty}\end{aligned}\tag{2.19}$$

for $u\in[0,1]$. This implies that

$$\|f'((1-u)A\otimes 1+u1\otimes B)\|\le\|f'\|_{I,\infty}$$

for $u\in[0,1]$, which gives

$$\begin{aligned}&\int_0^1|u-\lambda|\,\|f'((1-u)A\otimes 1+u1\otimes B)\|\,du \\ &\le\|f'\|_{I,\infty}\int_0^1|u-\lambda|\,du=\|f'\|_{I,\infty}\frac{(1-\lambda)^2+\lambda^2}{2} \\ &=\|1\otimes B-A\otimes 1\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|f'\|_{I,\infty}\end{aligned}$$

for all $\lambda\in[0,1]$, which proves (2.16).

3. RELATED RESULTS

In this section we give some norm trapezoid inequalities under various assumptions of convexity for the absolute value of the derivative $|f'|$ on I.

Theorem 3.1. *Assume that f is continuously differentiable on I with $|f'|$ is convex on I, A and B are selfadjoint operators with* $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$, *then*

$$\begin{aligned}&\left\|(1-\lambda)1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\| \\ &\le\|1\otimes B-A\otimes 1\|\left[p(1-\lambda)\|f'(A)\|+p(\lambda)\|f'(B)\|\right],\end{aligned}\tag{3.1}$$

for $\lambda\in[0,1]$, *where*

$$p(\lambda):=\frac{1}{6}\left(2\lambda^3-3\lambda+2\right).$$

In particular, we have the trapezoid inequality

$$\begin{aligned}&\left\|\frac{1\otimes f(B)+f(A)\otimes 1}{2}-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\| \\ &\le\frac{1}{8}\|1\otimes B-A\otimes 1\|\left(\|f'(A)\|+\|f'(B)\|\right).\end{aligned}\tag{3.2}$$

Proof. Since $|f'|$ is convex on I, then

$$|f'((1-u)t+us)|\le(1-u)|f'(t)|+u|f'(s)|$$

for all $t,s\in I$ and $u\in[0,1]$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}&\int_I\int_I|f'((1-u)t+us)|\,dE_t\otimes dF_s \\ &\le(1-u)\int_I\int_I|f'(t)|\,dE_t\otimes dF_s+u\int_I\int_I|f'(s)|\,dE_t\otimes dF_s,\end{aligned}$$

namely

$$\left|f'\left((1-u)A\otimes 1+u1\otimes B\right)\right|\leq(1-u)\left|f'(A)\right|\otimes 1+u\left|f'(B)\right|\otimes 1 \tag{3.3}$$

for all $u\in[0,1]$.

If we take the norm in (3.3), then we get

$$\begin{aligned}\left\|f'\left((1-u)A\otimes 1+u1\otimes B\right)\right\|&\leq\left\|(1-u)\left|f'(A)\right|\otimes 1+u\left|f'(B)\right|\otimes 1\right\|\\&\leq(1-u)\left\|f'(A)\right\|+u\left\|f'(B)\right\|\end{aligned} \tag{3.4}$$

for all $u\in[0,1]$.

By (2.18) and (3.4) we derive

$$\begin{aligned}&\left\|(1-\lambda)1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\|\\&\leq\|1\otimes B-A\otimes 1\|\int_0^1|u-\lambda|\left\|f'\left((1-u)A\otimes 1+u1\otimes B\right)\right\|du\\&\leq\|1\otimes B-A\otimes 1\|\\&\times\left[\left\|f'(A)\right\|\int_0^1|u-\lambda|(1-u)\,du+\left\|f'(B)\right\|\int_0^1 u|u-\lambda|\,du\right],\end{aligned} \tag{3.5}$$

for $\lambda\in[0,1]$.

Observe that, for $\lambda\in[0,1]$,

$$\int_0^1 u|u-\lambda|\,du=p(\lambda)\ \text{ and }\ \int_0^1(1-u)|u-\lambda|\,du=p(1-\lambda).$$

By utilizing (3.5) we derive (3.1).

We recall that the function $g:I\to\mathbb{R}$ is *quasi-convex*, if

$$g((1-\lambda)t+\lambda s)\leq\max\{g(t),g(s)\}=\frac{1}{2}\left(g(t)+g(s)+|g(t)-g(s)|\right)$$

for all $t,s\in I$ and $\lambda\in[0,1]$.

Theorem 3.2. *Assume that f is continuously differentiable on I with $|f'|$ is quasi-convex on I, A and B are selfadjoint operators with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$, then*

$$\begin{aligned}&\left\|(1-\lambda)1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\|\\&\leq\frac{1}{2}\|1\otimes B-A\otimes 1\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\\&\times\left(\left\|\left|f'(A)\right|\otimes 1+1\otimes\left|f'(B)\right|\right\|+\left\|\left|f'(A)\right|\otimes 1-1\otimes\left|f'(B)\right|\right\|\right).\end{aligned} \tag{3.6}$$

In particular,

$$\begin{aligned}&\left\|\frac{1\otimes f(B)+f(A)\otimes 1}{2}-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\|\\&\leq\frac{1}{8}\|1\otimes B-A\otimes 1\|\\&\times\left(\left\|\left|f'(A)\right|\otimes 1+1\otimes\left|f'(B)\right|\right\|+\left\|\left|f'(A)\right|\otimes 1-1\otimes\left|f'(B)\right|\right\|\right).\end{aligned} \tag{3.7}$$

Proof. Since $|f'|$ is quasi-convex on I, then we get

$$\left|f'((1-u)t+us)\right|\leq\frac{1}{2}\left(\left|f'(t)\right|+\left|f'(s)\right|+\left|\left|f'(t)\right|-\left|f'(s)\right|\right|\right)$$

for all for $u\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}&\int_I\int_I\left|f'((1-u)t+us)\right|dE_t\otimes dF_s\\&\leq\frac{1}{2}\int_I\int_I\left(\left|f'(t)\right|+\left|f'(s)\right|+\left|\left|f'(t)\right|-\left|f'(s)\right|\right|\right)dE_t\otimes dF_s\end{aligned}$$

namely

$$\begin{aligned}&\left|f'\left((1-u)A\otimes 1+u1\otimes B\right)\right|\\&\leq\frac{1}{2}\left(\left|f'(A)\right|\otimes 1+1\otimes\left|f'(B)\right|+\left|\left|f'(A)\right|\otimes 1-1\otimes\left|f'(B)\right|\right|\right)\end{aligned}$$

for all for $u\in[0,1]$.

If we take the norm, then we get

$$\begin{aligned}&\left\|f'\left((1-u)A\otimes 1+u1\otimes B\right)\right\|\\&\leq\frac{1}{2}\left(\left\|\left|f'(A)\right|\otimes 1+1\otimes\left|f'(B)\right|\right\|+\left\|\left|f'(A)\right|\otimes 1-1\otimes\left|f'(B)\right|\right\|\right)\end{aligned}\tag{3.8}$$

for all for $u\in[0,1]$.

By (2.18) and (3.8)

$$\begin{aligned}&\left\|(1-\lambda)1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\|\\&\leq\left\|1\otimes B-A\otimes 1\right\|\int_0^1\left|u-\lambda\right|\left\|f'\left((1-u)A\otimes 1+u1\otimes B\right)\right\|du\\&\leq\frac{1}{2}\left\|1\otimes B-A\otimes 1\right\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\\&\times\left(\left\|\left|f'(A)\right|\otimes 1+1\otimes\left|f'(B)\right|\right\|+\left\|\left|f'(A)\right|\otimes 1-1\otimes\left|f'(B)\right|\right\|\right)\end{aligned}$$

for all $\lambda\in[0,1]$ and the inequality (3.6) is proved.

4. EXAMPLES

It is known that if U and V are commuting, i.e. $UV=VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V)=\exp(V)\exp(U)=\exp(U+V).$$

Also, if U is invertible and $a,b\in\mathbb{R}$ with $a<b$ then

$$\int_a^b\exp(tU)\,dt=U^{-1}\left[\exp(bU)-\exp(aU)\right].$$

Moreover, if U and V are commuting and $V-U$ is invertible, then

$$\begin{aligned}\int_0^1\exp\left((1-s)U+sV\right)ds&=\int_0^1\exp\left(s(V-U)\right)\exp(U)\,ds\\&=\left(\int_0^1\exp\left(s(V-U)\right)ds\right)\exp(U)\\&=(V-U)^{-1}\left[\exp(V)-\exp(U)\right].\end{aligned}$$

Since the operators $U=A\otimes 1$ and $V=1\otimes B$ are commutative and if $1\otimes B-A\otimes 1$ is invertible, then

$$\begin{aligned}&\int_0^1\exp\left((1-u)A\otimes 1+u1\otimes B\right)du\\&=(1\otimes B-A\otimes 1)^{-1}\left[\exp(1\otimes B)-\exp(A\otimes 1)\right].\end{aligned}$$

If A, B are selfadjoint operators with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset[m,M]$, with $m<M$ real numbers and $1\otimes B-A\otimes 1$ is invertible, then by (2.16)

$$\begin{aligned}&\|(1-\lambda)\exp(A)\otimes 1+\lambda 1\otimes\exp(B)\\&-(1\otimes B-A\otimes 1)^{-1}\left[\exp(1\otimes B)-\exp(A\otimes 1)\right]\|\\&\leq\exp(M)\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\left\|1\otimes B-A\otimes 1\right\|,\end{aligned}\tag{4.1}$$

for $\lambda\in[0,1]$.

Since for $f(t) = \exp t$, $t \in \mathbb{R}$, $|f'|$ is convex, then by Theorem 3.1 we get

$$\begin{aligned} &\left\|(1-\lambda)\exp(A)\otimes 1+\lambda 1\otimes\exp(B)\right. \\ &\left.-(1\otimes B-A\otimes 1)^{-1}\left[\exp(1\otimes B)-\exp(A\otimes 1)\right]\right\| \\ &\leq \frac{1}{2}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\| \\ &\times\left(\|\exp(A)\otimes 1+1\otimes\exp(B)\|+\|\exp(A)\otimes 1-1\otimes\exp(B)\|\right) \end{aligned} \tag{4.2}$$

for $\lambda \in [0,1]$.

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RESEARCH ARTICLE

Positive Toeplitz Operators from a Weighted Harmonic Bloch Space into Another

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ABSTRACT
In the present paper, we define positive general Toeplitz operators between weighted harmonic Bloch spaces b^∞_α on the unit ball of $\mathbb{R}^n$ for the full range of parameter $\alpha \in \mathbb{R}$, where symbols are positive Borel measures on the unit ball of $\mathbb{R}^n$. We characterize the boundedness and compactness of Toeplitz operators from one weighted harmonic Bloch space into another in terms of Carleson measures and vanishing Carleson measures. Recently, in Doğan (2022), positive symbols of bounded and compact general Toeplitz operators between harmonic Bergman-Besov spaces are completely characterized in term of Carleson measures and vanishing Carleson measures. Our results extend those known for harmonic Bloch space.

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Keywords: toeplitz operator, harmonic Bergman-Besov space, weighted harmonic Bloch space, carleson measure

1. INTRODUCTION

Let $n \geq 2$ be an integer and $\mathbb{B} = \mathbb{B}_n$ be the open unit ball of $\mathbb{R}^n$. Let ν be the normalized Lebesgue measure on $\mathbb{B}$. For $\alpha \in \mathbb{R}$, we define the weighted measures ν_α on $\mathbb{B}$ by

$$d\nu_\alpha(x) = \frac{1}{V_\alpha}(1-|x|^2)^\alpha d\nu(x).$$

These measures are finite only when $\alpha > -1$ and in this case we select the constant V_α so that $\nu_\alpha(\mathbb{B}) = 1$. Naturally $V_0 = 1$. If $\alpha \leq -1$, we set $V_\alpha = 1$. For $0 < p < \infty$, we denote the Lebesgue classes with respect to ν_α by L^p_α and the corresponding norms by $\|\cdot\|_{L^p_\alpha}$.

Let $h(\mathbb{B})$ be the space of all complex-valued harmonic functions on $\mathbb{B}$ endowed with the topology of uniform convergence on compact subsets. For $\alpha > -1$, the harmonic Bergman space b^p_α is defined as $b^p_\alpha = L^p_\alpha \cap h(\mathbb{B})$ with norm $\|\cdot\|_{L^p_\alpha}$. When $p = 2$, b^2_α is a reproducing kernel Hilbert space endowed with the inner product $[f,g]_{b^2_\alpha} = \int_{\mathbb{B}} f\overline{g}\, d\nu_\alpha(x)$ and with the reproducing kernel $R_\alpha(x,y)$ such that $f(x) = [f(.), R_\alpha(x,\cdot)]_{b^2_\alpha}$ for every $f \in b^2_\alpha$ and $x \in \mathbb{B}$. R_α is real-valued and symmetric in its variables. The homogeneous expansion of $R_\alpha(x,y)$ is given in the $\alpha > -1$ part of the formulas (2) and (3) below (see Djrbashian and Shamoian (1988), Gergün et al. (2016)).

The well-known harmonic Bloch space b is consists of all $f \in h(\mathbb{B})$ such that

$$\sup_{x\in\mathbb{B}}(1-|x|^2)|\nabla f(x)|$$

is finite. Let L^∞ be the Lebesgue class of essentially bounded functions on $\mathbb{B}$. For $\alpha \in \mathbb{R}$ we define

$$L^\infty_\alpha = \{\varphi : (1-|x|^2)^\alpha \varphi(x) \in L^\infty\},$$

so that $L^\infty_0 = L^\infty$ and with norm $\|\varphi\|_{L^\infty_\alpha} = \|(1-|x|^2)^\alpha\varphi(x)\|_{L^\infty}$. For $\alpha > 0$, the weighted harmonic Bloch space b^∞_α is $h(\mathbb{B}) \cap L^\infty_\alpha$ endowed with the norm $\|\cdot\|_{L^\infty_\alpha}$, and is clearly imbedded in L^∞_α by the inclusion i.

For $\alpha > 0$, the harmonic Bergman projection $Q_\alpha : L^\infty_\alpha \to b^\infty_\alpha$ is given by the integral operator

$$Q_\alpha f(x) = \frac{1}{V_\alpha}\int_{\mathbb{B}} R_\alpha(x,y)f(y)(1-|y|^2)^\alpha d\nu(y) \quad (f \in L^\infty_\alpha). \tag{1}$$

It has a significant importance in the theory and the question when $Q_\alpha : L^\infty_\beta \to b^\infty_\beta$ is bounded is studied in many sources such

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as Choe et al. (2001); Jevtić and Pavlović (1999); Ligocka (1987) for $\beta = 0$ and Ren and Kähler (2003) with a different integral representation valid for $\beta > -1$. Then we define the Toeplitz operator ${}_{\alpha}T_{\theta} : b^{\infty}_{\alpha} \to b^{\infty}_{\alpha}$ with symbol $\theta \in L^1$ by ${}_{\alpha}T_{\theta} = Q_{\alpha} M_{\theta} i$, where M_{θ} is the operator of multiplication by θ. For a finite complex Borel measure μ on $\mathbb{B}$, the Toeplitz operator ${}_{\alpha}T_{\mu}$ is defined by

$$ {}_{\alpha}T_{\mu} f(x) = \frac{1}{V_{\alpha}} \int_{\mathbb{B}} R_{\alpha}(x,y) f(y)(1-|y|^2)^{\alpha} d\mu(y) $$

for $f \in L^{\infty}_{\alpha}$. The operator ${}_{\alpha}T_{\mu}$ is more general and reduces to ${}_{\alpha}T_{\theta}$ when $d\mu = \theta dv$. Toeplitz operators have been studied extensively on the harmonic Bergman spaces by many authors. Particularly, the boundedness and compactness of Toeplitz operators with positive symbols are completely characterized in term of Carleson measures as in Miao (1998), Miao (1997) on the ball and in Choe et al. (2004a) on smoothly bounded domains. The Boundedness and compactness of Toeplitz operators with positive symbols from a harmonic Bergman space into another are characterized in Choe et al. (2004b) on smoothly bounded domains and in Choe et al. (2002) on the half space.

The harmonic Bergman b^p_{α} and Bloch b^{∞}_{α} spaces can be extended to all real α. These are studied comprehensively in Gergün et al. (2016) and Doğan and Üreyen (2018), respectively. We call the extended set b^p_{α} $(\alpha \in \mathbb{R})$ harmonic Bergman-Besov spaces and the corresponding reproducing kernels $R_{\alpha}(x,y)$ $(\alpha \in \mathbb{R})$ harmonic Bergman-Besov kernels. The homogeneous expansion of $R_{\alpha}(x,y)$ can be expressed in terms of zonal harmonics $Z_k(x,y)$

$$ R_{\alpha}(x,y) = \sum_{k=0}^{\infty} \gamma_k(\alpha) Z_k(x,y) \quad (\alpha \in \mathbb{R},\ x,y \in \mathbb{B}), \tag{2} $$

where (see (Gergün et al. 2009, Theorem 3.7), (Gergün et al. 2016, Theorem 1.3))

$$ \gamma_k(\alpha) := \begin{cases} \dfrac{(1+n/2+\alpha)_k}{(n/2)_k}, & \text{if } \alpha > -(1+n/2); \\[2ex] \dfrac{(k!)^2}{(1-(n/2+\alpha))_k (n/2)_k}, & \text{if } \alpha \leq -(1+n/2), \end{cases} \tag{3} $$

and $(a)_b$ is the Pochhammer symbol. For further details about zonal harmonics, see (Axler et al. 2001, Chapter 5).

By using the radial differential operators D^t_s $(s,t \in \mathbb{R})$ introduced in Gergün et al. (2009) and Gergün et al. (2016), we can define the weighted harmonic Bloch spaces b^{∞}_{α} for all $\alpha \in \mathbb{B}$. These operators are compatible with reproducing kernels and yet mapping $h(\mathbb{B})$ onto itself. We present the fundamental properties of D^t_s in Section 2. The linear transformation I^t_s is defined by

$$ I^t_s f(x) := (1-|x|^2)^t D^t_s f(x), $$

for $f \in h(\mathbb{B})$.

Definition 1.1. For $\alpha \in \mathbb{R}$, we define the weighted harmonic Bloch space b^{∞}_{α} to consist of all $f \in h(\mathbb{B})$ for which $I^t_s f$ belongs to L^{∞}_{α} for some s and t satisfying (see Doğan and Üreyen (2018))

$$ \alpha + t > 0. \tag{4} $$

The quantity

$$ \|f\|_{b^{\infty}_{\alpha}} = \|I^t_s f\|_{L^{\infty}_{\alpha}} = \sup_{x \in \mathbb{B}} (1-|x|^2)^{\alpha+t} |D^t_s f(x)| < \infty, $$

defines a norm on b^{∞}_{α} for any such $s,t \in \mathbb{R}$.

Note that this definition is independent of s,t under (4), and the norms in these spaces are all equivalent. Therefore the operator I^t_s isometrically imbeds b^{∞}_{α} into L^{∞}_{α} for a given pair s,t if and only if (4) holds.

Harmonic Bergman-Besov projections Q_s that map Lebesgue classes boundedly onto weighted Bloch spaces b^{∞}_{α} can be identified exactly as in the case of $\alpha > 0$ by

$$ s > \alpha - 1. \tag{5} $$

Then I^t_s is a right inverse to Q_s. See Doğan and Üreyen (2018) for more details.

Let $\alpha \in \mathbb{R}$, s and t satisfing (5) and (4), and a measurable function θ on $\mathbb{B}$ be given. Then Q_s forces us to define Toeplitz operators on all b^{∞}_{α} as follows. We define the Toeplitz operator ${}_{s,t}T_{\theta} : b^{\infty}_{\alpha} \to b^{\infty}_{\alpha}$ with symbol θ by ${}_{s,t}T_{\theta} = Q_s M_{\theta} I^t_s$. Explicitly,

$$ {}_{s,t}T_{\theta} f(x) = \int_{\mathbb{B}} R_s(x,y)\theta(y) I^t_s f(y) dv_s(y) \quad (f \in b^{\infty}_{\alpha}). $$

We see that ${}_{s,t}T_{\theta}$ makes sense if $\theta \in L^1_{s-\alpha}$. When $\alpha > 0$, we can take $t = 0$ and a value of s satisfying (5) is $s = \alpha$. Then I^0_{α} is inclusion, and ${}_{s,t}T_{\theta}$ reduces to the classical Toeplitz operator ${}_{\alpha}T_{\theta} = Q_{\alpha} M_{\theta} i$ on b^{∞}_{α}, $\alpha > 0$. We use the word classical to mean a Toeplitz operator with $i = I_\alpha^0$. It is possible to take $s \neq \alpha$ also when $\alpha > -1$. Thus we have more general Toeplitz operators defined via I_s^t strictly on harmonic Bloch spaces too. It turns out that the properties of Toeplitz operators studied in this paper are independent of s, t under (5) and (4).

Since the integral form for ${}_{s,t}T_\theta$ is obtained, we can now define general Toeplitz operators on b_α^∞ with symbol μ. Let α, and s and t satisfing (5) and (4) be given. We let

$$d\kappa(y) = (1 - |y|^2)^{s+t} \, d\mu(y)$$

and define

$$\begin{aligned} {}_{s,t}T_\mu f(x) &= \frac{1}{V_s} \int_{\mathbb{B}} R_s(x,y) I_s^t f(y) (1 - |y|^2)^s \, d\mu(y) \\ &= \frac{1}{V_s} \int_{\mathbb{B}} R_s(x,y) D_s^t f(y) d\kappa(y). \end{aligned}$$

The operator ${}_{s,t}T_\mu$ is more general and reduces to ${}_{s,t}T_\theta$ when $d\mu = \theta d\nu$. It makes sense when

$$d\psi(y) = (1 - |y|^2)^{-(\alpha+t)} d\kappa(y) = (1 - |y|^2)^{s-\alpha} d\mu(y)$$

is finite. Note that μ need not be finite in conformity with that α is unrestricted.

