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Contents

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Research Article

On some general integral formulae

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ABSTRACT. We repeat and reformulate some more or less known general integral formulae and deduce from them some applications in a concise way. We then present some general double integral formulae which play an essential role in the calculation of fundamental solutions to homogeneous elliptic operators. In particular, this yields generalizations of definite integrals found in standard integral tables. In the final section, the area of an ellipsoidal hypersurface in $\mathbf{R}^n$ is represented by a hyperelliptic integral.

Keywords: Leray's formula, elliptic integrals, definite double integrals.

2020 Mathematics Subject Classification: 26A42, 33E05, 35E05, 44A10, 46F10.

1. INTRODUCTION AND NOTATION

By "general integral formulae", we understand here integral formulae containing "arbitrary" functions, i.e., formulae that hold at least for functions in a space of infinite dimension. E.g., Frullani's formula

$$\int_0^\infty \frac{f(ax)-f(bx)}{x} dx = f(0)\log(\frac{b}{a}), \quad a>0,\ b>0$$

holds for each temperate test function $f \in \mathcal{S}(\mathbf{R}^1)$, but of course also in a much more general context, see [17]. In contrast, the special case

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x}\, \mathrm{d}x = \log\Big(\frac{b}{a}\Big), \quad a>0,\ b>0,$$

of Frullani's formula is just a special definite integral.

Besides the many integral representations (Cauchy, Bochner–Martinelli, Leray–Koppelmann etc.) in complex analysis, see, e.g., [1], there is a host of general integral formulae in real analysis contained in integral tables, see, e.g., [2, 13.2 Schlömilch's Transformation, p. 251], [6, pp. 7, 63, 117, 129, 227 307], [9, pp. 93, 96, 98, 102, 107,109, 110, 114, 119, 121, 123, 125, 126, 130], [12, pp. 6–8], [15, Thms. 1–6, pp. 125–134].

The aim of this article is to attract attention to some general integral formulae in real analysis, to their connection with integrals over δ-measures (see Section 2) and to some applications (see Section 3). In Section 4, we present a general integral formula for double integrals, which earlier enabled to represent fundamental solutions of the homogeneous elliptic operators $\partial_x^4 + \partial_y^4 + \partial_z^4 + 2a\partial_x 2\partial_y^2 + 2b\partial_x^2\partial_z^2 + 2c\partial_y^2\partial_z^2$, see [20]. Section 5 is devoted to the calculation of the (hypersurface) area of an ellipsoidal hypersurface in $\mathbf{R}^n$. In dimensions $n \geq 4$ and for generic diameters, this yields a hyper-elliptic integral not reducible to elliptic ones.

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Let us introduce some notation. The inner product of $x, \xi \in \mathbf{R}^n$ is denoted by $x\xi$. We employ the standard notation for distributions as in [18] and, in particular, we denote the Heaviside function by Y, see [18, p. 36]. We write δ_s for the delta distribution with support in $s \in \mathbf{R}$, i.e., $\delta_s = \frac{\mathrm{d}}{\mathrm{d}x} Y(x-s)$.

The Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \int_{\mathbf{R}^n} \mathrm{e}^{-\mathrm{i}\xi x} f(x)\,\mathrm{d}x$$

for $f \in L^1(\mathbf{R}^n)$ and extended to the space of temperate distributions $\mathcal{S}'(\mathbf{R}^n)$ by continuity.

The pull-back $h^*T = T \circ h \in \mathcal{D}'(\Omega)$ of a distribution T in one variable t with respect to a submersive C^∞ function $h : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, is defined as in [16, Def. 1.2.12, p. 19], i.e.,

$$\langle \phi, h^*T \rangle = \Big\langle \frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_\Omega Y(t - h(x))\phi(x)\,\mathrm{d}x \Big), T \Big\rangle, \quad \phi \in \mathcal{D}(\Omega).$$

2. An integral formula of W. Gröbner and N. Hofreiter and its companion

In [12, Equ. 031.13f], the formula

$$\int_0^\infty f\Big(\xi x + \frac{\eta}{x}\Big)\frac{\mathrm{d}x}{x} = 2\int_{2\sqrt{\xi\eta}}^\infty \frac{f(u)}{\sqrt{u^2 - 4\xi\eta}}\,\mathrm{d}u, \quad \xi > 0,\ \eta > 0 \tag{2.1}$$

is stated. A companion formula holds for ξ, η of opposite sign:

$$\int_0^\infty f\Big(\xi x + \frac{\eta}{x}\Big)\frac{\mathrm{d}x}{x} = \int_{-\infty}^\infty \frac{f(u)}{\sqrt{u^2 - 4\xi\eta}}\,\mathrm{d}u, \quad \xi\eta < 0. \tag{2.2}$$

(Obviously, suitable conditions on the function f must be imposed in order to ensure the existence of the improper integrals.)

An important application of the above formulae is the Fourier transform of Riemann's singularity function $Y(x)x^{-1}\mathrm{e}^{-\mathrm{i}\eta/x}$, $\eta \in \mathbf{R}\backslash\{0\}$. In fact, if $f(u) = \mathrm{e}^{-\mathrm{i}u}$, then formulae (2.1) and (2.2) yield, by means of the well-known integral representations of the Bessel functions J_0, N_0, K_0,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\mathrm{i}(\xi x + \eta/x)}\frac{\mathrm{d}x}{x} &= 2Y(\xi\eta)\int_{2\sqrt{\xi\eta}}^\infty \frac{\mathrm{e}^{-\mathrm{i}u}}{\sqrt{u^2-4\xi\eta}}\,\mathrm{d}u + Y(-\xi\eta)\int_{-\infty}^\infty \frac{\mathrm{e}^{-\mathrm{i}u}}{\sqrt{u^2-4\xi\eta}}\,\mathrm{d}u \\ &= -\pi Y(\xi\eta)\big[N_0(2\sqrt{\xi\eta}) + \mathrm{i}J_0(2\sqrt{\xi\eta})\big] + 2Y(-\xi\eta)K_0(2\sqrt{-\xi\eta}),\end{aligned}$$

i.e.,

$$\begin{aligned}\mathcal{F}_x\big(Y(x)x^{-1}\mathrm{e}^{-\mathrm{i}\eta/x}\big)(\xi) &= \mathcal{F}_{xy}\big(Y(x)\delta(xy-1)\big)(\xi,\eta) \\ &= -\pi Y(\xi\eta)\big[N_0(2\sqrt{\xi\eta}) + \mathrm{i}J_0(2\sqrt{\xi\eta})\big] + 2Y(-\xi\eta)K_0(2\sqrt{-\xi\eta}).\end{aligned}$$

If we extend the integral formulae (2.1) and (2.2) to the negative axis by using the equation

$$-\int_{-\infty}^0 f\Big(\xi x + \frac{\eta}{x}\Big)\frac{\mathrm{d}x}{x} = \int_0^\infty f\Big(-\xi x - \frac{\eta}{x}\Big)\frac{\mathrm{d}x}{x},$$

we arrive at the following proposition.

Proposition 2.1. *Let f be a continuous function on $\mathbf{R}$ such that the integral $\int_{-\infty}^\infty f(u)\,\mathrm{d}u/(1+|u|)$ is convergent in the sense that*

$$\lim_{M\to-\infty}\lim_{N\to\infty}\int_M^N f(u)\,\frac{\mathrm{d}u}{1+|u|}$$

converges. Set $t_+^{-1/2} = Y(t)t^{-1/2}$ *for* $t \in \mathbf{R} \setminus \{0\}$. *Then the formula*

$$\int_{-\infty}^{\infty} f\Big(\xi x + \frac{\eta}{x}\Big) \frac{\mathrm{d}x}{|x|} = 2 \int_{-\infty}^{\infty} f(u)(u^2 - 4\xi\eta)_+^{-1/2} \, \mathrm{d}u \tag{2.3}$$

holds for all $\xi, \eta \in \mathbf{R} \setminus \{0\}$.

Proof. The application

$$\mathbf{R} \setminus \{0\} \longrightarrow \mathbf{R} : x \longmapsto u = \xi x + \frac{\eta}{x}$$

has the range $\{u \in \mathbf{R};\ u^2 \geq 4\xi\eta\}$ and it covers this range twice. Furthermore,

$$\Big|\frac{\mathrm{d}u}{\mathrm{d}x}\Big| = \Big|\xi - \frac{\eta}{x^2}\Big| = \frac{1}{|x|}\Big|\xi x - \frac{\eta}{x}\Big| = \frac{\sqrt{u^2 - 4\xi\eta}}{|x|}$$

and hence formula (2.3) follows from substitution. We observe that the integral on the left-hand side of formula (2.3) has to be interpreted as the limit

$$\lim_{M,N \to \infty} \int_{M^{-1} < |x| < N} f\Big(\xi x + \frac{\eta}{x}\Big) \frac{\mathrm{d}x}{|x|},$$

and this limit converges due to the conditional convergence of the integral $\int_{-\infty}^{\infty} f(u) \, \mathrm{d}u/(1 + |u|)$. □

Let us remark that, vice versa, formula (2.3) implies the equations in (2.1) and (2.2). In fact, if ξ, η are positive, then we simply set $f(u) = 0$ for $u < 0$; if $\xi\eta < 0$, we first observe that the integral $\int_0^\infty f(\xi x + \eta/x)\mathrm{d}x/x$ depends only on the value of the product $\xi\eta$ as shown by applying the substitutions $x \mapsto cx$, $c > 0$, and $x \mapsto x^{-1}$, respectively, in this integral. Hence

$$\int_{-\infty}^{0} f\Big(\xi x + \frac{\eta}{x}\Big) \frac{\mathrm{d}x}{|x|} = \int_0^\infty f\Big(-\xi x - \frac{\eta}{x}\Big) \frac{\mathrm{d}x}{x} = \int_0^\infty f\Big(\xi x + \frac{\eta}{x}\Big) \frac{\mathrm{d}x}{x}$$

holds for $\xi\eta < 0$.

Let us next explain the connection of the integral on the left-hand side of formula (2.3) with the measures $\delta_s(xy)$ supported by the hyperbolas $xy = s$ in $\mathbf{R}^2$, $s \in \mathbf{R} \setminus \{0\}$. As distributions, these measures are defined as

$$\begin{aligned}
\langle \phi, \delta_s(xy) \rangle &= \frac{\mathrm{d}}{\mathrm{d}s} \int_{\mathbf{R}^2} \phi(x,y) Y(s - xy) \, \mathrm{d}x\mathrm{d}y \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \int_{-\infty}^{\infty} \Big[Y(x) \int_{-\infty}^{s/x} \phi(x,y) \, \mathrm{d}y + Y(-x) \int_{s/x}^{\infty} \phi(x,y) \, \mathrm{d}y \Big] \mathrm{d}x \\
&= \int_{\mathbf{R}} \phi\Big(x, \frac{s}{x}\Big) \frac{\mathrm{d}x}{|x|}, \qquad \phi \in \mathcal{D}(\mathbf{R}^2),\ s \in \mathbf{R} \setminus \{0\}.
\end{aligned} \tag{2.4}$$

Incidentally, we observe that the absolute value in $|x|$ is missing in the well-known textbook [10], which has so many merits and so few flaws, see [10, Ch. III, Section 1.3, Ex. 3, Equ. (4), p. 223]. Note that a different, ad hoc definition of the symbol $\delta_s(xy)$—while, strictly logically, being possible—is not in agreement with the usual definitions of the composition of functions and of the pull-back of distributions. In fact, for a different determination of $\delta_s(xy) \in \mathcal{D}'(\mathbf{R}^2_{xy})$, $s \in \mathbf{R} \setminus \{0\}$, the equation

$$\delta_s(xy) = \lim_{\epsilon \searrow 0} \frac{1}{2\epsilon} Y(\epsilon - |xy - s|), \qquad s \in \mathbf{R} \setminus \{0\} \tag{2.5}$$

does not hold in $\mathcal{D}'(\mathbf{R}^2)$ as it should due to $\delta_s = \lim_{\epsilon \searrow 0} Y(\epsilon - |t - s|)/(2\epsilon)$ in $\mathcal{D}'(\mathbf{R}^1_t)$. (Equation (2.5) also shows that $\delta_s(xy)$ must be a *positive* Radon measure, in contrast to the determination in [10].) Similarly, the definition $Y' := -\delta$ might, strictly logically, be correct, but it would not make much sense either. We finally observe that $\delta_0(xy) = \delta(xy)$ cannot be defined unambiguously since the mapping $h(x, y) = xy$ is not submersive for $xy = 0$, i.e., on $h^{-1}(\operatorname{supp} T)$ for $T = \delta \in \mathcal{D}'(\mathbf{R}^1)$. Also, the limit in (2.5) diverges in $\mathcal{D}'(\mathbf{R}^2)$ if $s = 0$.

Note that we can apply the measure $\delta_s(xy)$ not only to test functions $\phi \in \mathcal{D}(\mathbf{R}^2)$, but to each continuous function $\phi(x, y)$ such that $\phi(x, y)(|x| + |y|)^\epsilon$ is bounded on the hyperbola $xy = s$ for some positive ϵ. Therefore

$$\langle f(\xi x + \eta y), \delta_s(xy)\rangle = \int_{-\infty}^{\infty} f\Big(\xi x + \frac{\eta s}{x}\Big)\frac{\mathrm{d}x}{|x|} \tag{2.6}$$

holds, e.g., for $f \in \mathcal{S}(\mathbf{R})$ and $\xi, \eta, s \in \mathbf{R} \setminus \{0\}$. Upon replacing η by ηs in Proposition 2.1, formula (2.6) leads to the following proposition.

