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AN OSTROWSKI TYPE TENSORIAL NORM INEQUALITY FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space. Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I} |f'(t)| < \infty$ and A, B are selfadjoint operators with $Sp(A)$, $Sp(B) \subset I$, then

$$\left\| f((1-\lambda) A \otimes 1 + \lambda 1 \otimes B) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) \, du \right\|$$
$$\leq \|f'\|_{I,\infty} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|1 \otimes B - A \otimes 1\|$$

for $\lambda \in [0,1]$. In particular, we have the midpoint inequality

$$\left\| f\left(\frac{A \otimes 1 + 1 \otimes B}{2} \right) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) \, du \right\|$$
$$\leq \frac{1}{4} \|f'\|_{I,\infty} \|1 \otimes B - A \otimes 1\|.$$

1. Introduction

In 1938, A. Ostrowski [13], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

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for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

If we take $x = \frac{a+b}{2}$, we get the *midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{4}\left\|f'\right\|_\infty (b-a),$$

with $\frac{1}{4}$ as best possible constant.

In order to extend this result for tensorial products of selfadjoint operators and norms, we need the following preparations.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.2}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \ge (\le)\, f(s)\, f(t) \text{ for all } s,t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [10, p. 173]

$$f(A \otimes B) \ge (\le)\, f(A) \otimes f(B) \text{ for all } A,\ B \ge 0. \tag{1.3}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.4}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B=B\#A \text{ and } (A\#B)\otimes(B\#A)=(A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [14] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B) &\leq \frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right] \qquad (1.5)\\ &\leq \frac{1}{2}(A\otimes B+B\otimes A)\end{aligned}$$

for $A,\ B>0$ and $\alpha\in[0,1]$. For other similar results, see [1], [3] and [8]-[11].

Motivated by the above results, if f is continuously differentiable on I with $\|f'\|_{I,\infty}:=\sup_{t\in I}|f'(t)|<\infty$ and $A,\ B$ are selfadjoint operators with $Sp(A)$, $Sp(B)\subset I$, then

$$\begin{aligned}&\left\|f((1-\lambda)A\otimes 1+\lambda 1\otimes B)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\|\\ &\leq \|f'\|_{I,\infty}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\|\end{aligned}$$

for $\lambda\in[0,1]$.

In particular, we have the midpoint inequality

$$\begin{aligned}&\left\|f\left(\frac{A\otimes 1+1\otimes B}{2}\right)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\|\\ &\leq \frac{1}{4}\|f'\|_{I,\infty}\|1\otimes B-A\otimes 1\|.\end{aligned}$$

2. Main Results

Recall the following property of the tensorial product

$$(AC)\otimes(BD)=(A\otimes B)(C\otimes D) \qquad (2.1)$$

that holds for any $A,B,C,D\in B(H)$.

If we take $C=A$ and $D=B$, then we get

$$A^2\otimes B^2=(A\otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n\otimes B^n=(A\otimes B)^n \text{ for natural } n\geq 0. \qquad (2.2)$$

In particular

$$A^n\otimes 1=(A\otimes 1)^n \text{ and } 1\otimes B^n=(1\otimes B)^n \qquad (2.3)$$

for all $n\geq 0$.

We also observe that, by (2.1), the operators $A\otimes 1$ and $1\otimes B$ are commutative and

$$(A\otimes 1)(1\otimes B)=(1\otimes B)(A\otimes 1)=A\otimes B. \qquad (2.4)$$

Moreover, for two natural numbers $m,\ n$ we have

$$(A\otimes 1)^m(1\otimes B)^n=(1\otimes B)^n(A\otimes 1)^m=A^m\otimes B^n. \qquad (2.5)$$

We have the following representation results for continuous functions:

Lemma 2.1. *Assume A and B are selfadjoint operators with $Sp(A) \subset I$ and $Sp(B) \subset J$. Let f, h be continuous on I, g, k continuous on J and φ continuous on an interval K that contains the sum of the intervals $h(I) + k(J)$, then*

$$(f(A) \otimes 1 + 1 \otimes g(B)) \varphi(h(A) \otimes 1 + 1 \otimes k(B)) \tag{2.6}$$
$$= \int_I \int_J (f(t) + g(s)) \varphi(h(t) + k(s)) dE_t \otimes dF_s,$$

where A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_J s dF(s). \tag{2.7}$$

Proof. By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^n$ with n any natural number.

For natural number $n \geq 1$ we have

$$\mathcal{K} := \int_I \int_J (f(t) + g(s)) (h(t) + k(s))^n dE_t \otimes dF_s \tag{2.8}$$
$$= \int_I \int_J (f(t) + g(s)) \sum_{m=0}^{n} C_n^m [h(t)]^m [k(s)]^{n-m} dE_t \otimes dF_s$$
$$= \sum_{m=0}^{n} C_n^m \int_I \int_J (f(t) + g(s)) [h(t)]^m [k(s)]^{n-m} dE_t \otimes dF_s$$
$$= \sum_{m=0}^{n} C_n^m \left[\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} dE_t \otimes dF_s\right.$$
$$\left.+ \int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} dE_t \otimes dF_s\right].$$

Observe that

$$\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} dE_t \otimes dF_s$$
$$= f(A) [h(A)]^m \otimes [k(B)]^{n-m} = (f(A) \otimes 1) \left([h(A)]^m \otimes [k(B)]^{n-m}\right)$$
$$= (f(A) \otimes 1) ([h(A)]^m \otimes 1) \left(1 \otimes [k(B)]^{n-m}\right)$$
$$= (f(A) \otimes 1) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m}$$

and

$$\int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} dE_t \otimes dF_s$$
$$= [h(A)]^m \otimes \left(g(B) [k(B)]^{n-m}\right) = (1 \otimes g(B)) \left([h(A)]^m \otimes [k(B)]^{n-m}\right)$$
$$= (1 \otimes g(B)) ([h(A)]^m \otimes 1) \left(1 \otimes [k(B)]^{n-m}\right)$$
$$= (1 \otimes g(B)) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m},$$

with $h(A) \otimes 1$ and $1 \otimes k(B)$ commutative.

Therefore

$$\begin{aligned}\mathcal{K} &= (f(A)\otimes 1 + 1\otimes g(B)) \sum_{m=0}^{n} C_n^m (h(A)\otimes 1)^m (1\otimes k(B))^{n-m} \\ &= (f(A)\otimes 1 + 1\otimes g(B)) (h(A)\otimes 1 + 1\otimes k(B))^n ,\end{aligned}$$

for which the commutativity of $h(A)\otimes 1$ and $1\otimes k(B)$ has been employed. □

Theorem 2.2. *Assume that f is continuously differentiable on I, A and B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\begin{aligned}& f((1-\lambda)A\otimes 1 + \lambda 1\otimes B) - \int_0^1 f((1-u)A\otimes 1 + u1\otimes B)\,du \qquad (2.9)\\ &= \lambda^2 (1\otimes B - A\otimes 1) \\ &\times \int_0^1 u f'((1-u\lambda)A\otimes 1 + u\lambda 1\otimes B)\,du \\ &- (1-\lambda)^2 (1\otimes B - A\otimes 1) \\ &\times \int_0^1 u f'(u(1-\lambda)A\otimes 1 + (1-(1-\lambda)u)1\otimes B)\,du,\end{aligned}$$

for all $\lambda \in [0,1]$. In particular, for $\lambda = \frac{1}{2}$, we have the midpoint identity

$$\begin{aligned}& f\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) - \int_0^1 f((1-u)A\otimes 1 + u1\otimes B)\,du \qquad (2.10)\\ &= \frac{1}{4}(1\otimes B - A\otimes 1)\int_0^1 u\left[f'\left(\left(1-\frac{u}{2}\right)A\otimes 1 + \frac{u}{2}1\otimes B\right)\right. \\ &\left.-f'\left(\frac{u}{2}A\otimes 1 + \left(1-\frac{u}{2}\right)1\otimes B\right)\right]du.\end{aligned}$$

Proof. We start to the Montgomery identity for real valued absolutely continuous functions on $[a,b]$ that can be easily proved integrating by parts in the right side of the equality,

$$(b-a)f(x) - \int_a^b f(t)\,dt = \int_a^x (t-a)f'(t)\,dt + \int_x^b (t-b)f'(t)\,dt \qquad (2.11)$$

for $a \le x \le b$.

