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Extension of Synthetic Division and Its Applications

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Abstract

This study focused on "Extension of synthetic division and its applications". The study was designed to show synthetic division and its extension and to point out the applications of synthetic division and its extension. The study found out that the concepts of polynomial and rational expressions in single variables are basic concepts to deal extension of synthetic division and its applications. Using the preliminary concepts, the concept of synthetic division is extended in this study. Also, the study found out that an extension of synthetic division is used for finding the oblique asymptote of the graph of a rational function, evaluating the integration of some rational functions, representing polynomial expression by factorial function in numerical analysis, and so on.

1. Introduction

Division is one of the arithmetic operations [1]. The division of two real numbers, saying a and b, is denoted by $a \div b$ *or* $\frac{a}{b}$ provided that b is different from zero. From this expression, a and b are said to be dividend and divisor, respectively [2]. The task of division of real numbers$(a \div b;\ b \neq 0)$ is finding real numbers k *and* t that satisfy the equation $a = kb + t$ [3]. From this equation, k and t are said to be quotient and remainder of the division, respectively [4]. Since human beings use the concept of division of real numbers for their day $-$ to $-$day activities frequently, they upgraded this concept beyond the set of real numbers. Polynomial division is one of this upgrading. In this division, the quotient and the remainder are polynomial expressions [5]. There are different techniques to perform polynomial division. Among these techniques, long division and synthetic division are the most known techniques. Long division uses variable $-$ wise division whereas synthetic division uses coefficient $-$ wise division [6]. This study is mainly focused on extension of synthetic division as well as its applications.
The main objectives of this study are

- to show synthetic division.
- to show the extension of synthetic division.
- to point out the applications of synthetic division and its extension.

2. Preliminaries

Polynomial and rational expressions in single variable are the preliminary concepts to deal synthetic division and its extension [7].

Definition 2.1 ([8]). *A polynomial expression in a single variable x of degree n is an expression in the form of*

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \tag{2.1}$$

where n is a whole number and $a_n \neq 0$.

Remark 2.2. *From equation (2.1),*

- $a_n,\ a_{n-1},\ \ldots,\ a_1$ *and* a_0 are said to be coefficients of a polynomial expression.

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- $a_n x^n$ is said to be the leading term of a polynomial expression.
- a_n is said to be the leading coefficient of a polynomial expression.
- a_0 is said to be the constant term of a polynomial expression.
- For any doing in this paper, we use the set of complex numbers as a domain of a polynomial expression.

Example 2.3. *x^3+1 and 3 are polynomial expressions in x of degree 3 and 0, respectively but 0 is a polynomial expression in x with no degree.*

Definition 2.4 ([9]). *A complex number r is said to be a zero of a polynomial expression P(x) if P(r) = 0.*

Theorem 2.5 (**Linear Factorization Theorem**, [10]). *Let $a_n \neq 0$ be a leading coefficient of a polynomial expression $P(x)$ and the zeros of $P(x)$ with their multiplicities are listed in the following table (k is a natural number). Then the factorization form of $P(x)$ is $P(x) = a_n\ (x-r_1)^{m_1}\ (x-r_2)^{m_2}\ (x-r_3)^{m_3}\ \ldots\ (x-r_k)^{m_k}$.*

zeros	Multiplicities
r_1	m_1
r_2	m_2
r_3	m_3
$\vdots$	$\vdots$
r_k	m_k

Table 1: The zeros of P(x) with their multiplicities.

Definition 2.6 ([11]). *A rational expression in a single variable x is an expression in the form of*

$$\frac{P(x)}{Q(x)}, \tag{2.2}$$

where P(x) and Q(x) are polynomial expressions in x and Q(x)≠0.

Remark 2.7. *From equation (2.2),*

- $P(x)$ is said to be numerator of rational expression.
- $Q(x)$ is said to be denominator of rational expression.
- If the degree of $P(x)$ is less than the degree of $Q(x)$, then the rational expression is said to be proper rational expression.
- If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then the rational expression is said to be improper rational expression.

Example 2.8. *$\frac{x^3+1}{x-1}$ and x^2+3 are rational expressions in x but $\frac{x^{0.5}}{x-1}$ is not rational expression because $x^{0.5}$ is not a polynomial expression.*

Theorem 2.9 ((**Polynomial Division Theorem**), [12]). *Let f(x) and d(x) are polynomial expressions such that d(x)≠0, and $\frac{f(x)}{d(x)}$ is improper rational expression. Then there exist unique polynomial expressions q(x) and r(x) such that*

$$f(x) = q(x)d(x) + r(x), \tag{2.3}$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$.

Remark 2.10. *From equation (2.3), $q(x)$ and $r(x)$ are the quotient and remainder of $f(x) \div d(x)$, respectively.*

3. Synthetic Division

Synthetic division is a way to divide a polynomial expression in x by the binomial $x-c$, where c is a constant [13]. In synthetic division, we follow the following steps

Step 1: Set up the synthetic division.
Step 2: Bring down the leading coefficient to the 2^{nd} cell of the bottom row.
Step 3: Multiply c by the value just written on the 2^{nd} cell of the bottom row, and then write this result on the 3^{rd} cell of the 3^{rd} row.
Step 4: Add the column created in Step 3, and then write this result on the 3^{rd} cell of the bottom row.
Step 5: Repeat Steps 3 and 4 until done.
Step 6: Write out the answer [14].

The numbers in the bottom row make up our coefficients of the quotient and remainder. The final value on the last cell of the bottom row is the remainder and the rests are the coefficients of the quotient [15].

Example 3.1. *Consider a rational expression $\frac{x^3+1}{x-2}$.*

x^3+1 means x^3+0x^2+0x+1 and the zero of $x-2$ is 2.

- Write the coefficients of x^3+0x^2+0x+1 and the zero of $x-2$ in the following manner.

Zero of the divisor				
$\boxed{2}$	1	0	0	1

Table 2: Step-1 of synthetic division for Example 3.1.

- Bring down the leading coefficient, 1 to the 2^{nd} cell of the bottom row. That is

Zero of the divisor				
$\boxed{2}$	1	0	0	1
	1			

Table 3: Step-2 of synthetic division for Example 3.1.

- Multiply 1 by the zero of the divisor, 2. This is equal to 2. Then write 2 on the 3^{rd} cell of the 3^{rd} row. That is

Zero of the divisor				
$\boxed{2}$	1	0	0	1
		2		
	1			

Table 4: Step-3 of synthetic division for Example 3.1.

- Add 0 and 2. This is equal to 2. Then write 2 on the 3^{rd} cell of the bottom row. That is

Zero of the divisor				
$\boxed{2}$	1	0	0	1
		2		
	1	2		

Table 5: Step-4 of synthetic division for Example 3.1.

- Multiply 2 by the zero of the divisor, 2. This is equal to 4. Then write 4 on the 4^{th} cell of the 3^{rd} row. That is

Zero of the divisor				
$\boxed{2}$	1	0	0	1
		2	4	
	1	2		

Table 6: Step-5 of synthetic division for Example 3.1.

- Add 0 and 4. This is equal to 4. Then write 4 on the 4^{th} cell of the bottom row. That is

Zero of the divisor				
$\boxed{2}$	1	0	0	1
		2	4	
	1	2	4	

Table 7: Step-6 of synthetic division for Example 3.1.

- Multiply 4 by the zero of the divisor, 2. This is equal to 8. Then write 8 on the 5^{th} cell of the 3^{rd} row. That is

Zero of the divisor				
$\boxed{2}$	1	0	0	1
		2	4	8
	1	2	4	

Table 8: Step-7 of synthetic division for Example 3.1.

- Add 1 and 8. This is equal to 9. Then write 9 on the 5^{th} cell of the bottom row. That is

Zero of the divisor				
$\boxed{2}$	1	0	0	1
		2	4	8
	1	2	4	$\boxed{\mathbf{9}=r}$

Table 9: Step-8 of synthetic division for Example 3.1.

Therefore, 9 is the remainder and 1, 2, and 4 are the coefficients of the quotient of the given polynomial division. That is, $r(x) = 9$ and $q(x) = x^2 + 2x + 4$.

Let $\frac{f(x)}{d(x)}$ be improper rational expression such that $d(x) = t(x-c)$; $t \neq 0$. Assume that the quotient and remainder of this rational expression are $q(x)$ and $r(x)$, respectively.

$$\frac{f(x)}{d(x)} = \frac{f(x)}{t(x-c)} = q(x) + \frac{r(x)}{t(x-c)} \Longrightarrow \frac{1}{t}\,\frac{f(x)}{x-c} = \frac{1}{t}\left[tq(x) + \frac{r(x)}{x-c}\right] \Longrightarrow \frac{f(x)}{x-c} = t\,q(x) + \frac{r(x)}{x-c}$$

From the preceding equation, we can conclude that

- The quotient of $\frac{f(x)}{x-c}$ is t times of the quotient of $\frac{f(x)}{t(x-c)}$.
- The remainder of $\frac{f(x)}{x-c}$ is the same as the remainder of $\frac{f(x)}{t(x-c)}$.

Remark 3.2. *Two proportional improper rational expressions have the same quotient but not the same remainder.*

4. Extension of Synthetic Division

Let $\frac{P(x)}{Q(x)}$ be improper rational expression and $Q(x) = (x-a_1)(x-a_2)(x-a_3)\ \ldots\ (x-a_n)$; each $a_i\,'s$ is a constant.

- $\frac{P(x)}{x-a_1} = q_1(x) + \frac{r_1}{x-a_1}$ where $q_1(x)$ and r_1 are the quotient and remainder of $\frac{P(x)}{x-a_1}$, respectively.
- Multiply the preceding equation by $\frac{1}{x-a_2}$. Then we get

$$\frac{P(x)}{(x-a_1)(x-a_2)} = \frac{q_1(x)}{x-a_2} + \frac{r_1}{(x-a_1)(x-a_2)} \implies \frac{P(x)}{(x-a_1)(x-a_2)} = q_2(x) + \frac{r_2}{x-a_2} + \frac{r_1}{(x-a_1)(x-a_2)},$$

  where $q_2(x)$ and r_2 are the quotient and remainder of $\frac{q_1(x)}{x-a_2}$, respectively.

$$\implies \frac{P(x)}{(x-a_1)(x-a_2)} = q_2(x) + \frac{r_2(x-a_1)+r_1}{(x-a_1)(x-a_2)},$$

- Multiply the preceding equation by $\frac{1}{x-a_3}$. Then we get

$$\frac{P(x)}{(x-a_1)(x-a_2)(x-a_3)} = \frac{q_2(x)}{x-a_3} + \frac{r_2(x-a_1)+r_1}{(x-a_1)(x-a_2)(x-a_3)}$$
$$\implies \frac{P(x)}{(x-a_1)(x-a_2)(x-a_3)} = q_3(x) + \frac{r_3}{x-a_3} + \frac{r_2(x-a_1) + r_1}{(x-a_1)(x-a_2)(x-a_3)},$$

  where $q_3(x)$ and r_3 are the quotient and remainder of $\frac{q_2(x)}{x-a_3}$, respectively.

$$\implies \frac{P(x)}{(x-a_1)(x-a_2)(x-a_3)} = q_3(x) + \frac{r_3(x-a_1)(x-a_2) + r_2(x-a_1) + r_1}{(x-a_1)(x-a_2)(x-a_3)}$$

- Using similar pattern

$$\frac{P(x)}{(x-a_1)(x-a_2)(x-a_3)\ \ldots\ (x-a_n)} = q_n(x) + \frac{r_1 + r_2(x-a_1) + r_3(x-a_1)(x-a_2) + \ldots + r_n(x-a_1)(x-a_2)\ \ldots\ (x-a_{n-1})}{(x-a_1)(x-a_2)(x-a_3)\ \ldots\ (x-a_n)}$$

Hence, $q(x) = q_n(x)$ is the quotient of $\frac{P(x)}{Q(x)}$ and $r(x) = r_1 + r_2(x-a_1) + r_3(x-a_1)(x-a_2) + \ldots + r_n(x-a_1)(x-a_2)\ \ldots\ (x-a_{n-1})$ is the remainder of $\frac{P(x)}{Q(x)}$. To find $q_1(x),\ q_2(x),\ \ldots,\ q_n(x),\ r_1,\ r_2,\ \ldots,$ *and* r_n, we use synthetic division successively in the following manner.

Zeros of the divisor	
$\boxed{a_1}$	**Coefficients of** $P(x)$
	This line consists numbers which are obtained using steps of synthetic division
	Coefficients of $q_1(x)$ *and* r_1
$\boxed{a_2}$	**Coefficients of** $q_1(x)$
	This line consists numbers which are obtained using steps of synthetic division
	Coefficients of $q_2(x)$ *and* r_2
$\vdots$	$\vdots$
	Coefficients of $q_{n-1}(x)$ & r_{n-1}
$\boxed{a_n}$	**Coefficients of** $q_{n-1}(x)$
	This line consists numbers which are obtained using steps of synthetic division
	Coefficients of $q_n(x)$ *and* r_n

Table 10: Steps of extension of synthetic division.

Hence, such type of successive use of synthetic division is called **extension of synthetic division**.

Remark 4.1. *The order of* $a_1,\ a_2,\ldots$ & a_n *in the preceding table does not affect the desired results.*

Example 4.2. *Let us find the quotient and remainder of rational expressions* $\frac{x^5+x+1}{(x^2-1)(x+2)}$, $\frac{x^4+x}{x^2+1}$ *and* $\frac{x^4+x}{x^2+x}$ *using extension of synthetic division.*

(a) $\frac{x^5+x+1}{(x^2-1)(x+2)} = \frac{x^5+0x^4+0x^3+0x^2+x+1}{(x+2)(x+1)(x-1)}$. Hence, the coefficients of a polynomial expression in the numerator of this rational expression are 1, 0, 0, 0, 1 & 1, and the zeros of a polynomial expression in the denominator of this rational expression are -2, -1 & 1. Now let us construct a table.

Zeros of the divisor						
$\boxed{\mathbf{-2}}$	1	0 -2	0 4	0 -8	1 16	1 -34
	1	-2	4	-8	17	$\boxed{\mathbf{-33}=r_1}$
$\boxed{\mathbf{-1}}$	1	-2 -1	4 3	-8 -7	17 15	
	1	-3	7	-15	$\boxed{\mathbf{32}=r_2}$	
$\boxed{\mathbf{1}}$	1	-3 1	7 -2	-15 5		
	1	-2	5	$\boxed{\mathbf{-10}=r_3}$		

Table 11: Steps of synthetic division for Example 4.2 (a).

Therefore,

- the quotient is $q(x) = x^2 - 2x + 5$.
- the remainder is

$$r(x) = r_1 + r_2(x+2) + r_3(x+2)(x+1) = -33 + 32(x+2) - 10(x+2)(x+1)$$
$$\Longrightarrow r(x) = -10x^2 + 2x + 11$$

(b) $\frac{x^4+x}{x^2+1} = \frac{x^4+0x^3+0x^2+x+0}{(x+i)(x-i)}$. Hence, the coefficients of a polynomial expression in the numerator of this rational expression are 1, 0, 0, 1 & 0, and the zeros of a polynomial expression in the denominator of this rational expression are $-i$ & i. Now let us construct a table.

Zeros of the divisor					
$\boxed{-\mathbf{i}}$	1	0 $-$i	0 -1	1 i	0 1$-$i
	1	$-$i	-1	1 + i	$\boxed{\mathbf{1-i}=r_1}$
$\boxed{\mathbf{i}}$	1	$-$i i	-1 0	1 + i $-$i	
	1	0	-1	$\boxed{\mathbf{1}=r_2}$	

Table 12: Steps of synthetic division for Example 4.2 (b).

Therefore,

- the quotient is $q(x) = x^2 - 1$.
- the remainder is $r(x) = r_1 + r_2(x+i) = 1\text{-}i + 1(x+i) \Longrightarrow r(x) = x + 1$

(c) $\frac{x^4+x}{x^2+x} = \frac{x^4+0x^3+0x^2+x+0}{x(x+1)}$. Hence, the coefficients of a polynomial expression in the numerator of this rational expression are 1, 0, 0, 1 & 0, and the zeros of a polynomial expression in the denominator of this rational expression are 0 & -1. Now let us construct a table.

Zeros of the divisor					
$\boxed{\mathbf{0}}$	1	0 0	0 0	1 0	0 0
	1	0	0	1	$\boxed{\mathbf{0}=r_1}$
$\boxed{-\mathbf{1}}$	1	0 -1	0 1	1 -1	
	1	-1	1	$\boxed{\mathbf{0}=r_2}$	

Table 13: Steps of synthetic division for Example 4.2 (c).

Therefore,

- the quotient is $q(x) = x^2 - x + 1$.
- the remainder is $r(x) = r_1 + r_2 x = 0 + 0x \Longrightarrow r(x) = 0$. Hence, a polynomial expression in the denominator of the given rational expression is a factor of a polynomial expression in the numerator of the given rational expression.

Remark 4.3. *Let $\frac{f(x)}{d(x)}$ be improper rational expression such that $d(x) = (a_1x - c_1)(a_2x - c_2)(a_3x - c_3) \ldots (a_nx - c_n)$; each c_i's is a constant & each a_i's is a non $-$ zero constant, then*

- *the quotient of $\frac{f(x)}{d(x)}$ is $\frac{1}{a_1\, a_2\, a_3\, \ldots\, a_n}$ of the quotient of $\frac{f(x)}{(x-\frac{c_1}{a_1})(x-\frac{c_2}{a_2})(x-\frac{c_3}{a_3})\ldots(x-\frac{c_n}{a_n})}$.*
- *the remainder of $\frac{f(x)}{d(x)}$ is the same as the remainder of $\frac{f(x)}{(x-\frac{c_1}{a_1})(x-\frac{c_2}{a_2})(x-\frac{c_3}{a_3})\ldots(x-\frac{c_n}{a_n})}$.*

5. Applications of Extension of Synthetic Division

Extension of synthetic division has applications for

- finding the oblique asymptote of the graph of a rational function
- evaluating the integration of some rational functions
- representing polynomial expression by factorial function in numerical analysis
- finding an equation of a line tangent to a graph of polynomial function at a point
- finding the greatest common factor of two polynomial expressions.

5.1. Oblique asymptote of the graph of a rational function

Extension of synthetic division is applicable to find oblique asymptote of the graph of a rational function. The graph of a rational function has oblique asymptote if the degree of a polynomial expression in the numerator exceeds the degree of a polynomial expression in the denominator by one.
Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function. If the degree of P(x) exceeds the degree of $Q(x)$ by one, then the oblique asymptote of the graph of $f(x)$ is a line$y = q(x)$ where $q(x)$ is a quotient of $P(x) \div Q(x)$.

Example 5.1. *Let us find the oblique asymptote of the graph of f(x) =* $\frac{x^3}{x^2-25}$ *using extension of synthetic division.*

Since the degree of x^3 exceeds the degree of x^2-25 by 1, then the graph of $f(x)$ has oblique asymptote.
$x^3 = x^3 + 0x^2 + 0x + 0$ and $x^2-25 = (x+5)(x-5)$.

Zeros of the divisor				
$\boxed{-5}$	1	0 -5	0 25	0 -125
	1	-5	25	$\boxed{-125 = r_1}$
$\boxed{\mathbf{5}}$	1	-5 5	25 0	
	1	0	$\boxed{\mathbf{25} = r_2}$	

Table 14: Steps of synthetic division for Example 5.1.

From the preceding table, $q(x) = x$. Hence, the oblique asymptote of the graph of the given function is a line $y = x$.

5.2. Integration of some rational functions

If the integrand of a given integration is improper rational expression, then extension of synthetic division has application in a time of evaluation of the given integration using a method of integration by partial fractions.

Example 5.2. *Consider* $\int \frac{x^4}{x^2-2x-3}\, dx$.

To evaluate this integral using a method of integration by partial fractions, the integrand must be expressed as the sum of polynomial expression and proper rational expression. To do this, we apply extension of synthetic division.
$x^4 = x^4 + 0x^3 + 0x^2 + 0x + 0$ and $x^2-2x-3 = (x+1)(x-3)$.

Zeros of the divisor					
$\boxed{-1}$	1	0 -1	0 1	0 -1	0 1
	1	-1	1	-1	$\boxed{\mathbf{1} = r_1}$
$\boxed{\mathbf{3}}$	1	-1 3	1 6	-1 21	
	1	2	7	$\boxed{\mathbf{20} = r_2}$	

Table 15: Steps of synthetic division for Example 5.2.

From the preceding table,

- the quotient of $\frac{x^4}{x^2-2x-3}$ is $q(x) = x^2 + 2x + 7$.
- the remainder of $\frac{x^4}{x^2-2x-3}$ is $r(x) = 1+20(x+1) = 20x+21$.

Hence,

$$\frac{x^4}{x^2-2x-3} = x^2 + 2x + 7 + \frac{20x + 21}{x^2-2x-3} \Longrightarrow \frac{x^4}{x^2-2x-3} = x^2 + 2x + 7 + \frac{81/4}{x-3} + \frac{-1/4}{x+1}$$

since $\frac{20x + 21}{x^2-2x-3}$ is decomposed as $\frac{81/4}{x-3} + \frac{-1/4}{x+1}$ using the concept of partial fraction decomposition.

$$\int \frac{x^4}{x^2-2x-3} dx = \int (x^2 + 2x + 7) dx + \int \frac{81/4}{x-3}\, dx + \int \frac{-1/4}{x+1} dx.$$

After this, the evaluations of these integrals are too simple.

5.3. Representation of polynomial expression by factorial function

Extension of synthetic division has application in a time of expression of a polynomial expression by factorial function.
A product of the form $x(x-h)(x-2h)(x-3h)\ldots(x-(n-1)h)$ is called factorial function and denoted by $x^{(n)}$. The name of h is step size of $\{x, x-h, x-2h, x-3h, \ldots, x-(n-1)h\}$. Thus $x^{(n)} = x(x-h)(x-2h)(x-3h)\ldots(x-(n-1)h)$. If $h=1$, then $x^{(n)} = x(x-1)(x-2)(x-3)\ldots(x-(n-1))$.

