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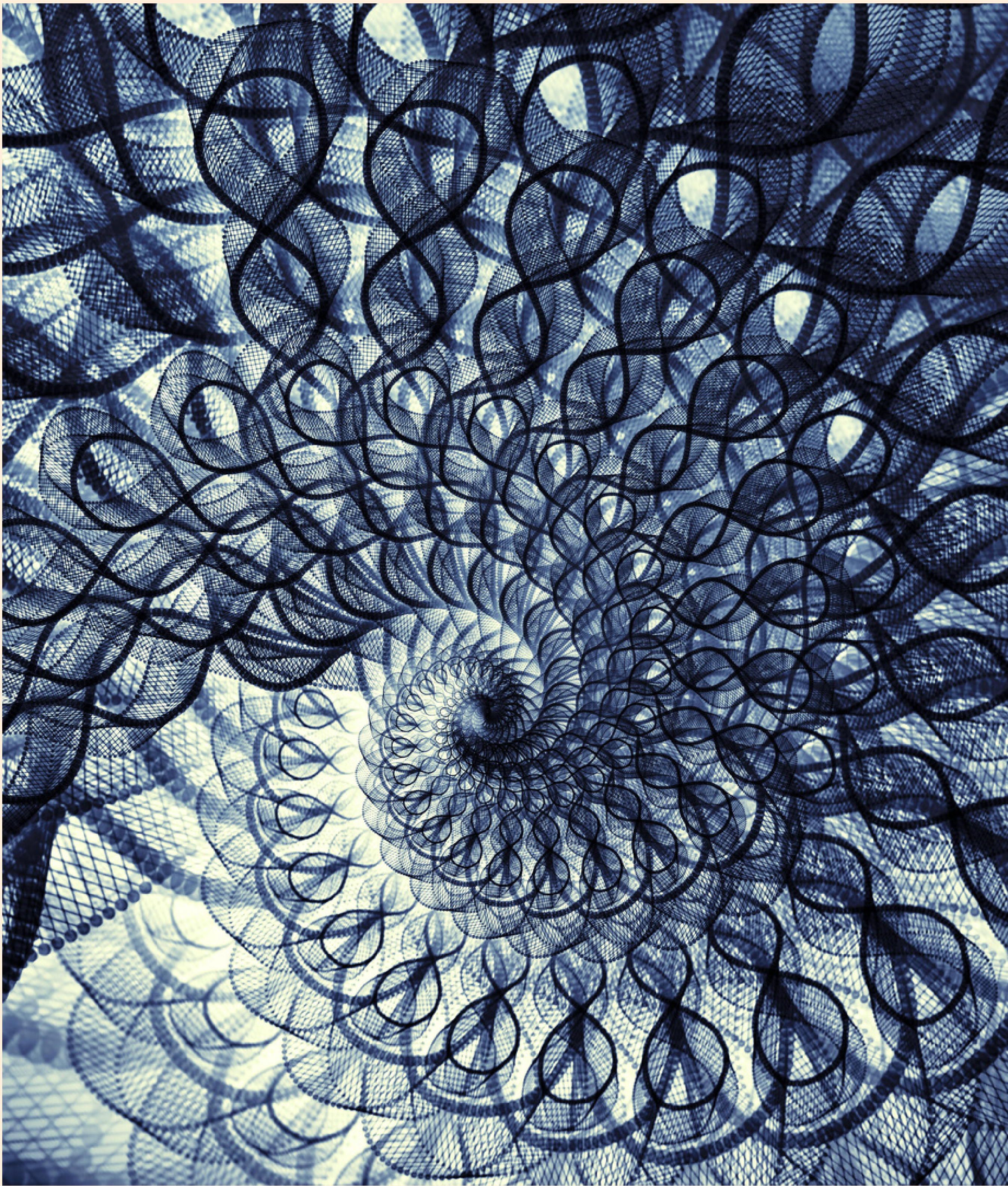

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On Characterization of Smarandache Curves Constructed by Modified Orthogonal Frame

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Abstract
In this study, we investigate Smarandache curves constructed by a space curve with a modified orthogonal frame. Firstly, the relations between the Frenet frame and the modified orthogonal frame are summarized. Later, the Smarandache curves based on the modified orthogonal frame are obtained. Finally, the tangent, normal, binormal vectors and the curvatures of the Smarandache curves are determined. A special curve known as the Gerono lemniscate curve whose curvature is not differentiable, the principal normal and binormal vectors are discontinuous at zero point is considered as an example and the Smarandache curves of this curve are obtained by the aid of its modified orthogonal frame, and their graphics are given.

Keywords: *Gerono lemniscate curve, Modified orthogonal frame, Smarandache curves*

AMS Subject Classification (2020): *53A04; 14H50*

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1. Introduction

Curve theory is one of the most important and interesting research topics of differential geometry. Many studies have been done about curves in the scientific world and the characterizations of curves have been examined by considering different spaces. Even in prehistoric times, curves seem to have an important place in the fields of art and decoration. Curves are used frequently in many related fields such as computer graphics, animation, and modeling. In this study, we investigated the Smarandache curves using the modified orthogonal frame to give a new perspective to curves. The Smarandache curves are characterized using different frames in Euclidean and non-Euclidean spaces [1–9]. The Smarandache curves obtained from spacelike Salkowski and anti-Salkowski curves are given by Eren and Şenyurt in Minkowski space [10–13]. Also, the Smarandache curves are characterized using the positional adapted frame by Özen et al. [14, 15]. However, the Serret-Frenet frame is insufficient at points where the curvature of the space curve is zero. Because at points where the curvature is zero, the principal normal and binormal vector of a space curve becomes discontinuous. Sasai has defined the modified orthogonal frame as an alternative to the Frenet frame to solve this problem [16]. Then, the modified orthogonal frame was defined by

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Bükçü and Karaca for the curvature and the torsion of non-zero space curves in Minkowski 3-space [17]. This study aims to investigate the geometric properties of the Smarandache curves according to the modified orthogonal frame. First of all, the equations of the Smarandache curves according to the modified orthogonal frame are obtained. Then, the graphs of the obtained Smarandache curves are drawn. Therefore, it is aimed to contribute to the world of science with the newly obtained curves.

2. Preliminaries

In Euclidean 3-space, Euclidean inner product is given by $<,>= d\alpha_1^2 + d\alpha_2^2 + d\alpha_3^2$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in E^3$. Norm of a vector $\alpha \in E^3$ is $\|\alpha\| = \sqrt{<\alpha, \alpha>}$. For any the space curve α, if $\|\alpha'(s)\| = 1$, then the curve α is unit speed curve in Euclidean 3-space. Let α be a moving space curve with respect to the arc-length s in Euclidean 3-space E^3. t, n, and b are tangent, principal normal, and binormal unit vectors at $\alpha(s)$ point of the curve α, respectively. Then, there exists an orthogonal frame $\{t, n, b\}$ which satisfies the Frenet-Serret equation

$$\begin{aligned} t' &= \kappa n, \\ n' &= -\kappa t + \tau b, \\ b' &= -\tau n \end{aligned}$$

where κ and τ are the curvature and the torsion of the space curve α, respectively. For the reason that the principal normal and binormal vectors in the Frenet frame of a space curve are discontinuous at the points where the curvature is zero, the modified orthogonal frame was introduced by Sasai as an alternative to the Frenet frame. In this sense, we assume that the curvature $\kappa(s)$ of the space curve α is not zero and then we define the modified orthogonal frame $\{T, N, B\}$ as follow:

$$T = \frac{d\alpha}{ds}, \quad N = \frac{dT}{ds}, \quad B = T \wedge N$$

where $T \wedge N$ is the vector product of T and N. The relations between the modified orthogonal frame $\{T, N, B\}$ and Serret-Frenet frame $\{t, n, b\}$ at non-zero points of κ are

$$T = t, \quad N = \kappa n, \quad B = \kappa b.$$

From these equations, it is known that the differentiation of the elements of the modified orthogonal frame $\{T, N, B\}$ satisfy

$$\begin{aligned} T'(s) &= N(s), \\ N'(s) &= -\kappa^2 T(s) + \frac{\kappa'}{\kappa} N(s) + \tau B(s), \\ B'(s) &= -\tau N(s) + \frac{\kappa'}{\kappa} B(s) \end{aligned}$$

where κ' denotes the differentiation of the curvature with respect to the arc-length parameter s and $\tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}$ is the torsion of the space curve α. Moreover, the modified orthogonal frame $\{T, N, B\}$ satisfies

$$\langle T, T\rangle = 1,\ \langle N, N\rangle = \langle B, B\rangle = \kappa^2,\ \langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0.$$

3. Smarandache curves constructed by modified orthogonal frame

In this section, we investigate the Smarandache curves according to the modified orthogonal frame $\{T, N, B\}$ in Euclidean 3-space. Let $\alpha = \alpha(s)$ be unit speed regular curve with arc-length parameter s.

Definition 3.1. Let α be a space curve with modified orthogonal frame $\{T, N, B\}$, then the Smarandache curve obtained from the unit vectors T and N of the curve α can be defined as

$$\beta_1(s^*) = \frac{1}{\sqrt{2}} (T(s) + N(s)) \tag{3.1}$$

such that s^* is the arc-length parameter of the Smarandache curve β_1.

Now, we investigate the Frenet apparatus of the Smarandache curve β_1 obtained from the curve α. Taking the differential of the equation (3.1) according to s, we get

$$\beta_1' = \frac{d\beta_1}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(T'(s) + N'(s)\right)$$

and

$$T_{\beta_1}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-\kappa^2 T + \left(1 + \frac{\kappa'}{\kappa}\right)N + \tau B\right)$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{\kappa^4 + \left(1 + \frac{\kappa'}{\kappa}\right)^2 + \tau^2}$$

or

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\rho_1, \quad \rho_1 = \sqrt{\kappa^4 + \left(1 + \frac{\kappa'}{\kappa}\right)^2 + \tau^2}. \tag{3.2}$$

So, the tangent vector of the Smarandache curve β_1 is written as follows:

$$T_{\beta_1} = \frac{\left(-\kappa^2 T + \left(1 + \frac{\kappa'}{\kappa}\right)N + \tau B\right)}{\rho_1}. \tag{3.3}$$

By differentiating the equation (3.3) with respect to s, we obtain

$$\frac{dT_{\beta_1}}{ds^*}\frac{ds^*}{ds} = \frac{\lambda_1 T + \eta_1 N + \mu_1 B}{\kappa {\rho_1}^2} \tag{3.4}$$

where

$$\begin{aligned}
\lambda_1 &= \kappa^2\left(-\rho_1\left(\kappa + 3\kappa'\right) + \kappa {\rho'}_1\right), \\
\eta_1 &= -\kappa^3\rho_1 - \kappa'{\rho'}_1 - \kappa\left(\rho_1\tau^2 + {\rho'}_1\right) + \rho_1\left(\kappa' + \kappa''\right), \\
\mu_1 &= 2\rho_1\tau\kappa' + \kappa\left(-\tau{\rho'}_1 + \rho_1\left(\tau + \tau'\right)\right).
\end{aligned}$$

Substituting the equation (3.2) into the equation (3.4), we get

$$T'_{\beta_1} = \frac{\sqrt{2}}{\kappa{\rho_1}^3}\left(\lambda_1 T + \eta_1 N + \mu_1 B\right).$$

Then, the curvature and the normal vector of the Smarandache curve β_1 are

$$\kappa_{\beta_1} = \|{T'}_{\beta_1}\| = \frac{\sqrt{2\left({\lambda_1}^2 + {\eta_1}^2 + {\mu_1}^2\right)}}{\kappa{\rho_1}^3}$$

and

$$N_{\beta_1} = \frac{\lambda_1 T + \eta_1 N + \mu_1 B}{\sqrt{{\lambda_1}^2 + {\eta_1}^2 + {\mu_1}^2}}, \tag{3.5}$$

respectively. From the equations (3.3) and (3.5), the binormal vector of the Smarandache curve β_1 is found as

$$B_{\beta_1} = \frac{1}{\rho_1 q_1}\left(\left(-\eta_1\tau + \mu_1\left(1 + \frac{\kappa'}{\kappa}\right)\right)T + \left(\lambda_1\tau + \mu_1\kappa^2\right)N - \left(\lambda_1\left(1 + \frac{\kappa'}{\kappa}\right) + \eta_1\kappa^2\right)B\right),$$

where $q_1 = \sqrt{{\lambda_1}^2 + {\eta_1}^2 + {\mu_1}^2}$. To calculate the torsion of the curve, we differentiate the curve β_1'

$$\beta_1'' = \frac{\vartheta_1 T + \sigma_1 N + \omega_1 B}{\sqrt{2}\kappa}$$

where

$$\begin{aligned}
\vartheta_1 &= -\kappa^2\left(\kappa + 3\kappa'\right), \\
\sigma_1 &= -\kappa\left(\kappa^2 + \tau^2\right) + \kappa' + \kappa'', \\
\omega_1 &= 2\tau\kappa' + \kappa\left(\tau + \tau'\right)
\end{aligned}$$

and similarly

$$\beta_1''' = \frac{1}{\sqrt{2}\kappa}\left(\varsigma_1 T + \xi_1 N + \zeta_1 B\right)$$

where

$$\begin{aligned}
\varsigma_1 &= \kappa\left(\kappa^4 + \kappa^2\tau^2 - 3\kappa'^2 - \kappa\left(3\kappa' + 4\kappa''\right)\right),\\
\xi_1 &= -\kappa^3\tau + \kappa'\left(2\tau + 3\tau'\right) + 3\tau\kappa'' + \kappa\left(-\tau^3 + \tau' + \tau''\right),\\
\zeta_1 &= -\left(\kappa^3 + 6\kappa^2\kappa' + 3\tau^2\kappa' + \kappa\tau\left(\tau + 3\tau'\right) - \kappa'' - \kappa^3\right).
\end{aligned}$$

As a result, we get the torsion of the Smarandache curve β_1 as follows:

$$\tau_{\beta_1} = \frac{\sqrt{2}\left(\left(\omega_1\left(1+\frac{\kappa'}{\kappa}\right) - \sigma_1\tau\right)\varsigma_1 + \left(\omega_1\kappa^2 - \vartheta_1\tau\right)\xi_1 - \left(\sigma_1\kappa^2 + \vartheta_1\left(1+\frac{\kappa'}{\kappa}\right)\right)\zeta_1\right)}{\left(\omega_1\left(1+\frac{\kappa'}{\kappa}\right) - \sigma_1\tau\right)^2 + \left(\omega_1\kappa^2 - \vartheta_1\tau\right)^2 + \left(\sigma_1\kappa^2 + \vartheta_1\left(1+\frac{\kappa'}{\kappa}\right)\right)^2}.$$

Definition 3.2. Let α be a space curve with modified orthogonal frame $\{T, N, B\}$, then the Smarandache curve obtained from the unit vectors T and B of the curve α can be defined as

$$\beta_2(s^*) = \frac{1}{\sqrt{2}}\left(T(s) + B(s)\right). \tag{3.6}$$

Here s^* is the arc-length parameter of the Smarandache curve β_2.

We research the Frenet apparatus of the Smarandache β_2 obtained from the curve α. Taking the differential of the equation (3.6) according to s, we get

$$\beta_2' = \frac{d\beta_2}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(T'(s) + B'(s)\right)$$

and

$$T_{\beta_2}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(\left(1-\tau\right)N + \frac{\kappa'}{\kappa}B\right)$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{\left(1-\tau\right)^2 + \left(\frac{\kappa'}{\kappa}\right)^2}$$

or

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\rho_2, \quad \rho_2 = \sqrt{\left(1-\tau\right)^2 + \left(\frac{\kappa'}{\kappa}\right)^2}. \tag{3.7}$$

So, the tangent vector of the Smarandache curve β_2 is written as follows

$$T_{\beta_2} = \frac{\left(\left(1-\tau\right)N + \frac{\kappa'}{\kappa}B\right)}{\rho_2}. \tag{3.8}$$

By differentiating the equation (3.8) with respect to s, we obtain

$$\frac{dT_{\beta_2}}{ds^*}\frac{ds^*}{ds} = \frac{\lambda_2 T + \eta_2 N + \mu_2 B}{\kappa{\rho_2}^2} \tag{3.9}$$

where

$$\begin{aligned}
\lambda_2 &= \rho_2\kappa^3(-1+\tau),\\
\eta_2 &= \rho_2(\kappa' - \kappa\tau') + \kappa(-1+\tau){\rho_2}',\\
\mu_2 &= \kappa'{\rho_2}' - \rho_2(\kappa(-1+\tau)\tau + \kappa''.
\end{aligned}$$

Substituting the equation (3.7) into the equation (3.9), we get

$$T_{\beta_2}' = \frac{\sqrt{2}}{\kappa{\rho_2}^3}\left(\lambda_2 T + \eta_2 N + \mu_2 B\right).$$

Then, the curvature and the normal vector of the Smarandache curve β_2 are

$$\kappa_{\beta_2} = \|T'_{\beta_2}\| = \frac{\sqrt{2\left({\lambda_2}^2 + {\eta_2}^2 + {\mu_2}^2\right)}}{\kappa {\rho_2}^3}$$

and

$$N_{\beta_2} = \frac{\lambda_2 T + \eta_2 N + \mu_2 B}{\sqrt{{\lambda_2}^2 + {\eta_2}^2 + {\mu_2}^2}}, \tag{3.10}$$

respectively. From the equations (3.8) and (3.10), the binormal vector of the Smarandache curve β_2 is found as

$$B_{\beta_2} = \frac{1}{\rho_2 q_2}\left(\left(-\eta_2 \frac{\kappa'}{\kappa} + \mu_2 (1-\tau)\right) T + \lambda_2 \frac{\kappa'}{\kappa} N + \lambda_2 (\tau - 1) B\right)$$

where $q_2 = \sqrt{{\lambda_2}^2 + {\eta_2}^2 + {\mu_2}^2}$. To calculate the torsion of the curve, we differentiate the curve β_2' and we get

$$\beta_2'' = \frac{\vartheta_2 T + \sigma_2 N + \omega_2 B}{\sqrt{2}\kappa}$$

where

$$\begin{aligned}
\vartheta_2 &= \kappa^3(-1+\tau),\\
\sigma_2 &= \kappa' - \kappa\tau',\\
\omega_2 &= \kappa(-1+\tau)\tau + \kappa''
\end{aligned}$$

and similarly

$$\beta_2''' = \frac{1}{\sqrt{2}\kappa}\left(\varsigma_2 T + \xi_2 N + \zeta_2 B\right)$$

where

$$\begin{aligned}
\varsigma_2 &= \kappa^2\kappa'(-3+2\tau) + 2\tau',\\
\xi_2 &= \kappa^3(-1+\tau) - \kappa'\tau' + (1+\tau)\kappa'' + \kappa\left((-1+\tau)\tau^2 - \tau''\right),\\
\zeta_2 &= \kappa\tau'(1-3\tau) + (-2+\tau)\tau\kappa' + \kappa''').
\end{aligned}$$

As a result, we get the torsion of the Smarandache curve β_2 as follows

$$\tau_{\beta_2} = \frac{\sqrt{2}\left(\varsigma_2\left(\omega_2(1-\tau) - \sigma_2\frac{\kappa'}{\kappa}\right) + \xi_2\vartheta_2\frac{\kappa'}{\kappa} + \zeta_2\vartheta_2(\tau-1)\right)}{\left(\omega_2(1-\tau) - \sigma_2\frac{\kappa'}{\kappa}\right)^2 + \left(\vartheta_2\frac{\kappa'}{\kappa}\right)^2 + \left(\vartheta_2(\tau-1)\right)^2}.$$

Definition 3.3. Let α be a space curve with modified orthogonal frame $\{T, N, B\}$, then the Smarandache curve obtained from the unit vectors N and B of the curve α can be defined as

$$\beta_3(s^*) = \frac{1}{\sqrt{2}}\left(N(s) + B(s)\right) \tag{3.11}$$

such that s^* is the arc-length parameter of the Smarandache curve β_3.

We investigate the Frenet apparatus of the Smarandache curve β_3 obtained from the curve α. Taking the differential of the equation (3.11) according to s, we get

$$\beta_3' = \frac{d\beta_3}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(N'(s) + B'(s)\right)$$

and

$$T_{\beta_3}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-\kappa^2 T + \left(\frac{\kappa'}{\kappa} - \tau\right) N + \left(\frac{\kappa'}{\kappa} + \tau\right) B\right)$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{\kappa^4 + 2\left(\left(\frac{\kappa'}{\kappa}\right)^2 + \tau^2\right)}$$

or

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\rho_3, \quad \rho_3 = \sqrt{\kappa^4 + 2\left(\left(\frac{\kappa'}{\kappa}\right)^2 + \tau^2\right)}. \tag{3.12}$$

So, the tangent vector of the Smarandache curve β_3 is written as follows:

$$T_{\beta_3} = \frac{\left(-\kappa^2 T + \left(\frac{\kappa'}{\kappa} - \tau\right) N + \left(\frac{\kappa'}{\kappa} + \tau\right) B\right)}{\rho_3}. \tag{3.13}$$

By differentiating the equation (3.13) with respect to s, we obtain

$$\frac{dT_{\beta_3}}{ds^*}\frac{ds^*}{ds} = \frac{\lambda_3 T + \eta_3 N + \mu_3 B}{\kappa {\rho_3}^2} \tag{3.14}$$

where

$$\begin{aligned}
\lambda_3 &= -3\kappa^2\rho_3\kappa' + \kappa^3\left({\rho_3}' + \tau\rho_3\right), \\
\eta_3 &= -2\kappa'\tau\rho_3 - {\rho_3}' + \kappa(-\rho_3(\tau^2 + \tau') + \tau{\rho_3}') - \rho_3\left(\kappa^3 + \kappa''\right), \\
\mu_3 &= \kappa'(2\tau\rho_3 - {\rho_3}') + \kappa(\rho_3(-\tau^2 + \tau') - \tau{\rho_3}') + \rho_3\kappa''.
\end{aligned}$$

Substituting the equation (3.12) into the equation (3.14), we get

$$T'_{\beta_3} = \frac{\sqrt{2}}{\kappa{\rho_3}^3}\left(\lambda_3 T + \eta_3 N + \mu_3 B\right).$$

Then, the curvature and the normal vector of the Smarandache curve β_3 are

$$\kappa_{\beta_3} = \|{T'}_{\beta_3}\| = \frac{\sqrt{2\left({\lambda_3}^2 + {\eta_3}^2 + {\mu_3}^2\right)}}{\kappa{\rho_3}^3}$$

and

$$N_{\beta_3} = \frac{\lambda_3 T + \eta_3 N + \mu_3 B}{\sqrt{{\lambda_3}^2 + {\eta_3}^2 + {\mu_3}^2}}, \tag{3.15}$$

respectively. From the equations (3.13) and (3.15), the binormal vector of the Smarandache curve β_3 is found as

$$B_{\beta_3} = \frac{1}{\rho_3 q_3}\left(\left(-\eta_3\left(\frac{\kappa'}{\kappa} + \tau\right) + \mu_3\left(\frac{\kappa'}{\kappa} - \tau\right)\right)T + \left(\lambda_3\left(\frac{\kappa'}{\kappa} + \tau\right) + \mu_3\kappa^2\right)N - \left(\lambda_3\left(\frac{\kappa'}{\kappa} - \tau\right) + \eta_3\kappa^2\right)B\right)$$

where $q_3 = \sqrt{{\lambda_3}^2 + {\eta_3}^2 + {\mu_3}^2}$. To calculate the torsion of the curve, we differentiate the equation of the curve β'_3

$$\beta''_3 = \frac{\vartheta_3 T + \sigma_3 N + \omega_3 B}{\sqrt{2}\kappa}$$

where

$$\begin{aligned}
\vartheta_3 &= \kappa^3\tau - 3\kappa^2\kappa' \\
\sigma_3 &= -\kappa^3 - 2\tau\kappa' - \kappa(\tau^2 + \tau') + \kappa'' \\
\omega_3 &= +2\tau\kappa' + \kappa(-\tau^2 + \tau') + \kappa''
\end{aligned}$$

and similarly

$$\beta'''_3 = \frac{1}{\sqrt{2}\kappa}\left(\varsigma_3 T + \xi_3 N + \zeta_3 B\right)$$

where

$$\begin{aligned}
\varsigma_3 &= \kappa^5 + \kappa^3(\tau^2 + 2\tau') + 4\kappa^2(\tau\kappa' - \kappa'') - 3\kappa{\kappa'}^2, \\
\xi_3 &= \kappa^3\tau - 6\kappa^2\kappa' + \kappa(\tau^3 - 3\tau\tau' - \tau'') + (-3\kappa'(\tau^2 + \tau') - 3\tau\kappa'' + \kappa'''), \\
\zeta_3 &= -\kappa^3\tau + 3\kappa'(-\tau^2 + \tau') + 3\tau\kappa'' + \kappa(-\tau^3 - 3\tau\tau' + \tau'') + \kappa'''.
\end{aligned}$$

As a result, we get the torsion of the Smarandache curve β_3 as follows

$$\tau_{\beta_3}=\frac{\sqrt{2}\left(\left(\omega_3\left(\frac{\kappa'}{\kappa}-\tau\right)+\sigma_3\left(\frac{\kappa'}{\kappa}+\tau\right)\right)\varsigma_3+\left(\omega_3\kappa^2-\vartheta_3\left(\frac{\kappa'}{\kappa}+\tau\right)\right)\xi_3-\left(\sigma_3\kappa^2+\vartheta_3\left(\frac{\kappa'}{\kappa}-\tau\right)\right)\zeta_3\right)}{\left(\omega_3\left(\frac{\kappa'}{\kappa}-\tau\right)+\sigma_3\left(\frac{\kappa'}{\kappa}+\tau\right)\right)^2+\left(\omega_3\kappa^2-\vartheta_3\left(\frac{\kappa'}{\kappa}+\tau\right)\right)^2+\left(\sigma_3\kappa^2+\vartheta_3\left(\frac{\kappa'}{\kappa}-\tau\right)\right)^2}.$$

Definition 3.4. Let α be a space curve with modified orthogonal frame $\{T,N,B\}$, then the Smarandache curve obtained from the unit vectors T, N and B of the curve α can be defined as

$$\beta_4(s^*)=\frac{1}{\sqrt{3}}\left(T(s)+N(s)+B\left(s\right)\right). \tag{3.16}$$

s^* is the arc-length parameter of the Smarandache curve β_4.