If we use the change of variable $t = (1-u)a + ux$, then we have $dt = (x-a)\,du$ and

$$\int_a^x (t-a)f'(t)\,dt = (x-a)^2 \int_0^1 uf'((1-u)a+ux)\,du.$$

If we use the change of variable $t = (1-u)x + ub$, then we have $dt = (b-x)\,du$ and

$$\int_x^b (t-b)f'(t)\,dt = -(b-x)^2 \int_0^1 (1-u)f'((1-u)x+ub)\,du.$$

By (2.11) we get

$$
\begin{aligned}
&(b-a) f(x)-(b-a) \int_0^1 f((1-u) a+u b) d u \qquad (2.12)\\
&=(x-a)^2 \int_0^1 u f^{\prime}((1-u) a+u x) d u \\
&-(b-x)^2 \int_0^1(1-u) f^{\prime}((1-u) x+u b) d u .
\end{aligned}
$$

If we take $x=(1-\lambda) a+\lambda b$, $\lambda \in[0,1]$ in (2.12), then we get

$$
\begin{aligned}
&(b-a) f((1-\lambda) a+\lambda b)-(b-a) \int_0^1 f((1-u) a+u b) d u \qquad (2.13)\\
&=(b-a)^2 \lambda^2 \int_0^1 u f^{\prime}((1-u) a+u[(1-\lambda) a+\lambda b]) d u \\
&-(b-a)^2(1-\lambda)^2 \int_0^1(1-u) f^{\prime}((1-u)[(1-\lambda) a+\lambda b]+u b) d u \\
&=(b-a)^2 \lambda^2 \int_0^1 u f^{\prime}((1-u \lambda) a+u \lambda b) d u \\
&-(b-a)^2(1-\lambda)^2 \int_0^1(1-u) f^{\prime}((1-u)(1-\lambda) a+(\lambda+(1-\lambda) u) b) d u .
\end{aligned}
$$

Therefore, for all $a, b \in I$ and $\lambda \in[0,1]$,

$$
\begin{aligned}
&f((1-\lambda) a+\lambda b)-\int_0^1 f((1-u) a+u b) d u \qquad (2.14)\\
&=(b-a) \lambda^2 \int_0^1 u f^{\prime}((1-u \lambda) a+u \lambda b) d u \\
&-(b-a)(1-\lambda)^2 \int_0^1(1-u) f^{\prime}((1-u)(1-\lambda) a+(\lambda+(1-\lambda) u) b) d u \\
&=\lambda^2(b-a) \int_0^1 u f^{\prime}((1-u \lambda) a+u \lambda b) d u \\
&-(1-\lambda)^2(b-a) \int_0^1 u f^{\prime}(u(1-\lambda) a+(1-(1-\lambda) u) b) d u,
\end{aligned}
$$

where for the last equality we change the variable $1-u$ with u in the second previous integral.

Assume that A and B have the spectral resolutions

$$
A=\int_I t d E(t) \text { and } B=\int_I s d F(s) .
$$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$ in (2.15), then we get

$$\begin{aligned}
&\int_I \int_I f((1-\lambda)t+\lambda s)\,dE_t \otimes dF_s \\
&-\int_I \int_I \left(\int_0^1 f((1-u)t+us)\,du\right) dE_t \otimes dF_s \\
&= \lambda^2 \int_I \int_I \left((s-t)\int_0^1 uf'((1-u\lambda)t+u\lambda s)\,du\right) dE_t \otimes dF_s \\
&-(1-\lambda)^2 \\
&\times \int_I \int_I \left((s-t)\int_0^1 uf'(u(1-\lambda)t+(1-(1-\lambda)u)s)\,du\right) dE_t \otimes dF_s,
\end{aligned} \tag{2.15}$$

for all $\lambda \in [0,1]$.

By utilizing the Fubini's theorem and Lemma 2.1 for appropriate choices of the functions involved, we have successively

$$\int_I \int_I f((1-\lambda)t+\lambda s)\,dE_t \otimes dF_s = f((1-\lambda)A\otimes 1+\lambda 1\otimes B),$$

$$\begin{aligned}
&\int_I \int_I \left(\int_0^1 f((1-u)t+us)\,du\right) dE_t \otimes dF_s \\
&= \int_0^1 \left(\int_I \int_I f((1-u)t+us)\,dE_t \otimes dF_s\right) du \\
&= \int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du,
\end{aligned}$$

$$\begin{aligned}
&\int_I \int_I \left((s-t)\int_0^1 uf'((1-u\lambda)t+u\lambda s)\,du\right) dE_t \otimes dF_s \\
&= \int_0^1 u\left(\int_I \int_I (s-t)f'((1-u\lambda)t+u\lambda s)\,dE_t \otimes dF_s\right) du \\
&= (1\otimes B-A\otimes 1)\int_0^1 uf'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\,du
\end{aligned}$$

and

$$\begin{aligned}
&\int_I \int_I \left((s-t)\int_0^1 uf'(u(1-\lambda)t+(1-(1-\lambda)u)s)\,du\right) dE_t \otimes dF_s \\
&= \int_0^1 u\left(\int_I \int_I (s-t)f'((1-\lambda)t+(1-(1-\lambda)u)s)\,dE_t \otimes dF_s\right) du \\
&= (1\otimes B-A\otimes 1)\int_0^1 u\left(f'((1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B)\right) du.
\end{aligned}$$

By employing (2.15), we then get the desired result (2.9). □

Theorem 2.3. *Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I}|f'(t)| < \infty$ and A, B are selfadjoint operators with $Sp(A)$, $Sp(B) \subset I$,* *then*

$$\left\| f\left(\left(1-\lambda \right) A\otimes 1+\lambda 1\otimes B\right) -\int_{0}^{1}f\left(\left(1-u\right) A\otimes 1+u1\otimes B\right) du\right\| \tag{2.16}$$
$$\leq \left\| f^{\prime }\right\| _{I,\infty }\left[\frac{1}{4}+\left(\lambda -\frac{1}{2}\right) ^{2}\right] \left\| 1\otimes B-A\otimes 1\right\|$$

for $\lambda \in \left[0,1\right] .$

In particular, we have the midpoint inequality

$$\left\| f\left(\frac{A\otimes 1+1\otimes B}{2}\right) -\int_{0}^{1}f\left(\left(1-u\right) A\otimes 1+u1\otimes B\right) du\right\| \tag{2.17}$$
$$\leq \frac{1}{4}\left\| f^{\prime }\right\| _{I,\infty }\left\| 1\otimes B-A\otimes 1\right\| .$$