The *holomorphic* counterpart of our characterizations from a Dirichlet space into itself have been obtained in Alpay and Kaptanoğlu (2007). Recently, in Doğan (2022), positive symbols of bounded and compact general Toeplitz operators between harmonic Bergman-Besov spaces are completely characterized in term of Carleson measures. In the present paper, we consider the positive Toeplitz operator ${}_{s,t}T_\mu$ and characterize those that are bounded and compact from a weighted harmonic Bloch space $b_{\alpha_1}^\infty$ into another $b_{\alpha_2}^\infty$ for $\alpha_1, \alpha_2 \in \mathbb{R}$. Our main tool is Carleson measure.

Suppose μ is a positive Borel measure on $\mathbb{B}$. For $\alpha > -1$, we say that μ is a α-Carleson measure if the inclusion $i : b_\alpha^p \to L^p(\mu)$ is bounded, that is, if

$$\left(\int_{\mathbb{B}} |f(x)|^p \, d\mu(x) \right)^{1/p} \lesssim \|f\|_{b_\alpha^p}, \qquad (f \in b_\alpha^p).$$

As is usual with Carleson measure theorems, the property of being an α-Carleson measure is independent of p, because Theorem 3.1 is true for any p. However, it depends on $\alpha > -1$. So for a fixed α, an α-Carleson measure for one b_α^p is a Carleson measure for all b_α^p with the same α. We can now state our main result.

Theorem 1.2. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$. Suppose that $\alpha_1 + t > 0$, $\alpha_2 + t > 0$ and*

$$s > \alpha_i - 1, \qquad i = 1, 2. \tag{6}$$

Let

$$\gamma = s + t + \alpha_1 - \alpha_2.$$

Let μ be a positive Borel measure on $\mathbb{B}$ and $d\kappa(y) = (1 - |y|^2)^{s+t} d\mu(y)$. Then the following are equivalent:

(i) ${}_{s,t}T_\mu : b_{\alpha_1}^\infty \to b_{\alpha_2}^\infty$ *is bounded.*
(ii) *κ is a γ-Carleson measure.*

In order to characterize compact Toeplitz operators ${}_{s,t}T_\mu$ with positive μ from weighted harmonic Bloch spaces $b_{\alpha_1}^\infty$ into another $b_{\alpha_2}^\infty$ for all $\alpha_1, \alpha_2 \in \mathbb{R}$, we present the notion of vanishing α-Carleson measures. If, for any sequence $\{f_k\}$ in b_α^p with $f_k \to 0$ uniformly on each compact subset of $\mathbb{B}$ and $\|f_k\|_{b_\alpha^p} \leq 1$, where

$$\lim_{k \to \infty} \int_{\mathbb{B}} |f_k(x)|^p \, d\mu(x) = 0,$$

then $\mu \geq 0$ is called vanishing α-Carleson measure. One can see from Theorem 3.3 that the notion of vanishing α-Carleson measures on b_α^p is also independent of p.

Theorem 1.3. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$. Let s, t, γ and κ be as in Theorem 1.2. Then the following are equivalent:*

(i) ${}_{s,t}T_\mu : b_{\alpha_1}^\infty \to b_{\alpha_2}^\infty$ *is compact.*
(ii) *κ is a vanishing γ-Carleson measure.*

The proofs of our results are inspired by the work of Pau and Zhao (2015), where bounded and compact classical Toeplitz operators between holomorphic weighted Bergman spaces are characterized.

We briefly summarize the notation and some preliminary material in Section 2. Section 3 is devoted to recall some characterizations of (vanishing) α-Carleson measures. We will give the proof of our main results, Theorems 1.2 and 1.3, in Section 4.

Throughout the paper, for two positive expressions A and B, $A \lesssim B$ means that there exists a positive constant C, whose exact value is inessential, such that $A \le CB$. We also use $A \sim B$ to mean both $A \lesssim B$ and $B \lesssim A$.

2. NOTATION AND PRELIMINARIES

The Pochhammer symbol $(a)_b$ is defined by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)},$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function. Stirling formula provides

$$\frac{(a)_c}{(b)_c} \sim c^{a-b} \quad (c \to \infty). \tag{7}$$

2.1. Pseudo-hyperbolic Metric

For any $a \in \mathbb{B}$ with $a \neq 0$, the Möbius transformation on $\mathbb{B}$ that exchanges the points 0 and a is

$$\varphi_a(x) = \frac{(1-|a|^2)(a-x) + |a-x|^2 a}{[x,a]^2}.$$

Here we use the abbreviation

$$[x,a] = \sqrt{1 - 2x \cdot a + |x|^2|a|^2},$$

where $x \cdot a$ denotes the usual inner product in $\mathbb{R}^n$. Note that $[x,x] = 1 - |x|^2$. The pseudo-hyperbolic distance on $\mathbb{B}$ between $x, y \in \mathbb{B}$ is defined by

$$\rho(x,y) = |\varphi_x(y)| = \frac{|x-y|}{[x,y]}.$$

We need the following lemma from (Choe et al. 2008, Lemma 2.2).

Lemma 2.1. *If $a, x, y \in \mathbb{B}$, then*

$$\frac{1-\rho(x,y)}{1+\rho(x,y)} \le \frac{[x,a]}{[y,a]} \le \frac{1+\rho(x,y)}{1-\rho(x,y)}.$$

The following lemma shows that if $x, y \in \mathbb{B}$ are close in the pseudo-hyperbolic metric, then certain quantities are comparable. Its proof clearly follows from Lemma 2.1.

Lemma 2.2. *Let $0 < \delta < 1$. Then*

$$[x,y] \sim 1 - |x|^2 \sim 1 - |y|^2,$$

for all $x, y \in \mathbb{B}$ with $\rho(x,y) < \delta$.

For $x \in \mathbb{B}$, and $0 < \delta < 1$, the pseudo-hyperbolic ball with center x and radius δ is given by $E_\delta(x)$. We note that $E_\delta(x)$ is an Euclidean ball with center at c and radius r, where

$$c = \frac{(1-\delta^2)x}{1-\delta^2|x|^2} \qquad \text{and} \qquad r = \frac{(1-|x|^2)\delta}{1-\delta^2|x|^2}.$$

So, we have $\nu(E_\delta(x)) \sim (1-|x|^2)^n$ for fixed $0 < \delta < 1$. More generally, for $\alpha \in \mathbb{R}$, by Lemma 2.2

$$\nu_\alpha(E_\delta(x)) = \frac{1}{V_\alpha} \int_{E_\delta(x)} (1-|y|^2)^\alpha \, d\nu(y) \sim (1-|x|^2)^\alpha \nu(E_\delta(x)) \sim (1-|x|^2)^{\alpha+n}. \tag{8}$$

Let $\{a_k\}$ be a sequence of points in $\mathbb{B}$ and $0 < \delta < 1$. We say that $\{a_k\}$ is δ-separated if $\rho(a_j, a_k) \ge \delta$ for all $j \neq k$. See Luecking (1993) for a proof of the following lemma.

Lemma 2.3. *For fixed $0 < \delta < 1$, There exists a sequence of points $\{a_k\}$ in $\mathbb{B}$ such that the following hold.*

(i) $\{a_k\}$ *is δ-separated.*

(ii) $\displaystyle\bigcup_{k=1}^{\infty} E_\delta(a_k) = \mathbb{B}$.

(iii) There exists a positive integer N such that each $x \in \mathbb{B}$ is contained in at most N of the balls $E_\delta(a_k)$.

From now on, whenever we use representation like $\widehat{\mu}_{\alpha,\delta}(a_k)$, the sequence $\{a_k\} = \{a_k(\delta)\}$ will refer to the sequence chosen in Lemma 2.3 at all times.

2.2. The Radial Differential Operators D_s^t

If $f \in h(\mathbb{B})$, then f has a homogeneous expansion $f(x) = \sum_{k=0}^{\infty} f_k(x)$ with homogeneous harmonic polynomials f_k of degree k. The series converges absolutely and uniformly on compact subsets of $\mathbb{B}$.

For every $\alpha \in \mathbb{R}$, $R_\alpha(x,y)$ is harmonic as a function of either of its variables on $\overline{\mathbb{B}}$. We have by (7)

$$\gamma_k(\alpha) \sim k^{1+\alpha} \quad (k \to \infty) \tag{9}$$

for every $\alpha \in \mathbb{R}$. By using the coefficients $\gamma_k(\alpha)$ in the Bergman-Besov kernels, we define the radial differential operators D_s^t of order t.

Definition 2.4. Let $f = \sum_{k=0}^{\infty} f_k \in h(\mathbb{B})$ be given by its homogeneous expansion. For $s, t \in \mathbb{R}$ we define D_s^t of order t by

$$D_s^t f := \sum_{k=0}^{\infty} \frac{\gamma_k(s+t)}{\gamma_k(s)} f_k.$$

By (9), $\gamma_k(s+t)/\gamma_k(s) \sim k^t$ for any s, t. For every $s \in \mathbb{R}$, $D_s^0 = I$, the identity. The additive property $D_{s+t}^z D_s^t = D_s^{z+t}$ of D_s^t implies that it is invertible with two-sided inverse

$$D_{s+t}^{-t} D_s^t = D_s^t D_{s+t}^{-t} = I. \tag{10}$$

For every $s, t \in \mathbb{R}$, the operator $D_s^t : h(\mathbb{B}) \to h(\mathbb{B})$ is continuous in the topology of uniform convergence on compact subsets (see (Gergün et al. 2016, Theorem 3.2)). The operator D_s^t is constructed so that in all cases

$$D_s^t R_s(x,y) = R_{s+t}(x,y), \tag{11}$$

where differentiation is performed on one of the variables.

One of the most crucial properties about the map D_s^t is that it enables us to pass from one Bloch space to another. Moreover, we have the following isomorphism. For a proof see (Doğan and Üreyen 2018, Proposition 4.6).

Lemma 2.5. *Given α, for any $s, t \in \mathbb{R}$, the map $D_s^t : b_\alpha^\infty \to b_{\alpha+t}^\infty$ is an isomorphism.*

The following duality result is (Doğan and Üreyen 2018, Theorem 5.4).

Theorem 2.6. *Let $q \in \mathbb{R}$. Pick s', t' such that*

$$s' > q,$$
$$q + t' > -1.$$

The dual of b_q^1 can be identified with b_α^∞ (for any $\alpha \in \mathbb{R}$) under the pairing

$$\langle f, g \rangle = \int_{\mathbb{B}} I_{s'}^{t'} f \, \overline{I_{t'+q+\alpha}^{s'-q-\alpha} g} \, d\nu_{q+\alpha}, \qquad (f \in b_q^1, \ g \in b_\alpha^\infty).$$

2.3. Estimates on Harmonic Bergman-Besov Kernels

In case $\alpha > -1$, Bergman Kernels $R_\alpha(x,y)$ are real-valued and well-studied by many authors. The curious reader is referred to Gergün et al. (2016) for extension of these properties to all $\alpha \in \mathbb{R}$.

We have the following pointwise upper bounds on the Bergman-Besov kernels. For a proof see Coifman and Coifman (1980); Ren (2003) when $\alpha > -1$ and Gergün et al. (2016) when $\alpha \in \mathbb{R}$.

Lemma 2.7. *Let $\alpha \in \mathbb{R}$. For all $x, y \in \mathbb{B}$,*

$$|R_\alpha(x,y)| \lesssim \begin{cases} \dfrac{1}{[x,y]^{\alpha+n}}, & \text{if } \alpha > -n; \\ 1 + \log \dfrac{1}{[x,y]}, & \text{if } \alpha = -n; \\ 1, & \text{if } \alpha < -n. \end{cases}$$

The next result shows that the first part of the above estimate continues to hold when x and y are close enough in the pseudo-hyperbolic distance. It can be proved in just the same way as (Miao 1998, Proposition 5).

Lemma 2.8. *Assume $\alpha > -n$. Then there exists a $\delta \in (0,1)$ such that*

$$R_\alpha(x,y) \sim \frac{1}{(1-|x|^2)^{\alpha+n}}$$

whenever $x \in \mathbb{B}$ and $y \in E_\delta(x)$.

3. CARLESON MEASURES

Carleson measures on more general domains have been well studied by many authors. In this subsection we will recollect some characterizations of (vanishing) α-Carleson measures for b^p_α $(\alpha > -1)$ in terms of the averaging functions.

Let $0 < \delta < 1$, the averaging function $\widehat{\mu}_\delta$ of μ is defined by

$$\widehat{\mu}_\delta(x) = \frac{\mu(E_\delta(x))}{\nu(E_\delta(x))} \qquad (x \in \mathbb{B}).$$

Also, for general case $\alpha \in \mathbb{R}$ we define

$$\widehat{\mu}_{\alpha,\delta}(x) := \frac{\mu(E_\delta(x))}{\nu_\alpha(E_\delta(x))} \qquad (x \in \mathbb{B}).$$

By (8), $\widehat{\mu}_{\alpha,\delta}(x) \sim \mu(E_\delta(x))/(1-|x|^2)^{\alpha+n}$.

Now, we cite the next characterization of Carleson measures in terms of averaging functions which justify the fact that the notion of α-Carleson measures on b^p_α depend only on α.

Theorem 3.1. *Assume μ is a positive Borel measure on $\mathbb{B}$, $0 < p < \infty$ and $\alpha > -1$. The following are equivalent:*

(a) μ is a α-Carleson measure.
(b) $\widehat{\mu}_{\alpha,\delta} \lesssim 1$ for some (every) $0 < \delta < 1$.

Notice that the condition (b) is equivalent to

$$\mu(E_\delta(x)) \lesssim (1-|x|^2)^{\alpha+n} \quad \text{for some (every) } 0 < \delta < 1.$$

Proof. For the case $\alpha = 0$, equivalence of (a) and (b) is given in (Choe et al. 2004a, Theorem 3.5) for bounded smooth domains. The proof works just as well for general α too.

We also need the following proposition. Its proof is similar to that of (Doğan 2022, Proposition 3.6), but for the sake of completeness, we give the simplified version of it.

Proposition 3.2. *Let μ be a positive Borel measure on $\mathbb{B}$. Let $\alpha_1 > 0$ and $-1 < \alpha_2 < \infty$ and let*

$$\varrho = \alpha_1 + \alpha_2.$$

If μ is a ϱ-Carleson measure, then

$$\int_{\mathbb{B}} |f(x)||g(x)|\, d\mu(x) \lesssim \|f\|_{b^\infty_{\alpha_1}} \|g\|_{b^1_{\alpha_2}} \qquad (f \in b^\infty_{\alpha_1}, g \in b^1_{\alpha_2}).$$

Proof. First, for $f \in b^\infty_{\alpha_1}$,

$$\int_{\mathbb{B}} |f(x)||g(x)|\, d\mu(x) \leq \|f\|_{b^\infty_{\alpha_1}} \int_{\mathbb{B}} |g(x)|(1-|x|^2)^{-\alpha_1}\, d\mu(x).$$

Next, by Theorem 3.1, if μ is a $\varrho = \alpha_1 + \alpha_2$ Carleson measure, that is, $\mu(E_\delta(x)) \lesssim (1-|x|^2)^{\alpha_1+\alpha_2+n}$, then $(1-|x|^2)^{-\alpha_1}\, d\mu(x)$ is an α_2-Carleson measure since by Lemma 2.2,

$$\int_{E_\delta(x)} (1-|y|^2)^{-\alpha_1} d\mu(y) \sim (1-|x|^2)^{-\alpha_1} \mu(E_\delta(x)) \lesssim (1-|x|^2)^{\alpha_2+n}.$$

Thus by the definition of a Carleson measure

$$\int_{\mathbb{B}} |g(x)|(1-|x|^2)^{-\alpha_1}\, d\mu(x) \lesssim \|g\|_{b^1_{\alpha_2}}$$

for all $g \in b^1_{\alpha_2}$, which concludes the proof.

We next present a characterization of vanishing α-Carleson measures.

Theorem 3.3. *Let μ be a positive Borel measure on $\mathbb{B}$, $0 < p < \infty$ and $\alpha > -1$. The following are equivalent:*

(a) μ is a vanishing α-Carleson measure.
(b) $\lim_{|x|\to 1^-} \widehat{\mu}_{\alpha,\varepsilon}(x) = 0$ for some (every) $0 < \varepsilon < 1$.
(c) $\lim_{k\to\infty} \widehat{\mu}_{\alpha,\delta}(a_k) = 0$ for some (every) $0 < \delta < 1$.

Proof. For the case $\alpha = 0$, equivalence of (a), (b) and (c) is given in (Choe et al. 2004b, Theorem 3.5) for bounded smooth domains. It works just as well for general α too.

4. BOUNDEDNESS AND COMPACTNESS OF TOEPLITZ OPERATORS

Our goal in this section is to prove Theorems 1.2 and 1.3. Before that we introduce a helpful relation for transforming certain problems for general Toeplitz operators between b^∞_α, $\alpha \in \mathbb{R}^n$ to similar problems for classical Toeplitz operators between b^∞_α when $\alpha > 0$. The harmonic and holomorphic Bergman-Besov-space versions are in Doğan (2022) and Alpay and Kaptanoğlu (2007), respectively.

Theorem 4.1. *We have $D^t_s({}_{s,t}T_\mu) = ({}_{s+t}T_\kappa)D^t_s$, where*

$$ {}_{s+t}T_\kappa f(x) = \frac{1}{V_s}\int_{\mathbb{B}} R_{s+t}(x,y) f(y)\, d\kappa(y)$$

is the classical Toeplitz operator from $b^\infty_{\alpha_1+t}$ to $b^\infty_{\alpha_2+t}$. Consequently,

$$({}_{s,t}T_\mu) = D^{-t}_{s+t}({}_{s+t}T_\kappa)D^t_s, \qquad ({}_{s+t}T_\kappa) = D^t_s({}_{s,t}T_\mu)D^{-t}_{s+t}.$$

Proof. By differentiation under the integral sign and (11), we have

$$\begin{aligned} D^t_s({}_{s,t}T_\mu f)(x) &= \frac{1}{V_s}\int_{\mathbb{B}} R_{s+t}(x,y) D^t_s f(y)\, d\kappa(y) \\ &= ({}_{s+t}T_\kappa)(D^t_s f)(x) \qquad (f \in b^\infty_{\alpha_1}). \end{aligned}$$

For the other statements, we note that $(D^t_s)^{-1} = D^{-t}_{s+t}$ by (10).

By Theorem 4.1, ${}_{s,t}T_\mu$ is bounded from $b^\infty_{\alpha_1}$ to $b^\infty_{\alpha_2}$ if and only if ${}_{s+t}T_\kappa$ is bounded from $b^\infty_{\alpha_1+t}$ to $b^\infty_{\alpha_2+t}$. With all these preliminary works, we have laid the groundwork for proving our main results.

4.1. Proof of Theorem 1.2

(i) Implies (ii). Let ${}_{s,t}T_\mu : b^\infty_{\alpha_1} \to b^\infty_{\alpha_2}$ be bounded. First note that $[x,y] \gtrsim (1-|x|^2)$ and $[x,y] \gtrsim (1-|y|^2)$ for $x, y \in \mathbb{B}$. Then fix $x \in \mathbb{B}$ and consider $R_{s+t}(x,.)$. Under the condition $n+s+t > \alpha_1 + t$ provided by (6), it is elementary to verify using Lemma 2.7 that $R_{s+t}(x,.) \in b^\infty_{\alpha_1+t}$ with

$$\|R_{s+t}(x,.)\|_{b^\infty_{\alpha_1+t}} \lesssim \sup_{y\in\mathbb{B}} \frac{(1-|y|^2)^{\alpha_1+t}}{[x,y]^{n+s+t}} \lesssim \sup_{y\in\mathbb{B}} \frac{(1-|y|^2)^{\alpha_1+t}}{(1-|y|^2)^{\alpha_1+t}(1-|x|^2)^{n+s-\alpha_1}} = (1-|x|^2)^{\alpha_1-(n+s)}.$$

Take $\delta = \delta_0$ where δ_0 is the number made available by Lemma 2.8. We have by Lemma 2.2 and Lemma 2.8

$$\begin{aligned} \kappa(E_\delta(x)) &\lesssim \frac{V_{\alpha_1}}{V_s}(1-|x|^2)^{2(n+s+t)}\int_{E_\delta(x)} |R_{s+t}(x,y)|^2 d\kappa(y) \\ &\lesssim \frac{V_{\alpha_1}}{V_s}(1-|x|^2)^{2(n+s+t)}\int_{\mathbb{B}} |R_{s+t}(x,y)|^2 d\kappa(y) \\ &= (1-|x|^2)^{2(n+s+t)}\, {}_{s+t}T_\kappa[R_{s+t}(x,.)](x), \end{aligned}$$

and therefore

$$\begin{aligned} \widehat{\kappa}_{\gamma,\delta}(x) &= \frac{\kappa(E_\delta(x))}{\nu_\gamma(E_\delta(x))} \\ &\lesssim (1-|x|^2)^{2(n+s+t)-(n+\gamma)}\, {}_{s+t}T_\kappa[R_{s+t}(x,.)](x). \end{aligned}$$

On the other hand, by the definition of b^∞_α, $\alpha > 0$, the boundedness of the Toeplitz operator ${}_{s+t}T_\kappa$ and an inequality above, we obtain

$$\begin{aligned} {}_{s+t}T_\kappa[R_{s+t}(x,.)](x) &= |{}_{s+t}T_\kappa[R_{s+t}(x,.)](x)| \\ &\lesssim (1-|x|^2)^{-t-\alpha_2}\|{}_{s+t}T_\kappa[R_{s+t}(x,.)]\|_{b^\infty_{\alpha_2+t}} \\ &\lesssim (1-|x|^2)^{-t-\alpha_2}\|{}_{s+t}T_\kappa\|\|R_{s+t}(x,.)\|_{b^\infty_{\alpha_1+t}} \\ &\lesssim (1-|x|^2)^{-t-\alpha_2-n-s+\alpha_1}\|{}_{s+t}T_\kappa\|, \end{aligned}$$

where $\|{}_{s+t}T_\kappa\|$ denotes the operator norm of ${}_{s+t}T_\kappa : b^\infty_{\alpha_1+t} \to b^\infty_{\alpha_2+t}$. By bringing these estimates together, we conclude that

$$\widehat{\kappa}_{\gamma,\delta}(x) \lesssim \|{}_{s+t}T_\kappa\|.$$

By Theorem 3.1 this means that κ is a γ-Carleson measure.