Proposition 2.2. *We set, as before, $t_+^{-1/2} = Y(t)t^{-1/2}$ for $t \in \mathbf{R} \setminus \{0\}$ and assume that $\xi, \eta \in \mathbf{R} \setminus \{0\}$. Then the equation*

$$\int_{\mathbf{R}^2} F(\xi x + \eta y, xy)\,\mathrm{d}x\mathrm{d}y = 2\int_{\mathbf{R}^2} F(u, s)(u^2 - 4\xi\eta s)_+^{-1/2}\,\mathrm{d}u\mathrm{d}s \tag{2.7}$$

holds for each measurable function $F : \mathbf{R}^2 \to \mathbf{C}$ such that the integral on the right-hand side of (2.7) is absolutely convergent.

Proof. We first note that the substitution

$$\mathbf{R}^2 \longrightarrow \mathbf{R}^2 : (x, y) \longmapsto (u, s) = (\xi x + \eta y, xy)$$

covers twice its range $\{(u, s);\ u^2 \geq 4\xi\eta s\}$ and has the Jacobian $\xi x - \eta y = \pm\sqrt{u^2 - 4\xi\eta s}$. Hence equation (2.7) holds for $F \in \mathcal{D}(\{(u, s);\ u^2 \neq 4\xi\eta s\})$ and consequently, by density, also for all measurable F making one (and hence both) of the integrals in (2.7) absolutely convergent. $\square$

3. Generalization to $\mathbf{R}^{n+1}$. The formulae of J.Leray, J.Faraut and K. Harzallah

Let us generalize now Proposition 2.2 to $n + 1$ dimensions by considering the Lorentz form $t^2 - |x|^2$, $t \in \mathbf{R}$, $x \in \mathbf{R}^n$, instead of the form $(x, y) \mapsto xy$ on $\mathbf{R}^2$.

Proposition 3.3. *Let $\tau \in \mathbf{R}$, $\xi \in \mathbf{R}^{n+1}$ such that $\tau > |\xi|$ and set $\rho = \sqrt{\tau^2 - |\xi|^2}$ and $t_+^{n/2-1} = Y(t)t^{n/2-1}$ for $t \in \mathbf{R}$. We assume that $F : \mathbf{R}^2 \to \mathbf{C}$ is measurable and that the integral $\int_{\mathbf{R}^2} |F(u, s)|[u^2 - \rho^2 s]_+^{n/2-1}\mathrm{d}u\mathrm{d}s$ is finite. Then*

$$\int_{\mathbf{R}^{n+1}} F(\tau t + \xi x, t^2 - |x|^2)\,\mathrm{d}t\mathrm{d}x = \frac{\pi^{n/2}\rho^{1-n}}{\Gamma(\frac{n}{2})}\int_{\mathbf{R}^2} F(u, s)[u^2 - \rho^2 s]_+^{n/2-1}\mathrm{d}u\mathrm{d}s. \tag{3.8}$$

Proof. Upon using a Lorentz transformation (which automatically preserves volumes), we can replace (τ, ξ) by $(\rho, 0)$. We assume first that F belongs to $\mathcal{C}(\mathbf{R}^2)$ and has compact support. Using polar coordinates $x = r\omega$, $r > 0$, $\omega \in \mathbf{S}^{n-1}$, the substitutions $u = \rho t$ and $s = \rho^{-2}u^2 - r^2$, $\mathrm{d}s = -2r\mathrm{d}r$, $r = (\rho^{-2}u^2 - s)^{1/2}$, and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbf{R}^{n+1}} F(\rho t, t^2 - |x|^2)\,\mathrm{d}t\mathrm{d}x &= \frac{2\pi^{n/2}}{\rho\Gamma(\frac{n}{2})}\int_{-\infty}^{\infty}\Big[\int_0^{\infty} F\Big(u, \frac{u^2}{\rho^2} - r^2\Big)r^{n-1}\mathrm{d}r\Big]\mathrm{d}u \\
&= \frac{\pi^{n/2}}{\rho\Gamma(\frac{n}{2})}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{u^2/\rho^2} F(u, s)\Big(\frac{u^2}{\rho^2} - s\Big)^{n/2-1}\mathrm{d}s\Big]\mathrm{d}u \\
&= \frac{\pi^{n/2}\rho^{1-n}}{\Gamma(\frac{n}{2})}\int_{\mathbf{R}^2} F(u, s)[u^2 - \rho^2 s]_+^{n/2-1}\mathrm{d}u\mathrm{d}s.
\end{aligned}$$

As in Proposition 2.2, the proof is completed by a density argument. □

From equation (3.8) in Proposition 3.3, we can easily deduce Leray's formula for the Laplace transform of Lorentz invariant functions on the cone $C = \{(t,x) \in \mathbf{R}^{n+1};\ t \geq |x|\}$, see [14, Equ. (19.11), p. 41], [13, Thm. 1, p. 53], [19].

Proposition 3.4. *Let $(\tau,\xi) \in C$ and set $\rho = \sqrt{\tau^2 - |\xi|^2}$. We assume that $g : [0,\infty) \longrightarrow \mathbf{C}$ is measurable such that $\int_0^\infty |g(s)| K_{(n-1)/2}(\rho\sqrt{s}) s^{(n-1)/4}\,\mathrm{d}s$ is finite. Then*

$$\int_C \mathrm{e}^{-(\tau t+\xi x)} g(t^2 - |x|^2)\,\mathrm{d}t\mathrm{d}x = \int_0^\infty K_{(n-1)/2}(\rho\sqrt{s})\Big(\frac{2\pi\sqrt{s}}{\rho}\Big)^{(n-1)/2} g(s)\,\mathrm{d}s. \tag{3.9}$$

Proof. We set $g(s) = 0$ for $s < 0$ and $F(u,s) = Y(u)\mathrm{e}^{-u}g(s)$. Then the function $F(\tau t + \xi x, t^2 - |x|^2)$ coincides with $\mathrm{e}^{-(\tau t - \xi x)} g(t^2 - |x|^2)$ on C and it vanishes on $\mathbf{R}^{n+1} \setminus C$. Hence we can apply Proposition 3.3, and [12, Equ. 313.23] implies

$$\begin{aligned}\int_C \mathrm{e}^{-(\tau t+\xi x)} g(t^2 - |x|^2)\,\mathrm{d}t\mathrm{d}x &= \frac{\pi^{n/2}\rho^{1-n}}{\Gamma(\frac{n}{2})}\int_0^\infty g(s)\Big[\int_{\rho\sqrt{s}}^\infty \mathrm{e}^{-u}(u^2-\rho^2 s)^{n/2-1}\mathrm{d}u\Big]\mathrm{d}s\\ &= \int_0^\infty K_{(n-1)/2}(\rho\sqrt{s})\Big(\frac{2\pi\sqrt{s}}{\rho}\Big)^{(n-1)/2} g(s)\,\mathrm{d}s.\end{aligned}$$

This completes the proof. □

We remark that Leray's formula is the analogue of Poisson–Bochner's formula for the Fourier transform of radially invariant distributions, see [18, Equ. (VII, 7; 22), p. 259].

Examples. We can derive Faraut–Harzallah's formula for the Laplace transform of powers of Lorentz distances [7, Prop. III.9, p. 43] from formula (3.9) above by setting $g(s) = s^{(\mu-n-1)/2}$, $\mu \in \mathbf{C}$, $\operatorname{Re}\mu > n-1$. This yields

$$\begin{aligned}\int_C \mathrm{e}^{-(\tau t+\xi x)}(t^2 - |x|^2)^{(\mu-n-1)/2}\,\mathrm{d}t\mathrm{d}x &= \Big(\frac{2\pi}{\rho}\Big)^{(n-1)/2}\int_0^\infty K_{(n-1)/2}(\rho\sqrt{s})s^{(2\mu-n-3)/4}\,\mathrm{d}s\\ &= 2\Big(\frac{2\pi}{\rho}\Big)^{(n-1)/2}\int_0^\infty K_{(n-1)/2}(\rho\sigma)\sigma^{\mu-(n+1)/2}\,\mathrm{d}\sigma\\ &= \frac{2^{\mu-1}\pi^{(n-1)/2}\Gamma(\frac{\mu}{2})\Gamma(\frac{1+\mu-n}{2})}{(\tau^2-|\xi|^2)^{\mu/2}}, \qquad \tau > |\xi|,\end{aligned}$$

by [11, Equ. 6.561.16]. Let us remark that the special case of $\mu = n+1$ furnishes Exercise 1 in [4, p. 174].

Let us also explain how Proposition 3.3 is connected with a formula in [3]. If we set $n = 2$ and apply formula (3.8), using a limit process, to the distribution $F(u,s) = Y(u)f(u)\delta_1(s)$ for $f \in \mathcal{C}(\mathbf{R})$ with compact support, then we obtain

$$\int_0^\infty \langle f(\tau t + \xi x), \delta(t^2 - |x|^2 - 1)\rangle\,\mathrm{d}t = \frac{\pi}{\rho}\int_\rho^\infty f(u)\,\mathrm{d}u \tag{3.10}$$

for $(\tau,\xi) \in \mathbf{R}^3$ with $\tau > |\xi|$ and $\rho = \sqrt{\tau^2 - |\xi|^2}$.

Due to

$$Y(t)\delta(t^2 - |x|^2 - 1) = \frac{1}{2\sqrt{1+|x|^2}}\,\delta\big(t - \sqrt{1+|x|^2}\big),$$

we infer that

$$\int_{\mathbf{R}^2} f\big(\tau\sqrt{1+|x|^2} + \xi x\big)\frac{\mathrm{d}x}{\sqrt{1+|x|^2}} = \frac{2\pi}{\rho}\int_\rho^\infty f(u)\,\mathrm{d}u.$$

Finally, employing the parametrization $x_1 = \cosh\alpha \sinh\beta$, $x_2 = \sinh\alpha$, $t = \sqrt{1+|x|^2} = \cosh\alpha\cosh\beta$ of the upper shell $t > 0$ of the hyperboloid $t^2 = 1 + |x|^2$ and taking account of $\mathrm{d}x = \cosh^2\alpha\cosh\beta\,\mathrm{d}\alpha\mathrm{d}\beta$, we arrive at

$$\int_{\mathbf{R}^2} f\big(\tau\cosh\alpha\cosh\beta + \xi_1\cosh\alpha\sinh\beta + \xi_2\sinh\alpha\big)\cosh\alpha\,\mathrm{d}\alpha\mathrm{d}\beta = \frac{2\pi}{\rho}\int_\rho^\infty f(u)\,\mathrm{d}u, \quad \tau > |\xi|,\ \rho = \sqrt{\tau^2 - |\xi|^2},$$

which is formula 3.1.4.1 in [3].

4. Algebraic double integrals and "elliptic arctan-integrals"

In [16], we employed the formula

$$\partial_3 E(x_1, 1, x_3) = -\frac{1}{4\pi^2}\int_0^{x_3}\mathrm{d}\lambda\int_{-\infty}^\infty \frac{\mathrm{d}\alpha}{P(\alpha, -\lambda - x_1\alpha, 1)} \tag{4.11}$$

in order to represent the (uniquely determined) even and homogeneous fundamental solution E of the homogeneous elliptic operator $P(\partial)$ of degree four and in three variables, see [16, Prop. 5.2.7, p. 357, and p. 359, line two from below].

Using formula (4.11), we calculated E in the cases of $P(\partial) = \partial_1^4 + \partial_2^4 + \partial_3^4$, see [16, Ex. 5.2.9, p. 359], and of $P(\partial) = \partial_1^4 + \partial_2^4 + \partial_3^4 + 2a\partial_1^2\partial_2^2$, $a > -1$, see [16, Ex. 5.2.11, p. 362]. For the operator $P(\partial) = \partial_1^4 + \partial_2^4 + \partial_3^4$, the fundamental solution E was first obtained in [8, p. 350]; for elliptic operators of the general form $P(\partial) = \sum_{j=1}^3\sum_{k=1}^3 c_{jk}\partial_j^2\partial_k^2$, this was done in [20, Prop. 3, p. 1198]. All these fundamental solutions can explicitly be represented by the complete elliptic integral of the first kind.