We investigate the Frenet apparatus of the Smarandache curve β_4 obtained from the curve α. Taking the differential of the equation (3.16) according to s, we get

$$\beta_4'=\frac{d\beta_4}{ds^*}\frac{ds^*}{ds}=\frac{1}{\sqrt{3}}\left(T'(s)+N'(s)+B'\left(s\right)\right)$$

and

$$T_{\beta_4}\frac{ds^*}{ds}=\frac{1}{\sqrt{3}}\left(-\kappa^2T+\left(\frac{\kappa'}{\kappa}-\tau+1\right)N+\left(\frac{\kappa'}{\kappa}+\tau\right)B\right)$$

where

$$\frac{ds^*}{ds}=\frac{1}{\sqrt{3}}\sqrt{\kappa^4+\left(\frac{\kappa'}{\kappa}-\tau+1\right)^2+\left(\frac{\kappa'}{\kappa}+\tau\right)^2}$$

or

$$\frac{ds^*}{ds}=\frac{1}{\sqrt{3}}\rho_4,\quad \rho_4=\sqrt{\kappa^4+\left(\frac{\kappa'}{\kappa}-\tau+1\right)^2+\left(\frac{\kappa'}{\kappa}+\tau\right)^2}. \tag{3.17}$$

So, the tangent vector of the Smarandache curve β_4 is written as follows:

$$T_{\beta_4}=\frac{\left(-\kappa^2T+\left(\frac{\kappa'}{\kappa}-\tau+1\right)N+\left(\frac{\kappa'}{\kappa}+\tau\right)B\right)}{\rho_4}. \tag{3.18}$$

By differentiating the equation (3.18) with respect to s, we obtain

$$\frac{dT_{\beta_4}}{ds^*}\frac{ds^*}{ds}=\frac{\lambda_4T+\eta_4N+\mu_4B}{\kappa{\rho_4}^2} \tag{3.19}$$

where

$$\begin{aligned}
\lambda_4&=-3\kappa^2\rho_4\kappa'+\kappa^3((-1+\tau)\rho_4+{\rho_4}'),\\
\eta_4&=\kappa'((1-2\tau)\rho_4-{\rho_4}')+\kappa(-\rho_4(\tau^2+\tau')-(1-\tau){\rho_4}')-\rho_4\left(\kappa^3-\kappa''\right),\\
\mu_4&=\kappa'(2\tau\rho_4-{\rho_4}')+\kappa(\rho_4(\tau(1-\tau)+\tau')-\tau{\rho_4}')+\rho_4\kappa''.
\end{aligned}$$

Substituting the equation (3.17) into the equation (3.19), we get

$$T'_{\beta_4}=\frac{\sqrt{3}}{\kappa{\rho_4}^3}\left(\lambda_4T+\eta_4N+\mu_4B\right).$$

Then, the curvature and the normal vector of the Smarandache curve β_4 are

$$\kappa_{\beta_4}=\|{T'}_{\beta_4}\|=\frac{\sqrt{3\left({\lambda_4}^2+{\eta_4}^2+{\mu_4}^2\right)}}{\kappa{\rho_4}^3}$$

and

$$N_{\beta_4}=\frac{\lambda_4T+\eta_4N+\mu_4B}{\sqrt{{\lambda_4}^2+{\eta_4}^2+{\mu_4}^2}}, \tag{3.20}$$

respectively. From the equations (3.18) and (3.20), the binormal vector of the Smarandache curve β_4 is found as

$$B_{\beta_4} = \frac{1}{\rho_4 q_4}\left(\begin{array}{l}\left(\left(-\eta_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)\right)T+\left(\lambda_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\kappa^2\right)N\\ -\left(\lambda_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)+\eta_4\kappa^2\right)B\end{array}\right)$$

where $q_4 = \sqrt{{\lambda_4}^2+{\eta_4}^2+{\mu_4}^2}$. To calculate the torsion of the curve, we differentiate the curve β_4'

$$\beta_4'' = \frac{\vartheta_4 T + \sigma_4 N + \omega_4 B}{\sqrt{3}\kappa}$$

where

$$\begin{aligned}\vartheta_4 &= \kappa^3(\tau-1)-3\kappa^2\kappa',\\ \sigma_4 &= -\kappa^3+\kappa'(1-2\tau)-\kappa(\tau^2+\tau')+\kappa'',\\ \omega_4 &= 2\tau\kappa'+\kappa(\tau(1-\tau)+\tau')+\kappa''\end{aligned}$$

and similarly

$$\beta_4''' = \frac{1}{\sqrt{3}\kappa}\left(\varsigma_4 T + \xi_4 N + \zeta_4 B\right)$$

where

$$\begin{aligned}\varsigma_4 &= \kappa^5+\kappa^3(\tau^2+2\tau')+\kappa^2((-3+4\tau)\kappa'-4\kappa'')-3\kappa{\kappa'}^2,\\ \xi_4 &= \kappa^3(-1+\tau)-6\kappa^2\kappa'+\kappa(\tau((-1+\tau)\tau-3\tau')-\tau'')+(-3\kappa'(\tau^2+\tau')+\kappa''-3\tau\kappa''+\kappa'''),\\ \zeta_4 &= -\kappa^3\tau+(-\tau^3+\tau'-3\tau\tau'+\tau'')+\kappa'((2-3\tau)\tau+3\tau')+3\tau\kappa''+\kappa'''.\end{aligned}$$

As a result, we get the torsion of the Smarandache curve β_4 as follows:

$$\tau_{\beta_4} = \frac{\sqrt{3}\left(\left(-\eta_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)\right)\varsigma_4+\left(\lambda_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\kappa^2\right)\xi_4-\left(\lambda_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)+\eta_4\kappa^2\right)\zeta_4\right)}{\left(-\eta_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)\right)^2+\left(\lambda_4\left(\frac{\kappa'}{\kappa}+\tau\right)+\mu_4\kappa^2\right)^2+\left(\lambda_4\left(\frac{\kappa'}{\kappa}-\tau+1\right)+\eta_4\kappa^2\right)^2}.$$

Example 3.1. Let's plot the graphics of the Smarandache curves based on the modified orthogonal frame of the eight curve which is known as Gerono lemniscate curve [18]. The parametric equation of this curve is given by

$$\alpha(s) = (\sin(s), \sin(s)\cos(s), s).$$

The elements of the Frenet trihedron of the curve $\alpha(s)$ are obtained as

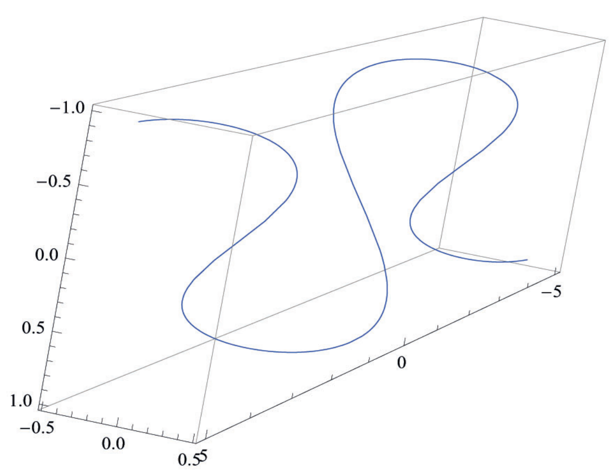


Figure 1. The Gerono lemniscate curve

$$t(s)=\frac{\left(\sqrt{2}\cos(s),\sqrt{2}\cos(2s),\sqrt{2}\right)}{\sqrt{4+\cos(2s)+\cos(4s)}},$$
$$n(s)=\frac{\left((-1+4\cos(2s)+\cos(4s))\sin(s),-\sin(2s)(6+\cos(2s)),\sin(2s)+2\sin(4s)\right)}{\sqrt{4+\cos(2s)+\cos(4s)}\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}},$$
$$b(s)=\frac{(4\sin(2s),-2\sin(s),-3\sin(s)-\sin(3s))}{\sqrt{2}\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}.$$

The curvature of the curve $\alpha(s)$ is found as

$$\kappa(s)=\frac{2\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}{(4+\cos(2s)+\cos(4s))^{3/2}}.$$

Besides the curvature $\kappa(s)=\frac{2\sqrt{(27+24\cos(2s)+\cos(4s))\sin(s)^2}}{(4+\cos(2s)+\cos(4s))^{3/2}}$ is not differentiable, the principal normal and binormal vectors are discontinuous at $s=0$ since $n_+\neq n_-$ and $b_+\neq b_-$ for $n_+=\lim\limits_{s\to0^+}n(s)$, $n_-=\lim\limits_{s\to0^-}n(s)$ and $b_+=\lim\limits_{s\to0^+}b(s)$, $b_-=\lim\limits_{s\to0^-}b(s)$. Looking for a solution to this problem, Sasai has defined the modified orthogonal frame as an alternative to the Frenet frame. The elements of the modified orthogonal frame of the curve $\alpha(s)$ are obtained as

$$T(s)=\frac{\sqrt{2}(\cos(s),\cos(2s),1)}{\sqrt{4+\cos(2s)+\cos(4s)}},$$
$$N(s)=\frac{2\left(\sin(s)(-1+4\cos(2s)+\cos(4s)),-\sin(2s)(6+\cos(2s)),(\sin(2s)+2\sin(4s))\right)}{(4+\cos(2s)+\cos(4s))^2},$$
$$B(s)=\frac{\sqrt{2}\left(4\sin(2s),-2\sin(s),-(3\sin(s)+\sin(3s))\right)}{(4+\cos(2s)+\cos(4s))^{3/2}}.$$

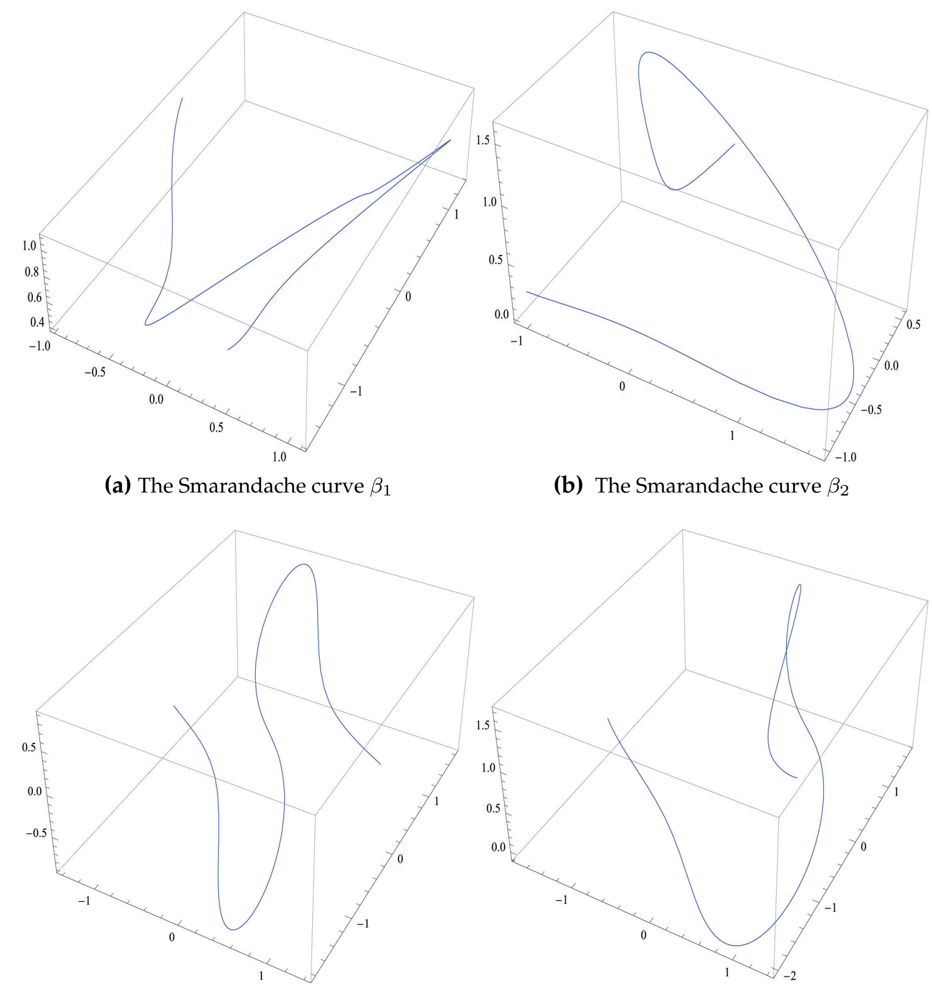
The Smarandache curves β_1,β_2,β_3 and β_4 obtained from the curve α are given as

$$\beta_1=\begin{pmatrix}\frac{\sqrt{2}\cos(s)}{\sqrt{4+\cos(2s)+\cos(4s)}}+\frac{-6\sin(s)+3\sin(3s)+\sin(5s)}{(4+\cos(2s)+\cos(4s))^2},\frac{\sqrt{2}\cos(2s)}{\sqrt{4+\cos(2s)+\cos(4s)}}\\-\frac{2\sin(2s)(6+\cos(2s))}{(4+\cos(2s)+\cos(4s))^2},\frac{\sqrt{2}}{\sqrt{4+\cos(2s)+\cos(4s)}}+\frac{2(\sin(2s)+2\sin(4s))}{(4+\cos(2s)+\cos(4s))^2}\end{pmatrix},$$

$$\beta_2=\begin{pmatrix}\frac{9\cos(s)+2\cos(3s)+\cos(5s)+8\sin(2s)}{\sqrt{2}(4+\cos(2s)+\cos(4s))^{3/2}},\frac{1+9\cos(2s)+\cos(4s)+\cos(6s)-4\sin(s)}{\sqrt{2}(4+\cos(2s)+\cos(4s))^{3/2}},\\\frac{2(4+\cos(2s)+\cos(4s)-3\sin(s)-\sin(3s))}{\sqrt{2}(4+\cos(2s)+\cos(4s))^{3/2}}\end{pmatrix},$$

$$\beta_3=\begin{pmatrix}\frac{2\sin(s)(4\cos(2s)+\cos(4s)-1)}{(4+\cos(2s)+\cos(4s))^2}+\frac{4\sqrt{2}\sin(2s)}{(4+\cos(2s)+\cos(4s))^{3/2}},\\\frac{-2\sin(s)(13\cos(s)+\cos(3s))}{(4+\cos(2s)+\cos(4s))^2}+\frac{2\sqrt{2}\sin(s)}{(4+\cos(2s)+\cos(4s))^{3/2}},\\\frac{2(\sin(2s)+2\sin(4s))}{(4+\cos(2s)+\cos(4s))^2}-\frac{\sqrt{2}(3\sin(s)+\sin(3s))}{(4+\cos(2s)+\cos(4s))^{3/2}}\end{pmatrix},$$

$$\beta_4 = \begin{pmatrix} \dfrac{\sqrt{2}\cos(s)}{\sqrt{4+\cos(2s)+\cos(4s)}} + \dfrac{2\sin(s)\,(4\cos(2s)+\cos(4s)-1)}{(4+\cos(2s)+\cos(4s))^2} + \dfrac{4\sqrt{2}\sin(2s)}{(4+\cos(2s)+\cos(4s))^{3/2}}, \\ \dfrac{\sqrt{2}\cos(2s)}{\sqrt{4+\cos(2s)+\cos(4s)}} - \dfrac{2\sin(s)\,(13\cos(s)+\cos(3s))}{(4+\cos(2s)+\cos(4s))^2} - \dfrac{2\sqrt{2}\sin(s)}{(4+\cos(2s)+\cos(4s))^{3/2}}, \\ \dfrac{\sqrt{2}}{\sqrt{4+\cos(2s)+\cos(4s)}} + \dfrac{2\,(\sin(2s)+2\sin(4s))}{(4+\cos(2s)+\cos(4s))^2} - \dfrac{\sqrt{2}\,(3\sin(s)+\sin(3s))}{(4+\cos(2s)+\cos(4s))^{3/2}} \end{pmatrix}.$$



(a) The Smarandache curve β_1 **(b)** The Smarandache curve β_2

(c) The Smarandache curve β_3 **(d)** The Smarandache curve β_4

Figure 2. The Smarandache curves for $s \in [-2, 2]$

4. Conclusion

In this paper, we investigate the geometric properties of the Smarandache curves with respect to the modified orthogonal frame. Sasai presented the modified orthogonal frame as an alternative to the Frenet frame. Because the principal normal and binormal vectors of the Frenet frame of a space curve become discontinuous at the points where the curvature is zero, However, the Smarandache curves have not been examined under these conditions yet. For this reason, this research is a new study in the geometry field.

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Rings Whose Certain Modules are Dual Self-CS-Baer

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Abstract
In this work, we characterize some rings in terms of dual self-CS-Baer modules (briefly, ds-CS-Baer modules). We prove that any ring R is a left and right artinian serial ring with $J^2(R) = 0$ iff $R \oplus M$ is ds-CS-Baer for every right R-module M. If R is a commutative ring, then we prove that R is an artinian serial ring iff R is perfect and every R-module is a direct sum of ds-CS-Baer R-modules. Also, we show that R is a right perfect ring iff all countably generated free right R-modules are ds-CS-Baer.

Keywords: *Dual self-CS-Baer module, Harada ring, Lifting module, Perfect ring, QF-ring, Serial ring*

AMS Subject Classification (2020): *16D10;16L30*

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1. Introduction

Throughout the paper, all rings will have an identity element and all modules will be unitary right modules unless otherwise stated. Let M be a module and N a submodule of M. Then $N \ll M$ means that N is a small submodule of M (namely, M is different from $N + K$ for every proper submodule K of M). $J(R)$ will denote the Jacobson radical of any ring R and $Rad(M)$ will denote the radical of any module M.

A module M is called *lifting* (or *satisfies* (D_1)), if every submodule N of M lies above a direct summand, that is, N contains a direct summand X of M such that $N/X \ll M/X$ (see [1] and [2]). A module M is said to be *dual self-CS-Baer* (briefly, *ds-CS-Baer*) if for every family $(f_i)_{i \in I}$ of homomorphisms $f_i : M \to M$, $\sum_{i \in I} Im(f_i)$ lies above a direct summand of M (see [3]). Clearly, every lifting module is ds-CS-Baer. Moreover, if R is a right Harada ring, then every injective right R-module is ds-CS-Baer. Because, remember that any ring R is called a right *Harada* ring if every injective right R-module is lifting (see [1]). Recall that any right R-module M is called *hollow*, if every proper submodule of M is small in M (see [2, Definition 4.1]) and it is called *local*, if it is hollow and $Rad(M) \neq M$. Note that M is local iff M is cyclic and has a unique maximal submodule (see [4, page 357]). It is not hard to see that every hollow module and so every local module is a lifting module.

In recent years, ds-CS-Baer modules and their related topics have been studied by Crivei, Keskin Tütüncü, Radu and Tribak (see for example [3], [5] and [6]). In this paper, we continue the study of ds-CS-Baer modules.

In section 2, we characterize some rings in terms of ds-CS-Baer modules. Among others, we mainly prove the followings:

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(A) Let R be a ring. Then R is an artinian serial ring with $J^2(R) = 0$ iff for every right R-module M, $R \oplus M$ is ds-CS-Baer (Theorem 2.1).

(B) Let R be a right self-injective ring. Then R is a QF-ring iff every injective right R-module is ds-CS-Baer (Theorem 2.3).

(C) Let R be a ring. Then R is a right perfect ring iff every free right R-module is ds-CS-Baer (Theorem 2.4).

(D) Let R be a commutative ring. Then R is semiperfect iff every cyclic R-module is ds-CS-Baer (Proposition 2.1).

(E) Let R be a commutative ring. Then R is an artinian serial ring iff R is perfect and every 2-f.p. R-module is a finite direct sum of ds-CS-Baer modules (Proposition 2.4).

2. Results

We first give the following easy observation.

Lemma 2.1. *Let R be a ring. Let M be a free right R-module. Then M is lifting iff it is ds-CS-Baer.*

Proof. Let M be a free right R-module. Then we can assume that $M = \oplus_{i \in I} R$. Now the result is obvious by the proof of [3, Proposition 9.4]. □

Let R be ring and M a module. M is called *uniserial* if its submodules are linearly ordered by inclusion and is called *serial* if it is a direct sum of uniserial submodules. The ring R is called *right* (left) *serial* if the right (left) R-module R_R (${}_RR$) is serial. Also R is called *artinian serial* if it is both right and left artinian serial. By [4, Theorem 32.3], we know that if R is an artinian serial ring, then every right R-module and every left R-module is a direct sum of uniserial R-modules.

Now, we characterize artinian serial rings with $J^2(R) = 0$ via ds-CS-Baer modules.

Theorem 2.1. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is an artinian serial ring with $J^2(R) = 0$.*

(2) *Every right R-module is lifting.*

(3) *For every right R-module M, $R \oplus M$ is lifting.*

(4) *For every right R-module M, $R \oplus M$ is ds-CS-Baer.*

Proof. (1) ⇔ (2): It is satisfied by [1, 29.10].
(3) ⇔ (4): It is proved in [3, Proposition 9.4].
(2) ⇒ (3): It is clear.
(3) ⇒ (2): It is clear since lifting property is preserved by direct summands (see for example [1, Lemma 22.6]). □

The next result is a consequence of Theorem 2.1.

