FUNDAMENTAL JOURNAL OF MATHEMATICS AND APPLICATIONS

VOLUME VII ISSUE I

www.dergipark.org.tr/en/pub/fujma

ISSN 2645-8845

VOLUME 7 ISSUE 1
ISSN 2645-8845

March 2024
www.dergipark.org.tr/en/pub/fujma

FUNDAMENTAL JOURNAL OF MATHEMATICS AND APPLICATIONS

Contents

Fundamental Journal of Mathematics and Applications, 7(1) (2024), 1-11
Research Paper / Open Access

Fundamental Journal of Mathematics and Applications

ISSN Online: 2645-8845
www.dergipark.org.tr/en/pub/fujma
https://doi.org/10.33401/fujma.1362681

Some Refinements and Reverses of Callebaut's Inequality for Isotonic Functionals via a Result Due to Cartwright and Field

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Article Information

Keywords: Isotonic functionals; Schwarz's inequality; Callebaut's inequality; Integral inequalities; Discrete inequalities

AMS 2020 Classification: 26D15; 26D10

Abstract

In this paper we obtain some refinements and reverses of Callebaut's inequality for isotonic functionals via a result of Young's inequality due to Cartwright and Field.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1, t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalized if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis that enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [1], [2] and [3]). For other inequalities for isotonic functionals see [4], and [5]-[15].
We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second, with $g = \{g_k\}_{k \in E}$ and $p_k \geq 0$, $k \in E$.
We have the following inequality that provides a refinement and a reverse for the celebrated *Young's inequality*

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}} \leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \leq \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}} \tag{1.1}$$

Received: 19-09-2023 **Revised:** 01-12-2023 **Accepted:** 27-12-2023 **Online:** 29-03-2024 **Published:** 31-03-2024

for any $a,\, b>0$ and $\nu\in[0,1]$.
This result was obtained in 1978 by Cartwright and Field [16] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.
The functional version of *Callebaut's inequality* states that

$$A^2(fg)\le A\left(f^{2-\nu}g^{\nu}\right)A\left(f^{\nu}g^{2-\nu}\right)\le A\left(f^2\right)A\left(g^2\right) \tag{1.2}$$

provided that $f^2,\, g^2,\, f^{2-\nu}g^{\nu},\, f^{\nu}g^{2-\nu},\, fg\in L$ for some $\nu\in[0,2]$. For the discrete and integral of one real variable versions see [17].
In this paper we obtain some inequalities for isotonic functionals via the reverse and refinement of Young's inequality (1.1) that are related to the second part of Callebaut's inequality (1.2). Applications for integrals and n-tuples of real numbers are also provided.

2. On Callebaut's Inequality

We have the following result that provides a refinement and reverse of Callebaut's second inequality:

Theorem 2.1. *Let $A,\, B: L\to\mathbb{R}$ be two normalized isotonic functionals. If $f,\, g: E\to\mathbb{R}$ are such that, $f^2,\, g^2,\, \frac{g^4}{f^2},\, f^{2(1-\nu)}g^{2\nu},\, f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$, and*

$$0<m\le\frac{f}{g}\le M<\infty \tag{2.1}$$

for real numbers $M>m>0$, then

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)m^2\left(A\left(\frac{g^4}{f^2}\right)B\left(f^2\right)+A\left(f^2\right)B\left(\frac{g^4}{f^2}\right)-2\right) &\le (1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right)\\ &\quad -A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right)\\ &\le \frac{1}{2}\nu(1-\nu)M^2\left(A\left(\frac{g^4}{f^2}\right)B\left(f^2\right)+A\left(f^2\right)B\left(\frac{g^4}{f^2}\right)-2\right).\end{aligned} \tag{2.2}$$

Proof. Since $ab=\min\{a,b\}\max\{a,b\}$ for any $a,b>0$, then from (1.1) we have

$$\frac{1}{2}\nu(1-\nu)\min\{a,b\}\frac{(b-a)^2}{ab}\le(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\le\frac{1}{2}\nu(1-\nu)\max\{a,b\}\frac{(b-a)^2}{ab},$$

where $\nu\in[0,1]$. This can be written as

$$\frac{1}{2}\nu(1-\nu)\min\{a,b\}\left(\frac{b}{a}+\frac{a}{b}-2\right)\le(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\le\frac{1}{2}\nu(1-\nu)\max\{a,b\}\left(\frac{b}{a}+\frac{a}{b}-2\right), \tag{2.3}$$

for any $a,\, b>0$.
Let $x,\, y\in E$ such that $g(x),\, g(y)\neq 0$. If we use the inequalities (2.3) for

$$a=\frac{f^2(x)}{g^2(x)},\ b=\frac{f^2(y)}{g^2(y)}\in\left[m^2,M^2\right]$$

then we get

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)m^2\left(\frac{f^2(y)}{g^2(y)}\frac{g^2(x)}{f^2(x)}+\frac{f^2(x)}{g^2(x)}\frac{g^2(y)}{f^2(y)}-2\right) &\le (1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}-\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\\ &\le \frac{1}{2}\nu(1-\nu)M^2\left(\frac{f^2(y)}{g^2(y)}\frac{g^2(x)}{f^2(x)}+\frac{f^2(x)}{g^2(x)}\frac{g^2(y)}{f^2(y)}-2\right),\end{aligned} \tag{2.4}$$

where $\nu\in[0,1]$.
If we multiply (2.4) by $g^2(x)g^2(y)$, then we get

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)m^2\left(f^2(y)\frac{g^4(x)}{f^2(x)}+f^2(x)\frac{g^4(y)}{f^2(y)}-2\right) &\le (1-\nu)g^2(y)f^2(x)+\nu f^2(y)g^2(x)\\ &\quad -f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}(x)g^{2\nu}(x)\\ &\le \frac{1}{2}\nu(1-\nu)M^2\left(f^2(y)\frac{g^4(x)}{f^2(x)}+f^2(x)\frac{g^4(y)}{f^2(y)}-2\right),\end{aligned} \tag{2.5}$$

which holds for any $x,\, y\in E$.
Fix $y\in E$. Then by (2.5) we have in the order of L that

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)m^2\left(f^2(y)\frac{g^4}{f^2}+\frac{g^4(y)}{f^2(y)}f^2-2\right) &\leq (1-\nu)g^2(y)f^2+\nu f^2(y)g^2-f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}g^{2\nu} \\ &\leq \frac{1}{2}\nu(1-\nu)M^2\left(f^2(y)\frac{g^4}{f^2}+\frac{g^4(y)}{f^2(y)}f^2-2\right).\end{aligned} \tag{2.6}$$

If we take the functional A in (2.6), then we get

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)m^2\left(f^2(y)A\left(\frac{g^4}{f^2}\right)+\frac{g^4(y)}{f^2(y)}A\left(f^2\right)-2\right) &\leq (1-\nu)g^2(y)A\left(f^2\right)+\nu f^2(y)A\left(g^2\right)-f^{2\nu}(y)g^{2(1-\nu)}(y)A\left(f^{2(1-\nu)}g^{2\nu}\right) \\ &\leq \frac{1}{2}\nu(1-\nu)M^2\left(f^2(y)A\left(\frac{g^4}{f^2}\right)+\frac{g^4(y)}{f^2(y)}A\left(f^2\right)-2\right),\end{aligned}$$

for any $y\in E$.
If we write this inequality in the order of L, then we have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)m^2\left(A\left(\frac{g^4}{f^2}\right)f^2+A\left(f^2\right)\frac{g^4}{f^2}-2\right) &\leq (1-\nu)A\left(f^2\right)g^2+\nu A\left(g^2\right)f^2-A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)} \\ &\leq \frac{1}{2}\nu(1-\nu)M^2\left(A\left(\frac{g^4}{f^2}\right)f^2+A\left(f^2\right)\frac{g^4}{f^2}-2\right),\end{aligned}$$

and by taking the functional B we deduce the desired result (2.2). $\square$

Corollary 2.2. *Let $A:L\to\mathbb{R}$ be a normalized isotonic functional. If $f,\, g:E\to\mathbb{R}$ are such that $f^2,\, g^2,\, \frac{g^4}{f^2},\, f^{2(1-\nu)}g^{2\nu},\, f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and the condition (2.1) holds, then*

$$\begin{aligned}\nu(1-\nu)m^2\left(A\left(\frac{g^4}{f^2}\right)A\left(f^2\right)-1\right) &\leq A\left(f^2\right)A\left(g^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\leq \nu(1-\nu)M^2\left(A\left(\frac{g^4}{f^2}\right)A\left(f^2\right)-1\right).\end{aligned} \tag{2.7}$$

In particular, if $f^2,\, g^2,\, \frac{g^4}{f^2},\, fg\in L$ and the condition (2.1) holds, then

$$\begin{aligned}\frac{1}{4}m^2\left(A\left(\frac{g^4}{f^2}\right)A\left(f^2\right)-1\right) &\leq A\left(f^2\right)A\left(g^2\right)-A^2(fg) \\ &\leq \frac{1}{4}M^2\left(A\left(\frac{g^4}{f^2}\right)A\left(f^2\right)-1\right).\end{aligned} \tag{2.8}$$

The following result also holds:

Theorem 2.3. *Let $A,\, B:L\to\mathbb{R}$ be two normalized isotonic functionals. If $f,\, g:E\to\mathbb{R}$ are such that $f\geq 0,\, g>0,\, f^2,\, g^2,\, \frac{f^4}{g^2},\, f^{2(1-\nu)}g^{2\nu},\, f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and the condition (2.1) holds, then*

$$\begin{aligned}\frac{1}{2M^2}\nu(1-\nu)\left(A\left(g^2\right)B\left(\frac{f^4}{g^2}\right)-2A\left(f^2\right)B\left(f^2\right)+A\left(\frac{f^4}{g^2}\right)B\left(g^2\right)\right) &\leq (1-\nu)A\left(f^2\right)B\left(g^2\right)+\nu A\left(g^2\right)B\left(f^2\right) \\ &\quad -A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\leq \frac{1}{2m^2}\nu(1-\nu) \\ &\quad \times\left(A\left(g^2\right)B\left(\frac{f^4}{g^2}\right)-2A\left(f^2\right)B\left(f^2\right)+A\left(\frac{f^4}{g^2}\right)B\left(g^2\right)\right).\end{aligned} \tag{2.9}$$

Proof. For any $x,\, y\in E$ we have

$$m^2\leq\frac{f^2(x)}{g^2(x)},\frac{f^2(y)}{g^2(y)}\leq M^2.$$

If we use the inequality (1.1) for

$$a=\frac{f^2(x)}{g^2(x)},\; b=\frac{f^2(y)}{g^2(y)},$$

then we get

$$\begin{aligned}\frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^2(y)}{g^2(y)}-\frac{f^2(x)}{g^2(x)}\right)^2 &\le (1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}-\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\\ &\le \frac{1}{2m^2}\nu(1-\nu)\left(\frac{f^2(y)}{g^2(y)}-\frac{f^2(x)}{g^2(x)}\right)^2\end{aligned}$$

for any x, $y\in E$.
This can be written as

$$\begin{aligned}\frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^4(y)}-2\frac{f^2(y)}{g^2(y)}\frac{f^2(x)}{g^2(x)}+\frac{f^4(x)}{g^4(x)}\right) &\le (1-\nu)\frac{f^2(x)}{g^2(x)}+\nu\frac{f^2(y)}{g^2(y)}-\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu}\left(\frac{f^2(y)}{g^2(y)}\right)^{\nu} \quad (2.10)\\ &\le \frac{1}{2m^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^4(y)}-2\frac{f^2(y)}{g^2(y)}\frac{f^2(x)}{g^2(x)}+\frac{f^4(x)}{g^4(x)}\right).\end{aligned}$$

Now, if we multiply (2.10) by $g^2(x)g^2(y)>0$ then we get

$$\begin{aligned}\frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}g^2(x)-2f^2(y)f^2(x)+\frac{f^4(x)}{g^2(x)}g^2(y)\right) &\le (1-\nu)g^2(y)f^2(x)+\nu f^2(y)g^2(x) \quad (2.11)\\ &\quad -f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}(x)g^{2\nu}(x)\\ &\le \frac{1}{2m^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}g^2(x)-2f^2(y)f^2(x)+\frac{f^4(x)}{g^2(x)}g^2(y)\right)\end{aligned}$$

for any x, $y\in E$.
Fix $y\in E$. Then by (2.11) we have in the order of L that

$$\begin{aligned}\frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}g^2-2f^2(y)f^2+g^2(y)\frac{f^4}{g^2}\right) &\le (1-\nu)g^2(y)f^2+\nu f^2(y)g^2-f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}g^{2\nu} \quad (2.12)\\ &\le \frac{1}{2m^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}g^2-2f^2(y)f^2+g^2(y)\frac{f^4}{g^2}\right).\end{aligned}$$

If we take the functional A in (2.12), then we get

$$\begin{aligned}\frac{1}{2M^2}\nu(1-\nu)\left(\frac{f^4(y)}{g^2(y)}A(g^2)-2f^2(y)A(f^2)+g^2(y)A\left(\frac{f^4}{g^2}\right)\right) &\le (1-\nu)g^2(y)A(f^2)+\nu f^2(y)A(g^2) \quad (2.13)\\ &\quad -f^{2\nu}(y)g^{2(1-\nu)}(y)A\left(f^{2(1-\nu)}g^{2\nu}\right)\\ &\le \frac{1}{2m^2}\nu(1-\nu)\\ &\quad\times\left(\frac{f^4(y)}{g^2(y)}A(g^2)-2f^2(y)A(f^2)+g^2(y)A\left(\frac{f^4}{g^2}\right)\right)\end{aligned}$$

for any $y\in E$.
This inequality can be written in the order of L as

$$\begin{aligned}\frac{1}{2M^2}\nu(1-\nu)\left(A(g^2)\frac{f^4}{g^2}-2A(f^2)f^2+A\left(\frac{f^4}{g^2}\right)g^2\right) &\le (1-\nu)A(f^2)g^2+\nu A(g^2)f^2-A\left(f^{2(1-\nu)}g^{2\nu}\right)f^{2\nu}g^{2(1-\nu)} \quad (2.14)\\ &\le \frac{1}{2m^2}\nu(1-\nu)\left(A(g^2)\frac{f^4}{g^2}-2A(f^2)f^2+A\left(\frac{f^4}{g^2}\right)g^2\right).\end{aligned}$$

Now, if we take the functional B in (2.14), then we get the desired result (2.9). □

Corollary 2.4. *Let $A:L\to\mathbb{R}$ be a normalized isotonic functional. If f, $g:E\to\mathbb{R}$ are such that $f\ge 0$, $g>0$, f^2, g^2, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L$ for some $\nu\in[0,1]$ and the condition (2.1) is valid, then*

$$\begin{aligned}\frac{1}{M^2}\nu(1-\nu)\left(A(g^2)A\left(\frac{f^4}{g^2}\right)-A^2(f^2)\right) &\le A(f^2)A(g^2)-A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right) \quad (2.15)\\ &\le \frac{1}{m^2}\nu(1-\nu)\left(A(g^2)A\left(\frac{f^4}{g^2}\right)-A^2(f^2)\right).\end{aligned}$$

In particular, if $f^2,\ g^2,\ \frac{f^4}{g^2},\ fg \in L$ *and the condition (2.1) is valid, then we have*

$$\begin{aligned} \frac{1}{4M^2}\left(A\left(g^2\right)A\left(\frac{f^4}{g^2}\right)-A^2\left(f^2\right)\right) &\leq A\left(f^2\right)A\left(g^2\right)-A^2\left(fg\right) \\ &\leq \frac{1}{4m^2}\left(A\left(g^2\right)A\left(\frac{f^4}{g^2}\right)-A^2\left(f^2\right)\right). \end{aligned} \tag{2.16}$$

3. Other Related Results

If we write the inequality (1.1) for $a=1$ and $b=x$ we get

$$\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{x,1\}} \leq 1-\nu+\nu x-x^\nu \leq \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{x,1\}} \tag{3.1}$$

for any $x>0$ and for any $\nu\in[0,1]$.
If $x\in[t,T]\subset(0,\infty)$, then $\max\{x,1\}\leq\max\{T,1\}$ and $\min\{t,1\}\leq\min\{x,1\}$ and by (3.1) we get

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)\frac{\min_{x\in[t,T]}(x-1)^2}{\max\{T,1\}} &\leq \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{T,1\}} \\ &\leq 1-\nu+\nu x-x^\nu \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{t,1\}} \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{\max_{x\in[t,T]}(x-1)^2}{\min\{t,1\}} \end{aligned} \tag{3.2}$$

for any $x\in[t,T]$ and for any $\nu\in[0,1]$.
Observe that

$$\min_{x\in[t,T]}(x-1)^2=\begin{cases}(T-1)^2 \text{ if } T<1,\\ 0 \text{ if } t\leq 1\leq T,\\ (t-1)^2 \text{ if } 1<t\end{cases}$$

and

$$\max_{x\in[t,T]}(x-1)^2=\begin{cases}(t-1)^2 \text{ if } T<1,\\ \max\left\{(t-1)^2,(T-1)^2\right\} \text{ if } t\leq 1\leq T,\\ (T-1)^2 \text{ if } 1<t.\end{cases}$$

Then

$$c(t,T):=\frac{\min_{x\in[t,T]}(x-1)^2}{\max\{T,1\}}=\begin{cases}(T-1)^2 \text{ if } T<1,\\ 0 \text{ if } t\leq 1\leq T,\\ \frac{(t-1)^2}{T} \text{ if } 1<t\end{cases} \tag{3.3}$$

and

$$C(t,T):=\frac{\max_{x\in[t,T]}(x-1)^2}{\min\{t,1\}}=\begin{cases}\frac{(t-1)^2}{t} \text{ if } T<1,\ \frac{1}{t}\max\left\{(t-1)^2,(T-1)^2\right\} \text{ if } t\leq 1\leq T,\\ (T-1)^2 \text{ if } 1<t.\end{cases} \tag{3.4}$$

Using the inequality (3.2) we have

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)c(t,T) &\leq \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{T,1\}}\leq 1-\nu+\nu x-x^\nu \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{t,1\}}\leq\frac{1}{2}\nu(1-\nu)C(t,T) \end{aligned} \tag{3.5}$$

for any $x\in[t,T]$ and for any $\nu\in[0,1]$.
Now, if $a,\ b>0$ and assume that $\frac{b}{a}\in[t,T]$, then by (3.5) we get

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)c(t,T)a &\leq \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{T,1\}a} \\ &\leq (1-\nu)a+\nu b-b^\nu a^{1-\nu} \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{t,1\}a}\leq\frac{1}{2}\nu(1-\nu)C(t,T)a \end{aligned} \tag{3.6}$$

for any $\nu\in[0,1]$, where $c(t,T)$ and $C(t,T)$ are defined by (3.3) and (3.4), respectively.

Theorem 3.1. *Let A, $B : L \to \mathbb{R}$ be two normalized isotonic functionals. If f, $g : E \to \mathbb{R}$ are such that, f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and the condition (2.1) holds, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(A\left(\frac{g^4}{f^2}\right)B\left(\frac{f^4}{g^2}\right) - 2A\left(g^2\right)B\left(f^2\right) + A\left(f^2\right)B\left(g^2\right)\right) \\ &\leq (1-\nu)A\left(f^2\right)B\left(g^2\right) + \nu A\left(g^2\right)B\left(f^2\right) - A\left(f^{2(1-\nu)}g^{2\nu}\right)B\left(f^{2\nu}g^{2(1-\nu)}\right) \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(A\left(\frac{g^4}{f^2}\right)B\left(\frac{f^4}{g^2}\right) - 2A\left(g^2\right)B\left(f^2\right) + A\left(f^2\right)B\left(g^2\right)\right) \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2 A\left(f^2\right)B\left(g^2\right). \end{aligned} \tag{3.7}$$

Proof. For any x, $y \in E$ we have

$$m^2 \leq \frac{f^2(x)}{g^2(x)}, \frac{f^2(y)}{g^2(y)} \leq M^2.$$

Consider

$$a = \frac{f^2(x)}{g^2(x)},\ b = \frac{f^2(y)}{g^2(y)},$$

then $\frac{b}{a} \in \left[\frac{m^2}{M^2}, \frac{M^2}{m^2}\right]$ and by (3.6) we get

$$\begin{aligned} 0 &\leq \frac{1}{2}\nu(1-\nu)\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2}{\frac{M^2}{m^2}\frac{f^2(x)}{g^2(x)}} \\ &\leq (1-\nu)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu} \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2}{\frac{m^2}{M^2}\frac{f^2(x)}{g^2(x)}} \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\max\left\{\left(\frac{m^2}{M^2}-1\right)^2, \left(\frac{M^2}{m^2}-1\right)^2\right\}\frac{f^2(x)}{g^2(x)} \\ &= \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2\frac{f^2(x)}{g^2(x)} \end{aligned}$$

for any x, $y \in E$ and $\nu \in [0,1]$.
This inequality is equivalent to

$$\begin{aligned} 0 &\leq \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2 g^2(x)}{f^2(x)} \\ &\leq (1-\nu)\frac{f^2(x)}{g^2(x)} + \nu\frac{f^2(y)}{g^2(y)} - \left(\frac{f^2(y)}{g^2(y)}\right)^{\nu}\left(\frac{f^2(x)}{g^2(x)}\right)^{1-\nu} \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\frac{\left(\frac{f^2(y)}{g^2(y)} - \frac{f^2(x)}{g^2(x)}\right)^2 g^2(x)}{f^2(x)} \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2\frac{f^2(x)}{g^2(x)} \end{aligned} \tag{3.8}$$

for any $x, y \in E$ and $\nu \in [0,1]$.

Now, if we multiply (3.8) by $g^2(x)g^2(y) > 0$ then we get

$$\begin{aligned}
0 &\leq \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\frac{\left(\frac{f^2(y)}{g^2(y)}-\frac{f^2(x)}{g^2(x)}\right)^2 g^4(x)g^2(y)}{f^2(x)} \\
&\leq (1-\nu)g^2(y)f^2(x)+\nu f^2(y)g^2(x)-f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}(x)g^{2\nu}(x) \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\frac{\left(\frac{f^2(y)}{g^2(y)}-\frac{f^2(x)}{g^2(x)}\right)^2 g^4(x)g^2(y)}{f^2(x)} \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2 f^2(x)g^2(y)
\end{aligned} \tag{3.9}$$

for any $x, y \in E$ and $\nu \in [0,1]$.
Observe that

$$\begin{aligned}
\frac{\left(\frac{f^2(y)}{g^2(y)}-\frac{f^2(x)}{g^2(x)}\right)^2 g^4(x)g^2(y)}{f^2(x)} &= \frac{\left(\frac{f^4(y)}{g^4(y)}-2\frac{f^2(y)}{g^2(y)}\frac{f^2(x)}{g^2(x)}+\frac{f^4(x)}{g^4(x)}\right)g^4(x)g^2(y)}{f^2(x)} \\
&= \frac{\frac{f^4(y)g^4(x)}{g^2(y)}-2f^2(y)f^2(x)g^2(x)+f^4(x)g^2(y)}{f^2(x)} \\
&= \frac{f^4(y)g^4(x)}{g^2(y)f^2(x)}-2f^2(y)g^2(x)+f^2(x)g^2(y)
\end{aligned}$$

and by (3.9) we get

$$\begin{aligned}
0 &\leq \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(\frac{f^4(y)g^4(x)}{g^2(y)f^2(x)}-2f^2(y)g^2(x)+f^2(x)g^2(y)\right) \\
&\leq (1-\nu)g^2(y)f^2(x)+\nu f^2(y)g^2(x)-f^{2\nu}(y)g^{2(1-\nu)}(y)f^{2(1-\nu)}(x)g^{2\nu}(x) \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{f^4(y)g^4(x)}{g^2(y)f^2(x)}-2f^2(y)g^2(x)+f^2(x)g^2(y)\right) \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2 f^2(x)g^2(y)
\end{aligned}$$

for any $x, y \in E$ and $\nu \in [0,1]$.
Now, if we use a similar argument to the one from the proof of Theorem 2.1 we deduce the desired result (3.7). □

Corollary 3.2. *Let $A : L \to \mathbb{R}$ be a normalized isotonic functional. If $f, g : E \to \mathbb{R}$ are such that, f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L$ for some $\nu \in [0,1]$ and the condition (2.1) holds, then*

$$\begin{aligned}
0 &\leq \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)-A\left(g^2\right)A\left(f^2\right)\right) \\
&\leq A\left(f^2\right)A\left(g^2\right)-A\left(f^{2(1-\nu)}g^{2\nu}\right)A\left(f^{2\nu}g^{2(1-\nu)}\right) \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)-A\left(g^2\right)A\left(f^2\right)\right) \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2 A\left(f^2\right)A\left(g^2\right).
\end{aligned} \tag{3.10}$$

In particular, if f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $fg \in L$, then we have

$$\begin{aligned}
0 &\leq \frac{1}{8}\frac{m^2}{M^2}\left(A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)-A\left(g^2\right)A\left(f^2\right)\right) \\
&\leq A\left(f^2\right)A\left(g^2\right)-A^2(fg) \\
&\leq \frac{1}{8}\frac{M^2}{m^2}\left(A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)-A\left(g^2\right)A\left(f^2\right)\right) \\
&\leq \frac{1}{8}\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2 A\left(f^2\right)A\left(g^2\right).
\end{aligned} \tag{3.11}$$

We observe that the inequality (3.11) can be written as

$$\begin{aligned} 0 &\leq \frac{1}{8}\frac{m^2}{M^2}\left(\frac{A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)}{A\left(g^2\right)A\left(f^2\right)}-1\right) \leq 1-\frac{A^2\left(fg\right)}{A\left(f^2\right)A\left(g^2\right)} \\ &\leq \frac{1}{8}\frac{M^2}{m^2}\left(\frac{A\left(\frac{g^4}{f^2}\right)A\left(\frac{f^4}{g^2}\right)}{A\left(g^2\right)A\left(f^2\right)}-1\right) \leq \frac{1}{8}\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2. \end{aligned} \tag{3.12}$$