(ii) Implies (i). Next, suppose κ is a γ-Carleson measure. Let

$$\alpha_2' = s - \alpha_2 > -1. \tag{12}$$

Since $\alpha_2' > -1$ and $\alpha_2 + t > 0$, applying Theorem 2.6 (with $q = \alpha_2'$, $\alpha = \alpha_2 + t$, $s' = \alpha_2' + \alpha_2 + t$ and $t' = 0$), we get that the dual of $b^1_{\alpha_2'}$ can be identified with $b^\infty_{\alpha_2+t}$ under each of the pairings

$$[f, g]_{b^2_{s+t}} = \int_{\mathbb{B}} f(x)\overline{g(x)}\, d\nu_{s+t}(x).$$

Let $f \in b^\infty_{\alpha_1+t}$ and $h \in b^1_{\alpha_2'}$. Fubini theorem and the reproducing formula (1.5) of Gergün et al. (2016), since $\alpha_2' > -1$ and $\alpha_2' < s + t$ by $\alpha_2 + t > 0$, yield

$$\begin{aligned}
[h, {}_{s+t}T_\kappa f]_{b^2_{s+t}} &= \frac{1}{V_s}\int_{\mathbb{B}} h(y)\int_{\mathbb{B}} R_{s+t}(x, y)\overline{f(x)}d\kappa(x)\, d\nu_{s+t}(y)\\
&= \frac{1}{V_s}\int_{\mathbb{B}}\left(\int_{\mathbb{B}} R_{s+t}(x, y)h(y)d\nu_{s+t}(y)\right)\overline{f(x)}\, d\kappa(x)\\
&= \frac{1}{V_s}\int_{\mathbb{B}} h(x)\overline{f(x)}\, d\kappa(x).
\end{aligned}$$

The γ in the statement of the theorem is

$$\gamma = \alpha_1 + t + \alpha_2'.$$

Thus, by Proposition 3.2,

$$|[h, {}_{s+t}T_\kappa f]_{b^2_{s+t}}| \lesssim \int_{\mathbb{B}} |h(x)||f(x)|\, d\kappa(x) \lesssim \|f\|_{b^\infty_{\alpha_1+t}}\|h\|_{b^1_{\alpha_2'}}.$$

By duality we have

$$\begin{aligned}
\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} &\lesssim \sup_{\|h\|_{b^1_{\alpha_2'}}\le 1} |[h, {}_{s+t}T_\kappa f_k]_{b^2_{s+t}}|\\
&\lesssim \|f\|_{b^\infty_{\alpha_1+t}}.
\end{aligned}$$

Hence ${}_{s+t}T_\kappa$ is bounded from $b^\infty_{\alpha_1+t}$ to $b^\infty_{\alpha_2+t}$.

4.2. Proof of Theorem 1.3

Before going to the proof, it is worth noting that by Theorem 4.1, ${}_{s,t}T_\mu$ is compact from $b^\infty_{\alpha_1}$ to $b^\infty_{\alpha_2}$ if and only if ${}_{s+t}T_\kappa$ is compact from $b^\infty_{\alpha_1+t}$ to $b^\infty_{\alpha_2+t}$. For the proof of Theorem 1.3 we need the following lemma.

Lemma 4.2. *Let $0 < \alpha_1, \alpha_2 < \infty$ and s, t, γ and κ be as in Theorem 1.2. Let ${}_{s+t}T_\kappa$ be a bounded linear operator from $b^\infty_{\alpha_1}$ into $b^\infty_{\alpha_2}$. Then ${}_{s+t}T_\kappa$ is compact if and only if $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2}} \to 0$ as $k \to \infty$ whenever $\{f_k\}$ is a bounded sequence in $b^\infty_{\alpha_1}$ that converges to 0 uniformly on compact subsets of $\mathbb{B}$.*

Proof. The necessity being obvious we will only prove the sufficiency part of the equivalence above. Suppose $\{f_k\}$ is a bounded sequence in $b^\infty_{\alpha_1}$. Note that if $\alpha > 0$, we have by (Doğan and Üreyen 2018, Corollary 5.3)

$$|u(x)| \lesssim \frac{\|u\|_{b^\infty_\alpha}}{(1 - |x|^2)^\alpha} \tag{13}$$

for all $u \in b^\infty_\alpha$ and $x \in \mathbb{B}$. Accordingly, it is uniformly bounded on each compact subset of $\mathbb{B}$ by (13) and thus it is a normal family (see (Axler et al. 2001, Theorem 2.6)). That is, there exists a subsequence of $\{f_k\}$ that converges uniformly on compact subsets of $\mathbb{B}$ to a bounded harmonic function f on $\mathbb{B}$; for simplicity we denote this subsequence by $\{f_k\}$ as well. The sequence $\{f_k - f\}$ is therefore bounded in $b^\infty_{\alpha_1}$ and converges to 0 uniformly on compact subsets of $\mathbb{B}$. By assumption $\|{}_{s+t}T_\kappa(f_k - f)\|_{b^\infty_{\alpha_2}} \to 0$ as $k \to \infty$. This implies that the subsequence $\{{}_{s+t}T_\kappa f_k\}$ converges in $b^\infty_{\alpha_2}$ (to ${}_{s+t}T_\kappa f$). The proof is complete.

(i) Implies (ii). Since ${}_{s+t}T_\kappa$ is compact, then $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} \to 0$ whenever $\{f_k\}$ is a bounded sequence in $b^\infty_{\alpha_1+t}$ that converges to 0 uniformly on compact subsets of $\mathbb{B}$ by Lemma 4.2. Let $\{a_k\} \subset \mathbb{B}$ with $|a_k| \to 1^-$ and consider the functions

$$f_k(x) = (1 - |a_k|^2)^{n+s-\alpha_1} R_{s+t}(x, a_k).$$

Under the assumptions on s and Lemma 2.7, since $[x, y] \gtrsim (1-|x|^2)$ and $[x, y] \gtrsim (1-|y|^2)$ for $x, y \in \mathbb{B}$, we get $\sup_k \|f_k\|_{b^\infty_{\alpha_1+t}} < \infty$, and it is clear that f_k converges to 0 uniformly on compact subsets of $\mathbb{B}$. Thus $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} \to 0$. Therefore, proceeding as in (i)

Implies (ii) in Theorem 1.2, for any $\delta > 0$, we obtain

$$\begin{aligned}\widehat{\kappa}_{\gamma,\delta}(a_k) &\lesssim (1-|a_k|^2)^{2(n+s+t)-(n+\gamma)}|{}_{s+t}T_\kappa[R_{s+t}(a_k,.)](a_k)| \\ &= (1-|a_k|^2)^{2(n+s+t)-(n+\gamma)-(n+s-\alpha_1)}|{}_{s+t}T_\kappa f_k(a_k)| \\ &= (1-|a_k|^2)^{\alpha_2+t}|{}_{s+t}T_\kappa f_k(a_k)| \\ &\lesssim \|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} \to 0.\end{aligned}$$

Hence, by Theorem 3.3, the measure κ is a vanishing γ-Carleson measure.

(ii) Implies (i). Finally, assume that κ is a vanishing γ-Carleson measure. In particular, it is a γ-Carleson measure and thus ${}_{s+t}T_\kappa : b^\infty_{\alpha_1+t} \to b^\infty_{\alpha_2+t}$ is bounded by Theorem 1.2. To show that the operator ${}_{s+t}T_\kappa$ is compact, we must prove that $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} \to 0$ whenever $\{f_k\}$ is a bounded sequence in $b^\infty_{\alpha_1+t}$ converging to 0 uniformly on compact subsets of $\mathbb{B}$ by Lemma 4.2. Similarly, as in the proof of Theorem 1.2, by duality we have (the number α_2' being the one defined by (12)

$$\begin{aligned}\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+t}} &\lesssim \sup_{\|h\|_{b^1_{\alpha_2'}}\le 1} |[h, {}_{s+t}T_\kappa f_k]_{b^2_{s+t}}| \\ &\le \sup_{\|h\|_{b^1_{\alpha_2'}}\le 1} \int_{\mathbb{B}} |f_k(x)||h(x)|d\kappa(x).\end{aligned}$$

Let $0 < \delta < 1$. Since $E_{\delta/2}(x)$ is also a Euclidean ball with center at $c = (1-(\delta/2)^2)x/(1-(\delta/2)^2|x|^2)$ and its radius behaves like $1-|x|^2$ when $\delta/2$ is fixed, (Doğan 2020, Lemma 3.3) implies that

$$|f_k(x)h(x)| \lesssim \frac{1}{r^n}\int_{B(x,r)} |f_k(y)h(y)|d\nu(y),$$

whenever $B(x,r) = \{y : |y-x| < r\} \subset E_{\delta/2}(x)$ for all $x \in \mathbb{B}$. This directly leads to the estimate

$$|f_k(x)h(x)| \lesssim \frac{1}{(1-|x|^2)^{n+\gamma}}\int_{E_{\delta/2}(x)} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y) \quad (x \in \mathbb{B}).$$

Note that $E_{\delta/2}(x) \subset E_\delta(a)$ for $a \in \mathbb{B}$ and $x \in E_{\delta/2}(a)$. Let $E_{\delta/2}(a_i)$ be the balls related to the sequence $\{a_i\} = \{a_i(\delta/2)\}$ in Lemma 2.3. So we obtain

$$\begin{aligned}|f_k(x)h(x)| &\lesssim \frac{1}{(1-|x|^2)^{n+\gamma}}\int_{E_{\delta/2}(x)} |f(y)h(y)|(1-|y|^2)^\gamma d\nu(y) \\ &\lesssim \frac{1}{(1-|x|^2)^{n+\gamma}}\int_{E_\delta(a_i)} |f(y)h(y)|(1-|y|^2)^\gamma d\nu(y), \quad x \in E_{\delta/2}(a_i)\end{aligned}$$

for $i = 1, 2, \ldots$. Then Lemma 2.3 and Lemma 2.2 yield

$$\begin{aligned}&\int_{\mathbb{B}} |f_k(x)h(x)|\, d\kappa(x) \\ &\lesssim \sum_{i=1}^\infty \int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x) \\ &\lesssim \sum_{i<j}\int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x) + \sum_{i\ge j}\int_{E_\delta(a_i)} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y)\int_{E_{\delta/2}(a_i)}\frac{d\kappa(x)}{(1-|x|^2)^{n+\gamma}} \\ &\lesssim \sum_{i<j}\int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x) + \sum_{i\ge j}\frac{\kappa(E_{\delta/2}(a_i))}{(1-|a_i|^2)^{n+\gamma}}\int_{E_\delta(a_i)} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y) \\ &\lesssim \sum_{i<j}\int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x) + \sup_{i\ge j}\widehat{\kappa}_{\gamma,\delta}(a_i)\sum_{i\ge j}\int_{E_\delta(a_i)} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y) \\ &\lesssim \sum_{i<j}\int_{E_{\delta/2}(a_i)} |f_k(x)h(x)|\, d\kappa(x) + N\sup_{i\ge j}\widehat{\kappa}_{\gamma,\delta}(a_i)\int_{\mathbb{B}} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y)\end{aligned}$$

for any j where N denotes the number provided by Lemma 2.3. Fix j and let $k \to \infty$. Since f_k converges to 0 uniformly on each

$E_{\delta/2}(a_i)$, the $i < j$ terms go to 0. The result is

$$\begin{aligned}\limsup_k \sup_{\|h\|_{b^1_{\alpha_2'}}\le 1} \int_{\mathbb{B}} |f_k(x)h(x)|\, d\kappa(x) &\lesssim \sup_{i\ge j} \widehat{\kappa}_{\gamma,\delta}(a_i) \sup_k \sup_{\|h\|_{b^1_{\alpha_2'}}\le 1} \int_{\mathbb{B}} |f_k(y)h(y)|(1-|y|^2)^\gamma d\nu(y) \\ &\lesssim \sup_{i\ge j} \widehat{\kappa}_{\gamma,\delta}(a_i) \sup_k \|f_k\|_{b^\infty_{\alpha_1+t}} \sup_{\|h\|_{b^1_{\alpha_2'}}\le 1} \int_{\mathbb{B}} |h(y)| d\nu_{\alpha_2'}(x) \\ &\lesssim \sup_{i\ge j} \widehat{\kappa}_{\gamma,\delta}(a_i) \sup_k \|f_k\|_{b^\infty_{\alpha_1+t}}\end{aligned}$$

for each j. Now let $j \to \infty$. Since $b^\infty_{\alpha_1+t}$-norms of f_k are bounded and $\sup_{i\ge j} \widehat{\kappa}_{\gamma,\delta}(a_i) \to 0$ by assumption, it follows that

$$\sup_{\|h\|_{b^1_{\alpha_2'}}\le 1} \int_{\mathbb{B}} |f_k(x)||h(x)| d\kappa(x) \to 0.$$

Thus, $\|{}_{s+t}T_\kappa f_k\|_{b^\infty_{\alpha_2+p_2 t}} \to 0$, finishing the proof.

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RESEARCH ARTICLE

ζ(Ric)-vector fields on doubly warped product manifolds

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ABSTRACT
We investigate ζ(Ric)-vector fields on doubly warped product manifolds. We obtain some results when the vector field is also ζ(Ric) on factor manifolds. We prove that if a vector field is a ζ(Ric)-vector field on a doubly warped product manifold, it is also a ζ(Ric)-vector field on the factor manifolds under certain conditions. Also, we show that a vector field on a doubly warped product manifold can be a ζ(Ric)-vector field with some conditions. Moreover we give two important applications of this concept in the Lorentzian settings, which are the doubly warped product generalized Robertson-Walker space-time and doubly warped product standard static space-time.

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Keywords: ζ(Ric)-vector field, warped product manifold, standard static space-times, generalized Robertson-Walker space-times

1. INTRODUCTION

There are many special types of smooth vector fields in the literature such as Killing, conformal, concircular, etc. The existence of any special type of vector field can directly influence the geometry of the manifold on which it is defined. For example, any Riemannian manifold with non-zero concircular vector field is a locally warped product (see Chen (2015)). Also, the topological property of a Riemannian manifold can influence the form of a vector field defined in that manifold. For instance, every affine vector field is Killing (see Kobayashi (1995)) on a compact and orientable Riemannian manifold. Moreover, the existence of a vector field and the algebraic topological property of the manifold on which it is defined are closely related.
The notion of ζ(Ric)-vector fields was first defined by Hinterleitner and Kiosak (2008), then many geometers have studied these types of vector fields in several kinds of differentiable structures (see De et al. (2021), Hinterleitner and Kiosak (2009), Kırık and Özen Zengin (2015), Kırık and Özen Zengin (2015), Kırık and Özen Zengin (2019), Özen Zengin and Kırık (2013)).

The concept of warped product manifolds introduced by Bishop and O'Neill Bishop and O'Neill (1969) to investigate Riemannian manifolds with negative sectional curvature. This is the concept that describes the geometry of many significant relativistic space-time, which has a wide range of uses in both differential geometry and mathematical physics (Bishop and O'Neill (1969), O'Neill (1983)).

In the present paper, we consider ζ(Ric)-vector fields on doubly warped product manifolds. We obtain that if a vector field is a ζ(Ric)-vector field on a doubly warped product manifold, it is also a ζ(Ric)-vector field on the factor manifolds under certain conditions. Moreover, we show that a vector field on a doubly warped product manifold can be a ζ(Ric)-vector field with some conditions. Finally, considering ζ(Ric)-vector fields on a doubly warped product generalized Robertson-Walker space-time and doubly warped product standard static space-time, we get some results.

2. DOUBLY WARPED PRODUCT MANIFOLDS WITH ζ(RIC)-VECTOR FIELDS

A *doubly warped product* Ehrlich (1974) ${}_{f_2}M_1 \times_{f_1} M_2$ of (M_1, g_1) and (M_2, g_2) is the product manifold $M = M_1 \times M_2$ and it has the following metric:

$$g = (f_2 \circ \sigma)^2 \sigma_1^*(g_1) + (f_1 \circ \sigma)^2 \sigma_2^*(g_2), \tag{1}$$

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where (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds and $f_1 \in C^\infty(M_1)$, $f_2 \in C^\infty(M_2)$. σ_1 and σ_2 are defined as canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. For $i = 1, 2$, $\sigma_i^*(g_i)$ is the pullback of g_i via σ_i. We say that f_i is a *warping function* of $({}_{f_2}M_1 \times_{f_1} M_2, g)$. If f_1 or f_2 is constant, then the manifold is a *warped product* Bishop and O'Neill (1969). Also, we get a *direct product manifold* Chen (2017) when both f_1 and f_2 are constant.

Let $({}_{f_2}M_1 \times_{f_1} M_2, g)$ be a doubly warped product manifold. In this study, the same notation will be used for a vector field and for its lift. It is also true for a metric and its pullback. Because each σ_i is a (positive) homothety, the connection is preserved. Also, we can use the same notation for a connection on M_i and for its pullback via σ_i. For $({}_{f_2}M_1 \times_{f_1} M_2, g)$, the covariant derivative formulas Ehrlich (1974) are obtained as follows:

$$\nabla_Z T = \nabla^1_Z T - g(Z, T)\nabla(\ln(f_2 \circ \pi_2)), \tag{2}$$

$$\nabla_Z W = \nabla_W Z = W(\ln(f_2 \circ \pi_2))Z + Z(\ln(f_1 \circ \pi_1))W, \tag{3}$$

$$\nabla_V W = \nabla^2_V W - g(V, W)\nabla(\ln(f_1 \circ \pi_1)), \tag{4}$$

for $Z, T \in \mathfrak{L}(M_1)$ and $V, W \in \mathfrak{L}(M_2)$. Here ∇ and ∇^i are the Levi-Civita connections of ${}_{f_2}M_1 \times_{f_1} M_2$ and M_i respectively, for $i \in \{1, 2\}$. Also, we use the notation $\mathfrak{L}(M_i)$ for the set of lifts of vector fields on M_i. On the other hand, we obtain $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally umbilical submanifolds and their mean curvature vector fields are closed in ${}_{f_2}M_1 \times_{f_1} M_2$ Gutierrez and Olea (2012). Here $p_1 \in M_1$ and $p_2 \in M_2$.

Remark 2.1. Here, $l = \ln f_2$ (resp. $k = \ln f_1$) and for the function l (resp. k) and its pullback $l \circ \sigma_2$ (resp. $k \circ \sigma_1$), the same symbol is used from now on.

Let $\mathcal{S}$, $\mathcal{S}^1$ and $\mathcal{S}^2$ be the lifts of Ricci curvature tensors of $({}_{f_2}M_1 \times_{f_1} M_2, g)$, (M_1, g_1) and (M_2, g_2) respectively. Then, the followings are hold:

Lemma 2.2. *Blaga and Taştan (2022) Let $Z, T \in \mathfrak{L}(M_1)$ and $V, W \in \mathfrak{L}(M_2)$. Then, we have*

$$\mathcal{S}(Z, T) = \mathcal{S}^1(Z, T) - \tfrac{m_2}{f_1} h_1^{f_1}(Z, T) - g(Z, T)\Delta l, \tag{5}$$

$$\mathcal{S}(Z, V) = (m_1 + m_2 - 2)Z(k)V(l), \tag{6}$$

$$\mathcal{S}(V, W) = \mathcal{S}^2(V, W) - \tfrac{m_1}{f_2} h_2^{f_2}(V, W) - g(V, W)\Delta k, \tag{7}$$

where Δ is the Laplacian operator on $({}_{f_2}M_1 \times_{f_1} M_2, g)$, $m_i = dim(M_i)$ for $i \in \{1, 2\}$ and $h_1^{f_1}(Z, T) = ZT(f_1) - (\nabla^1_Z T)(f_1)$ and $h_2^{f_2}(V, W) = VW(f_2) - (\nabla^2_V W)(f_2)$.

Now, we recall the definition of $\zeta(\mathrm{Ric})$ vector field defined by Hinterleitner and Kiosak (2008).

Definition 2.3. A vector field ζ is called $\zeta(\mathrm{Ric})$ if for any vector field X on a Riemannian manifold (M^m, g) the equation

$$\nabla_X \zeta = \mu Q X \tag{8}$$

holds, where ∇ is the Levi-Civita connection of the metric g, Q is the Ricci operator of the Ricci tensor $\mathcal{S}$ of M and μ is a constant.

For a doubly warped product manifold, we give the main theorem about $\zeta(\mathrm{Ric})$-vector fields as follows:

Theorem 2.4. *Let the vector field $\zeta = \zeta_1 + \zeta_2$ be $\zeta(\mathrm{Ric})$ on $(M =_{f_2} M_1 \times_{f_1} M_2, g)$ for $i = 1, 2$, $\zeta_i \in \mathfrak{L}(M_i)$. Then, we have*

(i) *The vector field ζ_1 is $\zeta_1(\mathrm{Ric})$ on $M_1 \Leftrightarrow$*

$$\tfrac{\mu m_2}{f_1} h_1^{f_1}(Z, T) + \Big\{\mu \Delta l + \zeta_2(l)\Big\} g(Z, T) = 0, \tag{9}$$

(ii) *The vector field ζ_2 is $\zeta_2(\mathrm{Ric})$ on $M_2 \Leftrightarrow$*

$$\tfrac{\mu m_1}{f_2} h_2^{f_2}(V, W) + \Big\{\mu \Delta k + \zeta_1(k)\Big\} g(V, W) = 0, \tag{10}$$

where $Z, T \in \mathfrak{L}(M_1)$ and $V, W \in \mathfrak{L}(M_2)$.