Proof. If we take the operator norm and use the triangle inequality, we get

$$\left\| f\left(\left(1-\lambda \right) A\otimes 1+\lambda 1\otimes B\right) -\int_{0}^{1}f\left(\left(1-u\right) A\otimes 1+u1\otimes B\right) du\right\| \tag{2.18}$$
$$\leq \lambda ^{2}\left\| 1\otimes B-A\otimes 1\right\|$$
$$\times \left\| \int_{0}^{1}uf^{\prime }\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) du\right\|$$
$$+\left(1-\lambda \right) ^{2}\left\| 1\otimes B-A\otimes 1\right\|$$
$$\times \left\| \int_{0}^{1}uf^{\prime }\left(u\left(1-\lambda \right) A\otimes 1+\left(1-\left(1-\lambda \right) u\right) 1\otimes B\right) du\right\| ,$$

for all $\lambda \in \left[0,1\right] .$

By the properties of the integral and norm, we have

$$\left\| \int_{0}^{1}uf^{\prime }\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) du\right\| \tag{2.19}$$
$$\leq \int_{0}^{1}u\left\| f^{\prime }\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) \right\| du$$

and

$$\left\| \int_{0}^{1}uf^{\prime }\left(u\left(1-\lambda \right) A\otimes 1+\left(1-\left(1-\lambda \right) u\right) 1\otimes B\right) du\right\| \tag{2.20}$$
$$\leq \int_{0}^{1}u\left\| f^{\prime }\left(u\left(1-\lambda \right) A\otimes 1+\left(1-\left(1-\lambda \right) u\right) 1\otimes B\right) \right\| du.$$

Observe that, by Lemma 2.1

$$\left\vert f^{\prime }\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) \right\vert =\int_{I}\int_{I}\left\vert f^{\prime }\left(\left(1-u\lambda \right) t+u\lambda s\right) \right\vert dE_{t}\otimes dF_{s}$$

for $u,\lambda \in \left[0,1\right] .$

Since

$$\left\vert f^{\prime }\left(\left(1-u\lambda \right) t+u\lambda s\right) \right\vert \leq \left\| f^{\prime }\right\| _{I,\infty }$$

for $u,\lambda \in \left[0,1\right]$ and $t,s\in I.$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\begin{aligned} &\left| f'\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) \right| \\ &=\int_I \int_I \left| f'\left(\left(1-u\lambda \right) t+u\lambda s\right) \right| dE_t\otimes dF_s\leq \left\Vert f'\right\Vert_{I,\infty} \int_I \int_I dE_t\otimes dF_s \\ &=\left\Vert f'\right\Vert_{I,\infty} \end{aligned} \tag{2.21}$$

for $u,\lambda \in [0,1]$. This implies that

$$\left\Vert f'\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) \right\Vert \leq \left\Vert f'\right\Vert_{I,\infty}$$

for $u,\lambda \in [0,1]$ which gives

$$\int_0^1 u\left\Vert f'\left(\left(1-u\lambda \right) A\otimes 1+u\lambda 1\otimes B\right) \right\Vert du\leq \left\Vert f'\right\Vert_{I,\infty} \int_0^1 udu=\frac{1}{2}\left\Vert f'\right\Vert_{I,\infty}$$

Similarly, we have

$$\int_0^1 u\left\Vert f'\left(u\left(1-\lambda \right) A\otimes 1+\left(1-\left(1-\lambda \right) u\right) 1\otimes B\right) \right\Vert du\leq \frac{1}{2}\left\Vert f'\right\Vert_{I,\infty}.$$

By (2.18)-(2.20) we derive

$$\begin{aligned} &\left\Vert f\left(\left(1-\lambda \right) A\otimes 1+\lambda 1\otimes B\right) -\int_0^1 f\left(\left(1-u\right) A\otimes 1+u1\otimes B\right) du\right\Vert \\ &\leq \frac{1}{2}\left\Vert f'\right\Vert_{I,\infty} \left\Vert 1\otimes B-A\otimes 1\right\Vert \left[\lambda^2+\left(1-\lambda \right)^2\right] \\ &=\left\Vert 1\otimes B-A\otimes 1\right\Vert \left[\frac{1}{4}+\left(\lambda -\frac{1}{2}\right)^2\right] \left\Vert f'\right\Vert_{I,\infty}, \end{aligned}$$

which proves (2.16). □

3. Related Results

We start by the following result:

Theorem 3.1. *Assume that f is continuously differentiable on I with $|f'|$ is convex on I, A and B are selfadjoint operators with $Sp(A)$, $Sp(B) \subset I$, then*

$$\begin{aligned} &\left\Vert f\left(\left(1-\lambda \right) A\otimes 1+\lambda 1\otimes B\right) -\int_0^1 f\left(\left(1-u\right) A\otimes 1+u1\otimes B\right) du\right\Vert \\ &\leq \left\Vert 1\otimes B-A\otimes 1\right\Vert \left[p\left(1-\lambda \right) \left\Vert f'\left(A\right) \right\Vert +p\left(\lambda \right) \left\Vert f'\left(B\right) \right\Vert \right], \end{aligned} \tag{3.1}$$

for $\lambda \in [0,1]$, where

$$p\left(\lambda \right) =\frac{1}{3}\left[\lambda^3-\left(1-\lambda \right)^3\right] +\frac{1}{2}\left(1-\lambda \right)^2,\ \lambda \in [0,1].$$

In particular, for $\lambda = \frac{1}{2}$, we get the midpoint inequality:

$$\begin{aligned} &\left\Vert f\left(\frac{A\otimes 1+1\otimes B}{2}\right) -\int_0^1 f\left(\left(1-u\right) A\otimes 1+u1\otimes B\right) du\right\Vert \\ &\leq \frac{1}{8}\left\Vert 1\otimes B-A\otimes 1\right\Vert \left[\left\Vert f'\left(A\right) \right\Vert +\left\Vert f'\left(B\right) \right\Vert \right]. \end{aligned} \tag{3.2}$$

Proof. Since $|f'|$ is convex on I, then we get

$$|f'((1-u\lambda)t+u\lambda s)| \leq (1-u\lambda)|f'(t)| + u\lambda |f'(s)|$$

for all for $u, \lambda \in [0,1]$ and $t, s \in I$.

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\begin{aligned} &|f'((1-u\lambda)A\otimes 1 + u\lambda 1\otimes B)| \\ &= \int_I \int_I |f'((1-u\lambda)t+u\lambda s)| \, dE_t \otimes dF_s \\ &\leq \int_I \int_I [(1-u\lambda)|f'(t)| + u\lambda |f'(s)|] \, dE_t \otimes dF_s \\ &= (1-u\lambda)|f'(A)| \otimes 1 + u\lambda 1 \otimes |f'(B)| \end{aligned} \tag{3.3}$$

for all for $u, \lambda \in [0,1]$.