In the following, let us repeat some steps in these calculations starting from formula (4.11). We assume that $x_3 > 0$. Substitution of the variables

$$\alpha = t\sqrt[4]{\mu},\ \lambda = \sqrt[4]{\mu},\ \frac{\partial(\alpha,\lambda)}{\partial(t,\mu)} = \begin{pmatrix} \sqrt[4]{\mu} & \frac{t}{4}\mu^{-3/4} \\ 0 & \frac{1}{4}\mu^{-3/4}\end{pmatrix}$$

leads to

$$\partial_3 E(x_1, 1, x_3) = -\frac{1}{16\pi^2}\int_0^{x_3^4}\frac{\mathrm{d}\mu}{\sqrt{\mu}}\int_{-\infty}^\infty\frac{\mathrm{d}t}{P(t\sqrt[4]{\mu}, -(1+tx_1)\sqrt[4]{\mu}, 1)}.$$

In the case of the operator $P(\partial) = \partial_1^4 + \partial_2^4 + \partial_3^4 + 2a\partial_1^2\partial_2^2$, $a > -1$, we obtain

$$P(t\sqrt[4]{\mu}, -(1+tx_1)\sqrt[4]{\mu}, 1) = Q(t)\mu + 1,$$

where $Q(t)$ is a polynomial of degree four fulfilling $Q(t) > 0$ for $t \in \mathbf{R}$. (In the notation, we suppressed the dependence of the coefficients of Q on x_1.) Inverting the order of integrations and substituting $u = \sqrt{Q(t)\mu}$ results in

$$\partial_3 E(x_1, 1, x_3) = -\frac{1}{8\pi^2}\int_{-\infty}^\infty \frac{\arctan\big(x_3^2\sqrt{Q(t)}\big)}{\sqrt{Q(t)}}\,\mathrm{d}t.$$

By these considerations, we want to motivate our treatment of integrals of the form

$$\begin{aligned} I := \int_{-\infty}^\infty \frac{\arctan\big(\gamma\sqrt{Q(t)}\big)}{\sqrt{Q(t)}}\,\mathrm{d}t &= \frac{1}{2}\int_0^{\gamma^2}\frac{\mathrm{d}\mu}{\sqrt{\mu}}\int_{-\infty}^\infty\frac{\mathrm{d}t}{Q(t)\mu+1} \\ &= \frac{1}{2}\int_{\gamma^{-2}}^\infty\frac{\mathrm{d}\mu}{\sqrt{\mu}}\int_{-\infty}^\infty\frac{\mathrm{d}t}{Q(t)+\mu}, \quad \gamma > 0. \end{aligned} \tag{4.12}$$

As will be seen in Corollary 4.2 below, I in formula (4.12) can be expressed as an elliptic integral of the first kind and we shall call it therefore an "elliptic arctan-integral".

Let us first explain the basic idea of the evaluation of I in the simpler case of the biquadratic $Q(t) = t^4 + pt^2 + r$. We shall assume that $r > 0$ and $p > -2\sqrt{r}$, which are the conditions that the polynomial Q is positive on the real axis. If, additionally, $0 < r \leq p^2/4$ and if we set $\lambda = \sqrt{r}$, we can write Q in the form $Q(t) = (t^2 + a^2)(t^2 + b^2)$ with $a > 0$, $b > 0$ and hence $ab = \lambda$ and $a + b = \sqrt{a^2 + b^2 + 2ab} = \sqrt{p + 2\lambda}$. Therefore [12, Equ. 141.14] yields

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t^4 + pt^2 + \lambda^2} = \frac{\pi}{\lambda\sqrt{p + 2\lambda}}, \tag{4.13}$$

and this equation persists for all $\lambda > 0$ and $p > -2\lambda$ by analytic continuation.

Inserting formula (4.13) into (4.12) and substituting $\lambda = \sqrt{\mu + r}$, then implies

$$\begin{aligned} I &= \frac{1}{2} \int_{\gamma^{-2}}^{\infty} \frac{\mathrm{d}\mu}{\sqrt{\mu}} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{Q(t) + \mu} = \frac{\pi}{2} \int_{\gamma^{-2}}^{\infty} \frac{\mathrm{d}\mu}{\sqrt{\mu}\sqrt{\mu + r}\sqrt{p + 2\sqrt{\mu + r}}} \\ &= \frac{\pi}{\sqrt{2}} \int_{\sqrt{r+\gamma^{-2}}}^{\infty} \frac{\mathrm{d}\lambda}{\sqrt{\lambda^2 - r}\sqrt{\lambda + p/2}}. \end{aligned} \tag{4.14}$$

By using formula 3.131.8 in [11], we can then represent I by an elliptic integral of the first kind, i.e., by

$$F(\varphi, k) = \int_0^{\varphi} \frac{\mathrm{d}\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}, \quad 0 \leq k < 1, \ \varphi \in \mathbf{R}.$$

This implies the following proposition.

Proposition 4.5. *Let $\gamma > 0$, $r > 0$, $p > -2\sqrt{r}$ and set $Q(t) = t^4 + pt^2 + r$. Then*

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\arctan\left(\gamma\sqrt{Q(t)}\right)}{\sqrt{Q(t)}} \, \mathrm{d}t \\ &= \begin{cases} \dfrac{\pi}{\sqrt[4]{r}} F\left(\arcsin\sqrt{\dfrac{2\sqrt{r}}{\sqrt{r} + \sqrt{r + \gamma^{-2}}}}, \sqrt{\dfrac{\sqrt{r} - p/2}{2\sqrt{r}}}\right) : -2\sqrt{r} < p \leq 2\sqrt{r}, \\ \dfrac{2\pi}{\sqrt{p + 2\sqrt{r}}} F\left(\arcsin\sqrt{\dfrac{p + 2\sqrt{r}}{p + 2\sqrt{r + \gamma^{-2}}}}, \sqrt{\dfrac{p - 2\sqrt{r}}{p + 2\sqrt{r}}}\right) : p \geq 2\sqrt{r}. \end{cases} \end{aligned} \tag{4.15}$$

Let us observe that the limit case $\gamma \to \infty$ yields

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{t^4 + pt^2 + r}} = \begin{cases} \dfrac{2}{\sqrt[4]{r}} \mathbf{K}\left(\sqrt{\dfrac{\sqrt{r} - p/2}{2\sqrt{r}}}\right) : -2\sqrt{r} < p \leq 2\sqrt{r}, \\ \dfrac{4}{\sqrt{p + 2\sqrt{r}}} \mathbf{K}\left(\sqrt{\dfrac{p - 2\sqrt{r}}{p + 2\sqrt{r}}}\right) : p \geq 2\sqrt{r}. \end{cases}$$

(As usual the function $\mathbf{K}$ denotes the complete elliptic integral, i.e., $\mathbf{K}(k) = F(\frac{\pi}{2}, k), 0 \leq k < 1$.) The upper formula is in accordance with [12, Equ. 222.2c] upon using the substitution $x = t^2$.

More generally as in Proposition 4.5, we can replace the integrand $\mu^{-1/2}$ in formula (4.14) by a function $f(\mu)$ and use formula (4.13) in order to represent the double integral

$$\int_{\mu_1}^{\mu_2} f(\mu) \, \mathrm{d}\mu \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t^4 + pt^2 + r + \mu}, \quad 0 < \mu_1 < \mu_2,$$

by a simple one. If we substitute $\lambda = \sqrt{\mu + r}$ as before and set $z = p + 2\lambda$, then we obtain the following proposition.

Proposition 4.6. *Let $r > 0$, $p > -2\sqrt{r}$, $0 < \mu_1 < \mu_2$ and $f \in L^1([\mu_1, \mu_2])$. Then*

$$\int_{\mu_1}^{\mu_2} f(\mu)\,\mathrm{d}\mu \int_{-\infty}^{\infty} \frac{dt}{t^4 + pt^2 + r + \mu} = \pi \int_{p+2\sqrt{r+\mu_1}}^{p+2\sqrt{r+\mu_2}} f\Big(\frac{(z-p)^2}{4} - r\Big)\,\frac{\mathrm{d}z}{\sqrt{z}}. \tag{4.16}$$

Proposition 4.6 can be generalized to general positive quartics $Q(t) = t^4 + pt^2 + qt + r$. The corresponding result, i.e., Equ. (5) in [20, p. 1197], is a special case of [20, Prop. 2, p. 1196] and we just quote it in the next proposition.

Proposition 4.7. *Let $p, q, r \in \mathbf{R}$ such that the quartic $Q(t) = t^4 + pt^2 + qt + r$ is positive for each real t. Let $0 < \mu_1 < \mu_2 \le \infty$ and $f : (\mu_1, \mu_2) \to \mathbf{C}$ such that $\mu^{-3/4} f(\mu) \in L^1((\mu_1, \mu_2))$. Then*

$$\int_{\mu_1}^{\mu_2} f(\mu)\,\mathrm{d}\mu \int_{-\infty}^{\infty} \frac{dt}{Q(t) + \mu} = \pi \int_{z_1}^{z_2} f\big(\mu(z)\big)\,\frac{\mathrm{d}z}{\sqrt{z}}, \tag{4.17}$$

where

$$\mu(z) = \frac{(z-p)^2}{4} - r + \frac{q^2}{4z}$$

and $z_{1,2}$ denote the largest real roots of $\mu(z) = \mu_{1,2}$, respectively.

If we use the function $f(\mu) = \mu^{-1/2}$ in Proposition 4.7, we come back to elliptic arctan-integrals and we can generalize in this way Proposition 4.5.

Corollary 4.1. *Let $\gamma > 0$ and $p, q, r \in \mathbf{R}$ such that the quartic $Q(t) = t^4 + pt^2 + qt + r$ is positive for each real t. Then*

$$I = \int_{-\infty}^{\infty} \frac{\arctan\big(\gamma\sqrt{Q(t)}\big)}{\sqrt{Q(t)}}\,\mathrm{d}t = \pi \int_{z_1}^{\infty} \frac{\mathrm{d}z}{\sqrt{(z-p)^2 z - 4rz + q^2}}, \tag{4.18}$$

where z_1 is the largest real root of the cubic $(z-p)^2 z - 4(r + \gamma^{-2})z + q^2$.

Note that the right-hand side of equation (4.18) is an elliptic integral in Weierstraß' normal form. In particular, if the quartic Q has the form

$$Q(t) = \big[(t-t_1)^2 + u_1^2\big]\big[(t-t_2)^2 + u_2^2\big], \quad t_1, t_2 \in \mathbf{R},\ u_1 > 0, u_2 > 0, \tag{4.19}$$

then we can represent the integral I by the elliptic integral $F(\varphi, k)$ of the first kind.

Corollary 4.2. *Let $\gamma > 0$ and Q be as in equation (4.19). Then*

$$\begin{aligned} I = \int_{-\infty}^{\infty} \frac{\arctan\big(\gamma\sqrt{Q(t)}\big)}{\sqrt{Q(t)}}\,\mathrm{d}t &= \frac{2\pi}{\sqrt{(t_1-t_2)^2 + (u_1+u_2)^2}} \\ &\times F\Bigg(\arcsin\sqrt{\frac{(t_1-t_2)^2 + (u_1+u_2)^2}{(t_1-t_2)^2 + z_1}}, \sqrt{\frac{(t_1-t_2)^2 + (u_1-u_2)^2}{(t_1-t_2)^2 + (u_1+u_2)^2}}\Bigg), \end{aligned} \tag{4.20}$$

where z_1 is the largest real root of the equation

$$\big[z + (t_1-t_2)^2\big]\big[z - (u_1-u_2)^2\big]\big[z - (u_1+u_2)^2\big] = 4\gamma^{-2} z.$$

Proof. By translation the integral I depends only on the difference $t_1 - t_2$ and hence we can assume that $t_2 = -t_1$. Then $Q(t) = t^4 + pt^2 + qt + r$, where $p = -2t_1^2 + u_1^2 + u_2^2$, $q = 2t_1(u_1^2 - u_2^2)$, $r = (t_1^2 + u_1^2)(t_1^2 + u_2^2)$. This implies that the cubic

$$\begin{aligned}(z-p)^2 z - 4rz + q^2 &= z^3 + 2(2t_1^2 - u_1^2 - u_2^2)z^2 \\ &\quad + \big[(u_1^2 - u_2^2)^2 - 8t_1^2(u_1^2 + u_2^2)\big]z + 4t_1^2(u_1^2 - u_2^2)^2 \\ &= \big[z + 4t_1^2\big]\big[z - (u_1 - u_2)^2\big]\big[z - (u_1 + u_2)^2\big]\end{aligned}$$

has the three real roots $(u_1 + u_2)^2 > (u_1 - u_2)^2 > -(t_1 - t_2)^2$. Hence, similarly as in the proof of Proposition 4.1, formula 3.131.8 in [11] implies the result. □

We remark that Corollary 4.2 generalizes Proposition 4.5. In fact, if $Q(t) = (t^2 + u_1^2)(t^2 + u_2^2)$, i.e., if $t_1 = t_2 = 0$ in (4.19), then $p = u_1^2 + u_2^2, q = 0, r = u_1^2 u_2^2$ and formula (4.20) yields the lower formula on the right-hand side of (4.17). On the other hand, if $Q(t) = [(t-t_1)^2 + u_1^2][(t+t_1)^2 + u_1^2]$, i.e., if $t_2 = -t_1$ and $u_1 = u_2$ in (4.19), then $p = 2(u_1^2 - t_1^2), q = 0, r = (t_1^2 + u_1^2)^2$ and formula (4.20) yields the upper formula on the right-hand side of (4.17).

As before, the limit $\gamma \to \infty$ yields a *complete* elliptic integral since $z_1 \to (u_1 + u_2)^2$ for $\gamma \to \infty$. Hence

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{Q(t)}} = \frac{4}{\sqrt{(t_1 - t_2)^2 + (u_1 + u_2)^2}} \mathbf{K}\left(\sqrt{\frac{(t_1 - t_2)^2 + (u_1 - u_2)^2}{(t_1 - t_2)^2 + (u_1 + u_2)^2}}\right). \tag{4.21}$$

Note that the representation of $\int_{\mathbf{R}} \mathrm{d}t/\sqrt{Q(t)}$ in [12, Equ. 223.2e] is more complicated.