Proof. Let the vector field ζ be $\zeta(\mathrm{Ric})$ on M. Then, we get $\mu\mathcal{S}(Z,T) = g(\nabla_Z\zeta, T)$ for all $Z, T \in \mathfrak{L}(M_1)$. From (5), we get

$$\mu\mathcal{S}(Z,T) = \mu\Big\{\mathcal{S}^1(Z,T) - \tfrac{m_2}{f_1}h_1^{f_1}(Z,T) - g(Z,T)\Delta l\Big\}.$$

Hence, from (2), we obtain

$$\mu\,\mathcal{S}^1(Z,T) - \mu\tfrac{m_2}{f_1}h_1^{f_1}(Z,T) - \mu g(Z,T)\Delta l$$

$$= g(\nabla^1_Z\zeta_1 - g(Z,\zeta_1)\nabla l, T) + g(Z(k)\zeta_2 + \zeta_2(l)Z, T).$$

Thus, we have

$$\begin{aligned}\mu\,\mathcal{S}^1(Z,T) = {} & g(\nabla^1_Z\zeta_1, T) + \tfrac{\mu m_2}{f_1}h_1^{f_1}(Z,T) + \mu g(Z,T)\Delta l\\ & - g(Z,\zeta_1)g(\nabla l, T) + Z(k)g(\zeta_2, T) + \zeta_2(l)g(Z,T)\\ = {} & g(\nabla^1_Z\zeta_1, T) + \tfrac{\mu m_2}{f_1}h_1^{f_1}(Z,T) + \Big\{\mu\Delta l + \zeta_2(l)\Big\}g(Z,T).\end{aligned}$$

This concludes the first assertion.

Regarding the second assertion, we have $\mu\mathcal{S}(V,W) = g(\nabla_V\zeta, W)$ for all $V, W \in \mathfrak{L}(M_2)$, since the vector field ζ is $\zeta(\mathrm{Ric})$ on M. Using (7), we get

$$\mu\mathcal{S}(V,W) = \mu\Big\{\mathcal{S}^2(V,W) - \tfrac{m_1}{f_2}h_2^{f_2}(V,W) - g(V,W)\Delta k\Big\}.$$

Hence, using (4) we obtain

$$\mu\,\mathcal{S}^2(V,W) - \mu\tfrac{m_1}{f_2}h_2^{f_2}(V,W) - \mu g(V,W)\Delta k$$

$$= g(\nabla^2_V\zeta_2 - g(V,\zeta_2)\nabla k, W) + g(V(l)\zeta_1 + \zeta_1(k)V, W).$$

After some calculations, we obtain

$$\begin{aligned}\mu\,\mathcal{S}^2(V,W) & = g(\nabla^2_V\zeta_2, W) + \tfrac{\mu m_1}{f_2}h_2^{f_2}(V,W) + \mu g(V,W)\Delta k\\ & \quad - g(V,\zeta_2)g(\nabla k, W) + V(l)g(\zeta_1, W) + \zeta_1(k)g(V,W)\\ & = g(\nabla^2_V\zeta_2, W) + \tfrac{\mu m_1}{f_2}h_2^{f_2}(V,W) + \Big\{\mu\Delta k + \zeta_1(k)\Big\}g(V,W).\end{aligned}$$

Thus the assertion is hold.

Theorem 2.5. *Let the vector field $\zeta = \zeta_1 + \zeta_2$ be defined on a doubly warped product $(M =_{f_2} M_1 \times_{f_1} M_2, g)$, where $\zeta_i \in \mathfrak{L}(M_i)$, for $i = 1, 2$. If*

$$\begin{aligned}\mu\,\mathcal{S}^1(X_1,Y_1) & = g(\nabla^1_{X_1}\zeta_1, Y_1) + \tfrac{\mu m_2}{f_1}h_1^{f_1}(X_1,Y_1) + \mu g(X_1,Y_1)\Delta l\\ & \quad - g(X_1,\zeta_1)Y_2(l) + \zeta_2(l)g(X_1,Y_1) + X_2(l)g(\zeta_1,Y_1)\\ & \quad - \mu(m_1+m_2-2)X_1(k)Y_2(l)\end{aligned} \tag{11}$$

and

$$\begin{aligned}\mu\,\mathcal{S}^2(X_2,Y_2) & = g(\nabla^2_{X_2}\zeta_2, Y_2) + \tfrac{\mu m_1}{f_2}h_2^{f_2}(X_2,Y_2) + \mu g(X_2,Y_2)\Delta k\\ & \quad - g(X_2,\zeta_2)Y_1(k) + \zeta_1(k)g(X_2,Y_2) + X_1(k)g(\zeta_2,Y_2)\\ & \quad - \mu(m_1+m_2-2)X_2(l)Y_1(k),\end{aligned} \tag{12}$$

hold, then the vector field ζ is $\zeta(\mathrm{Ric})$ with scalar μ, where $X_1, Y_1 \in \mathfrak{L}(M_1)$ and $X_2, Y_2 \in \mathfrak{L}(M_2)$.

Proof. Let $T, W \in \mathfrak{X}(M)$, where $T = X_1 + X_2$ and $W = Y_1 + Y_2$. Suppose that the vector field ζ is $\zeta(\mathrm{Ric})$ on M with scalar μ. Then, $\mu\mathcal{S}(T,W) = g(\nabla_T\zeta, W)$. Using (5) and (7), we have

$$\mu\mathcal{S}(X_1+X_2, Y_1+Y_2) = g(\nabla_{X_1+X_2}(\zeta_1+\zeta_2), Y_1+Y_2).$$

Then, we have

$$\mu\Big\{\mathcal{S}(X_1,Y_1)+\mathcal{S}(X_1,Y_2)+\mathcal{S}(X_2,Y_1)+\mathcal{S}(X_2,Y_2)\Big\}$$
$$= g(\nabla^1_{X_1}\zeta_1 - g(X_1,\zeta_1)\nabla l + X_1(k)\zeta_2 + \zeta_2(l)X_1 + \zeta_1(k)X_2 + X_2(l)\zeta_1$$
$$+ \nabla^2_{X_2}\zeta_2 - g(X_2,\zeta_2)\nabla k, Y_1+Y_2).$$

Hence, we obtain

$$\begin{aligned}
&\mu\Big\{\mathcal{S}^1(X_1,Y_1) - \tfrac{m_2}{f_1}h_1^{f_1}(X_1,Y_1) - g(X_1,Y_1)\Delta l \\
&+(m_1+m_2-2)X_1(k)Y_2(l) + (m_1+m_2-2)X_2(l)Y_1(k) \\
&+\mathcal{S}^2(X_2,Y_2) - \frac{m_1}{f_2}h_2^{f_2}(X_2,Y_2) - g(X_2,Y_2)\Delta k\Big\} \\
&= g(\nabla^1_{X_1}\zeta_1,Y_1) - g(X_1,\zeta_1)g(\nabla l,Y_2) + X_1(k)g(\zeta_2,Y_2) \\
&+\zeta_2(l)g(X_1,Y_1) + \zeta_1(k)g(X_2,Y_2) + X_2(l)g(\zeta_1,Y_1) \\
&+g(\nabla^2_{X_2}\zeta_2,Y_2) - g(X_2,\zeta_2)g(\nabla k,Y_1).
\end{aligned} \tag{13}$$

If the equations (11) and (12) hold, the assertion is hold from (13), which completes the proof.

In the remaining part, we give the definitions of a standard static space-time (SSS-T) and a generalized Robertson-Walker space-time (GRW). Let $(M_2,\ g_2)$ be an m_2-dimensional Riemannian manifold and J is an open connected interval of $\mathbb{R}$. If a $(m_2+1)-$ dimensional doubly warped product $\bar{M} = {}_{f_2}J\times_{f_1} M_2$ has the metric tensor

$$\bar{g} = -(f_2^2)dt^2 \oplus (f_1^2)g_2,$$

then it is called a *doubly warped product generalized Robertson-Walker space-time*. Here, $f_1 \in C^\infty(J)$ and $f_2 \in C^\infty(M_2)$, respectively and dt^2 is defined as the usual Euclidean metric tensor on J. For more details, see Flores and Sánchez (1974), Sánchez (1999), Sánchez (1998).

The following lemma is the direct consequences of (2)~(4), see also El-Sayied et al. (2020), pp. 3775.

Lemma 2.6. *Let $(\bar{M} = {}_{f_2}J\times_{f_1} M_2,\ \bar{g})$ be a doubly warped product generalized Robertson-Walker space-time and $U,V\in\mathfrak{L}(M_2)$. Then we have*

$$\nabla_{\partial t}\partial t = f_2^2\nabla l, \tag{14}$$

$$\nabla_V\partial t = \nabla_{\partial t}V = V(l)\partial t + k'V, \tag{15}$$

$$\nabla_U V = \nabla^2_U V - \bar{g}(U,V)\nabla k, \tag{16}$$

for the components of the Levi-Civita connection of $\bar{M}$.

From Lemma 2.2, we get the following result directly, see also El-Sayied et al. (2020), pp. 3775.

Lemma 2.7. *Let $(\bar{M} = {}_{f_2}J\times_{f_1} M_2,\ \bar{g})$ be a doubly warped product generalized Robertson-Walker space-time. Then we have*

$$\mathcal{S}(\partial t,\partial t) = (-k'' + (k')^2)m_2 + f_2^2\Delta l - \bar{g}(\nabla l,\nabla l), \tag{17}$$

$$\mathcal{S}(\partial t,U) = k'U(l)(m_2-1), \tag{18}$$

$$\mathcal{S}(U,V) = f_1^2\mathcal{S}^2(U,V), \tag{19}$$

for the non-zero components of the Ricci tensor of $\bar{M}$, where $U,V\in\mathfrak{L}(M_2)$.

Remark 2.8. The vector field $h\partial t$ is a ζ_1(Ric)-vector field on $(J,-dt^2)$ such that $h\in C^\infty(J) \Leftrightarrow h' = 0$ on J. Here, " $'$ " is the derivative with respect to "t" on J.

Theorem 2.9. *Let the vector field $\bar{\zeta} = h\partial t + \zeta_2$ be $\bar{\zeta}(\mathrm{Ric})$ on a doubly warped product GRW space-time of the form $(\bar{M} =_{f_2} J \times_{f_1} M_2, \bar{g})$ with scalar μ and $U, V \in \mathfrak{L}(M_2)$. Then, the following conditions hold:*

$$\mu\left\{(k'' + (k')^2)m_2 + f_2^2\Delta l - |\nabla l|^2\right\} = (h' + \zeta_2(l))\bar{g}(\partial t, \partial t) \tag{20}$$

and

$$\mu f_1^2 \mathcal{S}^2(U, V) = hk'\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V) \tag{21}$$

or ζ_2 is a $\zeta_2(\mathrm{Ric})$-vector field if and only if $hk' = 0$.

Proof. Let the vector field $\bar{\zeta}$ be $\bar{\zeta}(\mathrm{Ric})$ on $\bar{M}$. Then, for all $T, W \in \mathfrak{X}(\bar{M})$, $\mu\mathcal{S}(T, W) = \bar{g}(\nabla_T\bar{\zeta}, W)$. Hence, we get $\mu\mathcal{S}(\partial t, \partial t) = \bar{g}(\nabla_{\partial t}\bar{\zeta}, \partial t)$ for $T = \partial t$, $W = \partial t$. Using (17), we obtain

$$\mu\left\{(k'' + (k')^2)m_2 - k'\partial t(l) + f_2^2\Delta l - \bar{g}(\nabla l, \nabla l)\right\} = \bar{g}(\nabla_{\partial t}(h\partial t) + \nabla_{\partial t}\zeta_2, \partial t).$$

Since $\partial t(l) = 0$, we obtain

$$\begin{aligned}\mu\left\{(k'' + (k')^2)m_2 + f_2^2\Delta l - |\nabla l|^2\right\} &= \bar{g}(\partial(h)\partial t + h\nabla_{\partial t}\partial t + \nabla_{\partial t}\zeta_2, \partial t)\\ &= \bar{g}(h'\partial t + hf_2^2\nabla l + \zeta_2(l)\partial t + k'\zeta_2, \partial t)\\ &= h'\bar{g}(\partial t, \partial t) + \zeta_2(l)\bar{g}(\partial t, \partial t).\end{aligned}$$

Hence, we get

$$\mu\left\{(k'' + (k')^2)m_2 + f_2^2\Delta l - |\nabla l|^2\right\} = (h' + \zeta_2(l))\bar{g}(\partial t, \partial t), \tag{22}$$

which proves (20). Since $\mathcal{S}(U, V) = \bar{g}(\nabla_U\bar{\zeta}, V)$ for $U, V \in \mathfrak{L}(M_2)$, using (19), we get

$$\begin{aligned}\mu f_1^2\mathcal{S}^2(U, V) = \;& \bar{g}(\nabla_U(h\partial t) + \nabla_U\zeta_2, V)\\ &= \bar{g}(U(h)\partial t + h\nabla_U\partial t + \nabla_U\zeta_2, V)\\ &= \bar{g}(h(U(l)\partial t + k'U) + \nabla_U^2\zeta_2 - \bar{g}(U, \zeta_2)\nabla k, V)\\ &= hU(l)\bar{g}(\partial t, V) + hk'\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V) - \bar{g}(U, \zeta_2)\bar{g}(\nabla k, V)\\ &= hk'\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V).\end{aligned} \tag{23}$$

Thus, we have (21) from (23). On the other hand, using (23), we get

$$\mu\mathcal{S}^2(U, V) = hk'g_2(U, V) + g_2(\nabla_U^2\zeta_2, V). \tag{24}$$

Then, the vector field ζ_2 is $\zeta_2(\mathrm{Ric})$ on $M_2 \Leftrightarrow$ the condition $hk' = 0$ is satisfied in (24), i.e. $k' = 0$ or $h = 0$. Hence, $\bar{M}$ is a GRW space-time or $\bar{\zeta} = \zeta_2$, where ζ_2 is also $\zeta_2(\mathrm{Ric})$-vector field on $\bar{M}$. The proof is completed.

If a $(m_2 + 1)-$dimensional doubly warped product $\bar{M} =_{f_1} J \times_{f_2} M_2$ has a metric tensor

$$\bar{g} = -(f_1^2)dt^2 \oplus (f_2^2)g_2,$$

then it is called a *doubly warped product SSS-T*, where (M_2, g_2) be an m_2-dimensional Riemannian manifold, here $f_1 \in C^\infty(M_2)$ and $f_2 \in C^\infty(J)$. Also dt^2 is defined as the usual Euclidean metric tensor on J, where J is an open connected interval of $\mathbb{R}$. For more details about standard static space-times, see Allison (1988)-Besse (2007)). From (2)~(4), we have:

Lemma 2.10. *Let $(\bar{M} =_{f_1} J \times_{f_2} M_2, \bar{g})$ be a doubly warped product SSS-T. Then we have*

$$\nabla_{\partial t}\partial t = 2\dot{k}\partial t + f_1^2\nabla k, \tag{25}$$

$$\nabla_V\partial t = \nabla_{\partial t}V = V(k)\partial t + \partial t(l)V, \tag{26}$$

$$\nabla_U V = \nabla_U^2 V - \bar{g}(U, V)\nabla l, \tag{27}$$

for the components of Levi-Civita connection of $\bar{M}$, where $U, V \in \mathfrak{L}(M_2)$. Here, " $\dot{}$ " is the derivative with respect to ∇^2.

From Lemma 2.2, we get the following result directly.

Lemma 2.11. *Let $(\bar{M} =_{f_1} J \times_{f_2} M_2, \bar{g})$ be a doubly warped product SSS-T. Then we have*

$$\mathcal{S}(\partial t, \partial t) = -m_2(-l^{'} + (l')^2 - 2l^{'}\dot{k}) + f_1^2\bar{g}(\nabla k, \nabla k) + f_1^2\Delta k, \tag{28}$$

$$\mathcal{S}(\partial t, U) = U(k)(1 - l^{'}m_2), \tag{29}$$

$$\mathcal{S}(U, V) = f_2^2\mathcal{S}^2(U, V), \tag{30}$$

for the non-zero components of the Ricci tensor of $\bar{M}$, where $U, V \in \mathfrak{L}(M_2)$.

Theorem 2.12. *Let the vector field $\bar{\zeta} = h\partial t + \zeta_2$ be $\bar{\zeta}(\mathrm{Ric})$ on a doubly warped product SSS-T of the form $(\bar{M} =_{f_1} J \times_{f_2} M_2, \bar{g})$ with scalar μ. Then, we have*

$$\begin{aligned}\mu\Big\{ - m_2\Big(- l^{'} + (l')^2 - 2l^{'}\dot{k}\Big) + f_1^2\Big(|\nabla k|^2 + \Delta k\Big)\Big\} \\ = \Big\{\dot{h} + 2h\dot{k} + \zeta_2(k)\Big\}|\partial t|^2,\end{aligned} \tag{31}$$

and

$$\mu f_2^2\mathcal{S}^2(U, V) = \partial t(l)\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V) \tag{32}$$

or the vector field ζ_2 is $\zeta_2(\mathrm{Ric})$ on $M_2 \Leftrightarrow \partial t(l) = 0$, namely $\bar{M}$ is a SSS-T.

Proof. Let the vector field $\bar{\zeta}$ be $\bar{\zeta}(\mathrm{Ric})$ on $\bar{M}$. Then, $\mu\mathcal{S}(T, W) = \bar{g}(\nabla_T\bar{\zeta}, W)$ for all $T, W \in \mathfrak{X}(\bar{M})$. It follows that $\mu\mathcal{S}(\partial t, \partial t) = \bar{g}(\nabla_{\partial t}\bar{\zeta}, \partial t)$. Hence, using (28) we get

$$\begin{aligned}&\mu\Big\{ - m_2(-l^{'} + (l^{'})^2 - 2l^{'}\dot{k}) + f_1^2\bar{g}(\nabla k, \nabla k) + f_1^2\Delta k\Big\} \\ &= \bar{g}(\nabla_{\partial t}(h\partial t) + \nabla_{\partial t}\zeta_2, \partial t) \\ &= \bar{g}(\partial t(h)\partial t + h\nabla_{\partial t}\partial t + \nabla_{\partial t}\zeta_2, \partial t) \\ &= h^{'}\bar{g}(\partial t, \partial t) + h\bar{g}(2\dot{k}\partial t + f_1^2\nabla k, \partial t) + \bar{g}(\zeta_2(k)\partial t + \partial t(l)\zeta_2, \partial t) \\ &= h^{'}\bar{g}(\partial t, \partial t) + 2\dot{k}h\bar{g}(\partial t, \partial t) + hf_1^2\bar{g}(\nabla k, \partial t) + \zeta_2(k)\bar{g}(\partial t, \partial t) + \partial(l)\bar{g}(\zeta_2, \partial t) \\ &= \Big\{h^{'} + 2h\dot{k} + \zeta_2(k)\Big\}|\partial t|^2.\end{aligned}$$

Hence, we get (31). Since $\mu\mathcal{S}(U, V) = \bar{g}(\nabla_U\bar{\zeta}, V)$, using (30), we get

$$\begin{aligned}\mu f_2^2\,\mathcal{S}^2(U, V) &= \bar{g}(\nabla_U(h\partial t) + \nabla_U\zeta_2, V) \\ &= \bar{g}(h(U(l)\partial t + \dot{k}U) + \nabla_U\zeta_2 - \bar{g}(U, \zeta_2)\nabla k, V) \\ &= \bar{g}(hU(l)\partial t + \partial t(l)U, V) + \bar{g}(\nabla_U^2\zeta_2 - \bar{g}(U, \zeta_2)\nabla l, V) \\ &= \partial t(l)\bar{g}(U, V) + \bar{g}(\nabla_U^2\zeta_2, V),\end{aligned} \tag{33}$$

for $U, V \in \mathfrak{L}(M_2)$. Thus, we have (32) from (33). Then, using (33), we obtain

$$\mu\mathcal{S}^2(U, V) = \partial t(l)g_2(U, V) + g_2(\nabla_U^2\zeta_2, V). \tag{34}$$

Thus, the vector field ζ_2 is $\zeta_2(\mathrm{Ric})$ on $M_2 \Leftrightarrow$ the condition $\partial t(l) = 0$ is satisfied in (34), i.e. l is constant. It follows that $\bar{M}$ is a SSS-T and hence, the proof is completed.

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RESEARCH ARTICLE

A New biased estimator and variations based on the Kibria Lukman Estimator

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ABSTRACT
One of the problems encountered in linear regression models is called multicollinearity problem which is an approximately linear relationship between the explanatory variables. This problem causes the estimated parameter values to be highly sensitive to small changes in the data. In order to reduce the impact of this problem on the model parameters, alternative biased estimators to the ordinary least squares estimator have been proposed in the literature. In this study, we propose a new biased estimator that can be an alternative to existing estimators. The superiority of this estimator over other biased estimators is analyzed in terms of matrix mean squared error. In addition, two different Monte Carlo simulation experiments are carried out to examine the performance of the biased estimators under consideration. A numerical example is given to evaluate the performance of the proposed estimator against other biased estimators.

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Keywords: Liu estimator, Liu-type estimator, multicollinearity, Ridge estimator

1. INTRODUCTION

Regression analysis is one of the most widely used statistical techniques to explain the statistical relationship between explanatory and response variables using a model. Let us consider the following linear regression model:

$$Y = X\beta + \varepsilon \tag{1}$$

where Y is an $n \times 1$ vector of dependent variables, X is an $n \times p$ full column rank matrix of n observations on p independent explanatory variables, β is a $p \times 1$ vector of unknown parameters and ε is an $n \times 1$ vector of random errors which are distributed as Normal with mean vector 0 and covariance matrix $\sigma^2 I$. The Ordinary Least Squares (OLS) estimator of β is given by

$$\hat{\beta}_{OLS} = (X'X)^{-1} X'Y \tag{2}$$

where the covariance matrix of $\hat{\beta}_{OLS}$ is obtained as $cov\left(\hat{\beta}_{OLS}\right) = \sigma^2 (X'X)^{-1}$. According to the Gauss-Markov Theorem, the OLS estimator of the parameter vector β is the best linear unbiased estimator. In other words, we mean that $\hat{\beta}_{OLS}$ has the smallest variance among the class of all unbiased estimators that are linear combinations of the data. However, if there is an approximate relationship between the explanatory variables close to linear dependence, a biased estimator with a smaller variance may be found. This situation, i.e. a relationship close to linear dependence between explanatory variables, is called the multicollinearity problem in regression analysis. In the case of multicollinearity in the model, a very small change in the matrix X results in a very large change in matrix $(X'X)^{-1}$. Therefore, some values in the parameter vector of the OLS estimator will have a large variance. If there is multicollinearity in the linear regression model, then the OLS estimator given by (2) is again the best-unbiased estimator. However, since the variance of the OLS estimator will be very large, it will tend to produce unstable results. Although there are methods to overcome this situation by reducing the variables, alternative approaches can be used to solve the multicollinearity problem by keeping all explanatory variables in the model. Another method for solving this problem is to use biased estimators that can minimize parameter variances. For more detailed information about these proposed biased estimators in linear regression models, researchers can review the articles Hoerl and Kennard (1970),Liu (1993),Liu (2003),Kibria (2003),Özkale and Kaçıranlar (2007),Sakallıoğlu and Kaçıranlar (2008),Yang and Chang (2010),Kurnaz and Akay (2015),Kurnaz and Akay (2018),Qasim et al.