4. Applications for Integrals

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ -a.e. (almost every) $x\in\Omega$ and $p\geq 1$ consider the Lebesgue space

$$L_w^p(\Omega,\mu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|^p w(x)\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.
Let f, g be μ-measurable functions with the property that there exists the constants M, $m>0$ such that

$$0<m\leq\frac{f}{g}\leq M<\infty \ \mu\text{-almost everywhere (a.e.) on } \Omega. \tag{4.1}$$

If f^2, g^2, $\frac{g^4}{f^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L_w(\Omega,\mu)$ for some $\nu\in[0,1]$ and the condition (4.1) holds, then by (2.7) we have

$$\begin{aligned} \nu(1-\nu)m^2\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega wf^2d\mu-1\right) &\leq \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu-\int_\Omega wf^{2(1-\nu)}g^{2\nu}d\mu\int_\Omega wf^{2\nu}g^{2(1-\nu)}d\mu \\ &\leq \nu(1-\nu)M^2\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega wf^2d\mu-1\right). \end{aligned} \tag{4.2}$$

In particular, if f^2, g^2, $\frac{g^4}{f^2}$, $fg\in L_w(\Omega,\mu)$ and the condition (4.1) holds, then

$$\begin{aligned} \frac{1}{4}m^2\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega wf^2d\mu-1\right) &\leq \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu-\left(\int_\Omega wfgd\mu\right)^2 \\ &\leq \frac{1}{4}M^2\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega wf^2d\mu-1\right). \end{aligned} \tag{4.3}$$

If f^2, g^2, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)}\in L_w(\Omega,\mu)$ for some $\nu\in[0,1]$ and the condition (4.1) holds, then by (2.15) we have

$$\begin{aligned} \frac{1}{M^2}\nu(1-\nu)\left(\int_\Omega wg^2d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\left(\int_\Omega wf^2d\mu\right)^2\right) &\leq \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu-\int_\Omega wf^{2(1-\nu)}g^{2\nu}d\mu\int_\Omega wf^{2\nu}g^{2(1-\nu)}d\mu \\ &\leq \frac{1}{m^2}\nu(1-\nu)\left(\int_\Omega wg^2d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\left(\int_\Omega wf^2d\mu\right)^2\right). \end{aligned} \tag{4.4}$$

In particular, if f^2, g^2, $\frac{f^4}{g^2}$, $fg\in L_w(\Omega,\mu)$ and the condition (4.1) is valid, then we have

$$\begin{aligned} \frac{1}{4M^2}\left(\int_\Omega wg^2d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\left(\int_\Omega wf^2d\mu\right)^2\right) &\leq \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu-\left(\int_\Omega wfgd\mu\right)^2 \\ &\leq \frac{1}{4m^2}\left(\int_\Omega wg^2d\mu\int_\Omega w\frac{f^4}{g^2}d\mu-\left(\int_\Omega wf^2d\mu\right)^2\right). \end{aligned} \tag{4.5}$$

If f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $f^{2(1-\nu)}g^{2\nu}$, $f^{2\nu}g^{2(1-\nu)} \in L_w(\Omega,\mu)$ for some $\nu \in [0,1]$, and the condition (4.1) holds, then

$$\begin{aligned} 0 &\le \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega w\frac{f^4}{g^2}d\mu - \int_\Omega wg^2d\mu\int_\Omega wf^2d\mu\right) \\ &\le \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu - \int_\Omega wf^{2(1-\nu)}g^{2\nu}d\mu\int_\Omega wf^{2\nu}g^{2(1-\nu)}d\mu \\ &\le \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega w\frac{f^4}{g^2}d\mu - \int_\Omega wg^2d\mu\int_\Omega wf^2d\mu\right) \\ &\le \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2\int_\Omega wf^2d\mu\int_\Omega wg^2d\mu. \end{aligned} \tag{4.6}$$

In particular, if f^2, g^2, $\frac{g^4}{f^2}$, $\frac{f^4}{g^2}$, $fg \in L_w(\Omega,\mu)$, then we have

$$\begin{aligned} 0 &\le \frac{1}{8}\frac{m^2}{M^2}\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega w\frac{f^4}{g^2}d\mu - \int_\Omega wg^2d\mu\int_\Omega wf^2d\mu\right) \\ &\le \int_\Omega wf^2d\mu\int_\Omega wg^2d\mu - \left(\int_\Omega wfgd\mu\right)^2 \\ &\le \frac{1}{8}\frac{M^2}{m^2}\left(\int_\Omega w\frac{g^4}{f^2}d\mu\int_\Omega w\frac{f^4}{g^2}d\mu - \int_\Omega wg^2d\mu\int_\Omega wf^2d\mu\right) \\ &\le \frac{1}{8}\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2\int_\Omega wf^2d\mu\int_\Omega wg^2d\mu. \end{aligned} \tag{4.7}$$

5. Applications for Real Numbers

We consider the n-tuples of positive numbers $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ and the probability distribution $p=(p_1,...,p_n)$, i.e. $p_i \ge 0$ for any $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$. If there exist the constants m, $M>0$ such that

$$0 < m \le \frac{a_i}{b_i} \le M < \infty \text{ for any } i \in \{1,...,n\}, \tag{5.1}$$

then by (4.2) and (4.3) for the counting discrete measure, we have

$$\begin{aligned} \nu(1-\nu)m^2\left(\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}\sum_{i=1}^n p_ia_i^2 - 1\right) &\le \sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2 - \sum_{i=1}^n p_ia_i^{2(1-\nu)}b_i^{2\nu}\sum_{i=1}^n p_ia_i^{2\nu}b_i^{2(1-\nu)} \\ &\le \nu(1-\nu)M^2\left(\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}\sum_{i=1}^n p_ia_i^2 - 1\right) \end{aligned} \tag{5.2}$$

for any $\nu \in [0,1]$ and

$$\begin{aligned} \frac{1}{4}m^2\left(\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}\sum_{i=1}^n p_ia_i^2 - 1\right) &\le \sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2 - \left(\sum_{i=1}^n p_ia_ib_i\right)^2 \\ &\le \frac{1}{4}M^2\left(\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}\sum_{i=1}^n p_ia_i^2 - 1\right). \end{aligned} \tag{5.3}$$

If $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ satisfy (5.1), then by (4.4) and (4.5) for the counting discrete measure, we have

$$\begin{aligned} \frac{1}{M^2}\nu(1-\nu)\left(\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_i\frac{a_i^4}{b_i^2} - \left(\sum_{i=1}^n p_ia_i^2\right)^2\right) &\le \sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2 - \sum_{i=1}^n p_ia_i^{2(1-\nu)}b_i^{2\nu}\sum_{i=1}^n p_ia_i^{2\nu}b_i^{2(1-\nu)} \\ &\le \frac{1}{m^2}\nu(1-\nu)\left(\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_i\frac{a_i^4}{b_i^2} - \left(\sum_{i=1}^n p_ia_i^2\right)^2\right) \end{aligned} \tag{5.4}$$

for any $\nu \in [0,1]$ and

$$\begin{aligned} \frac{1}{4M^2}\left(\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_i\frac{a_i^4}{b_i^2} - \left(\sum_{i=1}^n p_ia_i^2\right)^2\right) &\le \sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2 - \left(\sum_{i=1}^n p_ia_ib_i\right)^2 \\ &\le \frac{1}{4m^2}\left(\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_i\frac{a_i^4}{b_i^2} - \left(\sum_{i=1}^n p_ia_i^2\right)^2\right). \end{aligned} \tag{5.5}$$

If $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ satisfy (5.1), then by (4.6) and (4.7) for the counting discrete measure, we have

$$\begin{aligned} 0 &\leq \frac{1}{2}\nu(1-\nu)\frac{m^2}{M^2}\left(\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i\frac{a_i^4}{b_i^2}-\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_ia_i^2\right) \\ &\leq \sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2-\sum_{i=1}^n p_ia_i^{2(1-\nu)}b_i^{2\nu}\sum_{i=1}^n p_ia_i^{2\nu}b_i^{2(1-\nu)} \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i\frac{a_i^4}{b_i^2}-\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_ia_i^2\right) \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2\sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2 \end{aligned} \tag{5.6}$$

for any $\nu\in[0,1]$ and

$$\begin{aligned} 0 &\leq \frac{1}{8}\frac{m^2}{M^2}\left(\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i\frac{a_i^4}{b_i^2}-\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_ia_i^2\right) \\ &\leq \sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2-\left(\sum_{i=1}^n p_ia_ib_i\right)^2 \\ &\leq \frac{1}{8}\frac{M^2}{m^2}\left(\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}\sum_{i=1}^n p_i\frac{a_i^4}{b_i^2}-\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_ia_i^2\right) \\ &\leq \frac{1}{8}\frac{M^2}{m^2}\left(\frac{M^2}{m^2}-1\right)^2\sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2. \end{aligned} \tag{5.7}$$

6. Conclusion

In this paper, by making use of some reverses and refinements of Young's inequality (1.1),we obtained some inequalities for isotonic functionals that are related to the second part of Callebaut's inequality (1.2). Natural applications for integrals and n-tuples of real numbers were also provided.

Declarations

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Conflict of Interest Disclosure: The author declares no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

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Fundamental Journal of Mathematics and Applications (FUJMA), (Fundam. J. Math. Appl.)
https://dergipark.org.tr/en/pub/fujma

How to cite this article: S.S. Dragomir, *Some refinements and reverses of Callebaut's inequality for isotonic functionals via a result due to Cartwright and Field*, Fundam. J. Math. Appl., **7**(1) (2024), 1-11. DOI 10.33401/fujma.1362681

Fundamental Journal of Mathematics and Applications, 7(1) (2024), 12-25
Research Paper / Open Access

FUJMA

Fundamental Journal of Mathematics and Applications

ISSN Online: 2645-8845
www.dergipark.org.tr/en/pub/fujma
https://doi.org/10.33401/fujma.1378534

On a Generalized Mittag-Leffler Function and Fractional Integrals

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Article Information

Keywords: Generating functions; Mittag-Leffler function; Riemann-Liouville fractional integrals; Special functions

AMS 2020 Classification: 26A33; 33E12

Abstract

The object of this paper is to study a generalized Mittag-Leffler function and a modified general class of functions which is reducible to several special functions. convergent conditions of these functions are discussed. Some results pertaining to the generalized Mittag-Leffler function and generating relations involving these functions are derived. Further, fractional integrals involving these functions are achieved. Some illustrative exclusive cases of the results are presented.

1. Introduction

In 1906, Barnes [1] presented a function. In 1940, Wright [2] presented another function. In 1971, Prabhakar [3] studied an extended Mittag-Leffler function which is a particular case of the Wright's function. Recently in 2021, Srivastava [4] presented a more general function which is reducible to the Mittag-Leffler function by giving a suitable value to the general function involved therein. In this paper, two more general functions are presented which are reducible to the Srivastava's function defined by (1.6) and generalized Hurwitz-Lerch zeta function [5].
Now, we present some relevant definitions.

Definition 1.1. *A Swedish Scholar namely Magnus Gustaf "Gösta" Mittag-Leffler introduced his function [6], named after his name, as follows:*

$$E_{\delta}(x)=\sum_{c=0}^{\infty}\frac{x^{c}}{\Gamma(\delta c+1)}, \tag{1.1}$$

where $Re(\delta)>0$.

Definition 1.2. *Another Swedish Scholar namely Anders Wiman [7] presented a more general function as follows (see also [8], [9]):*

$$E_{\delta,\,\rho}(x)=\sum_{c=0}^{\infty}\frac{x^{c}}{\Gamma(\delta c+\rho)}, \tag{1.2}$$

where $\delta,\ \rho\in\mathbb{C}$ *and* $Re(\delta)>0$.

It is obvious that when $\rho=1$ in (1.2), it becomes (1.1).

Definition 1.3. *A British Scholar namely Ernest William Barnes [1] presented his function as follows:*

$$E^{\xi}_{\delta,\,\rho}(s;\ x)=\sum_{c=0}^{\infty}\frac{x^{c}}{(c+\xi)^{s}\,\Gamma(\delta c+\rho)}, \tag{1.3}$$

where $\delta,\ \rho\in\mathbb{C}$ *and* $Re(\delta)>0$

Received: 19-10-2023 **Revised:** 13-12-2023 **Accepted:** 27-12-2023 **Online:** 29-03-2024 **Published:** 31-03-2024

It is obvious that when $s = 0$ in (1.3), it becomes (1.2).

Definition 1.4. *Another British Scholar namely Sir Edward Maitland Wright [2], presented his function as follows:*

$$E_{\delta,\,\rho}(\Phi;\, x) = \sum_{c=0}^{\infty} \frac{\Phi(c)}{\Gamma(\delta c+\rho)} x^c, \tag{1.4}$$

where $F(c)$ is a general function and $\delta,\ \rho \in \mathbb{C}$, $Re(\delta) > 0$.

It is obvious that when $\Phi(c) = \frac{1}{(c+\xi)^s}$ in (1.4), it becomes (1.3).

Definition 1.5. *If we substitute $\Phi(c) = \frac{(\zeta)_c}{c!}$ in (1.4), we achieve the extended Mittag-Leffler function as follows:*

$$E^{\zeta}_{\delta,\,\rho}(x) = \sum_{c=0}^{\infty} \frac{(\zeta)_c\, x^c}{\Gamma(\delta c+\rho)\; c!}, \tag{1.5}$$

where $\zeta,\ \delta,\ \rho \in \mathbb{C}$, $Re(\zeta) > 0$, $Re(\delta) > 0$ and

$$(a)_c = \frac{\Gamma(a+c)}{\Gamma(a)},$$

that is

$$(a)_0 = 1,\ \ (a)_c = a(a+1)(a+2)...(a+c-1),$$

where $c = 1,\ 2,\ 3,\ \dots$.

It is obvious that when $\zeta = 1$ in (1.5), it becomes (1.2). Indian Scholar namely Tilak Raj Prabhakar [3] studied (1.5). Some more exclusive cases of (1.4) have been considered and studied, among others, by Kamarujjama et al. [10], Khan and Ahmed [11], [12], Khan and Khan [13], Khan et al. [14], Shukla and Prajapati [15] and Salim [16].

Definition 1.6. *Recently, a Canadian Scholar of Indian origin namely Hari Mohan Srivastava [4], [17] presented his function as follows:*

$$E_{\delta,\,\rho}(\Phi;\, x;\, s,\, \xi) = \sum_{c=0}^{\infty} \frac{\Phi(c)}{(c+\xi)^s\; \Gamma(\delta c+\rho)} x^c, \tag{1.6}$$

where $\delta,\ \rho \in \mathbb{C}$ and $Re(\delta) > 0$.

It is obvious that when $\Phi(c) = \frac{(\zeta)_c}{c!}$, $s = 0$ in (1.6), it becomes (1.5). When $s = 0$ in (1.6), it becomes (1.4) and when $\Phi(c) = (\zeta)_c,\ \delta = 1,\ \rho = 1$, it becomes Goyal-Laddha zeta function [18].

Definition 1.7. *Two More general functions are hereby presented as follows:*

$$E_{\alpha,\,\beta,\,\delta}(x,\ s,\ \rho) = \sum_{c=0}^{\infty} \frac{\phi(c)\, x^c}{(\delta c+\rho)^s\; \Gamma(\alpha c+\beta)}, \tag{1.7}$$

where $\delta,\ \rho,\ s,\ \alpha,\ \beta \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(s) \geq 0$,
and

$$E_{\alpha,\,\beta,\,\delta,\,\eta}(x,\ s,\ \rho) = \sum_{c=0}^{\infty} \frac{\phi(c)\, x^c}{(\rho+\delta c x^{\eta})^s\; \Gamma(\alpha c+\beta)}, \tag{1.8}$$

where $\delta,\ \rho,\ s,\ \alpha,\ \beta \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(s) \geq 0$, $\eta \geq 0$.

It is obvious that when $\delta = 1$ in (1.7), it becomes (1.6) and when $\eta = 0$ in (1.8), it becomes (1.7). If we put $\phi(c) = (\mu)_c,\ \alpha = 1$ and $\beta = 1$ in (1.8), it becomes the generalized Hurwitz-Lerch zeta function [5].

Definition 1.8. *If we assign*

$$\phi(c) = \frac{(\mu)_c}{c!}$$

in (1.7), we achieve a more generalized Mittag-Leffler function as follows:

$$E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\ s,\ \rho) = \sum_{c=0}^{\infty} \frac{(\mu)_c\, x^c}{(\delta c+\rho)^s\; \Gamma(\alpha c+\beta)\; c!}, \tag{1.9}$$

where $Re(\mu) > 0$, $Re(\delta) > 0$, $Re(\rho) > 0$, $Re(s) \geq 0$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $|x| \leq 1$ and $(a)_c$ is defined in (1.5).

It is obvious that when $s=0$ in (1.9), it becomes (1.5).
The function (1.9) is studied in this paper.

Remark 1.9. *Other generalized Mittag-Leffler functions and generalized Hurwitz-Lerch zeta functions can be achieved by assigning suitable values to $\phi(c)$ in (1.7) and (1.8). Three such examples are given here.*
(i) If we assign $s=0$, $\phi(c)=\frac{\Gamma(\zeta c+\mu)}{\Gamma(\delta c+\rho)}$ in (1.7), we achieve the following generalized Mittag-Leffler function:

$$E_{\alpha,\,\beta,\,\delta,\,\rho}^{\zeta,\,\mu}(x)=\sum_{c=0}^{\infty}\frac{\Gamma(\zeta c+\mu)\,x^{c}}{\Gamma(\alpha c+\beta)\,\Gamma(\delta c+\rho)}, \tag{1.10}$$

where $Re(\mu)>0$, $Re(\alpha)>0$, $Re(\beta)>0$, $Re(\delta)>0$, $Re(\rho>0$.
It is obvious that when $\zeta=\delta$, $\mu=\rho$ in (1.10), it becomes (1.2). Using (2.2), it may be ascertained that the series in (1.10) is absolutely convergent when $|\alpha+\delta|>|\zeta|$ and $|x|\leq 1$.
(ii) If we assign $\phi(c)=\Gamma(\zeta c+\mu)$ in (1.7) and (1.8), we achieve the following generalized Hurwitz-Lerch zeta functions:

$$\phi_{\zeta,\,\mu}^{\delta,\,\alpha,\,\beta}(x,\,s,\,\rho)=\sum_{c=0}^{\infty}\frac{\Gamma(\zeta c+\mu)\,x^{c}}{(\rho+\delta c)^{s}\,\Gamma(\alpha c+\beta)}, \tag{1.11}$$

where $\zeta,\,\mu,\,\delta,\,\rho,\alpha,\,\beta,\,s\in\mathbb{C}$, $Re(\zeta)>0$, $Re(\delta)>0$, $Re(\alpha)>0$, $Re(s)\geq 0$,
and

$$\phi_{\zeta,\,\mu}^{\delta,\,\eta,\,\alpha,\,\beta}(x,\,s,\,\rho)=\sum_{c=0}^{\infty}\frac{\Gamma(\zeta c+\mu)\,x^{c}}{(\rho+\delta c x^{\eta})^{s}\,\Gamma(\alpha c+\beta)}, \tag{1.12}$$

where $\zeta,\,\mu,\,\alpha,\,\beta,\,\delta,\,\rho,\,s\in\mathbb{C}$, $Re(\zeta)>0$, $Re(\alpha)>0$, $Re(s)\geq 0$, $\eta\geq 0$.
It is obvious that when $\zeta=\alpha$, $\mu=\beta$, $\delta=1$ in (1.11), it becomes the Hurwitz-Lerch zeta function [19], (p. 27, Eq. (1)) and when $\eta=0$ in (1.12), it becomes (1.11). Using (2.2), it may be ascertained that series in (1.11) and (1.12) are absolutely convergent when $|\alpha|>|\zeta|$ and $|x|\leq 1$.

Lemma 1.10. *The function $E_{\alpha,\,\beta,\,\delta}^{\mu}(x,\,s,\rho)$ expressed by (1.9) is represented as an integral as follows:*

$$E_{\alpha,\,\beta,\,\delta}^{\mu}(x,\,s,\rho)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}\,e^{-\rho t}\,E_{\alpha,\,\beta}^{\mu}(xe^{-\delta t})dt, \tag{1.13}$$

where $E_{\alpha,\,\beta}^{\mu}(xe^{-\delta t})$ is expressed by (1.5) and $Re(\mu)>0$, $Re(\alpha)>0$, $Re(\beta)>0$, $Re(\delta)>0$, $Re(s)>0$, $|x|\leq 1$.

Proof. Assigning $p=(\delta c+\rho)$ in [19], (p. 1, Eq. (5))

$$p^{-s}=\frac{1}{\Gamma(s)}\int_{0}^{\infty}e^{-pt}\,t^{s-1}dt,\ \ Re(s)>0,$$

we achieve

$$(\delta c+\rho)^{-s}=\frac{1}{\Gamma(s)}\int_{0}^{\infty}e^{-(\delta c+\rho)t}\,t^{s-1}dt,\ \ Re(s)>0.$$

Now, from (1.9), we achieve

$$E_{\alpha,\,\beta,\,\delta}^{\mu}(x,\,s,\rho)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}t^{s-1}\,e^{-\rho t}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_{c}(xe^{-\delta t})^{c}}{\Gamma(\alpha c+\beta)\;c!}\right\}dt$$

and applying (1.5), we easily procure (1.13). $\square$

Remark 1.11. *If we assign $\alpha=1$ in (1.13), we procure the expression:*

$$E_{\beta,\,\delta}^{\mu}(x,\,s,\rho)=\frac{1}{\Gamma(s)\,\Gamma(\beta)}\int_{0}^{\infty}t^{s-1}\,e^{-\rho t}\,{}_{1}F_{1}(\mu;\,\beta;\,xe^{-\delta t})dt,$$

where ${}_{1}F_{1}(\mu;\,\beta;\,xe^{-\delta t})$ is the confluent hypergeometric function [19] and $\mathrm{Re}(\mu)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\delta)>0$, $\mathrm{Re}(s)>0$, $|x|\leq 1$.

Definition 1.12. *Riemann-Liouville's fractional integral of order ω of $f(t)$ is given as follows [20]:*

$$I_x^{\omega}\{f(t)\} = \frac{1}{\Gamma(\omega)}\int_0^x (x-t)^{\omega-1} f(t)dt, \tag{1.14}$$

where ω, x are complex variables and $Re(\omega) > 0$.

Definition 1.13. *A modified general class of functions is hereby presented as follows:*

$$\begin{aligned} V_n^{\lambda}(z) &= V_n^{\lambda,\, h_m,\, c,\, d,\, g_j}[p,\ \tau,\ k,\ w,\ q,\ \rho_m,\ k_m,\ \gamma_j,\ a_j,\ b_r,\ \eta,\ \alpha,\ \beta,\ \delta;\ z] \\ &= \lambda \sum_{n=0}^{\infty} \frac{(p)^n \prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(z/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}, \end{aligned} \tag{1.15}$$

where

(i) $p,\ k,\ w$ and $q \in \mathbb{R}$.
(ii) t, s and $u \in \mathbb{N}$.
(iii) $h_m,\ \rho_m,\ k_m,\ c,\ d,\ g_j,\ \gamma_j,\ a_j,\ \eta,\ \alpha,\ \beta,\ \delta,\ b_r$ and $\tau \in \mathbb{C}$. d may be considered as real or complex.
(iv) $Re(h_m) > 0$, $Re(\rho_m) > 0$, $Re(g_j) > 0$, $Re(\gamma_j) > 0$, $Re(d) > 0$, z being a variable and λ being an arbitrary constant.
(v) The series in (1.15) is absolutely convergent when $|\alpha\delta+\gamma_j| > |\rho_m|$ and $|p(z/2)^k| \le 1$.

Remark 1.14. *On substituting $\rho_m = 1,\ c = d,\ \eta = \alpha$ and $\gamma_j = 1$ in (1.15), it becomes the general class of functions defined in [?], [5].*

Remark 1.15. *If we assign $p = 2,\ k = 1,\ c = d = 1,\ \tau = 1,\ w = 0,\ q = 0,\ \alpha = \eta = 1,\ \beta = -1,\ \delta = 1,\ b_1 = -1,\ r = 1,\ k_m = 0,\ a_j = 0$ and $\lambda = \frac{\prod_{m=1}^{t}\Gamma(h_m)}{\prod_{j=1}^{s}\Gamma(g_j)}$ in (1.15), it becomes the Wright's generalized hypergeometric function as follows [19], (p. 183):*

$${}_t\Psi_s\left[\begin{matrix}(h_m,\ \rho_m)_{1,\ t}\\ (g_j,\ \gamma_j)_{1,\ s}\end{matrix};\ z\right] = \sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}\Gamma(h_m+n\rho_m)}{\prod_{j=1}^{s}\Gamma(g_j+n\gamma_j)}\frac{z^n}{n!} \tag{1.16}$$

2. Convergence conditions of (1.9) and (1.15)

Here convergence conditions of the series in (1.9) and (1.15) are discussed.