5. Representation of hypersurface areas by volume integrals

If the hypersurface M in $\mathbf{R}^n$ is given by $M = f^{-1}(1)$ for a homogeneous function f, then the area of M can be represented by a volume integral:

Proposition 5.8. *Let $f : \mathbf{R}^n \setminus \{0\} \longrightarrow (0, \infty)$ be $\mathcal{C}^1$ and homogeneous of degree $\lambda > 0$ and set $M = f^{-1}(1)$. Then the hypersurface area $\Sigma(M)$ of M is given by*

$$\Sigma(M) = \frac{\lambda + n - 1}{\lambda} \int_{\{x \in \mathbf{R}^n;\, f(x) < 1\}} |\nabla f(x)|\, \mathrm{d}x. \tag{5.22}$$

Proof. Let $\mathrm{d}\sigma$ denote the surface measure on M and $\nu = \nabla f / |\nabla f|$ the outward unit normal. Due to Euler's equation, we have $x \cdot \nabla f(x) = \lambda f(x) = \lambda$ if $x \in M$ and $x \cdot \nabla |\nabla f|(x) = (\lambda - 1)|\nabla f|(x)$ for $x \in \mathbf{R}^n \setminus \{0\}$. Hence

$$\operatorname{div}(x|\nabla f|) = n|\nabla f| + x \cdot \nabla |\nabla f| = (n + \lambda - 1)|\nabla f|.$$

Therefore Gauß' divergence theorem yields

$$\begin{aligned}\Sigma(M) &= \int_M \mathrm{d}\sigma = \frac{1}{\lambda}\int_M x \cdot \nabla f \,\mathrm{d}\sigma = \frac{1}{\lambda}\int_M x|\nabla f| \cdot \nu \,\mathrm{d}\sigma \\ &= \frac{1}{\lambda}\int_{f(x)<1} \operatorname{div}(x|\nabla f|)\,\mathrm{d}x = \frac{\lambda + n - 1}{\lambda}\int_{f(x)<1} |\nabla f(x)|\,\mathrm{d}x.\end{aligned}$$

□

We shall apply formula (5.22) in order to show that the area of an ellipsoidal hypersurface in $\mathbf{R}^n$ can be represented by a hyperelliptic integral.

Proposition 5.9. *Let $n \geq 2$ and a_i, $i = 1, \dots, n$, be positive numbers and set*

$$M = \Big\{x \in \mathbf{R}^n; \sum_{i=1}^n \frac{x_i^2}{a_i^2} = 1\Big\}.$$

Then its hypersurface area is given by

$$\Sigma(M) = \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})}\Big(\prod_{j=1}^{n-1} a_j^2\Big)\int_0^\infty \Big(\sum_{j=1}^{n-1}\frac{1}{s+a_j^2}\Big)\frac{\sqrt{s+a_n^2}\,\mathrm{d}s}{\sqrt{s\prod_{j=1}^{n-1}(s+a_j^2)}}. \tag{5.23}$$

Proof. The function $f(x) = \sum_{i=1}^n x_i^2/a_i^2$ is homogeneous of degree $\lambda = 2$ and $|\nabla f| = 2(\sum_{i=1}^n x_i^2/a_i^4)^{1/2}$. Hence formula (5.22) in Proposition 5.8 implies, upon substituting $y_i = a_i x_i$, $i = 1, \dots n$,

$$\Sigma(M) = (n+1)\int_{f(x)<1}\sqrt{\sum_{i=1}^n \frac{x_i^2}{a_i^4}}\,\mathrm{d}x = (n+1)\Big(\prod_{i=1}^n a_i\Big)\int_{|y|<1}\sqrt{\sum_{i=1}^n \frac{y_i^2}{a_i^2}}\,\mathrm{d}y.$$

With the further substitution $y_n = t(\sum_{j=1}^{n-1} y_j^2/a_j^2)^{1/2}$, we then obtain

$$\Sigma(M) = 2(n+1)\Big(\prod_{i=1}^n a_i\Big)\int_0^\infty \sqrt{1+\frac{t^2}{a_n^2}}\,\mathrm{d}t\int_{E_t}\sum_{j=1}^{n-1}\frac{y_j^2}{a_j^2}\,\mathrm{d}y', \tag{5.24}$$

where the inner integral runs over the ellipsoid

$$E_t = \Big\{y' \in \mathbf{R}^{n-1}; \sum_{j=1}^{n-1}\frac{y_j^2}{A_j^2} \leq 1\Big\}, \quad A_j = \frac{a_j}{\sqrt{t^2+a_j^2}},\ j = 1, \dots, n-1,$$

and represents a sum of moments of second order thereof.

The calculation of such moments is quite straight-forward. We present it here just for completeness. Evidently, it suffices to consider the summand y_{n-1}^2/a_{n-1}^2 in the inner integral on the right-hand side of formula (5.24). Substituting $y_j = A_j u_j$, $j = 1, \dots, n-1$, and setting $u'' = (u_1, \dots, u_{n-2})$ we obtain

$$\int_{E_t}\frac{y_{n-1}^2}{a_{n-1}^2}\,\mathrm{d}y' = \frac{A_{n-1}^2}{a_{n-1}^2}\Big(\prod_{j=1}^{n-1} A_j\Big)\int_{|u'|<1} u_{n-1}^2\,\mathrm{d}u'$$

and

$$\begin{aligned}\int_{|u'|<1} u_{n-1}^2\,\mathrm{d}u' &= 2\int_0^1 u_{n-1}^2\,\mathrm{d}u_{n-1}\int_{|u''|^2<1-u_{n-1}^2}\mathrm{d}u'' \\ &= \frac{2\pi^{n/2-1}}{\Gamma(\frac{n}{2})}\int_0^1 u_{n-1}^2(1-u_{n-1}^2)^{n/2-1}\,\mathrm{d}u_{n-1} \\ &= \frac{2\pi^{n/2-1}}{\Gamma(\frac{n}{2})}\cdot\frac{1}{2}B\Big(\frac{3}{2},\frac{n}{2}\Big) = \frac{\pi^{(n-1)/2}}{2\,\Gamma(\frac{n+3}{2})}.\end{aligned}$$

Altogether this yields

$$\Sigma(M) = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})}\Big(\prod_{j=1}^{n-1} a_j^2\Big)\int_0^\infty\Big(\sum_{j=1}^{n-1}\frac{1}{t^2+a_j^2}\Big)\frac{\sqrt{t^2+a_n^2}\,\mathrm{d}t}{\prod_{j=1}^{n-1}\sqrt{t^2+a_j^2}}.$$

The final substitution $s = t^2$ then leads to formula (5.23) and thus concludes the proof. $\square$

We remark that the integral in formula (5.23) is an elliptic integral for $n = 2$ and for $n = 3$, but is hyperelliptic and not elliptic in dimensions $n \geq 4$ if the diameters $2a_i$, $i = 1, \dots, n$, are generic positive real numbers. The representation of the length of an ellipse $(n = 2)$ and of the surface area of an ellipsoid $(n = 3)$, respectively, by elliptic integrals is known since the times of Legendre, see [5, Problem 1, p. 265, Problem 15, p. 279].

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Research Article

Extensions of the operator Bellman and operator Hölder type inequalities

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ABSTRACT. In this paper, we employ the concept of operator means as well as some operator techniques to establish new operator Bellman and operator Hölder type inequalities. Among other results, it is shown that if $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ are continuous fields of positive invertible operators in a unital C^*-algebra $\mathscr{A}$ such that $\int_\Omega A_t\,d\mu(t) \le I_{\mathscr{A}}$ and $\int_\Omega B_t\,d\mu(t) \le I_{\mathscr{A}}$, and if ω_f is an arbitrary operator mean with the representing function f, then

$$\left(I_{\mathscr{A}} - \int_\Omega (A_t \omega_f B_t)\,d\mu(t)\right)^p \ge \left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right)$$

for all $0 < p \le 1$, which is an extension of the operator Bellman inequality.

Keywords: Bellman inequality, Cauchy-Schwarz inequality, Hölder inequality, operator mean, Hadamard product, continuous field of operators, C^*-algebra

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathscr{L}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity $I_{\mathscr{H}}$. An operator $A \in \mathscr{L}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathscr{H}$ and in this case we write $A \ge 0$. We write $A > 0$ if A is a positive invertible operator. The set of all positive invertible operators is denoted by $\mathscr{L}(\mathscr{H})_+$. For self-adjoint operators $A, B \in \mathscr{L}(\mathscr{H})$, we say $A \le B$ if $B - A \ge 0$. Also, an operator $A \in \mathscr{L}(\mathscr{H})$ is said to be contraction, if $A^*A \le I_{\mathscr{H}}$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and $I_{\mathscr{H}}$. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \ge g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \ge g(A)$.

Let f be a continuous real valued function defined on an interval J. It is called operator monotone on J if $A \le B$ implies $f(A) \le f(B)$ for all self-adjoint operators $A, B \in \mathscr{L}(\mathscr{H})$ with spectra in J. It is said to be operator concave on J if $\lambda f(A) + (1-\lambda)f(B) \le f(\lambda A + (1-\lambda)B)$ for all self-adjoint operators $A, B \in \mathscr{L}(\mathscr{H})$ with spectra in J and all $\lambda \in [0,1]$, see, e.g., [10]. Every nonnegative continuous function f is operator monotone on $[0, +\infty)$ if and only if f is operator concave on $[0, +\infty)$, see [11, Theorem 8.1]. A map Ψ on $\mathscr{L}(\mathscr{H})$ is called positive if $\Psi(A) \ge 0$ whenever $A \ge 0$ and is said to be unital if $\Psi(I_{\mathscr{H}}) = I_{\mathscr{H}}$. If Ψ is a unital positive linear map and f is an operator concave function on an interval J, then

$$f(\Psi(A)) \ge \Psi(f(A)) \qquad \text{(Davis-Choi-Jensen's inequality)} \tag{1.1}$$

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for every self-adjoint operator A on $\mathscr{H}$, whose spectrum is contained in J, see also [11, 17]. Let A and B be bounded linear operators on a Hilbert space $\mathscr{H}$. The operator $A\otimes B$ on $\mathscr{H}\otimes\mathscr{H}$ is defined by $(A\otimes B)(x\otimes y)=Ax\otimes By$ for every $x,y\in\mathscr{H}$. From this definition, it is clear that the tensor product of positive operators is positive. Furthermore, for operators $A,B,C,D\in\mathscr{L}(\mathscr{H})$, by the definition of the tensor product, we have $(A\otimes B)(C\otimes D)=AC\otimes BD$ and if A and B are positive, then $(A\otimes B)^r=A^r\otimes B^r$ for all $r\geq 0$. For a given orthonormal basis $\{e_j\}$ of a Hilbert space $\mathscr{H}$, the Hadamard product $A\circ B$ of two operators $A,B\in\mathscr{L}(\mathscr{H})$ is defined by $\langle A\circ Be_i,e_j\rangle=\langle Ae_i,e_j\rangle\langle Be_i,e_j\rangle$. It is known that the Hadamard product can be presented by filtering the tensor product $A\otimes B$ through a positive linear map. In fact, $A\circ B=U^*(A\otimes B)U$, where $U:\mathscr{H}\to\mathscr{H}\otimes\mathscr{H}$ is the isometry defined by $Ue_j=e_j\otimes e_j$, see [3, 4, 9, 23].

The axiomatic theory for operator means of positive invertible operators has been developed by Kubo and Ando [16]. A binary operation ρ on $\mathscr{L}(\mathscr{H})_+$ is called an operator mean, if the following conditions are satisfied:

(i) $A\leq C$ and $B\leq D$ imply $A\,\rho\,B\leq C\,\rho\,D$;
(ii) $A_n\downarrow A$ and $B_n\downarrow B$ imply $A_n\rho B_n\downarrow A\,\rho\,B$, where $A_n\downarrow A$ means that $A_1\geq A_2\geq\cdots$ and $A_n\to A$ as $n\to\infty$ in the strong operator topology;
(iii) $T^*(A\,\rho\,B)T\leq(T^*AT)\rho(T^*BT)$ $(T\in\mathscr{L}(\mathscr{H}))$;
(iv) $I_{\mathscr{H}}\,\rho\,I_{\mathscr{H}}=I_{\mathscr{H}}$.

It is easy to see that $T^*(A\,\rho\,B)T=(T^*AT)\,\rho\,(T^*BT)$ for all invertible operators T. In particular, $(\alpha A\,\rho\,\alpha B)=\alpha\,(A\,\rho\,B)\,,\ (\alpha\geq 0)$. There exists an affine order isomorphism between the class of operator means and the class of positive operator monotone functions f defined on $(0,\infty)$ via $f(t)I_{\mathscr{H}}=I_{\mathscr{H}}\,\rho\,(tI_{\mathscr{H}})$ $(t>0)$ with $f(1)=1$. In addition,

$$A\,\rho\,B=A^{\frac{1}{2}}f(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$$

for all $A,B\in\mathscr{L}(\mathscr{H})_+$. The operator monotone function f is called the representing function of ρ. If f and g are the representing functions of the operator means ρ_f and ρ_g, respectively, then $f\leq g$ on $(0,+\infty)$ if and only if $(A\,\rho_f\,B)\leq(A\,\rho_g\,B)$ for all positive invertible operators A and B. The functions $f_{\sharp_\mu}(t)=t^\mu$, $f_{\nabla_\mu}(t)=(1-\mu)+\mu t$, and $f_{!_\mu}(t)=\left(\frac{(1-\mu)+t^{-1}\mu}{2}\right)^{-1}$ on $(0,\infty)$ give the operator weighted geometric mean $A\sharp_\mu B=A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\mu}A^{\frac{1}{2}}$, the operator weighted arithmetic mean $A\nabla_\mu B=(1-\mu)A+\mu B$, and the operator weighted harmonic mean $A!_\mu B=\left(\frac{(1-\mu)A^{-1}+\mu B^{-1}}{2}\right)^{-1}$, respectively, for all $\mu\in(0,1)$. An operator mean ρ is symmetric if $A\rho B=B\rho A$ for all $A,B\in\mathscr{L}(\mathscr{H})_+$. For a symmetric operator mean ρ, a parametrized operator mean ρ_t, $0\leq t\leq 1$, is called an interpolational path for ρ if it satisfies

(1) $A\,\rho_0\,B=A$, $A\rho_{1/2}B=A\,\rho\,B$, and $A\,\rho_1\,B=B$;
(2) $(A\,\rho_p\,B)\,\rho\,(A\,\rho_q\,B)=A\,\rho_{\frac{p+q}{2}}\,B$ for all $p,q\in[0,1]$;
(3) The map $t\in[0,1]\mapsto A\,\rho_t\,B$ is norm continuous for each A and B.