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(2020),Lukman et al. (2019),Lukman et al. (2020), Ahmad and Aslam (2022), Zeinal and Azmoun (2023), Üstündağ et al. (2021), Aslam and Ahmad (2022),Babar and Chand (2022),Dawoud (2022), Qasim et al. (2022), Shewa and Ugwuowo (2023).

There are many biased estimators proposed in the literature to minimize the problems arising from collinearity. Among these estimators, the Ridge Estimator (RE) proposed by Hoerl and Kennard (1970) and the Liu Estimator (LE) proposed by Liu (1993) are widely preferred. The RE is defined by

$$\hat{\beta}_{RE} = (X'X + kI)^{-1} X'Y, \qquad k > 0 \tag{3}$$

where k is a biasing parameter. On the other hand, LE, which combines the advantages of the RE and Stein (1956) estimators, is defined as follows:

$$\hat{\beta}_{LE} = (X'X + I)^{-1} \left(X'Y + d\hat{\beta}_{OLS}\right), \quad 0 < d < 1 \tag{4}$$

where d is a biasing parameter. Stein (1956) defined the Stein estimator as follows: $\hat{\beta}_S = c\hat{\beta}_{OLS}$ where $0 < c < 1$.

However, although RE and LE are the first-choice estimators due to collinearity in the linear regression model, these estimators have several disadvantages. To utilize the advantageous features of both RE and LE, the researchers created estimators with two biasing parameters k and d. For example, Liu (2003) introduced an estimator that is dependent on k and d as follows:

$$\hat{\beta}_{LTE} = (X'X + kI)^{-1} \left(X'Y - d\hat{\beta}^*\right), \quad k > 0, \quad -\infty < d < \infty \tag{5}$$

where $\hat{\beta}^*$ can be any estimator of β. The estimator in (5) is known as the Liu-type estimator. The OLS method is used to produce this estimator after adding $\left(-d \big/ k^{1/2}\right)\beta^* = k^{1/2}\beta + \varepsilon'$ to the model (1). As an alternative, Özkale and Kaçıranlar (2007) developed the following Two-parameter Estimator (TPE):

$$\hat{\beta}_{TPE} = (X'X + kI)^{-1} \left(X'Y + kd\hat{\beta}_{OLS}\right), \quad k > 0, \quad 0 < d < 1, \tag{6}$$

where k and d are two biasing parameters. The TPE is a general estimator which includes the OLS, RE, and LE as special cases.

Kurnaz and Akay (2015) presented a general Liu-type estimator as an alternative to the estimators previously introduced. This estimator includes estimators (2), (3), (4), (5), and (6) as special cases as follows:

$$\hat{\beta}_{NLTE} = (X'X + kI)^{-1} \left(X'Y + f(k)\hat{\beta}^*\right), \quad k > 0 \tag{7}$$

where $\hat{\beta}^*$ is any estimator of β and $f(k)$ is a continuous function of the biasing parameter k. Similarly, NLTE is obtained by augmenting $\frac{f(k)}{k^{1/2}}\hat{\beta}^* = k^{1/2}\beta + \varepsilon'$ to (1) and then using OLS method. For example, if $f(k) = -k$ and $\hat{\beta}^* = \hat{\beta}_{OLS}$, the KL estimator given by Kibria and Lukman (2020) is obtained. The KL estimator, which is a special case of the estimator (7), is defined as follows:

$$\hat{\beta}_{KL} = (X'X + kI)^{-1} (X'X - kI)\hat{\beta}_{OLS}, \quad k > 0 \tag{8}$$

where k is a biasing parameter. On the other hand, Qasim et al. (2022) proposed the Two-step shrinkage (TSS) estimator in the presence of multicollinearity as follows:

$$\hat{\beta}_{TSS} = (X'X + kI)^{-1} (X'X - kdI)\hat{\beta}_{OLS}, \qquad k > 0, \ 0 \le d < 1 \tag{9}$$

where k and d are two biasing parameters. Note that this estimator given in (9) can be obtained by taking $f(k) = -kd$ and $\hat{\beta}^* = \hat{\beta}_{OLS}$ in (7). On the other hand, when we take $f(k) = \frac{k}{d}$ where $d \in R - \{0\}$ and $\hat{\beta}^* = \hat{\beta}_{LE}$ in (7), a new two-parameter estimator proposed by Üstündağ et al. (2021) is obtained as follows:

$$\hat{\beta}_{STO} = (X'X + kI)^{-1} \left(X'Y + \frac{k}{d}\hat{\beta}_{LE}\right), \ k > 0, \ d > 1 \tag{10}$$

where k and d are two biasing parameters. Furthermore, Sakallıoğlu and Kaçıranlar (2008) proposed another biased estimator based on RE which is given by

$$\hat{\beta}_{SK} = (X'X + I)^{-1} \left(X'Y + d\hat{\beta}_{RE}\right), \qquad k > 0, \quad -\infty < d < \infty \tag{11}$$

where k and d are two biasing parameters. This estimator given in (11) is a general estimator that includes the OLS, RE, and LEs as special cases. Also, this estimator is obtained by augmenting the equation $d\hat{\beta}_{RE} = \beta + \varepsilon'$ to (1) and using the OLS method. Also, Yang and Chang (2010) proposed a new biased estimator based on RE as follows:

$$\hat{\beta}_{YC} = (X'X + I)^{-1} (X'X + dI)\hat{\beta}_{RE}, \qquad k > 0, \quad 0 < d < 1 \tag{12}$$

where k and d are two biasing parameters. The estimator given in (12) is obtained by augmenting $(d - k)\hat{\beta}_{RE} = \beta + \varepsilon'$ to (1) and using the OLS method. Also, the YC estimator is a general estimator that includes OLS, RE, and LE as special cases.

On the other hand, Idowu et al. (2023) modified the LE provided by (4). They used the KL estimator provided by (8) in place of the OLS estimator in LE. The estimator is called LKL by Idowu et al. (2023) is given as follows:

$$\hat{\beta}_{LKL} = (X'X + I)^{-1} (X'X + dI)\hat{\beta}_{KL}, \quad k > 0, \ 0 < d < 1 \tag{13}$$

where k and d are two biasing parameters.

One of the common features of the estimators we consider is that they are defined based on Ridge, Liu, or Liu-type estimators with a modification on these estimators. Another important point here is that all estimators we have considered depend on the OLS estimator. Therefore, to reduce the problems that may arise due to collinearity, a new estimator is obtained by replacing the OLS estimator with a more powerful estimator. The estimators obtained in this case usually depend on the biasing parameters k and d.

In the literature, there are many estimators for linear regression models based on the biasing parameters k and d. Some of these estimators are as follows: LTE, SK, YC, TSS, TPE, STO, and LKL estimators. However, one of the major problems for these estimators is that it is also difficult to find optimal estimates of these biasing parameters (Liu (2003)), (Özkale and Kaçıranlar (2007)), (Sakallıoğlu and Kaçıranlar (2008)), (Yang and Chang (2010), (Ahmad and Aslam (2022)), (Aslam and Ahmad (2022)), (Qasim et al. (2022)), (Shewa and Ugwuowo (2023)). Therefore, our first objective in this study is to achieve a new estimator with a single biasing parameter by modifying the existing estimators. Another objective is to investigate the performance of this estimator with other estimators through different simulation studies.

The article is organized as follows. In Section 2, the proposed biased estimator is introduced. In Section 3, the proposed estimator is compared with the NLTE under the MMSE sense. Two Monte Carlo simulation studies are designed to evaluate the performances of the considered estimators in Section 4. In Section 5, the performance evaluation of all considered estimators is given in the Portland cement data. Finally, some conclusions are given in Section 6.

2. A NEW BIASED ESTIMATOR

In recent years, researchers have focused especially on the KL estimator proposed by Kibria and Lukman (2020). In the literature, they have proposed new estimators based on the KL estimator Dawoud (2022), Idowu et al. (2023), Shewa and Ugwuowo (2023). In this study, in order to take the performance of the KL estimator one step further, the RE estimator will be used instead of the OLS estimator in the KL estimator. In other words, the KL estimator is obtained by augmenting $-\sqrt{k}\hat{\beta}_{OLS} = \sqrt{k}\beta + \varepsilon'$ to (1) and then using the OLS method. As an alternative to this constraint, let us consider the constraint as follows: $-2\sqrt{k}\hat{\beta}_{RE} = \sqrt{k}\beta + \varepsilon'$. In this case, the estimator is obtained as follows:

$$\hat{\beta}_{KLR} = (X'X + kI)^{-1}(X'X - kI)(X'X + kI)^{-1} X'Y, \quad k > 0 \tag{14}$$

where k is a biasing parameter. This estimator given in (14) is called KLR. Let us consider the following objective function:

$$L(\beta) = (y - X\beta)'(y - X\beta) + \left((\beta - \hat{\beta}_{KLR})'(\beta - \hat{\beta}_{KLR}) - c\right) \tag{15}$$

where $\hat{\beta}_{KLR}$ is the KLR estimator given in (14). When Equation (15) is differentiated with respect to β, the following equation is obtained:

$$(X'X + I)\beta = X'Y + \hat{\beta}_{KLR}. \tag{16}$$

Solving the system given in (16) with respect to the parameter β, yields the following estimator:

$$\begin{aligned} \hat{\beta}_{LKLR} &= (X'X + I)^{-1}\left(X'Y + \hat{\beta}_{KLR}\right), \qquad k > 0 \\ \hat{\beta}_{LKLR} &= (X'X + I)^{-1}\left((X'X) + (X'X + kI)^{-1}(X'X - kI)(X'X + kI)^{-1}(X'X)\right)(X'X)^{-1} X'Y \end{aligned} \tag{17}$$

where k is a biasing parameter. We can obtain the estimator given in (17) estimator by augmenting $\hat{\beta}_{KLR} = \beta + \varepsilon'$ to model (1) and using the OLS method.

We rewrite the model (1) in canonical form

$$Y = Z\alpha + \varepsilon \tag{18}$$

where $Z = XQ$, $\alpha = Q'\beta$ and Q is the orthogonal matrix. The columns of the orthogonal matrix Q are the eigenvectors of $X'X$. Then $Z'Z = Q'X'XQ = \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_p)$ where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p \geq 0$ are the ordered eigenvalues of $X'X$. For model (18), we can rewrite the proposed estimators in canonical form as follows:

$$\hat{\alpha}_{LKLR} = (\Lambda + I)^{-1}\left(\Lambda + (\Lambda + kI)^{-1}(\Lambda - kI)(\Lambda + kI)^{-1}\Lambda\right)\hat{\alpha}_{OLS} \tag{19}$$

where $\hat{\alpha}_{OLS} = \Lambda^{-1}Z'y$.

We compute the biasing vector and variance-covariance matrix of the estimator $\hat{\alpha}_{LKLR}$:

$$var\left(\hat{\alpha}_{LKLR}\right)=cov\left(\left(\Lambda+I\right)^{-1}\left(\Lambda+\left(\Lambda+kI\right)^{-1}\left(\Lambda-kI\right)\left(\Lambda+kI\right)^{-1}\Lambda\right)\hat{\alpha}_{OLS}\right)$$
$$=\sigma^{2}\left(\Lambda+I\right)^{-1}\left(\Lambda+\left(\Lambda+kI\right)^{-1}\left(\Lambda-kI\right)\left(\Lambda+kI\right)^{-1}\Lambda\right)\Lambda^{-1}\left(\Lambda+\left(\Lambda+kI\right)^{-1}\left(\Lambda-kI\right)\left(\Lambda+kI\right)^{-1}\Lambda\right)\left(\Lambda+I\right)^{-1}$$
$$bias\left(\hat{\alpha}_{LKLR}\right)=E\left(\hat{\alpha}_{LKLR}\right)-\alpha=E\left[\left(\Lambda+I\right)^{-1}\left(\Lambda+\left(\Lambda+kI\right)^{-1}\left(\Lambda-kI\right)\left(\Lambda+kI\right)^{-1}\Lambda\right)\hat{\alpha}_{OLS}\right]-\alpha$$
$$=\left(\left(\Lambda+I\right)^{-1}\left(\Lambda+\left(\Lambda+kI\right)^{-1}\left(\Lambda-kI\right)\left(\Lambda+kI\right)^{-1}\Lambda\right)-I\right)\alpha$$

The MMSE and SMSE of an estimator $\tilde{\beta}$ are defined as:

$$\begin{aligned}&MMSE\left(\tilde{\beta}\right)=var\left(\tilde{\beta}\right)+\left[bias\left(\tilde{\beta}\right)\right]\left[bias\left(\tilde{\beta}\right)\right]'\\&SMSE\left(\tilde{\beta}\right)=tr\left(MMSE\left(\tilde{\beta}\right)\right)=tr\left(var\left(\tilde{\beta}\right)\right)+bias\left(\tilde{\beta}\right)'bias\left(\tilde{\beta}\right).\end{aligned}\tag{20}$$

where $var\left(\tilde{\beta}\right)$ is the variance-covariance matrix and $bias\left(\tilde{\beta}\right)=E\left(\tilde{\beta}\right)-\beta$ is the biasing vector.
Let $\tilde{\beta}_1$ and $\tilde{\beta}_2$be any two estimators of parameter β. Then, $\tilde{\beta}_2$ is superior to $\tilde{\beta}_1$ with respect to the MMSE criterion if and only if $MMSE\left(\tilde{\beta}_1\right)-MMSE\left(\tilde{\beta}_2\right)$ is a positive definite (pd) matrix. If $MMSE\left(\tilde{\beta}_1\right)-MMSE\left(\tilde{\beta}_2\right)$ is a non-negative definite matrix, then $SMSE\left(\tilde{\beta}_1\right)-SMSE\left(\tilde{\beta}_2\right)\geq 0$. But, the reverse is not always true (Theobald (1974)). Because of the relation of $\alpha=Q'\beta$; $\hat{\beta}_{OLS}$, $\hat{\beta}_{RE}$, $\hat{\beta}_{LE}$, $\hat{\beta}_{NLTE}$, $\hat{\beta}_{SK}\left(k,d\right)$, $\hat{\beta}_{YC}\left(k,d\right)$ and $\hat{\beta}_{LKLR}\left(k\right)$ have the same mean squared error values as $\hat{\alpha}_{OLS}$, $\hat{\alpha}_{RE}$, $\hat{\alpha}_{LE}$, $\hat{\alpha}_{NLTE}$, $\hat{\alpha}_{SK}\left(k,d\right)$, $\hat{\alpha}_{YC}\left(k,d\right)$, and $\hat{\alpha}_{LKLR}\left(k\right)$, respectively.
In general, the theorems used to compare the two biased estimators are given below.

Theorem 2.1. *Farebrother (2022): Let A be a positive definite matrix, namely $A>0$, and c be a nonzero vector. Then, $A-cc'>0$ if and only if $c'A^{-1}c<1$.*

Theorem 2.2. *Trenkler and Toutenburg (1990): Let $\tilde{\beta}_l=B_lY,\;\; l=1,2$ be two homogeneous linear estimators of β and C be a positive definite matrix, where $B_1B_1'-B_2B_2'$. Then $MMSE\left(\tilde{\beta}_1\right)-MMSE\left(\tilde{\beta}_2\right)>0$ iff $bias\left(\tilde{\beta}_2\right)'\left(\sigma^2C+bias\left(\tilde{\beta}_1\right)bias\left(\tilde{\beta}_1\right)'\right)^{-1}bias\left(\tilde{\beta}_2\right)<1$.*

3. SUPERIORITY OF THE PROPOSED ESTIMATOR

In this section, the proposed estimator is compared with OLS, RE, LE, and KL estimators based on the MMSE sense. However, a more general theorem is given here by considering the NLTE which includes OLS, RE, LE, and KL estimators. To compare KLKR and NLTE estimators, let us first calculate the MMSE matrices of both estimators.

The *MMSE* of $\hat{\alpha}_{NLTE}=A_1Y$ and $\hat{\alpha}_{LKLR}=A_2Y$ are given as follows:

$$\begin{aligned}&MMSE\left(\hat{\alpha}_{NLTE}\right)=var\left(\hat{\alpha}_{NLTE}\right)+bias\left(\hat{\alpha}_{NLTE}\right)bias\left(\hat{\alpha}_{NLTE}\right)'\\&=\sigma^2A_1A_1'+\left(A_1Z-I\right)\alpha\alpha'\left(A_1Z-I\right)'\\&=\sigma^2\left(\Lambda+kI\right)^{-1}\left(\Lambda+f\left(k\right)I\right)\Lambda^{-1}\left(\Lambda+f\left(k\right)I\right)\left(\Lambda+kI\right)^{-1}\\&+\left(f\left(k\right)-k\right)^2\left(\Lambda+kI\right)^{-1}\alpha\alpha'\left(\Lambda+kI\right)^{-1}\end{aligned}\tag{21}$$

$$\begin{aligned}&MMSE\left(\hat{\alpha}_{LKLR}\right)=var\left(\hat{\alpha}_{LKLR}\right)+bias\left(\hat{\alpha}_{LKLR}\right)bias\left(\hat{\alpha}_{LKLR}\right)'\\&=\sigma^2A_2A_2'+\left(A_2Z-I\right)\alpha\alpha'\left(A_2Z-I\right)'\\&=\sigma^2\left(\Lambda+I\right)^{-1}\left(\Lambda+\left(\Lambda+kI\right)^{-1}\left(\Lambda-kI\right)\left(\Lambda+kI\right)^{-1}\Lambda\right)\Lambda^{-1}\left(\Lambda+\left(\Lambda+kI\right)^{-1}\left(\Lambda-kI\right)\left(\Lambda+kI\right)^{-1}\Lambda\right)\left(\Lambda+I\right)^{-1}\\&+\left(\left(\Lambda+I\right)^{-1}\left(\Lambda+\left(\Lambda+kI\right)^{-1}\left(\Lambda-kI\right)\left(\Lambda+kI\right)^{-1}\Lambda\right)-I\right)\alpha\alpha'\left(\left(\Lambda+I\right)^{-1}\left(\Lambda+\left(\Lambda+kI\right)^{-1}\left(\Lambda-kI\right)\left(\Lambda+kI\right)^{-1}\Lambda\right)-I\right)'\end{aligned}\tag{22}$$

Then, we can give the following theorem:

Theorem 3.1. *Let be $k>0$ and $\left|\lambda_j+f\left(k\right)\right|\left(\lambda_j+1\right)\left(\lambda_j+k\right)>\lambda_j\left|\left(\lambda_j+k\right)^2+\lambda_j-k\right|$ where $j=1,2,\ldots,p+1$. Then, $MMSE\left(\hat{\alpha}_{NLTE}\right)-MMSE\left(\hat{\alpha}_{LKLR}\right)>0$ if and only if*

$$bias\left(\hat{\alpha}_{NLTE}\right)'\left[\sigma^2\left(A_1A_1'-A_2A_2'\right)+bias\left(\hat{\alpha}_{LKLR}\right)bias\left(\hat{\alpha}_{LKLR}\right)'\right]^{-1}bias\left(\hat{\alpha}_{NLTE}\right)<1\tag{23}$$

where $bias\left(\hat{\alpha}_{NLTE}\right)=\left(f\left(k\right)-k\right)\left(\Lambda+kI\right)^{-1}\alpha$.

Proof. Using (21) and (22), we obtain

$$
\begin{aligned}
& var\left(\hat{\alpha}_{NLTE}\right) - var\left(\hat{\alpha}_{LKLR}\right) = \sigma^2\left[A_1A_1' - A_2A_2'\right] \\
& = \sigma^2\left[(\Lambda + kI)^{-1}\left(\Lambda + f(k)I\right)\Lambda^{-1}\left(\Lambda + f(k)I\right)(\Lambda + kI)^{-1}\right. \\
& \left. -(\Lambda + I)^{-1}\left(\Lambda + (\Lambda + kI)^{-1}(\Lambda - kI)(\Lambda + kI)^{-1}\Lambda\right)\Lambda^{-1}\left(\Lambda + (\Lambda + kI)^{-1}(\Lambda - kI)(\Lambda + kI)^{-1}\Lambda\right)(\Lambda + I)^{-1}\right] \\
& = \sigma^2\, diag\left\{\frac{\left(\lambda_j + f(k)\right)^2}{\lambda_j\left(\lambda_j + k\right)^2} - \frac{\lambda_j\left(\left(\lambda_j + k\right)^2 + \lambda_j - k\right)^2}{\left(\lambda_j + 1\right)^2\left(\lambda_j + k\right)^4}\right\}_{j=1}^{p+1}.
\end{aligned}
$$

We can observe that $A_1A_1' - A_2A_2' > 0$ if and only if $\left(\lambda_j + f(k)\right)^2\left(\lambda_j + 1\right)^2\left(\lambda_j + k\right)^2 - \lambda_j^2((\lambda_j + k)^2 + \lambda_j - k)^2 > 0$ where $j = 1, 2, ..., p + 1$. Therefore, $A_1A_1' - A_2A_2'$ is the pd matrix. By Theorem 2.2, the proof is completed.