Example 5.3. *Let us represent a polynomial expression $x^4-12x^3+42x^2-30x+9$ by factorial function using the help of extension of synthetic division(assuming $h=1$).*

Suppose

$$x^4-12x^3+42x^2-30x+9 = Ax^{(4)}+Bx^{(3)}+Cx^{(2)}+Dx^{(1)}+E$$
$$\Longrightarrow x^4-12x^3+42x^2-30x+9 = Ax(x-1)(x-2)(x-3)+Bx(x-1)(x-2)+Cx(x-1)+Dx+E$$

Let us divide both sides of the preceding equation by x$(x-1)(x-2)(x-3)(x-4)$. Then we get

$$\frac{x^4-12x^3+42x^2-30x+9}{x(x-1)(x-2)(x-3)(x-4)} = \frac{Ax(x-1)(x-2)(x-3)+Bx(x-1)(x-2)+Cx(x-1)+Dx+E}{x(x-1)(x-2)(x-3)(x-4)}.$$

Thus, 0 and $Ax(x-1)(x-2)(x-3)+Bx(x-1)(x-2)+Cx(x-1)+Dx+E$ are the quotient and the remainder of $\frac{x^4-12x^3+42x^2-30x+9}{x(x-1)(x-2)(x-3)(x-4)}$, respectively. To find all the unknown coefficients, let us apply extension of synthetic division.

$$\Longrightarrow A = r_5,\ B = r_4,\ C = r_3,\ D = r_2 \textit{ and } E = r_1.$$

Zeros of the divisor					
0	1	−12 0	42 0	−30 0	9 0
	1	−12	42	−30	**9 =** r_1 **= E**
1	1	−12 1	42 −11	−30 31	
	1	−11	31	**1 =**r_2 **= D**	
2	1	−11 2	31 −18		
	1	−9	**13 =** r_3 **= C**		
3	1	−9 3			
	1	**−6 =** r_4 **= B**			
4	1				
	1 = r_5 **= A**				

Table 16: Steps of synthetic division for Example 5.3.

From the preceding table, the factorial representation of the given polynomial expression is $x^{(4)}-6x^{(3)}+13x^{(2)}+x^{(1)}+9$.

5.4. Equation of a line tangent to a graph of polynomial function at a point

Extension of synthetic division has application to find the equation of a line which is tangent to the graph of a polynomial function $f(x)$ at a point $(r, f(r))$.
The equation of a line which is tangent to the graph of a polynomial function $f(x)$ at a point $(r, f(r))$ is $y = r(x)$ where $r(x)$ is the remainder of $f(x) \div (x-r)^2$.

Example 5.4. *Let us find the equation of a line which is tangent to the graph of a polynomial function $f(x) = x^5-1$ at a point $(1, f(1))$ using the concept of extension of synthetic division.*

The equation of a line which is tangent to the graph of a given polynomial function $f(x)$ at a point $(1, f(1))$ is the remainder of $f(x) \div (x-1)^2$.

Zeros of the divisor						
1	1	0 1	0 1	0 1	0 1	−1 1
	1	1	1	1	1	**0 =** r_1
1	1	1 1	1 2	1 3	1 4	
	1	2	3	4	**5 =** r_2	

Table 17: Steps of synthetic division for Example 5.4.

From the preceding table, $r(x) = 0+5(x\text{-}1) = 5x-5$. Hence, the equation of a line which is tangent to the graph of a polynomial function $f(x) = x^5-1$ at a point $(1, f(1))$ is $y(x) = 5x-5$.

5.5. Greatest common factor (GCF) of two polynomial expressions

Using the concept of Euclidean algorithm theorem and extension of synthetic division, we can find GCF of two polynomial expressions.
Euclidean algorithm theorem: Let $f(x)$ and $g(x)$ are polynomial expressions such that $g(x) \neq 0$ and degree of $f(x) \geq$ degree of $g(x)$. Apply polynomial division theorem successively as follows

$$f(x) = q_0(x)g(x) + r_0(x) \quad \text{with degree of } g(x) > \text{degree of } r_0(x)$$
$$g(x) = q_1(x)r_0(x) + r_1(x) \quad \text{with degree of } r_0(x) > \text{degree of } r_1(x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$r_{n-2}(x) = q_n(x)r_{n-1}(x) + r_n(x) \quad \text{with degree of } r_{n-1}(x) > \text{degree of } r_n(x)$$
$$r_{n-1}(x) = q_{n+1}(x)r_n(x) + 0.$$

Therefore, $GCF[f(x), g(x)] = \frac{1}{c}r_n(x)$ where c is the leading coefficient of $r_n(x)$ [6].
We use extension of synthetic division to find $q_0(x), q_1(x), q_2(x), \ldots, q_{n+1}(x), r_0(x), r_1(x), r_2(x), \ldots$ and $r_n(x)$.

Example 5.5. *Consider polynomial expressions $x^4 - x^3$ and $x^3 - x$.*

Firstly Let us find the quotient and the remainder of $x^4 - x^3 \div x^3 - x$ using extension of synthetic division.

Zeros of the divisor					
$\boxed{0}$	1	-1	0	0	0
		0	0	0	0
	1	-1	0	0	$\boxed{0 = r_1}$
$\boxed{-1}$	1	-1	0	0	
		-1	2	-2	
	1	-2	2	$\boxed{-2 = r_2}$	
$\boxed{1}$	1	-2	2		
		1	-1		
	1	-1	$\boxed{1 = r_3}$		

Table 18: Step-1 of synthetic division for Example 5.5.

From the preceding table, $x-1$ and $x(x-1)$ are the quotient and the remainder of $x^4 - x^3 \div x^3 - x$, respectively. Hence, $x^4 - x^3 = (x-1)(x^3 - x) + x(x-1)$.
Secondly Let us find the quotient and remainder of $x^3 - x \div$ [x(x−1)] using extension of synthetic division.

Zeros of the divisor				
$\boxed{0}$	1	0	-1	0
		0	0	0
	1	0	-1	$\boxed{0 = r_1}$
$\boxed{1}$	1	0	-1	
		1	1	
	1	1	$\boxed{0 = r_2}$	

Table 19: Step-2 of synthetic division for Example 5.5.

From the preceding table, $x+1$ and 0 are the quotient and remainder of $x^3 - x \div [x(x-1)]$, respectively. Hence, $x^3 - x = (x+1)[x(x-1)] + 0$. Therefore, the GCF of the given two polynomial expressions is $\frac{1}{1}x(x-1) = x^2 - x$ since the leading coefficient of $x(x-1)$ is 1.

6. Conclusion

Using variable − wise division(long division), we can able to find the quotient and remainder of improper rational expression $\frac{P(x)}{Q(x)}$. Sometimes, this division is more complex and tedious. To find the quotient and remainder of $\frac{P(x)}{Q(x)}$ in easiest and shortest way, we need to have an easiest and simplest way. Extension of synthetic division is among of them. To deal the extension of synthetic division, we need to have the concepts of polynomial and rational expressions as preliminary concepts. This division has great applications in different ares of Mathematics as stated earlier. We would like to recommend that mathematics teachers should use the extension of synthetic division for polynomial divisions in their teaching-learning tasks.

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Some Identities and Generating Functions for Bidimensional Balancing and Cobalancing Sequences

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Abstract

This article studies the bidimensional versions of four number sequences: balancing, Lucas-balancing, Lucas-cobalancing and cobalancing. Some of the identities and generating functions were presented.

1. Introduction

The study of numerical sequences has been the subject of a lot of research in recent years. Much emphasis has been placed on the Fibonacci, balancing, Lucas-balancing, cobalancing and Lucas-cobalancing sequences, among others. In [1], some identities were deduced for the Pell, Pell-Lucas and balancing numbers and some relationships between them. Some formulas for sums, divisibility properties, perfect squares and Pythagorean triples involving these numbers were also studied. In [2], some combinatorial expressions of balancing and Lucas-balancing numbers were established and some of their properties were investigated. Finally, in [3], a brief study was made of the limits for reciprocal sums involving terms from balancing and Lucas-balancing sequences. Many properties and identities of sequences are established using the so-called Binet formula for these sequences. This formula is known to be an explicit formula used to determine any term of a specific numerical sequence without having to resort to its previous terms. In 1985, the mathematician Levesque, in [4], deduced this formula for a linear recurrence of m-th order ($m \in \mathbb{N}$). Binet's formula for the sequence of balancing numbers B_n, Lucas-balancing numbers C_n, Lucas-cobalancing numbers c_n and cobalancing numbers b_n, is given, respectively, by:

$$B_n = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

$$C_n = \frac{r_1^n + r_2^n}{2},$$

$$c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2},$$

$$b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2},$$

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with $r_1 = \alpha_1^2 = 3+2\sqrt{2}$ and $r_2 = \alpha_2^2 = 3-2\sqrt{2}$, where $\alpha_1 = \sqrt{3+2\sqrt{2}}$ and $\alpha_2 = \sqrt{3-2\sqrt{2}}$.
In this article, our objective is to study the bidimensional versions of these four number sequences, presenting some identities and generating functions.
There are already many works on the bidimensional, tridimensional and n-dimensional versions of some number sequences. For example, in [5], we can find a brief approach to the Leonardo sequence, as well as a discussion related to the bidimensional recurrence relations of this type of number from its unidimensional model. In [6], we have a study of bidimensional and tridimensional recursive relations defined from the unidimensional recursive model of the Narayana sequence. In [7], we can find a study of bidimensional and tridimensional identities for the Fibonacci numbers in complex form. In [8], the authors introduced the gaussian Fibonacci numbers and the bidimensional recurrence relations of this sequence. In [9], we have a brief study on the bidimensional extensions of the balancing and Lucas-balancing numerical sequences and, in particular, in this study, the authors define, respectively, the bidimensional recurrence relations of these two numerical sequences, as follows:
• The bidimensional numerical sequence balancing $B_{(n,m)}$ satisfies the following recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} B_{(n+1,m)} = 6B_{(n,m)} - B_{(n-1,m)}, \\ B_{(n,m+1)} = 6B_{(n,m)} - B_{(n,m-1)}, \end{cases} \tag{1.1}$$

with the initial conditions $B_{(0,\ 0)} = 0$, $B_{(1,\ 0)} = 1$, $B_{(0,\ 1)} = i$, $B_{(1,\ 1)} = 1+i$, where $i^2 = -1$.
• The bidimensional numerical sequence Lucas-balancing $C_{(n,m)}$ satisfies the following recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} C_{(n+1,m)} = 6C_{(n,m)} - C_{(n-1,m)}, \\ C_{(n,m+1)} = 6C_{(n,m)} - C_{(n,m-1)}, \end{cases}$$

with the initial conditions $C_{(0,\ 0)} = 1$, $C_{(1,\ 0)} = 3$, $C_{(0,\ 1)} = 1+i$, $C_{(1,1)} = 3+i$, where $i^2 = -1$.
Finally, in [10], we have the introduction of a new bidimensional version of the cobalancing and Lucas-cobalancing numbers, where we can find the study of some properties and identities satisfied by these new bidimensional sequences. In this study, the authors define, respectively, the bidimensional recurrence relations of these sequences, as follows:
• The bidimensional numerical sequence Lucas-cobalancing $c_{(m,n)}$ satisfies the following recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} c_{(n+1,m)} = 6c_{(n,m)} - c_{(n-1,m)}, \\ c_{(n,m+1)} = 6c_{(n,m)} - c_{(n,m-1)}, \end{cases} \tag{1.2}$$

with the initial conditions $c_{(0,0)} = 1$, $c_{(1,0)} = 7$, $c_{(0,1)} = 1+i$, $c_{(1,1)} = 7+i$, where $i^2 = -1$.
• The bidimensional sequence of the cobalancing numbers $b_{(m,n)}$, satisfies the recurrence relation

$$b_{(n,m)} = \frac{1}{8}c_{(n+1,m)} - \frac{3}{8}c_{(n,m)} - \frac{1}{2}, \ \forall n,m \in \mathbb{N}_0, \tag{1.3}$$

with the initial conditions $b_{(0,0)} = 0$, $b_{(1,0)} = 2$, $b_{(0,1)} = -\frac{1}{4}i$, $b_{(1,1)} = 2+\frac{1}{4}i$, where $i^2 = -1$.

It was, therefore, these and other works that served as motivation for our study, which consists of continuing to approach the bidimensional version of balancing and cobalancing numbers, investigating topics related to these sequences.
The following results are included in [9] and they are one of the main tools of this work, being used in the various proofs of this paper. To clarify this article, we have decided to include these results in what follows. Thus:

Lemma 1.1 (Lemma 3.2 in [9]). *The following properties are true for every non-negative integers n and m:*

1. $B_{(n,0)} = B_n$;
2. $B_{(0,m)} = B_m i$;
3. $B_{(n,1)} = B_n + (B_n - B_{n-1})i$;
4. $B_{(1,m)} = (B_m - B_{m-1}) + B_m i$.

Lemma 1.2 (Theorem 3.3 in [9]). *For non-negative integers n, m, the bidimensional balancing numbers are described as follows:*

$$B_{(n,m)} = B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i. \tag{1.4}$$

Lemma 1.3 (Theorem 5.3 in [9]). *For non-negative integers n, m, the bidimensional Lucas-balancing numbers are described as follows:*

$$C_{(n,m)} = C_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i. \tag{1.5}$$

This article is structured as follows: in the next section, the Catalan and Cassini identities of the bidimensional versions of the balancing, Lucas-balancing, Lucas-cobalancing and cobalancing numerical sequences will be presented. Section 3 is dedicated to the study of the generating functions of these four sequences. Finally, Section 4 presents a brief conclusion on this subject.

2. Some Identities

In this section we study some identities involving the bidimensional sequences already mentioned in the previous section. These identities are the Catalan identity (sometimes called the Simson identity) and the Cassini identity. It should be noted that Cassini's identity is a particular case of Catalan's identity. Each subsection is dedicated to one of the bidimensional versions of the sequences mentioned above.

2.1. For the sequence $\{B_{(n,m)}\}_{n,m\geq 0}$

In this subsection we present the results for the bidimensional balancing sequence relating to Catalan's identity and Cassini's identity.

Proposition 2.1 (Catalan's identity)**.** *For any nonzero positive integer r and for all positive integers n,m, the following identity concerning the numerical sequence balancing in the bidimensional version is valid:*

$$B_{(n-r,m)}B_{(n+r,m)}-\left(B_{(n,m)}\right)^2=-B_r^2X_m^2-d_rB_m^2+e_rX_mB_mi, \tag{2.1}$$

where $d_r=X_{n+r}X_{n-r}-X_n^2$ e $e_r=X_{n+r}B_{n-r}+X_{n-r}B_{n+r}-2B_nX_n$.

Proof. Applying Lemma 1.2 (Theorem 3.3 in [9]) to the expressions $B_{(n-r,m)}$ and $B_{(n+r,m)}$, calling $X_i=B_i-B_{i-1}$, $i\geq 0$, multiplying both terms and using expression (2.1) from Proposition 2.1 in [**?**], we get

$$B_{(n-r,m)}B_{(n+r,m)}=\left(B_n^2-B_r^2\right)X_m^2-X_{n+r}X_{n-r}B_m^2+(X_{n+r}B_{n-r}+X_{n-r}B_{n+r})X_mB_mi. \tag{2.2}$$

Once again, taking into account Lemma 1.2 (Theorem 3.3 in [9] and using the perfect square trinomial, we obtain

$$\left(B_{(n,m)}\right)^2=B_n^2X_m^2-X_n^2B_m^2+2B_nX_nX_mB_mi. \tag{2.3}$$

Subtracting the identities (2.2) and (2.3), the result follows. □

As we have already mentioned, Cassini's identity is a particular case of Catalan's identity, making $r=1$. So we get:

Corollary 2.2 (Cassini's identity)**.** *When $r=1$, for the bidimensional balancing sequence, we have Cassini's identity which consists of the following identity:*

$$B_{(n-1,m)}B_{(n+1,m)}-\left(B_{(n,m)}\right)^2=-X_m^2-d_1B_m^2+e_1X_mB_mi, \tag{2.4}$$

where $d_1=X_{n+1}X_{n-1}-X_n^2$ and $e_1=X_{n+1}B_{n-1}+X_{n-1}B_{n+1}-2B_nX_n$.

2.2. For the sequence $\{C_{(n,m)}\}_{n,m\geq 0}$

In this subsection, we present results similar to those in the previous subsection for the bidimensional Lucas-balancing sequence related to the Catalan's identity and Cassini's identity.

Proposition 2.3 (Catalan's identity)**.** *For any nonzero positive integer r and for all positive integers n,m, the following identity related to the bidimensional Lucas-balancing number is true:*

$$C_{(n-r,m)}C_{(n+r,m)}-\left(C_{(n,m)}\right)^2=\left(C_r^2-1\right)X_m^2-d_rB_m^2+\tilde{e}_rX_mB_mi,$$

where d_r has already been previously defined and $\tilde{e}_r=C_{n-r}X_{n+r}+C_{n+r}X_{n-r}-2C_nX_n$.

Proof. Using Lemma 1.2 (Theorem 3.3 in [9]) in the expressions $B_{(n-r,m)}$ and $B_{(n+r,m)}$, considering $X_i=B_i-B_{i-1}$, $i\geq 0$, multiplying both terms and taking into account expression (2.2) of Proposition 2.1 in [**?**], we obtain

$$C_{(n-r,m)}C_{(n+r,m)}=\left(C_n^2+C_r^2-1\right)X_m^2-X_{n-r}X_{n+r}B_m^2+X_mB_mi. \tag{2.5}$$

Considering Lemma 1.3 (Theorem 5.3 in [9]) and using the perfect square trinomial, we get

$$\left(C_{(n,m)}\right)^2=C_n^2X_m^2-X_n^2B_m^2+2C_nX_nX_mB_mi. \tag{2.6}$$

The result follows, subtracting identities (2.5) and (2.6). □

The Cassini's identity for this sequence of numbers is defined by:

Corollary 2.4 (Cassini's identity)**.** *When $r=1$ we have the Cassini's identity for the bidimensional Lucas-balancing sequence:*

$$C_{(n-1,m)}C_{(n+1,m)}-\left(C_{(n,m)}\right)^2=8X_m^2-d_1B_m^2+\tilde{e}_1X_mB_mi,$$

where d_1 has already been defined and $\tilde{e}_1=C_{n-1}X_{n+1}+C_{n+1}X_{n-1}-2C_nX_n$.

2.3. For the sequence $\{c_{(n,m)}\}_{n,m\geq 0}$

In this subsection we present results similar to those in the previous subsections on the bidimensional Lucas-cobalancing sequence involving Catalan's and Cassini's identities. We will omit the respective proofs, since they are similar to the previous ones.

Proposition 2.5 (Catalan's identity)**.** *For any nonzero positive integer r and for all positive integers n,m, the following identity related to bidimensional Lucas-cobalancing numbers is valid:*

$$c_{(n-r,m)}c_{(n+r,m)}-\left(c_{(n,m)}\right)^2=f_rX_m^2-d_rB_m^2+g_rX_mB_mi,$$

where $f_r=c_{n-r+1}c_{n+r+1}-c_{n+1}^2$, d_r already known and $g_r=c_{n-r+1}X_{n+r}+c_{n+r+1}X_{n-r}-2c_{n+1}X_n$.

Cassini's identity for this numerical sequence is given by:

Corollary 2.6 (Cassini's identity)**.** *When $r=1$ we have the identity for the bidimensional Lucas-cobalancing sequence:*

$$c_{(n-1,m)}c_{(n+1,m)}-\left(c_{(n,m)}\right)^2=f_1X_m^2-d_1B_m^2+g_1X_mB_mi,$$

where $f_1=c_nc_{n+2}-c_{n+1}^2$, d_1 already presented and $g_1=c_nX_{n+1}-2c_{n+1}X_n+c_{n+2}X_{n-1}$.

2.4. For the sequence $\{b_{(n,m)}\}_{n,m\geq 0}$

In this subsection we present the results for the bidimensional cobalancing numerical sequence involving Catalan's and Cassini's identities.