Corollary 2.1. *Let R be a ring. Then R is an artinian serial ring with $J^2(R) = 0$ iff every (finitely generated) right R-module is ds-CS-Baer.*

Proof. This follows from [7, Theorem 3.15], [3, Proposition 9.4] and Theorem 2.1 and the fact that being ds-CS-Baer or lifting is preserved by taking direct summands.

□

Remark 2.1. The left-handed versions of Theorem 2.1 and Corollary 2.1 are equal to being artinian serial ring with $J^2(R) = 0$.

A finitely generated right R-module M is said to be *finitely presented* in case in every exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F finitely generated and free the kernel K is also finitely generated. An exact sequence of right R-modules

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

is called a *minimal projective presentation* of M in case P_1 and P_0 are finitely generated projective and $\mathrm{Ker} f \ll P_1$ and $\mathrm{Im} f \ll P_0$. Let M a finitely presented right R-module with no nonzero projective direct summands. Following [4], M is called a *2-f.p. module* if there are primitive idempotents e, e_1 and e_2 of R and there is a minimal projective presentation

$$eR \longrightarrow e_1R \oplus e_2R \longrightarrow M \longrightarrow 0.$$

Therefore a 2-f.p. module is both 2-primitive generated and finitely presented.

Recall from [8] that a module M is called w-local if it has a unique maximal submodule. Clearly, a module M is local if and only if M is a cyclic w-local module.

Next, we can give the following.

Theorem 2.2. *Let R be a ring. Consider the following statements:*

(1) *R is serial and every direct sum of two ds-CS-Baer right R-modules and every direct sum of two ds-CS-Baer left R-modules is ds-CS-Baer.*

(2) *Every finitely presented right R-module and finitely presented left R-module is ds-CS-Baer.*

(3) *Every 2-generated finitely presented right R-module and 2-f.p. left R-module is ds-CS-Baer.*

(4) *R is semiperfect and every 2-f.p. right R-module and 2-f.p. left R-module is ds-CS-Baer.*

Then (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof. (1) ⇒ (2): Let M be a finitely presented right R-module and N a finitely presented left R-module. By [9, Corollary 3.4], M and N are finite direct sum of cyclic w-local submodules. In particular, they are finite direct sum of local submodules. Since local modules are lifting, they are also ds-CS-Baer. Therefore M and N are ds-CS-Baer by (1).

(2) ⇒ (3) ⇒ (4): These are clear by definitions and [3, Proposition 5.9]. □

Inspired by Theorem 2.1, we give the following theorem that characterizes QF-rings. First, remember that any ring R is called a *QF-ring*, if R is noetherian and injective as a left (or right) R-module (see for example [4, page 333]).

Theorem 2.3. *Let R be a right self-injective ring. Then the following assertions are equivalent:*

(1) *R is a QF-ring.*

(2) *R is a right Harada ring.*

(3) *For every injective right R-module M, $R \oplus M$ is lifting.*

(4) *For every injective right R-module M, $R \oplus M$ is ds-CS-Baer.*

(5) *Every injective right R-module is ds-CS-Baer.*

Proof. (1) ⇔ (2): It is clear by [1, 28.10 and 28.16].

(3) ⇔ (4): It is clear by [3, Proposition 9.4].

(2) ⇒ (3): Let M be an injective right R-module. By hypothesis, $R \oplus M$ is an injective right R-module. Since R is right Harada, it follows that $R \oplus M$ is lifting.

(3) ⇒ (2): Let M be an injective right R-module. By (3), $R \oplus M$ is lifting. Therefore, M is lifting. Hence, R is a right Harada ring.

(4) ⇔ (5): It is clear. □

In the following, we characterize right perfect rings in terms of ds-CS-Baer modules. Firstly, remember that any module M is called *$\oplus$-supplemented*, if for every submodule N of M there exists a direct summand K of M with $M = N + K$ and $N \cap K$ small in K. This notion is a generalization of lifting modules (see [2]).

Theorem 2.4. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is a right perfect ring.*

(2) *$R^{(\mathbb{N})}$ is a ds-CS-Baer right R-module.*

(3) *Every countably generated free right R-module is ds-CS-Baer.*

(4) *Every free right R-module is ds-CS-Baer.*

Proof. (1) $\Rightarrow$ (2): Assume that R is a right perfect ring. Consider the right R-module $M = R^{(\mathbb{N})}$. By [2, Theorem 4.41], M is lifting, and so it is ds-CS-Baer by definitions.

(2) $\Rightarrow$ (1): Assume that the right R-module $R^{(\mathbb{N})}$ is ds-CS-Baer. Since it is free, by Lemma 2.1, it is lifting. Hence it is $\oplus$-supplemented. Therefore, R is a right perfect ring by [7, Theorem 2.10].

(1) $\Rightarrow$ (4): Let M be a free right R-module. Then M is projective. So, M is lifting by [2, Theorem 4.41]. Thus, M is ds-CS-Baer by definitions.

(4) $\Rightarrow$ (1): Assume that every free right R-module is ds-CS-Baer. Then every free right R-module is lifting by Lemma 2.1. By [2, Theorem 4.41], R is a right perfect ring.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2): These are clear. □

Next, we give a characterization of commutative semiperfect rings in terms of cyclic dual self-CS-Baer modules.

Proposition 2.1. *Let R be a commutative ring. Then R is semiperfect iff every cyclic R-module is ds-CS-Baer.*

Proof. Let R be a semiperfect ring. Let M be a cyclic R-module. Assume that $M = xR$, where $x \in M$. We know that $M \cong R/I$, for some ideal I of R. By [1, 4.9 (1)], since I is fully invariant in R, R/I is quasi-projective and hence M is quasi-projective. Then by [2, Theorem 4.41], M is lifting and so M is ds-CS-Baer.

Conversely, assume that every cyclic R-module is ds-CS-Baer. Then R is a ds-CS-Baer R-module. Therefore by [3, Proposition 5.9], R is semiperfect. □

Now, we give a characterization of commutative semiperfect FGC-rings. Let R be a commutative ring. R is called an *FGC-ring*, if every finitely generated R-module is a direct sum of cyclic modules (see [10]).

Proposition 2.2. *Let R be a commutative ring. Then the following assertions are equivalent:*

(1) *Every finitely generated R-module is $\oplus$-supplemented.*

(2) *Every finitely generated R-module is a finite direct sum of ds-CS-Baer modules.*

(3) *R is a semiperfect FGC-ring.*

(4) *R is a direct sum of almost maximal valuation rings.*

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): These are proved in [7, Proposition 2.8].

(1) $\Rightarrow$ (2): Let M be a finitely generated R-module. By (1), M is $\oplus$-supplemented. By [7, Corollary 2.6], $M = \oplus_{i=1}^{n} x_i R$. Note that each $x_i R$ is quasi-projective since R is commutative. Therefore by [2, Theorem 4.41], each $x_i R$ is lifting and so ds-CS-Baer.

(2) $\Rightarrow$ (1): Let M be a finitely generated R-module. By (2), $M = \oplus_{i=1}^{n} x_i R$, where each $x_i R$ is ds-CS-Baer. By [3, Proposition 5.12], each $x_i R$ is lifting and hence $\oplus$-supplemented. Therefore by [11, Theorem 1.4], M is $\oplus$-supplemented. □

Corollary 2.2. *Let R be a commutative indecomposable ring. Then R is an almost maximal valuation ring iff every finitely generated R-module is a direct sum of cyclic ds-CS-Baer R-modules.*

Next, we characterize commutative serial rings via direct sums of cyclic ds-CS-Baer modules.

Proposition 2.3. *Let R be a commutative ring. Then the following assertions are equivalent:*

(1) *R is serial.*

(2) *R is semiperfect and every 2.f.p. R-module is $\oplus$-supplemented.*

(3) *R is semiperfect and every finitely presented R-module is a finite direct sum of ds-CS-Baer modules.*

(4) *R is semiperfect and every 2-generated finitely presented R-module is a finite direct sum of ds-CS-Baer modules.*

(5) *R is semiperfect and every 2-f.p. R-module is a finite direct sum of ds-CS-Baer modules.*

Proof. (1) $\Leftrightarrow$ (2): This follows from [7, Theorem 3.5].

(1) $\Rightarrow$ (3): Clearly, R is semiperfect. Now, let M be a finitely presented R-module. Note that M is finitely generated. By [9, Corollary 3.4], $M = \oplus_{i=1}^{n} M_i$, where each M_i is w-local and cyclic. Note that each M_i $(1 \leq i \leq n)$ is a local module. Hence each M_i is ds-CS-Baer.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5): These are clear.

(5) $\Rightarrow$ (2): Let M be a 2-f.p. R-module. By (5), $M = \oplus_{i=1}^{n} M_i$, where each M_i is a cyclic ds-CS-Baer R-module. By [3, Proposition 5.12], each M_i is lifting and hence $\oplus$-supplemented. Hence M is $\oplus$-supplemented by [11, Theorem 1.4]. □

Finally, we characterize commutative artinian serial rings as follows.

Proposition 2.4. *Let R be a commutative ring. Then the following assertions are equivalent:*

(1) *R is an artinian serial ring.*

(2) *R is perfect and every 2-f.p. R-module is $\oplus$-supplemented.*

(3) *R is perfect and every R-module is a direct sum of ds-CS-Baer modules.*

(4) *R is perfect and every countably generated R-module is a direct sum of ds-CS-Baer modules.*

(5) *R is perfect and every finitely presented R-module is a finite direct sum of ds-CS-Baer modules.*

(6) *R is perfect and every 2-f.p. R-module is a finite direct sum of ds-CS-Baer modules.*

Proof. (1) $\Leftrightarrow$ (2): It is proved in [7, Corollary 3.13].

(1) $\Rightarrow$ (3): By [4, Corollary 28.8], R is a perfect ring. Now, let M be any R-module. By [4, Theorem 32.3], $M = \oplus_{i \in I} M_i$, where each M_i is uniserial. Clearly every uniserial module is hollow. Since R is perfect, then each M_i has small radical (see [4, Remark 28.5]). Therefore, each M_i is local, and so cyclic. Hence M is a direct sum of cyclic ds-CS-Baer modules.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6): These are clear.

(6) $\Rightarrow$ (2): Let M be a 2-f.p. R-module. By (6), $M = \oplus_{i=1}^{n} M_i$, where each M_i is a cyclic ds-CS-Baer R-module. By [3, Proposition 5.12], each M_i is lifting and hence $\oplus$-supplemented. Therefore M is $\oplus$-supplemented by [11, Theorem 1.4]. □

Propositions 2.3 and 2.4 are not true over noncommutative rings as we see in the following example.

Example 2.1. (see [7, Example 3.16]) Let R be a local artinian ring with Jacobson radical $J(R)$ such that $J^2(R) = 0, Q = R/J(R)$ is commutative, $\dim({}_Q J(R)) = 1$ and $\dim(J(R)_Q) = 2$. Then R is left serial but not right serial. Let $J(R) = uR \oplus vR$. $A_1 = R/J(R), A_2 = R/uR$ and $A_3 = R_R$ are the only three isomorphism types of indecomposable right R-modules. Here each A_i is lifting and hence ds-CS-Baer. Note that every right R-module is a direct sum of indecomposable modules, and hence a direct sum of cyclic ds-CS-Baer modules. However, R is not a serial ring.

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Investigation of the Spectrum of Nonself-Adjoint Discontinuous Sturm-Liouville Operator

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Abstract
In this paper, we study nonself-adjoint Sturm-Liouville operator containing both the discontinuous coefficient and discontinuity conditions at some point on the positive half-line. The eigenvalues and the spectral singularities of this problem are examined and it is proved that this problem has a finite number of spectral singularities and eigenvalues with finite multiplicities under two different additional conditions. Furthermore, the principal functions corresponding to the eigenvalues and the spectral singularities of this operator are determined.

Keywords: *Discontinuous coefficient, Discontinuity conditions, Eigenvalues and spectral singularities, Nonself-adjoint Sturm-Liouville operator, Principal functions*

AMS Subject Classification (2020): *34B24; 34L05; 47A10*

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1. Introduction

The development of discontinuous boundary value problems has been great interest recently. It has an important role and making progress in the different field of mathematics and engineering such as mechanics, mathematical physics, geophysics (see [1–4]) and etc. Therefore, discontinuous Sturm-Liouville problems have attracted attention and numerous studies have been done on this subject. The difference between this study from others is that the nonself-adjoint discontinuous Sturm-Liouville problem which includes both a discontinuous coefficient and the discontinuity conditions at the point on the positive half line is investigated. Namely, we take into account the following nonself-adjoint problem created by the Sturm-Liouville equation with discontinuous coefficient

$$\ell(\varphi) = -\varphi'' + q(\xi)\varphi = \mu^2 \rho(\xi)\varphi, \quad \xi \in (0, a) \cup (a, \infty), \tag{1.1}$$

with the discontinuity conditions

$$\varphi(a-0) = \alpha\varphi(a+0), \qquad \varphi'(a-0) = \alpha^{-1}\varphi'(a+0) \tag{1.2}$$

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and boundary condition

$$\varphi(0) = 0, \tag{1.3}$$

where $0 < \alpha \neq 1$, μ is a complex parameter, $\rho(\xi)$ is the piecewise continuous functions

$$\rho(\xi) = \begin{cases} \beta^2, & 0 < \xi < a, \\ 1, & a < \xi < \infty \end{cases}$$

with $0 < \beta \neq 1$, $q(\xi)$ is a complex-valued function and satisfies the condition

$$\int_0^\infty \xi |q(\xi)| d\xi < \infty. \tag{1.4}$$

The spectral theory of nonself-adjoint operator in the classical case (i.e., $\rho(\xi) \equiv 1$ and $\alpha = 1$) was studied by Naimark [5, 6]. He shows that some poles of the resolvent kernel are not the eigenvalues of the operator and belong to the continuous spectrum; moreover, these poles are called spectral singularities and were first introduced by Schwartz [7]. In the self-adjoint case, the operator has a finite number of eigenvalues under the condition (1.4) (see [8]); however, in the nonself-adjoint case, the operator has a finite number of eigenvalues under the additional restriction. For example, the condition

$$\sup_{0 \leq \xi < \infty} \{|q(\xi)| \exp(\varepsilon \xi)\} < \infty, \quad \varepsilon > 0$$

was introduced by Naimark (see [5]) and it is shown that the number of eigenvalues is finite under this condition. Then, Pavlov weakened this additional condition as follows (see [9]):

$$\sup_{0 \leq \xi < \infty} \left\{|q(\xi)| \exp(\varepsilon \sqrt{\xi})\right\} < \infty, \quad \varepsilon > 0$$

and demonstrates that the operator has a finite number of eigenvalues. Moreover, Adıvar and Akbulut [10] obtain that the operator has a finite number of the eigenvalues under the following additional condition:

$$\sup_{0 \leq \xi < \infty} \left\{|q(\xi)| \exp\left(\varepsilon \xi^\delta\right)\right\} < \infty, \quad \varepsilon > 0, \ \frac{1}{2} \leq \delta < 1.$$

Note that for any $0 < \delta < \frac{1}{2}$, the condition does not provide that the number of eigenvalues is finite (see [11]). The spectral singularities have an essential role in the spectral analysis of the nonself-adjoint operator and Lyantse [12, 13] investigated the influence of the spectral singularities in the spectral expansion with respect to the principal functions of the operator. The investigations on the spectrum, principal functions and the spectral expansion with respect to the principal functions of the nonself-adjoint operator are very attractive and there are many works on the nonself-adjoint operator under different boundary conditions (see [14–22] and the references therein). Moreover, the nonself-adjoint operator with discontinuous coefficient is studied in [10], some spectral properties of the impulsive Sturm-Liouville operator is worked in [23].

To purpose of this study is to investigate the spectrum and the principal functions of the nonself-adjoint discontinuous problem (1.1)-(1.3). In examining this problem, we use new Jost solution of the equation (1.1) with discontinuity condition (1.2). The presence of the discontinuous parameter $\rho(\xi)$ and the discontinuity condition (1.2) strongly influence the structure of the representation of the Jost solution, so the triangular property of the Jost solution representation is lost and the kernel function has a discontinuity along the line $s = \beta(a - \xi) + a$ for $\xi \in (0, a)$ (see [24]).

2. Preliminaries

Assume that the function $e(\xi, \mu)$ satisfies the equation (1.1), discontinuity conditions (1.2) and condition at infinity

$$\lim_{\xi \to \infty} e^{-i\mu\xi} e(\xi, \mu) = 1.$$

Note that the function $e(\xi, \mu)$ is defined as a Jost solution of equation (1.1).

Theorem 2.1. *Let a complex-valued function $q(\xi)$ satisfies equation (1.4). Then for all μ from the closed upper half-plane, there exists the Jost solution $e(\xi,\mu)$ of equation (1.1) with discontinuity conditions (1.2), it is unique and representable in the form*

$$e(\xi,\mu)=e_0(\xi,\mu)+\int_{\tau(\xi)}^{\infty}k(\xi,s)e^{i\mu s}ds, \tag{2.1}$$

where

$$e_0(\xi,\mu)=\begin{cases}e^{i\mu\xi}, & \xi>a,\\ \theta^+e^{i\mu(\beta(\xi-a)+a)}+\theta^-e^{i\mu(-\beta(\xi-a)+a)}, & 0<\xi<a,\end{cases}$$

with $\theta^{\pm}=\frac{1}{2}\left(\alpha\pm\frac{1}{\alpha\beta}\right)$ and $\theta^++|\theta^-|>1$,

$$\tau(\xi)=\begin{cases}\xi, & \xi>a,\\ \beta(\xi-a)+a, & 0<\xi<a,\end{cases}$$

the kernel function $k(\xi,.)\in L_1(\tau(\xi),\infty)$ for each fixed $\xi\in(0,a)\cup(a,\infty)$ and satisfies the inequality

$$\int_{\tau(\xi)}^{\infty}|k(\xi,s)|ds\le e^{c\sigma_1(\xi)}-1,\quad \sigma_1(\xi)=\int_{\xi}^{\infty}t|q(t)|dt,\quad c=\theta^++|\theta^-|. \tag{2.2}$$

Remark 2.1. The above theorem is proved in [24] when the $q(\xi)$ is real valued function. In case the $q(\xi)$ is complex valued function, the theorem is proved in the same way.

Lemma 2.1. *The following estimate holds:*

$$|k(\xi,s)|\le\frac{c}{2}\sigma\left(\frac{\tau(\xi)+s}{2}\right)e^{(c+1)\sigma_1(\xi)},\quad c=\theta^++|\theta^-|. \tag{2.3}$$

Proof. The function $k(\xi,s)$ is in the form for $0<\xi<a$:

$$\begin{aligned}k(\xi,s) &= k_0(\xi,s)+\frac{1}{2\beta}\int_{\xi}^{a}q(\zeta)\int_{s-\beta(\zeta-\xi)}^{s+\beta(\zeta-\xi)}k(\zeta,u)dud\zeta+\frac{\theta^+}{2}\int_{a}^{\infty}q(\zeta)\int_{s-\zeta+\beta(\xi-a)+a}^{s+\zeta+\beta(a-\xi)-a}k(\zeta,u)dud\zeta\\ &\quad-\frac{\theta^-}{2}\int_{a}^{\beta(a-\xi)+a}q(\zeta)\int_{s+\zeta+\beta(\xi-a)-a}^{s-\zeta+\beta(a-\xi)+a}k(\zeta,u)dud\zeta+\frac{\theta^-}{2}\int_{\beta(a-\xi)+a}^{\infty}q(\zeta)\int_{s-\zeta+\beta(a-\xi)+a}^{s+\zeta+\beta(\xi-a)-a}k(\zeta,u)dud\zeta,\end{aligned}$$

where for $\beta(\xi-a)+a<s<\beta(a-\xi)+a$

$$k_0(\xi,s)=\frac{\theta^+}{2\beta}\int_{\frac{s+\beta(\xi+a)-a}{2\beta}}^{a}q(\zeta)d\zeta+\frac{\theta^-}{2\beta}\int_{\frac{\beta(\xi+a)+a-s}{2\beta}}^{a}q(\zeta)d\zeta+\frac{\theta^+}{2}\int_{a}^{\infty}q(\zeta)d\zeta-\frac{\theta^-}{2}\int_{a}^{\frac{s+\beta(a-\xi)+a}{2}}q(\zeta)d\zeta, \tag{2.4}$$

and for $\beta(a-\xi)+a<s<\infty$

$$k_0(\xi,s)=\frac{\theta^+}{2}\int_{\frac{s+\beta(\xi-a)+a}{2}}^{\infty}q(\zeta)d\zeta+\frac{\theta^-}{2}\int_{\frac{s+\beta(a-\xi)+a}{2}}^{\infty}q(\zeta)d\zeta, \tag{2.5}$$

and for the kernel $k(\xi,s)$ has the form for $\xi>a$

$$k(\xi,s)=k_0(\xi,s)+\frac{1}{2}\int_{\xi}^{\infty}q(\zeta)\int_{s-\zeta+\xi}^{s+\zeta-\xi}k(\zeta,u)dud\zeta,$$

where

$$k_0(\xi,s)=\frac{1}{2}\int_{\frac{\xi+s}{2}}^{\infty}q(\zeta)d\zeta.$$

When $\xi>a$, we face the classical case (see [6]) and we have

$$|k(\xi,s)|\le\frac{1}{2}e^{\sigma_1(\xi)}\sigma\left(\frac{\xi+s}{2}\right).$$

Now, let us examine the case $0<\xi<a$. Set

$$\begin{aligned} k_m(\xi,s) &= \frac{1}{2\beta}\int_{\xi}^{a} q(\zeta)\int_{s-\beta(\zeta-\xi)}^{s+\beta(\zeta-\xi)} k_{m-1}(\zeta,u)du d\zeta \\ &+\frac{\theta^+}{2}\int_{a}^{\infty} q(\zeta)\int_{s-\zeta+\beta(\xi-a)+a}^{s+\zeta+\beta(a-\xi)-a} k_{m-1}(\zeta,u)du d\zeta \\ &-\frac{\theta^-}{2}\int_{a}^{\beta(a-\xi)+a} q(\zeta)\int_{s+\zeta+\beta(\xi-a)-a}^{s-\zeta+\beta(a-\xi)+a} k_{m-1}(\zeta,u)du d\zeta \\ &+\frac{\theta^-}{2}\int_{\beta(a-\xi)+a}^{\infty} q(\zeta)\int_{s-\zeta+\beta(a-\xi)+a}^{s+\zeta+\beta(\xi-a)-a} k_{m-1}(\zeta,u)du d\zeta, \quad m=1,2... \end{aligned}$$

and $k_0(\xi,s)$ is determined by the formulas (2.4) and (2.5). Then, we obtain for $0<\xi<a$:

$$|k_0(\xi,s)| \leq \frac{c}{2}\sigma\left(\frac{s+\beta(\xi-a)+a}{2}\right),$$

$$|k_m(\xi,s)| \leq \frac{c}{2}\sigma\left(\frac{s+\beta(\xi-a)+a}{2}\right)\frac{(c+1)^m(\sigma_1(\xi))^m}{m!}.$$

This implies that the series $\sum_{m=0}^{\infty} k_m(\xi,s)$ converges and its sum $k(\xi,s)$ satisfies the inequality

$$|k(\xi,s)| \leq \frac{c}{2}\sigma\left(\frac{\beta(\xi-a)+a+s}{2}\right)e^{(c+1)\sigma_1(\xi)}, \quad 0<\xi<a.$$

Moreover, since for $\xi>a$

$$|k(\xi,s)| \leq \frac{1}{2}e^{\sigma_1(\xi)}\sigma\left(\frac{\xi+s}{2}\right),$$

we obtain that for $\xi\in(0,a)\cup(a,\infty)$ the inequality (2.3) is valid. □

Now, we define $\hat{e}(\xi,\mu)$ as the solution of the equation (1.1) with discontinuity conditions (1.2) and satisfies

$$\lim_{\xi\to\infty} e^{i\mu\xi}\hat{e}(\xi,\mu)=1$$

and when $q(\xi)\equiv 0$ in equation (1.1), the solution has the form:

$$\hat{e}_0(\xi,\mu)=\begin{cases} e^{-i\mu\xi}, & \xi>a, \\ \theta^+e^{-i\mu(-\beta(a-\xi)+a)}+\theta^-e^{-i\mu(\beta(a-\xi)+a)}, & 0<\xi<a. \end{cases} \tag{2.6}$$

The Wronskian of the solutions $e(\xi,\mu)$ and $\hat{e}(\xi,\mu)$ is obtained as

$$w[e(\xi,\mu),\hat{e}(\xi,\mu)]=-2i\mu, \quad Im\mu>0.$$

3. The eigenvalues and spectral singularities

Denote the boundary value problem (1.1)-(1.3) by an operator L operating on the Hilbert space $L_{2,\rho}(0,\infty)$. The values $\lambda=\mu^2$ for which the operator L has a non-zero solution are said eigenvalues and the corresponding nontrivial solutions are defined as eigenfunctions.