Theorem 2.1. *If $Re(\mu) > 0$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\delta) > 0$, $Re(\rho) > 0$, $Re(s) \ge 0$ and $|x| \le 1$, then series in (1.9) is absolutely convergent.*

Proof. D' Alembert's ratio test is applied to prove the theorem. Taking

$$U_c(x) = \frac{(\mu)_c\, x^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\, c!}.$$

Then

$$U_{c+1}(x) = \frac{(\mu)_{c+1}\, x^{c+1}}{(\delta c+\rho+\delta)^s\,\Gamma(\alpha c+\beta+\alpha)\ (c+1)!}$$

and on simplification

$$\left|\frac{U_{c+1}(x)}{U_c(x)}\right| = \left|\frac{\mu+c}{c+1}\,\frac{(\delta c+\rho)^s}{(\delta c+\delta+\rho)^s}\,\frac{\Gamma(\alpha c+\beta)}{\Gamma(\alpha c+\beta+\alpha)}x\right|. \tag{2.1}$$

Applying in (2.1), the result [19], (p. 5, Eq. (2)):

$$\frac{\Gamma(a)}{\Gamma(a+b)} = e^{\gamma\, b}\prod_{n=0}^{\infty}\left(1+\frac{b}{a+n}\right)e^{\frac{-b}{1+n}}, \tag{2.2}$$

where $\gamma(=0.58)$ is the Euler constant, (2.1) becomes

$$\left|\frac{U_{c+1}(x)}{U_c(x)}\right| = \left|\frac{\mu+c}{c+1}\,\frac{(\delta c+\rho)^s}{(\delta c+\delta+\rho)^s}\ e^{\gamma\,\alpha}\prod_{n=0}^{\infty}\left(\frac{n+\alpha c+\beta+\alpha}{n+\alpha c+\beta}\right)e^{\frac{-\alpha}{1+n}}\ x\right|.$$

On simplification we procure

$$\left|\frac{U_{c+1}(x)}{U_c(x)}\right| = \left|\frac{\frac{\mu}{c}+1}{1+\frac{1}{c}}\frac{(1+\frac{\rho}{\delta c})^s}{(1+\frac{1}{c}+\frac{\rho}{\delta c})^s}e^{\gamma\,\alpha}\prod_{n=0}^{\infty}\left(\frac{1+\frac{\beta}{\alpha c}+\frac{n}{\alpha c}+\frac{1}{c}}{1+\frac{\beta}{\alpha c}+\frac{n}{\alpha c}}\right)e^{\frac{-\alpha}{1+n}}\,x\right|.$$

Now, it is observed that

$$\lim_{c\to\infty}\left|\frac{U_{c+1}(x)}{U_c(x)}\right| = \left|\,x\,e^{\gamma\,\alpha}\prod_{n=0}^{\infty}e^{\frac{-\alpha}{1+n}}\right|.$$

Therefore, series in (1.9) converges absolutely when

$$\left|x\,e^{\gamma\,\alpha}\prod_{n=0}^{\infty}e^{\frac{-\alpha}{1+n}}\right| < 1.$$

Or

$$|x| \le 1,$$

since

$$\left|e^{\gamma\,\alpha}\prod_{n=0}^{\infty}e^{\frac{-\alpha}{1+n}}\right| < 1,$$

provided that $\mathrm{Re}(\mu)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\delta)>0$, $\mathrm{Re}(\rho)>0$ and $\mathrm{Re}(s)\ge 0$. □

Theorem 2.2. *If $Re(\alpha)>0$, $Re(\delta)>0$, $|\alpha\delta+\gamma_j|>|\rho_m|$ and $|p(x/2)^k|\le 1$, then series in (1.15) converges absolutely.*

Proof. Taking

$$U_n(z) = \lambda\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(z/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}.$$

Then

$$U_{n+1}(z) = \lambda\frac{(p)^{n+1}\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+\rho_m+k_m}\right](c+\eta n+\eta+\beta)^{-\tau}(z/2)^{nk+k+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+\alpha\delta+b_r}\right]}$$

and on simplification we procure

$$\begin{aligned}\left|\frac{U_{n+1}(x)}{U_n(x)}\right| = \Bigg|&\prod_{m=1}^{t}\left\{\frac{\Gamma(h_m+n\rho_m+k_m)}{\Gamma(h_m+n\rho_m+k_m+\rho_m)}\right\}^{-1}\prod_{j=1}^{s}\left\{\frac{\Gamma(g_j+n\gamma_j+a_j)}{\Gamma(g_j+n\gamma_j+a_j+\gamma_j)}\right\}\\ &\times\prod_{r=1}^{u}\left\{\frac{\Gamma(d+n\alpha\delta+b_r)}{\Gamma(d+n\alpha\delta+b_r+\alpha\delta)}\right\}\left\{\frac{(c+n\eta+\beta)}{(c+n\eta+\beta+\eta)}\right\}^{\tau}p(z/2)^k\Bigg|.\end{aligned} \tag{2.3}$$

On applying (2.2), (2.3) becomes

$$\begin{aligned}\left|\frac{U_{n+1}(x)}{U_n(x)}\right| = \Bigg|&\prod_{m=1}^{t}\left\{e^{\gamma\rho_m}\prod_{l=0}^{\infty}\left(\frac{\frac{h_m}{n\rho_m}+1+\frac{k_m}{n\rho_m}+\frac{l}{n\rho_m}+\frac{1}{n}}{\frac{h_m}{n\rho_m}+1+\frac{k_m}{n\rho_m}+\frac{l}{n\rho_m}}\right)e^{-\frac{\rho_m}{l+1}}\right\}^{-1}\prod_{j=1}^{s}e^{\gamma\gamma_j}\prod_{\rho=0}^{\infty}\left(\frac{\frac{g_j}{n\gamma_j}+1+\frac{a_j}{n\gamma_j}+\frac{\rho}{n\gamma_j}+\frac{1}{n}}{\frac{g_j}{n\gamma_j}+1+\frac{a_j}{n\gamma_j}+\frac{\rho}{n\gamma_j}}\right)e^{-\frac{\gamma_j}{\rho+1}}\\ &\times\prod_{r=1}^{u}e^{\gamma\alpha\delta}\prod_{\varepsilon=0}^{\infty}\left(\frac{\frac{d}{n\alpha\delta}+1+\frac{b_r}{n\alpha\delta}+\frac{\gamma}{n\alpha\delta}+\frac{1}{n}}{\frac{d}{n\alpha\delta}+1+\frac{b_r}{n\alpha\delta}+\frac{\gamma}{n\alpha\delta}}\right)e^{-\frac{\alpha\delta}{\varepsilon+1}}\left(\frac{\frac{c}{n\eta}+1+\frac{\beta}{n\eta}}{\frac{c}{n\eta}+1+\frac{\beta}{n\eta}+\frac{1}{n}}\right)^{\tau}p(z/2)^k\Bigg|,\end{aligned}$$

where γ is given with (2.2).
Now, it is observed that

$$\lim_{n\to\infty}\left|\frac{U_{n+1}(x)}{U_n(x)}\right| = \left|\prod_{j=1}^{s}\prod_{m=1}^{t}\left\{e^{\gamma(\alpha\delta+\gamma_j-\rho_m)}\prod_{\varepsilon=0}^{\infty}\prod_{\rho=0}^{\infty}\prod_{l=0}^{\infty}e^{-\left(\frac{\alpha\delta}{\varepsilon+1}+\frac{\gamma_j}{\rho+1}-\frac{\rho_m}{l+1}\right)}\right\}p(z/2)^k\right|$$

Therefore, series in (1.15) converges absolutely when

$$\left|\prod_{j=1}^{s}\prod_{m=1}^{t}\left\{e^{\gamma(\alpha\delta+\gamma_j-\rho_m)}\prod_{\varepsilon=0}^{\infty}\prod_{\rho=0}^{\infty}\prod_{l=0}^{\infty}e^{-\left(\frac{\alpha\delta}{\varepsilon+1}+\frac{\gamma_j}{\rho+1}-\frac{\rho_m}{l+1}\right)}\right\}p(z/2)^k\right|<1.$$

Or

$$|p(z/2)^k|\leq 1,$$

since

$$\left|\prod_{j=1}^{s}\prod_{m=1}^{t}\left\{e^{\gamma(\alpha\delta+\gamma_j-\rho_m)}\prod_{\varepsilon=0}^{\infty}\prod_{\rho=0}^{\infty}\prod_{l=0}^{\infty}e^{-\left(\frac{\alpha\delta}{\varepsilon+1}+\frac{\gamma_j}{\rho+1}-\frac{\rho_m}{l+1}\right)}\right\}\right|<1,$$

provided $|\alpha\delta+\gamma_j|>|\rho_m|$. □

3. Generating relations

Here we drive some generating relations pertaining to (1.9) and (1.15).

Theorem 3.1. *If* $|t|<|\rho|$, $Re(\mu)>0$ *along with conditions associated with* (1.9)*, we procure the generating relation as follows:*

$$\sum_{n=0}^{\infty}(\xi)_n\,E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\ \xi+n,\ \rho)\frac{t^n}{n!}=E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\ \xi,\ \rho-t). \tag{3.1}$$

Proof. Applying (1.9), we find

$$\begin{aligned}\sum_{n=0}^{\infty}(\xi)_n\,E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\ \xi+n,\ \rho)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}(\xi)_n\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho)^{\xi+n}\,\Gamma(\alpha c+\beta)\,c!}\frac{t^n}{n!}\\&=\sum_{c=0}^{\infty}\frac{1}{\Gamma(\alpha c+\beta)(\delta c+\rho)^{\xi}}\left[\sum_{n=0}^{\infty}(\xi)_n\left(\frac{t}{\delta c+\rho)}\right)^n\frac{1}{n!}\right](\mu)_c\frac{x^c}{c!}.\end{aligned} \tag{3.2}$$

Applying in (3.2), the result

$$\sum_{n=0}^{\infty}\frac{(\xi)_n\,x^n}{n!}=(1-x)^{-\xi},\quad |x|<1, \tag{3.3}$$

we procure

$$\begin{aligned}\sum_{n=0}^{\infty}(\xi)_n\,E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\ \xi+n,\ \rho)\frac{t^n}{n!}&=\sum_{c=0}^{\infty}\frac{1}{\Gamma(\alpha c+\beta)(\delta c+\rho)^{\xi}}\left(1-\frac{t}{\delta c+\rho)}\right)^{-\xi}(\mu)_c\frac{x^c}{c!}\\&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho-t)^{\xi}\,\Gamma(\alpha c+\beta)c!}.\end{aligned}$$

Using (1.9), (3.1) is arrived at, provided $|t|<|\rho|$. □

Theorem 3.2. *If* $|\rho|>|t|$, $Re(u+\xi)>Re(v)>0$ *along with conditions associated with* (1.9)*, we procure the bilateral generating function as follows:*

$$\sum_{n=0}^{\infty}\frac{(\xi)_n(u)_n}{(v)_n}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(x,\ \xi+u-v+n,\ \rho)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{(\mu)_n\,x^n}{(\delta n+\rho)^{\xi+u-v}\,\Gamma(\alpha n+\beta)\,n!}\,{}_2F_1\left(\xi,\ u;\ v;\ \frac{t}{\delta n+\rho}\right),$$

where ${}_2F_1\,(a,\ b;\ v;\ z)$ *represents the hypergeometric function [19] (p. 56, Eq. (2)).*

Proof. applying (1.9), we procure

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\xi)_n(u)_n}{(v)_n}\,E^{\mu}_{\alpha,\,\beta,\,\delta}(z,\ \xi+u-v+n,\ \rho)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\frac{(\xi)_n(u)_n}{(v)_n}\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho)^{\xi+u-v+n}\,\Gamma(\alpha c+\beta)\,c!}\frac{t^n}{n!}\\&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho)^{\xi+u-v}\,\Gamma(\alpha c+\beta)\,c!}\left[\sum_{n=0}^{\infty}\frac{(\xi)_n(u)_n}{(v)_n}\left(\frac{t}{\delta c+\rho}\right)^n\frac{1}{n!}\right]\\&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,x^c}{(\delta c+\rho)^{\xi+u-v}\,\Gamma(\alpha c+\beta)\,c!}\,{}_2F_1\left(\xi,\ u;\ v;\ \frac{t}{\delta c+\rho}\right)\\&=\sum_{n=0}^{\infty}\frac{(\mu)_n\,x^n}{(\delta n+\rho)^{\xi+u-v}\,\Gamma(\alpha n+\beta)\,n!}\,{}_2F_1\left(\xi,\ u;\ v;\ \frac{t}{\delta n+\rho}\right),\end{aligned}$$

provided $|\rho|>|t|$ and $\mathrm{Re}(u+\xi)>\mathrm{Re}(v)>0$. □

Theorem 3.3. *If conditions associated with (1.9) and (1.15) are satisfied, we procure the bilateral generating function as follows:*

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\, V^{\lambda}_n(y)\,\frac{t^n}{n!} = E^{\mu}_{\gamma,\,\sigma,\,\eta}\left(x,\,\varepsilon,\,\rho-pt\left(\frac{y}{2}\right)^k\right)\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}. \tag{3.4}$$

Proof. Applying (1.15), it is procured that

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\, V^{\lambda}_n(y)\,\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\,\frac{t^n}{n!}$$
$$\times\lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\left(\frac{y}{2}\right)^{nk+dw+q}. \tag{3.5}$$

Applying (1.9) in (3.5), we find

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\, V^{\lambda}_n(y)\,\frac{t^n}{n!} = \lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}\left(\frac{y}{2}\right)^{nk+dw+q}$$
$$\times\left[\sum_{n=0}^{\infty}(\varepsilon)_n\left\{\sum_{\omega=0}^{\infty}\frac{(\mu)_\omega\, x^\omega}{(\eta\omega+\rho)^{\varepsilon+n}\,\Gamma(\gamma\omega+\sigma)\,\omega!}\right\}\frac{t^n}{n!}\right]$$
$$=\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}$$
$$\times\sum_{\omega=0}^{\infty}\frac{(\mu)_\omega\, x^\omega}{(\eta\omega+\rho)^{\varepsilon}\,\Gamma(\gamma\omega+\sigma)\,\omega!}\left[\sum_{n=0}^{\infty}(\varepsilon)_n\left\{\frac{pt(\frac{y}{2})^k}{\eta\omega+\rho}\right\}^n\frac{1}{n!}\right]. \tag{3.6}$$

Applying (3.3) in (3.6), we find

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma,\,\eta}(x,\,\varepsilon+n,\,\rho)\, V^{\lambda}_n(y)\,\frac{t^n}{n!} = \left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}$$
$$\times\sum_{\omega=0}^{\infty}\frac{(\mu)_\omega\, x^\omega}{(\eta\omega+\rho)^{\varepsilon}\,\Gamma(\gamma\omega+\sigma)\,\omega!}\left\{1-\frac{pt(\frac{y}{2})^k}{\eta\omega+\rho}\right\}^{-\varepsilon}$$
$$=\left(\frac{y}{2}\right)^{dw+q}\lambda\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{t}[(h_m)_{n\rho_m+k_m}](c+\eta n+\beta)^{-\tau}}{\prod_{j=1}^{s}[(g_j)_{n\gamma_j+a_j}]\prod_{r=1}^{u}[(d)_{\alpha n\delta+b_r}]}$$
$$\times\sum_{\omega=0}^{\infty}\frac{(\mu)_\omega\, x^\omega}{\left(\eta\omega+\rho-pt(\frac{y}{2})^k\right)^{\varepsilon}\,\Gamma(\gamma\omega+\sigma)\,\omega!}. \tag{3.7}$$

On applying (1.9), (3.7) easily approaches to (3.4). □

4. Special cases of the generating relation (3.4)

Here some special cases of (3.4) are achieved.

(i) On taking $p=-2,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ g_1=1,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ c=d,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_1=0,\ a_1=0,\ a_2=0,\ \eta=\alpha,\ \beta=0,\ \delta=1,\ b_1=0$ and $\lambda=\frac{1}{\Gamma(d)}$ in (3.4), the modified general class of functions takes the form of Wright's generalized Bessel function [21] and we procure the generating relation as follows:

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma,\,\alpha}(x,\,\varepsilon+n,\,\rho)\, J^{\alpha}_d(y)\,\frac{t^n}{n!} = \frac{1}{\Gamma(1+\alpha n+d)\,n!}\,E^{\mu}_{\gamma,\,\sigma,\,\alpha}(x,\,\varepsilon,\,\rho+ty),$$

where $J^{\alpha}_d(y)$ represents the Wright's generalized Bessel function.

(ii) On taking $p=-1,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ k_1=0,\ g_1=1,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ c=d=\frac{1}{2},\ \tau=1,\ k=2,\ w=2,\ q=0,\ a_1=0,\ a_2=-1,\ \beta=-\frac{1}{2},\ \delta=1,\ b_1=1,\ \eta=\alpha=1$ and $\lambda=1$ in (3.4), the modified general class of functions takes the form of sine function and we procure the generating relation as follows:

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma}(x,\ \varepsilon+n,\ \rho)\ sin\ y\ \frac{t^n}{n!}=\frac{y\sqrt{\pi}}{2\,\Gamma(\frac{3}{2}+n)\ n!}\ E^{\mu}_{\gamma,\,\sigma}\left(x,\ \varepsilon,\ \rho+t\left(\frac{y}{2}\right)^2\right).$$

(iii) On taking $p=-1,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ k_1=0,\ g_1=1,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ c=d=\frac{1}{2},\ \tau=1,\ k=2,\ w=0,\ q=0,\ a_1=0,\ a_2=-1,\ \beta=-\frac{1}{2},\ \delta=1,\ b_1=0,\ \eta=\alpha=1$ and $\lambda=1$ in (3.4), the modified general class of functions takes the form of cosine function and we procure the generating relation as follows:

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma}(x,\ \varepsilon+n,\ \rho)\ cos\ y\ \frac{t^n}{n!}=\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}+n)\ n!}\ E^{\mu}_{\gamma,\,\sigma}\left(x,\ \varepsilon,\ \rho+t\left(\frac{y}{2}\right)^2\right).$$

(iv) On taking $p=-2,\ u=1,\ c=d=1,\ \rho_m=1,\ \gamma_j=1,\ t=P,\ s=Q,\ \tau=1,\ k=1,\ w=0,\ q=0,\ k_m=0,\ a_j=0,\ b_1=-1,\ \eta=\alpha=1,\ \beta=-1,\ \delta=1$ and $\lambda=\frac{\prod_{m=1}^{P}\Gamma(h_m)}{\prod_{j=1}^{Q}\Gamma(g_j)}$ in (3.4), the general class of functions takes the form of MacRobert's $E-$ function [19], (p. 203, Eq. (1)) and we procure the generating relation as follows:

$$\sum_{n=0}^{\infty}(\varepsilon)_n\, E^{\mu}_{\gamma,\,\sigma}(x,\ \varepsilon+n,\ \rho)\ E\left[P;\ (h_P);\ Q;\ (g_Q);\ \frac{1}{y}\right]\ \frac{t^n}{n!}=\frac{\prod_{m=1}^{P}\Gamma(h_m)}{\prod_{j=1}^{Q}\Gamma(g_j)}\sum_{n=0}^{\infty}\frac{\prod_{m=1}^{P}(h_m)_n}{\prod_{j=1}^{Q}\ (g_j)_n}\ E^{\mu}_{\gamma,\,\sigma}(x,\ \varepsilon,\ \rho+yt).$$

where $E\,[P;\ (h_P);\ Q;\ (g_Q);\ z]$ represents the MacRobert's $E-$ function.

Remark 4.1. *Other spacial cases of the generating relation* (3.4) *may be procured using the substitutions of section 7.*

5. Some results pertaining to (1.9)

Here we establish some results pertaining to (1.9) associated with differentiation and integration.

Theorem 5.1. *If* $Re(\mu)>0$, $Re(\varepsilon)>0$ *along with conditions associated with* (1.9)*, the result procured is as follows:*

$$\frac{1}{\Gamma(\varepsilon)}\sum_{c=0}^{\infty}\frac{(\mu)_c\ x^c}{(\delta c+\rho)^s\ \Gamma(\alpha c+\beta)\ c!}\int_0^1 t^{\alpha c+\beta-1}(1-t)^{\varepsilon-1}dt=E^{\mu}_{\alpha,\ \beta+\varepsilon,\ \delta}(x,\ s,\ \rho). \tag{5.1}$$

Proof. (5.1) may easily be proved using Beta integral. □

Assigning $s=0$ in (5.1), it becomes a result procured by Prabhakar [3].

Theorem 5.2. *If* $a>0$ *along with conditions associated with* (1.9)*, the result procured is as follows:*

$$E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\alpha},\ s,\ \rho)=x\ \frac{d}{dx}E^{\mu}_{\alpha,\ \beta+1,\ \delta}(ax^{\alpha},\ s,\ \rho)+\beta\ E^{\mu}_{\alpha,\ \beta+1,\ \delta}(ax^{\alpha},\ s,\ \rho). \tag{5.2}$$

Proof. Applying (1.9), we procure

$$\begin{aligned}\frac{d}{dx}E^{\mu}_{\alpha,\ \beta+1,\ \delta}(ax^{\alpha},\ s,\ \rho)&=\sum_{c=0}^{\infty}\frac{(\mu)_c}{(\delta c+\rho)^s\ \Gamma(\alpha c+\beta+1)\ c!}\frac{d}{dx}(ax^{\alpha})^c\\&=\frac{1}{x}\sum_{c=0}^{\infty}\frac{(\mu)_c\ a^c(\alpha c)\ x^{\alpha c}}{(\delta c+\rho)^s\ (\alpha c+\beta)\ \Gamma(\alpha c+\beta)\ c!}\\&=\frac{1}{x}\sum_{c=0}^{\infty}\frac{(\mu)_c\ a^c\{(\alpha c+\beta)-\beta\}\ x^{\alpha c}}{(\delta c+\rho)^s\ (\alpha c+\beta)\ \Gamma(\alpha c+\beta)\ c!}\\&=\frac{1}{x}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_c\ a^c(\alpha c+\beta)\ x^{\alpha c}}{(\delta c+\rho)^s\ (\alpha c+\beta)\ \Gamma(\alpha c+\beta)\ c!}-\beta\sum_{c=0}^{\infty}\frac{(\mu)_c\ a^c\ x^{\alpha c}}{(\delta c+\rho)^s\ (\alpha c+\beta)\ \Gamma(\alpha c+\beta)\ c!}\right\}\\&=\frac{1}{x}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_c\ (ax^{\alpha})^c}{(\delta c+\rho)^s\ \Gamma(\alpha c+\beta)\ c!}-\beta\sum_{c=0}^{\infty}\frac{(\mu)_c\ (ax^{\alpha})^c}{(\delta c+\rho)^s\ \Gamma(\alpha c+\beta+1)\ c!}\right\}\end{aligned} \tag{5.3}$$

On applying (1.9), (5.3) arrives at

$$x\ \frac{d}{dx}E^{\mu}_{\alpha,\ \beta+1,\ \delta}(ax^{\alpha},\ s,\ \rho)=E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\alpha},\ s,\ \rho)-\beta\ E^{\mu}_{\alpha,\ \beta+1,\ \delta}(ax^{\alpha},\ s,\ \rho).$$

On simplification, (5.2) is arrived at. □

Corollary 5.3. *Assigning $\beta=\beta+\varepsilon$ in (5.2), the result obtained is as follows:*

$$E^{\mu}_{\alpha,\ \beta+\varepsilon,\ \delta}(ax^{\alpha},\ s,\ \rho)=x\,\frac{d}{dx}E^{\mu}_{\alpha,\ \beta+\varepsilon+1,\ \delta}(ax^{\alpha},\ s,\ \rho)+(\beta+\varepsilon)\,E^{\mu}_{\alpha,\ \beta+\varepsilon+1,\ \delta}(ax^{\alpha},\ s,\ \rho). \tag{5.4}$$

Theorem 5.4. *If $a>0$ along with conditions associated with (1.9), the result procured is as follows:*

$$E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\delta},\ s-1,\ \rho)=x\frac{d}{dx}E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\delta},\ s,\ \rho)+\rho\,E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\alpha},\ s,\ \rho). \tag{5.5}$$

Proof. Applying (1.9), we procure

$$\begin{aligned}\frac{d}{dx}E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\delta},\ s,\ \rho)&=\sum_{c=0}^{\infty}\frac{(\mu)_c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\frac{d}{dx}(ax^{\delta})^c\\&=\frac{1}{x}\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\delta c)\,x^{\delta c}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\\&=\frac{1}{x}\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c\{(\delta c+\rho)-\rho\}\,x^{\delta c}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\\&=\frac{1}{x}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\delta c+\rho)\,x^{\delta c}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}-\rho\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c\,x^{\delta c}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\right\}\\&=\frac{1}{x}\left\{\sum_{c=0}^{\infty}\frac{(\mu)_c\,(ax^{\delta})^c}{(\delta c+\rho)^{s-1}\,\Gamma(\alpha c+\beta)\,c!}-\rho\sum_{c=0}^{\infty}\frac{(\mu)_c\,(ax^{\delta})^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\right\}\end{aligned} \tag{5.6}$$

On applying (1.9), (5.6) becomes

$$x\,\frac{d}{dx}E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\delta},\ s,\ \rho)=E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\delta},\ s-1,\ \rho)-\rho\,E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\delta},\ s,\ \rho).$$

On simplification, (5.5) is easily arrived at. □

Theorem 5.5. *Along with conditions associated with (1.9), for any $n\in\mathbb{N}$ the result procured is as follows:*

$$\left(\frac{d}{dx}\right)^n\left\{x^{\beta-1}E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\alpha},\ s,\ \rho)\right\}=x^{\beta-n-1}\,E^{\mu}_{\alpha,\ \beta-n,\ \delta}(ax^{\alpha},\ s,\ \rho). \tag{5.7}$$

Proof. we find

$$\begin{aligned}\frac{d}{dx}\left\{x^{\beta-1}E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\alpha},\ s,\ \rho)\right\}&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}\frac{d}{dx}x^{\alpha c+\beta-1}\\&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\alpha c+\beta-1)\,x^{\alpha c+\beta-2}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta)\,c!}.\end{aligned} \tag{5.8}$$

Applying $\Gamma(x)=(x-1)\Gamma(x-1)$ in (5.8), we procure

$$\begin{aligned}\frac{d}{dx}\left\{x^{\beta-1}E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\alpha},\ s,\ \rho)\right\}&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c(\alpha c+\beta-1)\,x^{\alpha c+\beta-2}}{(\delta c+\rho)^s\,(\alpha c+\beta-1)\,\Gamma(\alpha c+\beta-1)\,c!}\\&=\sum_{c=0}^{\infty}\frac{(\mu)_c\,a^c\,x^{\alpha c+\beta-2}}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta-1)\,c!}\\&=x^{\beta-2}\sum_{c=0}^{\infty}\frac{(\mu)_c\,(ax^{\alpha})^c}{(\delta c+\rho)^s\,\Gamma(\alpha c+\beta-1)\,c!}\\&=x^{\beta-2}\,E^{\mu}_{\alpha,\ \beta-1,\ \delta}(ax^{\alpha},\ s,\ \rho)\end{aligned}$$

Similarly, we get

$$\left(\frac{d}{dx}\right)^2\left\{x^{\beta-1}E^{\mu}_{\alpha,\ \beta,\ \delta}(ax^{\alpha},\ s,\ \rho)\right\}=x^{\beta-3}\,E^{\mu}_{\alpha,\ \beta-2,\ \delta}(ax^{\alpha},\ s,\ \rho).$$

Following the same process we procure (5.7). □

Assigning $s=0$ in (5.7), a result of Kilbas, Saigo and Saxena [8] is procured.