Proposition 2.7 (Catalan's identity)**.** *For any nonzero positive integer r and for all positive integers n,m, the following identity for bidimensional cobalancing numbers is true:*

$$b_{(n-r,m)}b_{(n+r,m)}-\left(b_{(n,m)}\right)^2=\frac{1}{64}s_rX_m^2-\frac{1}{8}t_rX_m-\frac{1}{64}u_rB_m^2+\frac{1}{32}v_rX_mB_mi-\frac{1}{8}z_rB_mi,$$

where

$$s_r=(c_{n-r+2}-3c_{n-r+1})(c_{n+r+2}-3c_{n+r+1})-c_{n+2}+3c_{n+1};$$

$$t_r=\frac{c_{n-r+2}c_{n+r+2}-3(c_{n-r+1}+c_{n+r+1})}{2}-c_{n+2}+3c_{n+1};$$

$$u_r=(X_{n-r+2}-3X_{n-r})+(X_{n+r+2}-3X_{n+r})-(X_{n+2}-3X_n)^2;$$

$$v_r=\frac{(c_{n-r+2}-3c_{n-r+1})(X_{n+r+2}-3X_{n+r})+(c_{n+r+2}-3c_{n+r+1})(X_{n-r+2}-3X_{n-r})}{2}-(c_{n+2}-3c_{n+1})(X_{n+2}-3X_n)$$

and

$$z_r=\frac{X_{n-r+2}+X_{n+r+2}-3(X_{n-r}+X_{n+r})}{2}-X_{n+2}+3X_n.$$

Proof. Using the identity (1.3) for the expressions $b_{(n-r,m)}$ and $b_{(n+r,m)}$, making $X_i=B_i-B_{i-1}$, $i\geq 0$ and multiplying both terms, we get

$$b_{(n-r,m)}b_{(n+r,m)}=\frac{1}{64}h_rX_m^2-\frac{1}{16}j_rX_m-\frac{1}{64}l_rB_m^2+\frac{1}{64}p_rX_mB_mi-\frac{1}{16}q_rB_mi+\frac{1}{4}, \tag{2.7}$$

where

$$h_r=(c_{n-r+2}-3c_{n-r+1})(c_{n+r+2}-3c_{n+r+1});$$

$$j_r=c_{n-r+2}c_{n+r+2}-3(c_{n-r+1}+c_{n+r+1});$$

$$l_r=(X_{n-r+2}-3X_{n-r})+(X_{n+r+2}-3X_{n+r});$$

$$p_r=(c_{n-r+2}-3c_{n-r+1})(X_{n+r+2}-3X_{n+r})+(c_{n+r+2}-3c_{n+r+1})(X_{n-r+2}-3X_{n-r})$$

and

$$q_r=X_{n-r+2}+X_{n+r+2}-3(X_{n-r}+X_{n+r}).$$

Once again, taking into account expression (1.3) and applying the perfect square trinomial, we obtain

$$\left(b_{(n,m)}\right)^2=\frac{1}{64}\tilde{h}_rX_m^2-\frac{1}{8}\tilde{j}_rX_m-\frac{1}{64}\tilde{l}_rB_m^2+\frac{1}{32}\tilde{p}_rX_mB_mi-\frac{1}{8}\tilde{q}_rB_mi+\frac{1}{4}, \tag{2.8}$$

where $\tilde{h}_r=(c_{n+2}-3c_{n+1})^2$, $\tilde{j}_r=c_{n+2}-3c_{n+1}$, $\tilde{l}_r=(X_{n+2}-3X_n)^2$, $\tilde{p}_r=(c_{n+2}-3c_{n+1})(X_{n+2}-3X_n)$ and $\tilde{q}_r=X_{n+2}-3X_n$.
Subtracting (2.7) from (2.8), the result follows. □

The Cassini's identity for this number sequence is given by:

Corollary 2.8 (Cassini's identity)**.** *When $r=1$ we have the identity for the bidimensional cobalancing sequence:*

$$b_{(n-1,m)}b_{(n+1,m)}-\left(b_{(n,m)}\right)^2=\frac{1}{64}s_1X_m^2-\frac{1}{8}t_1X_m-\frac{1}{64}u_1B_m^2+\frac{1}{32}v_1X_mB_mi-\frac{1}{8}z_1B_mi,$$

where

$$s_1=(c_{n+1}-3c_n)\,c_{n+3}-(c_{n+1}-3c_n+1)\,c_{n+2}+3c_{n+1};$$

$$t_1=\frac{c_{n+2}+(c_{n+3}+6)\,c_{n+1}-3c_n}{2};$$

$$u_1=X_{n+3}-2X_{n+1}-3X_{n-1}-(X_{n+2}-3X_n)^2;$$

$$v_1=\frac{(c_{n+1}-3c_n)\,X_{n+3}-2\,(c_{n+2}-3c_{n+1})\,(X_{n+2}-3X_n)-(c_{n+3}-3\,(c_{n+2}-c_{n+1}-3))\,X_{n+1}}{2}$$

and

$$z_1=\frac{X_{n+3}-2\,(X_{n+2}+X_{n+1})+3\,(2X_n-X_{n-1})}{2}.$$

3. Generating Functions

In this section we study the generating functions of the four sequences mentioned above, in their bidimensional versions. We use the definition of the ordinary generating function of any sequence $\{a_n\}_n$ which is given by $G_{(a_n;x)}=\sum_{n=0}^{\infty}a_nx^n$, where x is any positive integer and n is any natural number, except for the case of bidimensional cobalancing numbers. So let's start with the case of balancing numbers in their bidimensional version.

Proposition 3.1. *For any positive integer x and for all natural numbers m,n, the following generating functions for the balancing numbers in the bidimensional version are valid:*

1. $G_{(B_{(n,\mathbf{m})};x)}=\frac{B_mi+((B_m-B_{m-1})-5B_mi)x}{1-6x+x^2}$, *where m is fixed;*
2. $G_{(B_{(\mathbf{n},m)};x)}=\frac{B_n-(5B_n-(B_n-B_{n-1})i)x}{1-6x+x^2}$, *where n is fixed.*

Proof. 1 The proof is done by fixing m. By the definition of the ordinary generating function, we obtain

$$G_{(B_{(n,\mathbf{m})};x)}=\sum_{n=0}^{\infty}B_{(n,m)}x^n=B_{(0,m)}x^0+B_{(1,m)}x+B_{(2,m)}x^2+\sum_{n=3}^{\infty}B_{(n,m)}x^n.$$

Thus, by Lemma 1.1, items 2 and 4 (Lemma 3.2, items 2 and 4 in [9]) and by (1.1), we get

$$G_{(B_{(n,\mathbf{m})};x)}=B_mi+((B_m-B_{m-1})+B_mi)\,x+\left(6B_{(1,m)}-B_{(0,m)}\right)x^2+x^3\sum_{n=3}^{\infty}B_{(n,m)}x^{n-3}.$$

Once again, by Lemma 1.1, items 4 and 2 (Lemma 3.2, items 4 and 2 in [9] and the fact that $\sum_{i=s}^{t}x_i=\sum_{i=s+p}^{t+p}x_{i-p}$, we obtain

$$G_{(B_{(n,\mathbf{m})};x)}=B_mi+(B_m-B_{m-1})\,x+B_mix+6\,((B_m-B_{m-1})+B_mi)\,x^2-B_mix^2+x^3\sum_{n=2}^{\infty}B_{(n+1,m)}x^{n-2}.$$

Again, by (1.1) and using one of the properties of summation, we get

$$\begin{aligned}
G_{(B_{(n,\mathbf{m})};x)}&=B_mi+(B_m-B_{m-1})\,x+B_mix+6\,(B_m-B_{m-1})\,x^2+5B_mix^2+x^3\sum_{n=2}^{\infty}\left(6B_{(n,m)}-B_{(n-1,m)}\right)x^{n-2}\\
&=B_mi+(B_m-B_{m-1})\,x+B_mix+6\,(B_m-B_{m-1})\,x^2+5B_mix^2+6x^3\sum_{n=2}^{\infty}B_{(n,m)}x^{n-2}-x^3\sum_{n=2}^{\infty}B_{(n-1,m)}x^{n-2}\\
&=B_mi+(B_m-B_{m-1})\,x+B_mix+6\,(B_m-B_{m-1})\,x^2+5B_mix^2+6x^3x^{-2}\sum_{n=2}^{\infty}B_{(n,m)}x^{n}-x^3x^{-1}\sum_{n=2}^{\infty}B_{(n-1,m)}x^{n-1}\\
&=B_mi+(B_m-B_{m-1})\,x+B_mix+6\,(B_m-B_{m-1})\,x^2+5B_mix^2+6x\left(\sum_{n=0}^{\infty}B_{(n,m)}x^n-B_{(0,m)}-B_{(1,m)}x\right)-x^2\sum_{n=1}^{\infty}B_{(n,m)}x^n.
\end{aligned}$$

And once again, by items 2 and 4 of Lemma 1.1 (items 2 and 4 of Lemma 3.2 in [9]), we obtain

$$\begin{aligned}
G_{(B_{(n,\mathbf{m})};x)}&=B_mi+(B_m-B_{m-1})\,x+B_mix+6\,(B_m-B_{m-1})\,x^2+5B_mix^2+6xG_{(B_{(n,\mathbf{m})};x)}-6xB_mi-6\,((B_m-B_{m-1})+B_mi)\,x^2\\
&\quad-x^2\left(\sum_{n=0}^{\infty}B_{(n,m)}x^n-B_{(0,m)}\right)\\
&=B_mi+(B_m-B_{m-1})\,x+B_mix+6\,(B_m-B_{m-1})\,x^2+5B_mix^2+6xG_{(B_{(n,\mathbf{m})};x)}-6xB_mi-6\,(B_m-B_{m-1})\,x^2-6B_mix^2\\
&\quad-x^2G_{(B_{(n,\mathbf{m})};x)}+B_mix^2.
\end{aligned}$$

Then come:

$$\left(1-6x+x^2\right)G_{(B_{(n,\mathbf{m})};x)} = B_m i + (B_m - B_{m-1})x - 5B_m ix.$$

Therefore,

$$G_{(B_{(n,\mathbf{m})};x)} = \frac{B_m i + ((B_m - B_{m-1}) - 5B_m i)x}{1-6x+x^2}.$$

2 The proof is done by fixing n. Using the definition of the generating function, we obtain

$$G_{(B_{(\mathbf{n},m)};x)} = \sum_{m=0}^{\infty} B_{(n,m)}x^m = B_{(n,0)}x^0 + B_{(n,1)}x + B_{(n,2)}x^2 + \sum_{m=3}^{\infty} B_{(n,m)}x^m.$$

Thus, using items 1 and 3 of Lemma 1.1 (items 1 and 3 of Lemma 3.2 in [9]) and by (1.1), we get

$$G_{(B_{(\mathbf{n},m)};x)} = B_n + (B_n + (B_n - B_{n-1})i)x + \left(6B_{(n,1)} - B_{(n,0)}\right)x^2 + \sum_{m=3}^{\infty} B_{(n,m)}x^m.$$

Once again, by items 3 and 1 of Lemma 1 (items 3 and 1 of Lemma 3.2 in [9]), by one of the properties of summations and also by (1.1), we get

$$\begin{aligned}
G_{(B_{(\mathbf{n},m)};x)} &= B_n + B_n x + (B_n - B_{n-1})ix + (6(B_n + (B_n - B_{n-1})i) - B_n)x^2 + x^3\sum_{m=3}^{\infty} B_{(n,m)}x^{m-3} \\
&= B_n + B_n x + (B_n - B_{n-1})ix + 6B_n x^2 + 6(B_n - B_{n-1})ix^2 - B_n x^2 + x^3\sum_{m=2}^{\infty} B_{(n,m+1)}x^{m-2} \\
&= B_n + B_n x + (B_n - B_{n-1})ix + 5B_n x^2 + 6(B_n - B_{n-1})ix^2 + x^3\sum_{m=2}^{\infty}\left(6B_{(n,m)} - B_{(n,m-1)}\right)x^{m-2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
G_{(B_{(\mathbf{n},m)};x)} &= B_n + B_n x + (B_n - B_{n-1})ix + 5B_n x^2 + 6(B_n - B_{n-1})ix^2 + x^3\left(6\sum_{m=2}^{\infty} B_{(n,m)}x^{m-2} - \sum_{m=2}^{\infty} B_{(n,m-1)}x^{m-2}\right) \\
&= B_n + B_n x + (B_n - B_{n-1})ix + 5B_n x^2 + 6(B_n - B_{n-1})ix^2 + 6x^3x^{-2}\sum_{m=2}^{\infty} B_{(n,m)}x^m - x^3\sum_{m=2}^{\infty} B_{(n,m-1)}x^{m-2} \\
&= B_n + B_n x + (B_n - B_{n-1})ix + 5B_n x^2 + 6(B_n - B_{n-1})ix^2 + 6x\sum_{m=2}^{\infty} B_{(n,m)}x^m - x^3x^{-1}\sum_{m=2}^{\infty} B_{(n,m-1)}x^{m-2} \\
&= B_n + B_n x + (B_n - B_{n-1})ix + 5B_n x^2 + 6(B_n - B_{n-1})ix^2 + 6x\left(\sum_{m=0}^{\infty} B_{(n,m)}x^m - B_{(n,0)} - B_{(n,1)}x\right) - x^2\sum_{m=2}^{\infty} B_{(n,m-1)}x^{m-1}.
\end{aligned}$$

Once again, using items 1 and 3 of Lemma 1.1 (items 1 and 3 of Lemma 3.2 in [9]) and by one of the properties of summations, we get

$$\begin{aligned}
G_{(B_{(\mathbf{n},m)};x)} &= B_n + B_n x + (B_n - B_{n-1})ix + 5B_n x^2 + 6(B_n - B_{n-1})ix^2 + 6x\left(\sum_{m=0}^{\infty} B_{(n,m)}x^m - B_n - (B_n + (B_n - B_{n-1})i)x\right) \\
&\quad - x^2\sum_{m=1}^{\infty} B_{(n,m)}x^m.
\end{aligned}$$

So,

$$\begin{aligned}
G_{(B_{(\mathbf{n},m)};x)} &= B_n + B_n x + (B_n - B_{n-1})ix + 5B_n x^2 + 6(B_n - B_{n-1})ix^2 + 6xG_{(B_{(\mathbf{n},m)};x)} - 6xB_n - 6(B_n + (B_n - B_{n-1})i)x^2 \\
&\quad - x^2\left(\sum_{m=0}^{\infty} B_{(n,m)}x^m - B_{(n,0)}\right).
\end{aligned}$$

And once again, by items Lemma 1.1, item 1 (Lemma 3.2, item 1 in [9]), we get

$$G_{(B_{(\mathbf{n},m)};x)} = B_n - 5B_n x + (B_n - B_{n-1})ix + 5B_n x^2 + 6(B_n - B_{n-1})ix^2 + 6xG_{(B_{(\mathbf{n},m)};x)} - 6B_n x^2 - 6(B_n - B_{n-1})ix^2 - x^2G_{(B_{(\mathbf{n},m)};x)} + B_n x^2.$$

Thus,

$$G_{(B_{(\mathbf{n},m)};x)} = \frac{B_n - (5B_n - (B_n - B_{n-1})i)x}{1-6x+x^2}.$$

□

The following results concern the generating functions for the cases of the bidimensional versions of the Lucas-balancing and Lucas-cobalancing numbers.
Since the demonstrations are similar to those in the previous case, we omit their proofs here.

Proposition 3.2. *For any positive integer x and for all natural numbers m,n, the following generating functions related to bidimensional Lucas-balancing numbers are true:*

1. $G_{(C_{(n,\boldsymbol{m})};x)} = \frac{((B_m-B_{m-1})+B_m i)-(3(B_m-B_{m-1})+5B_m i)x}{1-6x+x^2}$, *where m is fixed;*
2. $G_{(C_{(\boldsymbol{n},m)};x)} = \frac{C_n+((B_n-B_{n-1})-5C_n)x}{1-6x+x^2}$, *where n is fixed.*

Proposition 3.3. *For any positive integer x and for all natural numbers m,n, the following generating functions for bidimensional Lucas-cobalancing numbers are valid:*

1. $G_{(c_{(n,\boldsymbol{m})};x)} = \frac{B_{(1,m)}-\left(5B_{(1,m)}-6(B_m-B_{m-1})\right)x}{1-6x+x^2}$, *where m is fixed;*
2. $G_{(c_{(\boldsymbol{n},m)};x)} = \frac{C_n+((B_n-B_{n-1})-5C_n)x}{1-6x+x^2}$, *where n is fixed.*

The next result we present concerns the generating functions of cobalancing numbers in the bidimensional version. In this case, let's consider that x is a positive integer greater than 1.

Proposition 3.4. *For any positive integer x and for all natural numbers m,n, the following generating functions related to cobalancing numbers in their bidimensional version are valid:*

1. $G_{(b_{(n,\boldsymbol{m})};x)} = \frac{1}{8x}\left((1-3x)\,G_{(c_{(n,\boldsymbol{m})};x)} - B_{(1,\boldsymbol{m})}\right) - \frac{1}{2(1-x)}$, *where m is fixed;*
2. $G_{(b_{(\boldsymbol{n},m)};x)} = \frac{1}{8}\left(G_{(c_{(\boldsymbol{n}+1,m)};x)} - 3G_{(c_{(\boldsymbol{n},m)};x)} + c_{\boldsymbol{n}+2}\right) - \frac{1}{2(1-x)}$, *where n is fixed.*

Proof. 1 The proof is done by fixing m. By (1.2),

$$G_{(b_{(n,\mathbf{m})};x)} = \sum_{n=0}^{\infty}\left(\frac{1}{8}c_{(n+1,\mathbf{m})} - \frac{3}{8}c_{(n,\mathbf{m})} - \frac{1}{2}\right)x^n.$$

So,

$$G_{(b_{(n,\mathbf{m})};x)} = \frac{1}{8}\sum_{n=0}^{\infty}c_{(n+1,m)}x^n - \frac{3}{8}\sum_{n=0}^{\infty}c_{(n,m)}x^n - \frac{1}{2}\sum_{n=0}^{\infty}x^n.$$

By one of the properties of summations and the fact that $\sum_{k=0}^{\infty}ar^k = \frac{a}{1-r}$ for $|r|<1$, we get

$$G_{(b_{(n,\mathbf{m})};x)} = \frac{1}{8}\sum_{n=1}^{\infty}c_{(n,\mathbf{m})}x^{n-1} - \frac{3}{8}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{2}\left(\frac{1}{1-x}\right).$$

So come: 1 (Lemma 3.2, item 1 in [9]), we get

$$\begin{aligned}G_{(b_{(n,\mathbf{m})};x)} &= \frac{1}{8}x^{-1}\sum_{n=1}^{\infty}c_{(n,\mathbf{m})}x^n - \frac{3}{8}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{2(1-x)}\\ &= \frac{1}{8}x^{-1}\left(\sum_{n=0}^{\infty}c_{(n,\mathbf{m})}x^n - c_{(0,\mathbf{m})}\right) - \frac{3}{8}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{2(1-x)}.\end{aligned}$$

By Lemma 2.2, item (b) in [10], we get

$$G_{(b_{(n,\mathbf{m})};x)} = \frac{1}{8}x^{-1}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{8}x^{-1}B_{(1,\mathbf{m})} - \frac{3}{8}G_{(c_{(n,\mathbf{m})};x)} - \frac{1}{2(1-x)}.$$

Therefore,

$$G_{(b_{(n,\mathbf{m})};x)} = \frac{1}{8x}\left((1-3x)\,G_{(c_{(n,\mathbf{m})};x)} - B_{(1,\mathbf{m})}\right) - \frac{1}{2(1-x)}.$$

2 The proof is done by fixing n. By (1.2),

$$G_{(b_{(\mathbf{n},m)};x)} = \sum_{m=0}^{\infty}\left(\frac{1}{8}c_{(\mathbf{n}+1,m)} - \frac{3}{8}c_{(\mathbf{n},m)} - \frac{1}{2}\right)x^m.$$

Hence,

$$G_{(b_{(\mathbf{n},m)};x)} = \frac{1}{8}\sum_{m=0}^{\infty}c_{(\mathbf{n}+1,m)}x^m - \frac{3}{8}\sum_{m=0}^{\infty}c_{(\mathbf{n},m)}x^m - \frac{1}{2}\sum_{m=0}^{\infty}x^m.$$

Applying one of the properties of summations and the fact that $\sum_{k=0}^{\infty}ar^k = \frac{a}{1-r}$ for $|r|<1$, we obtain

$$G_{(b_{(\mathbf{n},m)};x)} = \frac{1}{8}\sum_{m=0}^{\infty}c_{(n+1,m+1)}x^{m+1} + \frac{1}{8}c_{(\mathbf{n}+1,0)} - \frac{3}{8}G_{(c_{(\mathbf{n},m)};x)} - \frac{1}{2}\left(\frac{1}{1-x}\right).$$

By item (a) of Lemma 2.2 in [10], we get

$$G_{(b_{(\mathbf{n},m)};x)} = \frac{1}{8}G_{(c_{(\mathbf{n+1},m+1)};x)} - \frac{3}{8}G_{(c_{(\mathbf{n},m)};x)} - \frac{1}{2(1-x)} + \frac{1}{8}c_{\mathbf{n}+1}.$$

Thus,

$$G_{(b_{(\mathbf{n},m)};x)} = \frac{1}{8}\left(G_{(c_{(\mathbf{n}+1,m)};x)} - 3G_{(c_{(\mathbf{n},m)};x)} + c_{\mathbf{n}+2}\right) - \frac{1}{2(1-x)}.$$

□

4. Conclusion

This article continues work related to the bidimensional version of some numerical sequences. The results presented in this paper are considered a contribution to the field of mathematics and offer an opportunity for researchers interested in this topic of number sequences to spend some time studying them. As future work, we plan to study the respective Binet's formulas.

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Lacunary Invariant Statistical Convergence in Fuzzy Normed Spaces

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Abstract

In the study done here regarding the theory of summability, we introduce some new concepts in fuzzy normed spaces. First, at the beginning of the original part of our study, we define the lacunary invariant statistical convergence. Then, we examine some characteristic features like uniqueness, linearity of this new notion and give its important relation with pre-given concepts.

1. Introduction and Definitions

First, we note some basic information available in the literature for a better understanding of our work and to use for the definitions of new concepts that we will give in the original chapter. The convergence of sequences in real numbers was generalized to statistical convergence by Schoenberg [1] and Fast [2]. Several features have been studied like being subspace of bounded sequence space by Salat [3], statistical Cauchy sequence by Fridy [4], statistical convergence and its equivalence of strong p-Cesaro summability for bounded sequences by Connor [5] and so on.
Lacunary convergence with the relation to strong Cesàro summability was studied by Freedman et al. [6]. Also Das and Patel [7] investigated this issue comprehensively. Fridy and Orhan [8,9] contributed to the literature about lacunary statistical convergence. Additionally, Ulusu and Nuray have been studied on this issue [10–12].
Banach limit was first introduced by Banach [13]. In case all Banach limits are equal for a given bounded sequence, Lorentz [14] called that almost convergence. Later, as a generalization of Banach limit and almost convergence, the notions of invariant mean and invariant convergence were presented by Raimi et al. [15,16]. Also, it has been studied by several authors [17–21]. Especially, Savaş and Nuray [22,23] proved important theorems in their studies.
The definitions of concepts such as statistical convergence, Banach limit, invariant mean, invariant convergence, lacunary sequence and lacunary convergence are not given here, and the references are based on the studies mentioned above.
Zadeh [24] proposed fuzzy set as a new concept to study on imprecise phenomena. A fuzzy set having certain properties was described as a fuzzy number [25,26]. The literature includes studies on concepts such as fuzzy topological spaces [27–29], fuzzy metric [30,31], fuzzy norm [32,33].
Now, based on these studies, the definition of a fuzzy number, arithmetic operations on fuzzy numbers, convergence on fuzzy numbers sequence, and fuzzy norm introduced by Felbin [32] will be given.
A fuzzy number u is a fuzzy set provided that
(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
(ii) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$;
(iii) u is upper semi-continuous;
(iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is a compact set.