If we take the norm in (3.3), then we get

$$\begin{aligned} &\|f'((1-u\lambda)A\otimes 1 + u\lambda 1\otimes B)\| \\ &\leq \|(1-u\lambda)|f'(A)| \otimes 1 + u\lambda 1 \otimes |f'(B)|\| \\ &\leq (1-u\lambda)\||f'(A)| \otimes 1\| + u\lambda \|1 \otimes |f'(B)|\| \\ &= (1-u\lambda)\|f'(A)\| + u\lambda \|f'(B)\|. \end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned} &\int_0^1 u \|f'((1-u\lambda)A\otimes 1 + u\lambda 1\otimes B)\| \, du \\ &\leq \|f'(A)\| \int_0^1 u(1-u\lambda) \, du + \|f'(B)\| \lambda \int_0^1 u^2 du \\ &= \left(\frac{1}{2} - \frac{\lambda}{3}\right)\|f'(A)\| + \|f'(B)\| \frac{\lambda}{3}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_0^1 u \|f'(u(1-\lambda)A\otimes 1 + (1-(1-\lambda)u)1\otimes B)\| \, du \\ &\leq \int_0^1 u [u(1-\lambda)\|f'(A)\| + (1-(1-\lambda)u)\|f'(B)\|] \\ &= \frac{1}{3}(1-\lambda)\|f'(A)\| + \left(\frac{1}{2} - \frac{(1-\lambda)}{3}\right)\|f'(B)\|. \end{aligned}$$

From (2.18) we get

$$\begin{aligned} &\left\| f((1-\lambda)A\otimes 1 + \lambda 1\otimes B) - \int_0^1 f((1-u)A\otimes 1 + u1\otimes B) \, du \right\| \\ &\leq \lambda^2 \|1\otimes B - A\otimes 1\| \left[\left(\frac{1}{2} - \frac{\lambda}{3}\right)\|f'(A)\| + \|f'(B)\| \frac{\lambda}{3}\right] \\ &+ (1-\lambda)^2 \|1\otimes B - A\otimes 1\| \\ &\times \left[\frac{1}{3}(1-\lambda)\|f'(A)\| + \left(\frac{1}{2} - \frac{(1-\lambda)}{3}\right)\|f'(B)\|\right] \end{aligned}$$

$$
\begin{aligned}
&= \|1 \otimes B - A \otimes 1\| \\
&\times \left\{ \lambda^2 \left[\left(\frac{1}{2} - \frac{\lambda}{3} \right) \|f'(A)\| + \|f'(B)\| \frac{\lambda}{3} \right] \right. \\
&\left. + (1-\lambda)^2 \left[\frac{1}{3} (1-\lambda) \|f'(A)\| + \left(\frac{1}{2} - \frac{(1-\lambda)}{3} \right) \|f'(B)\| \right] \right\} \\
&= \|1 \otimes B - A \otimes 1\| \\
&\times \left\{ \left[\frac{1}{3} (1-\lambda)^3 + \lambda^2 \left(\frac{1}{2} - \frac{\lambda}{3} \right) \right] \|f'(A)\| \right. \\
&\left. + \left[\frac{1}{3} \lambda^3 + (1-\lambda)^2 \left(\frac{1}{2} - \frac{1-\lambda}{3} \right) \right] \|f'(B)\| \right\},
\end{aligned}
$$

which gives the desired result (3.1). □

We recall that the function $g : I \to \mathbb{R}$ is *quasi-convex*, if

$$g((1-\lambda)t + \lambda s) \leq \max\{g(t), g(s)\} = \frac{1}{2}(g(t) + g(s) + |g(t) - g(s)|)$$

for all $t, s \in I$ and $\lambda \in [0,1]$.

Theorem 3.2. *Assume that f is continuously differentiable on I with $|f'|$ is quasi-convex on I, A and B are selfadjoint operators with $Sp(A)$, $Sp(B) \subset I$, then*

$$
\begin{aligned}
&\left\| f((1-\lambda) A \otimes 1 + \lambda 1 \otimes B) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) \, du \right\| \quad (3.5) \\
&\leq \frac{1}{2} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|1 \otimes B - A \otimes 1\| \\
&\times (\||f'(A)| \otimes 1 + 1 \otimes |f'(B)|\| + \||f'(A)| \otimes 1 - 1 \otimes |f'(B)|\|)
\end{aligned}
$$

for $\lambda \in [0,1]$.

In particular, we have the midpoint inequality:

$$
\begin{aligned}
&\left\| f\left(\frac{A \otimes 1 + 1 \otimes B}{2} \right) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) \, du \right\| \quad (3.6) \\
&\leq \frac{1}{8} \|1 \otimes B - A \otimes 1\| \\
&\times (\||f'(A)| \otimes 1 + 1 \otimes |f'(B)|\| + \||f'(A)| \otimes 1 - 1 \otimes |f'(B)|\|).
\end{aligned}
$$

Proof. Since $|f'|$ is quasi-convex on I, then we get

$$|f'((1-u\lambda)t + u\lambda s)| \leq \frac{1}{2}(|f'(t)| + |f'(s)| + ||f'(t)| - |f'(s)||)$$

for all for $u, \lambda \in [0,1]$ and $t, s \in I$.

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$
\begin{aligned}
&|f'((1-u\lambda) A \otimes 1 + u\lambda 1 \otimes B)| \quad (3.7) \\
&= \int_I \int_I |f'((1-u\lambda)t + u\lambda s)| \, dE_t \otimes dF_s \\
&\leq \frac{1}{2} \int_I \int_I (|f'(t)| + |f'(s)| + ||f'(t)| - |f'(s)||) \, dE_t \otimes dF_s \\
&= \frac{1}{2} (|f'(A)| \otimes 1 + 1 \otimes |f'(B)| + ||f'(A)| \otimes 1 - 1 \otimes |f'(B)||)
\end{aligned}
$$

for all for $u,\lambda\in[0,1]$.

If we take the norm, then we get

$$\begin{aligned}&\|f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\|\\&\leq\frac{1}{2}\|(|f'(A)|\otimes 1+1\otimes|f'(B)|+||f'(A)|\otimes 1-1\otimes|f'(B)||)\|\\&\leq\frac{1}{2}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)\end{aligned}$$

for all for $u,\lambda\in[0,1]$.

Therefore

$$\begin{aligned}&\int_0^1 u\|f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\|du\\&\leq\frac{1}{2}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)\int_0^1 udu\\&=\frac{1}{4}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)\end{aligned}$$

and, in a similar way

$$\begin{aligned}&\int_0^1 u\|f'(u(1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B)\|du\\&\leq\frac{1}{4}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|).\end{aligned}$$

By utilizing (2.18) we then get

$$\begin{aligned}&\left\|f((1-\lambda)A\otimes 1+\lambda 1\otimes B)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)du\right\|\\&\leq\lambda^2\|1\otimes B-A\otimes 1\|\\&\times\frac{1}{4}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)\\&+(1-\lambda)^2\|1\otimes B-A\otimes 1\|\\&\times\frac{1}{4}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)\\&=\frac{1}{4}\left(\lambda^2+(1-\lambda)^2\right)\|1\otimes B-A\otimes 1\|\\&\times(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)\\&=\frac{1}{2}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\|\\&\times(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|),\end{aligned}$$

which proves the desired inequality (3.5). □

4. Examples

It is known that if U and V are commuting, i.e. $UV=VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V)=\exp(V)\exp(U)=\exp(U+V).$$

Also, if U is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tU)\,dt = U^{-1}\left[\exp(bU) - \exp(aU)\right].$$

Moreover, if U and V are commuting and $V - U$ is invertible, then

$$\begin{aligned}
\int_0^1 \exp((1-s)U + sV)\,ds &= \int_0^1 \exp(s(V-U))\exp(U)\,ds \\
&= \left(\int_0^1 \exp(s(V-U))\,ds\right)\exp(U) \\
&= (V-U)^{-1}\left[\exp(V-U) - I\right]\exp(U) \\
&= (V-U)^{-1}\left[\exp(V) - \exp(U)\right].
\end{aligned}$$

Since the operators $U = A\otimes 1$ and $V = 1\otimes B$ are commutative and if $1\otimes B - A\otimes 1$ is invertible, then

$$\begin{aligned}
&\int_0^1 \exp((1-u)A\otimes 1 + u1\otimes B)\,du \\
&= (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right].
\end{aligned}$$

If A, B are selfadjoint operators with $Sp(A)$, $Sp(B) \subset [m, M]$ and $1\otimes B - A\otimes 1$ is invertible, then by (2.16)

$$\begin{aligned}
&\|\exp((1-\lambda)A\otimes 1 + \lambda 1\otimes B) \\
&- (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\Big\| \\
&\le \exp(M)\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\|,
\end{aligned} \tag{4.1}$$

for $\lambda \in [0, 1]$.