Consider $\tilde{e}(\xi,\mu)=e(\xi,-\mu)$ with $Im\mu\leq 0$ and the Wronskian of $e(\xi,\mu)$ and $\tilde{e}(\xi,\mu)$ is in the form:

$$w[e(\xi,\mu),\tilde{e}(\xi,\mu)]=-2i\mu, \quad Im\mu=0. \tag{3.1}$$

Let us describe $s(\xi,\mu)$ as the solution of the equation (1.1) under the discontinuity conditions (1.2) and the initial conditions

$$s(0,\mu)=0, \quad s'(0,\mu)=1.$$

It is obtained that

$$s(\xi,\mu)=\frac{\hat{e}(0,\mu)e(\xi,\mu)-e(0,\mu)\hat{e}(\xi,\mu)}{2i\mu}, \quad Im\mu>0. \tag{3.2}$$

The following lemma is proved in the same way as in [6]:

Lemma 3.1. *1. The nonself-adjoint operator L does not have positive eigenvalues.*

2. The necessary and sufficient conditions that $\lambda \neq 0$ be an eigenvalue of L are that

$$e(0,\mu) = 0, \;\; \lambda = \mu^2, \;\; Im\mu > 0.$$

3. The set of eigenvalues of L is bounded, is no more than countable and its limit points can lie only on the half-axis $\lambda \geq 0$.

All numbers λ of the form $\lambda = \mu^2$, $Im\mu > 0$, $e(0,\mu) \neq 0$ belongs to the resolvent set of L. Assume that $\lambda = \mu^2$ belongs to the resolvent set of L. Then, the resolvent operator $R_{\mu^2} = \left(L - \mu^2 I\right)^{-1}$ exists and has the following representation:

$$R_{\mu^2}(L) = \int_0^\infty r(\xi, s; \mu^2) f(s) ds,$$

where

$$r(\xi, s; \mu^2) = \begin{cases} \frac{\hat{e}(0,\mu)e(\xi,\mu)e(s,\mu)}{2i\mu e(0,\mu)} - \frac{\hat{e}(\xi,\mu)e(s,\mu)}{2i\mu}, & \xi < s < \infty, \\ \\ \frac{\hat{e}(0,\mu)e(\xi,\mu)e(s,\mu)}{2i\mu e(0,\mu)} - \frac{e(\xi,\mu)\hat{e}(s,\mu)}{2i\mu}, & 0 < s < \xi. \end{cases}$$

Note that all number $\lambda > 0$ belongs to the continuous spectrum of L (see [6]).

The spectral singularities is defined as the poles of the kernel function of the resolvent operator and belong to the continuous spectrum. The operator L which has the compact set of spectral singularities, has zero measure in the sense of Lebesgue. This is provided from the boundary uniqueness theorem of analytic functions [25] (also, see [10]).

Denote the eigenvalues and spectral singularities of the operator L, respectively, as follows:

$$\sigma_d(L) = \left\{\lambda : \lambda = \mu^2, \; Im\mu > 0, \; e(0,\mu) = 0\right\},$$

$$\sigma_{ss}(L) = \left\{\lambda : \lambda = \mu^2, \; Im\mu = 0, \; \mu \neq 0, \; e(0,\mu) = 0\right\}.$$

Moreover, the multiplicity of the corresponding eigenvalue and spectral singularity of L is called the multiplicity of the zero of $e(0,\mu)$.

3.1 The finiteness of eigenvalues and spectral singularities

Now, we will demonstrate that the nonself-adjoint operator L has a finite number of eigenvalues and spectral singularities under the two different additional restrictions, respectively.

Additional restriction 1:

$$\int_0^\infty e^{\epsilon\xi} |q(\xi)| d\xi < \infty, \;\; \epsilon > 0, \tag{3.3}$$

This condition is introduced by M. A. Naimark (see [6]).

Theorem 3.1. *Assume that the condition (3.3) is valid. Then, the operator L has finite number of eigenvalues and spectral singularities with finite multiplicity.*

Proof. The condition (3.3) implies that

$$\sigma(\xi) = \int_\xi^\infty |q(t)| dt \leq C_\epsilon e^{-\epsilon\xi},$$

$$\sigma_1(\xi) = \int_\xi^\infty t|q(t)| dt \leq C_{\epsilon'} e^{-\epsilon'\xi},$$

where $C_\epsilon > 0$, $C_{\epsilon'} > 0$ and $0 < \epsilon' < \epsilon$ (see [6]). Using these relations and the estimate (2.3), we have

$$|k(\xi, s)| \leq C \exp\left\{-\epsilon\left(\frac{\tau(\xi) + s}{2}\right)\right\}, \tag{3.4}$$

where $C = \frac{c}{2} c_\epsilon e^{(c+1)d_\epsilon}$, $c = \theta^+ + |\theta^-| > 1$, $c_\epsilon > 0$ and $d_\epsilon > 0$. It is obtained from (3.4) that the function $e(0,\mu)$ has an analytic continuation from the real axis to the half plane $Im\mu > -\frac{\epsilon}{2}$. Then, there is no limit points of the sets of the eigenvalues $\sigma_d(L)$ and the spectral singularities $\sigma_{ss}(L)$ on the positive real line. Since $\sigma_d(L)$ and $\sigma_{ss}(L)$ are bounded and $e(0,\mu)$ is holomorphic in the half plane $Im\mu > -\frac{\epsilon}{2}$, the operator L has finite number of eigenvalues and spectral singularities with finite multiplicity. □

Additional restriction 2:

$$\sup_{0\le\xi<\infty}\left\{\exp(\epsilon\xi^{\delta})|q(\xi)|\right\}<\infty,\ \ \epsilon>0,\ \ \frac{1}{2}\le\delta<1. \tag{3.5}$$

To prove the finiteness of the eigenvalues and spectral singularities under the condition (3.5), firstly we define the set of zeros of $e(0,\mu)$ in the closed upper half plane $Im\mu\ge 0:$

$$M_1:=\{\mu:\ \mu\in\mathbb{C}_+,\ e(0,\mu)=0\},\quad M_2:=\{\mu:\ \mu\in\mathbb{R},\ \mu\ne 0,\ e(0,\mu)=0\},$$

moreover, define the sets of all limit points of M_1 and M_2 as M_3 and M_4, respectively and the set of all zeros of $e(0,\mu)$ with infinite multiplicity as M_5. We have

$$M_1\cap M_5=\emptyset,\ \ M_3\subset M_2,\ \ M_4\subset M_2,\ \ M_5\subset M_2$$

from the uniqueness theorem of analytic functions (see [26]) and

$$M_3\subset M_5,\quad M_4\subset M_5 \tag{3.6}$$

from the continuity of all derivatives of the function $e(0,\mu)$ up to the real axis.

Lemma 3.2. *Assume that the condition (3.5) is satisfied, then $M_5=\emptyset$.*

Proof. To prove this lemma, we use the following theorem (see [9], also [10, 14]): Suppose that the function ψ is holomorphic function on the upper half plane without real line and all of its derivatives are also continuous on the real axis, and there exists $T>0$ such that

$$|\psi^{(m)}(z)|\le K_m,\quad m=0,1,...\ z\in\mathbb{C}_+,\ |z|<2T, \tag{3.7}$$

and

$$\left|\int_{-\infty}^{-T}\frac{\ln|\psi(\xi)|}{1+\xi^2}d\xi\right|<\infty,\quad \left|\int_{T}^{\infty}\frac{\ln|\psi(\xi)|}{1+\xi^2}d\xi\right|<\infty. \tag{3.8}$$

If the set Q with linear Lebesgue measure zero is the set of all zeros of the function ψ with infinite multiplicity and if

$$\int_0^h \ln F(s)d\mu(Q_s)=-\infty, \tag{3.9}$$

then $\psi(z)\equiv 0$, where $F(s)=\inf_m \frac{K_m s^m}{m!}$, $m=0,1,...$, $\mu(Q_s)$ is the linear Lebesgue measure of s-neighborhood of Q and h is an arbitrary positive constant.

Now, it is obtained from the relation (2.3) and the condition (3.5) that

$$|k(\xi,s)|\le\widetilde{C}\exp\left\{-\epsilon\left(\frac{\tau(\xi)+s}{2}\right)^{\delta}\right\},\ \ \widetilde{C}=\frac{c}{2}c_\epsilon e^{(c+1)c_\epsilon},\ \ c=\theta^++|\theta^-|>1.$$

Then, the function $e(0,\mu)$ is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis and we have

$$\left|\frac{d^m e(0,\mu)}{d\mu^m}\right|\le K_m,\ \mu\in\overline{\mathbb{C}}_+,\ m=1,2,..., \tag{3.10}$$

where

$$K_m=\widetilde{C}(\beta a+a)^m\left\{1+\int_0^\infty s^m\exp\left\{-\epsilon\left(\frac{s}{2}\right)^{\delta}\right\}ds\right\},\ \ m=1,2,...$$

Moreover, since the set of zeros of $e(0,\mu)$ is bounded, for sufficiently large T the function $e(0,\mu)$ satisfies the condition (3.8). Thus, it follows from this fact and the relation (3.10) that $e(0,\mu)$ provides the conditions (3.7) and (3.8). Since the function $e(0,\mu)\ne 0$, from (3.9), we have

$$\int_0^h \ln F(s)d\mu(M_{5,s})>-\infty, \tag{3.11}$$

where $F(s) = \inf_m \frac{K_m s^m}{m!}$ and $\mu(M_{5,s})$ is the Lebesgue measure of the s-neighborhood of M_5. The following estimate holds:

$$K_m \le \left(\widetilde{C}(\beta a + a)^m + Dd^m\right) m^m m!, \tag{3.12}$$

where $D = 4\frac{\widetilde{C}e}{\delta}\epsilon^{-\frac{1}{\delta}}(m+1)$ and $d = 4(\beta a + a)\epsilon^{-\frac{1}{\delta}}$. In fact, we can write

$$\begin{aligned} K_m &= \widetilde{C}(\beta a + a)^m \left\{1 + \int_0^\infty s^m \exp\left\{-\epsilon\left(\frac{s}{2}\right)^\delta\right\} ds\right\} \\ &\le \widetilde{C}(\beta a + a)^m \left\{1 + \frac{2^{(m+1)}}{\delta}\epsilon^{-\frac{(m+1)}{\delta}}(2m+2)^{m+1} m!\right\} \\ &\le \widetilde{C}(\beta a + a)^m \left\{1 + \frac{2^{2(m+1)}}{\delta}\epsilon^{-\frac{(m+1)}{\delta}}\left(1 + \frac{1}{m}\right)^m (m+1) m^m m!\right\} \\ &\le \left(\widetilde{C}(\beta a + a)^m + Dd^m\right) m^m m!. \end{aligned}$$

Putting the estimate (3.12) into $F(s)$, we get

$$\begin{aligned} F(s) &\le \widetilde{C}\inf_m \{(\beta a + a)^m m^m s^m\} + D \inf_m \{d^m m^m s^m\} \\ &\le \widetilde{C}\exp\left\{-(\beta a + a)^{-1} s^{-1} e^{-1}\right\} + D\exp\left\{-d^{-1}s^{-1}e^{-1}\right\}. \end{aligned} \tag{3.13}$$

Then, taking into account (3.11) and (3.13), we have

$$\int_0^h \frac{1}{s} d\mu(M_{5,s}) < \infty.$$

This inequality is valid for an arbitrary s if and only if $d\mu(M_{5,s}) = 0$ or $M_5 = \emptyset$. □

Theorem 3.2. *If the condition (3.5) is satisfied, then the operator L has finite number of eigenvalues and spectral singularities with finite multiplicity.*

Proof. It follows from (3.6) and Lemma 3.2 that $M_3 = \emptyset$ and $M_4 = \emptyset$. For this reason, the bounded sets M_1 and M_2 do not have limit points. Thus, the finiteness of the sets of eigenvalues $\sigma_d(L)$ and spectral singularities $\sigma_{ss}(L)$ are found. Moreover, due to $M_5 = \emptyset$, the eigenvalues and spectral singularities have finite multiplicities. □

4. Principal functions

In this section, we examine the principal functions of the nonself-adjoint operator L. Now, assume that the condition (3.5) is provided.

Denote $\mu_1, \mu_2, ..., \mu_\ell$ by the zeros of $e(0,\mu)$ in $\mathbb{C}_+$ with multiplicities $n_1, n_2, ..., n_\ell$, respectively (note that $\mu_1^2, \mu_2^2, ..., \mu_\ell^2$ are the eigenvalues of the operator L). We can write

$$\left\{\frac{d^m}{d\mu^m} W[e(\xi,\mu), s(\xi,\mu)]\right\}_{\mu=\mu_j} = \left\{\frac{d^m}{d\mu^m} e(0,\mu)\right\}_{\mu=\mu_j} = 0 \tag{4.1}$$

for $m = 0, 1, ..., n_j - 1$, $j = \overline{1,\ell}$. In case of $m = 0$, we have

$$e(\xi,\mu_j) = \kappa_0(\mu_j) s(\xi,\mu_j), \;\; \kappa_0(\mu_j) \ne 0, \;\; j = \overline{1,\ell}. \tag{4.2}$$

Lemma 4.1. *The following relation*

$$\left\{\frac{\partial^m}{\partial\mu^m} e(\xi,\mu)\right\}_{\mu=\mu_j} = \sum_{i=0}^m \binom{m}{i} \kappa_{m-i} \left\{\frac{\partial^i}{\partial\mu^i} s(\xi,\mu)\right\}_{\mu=\mu_j} \tag{4.3}$$

is valid for $m = \overline{0, n_j - 1}$, $j = \overline{1,\ell}$ and here $\kappa_0, \kappa_1 ..., \kappa_m$ depend on μ_j.

Proof. To prove of this theorem, we use the mathematical induction. Consider $m = 0$. It follows from the relation (4.2) that the proof is trivial. Now, suppose that the formula (4.3) holds for m_0 such that $0 < m_0 \leq n_j - 2$. That is,

$$\left\{ \frac{\partial^{m_0}}{\partial \mu^{m_0}} e(\xi, \mu) \right\}_{\mu=\mu_j} = \sum_{i=0}^{m_0} \binom{m_0}{i} \kappa_{m_0-i} \left\{ \frac{\partial^i}{\partial \mu^i} s(\xi, \mu) \right\}_{\mu=\mu_j}. \tag{4.4}$$

Then, we will show that the formula (4.3) is satisfied for $m_0 + 1$. If $\varphi(\xi, \mu)$ is the solution of (1.1), then we find

$$\left\{ -\frac{d^2}{d\xi^2} + q(\xi) - \mu^2 \rho(\xi) \right\} \frac{\partial^m}{\partial \mu^m} \varphi(\xi, \mu) = 2\mu m \rho(\xi) \frac{\partial^{m-1}}{\partial \mu^{m-1}} \varphi(\xi, \mu) + m(m-1)\rho(\xi) \frac{\partial^{m-2}}{\partial \mu^{m-2}} \varphi(\xi, \mu). \tag{4.5}$$

Since the functions $e(\xi, \mu)$ and $s(\xi, \mu)$ are solutions of the equation (1.1), using (4.4) and (4.5), we calculate

$$\left\{ -\frac{d^2}{d\xi^2} + q(\xi) - \mu_j^2 \rho(\xi) \right\} h_{m_0+1}(\xi, \mu_j) = 0,$$

where

$$h_{m_0+1}(\xi, \mu_j) = \left\{ \frac{\partial^{m_0+1}}{\partial \mu^{m_0+1}} e(\xi, \mu) \right\}_{\mu=\mu_j} - \sum_{i=0}^{m_0+1} \binom{m_0+1}{i} \kappa_{m_0+1-i} \left\{ \frac{\partial^i}{\partial \mu^i} s(\xi, \mu) \right\}_{\mu=\mu_j}.$$

It follows from (4.1) that

$$W[h_{m_0+1}(\xi, \mu_j), s(\xi, \mu_j)] = \left\{ \frac{d^{m_0+1}}{d\mu^{m_0+1}} W[e(\xi, \mu), s(\xi, \mu)] \right\}_{\mu=\mu_j} = 0. \tag{4.6}$$

Then, this shows that

$$h_{m_0+1}(\xi, \mu_j) = \kappa_{m_0+1}(\mu_j) s(\xi, \mu_j), \quad j = \overline{1, \ell}.$$

Consequently, we obtain that the formula (4.3) is satisfied for $m = m_0 + 1$. □

Now, we define the functions

$$\psi_m(\xi, \lambda_j) = \left\{ \frac{\partial^m}{\partial \mu^m} e(\xi, \mu) \right\}_{\mu=\mu_j} = \sum_{i=0}^{m} \binom{m}{i} \kappa_{m-i} \left\{ \frac{\partial^i}{\partial \mu^i} s(\xi, \mu) \right\}_{\mu=\mu_j} \tag{4.7}$$

for $m = \overline{0, n_j - 1}$, $j = \overline{1, \ell}$ and $\lambda_j = \mu_j^2$.

Theorem 4.1. $\psi_m(\xi, \lambda_j) \in L_{2,\rho}(0, \infty),\ m = \overline{0, n_j - 1},\ j = \overline{1, \ell}.$

Proof. Since

$$|k(\xi, s)| \leq \widetilde{C} \exp\left\{ -\epsilon \left(\frac{\tau(\xi) + s}{2} \right)^{\delta} \right\}, \quad \widetilde{C} = \frac{c}{2} c_\epsilon e^{(c+1)c_\epsilon}, \quad c = \theta^+ + |\theta^-| > 1,$$

using the integral representation (2.1), we have for $0 < \xi < a$

$$\begin{aligned} \left| \left\{ \frac{\partial^m}{\partial \mu^m} e(\xi, \mu) \right\}_{\mu=\mu_j} \right| \quad &\leq \quad \xi^m \theta^+ e^{-Im\mu_j \xi} + (\beta(a-\xi) + a)^m |\theta^-| e^{-Im\mu_j(\beta(a-\xi)+a)} \\ &+ \widetilde{C} \int_{\beta(\xi-a)+a}^{\infty} s^n \exp\left\{ -\epsilon \left(\frac{\beta(\xi - a) + a + s}{2} \right)^{\delta} \right\} e^{-Im\mu_j s} ds \end{aligned} \tag{4.8}$$

and for $a < \xi < \infty$

$$\left| \left\{ \frac{\partial^m}{\partial \mu^m} e(\xi, \mu) \right\}_{\mu=\mu_j} \right| \leq \xi^m e^{-Im\mu_j \xi} + \widetilde{C} \int_{\xi}^{\infty} s^m \exp\left\{ -\epsilon \left(\frac{\xi + s}{2} \right)^{\delta} \right\} e^{-Im\mu_j s} ds. \tag{4.9}$$

Since $\lambda_j = \mu_j^2$, $j = \overline{1, \ell}$ are eigenvalues of the operator L, it is obtained from (4.8) and (4.9) for $Im\mu_j > 0$ that

$$\left\{ \frac{\partial^m}{\partial \mu^m} e(\xi, \mu) \right\}_{\mu=\mu_j} \in L_{2,\rho}(0, \infty), \quad m = \overline{0, n_j - 1}, \quad j = \overline{1, \ell}.$$

Consequently, from (4.7) we have $\psi_m(\xi, \lambda_j) \in L_{2,\rho}(0, \infty)$, $m = \overline{0, n_j - 1}$, $j = \overline{1, \ell}$. □

Definition 4.1. The functions $\psi_0(\xi,\lambda_j)$, $\psi_1(\xi,\lambda_j)$,...,$\psi_{n_j-1}(\xi,\lambda_j)$ are called the principal functions associated with eigenvalues $\lambda_j=\mu_j^2$, $j=\overline{1,\ell}$ of the operator L. The function $\psi_0(\xi,\lambda_j)$ is the eigenfunction, $\psi_1(\xi,\lambda_j)$, $\psi_2(\xi,\lambda_j)$,..., $\psi_{n_j-1}(\xi,\lambda_j)$ are the associated functions of $\psi_0(\xi,\lambda_j)$ corresponding to eigenvalue λ_j.

Now, we define the spectral singularities of L: $\mu_{\ell+1},\mu_{\ell+2},...,\mu_p$ are the zeros of the function $e(0,\mu)$ in $\mathbb{R}-\{0\}$ with multiplicities $n_{\ell+1},n_{\ell+2},...,n_p$, respectively. Then, using the similar way in Lemma 4.1, we obtain

$$\left\{\frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu)\right\}_{\mu=\mu_r}=\sum_{j=0}^{\eta}\begin{pmatrix}\eta\\ j\end{pmatrix}\tau_{\eta-j}(\mu_r)\left\{\frac{\partial^j}{\partial\mu^j}s(\xi,\mu)\right\}_{\mu=\mu_r} \tag{4.10}$$

for $\eta=\overline{0,n_r-1}$, $r=\ell+1,\ell+2,...,p$. Denote the functions

$$\psi_\eta(\xi,\lambda_r)=\left\{\frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu)\right\}_{\mu=\mu_r}=\sum_{j=0}^{\eta}\begin{pmatrix}\eta\\ j\end{pmatrix}\tau_{\eta-j}(\mu_r)\left\{\frac{\partial^j}{\partial\mu^j}s(\xi,\mu)\right\}_{\mu=\mu_r} \tag{4.11}$$

for $\eta=\overline{0,n_r-1}$, $r=\ell+1,\ell+2,...,p$ and $\lambda_j=\mu_j^2$.

Theorem 4.2. *The functions $\psi_\eta(\xi,\lambda_r)$ do not belong to $L_{2,\rho}(0,\infty)$, $\eta=\overline{0,n_r-1}$, $r=\ell+1,\ell+2,...,p$.*

Proof. Take into account the relations (4.8) and (4.9) for $\mu=\mu_r$, $r=\ell+1,\ell+2,...,p$ and since $Im\mu_r=0$ for the spectral singularities, we have

$$\left\{\frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu)\right\}_{\mu=\mu_r}\notin L_{2,\rho}(0,\infty),\ \eta=\overline{0,n_r-1},\ r=\ell+1,\ell+2,...,p.$$

As a result, from the definition of the functions (4.11), we find $\psi_\eta(\xi,\lambda_r)\notin L_{2,\rho}(0,\infty)$, $\eta=\overline{0,n_r-1}$, $r=\ell+1,\ell+2,...,p$. □

Now, we introduce the Hilbert spaces

$$H_{\zeta,\rho}=\left\{f:\|f\|_{\zeta,\rho}<\infty\right\},\ \ H_{-\zeta,\rho}=\left\{f:\|f\|_{-\zeta,\rho}<\infty\right\},\ \ \zeta=1,2,...$$

with the norms

$$\|f\|^2_{\zeta,\rho}=\int_0^\infty(1+\tau(s))^{2\zeta}|f(s)|^2\rho(s)ds,\quad \|f\|^2_{-\zeta,\rho}=\int_0^\infty(1+\tau(s))^{-2\zeta}|f(s)|^2\rho(s)ds,$$

respectively and evidently, $H_{0,\rho}=L_{2,\rho}(0,\infty)$.