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We will denote all fuzzy numbers by the set $\mathscr{L}(\mathbb{R})$. Every $r \in \mathbb{R}$ is also a fuzzy number denoted by $\tilde{r} = \tilde{r}(t)$ and its value is 1 when $t = r$ and 0 otherwise. So, $\mathbb{R}$ is included by $\mathscr{L}(\mathbb{R})$.
The α-level sets, partial ordering, arithmetic equations and supremum metric on $\mathscr{L}(\mathbb{R})$ are very important in fuzzy numbers and will be used in the operations performed in our study. Now let's give the definitions and features of these concepts.
The α-level set of $u \in \mathscr{L}(\mathbb{R})$ is given by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ cl\,\{x \in \mathbb{R} : u(x) > \alpha\}, & \text{if } \alpha = 0. \end{cases}$$

and is written as a non-empty interval $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ which is also bounded and closed for every $\alpha \in [0,1]$. Here, $[-\infty, \infty]$ is admissible. When $u(x) = 0$ for all $x < 0$, $u \in \mathscr{L}(\mathbb{R})$ is a non-negative fuzzy number. We will denote all non-negative fuzzy numbers by the set $\mathscr{L}^*(\mathbb{R})$. It is clearly understood that $\tilde{0} \in \mathscr{L}^*(\mathbb{R})$.
For $u, v \in \mathscr{L}(\mathbb{R})$ and all $\alpha \in [0,1]$, the partial ordering $\preceq$ in $\mathscr{L}(\mathbb{R})$ is given as following

$$u \preceq v \quad \text{iff} \quad u_\alpha^- \leq v_\alpha^- \quad \& \quad u_\alpha^+ \leq v_\alpha^+.$$

On $\mathscr{L}(\mathbb{R})$, arithmetic equations are defined as follows
(i) $(u \oplus v)(t) = \sup_{s \in \mathbb{R}} \{u(s) \wedge v(t-s)\}$,
(ii) $(u \odot v)(t) = \sup_{s \in \mathbb{R} s \neq 0} \{u(s) \wedge v(t/s)\}$,
(iii) $ru(t) = u(t/r)$ for $r \in \mathbb{R}^+$ and $0u(t) = \tilde{0}$,
for $u, v \in \mathscr{L}(\mathbb{R})$ and $t \in \mathbb{R}$.
Using α-level sets, arithmetic equations are given as follows
(i) $[u \oplus v]_\alpha = \left[u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+\right]$, for $u, v \in \mathscr{L}(\mathbb{R})$,
(ii) $[u \odot v]_\alpha = \left[u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+\right]$, for $u, v \in \mathscr{L}^*(\mathbb{R})$,
(iii) For $u \in \mathscr{L}(\mathbb{R})$,

$$[ru]_\alpha = r[u]_\alpha = \begin{cases} [ru_\alpha^-, ru_\alpha^+], & \text{if } r \geq 0, \\ [ru_\alpha^+, ru_\alpha^-], & \text{if } r < 0. \end{cases}$$

On $\mathscr{L}(\mathbb{R})$, it is described that a metric known as supremum metric;

$$\mathscr{D}(u,v) = \sup_{0 \leq \alpha \leq 1} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\},$$

for $u, v \in \mathscr{L}(\mathbb{R})$. Obviously,

$$\mathscr{D}\left(u, \tilde{0}\right) = \sup_{0 \leq \alpha \leq 1} \max\left\{\left|u_\alpha^-\right|, \left|u_\alpha^+\right|\right\} = \max\left\{\left|u_0^-\right|, \left|u_0^+\right|\right\}$$

and for $u \in \mathscr{L}^*(\mathbb{R})$, we obtain $\mathscr{D}\left(u, \tilde{0}\right) = u_0^+$.
In $\mathscr{L}(\mathbb{R})$, the sequence (u_n) is convergent to $u \in \mathscr{L}(\mathbb{R})$ if $\lim\limits_{n \to \infty} \mathscr{D}(u_n, u) = 0$. This convergence is denoted by $\mathscr{D} - \lim\limits_{n \to \infty} u_n = u$.
Now let's give the definition and features of fuzzy normed space.
For a vector space $\mathscr{X}$ over $\mathbb{R}$, consider $\|.\| : \mathscr{X} \to \mathscr{L}^*(\mathbb{R})$. For the symmetric and non-decreasing mappings $L, R : [0,1] \times [0,1] \to [0,1]$, let the conditions $L(0,0) = 0$ and $R(1,1) = 1$ be satisfied.
If the followings
(i) $\|x\| = \tilde{0}$ iff x is zero vector.
(ii) $\|rx\| = |r| \odot \|x\|$ for $r \in \mathbb{R}, x \in \mathscr{X}$.
(iii) For all $x, y \in \mathscr{X}$,
(a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, if $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ & $s+t \leq \|x+y\|_1^-$,
(b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, if $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ & $s+t \geq \|x+y\|_1^-$
hold, then $\|.\|$ is named fuzzy norm and the quadruple $(\mathscr{X}, \|.\|, L, R)$ is fuzzy normed space (FNS).
We substitute min and max for L and R in (iii), then we have

$$\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^- \quad \& \quad \|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+,$$

for $x, y \in \mathscr{X}$ and all $\alpha \in (0,1]$. Also, $\|.\|_\alpha^-$ and $\|.\|_\alpha^+$ satisfy the other norm conditions.
From now on, in our study, we take $(\mathscr{X}, \|.\|)$ as a FNS.
Let's take $\mathscr{X}$ as a topological structure. For any $\varepsilon > 0$ and all $\alpha \in [0,1]$, the $\varepsilon(\alpha)-$neighborhood of $x \in \mathscr{X}$ is the set

$$\mathscr{N}_{\varepsilon(\alpha)}^x = \{y \in \mathscr{X} : \|x-y\|_\alpha^+ < \varepsilon\}.$$

Recently, several convergence types have been studied on fuzzy normed spaces by a lot of authors [34–39].
If for each $\varepsilon > 0$, an $n_0 \in \mathbb{N}$ exists and satisfy

$$\mathscr{D}\left(\|x_n - x_0\|, \tilde{0}\right) = \sup_{\alpha \in [0,1]} \|x_n - x_0\|_\alpha^+ = \|x_n - x_0\|_0^+ < \varepsilon$$

for all $n > n_0$, then the sequence $(x_n) \subset \mathscr{X}$ is convergent to $x_0 \in \mathscr{X}$ and we write $x_n \xrightarrow{FN} x_0$. In other words, in terms of neighborhoods, it can be said this way: for all $\varepsilon > 0$, an $n_0 \in \mathbb{N}$ exists such that $x_n \in \mathscr{N}_{\varepsilon(0)}^{x_0}$, for $n > n_0$.
The lacunary sequence, which has been studied in many different spaces in the theory of summability in recent years, is well known in the literature, and its convergence types rather than its basic definition will be noted here.

Then after this, A lacunary sequence will be taken as $\theta=\{k_r\}$. For the sequence $(x_n)\subset\mathscr{X}$, if there is an $\ell\in\mathscr{X}$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\left(\sum_{k\in I_r}\mathscr{D}\left(\|x_k-\ell\|,\tilde{0}\right)\right)=0$$

holds, then it is lacunary summable to ℓ.

If for every $\varepsilon>0$, $\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\mathscr{D}\left(\|x_k-\ell\|,\tilde{0}\right)\geq\varepsilon\right\}\right|=0$ holds then the sequence $(x_n)\subset\mathscr{X}$ is lacunary st-convergent to ℓ, briefly $x_n\overset{FS_\theta}{\longrightarrow}\ell$.

The concepts of invariant mean and invariant convergence types are studied in many different spaces in summability theory and are well-known in the literature. Here, rather than its basic definition, some types of convergence, especially in fuzzy normed spaces, will be noted.

Now, let

$$t_{mn}=\frac{x_{\sigma(n)}+x_{\sigma^2(n)}+\cdots+x_{\sigma^m(n)}}{m}.$$

The bounded sequence $(x_n)\subset\mathscr{X}$ is invariant convergent to the ℓ iff $\lim_{m\to\infty}t_{mn}=\ell$ uniformly in n, namely $(\mathscr{D})-\lim_{m\to\infty}\|t_{mn}-\ell\|=\tilde{0}$, uniformly in n, that is, there exists an $m_0\in\mathbb{N}$ for every $\varepsilon>0$ such that

$$\mathscr{D}(\|t_{mn}-\ell\|,\tilde{0})=\sup_{\alpha\in[0,1]}\|t_{mn}-\ell\|_\alpha^+=\|t_{mn}-\ell\|_0^+<\varepsilon,$$

for all $m>m_0$ and every $n\in\mathbb{N}$, in other words, in terms of neighborhoods, it can be said this way: There exists an $m_0\in\mathbb{N}$ for every $\varepsilon>0$ such that $t_{mn}\in\mathscr{N}^{\ell}_{\varepsilon(0)}$ for all $m>m_0$ and every $n\in\mathbb{N}$. For this convergence, we write $x_n\overset{\sigma-FN}{\longrightarrow}\ell$.

If for every $\varepsilon>0$, $\lim_{m\to\infty}\frac{1}{m}\left|\left\{k\leq m:\|x_{\sigma^k(n)}-\ell\|_0^+\geq\varepsilon\right\}\right|=0$, uniformly in n, then the sequence $(x_n)\subset\mathscr{X}$ is invariant statistical convergent to ℓ and we write $x_n\overset{S_\sigma FN}{\longrightarrow}\ell$. The set of sequences that have this convergence is denoted by $S_\sigma FN$.

Any $(x_n)\subset X$ is lacunary invariant convergent to ℓ and denoted by $x_n\overset{\sigma-FN_\theta}{\longrightarrow}\ell$ iff

$$\lim_{r\to\infty}\mathscr{D}\left(\left\|\frac{1}{h_r}\sum_{k\in I_r}x_{\sigma^k(n)}-\ell\right\|,\tilde{0}\right)=\lim_{r\to\infty}\left\|\frac{1}{h_r}\sum_{k\in I_r}x_{\sigma^k(n)}-\ell\right\|_0^+=0,$$

uniformly in n. By $\sigma-FN_\theta$, we show the set of sequences have this convergence.

Any $(x_n)\subset\mathscr{X}$ is strongly lacunary invariant convergent to ℓ and denoted by $x_n\overset{[\sigma-FN]_\theta}{\longrightarrow}\ell$ iff

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\mathscr{D}\left(\left\|x_{\sigma^k(n)}-\ell\right\|,\tilde{0}\right)=\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+=0,$$

uniformly in n. By $[\sigma-FN]_\theta$, we show the set of sequences have this convergence.

2. Main Results

First, in the beginning of the original part of our study, we want to give $S_{\sigma\theta}FN$-convergence and $S_\sigma FN_\theta$-convergence, which have not been defined in the literature before.

Definition 2.1. *For a sequence $(x_n)\subset\mathscr{X}$, if for every $\varepsilon>0$ and uniformly in n,*

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\mathscr{D}\left(\left\|x_{\sigma^k(n)}-\ell\right\|,\tilde{0}\right)\geq\varepsilon\right\}\right|=\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geq\varepsilon\right\}\right|=0,$$

then the sequence $(x_n)\subset\mathscr{X}$ is lacunary invariant statistically convergent to ℓ and we write $x_n\overset{S_{\sigma\theta}FN}{\longrightarrow}\ell$. The set of sequences that have this convergence is denoted by $S_{\sigma\theta}FN$.

Definition 2.2. *Let*

$$t_{rn}=\frac{1}{h_r}\sum_{k\in I_r}x_{\sigma^k(n)}.$$

For a sequence $(x_n)\subset\mathscr{X}$, if for every $\varepsilon>0$ and uniformly in n,

$$\lim_{m\to\infty}\frac{1}{m}\left|\left\{r\leq m:\mathscr{D}\left(\|t_{rn}-\ell\|,\tilde{0}\right)\geq\varepsilon\right\}\right|=\lim_{m\to\infty}\frac{1}{m}\left|\left\{r\leq m:\|t_{rn}-\ell\|_0^+\geq\varepsilon\right\}\right|=0,$$

therefore, the sequence $(x_n)\subset\mathscr{X}$ is statistically lacunary invariant convergent to ℓ and we write $x_n\overset{S_\sigma FN_\theta}{\longrightarrow}\ell$. The set of sequences that have this convergence is denoted by $S_\sigma FN_\theta$.

Now we will give the theorem examining the relations between $[\sigma-FN]_\theta$ and $S_{\sigma\theta}FN$ with its proof.

Theorem 2.3. *For $0<q<\infty$ and a sequence $(x_n)\subset\mathscr{X}$, the followings hold:*

(i) *If $x_n\overset{[\sigma-FN]_\theta}{\longrightarrow}\ell$, then $x_n\overset{S_{\sigma\theta}FN}{\longrightarrow}\ell$.*

(ii) *If (x_n) is bounded sequence and $x_n\overset{S_{\sigma\theta}FN}{\longrightarrow}\ell$, then $x_n\overset{[\sigma-FN]_\theta}{\longrightarrow}\ell$.*

Proof. **(i)** According to our assumption, uniformly in n, we get

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+=0.$$

For every $\varepsilon>0$ and n, from the following inequality

$$\frac{1}{h_r}\sum_{k\in I_r}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geq\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_{\sigma^k(n)}-\ell\|_0^+\geq\varepsilon}}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+$$

$$\geq\frac{1}{h_r}\varepsilon\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geqslant\varepsilon\right\}\right|,$$

we obtain

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geqslant\varepsilon\right\}\right|=0,$$

uniformly in n. Thus, (x_n) is lacunary invariant statistically convergent to ℓ.
(ii) Let's presume that the bounded sequence $(x_n)\subset\mathscr{X}$ is lacunary invariant statistically convergent to ℓ. So, an $M>0$ exists such that

$$\|x_{\sigma^k(n)}-\ell\|_0^+<M$$

for every $k\in\mathbb{N}$ and all $n\in\mathbb{N}$. Also we have for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geqslant\varepsilon\right\}\right|=0,$$

uniformly in n. We know

$$\frac{1}{h_r}\sum_{k\in I_r}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_{\sigma^k(n)}-\ell\|_0^+\geq\varepsilon}}\left\|x_{\sigma^k(n)}-\ell\right\|_0^++\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \|x_{\sigma^k(n)}-\ell\|_0^+<\varepsilon}}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+$$

$$\leq\frac{1}{h_r}M\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell\right\|_0^+\geqslant\varepsilon\right\}\right|+\varepsilon,$$

for every n. Therefore, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left\|x_{\sigma^k(n)}-\ell\right\|_0^+=0,$$

uniformly in n. Hence, (x_n) is strongly lacunary invariant convergent to ℓ. □

Now, we will prove the theorem about the uniqueness of the limit.We will now prove the theorem about the uniqueness of the limit, which has an important place in summability theory.

Theorem 2.4. *Let $(x_n)\subset\mathscr{X}$ be a sequence. If $x_n\overset{S_{\sigma\theta}FN}{\longrightarrow}\ell$, in this case ℓ is unique.*

Proof. Let's presume that $x_n\overset{S_{\sigma\theta}FN}{\longrightarrow}\ell_1$, $x_n\overset{S_{\sigma\theta}FN}{\longrightarrow}\ell_2$ and $\ell_1\neq\ell_2$. Then for any given $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell_1\right\|_0^+\geqslant\frac{\varepsilon}{2}\right\}\right|=0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell_2\right\|_0^+\geqslant\frac{\varepsilon}{2}\right\}\right|=0,$$

uniformly in n. Put

$$N_1^r=\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell_1\right\|_0^+<\frac{\varepsilon}{2}\right\}\ \text{ and }\ N_2^r=\left\{k\in I_r:\left\|x_{\sigma^k(n)}-\ell_2\right\|_0^+<\frac{\varepsilon}{2}\right\}.$$

We know as $r\to\infty$,

$$\frac{\left|N_1^r\cap I_r\right|}{h_r}\to 1\text{ and }\frac{\left|N_2^r\cap I_r\right|}{h_r}\to 1. \tag{2.1}$$

Since $\ell_1\neq\ell_2$, $\|\ell_1-\ell_2\|_0^+\geq\varepsilon$ for some $\varepsilon>0$.
Obviously,

$$N_1^r\cap N_2^r=\emptyset\ \text{ and }\ N_1^r\cup N_2^r\subseteq I_r. \tag{2.2}$$

We can write

$$(N_1^r \cap I_r) \cup (N_2^r \cap I_r) = (N_1^r \cup N_2^r) \cap I_r \subseteq I_r$$

and

$$\frac{\left|N_1^r \cap I_r\right|}{h_r} + \frac{\left|N_2^r \cap I_r\right|}{h_r} \leq \frac{|I_r|}{h_r}, \text{ from (2.2)}.$$

Because of (2.1) we obtain

$$1+1 \leq 1 \text{ as } r \to \infty$$

which is the contradiction. Therefore, $\ell_1 = \ell_2$. □

Now we will give the theorems examining the linearity properties of lacunary invariant statistical convergence and their proofs. We will give these properties in two parts in the following theorem.

Theorem 2.5. *Let $x = (x_n)$, $y = (y_n)$ be sequences in $\mathscr{X}$ and assume that $x_n \overset{S_{\sigma\theta}FN}{\longrightarrow} \ell_1$ and $y_n \overset{S_{\sigma\theta}FN}{\longrightarrow} \ell_2$. In this case, we obtain the following hypotheses:*
(i) $x_n + y_n \overset{S_{\sigma\theta}FN}{\longrightarrow} \ell_1 + \ell_2$,
(ii) $(cx_n) \overset{S_{\sigma\theta}FN}{\longrightarrow} c\ell_1$ *where c is a scaler.*

Proof. **(i)** Let's presume that $x_n \overset{S_{\sigma\theta}FN}{\longrightarrow} \ell_1$ and $y_n \overset{S_{\sigma\theta}FN}{\longrightarrow} \ell_2$. Then, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| x_{\sigma^k(n)} - \ell_1 \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| y_{\sigma^k(n)} - \ell_2 \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| = 0,$$

uniformly in n. From the triangle inequality,

$$\left\| \left(x_{\sigma^k(n)} + y_{\sigma^k(n)} \right) - (\ell_1 + \ell_2) \right\|_0^+ \leq \left\| x_{\sigma^k(n)} - \ell_1 \right\|_0^+ + \left\| y_{\sigma^k(n)} - \ell_2 \right\|_0^+,$$

for any given $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \left(x_{\sigma^k(n)} + y_{\sigma^k(n)} \right) - \left(\ell_1 + \ell_2 \right) \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| \\
&\quad \leq \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| x_{\sigma^k(n)} - \ell_1 \right\|_0^+ + \left\| y_{\sigma^k(n)} - \ell_2 \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| \\
&\quad \leq \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| x_{\sigma^k(n)} - \ell_1 \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| + \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| y_{\sigma^k(n)} - \ell_2 \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right|.
\end{aligned}$$

So, we concluded that

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \left(x_{\sigma^k(n)} + y_{\sigma^k(n)} \right) - \left(\ell_1 + \ell_2 \right) \right\|_0^+ \geqslant \frac{\varepsilon}{2} \right\} \right| = 0,$$

that is,

$$x_n + y_n \overset{S_{\sigma\theta}FN}{\longrightarrow} \ell_1 + \ell_2.$$

(ii) Let c be a scaler. From the inequality

$$\frac{1}{h_r} \left| \left\{ k \in I_r : \left\| c x_{\sigma^k(n)} - c\ell_1 \right\|_0^+ \geqslant \varepsilon \right\} \right| \leq \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| x_{\sigma^k(n)} - \ell_1 \right\|_0^+ \geqslant \frac{\varepsilon}{|c|} \right\} \right|,$$

we obtain

$$(cx_n) \overset{S_{\sigma\theta}FN}{\longrightarrow} cL.$$

□

We give the following lemma without the proof. It can be proved like in [23].

Lemma 2.6. *Let $(x_n) \subset \mathscr{X}$ be a sequence. Presume for given $\varepsilon_1 > 0$ and for all $\varepsilon > 0$, n_0 and m_0 exist such that for all $n \geq n_0$ and $m \geq m_0$,*

$$\frac{1}{m} \left| \left\{ k \leq m : \| x_{\sigma^k(n)} - \ell \|_0^+ \geq \varepsilon \right\} \right| < \varepsilon_1,$$

then (x_n) is invariant statistical convergent to ℓ.

Finally, we will show the relation between invariant statistical convergence and lacunary invariant statistical convergence with the following theorem.