6. Applications

Here fractional integral (1.14) is applied to procure images of (1.9) and (1.15), and finally to gain integrals involving special functions.

Theorem 6.1. *If $Re(\varepsilon) > 0$ along with conditions associated with (1.9), the result procured is as follows:*

$$I_x^{\varepsilon}\left\{t^{\beta-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho)\right\} = x^{\varepsilon+\beta-1}\, E_{\alpha,\,\beta+\varepsilon,\,\delta}^{\mu}(ax^{\alpha},\, s,\, \rho). \tag{6.1}$$

Proof. On applying (1.14), it is procured that

$$I_x^{\varepsilon}\left\{t^{\beta-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho)\right\} = \frac{1}{\Gamma(\varepsilon)}\int_0^x t^{\beta-1}(x-t)^{\varepsilon-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho)dt. \tag{6.2}$$

Use of (1.9) in (6.2) gives

$$I_x^{\varepsilon}\left\{t^{\beta-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho)\right\} = \frac{1}{\Gamma(\varepsilon)}\int_0^x t^{\beta-1}(x-t)^{\varepsilon-1}\,\sum_{k=0}^{\infty}\frac{(\mu)_k\,(at^{\alpha})^k}{(\delta k+\rho)^s\,\Gamma(\alpha k+\beta)\,k!}dt.$$

Conditions associated with (1.9) permit to interchange the order of integration and summation and it is done to gain

$$I_x^{\varepsilon}\left\{t^{\beta-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho)\right\} = \frac{1}{\Gamma(\varepsilon)}\sum_{k=0}^{\infty}\frac{(\mu)_k\,a^k}{(\delta k+\rho)^s\,\Gamma(\alpha k+\beta)\,k!}\int_0^x t^{\alpha k+\beta-1}(x-t)^{\varepsilon-1}dt. \tag{6.3}$$

Applying in (6.3), the result [20], (p. 185, Eq. (7))

$$\int_0^x y^{b-1}(x-y)^{a-1}dy = x^{a+b-1}\,\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}, \tag{6.4}$$

where $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$, to gain

$$I_x^{\varepsilon}\left\{t^{\beta-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho)\right\} = \frac{1}{\Gamma(\varepsilon)}\sum_{k=0}^{\infty}\frac{(\mu)_k\,a^k}{(\delta k+\rho)^s\,\Gamma(\alpha k+\beta)\,k!}x^{\varepsilon+\alpha k+\beta-1}\,\frac{\Gamma(\varepsilon)\,\Gamma(\alpha k+\beta)}{\Gamma(\varepsilon+\alpha k+\beta)}.$$

On simplifying, it is procured that

$$I_x^{\varepsilon}\left\{t^{\beta-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho)\right\} = x^{\varepsilon+\beta-1}\sum_{k=0}^{\infty}\frac{(\mu)_k\,(ax^{\alpha})^k}{(\delta k+\rho)^s\,\Gamma(\alpha k+\beta+\varepsilon)\,k!}. \tag{6.5}$$

Now, use of (1.9) in (6.5) completes the proof. □

Corollary 6.2. *From (6.1) and (6.2), it is found that*

$$\frac{1}{\Gamma(\varepsilon)}\int_0^x t^{\beta-1}(x-t)^{\varepsilon-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho)dt = x^{\varepsilon+\beta-1}\, E_{\alpha,\,\beta+\varepsilon,\,\delta}^{\mu}(ax^{\alpha},\, s,\, \rho). \tag{6.6}$$

Corollary 6.3. *$x = 1$ in (6.6) gives the Eulerian integral as follows:*

$$\frac{1}{\Gamma(\varepsilon)}\int_0^1 t^{\beta-1}(1-t)^{\varepsilon-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho)dt = E_{\alpha,\,\beta+\varepsilon,\,\delta}^{\mu}(a,\, s,\, \rho). \tag{6.7}$$

Further, on assigning $t = \frac{x-u}{y-u}$ and $a = \lambda(y-u)^{\alpha}$ in (6.7), an interesting integral is procured as follows:

$$\frac{1}{\Gamma(\varepsilon)}\int_u^y (x-u)^{\beta-1}(y-x)^{\varepsilon-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}\{\lambda(x-u)^{\alpha},\, s,\, \rho\}dx = (y-u)^{\beta+\varepsilon-1}\, E_{\alpha,\,\beta+\varepsilon,\,\delta}^{\mu}\{\lambda(y-u)^{\alpha},\, s,\, \rho\}$$

and on assigning $t = \frac{y-x}{y-u}$ and $a = \lambda(y-u)^{\alpha}$ in (6.7), an interesting integral is procured as follows:

$$\frac{1}{\Gamma(\varepsilon)}\int_u^y (y-x)^{\beta-1}(x-u)^{\varepsilon-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}\{\lambda(y-x)^{\alpha},\, s,\, \rho\}dx = (y-u)^{\beta+\varepsilon-1}\, E_{\alpha,\,\beta+\varepsilon,\,\delta}^{\mu}\{\lambda(y-u)^{\alpha},\, s,\, \rho\}.$$

On assigning $s = 0$ in these integrals, Prabhakar's [3] integrals are achieved.

Corollary 6.4. *If $a > 0$, $Re(\varepsilon) > 0$ along with the conditions associated with (1.9), the result is procured as follows:*

$$I_x^{\varepsilon}\left\{t^{\beta-1}\, E_{\alpha,\,\beta,\,\delta}^{\mu}(at^{\alpha},\, s,\, \rho);\, x\right\} = x^{\varepsilon+\beta-1}\left\{x\frac{d}{dx}E_{\alpha,\,\beta+\varepsilon+1,\,\delta}^{\mu}(ax^{\alpha},\, s,\, \rho) + (\beta+\varepsilon)\, E_{\alpha,\,\beta+\varepsilon+1,\,\delta}^{\mu}(ax^{\alpha},\, s,\, \rho)\right\}. \tag{6.8}$$

Putting the value of $E^{\mu}_{\alpha,\ \beta+\varepsilon,\ \delta}(ax^{\alpha},\ s,\ \rho)$ from (5.4) in (6.1), (6.8) is proved.

Theorem 6.5. *If $Re(\varepsilon)>0$ along with conditions associated with (1.9) and (1.15), the result procured is as follows:*

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,V_n^{\lambda}\{b(x-z)^{\sigma}\}\right]=\frac{\lambda}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(b/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}$$
$$\times\Gamma\{\sigma(nk+dw+q)+\varepsilon\}\,x^{\sigma(nk+dw+q)+\varepsilon+\gamma-1}$$
$$\times E^{\mu}_{\sigma,\ \gamma+\sigma(nk+dw+q)+\varepsilon,\ \omega}(ax^{\sigma},\ y,\ \rho). \tag{6.9}$$

Proof. Applying (1.14), it is procured that

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,V_n^{\lambda}\{b(x-z)^{\sigma}\}\right]=\frac{1}{\Gamma(\varepsilon)}\int_0^x z^{\gamma-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,V_n^{\lambda}\{b(x-z)^{\sigma}\}(x-z)^{\varepsilon-1}dz. \tag{6.10}$$

Use of (1.9) and (1.15) in (6.10), and interchange of the order of integration and summations permitted by the conditions associated therein, give the l.h.s of (6.10) (supposing L) as follows:

$$L=\lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(b/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}\sum_{v=0}^{\infty}\frac{(\mu)_v\,a^v}{(\omega v+\rho)^y\,\Gamma(\sigma v+\gamma)\,v!}$$
$$\times\frac{1}{\Gamma(\varepsilon)}\int_0^x z^{\sigma v+\gamma-1}(x-z)^{\sigma(nk+dw+q)+\varepsilon-1}dz. \tag{6.11}$$

On evaluation of z-integral in (6.11) using (6.4), it is procured that

$$L=\lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(b/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}\sum_{v=0}^{\infty}\frac{(\mu)_v\,a^v}{(\omega v+\rho)^y\,\Gamma(\sigma v+\gamma)\,v!}$$
$$\times\frac{1}{\Gamma(\varepsilon)}x^{\sigma(nk+dw+q)+\varepsilon+\sigma v+\gamma-1}\,\frac{\Gamma(\sigma v+\gamma)\Gamma\{\sigma(nk+dw+q)+\varepsilon\}}{\Gamma\{\sigma v+\gamma+\sigma(nk+dw+q)+\varepsilon\}}. \tag{6.12}$$

Now, use of (1.9) in (6.12) completes the proof. □

Corollary 6.6. *From (6.9) and (6.10), it is found that*

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,V_n^{\lambda}\{b(x-z)^{\sigma}\}dz=\lambda\sum_{n=0}^{\infty}\frac{(p)^n\prod_{m=1}^{t}\left[(h_m)_{n\rho_m+k_m}\right](c+\eta n+\beta)^{-\tau}(b/2)^{nk+dw+q}}{\prod_{j=1}^{s}\left[(g_j)_{n\gamma_j+a_j}\right]\prod_{r=1}^{u}\left[(d)_{\alpha n\delta+b_r}\right]}$$
$$\times\Gamma\{\sigma(nk+dw+q)+\varepsilon\}\,x^{\sigma(nk+dw+q)+\varepsilon+\gamma-1}$$
$$\times E^{\mu}_{\sigma,\ \gamma+\sigma(nk+dw+q)+\varepsilon,\ \omega}(ax^{\sigma},\ y,\ \rho).$$

7. Special cases of (6.9)

Here some special cases of (6.9) are achieved.

(i) On assigning $p=-1,\ t=1,\ \beta=0,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ k_1=0,\ \tau=1,\ g_1=1,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ k=2,\eta=\alpha=1,\ a_1=0,\ a_2=0,\ q=0,\ c=d,\ w=1,\ b_1=0,\ \delta=1$ and $\lambda=1/\Gamma(d)$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(az^{\sigma},\ y,\ \rho)\,J_d\{b(x-z)^{\sigma}\}\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+d}\,\Gamma\{\sigma(2n+d)+\varepsilon\}}{\Gamma(1+d+n)\,n!}$$
$$\times x^{\sigma(2n+d)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\ \gamma+\sigma(2n+d)+\varepsilon,\ \omega}(ax^{\sigma},\ y,\ \rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\ \gamma,\ \omega}(at^{\sigma},\ y,\ \rho)\,J_d\{b(x-z)^{\sigma}\}dz=\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+d}\,\Gamma\{\sigma(2n+d)+\varepsilon\}}{\Gamma(1+d+n)\,n!}$$
$$\times x^{\sigma(2n+d)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\ \gamma+\sigma(2n+d)+\varepsilon,\ \omega}(ax^{\sigma},\ y,\ \rho),$$

where $J_d(z)$ represents the Bessel function of the first kind [22], (p. 4, Eq. (2)).

(ii) On assigning $p=-1,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ k_1=0,\ \tau=1,\ g_1=3/2,\ g_2=1,\ \gamma_1=1,\ \gamma_2=1,\ k=2,\ a_1=0,\ a_2=0,\ w=1,\ c=d,\ q=1,\ b_1=1/2,\ \eta=\alpha=1,\ \beta=1/2,\ \delta=1$ and $\lambda=1/\{\Gamma(d)\Gamma(3/2)\}$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,H_d\{b(x-z)^{\sigma}\}\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+d+1}\,\Gamma\{\sigma(2n+d+1)+\varepsilon\}}{\Gamma(n+\frac{3}{2})\,\Gamma(d+n+\frac{3}{2})}\times x^{\sigma(2n+d+1)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\,\gamma+\sigma(2n+d+1)+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,H_d\{b(x-z)^{\sigma}\}dz=\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+d+1}\,\Gamma\{\sigma(2n+d+1)+\varepsilon\}}{\Gamma(n+\frac{3}{2})\,\Gamma(d+n+\frac{3}{2})}\times x^{\sigma(2n+d+1)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\,\gamma+\sigma(2n+d+1)+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where $H_d(z)$ represents the Struve's function [22], (p. 38, Eq. (55)).

(iii) On assigning $p=-1,\ t=1,\ s=2,\ u=1,\ h_1=1,\ \rho_1=1,\ \eta=\alpha=1,\ k_1=0,\ \beta=-1,\ k=2,\ g_1=(\zeta+\xi+3)/2,\ g_2=(\zeta-\xi+3)/2,\gamma_1=1,\ \gamma_2=1,\ c=d=1,\ \tau=1,\ w=\zeta,\ q=1,\ a_1=0,\ a_2=0,\ b_1=-1,\ \delta=1$ and $\lambda=2^{\zeta+1}/(\zeta\pm\xi+1)$ in (6.9), the result is procured as follow:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,s_{\zeta,\,\xi}\{b(x-z)^{\sigma}\}\right]=\frac{2^{\zeta+1}}{(\zeta\pm\xi+1)}\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+\zeta+1}\,\Gamma\{\sigma(2n+\zeta+1)+\varepsilon\}}{\left(\frac{\zeta\pm\xi+3}{2}\right)_n}\times x^{\sigma(2n+\zeta+1)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\,\gamma+\sigma(2n+\zeta+1)+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,s_{\zeta,\,\xi}\{b(x-z)^{\sigma}\}dz=\frac{2^{\zeta+1}}{(\zeta\pm\xi+1)}\sum_{n=0}^{\infty}\frac{(-1)^n(b/2)^{2n+\zeta+1}\,\Gamma\{\sigma(2n+\zeta+1)+\varepsilon\}}{\left(\frac{\zeta\pm\xi+3}{2}\right)_n}\times x^{\sigma(2n+\zeta+1)+\varepsilon+\gamma-1}\,E^{\mu}_{\sigma,\,\gamma+\sigma(2n+\zeta+1)+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where $s_{\zeta,\,\xi}(z)$ represents the Lommel's function [22], (p. 40, Eq. (69)).

(iv) On assigning $p=2,\ t=1,\ s=1,\ u=1,\ h_1=h,\ \rho_1=1,\ k_1=0,\ g_1=1,\ \gamma_1=1,\ \eta=\alpha,\ a_1=0,\ c=d,\ \beta=-1,\ \tau=1,\ k=1,\ w=0,\ q=0,\ \delta=1,\ b_1=-1$ and $\lambda=1/\Gamma(d)$ in (3.1), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,E^{h}_{\alpha,\,d}\{b(x-z)^{\sigma}\}\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(h)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{\Gamma(\alpha n+d)\,n!}x^{\sigma n+\varepsilon+\gamma-1}E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,E^{h}_{\alpha,\,d}\{b(x-z)^{\sigma}\}dz=\sum_{n=0}^{\infty}\frac{(h)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{\Gamma(\alpha n+d)\,n!}x^{\sigma n+\varepsilon+\gamma-1}E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where $E^{h}_{\alpha,\,d}(z)$ represents (1.5).

(v) On assigning $p=2,\ t=1,\ s=1,\ u=1,\ h_1=h,\ \rho_1=1,\ \eta=\alpha=1,\ k_1=0,\ c=d,\ \beta=0,\ k=1,\ w=0,\ q=0,\ g_1=1,\ \gamma_1=1,\ a_1=0,\ b_1=0,\ \delta=0$ and $\lambda=1$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,\phi_h\{b(x-z)^{\sigma},\,\tau,\,d\}\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(h)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{(d+n)^{\tau}\,n!}x^{\sigma n+\varepsilon+\gamma-1}\times E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,\phi_h\{b(x-z)^{\sigma},\,\tau,\,d\}dz=\sum_{n=0}^{\infty}\frac{(h)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{(d+n)^{\tau}\,n!}x^{\sigma n+\varepsilon+\gamma-1}\times E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where $\phi_h\{z,\,\tau,\,d\}$ represents for the Goyal-Laddha zeta function [18].

(vi) On assigning $p=2,\ u=1,\ \rho_m=1,\ c=d=1,\ \eta=\alpha=1,\ \tau=1,\ \beta=-1,\ k=1,\ w=0,\ q=0,\ k_m=0,\ \gamma_j=1,\ a_j=0,\ b_1=-1,\ \delta=1$ and $\lambda=1$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,{}_tF_s\left(h_t;\;g_s;\;\;b(x-z)^{\sigma}\right)\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{\prod\limits_{m=1}^{P}(h_m)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{\prod\limits_{j=1}^{Q}(g_j)_n\,n!}x^{\sigma n+\varepsilon+\gamma-1}$$
$$\times E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,{}_PF_Q\left(h_P;\,g_Q;\;\;b(x-z)^{\sigma}\right)dz=\sum_{n=0}^{\infty}\frac{\prod\limits_{m=1}^{P}(h_m)_n\,b^n\,\Gamma(\sigma n+\varepsilon)}{\prod\limits_{j=1}^{Q}(g_j)_n\,n!}x^{\sigma n+\varepsilon+\gamma-1}$$
$$\times E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where ${}_tF_s\left(h_t;\;g_s;\;z\right)$ represents the generalized hypergeometric function [19], (p. 182, Eq. (1)).

(vii) On assigning $p=2,\ k=1,\ c=d=1,\ \tau=1,\ w=0,\ q=0,\ \alpha=\eta=1,\ \beta=-1,\ \delta=1,\ b_1=-1,\ r=1,\ k_m=0,\ a_j=0$ and $\lambda=\frac{\prod_{m=1}^{t}\Gamma(h_m)}{\prod_{j=1}^{s}\Gamma(g_j)}$ in (6.9), the result is procured as follows:

$$I_x^{\varepsilon}\left[z^{\gamma-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,{}_t\Psi_s\left[\begin{matrix}(h_m,\,\rho_m)_{1,\,t}\\(g_j,\,\gamma_j)_{1,\,s}\end{matrix};\,b(x-z)^{\sigma}\right]\right]=\frac{1}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{\prod\limits_{m=1}^{t}\Gamma(h_m+n\rho_m)\,b^n\,\Gamma(\sigma n+\varepsilon)}{\prod\limits_{j=1}^{s}\Gamma(g_j+n\gamma_j)\,n!}x^{\sigma n+\varepsilon+\gamma-1}$$
$$E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

Hence

$$\int_0^x z^{\gamma-1}(x-z)^{\varepsilon-1}\,E^{\mu}_{\sigma,\,\gamma,\,\omega}(az^{\sigma},\,y,\,\rho)\,{}_t\Psi_s\left[\begin{matrix}(h_m,\,\rho_m)_{1,\,t}\\(g_j,\,\gamma_j)_{1,\,s}\end{matrix};\,b(x-z)^{\sigma}\right]dz=\sum_{n=0}^{\infty}\frac{\prod\limits_{m=1}^{t}\Gamma(h_m+n\rho_m)\,b^n\,\Gamma(\sigma n+\varepsilon)}{\prod\limits_{j=1}^{s}\Gamma(g_j+n\gamma_j)\,n!}x^{\sigma n+\varepsilon+\gamma-1}$$
$$E^{\mu}_{\sigma,\,\gamma+\sigma n+\varepsilon,\,\omega}(ax^{\sigma},\,y,\,\rho).$$

where ${}_t\Psi_s\left[\begin{matrix}(h_m,\,\rho_m)_{1,\,t}\\(g_j,\,\gamma_j)_{1,\,s}\end{matrix};\,z\right]$ represents the Wright's generalized hypergeometric function (1.16) [19], (p. 183).

Remark 7.1. *Other special cases of* (6.9) *can be obtained using the substitutions of section 4.*

8. Conclusion

Two general functions reducible to Mittag-Leffler function and Riemann-zeta function, and a modified general class of functions reducible to several special functions have been represented and defined in this paper, and their convergence conditions have been discussed. Generating relations and fractional integrals involving new defined functions have been achieved. Some particular cases of the results have been achieved. Similar results may be obtained involving (1.10). A further study of the fractional integral operator defined by (1.14) may be carried out with the generalized Mittag-Leffler function defined by (1.9) in the kernel and its integral transforms may be studied. Moreover, composition relations between the fractional integral operator defined by (1.14) and integral operator with the generalized Mittag-Leffler function defined by (1.9) in the kernel may be obtained.

Declarations

Acknowledgements: The author is highly thankful to the editor and anonymous worthy reviewers for their valuable comments and suggestions, and Dr. O. P. Garg, Ex-reader, Department of Mathematics, M. M. H. College, Ghaziabad, India for his constant encouragement.

Conflict of Interest Disclosure: The author declares no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

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Fundamental Journal of Mathematics and Applications (FUJMA), (Fundam. J. Math. Appl.)
https://dergipark.org.tr/en/pub/fujma

How to cite this article: V. Kumar, *On a generalized Mittag-Leffler function and fractional integrals*, Fundam. J. Math. Appl., **7**(1) (2024), 12-25. DOI 10.33401/fujma.1378534

Fundamental Journal of Mathematics and Applications, 7(1) (2024), 26-34
Research Paper / Open Access

FUJMA

Fundamental Journal of Mathematics and Applications

ISSN Online: 2645-8845
www.dergipark.org.tr/en/pub/fujma
https://doi.org/10.33401/fujma.1389595

New Exact and Numerical Experiments for the Caudrey-Dodd-Gibbon Equation

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Article Information

Keywords: Collocation; Caudrey-Dodd-Gibbon Equation; Direct algebraic method; Septic B-spline

AMS 2020 Classification: 65N30; 74S05; 76B25

Abstract

In this study, an exact and a numerical method namely direct algebraic method and collocation finite element method are proposed for solving soliton solutions of a special form of fifth-order KdV (fKdV) equation that is of particular importance: Caudrey-Dodd-Gibbon (CDG) equation. For these aims, homogeneous balance method and septic B-spline functions are used for exact and numerical solutions, respectively. Next, it is proved by applying von-Neumann stability analysis that the numerical method is unconditionally stable. The error norms L_2 and L_∞ have been computed to control proficiency and conservation properties of the suggested algorithm. The obtained numerical results have been listed in the tables. The graphs are modelled so that easy visualization of properties of the problem. Also, the obtained results indicate that our method is favourable for solving such problems.

1. Introduction

The fifth-order KdV-type (fKdV) equation has the following form

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{xxxxx} = 0, \tag{1.1}$$

where α, β and γ are arbitrary positive parameters [1]-[4]. The fKdV equation (1.1) identifies motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice [5]-[14], and has many physical applications in fields as diverse as nonlinear optics and quantum mechanics. These parameters greatly modify affect the characteristics of the equation. For example, if $\alpha = 180$, $\beta = 30$, and $\gamma = 30$ are taken the following CDG equation

$$u_t + 180u^2 u_x + 30u_x u_{xx} + 30uu_{xxx} + u_{xxxxx} = 0, \tag{1.2}$$

is obtained. It is well-known that equation is fully integrable. That means that it has multiple-soliton solutions [15]. The CDG equation owns the Painleve´ property as verified by Weiss in [16]. The equation can be found out to be solved by several methods, among other methods in the literature; Hirota's bilinear method [17] Hirota's direct method [15], Riccati equation method [18], tanh method [19], exp-function method [20, 21], collocation finite element approach [22].

The paper has been designated as follows: Analytical solutions of the equation are shown in Section 2 along with the graphs. In Section 3, construction of the numerical method has been done. Section 4 contains stability analysis of the numerical technique. Test problems taken from the literature have been solved and the obtained results are given in the tabular form as well as plotted graphically in Section 5. The article ends with the conclusions.