In particular,

$$\begin{aligned}
&\left\|\exp\left(\frac{A\otimes 1 + 1\otimes B}{2}\right)\right. \\
&\left. - (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\right\| \\
&\le \frac{1}{4}\exp(M)\|1\otimes B - A\otimes 1\|.
\end{aligned} \tag{4.2}$$

Since for $f(t) = \exp t$, $t \in \mathbb{R}$, $|f'|$ is convex, then by Theorem 3.1 we get

$$\begin{aligned}
&\|\exp((1-\lambda)A\otimes 1 + \lambda 1\otimes B) \\
&- (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\Big\| \\
&\le \frac{1}{2}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\| \\
&\times (\|\exp(A)\otimes 1 + 1\otimes \exp(B)\| + \|\exp(A)\otimes 1 - 1\otimes \exp(B)\|)
\end{aligned} \tag{4.3}$$

for $\lambda \in [0, 1]$.

In particular,

$$\left\|\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)\right. \tag{4.4}$$
$$\left.-\left(1\otimes B-A\otimes 1\right)^{-1}\left[\exp\left(1\otimes B\right)-\exp\left(A\otimes 1\right)\right]\right\|$$
$$\leq\frac{1}{8}\left\|1\otimes B-A\otimes 1\right\|$$
$$\times\left(\left\|\exp\left(A\right)\otimes 1+1\otimes\exp\left(B\right)\right\|+\left\|\exp\left(A\right)\otimes 1-1\otimes\exp\left(B\right)\right\|\right)$$

provided that $1\otimes B-A\otimes 1$ is invertible.

5. Conclusion

In this paper we established various Ostrowski type tensorial norm inequalities for continuous functions of selfadjoint operators in Hilbert spaces. Some examples for the operator exponential are also given.

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, Lin. Alg. & Appl. 26 (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, Proc. Amer. Math. Soc. 128 (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, Math. Japon. 42 (1995), 265-272.

[4] N. S. Barnett, P. Cerone and S. S. Dragomir, Some new inequalities for Hermite-Hadamard divergence in information theory. in Stochastic Analysis and Applications. Vol. 3, 7–19, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint RGMIA Res. Rep. Coll. 5 (2002), No. 4, Art. 8, 11 pp. [Online https://rgmia.org/papers/v5n4/NIHHDIT.pdf]

[5] S. S. Dragomir, Bounds for the normalized Jensen functional, Bull. Austral. Math. Soc. 74(3)(2006), 417-478.

[6] S. S. Dragomir, Some tensorial Hermite-Hadamard type inequalities for convex functions of selfadjoint operators in Hilbert spaces, Preprint RGMIA Res. Rep. Coll. 25 (2022), Art. 90, 14 pp. [Online https://rgmia.org/papers/v25/v25a90.pdf]

[7] A. Korányi. On some classes of analytic functions of several variables. Trans. Amer. Math. Soc., 101 (1961), 520–554.

[8] A. Ebadian, I. Nikoufar and M. E. Gordji, Perspectives of matrix convex functions, Proc. Natl. Acad. Sci. USA, 108 (2011), no. 18, 7313–7314.

[9] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. Math. Jpn. 41 (1995), 531-535

[10] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space, Element, Zagreb, 2005.

[11] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, Scient. Math.1 (1998), No. 2, 237-241.

[12] I. Nikoufar and M. Shamohammadi, The converse of the Loewner–Heinz inequality via perspective, Lin. & Multilin. Alg., 66 (2018), N0. 2, 243–249.

[13] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, Comment. Math. Hel, 10 (1938), 226-227.

[14] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, Lin. Alg. & Appl. 420 (2007), 433-440.

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NEW WAVE SOLUTIONS OF TIME FRACTIONAL CHAFEE-INFANTE EQUATION WITH BETA DERIVATIVE

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Abstract. In this article, we discuss the exact solutions forthe Chafee-Infante equation involving beta fractional derivative. Beta fractional derivative which is a local derivative, is a modification of conformable fractional derivative. Using the Modified Kudryashov Method, we obtain the general solution of the time fractional Chafee-Infante equation with the help of Wolfram Mathematica. We use chain rule and wave transform to convert the equation into integer order nonlinear ordinary differential equation. Hence, we don't need any discretization, normalization, or reduction. Moreover, 3D graphical representations are given. With the help of these representations, we can have an idea on the physical and geometrical behavior of the solutions.

1. Introduction

Differential equations are used for modeling problems in many areas of science and have been developed from centuries ago. The definition of fractional derivative originated in the correspondence between L'Hospital and Leibniz in 1695, based on their work on derivatives and integrals. L'Hospital asked Leibniz how to extend the integer derivative to a fractional order, which drew the attention of many mathematicians and led to numerous studies on fractional order derivatives. In the fields of mathematics, physics, and engineering, studies on fractional derivatives can provide much more accurate results in modeling and solving problems. Fractional derivatives are used in control theory of dynamic systems [1], formulating and solving viscoelasticity problems [2], modeling the mechanical [3] and electrical properties of materials, describing properties of various materials in electrochemistry [4]. Additionally, it can be observed that fractional calculus is used in fields such as control theory [5], heat conduction [6], electricity [7] and etc.

Recently, many articles have been published using fractional derivatives. For instance, Yalcinkaya et al.[8] have achieved soliton and periodic wave solutions for the long wave equation (SRLW) and Ostrovsky equation (OE) emerging as a model

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in ocean engineering using Beta derivative in their articles. Uddin et al. [9] have investigated the inclined plane waves with the dynamic behaviors of the gradient and evolution of the beta derivative, along with a nonlinear Schrödinger equation (NLSE) in their article. In another study, Ozkan et al. [10] obtained exact solutions for the Benjamin-Bona-Mahony equation and the Schrödinger equation using the F-expansion method. Farah et al. [11] studied the effects of white noise containing Beta derivative.

2. Preliminaries

This section mentions fractional derivatives [12] and provides the Beta derivative [13, 14]. Anomalous diffusion processes are extensively observed in physics, chemistry, and biology. Integer-order fractional diffusion equations have been relied upon to characterize anomalous diffusion processes and have succeeded tremendously. However, it has been found that integer-order fractional diffusion equations cannot characterize certain complex diffusion processes, such as in-homogeneous or heterogeneous diffusion processes. Additionally, when considering diffusion processes in porous media, if the medium structure or external environment changes over time, the model based on the integer-order fractional diffusion equation cannot effectively characterize such cases. Nevertheless, in some biological diffusion processes, the concentration of particles determines the diffusion model. To solve the above problems, it is proposed to use fractional-order diffusion equation models.