Theorem 2.7. *$S_{\sigma\theta}FN = S_{\sigma}FN$ for every lacunary sequence θ.*

Proof. Let the sequence $(x_n) \in S_{\sigma\theta}FN$. According to definition, for all $\varepsilon > 0$ and for any $\varepsilon_1 > 0$, r_0 and ℓ exist such that

$$\frac{1}{h_r}\left|\left\{0 < k \leq h_r : \left\|x_{\sigma^k(n)} - \ell\right\|_0^+ \geq \varepsilon\right\}\right| < \varepsilon_1,$$

for $r \geq r_0$ and $n = \sigma^{k_r - 1}(n^{'})$ and $n^{'} \geq 0$. Let $m \geq h_r$, write $m = th_r + s$ where $0 \leq s \leq h_r$ and t is a integer. Since $m \geq h_r$, $t \geq 1$. Now

$$\begin{aligned}
&\frac{1}{m}\left|\left\{0 < k \leq m : \|x_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{m}\left|\left\{0 < k \leq (t+1)h_r : \|x_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{m}\sum_{i=1}^{t}\left|\left\{ih_r < k \leq (i+1)h_r : \|x_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{m}(t+1)\,h_r\varepsilon_1 \\
&\leq \frac{2th_r\varepsilon_1}{m}, \quad (t \geq 1)
\end{aligned}$$

for $\frac{h_r}{m} \leq 1$ and since $\frac{th_r}{m} \leq 1$,

$$\frac{1}{m}\left|\left\{0 < k \leq m : \|x_{\sigma^k(n)} - \ell\|_0^+ \geq \varepsilon\right\}\right| \leq 2\varepsilon_1.$$

Then by Lemma, $S_{\sigma\theta}FN \subseteq S_{\sigma}FN$. Also, obviously $S_{\sigma}FN \subseteq S_{\sigma\theta}FN$. We concluded that $S_{\sigma\theta}FN = S_{\sigma}FN$. □

3. Conclusion

In the Fuzzy normed spaces, using the lacunary sequence, we introduce some new concepts in summability. In this sense, firstly, we define the lacunary invariant statistical convergence. Then, we examine some characteristic features like uniqueness, linearity of this new notion and give its important relation with pre-given concepts. In the future, these studies are also debatable in terms of regularly convergence for double sequences.

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$\mathscr{Z}^*$-Tensor on $N(k)$-Contact Metric Manifolds Admitting Ricci Soliton Type Structure

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Abstract

The main goal of this manuscript is to investigate the properties of $N(k)$-contact metric manifolds admitting a $\mathscr{Z}^*$-tensor. We prove the necessary conditions for which $N(k)$-contact metric manifolds endowed with a $\mathscr{Z}^*$-tensor are Einstein manifolds. In this sequel, we accomplish that an $N(k)$-contact metric manifold endowed with a $\mathscr{Z}^*$-tensor satisfying $\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta})\cdot\overset{\star}{\mathscr{R}}=0$ is either locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or an Einstein manifold. We also prove the condition for which an $N(k)$-contact metric manifold endowed with a $\mathscr{Z}^*$-tensor is a Sasakian manifold. To validate some of our results, we construct a non-trivial example of an $N(k)$-contact metric manifold.

1. Introduction

In 1988, Tanno [1] has initiated the concept of k-nullity distribution of a contact metric manifold. A contact metric manifold with ξ belonging to the k-nullity distribution is said to be $N(k)$-contact metric manifold (briefly, $N(k)$-(CMM)$_{2n+1}$). Blair et al. [2] generalized this idea on a contact manifold with ξ belongs to a (k,μ)-nullity distribution, where k and μ are real constants. In particular, if μ=0, then the (k,μ)-nullity distribution reduces to a k-nullity distribution. For more details see, ([3]- [11]).

The notion of Ricci soliton (RS) on Riemannian manifold $(\Theta,\hat{\mathfrak{g}})$ of dimension m is defined by [12, 13]:

$$\frac{1}{2}\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}}+\overset{\star}{\mathscr{S}}+\lambda\hat{\mathfrak{g}}=0, \tag{1.1}$$

where $\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}}$ is the Lie derivative of the Riemannian metric $\hat{\mathfrak{g}}$ along the vector field $\overset{\star}{V}$, $\overset{\star}{\mathscr{S}}$ is the Ricci tensor and λ is a real constant. In whole manuscript, an RS is denoted as $(\Theta,\hat{\mathfrak{g}},\overset{\star}{V},\lambda)$. Metrics satisfying (1.1) are interesting and useful in physics and are often referred to as quasi-Einstein metrics [14, 15]. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial\hat{\mathfrak{g}}}{\partial t}=-2\overset{\star}{\mathscr{S}}$, projected from the space of metrics onto its quotient modulo diffeomorphisms and scaling. An RS will be expanding, steady, or shrinking depending on $\lambda>0$, $\lambda=0$ or $\lambda<0$. Ricci solitons have been studied by several authors such as ([16]- [28]).

According to Mantica and Molinari [29], a generalized symmetric $\mathscr{Z}^*$-tensor of type (0, 2) is given by

$$\mathscr{Z}^*=\overset{\star}{\mathscr{S}}+\dot{\varphi}\hat{\mathfrak{g}}, \tag{1.2}$$

where $\dot{\varphi}$ is an arbitrary function. In References ([30]- [36]) various properties of the $\mathscr{Z}^*$-tensor were pointed out. In particular cases, the $\mathscr{Z}^*$-tensor have the several importance on $(\Theta,\hat{\mathfrak{g}})$. For example,

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1. if $\mathscr{Z}^*{}_{ij}$=0 (i.e, $\mathscr{Z}^*$-flat), then $(\Theta, \hat{\mathfrak{g}})$ reduces to an Einstein manifold [37],
2. if $\nabla_k\mathscr{Z}^*{}_{ij}$=$\lambda_k\mathscr{Z}^*{}_{ij}$(i.e., $\mathscr{Z}^*$-recurrent), then $(\Theta, \hat{\mathfrak{g}})$ reduces to a GRR manifold,
3. if $\nabla_k\mathscr{Z}^*{}_{ij}$=$\nabla_i\mathscr{Z}^*{}_{kl}$ (i.e., Codazzi tensor), then we find $\nabla_k\widetilde{\mathscr{R}}_{ij}$-$\nabla_i\widetilde{\mathscr{R}}_{kj}$=$\frac{1}{2(\widetilde{n}-1)}(g_{ij}\nabla_k - g_{kj}\nabla_i)\tau$ [38],
4. the relation between the $\mathscr{Z}^*$-tensor and the energy-stress tensor of Einstein's equations with cosmological constant Γ is $\mathscr{Z}^*{}_{kj}$=$\widetilde{\kappa}\mathscr{T}^*{}_{kj}$ [39], where ϕ=-$\frac{\tau}{2}+\Gamma$ and $\widetilde{\kappa}$ is the gravitational constant. In this case, the $\mathscr{Z}^*$-tensor may be considered as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ. The vacuum solution ($\mathscr{Z}^*$=0) determines an Einstein space Γ=$(\frac{(\widetilde{n}-2)}{2\widetilde{n}})\tau$; the conservation of TEM ($\nabla^l\mathscr{T}^*{}_{kl}$=0) gives ($\nabla_j\mathscr{T}^*{}_{kl}$=0) then this space-time gives the conserved energy-momentum density.

A new curvature tensor $\overset{\star}{\mathscr{Q}}$ of type $(1,3)$ on $(\Theta, \hat{\mathfrak{g}})$, $n>2$ is defined as

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3 = \overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3 - \frac{\breve{\Psi}}{n-1}[\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1 - \hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_2], \tag{1.3}$$

is known as $\mathscr{Q}$-curvature tensor [40], where $\breve{\Psi}$ is an arbitrary scalar function. If $\breve{\Psi}$=$\frac{\breve{\overset{\star}{\kappa}}}{n}$, where $\breve{\overset{\star}{\kappa}}$ is the scalar curvature, then $\mathscr{Q}$-curvature tensor reduces to concircular curvature tensor $\overset{\star}{\mathscr{C}}$ [41]. For more details about $\mathscr{Q}$-curvature tensor, see [42,43]).

With the help of (1.1) and (1.2), we define:

Definition 1.1. *A Riemannian metric $\hat{\mathfrak{g}}$ is called a $\mathscr{Z}^*$-soliton if*

$$\frac{1}{2}\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}} + \mathscr{Z}^* + \lambda\hat{\mathfrak{g}} = 0, \tag{1.4}$$

where $\mathfrak{L}$ is the Lie derivative and λ a real scalar. If $\overset{\star}{V}$ is the gradient of f, $\mathscr{Z}^*$-soliton is referred to as a gradient $\mathscr{Z}^*$-soliton and then equation (1.4) simplifies to

$$\nabla^2 f + \mathscr{Z}^* + \lambda\hat{\mathfrak{g}} = 0,$$

where the Hessian of the function f is $\nabla^2 f$.
As per above sequel, we obtain some results by using the $\mathscr{Z}^*$-tensor on $N(k)$-(CMM)$_{2n+1}$ with (RS)$_{2n+1}$. After the introduction, Section 2, deals with some basic concept of $N(k)$-(CMM)$_{2n+1}$. We also examine $N(k)$-(CMM)$_{2n+1}$ with conditions $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{Z}^*$=0, $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{Q}}$=0, $((\hat{\zeta}\wedge_{\mathscr{Z}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0 and $\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}}$=0 in the Sections 3, 4, 5 and 6, respectively. In Section 7, we categorized $N(k)$-(CMM)$_{2n+1}$ which satisfy the conditions $\overset{\star}{\mathscr{Q}}.h$=0, $h.\overset{\star}{\mathscr{Q}}$=0. In the Section 8, we deal with $\mathscr{Z}^\star$-recurrent on $N(k)$-(CMM)$_{2n+1}$. Finally, an appropriate example establishes the existence of a $\mathscr{Z}^*$-soliton on a $N(k)$-(CMM)$_3$ which validates some of our results.

2. Preliminaries

A contact metric manifold $(\Theta, \hat{\mathfrak{g}})$ of dimension $m(=2n+1), (n>1)$ is a quadruple $(\hat{\varphi}, \hat{\zeta}, \hat{\eta}, \hat{\mathfrak{g}})$, where $\hat{\varphi}$ is a (1, 1)-tensor field, $\hat{\zeta}$ is a vector field, $\hat{\eta}$ is a 1-form on $(\Theta, \hat{\mathfrak{g}})$ and $\hat{\mathfrak{g}}$ is a Riemannian metric, such that

$$\hat{\varphi}^2\mathscr{G}_1 = -\mathscr{G}_1 + \hat{\eta}(\mathscr{G}_1)\hat{\zeta}, \quad \hat{\eta}(\hat{\zeta}) = 1, \quad \hat{\varphi}\hat{\zeta} = 0, \quad \hat{\eta}\circ\hat{\varphi} = 0, \tag{2.1}$$

$$\hat{\mathfrak{g}}(\hat{\varphi}\mathscr{G}_1,\hat{\varphi}\mathscr{G}_2) = \hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2) - \hat{\eta}(\mathscr{G}_1)\hat{\eta}(\mathscr{G}_2), \tag{2.2}$$

$$\hat{\mathfrak{g}}(\mathscr{G}_1,\hat{\varphi}\mathscr{G}_2) = -\hat{\mathfrak{g}}(\hat{\varphi}\mathscr{G}_1,\mathscr{G}_2), \quad \hat{\mathfrak{g}}(\mathscr{G}_1,\hat{\zeta}) = \hat{\eta}(\mathscr{G}_1) \tag{2.3}$$

for all vector field $\mathscr{G}_1, \mathscr{G}_2 \in \Gamma(\Theta)$. On $(\Theta, \hat{\mathfrak{g}})$, a $(1,1)$-tensor field h is defined by h=$\frac{1}{2}\mathfrak{L}_{\hat{\zeta}}\hat{\varphi}$, which is symmetric and satisfies (see [44,45])

$$h\hat{\varphi} = -\hat{\varphi}h,\ Tr.h = Tr.,\ \hat{\varphi}h = 0,\ h\hat{\zeta} = 0,$$

$$\nabla_{\mathscr{G}_1}\hat{\zeta} = -\hat{\varphi}\mathscr{G}_1 - \hat{\varphi}h\mathscr{G}_1, \tag{2.4}$$

$$\hat{\mathfrak{g}}(h\mathscr{G}_1,\mathscr{G}_2) = \hat{\mathfrak{g}}(\mathscr{G}_1,h\mathscr{G}_2), \tag{2.5}$$

$$\hat{\eta}(h\mathscr{G}_1) = 0. \tag{2.6}$$

In 1995, Blair et al. introduced the notion of $N(k,\mu)$-(CMM), for real numbers k and μ as a distribution [2,46]

$$\begin{aligned} N(k,\mu): p \to N_p(k,\mu) = [\mathscr{G}_3 \in T_p\Theta : &\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3 \\ &= (kI+\mu h)(\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1 - \hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_2)]. \end{aligned}$$

If μ=0, the (k,μ)-nullity distribution reduces to k-nullity distributions and defined as [1,47]

$$N(k): p \rightarrow N_p(k) = [\mathscr{G}_3 \in T_p\Theta : \overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3 \\ = k\{\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1 - \hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_2\}], \tag{2.7}$$

where k is constant. In particular, if k=1, then $(\Theta,\hat{\mathfrak{g}})$ is Sasakian and if k=0, then $(\Theta,\hat{\mathfrak{g}})$ is locally isometric to the product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for n=1 [2,47]. In $N(k)$-(CMM)$_{2n+1}$, we have

$$h^2 = (k-1)\hat{\varphi}^2, \quad k \leq 1, \tag{2.8}$$

$$(\nabla_{\mathscr{G}_1}\hat{\varphi})\mathscr{G}_2 = \hat{\mathfrak{g}}(\mathscr{G}_1+h\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta} - \hat{\eta}(\mathscr{G}_2)(\mathscr{G}_1+h\mathscr{G}_1),$$

$$\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta} = k[\hat{\eta}(\mathscr{G}_2)\mathscr{G}_1 - \hat{\eta}(\mathscr{G}_1)\mathscr{G}_2], \tag{2.9}$$

$$\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\hat{\zeta})\mathscr{G}_2 = k[\hat{\eta}(\mathscr{G}_2)\mathscr{G}_1 - \hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta}], \tag{2.10}$$

$$\overset{\star}{\mathscr{R}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_2 = k[\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta} - \hat{\eta}(\mathscr{G}_2)\mathscr{G}_1], \tag{2.11}$$

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2) = 2(n-1)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2) + 2(n-1)\hat{\mathfrak{g}}(h\mathscr{G}_1,\mathscr{G}_2) + 2(nk-(n-1))\hat{\eta}(\mathscr{G}_1)\hat{\eta}(\mathscr{G}_2), \tag{2.12}$$

$$\overset{\star}{\mathscr{S}}(\hat{\varphi}\mathscr{G}_1,\hat{\varphi}\mathscr{G}_2) = \overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2) - 2nk\hat{\eta}(\mathscr{G}_1)\hat{\eta}(\mathscr{G}_2) - 4(n-1)\hat{\mathfrak{g}}(h\mathscr{G}_1,\mathscr{G}_2),$$

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\hat{\zeta}) = 2nk\hat{\eta}(\mathscr{G}_1), \tag{2.13}$$

$$\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_2) = (2nk+\varphi)\hat{\eta}(\mathscr{G}_2), \tag{2.14}$$

$$\mathscr{Z}^*(\hat{\zeta},\hat{\zeta}) = (2nk+\varphi) \tag{2.15}$$

for any vector field $\mathscr{G}_1,\mathscr{G}_2 \in \Gamma(\Theta)$.
Now, we recall some propositions, which will be used later on as follows:

Lemma 2.1 ([48]). *A contact metric manifold $\Theta(\hat{\varphi},\hat{\zeta},\hat{\eta},\hat{\mathfrak{g}})$ fulfills the criteria $\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\zeta}$=0 for all $\mathscr{G}_1,\mathscr{G}_2$ is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$.*

3. $N(k)$-(CMM)$_{2n+1}$ Admitting $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{Z}^*$=0

The condition $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{Z}^*$=0 on $(\Theta,\hat{\mathfrak{g}})$ implies that

$$\mathscr{Z}^*(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_2,\mathscr{G}_3) + \mathscr{Z}^*(\mathscr{G}_2,\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_3) = 0. \tag{3.1}$$

Using (1.3), (2.11), (2.14), and (2.15) in (3.1), we obtain

$$\left(k-\frac{\breve{\Psi}}{2n}\right)[(2nk+\varphi)\hat{\eta}(\mathscr{G}_3)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2) + (2nk+\varphi)\hat{\eta}(\mathscr{G}_2)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3) - \hat{\eta}(\mathscr{G}_2)\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_3) - \hat{\eta}(\mathscr{G}_3)\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_2)] = 0. \tag{3.2}$$

Putting $\mathscr{G}_3 = \hat{\zeta}$ in (3.2) and using (1.2), (2.3) and (2.14), we find

$$\left(k-\frac{\breve{\Psi}}{2n}\right)[2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2) - \overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)] = 0,$$

which implies that either k=$\frac{\breve{\Psi}}{2n}$, or $\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)$=$2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)$. If $k \neq \frac{\breve{\Psi}}{2n}$, then one can get

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2) = 2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2). \tag{3.3}$$

So, we have:

Theorem 3.1. *An $N(k)$-(CMM)$_{2n+1}$ admitting $\mathscr{L}^*$-tensor fulfills the criteria $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{L}^*$=0 is an Einstein manifold provided $k \neq \frac{\breve{\Psi}}{2n}$.*

Corollary 3.2. *An $N(k)$-(CMM)$_{2n+1}$ admitting $\mathscr{L}^*$-tensor satisfying the condition $\overset{\star}{\mathscr{C}}(\hat{\zeta},\mathscr{G}_1).\mathscr{L}^*$=0 is an Einstein manifold provided $k \neq \frac{\overset{\star}{\kappa}}{2n(2n+1)}$.*

Again from (1.2), (1.4) and (3.3), we have

$$\frac{1}{2}\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)+[2nk+\dot{\varphi}+\lambda]\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)=0. \tag{3.4}$$

Taking $\mathscr{G}_1$=$\mathscr{G}_2$=e_i in (3.4) and summing over i, $(1\leq i\leq 2n+1)$, we get

$$\frac{1}{2}\mathfrak{L}_{\overset{\star}{V}}\hat{\mathfrak{g}}(e_i,e_i)+[2nk+\dot{\varphi}+\lambda]\hat{\mathfrak{g}}(e_i,e_i)=0$$

which is equivalent to

$$div(\overset{\star}{V})+[2nk+\dot{\varphi}+\lambda](2n+1)=0. \tag{3.5}$$

If $\overset{\star}{V}$ is solenoidal that is, $div(\overset{\star}{V})$=0, then (3.5) reduces to

$$\lambda=-(2nk+\dot{\varphi}).$$

Also if $\overset{\star}{V}$=$grad(f)$. So from (3.5), we yield

$$\nabla(f)=-[2nk+\dot{\varphi}+\lambda](2n+1),$$

where $\nabla(f)$ is the Laplacian of smooth function f. Thus we conclude:

Corollary 3.3. *An $N(k)$-(CMM)$_{2n+1}$ admitting gradient $\mathscr{L}^*$-soliton fulfills the criteria $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{L}^*$=0, then*

$$\nabla(f)=-[2nk+\dot{\varphi}+\lambda](2n+1)$$

provided $k \neq \frac{\breve{\Psi}}{2n}$.

Corollary 3.4. *An $N(k)$-(CMM)$_{2n+1}$ with $\mathscr{L}^*$-soliton satisfies the condition $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\mathscr{L}^*$=0, where $\overset{\star}{V}$ is solenoidal, then the soliton is increasing, stable, or reducing depending on $\dot{\varphi}<-2nk$, $\dot{\varphi}$=$2nk$, or $\dot{\varphi}>2nk$.*

4. $N(k)$-(CMM)$_{2n+1}$ With $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{Q}}$=0

The condition $(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{Q}})(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4$=0 on (Θ,g) implies that

$$\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4-\overset{\star}{\mathscr{Q}}(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4-\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_3)\mathscr{G}_4-\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_4=0. \tag{4.1}$$

Also from (2.7) and (1.3) we have

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_2], \tag{4.2}$$

$$\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_2)\mathscr{G}_3=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}-\hat{\eta}(\mathscr{G}_3)\mathscr{G}_2], \tag{4.3}$$

$$\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}-\hat{\eta}(\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\mathscr{G}_1], \tag{4.4}$$

$$\overset{\star}{\mathscr{Q}}(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4-\hat{\eta}(\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_4], \tag{4.5}$$

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_3)\mathscr{G}_4=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4-\hat{\eta}(\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_1)\mathscr{G}_4], \tag{4.6}$$

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1)\mathscr{G}_4=\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}-\hat{\eta}(\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1]. \tag{4.7}$$

Using (4.3), (4.4), (4.5), (4.6) and (4.7) in (4.1), we get

$$\begin{aligned}\left(k-\frac{\breve{\Psi}}{2n}\right)&[\hat{\mathfrak{g}}(\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}-\hat{\eta}(\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\mathscr{G}_1-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4\\&+\hat{\eta}(\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_4-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4+\hat{\eta}(\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_1)\mathscr{G}_4\\&-\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}+\hat{\eta}(\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1]=0.\end{aligned} \tag{4.8}$$

Taking the inner product of (4.8) with $\hat{\zeta}$ and using (4.1), (4.3), we find

Theorem 4.1. *An $N(k)$-(CMM)$_{2n+1}$ always fulfills the condition $\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{Q}}=0$, provided $k\neq\frac{\breve{\Psi}}{2n}$.*

Corollary 4.2. *An $N(k)$-(CMM)$_{2n+1}$ always satisfy the condition $\overset{\star}{\mathscr{C}}(\hat{\zeta},\mathscr{G}_1).\overset{\star}{\mathscr{C}}=0$, provided $k\neq\frac{\overset{\star}{\breve{\kappa}}}{2n(2n+1)}$.*

5. $N(k)$-(CMM)$_{2n+1}$ Satisfying $((\hat{\zeta}\wedge_{\mathscr{L}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0

Let the condition $((\hat{\zeta}\wedge_{\mathscr{L}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4$=0 holds on $(\Theta,\hat{\mathfrak{g}})$. Then we have

$$\begin{aligned}&\mathscr{L}^*(\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}-\mathscr{L}^*(\hat{\zeta},\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\mathscr{G}_1-\mathscr{L}^*(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4+\mathscr{L}^*(\hat{\zeta},\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_4\\&-\mathscr{L}^*(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4+\mathscr{L}^*(\hat{\zeta},\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_1)\mathscr{G}_4-\mathscr{L}^*(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}+\mathscr{L}^*(\hat{\zeta},\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1=0.\end{aligned}\tag{5.1}$$

Using (1.2) and (2.14) in (5.1), we get

$$\begin{aligned}&\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}+\dot{\varphi}\hat{\mathfrak{g}}((\mathscr{G}_1,\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\hat{\zeta}-(2nk+\dot{\varphi})\hat{\eta}(\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_4)\mathscr{G}_1-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4\\&-\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_3)\mathscr{G}_4+(2nk+\dot{\varphi})\hat{\eta}(\mathscr{G}_2)\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_3)\mathscr{G}_4-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4-\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\hat{\zeta})\mathscr{G}_4\\&+(2nk+\dot{\varphi})\hat{\eta}(\mathscr{G}_3)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_1)\mathscr{G}_4-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}-\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\hat{\zeta}+(2nk+\dot{\varphi})\hat{\eta}(\mathscr{G}_4)\overset{\star}{\mathscr{Q}}(\mathscr{G}_2,\mathscr{G}_3)\mathscr{G}_1=0.\end{aligned}\tag{5.2}$$

Taking inner product of (5.2) with $\hat{\zeta}$ and using (4.2) and (4.3), we obtain

$$\begin{aligned}\left(k-\frac{\breve{\Psi}}{2n}\right)&[\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)+\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_3)\hat{\eta}(\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)-\dot{\varphi}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\hat{\eta}(\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)\\&+(2nk+\dot{\varphi})\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_3)\hat{\eta}(\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)-(2nk+\dot{\varphi})\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)]=0.\end{aligned}\tag{5.3}$$

For, fix $\mathscr{G}_3=\hat{\zeta}$ in (5.3) and using (2.3), we get

$$\left(k-\frac{\breve{\Psi}}{2n}\right)[\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)-2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)\hat{\eta}(\mathscr{G}_4)]=0.$$

So, we mention the result:

Theorem 5.1. *An $N(k)$-(CMM)$_{2n+1}$ admitting $\mathscr{L}^*$-tensor satisfying the criteria $((\hat{\zeta}\wedge_{\mathscr{L}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0, is an Einstein manifold provided $k\neq\frac{\breve{\Psi}}{2n}$.*

Corollary 5.2. *An $N(k)$-(CMM)$_{2n+1}$ admitting $\mathscr{L}^*$-tensor satisfying the condition $((\hat{\zeta}\wedge_{\mathscr{L}^*}\mathscr{G}_1).\overset{\star}{\mathscr{C}})$=0, is an Einstein manifold provided $k\neq\frac{\overset{\star}{\breve{\kappa}}}{2n(2n+1)}$.*

Likewise Section 3, we state the followings:

Corollary 5.3. *If a gradient $\mathscr{L}^*$-soliton $(g,\overset{\star}{V},\lambda)$ on $N(k)$-(CMM)$_{2n+1}$ satisfies the criteria $((\hat{\zeta}\wedge_{\mathscr{L}^*}\mathscr{G}_1).\overset{\star}{\mathscr{C}})$=0, then*

$$\nabla(f)=-[2nk+\dot{\varphi}+\lambda](2n+1)$$

provided $k\neq\frac{\overset{\star}{\breve{\kappa}}}{2n(2n+1)}$.