Received: 11-11-2023 **Revised:** 15-03-2024 **Accepted:** 22-03-2024 **Online:** 29-03-2024 **Published:** 31-03-2024

2. Analytical solutions

Here, we implement the direct algebraic method to the converted ODE of the investigated model by employing $u(x,t) = \mathcal{U}(\xi)$, $\xi = x + ct$, which is given by

$$c\mathcal{U}' + \alpha\mathcal{U}^2\mathcal{U}' + \beta\mathcal{U}'\mathcal{U}'' + \gamma\mathcal{U}\mathcal{U}^{(3)} + \mathcal{U}^{(5)} = 0. \tag{2.1}$$

Applying the homogeneous balance rule along with the method's framework, one gets the next general solutions of the ODE:

$$\mathcal{U}(\xi) = \sum_{i=0}^{n} a_i\,\phi(\xi)^i = a_2\,\phi(\xi)^2 + a_1\,\phi(\xi) + a_0, \tag{2.2}$$

where a_0, a_1, a_2 are arbitrary constants to be determined later. Using Eq.(2.2) along with the ODE (2.1) and the employed method's framework, obtain the values of the above-shown parameters as follows:
Set I

$$a_0 \to \frac{2a_2 d}{3}, a_1 \to 0, c \to \frac{2}{3}\left(a_2\beta d^2 + 36d^2\right), \alpha \to -\frac{6\left(a_2\beta + 2a_2\gamma + 60\right)}{a_2^2}. \tag{2.3}$$

Set II

$$a_1 \to 0, a_2 \to -\frac{60}{\beta+\gamma}, c \to \frac{-a_0^2\beta^2\gamma - 2a_0^2\beta\gamma^2 - a_0^2\gamma^3 - 80a_0\beta\gamma d - 80a_0\gamma^2 d - 160\beta d^2 - 1360\gamma d^2}{10(\beta+\gamma)}, \tag{2.4}$$

$$\alpha \to \frac{1}{10}\gamma(\beta+\gamma). \tag{2.5}$$

Set III

$$a_0 \to -\frac{40d}{\gamma}, a_1 \to 0, a_2 \to -\frac{60}{\gamma}, c \to \frac{8\left(3\gamma d^2 - 5\beta d^2\right)}{\gamma}, \alpha \to \frac{1}{10}\gamma(\beta+\gamma). \tag{2.6}$$

Thus, the soliton wave solutions of the investigated model are constructed by
for $b < 0$, we get

$$u_{\mathrm{I},1} = \frac{1}{3}a_2\left(3b\tan^2\left(\sqrt{b}\left(\frac{2}{3}t\left(a_2\beta d^2 + 36d^2\right) + x\right)\right) + 2d\right), \tag{2.7}$$

$$u_{\mathrm{I},2} = \frac{1}{3}a_2\left(3b\cot^2\left(\sqrt{b}\left(\frac{2}{3}t\left(a_2\beta d^2 + 36d^2\right) + x\right)\right) + 2d\right), \tag{2.8}$$

$$u_{\mathrm{II},1} = a_0 - \frac{60b\tan^2\left(\sqrt{b}\left(\frac{t\left(-a_0^2\beta^2\gamma - 2a_0^2\beta\gamma^2 - a_0^2\gamma^3 - 80a_0\beta\gamma d - 80a_0\gamma^2 d - 160\beta d^2 - 1360\gamma d^2\right)}{10(\beta+\gamma)} + x\right)\right)}{\beta+\gamma}, \tag{2.9}$$

$$u_{\mathrm{II},2} = a_0 - \frac{60b\cot^2\left(\sqrt{b}\left(\frac{t\left(-a_0^2\beta^2\gamma - 2a_0^2\beta\gamma^2 - a_0^2\gamma^3 - 80a_0\beta\gamma d - 80a_0\gamma^2 d - 160\beta d^2 - 1360\gamma d^2\right)}{10(\beta+\gamma)} + x\right)\right)}{\beta+\gamma}, \tag{2.10}$$

$$u_{\mathrm{III},1} = -\frac{20\left(3b\tan^2\left(\sqrt{b}\left(\frac{8t\left(3\gamma d^2 - 5\beta d^2\right)}{\gamma} + x\right)\right) + 2d\right)}{\gamma}, \tag{2.11}$$

$$u_{\mathrm{III},2} = -\frac{20\left(3b\cot^2\left(\sqrt{b}\left(\frac{8t\left(3\gamma d^2 - 5\beta d^2\right)}{\gamma} + x\right)\right) + 2d\right)}{\gamma}. \tag{2.12}$$

For $b > 0$, we get

$$u_{\mathrm{I},3} = \frac{1}{3}a_2\left(3b\tan^2\left(\sqrt{b}\left(\frac{2}{3}t\left(a_2\beta d^2 + 36d^2\right) + x\right)\right) + 2d\right), \tag{2.13}$$

$$u_{\mathrm{I},4}=\frac{1}{3}a_2\left(3b\cot^2\left(\sqrt{b}\left(\frac{2}{3}t\left(a_2\beta d^2+36d^2\right)+x\right)\right)+2d\right),$$

$$u_{\mathrm{II},3}=a_0-\frac{60b\tan^2\left(\sqrt{b}\left(\frac{t\left(-a_0^2\beta^2\gamma-2a_0^2\beta\gamma^2-a_0^2\gamma^3-80a_0\beta\gamma d-80a_0\gamma^2d-160\beta d^2-1360\gamma d^2\right)}{10(\beta+\gamma)}+x\right)\right)}{\beta+\gamma}, \tag{2.14}$$

$$u_{\mathrm{II},4}=a_0-\frac{60b\cot^2\left(\sqrt{b}\left(\frac{t\left(-a_0^2\beta^2\gamma-2a_0^2\beta\gamma^2-a_0^2\gamma^3-80a_0\beta\gamma d-80a_0\gamma^2d-160\beta d^2-1360\gamma d^2\right)}{10(\beta+\gamma)}+x\right)\right)}{\beta+\gamma}, \tag{2.15}$$

$$u_{\mathrm{III},3}=-\frac{20\left(3b\tan^2\left(\sqrt{b}\left(\frac{8t\left(3\gamma d^2-5\beta d^2\right)}{\gamma}+x\right)\right)+2d\right)}{\gamma}, \tag{2.16}$$

$$u_{\mathrm{III},4}=-\frac{20\left(3b\cot^2\left(\sqrt{b}\left(\frac{8t\left(3\gamma d^2-5\beta d^2\right)}{\gamma}+x\right)\right)+2d\right)}{\gamma}. \tag{2.17}$$

For $b=0$, we get

$$u_{\mathrm{I},5}=a_2\left(\frac{1}{\left(\frac{2}{3}t\left(a_2\beta d^2+36d^2\right)+x\right)^2}+\frac{2d}{3}\right), \tag{2.18}$$

$$u_{\mathrm{II},5}=a_0-\frac{60}{(\beta+\gamma)\left(\frac{t\left(-a_0^2\beta^2\gamma-2a_0^2\beta\gamma^2-a_0^2\gamma^3-80a_0\beta\gamma d-80a_0\gamma^2d-160\beta d^2-1360\gamma d^2\right)}{10(\beta+\gamma)}+x\right)^2}, \tag{2.19}$$

$$u_{\mathrm{III},5}=\frac{20\left(-\frac{3}{\left(\frac{8t\left(3\gamma d^2-5\beta d^2\right)}{\gamma}+x\right)^2}-2d\right)}{\gamma}. \tag{2.20}$$

The following figures belong to each exact solution family:

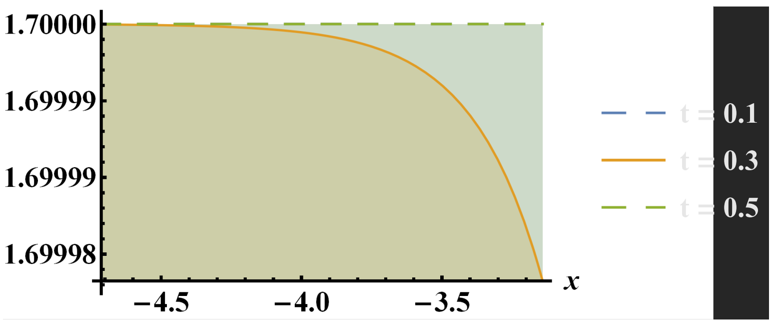

Figure 1: Graph of Set I.

3. Numerical scheme for the model problem

In this section, Eq. (1.2) has been solved by using the septic B-spline collocation method with the following boundary and initial conditions

$$\begin{array}{ll} u(a,t)=0, & u(b,t)=0,\\ u_x(a,t)=0, & u_x(b,t)=0,\\ u_{xx}(a,t)=0, & u_{xx}(b,t)=0,\\ u(x,0)=f(x), & a\le x\le b. \end{array} \tag{3.1}$$

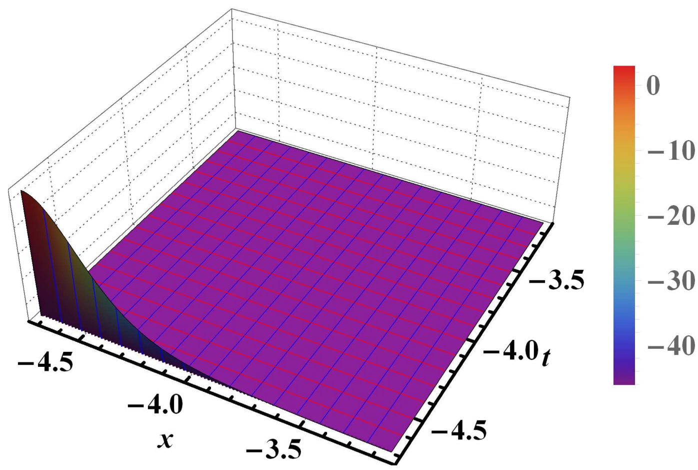

Figure 2: Graph of Set II.

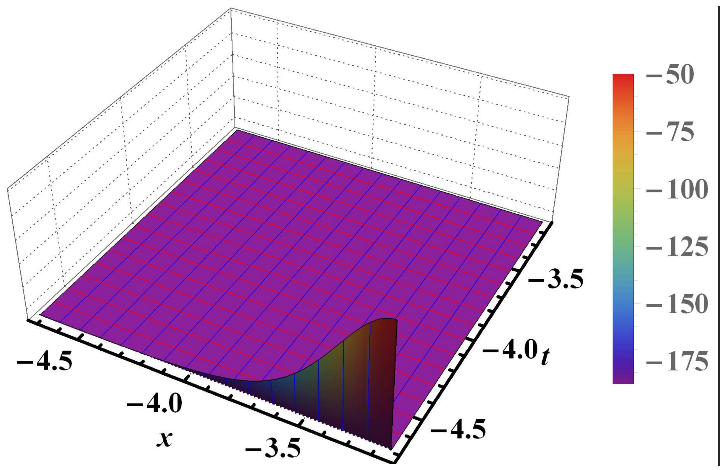

Figure 3: Graph of Set III.

Septic B-spline functions $\phi_m(x)$, $m=-3(1)N+3$, at the nodes x_m are given over the solution interval $[a,\ b]$ by Prenter [23]. In collocation method, $u_{numeric}(x,t)$ corresponding to the $u_{exact}(x,t)$ can be given as a linear combination of septic B-splines as follows [24]

$$u_N(x,t)=\sum_{m=-3}^{N+3}\phi_m(x)\sigma_m(t). \tag{3.2}$$

Implementing the following transformation $h\rho = x-x_m, \quad 0\le\rho\le 1$ to specific region $[x_m, x_{m+1}]$, the region turns to an interval of $[0,1]$ [25]. Thus the septic B-spline functions in the new region $[0,1]$ are obtained as follows:

$$\begin{aligned}
\phi_{m-3} &= 1-7\rho+21\rho^2-35\rho^3+35\rho^4-21\rho^5+7\rho^6-\rho^7,\\
\phi_{m-2} &= 120-392\rho+504\rho^2-280\rho^3+84\rho^5-42\rho^6+7\rho^7,\\
\phi_{m-1} &= 1191-1715\rho+315\rho^2+665\rho^3-315\rho^4-105\rho^5+105\rho^6-21\rho^7,\\
\phi_{m} &= 2416-1680\rho+560\rho^4-140\rho^6+35\rho^7,\\
\phi_{m+1} &= 1191+1715\rho+315\rho^2-665\rho^3-315\rho^4+105\rho^5+105\rho^6-35\rho^7,\\
\phi_{m+2} &= 120+392\rho+504\rho^2+280\rho^3-84\rho^5-42\rho^6+21\rho^7,\\
\phi_{m+3} &= 1+7\rho+21\rho^2+35\rho^3+35\rho^4+21\rho^5+7\rho^6-\rho^7,\\
\phi_{m+4} &= \rho^7.
\end{aligned} \tag{3.3}$$

Using the equalities given by (3.2) and (3.3), the following expressions are obtained:

$$\begin{aligned}
u_N(x_m,t) &= \rho_{m-3}+120\rho_{m-2}+1191\rho_{m-1}+2416\rho_m+1191\rho_{m+1}+120\rho_{m+2}+\rho_{m+3},\\
u'_m &= \tfrac{7}{h}(-\rho_{m-3}-56\rho_{m-2}-245\rho_{m-1}+245\rho_{m+1}+56\rho_{m+2}+\rho_{m+3}),\\
u''_m &= \tfrac{42}{h^2}(\rho_{m-3}+24\rho_{m-2}+15\rho_{m-1}-80\rho_m+15\rho_{m+1}+24\rho_{m+2}+\rho_{m+3}),\\
u'''_m &= \tfrac{210}{h^3}(-\rho_{m-3}-8\rho_{m-2}+19\rho_{m-1}-19\rho_{m+1}+8\rho_{m+2}+\rho_{m+3}),\\
u^{iv}_m &= \tfrac{840}{h^4}(\rho_{m-3}-9\rho_{m-1}+16\rho_m-9\rho_{m+1}+\rho_{m+3}),\\
u^{v}_m &= \tfrac{2520}{h^5}(-\rho_{m-3}+4\rho_{m-2}-5\rho_{m-1}+5\rho_{m+1}-4\rho_{m+2}+\rho_{m+3}).
\end{aligned} \tag{3.4}$$

Now, putting (3.2) and (3.4) into Eq.(1.2) and simplifying, the following system of ODEs are reached:

$$\begin{array}{l}\dot{\rho}_{m-3}+120\dot{\rho}_{m-2}+1191\dot{\rho}_{m-1}+2416\dot{\rho}_{m}+1191\dot{\rho}_{m+1}+120\dot{\rho}_{m+2}+\dot{\rho}_{m+3}\\ +(180Z_{m1}+30Z_{m2})\frac{7}{h}(-\rho_{m-3}-56\rho_{m-2}-245\rho_{m-1}+245\rho_{m+1}+56\rho_{m+2}+\rho_{m+3})\\ +30Z_{m3}\frac{210}{h^3}(-\rho_{m-3}-8\rho_{m-2}+19\rho_{m-1}-19\rho_{m+1}+8\rho_{m+2}+\rho_{m+3})\\ +\frac{2520}{h^5}(-\rho_{m-3}+4\rho_{m-2}-5\rho_{m-1}+5\rho_{m+1}-4\rho_{m+2}+\rho_{m+3})=0,\end{array} \tag{3.5}$$

where $\dot{\rho}=\frac{d\sigma}{dt}$,

$$\begin{array}{l}Z_{m1}=u^2=(\rho_{m-3}+120\rho_{m-2}+1191\rho_{m-1}+2416\rho_{m}+1191\rho_{m+1}+120\rho_{m+2}+\rho_{m+3})^2,\\ Z_{m2}=u_{xx}=\frac{42}{h^2}(\rho_{m-3}+24\rho_{m-2}+15\rho_{m-1}-80\rho_{m}+15\rho_{m+1}+24\rho_{m+2}+\rho_{m+3}),\\ Z_{m3}=u=\rho_{m-3}+120\rho_{m-2}+1191\rho_{m-1}+2416\rho_{m}+1191\rho_{m+1}+120\rho_{m+2}+\rho_{m+3}.\end{array}$$

If Crank-Nicolson scheme and forward difference approximation which are defined below is used respectively in Eq.(3.5)

$$\rho_i=\frac{\rho_i^{n+1}+\rho_i^n}{2},\ \dot{\rho}_i=\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} \tag{3.6}$$

the following iteration equation is obtained

$$\begin{array}{l}\lambda_1\rho_{m-3}^{n+1}+\lambda_2\rho_{m-2}^{n+1}+\lambda_3\rho_{m-1}^{n+1}+\lambda_4\rho_{m}^{n+1}+\lambda_5\rho_{m+1}^{n+1}+\lambda_6\rho_{m+2}^{n+1}+\lambda_7\rho_{m+3}^{n+1}\\ =\lambda_7\rho_{m-3}^{n}+\lambda_6\rho_{m-2}^{n}+\lambda_5\rho_{m-1}^{n}+\lambda_4\rho_{m}^{n}+\lambda_3\rho_{m+1}^{n}+\lambda_2\rho_{m+2}^{n}+\lambda_1\rho_{m+3}^{n},\end{array} \tag{3.7}$$

where

$$\begin{array}{l}\lambda_1=[1-E-T-M],\\ \lambda_2=[120-56E-8T+4M],\\ \lambda_3=[1191-245E+19T-5M],\\ \lambda_4=[2416],\\ \lambda_5=[1191+245E-19T+5M],\\ \lambda_6=[120+56E+8T-4M],\\ \lambda_7=[1+E+T+M],\\ E=\frac{\varpi}{2}\Delta t,\quad T=\frac{\varkappa}{2}\Delta t,\quad M=\frac{2520}{2h^5}\Delta t,\\ \varpi=[180Z_{m1}+30Z_{m2}],\\ \varkappa=[\frac{6300}{h^3}Z_{m3}].\end{array} \tag{3.8}$$

By eliminating the unknown parameters $\rho_{-3},\rho_{-2},\rho_{-1},\rho_{N+1},\rho_{N+2}$, and ρ_{N+3} which are not in the solution region of the problem, the system of equations given by (3.7) becomes solvable. This procedure can be easily done using the values of u and boundary conditions, and then the following system

$$R\mathbf{d}^{\mathbf{n+1}}=S\mathbf{d}^{\mathbf{n}} \tag{3.9}$$

is obtained where $d^n=(\rho_0,\rho_1,...,\rho_N)^T$.

4. Stability Analysis

For the stability analysis, Von Neumann technique has been used. In a typical amplitude mode, we can define the magnification factor ξ of the error as follows [26, 27]:

$$\rho_m^n=\xi^n e^{imkh}. \tag{4.1}$$

Using (4.1) into the (3.7),

$$\xi=\frac{\rho_1-i\rho_2}{\rho_1+i\rho_2}, \tag{4.2}$$

is obtained and in which

$$\begin{array}{l}\rho_1=2\cos(3kh)+240\cos(2kh)+2382\cos(kh)+2416,\\ \rho_2=(2M+2T+2E)\sin(3kh),\end{array} \tag{4.3}$$

so that $|\xi|=1$, which proves unconditional stability of the linearized numerical scheme for the CDG equation.

5. Numerical Experiments and Discussions

In this section, the proposed scheme is applied for solution of CDG equation for different values of the time and space division and we approximate them using the described scheme. Error norms, namely L_2 and L_∞, are used in order to check the method [28, 29]:

$$L_2 = \left\|u^{exact} - u_N\right\|_2 \simeq \sqrt{h\sum_{j=1}^{N}\left|u_j^{exact} - (u_N)_j\right|^2}, \tag{5.1}$$

and

$$L_\infty = \left\|u^{exact} - u_N\right\|_\infty \simeq \max_j \left|u_j^{exact} - (u_N)_j\right|, \quad j = 1,2,\ldots,N. \tag{5.2}$$

The CDG equation has an exact solution of the form [22]

$$u(x,t) = \frac{k^2\exp(k(x-k^4t))}{(1+\exp(k(x-k^4t)))^2}, \tag{5.3}$$

and the equation will be examined with the boundary-initial condition which is

$$u(x,0) = f(x) = \frac{k^2\exp(kx)}{(1+\exp(kx))^2}, \tag{5.4}$$

where $k = 1$ and $u \to 0$ as $x \to \pm\infty$.

To prove accuracy of our numerical algorithm, interval of the problem is chosen as $[-15, 15]$ and up to time $t = 1$. In simulation calculations in terms of compliance comply with the literature, as common values $\Delta t = 0.0004$ and 0.0001 with $h = 0.5$ and 0.05 are chosen. In Tables $(1-3)$, values of the error norms L_2 and L_∞ calculated over these values for time levels and step sizes are presented. So, it can be seen more clearly how the amount of collocation points have an effect on the method. When tables are examined, the calculated error norms L_2 and L_∞ are obtained to be marginally small. It is clear that the minimum L_∞ error norm 2.4892×10^{-5} with the parameters $\Delta t = 0.0001$ and $h = 0.05$. These errors hardly change as time progresses. Moreover, it can be said from the tables that the values of the error norms are compatible with the exact solution and the numerical solution, and the method is quite efficient. Two and three dimensional forms of bell-shaped solitary wave solutions produced from $t = 0$ to $t = 1$ are clearly seen in Figure (4). Besides, the contour line for the movement of the individual wave is plotted in Figure (4). It can be indicated that the wave maintains its amplitude and shape as time passes from these figures. Also, error distribution is shown at $t = 1$ for different values of h and Δt in Figure (5).

Table 1: Error norms for $k = 0.01$ and different values of h and Δt.

	$\Delta t = 0.0004, h = 0.5$		$\Delta t = 0.0001, h = 0.05$	
t	L_2	L_∞	L_2	L_∞
0.1	.0000494593	.0000249293	.0000414945	.0000270235
0.2	.0000532876	.0000252706	.0000465528	.0000248927
0.3	.0000532946	.0000257885	.0000497414	.0000271132
0.4	.0000537765	.0000249308	.0000557418	.0000303556
0.5	.0000544981	.0000255910	.0000617223	.0000337686
0.6	.0000582073	.0000249191	.0000619409	.0000441115
0.7	.0000563601	.0000249129	.0000679581	.0000468114
0.8	.0000553124	.0000249021	.0000803376	.0000475394
0.9	.0000559314	.0000256542	.0000912949	.0000595731
1.0	.0000587193	.0000256739	.0001058028	.0000597128

6. Conclusion

In this study, two important goals have been executed: Generating the direct algebraic method for obtaining exact solutions of the CDG equation and based on septic B-spline approximation, a collocation method has been introduced and performed for the numerical solution of CDG equation by taking into consideration different parameter values of test problem. The von Neumann method has been applied rigorously to check stability of the numerical scheme and the method has been proved to be unconditionally stable. The algorithm is run with a single solitary wave motion whose exact solution is known to perform

Table 2: The error norms for $k = 0.01$, $t = 0.0001$ and various values of h.

h	L_2	L_∞
0.25	.0000419227	.0000276440
0.1	.0000337090	.0000230659
0.01	.0000339219	.0000366059
0.05	.0000317705	.0000242181
0.025	.0000335113	.0000341147
1.0	.0000510913	.0000283967

Table 3: The error norms for$k = 0.01$,$h = 0.1$ and various values of Δt.

Δt	L_2	L_∞
0.04	.0000431579	.0000492631
0.02	.0000405068	.0000491365
0.01	.0000380355	.0000489782
0.001	.0000266854	.0000293648
0.005	.0000323331	.0000310227
0.0025	.0000298689	.0000306730
0.00125	.0000274591	.0000298586

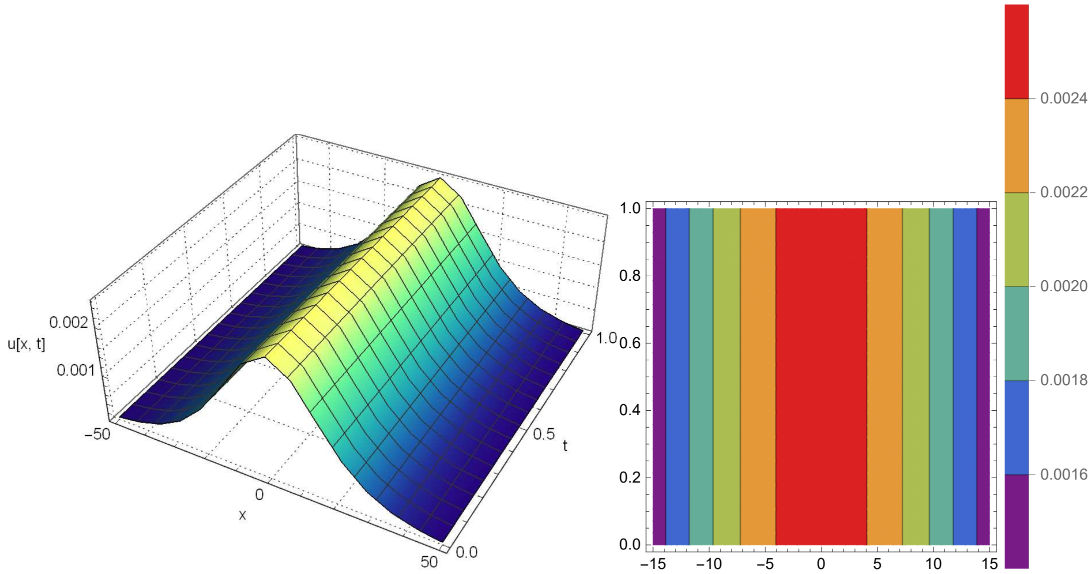

Figure 4: Motion of single solitary wave and its contour line for $\Delta t = 0.0004$ and $h = 0.5$.

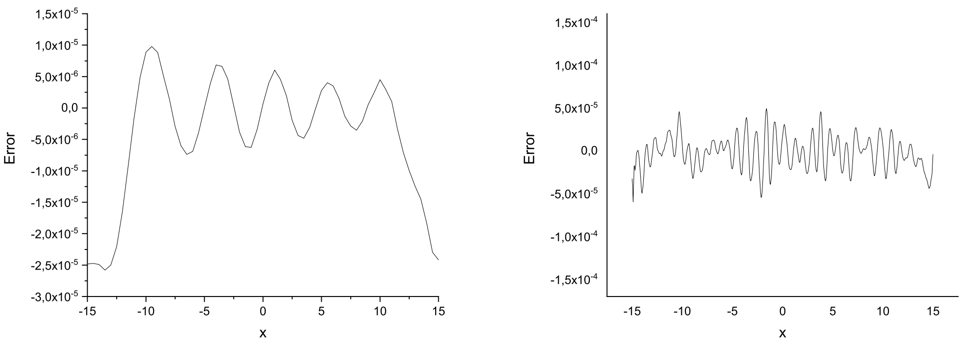

Figure 5: Error distributions at $t = 1$ for the parameters with $h = 0.05$; $\Delta t = 0.0004$; $h = 0.05$ and $\Delta t = 0.0001$.

numerical experiments. The obtained solutions from both methods are plotted graphically to check the dynamical behavior of the solutions. The reliability and efficiency of the numerical method have been evaluated using L_2 and L_∞ error norms and it can be seen that the obtained results are quite good. Finally, it is said that the approach applied in this study can be easily applied to other nonlinear evolutions and good results can be achieved.

Declarations

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: Conceptualization, S.B.G.K.; methodology D.Y.S.; validation, S.B.G.K.; investigation, S.B.G.K. and D.Y.S.; resources, D.Y.S.; data curation, S.B.G.K.; writing—original draft preparation, D.Y.S.; writing—review and editing, S.B.G.K. and D.Y.S.; supervision, S.B.G.K. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest Disclosure: The authors declare no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

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Fundamental Journal of Mathematics and Applications (FUJMA), (Fundam. J. Math. Appl.)
https://dergipark.org.tr/en/pub/fujma

How to cite this article: S.B.G. Karakoç and D. Yıldırım Sucu, *New exact and numerical experiments for the Caudrey-Dodd-Gibbon equation*, Fundam. J. Math. Appl., **7**(1) (2024), 26-34. DOI 10.33401/fujma.1389595

Fundamental Journal of Mathematics and Applications, 7(1) (2024), 35-52
Research Paper / Open Access

Fundamental Journal of Mathematics and Applications

ISSN Online: 2645-8845
www.dergipark.org.tr/en/pub/fujma
https://doi.org/10.33401/fujma.1423906

On an $(\imath, x_0)$-Generalized Logistic-Type Function

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Article Information

Keywords: Activation function; Logistic function; Matplotlib; neural networks; NumPy; Sigmoid function; Soft computing; Statistics; Survival analysis; SymPy.

AMS 2020 Classification: 05C38; 15A15; 05A15; 15A18

Abstract

In this article, some mathematical properties of $(\imath, x_0)$-generalized logistic-type function are presented. This four-parameter generalized function can be considered as a statistical phenomenon enhancing more vigorous survival analysis models. Moreover, the behaviors of the relevant parametric functions obtained are examined with graphics using computer programming language Python 3.9.