4. SELECTION OF BIASING PARAMETER

In general, the performance of estimators depends on the biasing parameters. There are many techniques for estimating biasing parameters. However, among researchers, values that can minimize the SMSE function are often suggested as estimators of the biasing parameter. Firstly, to find the optimal biasing parameter k, we take the derivative of $h(k) = SMSE\left(\hat{\beta}_{LKLR}\right)$ with respect to k where $SMSE\left(\hat{\beta}_{LKLR}\right)$ is given as follows:

$$
SMSE\left(\hat{\beta}_{LKLR}\right) = \sum_{j=1}^{p+1} \frac{\left(\lambda_j\left(\lambda_j + k\right)^2 + \left(\lambda_j - k\right)\lambda_j\right)^2\sigma^2}{\lambda_j\left(\lambda_j + 1\right)^2\left(\lambda_j + k\right)^2} + \left(\frac{\lambda_j\left(\lambda_j + k\right)^2 + \left(\lambda_j - k\right)\lambda_j}{\left(\lambda_j + 1\right)\left(\lambda_j + k\right)^2} - 1\right)^2\alpha_j^2
$$

Then, we find $h'(k)$ as follows differentiating $h(k)$ with respect to k:

$$
h'(k) = \sum_{j=1}^{p+1} \frac{2\lambda_j\left(k - 3\lambda_j\right)\left(-k\alpha_j^2\left(k + 3\lambda_j\right) + \sigma^2\left((-1 + k)k + (1 + 2k)\lambda_j + \lambda_j^2\right)\right)}{\left(1 + \lambda_j\right)^2\left(k + \lambda_j\right)^5}
$$

When it is accepted $h'(k) = 0$, we have:

$$
\begin{aligned}
k_1 &= 3\lambda_j \\
k_2 &= \frac{\sigma^2 - 2\sigma^2\lambda_j + 3\alpha_j^2\lambda_j - \sqrt{\sigma^4 - 8\sigma^4\lambda_j + 10\sigma^2\alpha_j^2\lambda_j - 8\sigma^2\alpha_j^2\lambda_j^2 + 9\alpha_j^4\lambda_j^2}}{2\left(\sigma^2 - \alpha_j^2\right)} \\
k_3 &= \frac{\sigma^2 - 2\sigma^2\lambda_j + 3\alpha_j^2\lambda_j + \sqrt{\sigma^4 - 8\sigma^4\lambda_j + 10\sigma^2\alpha_j^2\lambda_j - 8\sigma^2\alpha_j^2\lambda_j^2 + 9\alpha_j^4\lambda_j^2}}{2\left(\sigma^2 - \alpha_j^2\right)}
\end{aligned}
$$

where $i = 1, 2, ..., p+1$. Unfortunately, the k value depends on σ^2 and α_j^2. For practical purposes, we replace them with their unbiased estimators $\hat{\sigma}^2$ and $\hat{\alpha}_j^2$ to find the estimators of the biasing parameter k. Based on the simulation results, we can use the following estimators to estimate the biasing parameter k: $\hat{k}_{LKLR\ I} = \frac{3\max\left(\lambda_j\right)}{p}$, $\hat{k}_{LKLR\ II} = 3\ median\left(\lambda_j\right)$, $\hat{k}_{LKLR\ III} = \frac{\hat{\sigma}^2}{\left(\prod_{j=1}^{p+1}\hat{\alpha}_j^2\right)^{\frac{1}{p+1}}}$ where $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2}{n-p-1}$.

5. THE MONTE CARLO SIMULATION STUDIES

In this section, the performance of the proposed biased estimator is compared with other existing estimators using two different Monte Carlo simulation designs. In the first design, we investigated the effects of sample size (n), the degree of the collinearity (ρ), the number of explanatory variables (p), and the variance $\left(\sigma^2\right)$ on the performances of OLS, RE, LE, LTE, SK, YC, KL, TSS, STO, LKL, and LKLR estimators. In the second simulation design, we examined RE, LE, KL, and LKLR performances for each of n, p, ρ, and σ^2 values at certain values of k. For both simulation designs, we generate the explanatory variables by following McDonald and Galarneau (1975) and Kibria (2003) as

$$
x_{ij} = \left(1 - \rho^2\right)^{1/2} u_{ij} + \rho u_{ip+1}, \quad i = 1, 2, .., n, \quad j = 1, 2, ..., p \tag{24}
$$

where u_{ij} are independent standard normal pseudo-random numbers. ρ is specified so that the correlation between any two variables is given by ρ^2. These variables are standardized such that $X'X$ is a correlation matrix. Investigations are conducted on

Table 1. The EMSE values of the estimators for the model when $p = 4$.

σ^2	n	ρ	OLS	RE	LE	YCI	YCII	SK	LTE	KL	TSS	TPE	STO	LKL	LKLRI	LKLRII	LKLRIII
1	50	0.8	7.73	1.455	0.962	4.417	3.178	2.898	3.058	4.386	7.387	3.104	338.930	4.386	0.644**	**0.632***	0.883***
5	50	0.8	37.874	5.058	4.35	21.046	15.353	13.585	14.322	21.185	35.526	14.056	469.622	21.185	2.949**	**2.935***	3.802***
10	50	0.8	74.525	8.791	8.385	40.745	29.257	25.808	27.145	41.073	69.530	26.512	5596.336	41.073	5.655**	**5.630***	7.02***
1	50	0.9	17.546	2.339	0.633	9.476	6.847	6.179	6.426	9.485	16.794	6.466	22.261	9.485	0.392**	**0.275***	0.486***
5	50	0.9	84.246	7.491	2.725	44.749	32.293	28.924	29.885	44.912	78.449	29.614	4357299.439	44.912	1.764**	**1.348***	1.868***
10	50	0.9	174.597	13.952	5.543	94.22	67.023	60.561	62.517	94.497	162.223	61.774	761.113	94.497	3.588***	**2.738***	3.568**
1	50	0.95	39.226	3.87	0.44	20.961	15.063	13.596	14.051	21.005	37.523	14.045	43.351	21.005	**0.302***	0.328**	0.368***
5	50	0.95	196.097	13.716	1.914	104.112	74.761	67.173	69.443	104.3	182.348	69.066	423.743	104.3	**1.375***	1.675**	1.704***
10	50	0.95	413.446	26.2	3.705	225.051	163.637	146.813	152.39	225.5	384.367	151.422	56193.545	225.5	**2.617***	3.213***	2.964**
1	100	0.8	8.749	1.487	0.867	4.833	3.413	3.119	3.235	4.789	8.313	3.305	168.014	4.789	0.556**	**0.511***	0.73***
5	100	0.8	43.477	5.28	3.927	23.714	16.744	15.269	15.649	23.771	40.315	15.479	150.473	23.771	2.553**	**2.436***	3.121***
10	100	0.8	88.483	9.815	7.809	48.594	34.656	31.326	32.189	48.788	81.857	31.732	10580.962	48.788	5.024**	**4.784***	6.248***
1	100	0.9	18.385	2.316	0.618	9.773	6.936	6.32	6.501	9.768	17.498	6.558	39.603	9.768	0.39**	**0.272***	0.435***
5	100	0.9	91.472	8.055	2.701	48.369	34.702	31.355	32.02	48.448	84.475	31.810	859.464	48.448	1.745***	**1.286***	1.725**
10	100	0.9	188.796	15.185	5.451	102.537	72.716	66.116	67.663	102.714	174.134	67.057	2450.396	102.714	3.56***	**2.645***	3.438**
1	100	0.95	34.568	3.559	0.484	18.518	13.351	12.01	12.454	18.554	33.1	12.456	57.270	18.554	0.319**	**0.285***	0.375***
5	100	0.95	168.844	11.045	2.01	90.465	64.932	57.955	60.116	90.679	157.235	59.742	2785.722	90.679	1.392***	**1.360***	1.383**
10	100	0.95	338.382	20.712	4.089	181.26	129.603	116.701	120.421	181.718	314.091	119.547	3498.758	181.718	2.879**	2.881***	**2.853***
1	200	0.8	8.405	1.469	0.859	4.589	3.234	2.963	3.078	4.549	7.987	3.149	772.531	4.549	0.556**	**0.516***	0.739***
5	200	0.8	42.94	5.009	4.005	23.356	16.556	14.94	15.398	23.432	39.881	15.228	330.363	23.432	2.619**	**2.510***	3.195***
10	200	0.8	85.782	9.478	7.802	46.688	32.904	29.733	30.638	46.879	79.325	30.147	4292.141	46.879	5.054**	**4.846***	6.073***
1	200	0.9	16.174	2.16	0.662	8.6	6.11	5.568	5.728	8.585	15.385	5.783	41.577	8.585	0.41**	**0.294***	0.468***
5	200	0.9	84.006	7.882	2.974	45.71	32.927	29.939	30.554	45.76	77.806	30.355	465.148	45.76	1.897**	**1.405***	1.963***
10	200	0.9	167.663	14.1	5.887	91.208	65.68	59.698	60.985	91.367	154.677	60.403	611.768	91.367	3.778**	**2.858***	3.78***
1	200	0.95	28.83	3.187	0.518	15.428	11.019	10.025	10.28	15.442	27.526	10.302	74.288	15.442	0.331**	**0.259***	0.348***
5	200	0.95	139.313	10.602	2.199	73.591	52.831	47.801	48.904	73.712	128.824	48.608	3536.723	73.712	1.495***	**1.283***	1.455**
10	200	0.95	286.894	19.693	4.397	153.373	109.879	98.912	101.520	153.66	264.558	100.755	471.594	153.66	2.993***	**2.568***	2.915**

four distinct sets of correlations that correspond to $\rho = 0.8, 0.9$ and 0.95. The response variable is generated by

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_p x_{pi} + \varepsilon_i, \quad i = 1, 2, \ldots, n$$

where $\varepsilon_i \sim N\left(0, \sigma^2\right)$ and β_0 is equal to zero. The values of σ^2 are 1, 5, and 10 for various comparisons of the error term. For each set of explanatory variables, the parameter vector β is chosen as the normalized eigenvector corresponding to the largest eigenvalue of $X'X$ so that $\beta'\beta = 1$. The sample sizes n are 50, 100, and 200. The number of explanatory variables is chosen as $p = 4,\ 8,$ and 12.

For the simulation and application sections, we use the estimator proposed by Kibria (2003) to estimate the parameter k in RE, as follows: $\hat{k}_{RE} = \frac{\hat{\sigma}^2}{\left(\prod_{j=1}^{p+1} \hat{\alpha}_j^2\right)^{\frac{1}{p+1}}}$ where $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n-p-1}$. Based on the results given by Qasim et al. (2020), we use the best estimation of d in LE as $\hat{d}_{LE} = \max\left(0, \min\left(\frac{\hat{\alpha}_j^2 - \hat{\sigma}^2}{\max\left(\frac{\hat{\sigma}^2}{\lambda_j}\right) + \hat{\alpha}_{\max}^2}\right)\right)$. Moreover, we used the best estimators and iterative techniques recommended by Liu (2003), Özkale and Kaçıranlar (2007), Sakallıoğlu and Kaçıranlar (2008), Yang and Chang (2010), Huang and Yang (2014), Kibria and Lukman (2020), Üstündağ et al. (2021), Qasim et al. (2022), Idowu et al. (2023) to determine the estimates of the biasing parameters for the LTE, SK, YC, KL, TSS, STO, TPE, and LKL estimators.

The performance of the estimated MSEs (EMSEs) is used as a basis for comparing the proposed estimators, which are calculated for an estimator $\hat{\beta}$ of β as

$$EMSE\left(\hat{\beta}\right) = \frac{1}{N}\sum_{r=1}^{N}\left(\hat{\beta}_r - \beta\right)'\left(\hat{\beta}_r - \beta\right) \tag{25}$$

where $\left(\hat{\beta}_r - \beta\right)$ is the difference between the estimated and true parameter vectors at rth replication and N is the number of replications. The experiment was repeated 2000 times for each case of n, p, σ^2, and ρ by creating response variables. The computations were performed in R programming language. The results are given in Tables 1-3 where the first, second, and third best EMSE values in each row are indicated by the signs (*), (**), and (***).

In all 81 scenarios in Tables 1-3, the proposed estimator outperformed all other available estimators according to the EMSE criterion. However, all considered estimators exhibited different behaviors in different scenarios. The following observations can be obtained from Tables 1-3:

1. When the number of variables in the model is gradually increased while keeping ρ, n, and σ^2 constant, an increase is observed in the EMSE values of all estimators in general. However, this increase is much lower in the proposed estimator.
2. When the correlation ρ between the variables in the model is increased while keeping $n, p,$ and σ^2 constant, the EMSE values of some estimators increased while the EMSE values of some estimators systematically decreased. The EMSE of the proposed estimator tends to decrease as the correlation coefficient increases.

Table 2. The EMSE values of the estimators for the model when $p = 8$.

σ^2	n	ρ	**OLS**	**RE**	**LE**	**YCI**	**YCII**	**SK**	**LTE**	**KL**	**TSS**	**TPE**	**STO**	**LKL**	**LKLRI**	**LKLRII**	**LKLRIII**
1	50	0.8	17.927	1.901	1.615	10.916	5.893	5.524	5.87	10.975	17.701	5.902	7751.854	10.975	0.966**	**0.950***	1.186***
5	50	0.8	89.699	7.356	7.997	53.873	29.153	27.141	28.682	54.309	87.72	28.115	56086.283	54.309	4.827**	**4.756***	5.610***
10	50	0.8	177.684	13.807	15.598	105.388	57.253	52.721	56.191	106.26	173.343	54.862	4699.803	106.26	9.382**	**9.241***	10.812***
1	50	0.9	41.839	3.719	1.065	25.14	13.327	12.367	13.389	25.312	41.371	13.231	27424.305	25.312	0.570**	**0.320***	0.576***
5	50	0.9	206.925	13.664	5.103	123.075	65.337	59.351	65.216	124.027	202.052	64.073	17058.325	124.027	2.739***	**1.531***	2.432**
10	50	0.9	412.283	25.184	10.189	245.835	130.042	117.648	129.814	247.748	401.849	127.342	10383.83	247.748	5.530***	**3.176***	4.754**
1	50	0.95	93.092	7.164	0.666	56.037	30.042	27.194	30.369	56.422	92.182	29.828	510.059	56.422	0.371**	**0.353***	0.524***
5	50	0.95	459.191	24.519	3.156	277.59	147.993	131.557	148.453	279.665	449.73	145.97	565.976	279.665	1.772**	**1.700***	1.912***
10	50	0.95	917.846	45.27	6.246	551.874	291.829	258.909	293.122	555.681	896.501	288.224	2039.893	555.681	3.512***	**3.393***	3.455**
1	100	0.8	24.214	2.32	1.384	14.402	7.547	7.074	7.554	14.48	23.922	7.546	152315.676	14.48	0.771**	**0.650***	0.866***
5	100	0.8	123.027	9.379	6.943	73.791	38.853	36.518	38.553	74.252	120.277	37.947	8893.613	74.252	3.853**	**3.249***	4.176***
10	100	0.8	244.546	16.161	13.779	146.193	76.354	71.107	75.404	147.09	238.722	74.086	6325.787	147.09	7.649**	**6.472***	8.049***
1	100	0.9	37.573	3.45	1.061	22.402	11.361	11.022	11.334	22.445	37.105	11.365	2624.188	22.445	0.561***	**0.273***	0.469**
5	100	0.9	194.263	13.369	5.364	117.333	60.514	58.269	60.176	117.596	189.324	59.737	6023.231	117.596	2.868***	**1.414***	2.148**
10	100	0.9	386.218	26.721	10.581	232.783	121.565	117.205	120.897	233.363	375.623	119.85	11982.27	233.363	5.612***	**2.715***	4.449**
1	100	0.95	69.842	5.487	0.709	41.58	20.968	20.245	20.912	41.705	69.033	20.833	553.66	41.705	0.384***	**0.191***	0.256**
5	100	0.95	359.95	22.867	3.521	218.104	112.674	108.295	112.261	218.593	351.106	111.435	706.995	218.593	1.931***	**0.975***	1.209**
10	100	0.95	716.945	40.935	6.949	431.688	223.182	213.159	222.355	432.848	697.615	220.667	1132.808	432.848	3.829***	**1.975***	2.289**
1	200	0.8	17.411	1.884	1.591	10.496	5.377	5.245	5.366	10.5	17.132	5.493	24916.018	10.5	0.912**	**0.866***	1.067***
5	200	0.8	88.456	7.555	7.965	53.407	27.722	26.929	27.416	53.526	86.118	27.25	1506.584	53.526	4.589**	**4.380***	5.231***
10	200	0.8	177.523	14.336	15.978	107.477	55.476	53.841	54.734	107.653	172.516	54.213	1215.601	107.653	9.220**	**8.804***	10.45***
1	200	0.9	40.78	3.737	1.04	24.526	12.633	12.329	12.596	24.552	40.305	12.638	55943.504	24.552	0.551***	**0.240***	0.427**
5	200	0.9	201.348	13.602	5.054	120.901	61.766	59.923	61.307	121.119	196.316	60.954	1500.696	121.119	2.710***	**1.195***	1.866**
10	200	0.9	409.144	26.535	10.183	245.851	126.485	122.861	125.571	246.342	398.247	124.691	825.275	246.342	5.428***	**2.368***	3.712**
1	200	0.95	75.512	5.908	0.703	45.297	23.2	22.203	23.202	45.427	74.645	23.093	171.987	45.427	0.384***	**0.221***	0.274**
5	200	0.95	380.004	23.253	3.476	228.965	117.798	113.149	117.327	229.502	370.325	116.474	4790.433	229.502	1.934***	**1.108***	1.223**
10	200	0.95	753.565	41.767	6.857	451.33	229.927	221.459	228.576	452.452	732.852	226.918	1004.939	452.452	3.796***	**2.139***	2.252**

Table 3. The EMSE values of the estimators for the model when $p = 12$.

σ^2	n	ρ	**OLS**	**RE**	**LE**	**YCI**	**YCII**	**SK**	**LTE**	**KL**	**TSS**	**TPE**	**STO**	**LKL**	**LKLRI**	**LKLRII**	**LKLRIII**
1	50	0.8	36.987	2.821	2.182	23.621	11.626	10.354	11.637	23.865	36.751	11.434	297772.445	23.865	1.206**	**1.134***	1.403***
5	50	0.8	181.743	11.791	10.837	115.281	56.311	50.591	55.959	116.44	179.451	54.33	6352.978	116.44	6.019**	**5.678***	6.724***
10	50	0.8	359.223	22.14	21.439	226.952	109.898	98.497	109.114	229.446	354.314	105.785	36819.019	229.446	11.937**	**11.268***	13.324***
1	50	0.9	101.156	6.493	1.236	64.183	30.847	26.135	31.514	64.93	100.689	30.597	25701.481	64.930	0.623***	**0.277***	0.587**
5	50	0.9	504.845	26.812	6.173	321.946	156.031	131.887	158.225	325.534	499.227	153.804	188521.021	325.534	3.121***	**1.388***	2.349**
10	50	0.9	1023.08	51.434	12.291	653.041	316.387	263.471	320.556	660.255	1010.527	311.849	5239.590	660.255	6.206***	**2.743***	4.713**
1	50	0.95	162.466	10.121	0.902	103.233	50.001	41.96	51.299	104.375	161.757	49.694	718183.886	104.375	0.459**	**0.411***	0.719***
5	50	0.95	816.49	40.696	4.435	515.358	245.943	203.666	250.67	520.759	807.093	243.701	10870.123	520.759	2.262**	**2.036***	2.762***
10	50	0.95	1634.04	81.385	9.039	1043.321	501.717	417.399	510.434	1053.802	1613.713	496.017	12131.320	1053.802	4.620**	**4.095***	5.652***
1	100	0.8	32.046	2.643	2.227	20.287	9.363	8.959	9.364	20.385	31.819	9.398	8129.905	20.385	1.192**	**1.081***	1.300***
5	100	0.8	160.081	10.991	11.079	101.783	47.11	44.972	46.948	102.207	157.865	46.311	3839.132	102.207	5.970**	**5.427***	6.343***
10	100	0.8	320.861	20.747	22.252	204.985	94.66	90.554	94.2	205.84	316.083	92.863	88772.183	205.84	11.983**	**10.897***	12.690***
1	100	0.9	80.278	5.873	1.342	51.14	23.535	22.366	23.737	51.347	79.841	23.512	46913.183	51.347	0.667***	**0.228***	0.409**
5	100	0.9	395.126	22.718	6.544	250.47	114.084	107.973	114.771	251.494	389.874	113.33	1915.609	251.494	3.266***	**1.125***	1.742**
10	100	0.9	803.981	46.594	13.376	514.05	236.623	224.291	237.897	515.851	792.452	234.817	4297.482	515.851	6.694***	**2.329***	3.744**
1	100	0.95	142.346	9.444	0.909	90.366	41.55	38.237	42.151	91.012	141.738	41.353	2048.883	91.012	0.457***	**0.306***	0.423**
5	100	0.95	709.597	38.951	4.464	448.784	208.094	188.448	210.461	452.13	701.586	206.643	946.410	452.13	2.281***	**1.593***	1.840**
10	100	0.95	1429.302	70.908	8.882	903.696	418.151	376.611	422.328	909.787	1411.304	414.951	2386.412	909.787	4.493***	**3.114***	3.381**
1	200	0.8	32.059	2.627	2.222	20.415	9.35	9.113	9.354	20.47	31.803	9.452	7463.637	20.470	1.177**	**1.039***	1.235***
5	200	0.8	158.478	10.304	10.911	100.461	45.229	44	45.137	100.726	155.998	44.809	8605.372	100.726	5.795**	**5.124***	5.953***
10	200	0.8	314.225	19.312	21.62	198.306	88.923	86.551	88.606	198.833	308.752	87.761	27821.477	198.833	11.457**	**10.126***	11.752***
1	200	0.9	71.274	5.27	1.422	45.459	20.749	19.923	20.892	45.581	70.877	20.792	485542.076	45.581	0.703***	**0.231***	0.407**
5	200	0.9	347.253	20.583	6.95	221.085	99.149	95.303	99.532	221.68	342.463	98.611	6795816.344	221.680	3.457***	**1.159***	1.822**
10	200	0.9	710.097	39.336	14.265	453.976	205.42	197.836	205.874	455.257	699.539	203.932	306580.259	455.257	7.107***	**2.401***	3.650**
1	200	0.95	134.639	8.946	0.87	85.746	38.286	36.883	38.54	85.943	133.913	38.279	2037.410	85.943	0.439***	0.319**	**0.303***
5	200	0.95	670.351	36.382	4.317	423.068	189.037	182.964	189.833	423.85	660.96	188.341	663.181	423.850	2.185***	1.566**	**1.281***
10	200	0.95	1381.718	69.995	8.804	884.071	396.959	381.753	398.948	886.431	1360.706	395.945	5310.917	886.431	4.451***	3.242**	**2.565***

Table 4. The estimated parameter values and the estimated variance values of the estimators