Corollary 5.4. *If a gradient $\mathscr{L}^*$-soliton $(g,\overset{\star}{V},\lambda)$ on $N(k)$-(CMM)$_{2n+1}$ satisfies the condition $((\hat{\zeta}\wedge_{\mathscr{L}^*}\mathscr{G}_1).\overset{\star}{\mathscr{C}})$=0, where $\overset{\star}{V}$ is solenoidal, then the soliton is increasing, stable, or reducing depending on $\dot{\varphi}<-2nk$, $\dot{\varphi}$=2nk, or $\dot{\varphi}>2nk$.*

Corollary 5.5. *An $N(k)$-(CMM)$_{2n+1}$ admits gradient $\mathscr{L}^*$-soliton $(g,\overset{\star}{V},\lambda)$ fulfills the criteria $((\hat{\zeta}\wedge_{\mathscr{L}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0, where $\overset{\star}{V}$ is the gradient of a smooth function f, then we have*

$$\nabla(f)=-[2nk+\dot{\varphi}+\lambda](2n+1)$$

provided $k\neq\frac{\breve{\Psi}}{2n}$.

Corollary 5.6. *An $N(k)$-(CMM)$_{2n+1}$ with gradient $\mathscr{L}^*$-soliton $(g,\overset{\star}{V},\lambda)$ satisfying the condition $((\hat{\zeta}\wedge_{\mathscr{L}^*}\mathscr{G}_1).\overset{\star}{\mathscr{Q}})$=0, where $\overset{\star}{V}$ is solenoidal, then the soliton is expanding, steady or shrinking according as $\dot{\varphi}<-2nk$, $\dot{\varphi}$=2nk, or $\dot{\varphi}>2nk$.*

6. $N(k)$-(CMM)$_{2n+1}$ Satisfying $\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}}$=0

We suppose that $(\Theta,\hat{\mathfrak{g}})$ satisfies the below the condition

$$(\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3=0, \tag{6.1}$$

which implies that

$$(\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3=((\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta}).\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3, \tag{6.2}$$

where the endomorphism $(\mathscr{G}_1\wedge_{\mathscr{Z}^*}\mathscr{G}_4)\mathscr{G}_5$ is defined as

$$(\mathscr{G}_1\wedge_{\mathscr{Z}^*}\mathscr{G}_4)\mathscr{G}_5=\mathscr{Z}^*(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_1-\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_5)\mathscr{G}_4. \tag{6.3}$$

Now, from (6.2) we have

$$\begin{aligned}(\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3=&((\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta})\overset{\star}{\mathscr{R}})(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3-\overset{\star}{\mathscr{R}}((\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta})\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3\\&-\overset{\star}{\mathscr{R}}(\mathscr{G}_4,(\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta})\mathscr{G}_5)\mathscr{G}_3-\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)(\mathscr{G}_1\wedge_{\mathscr{Z}^*}\hat{\zeta})\mathscr{G}_3.\end{aligned} \tag{6.4}$$

Also, in view of (6.1), (6.3) and (6.4) we get

$$\begin{aligned}&\mathscr{Z}^*(\hat{\zeta},\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3)\mathscr{G}_1-\mathscr{Z}^*(\mathscr{G}_1,\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_3)\hat{\zeta}-\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_4)\overset{\star}{\mathscr{R}}(\mathscr{G}_1,\mathscr{G}_5)\mathscr{G}_3+\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_4)\overset{\star}{\mathscr{R}}(\hat{\zeta},\mathscr{G}_5)\mathscr{G}_3\\&-\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_5)\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_1)\mathscr{G}_3+\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_5)\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\hat{\zeta})\mathscr{G}_3-\mathscr{Z}^*(\hat{\zeta},\mathscr{G}_3)\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)\mathscr{G}_1+\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_3)\overset{\star}{\mathscr{R}}(\mathscr{G}_4,\mathscr{G}_5)\hat{\zeta}=0.\end{aligned} \tag{6.5}$$

Using (1.2), (2.9), (2.10), (2.12) and (2.14) in (6.5) and then taking the inner product with $\hat{\zeta}$, we obtain

$$\begin{aligned}&k[-\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_5)\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)-\dot{\varphi}\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_5)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)+\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_5)\\&+\dot{\varphi}\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_5)-(2nk+\dot{\varphi})\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_5)+(2nk+\dot{\varphi})\hat{\eta}(\mathscr{G}_3)\hat{\eta}(\mathscr{G}_4)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_5)]=0.\end{aligned} \tag{6.6}$$

Putting $\mathscr{G}_5=\hat{\zeta}$ in (6.6), we get

$$k[-\hat{\eta}(\mathscr{G}_3)\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)+2nk\hat{\eta}(\mathscr{G}_3)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)]=0. \tag{6.7}$$

Again putting $\mathscr{G}_3=\hat{\zeta}$ in (6.7), we find

$$k[-\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)+2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4)]=0,$$

which implies that either k=0 or,

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_4)=2nk\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_4).$$

Now, if k=0, then in view of (2.9) and Proposition 2.1, we state the following results:

Theorem 6.1. *If an $N(k)$-(CMM)$_{2n+1}$ admitting $\mathscr{Z}^*$-tensor fulfills the criteria $\mathscr{Z}^*(\mathscr{G}_1,\hat{\zeta}).\overset{\star}{\mathscr{R}}$=0, then $(\Theta,\hat{\mathfrak{g}})$ is either locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or the manifold is an Einstein.*

Corollary 6.2. *A $\mathscr{Z}^*$-soliton $(g,\overset{\star}{V},\lambda)$ on locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$, is reducing, stable or increasing depending upon the sign of scalar curvature.*

7. $N(k)$-(CMM)$_{2n+1}$ Equipped With $\overset{\star}{\mathscr{Q}}.h=0, h.\overset{\star}{\mathscr{Q}}$=0

The condition $(\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2).h)\mathscr{G}_3$=0 on $(\Theta,\hat{\mathfrak{g}})$ implies that

$$\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)h\mathscr{G}_3-h(\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3)=0 \tag{7.1}$$

for any $\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3\in\Gamma(\Theta)$. Putting $\mathscr{G}_1=\hat{\zeta}$ in (7.1), we have

$$\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_2)h\mathscr{G}_3-h(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_2)\mathscr{G}_3)=0. \tag{7.2}$$

Using (2.6), (4.3) in (7.2), we obtain

$$\left(k-\frac{\breve{\Psi}}{2n}\right)[\hat{\mathfrak{g}}(\mathscr{G}_2,h\mathscr{G}_3)\hat{\zeta}+\hat{\eta}(\mathscr{G}_3)h\mathscr{G}_2]=0. \tag{7.3}$$

Replacing $\mathscr{G}_3$ by $h\mathscr{G}_3$ in (7.3) and using (2.1), (2.2), (2.6), (2.8), we obtain

$$-(k-\frac{\breve{\Psi}}{2n})(k-1)\hat{\mathfrak{g}}(\hat{\varphi}\mathscr{G}_2,\hat{\varphi}\mathscr{G}_3)=0$$

and hence

$$\left(k-\frac{\breve{\Psi}}{2n}\right)(k-1)d\hat{\eta}(\hat{\varphi}\mathscr{G}_2,\mathscr{G}_3)=0,$$

which implies that either k=1, or $(k-\frac{\breve{\Psi}}{2n})d\hat{\eta}(\hat{\varphi}\mathscr{G}_2,\mathscr{G}_3)$=0. Thus we state:

Theorem 7.1. *If an $N(k)$-(CMM)$_{2n+1}$ satisfies the criteria $\overset{\star}{\mathscr{Q}}.h$=0, then $(\Theta,\hat{\mathfrak{g}})$ is Sasakian manifold provided $k\neq\frac{\breve{\Psi}}{2n}$.*

Next, we assume that $N(k)$-(CMM)$_{2n+1}$ fits the criteria $(h.\overset{\star}{\mathscr{Q}})(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3$=0, that is

$$h(\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3)-\overset{\star}{\mathscr{Q}}(h\mathscr{G}_1,\mathscr{G}_2)\mathscr{G}_3-\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,h\mathscr{G}_2)\mathscr{G}_3-\overset{\star}{\mathscr{Q}}(\mathscr{G}_1,\mathscr{G}_2)h\mathscr{G}_3=0 \tag{7.4}$$

for any $\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3\in\Gamma(\Theta)$. Putting $\mathscr{G}_1=\hat{\zeta}$ in (7.4) and using $h\hat{\zeta}=0$, we are leads to

$$h(\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_2)\mathscr{G}_3)-\overset{\star}{\mathscr{Q}}(\hat{\zeta},h\mathscr{G}_2)\mathscr{G}_3-\overset{\star}{\mathscr{Q}}(\hat{\zeta},\mathscr{G}_2)h\mathscr{G}_3=0. \tag{7.5}$$

Using (2.5), (2.6), (4.3) in (7.5), we find

$$-2(k-\frac{\breve{\Psi}}{2n})\hat{\mathfrak{g}}(h\mathscr{G}_2,\mathscr{G}_3)=0. \tag{7.6}$$

Replacing $\mathscr{G}_2$ by $h\mathscr{G}_2$ in (7.6) and by making use of (2.1), (2.2), (2.6), (2.8), the equation (7.6) reduces to

$$2(k-1)(k-\frac{\breve{\Psi}}{2n})\hat{\mathfrak{g}}(\hat{\phi}\mathscr{G}_2,\hat{\phi}\mathscr{G}_3)=0.$$

So, we conclude the results as:

Theorem 7.2. *If an $N(k)$-(CMM)$_{2n+1}$ satisfying the condition $h.\overset{\star}{\mathscr{Q}}$=0, then the $(\Theta,\hat{g})$ is Sasakian manifold, provided $k\neq\frac{\breve{\Psi}}{2n}$.*

In view of Theorem 7.1 and Theorem 7.2, we turn up the below outcome:

Corollary 7.3. *In an $N(k)$-(CMM)$_{2n+1}$ with $k\neq\frac{\breve{\Psi}}{2n}$, we have $\overset{\star}{\mathscr{Q}}.h=h.\overset{\star}{\mathscr{Q}}$.*

8. $\mathscr{Z}^*$-Recurrent on $N(k)$-(CMM)$_{2n+1}$

For $\mathscr{Z}^*$-recurrent on $(\Theta,\hat{\mathfrak{g}})$, we get

$$(\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{Z}^*)(\mathscr{G}_4,\mathscr{G}_5)=\hat{\eta}(\mathscr{G}_1)\mathscr{Z}^*(\mathscr{G}_4,\mathscr{G}_5). \tag{8.1}$$

Since, we have

$$(\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{Z}^*)(\mathscr{G}_4,\mathscr{G}_5)=\mathscr{G}_1\mathscr{Z}^*(\mathscr{G}_4,\mathscr{G}_5)-\mathscr{Z}^*(\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{G}_4,\mathscr{G}_5)-\mathscr{Z}^*(\mathscr{G}_4,\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{G}_5). \tag{8.2}$$

With the help of (8.1) and (8.2) we yield

$$\mathscr{G}_1\mathscr{Z}^*(\mathscr{G}_4,\mathscr{G}_5)-\mathscr{Z}^*(\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{G}_4,\mathscr{G}_5)-\mathscr{Z}^*(\mathscr{G}_4,\overset{*}{\nabla}_{\mathscr{G}_1}\mathscr{G}_5)=\hat{\eta}(\mathscr{G}_1)\mathscr{Z}^*(\mathscr{G}_4,\mathscr{G}_5). \tag{8.3}$$

Fix $\mathscr{G}_4=\mathscr{G}_5=\hat{\zeta}$ in (8.3) and using (2.1), (2.4), (2.14) and (2.15), we obtain

$$\mathscr{G}_1(2nk+\dot{\varphi})=\hat{\eta}(\mathscr{G}_1)(2nk+\dot{\varphi}).$$

We state the following:

Theorem 8.1. *In a $\mathscr{Z}^*$-recurrent $N(k)$-(CMM)$_{2n+1}$, we have*

$$\mathscr{G}_1(2nk+\dot{\varphi})=\hat{\eta}(\mathscr{G}_1)(2nk+\dot{\varphi}),$$

for all $\mathscr{G}_1\in\Gamma(\Theta)$.

A $\mathscr{Z}^*$-recurrent manifold is $\mathscr{Z}^*$-symmetric if and only if the 1-form $\hat{\eta}$ is zero. So we notice:

Corollary 8.2. *In a $\mathscr{Z}^*$-symmetric $N(k)$-(CMM)$_{2n+1}$, $2nk+\dot{\varphi}$=constant.*

Corollary 8.3. *If an $N(k)$-(CMM)$_{2n+1}$ is $\mathscr{Z}^*$-recurrent and if $2nk+\dot{\varphi}$ is constant, then either $2nk+\dot{\varphi}$=0 or, $(\Theta,\hat{\mathfrak{g}})$ reduces to a $\mathscr{Z}^*$-symmetric.*

Finally, we consider $\mathscr{Z}^*$-soliton with $\overset{\star}{V}=\hat{\zeta}$ on $N(k)$-(CMM)$_{2n+1}$. Then from (1.4), we have

$$\mathfrak{L}_{\hat{\zeta}}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)+2\mathscr{Z}^*(\mathscr{G}_1,\mathscr{G}_2)+2\lambda\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)=0. \tag{8.4}$$

Using (2.3) and (2.4), we find

$$\mathfrak{L}_{\hat{\zeta}}\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2)=-2\hat{\mathfrak{g}}(\hat{\phi}h\mathscr{G}_1,\mathscr{G}_2). \tag{8.5}$$

Now using (1.2), (8.5) in (8.4), we obtain

$$\overset{\star}{\mathscr{S}}(\mathscr{G}_1,\mathscr{G}_2)=-\hat{\mathfrak{g}}(h\mathscr{G}_1,\hat{\phi}\mathscr{G}_2)-(\dot{\varphi}+\lambda)\hat{\mathfrak{g}}(\mathscr{G}_1,\mathscr{G}_2). \tag{8.6}$$

In view of (2.13) and (8.6) we have

$$[2nk+\dot{\varphi}+\lambda]\eta(\mathscr{G}_1)=0,$$

which implies that

$$\lambda=-(2nk+\dot{\varphi}).$$

As per above, we mention the result:

Theorem 8.4. *If an $N(k)$-$(CMM)_{2n+1}$ admitting $\mathscr{Z}^*$-soliton, then we have*
(i) $\mathscr{Z}^*$-soliton is expanding if $\dot{\varphi} < -2nk$
(ii) $\mathscr{Z}^*$-soliton is shrinking if $\dot{\varphi} > -2nk$
(iii) $\mathscr{Z}^*$-soliton is steady if $\dot{\varphi} = -2nk$

Corollary 8.5. *A $\mathscr{Z}^*$-symmetric $N(k)$-$(CMM)_{2n+1}$ admitting $\mathscr{Z}^*$-soliton is always shrinking.*

Corollary 8.6. *A $\mathscr{Z}^*$-soliton on $\mathscr{Z}^*$-recurrent $N(k)$-$(CMM)_{2n+1}$ is always steady if $2nk+\dot{\varphi}$=constant.*

9. Example

Let a 3-dimensional manifold $\Theta = \{(r,s,t) \in \Re^3 : (r,s,t) \neq 0\}$, where (r,s,t) are standard coordinates in $\Re^3$. Let $(\vartheta_1, \vartheta_2, \vartheta_3)$ be the orthogonal system of vector fields at each point of Θ, defined as

$$\vartheta_1 = e^t \frac{\partial}{\partial r}, \ \vartheta_2 = e^t \frac{\partial}{\partial s}, \ \vartheta_3 = -\frac{\partial}{\partial t}$$

and

$$[\vartheta_1, \vartheta_2] = 0, \ [\vartheta_1, \vartheta_3] = \vartheta_1, \ [\vartheta_2, \vartheta_3] = \vartheta_2.$$

Let, we define the metric $\hat{\mathfrak{g}}$ as follows

$$\hat{\mathfrak{g}}_{ij} = \left\{ \begin{array}{ll} 0, & i \neq j = 1,2,3. \\ 1, & i = j \end{array} \right\}.$$

If $\hat{\eta}$ the 1-form have the significance

$$\hat{\eta}(\mathscr{G}_1) = \hat{\mathfrak{g}}(\mathscr{G}_1, \vartheta_1)$$

for any $\mathscr{G}_1 \in \Gamma(\Theta)$. Let $\hat{\varphi}$ be the $(1,1)$-tensor field defined by

$$\hat{\varphi}\,\vartheta_1 = 0, \qquad \hat{\varphi}\,\vartheta_2 = -\vartheta_3, \qquad \hat{\varphi}\,\vartheta_3 = \vartheta_2.$$

Making use of the linearity of $\hat{\varphi}$ and $\hat{\mathfrak{g}}$ we have

$$\hat{\eta}(\vartheta_1) = 1,$$
$$\hat{\varphi}^2(\mathscr{G}_1) = -\mathscr{G}_1 + \hat{\eta}(\mathscr{G}_1)\vartheta_1,$$
$$\hat{\mathfrak{g}}(\hat{\varphi}\mathscr{G}_1, \hat{\varphi}\mathscr{G}_2) = \hat{\mathfrak{g}}(\mathscr{G}_1, \mathscr{G}_2) - \hat{\eta}(\mathscr{G}_1)\hat{\eta}(\mathscr{G}_2),$$

for any $\mathscr{G}_1, \mathscr{G}_2 \in \Gamma(\Theta)$. Thus for ϑ_1=$\hat{\zeta}$ the structure $(\hat{\varphi}, \hat{\zeta}, \hat{\eta}, \hat{\mathfrak{g}})$ leads to a contact metric structure in $\Re^3$. We recall the Koszul's formula

$$2\hat{\mathfrak{g}}(\nabla_{\mathscr{G}_1}\mathscr{G}_2, \mathscr{G}_3) = \mathscr{G}_1(\hat{\mathfrak{g}}(\mathscr{G}_2, \mathscr{G}_3)) + \mathscr{G}_2(\hat{\mathfrak{g}}(\mathscr{G}_3, \mathscr{G}_1)) - \mathscr{G}_3(\hat{\mathfrak{g}}(\mathscr{G}_1, \mathscr{G}_2)) - \hat{\mathfrak{g}}(\mathscr{G}_1, [\mathscr{G}_2, \mathscr{G}_3]) - \hat{\mathfrak{g}}(\mathscr{G}_2, [\mathscr{G}_1, \mathscr{G}_3]) + \hat{\mathfrak{g}}(\mathscr{G}_3, [\mathscr{G}_1, \mathscr{G}_2]).$$

Making use Koszul's formula we have:

$$\left\{ \begin{array}{l} \nabla_{\vartheta_1}\vartheta_1 = -\vartheta_3, \quad \nabla_{\vartheta_1}\vartheta_2 = 0, \quad \nabla_{\vartheta_1}\vartheta_3 = \vartheta_1, \\ \nabla_{\vartheta_2}\vartheta_2 = 0, \quad \nabla_{\vartheta_2}\vartheta_3 = \vartheta_2, \quad \nabla_{\vartheta_3}\vartheta_1 = 0, \\ \nabla_{\vartheta_3}\vartheta_3 = 0, \quad \nabla_{\vartheta_2}\vartheta_1 = 0, \quad \nabla_{\vartheta_3}\vartheta_2 = 0. \end{array} \right.$$

Also we recall the following formula

$$\nabla_{\mathscr{G}_1}\vartheta_1 = -\hat{\varphi}\mathscr{G}_1 - \hat{\varphi}h\mathscr{G}_1.$$

Using above formula, one can easily calculate

$$h\vartheta_2 = -\vartheta_2, \ h\vartheta_3 = -\vartheta_3, \ h\vartheta_1 = 0.$$

The non-vanishing component of $\overset{\star}{\mathscr{R}}$ as follows:

$$\left\{ \begin{array}{lll} \overset{\star}{\mathscr{R}}(\vartheta_2, \vartheta_1)\vartheta_1 = \vartheta_2, & \overset{\star}{\mathscr{R}}(\vartheta_3, \vartheta_1)\vartheta_1 = \vartheta_3, & \overset{\star}{\mathscr{R}}(\vartheta_2, \vartheta_1)\vartheta_1 = \vartheta_2, \\ \overset{\star}{\mathscr{R}}(\vartheta_1, \vartheta_2)\vartheta_2 = \vartheta_1, & \overset{\star}{\mathscr{R}}(\vartheta_1, \vartheta_3)\vartheta_3 = \vartheta_1, & \overset{\star}{\mathscr{R}}(\vartheta_2, \vartheta_3)\vartheta_3 = \vartheta_2, \\ \overset{\star}{\mathscr{R}}(\vartheta_2, \vartheta_3)\vartheta_2 = -\vartheta_3, & \overset{\star}{\mathscr{R}}(\vartheta_1, \vartheta_3)\vartheta_3 = \vartheta_1, & \overset{\star}{\mathscr{R}}(\vartheta_3, \vartheta_1)\vartheta_1 = \vartheta_3. \end{array} \right.$$

We conclude that κ=1 and μ=0. Consequently ϑ_1=$\hat{\zeta} \in N(1,0)$-nullity distribution. Also the value of $\overset{\star}{\mathscr{S}}$ as below:

$$\overset{\star}{\mathscr{S}}(\vartheta_1, \vartheta_1) = \overset{\star}{\mathscr{S}}(\vartheta_2, \vartheta_2) = \overset{\star}{\mathscr{S}}(\vartheta_3, \vartheta_3) = 2. \tag{9.1}$$

In this case, equation (8.6) reduces to

$$\overset{\star}{\mathscr{S}}(\vartheta_1, \vartheta_1) = \overset{\star}{\mathscr{S}}(\vartheta_2, \vartheta_2) = \overset{\star}{\mathscr{S}}(\vartheta_3, \vartheta_3) = -(\lambda + \dot{\varphi}). \tag{9.2}$$

It is clear that from (9.1) and (9.2) that λ=$-(2+\dot{\varphi})$ and hence k=1, for n=1. Therefore, the Theorem 8.4 is verified.