1. Introduction

1.1. Motivation

Crudely put, the logistic and logistic-type functions play an important role in many scientific disciplines including probability and statistics, demography, machine learning, ecology, mathematical psychology and biology [1]. Actually, the logistic function has a long history dating back to the classical statistics and "belief neural networks" [2], [3]. It has a leading role in the logistic regression procedure, especially in terms of its statistical properties that we discuss here.While in early studies this appeared as the solution to a specific differential equation, it was later used as one of many possible smooth, monotonic "squash" functions that mapped real values to a limited range.

Over time, as a result of the increasing interest and need for learning concept and learning algorithms, the probabilistic properties of the logistic function have begun to be studied in depth. This orientation has led to more advanced learning methods. So, it has diversified and strengthened the connections between neural networks (NNs) and statistics.

Methods that preserve the logistic function offer a possibility in this context. So, as alternative methods to contingency table and general regression model; a simple artificial neural network architecture, a more comprehensive generalized additive model, or another flexible "approximate" model in logistic form may be a reason for preference. An example for generalized linear model is the generalization of logistic regression while probabilistic model for multi-class classification problem is a multinomial model. In this models, it is a reasonable approach to use a normalized exponential function as a logistic function, aka "softmax" function, which is defined below, and used intensively in the NNs literature [4], [5], [6].

Now, let $\sigma : \mathbb{R}^N \longmapsto (0,1)^N$ be a function defined by the formula

$$\sigma(j, z_1, z_2, ..., z_N) = \frac{e^{z_j}}{\sum_{j=1}^{N} e^{z_j}},$$

for $N \geq 1$. This function σ called as "unit softmax function" employs the classical exponential function to each of the inputs denoted by $z_1, z_2, ..., z_N$ and all these values are normalized by being divided by the sum of all the exponentials. The normalization process provides that the sum of the components of the output vector is 1. In addition, the softmax function

Received: 22-01-2024 **Revised:** 22-02-2024 **Accepted:** 04-03-2024 **Online:** 29-03-2024 **Published:** 31-03-2024

takes as inputs $z_1, z_2, ..., z_N$, and normalizes them into a probability distribution consisting of N probabilities proportional to the exponentials of the input numbers [7]. Moreover, this function takes values between 0 and 1. Here we give an $(\imath, x_0)$-generalized logistic-type function (also can be considered as a parametric generalization of softmax function) and also examine some mathematical properties such as convexity, sub-additivity, and multiplicativity.
The present paper includes four sections. In the following section, the construction of suggested $(\imath, x_0)$-generalized logistic-type function, and its analytical features are presented. After launching a brief introduction related to survival analysis; probability density function of related distribution, parametric exponential survival (PES) and parametric failure (hazard) rate (PFR) functions are given in the third section. We conclude the paper creating "ceteris paribus" graphics of these functions employing the computer programming language Python 3.9. Finally, we also add Python 3.9 codes as in Fig. 9 and Fig 10 at the end of the study to motivate readers to earn/develop her/his programming language ability.

2. Main Results

Let $\imath, \rho > 0$ be the parameters with $\xi > 1$; inspired by [8] and [9], we can consider an $(\imath, x_0)$-generalized logistic-type function as follows:

$$\Psi_{\rho,\imath}(x) = \frac{1}{1+\rho\xi^{-\imath(x-x_0)}} = \frac{\xi^{\imath(x-x_0)}}{\rho+\xi^{\imath(x-x_0)}}, \tag{2.1}$$

where $x, x_0 \in \mathbb{R}$.
The first and second derivatives of the $(\imath, x_0)$-generalized logistic-type function $\Psi_{\rho,\imath}$ are given as below: let $\imath, \rho > 0$ be the parameters, and $\xi > 1$

$$\begin{aligned}\Psi'_{\rho,\imath}(x) &= \left(\frac{1}{1+\rho\xi^{-\imath(x-x_0)}}\right)' = \rho\imath(\ln\xi)\,\xi^{-\imath(x-x_0)}\left(1+\rho\xi^{-\imath(x-x_0)}\right)^{-2} \\ &= \frac{\rho\imath(\ln\xi)}{\left(1+2\rho\xi^{-\imath(x-x_0)}+\rho^2\xi^{-2\imath(x-x_0)}\right)\xi^{\imath(x-x_0)}} \\ &= \frac{\rho\imath(\ln\xi)}{\left(\xi^{\imath(x-x_0)}+2\rho+\rho^2\xi^{-\imath(x-x_0)}\right)} = \rho\imath(\ln\xi)\left(\xi^{\imath(x-x_0)}+2\rho+\rho^2\xi^{-\imath(x-x_0)}\right)^{-1}\end{aligned}$$

for all $x, x_0 \in \mathbb{R}$.

Besides, taking the second derivative of (2.1) for $x \in \mathbb{R}$ we get

$$\Psi''_{\rho,\imath}(x) = \rho\imath^2\left(\ln^2\xi\right)\left(\xi^{\imath(x-x_0)}+2\rho+\rho^2\xi^{-\imath(x-x_0)}\right)^{-2}\left(\rho^2\xi^{-\imath(x-x_0)}-\xi^{\imath(x-x_0)}\right)$$

Since

$$\Psi''_{\rho,\imath}(x) > 0 \Longleftrightarrow \left(\rho^2\xi^{-\imath(x-x_0)}-\xi^{\imath(x-x_0)}\right) > 0$$

$$\Leftrightarrow \rho^2\xi^{-\imath(x-x_0)} > \xi^{\imath(x-x_0)}$$

$$\Leftrightarrow \rho^2 > \xi^{2\imath(x-x_0)} \Leftrightarrow |\rho| > \left|\xi^{\imath(x-x_0)}\right| \Leftrightarrow \rho > \xi^{\imath(x-x_0)},$$

and for $\rho > 0, \xi > 1$

$$\log_\xi \rho > \imath(x-x_0) \Leftrightarrow \frac{\log_\xi \rho}{\imath} + x_0 > x$$

is obtained.
Let $x < x_0 + \frac{\log_\xi \rho}{\imath} - 1$, then $x-1 < x+1 < x_0 + \frac{\log_\xi \rho}{\imath}$.
$\Psi'_{\rho,\imath}(x+1) > \Psi'_{\rho,\imath}(x-1)$. Thus $\Psi'_{\rho,\imath}(x)$ is positive and strictly increasing on $\left(-\infty,\ x_0 + \frac{\log_\xi \rho}{\imath}\right)$. Now, let $x > x_0 + \frac{\log_\xi \rho}{\imath} + 1$, then $x+1 > x-1 > x_0 + \frac{\log_\xi \rho}{\imath}$, and $\Psi'_{\rho,\imath}(x+1) < \Psi'_{\rho,\imath}(x-1)$. So $\Psi'_{\rho,\imath}(x)$ is strictly decreasing on $\left(x_0 + \frac{\log_\xi \rho}{\imath}, +\infty\right)$.

Proposition 2.1. *Let $\imath, \rho > 0$ be the parameters, $\xi > 1$, and $\Psi_{\rho,\imath}(x)$ be described as in (2.1). Now, let us take the first derivative:*

$$\begin{aligned}\Psi'_{\rho,\imath}(x) &= \rho\imath(\ln\xi)\,\xi^{-\imath(x-x_0)}\frac{1}{\left(1+\rho\xi^{-\imath(x-x_0)}\right)^2} \\ &= \imath(\ln\xi)\,\Psi_{\rho,\imath}(x)\left(1-\Psi_{\rho,\imath}(x)\right).\end{aligned} \tag{2.2}$$

By then, let's take the second derivative of the function $\Psi_{\rho,\iota}$:

$$\begin{aligned}
\Psi''_{\rho,\iota}(x) &= \iota(\ln\xi)\left(\Psi_{\rho,\iota}(x)-\Psi^2_{\rho,\iota}(x)\right)' \\
&= \iota(\ln\xi)\Psi'_{\rho,\iota}(x)\left(1-2\Psi_{\rho,\iota}(x)\right) \\
&= \iota^2\left(\ln^2\xi\right)\Psi_{\rho,\iota}(x)\left(1-\Psi_{\rho,\iota}(x)\right)\left(1-2\Psi_{\rho,\iota}(x)\right).
\end{aligned}$$

Thus the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ has the following properties:

$$\lim_{x\to+\infty}\Psi_{\rho,\iota}(x)=\lim_{x\to+\infty}\frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}=1,$$

$$\lim_{x\to-\infty}\Psi_{\rho,\iota}(x)=\lim_{x\to-\infty}\frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}=0,$$

$$\lim_{x\to x_0}\Psi_{\rho,\iota}(x)=\lim_{x\to x_0}\frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}=\frac{1}{1+\rho};\rho>0,$$

$$\lim_{x\to x_0}\Psi'_{\rho,\iota}(x)=\lim_{x\to x_0}\frac{\rho\iota(\ln\xi)}{\xi^{\iota(x-x_0)}\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2}=\frac{\rho\iota(\ln\xi)}{(1+\rho)^2},$$

$$\lim_{x\to-\infty}\Psi'_{\rho,\iota}(x)=\lim_{x\to-\infty}\rho\iota(\ln\xi)\ \frac{1}{\xi^{\iota(x-x_0)}+\rho^2\xi^{-\iota(x-x_0)}+2\rho}=0,$$

and

$$\int\Psi_{\rho,\iota}(x)\,dx=\int\frac{\xi^{\iota(x-x_0)}}{\rho+\xi^{\iota(x-x_0)}}dx=\frac{1}{\iota(\ln\xi)}\ln\left(\rho+\xi^{\iota(x-x_0)}\right)+C,\ C\text{ is a constant.} \tag{2.3}$$

Remark 2.2. *Additionally, if $\xi=e$, then (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ acts like an $\iota-$generalization of softplus function (see [1]). The derivative of (2.3) yields the $\iota-$generalized logistic-type function.*

Proposition 2.3. *From (2.2), $\Psi_{\rho,\iota}(x)$ is increasing and positive on $\left(-\infty,\ x_0+\frac{\log_\xi\rho}{\iota}\right)$. Furthermore, $l:=\Psi_{\rho,\iota}(x)$ is a solution to the initial value problem*

$$\left\{l'=\iota(\ln\xi)\,l(1-l),\ l(x_0)=\frac{1}{\rho+1};\ \rho>0.\right.$$

Theorem 2.4. *The (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ satisfies the following inequality:*

$$\Psi_{\rho,\iota}(x+y)<\Psi_{\rho,\iota}(x)+\Psi_{\rho,\iota}(y),$$

for $x_0\geq 0$, $\iota,\rho>0$, $x,y\in(-\infty,0)$, and also $x,y\in\left(x_0+\frac{\log_\xi\rho}{\iota},+\infty\right)$. In other words, the function $\Psi_{\rho,\iota}$ is sub-additive on $(-\infty,0)\cup\left(x_0+\frac{\log_\xi\rho}{\iota},+\infty\right)$.

Proof. We need to prove the cases $x,y\in(-\infty,0)$ and $x,y\in\left(x_0+\frac{\log_\xi\rho}{\iota},+\infty\right)$, respectively.
The case $x=y=0$ is straightforward.
For any fixed y: we obtain

$$\begin{aligned}
\varphi_{\rho,\iota}(x,y) &: = \Psi_{\rho,\iota}(x+y)-\Psi_{\rho,\iota}(x)-\Psi_{\rho,\iota}(y) \\
&= \frac{1}{1+\rho\xi^{-\iota(x+y-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(y-x_0)}} \\
&= \frac{\xi^{\iota(x+y-x_0)}}{\xi^{\iota(x+y-x_0)}+\rho}-\frac{\xi^{\iota(x-x_0)}}{\xi^{\iota(x-x_0)}+\rho}-\frac{\xi^{\iota(y-x_0)}}{\xi^{\iota(y-x_0)}+\rho},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial x}\varphi_{\rho,\iota}(x,y) &= \frac{\partial}{\partial x}\left(\frac{1}{1+\rho\xi^{-\iota(x+y-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(y-x_0)}}\right)\\
&= \frac{-\left(\rho\xi^{-\iota(x+y-x_0)}(-\iota)(\ln\xi)\right)}{\left(1+\rho\xi^{-\iota(x+y-x_0)}\right)^2}-\left(\frac{-\left(\rho\xi^{-\iota(x-x_0)}(-\iota)(\ln\xi)\right)}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2}\right)\\
&= \frac{\iota\rho(\ln\xi)\xi^{-\iota(x+y-x_0)}}{\left(1+\rho\xi^{-\iota(x+y-x_0)}\right)^2}-\frac{\iota\rho(\ln\xi)\xi^{-\iota(x-x_0)}}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2}.
\end{aligned}$$

For $\Psi'_{\rho,\iota}(x)$ is decreasing on $(x_0,+\infty)$, hence $\Psi_{\rho,\iota}(x)$ is decreasing on the same interval. Then for $x,y\in(x_0,+\infty)$, we can have

$$\begin{aligned}
\varphi_{\rho,\iota}(x,y) &< \varphi_{\rho,\iota}\left(x,x_0+\frac{\log_\xi\rho}{\iota}\right)\\
&= \lim_{x\to x_0+\frac{\log_\xi\rho}{\iota}}\varphi_{\rho,\iota}\left(x,x_0+\frac{\log_\xi\rho}{\iota}\right)\\
&= \varphi_{\rho,\iota}\left(x_0+\frac{\log_\xi\rho}{\iota},x_0+\frac{\log_\xi\rho}{\iota}\right)\\
&= \Psi_{\rho,\iota}\left(2x_0+\frac{2\log_\xi\rho}{\iota}\right)-\Psi_{\rho,\iota}\left(x_0+\frac{\log_\xi\rho}{\iota}\right)-\Psi_{\rho,\iota}\left(x_0+\frac{\log_\xi\rho}{\iota}\right)\\
&= \frac{1}{1+\rho\xi^{-\iota\left(2x_0+\frac{2\log_\xi\rho}{\iota}-x_0\right)}}-\frac{2}{1+\rho\xi^{-\iota\left(x_0+\frac{\log_\xi\rho}{\iota}-x_0\right)}}\\
&= \frac{1}{1+\rho\xi^{-\iota\left(x_0+\frac{2\log_\xi\rho}{\iota}\right)}}-\frac{2}{1+\rho\frac{1}{\rho}}=\frac{1}{1+\rho\xi^{-\iota\left(x_0+\frac{2\log_\xi\rho}{\iota}\right)}}-1\\
&= -\frac{\rho\xi^{-\iota\left(x_0+\frac{2\log_\xi\rho}{\iota}\right)}}{1+\rho\xi^{-\iota\left(x_0+\frac{2\log_\xi\rho}{\iota}\right)}}<0.
\end{aligned}$$

Thus $\varphi_{\rho,\iota}$ is increasing on $\left(-\infty,x_0+\frac{\log_\xi\rho}{\iota}\right)$.
We have

$$\begin{aligned}
\varphi_{\rho,\iota}(x,y) &< \varphi_{\rho,\iota}(x,0)=\lim_{x\to 0}\varphi_{\rho,\iota}(x,0)\\
&= \lim_{x\to 0}\left(\Psi_{\rho,\iota}(x+0)-\Psi_{\rho,\iota}(x)-\Psi_{\rho,\iota}(0)\right)\\
&= \lim_{x\to 0}\left\{\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(x-x_0)}}-\frac{1}{1+\rho\xi^{-\iota(-x_0)}}\right\}\\
&= \lim_{x\to 0}-\frac{1}{1+\rho\xi^{\iota x_0}}<0.
\end{aligned}$$

□

Remark 2.5. *In Theorem 2.4; if we take $x_0=0,\iota>0,\xi=e$, and $\rho=1$ then $\varphi_{\rho,\iota}(x,y)$ becomes sub-additive on $(-\infty,+\infty)$.*

For $\iota>0$, $x_0\in(-\infty,+\infty)$, and $y\in(0,+\infty)$ the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ fulfills the followings:

(i)

$$1<\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)}<\xi^{\iota y},\ \forall x\in(-\infty,+\infty),$$

(ii)

$$\frac{2\xi^{\iota y}}{1+\xi^{\iota y}}<\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)}<\xi^{\iota y},\forall x\in\left(-\infty,x_0+\frac{\log_\xi\rho}{\iota}\right),$$

(iii)

$$1<\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)}<\frac{2\xi^{\iota y}}{1+\xi^{\iota y}},\forall x\in\left(x_0+\frac{\log_\xi\rho}{\iota},+\infty\right).$$

Proof. Since for all $x\in(-\infty,+\infty)$,

$$\begin{aligned}\left(\frac{\Psi'_{\rho,\iota}(x)}{\Psi_{\rho,\iota}(x)}\right)' &= \left(\iota(\ln\xi)\left(1-\Psi_{\rho,\iota}(x)\right)\right)'=\iota(\ln\xi)\left(1-\Psi_{\rho,\iota}(x)\right)'\\ &= -\frac{\rho\iota^2\left(\ln^2\xi\right)}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2\xi^{\iota(x-x_0)}}<0.\end{aligned}$$

Then

$$\left(\frac{\Psi'_{\rho,\iota}(x)}{\Psi_{\rho,\iota}(x)}\right)'<-\frac{\rho\iota^2\left(\ln^2\xi\right)}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2}<0,\forall x\in(-\infty,+\infty).$$

Hence, the function $\frac{\Psi'_{\rho,\iota}(x)}{\Psi_{\rho,\iota}(x)}$ is decreasing on $(-\infty,+\infty)$.
Let

$$\aleph(x):=\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)},x\in(-\infty,+\infty),$$

and

$$\nu(x)=\log_e\aleph(x)=\ln\aleph(x).$$

So

$$\begin{aligned}\aleph'(x) &= \frac{\Psi'_{\rho,\iota}(x+y)\Psi_{\rho,\iota}(x)-\Psi'_{\rho,\iota}(x)\Psi_{\rho,\iota}(x+y)}{\Psi^2_{\rho,\iota}(x)}\\ &= \frac{\Psi'_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)}-\frac{\Psi'_{\rho,\iota}(x)\Psi_{\rho,\iota}(x+y)}{\Psi^2_{\rho,\iota}(x)},\end{aligned}$$

and also one has

$$\begin{aligned}\nu'(x) &= \frac{\Psi'_{\rho,\iota}(x+y)\Psi_{\rho,\iota}(x)-\Psi'_{\rho,\iota}(x)\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(x+y)}\\ &= \frac{\Psi'_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x+y)}-\frac{\Psi'_{\rho,\iota}(x)}{\Psi_{\rho,\iota}(x)}<0.\end{aligned}$$

Therefore, $\nu(x)$ and $\aleph(x)$ are both decreasing.
Accordingly,

$$\begin{aligned}\lim_{x\to+\infty}\aleph(x) &= \lim_{x\to+\infty}\frac{\Psi_{\rho,\iota}(x+y)}{\Psi_{\rho,\iota}(x)}\\ &= \lim_{x\to+\infty}\left(\frac{1+\rho\xi^{-\iota(x-x_0)}}{1+\rho\xi^{-\iota(x+y-x_0)}}\right)\frac{\xi^{\iota(x-x_0)}}{\xi^{\iota(x-x_0)}}\\ &= \lim_{x\to+\infty}\frac{\xi^{\iota(x-x_0)}+\rho}{\xi^{\iota(x-x_0)}+\rho\xi^{-\iota y}}=1,\end{aligned}$$

and

$$\lim_{x\to-\infty}\aleph(x)=\lim_{x\to-\infty}\frac{1+\rho\xi^{-\iota(x-x_0)}}{1+\rho\xi^{-\iota(x+y-x_0)}}=\xi^{\iota y},$$

$$1=\lim_{x\to+\infty}\aleph(x)<\aleph(x)<\lim_{x\to-\infty}\aleph(x)=\xi^{\iota y},x\in(-\infty,+\infty),$$

and also

$$\lim_{x\to x_0+\frac{\log_\xi \rho}{\iota}} \frac{1+\rho\xi^{-\iota(x-x_0)}}{1+\rho\xi^{-\iota(x+y-x_0)}} = \frac{2\xi^{\iota y}}{1+\xi^{\iota y}},$$

$$1=\lim_{x\to+\infty}\aleph(x)<\aleph(x)<\lim_{x\to x_0+\frac{\log_\xi \rho}{\iota}}\aleph(x)=\frac{2\xi^{\iota y}}{1+\xi^{\iota y}}, x\in\left(x_0+\frac{\log_\xi \rho}{\iota},+\infty\right).$$

Corollary 2.6. *For $\iota>0$ and $x_0\in(-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ yields inequalities below:*

$$1<\frac{\Psi_{\rho,\iota}\left(x+\frac{1}{\iota}\right)}{\Psi_{\rho,\iota}(x)}<\xi; x\in(-\infty,+\infty),$$

$$\frac{2\rho\xi}{1+\rho\xi}<\frac{\Psi_{\rho,\iota}\left(x+\frac{1}{\iota}\right)}{\Psi_{\rho,\iota}(x)}<\xi;\ x\in(-\infty,x_0),$$

and also

$$1<\frac{\Psi_{\rho,\iota}\left(x+\frac{1}{\iota}\right)}{\Psi_{\rho,\iota}(x)}<\frac{2\rho\xi}{1+\rho\xi}.$$

□

Corollary 2.7. *(see [10], [11], [12]) Let S be an open subinterval of $(0,\infty)$, and let $g:S\longrightarrow(0,\infty)$ be differentiable. g is AH-convex (concave)$\Longleftrightarrow \frac{g'(x)}{g^2(x)}$ is increasing (decreasing).*

Theorem 2.8. *For $\iota>0$ and $x_0\in[0,\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ is AH-concave on $(x_0,+\infty)$. Namely,*

$$\Psi_{\rho,\iota}\left(\frac{x+y}{2}\right)\geq\frac{2\Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y)}{\Psi_{\rho,\iota}(x)+\Psi_{\rho,\iota}(y)}, x\in(x_0,+\infty).$$

Proof. Let us take

$$\Psi'_{\rho,\iota}(x)=\frac{\iota\rho(\ln\xi)\xi^{-\iota(x-x_0)}}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2},$$

and

$$\Psi^2_{\rho,\iota}(x)=\left(1+\rho\xi^{-\iota(x-x_0)}\right)^{-2}.$$

Then

$$\begin{aligned}\left(\frac{\Psi'_{\rho,\iota}(x)}{\Psi^2_{\rho,\iota}(x)}\right)' &= \left(\iota\rho(\ln\xi)\xi^{-\iota(x-x_0)}\right)' \\ &= -\iota^2\left(\ln^2\xi\right)\xi^{-\iota(x-x_0)}<0.\end{aligned}$$

One has the desired result by Corollary 2.7. □

Theorem 2.9. *For $\iota>0$ and $x_0\in(-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ is logarithmically concave on $(-\infty,+\infty)$. Namely, for all $x,y\in(-\infty,+\infty)$; $z,p>1$ and $\frac{1}{z}+\frac{1}{p}=1$, the following inequality holds:*

$$\Psi_{\rho,\iota}\left(\frac{x}{z}+\frac{y}{p}\right)\geq\left[\Psi_{\rho,\iota}(x)\right]^{\frac{1}{z}}\left[\Psi_{\rho,\iota}(y)\right]^{\frac{1}{p}}. \tag{2.4}$$

Proof. Let

$$D_{\rho,\iota}\ (x) := \ln \Psi_{\rho,\iota}(x) = \log_e \Psi_{\rho,\iota}(x) = \log_e \left(\frac{1}{1+\rho\xi^{-\iota(x-x_0)}} \right),$$

$$\ln\left(\frac{1}{1+\rho\xi^{-\iota(x-x_0)}} \right) = \ln 1 - \ln\left(1+\rho\xi^{-\iota(x-x_0)}\right),$$

thus

$$D_{\rho,\iota}\ (x) = -\ln\left(1+\rho\xi^{-\iota(x-x_0)}\right).$$

Now take the first derivative of $D_{\rho,\iota}$,

$$D'_{\rho,\iota}\ (x) = \iota\rho\left(\ln\xi\right)\frac{\xi^{-\iota(x-x_0)}}{1+\rho\xi^{-\iota(x-x_0)}},$$

and also the second derivative of $D_{\rho,\iota}$ yields the following:

$$D''_{\rho,\iota}\ (x) = -\iota^2\rho\left(\ln^2\xi\right)\frac{\xi^{-\iota(x-x_0)}}{\left(1+\rho\xi^{-\iota(x-x_0)}\right)^2} < 0,$$

which indicates the inequality in (2.4). □

Theorem 2.10. *For $\iota > 0$ and $x_0 \in (-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ verifies the following inequalities:*

$$\Psi^2_{\rho,\iota}(x+y) \geq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y);\ x,y \in [0,+\infty),$$

and

$$\Psi^2_{\rho,\iota}(x+y) \leq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y);\ x,y \in (-\infty,0].$$

Furthermore, for $x = y = 0$, equality is satisfied.

Proof. For $\iota > 0$, $x,y \in [0,+\infty)$; $x+y \geq x$ and $x+y \geq y$ are valid. Since $\Psi_{\rho,\iota}(x)$ is increasing,

$$\Psi_{\rho,\iota}(x+y) \geq \Psi_{\rho,\iota}(x), \tag{2.5}$$

and

$$\Psi_{\rho,\iota}(x+y) \geq \Psi_{\rho,\iota}(y). \tag{2.6}$$

So, the product of (2.5) and (2.6) demonstrates the first inequality. Using the similar mindset, the second one may be proved. □

Theorem 2.11. *For $\iota > 0$ and $x_0 \in (-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ satisfies the inequalities below:*

$$\Psi^2_{\rho,\iota}(xy) \leq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y); x,y \in (0,1],$$

and

$$\Psi^2_{\rho,\iota}(xy) \geq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y);\ x,y \in [1,+\infty).$$

Proof. For $x,y \in (0,1]$, $xy \leq x$ and $xy \leq y$ are true.
As $\Psi_{\rho,\iota}(x)$ is increasing,

$$\Psi_{\rho,\iota}(x) \geq \Psi_{\rho,\iota}(xy) > 0,$$

and

$$\Psi_{\rho,\iota}(y) \geq \Psi_{\rho,\iota}(xy) > 0.$$

are satisfied. Furthermore, product of these two inequalities yields:

$$\Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y) \geq \Psi^2_{\rho,\iota}(xy).$$

Namely,

$$\Psi^2_{\rho,\iota}(xy) \leq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y)$$

is obtained.
Since for $x,y \in [1,+\infty)$, there exist $xy \geq x, xy \geq y$ and $\Psi_{\rho,\iota}(x)$ is increasing. Then

$$\Psi_{\rho,\iota}(x) \leq \Psi_{\rho,\iota}(xy),$$

and

$$\Psi_{\rho,\iota}(y) \leq \Psi_{\rho,\iota}(xy).$$

Multiplication of the last two inequalities gives the following:

$$\Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y) \leq \Psi^2_{\rho,\iota}(xy).$$

Below, one has the desired inequality:

$$\Psi^2_{\rho,\iota}(xy) \geq \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y).$$

□

Theorem 2.12. *For $\iota > 0$ and $x_0 \in (-\infty,+\infty)$, the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ is supermultiplicative on $(1,+\infty)$.*

$$\Psi_{\rho,\iota}(xy) > \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y);\ x,y \in (-\infty,+\infty)$$

holds.