The power means $Am_rB=A^{\frac{1}{2}}\left(\frac{I_{\mathscr{H}}+(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^r}{2}\right)^{\frac{1}{r}}A^{\frac{1}{2}}$ are some typical interpolational means for $r\in[-1,1]$. Their interpolational paths are

$$Am_{r,t}B=A^{\frac{1}{2}}\left((1-t)I_{\mathscr{H}}+t(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^r\right)^{\frac{1}{r}}A^{\frac{1}{2}}\qquad(t\in[0,1]).$$

In particular, $Am_{1,t}B=A\nabla_tB=(1-t)A+tB$, $Am_{0,t}B=A\sharp_tB$, and $Am_{-1,t}B=A!_tB=\left((1-t)A^{-1}+tB^{-1}\right)^{-1}$. If Ψ is a unital positive linear map on $\mathscr{L}(\mathscr{H})$ and ω is an operator mean, then we have

$$\Psi(A\,\omega\,B) \le \Psi(A)\,\omega\Psi(B) \tag{1.2}$$

for all positive invertible operators A and B, see [11, Theorem 5.8]. For more information about operator means, see [11, 16].

The classical Hölder inequality asserts that

$$\left(\sum_{j=1}^{n} x_j\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} y_j\right)^{\frac{1}{q}} \ge \sum_{j=1}^{n} x_j^{\frac{1}{p}} y_j^{\frac{1}{q}}, \tag{1.3}$$

where x_j, y_j $(1 \le j \le n)$ are positive real numbers and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $p = q = 2$ the above inequality states that the celebrated Cauchy-Schwarz inequality.

Let $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \le j \le n)$ and ω be an operator mean. Then the operator mean ω is concave on pairs of positive invertible operators i.e.,

$$\left(\sum_{j=1}^{n} A_j\right) \omega \left(\sum_{j=1}^{n} B_j\right) \ge \sum_{j=1}^{n} (A_j\,\omega\,B_j), \tag{1.4}$$

where for the weighted operator mean is an extension of the operator Hölder inequality as follows

$$\left(\sum_{j=1}^{n} A_j\right) \sharp_\nu \left(\sum_{j=1}^{n} B_j\right) \ge \sum_{j=1}^{n} (A_j \sharp_\nu B_j) \qquad \text{for all } 0 \le \nu \le 1. \tag{1.5}$$

As a special case of the inequality (1.4), we have

$$(A + B)\,\omega\,(C + D) \ge (A\,\omega\,C) + (B\,\omega\,D) \tag{1.6}$$

for all positive invertible operators A, B, C, D and an operator mean ω, see [11, Theorem 5.7].

Bellman [6] proved that if p is a positive integer and a, b, a_j, b_j $(1 \le j \le n)$ are positive real numbers such that $\sum_{j=1}^{n} a_j^p \le a^p$ and $\sum_{j=1}^{n} b_j^p \le b^p$, then

$$\left((a + b)^p - \sum_{j=1}^{n} (a_j + b_j)^p\right)^{1/p} \ge \left(a^p - \sum_{j=1}^{n} a_j^p\right)^{1/p} + \left(b^p - \sum_{j=1}^{n} b_j^p\right)^{1/p}.$$

A multiplicative analogue of this inequality for $p = 2$ is due to Aczél, see [1] and its operator version in [20]. Popoviciu [22] extended Aczél's inequality for $p \ge 1$. During the last decades, several generalizations, refinements, and applications of the Bellman inequality in various settings have been given and some results related to integral inequalities are presented, see [1, 3, 5, 6, 7, 8, 12, 15, 18, 19, 20, 25].

In [19], the authors showed the following generalization of the operator Bellman inequality

$$\left(I_{\mathscr{H}} - \left(\sum_{j=1}^{n} A_j\,\omega_f\,B_j\right)\right)^p \ge \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \omega_{f^p} \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right), \tag{1.7}$$

where A_j, B_j $(1 \le j \le n)$ are positive invertible operators such that $\sum_{j=1}^{n} A_j \le I_{\mathscr{H}}$, $\sum_{j=1}^{n} B_j \le I_{\mathscr{H}}$, ω_f is a mean with the representing function f and $0 < p \le 1$.

Let $\mathscr{A}$ be a C^*-algebra of operators acting on a Hilbert space, let Ω be a locally compact Hausdorff space, and let $\mu(t)$ be a Radon measure on Ω. A field $(A_t)_{t\in\Omega}$ of operators in $\mathscr{A}$ is called a *continuous field of operators* if the function $t \mapsto A_t$ is norm continuous on Ω and the function $t \mapsto \|A_t\|$ is integrable. One can form the Bochner integral $\int_\Omega A_t\, d\mu(t)$, which is the unique element in $\mathscr{A}$ such that

$$\varphi\left(\int_\Omega A_t\, d\mu(t)\right) = \int_\Omega \varphi(A_t)\, d\mu(t) \tag{1.8}$$

for every linear functional φ in the norm dual $\mathscr{A}^*$ of $\mathscr{A}$, see [13]. Let $\mathcal{C}(\Omega, \mathscr{A})$ denote the set of bounded continuous functions on Ω with values in $\mathscr{A}$, which is a C^*-algebra under the pointwise operations and the norm $\|(A_t)\| = \sup_{t\in\Omega} \|A_t\|$, see [13].

In this paper, by the concept of operator means, we obtain a refinement of the inequalities (1.2). By using this refinement, we present some refinements of the operator Hölder inequality (1.5) and the operator Bellman inequality (1.7) for positive invertible operators. Furthermore, we generalize and refine some derived results for *continuous fields of operators* in a C^*-algebra $\mathscr{A}$.

2. Refinements of some Generalized operator inequalities

In this section, by the concept of operator means, we present some refinements of the operator Hölder inequality and the operator Bellman inequality. We need the following lemmas to illustrate our result.

Lemma 2.1 ([18]). *Let $A, B \in \mathscr{L}(\mathscr{H})_+$ be such that A is contraction, let h be a nonnegative operator monotone function on $[0, +\infty)$, and let ω_f be an operator mean with the representing function f. Then*

$$A\, \omega_{hof} B \le h(A\, \omega_f B).$$

In the following lemma, we present an operator inequality for three arbitrary operator means.

Lemma 2.2. *Let σ, τ, ρ be three arbitrary operator means such that $\sigma \le \tau$ or $\tau \le \sigma$. Then*

$$A \le (A\, \sigma\, B)\, \rho\, (A\, \tau\, B) \le B \tag{2.9}$$

for all positive invertible operators A and B such that $A \le B$.

Proof. Assume that A and B are positive invertible operators such that $A \le B$. Applying the properties of operator means, we have

$$A = A\, \sigma\, A \le A\, \sigma\, B \le B\, \sigma\, B = B \quad \text{and} \quad A = A\, \tau\, A \le A\, \tau\, B \le B\, \tau\, B = B.$$

Moreover, if $\sigma \le \tau$, i.e., $A\, \sigma\, B \le A\, \tau\, B$, then

$$(A \le) \qquad A\, \sigma\, B \le (A\, \sigma\, B)\rho(A\, \tau\, B) \le A\, \tau\, B \qquad (\le B) \tag{2.10}$$

and if $\tau \le \sigma$, i.e., $A\, \tau\, B \le A\, \sigma\, B$, then

$$(A \le) \qquad A\, \tau\, B \le (A\, \sigma\, B)\, \rho\, (A\, \tau\, B) \le A\, \sigma\, B \qquad (\le B). \tag{2.11}$$

Combining inequalities (2.10) and (2.11), we get

$$A \le (A\, \sigma\, B)\, \rho\, (A\, \tau\, B) \le B,$$

as required. □

Remark 2.1. *Assume that $\sigma, \tau, \rho_1, \rho_2$ are arbitrary operator means such that $\sigma \le \tau$ or $\tau \le \sigma$ and A, B are positive invertible operators such that $A \le B$. Then, applying Lemma 2.1, we get*

$$A \le (A\, \sigma\, B)\, \rho_1\, (A\, \tau\, B) \le (A\, \sigma\, B)\, \rho_2\, (A\, \tau\, B) \le B,$$

where $\rho_1 \le \rho_2$. To see this, note that, if $\rho_1 \le \rho_2$, then for the positive invertible operators $A\, \sigma\, B$ and $A\, \tau\, B$, we have

$$(A\, \sigma\, B)\, \rho_1\, (A\, \tau\, B) \le (A\, \sigma\, B)\, \rho_2\, (A\, \tau\, B).$$

Moreover, by Lemma 2.1, we have

$$A \le (A\,\sigma\, B)\,\rho_1\,(A\,\tau\, B) \qquad \text{and} \qquad (A\,\sigma\, B)\,\rho_2\,(A\,\tau\, B) \le B$$

for arbitrary operator means σ, τ with $\sigma \le \tau$ or $\tau \le \sigma$. Combining the above inequalities, we get desired result.

Remark 2.2. *Assume that σ_f and σ_g are arbitrary operator means with the representing functions f and g, respectively, with $f \le g$ or $g \le f$. As a special case of Lemma 2.1 for $\rho = \nabla_\lambda$, $(0 \le \lambda \le 1)$, we have*

$$A \le A\sigma_{(1-\lambda)f+\lambda g}B \le B \tag{2.12}$$

for all positive invertible operators A and B such that $A \le B$. To see this, note that

$$\begin{aligned}(A\,\sigma_f\, B)\nabla_\lambda(A\,\sigma_g\, B) &= (1-\lambda)A^{\frac{1}{2}}f(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} + \lambda A^{\frac{1}{2}}g(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}\Big((1-\lambda)f(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}) + \lambda g(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})\Big)A^{\frac{1}{2}} \\ &= A\,\sigma_{(1-\lambda)f+\lambda g}\, B.\end{aligned}$$

Hence, by Lemma 2.1, we get

$$A \le (A\,\sigma_f\, B)\nabla_\lambda(A\,\sigma_g\, B) = A\,\sigma_{(1-\lambda)f+\lambda g}\, B \le B,$$

as required.

As an application of the above result, we have the next lemma, which is a refinement of the inequality (1.2).

Lemma 2.3. *Let $\sigma, \tau, \rho, \omega$ be arbitrary operator means such that $\sigma \le \tau$ or $\tau \le \sigma$, and let Ψ be a unital positive linear map on $\mathscr{L}(\mathscr{H})$. Then*

$$\begin{aligned}\big(\Psi(A)\,\omega\,\Psi(B)\big)^p &\ge \Big(\Psi^p(A\,\omega\, B)\,\sigma\,\big(\Psi(A)\,\omega\,\Psi(B)\big)^p\Big)\,\rho\,\Big(\Psi^p(A\,\omega\, B)\,\tau\,\big(\Psi(A)\,\omega\,\Psi(B)\big)^p\Big) \\ &\ge \Psi^p(A\,\omega\, B)\end{aligned} \tag{2.13}$$

for all positive invertible operators A, B and $0 < p \le 1$.

Proof. Applying the inequality (1.2) and the operator monotonicity of $g(t) = t^p, (0 < p \le 1)$, we have

$$\Psi^p(A\,\omega\, B) \le (\Psi(A)\,\omega\Psi(B))^p.$$

Replacing A by $\Psi^p(A\,\omega\, B)$ and B by $(\Psi(A)\,\omega\Psi(B))^p$, respectively, in the inequality (2.9), we have

$$\begin{aligned}\Psi^p(A\,\omega\, B) &\le \Big(\Psi^p(A\,\omega\, B)\,\sigma\,\big(\Psi(A)\,\omega\,\Psi(B)\big)^p\Big)\,\rho\,\Big(\Psi^p(A\,\omega\, B)\,\tau\,\big(\Psi(A)\,\omega\,\Psi(B)\big)^p\Big) \\ &\le \big(\Psi(A)\,\omega\,\Psi(B)\big)^p\end{aligned}$$

for all operator means $\sigma, \tau, \rho, \omega$ such that $\sigma \le \tau$ or $\tau \le \sigma$, as required. $\square$

In the first result of this section, we present a refinement of the operator Hölder inequality (1.4) as follows.