10. Conclusion

The exploration of the $\mathscr{Z}^*$-tensor in pseudo-Riemannian manifolds and space-times delves into their geometric characteristics, curvature patterns, and overall behavior using mathematical methods like differential forms. This research into such manifolds not only enhances our comprehension of geometric structures with limited symmetries but also has practical implications in various fields, including physics. For instance, Mantica and Molinari defined the $\mathscr{Z}^*$-tensor [29] in 2012 and introduced many interesting results and applications in physics. Thereafter many authors study various properties of these tensors ([49]- [51]). Inspired by these works we study some geometric properties of $N(k)-(CMM)_{2n+1}$, whose metrics are the $\mathscr{Z}^*$-soliton and deduce some interesting results.

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On Weighted Cauchy-Type Problem of Riemann-Liouville Fractional Differential Equations in Lebesgue Spaces with Variable Exponent

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Abstract

This paper aims to investigate the existence, uniqueness, and stability properties for a class of fractional weighted Cauchy-type problem in the variable exponent Lebesgue space $L^{p(\cdot)}$. The obtained results are set up by employing generalized intervals and piece-wise constant functions so that the $L^{p(\cdot)}$ is transformed into the classical Lebesgue spaces. Moreover, the usual Banach Contraction Principle is utilized, and the Ulam-Hyers (UH) stability is studied. At the final stage, we provide an example to support the accuracy of the obtained results.

1. Introduction

Lebesgue spaces with variable exponents were originally examined in Orlicz's work [1] in 1931 and then in Nakano's papers [2, 3]. More specifically, [2] provides a precise characterization that describes Musielak-Orlicz spaces, however, it appears that Orlicz is mostly focused on the completeness of the function spaces. Afterward, a Russian researcher named Sharapudinov in [4] individually improved variable exponent Lebesgue spaces (VELS) on the real line. In the early 1900s, Kováčik and Rákosník in [5] detailed the essential characteristics of Lebesgue and Sobolev spaces with variable exponents. Actually, this paper has a major effect on subsequent papers and was accepted as the norm reference providing the current basic properties. The authors offered a suitable counterpart of the Lebesgue spaces L^p and of the Sobolev spaces $W^{k,p}$ and proposed the concept of $L^{p(x)}$ for functions p accepting the values on $[1,\infty]$. They also provide an application of generalized Sobolev spaces $W^{k,p(x)}$ to partial differential equations involving Dirichlet conditions with coefficients of a variable growth. A decade later, Fan and Zhao [6] deduced the same features by applying different techniques.
The basic idea behind the VELS $L^{p(\cdot)}$ is to substitute a variable exponent measurable function (VEMF) $p(.)$ into the traditional constant exponent p in classical Lebesgue spaces (CLS). As a result, we naturally expect that $L^{p(\cdot)}$ becomes a generalization for CLS L^p. Though the concept seems to be complex and challenging, it has substantial effects and implications that perfectly represent several phenomena in image processing, optimization, electrorheological(ER) fluids, etc. See [7–11] and the references therein.
A lot of papers have been published concerning the existence and uniqueness of solutions of fractional differential equations (FDEs) in the space of continuous functions $C(\Lambda,\mathscr{R})$, whereas relatively fewer articles exist studying the existence and uniqueness of solutions of FDEs in $L^p(\Lambda,\mathscr{R})$ space of integrable functions. For example; by using the well-known monotone technique combined with the method of upper and lower solutions, Derbazi et al. [12] find the existence and uniqueness of maximal and minimal solutions in $C(J,\mathscr{R})$ to an initial value problem involving ψ-Caputo fractional derivative:

$$\begin{cases} D_a^{\alpha,\psi}\kappa(s) = f(s,\kappa(s)), \ s\in J, \\ \kappa(a) = a^*. \end{cases}$$

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The existence and uniqueness of p-integrable solution in $L^p(\alpha,\beta)$ space has been discussed in [13] for Caputo FDE with a boundary condition having the form:

$$\begin{cases} D_a^{\varpi}\kappa(s)=\Phi(s,\kappa(s)), \ s\in[\alpha,\beta], \\ \gamma\kappa(\alpha)+\mu\kappa(\beta)=c. \end{cases}$$

Agarwal et al. [14] proved the existence of L^p solutions of fractional order integral equations with abstract Volterra operators in separable Banach spaces. Arshad et al. [15] have studied local and global existence results by applying a compactness-type condition for L^p solutions for fractional integral equations in Banach spaces. The existence of the solutions of FDEs in $L^{p(.)}$ actually has received little attention since we are aware of several notable challenges in that space. Dongg et al. [16] employed the Riesz-Kolmogorov theorem to get the existence and uniqueness of solutions for a Cauchy problem involving FDEs in VELS. Some qualitative properties of a boundary value problem in [17] and a terminal value problem in [18] involving Riemann-Liouville(R-L) fractional operator were discussed in $L^{p(.)}$ space with variable exponent. In a very recent work [19], the results in [17, 18] have been generalized by discussing a multi-term fractional boundary value problem in VELS. See [20–22] for the most current works regarding the subject.
In this paper, we shall investigate the following problem involving weighted Cauchy type condition in order to obtain some qualitative properties in $L^{p(.)}(\Lambda,\mathscr{R})$:

$$\begin{cases} D_0^{\varpi}\kappa(s)=\vartheta(s,\kappa(\psi(s))), \ s\in\Lambda:=[0,1], \\ s^{1-\varpi}\kappa(s)|_{s=0}=\beta, \end{cases} \tag{1.1}$$

where $0<\varpi<1$, $\vartheta(.,\kappa(.))\in L^{p(.)}(\Lambda\times\mathscr{R},\mathscr{R})$, $\kappa\in L^{p(.)}(\Lambda,\mathscr{R})$ and $\psi:\Lambda\to\Lambda$, and D_0^{ϖ} denotes the left Riemann Liouville (R-L) FDE of order ϖ in $L^{p(.)}$ defined as (see [16, 23]):

$$(D_{0^+}^{\varpi}\kappa)(s)=\frac{1}{\Gamma(1-\varpi)}\frac{d}{ds}\int_0^t(s-\rho)^{-\varpi}\kappa(\rho)d\rho, \tag{1.2}$$

where $\Gamma(.)$ is the gamma function.
On the other hand, left-sided R-L FDE of order ϖ for function $\kappa(s)$ in $L^{p(.)}$ is given by

$$I_{0^+}^{\varpi}\kappa(s)=\frac{1}{\Gamma(\varpi)}\int_0^s(s-\rho)^{\varpi-1}\kappa(\rho)d\rho.$$

The outline of the paper is as follows: Fundamental concepts and helpful lemmas that are necessary for establishing the main results are introduced in Section 2. Critical results regarding the existence of solutions in the Lebesgue space of variable exponent for the problem (1.1), under certain conditions are established in the subsequent section. The UH stability of the solution is demonstrated in the following section. The last section is dedicated to a demonstrative case that supports the obtained results.

2. Mathematical Preliminaries

Definition 2.1 ([24], [23]). *By $L^p([\alpha,\beta],\mathscr{R})$, $1\le p<\infty$, we express the classical space of measurable functions $\Phi:[\alpha,\beta]\to\mathscr{R}$, provided with the norm*

$$\|\Phi\|_r=\left(\int_\alpha^\beta|\Phi(s)|^p ds\right)^{\frac{1}{p}}<\infty$$

and

$$\|\Phi\|_\infty=ess\ sup_{\alpha\le s\le\beta}|\Phi(s)| \quad if \quad r=\infty.$$

Lemma 2.2 ([23]). *Let $\Phi_1,\Phi_2\in L^p([\alpha,\beta],\mathscr{R})$, $1\le p<\infty$ and $\varpi,\beta>0$ then the following properties of the left RL fractional integral and R-L FDE are demonstrated.*
(1) $I_{\alpha^+}^{\varpi}I_{\alpha^+}^{\beta}f_1(s)=I_{\alpha^+}^{\varpi+\beta}\Phi_1(s)$
(2) $I_{\alpha^+}^{\varpi}[\Phi_1(s)+\Phi_2(s)]=I_{\alpha^+}^{\varpi}\Phi_1(s)+I_{\alpha^+}^{\varpi}\Phi_2(s)$
(3) $D_{\alpha^+}^{\varpi}I_{\alpha^+}^{\varpi}\Phi_1(s)=\Phi_1(s)$
(4) $\|I_{\alpha^+}^{\varpi}\Phi_1\|_p\le\frac{(\beta-\alpha)^{\varpi}}{\Gamma(\varpi+1)}\|\Phi_1\|_p$.

Lemma 2.3 ([23]). *If $\Phi\in L^p([\alpha,\beta],\mathscr{R})$, $1\le p<\infty$, $\varpi>0$, then $I_{\alpha^+}^{\varpi}\Phi\in L^p([\alpha,\beta],\mathscr{R})$.*

Lemma 2.4 ([23]). *Let $\varpi>0$, then the differential equation*

$$D_{\alpha^+}^{\varpi}\xi=0$$

has a unique solution

$$\xi(s)=c_1(s-\alpha)^{\varpi-1}+c_2(s-\alpha)^{\varpi-2}+...+c_n(s-\alpha)^{\varpi-n}$$

$c_\omega\in\mathscr{R}$, $1\le\omega\le n$, here $n=[\varpi]+1$.

Lemma 2.5 ([23]). *Let $\alpha>0$, $\xi\in L^1(\Lambda,\mathscr{R})$, $D_{\alpha^+}^{\varpi}\xi\in L^1(\Lambda,\mathscr{R})$, then*

$$I_{\alpha^+}^{\varpi}D_{\alpha^+}^{\varpi}\xi(s)=\xi(s)+c_1(s-a)^{\varpi-1}+c_2(s-\alpha)^{\varpi-2}+...+c_n(s-\alpha)^{\varpi-n}$$

where $c_\omega\in\mathscr{R}$, $1\le\omega\le n$, here $n=[\varpi]+1$.

We now recall the known Hölder inequality for integrals.

Lemma 2.6 ([25]). *Let p and ℓ satisfy $1<p<\infty, 1<\ell<\infty$, and $\frac{1}{p}+\frac{1}{\ell}=1$. If $\Phi_1 \in L^p(\Lambda,\mathscr{R})$ and $\Phi_2 \in L^\ell(\Lambda,\mathscr{R})$, then Φ_1,Φ_2 belongs to $L^1(\Lambda,\mathscr{R})$ and satisfies*

$$\int_\Lambda |\Phi_1\Phi_2|dx \leq \left[\int_\Lambda |\Phi_1|^p dx\right]^{\frac{1}{p}} \left[\int_\Lambda |\Phi_2|^\ell dx\right]^{\frac{1}{\ell}}.$$

Definition 2.7 ([26]). *Let $\Omega \subseteq \mathscr{R}^n$ be an open set in $\mathscr{R}^n$. By $L^{p(.)}(\Omega)$ we denote all space of measurable functions Φ on Ω such that*

$$I_{p(.)}(\Phi) = \int_\Omega |\Phi(s)|^{p(s)} ds < \infty,$$

where $p(s)$ is a VEMF on Ω with values in $[1,\infty)$. This is a Banach space given with the norm

$$\|\Phi\|_{p(.)} = \inf\{\eta > 0 : I_{p(.)}(\Phi/\eta) \leq 1\}.$$

We use the following notation:

$$p_- = \inf_{s\in\Omega} p(s), \quad p_+ = \sup_{s\in\Omega} p(s)$$

$\ell(.)$ the conjugate exponent of $p(.)$:

$$\ell(.) = \frac{p(.)}{p(.)-1},$$

$\mathscr{P}(\Omega)$ is defined as the set of bounded measurable functions $p(s):\Omega\to[1,\infty)$ while $\mathscr{P}^{log}(\Omega)$ designates the set of exponents $p\in\mathscr{P}(\Omega)$ satisfying the local Log condition:

$$|p(s)-p(\rho)| \leq \frac{A_r}{-log|s-\rho|}, |s-\rho| \leq \frac{1}{2}, \quad s,\rho\in\Omega,$$

where $A_p > 0$ is independent of t and ρ.
$\mathfrak{I}_{log}(\Omega)$ is the set off bounded exponents $\varpi:\Omega\to\mathscr{R}$ satisfying the local log condition.
$\mathbb{P}^{log}(\Omega)$ is a set consisting of exponents $p\in\mathscr{P}^{log}(\Omega)$ with $1<p_-\leq p_+<\infty$.
The following lemma is related to Hölder inequality in the variable exponent Lebesgue space $L^{p(.)}(\Omega)$.

Lemma 2.8 ([27]). *Let $\Omega\subseteq\mathscr{R}^n$ be an open set in $\mathscr{R}^n$ and $p(s),\ell(s)$ are two VEMF on Ω with values in $[1 \;\; \infty)$ where $1\leq p(s)\leq\infty$ and $\frac{1}{p(s)}+\frac{1}{\ell(s)}=1$. If $\Phi_1\in L^{p(.)}(\Lambda)$ and $\Phi_2\in L^{\ell(.)}(\Lambda)$, we have*

$$\int_\Omega |\Phi_1(s)\Phi_2(s)|ds \leq p\|\Phi_1\|_{p(.)}\|\Phi_2\|_{\ell(.)},$$

where $p = sup_{s\in\Omega}\frac{1}{p(s)} + sup_{s\in\Omega}\frac{1}{\ell(s)}$.

Theorem 2.9 ([16]). *Let $p(.)\in\mathscr{P}[0,M]$ and $0<\frac{1}{p_-}<\varpi<1$, then I_{0+}^{ϖ} is bounded in $L^{p(.)}([0,M],\mathscr{R})$.*

Definition 2.10 ([28]). *Let $\Lambda\subset\mathscr{R}$, Λ is named as a generalized interval if it is either an interval or $\{a_1\}$ or $\emptyset$.*
A finite set $\mathscr{P}$ is called a partition of Λ if each x inn Λ lies in exactly one of the generalized intervals E in $\mathscr{P}$.
A function $p:\Lambda\to\mathscr{R}$ is named by piece-wise constant as regards to partition $\mathscr{P}$ of Λ if for any $E\in\mathscr{P}$, p is constant on E.

Definition 2.11 ([29]). *The problem (1.1) is Ulam-Hyers(UH) stable if there exists $c_\vartheta>0$, such that for any $\varepsilon>0$ and for each solution $y\in L^p(\Lambda,\mathscr{R})$ of the following inequality*

$$|D_{0+}^{\varpi}y(s)-\vartheta(s,y(s))| \leq \varepsilon, \; s\in\Lambda \tag{2.1}$$

there exists a solution $\kappa\in L^p(\Lambda,\mathscr{R})$ of problem (1.1) with

$$|y(s)-\kappa(s)| \leq c_\vartheta\varepsilon, \; s\in\Lambda.$$

3. Existence and Uniqueness of Solutions

Let us begin with the following assumption:

(H1) Let the finite sequence of points $\{M_\omega\}_{\omega=0}^n$ satisfy $0=M_0<M_\omega<M_n=1$, and Λ_ω be defined as $\Lambda_\omega=(M_{\omega-1},\; M_\omega]$, $\omega=1,2,...,n$, $n\in\mathbb{N}$. Then $\mathscr{P}=\bigcup_{\omega=1}^n\Lambda_\omega$ would be a partition of the interval Λ.
For each $\omega=1,2,...,n$, the notation $\Upsilon_\omega=L^{p_\omega}(\Lambda_\omega,\mathscr{R})$ denotes the Banach space of VEMF from Λ_ω into $\mathscr{R}$ equipped with the norm:

$$\|\kappa\|_{\Upsilon_\omega} = \left(\int_{\Lambda_\omega}|\kappa|^{p_\omega}dx\right)^{\frac{1}{p_\omega}} < \infty,$$

where $1\leq\omega\leq n$.
Let $p(s):\Lambda\to[1,\infty)$ be a piece-wise constant function with regard to $\mathscr{P}$, i.e., $p(s)=\sum_{\omega=1}^n p_\omega I_\omega(s)$, where $1\leq p_\omega<\infty$ are constants and I_ω is the indicator of the interval Λ_ω, $\omega=1,2,...,n$

$$I_{\omega}(s)=\begin{cases}1, & for\ s\in\Lambda_{\omega},\\ 0, & for\ elsewhere.\end{cases}$$

So, for any $s\in\Lambda_{\omega}$, $1\le\omega\le n$, the left R-L FDE for the function defined by (1.2), can be written as

$$(D^{\varpi}_{0^+}\kappa)(s)=\frac{1}{\Gamma(1-\varpi)}\frac{d}{ds}\int_0^s(s-\rho)^{-\varpi}\kappa(\rho)d\rho$$

$$=\frac{1}{\Gamma(1-\varpi)}\Big(\sum_{\iota=1}^{\omega-1}\frac{d}{ds}\int_{M_{\iota-1}}^{M_\iota}(s-\rho)^{-\varpi}\kappa(\rho)d\rho+\frac{d}{ds}\int_{M_{\omega-1}}^{s}(s-\rho)^{-\varpi}\kappa(\rho)d\rho\Big). \tag{3.1}$$

Thus, the problem (1.1) can be explained for any $s\in\Lambda_{\omega}, 1\le\omega\le n$ in the form:

$$\frac{1}{\Gamma(1-\varpi)}\Big(\sum_{\iota=1}^{\omega-1}\frac{d}{ds}\int_{M_{\iota-1}}^{M_\iota}(s-\rho)^{-\varpi}\kappa(\rho)d\rho+\frac{d}{ds}\int_{M_{\omega-1}}^{s}(s-\rho)^{-\varpi}\kappa(\rho)d\rho\Big)=\vartheta(s,\kappa(\psi(s))) \tag{3.2}$$

Let the function $\kappa\in L^{p_\omega}(\Lambda_\omega)$ with $\kappa\equiv 0$ on $s\in[0,M_{\omega-1}]$ and it solves integral equation (3.2).
Then, (3.2) is reduced to

$$(D^{\varpi}_{M_{\omega-1}}\kappa)(s)=\vartheta(s,\kappa(\psi(s))),\ \ s\in\Lambda_\omega.$$

For any $1\le\omega\le n$, we look at the following auxiliary weighted Cauchy type problem of constant order :

$$\begin{cases}D^{\varpi}_{M_{\omega-1}}\kappa(s)=\vartheta(s,\kappa(\psi(s))),\ s\in\Lambda_\omega,\\ s^{1-\varpi}\kappa(s)|_{s=M_{\omega-1}}=b.\end{cases} \tag{3.3}$$

Lemma 3.1. *Let $1\le\omega\le n$ be a natural number, $0<\varpi<1$, $\vartheta\in L^{p_\omega}(\Lambda_\omega\times\mathscr{R},\mathscr{R})$. A function $\kappa_\omega\in\Upsilon_\omega$ is a solution of (3.3) if and only if $\kappa_\omega\in\Upsilon_\omega$ solves the integral equation*

$$\kappa_\omega(s)=bs^{\varpi-1}+\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\vartheta(\rho,\kappa_\omega(\psi(\rho)))d\rho \tag{3.4}$$

Proof. To show the necessity, we can write from (3.3)

$$s^{1-\varpi}\kappa_\omega(s)=b+s^{1-\varpi}I^{\varpi}_{M_{\omega-1}}\vartheta(s,\kappa_\omega(\psi(s))).$$

which implies

$$s^{1-\varpi}\kappa_\omega(s)|_{t=M_{\omega-1}}=b.$$

Also, applying $I^{1-\varpi}_{M_{\omega-1}}$ on both sides of (3.4), then

$$I^{1-\varpi}_{M_{\omega-1}}\kappa_\omega(s)=b_0+I_{M_{\omega-1}}\vartheta(s,\kappa_\omega(\psi(s))).$$

Differentiating both sides of order one, we achieve

$$D^{\varpi}_{M_{\omega-1}}\kappa_\omega(s)=\vartheta(s,\kappa_\omega(\psi(s)))$$

Inversely, let κ_ω be a solution of (3.3), by integrating both sides, then

$$I^{1-\varpi}\kappa_\omega(s)-I^{1-\varpi}\kappa_\omega(s)|_{t=0}=I^{1}_{M_{\omega-1}}\vartheta(s,\kappa_\omega(\psi(s))).$$

Operating by $I^{\varpi}_{M_{\omega-1}}$ on both sides of the last equation, we have

$$I\kappa_\omega(s)-I^{\varpi}C=I^{1+\varpi}_{M_{\omega-1}}\vartheta(s,\kappa_\omega(\psi(s))).$$

taking the ordinary derivative of the first order, it follows that

$$\kappa_\omega(s)-C_1s^{\varpi-1}=I^{\varpi}_{M_{\omega-1}}\vartheta(s,\kappa_\omega(\psi(s))),$$

By recalling the initial condition, we find that $C_1=b$, then we obtain (3.4), i.e., problem (3.3) and equation (3.4) are equivalent to each other. □

Banach Contraction Principle (BCP) is implemented to arrive at the conclusion of the following result.