Proof. For $0 < \Psi_{\rho,\iota}(u) < 1$, then

$$\Psi^2_{\rho,\iota}(u) < \Psi_{\rho,\iota}(u)$$

for $u \in (-\infty,+\infty)$. Since $\Psi_{\rho,\iota}$ is increasing, and $xy \geq x$, $xy > y$ on $(1,+\infty)$,

$$\Psi_{\rho,\iota}(xy) > \Psi^2_{\rho,\iota}(xy) > \Psi_{\rho,\iota}(x)\Psi_{\rho,\iota}(y)$$

is true. □

Presently, some sharp inequalities related to the (ι,x_0)-generalized logistic-type function $\Psi_{\rho,\iota}$ (the (ι,x_0)-generalized softplus activation function) are studied:

Theorem 2.13. *For $\iota > 0$ and $x_0 \in (-\infty,+\infty)$, the following inequalities are satisfied:*

$$\frac{\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}} < \ln\left(+\rho\xi^{\iota(x-x_0)}\right) < \ln(1+\rho) - \frac{1}{1+\rho} + \frac{\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}},$$

$$\ln(1+\rho) - \frac{1}{1+\rho} + \frac{\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}} < \ln\left(1+\rho\xi^{\iota(x-x_0)}\right),\ x \in (x_0,+\infty),$$

$$\frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}} < \ln\left(1+\rho\xi^{\iota(x-x_0)}\right),\ x \in (-\infty,+\infty). \tag{2.7}$$

Proof. Let us define

$$\Delta(x) := \ln\left(1+\rho\xi^{\iota(x-x_0)}\right) - \frac{\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}},\ x \in (-\infty,+\infty),$$

$$\Delta'(x) = \frac{\iota(\ln\xi)\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}\left(\rho - \frac{1}{1+\rho\xi^{\iota(x-x_0)}}\right) > 0,\ x \in (-\infty,+\infty).$$

So, $\Delta(x)$ is increasing on $(-\infty,+\infty)$.
For $x\in(-\infty,x_0)$,

$$0=\lim_{x\to-\infty}\Delta(x)<\Delta(x)<\lim_{x\to x_0}\Delta(x)=\ln(1+\rho)-\frac{1}{1+\rho},$$

which yields that the first inequality is valid.
For $x\in(x_0,+\infty)$,

$$\ln(1+\rho)-\frac{1}{1+\rho}=\lim_{x\to x_0}\Delta(x)<\Delta(x)<\lim_{x\to+\infty}\Delta(x)<+\infty,$$

which indicates that the second inequality is held.
Also, for $x\in(-\infty,+\infty)$,

$$0=\lim_{x\to-\infty}\Delta(x)<\Delta(x)<\lim_{x\to+\infty}\Delta(x)<+\infty,$$

which demonstrates that the third one is also satisfied. $\square$

Theorem 2.14. *For $\iota,\rho>0$, $x_0\in(-\infty,+\infty)$ and $x\in(-\infty,+\infty)$, the inequality*

$$\rho\xi^{\iota(x-x_0)}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)>0 \tag{2.8}$$

is provided.

Proof. Let

$$\Xi(x):=\rho\xi^{\iota(x-x_0)}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right),\ x\in(-\infty,+\infty),$$

and

$$\Xi'(x)=\rho\iota(\ln\xi)\xi^{\iota(x-x_0)}\left(\frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}\right)>0,$$

that indicates that $\Xi(x)$ is increasing on $(-\infty,+\infty)$. Hence, we get

$$\lim_{x\to-\infty}\Xi(x)=\lim_{x\to-\infty}\left(\rho\xi^{\iota(x-x_0)}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right)=0.$$

So

$$0=\lim_{x\to-\infty}\Xi(x)<\Xi(x)<\lim_{x\to-\infty}\left(\rho\xi^{\iota(x-x_0)}-\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right),$$

which verifies the last inequality, is valid. $\square$

Theorem 2.15. *For $\iota,\rho>0$, $\xi>1$; $x,x_0\in(-\infty,+\infty)$, let $m(x)=\left(1+\rho\xi^{\iota(x-x_0)}\right)^{\frac{1}{\rho\xi^{\iota(x-x_0)}}}$ and $k(x)=\left(1+\rho\xi^{\iota(x-x_0)}\right)^{1+\frac{1}{\rho\xi^{\iota(x-x_0)}}}$ be decreasing and increasing, respectively. Then the following inequalities hold:*

$$(1+\rho)\ln(1+\rho)\xi^{\iota(x-x_0)}<\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)<\rho\xi^{\iota(x-x_0)};\ x\in(-\infty,x_0),$$

$$\rho\xi^{\iota(x-x_0)}<\left(1+\rho\xi^{\iota(x-x_0)}\right)\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)<(1+\rho)\ln(1+\rho)\xi^{\iota(x-x_0)};x\in(-\infty,x_0),$$

and

$$\rho\xi^{\iota(x-x_0)}<\left(1+\rho\xi^{\iota(x-x_0)}\right)\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)<\left(1+\rho\xi^{\iota(x-x_0)}\right)\rho\xi^{\iota(x-x_0)};x\in(-\infty,+\infty).$$

Proof. For $x\in(-\infty,+\infty)$, $\iota,\rho>0$, and $\xi>1$,
let

$$\begin{aligned}M(x)&:=\ln(m(x))\\&=\ln\left(\left(1+\rho\xi^{\iota(x-x_0)}\right)^{\frac{1}{\rho\xi^{\iota(x-x_0)}}}\right)\\&=\frac{\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)}{\rho\xi^{\iota(x-x_0)}},\end{aligned}$$

and

$$\begin{aligned} K(x) &:= \ln(k(x)) \\ &= \ln\left(\left(1+\rho\xi^{\iota(x-x_0)}\right)^{1+\frac{1}{\rho\xi^{\iota(x-x_0)}}}\right) \\ &= \left(1+\frac{1}{\rho\xi^{\iota(x-x_0)}}\right)\ln\left(1+\rho\xi^{\iota(x-x_0)}\right). \end{aligned}$$

Now, taking the derivatives,

$$M'(x) = \frac{\iota(\ln\xi)}{\rho\xi^{\iota(x-x_0)}}\left(\frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}} - \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right) < 0.$$

Thus using (2.7), we conclude that $M(x)$ is decreasing and, accordingly, $m(x)$ is also decreasing.
In like manner,

$$K'(x) = \frac{\iota(\ln\xi)}{\rho\xi^{\iota(x-x_0)}}\left(\rho\xi^{\iota(x-x_0)} - \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right) > 0,$$

employing (2.8), it is obvious that $K(x)$ is increasing and so is $k(x)$.
Besides, let us take

$$\begin{aligned} M(x_0) &= \frac{\ln(1+\rho)}{\rho};\ K(x_0) = \left(1+\frac{1}{\rho}\right)\ln(1+\rho), \\ \lim_{x\to-\infty} M(x) &= 1;\ \lim_{x\to+\infty} M(x) = 0, \\ \lim_{x\to-\infty} K(x) &= 1;\ \lim_{x\to+\infty} K(x) = +\infty. \end{aligned}$$

For $x\in(-\infty, x_0)$, we obtain

$$\frac{\ln(1+\rho)}{\rho} = M(x_0) < M(x) < \lim_{x\to-\infty} M(x) = 1,$$

so that the first inequality is satisfied.
Again for $x\in(-\infty, x_0)$, we have

$$1 = \lim_{x\to-\infty} K(x) < K(x) < K(x_0) = \left(1+\frac{1}{\rho}\right)\ln(1+\rho),$$

which yields the second one.
Lastly, for $x\in(-\infty,+\infty)$, one has

$$0 = \lim_{x\to+\infty} M(x) < M(x) < \lim_{x\to-\infty} M(x) = 1,$$

which verifies the following

$$\ln\left(1+\rho\xi^{\iota(x-x_0)}\right) < \rho\xi^{\iota(x-x_0)}.$$

As well,

$$1 = \lim_{x\to-\infty} K(x) < K(x) < \lim_{x\to+\infty} K(x) = +\infty,$$

which again implies

$$\frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}} < \ln\left(1+\rho\xi^{\iota(x-x_0)}\right). \tag{2.9}$$

By the last two inequalities, the desired third inequality is proved. □

Theorem 2.16. *For $\iota > 0$ and $x_0 \in (-\infty, +\infty)$, set*

$$F(x) = \frac{\rho \xi^{\iota(x-x_0)} \ln\left(1+\rho \xi^{\iota(x-x_0)}\right)}{\rho \xi^{\iota(x-x_0)} - \ln\left(1+\rho \xi^{\iota(x-x_0)}\right)},\ x \in (-\infty, x_0).$$

It follows that, $F(x)$ is increasing and satisfies the inequality below:

$$2 < F(x) < \frac{\rho \ln(1+\rho)}{\rho - \ln(1+\rho)}.$$

Proof. First of all,

$$\lim_{x \to x_0} F(x) = \frac{\ln(2)}{\rho - \ln(2)},$$

and

$$\begin{aligned}
\lim_{x\to-\infty} F(x) &= \lim_{x\to-\infty} \frac{\ln\left(1+\rho\xi^{\iota(x-x_0)}\right) + \frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}}{1-\frac{1}{1+\rho\xi^{\iota(x-x_0)}}} \\
&= \lim_{x\to-\infty}\left(1+\frac{\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\left(1+\rho\xi^{\iota(x-x_0)}\right)}{\rho\xi^{\iota(x-x_0)}}\right) \\
&= 2.
\end{aligned}$$

Furthermore, let

$$\Lambda(x) := \rho\xi^{\iota(x-x_0)} \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)$$

and

$$\Upsilon(x) := \rho\xi^{\iota(x-x_0)} - \ln\left(1+\rho\xi^{\iota(x-x_0)}\right).$$

Hence

$$\begin{aligned}
\Lambda(-\infty) &= \lim_{x\to-\infty} \Lambda(x) \\
&= \lim_{x\to-\infty}\left(\rho\xi^{\iota(x-x_0)} \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon(-\infty) &= \lim_{x\to-\infty} \Upsilon(x) \\
&= \lim_{x\to-\infty}\left(\rho\xi^{\iota(x-x_0)} - \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right) \\
&= 0.
\end{aligned}$$

So,

$$\Lambda'(x) = \iota\rho(\ln\xi)\,\xi^{\iota(x-x_0)}\left(\ln\left(1+\rho\xi^{\iota(x-x_0)}\right) + \frac{\rho\xi^{\iota(x-x_0)}}{1+\rho\xi^{\iota(x-x_0)}}\right) > 0,$$

and

$$\Upsilon'(x) = \iota\rho(\ln\xi)\,\xi^{\iota(x-x_0)}\left(1-\frac{1}{1+\rho\xi^{\iota(x-x_0)}}\right) > 0.$$

Additionally,

$$\begin{aligned}
\left(\frac{\Lambda'(x)}{\Upsilon'(x)}\right)' &= \left(\frac{\ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\left(1+\rho\xi^{\iota(x-x_0)}\right)}{\rho\xi^{\iota(x-x_0)}}\right)' \\
&= \frac{\iota(\ln\xi)}{\rho\xi^{\iota(x-x_0)}}\left(\rho\xi^{\iota(x-x_0)} - \ln\left(1+\rho\xi^{\iota(x-x_0)}\right)\right) > 0.
\end{aligned}$$

In this manner, $\frac{\Lambda'(x)}{\Upsilon'(x)}$ is increasing, which elucidates that $\frac{\Lambda(x)}{\Upsilon(x)} = \digamma(x)$ is increasing, too (by the corollary 1.2 of [13]).

Ultimately, for $x \in (-\infty, x_0)$,

$$2 = \lim_{x \to -\infty} \digamma(x) < \digamma(x) < \lim_{x \to x_0} \digamma(x) = \frac{\rho \ln(1+\rho)}{\rho - \ln(1+\rho)}. \tag{2.10}$$

□

Proposition 2.17. *Let*

$$\xi := \frac{\rho \ln(1+\rho)}{\rho - \ln(1+\rho)},$$

the inequality (2.10) *can be written as*

$$\ln\left(1+\rho \xi^{\iota(x-x_0)}\right) < \frac{\xi \rho \xi^{\iota(x-x_0)}}{\xi + \rho \xi^{\iota(x-x_0)}},$$

and by inequality (2.9)

$$\frac{\rho \xi^{\iota(x-x_0)}}{1+\rho \xi^{\iota(x-x_0)}} < \ln\left(1+\rho \xi^{\iota(x-x_0)}\right) < \frac{\xi \rho \xi^{\iota(x-x_0)}}{\xi + \rho \xi^{\iota(x-x_0)}}$$

is observed.

3. A statistical interpretation of the (ι, x_0)-generalized logistic-type function in survival analysis

Survival Analysis is a subfield of statistics used to describe and measure data on the time until an event occurs. For example, it analyzes the expected time until failure in mechanical systems and death in biological organisms [14]. ”Time-to-event processes” are especially common in medical research because they provide more information than whether an event occurred or not [15]. In addition, ”reliability theory” or ”reliability analysis” are other names used for this area in engineering sciences. In this section, the (ι, x_0)-generalized logistic-type function

$$F(x,\gamma) := \Psi_{\rho,\iota}(x) = \frac{\xi^{\iota(x-x_0)}}{\rho + \xi^{\iota(x-x_0)}},\ \iota, \rho > 0;\ x_0 \in (-\infty, +\infty);\ \xi > 1,$$

is considered as a distribution function and the probability density function of the suggested distribution is

$$f(x,\gamma) = \Psi'_{\rho,\iota}(x) = \frac{\iota \rho (\ln \xi) \xi^{\iota(x-x_0)}}{\left(\rho + \xi^{\iota(x-x_0)}\right)^2},$$

where, $\gamma = (\iota, x_0)$ is the parameter set.

One of the common terms used in survival analysis is ”survival (reliability) function”. Primarily, we are deeply interested in parametric exponential version of this function and its graphical results in the sense of behavior of the function with respect to arbitrary chosen parameters: ξ, ι, x_0, and ρ. The distribution of survival times can be better predicted by a function such as the exponential function, which, create parametric survival models.

Now, we define parametric exponential survival (PES) and parametric failure (hazard) rate (PFR) functions, respectively as seen below:

$$\begin{aligned}
\overline{\Psi}(x,\gamma) &= 1 - F(x,\gamma) = \frac{\rho}{\rho + \xi^{\iota(x-x_0)}}, \\
h(x,\gamma) &= \frac{f(x,\gamma)}{\overline{\Psi}(x,\gamma)} = \frac{\iota (\ln \xi) \xi^{\iota(x-x_0)}}{\rho + \xi^{\iota(x-x_0)}}.
\end{aligned}$$

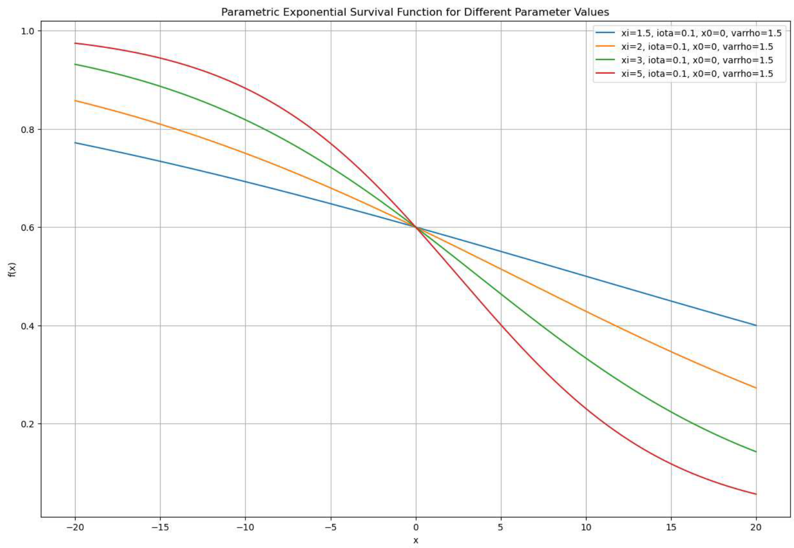

Figure 1: Behaviour of PES with respect to distinct parameter values of ξ.

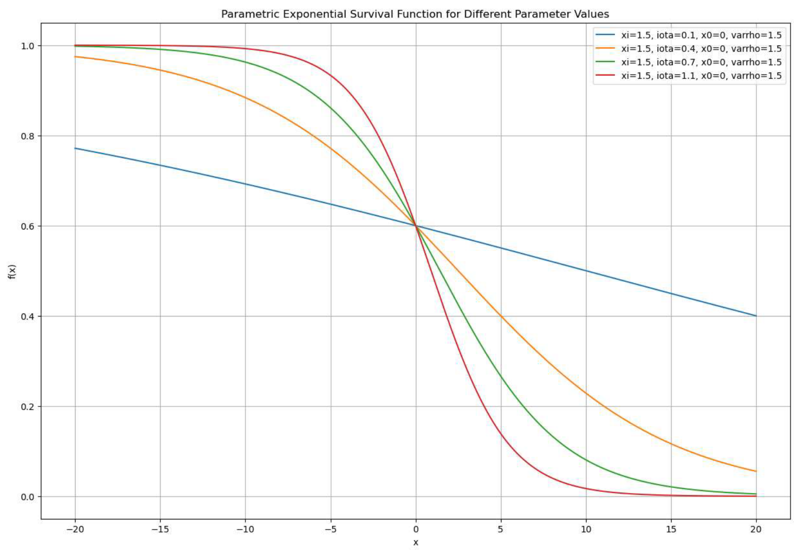

Figure 2: Behaviour of PES with respect to distinct parameter values of ι.

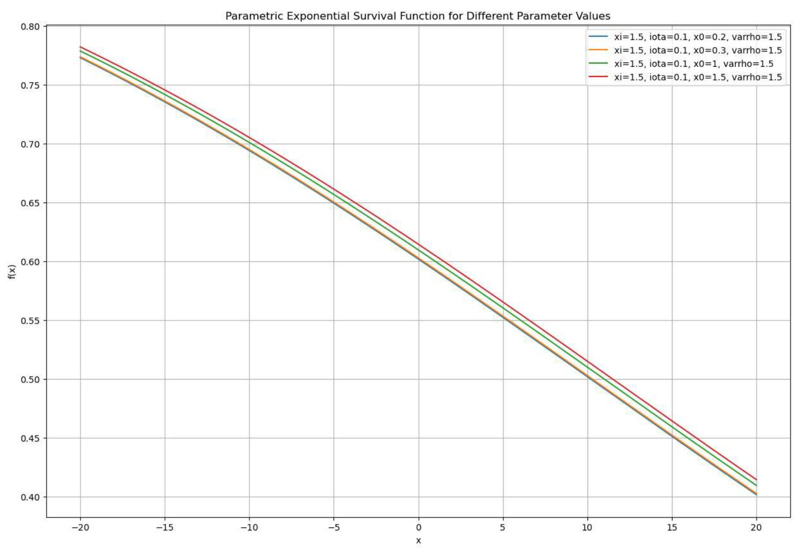

Figure 3: Behaviour of PES with respect to distinct parameter values of x_0.

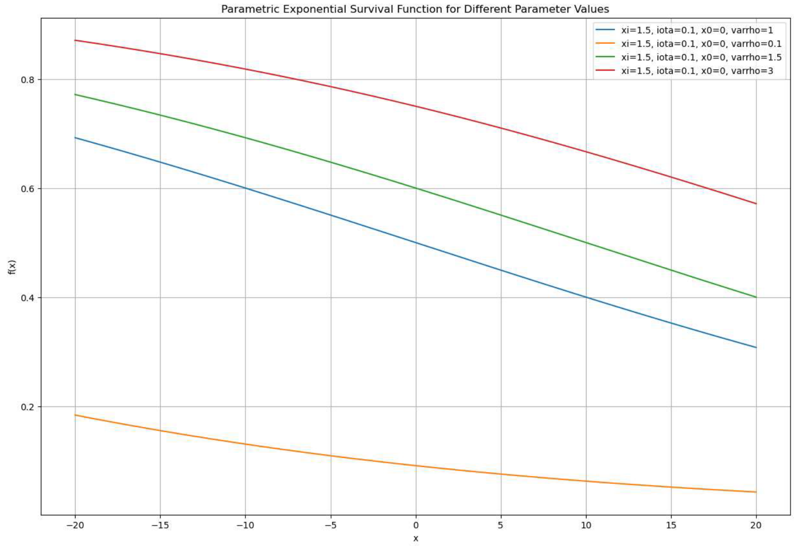

Figure 4: Behaviour of PES with respect to distinct parameter values of ρ.

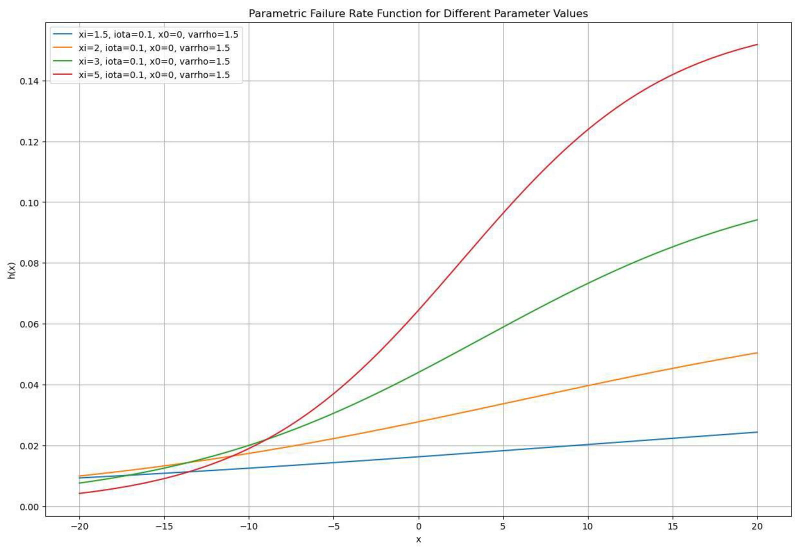

Figure 5: Behaviour of PFR with respect to arbitrary parameter values of ξ.

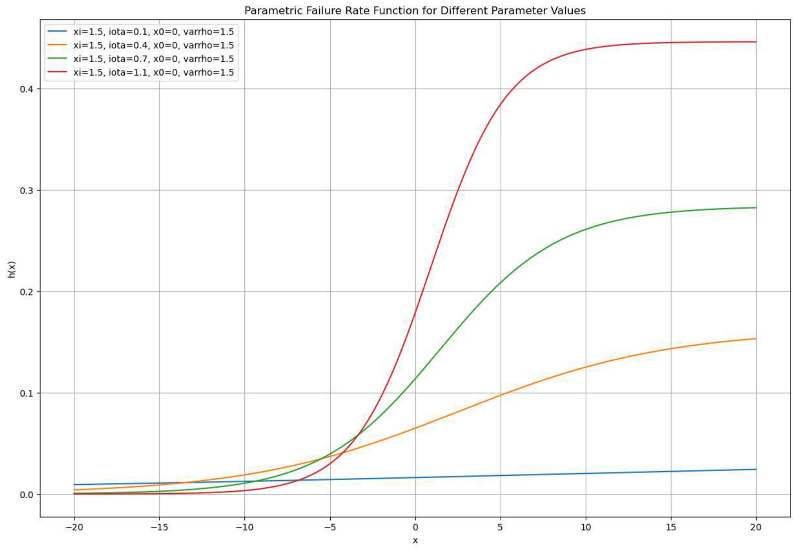

Figure 6: Behaviour of PFR with respect to arbitrary parameter values of ι.

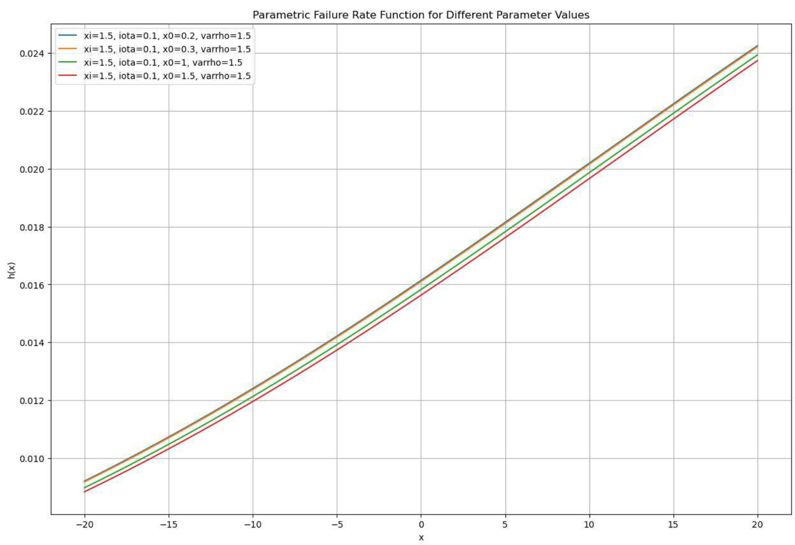

Figure 7: Behaviour of PFR with respect to arbitrary parameter values of x_0.