Let n_0 denotes the greatest of the multiplicities of the spectral singularities of L:

$$n_0=\max\left\{n_{\ell+1},n_{\ell+2},...,n_p\right\}.$$

We put

$$H_{+,\rho}=H_{(n_0+1),\rho},\quad H_-=H_{-(n_0+1),\rho}$$

Then, we have

$$H_{+,\rho}\subset L_{2,\rho}(0,\infty)\subset H_{-,\rho}$$

and for all $f\in H_{+,\rho}$, $\|f\|_{+,\rho}\geq\|f\|_\rho\geq\|f\|_{-,\rho}$, where $\|.\|_{\pm,\rho}=\|.\|_{\pm(n_0+1),\rho}$, $\|.\|_\rho=\|.\|_{0,\rho}$ (see [6]). We are particularly interested in the space $H_{\pm,\rho}$ because the space $H_{-,\rho}$ contains the principal functions for the spectral singularities. Now, we will prove this claim by using following lemma.

Lemma 4.2. *The following relation holds:*

$$\sup_{0\leq\xi<\infty}\frac{|e^{(n)}(\xi,\mu)|}{(1+\tau(\xi))^n}<\infty,\ \ e^{(n)}=\left(\frac{d}{d\mu}\right)^n e,\ \ Im\mu=0,\ \ n=0,1,2,... \tag{4.12}$$

Proof. Using the integral representation (2.1), we obtain for $Im\mu = 0$

$$|e^{(n)}(\xi,\mu)| \leq \xi^n\theta^+ + (\beta(a-\xi)+a)^n|\theta^-| + \widetilde{C}\int_{\beta(\xi-a)+a}^{\infty} s^n \exp\left\{-\epsilon\left(\frac{\beta(\xi-a)+a+s}{2}\right)^{\delta}\right\}ds,\ \ 0<\xi<a \tag{4.13}$$

and

$$|e^{(n)}(\xi,\mu)| \leq \xi^n + \widetilde{C}\int_{\xi}^{\infty} s^n \exp\left\{-\epsilon\left(\frac{\xi+s}{2}\right)^{\delta}\right\}ds,\quad a<\xi<\infty. \tag{4.14}$$

Then, taking into account (4.13) and (4.14), we find $\sup_{0\leq\xi<\infty}\frac{|e^{(n)}(\xi,\mu)|}{(1+\tau(\xi))^n} < \infty$ for $Im\mu = 0$. □

Theorem 4.3. $\psi_\eta(\xi,\lambda_r) \in H_{-(\eta+1),\rho}$, $\eta = \overline{0, n_r-1}$, $r = \ell+1, \ell+2, ..., p$.

Proof. Using the relation (4.12), we have

$$\left\|e^{(\eta)}(\xi,\mu)\right\|^2_{-(\eta+1),\rho} = \int_0^{\infty}|\frac{e^{(\eta)}(\xi,\mu)}{(1+\tau(\xi))^{\eta+1}}|^2\rho(\xi)d\xi < \infty.$$

That is, the functions $e^{(\eta)}(\xi,\mu) = \frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu) \in H_{-(\eta+1)}$ for $Im\mu = 0$ and $\eta = 0, 1, 2, ...$. Then, we get

$$\left\{\frac{\partial^\eta}{\partial\mu^\eta}e(\xi,\mu)\right\}_{\mu=\mu_r} \in H_{-(\eta+1),\rho}$$

for $Im\mu_r = 0$, $\eta = \overline{0, n_r-1}$ and $r = \ell+1, \ell+2, ..., p$. Consequently, it follows from the formula (4.11) that $\psi_\eta(\xi,\lambda_r) \in H_{-(\eta+1),\rho}$, $\eta = \overline{0, n_r-1}$, $r = \ell+1, \ell+2, ..., p$. □

Definition 4.2. The functions $\psi_0(\xi,\lambda_r), \psi_1(\xi,\lambda_r), ..., \psi_{n_r-1}(\xi,\lambda_r)$ are defined as the principal functions associated with the spectral singularities $\lambda_r = \mu_r^2$, $r = \ell+1, \ell+2, ..., p$ of operator L. The function $\psi_0(\xi,\lambda_r)$ is the generalized eigenfunction, $\psi_1(\xi,\lambda_r), ..., \psi_{n_r-1}(\xi,\lambda_r)$ are the generalized associated functions of $\psi_0(\xi,\lambda_r)$ corresponding to spectral singularity λ_r.

5. Conclusion

In this paper, we examine the spectrum and the principal functions of the nonself-adjoint discontinuous Sturm-Liouville operator which contains the discontinuous coefficient and the discontinuity conditions at the point on the positive half line. When examining the spectrum of the considered problem (1.1)-(1.3), we use the newly constructed Jost solution of the equation (1.1) with discontinuity condition (1.3). This solution is completely different from the classical Jost solution because of the presence of the discontinuous coefficient $\rho(\xi)$ and discontinuity condition (1.2). We point out that the triangular property of the Jost solution representation is lost and the kernel function has a discontinuity along the line $s = \beta(\xi-a)+a$ for $\xi \in (0,a)$. Under two different additional conditions, it is proved that the problem (1.1)-(1.3) has finite number of eigenvalues and spectral singularities with finite multiplicity. Finally using the additional restriction (3.5) which is weaker than the restriction (3.3), we determine the principal functions corresponding to the eigenvalues and the spectral singularities of the problem (1.1)-(1.3).

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On Contra πgs-Continuity

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Abstract
In this work, a novel form of contra continuity entitled as contra πgs-continuity is examined, which has connections to πgs-closed sets. Furthermore, correlations between contra πgs-continuity and several previously established forms of contra continuous functions are further explored, as well as basic features of contra πgs-continuous functions are disclosed.

Keywords: *πgs-closed sets, Contra πgs-continuity, Contra continuity*

AMS Subject Classification (2020): *54C08; 54C10; 54C0*

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1. Introduction

After defining semi-open sets [1] in 1963, Levine introduced the concept of g-closed sets [2] in 1970. This interesting new set type has led to the emergence of different types of generalized closed sets. Dontchev and Noiri defined πg-closed sets [3] in 2000. In 2006, Aslım et al. introduced the πgs-closed set [4] definition, which has an important place in this study, to the literature.

The idea of LC-continuous functions was first introduced and analyzed by Ganster and Reilly [5] in 1989. Dontchev [6] produced contra-continuity, as a more robust variant of LC-continuity in 1996. As a very interesting subject, contra continuous functions have continued to attract the attention of many researchers over the years. After Ekici gave the definition of contra πg-continuous functions [7] in 2008, contra πgs-continuous [8] functions were also defined in Caldas et al.'s studies in 2010, which essentially introduced and examined contra πgp-continuous functions [8].

The requirement that every open set in the codomain possesses a preimage that is πgs-closed in the domain identifies contra πgs-continuous functions [8]. A milder version of contra-continuity [6] and contra gs-continuity [9] is contra πgs-continuity. Crucial characteristics of contra πgs-continuous functions are also examined.

2. Preliminaries

Unless otherwise specified, topological spaces in this work always refer to on which no separation axioms are required; Ψ will stand for the topological space $(\Psi, \top)$ and Φ will stand for the topological space $(\Phi, \perp)$; $\aleph$ will

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stand for any subset of the space Ψ. The interior of $\aleph$ is indicated as $int(\aleph)$ and the closure of $\aleph$ in indicated as $cl(\aleph)$. Whenever $\aleph = int(cl(\aleph))$ (correspondingly, $\aleph = cl(int(\aleph))$), afterwards $\aleph$ is a regular closed set (correspondingly, regular open set) [10]. Whenever $\aleph \subset cl(int(\aleph))$, afterwards $\aleph$ is considered as a semi-open set [1]. Whenever $\aleph$ could be expressed as union of regular open sets, afterwards it is accepted as a δ-open set [11]. Complementary of semi-open set (correspondingly δ-open set) is introduced as semi-closed (correspondingly δ-closed). The intersection of whole semi-closed sets involving $\aleph$ is known as semi-closure [12] of $\aleph$ which is expressed by $scl(\aleph)$. Dually the semi-interior [12] of $\aleph$ is characterized as union of whole semi-open sets involved in $\aleph$ and indicated by $sint(\aleph)$.

$\nu \in \Psi$ is termed δ-cluster point [11] of $\aleph$, when $int(cl(F)) \cap \aleph \neq \emptyset$ for every $F \in O(\nu, \Psi)$, where $O(\nu, \Psi)$ stands for all open subsets of Ψ containing the point ν. Whole δ-cluster points of $\aleph$ composes δ-closure [11] of $\aleph$ that is shown with $cl_\delta(\aleph)$.

When $\aleph \subset cl(int(cl_\delta(\aleph)))$, then $\aleph$ is named as an e^*-open set [13]. We speak of an e^*-closed [13] set as complementary of an e^*-open. The e^*-closure [13] of $\aleph$ is the intersection of whole e^*-closed sets involving subset $\aleph$ and it is symbolized by $e^*\text{-}cl(\aleph)$.

Whenever $e^*\text{-}cl(F) \cap \aleph \neq \emptyset$ for each e^*-open set F involving point ν, afterwards ν is identified as e^*-θ-cluster point [14] of $\aleph$. The e^*-θ-closure [14] of $\aleph$ is the set of whole e^*-θ-cluster points of $\aleph$, and is expressed by $e^*\text{-}cl_\theta(\aleph)$. For $\aleph = e^*\text{-}cl_\theta(\aleph)$, then $\aleph$ is e^*-θ-closed [15]. $e^*\text{-}\theta C(\Psi)$ is the notion for the collection of whole e^*-θ-closed subsets of space Ψ.

When for every ν in $\aleph$, if there exists an e^*-open set F comprising ν such that $F \setminus \aleph$ is countable, then $\aleph$ is termed we^*-open [16]. A we^*-closed [16] set is the complementary of an we^*-open.

When $\aleph \subset cl(int(\aleph)) \cup int(cl(\aleph))$, subsequently $\aleph$ is named as b-open [17] (or sp-open [18] or γ-open [19]). A b-closed [17] (or γ-closed [20, 21]) set is the complementary of a b-open (or γ-open). The b-closure [17] (or γ-closure [20]) of $\aleph$ is expressed as $bcl(\aleph)$ (or $\gamma cl(\aleph)$) and it is the intersection of whole b-closed (or γ-closed) sets comprising $\aleph$. The set $\aleph$ is said to be pre-closed [22] if $cl(int(\aleph)) \subset \aleph$. The intersection of all pre-closed sets containing $\aleph$ is called pre-closure [20] of $\aleph$ and denoted by $pcl(\aleph)$.

A subset $\aleph$ of a space Ψ is characterized as a $\hat{g}$-closed [23] set, if $cl(\aleph) \subset F$, whenever F is a semi-open set satisfying the condition $\aleph \subset F$. $\hat{g}$-open sets [23] are the complement of $\hat{g}$-closed sets. When $bcl(\aleph) \subset F$ whenever $\aleph \subset F$ and F is a $\hat{g}$-open set in Ψ, $\aleph$ is a $b\hat{g}$-closed [24] set. A $b\hat{g}$-open [25] is the complementary of a $b\hat{g}$-closed set. When $scl(\aleph) \subset F$ whenever $\aleph \subset F$ and F is a $b\hat{g}$-open set in Ψ, $\aleph$ is called as a $sb\hat{g}$-closed [26] set.

π-open [27] corresponds to the finite union of regular open sets. π-closed represents the complementary of a π-open. When $\aleph \subset F$ and F is open (correspondingly, π-open), afterwards $\aleph$ is regarded as a generalized closed (briefly, g-closed) [2] (correspondingly, πg-closed [17]) if $cl(\aleph) \subset F$. g-open [24] (correspondingly, πg-open [7]) is the complementary of g-closed (correspondingly, πg-closed). While $\aleph \subset F$ and F is open (correspondingly, π-open), afterwards $\aleph$ is regarded to be generalized semi-closed (briefly, gs-closed) [28] (correspondingly, πgs-closed [4]) if $scl(\aleph) \subset F$. gs-open [24] (correspondingly, πgs-open) constitutes the complementary of a gs-closed (correspondingly, πgs-closed) set. If $pcl(\aleph) \subset F$ for all F which are π-open sets containing $\aleph$, then $\aleph$ is called as πgp-closed [29]. The set $\aleph$ is called as $\pi g\gamma$-closed [20], if $\gamma cl(\aleph) \subset F$ for all π-open sets F containing $\aleph$.

The entire πgs-closed (correspondingly, πgs-open, πgp-closed, $\pi g\gamma$-closed, gs-closed, gs-open, closed, semi-closed, semi-open, γ-open, π-open, πg-open, regular open, regular closed, g-closed, πg-closed, we^*-closed, e^*-closed, $e^*\theta$-closed, $b\hat{g}$-closed, $sb\hat{g}$-closed) subsets of Ψ are expressed by $\pi GSC(\Psi)$ (correspondingly, $\pi GSO(\Psi)$, $\pi GPC(\Psi)$, $\pi G\gamma C(\Psi)$, $GSC(\Psi)$, $GSO(\Psi)$, $C(\Psi)$, $SC(\Psi)$, $SO(\Psi)$, $\gamma O(\Psi)$, $\pi O(\Psi)$, $\pi GO(\Psi)$, $RO(\Psi)$, $RC(\Psi)$, $GC(\Psi)$, $\pi GC(\Psi)$, $we^*C(\Psi)$, $e^*C(\Psi)$, $e^*\theta C(\Psi)$, $b\hat{g}C(\Psi)$, $sb\hat{g}C(\Psi)$).

$\pi GSC(\nu, \Psi)$ (correspondingly, $\pi GSO(\nu, \Psi)$, $RO(\nu, \Psi)$, $C(\nu, \Psi)$, $SO(\nu, \Psi)$, $O(\nu, \Psi)$) means the collection of whole πgs-closed (correspondingly, πgs-open, regular open, closed, semi open, open) sets of Ψ comprising point $\nu \in \Psi$.

πgs-closure of the set $\aleph$ is denoted by $cl_{\pi gs}(\aleph)$, which is the intersection of whole πgs-closed sets involving $\aleph$. On the other hand, πgs-interior of a set $\aleph$ is expressed by $int_{\pi gs}(\aleph)$, which corresponds to the union of whole πgs-open sets included in $\aleph$.

Definition 2.1. A topological space Ψ is said to be:
(ι_i) strongly S-closed [6] while a finite subcover matching could found for each closed cover of Ψ,
(ι_{ii}) strongly countably S-closed [7] when a finite subcover matching found for each countable cover of Ψ consisting of closed sets,
(ι_{iii}) strongly S-Lindelöf [7] when a countable subcover matching could found for each closed cover of Ψ,
(ι_{iv}) ultra normal [30] if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets,
(ι_v) ultra Hausdorff [30] if for each couple of distinct points, ν_1 and ν_2 in Ψ there exist clopen sets $\aleph_1$ and $\aleph_2$ comprising ν_1 and ν_2 correspondingly, providing $N_1 \cap N_2 = \emptyset$ equality.

Definition 2.2. When $\aleph$ in Ψ is strongly S-closed as a subspace, then $\aleph$ is named strongly S-closed [6].

Definition 2.3. $\aleph$ in Ψ is called:
(ι_i) α-open [31] whenever $\aleph \subset int(cl(int(\aleph)))$,
(ι_{ii}) preopen [22] or nearly open [5] whenever $\aleph \subset int(cl(\aleph))$,
(ι_{iii}) β-open [32] or semi-preopen [33] whenever $\aleph \subset cl(int(cl(\aleph)))$.

Complement of an α-open (correspondingly, preopen, β-open) set is introduced as α-closed (correspondingly, preclosed, β-closed) set [7]. $\alpha O(\Psi)$ (correspondingly, $PO(\Psi), \beta O(\Psi)$) stands for the collection of whole α-open (correspondingly, preopen, β-open) subsets of Ψ.

Lemma 2.1. *Whenever* $\aleph \subset \Psi$,
(ι_i) $cl_{\pi gs}(\Psi \backslash \aleph) = \Psi \backslash int_{\pi gs}(\aleph)$;
(ι_{ii}) $\nu \in cl_{\pi gs}(\aleph) \Leftrightarrow \forall \digamma \in \pi GSO(\nu, \Psi), \aleph \cap \digamma \neq \emptyset$.

Proof. Before starting the proof, let's remind the definitions of πgs-interior and πgs-closure of a set in a topological space. Let $(\Psi, \top)$ be a topological space, $\aleph \subset \Psi$. Then, πgs-closure of $\aleph$ is $cl_{\pi gs}(\aleph) = \bigcap\{\Theta : \aleph \subset \Theta, \Theta \in \pi GSC(\Psi)\}$ and πgs-interior of $\aleph$ is $int_{\pi gs}(\aleph) = \bigcup\{\Game : \Game \subset \aleph, \Game \in \pi GSO(\Psi)\}$. Now we can start the proof.
(ι_i): We will complete the proof by showing that the sets claimed to be equal include each other.
Let $(\Psi, \top)$ be a topological space and $\aleph \subset \Psi$.
$(\Rightarrow)$: Let $\nu \in cl_{\pi gs}(\Psi \backslash \aleph)$. Assume that $\nu \notin \Psi \backslash int_{\pi gs}(\aleph)$. Since $\nu \in int_{\pi gs}(\aleph) = \bigcup\{\Game : \Game \subset \aleph, \Game \in \pi GSO(\Psi)\}$, it can be said that there exists a set $\digamma \in \pi GSO(\nu, \Psi)$ such that $\digamma \subset \aleph$. So $\Theta = \Psi \backslash \digamma \in \pi GSC(\Psi)$, $\nu \notin \Theta$ and $\Psi \backslash \aleph \subset \Theta$. This brings us to the contradiction $\nu \notin cl_{\pi gs}(\Psi \backslash \aleph)$ contrary to our assumption. Hence as a result $cl_{\pi gs}(\Psi \backslash \aleph) \subset \Psi \backslash int_{\pi gs}(\aleph)$.
$(\Leftarrow)$: Let $\nu \in \Psi \backslash int_{\pi gs}(\aleph)$. So it can be clearly seen that $\nu \notin int_{\pi gs}(\aleph) = \bigcup\{\Game : \Game \subset \aleph, \Game \in \pi GSO(\Psi)\}$. Then for all of the sets $\Game \in \pi GSO(\Psi)$ such that $\Game \subset \aleph$ we have $\nu \notin \Game$. This means that for all sets $\Psi \backslash \Game \in \pi GSC(\Psi)$ such that $\Psi \backslash \aleph \subset \Psi \backslash \Game$ we have $\nu \in \Psi \backslash \Game$. So $\nu \in cl_{\pi gs}(\Psi \backslash \aleph)$. Hence as a result $\Psi \backslash int_{\pi gs}(\aleph) \subset cl_{\pi gs}(\Psi \backslash \aleph)$.
Now we will give the proof of (ι_{ii}).
(ι_{ii}):
$(\Rightarrow)$: Let $\nu \in cl_{\pi gs}(\aleph)$. Assume that there exists a set $\Game \in \pi GSO(\nu, \Psi)$ such that $\Game \cap \aleph = \emptyset$. Under this assumption, for the set $\Theta = \Psi \backslash \Game$ it can be said that $\nu \notin \Theta$ and $\aleph \subset \Theta$. These results brings us to the contradiction $\nu \notin cl_{\pi gs}(\aleph)$ contrary to our assumption.
$(\Leftarrow)$: Let $\nu \in \Psi$ and let for all sets $\Game \in \pi GSO(\nu, \Psi)$ we have $\Game \cap \aleph \neq \emptyset$. Assume that $\nu \notin cl_{\pi gs}(\aleph)$. Then using (ι_i) we have $\nu \in \Psi \backslash cl_{\pi gs}(\aleph) = \Psi \backslash (\Psi \backslash int_{\pi gs}(\Psi \backslash \aleph)) = int_{\pi gs}(\Psi \backslash \aleph)$. So there exists a set $\digamma \in \pi GSO(\nu, \Psi)$ such that $\digamma \subset \Psi \backslash \aleph$, which means that $\digamma \cap \aleph = \emptyset$ which is a contradiction. So $\nu \in cl_{\pi gs}(\aleph)$.
Thus the proof is completed. □

While $\aleph$ is πgs-closed, then $cl_{\pi gs}(\aleph) = \aleph$. Typically, the opposite of this implication doesn't hold true, as demonstrated in the subsequent example:

Example 2.1. Consider the subset $\aleph = \{\nu_1, \nu_2\}$ of the set $\Psi = \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}$ and the topological space $(\Psi, \top)$, where $\top = \{\emptyset, \{\nu_1\}, \{\nu_2\}, \{\nu_1, \nu_2\}, \{\nu_1, \nu_2, \nu_3\}, \Psi\}$. Then the set $\aleph$ is an acceptable sample that fits the given situation just above, since $\aleph = cl_{\pi gs}(\aleph)$, while $\aleph \notin \pi GSC(\Psi)$.

$ker(\mho)$ [34] means $\bigcap\{\digamma \in \top : \mho \subset \digamma\}$ which is known as the kernel of $\mho$.

Lemma 2.2. *[35] The subsequent characteristics apply to subsets* $\digamma$ *and* $\mho$ *of* Ψ*:*
(ι_i) $\nu \in ker(\digamma) \Leftrightarrow (\forall \Theta \in C(\nu, \Psi))(\digamma \cap \Theta \neq \emptyset)$;
(ι_{ii}) $\digamma \subset ker(\digamma)$;
(ι_{iii}) $\digamma \in \Psi \Rightarrow \digamma = ker(\digamma)$;
(ι_{iv}) $\digamma \subset \mho \Rightarrow ker(\digamma) \subset ker(\mho)$.

3. Contra πgs-continuous functions

In this section, first the characterization of contra πgs-continuous functions is presented. Afterwards, the relationships between some types of contra continuous functions and contra πgs-continuous functions were examined. In addition, some new definitions in relation with πgs-open sets are given in order to examine various properties of contra πgs-continuous functions, and these properties are presented through theorems and results.

Definition 3.1. $\Delta : (\Psi, \top) \to (\Phi, \bot)$ is referred as contra πgs-continuous [8], whenever $\Delta^{-1}(\mho) \in \pi GSC(\Psi)$ for each $\mho \in \bot$.