Theorem 2.1. *Let $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \le j \le n)$ and $\sigma, \tau, \rho, \omega$ be arbitrary operator means such that $\sigma \le \tau$ or $\tau \le \sigma$. Then*

$$\left(\left(\sum_{j=1}^{n} A_j\right) \omega \left(\sum_{j=1}^{n} B_j\right)\right)^p$$
$$\ge \left[\left(\sum_{j=1}^{n}(A_j \omega B_j)\right)^p \sigma \left[\left(\sum_{j=1}^{n} A_j\right) \omega \left(\sum_{j=1}^{n} B_j\right)\right]^p\right] \rho \left[\left(\sum_{j=1}^{n}(A_j \omega B_j)\right)^p \tau \left[\left(\sum_{j=1}^{n} A_j\right) \omega \left(\sum_{j=1}^{n} B_j\right)\right]^p\right]$$
$$\ge \left(\sum_{j=1}^{n}(A_j \omega B_j)\right)^p$$

for $0 < p \le 1$.

Proof. Assume that $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \le j \le n)$ and $\sigma, \tau, \rho, \omega$ are arbitrary operator means with $\sigma \le \tau$ or $\tau \le \sigma$. Note that if $A_1 \oplus \cdots \oplus A_n$ and $B_1 \oplus \cdots \oplus B_n$ are two diagonal operator matrices, then by the definition of operator means, for the operator mean ω, we have

$$(A_1 \oplus \cdots \oplus A_n)\omega(B_1 \oplus \cdots \oplus B_n) = (A_1 \omega B_1) \oplus \cdots \oplus (A_n \omega B_n).$$

Replacing A by $A_1 \oplus \cdots \oplus A_n$ and B by $B_1 \oplus \cdots \oplus B_n$ in the inequality (2.13) and taking Ψ in the inequality (2.13) to be the unital positive linear map defined on the diagonal blocks of operators by $\Psi(A_1 \oplus \cdots \oplus A_n) = \frac{1}{n}\sum_{j=1}^{n} A_j$, we have the desired result. □

As a consequence of Theorem 2.1, we have a refinement of the operator Hölder inequality involving the weighted geometric mean.

Corollary 2.1. *Let $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \le j \le n)$ and σ, τ, ρ be arbitrary operator means such that $\sigma \le \tau$ or $\tau \le \sigma$. Then*

$$\begin{aligned}
&\left(\sum_{j=1}^{n}(A_j \sharp_\nu B_j)\right)^p \\
&\le \left[\left(\sum_{j=1}^{n}(A_j \sharp_\nu B_j)\right)^p \sigma \left[\left(\sum_{j=1}^{n} A_j\right) \sharp_\nu \left(\sum_{j=1}^{n} B_j\right)\right]^p\right] \\
&\quad \rho \left[\left(\sum_{j=1}^{n}(A_j \sharp_\nu B_j)\right)^p \tau \left[\left(\sum_{j=1}^{n} A_j\right)^p \sharp_\nu \left(\sum_{j=1}^{n} B_j\right)\right]^p\right] \\
&\le \left(\left(\sum_{j=1}^{n} A_j\right) \sharp_\nu \left(\sum_{j=1}^{n} B_j\right)\right)^p
\end{aligned} \tag{2.14}$$

for all $\nu \in [0,1]$ and $0 < p \le 1$. In particular, for $\tau = \sigma$, we have

$$\begin{aligned}
\left(\sum_{j=1}^{n}(A_j \sharp_\nu B_j)\right)^p &\le \left(\sum_{j=1}^{n}(A_j \sharp_\nu B_j)\right)^p \sigma \left[\left(\sum_{j=1}^{n} A_j\right) \sharp_\nu \left(\sum_{j=1}^{n} B_j\right)\right]^p \\
&\le \left(\left(\sum_{j=1}^{n} A_j\right) \sharp_\nu \left(\sum_{j=1}^{n} B_j\right)\right)^p
\end{aligned}$$

for all $\nu \in [0,1]$ *and* $0 < p \leq 1$.

Remark 2.3. *Note that if* $0 \leq s \leq t \leq 1$, *then* $A\sharp_s B \leq A\sharp_t B$ *for positive invertible operators* A *and* B *such that* $A \leq B$. *Therefore, for positive invertible operators* A_j, B_j $(1 \leq j \leq n)$ *with* $A_jB_j = B_jA_j$ $(1 \leq j \leq n)$ *and* $\sigma = \sharp_s$, $\rho = \nabla$, *and* $\tau = \sharp_t$ *in Corollary 2.1, we have*

$$
\begin{aligned}
&\sum_{j=1}^{n} A_j^{1-\nu} B_j^{\nu} \\
&\leq \frac{1}{2}\left[\left(\sum_{j=1}^{n} A_j^{1-\nu} B_j^{\nu}\right)^{1-s}\left(\sum_{j=1}^{n} A_j\right)^{(1-\nu)s}\left(\sum_{j=1}^{n} B_j\right)^{\nu s}\right] \\
&+\left[\left(\sum_{j=1}^{n} A_j^{1-\nu} B_j^{\nu}\right)^{1-t}\left(\sum_{j=1}^{n} A_j\right)^{(1-\nu)t}\left(\sum_{j=1}^{n} B_j\right)^{\nu t}\right] \\
&\leq \left(\sum_{j=1}^{n} A_j\right)^{(1-\nu)}\left(\sum_{j=1}^{n} B_j\right)^{\nu}
\end{aligned}
$$

for all $0 \leq s \leq t \leq 1$, *which is an extension and a refinement of the classical Hölder inequality.*

In the following result, we obtain a refinement of the generalized operator Bellman inequality (1.7).

Theorem 2.2. *Let* $A_j, B_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \leq j \leq n)$ *be such that* $\sum_{j=1}^{n} A_j \leq I_{\mathscr{H}}$, $\sum_{j=1}^{n} B_j \leq I_{\mathscr{H}}$, *and let* ω_f *be an operator mean with the representing function* f *and* $0 < p \leq 1$. *Then*

$$
\begin{aligned}
&\left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right) \omega_f^p \left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right) \\
\leq &\Bigg(\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\right)^{\mu} \\
&\rho\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^{n} A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^{n} B_j\right)\right)^{\nu} - \sum_{j=1}^{n}(A_j\omega_f B_j)\Bigg)^{p} \\
\leq &\left(I_{\mathscr{H}} - \sum_{j=1}^{n}(A_j\omega_f B_j)\right)^{p}
\end{aligned}
$$

for all arbitrary means ρ *and* $0 \leq \mu \leq \nu \leq 1$.

Proof. Applying Theorem 2.1 to $X_j, Y_j \in \mathscr{L}(\mathscr{H})_+$ $(1 \le j \le n+1)$ and to two arbitrary operator means ρ, ω_f, and to the weighted geometric means $\sharp_\mu$, and $\sharp_\nu$ such that $0 \le \mu \le \nu \le 1$, we get

$$\begin{aligned}
&\sum_{j=1}^{n+1}(X_j \omega Y_j)\\
&\le \left[\left(\sum_{j=1}^{n+1}(X_j\omega_f Y_j)\right)\sharp_\mu\left[\left(\sum_{j=1}^{n+1}X_j\right)\omega_f\left(\sum_{j=1}^{n+1}Y_j\right)\right]\right]\\
&\quad\rho\left[\left(\sum_{j=1}^{n+1}(X_j\omega_f Y_j)\right)\sharp_\nu\left[\left(\sum_{j=1}^{n+1}X_j\right)\omega_f\left(\sum_{j=1}^{n+1}Y_j\right)\right]\right]\\
&\le \left(\sum_{j=1}^{n+1}X_j\right)\omega_f\left(\sum_{j=1}^{n+1}Y_j\right). \qquad (2.15)
\end{aligned}$$

By putting $X_j = A_j, Y_j = B_j$ $(1 \le j \le n)$ $X_{n+1} = I_{\mathscr{H}} - \sum_{j=1}^n A_j$, and $Y_{n+1} = I_{\mathscr{H}} - \sum_{j=1}^n B_j$, and taking $\sigma = \sharp_\mu$ and $\sigma = \sharp_\nu$ in the inequalities (2.15), we get

$$\begin{aligned}
&\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\\
&\le \left[\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right)\sharp_\mu\,(I_{\mathscr{H}}\omega_f I_{\mathscr{H}})\right]\\
&\quad\rho\left[\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right)\sharp_\nu\,(I_{\mathscr{H}}\omega_f I_{\mathscr{H}})\right]\\
&\le (I_{\mathscr{H}}\omega_f I_{\mathscr{H}})
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\\
&\le \left[\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right)\sharp_\mu I_{\mathscr{H}}\right]\\
&\quad\rho\left[\left(\sum_{j=1}^{n}(A_j\omega_f B_j) + \left(I_{\mathscr{H}} - \sum_{j=1}^n A_j\right)\omega_f\left(I_{\mathscr{H}} - \sum_{j=1}^n B_j\right)\right)\sharp_\nu I_{\mathscr{H}}\right]\\
&\le I_{\mathscr{H}}, \qquad \text{for } 0 \le \mu \le \nu \le 1.
\end{aligned}$$

Using the definition of the operator means $\sharp_\mu$ and $\sharp_\nu$, we have

$$\begin{aligned}
&\sum_{j=1}^{n}(A_j\omega_f B_j)+\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\right)\omega_f\left(I_{\mathscr{H}}-\sum_{j=1}^{n}B_j\right)\\
\leq&\left(\sum_{j=1}^{n}(A_j\omega_f B_j)+\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\right)\omega_f\left(I_{\mathscr{H}}-\sum_{j=1}^{n}B_j\right)\right)^{\mu}\\
&\rho\left(\sum_{j=1}^{n}(A_j\omega_f B_j)+\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\right)\omega_f\left(I_{\mathscr{H}}-\sum_{j=1}^{n}B_j\right)\right)^{\nu}\\
\leq& I_{\mathscr{H}}
\end{aligned}$$

for all arbitrary means ρ and $0\leq\mu\leq\nu\leq 1$. Hence,

$$\begin{aligned}
&\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\right)\omega_f\left(I_{\mathscr{H}}-\sum_{j=1}^{n}B_j\right)\\
\leq&\left(\sum_{j=1}^{n}(A_j\omega_f B_j)+\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\right)\omega_f\left(I_{\mathscr{H}}-\sum_{j=1}^{n}B_j\right)\right)^{\mu}\\
&\rho\left(\sum_{j=1}^{n}(A_j\omega_f B_j)+\left(I-\sum_{j=1}^{n}A_j\right)\omega_f\left(I-\sum_{j=1}^{n}B_j\right)\right)^{\nu}-\sum_{j=1}^{n}(A_j\omega_f B_j)\\
\leq& I_{\mathscr{H}}-\sum_{j=1}^{n}(A_j\omega_f B_j),\qquad\text{for } 0\leq\mu\leq\nu\leq 1.
\end{aligned}$$

It follows from the operator monotonicity of $g(t)=t^p$ $(0<p\leq 1)$, the above inequalities, and Lemma 2.1 that

$$\begin{aligned}
&\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\right)\omega_{f^p}\left(I_{\mathscr{H}}-\sum_{j=1}^{n}B_j\right)\\
\leq&\left(\left(\sum_{j=1}^{n}(A_j\omega_f B_j)+\left(I_{\mathscr{H}}-\sum_{j=1}^{n}A_j\right)\omega_f\left(I_{\mathscr{H}}-\sum_{j=1}^{n}B_j\right)\right)^{\mu}\right.\\
&\left.\rho\left(\sum_{j=1}^{n}(A_j\omega_f B_j)+\left(I-\sum_{j=1}^{n}A_j\right)\omega_f\left(I-\sum_{j=1}^{n}B_j\right)\right)^{\nu}-\sum_{j=1}^{n}(A_j\omega_f B_j)\right)^{p}\\
\leq&\left(I_{\mathscr{H}}-\sum_{j=1}^{n}(A_j\omega_f B_j)\right)^{p}\qquad\text{for } 0\leq\mu\leq\nu\leq 1,
\end{aligned}$$

as required. $\square$

3. Some extensions for continuous fields of operators

Let $\mathscr{A}$ be a C^*-algebra of operators acting on a Hilbert space, let Ω be a compact Hausdorff space, and let $(A_t)_{t\in\Omega}$ be a continuous field of operators in $\mathscr{A}$. In this section, by using the concept of the *continuous fields of operators*, we present some results involving the operator Hölder type inequalities and the operator Bellman type inequalities.

We need following lemma to illustrate our results.