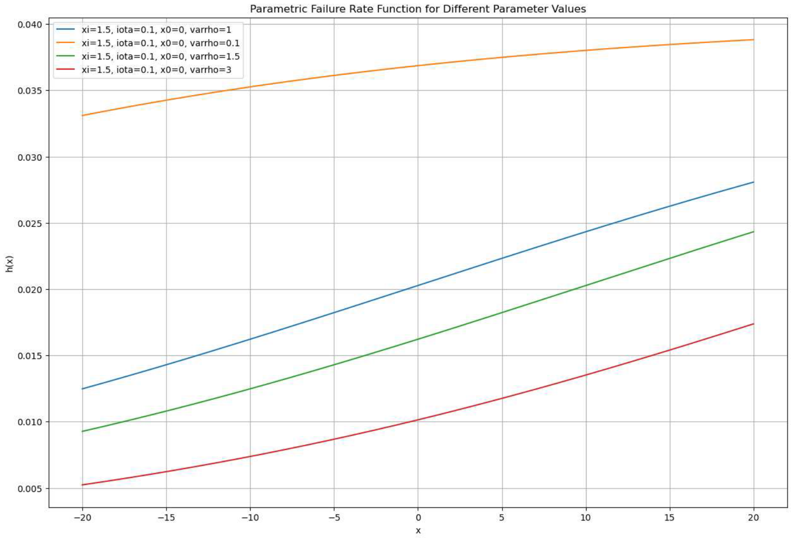

Figure 8: Behaviour of PFR with respect to arbitrary parameter values of ρ.

```python
#Parametric Exponential Survival Function for Different Parameter Values
# "xi">1, "varrho"> 0, "iota"> 0
#Investigation and Comparison of Parameters
# - "xi" = Maxinum value of the curve
# - x0 = Horizontal shifting parameter. It determines where the function starts.
# - "iota" = Slope parameter.It controls how quickly the function changes.(growth rate)

import numpy as np
import matplotlib.pyplot as plt

def survival(x,varrho,xi,iota,x0):
    return varrho / (varrho + pow(xi,iota*(x-x0)))

def plot_survival(xi_values,iota_values,x0_values,varrho_values):
    x_values = np.linspace(-20, 20, 1000)
    plt.figure(figsize=(15, 10))

    for xi in xi_values:
        for iota in iota_values:
            for x0 in x0_values:
                for varrho in varrho_values:
                    y_values = survival(x_values, varrho, xi,iota, x0)
                    label = f'xi={xi}, iota={iota}, x0={x0}, varrho={varrho}'
                    plt.plot(x_values, y_values, label=label)

    plt.title('Parametric Exponential Survival Function for Different Parameter Values')
    plt.xlabel('x')
    plt.ylabel('f(x)')
    plt.legend()
    plt.grid(True)
    plt.show()


# Different Parameter Values
xi_values = [1.5, 2, 3, 5]
iota_values =[ 0.1]
x0_values = [0]
varrho_values = [1.5]
plot_survival(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1,0.4,0.7,1.1]
x0_values = [0]
varrho_values = [1.5]
plot_survival(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1]
x0_values = [0.2,0.3,1,1.5]
varrho_values = [1.5]
plot_survival(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1]
x0_values = [0]
varrho_values = [1,0.1,1.5,3]
plot_survival(xi_values,iota_values,x0_values,varrho_values)
```

Figure 9: Algorithm 1.

```
#Parametric Failure Rate Function for Different Parameter Values
# "xi">1, "varrho" > 0, "iota" > 0
#Investigation and Comparison of Parameters
# - "xi" = Maxinum value of the curve
# - x0 = Horizontal shifting parameter. It determines where the function starts.
# - "iota" = Slope parameter.It controls how varrhouickly the function changes.(growth rate)

import numpy as np
import math
import matplotlib.pyplot as plt

def failure_rate(x,varrho,xi,iota,x0):
    return (iota*(math.log(xi)/math.log(math.e))*pow(xi,iota*(x-x0)))/(varrho+pow(xi,iota*(x-x0)))

def plot_failure_rate(xi_values,iota_values,x0_values,varrho_values):
    x_values = np.linspace(-20, 20, 1000)
    plt.figure(figsize=(15, 10))

    for xi in xi_values:
        for iota in iota_values:
            for x0 in x0_values:
                for varrho in varrho_values:
                    y_values = failure_rate(x_values, varrho, xi,iota, x0)
                    label = f'xi={xi}, iota={iota}, x0={x0}, varrho={varrho}'
                    plt.plot(x_values, y_values, label=label)

    plt.title('Parametric Failure Rate Function for Different Parameter Values')
    plt.xlabel('x')
    plt.ylabel('h(x)')
    plt.legend()
    plt.grid(True)
    plt.show()


# Different Parameter Values
xi_values = [1.5, 2, 3, 5]
iota_values =[ 0.1]
x0_values = [0]
varrho_values = [1.5]
plot_failure_rate(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1,0.4,0.7,1.1]
x0_values = [0]
varrho_values = [1.5]
plot_failure_rate(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1]
x0_values = [0.2,0.3,1,1.5]
varrho_values = [1.5]
plot_failure_rate(xi_values,iota_values,x0_values,varrho_values)

# Different Parameter Values
xi_values = [1.5]
iota_values =[ 0.1]
x0_values = [0]
varrho_values = [1,0.1,1.5,3]
plot_failure_rate(xi_values,iota_values,x0_values,varrho_values)
```

Figure 10: Algorithm 2.

Moreover, graphs of the PES functions are visualized in Fig. 1 - Fig. 4 using computer programming language Python 3.9, as you see [16].

In the second place, we focus on the PFR function of the proposed distribution with arbitrary parameter values obtained as in Fig. 5 - Fig. 8. This is the function that gives the steadily revised immediate probability of a critical event. With this feature, it finds applications in many disciplines such as health sciences, and mathematical psychology [17]. Also, while the PES function serves for surviving, the PFR function deals with the failing [18].

4. Conclusion

In this paper, some important inequalities like concavity, super multiplicativity, and sub-additivity of the (ι, x_0)-generalized logistic-type function have been proved. "Ceteris Paribus" plotting for parametric exponential survival (PES) and also parametric failure (hazard) rate (PFR) functions with four variables have been performed. Thus, when the survival function we parameterized is compared to a function that is not parametrized; we can say that the parametric one may provide more detailed and sophisticated modeling in survival analysis. In short, we may obtain higher accuracy values in the validation data of the models with the help of functions containing four parameters, that is, to make the models more robust.

Declarations

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions

Conflict of Interest Disclosure: The author declares no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

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How to cite this article: S. Karateke, *On an (ι,x_0)-generalized logistic-type function*, Fundam. J. Math. Appl., **7**(1) (2024), 35-52. DOI 10.33401/fujma.1423906

Fundamental Journal of Mathematics and Applications, 7(1) (2024), 53-58
Research Paper / Open Access

Fundamental Journal of Mathematics and Applications

ISSN Online: 2645-8845
www.dergipark.org.tr/en/pub/fujma
https://doi.org/10.33401/fujma.1424382

A Note On Kantorovich Type Operators Which Preserve Affine Functions

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Article Information

Keywords: Modulus of continuity; Rate of convergence; Voronovskaya theorem

AMS 2020 Classification: 41A25; 41A36

Abstract

The authors present an integral widening of operators which preserve affine functions. Influenced by the operators which preserve affine functions, we define the integral extension of these operators. We give quantitative type theorem using weighted modulus of continuity. Withal quantitative Voronovskaya theorem is aquired by classical modulus of continuity. When the moments of the operator are known, convergence results with the moments obtained for the Kantorovich form of the same operator is given.

1. Introduction

In mathematical analysis, studies on approximation by linear and positive operators retained its importance for many years. Recently many researchers have studied some generalizations of these operators, especially the Kantorovich form of Bernstein, Baskakov and Szàsz operators. Also they have studied some operators which preserve test functions, exponentials and affine functions (see [1]-[8]).

The Kantorovich version of Bernstein operators [9] defined by replacing the sample values $f\left(\frac{k}{n}\right)$ with the mean values of f in $\left[\frac{k}{n},\frac{k+1}{n}\right]$, namely for $x\in[0,1]$, $n\in\mathbb{N}$ and $f\in L_1[0,1]$, $P_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$, $k=0,1,\ldots,n$

$$K_n(f)(x)=(n+1)\sum_{k=0}^{n}P_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(t)dt. \tag{1.1}$$

Note that K_n is just reproduced 1. These operators provide us to switch a Lebesgue integrable function by means of its mean values on the sets $\left[\frac{k}{n},\frac{k+1}{n}\right]$.

General in use, such a $(L_n)_{n\geq 1}$ sequence of linear and positive operators are specified. In 2016, Agratini studied Kantorovich type operators which preserve affine functions ([2]). Inspire of these general operators which preserve affine functions, we study these operators on weighted spaces.

Let's describe the layout of this work. In first part, nodes and moments are given. The second part belongs to some approximation findings for the operators.

The purpose of this article is to show that if we know the moments of the operators, we find convergence results with the moments obtained for the Kantorovich type generalization of the same operator.

Received: 23-01-2024 **Revised:** 27-02-2024 **Accepted:** 11-03-2024 **Online:** 29-03-2024 **Published:** 31-03-2024

2. Properties of the operators

Througout the paper, we consider an interval $\mathbb{R}^+ = [0,\infty)$. In [9], we can see the Kantorovich form of the Bernstein operators as

$$K_n(f)(x) = (n+1)\sum_{k=0}^{n} P_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\ x \in [0,1],$$

where $f \in L_1[0,1]$. Let $C(\mathbb{R}^+)$ denotes the space of real-valued continuous functions on $\mathbb{R}^+$, now we give L_n operator which can be written as

$$L_n(f;x) = \sum_{k\in J_n} \lambda_{n,k}(x) f(x_{n,k}),\ x \in \mathbb{R}^+ \tag{2.1}$$

where $\lambda_{n,k} \in C(\mathbb{R}^+)$ and $\lambda_{n,k} \geq 0$ and $(n,k) \in \mathbb{N} \times J_n$. Also $(x_{n,k})_{k\in J_n}$ be set on the interval $\mathbb{R}^+$ where $J_n \subseteq \mathbb{N}$ is a set of indices. Now we consider nodes for each $n \in \mathbb{N}$,

$$x_{n,k+1} - x_{n,k} = u_n,\ k \in J_n$$

where $\lim\limits_{n\to\infty} u_n = 0$.
We take into about L_n operators given by (2.1) which preserve affine functions,

$$\sum_{k\in J_n} \lambda_{n,k}(x) = 1 \text{ and } \sum_{k\in J_n} \lambda_{n,k}(x) x_{n,k} = x,\ x \in \mathbb{R}^+.$$

Now let $u_n^* = \sup\limits_{n\in\mathbb{N}} u_n$. If $\mathbb{R}^+ = [0,\infty)$, then we set $A^* = [\frac{u_n^*}{2},\infty)$.

2.1. Auxiliary Results

We give some results which will be necessary for proofs of theorems. At first, we find some moments and central moments of

$$\widetilde{K_n}(f;x) = \frac{1}{u_n}\sum_{k\in J_n} \lambda_{n,k}(x) \int_{x_{n,k}}^{x_{n,k+1}} f(t)dt,\ x \in \mathbb{R}^+ \tag{2.2}$$

operators.

Lemma 2.1. *Let L_n defined by (2.1), $n \in \mathbb{N}$, $x \in A^*$ and $e_r(t) = t^r$ for*

$r = 1,\ 2,\ 3,\ 4$. Then we have

(i) $\widetilde{K_n}(e_0)(x) = 1$,
(ii) $\widetilde{K_n}(e_1)(x) = x + \frac{u_n}{2}$,
(iii) $\widetilde{K_n}(e_2)(x) = L_n(e_2)(x) + u_n x - \frac{u_n^2}{3}$,
(iv) $\widetilde{K_n}(e_3)(x) = L_n(e_3)(x) + \frac{3}{2}u_n L_n(e_2)(x) + u_n^2 x - \frac{u_n^3}{4}$,
(v) $\widetilde{K_n}(e_4)(x) = L_n(e_4)(x) + 2u_n L_n(e_3)(x) + 2u_n^2 L_n(e_2)(x) + u_n^3 x - \frac{u_n^4}{5}$.

Proof. (i) It is clear from the definition of the operator $\widetilde{K_n}$.
(ii)

$$\begin{aligned}
\widetilde{K_n}(e_1)(x) &= \frac{1}{u_n}\sum_{k\in J_n} \lambda_{n,k}(x) \int_{x_{n,k}}^{x_{n,k+1}} t\,dt \\
&= \frac{1}{u_n}\sum_{k\in J_n} \lambda_{n,k}(x) \frac{1}{2}\left(x_{n,k+1}^2 - x_{n,k}^2\right) \\
&= \frac{1}{u_n}\sum_{k\in J_n} \lambda_{n,k}(x) \frac{1}{2}\left(u_n^2 + 2u_n x_{n,k}\right) \\
&= \frac{u_n}{2} + x.
\end{aligned}$$

(iii)

$$\begin{aligned}
\widetilde{K_n}(e_2)(x) &= \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\frac{1}{3}\left(x_{n,k+1}^3 - x_{n,k}^3\right) \\
&= \frac{1}{3u_n}\sum_{k\in J_n}\lambda_{n,k}(x)u_n\left[\left(x_{n,k}+u_n\right)^2 + x_{n,k}\left(x_{n,k}+u_n\right)+x_{n,k}^2\right], \\
&= \sum_{k\in J_n}\lambda_{n,k}(x)x_{n,k}^2 + u_n x + \frac{u_n^2}{3} \\
&= L_n(e_2)(x) + u_n x + \frac{u_n^2}{3}.
\end{aligned}$$

(iv)

$$\begin{aligned}
\widetilde{K_n}(e_3)(x) &= \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\frac{1}{4}\left(x_{n,k+1}^4 - x_{n,k}^4\right) \\
&= \frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\left(x_{n,k+1}-x_{n,k}\right)\left(x_{n,k+1}+x_{n,k}\right)\left(x_{n,k+1}^2+x_{n,k}^2\right) \\
&= \frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\left[u_n(2x_{n,k}+u_n)(\left(u_n+x_{n,k}\right)^2+x_{n,k}^2)\right] \\
&= \sum_{k\in J_n}\lambda_{n,k}(x)x_{n,k}^3 + \frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)6u_n^2x_{n,k}^2 + \frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)4u_n^3x_{n,k} + \frac{1}{4u_n}\sum_{k\in J_n}\lambda_{n,k}(x)u_n^4 \\
&= L_n(e_3)(x) + \frac{3}{2}u_nL_n(e_2)(x) + u_n^2x - \frac{u_n^3}{4}.
\end{aligned}$$

(v) At that time, (v) can be calculated similarly.

□

Lemma 2.2. *Let $\varphi_x^n(t) = (t-x)^n, n = 0,1,2,...$ For the operator $\widetilde{K_n}$ given by (2.2) if we set $\zeta_{n,2}(x) = \widetilde{K_n}(\varphi_x^2(t);x)$ and $\zeta_{n,4}(x) = \widetilde{K_n}(\varphi_x^4(t);x)$, then we have*

$$\zeta_{n,2}(x) = L_n(e_{2,};x) + \frac{u_n^2}{3} - x^2,$$

$$\zeta_{n,4}(x) \;=\; L_n(e_4)(x) + (2u_n-4x)L_n(e_3)(x) + (2u_n^2 + -6xu_n + 6x^2)L_n(e_2)(x) + 4x^3u_nL_n(e_1)(x) - 3x^4L_n(e_0)(x) + u_n^4 - 2x^2u_n^2.$$

Proof. By using Lemma 1.1, we obtain

$$\begin{aligned}
\zeta_{n,2}(x) &= \widetilde{K_n}(\varphi_x^2(t);x) = L_n(e_{2,};x) + \frac{u_n^2}{3} + u_nx - 2x(x+\frac{u_n}{2}) + x^2 \\
&= L_n(e_{2,};x) + \frac{u_n^2}{3} - x^2.
\end{aligned}$$

Now let's calculate $\widetilde{K_n}(\varphi_x^4(t);x)$.

$$\begin{aligned}
\zeta_{n,4}(x) &= \widetilde{K_n}(\varphi_x^4(t);x) = \widetilde{K_n}(e_4,x) - 4\widetilde{K_n}(e_3,x)x + 6\widetilde{K_n}(e_2,x)x^2 - 4\widetilde{K_n}(e_1,x)x^3 + x^4\widetilde{K_n}(e_{0,},x) \\
&= L_n(e_{4,};x) + 2u_nL_n(e_{3,};x) + 2u_n^2L_n(e_{2,};x) + u_n^3x + u_n^4 - 4x(L_n(e_{3,};x) + \frac{3}{2}u_nL_n(e_{2,};x) + u_n^2x + \frac{u_n^3}{4}) \\
&\quad +6x^2(L_n(e_{2,};x) + u_nx + \frac{u_n^2}{3}) - 4x^3(\frac{u_n}{2} + L_n(e_{1,};x)) + x^4L_n(e_{0,};x)) \\
&= L_n(e_4)(x) + (2u_n-4x)L_n(e_3)(x) + (2u_n^2 + -6xu_n + 6x^2)L_n(e_2)(x) + 4x^3u_nL_n(e_1)(x) - 3x^4L_n(e_0)(x) + u_n^4 - 2x^2u_n^2,
\end{aligned}$$

so the desired result is achieved. □

3. Rate Of Convergence

In this part, setting $f \in \mathbb{R}^+$, approximation result is given for $\widetilde{K_n}$ operator. In [10] and [11], proof of Korovkin theorems are given.

Let $\mu(x)=1+x^2$ be a weight function and K_f be a positive constant depending of f, we define

$$B_\mu\left(\mathbb{R}^+\right)=\left\{f:\mathbb{R}^+\to\mathbb{R}:|f(x)|\leq K_f\ \mu(x)\right\}$$

and

$$C_\mu\left(\mathbb{R}^+\right)=C\left(\mathbb{R}^+\right)\cap B_\mu\left(\mathbb{R}^+\right).$$

Considering the space of functions

$$C_\mu^k\left(\mathbb{R}^+\right)=\left\{f\in C_\mu\left(\mathbb{R}^+\right):\lim_{x\to\infty}\frac{f(x)}{\mu(x)}=K_f<\infty\right\}.$$

Obviously $C_\mu^k\left(\mathbb{R}^+\right)\subset C_\mu\left(\mathbb{R}^+\right)\subset B_\mu\left(\mathbb{R}^+\right)$. Here the norm is defined as

$$\|f\|_\mu=\sup_{x\in\mathbb{R}^+}\frac{|f(x)|}{\mu(x)}.$$

If $f\in C_\mu^k\left(\mathbb{R}^+\right)$, then $\|L_n(f)\|_\mu\leq\|f\|_\mu$ .These results and Korovkin type theorems can be seen in [12, 10, 11].
Let $C^k\left(\mathbb{R}^+\right)$ be the subspace of all the functions $f\in C\left(\mathbb{R}^+\right)$ such that $\lim\limits_{x\to\infty}\frac{|f(x)|}{1+x^2}=k$, where k is a positive constant. For $f\in C^k\left(\mathbb{R}^+\right)$, weighted modulus of continuity is defined by

$$\Omega(f;\delta)=\sup_{|t-x|\leq\delta,\ x\in\mathbb{R}^+}\frac{|f(t)-f(x)|}{(1+x^2)(1+(t-x)^2)}. \tag{3.1}$$

Utilizing 3.1, we give quantitative type theorem.

Theorem 3.1. *If $f\in C_\mu^k\left(\mathbb{R}^+\right)$, then we have*

$$\left|\widetilde{K_n}(f;x)-f(x)\right|\leq 32(1+x^2)\Omega(f;\delta).$$

Proof. From the property of (3.1), we can write

$$\Omega(f;\lambda\delta)\leq 2(1+\lambda)(1+\delta^2)\Omega(f;\delta)$$

for positive λ (see in [13]). By the definition of $\Omega(f;\delta)$ for $f\in C_\mu^k\left(\mathbb{R}^+\right)$ and $x,\,t\in\mathbb{R}^+$ and $\delta>0$, the following inequality is satisfied:

$$|f(t)-f(x)|\leq 16\left(1+x^2\right)\Omega(f;\delta)\left(1+\frac{|t-x|^4}{\delta^4}\right) \tag{3.2}$$

and by using Lemma 1 and (3.2), we have

$$\left|\widetilde{K_n}(f;x)-f(x)\right|\leq f(x)\left|1-\widetilde{K_n}(1;x)\right|+\widetilde{K_n}(|f(t)-f(x)|;x).$$

Now applying (3.1) to $\widetilde{K_n}$,

$$\begin{aligned}\left|\widetilde{K_n}(f;x)-f(x)\right| &\leq \frac{1}{u_n}\sum_{k\in J_n}\lambda_{n,k}(x)\int_{x_{n,k}}^{x_{n,k+1}}|f(t)-f(x)|\,dt\\ &\leq 16\left(1+x^2\right)\Omega(f;\delta)\left(1+\frac{\zeta_{n,4}(x)}{\delta^4}\right),\end{aligned}$$

choosing $\delta=\sqrt[4]{\zeta_{n,4}(x)}$, it follows

$$\left|\widetilde{K_n}(f;x)-f(x)\right|\leq 32\left(1+x^2\right)\Omega\left(f;\sqrt[4]{\zeta_{n,4}(x)}\right),$$

so we obtain desired result. □

Let us denote by $\omega(f;\delta)$, the classical modulus of continuity defined as

$$\omega(f;\delta)=\sup_{|x-t|\leq\delta,x,t\in\mathbb{R}^+}|f(x)-f(t)|. \tag{3.3}$$

Theorem 3.2. *Let $f^{''} \in C(\mathbb{R})$ and $\omega\left(f^{''};\delta\right)$ is the modulus of contiuity of $f^{''}$ such as finite for $\delta > 0$. We have*

$$\left|\frac{1}{\zeta_{n,2}(x)}\left[\left(\widetilde{K_n}f\right)(x)-f(x)\right]-\frac{1}{2}f^{''}(x)\right| \leq \omega\left(f^{''};\frac{\sqrt{\zeta_{n,4}(x)}}{\sqrt{\zeta_{n,2}(x)}}\right).$$

Proof. By using the Taylor expansion at the fixed point x and (3.3) for $\xi \in [x,t]$, we obtain

$$\begin{aligned}
|h(t,x)| &= \left|f(t)-f(x)-\frac{f^{'}(x)}{1!}(t-x)-\frac{f^{''}(x)}{2!}(t-x)^2\right| \\
&= \frac{(t-x)^2}{2!}\left|f^{''}(\xi)-f^{''}(x)\right| \leq \frac{(t-x)^2}{2!}\omega\left(f^{''};|\xi-x|\right) \\
&\leq \frac{(t-x)^2}{2!}\omega\left(f^{''};|t-x|\right) \leq \frac{(t-x)^2}{2!}\left(1+\frac{|t-x|}{\delta}\right)\omega(f^{''};\delta) \\
&= \frac{1}{2}\left((t-x)^2+\frac{|t-x|^3}{\delta}\right)\omega(f^{''};\delta).
\end{aligned}$$

Now applying it to $\widetilde{K_n}$,we have

$$\begin{aligned}
\left|\left(\widetilde{K_n}h(\cdot,x)\right)(x)\right| &= \left|\left(\widetilde{K_n}f\right)(x)-f(x)-f^{'}(x)\ \zeta_{n,1}(x)-\frac{f^{''}(x)}{2}\zeta_{n,2}(x)\right| \\
&= \left|\left(\widetilde{K_n}f\right)(x)-f(x)-\frac{f^{''}(x)}{2}\ \zeta_{n,2}(x)\right| \leq (\widetilde{K_n}\,|h(f;\cdot,x)|)(x) \\
&\leq \frac{1}{2}\cdot\omega(f^{''};\delta)\left(\zeta_{n,2}(x)+\frac{(\widetilde{K_n}\,|e_1-x|^3)(x)}{\delta}\right) \\
&= \frac{\zeta_{n,2}(x)}{2}\cdot\omega(f^{''};\delta)\left(1+\frac{1}{\delta}\cdot\frac{(\widetilde{K_n}\,|e_1-x|^3)(x)}{\zeta_{n,2}(x)}\right).
\end{aligned}$$

If we choose

$$\delta = (\widetilde{K_n}\,|e_1-x|^3)(x)/\zeta_{n,2}(x)$$

and by using

$$(\widetilde{K_n}\,|e_1-x|^3)(x) \leq \sqrt{\zeta_{n,4}(x)}\cdot\sqrt{\zeta_{n,2}(x)},$$

inequality, we can write

$$\left|\left(\widetilde{K_n}f\right)(x)-f(x)-\frac{f^{''}(x)}{2}\sqrt{\zeta_{n,2}(x)}\right| \leq \sqrt{\zeta_{n,2}(x)}\,\omega\left(f^{''};\frac{\sqrt{\zeta_{n,4}(x)}}{\sqrt{\zeta_{n,2}(x)}}\right).$$

Thus we obtain

$$\left|\frac{1}{\sqrt{\zeta_{n,2}(x)}}\left(\widetilde{K_n}f\right)(x)-f(x)-\frac{1}{2}f^{''}(x)\right| \leq \omega\left(f^{''};\frac{\sqrt{\zeta_{n,4}(x)}}{\sqrt{\zeta_{n,2}(x)}}\right).$$

□

4. Conclusion

In this study, we showed that when the moments of an operator are known, some approximation theorems can be given for the Kantorovich type of the same operator using these moments.

Declarations

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions

Author's Contributions: Conceptualization, D.A.A.; methodology D.A.A.; validation, G.U.Y. investigation, D.A.A. and G.U.Y.; resources, D.A.A.; data curation, D.A.A.; writing—original draft preparation, D.A.A.; writing—review and editing, D.A.A. and G.U.Y.; supervision, G.U.Y. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest Disclosure: The authors declare no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

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Fundamental Journal of Mathematics and Applications (FUJMA), (Fundam. J. Math. Appl.)
https://dergipark.org.tr/en/pub/fujma

How to cite this article: D. Aydın Arı and G. Uğur Yılmaz, *A note on Kantorovich type operators which preserve affine functions*, Fundam. J. Math. Appl., **7**(1) (2024), 53-58. DOI 10.33401/fujma.1424382