Theorem 3.1. *Under the assumption $\pi GSO(\Psi)$ is closed under arbitrary unions (or likewise $\pi GSC(\Psi)$ is closed under arbitrary intersections), subsequent statements are coequal for $\Delta : (\Psi, \top) \to (\Phi, \bot)$.*
(ι_i) *Δ is contra πgs-continuous;*
(ι_{ii}) *$\mho \in C(\Phi) \Rightarrow \Delta^{-1}(\mho) \in \pi GSO(\Psi)$;*
(ι_{iii}) *$(\forall \nu \in \Psi)(\forall \Theta \in C(\Delta(\nu), \Phi))(\exists F \in \pi GSO(\nu, \Psi))(\Delta(F) \subset \Theta)$;*
(ι_{iv}) *$\aleph \subset \Psi \Rightarrow \Delta(cl_{\pi gs}(\aleph)) \subset ker(\Delta(\aleph))$;*
(ι_v) *$\Omega \subset \Phi \Rightarrow cl_{\pi gs}(\Delta^{-1}(\Omega)) \subset \Delta^{-1}(ker(\Omega))$.*

Proof. Let $\Delta : (\Psi, \top) \to (\Phi, \bot)$ be a function, where $(\Psi, \top)$ and $(\Phi, \bot)$ are two topological spaces and let $\pi GSO(\Psi)$ be closed under arbitrary unions (or likewise $\pi GSC(\Psi)$ be closed under arbitrary intersections).
$(\iota_i) \Rightarrow (\iota_{ii})$: Let $\Theta \in C(\Phi)$. Then $\Phi \backslash \Theta$ is open in Φ. Since Δ is contra πgs-continuous, $\Psi \backslash \Delta^{-1}(\Theta) = \Delta^{-1}(\Phi \backslash \Theta)$ is πgs-closed in Ψ. Therefore, $\Delta^{-1}(\Theta)$ is πgs-open in Ψ.
$(\iota_{ii}) \Rightarrow (\iota_i)$: Obvious.
$(\iota_i) \Rightarrow (\iota_{iii})$: Let $\nu \in \Psi$ and $\Theta \in C(\Delta(\nu), \Phi)$. Then by (ι_i), we have $\Delta^{-1}(\Theta) \in \pi GSO(\Psi)$. Choosing $F = \Delta^{-1}(\Theta)$ we obtain that $F \in \pi GSO(\nu, \Psi)$ and $\Delta(F) \subset \Theta$.
$(\iota_{iii}) \Rightarrow (\iota_{ii})$: Let $\Theta \in C(\Phi)$ and $\nu \in \Delta^{-1}(\Theta)$. Since $\Delta(\nu) \in \Theta$, by (ι_{iii}) there exist a πgs-open set $F_\nu \in \pi GSO(\nu, \Psi)$ such that $\Delta(F_\nu) \subset \Theta$. So we have $\nu \in F_\nu \subset \Delta^{-1}(\Theta)$ and hence $\Delta^{-1}(\Theta) = \bigcup\{F_\nu : \nu \in \Delta^{-1}(\Theta)\}$ is πgs-open in Ψ since $\pi GSO(\Psi)$ is closed under arbitrary unions.
$(\iota_{ii}) \Rightarrow (\iota_{iv})$: Let $\aleph$ be any subset of Ψ. Suppose that there exist an element μ of $\Delta(cl_{\pi gs}(\aleph))$ such that $\mu \notin ker(\Delta(\aleph))$. Then there exists an open set F of Φ such that $\Delta(\aleph) \subset F$ and $\mu \notin F$. Hence, there exists $\Theta = \Phi \backslash F \in C(\mu, \Phi)$ such that $\Delta(\aleph) \cap \Theta = \emptyset$ and $cl_{\pi gs}(\aleph) \cap \Delta^{-1}(\Theta) = \emptyset$. From here we obtain that $\Delta(cl_{\pi gs}(\aleph)) \cap \Theta = \emptyset$ and $\mu \notin \Delta(cl_{\pi gs}(\aleph))$ which is a contradiction.
$(\iota_{iv}) \Rightarrow (\iota_v)$: Let Ω be any subset of Φ. Then $\Delta^{-1}(\Omega) \subset \Psi$. By (ι_{iv}), $\Delta(cl_{\pi gs}(\Delta^{-1}(\Omega))) \subset ker(\Delta(\Delta^{-1}(\Omega))) \subset ker(\Omega)$. Hence, $cl_{\pi gs}(\Delta^{-1}(\Omega)) \subset \Delta^{-1}(ker(\Omega))$.
$(\iota_v) \Rightarrow (\iota_i)$: Let F be any open subset of Φ. Then by (ι_v) and by Lemma 2.2, $cl_{\pi gs}(\Delta^{-1}(F)) \subset \Delta^{-1}(ker(F)) = \Delta^{-1}(F)$. So we have $cl_{\pi gs}(\Delta^{-1}(F)) = \Delta^{-1}(F)$. Since $\pi GSO(\Psi)$ is closed under arbitrary unions, $\pi GSC(\Psi)$ is closed under arbitrary intersections and hence $\Delta^{-1}(F) = cl_{\pi gs}(\Delta^{-1}(F))$ is πgs-closed. □

Remark 3.1. Statements (ι_i) and (ι_{ii}) in Theorem 3.1 are identical even if $\pi GSO(\Psi)$ is not closed under arbitrary unions (or likewise, $\pi GSC(\Psi)$ is not closed under arbitrary intersections).

Definition 3.2. $\Delta : (\Psi, \top) \to (\Phi, \bot)$ is categorized as:
(ι_1) perfectly continuous [36] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in \top \cap C(\Psi))$,
(ι_2) RC-continuous [9] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in RC(\Psi))$,
(ι_3) strongly continuous [37] $:\Leftrightarrow (F \subset \Phi \Rightarrow \Delta^{-1}(F) \in \top \cap C(\Psi))$ (identically $(\aleph \subset \Psi \Rightarrow \Delta(cl(\aleph)) \subset \Delta(\aleph))$),
(ι_4) contra-continuous [6] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in C(\Psi))$,
(ι_5) contra-super-continuous [38]$:\Leftrightarrow (\forall \nu \in \Psi)(\forall \Theta \in C(\Delta(\nu), \Phi)(\exists F \in RO(\nu, \Psi))(\Delta(F) \subset \Theta)$,
(ι_6) contra-semicontinuous [9] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in SC(\Psi))$,
(ι_7) contra g-continuous [39] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in GC(\Psi))$,
(ι_8) contra gs-continuous [9] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in GSC(\Psi))$,
(ι_9) contra πg-continuous [7] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in \pi GC(\Psi))$,
(ι_{10}) contra we^*-continuous [16] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in we^*C(\Psi))$,
(ι_{11}) contra $e^*\theta$-continuous [40] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in e^*\theta C(\Psi))$,
(ι_{12}) contra e^*-continuous [41] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in e^*C(\Psi))$,
(ι_{13}) almost contra e^*-continuous [42] $:\Leftrightarrow (F \in RO(\Phi) \Rightarrow \Delta^{-1}(F) \in e^*C(\Psi))$,
(ι_{14}) almost contra $e^*\theta$-continuous [42] $:\Leftrightarrow (F \in RO(\Phi) \Rightarrow \Delta^{-1}(F) \in e^*\theta C(\Psi))$,
(ι_{15}) contra $b\hat{g}$-continuous [25] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in b\hat{g}C(\Psi))$,
(ι_{16}) contra sb$\hat{g}$-continuous [43] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in sb\hat{g}C(\Psi))$.
(ι_{17}) contra πgp-continuous function [8] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in \pi GPC(\Psi))$,
(ι_{18}) contra $\pi g\gamma$-continuous function [20] $:\Leftrightarrow (F \in \bot \Rightarrow \Delta^{-1}(F) \in \pi G\gamma C(\Psi))$.

Remark 3.2.

$$\begin{array}{rcccccccccc}
 & \iota_6 & \longleftarrow & \iota_4 & \longleftarrow & \iota_5 \longleftarrow \iota_2 \longleftarrow \iota_1 \longleftarrow \iota_3 \\
 & \downarrow & & \downarrow & & \\
 & \iota_8 & \longleftarrow & \iota_7 & & \\
 & \downarrow & & \downarrow & & \\
 & \text{contra } \pi\text{gs-continuous} & \leftarrow & \iota_9 & & \\
 & \downarrow & & \downarrow & & \\
 & \iota_{18} & \leftarrow & \iota_{17} & &
\end{array}$$

Remark 3.3. As can be seen from the samples below, reversibility of the consequences in the above diagram need not to be true.

Example 3.1. $\top = \{\emptyset, \{\nu_2\}, \{\nu_1, \nu_4\}, \{\nu_2, \nu_3\}, \{\nu_1, \nu_2, \nu_4\}, \{\nu_1, \nu_2, \nu_3, \nu_4\}, \Psi\}$ is the topology on $\Psi = \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}$. Since mappings under $\Delta : \Psi \to \Psi$ are listed as $\Delta(\nu_1) = \nu_1$, $\Delta(\nu_2) = \nu_2$, $\Delta(\nu_3) = \nu_3$, $\Delta(\nu_4) = \nu_5$, $\Delta(\nu_5) = \nu_4$ the contra πgs-continuity of Δ is evident. However, it is neither contra πg-continuous nor contra gs-continuous since $\Delta^{-1}(\{\nu_2\}) = \{\nu_2\} \notin \pi GC(\Psi)$ and $\Delta^{-1}(\{\nu_2\}) = \{\nu_2\} \notin \pi GSC(\Psi)$.

Example 3.2. Let $\Psi = \{\nu_1, \nu_2, \nu_3, \nu_4\}$, $\top = \{\emptyset, \{\nu_1\}, \{\nu_2\}, \{\nu_1, \nu_2\}, \Psi\}$. Match-ups for $\Delta : \Psi \to \Psi$ are

$$\Delta(\nu_1) = \Delta(\nu_2) = \Delta(\nu_3) = \nu_1, \Delta(\nu_4) = \nu_3.$$

Δ is contra πgs-continuous, but it is not contra $e^*\theta$-continuous since $\Delta^{-1}(\{\nu_1\}) = \Delta^{-1}(\{\nu_1, \nu_2\}) = \{\nu_1, \nu_2, \nu_3\}$ is not $e^*\theta$-closed w.r.t. $\top$.

Example 3.3. Given $\Psi = \{\nu_1, \nu_2, \nu_3, \nu_4\}$, $\top = \{\emptyset, \{\nu_1\}, \{\nu_2\}, \{\nu_1, \nu_2\}, \Psi\}$. Match-ups for $\Delta : \Psi \to \Psi$ are

$$\Delta(\nu_1) = \nu_3, \Delta(\nu_2) = \nu_1, \Delta(\nu_3) = \Delta(\nu_4) = \nu_4.$$

Although Δ is contra πgs-continuous, it is not almost contra e^*-continuous,since $\{\nu_1, \nu_3\}$ is regular open and $\Delta^{-1}(\{\nu_1, \nu_3\}) = \{\nu_1, \nu_2\}$ is not an e^*-closed. By checking the connections between these class of functions in [42] we can easily state that Δ cannot be almost contra $e^*\theta$-continuous, contra $e^*\theta$-continuous and contra e^*-continuous.

Example 3.4. $\top = \{\emptyset, \{\nu_1\}, \{\nu_2\}, \{\nu_2, \nu_1\}, \{\nu_3, \nu_1\}, \{\nu_1, \nu_3, \nu_2\}, \{\nu_1, \nu_2, \nu_4\}, \Psi\}$ is a topology on $\Psi = \{\nu_1, \nu_2, \nu_3, \nu_4\}$. Match-ups of $\Delta : \Psi \to \Psi$ are

$$\Delta(\nu_1) = \nu_1, \Delta(\nu_2) = \nu_2, \Delta(\nu_3) = \Delta(\nu_4) = \nu_4.$$

Since $\Delta^{-1}(\{\nu_1, \nu_2\}) = \Delta^{-1}(\{\nu_1, \nu_2, \nu_3\}) = \{\nu_1, \nu_2\} \notin \pi GSC(\Psi)$, Δ is not contra πgs-continuous. However, it is contra $e^*\theta$-continuous. So it is contra e^*-continuous, almost contra $e^*\theta$-continuous and almost contra e^*-continuous.

As seen from the examples above contra πgs-continuity does not require almost contra $e^*\theta$-continuity, almost contra e^*-continuity, contra $e^*\theta$-continuity and contra e^*-continuity. It is also clear that almost contra $e^*\theta$-continuity, almost contra e^*-continuity, contra $e^*\theta$-continuity and contra e^*-continuity does not require contra πgs-continuity. As another result we can state that contra we^*-continuity does not require contra πgs-continuity.

Research Question Does contra πgs-continuity require contra we^*-continuity?

Example 3.5. $\top = \{\emptyset, \{\nu_1\}, \{\nu_2\}, \{\nu_1, \nu_2\}, \{\nu_3, \nu_1\}, \{\nu_1, \nu_3, \nu_2\}, \{\nu_2, \nu_1, \nu_4\}, \Psi\}$ is a topology on $\Psi = \{\nu_1, \nu_2, \nu_3, \nu_4\}$. Match-ups of $\Delta : \Psi \to \Psi$ are

$$\Delta(\nu_1) = \nu_3, \Delta(\nu_2) = \nu_2, \Delta(\nu_3) = \nu_1, \Delta(\nu_4) = \nu_2$$

Δ is contra πgs-continuous, but it is not contra $b\hat{g}$-continuous since $\Delta^{-1}(\{\nu_1, \nu_3\}) = \{\nu_1, \nu_3\}$ is not $b\hat{g}$-closed. So it cannot be contra $sb\hat{g}$-continuous.

Example 3.6. $\top = \{\emptyset, \{\nu_1, \nu_5\}, \{\nu_2, \nu_4\}, \{\nu_1, \nu_2, \nu_4, \nu_5\}, \Psi\}$ is a topology on $\Psi = \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}$. Match-ups of $\Delta : \Psi \to \Psi$ are

$$\Delta(\nu_1) = \nu_1, \Delta(\nu_2) = \nu_2, \Delta(\nu_3) = \Delta(\nu_4) = \nu_3, \Delta(\nu_5) = \nu_5$$

Δ is contra $b\hat{g}$-continuous. However, since $\Delta^{-1}(\{\nu_1, \nu_2, \nu_4, \nu_5\}) = \{\nu_1, \nu_2, \nu_5\} \notin \pi GSC(\Psi)$, it is not contra πgs-continuous.

As seen from the examples above there is no relation between contra $b\hat{g}$-continuity and contra πgs-continuity. As another result we see that a contra πgs-continuity does not require contra $sb\hat{g}$-continuity.

Research Question Does contra $sb\hat{g}$-continuity require contra πgs-continuity?

Example 3.7. [8] Let $\top = \{\emptyset, \{\nu_1\}, \{\nu_2\}, \{\nu_1, \nu_2\}, \{\nu_3, \nu_2\}, \{\nu_3, \nu_2, \nu_1\}, \Psi\}$ and $\perp = \{\emptyset, \{\nu_1\}, \Psi\}$ be two topologies on $\Psi = \{\nu_1, \nu_2, \nu_3, \nu_4\}$. The identity function $\Delta : (\Psi, \top) \to (\Psi, \perp)$ is contra πgs-continuous, but it is not contra πgp-continuous.

Example 3.8. [8] Let $\top = \{\emptyset, \{\nu_2\}, \{\nu_3, \nu_2\}, \{\nu_1, \nu_4\}, \{\nu_1, \nu_2, \nu_4\}, \{\nu_1, \nu_2, \nu_4, \nu_3\}, \Psi\}$ and $\perp = \{\emptyset, \{\nu_4\}, \Psi\}$ be two topologies on $\Psi = \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}$. The identity function $\Delta : (\Psi, \top) \to (\Psi, \perp)$ is contra πgp-continuous and contra $\pi g\gamma$-continuous, but it is not contra πgs-continuous.

As seen from Example 3.7 and Example 3.8 there is no connection between contra πgp-continuity and contra πgs-continuity. Example 3.8 also shows that contra $\pi g\gamma$-continuity does not require contra πgs-continuity.

Theorem 3.2. *[4] Let $\aleph \subset \Psi$, afterwards $\aleph \in RO(\Psi)$ if and only if $\aleph \in \pi O(\Psi) \cap \pi GSC(\Psi)$.*

Definition 3.3. $\Delta : \Psi \to \Phi$ is called as:
(ι_1) π-continuous [3] :⇔ ($\digamma \in \perp \Rightarrow \Delta^{-1}(\digamma) \in \pi O(\Psi)$),
(ι_2) πg-continuous [3] :⇔ ($\digamma \in \perp \Rightarrow \Delta^{-1}(\digamma) \in \pi GO(\Psi)$),
(ι_3) πgs-continuous [4] :⇔ ($\digamma \in C(\Phi) \Rightarrow \Delta^{-1}(\digamma) \in \pi GSC(\Psi)$),
(ι_4) completely continuous [44] :⇔ ($\digamma \in \perp \Rightarrow \Delta^{-1}(\digamma) \in RO(\Psi)$).

Theorem 3.3. *Whenever $\Delta : \Psi \to \Phi$, afterwards the statement below is satisfied:*
Δ is contra πgs-continuous and π-continuous if and only if Δ is completely continuous.

Proof. Obvious from Theorem 3.2. □

Theorem 3.4. *Under the circumstance $\pi GSO(\Psi)$ is closed under arbitrary unions, it can be stated that whenever $\Delta : \Psi \to \Phi$ is contra πgs-continuous and Φ is regular, afterwards Δ is πgs-continuous.*

Definition 3.4. Whenever $\pi GSC(\Psi) \subset SC(\Psi)$ afterwards Ψ is accepted as πgs-$T_{\frac{1}{2}}$ [4].

Theorem 3.5. *Whenever Ψ is considered as πgs-$T_{\frac{1}{2}}$ space afterwards, contra πgs-continuity, contra-semicontinuity and contra gs-continuity of $\Delta : \Psi \to \Phi$ are identical.*

Proof. Assume that Ψ as a πgs-$T_{\frac{1}{2}}$ space. Since $SC(\Psi) \subset \pi GSC(\Psi)$, we have $SC(\Psi) = \pi GSC(\Psi)$. Using the relation $SC(\Psi) \subset GSC(\Psi)$, we obtain $\pi GSC(\Psi) \subset GSC(\Psi)$. Since $GSC(\Psi) \subset \pi GSC(\Psi)$, we have $GSC(\Psi) = \pi GSC(\Psi)$. Therefore $\pi GSC(\Psi) = SC(\Psi) = GSC(\Psi)$. □

Theorem 3.6. *For each $i \in I$, p_i stands for projection of $\prod \Phi_i$ onto Φ_i. If $\Delta : \Psi \to \prod \Phi_i$ is contra πgs-continuous, then $p_i \circ \Delta : \Psi \to \Phi_i$ is contra πgs-continuous for each $i \in I$.*

Proof. Since p_i is continuous and Δ is contra πgs-continuous, we can state that $p_i^{-1}(U_i)$ is open in $\prod \Upsilon_i$ for any $U_i \in \perp_i$ and $(p_i \circ \Delta)^{-1}(U_i) = \Delta^{-1}(p_i^{-1}(U_i)) \in \pi GSC(\Psi)$. Hereby, $p_i \circ \Delta$ is contra πgs-continuous. □

Definition 3.5. A topological space Ψ is said to be locally πgs-indiscrete if $\pi GSO(\Psi) \subset C(\Psi)$.

Theorem 3.7. *The fact that Ψ is locally πgs-indiscrete for contra πgs-continuous $\Delta : \Psi \to \Phi$ requires that Δ is continuous.*

Proof. Allow $\digamma \in \perp$. Since Δ is contra πgs-continuous, $\Delta^{-1}(\digamma) \in \pi GSC(\Psi)$. Since Ψ is locally πgs-indiscrete, $\Delta^{-1}(\digamma) \in \top$. □

Theorem 3.8. *Whenever Ψ is a πgs-$T_{\frac{1}{2}}$ for any $\Delta : \Psi \to \Phi$, afterwards following are equivalent :*
(ι_1) *Δ is completely continuous;*
(ι_2) *Δ is π-continuous and contra πgs-continuous;*
(ι_3) *Δ is π-continuous and contra gs-continuous;*
(ι_4) *Δ is π-continuous and contra-semicontinuous.*

Proof. Equivalence of (ι_2), (ι_3) and (ι_4) is obvious from Theorem 3.5 and the equivalence of (ι_1) and (ι_2) can be easily seen from Theorem 3.2. □

Definition 3.6. The topological space $(\Psi, \top)$ is called:
(ι_1) submaximal [45] $:\Leftrightarrow (\forall \aleph \subset \Psi)(cl(\aleph) = \Psi \Rightarrow \aleph \in \top)$,
(ι_2) extremally disconnected [45] $:\Leftrightarrow (\forall \aleph \subset \Psi)(\aleph \in \top \Rightarrow cl(\aleph) \in \top)$.

Definition 3.7. $\Delta : \Psi \to \Phi$ is called contra α-continuous [46] (correspondingly contra precontinuous [46], contra β-continuous [47], contra γ-continuous [48]) if the preimage of every open subsets of Φ is α-closed (correspondingly preclosed, β-closed, γ-closed) in Ψ.

Lemma 3.1. *For any $(\Psi, \top)$, if $\pi GSC(\Psi)$ is closed under finite unions then, πgs-$\top = \{U \subset \Psi : cl_{\pi gs}(\Psi \backslash U) = \Psi \backslash U\}$.*

Theorem 3.9. *Whenever Ψ is extremally disconnected, submaximal and πgs-$T_{\frac{1}{2}}$ for any $\Delta : \Psi \to \Phi$, afterwards the following are equivalent:*
(ι_1) Δ is contra πgs-continuous;
(ι_2) Δ is contra gs-continuous;
(ι_3) Δ is contra-semicontinuous;
(ι_4) Δ is contra-continuous;
(ι_5) Δ is contra precontinuous;
(ι_6) Δ is contra β-continuous;
(ι_7) Δ is contra α-continuous;
(ι_8) Δ is contra γ-continuous.

Proof. In an extremally disconnected submaximal space $(\Psi, \top)$,

$$\top = \alpha O(\Psi) = SO(\Psi) = PO(\Psi) = \gamma O(\Psi) = \beta O(\Psi).$$

From this fact we can say that $(\iota_3), (\iota_4), (\iota_5), (\iota_6), (\iota_7), (\iota_8)$ are equivalent. The equivalence of $(\iota_1), (\iota_2), (\iota_3)$ is obvious from Theorem 3.5. □

Theorem 3.10. *Whenever Ψ is said to be extremally disconnected, afterwards any $\Delta : \Psi \to \Phi$ is contra πgs-continuous and πgs-continuous.*

Definition 3.8. $\Delta : \Psi \to \Phi$ is said to be πgs-irresolute [4] if $\Delta^{-1}(F) \in \pi GSO(\Psi)$ for each $F \in \pi GSO(\Phi)$.

Theorem 3.11. *For $\Delta : \Psi \to \Phi$ and $\rho : \Phi \to \zeta$ following properties hold:*
(ι_1) If Δ is πgs-irresolute and ρ is contra πgs-continuous, then $\rho \circ \Delta$ is contra πgs-continuous;
(ι_2) If Δ is contra πgs-continuous and ρ is continuous, then $\rho \circ \Delta$ is contra πgs-continuous;
(ι_3) If Δ is contra πgs-continuous and ρ is RC-continuous, then $\rho \circ \Delta$ is πgs-continuous;
(ι_4) If Δ is πgs-continuous and ρ is contra continuous, then $\rho \circ \Delta$ is contra πgs-continuous;
(ι_5) If Δ is πgs-irresolute and ρ is RC-continuous (correspondingly contra π-continuous, contra-continuous, contra g-continuous, contra πg-continuous, contra-semicontinuous, contra gs-continuous), then $\rho \circ \Delta$ is contra πgs-continuous.

Definition 3.9. $\Delta : \Psi \to \Phi$ is characterized as πgs-open if $\Delta(\aleph)$ is πgs-open in Φ for each πgs-open subset $\aleph$ of Ψ.