In the Riemann-Liouville fractional derivative, an arbitrary function doesn't need to be continuous and differentiable at the origin. In Jumarie's definition which is the modified version of Riemann-Liouville fractional derivative, non continuous function needs to be differentiable; the fractional derivative of a constant is equal to zero, and more importantly, it removes singularity at the origin for all functions.

One of the biggest advantages of the Caputo fractional derivative is that it includes a fully specified initial condition and boundary conditions in formulating the problem. In addition, the derivative of any constant is zero. The Caputo derivative is a fractional operator used in modeling real-world problems.

Locally defined fractional derivatives are useful in studying the fractional differentiability properties of highly irregular functions that are not differentiable anywhere. These derivatives adhere to the Leibniz and chain rules of differentiation.

Fractional analysis can easily express the dependence of system analysis on previous processes comprehensively. However, integer arithmetic is insufficient in describing the past structure of the system due to its locality characteristics. Theoretical models expressed with fractional analysis are more compatible with experimental data than models expressed with integer arithmetic. It has been revealed that when defining complex physical mechanical problems, the model expressed with fractional analysis has a clearer physical meaning and a simpler expression.

Definition 2.1. *Let $a \in \mathbb{R}$ and $g : [a, \infty) \to \mathbb{R}$. The $\beta - th$ order fractional derivative of function g is defined as:*

$$D_t^{\beta}(g(t)) = \begin{cases} \lim_{\varepsilon \to 0} \dfrac{g\left(t + \varepsilon \left(t + \dfrac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - g(t)}{\varepsilon}, & t \geq 0,\ 0 < \beta \leq 1 \\ g(t), & t \geq 0,\ \beta = 0 \end{cases}$$

where Γ is the Gamma function. If the above limit exists, then it can be said as g is $\beta - th$ order differentiable is said to be. Notice that, for $\beta = 1$ $D_t^{\beta}(g(t)) = \dfrac{d}{dt}g(t)$.

Moreover, unlike other fractional derivatives, the β derivative of a function can be defined locally at a given point, like the first-order derivative [12].

3. A Brief Description of the Considered Method

This section summarizes the considered method as follows: First of all the homogeneous balance method which is used to find the homogeneous balance number is going to be expressed

3.1. **Homogeneous Balance Technique.** The homogeneous equilibrium number represents the upper bound of the solution series. In a nonlinear ordinary differential equation, a constant number is obtained between the highest-order linear term and the highest-degree nonlinear term. In an ordinary differential equation, let the highest order linear term be $\dfrac{d^q u}{d\xi^q}$ and the highest order nonlinear term be $u^p \left(\dfrac{d^r u}{d\xi^r}\right)^s$. If $u = \tau^N$ transform is made, homogeneous equilibrium relation is obtained as

$$N + q = Np + s(N + r)$$

where p, q, r, s are positive integers and N is a homogeneous equilibrium number. From this equation, N positive homogeneous equilibrium number is obtained [16].

3.2. **Modified Kudryashov Method.** In this part the Modified Kudryashov Method [15] is expressed briefly. The general form of a fractional nonlinear partial differential equation can be considered as follows:

$$F\left(u, \frac{\delta^\beta u}{\delta t^\beta}, \frac{\delta u}{\delta x}, \frac{\delta^\beta}{\delta t^\beta}\left(\frac{\delta^\beta u}{\delta t^\beta}\right), \frac{\delta^2 u}{\delta x^2}, \frac{\delta}{\delta x}\left(\frac{\delta^\beta u}{\delta t^\beta}\right), ...\right) = 0 \tag{3.1}$$

where $u = u(x,t)$ is the unknown function.

Step 1. With the help of wave transform $\xi = mx + \frac{n\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}$, and chain rule Equation 3.1

$$G(U, U', U'', ...) = 0 \tag{3.2}$$

is reduced to integer order nonlinear ordinary differential equation. Here, G is a polynomial that includes the transformed function U and its derivatives, and the expression U' represents the ordinary derivatives of U with respect to the new independent variable ξ.

Step 2. Assume that the solution of Equation 3.2 is defined as

$$U(\xi) = \sum_{i=1}^{N} a_i \phi^i(\xi) + a_0 \tag{3.3}$$

Here, $a_i (i = 1, 2, ..., N)$ are arbitrary constants to be determined later and N is the positive integer that can be evaluated by using the homogenous balance technique.

In Equation 3.3 the function $\phi(\xi)$ is regarded as the solution of the following differential equation.

$$\phi^{'}(\xi) = (\phi^2(\xi) - \phi(\xi)) \ln(k) \tag{3.4}$$

The general solution of the Equation 3.4 is

$$\phi(\xi) = \frac{1}{1 + dk^{\xi}} \tag{3.5}$$

where $k > 1$.

Step 3. If we substitute Equation 3.3 along with Equation 3.4 into Equation 3.2, and equate the coefficients of the functions $\phi^i(\xi)$ to zero for $a_i(i = 1, 2, ..., N)$, a system of algebraic equations depending on a_i, m, and n, are obtained. If this system is solved using the computer software Mathematica, the desired values can be found.

Step 4. If the obtained values obtained in Step 3 are substituted using Equations 3.3 and 3.5, the exact solutions of the non-linear fractional partial differential equation can be obtained.

4. Application of the Method

In this section, time fractional the Chafee-Infante differential equation [17] is considered where the fractional derivatives are in terms of beta derivative [10]. We use the modified Kudryashov method to find the exact solutions. The time-fractional Chafee-Infante equation can be expressed as follows

$$D_t^{\beta} u - D_x^2 u + \lambda(u^3 - u) = 0 \tag{4.1}$$

By using the chain rule and the wave transform Equation 4.1 turns into an integer order non-linear ordinary differential equation as

$$nU'(\xi) - m^2 U''(\xi) + \lambda((U(\xi))^3 - U(\xi)) = 0 \tag{4.2}$$

The solution of this nonlinear equation can be supposed as

$$U(\xi) = \sum_{i=0}^{N} a_i \phi^i(\xi)$$

where the function $\phi(\xi)$ is the solution of Equation 3.4. If the homogeneous balance principle is used in Equation 4.2 to find N, the following equation arises.

$$N + 2 = 3N$$

Hence we calculate $N = 1$. Hence the solution of Equation 4.2 can be supposed as follows:

$$u(\xi) = a_0 + a_1 \phi(\xi) \tag{4.3}$$

If Equation 4.3 is substituted into equation 4.2 with using Equation 3.4, then arranging the resulting expression with respect to powers of $\phi^i(\xi)$ and equating the coefficients of the corresponding function powers to zero, the following algebraic system of equations is obtained.

$$\begin{gathered} {a_0}^3 \lambda - a_0 \lambda = 0 \\ 3{a_0}^2 a_1 \lambda - a_1 m^2 \log^2(k) - a_1 n \log(k) - a_1 \lambda = 0 \\ 3a_0 {a_1}^2 \lambda + 3a_1 m^2 \log^2(k) + a_1 n \log(k) = 0 \\ {a_1}^3 \lambda - 2a_1 m^2 \log^2(k) = 0 \end{gathered} \tag{4.4}$$