	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$var(\hat{\beta})$	$SMSE(\hat{\beta})$
$\hat{\beta}_{OLS}$	62.4054	1.5511	0.5102	0.1019	–0.1441	4912.0902	
$\hat{\beta}_{RE}\left(\hat{k}_{RE}=1.4250\right)$	0.1003	2.1725	1.1568	0.7435	0.4882	0.0673	5.07197
$\hat{\beta}_{LE}\left(\hat{d}_{LE}=0\right)$	0.1230	2.1781	1.1552	0.7473	0.4871	0.0715	5.06501
$\hat{\beta}_{LTE}\left(\hat{k}_{LTE}=1.4250,\ \hat{d}_{LTE}=-0.6291\right)$	27.6066	1.8982	0.8713	0.4602	0.2091	959.5019	961.0631
$\hat{\beta}_{SK}\left(\hat{k}_{SK}=1.4250,\ \hat{d}_{SK}=493.7504\right)$	26.4790	8.5996	–0.6618	5.2740	–0.7883	878.0997	2620.2491
$\hat{\beta}_{YC\ I}\ (\hat{K}_1=0.0015,\ \hat{D}_1=0.9992)$	27.6068	1.9090	0.8688	0.4680	0.2075	959.5030	961.0595
$\hat{\beta}_{YC\ II}\ (\hat{K}_2=0.0008,\ \hat{D}_2=0.7206)$	27.6067	1.9052	0.8697	0.4653	0.2080	959.5027	961.0598
$\hat{\beta}_{TSS}\left(\hat{k}_{TSS}=0.5509\times10^{-3},\ \hat{d}_{TSS}=0.7920\right)$	27.6068	1.9091	0.8688	0.468	0.2075	959.5030	961.05953
$\hat{\beta}_{LKL}\left(\hat{k}_{LKL}=0.4714\times10^{-3},\ \hat{d}_{LKL}=1\right)$	27.6068	1.9091	0.8688	0.468	0.2075	959.5030	961.0595
$\hat{\beta}_{TPE}\left(\hat{k}_{TPE}=37.9673,\ \hat{d}_{TPE}=0.4420\right)$	27.6046	1.6898	0.9184	0.3167	0.2396	959.5464	962.9542
$\hat{\beta}_{STO}\left(\hat{k}_{STO}=29.4052,\ \hat{d}_{STO}=49148.7380\right)$	62.3251	1.2323	0.5835	–0.1196	–0.0962	4900.1245	4904.0676
$\hat{\beta}_{KL}\left(\hat{k}_{KL}=0.4714\times10^{-3}\right)$	27.6068	1.9091	0.8688	0.468	0.2075	959.5030	961.0595
$\hat{\beta}_{LKLR}\left(\hat{k}_{LKLR\ I}=26805.7236\right)$	0.1230	2.1780	1.1552	0.7473	0.4872	0.0715	5.0651
$\hat{\beta}_{LKLR}\left(\hat{k}_{LKLR\ II}=2429.8562\right)$	0.1230	2.1775	1.1554	0.7471	0.4872	0.0714	5.0655
$\hat{\beta}_{LKLR}\left(\hat{k}_{LKLR\ III}=1.4250\right)$	0.0701	2.1918	1.1527	0.7574	0.4857	0.0659	5.0606

3. The impact of model variance on the performance of estimators is quite high. In scenarios where n, p, and ρ are kept constant and the variance is increased, it is observed that the EMSE values of all existing estimators, including our proposed estimator, increase. However, the dramatic increase in model variance does not significantly reduce the performance of the proposed estimator.
4. It is observed that the change in the number of observations n does not have a significant effect on the estimators. The EMSE values of all estimators, including the proposed estimator, do not change significantly when the number of observations is increased.

As a result, the proposed LKLR estimator is not significantly affected by an increase in model variance, correlation between variables, or the number of variables in the model.

In the second simulation scheme, we investigate the performances of RE, LE, KL, and LKLR for each $n,\ p,\ \rho$, and σ^2. The purpose of this simulation is to examine the performances of RE, LE, KL, and LKLR at various values of the biasing parameter k according to EMSE values given in (25). The biasing parameter k is not estimated in the second simulation scheme. Only the EMSE values obtained by increasing k values in the range $[0.1,\ 1]$ by 0.1 are compared. We only consider the cases $\rho = 0.8,\ 0.9$, $n = 50,\ \ 200$, and $p = 4,\ \ 12$, and $\sigma^2 = 1,\ \ 10$. Depending on these $n, \rho, p,$ and σ^2 values, the explanatory variables are generated according to equation (24). For every value of k, the simulation is run 2000 times. The results are collectively presented graphically in Figures 1 and 2.
Figures 1 and 2 clearly show the effects of varying the biasing parameter between 0.1 and 1 on the EMSE values of the estimators. According to Figures 1 -2 , we can obtain the following results depending on each (n, ρ, p, σ^2).

1. The RE tends to decrease as k increases. But the decrease is lagging behind the other estimators for small values of the parameter k.
2. The EMSE values of LE have the best EMSE value at small values of the biasing parameter d, while it is observed that there is an increase with increasing values of d.
3. The EMSE values of the KL estimator first decrease and then increase as k values increase.
4. The proposed LKLR estimator has smaller EMSE values with increasing correlation between variables.

6. NUMERICAL EXAMPLE

In this section, we reconsider the dataset on Portland cement data which was analyzed by Hald (2022), Liu (2003), Sakallıoğlu and Kaçıranlar (2008), Yang and Chang (2010), Kurnaz and Akay (2018). In this data, the following four compounds are independent variables: tricalcium aluminate (x_1), tetracalcium silicate (x_2), tetracalcium alumino ferrite (x_3), and dicalcium silicate (x_4). The dependent variable y is the heat evolved in calories per gram of cement. We fit a linear regression model with an intercept to the data. Then, the eigenvalues of $X'X$ are $\lambda_1 = 44676.2059$, $\lambda_2 = 5965.4221$, $\lambda_3 = 809.9521$, $\lambda_4 = 105.4187$, and $\lambda_5 = 0.0012$. The condition number is approximately 3.66×10^7, therefore the matrix X is quite ill-conditioned.

The numerical results are summarized in Table 4. In addition, $\hat{\alpha}_{OLS}$ is substituted for α in order to calculate SMSE values. From Table 4, it can be observed that the estimated variance values and the SMSE values of LKLR I, LKLR II, and LKLR III yield appropriate results compared to other existing estimators.

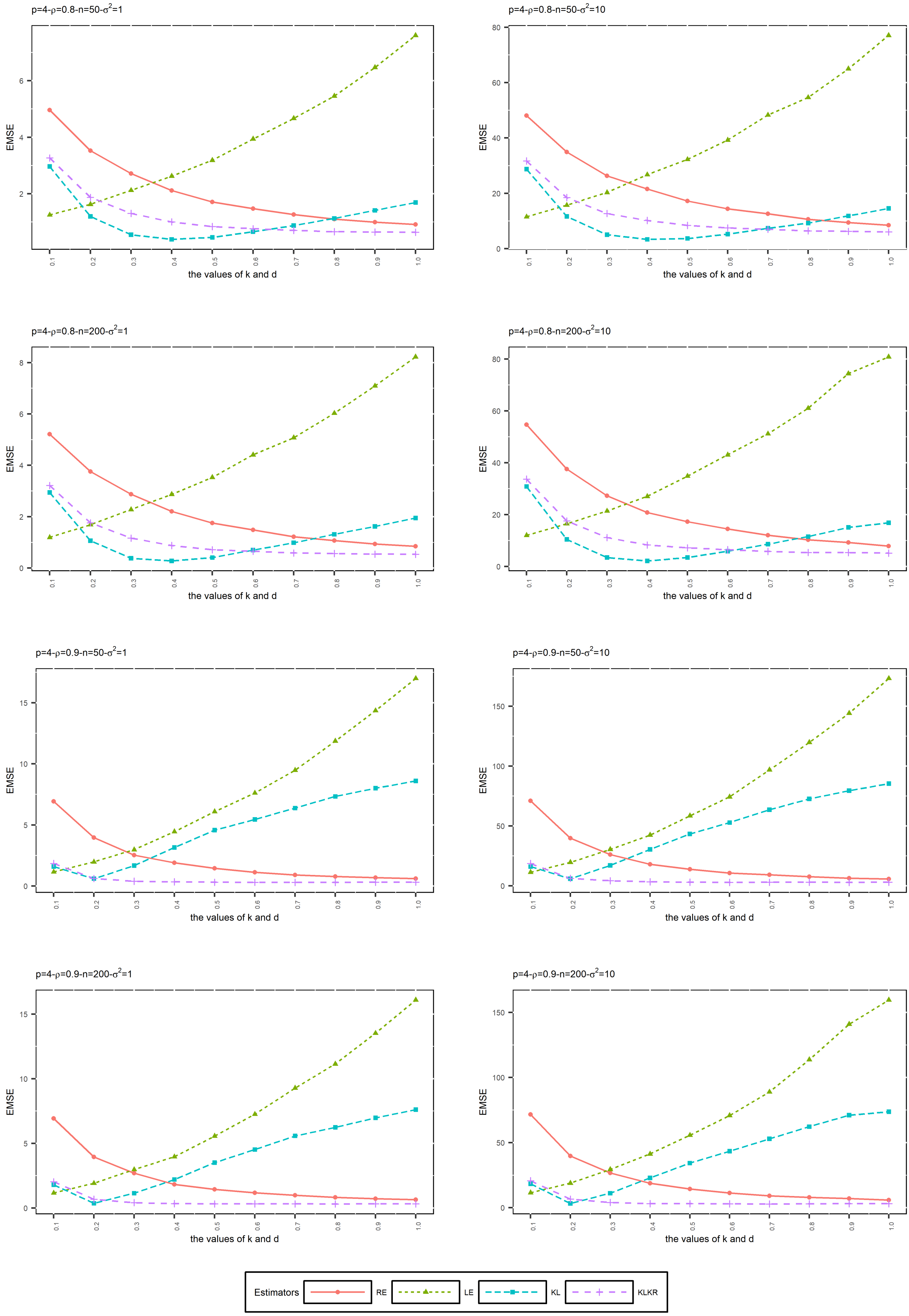

Figure 1. The EMSE values of RE, LE, KL, and LKLR as a function k and d where $p = 4$

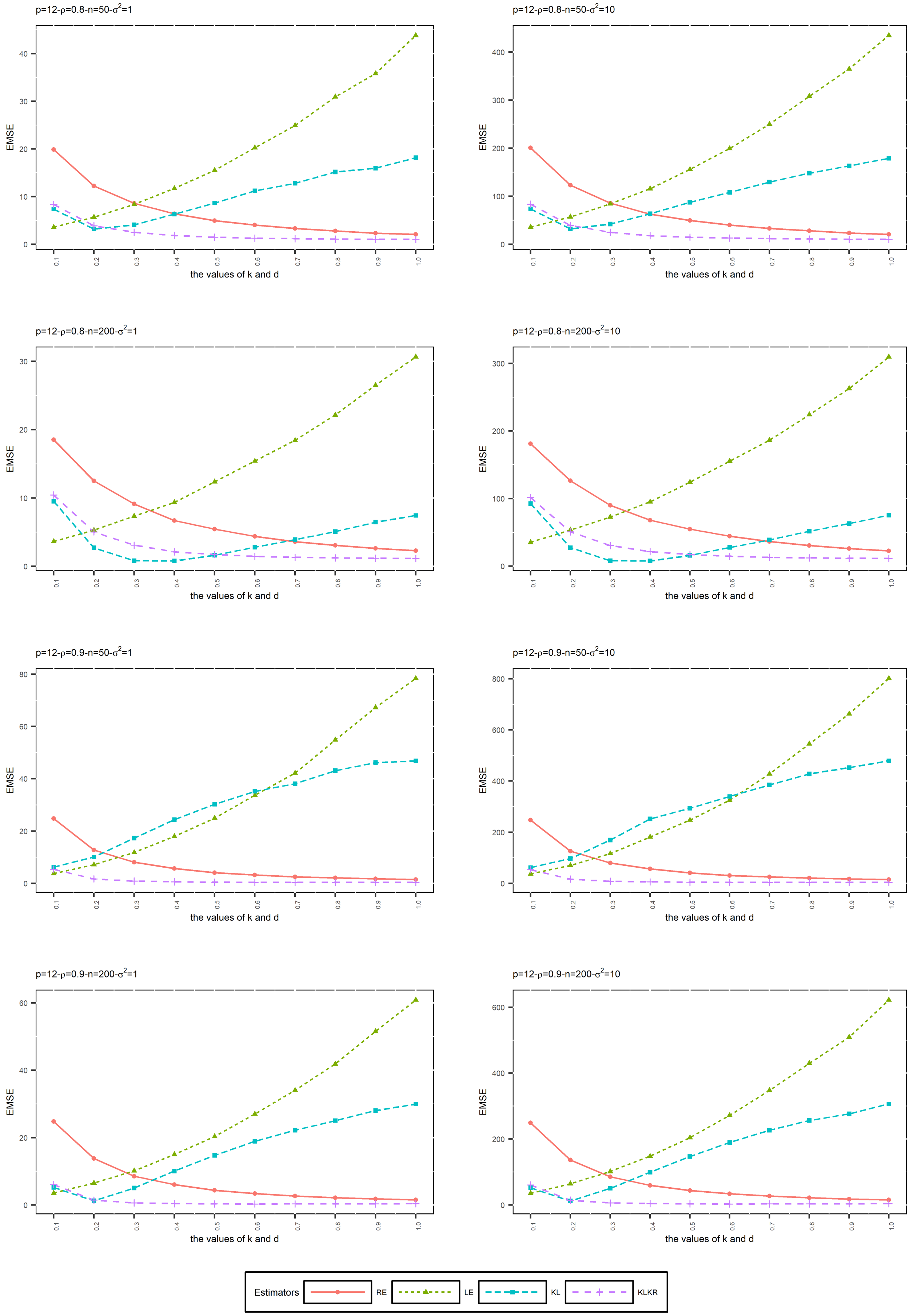

Figure 2. The EMSE values of RE, LE, KL, and LKLR as a function k and d where $p = 12$

7. CONCLUSION

In this study, a new biased estimator called LKLR is proposed in the presence of multicollinearity. This estimator has one biasing parameter as an alternative to estimators with two biasing parameters. New estimators are proposed to estimate the biasing parameter of the LKLR estimator. Simulation results show that the LKLR estimator performs better than standard estimators. In particular, $\hat{k}_{LKLR\ II}$ gave better results than other proposed biasing parameter estimators. We also examined the overall performance of other estimators with a single biasing parameter when k is in the range $[0.1, 1]$. Furthermore, the performance of the LKLR on Portland data is analyzed together with other existing estimators. Based on the results, a more robust estimator is obtained for increasing variance, variables, correlation, and number of observations than estimators with two biasing parameters. Finally, the LKLR is recommended when there is multicollinearity in the linear regression models.

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RESEARCH ARTICLE

Effectiveness of a One-fifth Hybrid Block Approach for Second Order Ordinary Differential Equations

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ABSTRACT
In this paper, a class of hybrid block methods for solving second order ordinary differential equations directly was developed. This class was obtained by interpolation and collocation techniques. The methods were analyzed based on the qualitative properties of linear multi-step methods and were found to be zero-stable, consistent and convergent with good region of absolute stability. The proposed methods were analyzed quantitatively and implemented on second order ordinary differential initial value problems. An improved performance of the new methods over existing methods in the literature was shown by solving five numerical examples. The results were presented in tabular form.

Mathematics Subject Classification (2020): 26A39, 26E70, 28B15, 46G10

Keywords: Block method, hybrid method, initial value problems, one-fifth step, convergence, stability

1. INTRODUCTION

Issac Newton discovered a large number of differential equations. He was responsible for large number of this type of equations in Physic and Mathematics. He is also responsible for the systematic development of model of motion. He has other discoveries like personal finance, electric circuits, behavior of musical instruments the logistic equation, electric magnetion, quantum chronodynamics oscillation, among others. A differential equation shows relationship between a function that is unknown and its respective derivatives. The order of the equation depends on the highest derivative of the the dependent function. Practically, this work contains extensive qualitative and quantitative analysis of a class of effective numerical methods used in approximating second order class of differential equations.
The solution of Initial Value Problems(IVPs) in Ordinary Differential Equations (ODEs) of the form

$$\begin{aligned} y' &= f(x, y, y' \ldots, y^{m-1}) \\ y(x_0) &= \eta_0, y'(x_0) = \eta_1, \ldots y^{m-1}(x_0) = \eta_{m-1}, \end{aligned} \tag{1}$$

with the interval $[x_0, x_1]$ has given rise to the methods of one step and multi-step methods. This is majorly attributed to Linear Multi-step Methods (LMMs). From literature, many scholars look for alternative methods of solving (1) and higher order differential equations without reducing them to systems of first order differential equations. Authors such as

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$$\left.\begin{aligned}
\alpha_0(z) &= 1-\frac{15}{2}z\\
\alpha_{\frac{2}{15}}(z) &= \frac{15}{2}z\\
\beta_0(z) &= -\frac{14}{675}z+\frac{1}{2}z^2-\frac{55}{12}z^3+\frac{75}{4}z^4-\frac{225}{8}z^5\\
\beta_{\frac{1}{15}}(z) &= -\frac{11}{225}z+\frac{15}{2}z^3-\frac{375}{8}z^4+\frac{675}{8}z^5\\
\beta_{\frac{2}{15}}(z) &= +\frac{1}{225}z-\frac{15}{4}z^3+\frac{75}{2}z^4-\frac{675}{8}z^5\\
\beta_{\frac{3}{15}}(z) &= -\frac{1}{675}z+\frac{5}{6}z^3-\frac{75}{8}z^4+\frac{225}{8}z^5
\end{aligned}\right\} \tag{2}$$

The first derivatives of the equations (2) give

$$\left.\begin{aligned}
\alpha_0'(z) &= -6\\
\alpha'_{\frac{2}{15}}(z) &= 6\\
\beta_0'(z) &= z-11z^2+48z^3-72z^4\\
\beta'_{\frac{1}{12}}(z) &= 18z^2-120z^3+216z^4\\
\beta'_{\frac{2}{12}}(z) &= -9z^2+96z^3-216z^4\\
\beta'_{\frac{3}{12}}(z) &= 2z^2-24z^3+72z^4
\end{aligned}\right\} \tag{3}$$

By evaluating the first derivative equation (2) together with (3) at points $p=0, \frac{1}{15}, \frac{2}{15}$ and $\frac{3}{15}$, we obtain

$$hy_n' = -\frac{5}{2}y_n+\frac{15}{2}y_{n+\frac{2}{5}}+h(-\frac{14}{675}f_n-\frac{11}{225}f_{n+\frac{1}{15}}-\frac{1}{1225}f_{n+\frac{2}{15}}+\frac{1}{675}f_{n+\frac{3}{15}}) \tag{4}$$

$$hy'_{n+\frac{1}{15}} = -\frac{5}{2}y_n+\frac{15}{2}y_{n+\frac{2}{5}}+h^2(\frac{23}{5400}f_n-\frac{7}{1800}f_{n+\frac{1}{15}}-\frac{17}{1800}f_{n+\frac{2}{15}}+\frac{7}{5400}f_{n+\frac{3}{15}}) \tag{5}$$

$$hy'_{n+\frac{2}{15}} = -\frac{5}{2}y_n+\frac{15}{2}y_{n+\frac{2}{5}}+h^2(\frac{1}{675}f_n-\frac{1}{25}f_{n+\frac{1}{15}}+\frac{2}{75}f_{n+\frac{2}{15}}-\frac{1}{675}f_{n+\frac{3}{15}}) \tag{6}$$

$$hy'_{n+\frac{2}{15}} = -\frac{5}{2}y_n+\frac{15}{2}y_{n+\frac{2}{5}}+h^2(\frac{23}{5400}f_n+\frac{7}{1800}f_{n+\frac{1}{15}}+\frac{143}{1800}f_{n+\frac{2}{15}}+\frac{127}{5400}f_{n+\frac{3}{15}}) \tag{7}$$

Next, we derive the block for a new one-fifth step hybrid method
In order to get the blocks for derivation of the block methods and to test for the zero stability, we combine equations (5), (6) and (7) and use their coefficients in the block form

$$\begin{bmatrix}1 & -\frac{1}{2} & 0\\ 0 & \frac{1}{2} & 1\\ 0 & -\frac{15}{2} & 0\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{15}}\\ y_{n+\frac{2}{15}}\\ y_{n+\frac{3}{15}}\end{bmatrix}=\begin{bmatrix}0 & 0 & -\frac{1}{2}\\ 0 & 0 & \frac{1}{2}\\ 0 & 0 & -\frac{15}{2}\end{bmatrix}\begin{bmatrix}y_{n-\frac{1}{15}}\\ y_{n-\frac{2}{15}}\\ y_n\end{bmatrix}+h\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}y'_{n-\frac{1}{15}}\\ y'_{n-\frac{2}{15}}\\ y'_n\end{bmatrix}$$

$$+h^2\begin{bmatrix}0 & 0 & -\frac{1}{5400}\\ 0 & 0 & \frac{1}{5400}\\ 0 & 0 & -\frac{14}{675}\end{bmatrix}\begin{bmatrix}f_{n-\frac{1}{15}}\\ f_{n-\frac{2}{15}}\\ f_n\end{bmatrix}+h^2\begin{bmatrix}-\frac{1}{540} & -\frac{1}{5400} & 0\\ -\frac{1}{450} & -\frac{7}{1800} & \frac{1}{2700}\\ -\frac{11}{225} & \frac{1}{225} & -\frac{1}{675}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{15}}\\ f_{n+\frac{2}{15}}\\ f_{n+\frac{3}{15}}\end{bmatrix} \tag{8}$$

After normalizing the equation (8) we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{15}} \\ y_{n+\frac{2}{15}} \\ y_{n+\frac{3}{15}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-\frac{1}{15}} \\ y_{n-\frac{2}{15}} \\ y_n \end{bmatrix} + h\begin{bmatrix} 0 & 0 & \frac{1}{15} \\ 0 & 0 & n+\frac{2}{15} \\ 0 & 0 & n+\frac{1}{5} \end{bmatrix}\begin{bmatrix} y'_{n-\frac{1}{15}} \\ y'_{n-\frac{2}{15}} \\ y'_n \end{bmatrix}$$

$$+h^2\begin{bmatrix} 0 & 0 & -\frac{13}{3000} \\ 0 & 0 & \frac{28}{10125} \\ 0 & 0 & -\frac{97}{81000} \end{bmatrix}\begin{bmatrix} f_{n-\frac{1}{15}} \\ f_{n-\frac{2}{15}} \\ f_n \end{bmatrix} + h^2\begin{bmatrix} -\frac{3}{250} & -\frac{3}{1000} & -\frac{1}{1500} \\ \frac{22}{3375} & -\frac{2}{3375} & \frac{2}{10125} \\ \frac{19}{13500} & -\frac{13}{2700} & \frac{1}{10125} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{15}} \\ f_{n+\frac{2}{15}} \\ f_{n+\frac{3}{15}} \end{bmatrix} \tag{9}$$

By rewriting equation (9) explicitly, we get:

$$\left.\begin{aligned} y_{n+\frac{1}{15}} &= y_n + \frac{1}{15}hy'_n + h^2(\frac{97}{81000}f_n + \frac{19}{3500}f_{n+\frac{1}{15}} + \frac{13}{27000}f_{n+\frac{2}{15}} + \frac{1}{10125}f_{n+\frac{3}{15}} \\ y_{n+\frac{2}{5}} &= y_n + \frac{2}{15}hy'_n + h^2(\frac{28}{10125}f_n + \frac{22}{3375}f_{n+\frac{1}{15}} - \frac{2}{3375}f_{n+\frac{2}{15}} + \frac{2}{10125}f_{n+\frac{3}{15}} \\ y_{n+\frac{1}{5}} &= y_n + \frac{3}{15}hy'_n + h^2(\frac{13}{3000}f_n + \frac{3}{250}f_{n+\frac{1}{15}} + \frac{13}{3000}f_{n+\frac{2}{15}} + \frac{1}{1500}f_{n+\frac{3}{15}}) \end{aligned}\right\} \tag{10}$$

Substituting $y_{n+\frac{2}{15}}$ of equation (10) into the equations (5), (6) and (7) gives

$$\left.\begin{aligned} y'_{n+\frac{1}{15}} &= y_n + h^2(\frac{1}{40}f_n + \frac{19}{360}f_{n+\frac{1}{15}} - \frac{1}{72}f_{n+\frac{2}{15}} + \frac{1}{360}f_{n+\frac{3}{15}}) \\ y'_{n+\frac{2}{15}} &= y_n + h^2(\frac{1}{45}f_n + \frac{4}{45}f_{n+\frac{1}{15}} - \frac{1}{45}f_{n+\frac{2}{15}}) \\ y'_{n+\frac{1}{5}} &= y_n + h^2(\frac{1}{4}f_n + \frac{3}{40}f_{n+\frac{1}{15}} + \frac{3}{40}f_{n+\frac{2}{15}} + \frac{1}{40}f_{n+\frac{3}{15}}) \end{aligned}\right\} \tag{11}$$

2. ANALYSIS OF THE METHODS

The methods have some basic properties which establish their validity. The properties: order error constant, consistency and zero stability reveal the nature of convergence of the methods.