Theorem 3.12. *$\Delta : \Psi \to \Phi$ and $\rho : \Phi \to \zeta$ be two functions and suppose that $\pi GSC(\Phi)$ is closed under arbitrary intersections. Whenever Δ is surjective πgs-open function and $\rho \circ \Delta$ is contra πgs-continuous, afterwards ρ is contra πgs-continuous.*

Proof. Suppose $\mu \in \Phi$ and $\Theta \in C(\rho(\mu), \zeta)$. Since Δ is surjective, existence of $\nu \in \Psi$ satisfying $\Delta(\nu) = \mu$ is clear. Naturally, $\Theta \in C(\rho \circ \Delta(\nu), \zeta)$. Since $\rho \circ \Delta$ is contra πgs-continuous, $\eth \in \pi GSO(\nu, \Psi)$ naturally appears satisfying $\rho \circ \Delta(\eth) \subset \Theta$ relation. Since Δ is πgs-open, $\Delta(\eth)$ is an element of $\pi GSO(\mu, \Phi)$. Hence, for each $\mu \in \Phi$ and for each $\Theta \in C(\rho(\mu), \zeta)$, existence of $\Delta(\eth) = F \in \pi GSO(\mu, \Phi)$ is natural satisfying $\rho(F) \subset \Theta$. By Theorem 3.1 ρ is contra πgs-continuous. □

Corollary 3.1. *Whenever $\pi GSC(\Phi)$ is closed under arbitrary intersections and $\Delta : \Psi \to \Phi$ is surjective πgs-irresolute and πgs-open, afterwards for any $\rho : \Phi \to \zeta$, $\rho \circ \Delta$ is contra πgs-continuous if and only if ρ is contra πgs-continuous.*

Proof. Obvious from Theorems 3.11 and 3.12. □

Definition 3.10. $\Delta : \Psi \to \Phi$ is characterized as weakly contra πgs-continuous whenever $\nu \in \Psi$ and $\Theta \in C(\Delta(\nu), \Phi)$, afterwards a set $F \in \pi GSO(\nu, \Psi)$ exists satisfying $int(\Delta(F)) \subset \Theta$.

Definition 3.11. A function $\Delta : \Psi \to \Phi$ is called as $(\pi gs\text{-}s)$-*open* whenever $\Delta(F) \in SO(\Phi)$ for all $F \in \pi GSO(\Psi)$.

Theorem 3.13. *Whenever $\Delta : \Psi \to \Phi$ is a weakly contra πgs-continuous and $(\pi gs\text{-}s)$-open and $\pi GSO(\Psi)$ is closed under arbitrary unions, afterwards Δ is contra πgs-continuous.*

Proof. Whenever $\nu \in \Psi$ and $\Theta \in C(\Delta(\nu), \Phi)$, with the weakly contra πgs-continuity of Δ, as a result the set $F \in \pi GSO(\nu, \Psi)$ appears satisfying $int(\Delta(F)) \subset \Theta$. Since Δ is $(\pi gs\text{-}s)$-*open*, $\Delta(F)$ is semi-open in Φ. Hence, $\Delta(F) \subset cl(int(\Delta(F))) \subset cl(\Theta) = \Theta$. □

Definition 3.12. $fr_{\pi gs}(\aleph)$ stands for πgs-frontier of $\aleph \in \Psi$ and characterized as $cl_{\pi gs}(\aleph) \cap cl_{\pi gs}(\Psi \backslash \aleph)$.

Theorem 3.14. *Let $\Delta : \Psi \to \Phi$ be a function. Whenever $\pi GSC(\Psi)$ is closed under arbitrary intersections then, the set of whole points $\nu \in \Psi$ at which Δ is not contra πgs-continuous is equal to $\bigcup\{fr_{\pi gs}(\Delta^{-1}(\Theta)) : \Theta \in C(\Delta(\nu), \Phi)\}$.*

Proof. Let ν be any element of Ψ at which Δ is not contra πgs-continuous. Then, there exists a closed subset Θ of Φ comprising $\Delta(\nu)$ such that $\Delta(F)$ is not contained in Θ for every $F \in \pi GSO(\nu, \Psi)$. So $F \cap (\Psi \backslash \Delta^{-1}(\Theta)) \neq \emptyset$. Then, we have $\nu \in cl_{\pi gs}(\Psi \backslash \Delta^{-1}(\Theta))$. Since $\nu \in \Delta^{-1}(\Theta) \subset cl_{\pi gs}(\Delta^{-1}(\Theta))$, $\nu \in fr_{\pi gs}(\Delta^{-1}(\Theta))$.
For the converse, assume that Δ is contra πgs-continuous at $\nu \in \Psi$ and $\Theta \in C(\Delta(\nu), \Phi)$. Naturally a set $F \in \pi GSO(\nu, \Psi)$ appears satisfying $F \subset \Delta^{-1}(\Theta)$. Therefore, $\nu \in int_{\pi gs}(\Delta^{-1}(\Theta))$. Hence, $\nu \notin fr_{\pi gs}(\Delta^{-1}(\Theta))$. □

Corollary 3.2. *For any $\Delta : \Psi \to \Phi$, whenever $\pi GSC(\Psi)$ is closed under arbitrary intersections, afterwards Δ is not contra πgs-continuous at ν if and only if $\Theta \in C(\Delta(\nu), \Phi)$ appears satisfying $\nu \in fr_{\pi gs}(\Delta^{-1}(\Theta))$.*

4. Preservation theorems

In this section, new separation axioms, connected spaces, compact spaces, covers and graphs related to πgs-open sets are defined and various results are presented by examining the properties of these new concepts.

Definition 4.1. Ψ is said to be πgs-T_1 whenever ν and μ in Ψ are distinct points, sets $F \in \pi GSO(\nu, \Psi)$ and $\mho \in \pi GSO(\mu, \Psi)$ naturally appears satisfying $\mu \notin F$ and $\nu \notin \mho$.

Definition 4.2. Ψ is said to be πgs-T_2 whenever ν and μ in Ψ are distinct points, sets $F \in \pi GSO(\nu, \Psi)$ and $\mho \in \pi GSO(\mu, \Psi)$ naturally appears satisfying $F \cap \mho = \emptyset$.

Theorem 4.1. *Under the assumption $\mho$ is an Uryshon space, whenever ν and μ are distinct points in Ψ a function $\Delta : \Psi \to \Phi$ naturally appears that is contra πgs-continuous at ν and μ and for which $\Delta(\nu) \neq \Delta(\mu)$, afterwards Ψ is πgs-T_2.*

Proof. Assume that ν and μ as distinct points in Ψ. Also, let $\Delta : \Psi \to \Phi$ be contra πgs-continuous at ν and μ such that $\Delta(\nu) \neq \Delta(\mu)$. Letting $\nu' = \Delta(\nu)$ and $\mu' = \Delta(\mu)$ with the knowlegde of Φ is Urysohn, existence of $\partial \in O(\nu', \Phi)$ and $F \in O(\mu', \Phi)$ guaranteed such that $cl(\partial) \cap cl(F) = \emptyset$. Since Δ is contra πgs-continuous at ν and μ, there exist πgs-open subsets $\aleph$ and Ω of Ψ comprising ν and μ, correspondingly, such that $\Delta(\aleph) \subset cl(\partial)$ and $\Delta(\Omega) \subset cl(F)$. Hereby, $\Delta(\aleph \cap \Omega) \subset \Delta(\aleph) \cap \Delta(\Omega) \subset cl(\partial) \cap cl(F) = \emptyset$ which implies that $\aleph \cap \Omega = \emptyset$. Hence, Ψ is πgs-T_2. □

Corollary 4.1. *Whenever $\Delta : \Psi \to \Phi$ is contra πgs-continuous injection and Φ is an Urysohn space, afterwards Ψ is πgs-T_2.*

Definition 4.3. The topological space Ψ is called as,
(ι_1) πgs-connected space $:\Leftrightarrow$ Ψ is not the union of two disjoint non-empty πgs-open sets,
(ι_2) gs-connected space [15] $:\Leftrightarrow$ Ψ is not the union of two disjoint non-empty gs-open sets.

Remark 4.1. Although πgs-connected spaces are gs-connected, the contrary implication is not valid in general.

Example 4.1. Let $\Psi = \{\nu, \mu\}$ and $\top = \{\emptyset, \{\nu\}, \Psi\}$. Ψ is gs-connected, but it is not πgs-connected since $\{\nu\}$ and $\{\mu\}$ are non-empty disjoint πgs-open subsets of Ψ.

Theorem 4.2. *For a topological space Ψ the following are equivalent:*
(ι_1) *Ψ is πgs-connected;*
(ι_2) *The only subsets of Ψ which are both πgs-open and πgs-closed are $\emptyset$ and Ψ;*
(ι_3) *Each πgs-continuous function of Ψ into a discrete space Φ with at least two points is a constant function.*

Proof. Firstly let Ψ be a topological space.
$(\iota_1) \Rightarrow (\iota_2)$ Suppose that $\aleph$ is a proper non-empty subset of Ψ which is both πgs-open and πgs-closed. Then, $\Psi\backslash\aleph$ is a proper non-empty subset of Ψ which is both πgs-open and πgs-closed, $\aleph \cap (\Psi\backslash\aleph) = \emptyset$ and $\aleph \cup (\Psi\backslash\aleph) = \Psi$. But this result contradicts with the πgs-connectedness of Ψ. Hence, the only subsets of Ψ which are both πgs-open and πgs-closed $\emptyset$ and Ψ.
$(\iota_2) \Rightarrow (\iota_1)$ Suppose that Ψ is not πgs-connected. Then as a result two non-empty disjoint πgs-open subsets $\aleph$ and Ω of Ψ appears such that $\aleph \cup \Omega = \Psi$. Since $\aleph = \Psi\backslash\Omega$ and $\Omega = \Psi\backslash\aleph$, $\aleph$ and Ω are proper non-empty subsets of Ψ which are both πgs-open and πgs-closed, but this is a contradiction. Hereby, Ψ is πgs-connected.
$(\iota_2) \Rightarrow (\iota_3)$ Let Φ be any discrete space with at least two elements and $\Delta : \Psi \to \Phi$ be any contra πgs-continuous function. Since Φ is discrete, $\{\mu\}$ is clopen in Φ for each $\mu \in \Phi$. Therefore, $\{\mu\}$ is both πgs-open and πgs-closed in Φ for each $\mu \in \Phi$. We also have $\Psi = \Delta^{-1}(\Phi) = \Delta^{-1}(\bigcup\{\{\mu\} : \mu \in \Phi\}) = \bigcup\{\Delta^{-1}(\{\mu\}) : \mu \in \Phi\}$. By (ι_2), $\Delta^{-1}(\{\mu\}) = \emptyset$ or $\Delta^{-1}(\{\mu\}) = \Psi$ for each $\mu \in \Phi$. If $\Delta^{-1}(\{\mu\}) = \emptyset$ for some $\mu \in \Phi$ then, Δ would not be a function anymore. If there exist at least two distinct elements a and b in Φ such that $\Delta^{-1}(\{a\}) = \Psi = \Delta^{-1}(\{b\})$, then Δ would not be a function anymore. Therefore, there exists only one element μ of Φ such that $\Delta^{-1}(\{\mu\}) = \Psi$, which means that $\Delta(\Psi) = \{\mu\}$. Hence, Δ is a constant function.
$(\iota_3) \Rightarrow (\iota_2)$ Let P be a non-empty set such that $P \in \pi GSO(\Psi) \cap \pi GSC(\Psi)$, Φ be any discrete space with at least two elements and contra πgs-continuous $\Delta : \Psi \to \Phi$ defined as $\Delta(P) = \{\varsigma\}$ and $\Delta(\Psi\backslash P) = \{\eta\}$, for distinct elements ς and η of Φ. Since Δ is constant by (ι_3), $\Psi\backslash P = \emptyset$. Therefore, $P = \Psi$. □

Theorem 4.3. *Let $\Delta : \Psi \to \Phi$ be a surjective contra πgs-continuous function. While Ψ is πgs-connected, Φ cannot be a discrete space.*

Proof. Assume Φ as a discrete space. Let $\aleph$ be any proper non-empty subset of Φ. Since $\aleph$ is clopen in Φ and $\Delta : \Psi \to \Phi$ is contra πgs-continuous surjection, $\Delta^{-1}(\aleph) \in \pi GSO(\Psi) \cap \pi GSC(\Psi)$ is a proper non-empty subset of Ψ. But this result contradicts with the πgs-connectedness of Ψ. Hence, Φ is not a discrete space. □

Theorem 4.4. *While whole contra πgs-continuous functions with a domain Ψ into any T_0 space Φ is constant, Ψ has to be πgs-connected.*

Proof. Assume that Ψ is not πgs-connected. So, at least one proper non-empty subset $\aleph \in \pi GSO(\Psi) \cap \pi GSC(\Psi)$ appears. Let $\Phi = \{\varsigma, \eta\}$ and $\perp = \{\emptyset, \{\varsigma\}, \{\eta\}, \Phi\}$. Let $\Delta : \Psi \to \Phi$ be a function such that $\Delta(\aleph) = \{\varsigma\}$ and $\Delta(\Psi\backslash\aleph) = \{\eta\}$. Then, Φ is a T_0 space and Δ is a contra πgs-continuous function which is not constant. But this is a contradiction. Hereby, Ψ has to be πgs-connected. □

Theorem 4.5. *Whenever $\Delta : \Psi \to \Phi$ is surjective contra πgs-continuous function and Ψ is πgs-connected, afterwards Φ has to be connected.*

Proof. Suppose that Φ as a disconnected space. So two non-empty disjoint open sets $\aleph$ and Ω of Φ appear, so that $\aleph \cup \Omega = \Phi$. So $\Delta^{-1}(\aleph) \neq \emptyset$, $\Delta^{-1}(\Omega) \neq \emptyset$, $\Delta^{-1}(\aleph) \cap \Delta^{-1}(\Omega) = \emptyset$, $\Delta^{-1}(\aleph) \cup \Delta^{-1}(\Omega) = \Psi$ since Δ is surjective. Since Δ is contra πgs-continuous, $\Delta^{-1}(\aleph)$ and $\Delta^{-1}(\Omega)$ are both πgs-open and πgs-closed in Ψ. Therefore, we reach the result that Ψ is not πgs-connected which is a contradiction. Hereby, Φ is connected. □

Theorem 4.6. *The projection functions $p_\Psi : \Psi \times \Phi \to \Psi$ and $p_\Phi : \Psi \times \Phi \to \Phi$ are πgs-irresolute.*

Proof. Let $p_\Psi : \Psi \times \Phi \to \Psi$ be the projection function from $\Psi \times \Phi$ onto Ψ and $\aleph$ be any πgs-closed subset of Ψ. Then, ${p_\Psi}^{-1}(\aleph) = \aleph \times \Phi$. Let $\digamma$ be any π-open subset of $\Psi \times \Phi$ involving $\aleph \times \Phi$. Then, there exists a π-open subset $\mho$ of Ψ involving $\aleph$ such that $\digamma = \mho \times \Phi$. Since $\aleph$ is πgs-closed in Ψ, $scl(\aleph) \subset \mho$. Therefore, $scl(\aleph) \times \Phi \subset \mho \times \Phi = \digamma$. Since $scl(\aleph \times \Phi) \subset scl(\aleph) \times \Phi$, we have $scl(\aleph \times \Phi) \subset \digamma$. So $\aleph \times \Phi = p_\Psi^{-1}(\aleph)$ is πgs-closed in $\Psi \times \Phi$. Hence, projection function $p_\Psi : \Psi \times \Phi \to \Psi$ is πgs-irresolute. The proof for the other projection function $p_\Phi : \Psi \times \Phi \to \Phi$ is similar. □

Theorem 4.7. *Whenever $\Delta : \Psi \to \Phi$ is a πgs-irresolute surjection and Ψ is πgs-connected, afterwards Φ has to be πgs-connected.*

Proof. Assume that Φ is not πgs-connected. Naturally, two non-empty disjoint πgs-open subsets $\digamma$ and Ω of Φ appears so that $\digamma \cup \Omega = \Phi$. Then $\Delta^{-1}(\digamma)$ and $\Delta^{-1}(\Omega)$ are non-empty πgs-open subsets of Ψ, since Δ is surjective and πgs-irresolute. Besides, $\emptyset = \Delta^{-1}(\digamma \cap \Omega) = \Delta^{-1}(\digamma) \cap \Delta^{-1}(\Omega)$ and $\Psi = \Delta^{-1}(\digamma) \cup \Delta^{-1}(\Omega)$. Therefore, we reach the result that Ψ is not πgs-connected which is a contradiction. Hereby, Φ is πgs-connected. □

Theorem 4.8. *Whenever the product space of two non-empty spaces is πgs-connected, each factor space has to be πgs-connected.*

Proof. Accept Ψ and Φ as non-empty topological spaces and the product space $\Psi \times \Phi$ as πgs-connected. Since the projection functions are πgs-irresolute and surjective, by Theorem 4.7, Ψ and Φ are πgs-connected. □

Definition 4.4. A topological space Ψ is called as:
(ι_1) πgs-compact if every πgs-open cover of Ψ has a finite subcover,
(ι_2) countably πgs-compact if every countable cover of Ψ by πgs-open sets has a finite subcover,
(ι_3) πgs-Lindelöf if every πgs-open cover of Ψ has a countable subcover.

Definition 4.5. $\aleph \in \Psi$ is characterized to be πgs-compact relative to Ψ whenever every πgs-open cover of $\aleph$ by πgs-open sets of Ψ has a finite subcover.

Theorem 4.9. *Whenever $\Delta : \Psi \to \Phi$ is contra πgs-continuous and $\aleph \subset \Psi$ is πgs-compact relative to Ψ, afterwards $\Delta(\aleph)$ has to be strongly S-closed.*

Proof. Let $\{\Theta_i : i \in I\}$ be a closed cover of $\Delta(\aleph)$ by closed subsets of the subspace $\Delta(\aleph)$. Then for each $i \in I$, there exits a closed set $\aleph_i$ in Φ such that $\Delta(\aleph) = \bigcup\{\Theta_i : i \in I\} = \bigcup\{\aleph_i \cap \Delta(\aleph) : i \in I\} = (\bigcup\{\aleph_i : i \in I\}) \cap \Delta(\aleph)$ and $\Theta_i = \aleph_i \cap \Delta(\aleph)$. Since for each $\nu \in \aleph$, we have $\Delta(\nu) \in \Delta(\aleph)$ and since Δ is contra πgs-continuous, for each $\nu \in \aleph$ there exists $i(\nu) \in I$ and there exists $F_\nu \in \pi GSO(\nu, \Psi)$ such that $\Delta(\nu) \in \aleph_{i(\nu)}$ and $\Delta(F_\nu) \subset \aleph_{i(\nu)}$. Then, $\{F_\nu : \nu \in \aleph\}$ is a cover of $\aleph$ by πgs-open sets of Ψ. Since $\aleph$ is πgs-compact relative to Ψ, there exists a finite subset $\aleph_0$ of $\aleph$ such that $\aleph \subset \bigcup\{F_\nu : \nu \in \aleph_0\}$. Then, we obtain $\Delta(\aleph) \subset \bigcup\{\aleph_{i(\nu)} : \nu \in \aleph_0\}$. Therefore, $\Delta(\aleph) = \Delta(\aleph) \cap (\bigcup\{\aleph_{i(\nu)} : \nu \in \aleph_0\}) = \bigcup\{\Delta(\aleph) \cap \aleph_{i(\nu)} : \nu \in \aleph_0\} = \bigcup\{\Theta_{i(\nu)} : \nu \in \aleph_0\}$ and this means that $\{\Theta_{i(\nu)} : \nu \in \aleph_0\}$ is a finite subcover of $\{\Theta_i : i \in I\}$. Hence, $\Delta(\aleph)$ is strongly S-closed. □

Corollary 4.2. *Whenever $\Delta : \Psi \to \Phi$ is a contra πgs-continuous surjection and Ψ is πgs-compact, afterwards Φ has to be strongly S-closed.*

Theorem 4.10. *Whenever the product space of two non-empty spaces is πgs-compact, afterwards each factor space has to be πgs-compact.*

Proof. Let $\Psi \times \Phi$ be the product space of the non-empty topological spaces Ψ and Φ and $\Psi \times \Phi$ be πgs-compact. Let $\{\eth_i : i \in I\}$ be any πgs-open cover of Ψ. Then, $\Psi \times \Phi = p_\Psi^{-1}(\Psi) = p_\Psi^{-1}(\bigcup\{\eth_i : i \in I\}) = \bigcup\{p_\Psi^{-1}(\eth_i) : i \in I\}$. Since p_Ψ is πgs-irresolute, $p_\Psi^{-1}(\eth_i) = \eth_i \times \Phi$ is πgs-open in $\Psi \times \Phi$ for each $i \in I$. Therefore, $\{\eth_i \times \Phi : i \in I\}$ is a πgs-open cover of $\Psi \times \Phi$. Since $\Psi \times \Phi$ is πgs-compact, there exists a finite subset I_0 of I such that $\bigcup\{\eth_i \times \Phi : i \in I_0\} = \Psi \times \Phi$. Then, $\Psi = p_\Psi(\Psi \times \Phi) = p_\Psi(\bigcup\{\eth_i \times \Phi : i \in I_0\}) = p_\Psi((\bigcup\{\eth_i : i \in I_0\}) \times \Phi) = \bigcup\{\eth_i : i \in I_0\}$. Hence, Ψ is πgs-compact. The proof for the space Φ is similar. □

Theorem 4.11. *Contra πgs-continuous images of πgs-Lindelöf (correspondingly countably πgs-compact) spaces are strongly S-Lindelöf (correspondingly strongly countably S-closed).*

Proof. Let Ψ be a πgs-Lindelöf space and $\Delta : \Psi \to \Phi$ be a surjective contra πgs-continuous function. Let $\{\Theta_i : i \in I\}$ be a closed cover of Φ. Since Δ is contra πgs-continuous, $\{\Delta^{-1}(\Theta_i) : i \in I\}$ is a πgs-open cover of Ψ. Since Ψ is πgs-Lindelöf, there exists a countable subset I_0 of I such that $\bigcup\{\Delta^{-1}(\Theta_i) : i \in I_0\} = \Psi$. Since Δ is surjective, $\Phi = \Delta(\Psi) = \Delta(\bigcup\{\Delta^{-1}(\Theta_i) : i \in I_0\}) = \bigcup\{\Delta(\Delta^{-1}(\Theta_i)) : i \in I_0\} = \bigcup\{\Theta_i : i \in I_0\}$ and then $\Phi = \bigcup\{\Theta_i : i \in I_0\}$. Hence, Φ is strongly S-Lindelöf. The proof for the contra πgs-continuous images of countably πgs-compact spaces is similar. □

Definition 4.6. The graph $G(\Delta)$ of $\Delta : \Psi \to \Phi$ is said to be a contra πgs-graph if for each (ν, μ) in $(\Psi \times \Phi) \backslash G(\Delta)$, there exist a set $\aleph$ in $\pi GSO(\nu, \Psi)$ and a set Ω in $C(\mu, \Phi)$ such that $(\aleph \times \Omega) \cap G(\Delta) = \emptyset$.