By solving the equation system is solved, the following results are acquired

Set 1. $n = \frac{3\lambda}{2\log(k)}$, $a_0 = -1$, $m = -\frac{\sqrt{\lambda}}{\sqrt{2}\log(k)}$, $a_1 = 1$

Set 2. $n = -\frac{3\lambda}{2\log(k)}$, $a_0 = 0$, $m = -\frac{\sqrt{\lambda}}{\sqrt{2}\log(k)}$, $a_1 = -1$

Set 3. $n = -\frac{3\lambda}{2\log(k)}$, $a_0 = 0$, $m = -\frac{\sqrt{\lambda}}{\sqrt{2}\log(k)}$, $a_1 = 1$

Set 4. $n = \frac{3\lambda}{2\log(k)}$, $a_0 = 1$, $m = -\frac{\sqrt{\lambda}}{\sqrt{2}\log(k)}$, $a_1 = -1$
Set 5. $n = \frac{3\lambda}{2\log(k)}$, $a_0 = -1$, $m = \frac{\sqrt{\lambda}}{\sqrt{2}\log(k)}$, $a_1 = 1$
Set 6. $n = -\frac{3\lambda}{2\log(k)}$, $a_0 = 0$, $m = \frac{\sqrt{\lambda}}{\sqrt{2}\log(k)}$, $a_1 = -1$
Set 7. $n = -\frac{3\lambda}{2\log(k)}$, $a_0 = 0$, $m = \frac{\sqrt{\lambda}}{\sqrt{2}\log(k)}$, $a_1 = 1$
Set 8. $n = \frac{3\lambda}{2\log(k)}$, $a_0 = 1$, $m = \frac{\sqrt{\lambda}}{\sqrt{2}\log(k)}$, $a_1 = -1$

If the obtained results are put into Equation 4.3 by using wave transform and Equality 3.5, general solutions of Time-Fractional Chafee-Infante Equation 4.1 are obtained as follows:

$$u_{1,2}(x,t) = \pm \frac{1}{dk^{\frac{3\lambda\left(\frac{1}{\Gamma(\beta)}+t\right)^{\beta}}{2\beta\log(k)} - \frac{\sqrt{\lambda}x}{\sqrt{2}\log(k)}} + 1} \mp 1$$

$$u_{3,4}(x,t) = \mp \frac{1}{dk^{-\frac{3\lambda\left(\frac{1}{\Gamma(\beta)}+t\right)^{\beta}}{2\beta\log(k)} - \frac{\sqrt{\lambda}x}{\sqrt{2}\log(k)}} + 1}$$

$$u_{5,6}(x,t) = \pm \frac{1}{dk^{\frac{3\lambda\left(\frac{1}{\Gamma(\beta)}+t\right)^{\beta}}{2\beta\log(k)} + \frac{\sqrt{\lambda}x}{\sqrt{2}\log(k)}} + 1} \mp 1$$

$$u_{7,8}(x,t) = \mp \frac{1}{dk^{\frac{\sqrt{\lambda}x}{\sqrt{2}\log(k)} - \frac{3\lambda\left(\frac{1}{\Gamma(\beta)}+t\right)^{\beta}}{2\beta\log(k)}} + 1}$$

5. Graphical Simulation

In this part, some graphical representations of some solutions are given. Figures 1 and 3 show kink soliton solutions, and Figures 2, 5, and 6 manifest singular kink solitons. In this way, we understand that soliton solutions which can be described as a self-reinforcing wave packet that maintains their shape while it propagates at a constant velocity have appeared.

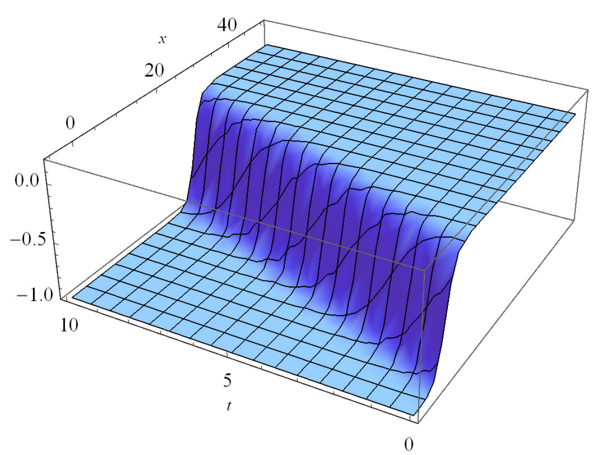

Figure 1. Graphichal representation of $u_1(x,t)$ for $\lambda = 1.7, \beta = 0.9, k = 2.7$, and $d = 2$

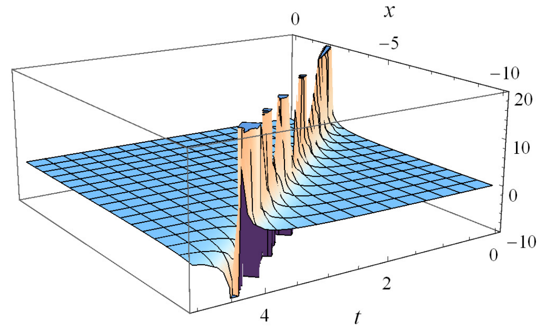

FIGURE 2. Graphichal representation of $u_2(x,t)$ for $\beta = 0.9, \lambda = 0.9, k = 2$, and $d = -1$

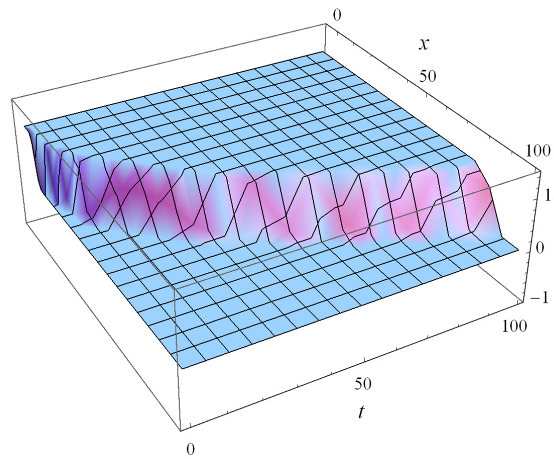

FIGURE 3. Graphichal representation of $u_4(x,t)$ for $\beta = 0.5, \lambda = 3.9, k = 9.9$, and $d = 10$

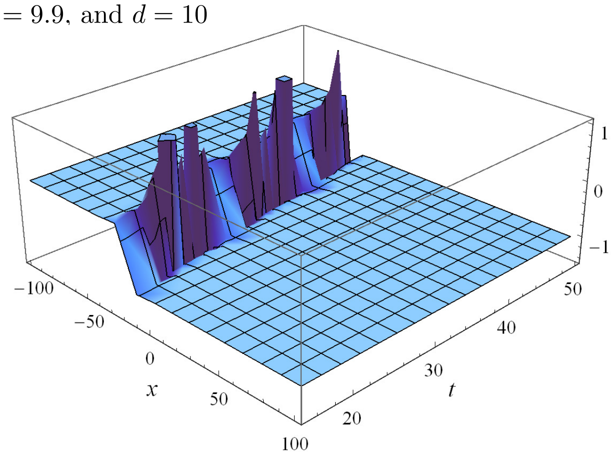

FIGURE 4. Graphichal representation of $u_5(x,t)$ for $\beta = 0.9, \lambda = 0.5, k = 1.9$, and $d = -1.8$