Lemma 3.4. *Let $\mathscr{A}$ be a C^*-algebra, Ω be a compact Hausdorff space equipped with a Radon measure μ, and let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators. Then*

$$\int_\Omega \int_\Omega (A_t \circ B_s)\, d\mu(t)\, d\mu(s) = \int_\Omega A_t\, d\mu(t) \circ \int_\Omega B_s\, d\mu(s) \qquad (A_t, B_s \in \mathscr{A}). \tag{3.16}$$

Proof. Assume that $\mathscr{A}$ is a C^*-algebra of operators acting on a Hilbert space, Ω is a compact Hausdorff space, and $(A_t)_{t\in\Omega}$ is a continuous field of operators in $\mathscr{A}$. Using [21, Page 78], since $\mathbf{A} : t \mapsto A_t$ is a continuous function from Ω to $\mathscr{A}$, for every operator $A_t \in \mathscr{A}$ and for every $\varepsilon > 0$, we can consider an element of the form

$$I_\lambda(A_t) = \Sigma_{k=1}^n \mathbf{A}(t_k)\mu(E_k) = \Sigma_{k=1}^n A_{t_k}\mu(E_k),$$

where the $E_k's$ form a partition of Ω into disjoint Borel subsets, and

$$t_k \in E_k \subseteq \{t \in \Omega : \|A_t - A_{t_k}\| \le \varepsilon\} \qquad (1 \le k \le n),$$

with $\lambda = \{E_1, \cdots, E_n, \varepsilon\}$. Then $(I_\lambda(A_t))_{\lambda\in\Lambda}$ is a uniformly convergent net to $\int_\Omega A_t d\mu(t)$. It follows from the norm continuity of the tensor product of two operators that for any operator $B \in \mathscr{A}$, we have

$$\int_\Omega (A_t \otimes B)\, d\mu(t) = \left(\int_\Omega A_t\, d\mu(t)\right) \otimes B. \tag{3.17}$$

Also, by using the definition of the Bochner integral for any operator $X \in \mathscr{A}$, we have $\int_\Omega (X^* A_t X) d\mu(t)$ $X^*\left(\int_\Omega A_t d\mu(t)\right) X$. Therefore, for an arbitrary operator $B \in \mathscr{A}$, we get

$$\begin{aligned}\int_\Omega (A_t \circ B)\, d\mu(t) &= \int_\Omega V^*(A_t \otimes B) V\, d\mu(t) = V^* \int_\Omega (A_t \otimes B)\, d\mu(t) V \\ &= V^*\left(\int_\Omega A_t\, d\mu(t) \otimes B\right) V = \int_\Omega A_t\, d\mu(t) \circ B \qquad (A_t, B \in \mathscr{A}),\end{aligned} \tag{3.18}$$

where $V : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is the isometry defined by $Ve_j = e_j \otimes e_j$, for a given orthonormal basis $\{e_j\}$ of the Hilbert space $\mathscr{H}$. Hence, we have

$$\begin{aligned}
\int_\Omega \int_\Omega (A_t \circ B_s)\, d\mu(t)\, d\mu(s) &= \int_\Omega \int_\Omega V^*(A_t \otimes B_s)V\, d\mu(t)\, d\mu(s) \\
&= \int_\Omega V^* \left(\int_\Omega (A_t \otimes B_s)\, d\mu(t) \right) V\, d\mu(s) \quad \text{(by (3.18))} \\
&= \int_\Omega V^* \left(\left(\int_\Omega A_t\, d\mu(t) \right) \otimes B_s \right) V\, d\mu(s) \quad \text{(by (3.17))} \\
&= \int_\Omega \left(\int_\Omega A_t\, d\mu(t) \right) \circ B_s\, d\mu(s) \\
&= \int_\Omega U^* \left(\left(\int_\Omega A_t\, d\mu(t) \right) \otimes B_s \right) U\, d\mu(s) \\
&= U^* \left(\int_\Omega \left(\int_\Omega A_t\, d\mu(t) \right) \otimes B_s\, d\mu(s) \right) U \quad \text{(by (3.18))} \\
&= U^* \left(\left(\int_\Omega A_t\, d\mu(t) \right) \otimes \left(\int_\Omega B_s\, d\mu(s) \right) \right) U \quad \text{(by (3.17))} \\
&= \int_\Omega A_t\, d\mu(t) \circ \int_\Omega B_s\, d\mu(s) \qquad \text{for } A_t, B_s \in \mathscr{A}.
\end{aligned}$$

□

The first result of this section is the Hölder inequality for *continuous fields of operators* involving an arbitrary operator mean. The main ideas of the following result are stimulated by [4].

Theorem 3.3. *Let $\mathscr{A}$ be a C^*-algebra, Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $C(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators, and let ω_f be an operator mean with the representing function f. Then*

$$\left(\int_\Omega A_s\, d\mu(s) \right) \omega_f \left(\int_\Omega B_s\, d\mu(s) \right) \geq \int_\Omega (A_s\, \omega_f\, B_s)\, d\mu(s). \tag{3.19}$$

Proof. For the *continuous fields of positive invertible operators* $\mathbf{A} = (A_t)_{t\in\Omega} \in \mathscr{A}$ and $\mathbf{B} = (B_t)_{t\in\Omega} \in \mathscr{A}$, we put the positive unital linear map

$$\Psi(S) = \int_\Omega Z^* S Z d\mu(t) \;\; (S \in \mathscr{A}),$$

where $Z = B_t^{\frac{1}{2}} \left(\int_\Omega B_s\, d\mu(s) \right)^{-\frac{1}{2}}$. Thus, we have

$$\begin{aligned}
&\left(\int_\Omega A_t\, d\mu(t) \right) \omega_f \left(\int_\Omega B_s\, d\mu(s) \right) \\
&= \left(\int_\Omega B_s\, d\mu(s) \right)^{\frac{1}{2}} f \left(\left(\int_\Omega B_s\, d\mu(s) \right)^{\frac{1}{2}} \int_\Omega A_t\, d\mu(t) \left(\int_\Omega B_s\, d\mu(s) \right)^{-\frac{1}{2}} \right) \left(\int_\Omega B_s\, d\mu(s) \right)^{\frac{1}{2}} \\
&= \left(\int_\Omega B_s\, d\mu(s) \right)^{\frac{1}{2}} f \left(\int_\Omega \left(\int_\Omega B_s\, d\mu(s) \right)^{-\frac{1}{2}} B_t^{\frac{1}{2}} (B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}) B_t^{\frac{1}{2}} \left(\int_\Omega B_s\, d\mu(s) \right)^{-\frac{1}{2}} d\mu(t) \right) \\
&\times \left(\int_\Omega B_s\, d\mu(t) \right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} f\left(\int_\Omega Z^* B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}} Z d\mu(t)\right)\left(\int_\Omega B_s\, d\mu(t)\right)^{\frac{1}{2}}\\
&= \left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} f\left(\Phi\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right)\right)\left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}}\\
&\geq \left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} \Phi\left(f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right)\right)\left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} \qquad \text{(by (1.1))}\\
&= \left(\int_\Omega B_s\, d\mu(s)\right)^{\frac{1}{2}} \left(\int_\Omega Z^* f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right) Z d\mu(t)\right)\left(\int_\Omega B_s\, d\mu(t)\right)^{\frac{1}{2}}\\
&= \int_\Omega B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}} d\mu(t)\\
&= \int_\Omega (A_t\, \omega_f\, B_t)\, d\mu(t),
\end{aligned}
$$

as required. □

Remark 3.4. *In the discrete case $\Omega = \{1, \cdots, n\}$, for positive invertible operators $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$, Theorem 3.4 enforces the inequality (1.4).*

Remark 3.5. *Assume that $\Omega = [0,1]$ is with the Lebesgue measure and $\mathscr{A} = \mathbb{R}$ is the real numbers. Then $\mathcal{C}([0,1], \mathbb{R})$ is the C^*-algebra involving all continuous real-valued functions over $[0,1]$. As a special case of Theorem 3.3, we have the integral version of the Hölder inequality as follows*

$$
\left(\int_a^b f(x)\, dx\right) \omega \left(\int_a^b g(x)\, dx\right) \geq \int_a^b (f(x)\, \omega\, g(x))\, dx,
$$

where $f, g \in \mathcal{C}([0,1], \mathbb{R})$ are positive functions and ω is an operator mean.

Using the inequality (2.9), we obtain a refinement of the Hölder inequality for *continuous fields of operators* (3.19) as follows.

Theorem 3.4. *Let $\mathscr{A}$ be a C^*-algebra, Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators, and let $\sigma, \tau, \rho, \omega$ be arbitrary operator means such that $\sigma \leq \tau$ or $\tau \leq \sigma$. Then*

$$
\begin{aligned}
\left(\int_\Omega A_t\, d\mu(t)\right) \omega \left(\int_\Omega B_s\, d\mu(s)\right) &\geq \left[\left(\int_\Omega (A_t\, \omega\, B_t)\, d\mu(t)\right) \sigma \left[\left(\int_\Omega A_t\, d\mu(t)\right) \omega \left(\int_\Omega B_s\, d\mu(s)\right)\right]\right]\\
&\qquad \rho \left[\left(\int_\Omega (A_t\, \omega\, B_t)\, d\mu(t)\right) \tau \left[\left(\int_\Omega A_t\, d\mu(t)\right) \omega \left(\int_\Omega B_s\, d\mu(s)\right)\right]\right]\\
&\geq \int_\Omega (A_t\, \omega\, B_t)\, d\mu(t).
\end{aligned}
$$

Proof. Using the inequality (3.19) and replacing

$$
A \;\text{ by }\; \int_\Omega (A_t\, \omega\, B_t)\, d\mu(t) \qquad \text{and} \qquad B \;\text{ by }\; \left(\int_\Omega A_t\, d\mu(t)\right) \omega \left(\int_\Omega B_s\, d\mu(s)\right)
$$

in the inequality (2.9), respectively, we get the desired result. □

In the next result, we obtain an inequality for *continuous fields of operators*.

Theorem 3.5. *Let $\mathscr{A}$ be a C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators such that $\int_\Omega A_t\, d\mu(t) \le A$ and $\int_\Omega B_t\, d\mu(t) \le B$ for some positive invertible operators $A, B \in \mathscr{A}$, and let ω_f be an arbitrary operator mean with the representing function f. Then*

$$\left((A\,\omega_f\,B) - \int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t)\right)^p \ge \left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(B - \int_\Omega B_t\,d\mu(t)\right) \tag{3.20}$$

for all $0 < p \le 1$.

Proof. Assume $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ are *continuous fields* of positive invertible operators such that $\int_\Omega A_t\, d\mu(t) \le A$ and $\int_\Omega B_t\, d\mu(t) \le B$ for some positive invertible operators $A, B \in \mathscr{A}$. Then we have

$$\begin{aligned}
A\,\omega_f\,B &= \left(A - \int_\Omega A_t\,d\mu(t) + \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(B - \int_\Omega B_t\,d\mu(t) + \int_\Omega B_t\,d\mu(t)\right)\\
&\ge \left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(B - \int_\Omega B_t\,d\mu(t)\right) + \left(\int_\Omega A_t\,d\mu(t)\,\omega_f\int_\Omega B_t\,d\mu(t)\right)\\
&\qquad\qquad \text{(by the inequality (1.6))}\\
&\ge \left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(B - \int_\Omega B_t\,d\mu(t)\right) + \left(\int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t)\right).
\end{aligned}$$

Hence, by the above inequality, the operator monotonicity of $f(t) = t^p$ $(0 < p \le 1)$ and Lemma 2.1, we get

$$\begin{aligned}
\left((A\,\omega_f\,B) - \int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t)\right)^p &\ge \left(\left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(B - \int_\Omega B_t\,d\mu(t)\right)\right)^p\\
&\ge \left(A - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(B - \int_\Omega B_t\,d\mu(t)\right),
\end{aligned}$$

as required. □

In the next result, by using Theorem 3.5, we have the operator Bellman inequality for *continuous fields* in a unital C^*-algebra.

Corollary 3.2. *Let $\mathscr{A}$ be a unital C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators such that $\int_\Omega A_t\, d\mu(t) \le I_{\mathscr{A}}$ and $\int_\Omega B_t\, d\mu(t) \le I_{\mathscr{A}}$, and let ω_f be an operator mean with the representing function f. Then*

$$\left(I_{\mathscr{A}} - \int_\Omega (A_t\omega_f B_t)\,d\mu(t)\right)^p \ge \left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right) \tag{3.21}$$

for all $0 < p \le 1$.

Remark 3.6. *Assume that $\mathcal{C}([0,1], \mathbb{R})$ is the C^*-algebra involving all continuous real-valued functions over $[0,1]$. As a special case of the inequality (3.21), we have the integral version of the Bellman inequality as follows*

$$\left(1 - \int_a^b (g(x)\omega_f h(x))\,dx\right)^p \ge \left(1 - \int_a^b g(x)\,dx\right)\omega_{f^p}\left(1 - \int_a^b h(x)\,dx\right) \qquad (0 < p \le 1),$$

where $f, g \in \mathcal{C}([0,1],\mathbb{R})$ are positive functions such that $\int_a^b g(x)\,dx \le 1$ and $\int_a^b h(x)\,dx \le 1$, and ω_f is an operator mean with the representing function f. In particular, for $\omega_f = \sharp_{\frac{1}{2}}$, we have

$$1-\int_a^b \sqrt{g(x)h(x)}\,dx \ge \sqrt{1-\int_a^b g(x)\,dx}\sqrt{1-\int_a^b h(x)\,dx}.$$

These two above inequalities are the integral version of the Bellman inequality (1.7).

In the next theorem, we present a refinement of the operator Bellman inequality (3.21) for *continuous fields of operators*.

Theorem 3.6. *Let $\mathscr{A}$ be a unital C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega,\mathscr{A})$ be continuous fields of positive invertible operators such that $\int_\Omega A_t\,d\mu(t) \le I_{\mathscr{A}}$, $\int_\Omega B_t\,d\mu(t) \le I_{\mathscr{A}}$, and let ω_f be an arbitrary operator mean with the representing function f. Then*

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega (A_t \omega_f B_t)\,d\mu(t)\right)^p \\
&\ge \left(\left(I_{\mathscr{A}} - \int_{\Omega_1} A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_{\Omega_1} B_t\,d\mu(t)\right) - \int_{\Omega_2} (A_t\,\omega_f\,B_t)\,d\mu(t)\right)^p \\
&\ge \left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right)
\end{aligned}$$

for all $0 < p \le 1$ and for two disjoint sets $\Omega_1, \Omega_2 \subseteq \Omega$ such that $\Omega = \Omega_1 \cup \Omega_2$.