Theorem 3.2. *Suppose that Lemma 3.1 is satisfied and we have a constant $M>0$ such that $|\vartheta(s,\kappa_1)-\vartheta(s,\kappa_2)|\le M|\kappa_1-\kappa_2|$, for any $\kappa_1,\ \kappa_2\in L^{p_\omega}(\Lambda_\omega)$ $s\in\Lambda_\omega$ and moreover the inequality*

$$W_{\varpi,M,p_\omega,M_{\omega-1},M_\omega}<1, \tag{3.5}$$

holds where

$$W_{\varpi,M,p_\omega,M_{\omega-1},M_\omega}=\Bigg[\Bigg(\frac{M}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{1}{\ell_\omega}}\Gamma(\varpi)}\Bigg)^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Bigg]^{\frac{1}{p_\omega}}.$$

Then, for every $1\le\omega\le n$ there exists a unique solution on Λ_ω for the problem (3.3).

Proof. We use a transformation for the problem (3.4) so that it returns to a fixed point problem. Let the operator

$$S : L^{p_\omega}(\Lambda_\omega, \mathscr{R}) \to L^{p_\omega}(\Lambda_\omega, \mathscr{R})$$

which is given by

$$(S\kappa_\omega)(s) = bs^{\varpi-1} + \frac{1}{\Gamma(\varpi)} \int_{M_{\omega-1}}^{s} (s-\rho)^{\varpi-1} \vartheta(\rho, \kappa_\omega(\psi(\rho)))d\rho.$$

BCP is used as the main tool to determine that S has a unique fixed point. To do that, let $\kappa_\omega, x_\omega \in L^{p_\omega}(\Lambda_\omega)$, then we have

$$\begin{aligned}
\|S(\kappa_\omega(s)) - S(x_\omega(s))\|^{p_\omega} &= \|\frac{1}{\Gamma(\varpi)} \int_{M_{\omega-1}}^{s} (s-\rho)^{\varpi-1}\big(\vartheta(\rho,\kappa_\omega(\rho)) - \vartheta(\rho, x_\omega(\rho))\big)d\rho\|^{p_\omega} \\
&= \int_{M_{\omega-1}}^{M_\omega} \Big|\frac{1}{\Gamma(\varpi)} \int_{M_{\omega-1}}^{s} (s-\rho)^{\varpi-1}\Big(\vartheta(\rho,\kappa_\omega(\rho)) - \vartheta(\rho, x_\omega(\rho))\Big)d\rho\Big|^{p_\omega} ds \\
&\leq \frac{1}{(\Gamma(\varpi))^{p_\omega}} \int_{M_{\omega-1}}^{M_\omega} \Big|\int_{M_{\omega-1}}^{s} (s-\rho)^{\varpi-1}\Big(\vartheta(\rho,\kappa_\omega(\rho)) - \vartheta(\rho, x_\omega(\rho))\Big)d\rho\Big|^{p_\omega} ds \\
&\leq \frac{M^{p_\omega}}{(\Gamma(\varpi))^{p_\omega}} \int_{M_{\omega-1}}^{M_\omega} \Big(\int_{M_{\omega-1}}^{s} (s-\rho)^{\varpi-1}\Big|\kappa_\omega(\rho)) - x_\omega(\rho))\Big|d\rho\Big)^{p_\omega} ds \\
&\leq \frac{M^{p_\omega}}{(\Gamma(\varpi))^{p_\omega}} \int_{M_{\omega-1}}^{M_\omega} \Bigg[\Big(\int_{M_{\omega-1}}^{s} (s-\rho)^{\ell_\omega(\varpi-1)}d\rho\Big)^{\frac{1}{\ell_\omega}} \times \Big(\int_{M_{\omega-1}}^{s} \Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega} d\rho\Big)^{\frac{1}{p_\omega}}\Bigg]^{p_\omega} ds.
\end{aligned}$$

Observe that we have utilized the Hölder Inequality. If we proceed with calculations

$$\begin{aligned}
\|S(\kappa_\omega(s)) - S(x_\omega(s))\|^{p_\omega} &\leq \frac{M^{p_\omega}}{(\Gamma(\varpi))^{p_\omega}} \int_{M_{\omega-1}}^{M_\omega} \frac{(s-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}}}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{p_\omega}{\ell_\omega}}} \Big(\int_{M_{\omega-1}}^{s} \Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega} d\rho\Big) ds \\
&\leq \Big[\frac{M}{(\ell_\omega(\varpi-1)+1)^{\frac{1}{\ell_\omega}}\Gamma(\varpi)}\Big]^{p_\omega} \int_{M_{\omega-1}}^{M_\omega} (s-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}} \times \Big(\int_{M_{\omega-1}}^{s} \Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega} d\rho\Big) ds.
\end{aligned}$$

After rearranging the integrals, we reach at

$$\int_{M_{\omega-1}}^{M_\omega} (s-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}} \Big(\int_{M_{\omega-1}}^{s} \Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega} d\rho\Big) ds = \int_{M_{\omega-1}}^{M_\omega} (s-M_{\omega-1})^{\theta_\omega} \sigma_\omega(s) ds = I,$$

where

$$\theta_\omega = \frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}, \quad \sigma_\omega(s) = \int_{M_{\omega-1}}^{s} \Big|\kappa_\omega(\rho) - x_\omega(\rho)\Big|^{p_\omega} d\rho.$$

Integrating by parts formula yields

$$\begin{aligned}
I &= \frac{(M_\omega - M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma_\omega(M_\omega) - \int_{M_{\omega-1}}^{M_\omega} \frac{(s-M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma'_\omega(s)ds \\
&= \frac{(M_\omega - M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma_\omega(M_\omega) - \int_{M_{\omega-1}}^{M_\omega} \frac{(s-M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma'_\omega(s)ds.
\end{aligned}$$

Since the integral

$$\int_{M_{\omega-1}}^{M_\omega} \frac{(s-M_{\omega-1})^{\theta_\omega+1}}{\theta_\omega+1}\sigma'_\omega(s)ds \geq 0$$

then,

$$\begin{aligned}
\|S(\kappa_\omega(s)) - S(x_\omega(s))\| &\leq \Bigg[\Bigg(\frac{M}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{1}{\ell_\omega}}\Gamma(\varpi)}\Bigg)^{p_\omega} \frac{(M_\omega - M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Bigg]^{\frac{1}{p_\omega}} [\sigma_\omega(M_\omega)]^{\frac{1}{p_\omega}} \\
&= \Bigg[\Bigg(\frac{M}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{1}{\ell_\omega}}\Gamma(\varpi)}\Bigg)^{p_\omega} \frac{(M_\omega - M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Bigg]^{\frac{1}{p_\omega}} \|\kappa_\omega - x_\omega\|_{\Upsilon_\omega}.
\end{aligned}$$

As a result, by (3.5), the operator S is a contraction. Therefore, by BCP, S has a unique fixed point $\widetilde{\kappa_i} \in L^{p_\omega}(\Lambda_\omega)$, that yields the unique solution of the problem (3.3). $\square$

We are now ready to prove the existence result for (1.1).
Let us consider the following condition:

(H2) There exist a constant $M > 0$ such that,
$|\vartheta(s,\kappa_1) - \vartheta(s,\kappa_2)| \leq M|\kappa_1 - \kappa_2|$, for any $\kappa_1,\ \kappa_2 \in L^{p(.)}(\Lambda)$ and $s \in \Lambda$.

Theorem 3.3. *Assume that (H1), (H2) and inequality (3.5) fulfill for all $1 \leq \omega \leq n$. Then, problem (1.1) has at most a solution in $L^{p(\cdot)}(\Lambda)$*

Proof. As mentioned in Theorem 3.2, for each $1 \leq \omega \leq n$, (3.3) possesses a unique solution $\widetilde{\kappa} \in \Upsilon_\omega$.
For any $1 \leq \omega \leq n$ we give the function as follow;

$$\kappa_\omega = \begin{cases} 0, & if\ s \in [0\ \ M_{\omega-1}], \\ \widetilde{\kappa}, & if\ s \in \Lambda_\omega. \end{cases}$$

Therefore, $\kappa_\omega \in L^p([0,\ \ M_{\omega-1}], \mathscr{R})$ is a solution for the integral equation (3.2) for $s \in \Lambda_\omega$ meaning that it solves (3.3) for $s \in \Lambda_\omega$.
Then the function:

$$\kappa(s) = \begin{cases} \kappa_1(s) \in L^{p_1}(\Lambda_1, \mathscr{R}), \\ \kappa_2(s) \in L^{p_2}(\Lambda_2, \mathscr{R}), \\ . \\ . \\ . \\ \kappa_n(s) \in L^{p_n}(\Lambda_n, \mathscr{R}). \end{cases}$$

is a unique solution of the problem (1.1) in $L^{p(\cdot)}(\Lambda)$. □

4. Ulam-Hyers Stability

Theorem 4.1. *Assume that* $(H1), (H2)$*, and inequality (3.5) hold. Then, (1.1) is* ***UH*** *stable .*

Proof. Take ε as an arbitrary positive number and the function $y(s)$ from $y \in L^{p_\omega}(\Lambda_\omega, \mathscr{R})$ satisfy inequality (2.1).
For any $\omega \in \{1,2,...,n\}$ we define the functions $y_1(s) \equiv y(s), s \in [0, M_1]$ and for $\omega = 2,3,...,n$

$$y_\omega(s) = \begin{cases} 0, & s \in [0, M_{\omega-1}], \\ \\ y(s), & s \in \Lambda_\omega. \end{cases} \tag{4.1}$$

According to equality (3.1) for any $\omega \in \{1,2,...,n\}$ and $t \in \Lambda_\omega$ we get

$$(D^{\varpi}_{0^+} y_\omega)(s) = \frac{1}{\Gamma(1-\varpi)} \frac{d}{ds} \int_{M_{\omega-1}}^{s} (s-\rho)^{-\varpi} y(\rho) d\rho.$$

Taking $I^{\varpi}_{M^+_{\omega-1}}$ of both sides of the inequality (2.1), we get

$$\left| y_\omega(s) - bs^{\varpi-1} - \frac{1}{\Gamma(\varpi)} \int_{M_{\omega-1}}^{s} (s-\rho)^{\varpi-1} \vartheta(\rho, y_\omega(\psi(\rho))) d\rho \right| \leq \varepsilon \frac{(s-M_{\omega-1})^{\varpi}}{\Gamma(\varpi+1)} \leq \varepsilon \frac{(M_\omega - M_{\omega-1})^{\varpi}}{\Gamma(\varpi+1)}.$$

According to Theorem 3.3, (1.1) has a unique solution $\kappa \in L^{p(\cdot)}(\Lambda)$ defined by $\kappa(s) = \kappa_\omega(s)$ for $s \in \Lambda_\omega,\ \omega = 1,2,...,n$, where

$$\kappa_\omega = \begin{cases} 0, & s \in [0, M_{\omega-1}], \\ \widetilde{\kappa}_\omega, & s \in \Lambda_\omega, \end{cases} \tag{4.2}$$

and $\widetilde{\kappa}_\omega \in \Upsilon_\omega$ is a unique solution of problem (3.3).
In view of Lemma 3.1, the integral equation

$$\widetilde{\kappa}_\omega(s) = bs^{\varpi-1} + \frac{1}{\Gamma(\varpi)} \int_{M_{\omega-1}}^{s} (s-\rho)^{\varpi-1} \vartheta(\rho, \widetilde{\kappa}_\omega(\psi(\rho))) d\rho \tag{4.3}$$

holds.
For $t \in \Lambda_\omega,\ \omega = 1,2,...,n$, by (4.1), (4.2) we have,

$$|y(s) - \kappa(s)| = |y(s) - \kappa_\omega(s)| = |y_\omega(s) - \widetilde{\kappa}_\omega(s)|$$

Then, by (4.3) we get

$$
\begin{aligned}
\|y-\kappa\|_{\Upsilon_\omega}^{p_\omega} &= \|y-\kappa_\omega\|_{\Upsilon_\omega}^{p_\omega} = \|y_\omega-\widetilde{\kappa}_\omega\|_{\Upsilon_\omega}^{p_\omega}\\
&= \|y_\omega(s)-bs^{\varpi-1}-\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\vartheta(\rho,\widetilde{\kappa}_\omega(\rho))d\rho\|_{\Upsilon\omega}^{p_\omega}\\
&\leq \|y_\omega(s)-bs^{\varpi-1}-\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}\vartheta(\rho,y_\omega(\rho))d\rho\|_{\Upsilon_\omega}^{p_\omega}+\|\frac{1}{\Gamma(\varpi)}\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}(\vartheta(\rho,y_\omega(\rho))-\vartheta(\rho,\widetilde{\kappa}_\omega(\rho)))d\rho\|_{\Upsilon_\omega}^{p_\omega}\\
&\leq \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{p_\omega}(\varpi+1)}+\frac{1}{\Gamma^{p_\omega}(\varpi)}\int_{M_{\omega-1}}^{M_\omega}(\int_{M_{\omega-1}}^{s}(s-\rho)^{\varpi-1}|\vartheta(\rho,y_\omega(\rho))-\vartheta(\rho,\widetilde{\kappa}_\omega(\rho))|d\rho)^{p_\omega}ds\\
&\leq \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{r_\omega}(\varpi+1)}+\frac{1}{\Gamma^{p_\omega}(\varpi)}\int_{M_{\omega-1}}^{M_\omega}\Big[\Big(\int_{M_{\omega-1}}^{s}(s-\rho)^{\ell_\omega(\varpi-1)}d\rho\Big)^{\frac{1}{\ell_\omega}}\times\Big(\int_{M_{\omega-1}}^{s}\Big|\vartheta(\rho,y_\omega(\rho))-\vartheta(\rho,\widetilde{\kappa}_\omega(\rho))\Big|^{p_\omega}d\rho\Big)^{\frac{1}{p_\omega}}\Big]^{p_\omega}ds\\
&\leq \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{p_\omega}(\varpi+1)}+\frac{1}{\Gamma^{p_\omega}(\varpi)}\int_{M_{\omega-1}}^{M_\omega}\frac{(s-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}}}{\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{p_\omega}{\ell_\omega}}}\Big(\int_{M_{\omega-1}}^{s}\Big|\vartheta(\rho,y_\omega(\rho))-\vartheta(\rho,\widetilde{\kappa}_\omega(\rho))\Big|^{p_\omega}d\rho\Big)ds\\
&\leq \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{p_\omega}(\varpi+1)}+\Big[\frac{M^{p_\omega}}{\Gamma^{p_\omega}(\varpi)\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{p_\omega}{\ell_\omega}}}\frac{(M_\omega-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Big]\|y_\omega-\widetilde{\kappa}_\omega\|_{\Upsilon_\omega}^{p_\omega}\\
&\leq \varepsilon^{p_\omega}\frac{(M_\omega-M_{\omega-1})^{\varpi p_\omega+1}}{\Gamma^{p_\omega}(\varpi+1)}+\tau\|y-\kappa\|_{\Upsilon_\omega}^{p_\omega},
\end{aligned}
$$

where

$$
\tau=\max_{\omega=1,2,\ldots,n}\Big[\frac{M^{p_\omega}}{\Gamma^{p_\omega}(\varpi)\Big(\ell_\omega(\varpi-1)+1\Big)^{\frac{p_\omega}{\ell_\omega}}}\frac{(M_\omega-M_{\omega-1})^{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}}{\frac{p_\omega(\ell_\omega(\varpi-1)+1)}{\ell_\omega}+1}\Big].
$$

Then,

$$
\|y-\kappa\|_{\Upsilon_\omega}\leq\frac{(M_\omega-M_{\omega-1})^{\frac{\varpi p_\omega+1}{p_\omega}}}{(1-\tau)^{\frac{1}{p_\omega}}\Gamma(\varpi+1)}\varepsilon.
$$

We obtain,

$$
\|y-\kappa\|_{p}\leq\frac{1}{\Gamma(\varpi+1)}\Big(\sum_{\omega=1}^{\omega=n}\frac{(M_\omega-M_{\omega-1})^{\frac{\varpi p_\omega+1}{p_\omega}}}{(1-\tau)^{\frac{1}{r_\omega}}}\Big)\varepsilon:=c_\vartheta\varepsilon.
$$

Therefore, the (1.1) is **UH** stable. □

5. Example

Consider the flowing fractional weighted Cauchy type problem:

$$
\begin{cases} D^{0.5}\kappa(s)=\frac{|\kappa(s)|}{(2+e^s)(1+\kappa(s))}, & s\in\Lambda:=[0,1],\\ s^{0.5}\kappa(s)=0. \end{cases} \tag{5.1}
$$

Let

$$
\vartheta(s,\psi(\kappa))=\frac{|\kappa(s)|}{(2+e^s)(1+\kappa(s))},\ s\in[0,1].
$$

Then, we have

$$
\begin{aligned}
|\vartheta(s,\psi(x))-\vartheta(s,\psi(\kappa))| &= \frac{1}{(2+e^s)}\Big|\frac{x}{1+x}-\frac{\kappa}{1+\kappa}\Big|\\
&=\frac{|x-\kappa|}{(2+e^s)(1+x)(1+\kappa)}\\
&\leq\frac{|x-\kappa|}{2+e^s}\\
&\leq\frac{1}{2}|x-\kappa|.
\end{aligned}
$$

Thus the condition (H2) is satisfied with $M=\frac{1}{2}$.
Let

$$
p(s)=\begin{cases} p_1=4, & if\ s\in[0,0.5],\\ p_2=5, & if\ s\in]0.5,1]. \end{cases} \tag{5.2}
$$

According to (3.3), we consider two auxiliary 3.3, the problem (5.1) is equivalent to the followings problems:

$$\begin{cases} D^{0.5}\kappa(s)=\frac{|\kappa(s)|}{(2+e^s)(1+\kappa(s))}, \ s\in\Lambda_1:=[0,0.5], \\ s^{0.5}\kappa(s)=0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} D^{0.5}\kappa(s)=\frac{|\kappa(s)|}{(2+e^s)(1+\kappa(s))}, \ s\in\Lambda_2:=]0.5,1], \\ s^{0.5}\kappa(s)=0. \end{cases} \tag{5.4}$$

Next, we demonstrate that (3.5) is satisfied for $\omega=1, p_1=4$. Indeed,

$$W_{\varpi,M,p_1,M_0,M_1}=0,172681927<1.$$

As a consequence, the inequality (3.5) is satisfied.
Thus, in light of Theorem (3.2), the (5.3) has a unique solution $\widetilde{\kappa}_1\in L^4(\Lambda_1,\mathscr{R})$.
We have revealed that the inequality (3.5) is valid for $\omega=2, p_2=5$. Indeed,

$$W_{\varpi,M,p_2,M_1,M_2}=0,202489255<1.$$

Then, the inequality (3.5) is fulfilled.
Taking into account Theorem 3.2, the (5.4) provides a unique solution. $\widetilde{\kappa}_2\in L^5(\Lambda_2,\mathscr{R})$.
Hence, in view of Theorem (3.3), the (5.1) possesses a unique solution.

$$\kappa(s)=\begin{cases} \widetilde{\kappa}_1(s)\in L^4(\Lambda_1,\mathscr{R}), \\ \kappa_2(s)\in L^5(\Lambda_2,\mathscr{R}), \end{cases}$$

where

$$\kappa_2(s)=\begin{cases} 0, \ \ s\in\Lambda_1, \\ \widetilde{\kappa}_2(s), \ \ s\in\Lambda_2. \end{cases}$$

Clearly, one can show that by Theorem 4.1, solution of problem (5.1) is **UH** stable.

6. Conclusion

We investigate some qualitative properties of a weighted Cauchy problem (1.1) in Lebesgue spaces with variable exponent $L^{p(\cdot)}$. Our main proofs are based on exploiting the generalized intervals and piece-wise constant functions that transform $L^{p(\cdot)}$ to the classical Lebesgue spaces. Additionally, we support the theoretical results by constructing a numerical example.
There have been only a few investigations conducted in this area due to the complex structure of the variable exponent Lebesgue spaces. As a result, the fundamental results provided in this paper offer several opportunities for further investigations.

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