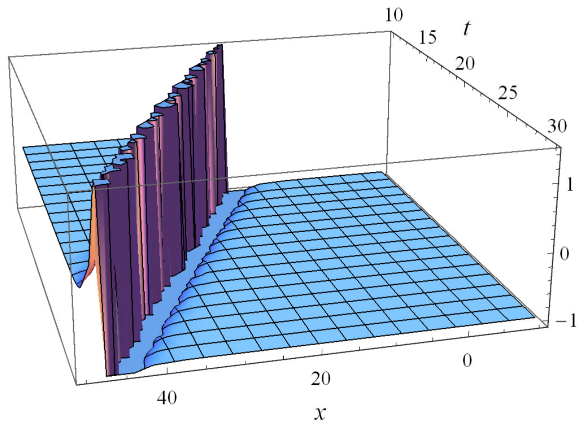

FIGURE 5. Graphichal representation of $u_6(x,t)$ for $\beta = 0.9, \lambda = 0.9, k = -3.9$, and $d = -3.5$

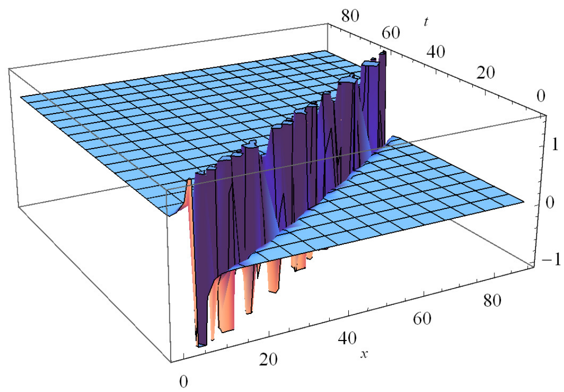

FIGURE 6. Graphichal representation of $u_7(x,t)$ for $\beta = 0.9, \lambda = 0.9, k = 20$, and $d = -0.5$

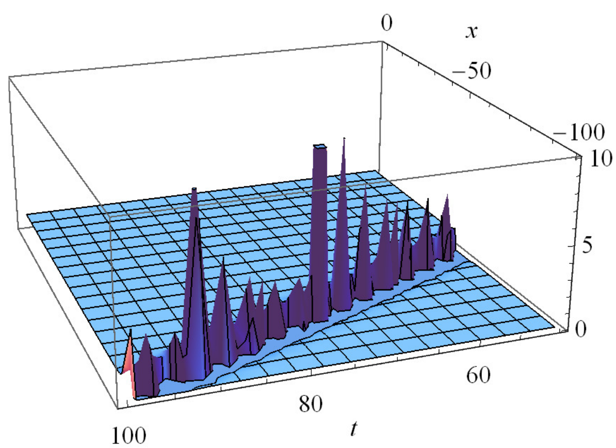

FIGURE 7. Graphichal representation of $u_8(x,t)$ for $\beta = 0.9, \lambda = 0.5, k = 2$, and $d = -0.1$

6. Conclusion

In this article, the analytical solutions for the time-fractional Chafee-Infante equation are obtained by using the Modified Kudryashov Method. Wave transform and chain rule give a chance to convert the fractional nonlinear partial differential equation into integer order nonlinear ordinary differential equation. Many methods can be applied to fractional partial differential equations which are suitable for integer-order ordinary differential equations. In this way, we can understand the nature of the solutions and the physical behavior of the solutions of considered equations. This may cause different insights to the scientists studying the mathematical models of real nature problems with fractional terms.

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References

[1] U. Younas, A. R. Seadawy, M. Younis, S. T. R. Rizvi, *Construction of Analytical Wave Solutions to the Conformable Fractional Dynamical System of Ion Sound and Langmuir Waves*, Waves in Random and Complex Media 32 (6) (2022) 2587-2605.

[2] S. Khavale, K. Gaiwad, *Two-Dimensional Generalized Magneto-Thermo-Viscoelasticity Problem for a Spherical Cavity With One Relaxation Time Using Fractional Derivative*, International Journal of Thermodynamics 25 (2) (2022) 89-97.

[3] W. Chen, H. Sun, X. Li, Fractional derivative modeling in mechanics and engineering, 1st Edtion, Springer, Singapore, 2022.

[4] K. B. Oldham, *Fractional Differential Equations in Electrochemistry*, Advances in Engineering Software 41 (1) (2010) 9-12.

[5] B. M. Vinagre, I. Podlubny, A. Hernandez, V. Feliu, *Some Approximations of Fractional Order Operators Used in Control Theory and Applications*, Fractional calculus and applied analysis 3 (3) (2000) 231-248.

[6] M. Khazayinejad, S. S. Nourazar, *On the Effect of Spatial Fractional Heat Conduction in MHD Boundary Layer Flow Using Gr-Fe3O4–H2O Hybrid Nanofluid*, International Journal of Thermal Sciences 172 (2022) 107265.

[7] I. Dassios, T. Kerçi, D. Baleanu, F. Milano, *Fractional-order Dynamical Model for Electricity Markets*, Mathematical Methods in the Applied Sciences 46 (7) (2023) 8349-8361.

[8] İ Yalçınkaya, H. Ahmad, O. Tasbozan, A. Kurt, *Soliton Solutions for Time Fractional Ocean Engineering Models With Beta Derivative*, Journal of Ocean Engineering and Science 7 (5) (2022) 444-448.

[9] M. F. Uddin, M. G. Hafez, S. A. Iqbal, *Dynamical Plane Wave Solutions for the Heisenberg Model of Ferromagnetic Spin Chains With Beta Derivative Evolution and Obliqueness*, Heliyon 8 (3) (2022) e09199 17 pages.

[10] E. M. Ozkan, *New Exact Solutions of Some Important Nonlinear Fractional Partial Differential Equations With Beta Derivative*, Fractal and Fractional 6 (3) (2022) 173.

[11] F. M. Al-Askar, C. Cesarano, W. W. Mohammed, *The Influence of White Noise and the Beta Derivative on the Solutions of the BBM Equation*, Axioms 12 (5) (2023) 447.

[12] A. Atangana, Derivative With a New Parameter: Theory, Methods and Applications, 1st Edition, Academic Press, 2015.

[13] M. A. Iqbal, M. A. Akbar, M. A. Islam, *The nonlinear wave dynamics of fractional foam drainage and Boussinesq equations with Atangana's beta derivative through analytical solutions*, Results in Physics 56 (2024) 107251.

[14] L. Tang, *Dynamical behavior and multiple optical solitons for the fractional Ginzburg–Landau equation with β-derivative in optical fibers*, Optical and Quantum Electronics, 56 (2) (2024) 175.

[15] D. Kumar, A. R. Seadawy, A. K. Joardar, *Modified Kudryashov Method via New Exact Solutions for Some Conformable Fractional Differential Equations Arising in Mathematical Biology*, Chinese Journal of Physics 56 (1) (2018) 75-85.

[16] O. Tasbozan, Y. Çenesiz, A. Kurt, *New Solutions for Conformable Fractional Boussinesq and Combined KdV-mKdV Equations Using Jacobi Elliptic Function Expansion Method*, The European Physical Journal Plus 131 (7) (2016) 1-14.

[17] M. Cina, A. Secer, M. Bayram, *Solving Nonlinear Fractional PDEs Using Novel Wavelet Collocation Method*, New Trends in Mathematical Sciences 10 (1) (2022).

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