Proof. Assume that $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ are *continuous fields* of positive invertible operators in a unital C^*-algebra $\mathscr{A}$ with $\int_\Omega A_t\,d\mu(t) \le I_{\mathscr{A}}$ and $\int_\Omega B_t\,d\mu(t) \le I_{\mathscr{A}}$. We have

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right) \\
&=\left(I_{\mathscr{A}} - \int_{\Omega_1} A_t\,d\mu(t) - \int_{\Omega_2} A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_{\Omega_1} B_t\,d\mu(t) - \int_{\Omega_2} B_t\,d\mu(t)\right) \\
&\le\left(I_{\mathscr{A}} - \int_{\Omega_1} A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_{\Omega_1} B_t\,d\mu(t)\right) - \int_{\Omega_2} (A_t\,\omega_f\,B_t)\,d\mu(t) \\
&\qquad\qquad \text{(by the inequality (3.20))} \\
&\le (I_{\mathscr{A}}\omega_f I_{\mathscr{A}}) - \int_{\Omega_1} (A_t\,\omega_f\,B_t)\,d\mu(t) - \int_{\Omega_2} (A_t\,\omega_f\,B_t)\,d\mu(t) \\
&\qquad\qquad \text{(by the inequality (3.20))} \\
&= I_{\mathscr{A}} - \int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t).
\end{aligned}$$

Hence, by the above inequalities, the operator monotonicity of $f(t) = t^p\,(0 < p \le 1)$ and Lemma 2.1, we have

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_{f^p}\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right) \\
&\qquad\le \left(\left(I_{\mathscr{A}} - \int_\Omega A_t\,d\mu(t)\right)\omega_f\left(I_{\mathscr{A}} - \int_\Omega B_t\,d\mu(t)\right)\right)^p \\
&\qquad\le \left(I_{\mathscr{A}} - \int_\Omega (A_t\,\omega_f\,B_t)\,d\mu(t)\right)^p,
\end{aligned}$$

as required. $\square$

In the next theorem, we present another refinement of the operator Bellman inequality (3.21) involving *continuous fields of operators*.

Theorem 3.7. *Let $\mathscr{A}$ be a unital C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators such that $\int_\Omega A_t\, d\mu(t) \leq I_{\mathscr{A}}$ and $\int_\Omega B_t\, d\mu(t) \leq I_{\mathscr{A}}$, let ω_f be an arbitrary operator mean with the representing function f, and let $\lambda : s \in \Omega \longmapsto \lambda_s \in [0, 1]$ be a measurable function. Then*

$$
\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega (A_s \omega_f B_s)\, d\mu(s)\right)^p \\
&\geq \left(\left(\left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s)\right) \omega_f \left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s)\right)\right) - \int_\Omega (1-\lambda_s)\big(A_s \omega_f B_s\big) d\mu(s)\right)^p \\
&\geq \left(I_{\mathscr{A}} - \int_\Omega A_s\, d\mu(s)\right) \omega_{f^p} \left(I_{\mathscr{A}} - \int_\Omega B_s\, d\mu(s)\right)
\end{aligned}
$$

for $0 < p \leq 1$.

Proof. Assume that $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ are *continuous fields* of positive invertible operators in a unital C^*-algebra $\mathscr{A}$ such that $\int_\Omega A_t\, d\mu(t) \leq I_{\mathscr{A}}$ and $\int_\Omega B_t\, d\mu(t) \leq I_{\mathscr{A}}$ and $\lambda_s \in [0, 1]$ $(s \in \Omega)$. First note that

$$
\int_\Omega A_s\, d\mu(s) = \int_\Omega (A_s \nabla_{\lambda_s} A_s)\, d\mu(s) = \int_\Omega \lambda_s A_s\, d\mu(s) + \int_\Omega (1-\lambda_s) A_s\, d\mu(s),
$$

$$
\int_\Omega B_s\, d\mu(s) = \int_\Omega (B_s \nabla_{\lambda_s} B_s)\, d\mu(s) = \int_\Omega \lambda_s B_s\, d\mu(s) + \int_\Omega (1-\lambda_s) B_s\, d\mu(s),
$$

$$
I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s) \geq \int_\Omega (1-\lambda_s) A_s\, d\mu(s) \geq 0,
$$

and

$$
I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s) \geq \int_\Omega (1-\lambda_s) B_s\, d\mu(s) \geq 0.
$$

Then,

$$
\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega A_s\, d\mu(s)\right) \omega_f \left(I_{\mathscr{A}} - \int_\Omega B_s\, d\mu(s)\right) \\
&= \left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s) - \int_\Omega (1-\lambda_s) A_s\, d\mu(s)\right) \omega_f \left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s) - \int_\Omega (1-\lambda_s) B_s\, d\mu(s)\right) \\
&\leq \left(\left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s)\right) \omega_f \left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s)\right)\right) - \int_\Omega \big[((1-\lambda_s)A_s)\omega_f((1-\lambda_s)B_s)\big] d\mu(s) \\
&\qquad\qquad \text{(by the inequality (3.20))} \\
&\leq \left(\left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s)\right) \omega_f \left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s)\right)\right) - \int_\Omega (1-\lambda_s)\big(A_s \omega_f B_s\big) d\mu(s) \\
&\qquad\qquad \text{(by the properties of means)}
\end{aligned}
$$

$$\leq \left((I_{\mathscr{A}} \omega_f I_{\mathscr{A}}) - \int_\Omega (\lambda_s A_s \omega_f \lambda_s B_s)\, d\mu(s) \right) - \int_\Omega (1-\lambda_s)\big(A_s \omega_f B_s\big) d\mu(s)$$
(by the inequality (3.20))
$$\leq I_{\mathscr{A}} - \int_\Omega \lambda_s (A_s \omega_f B_s)\, d\mu(s) - \int_\Omega (1-\lambda_s)\big(A_s \omega_f B_s\big) d\mu(s)$$
(by the properties of means)
$$= I_{\mathscr{A}} - \int_\Omega (A_s \omega_f B_s)\, d\mu(s).$$

Hence, the operator monotonicity of $f(t) = t^p$ $(0 < p \leq 1)$ Lemma 2.1, and the above inequalities imply that

$$\begin{aligned}
&\left(I_{\mathscr{A}} - \int_\Omega A_s\, d\mu(s)\right) \omega_{f^p} \left(I_{\mathscr{A}} - \int_\Omega B_s\, d\mu(s)\right) \\
&\leq \left(\left(I_{\mathscr{A}} - \int_\Omega A_s\, d\mu(s)\right) \omega_f \left(I_{\mathscr{A}} - \int_\Omega B_s\, d\mu(s)\right) \right)^p \\
&\leq \left(\left(\left(I_{\mathscr{A}} - \int_\Omega \lambda_s A_s\, d\mu(s)\right) \omega_f \left(I_{\mathscr{A}} - \int_\Omega \lambda_s B_s\, d\mu(s)\right) \right) - \int_\Omega (1-\lambda_s)\big(A_s \omega_f B_s\big) d\mu(s) \right)^p \\
&\leq \left(I_{\mathscr{A}} - \int_\Omega (A_s \omega_f B_s)\, d\mu(s)\right)^p
\end{aligned}$$

for $0 < p \leq 1$. This completes the proof. □

In the following result, we obtain the operator Hölder inequality involving the Hadamard product of operators. The main ideas of the next result are stimulated by [2, 24].

Theorem 3.8. *Let $\mathscr{A}$ be a unital C^*-algebra, let Ω be a compact Hausdorff space equipped with a Radon measure μ, and let $\mathbf{A} = (A_t)_{t\in\Omega}$ and $\mathbf{B} = (B_t)_{t\in\Omega}$ in $\mathcal{C}(\Omega, \mathscr{A})$ be continuous fields of positive invertible operators. Then*

$$\int_\Omega A_t\, d\mu(t) \circ \int_\Omega B_s\, d\mu(s) \geq \int_\Omega (A_t \sharp_\alpha B_t)\, d\mu(t) \circ \int_\Omega (A_s \sharp_{1-\alpha} B_s)\, d\mu(s)$$

for $0 \leq \alpha \leq 1$.

Proof. Assume that $a, b > 0$ and $X_t, X_s \in \mathscr{A}$ are positive invertible operators. The Heinz inequality [14] asserts that

$$a^{1-\nu} b^\nu + a^\nu b^{1-\nu} \leq a + b \quad \text{for } 0 \leq \nu \leq 1. \tag{3.22}$$

If we replace b by a^{-1} and take $\mu = 2\nu - 1$ (3.22), then we get

$$a^\mu + a^{-\mu} \leq a + a^{-1} \quad \text{for } 0 \leq \mu \leq 1.$$

Replacing a by the positive invertible operator $X_t \otimes X_s^{-1}$ in the above inequality, we get

$$X_t^\mu \otimes X_s^{-\mu} + X_t^{-\mu} \otimes X_s^\mu \leq X_t \otimes X_s^{-1} + X_t^{-1} \otimes X_s. \tag{3.23}$$

Multiplying both sides of (3.23) by the positive invertible operator $X_t^{\frac{1}{2}} \otimes X_s^{\frac{1}{2}}$, we have

$$X_t^{1+\mu} \otimes X_s^{1-\mu} + X_t^{1-\mu} \otimes X_s^{1+\mu} \leq X_t^2 \otimes I_{\mathscr{A}} + I_{\mathscr{A}} \otimes X_s^2.$$

Now, replacing μ by $2\alpha-1$, X_t by $X_t^{\frac{1}{2}}$, and X_s by $X_s^{\frac{1}{2}}$, respectively, in the above inequality, we get

$$X_t^{\alpha}\otimes X_s^{1-\alpha}+X_t^{1-\alpha}\otimes X_s^{\alpha}\leq X_t\otimes I_{\mathscr{A}}+I_{\mathscr{A}}\otimes X_s \quad \text{for } 0\leq\alpha\leq 1.$$

Now, setting $X_t=A_t^{-\frac{1}{2}}B_tA_t^{-\frac{1}{2}}$ and $X_s=A_s^{-\frac{1}{2}}B_sA_s^{-\frac{1}{2}}$, and then, multiplying by $A_t^{\frac{1}{2}}\otimes A_s^{\frac{1}{2}}$, in the above inequality, we get

$$(A_t\sharp_\alpha B_t)\otimes(A_s\sharp_{1-\alpha}B_s)+(A_t\sharp_{1-\alpha}B_t)\otimes(A_s\sharp_\alpha B_s)\leq A_t\otimes B_s+B_s\otimes A_t.$$

Therefore, for the Hadamard product, we have

$$(A_t\sharp_\alpha B_t)\circ(A_s\sharp_{1-\alpha}B_s)+(A_t\sharp_{1-\alpha}B_t)\circ(A_s\sharp_\alpha B_s)\leq A_t\circ B_s+B_s\circ A_t.$$

Taking the double integral over the above inequality, we get

$$\begin{aligned}&\int_\Omega\int_\Omega((A_t\sharp_\alpha B_t)\circ(A_s\sharp_{1-\alpha}B_s)+(A_t\sharp_{1-\alpha}B_t)\circ(A_s\sharp_\alpha B_s))\,d\mu(t)\,d\mu(s)\\ &\leq\int_\Omega\int_\Omega(A_t\circ B_s+B_s\circ A_t)\,\,d\mu(t)\,d\mu(s).\end{aligned}$$

Using Lemma 3.4, we have

$$\begin{aligned}&\int_\Omega\int_\Omega\big((A_t\sharp_\alpha B_t)\circ(A_s\sharp_{1-\alpha}B_s)+(A_t\sharp_{1-\alpha}B_t)\circ(A_s\sharp_\alpha B_s)\big)\,d\mu(t)\,d\mu(s)\\ =&\int_\Omega(A_t\sharp_\alpha B_t)\,d\mu(t)\circ\int_\Omega(A_s\sharp_{1-\alpha}B_s)\,d\mu(s)+\int_\Omega(A_t\sharp_{1-\alpha}B_t)\,d\mu(t)\circ\int_\Omega(A_s\sharp_\alpha B_s)\,d\mu(s)\\ =&2\int_\Omega(A_t\sharp_\alpha B_t)\,d\mu(t)\circ\int_\Omega(A_s\sharp_{1-\alpha}B_s)\,d\mu(s)\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\int_\Omega(A_t\circ B_s+B_s\circ A_t)\,d\mu(t)\,d\mu(s)\\ =&\left(\int_\Omega A_t\mu(t)\circ\int_\Omega B_s\,d\mu(s)\right)+\left(\int_\Omega B_s\mu(s)\circ\int_\Omega A_t\,d\mu(t)\right)\\ =&2\int_\Omega A_t\mu(t)\circ\int_\Omega B_s\,d\mu(s).\end{aligned}$$

Hence, we get

$$\int_\Omega(A_t\sharp_\alpha B_t)\,d\mu(t)\circ\int_\Omega(A_s\sharp_{1-\alpha}B_s)\,d\mu(s)\leq\int_\Omega A_t\mu(t)\circ\int_\Omega B_s\,d\mu(s),$$

as required. □

Remark 3.7. *In the discrete case $\Omega=\{1,\cdots,n\}$, for positive invertible operators $A_1,\cdots,A_n$ and $B_1,\cdots,B_n$, Theorem 3.8 enforces the inequality (1.4) for the Hadamard product.*

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