2.1. Order and Error Constant

We define the truncation error associated with equation (10) by the difference operator

$$\mathcal{L}\left(y\left(x,h\right)\right) = \sum_{j=0}^{k}\Big[\alpha_j y\left(x_n+jh\right) - \alpha_{vj}y\left(x_n+vjh\right) - h^2\beta_j y''\left(x_n+jh\right) - h^2\beta_{vj}y''\left(x_n+jh\right)\Big], \tag{12}$$

where $y(x)$ is an arbitrary test function which is continuously differentiable in the interval expanding (x) in Taylor series about x_n and collecting like terms in h and y gives

$$\mathcal{L}\left(y\left(x\right)\right) = C_0y(x) + C_1hy'(x) + C_2h^2y''(x) + C_3h^3y'(x) + \ldots + C_{p+3}3h^{p+3}y^{p+3}(x), \tag{13}$$

where the coefficient $C_q, q = 0, 1, 2, \ldots$ are given as

$$C_0 = \sum_{j=0}^{k}\alpha_j,$$

$$C_1 = \sum_{j=1}^{k}j\alpha_j,$$

$$C_2 = \frac{1}{2}\sum_{j=2}^{k}j^2\alpha_j,$$

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2)\sum_{j=1}^{k} j^q \beta_j j^{q-3}\right].$$

According to Henrici (1962) method (13) has order p if $C_0 = C_1 = C_2 = ...C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$
The $C_{p+2} \neq 0$ is called the error constant and $C_{p+2}h^{p+2}y^{p+2}(x)$ is the principal local truncation error at the point x_n. Using Taylor series expansion on equations (10) and (11) we get the order of the new proposed block methods respectively as $(4,4,4,4,4,4)$ with error constants as

$$\left(\frac{-31}{1366875000}, \frac{16}{170859375}, \frac{7}{16875000}, \frac{-1}{20250}, \frac{-4}{10125}, \frac{-1}{750}\right).$$

2.2. Zero stability of One-fifth Step-length For Second Order Differential Equation

In order to test for zero stability of the block method (10), we consider the matrix difference equation of the form

$$p^0 Y_{m+1} = p' y_m + h^2[Q^0 F_{m+1} + Q' F_m + hR'], \tag{14}$$

where

$$Y_{m+1} = [y_{n+\frac{1}{15}}, ...y_{n+\frac{1}{5}}]^T, Y_m = [y_{n-\frac{1}{15}}, ...y_n]^T F_{m+1} = [f_{n+\frac{1}{15}}, ...f_{n+\frac{1}{5}}]^T, F_m = [f_{n-\frac{1}{15}}, ...f_n]^T. \tag{15}$$

The matrices p^0, p', Q^0, Q' and $R^0$are the coefficients of equation (10) which defined as follows

$$p^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{16}$$

$$p' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

$$Q^0 = \begin{bmatrix} \frac{19}{3500} & \frac{13}{27000} & \frac{1}{10125} \\ \frac{22}{3375} & -\frac{2}{3375} & \frac{2}{10125} \\ \frac{3}{250} & \frac{13}{3000} & \frac{1}{1500} \end{bmatrix}, \tag{18}$$

$$Q' = \begin{bmatrix} 0 & 0 & \frac{97}{81000} \\ 0 & 0 & \frac{28}{10125} \\ 0 & 0 & \frac{13}{3000} \end{bmatrix}, \tag{19}$$

$$R' = \begin{bmatrix} 0 & 0 & \frac{1}{15} \\ 0 & 0 & \frac{2}{15} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}. \tag{20}$$

A block method is said to be zero stable if the roots

$$\left|[\lambda p^0 - p^1]\right| = 0$$

are sample with maximum modulus 1.
Now

$$\left|[\lambda p^0 - p']\right| = \left|\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right| = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 0 \tag{21}$$

implies that $\lambda^3 - \lambda^2 = 0$, This gives $\lambda = 0, 0, 1$.
Since the root have modulus less than or equal to one and are simple, the method in zero-stable

2.3. Consistency of One-fifth Step Length For Second Order Differential Equation.

The first and second characteristic polynomials of method (10) are given by Remark
condition (i) is the a sufficient condition for the associated block method to be consistent.
Consistency of the main method (10) The first and second characteristic polynomials of method (10) are given by

$$\rho(z) = z^{\frac{1}{5}} + \frac{1}{2}z^0 - \frac{3}{2}z^{\frac{2}{15}} \tag{22}$$

$$\sigma(z) = \frac{1}{5400}z^0 + \frac{1}{450}z^{\frac{1}{15}} + \frac{7}{1800}z^{\frac{2}{15}} + \frac{1}{2700}z^{\frac{3}{15}} \tag{23}$$

The method (10) is consistent if it satisfies the condition

a The order of the method is $\rho = 4 \geq 1$.

b

$$\alpha_0 = \frac{1}{2},$$

$$\alpha_{\frac{2}{15}} = \frac{-3}{2}$$

and

$$\alpha_{\frac{1}{5}} = 1.$$

Thus,

$$\sum_j \alpha_j = \frac{1}{2} - \frac{3}{2} + 1 = 0. \;\; j = 0, \frac{1}{15}, \frac{2}{15}$$

c

$$\rho(1) = \frac{1}{2} - \frac{3}{2} + 1 = 0$$

$$\rho'(z) = \frac{1}{5}z^{\frac{4}{5}} - \frac{1}{5}z^{\frac{13}{15}} = 0$$

for $z = 1$

$$\rho'(1) = \frac{1}{5}(1)^{\frac{4}{5}} - \frac{1}{5}(1)^{\frac{13}{15}} = 0$$

$$\rho(1) = \rho'(1) = 0.$$

Hence, this condition is satisfied

d

$$\rho''(z) = \frac{4}{25}z^{\frac{9}{5}} + \frac{13}{75}z^{\frac{28}{15}} = 0$$

$$\rho''(1) = \frac{4}{25}(1)^{\frac{9}{5}} + \frac{13}{75}(1)^{\frac{28}{15}} = 0$$

$$\sigma(1) = \frac{1}{5400}(1)^0 + \frac{1}{450}(1)^{\frac{1}{15}} + \frac{7}{1800}(1)^{\frac{2}{15}} + \frac{1}{2700}(1)^{\frac{1}{5}} = 150$$

and

$$2!\sigma(1) = 2\frac{1}{150}$$

$$\rho''(1) = 2!\sigma(1) = \frac{1}{75}$$

Hence the method is consistent.

2.4. Convergence

The convergence of the continuous implicit hybrid block method base on the basic properties discussed above with the fundamental theorem of Dahlquist for the linear multi-step methods. The theorem is stated below without proof.
Theorem 1
The necessary and sufficient condition for a linear multi-step method to be convergent is for it to be consistent and zero stable.

2.5. Region of Absolute Stability of the Block Method

The stability matrix for the method is defined as follows:

$$M(z) = V + zB(M - zA)U \tag{24}$$

and the stability function

$$p(\eta, z) = det(\eta I - M(z)). \tag{25}$$

Then, we represent the block method in form of

$$\begin{bmatrix} y_n \\ --- \\ y_{n+\frac{3}{15}} \end{bmatrix} = \begin{bmatrix} A & & U \\ --- & --- & --- \\ B & & V \end{bmatrix} \begin{bmatrix} h^2 f(y) \\ --- \\ y_{i-1} \end{bmatrix} \tag{26}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{97}{81000} & \frac{19}{13500} & \frac{-13}{27000} & \frac{1}{10125} \\ \frac{28}{10125} & \frac{22}{3375} & \frac{-2}{3375} & \frac{2}{10120} \\ \frac{13}{3000} & \frac{3}{250} & \frac{3}{1000} & \frac{1}{1500} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{97}{81000} & \frac{19}{13500} & \frac{-13}{27000} & \frac{1}{10125} \\ \frac{28}{10125} & \frac{22}{3375} & \frac{-2}{3375} & \frac{2}{10120} \end{bmatrix},$$

$$V = \begin{bmatrix} 0 & 1 \\ 0 & I \end{bmatrix}, U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & I \end{bmatrix}, f(y) = \begin{bmatrix} y_n \\ y_{n+\frac{1}{15}} \\ y_{n+\frac{2}{15}} \\ y_{n+\frac{3}{15}} \end{bmatrix}, Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{15}} \\ y_{n+\frac{3}{15}} \end{bmatrix}, Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{15}} y_{n+\frac{3}{15}} \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}.$$

This gives the stability polynomial of the one-fifth step method which was plotted below

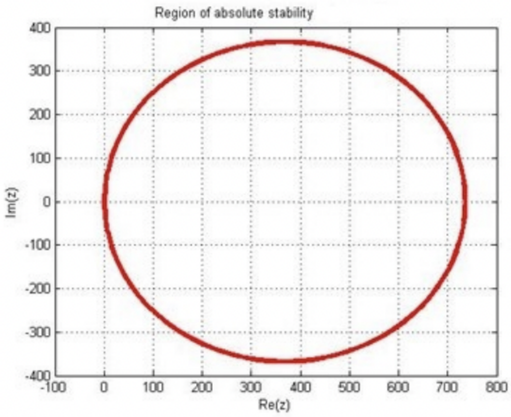

Figure 1: Region of Absolute stability of $\frac{1}{5}$ HBMS

3. NUMERICAL EXAMPLES

In this section, some numerical examples of second order ordinary differential equations are solved. The methods are implemented directly without using any starting value and with use of Matlab and Maple. The table below shows some notations that were use to present the numerical and graphical results obtained for some test problems by application of the proposed schemes.

Problem 1: Consider a Linear non-homogeneous test problem

$$y'' = 3y' + 8e^{2x},$$

$$y(0) = 1,$$

$$y'(0) = 1, h = 0.05$$

Exact solution:

$$y(x) = -4e^{2x} + 3e^{3x} + 2.$$

Problem 2: Consider a specially oscillatory test problem

$$y'' = -\lambda^2 y,$$

$$y(0) = 1, \ y'(0) = 2, \ h = 0.01, \ \lambda = 2$$

Exact solution:

$$y(x) = \cos 2x + \sin 2x.$$

Problem 3: Consider a singular non-homogeneous test problem

$$y'' = \frac{2y'}{x} + xe^x - y(1 + \frac{2}{x^2})$$

$$y(\frac{\pi}{2}) = 4 - \pi + \frac{1}{4}(e^{\frac{\pi}{2}})(\pi + 2)$$

$$y'(\frac{\pi}{2}) = \frac{\pi}{2}(8 + e^{\frac{\pi}{2}})$$

$$h = 0.003125$$

Exact solution:

$$y(x) = 2x\cos x + 4x\sin x + \frac{1}{2}xe^x.$$

Problem 4: We consider a linear homogeneous test problem

$$y'' = y',$$

$$y(0) = 0,\ y'(0) = -1,\ h = 0.1$$

Exact solution :

$$y(x) = 1 - e^x$$

Problem 5: We consider a non-linear non-homogeneous test problem

$$y'' = x(y')^2$$

$$y(0) = 1,\ y'(0) = \frac{1}{2},\ h = 0.1$$

Exact solution:

$$y(x) = 1 + \frac{1}{2}\ln(\frac{2+x}{2-x}).$$

3.1. Tabular Presentation of Numerical Results

Here we present numerical and errors results for $\frac{1}{5}HBM$ in tabular form below

Table 1: Numerical Results for Problem 1

x	Exact	Computed	$Errorinourmethod$	Error in $Areo$(2016)
0.0100	1.01979867335991	1.01979867335989	$1.8874E-14$	$8.4599E-14$
0.0200	1.03918944084761	1.03918944084754	$7.5939E-13$	$3.4861E-13$
0.0300	1.05816454641465	1.05816454641448	$1.7231E-13$	$7.8870E-13$
0.0400	1.07671640027179	1.07671640027149	$3.0509E-13$	$1.4004E-12$
0.0500	1.09483758192485	1.09483758192438	$4.7384E-13$	$2.1791E-12$
0.0600	1.11252084314279	1.11252084314211	$6.7768E-13$	$3.1208E-12$
0.0700	1.12975911085687	1.12975911085596	$9.1704E-13$	$4.2211E-12$
0.0800	1.14654548998987	1.14654548998868	$1.1884E-12$	$0.23146E-11$
0.0900	1.16287326621395	1.16287326621245	$1.4924E-12$	$6.8801E-12$
0.1000	1.17873590863630	1.17873590863448	$1.8268E-12$	$8.4293E-12$

Table 2: Numerical Results for Problem 2

x	Exact	Computed	Error in our method	Error in $Areo$(2016)
0.0050	1.00513852551049	1.00513852547870	$3.1790E-11$	$1.2349E-09$
0.0100	1.01055824175353	1.01055824162003	$8.4730E-11$	$2.6905E-09$
0.0150	1.01626544391208	1.01626544360366	$3.0842E-10$	$4.3738E-09$
0.0200	1.02226654286653	1.02226654230753	$5.5900E-10$	$6.2921E-09$
0.0250	1.02856806714980	1.02856806626307	$8.8673E-10$	$8.9697E-09$
0.0300	1.03517666493419	1.03517666364109	$1.2931E-09$	$1.0863E-08$
0.0350	1.04209910605025	1.04209910426619	$1.7841E-09$	$1.6463E-08$
0.0400	1.04934228403829	1.04934228168012	$2.3581E-09$	

Table 3: Numerical Results for Problem 3

x	Exact	Computed	Error in our method	Error in $Areo(2016)$
1.7000	10.95785118097658	10.95785118071406	$2.6252E-10$	
1.8000	11.63820762976944	11.63820762869096	$1.0785-09$	$4.0964E-09$
1.9000	12.31472912025427	12.31472911749325	$2.7610E-09$	$1.6840E-08$
2.0000	12.99859200531184	12.99859199968641	$5.6254E-09$	$0.43121E-08$
2.1000	13.70481572693030	13.70481571693856	$9.9917E-09$	$8.7867E-08$
2.2000	14.45259109075646	14.45259107457567	$1.6181E-08$	$1.5609E-07$
2.3000	15.26561176327774	15.26561173877142	$2.4506E-08$	$3.8289E-07$
2.4000	16.17241142639307	16.17241139112289	$3.5270E-08$	$5.5108E-07$
2.5000	17.20670978769302	17.20670973893582	$4.8757E-08$	$7.6182E-07$
2.6000	18.40777146832744	18.40777140309677	$6.5231E-08$	$1.0192E-06$

Table 4: Numerical Results for Problem 4

x	Exact	Computed	Error in our method	Error in Ramos et al(2016)
0.1000000	−0.105170918075647710	−0.105170918075646940	$7.77156117E-15$	−0.105170918075645880
0.2000000	−0.221402758160170080	−0.221402758160166880	$3.19189120E-14$	$5.441E-07$
0.3000000	−0.349858807576003410	−0.349858807575995860	$7.54951657E-14$	$9.114E-07$
0.4000	−0.491824697641270790	−0.491824697641256470	$1.43218770E-13$	$1.329E-06$
0.5000	−0.648721270700128640	−0.648721270700105430	$2.32036612E-13$	$1.447E-06$
0.6000	−0.822118800390509770	−0.822118800390473690	$3.60822483E-13$	$2.435E-06$
0.7000	−1.013752707470477500	−1.013752707470424700	$5.28466160E-13$	$3.153E-6$
0.8000	−1.225540928492468800	−1.225540928492394400	$7.43849426E-13$	$4.965E-06$
0.9000	−1.459603111156951200	−1.459603111156850200	$1.01030295E-12$	$4.948E-06$

Table 5: Numerical Results for Problem 5

x	Exact	Computed	Error in our method	Error in Ramos et al(2016)
0.1000000	1.050041729278491400	1.050041729278491800	$0.444089210E-15$	$1.18393E-10$
0.2000000	1.100335347731074900	1.100335347731074900	$0.000000E-00$	$2.3749E-10$
0.3000000	1.151140435936465000	1.151140435936462300	$0.26645E-14$	$4.2485E-10$
0.4000	1.202732554054079200	1.202732554054065400	$0.13766E-13$	$6.1628E-10$
0.5000	1.255412811882991000	1.255412811882949100	$0.41966E-13$	$1.0233E-09$
0.6000	1.309519604203106100	1.309519604203005100	$0.10103E-12$	$1.4483E-09$
0.7000	1.365443754271389100	1.365443754271173500	$2.1560E-013$	$2.5449E-09$
0.8000	1.423648930193593300	1.423648930193167200	$0.4261E-12$	$3.7221E-09$
0.9000	1.484700278594041300	1.484700278593238400	$0.80291E-12$	$7.3287E-08$

4. DISCUSSION OF RESULTS AND CONCLUSION

In this paper, we developed one-fifth order initial value problems directly with out reducing the system of first order differential equation. the method that was develop was test by using it to solve numerical examples which are linear,non linear and stiff initial value problems of second order ordinary differential equation. The table of results of our method is show below comparing the proposed method with the exalt and the existing method. In the table of result,the first,second and third example solved was compared with Areo & Rufai (2016) were the fourth and fifth example was compared with Ramos *et al.* (2016) the newly develop method performed better. Overall, in this paper, a class of numerical schemes are developed in which fractions was used as the step-lengths for second order ordinary differential equations. The resulting methods are consistent and zero stable, therefore it convergences. The methods have good region of absolute stability. The results of the problem show that the method is effective and accurate compared with Areo & Rufai (2016) and Ramos *et al.* (2016) methods.

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ERRATUM

Erratum to "Clairaut and Einstein conditions for locally conformal Kaehler submersions"

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We noticed a small error in the proof of Theorem 3.5 in our paper cited in the heading. Here, we explicitly explain some details. The mentioned theorem in the proof asserts that the Lee vector field of the total manifold of a locally conformal Kaehler submersion cannot be vertical. But this is not true in general. Therefore, that theorem is not valid, and unfortunately, that theorem slightly affects the validity of Theorem 3.5 of our paper Pirinççi et al. (2023). We would like to update this theorem and its proof as follows:

Theorem 3.5. Let $\pi : (M, J, g) \rightarrow (N, J', g')$ be a l.c.K. submersion with connected fibers. If a curve α is a horizontal geodesic on M with respect to both ∇ and $\tilde{\nabla}$, then the dimension of horizontal distribution is equal to 2 or the Lee vector field B of (M, J, g) is vertical.

Proof Let $\{X_1, ..., X_m\}$ be an orthonormal basis of the horizontal distribution of the submersion π at $p \in \pi^{-1}(q)$, where $q \in N$. Then there exist horizontal geodesic curves $\alpha_1, ..., \alpha_m$ such that $\dot{\alpha}_i = X_i$, $i = 1, ..., m$. Thus, for every $i = 1, ..., m$, we have

$$g(B, X_i)X_i = \frac{1}{2}B^h \tag{1}$$

from (5) and (16). Taking summation of the equation (1) over i, we obtain

$$\left(1 - \frac{m}{2}\right)B^h = 0.$$

Hence, it follows that $m = 2$ or $B^h = 0$. In the second case we find that B is vertical. □

Remark: The validity of Theorem 3.5 does not affect the other results in Pirinççi et al. (2023) in any way.

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