Theorem 4.12. *The following are equivalent for the graph $G(\Delta)$ of any $\Delta : \Psi \to \Phi$.*
(ι_1) *$G(\Delta)$ is contra πgs-graph;*
(ι_2) *For all $(\nu, \mu) \in (\Psi \times \Phi) \backslash G(\Delta)$, there exist a πgs-open set $\aleph \subset \Psi$ comprising ν and a closed set $\Omega \subset \Phi$ comprising μ such that $\Delta(\aleph) \cap \Omega = \emptyset$.*

Theorem 4.13. *Whenever $\Delta : \Psi \to \Phi$ is contra πgs-continuous and Φ is an Uryshon space, afterwards $G(\Delta)$ has to be a contra πgs-graph.*

Proof. For all $(\nu, \mu) \in (\Psi \times \Phi) \backslash G(\Delta)$, it is clear that $\Delta(\nu) \neq \mu$. Since Φ is Uryshon space, there exist open sets $\eth_\nu$ and $\eth_\mu$ in Φ comprising $\Delta(\nu)$ and μ, correspondingly, such that $cl(\eth_\nu) \cap cl(\eth_\mu) = \emptyset$. Since Δ is contra πgs-continuous, a $\aleph \in \pi GSO(\nu, \Psi)$ appears so that $\Delta(\aleph) \subset cl(\eth_\nu)$. Then, $\Delta(\aleph) \cap cl(\eth_\mu) = \emptyset$. Hereby, $G(\Delta)$ is contra πgs-graph. □

Theorem 4.14. *Let $\Delta : \Psi \to \Phi$ be a function and $\rho : \Psi \to \Psi \times \Phi$ be the graph function of Δ defined as $\rho(\nu) = (\nu, \Delta(\nu))$ for every $\nu \in \Psi$. If ρ is contra πgs-continuous, then Δ is contra πgs-continuous.*

Proof. For all open set $\digamma \subset \Phi$, it is clear that $\Psi \times \digamma$ is open in $\Psi \times \Phi$. Since ρ is a contra πgs-continuous function, $\Delta^{-1}(\digamma) = \rho^{-1}(\Psi \times \digamma)$ is πgs-closed in Ψ. Hence, Δ is contra πgs-continuous. □

Theorem 4.15. *Let $\Delta : \Psi \to \Phi$ and $\rho : \Psi \to \Phi$ be two contra πgs-continuous functions. If Φ is an Uryshon space and $\pi GSO(\Psi)$ is closed under finite intersections then, the set $E = \{\nu \in \Psi : \Delta(\nu) = \rho(\nu)\}$ is πgs-closed in Ψ.*

Proof. If we show that "$\nu \notin E \Rightarrow \nu \notin cl_{\pi gs}(E)$", then the theorem will be proved. Let $\nu \in \Psi \backslash E$. Then, $\Delta(\nu) \neq \rho(\nu)$. Since Φ is Uryshon, there exist open subsets $\digamma$ and $\mho$ of Φ comprising $\Delta(\nu)$ and $\rho(\nu)$, correspondingly, such that $cl(\digamma) \cap cl(\mho) = \emptyset$. Since Δ and ρ are contra πgs-continuous, $\Delta^{-1}(cl(\digamma))$ and $\rho^{-1}(cl(\mho))$ are πgs-open in Ψ. Let $\Delta^{-1}(cl(\digamma)) = \eth_1$ and $\rho^{-1}(cl(\mho)) = \eth_2$. Then, $\nu \in \eth_1 \cap \eth_2$. Let $\aleph = \eth_1 \cap \eth_2$. Since $\pi GSO(\Psi)$ is closed under finite intersections, $\aleph$ is a πgs-open set in Ψ comprising ν. So, $\Delta(\aleph) \cap \rho(\aleph) = \emptyset$. Hence, $\aleph \cap E = \emptyset$. By Lemma 2.1, $\nu \notin cl_{\pi gs}(E)$. □

Definition 4.7. For a subset $\aleph$ of space Ψ, if $cl_{\pi gs}(\aleph) = \Psi$ then $\aleph$ is said to be πgs-dense in Ψ.

Theorem 4.16. *Let $\Delta : \Psi \to \Phi$ and $\rho : \Psi \to \Phi$ be two functions. If*
(ι_1) Φ is an Uryshon space and $\pi GSO(\Psi)$ is closed under finite intersections,
(ι_2) Δ and ρ are contra πgs-continuous,
(ι_3) $\Delta = \rho$ on a πgs-dense subset $\aleph$ of Ψ,
then $\Delta = \rho$ on Ψ.

Proof. By Theorem 4.15, the set $E = \{\nu \in \Psi : \Delta(\nu) = \rho(\nu)\}$ is πgs-closed in Ψ. Since $\Delta = \rho$ on a πgs-dense subset $\aleph$, we have $\aleph \subset E$. Then, $\Psi = cl_{\pi gs}(\aleph) \subset cl_{\pi gs}(E) = E$. Hence, $E = \Psi$. □

Definition 4.8. Ψ is characterized to be weakly Hausdorff [49] if each element of Ψ is an intersection of regular closed sets.

Theorem 4.17. *Let $\Delta : \Psi \to \Phi$ be an injective contra πgs-continuous function. If Φ is weakly Hausdorff then, Ψ is πgs-T_1.*

Proof. Let ν and μ be any two elements in Ψ such that $\nu \neq \mu$. Since Δ is injective, $\Delta(\nu) \neq \Delta(\mu)$. Since Φ is weakly Hausdorff, regular closed subsets Θ_1 and Θ_2 of Φ comprising $\Delta(\nu)$ and $\Delta(\mu)$, correspondingly, appears such that $\Delta(\nu) \notin \Theta_2$ and $\Delta(\mu) \notin \Theta_1$. Since regular closed sets are closed and Δ is contra πgs-continuous, $\Delta^{-1}(\Theta_1)$ and $\Delta^{-1}(\Theta_2)$ are πgs-open subsets of Ψ comprising ν and μ, correspondingly, such that $\mu \notin \Delta^{-1}(\Theta_1)$ and $\nu \notin \Delta^{-1}(\Theta_2)$. Hence, Ψ is πgs-T_1. □

Theorem 4.18. *If $\Delta : \Psi \to \Phi$ is an injective function whose graph $G(\Delta)$ is contra πgs-graph then, Ψ is πgs-T_1.*

Proof. Let ν and μ be any two elements in Ψ such that $\nu \neq \mu$. Since Δ is injective, $(\nu, \Delta(\mu)) \in (\Psi \times \Phi) \backslash G(\Delta)$. Since $G(\Delta)$ is contra πgs-graph, there exists a πgs-open subset $\eth$ of Ψ and a closed subset Θ of Φ comprising ν and $\Delta(\mu)$, correspondingly, such that $\Delta(\eth) \cap \Theta = \emptyset$. Then $\Delta^{-1}(\Theta) \cap \eth = \emptyset$ and $\mu \notin \eth$. Similarly, since $(\Delta(\nu), \mu) \in (\Psi \times \Phi) \backslash G(\Delta)$, there exists a πgs-open subset Ω of Ψ comprising μ such that $\nu \notin \Omega$. Hence, Ψ is πgs-T_1. □

Theorem 4.19. *Let $\Delta : \Psi \to \Phi$ be an injective contra πgs-continuous function. Whenever Φ is an ultra Hausdorff space, Ψ has to be πgs-T_2.*

Proof. Let ν and μ be any two elements in Ψ such that $\nu \neq \mu$. Since Δ is injective, $\Delta(\nu) \neq \Delta(\mu)$. Since Φ is an ultra Hausdorff space, there exist disjoint clopen subsets $\eth_1$ and $\eth_2$ of Φ comprising $\Delta(\nu)$ and $\Delta(\mu)$, correspondingly. Then, $\Delta^{-1}(\eth_1)$ and $\Delta^{-1}(\eth_2)$ are disjoint subsets of Ψ comprising ν and μ, correspondingly, which are both πgs-open and πgs-closed in Ψ since Δ is contra πgs-continuous. Hence, Ψ is πgs-T_2. □

Definition 4.9. A space Ψ is said to be πgs-normal if each pair of non-empty disjoint closed sets can be seperated by disjoint πgs-open sets.

Theorem 4.20. *Let $\Delta : \Psi \to \Phi$ be an injective closed contra πgs-continuous function. If Φ is ultra normal, then Ψ is πgs-normal.*

Proof. Let Θ_1 and Θ_2 be any two non-empty disjoint closed subsets of Ψ. Since Δ is injective and closed, $\Delta(\Theta_1)$ and $\Delta(\Theta_2)$ are non-empty disjoint closed subsets of Φ. Since Φ is ultra normal, there exist disjoint clopen subsets $\eth_1$ and $\eth_2$ of Φ such that $\Delta(\Theta_1) \subset \eth_1$ and $\Delta(\Theta_2) \subset \eth_2$. Since Δ is contra πgs-continuous, $\Delta^{-1}(\eth_1)$ and $\Delta^{-1}(\eth_2)$ are disjoint πgs-open subsets of Ψ such that $\Theta_1 \subset \Delta^{-1}(\eth_1)$ and $\Theta_2 \subset \Delta^{-1}(\eth_2)$. Hence, Ψ is πgs-normal. □

5. Conclusion

It is understood from the studies of many researchers on contra continuity, which is one of the types of continuity that has been frequently studied recently as in the past, still arouses curiosity today. Researchers have not only examined various properties of the different types of contra continuous functions they have identified, but also examined the relationships between different contra continuities. In this study, we not only share the concept of contra πgs-continuity [8] related with πgs-open sets defined by Çaksu [4], but also investigated various properties of contra πgs-continuous functions and examined the relationships between different contra continuities. Remark 3.2 clearly shows that the concept of contra πgs-continuity is weaker than the concepts of contra πg-continuity [7], contra gs-continuity [9], contra g-continuity [39], contra semicontinuity [9], contra super continuity [38], contra continuity [6], strong contra continuity [37], perfect continuity [35] and RC continuity [9]. We also obtained important results by examining various properties related to separation axioms, connectedness, compactness, cover and graph concepts. We believe that our study will shed light on the studies researchers interested in contra continuous functions.

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On the q-Cesàro Bounded Double Sequence Space

Sezer Erdem*

Abstract
In this article, the new sequence space $\tilde{\mathcal{M}}_u^q$ is acquainted, described as the domain of the 4d (4-dimensional) q-Cesàro matrix operator, which is the q-analogue of the first order 4d Cesàro matrix operator, on the space of bounded double sequences. In the continuation of the study, the completeness of the new space is given and the inclusion relation related to the space is presented. In the last two parts, the duals of the space are determined and some matrix classes are acquired.

Keywords: *q-analogue, 4d q-Cesàro matrix, Double sequence space, Duals, Matrix transformations*

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1. Introduction

Obtaining q-analoques of known results has recently been found interesting by researchers. The q-analogue of a mathematical expression is the result that contains the parameter q and is more general than that expression, but reduces to the basic expression for $q \to 1$. According to the basic information about q-calculus acquired from [1], the q-analogue of any nonnegative number r is described as

$$[r]_q = \begin{cases} \dfrac{1-q^r}{1-q} & , \quad q \neq 1, \\ r & , \quad q = 1. \end{cases}$$

A little after the concept of convergence of double series with real terms (convergence in the Pringsheim's sense) introduced by Pringsheim [2], Hardy [3] introduced regular convergence, which also requires convergence according to each index. Zeltser [4] also contributed to these developments by comprehensively examining the topological structure of double sequences. The spaces of all double sequences that are convergent in the Pringsheim's sense ($\mathcal{C}_{\mathcal{P}}$), regularly convergent ($\mathcal{C}_r$), p-absolutely summable ($\mathcal{L}_p$) and bounded ($\mathcal{M}_u$) can be given as examples of the most basic double sequence spaces. It is known that a convergent double sequences in the Pringsheim's sense (shortly $\mathcal{P}$-convergent) need not be bounded. The space of bounded and $\mathcal{P}$-convergent double sequences is specifically denoted by $\mathcal{C}_{b\mathcal{P}}$. Additionally, for $p = 1$, the space $\mathcal{L}_p$ [5] is reduced to the space $\mathcal{L}_u$ [6]. The linear space of all double sequences with real terms is represented by Ω.

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If $u = (u_{lm}) \in \Omega$ is ϑ-convergent to a limit point M, in that case, it is expressed as $\vartheta - \lim_{l,m\to\infty} u_{lm} = M$ for $\vartheta \in \{\mathcal{P}, b\mathcal{P}, r\}$. Zeltser [6] described the double sequences $e^{rk} = (e^{rk}_{lm})$ by

$$e^{rk}_{lm} := \begin{cases} 1 & , \quad \text{if } (r,k) = (l,m) \\ 0 & , \quad \text{otherwise} \end{cases}$$

and e by $e = \sum_{r,k} e^{rk}$, where $\sum_{r,k} e^{rk} = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} e^{rk}$. If $b_{rklm} = 0$ for $l > r$ or $m > k$ or both, the 4d matrix $B = (b_{rklm})$ is called as *triangular matrix* and also if $b_{rkrk} \neq 0$, then the 4d triangular matrix B is named as *triangle* for all $r, k, l, m \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, 3, ...\}$.

Consider that $\Psi, \Lambda \in \Omega$, $u = (u_{lm}) \in \Psi$ and the 4d matrix $B = (b_{rklm})$. If $(Bu)_{rk} = \vartheta - \sum_{l,m} b_{rklm}u_{lm}$ (the B-transform of u) is in Λ, in that case B is called as a matrix mapping from Ψ into Λ and it is denoted by $B : \Psi \to \Lambda$ for all $u = (u_{lm}) \in \Psi$. Moreover, $B \in (\Psi : \Lambda)$ if and only if $B_{rk} \in \Psi^{\beta(\vartheta)}$ and $Bu \in \Lambda$, where $B_{rk} = (b_{rklm})_{l,m\in\mathbb{N}}$, $(\Psi : \Lambda) = \{B = (b_{rklm}) |\ B : \Psi \to \Lambda\}$ for all $r, k \in \mathbb{N}$ and $\Psi^{\beta(\vartheta)}$ is the $\beta(\vartheta)$ dual of Ψ.

The ϑ-summability domain $\Psi^{(\vartheta)}_B$ of the 4d matrix B is expressed as

$$\Psi^{(\vartheta)}_B := \left\{ u = (u_{lm}) \in \Omega : Bu := \left(\vartheta - \sum_{l,m} b_{rklm}u_{lm} \right)_{r,k\in\mathbb{N}} \text{exists, } Bu \in \Psi \subset \Omega \right\}. \tag{1.1}$$

The 4d matrix that transforms bounded and $\mathcal{P}$-convergent double sequences into $\mathcal{P}$-convergent double sequences with the same limit is called as RH regular [7, 8].

The double series spaces $\mathcal{BS}$ and $\mathcal{CS}_\vartheta$ spaces, whose sequences of partial sums are in the spaces $\mathcal{M}_u$ and $\mathcal{C}_\vartheta$, respectively, are described by Altay and Başar [9]. In addition to other related studies on single and double sequence spaces, some q-analogue studies and their references can be also expressed as [10–34].

Recently, Erdem and Demiriz [35] constructed a new double sequence space using the domain in $\mathcal{L}_p$ space of the 4d q-Cesàro matrix operator (q-analogue of the ordinary 4d Cesàro matrix) presented by Çinar and Et [36] and examined some algebraic and topological properties of this space.

As a continuation of the studies mentioned above, this article aims to acquaint the new double sequence space $\tilde{\mathcal{M}}^q_u$ as the domain of the 4d q-Cesàro matrix on the space $\mathcal{M}_u$, to examine its completeness, to determine its duals and to present some matrix mappings classes related aforementioned space.

2. q-Cesàro bounded double sequence space $\tilde{\mathcal{M}}^q_u$

In this section, the space $\tilde{\mathcal{M}}^q_u \in \Omega$ is constructed and we obtain that $\tilde{\mathcal{M}}^q_u$ is Banach space and linearly isomorphic to $\mathcal{M}_u$. Finally, an inclusion relation is presented about the space $\tilde{\mathcal{M}}^q_u$.

The 4d Cesàro matrix $C = (c_{rklm})$ of order one is given by

$$c_{rklm} := \begin{cases} \dfrac{1}{(r+1)(k+1)} & , \quad 0 \le l \le r\ ,\ 0 \le m \le k, \\ 0 & , \quad \text{otherwise,} \end{cases} \tag{2.1}$$

for all $r, k, l, m \in \mathbb{N}$. The 4d q-Cesàro matrix $C_{(1,1)}(q) = (c_{zntk}(q))$ that is the q-analogue of the matrix C and presented by Çinar and Et [36], is in the form below:

$$c_{rklm}(q) := \begin{cases} \dfrac{q^{l+m}}{[r+1]_q[k+1]_q} & , \quad 0 \le l \le r\ ,\ 0 \le m \le k, \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2.2}$$

In the same study, the authors showed that $C_{(1,1)}(q)$ is RH-regular for $q \geq 1$. The inverse $(C_{(1,1)}(q))^{-1}$ of the $C_{(1,1)}(q)$ is presented by

$$c^{-1}_{rklm}(q) := \begin{cases} (-1)^{r+k-(l+m)} \dfrac{[l+1]_q[m+1]_q}{q^{r+k}} & , \quad r-1 \le l \le r\ ,\ k-1 \le m \le k, \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2.3}$$

From the mentioned above, it can be seen that the $C_{(1,1)}(q)$-transform of a $u = (u_{tk}) \in \Omega$ is denoted by

$$\nu_{rk} := (C_{(1,1)}(q)u)_{rk} = \frac{1}{[r+1]_q[k+1]_q} \sum_{l,m=0}^{r,k} q^{l+m} u_{lm}, \quad (r, k \in \mathbb{N}). \tag{2.4}$$

It can be said that for the case $q \to 1$, $C_{(1,1)}(q)$ will be reduced to C.

Now, it is acquainted the set $\tilde{\mathcal{M}}_u^q$ of all q-Cesàro bounded double sequences by

$$\tilde{\mathcal{M}}_u^q = \left\{ u = (u_{lm}) \in \Omega : \sup_{r,k} \left| \frac{1}{[r+1]_q[k+1]_q} \sum_{l,m=0}^{r,k} q^{l+m} u_{lm} \right| < \infty \right\}.$$

Thus, $\tilde{\mathcal{M}}_u^q$ can be rephrased as $\tilde{\mathcal{M}}_u^q = (\mathcal{M}_u)_{C_{(1,1)}(q)}$ with the impression (1.1) and it can be called as q-Cesàro bounded double sequence space.

When q approaches 1, $\tilde{\mathcal{M}}_u^q$ is reduced to the space $\tilde{\mathcal{M}}_u$ presented in [37]. From now on, any term with a negative index will be ignored and assumed to be $q > 1$.

Theorem 2.1. *The set $\tilde{\mathcal{M}}_u^q$ is a Banach space with*

$$\|u\|_{\tilde{\mathcal{M}}_u^q} = \|C_{(1,1)}(q)u\|_{\mathcal{M}_u} = \left(\sup_{r,k \in \mathbb{N}} \left| \frac{1}{[r+1]_q[k+1]_q} \sum_{l,m=0}^{r,k} q^{l+m} u_{lm} \right| \right). \tag{2.5}$$

Proof. It is a known procedure to show that $\tilde{\mathcal{M}}_u^q$ is a normed linear space with (2.5) and it is omitted.

Consider the Cauchy sequence $u^{(n)} = \left(u_{lm}^{(n)}\right) \in \tilde{\mathcal{M}}_u^q$ for $n \in \mathbb{N}$. In that case, $\forall \varepsilon > 0$, $\exists M \in \mathbb{N}$ such that

$$\begin{aligned} \|u^{(n)} - u^{(z)}\|_{\tilde{\mathcal{M}}_u^q} &= \left(\sup_{r,k} \left| \frac{1}{[r+1]_q[k+1]_q} \sum_{l,m=0}^{r,k} q^{l+m} \left(u_{lm}^{(n)} - u_{lm}^{(z)} \right) \right| \right) \\ &= \left(\sup_{r,k} \left| \left(C_{(1,1)}(q)u^{(n)} \right)_{rk} - \left(C_{(1,1)}(q)u^{(z)} \right)_{rk} \right| \right) < \varepsilon \end{aligned} \tag{2.6}$$

for all $n, z > M$ and it is reached that $\left\{ \left(C_{(1,1)}(q)u^{(n)} \right)_{rk} \right\}_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{M}_u$. From the completeness of $\mathcal{M}_u$, $\left\{ \left(C_{(1,1)}(q)u^{(n)} \right)_{rk} \right\}_{n \in \mathbb{N}}$ converges and it can be written that $\left\{ \left(C_{(1,1)}(q)u^{(n)} \right)_{rk} \right\}_{n \in \mathbb{N}} \to \left(C_{(1,1)}(q)u \right)_{rk}$ for $n \to \infty$. In that case, we may define the sequence $\left(C_{(1,1)}(q)u \right)_{rk}$. After all of these, it must be proven that $\left(C_{(1,1)}(q)u \right)_{rk} \in \mathcal{M}_u$. From $\left\{ \left(C_{(1,1)}(q)u^{(n)} \right)_{rk} \right\}_{n \in \mathbb{N}} \in \mathcal{M}_u$, it is obtained that $\left(\sup_{r,k} \left| \left(C_{(1,1)}(q)u^{(n)} \right)_{rk} \right| \right) < \infty$. So, we see that $\left(C_{(1,1)}(q)u \right)_{rk} \in \mathcal{M}_u$ from

$$\begin{aligned} \| \left(C_{(1,1)}(q)u \right)_{rk} \|_{\mathcal{M}_u} &= \left(\sup_{r,k} \left| \left(C_{(1,1)}(q)u \right)_{rk} \right| \right) \\ &\le \left(\sup_{r,k} \left| \left(C_{(1,1)}(q)u^{(n)} \right)_{rk} - \left(C_{(1,1)}(q)u \right)_{rk} \right| \right) \\ &+ \left(\sup_{r,k} \left| \left(C_{(1,1)}(q)u^{(n)} \right)_{rk} \right| \right) < \infty \end{aligned}$$

by applying limit on (2.6) for $z \to \infty$. Consequently, $u \in \tilde{\mathcal{M}}_u^q$ and $\tilde{\mathcal{M}}_u^q$ is complete with $\|.\|_{\tilde{\mathcal{M}}_u^q}$. □

Theorem 2.2. *$\tilde{\mathcal{M}}_u^q$ is linearly norm isomorphic to $\mathcal{M}_u$.*

Proof. The linearity of the mapping described as $\Upsilon : \tilde{\mathcal{M}}_u^q \to \mathcal{M}_u$, $\Upsilon(u) = C_{(1,1)}(q)u$ is obvious for $u = (u_{lm}) \in \tilde{\mathcal{M}}_u^q$. Additionally, from the expression $\Upsilon(u) = 0 \Rightarrow u = 0$, Υ is injective.

Let us consider the sequences $\nu = (\nu_{lm}) \in \mathcal{M}_u$ and $u = (u_{lm})$ as follows:

$$u_{rk} = \frac{1}{q^{r+k}} \sum_{l=r-1}^{r} \sum_{m=k-1}^{k} (-1)^{r+k-(l+m)} [l+1]_q [m+1]_q \nu_{lm} \quad (r, k \in \mathbb{N}). \tag{2.7}$$

Then, from the equality

$$\begin{aligned}
\|u\|_{\tilde{\mathcal{M}}_u^q} &= \left(\sup_{r,k} \left| \frac{1}{[r+1]_q [k+1]_q} \sum_{l,m=0}^{r,k} q^{l+m} u_{lm} \right| \right) \\
&= \left(\sup_{r,k} \left| \frac{1}{[r+1]_q [k+1]_q} \sum_{l,m=0}^{r,k} q^{l+m} \sum_{i=l-1}^{l} \sum_{j=m-1}^{m} \frac{1}{q^{l+m}} (-1)^{l+m-(i+j)} [i+1]_q [j+1]_q \nu_{ij} \right| \right) \\
&= \left(\sup_{r,k} |\nu_{rk}| \right) = \|\nu\|_{\mathcal{M}_u} < \infty,
\end{aligned}$$

it is seen that Υ is surjective. Finally, since $\|u\|_{\tilde{\mathcal{M}}_u^q} = \|\nu\|_{\mathcal{M}_u}$, in that case Υ is norm keeping. □

Theorem 2.3. *The inclusion $\mathcal{M}_u \subset \tilde{\mathcal{M}}_u^q$ holds.*

Proof. Consider that $u = (u_{lm}) \in \mathcal{M}_u$. In that case, it can be written that $\sup_{l,m} |u_{lm}| < \delta$ for at least positive real number δ. Consequently, it is achieved that

$$\begin{aligned}
\|u\|_{\tilde{\mathcal{M}}_u^q} &= \sup_{r,k} \left| \frac{1}{[r+1]_q [k+1]_q} \sum_{l,m=0}^{r,k} q^{l+m} u_{lm} \right| \\
&\leq \sup_{r,k} \left| \frac{1}{[r+1]_q [k+1]_q} \sum_{l,m=0}^{r,k} q^{l+m} \right| |u_{lm}| \\
&\leq \delta \sup_{r,k} \left| \frac{1}{[r+1]_q [k+1]_q} \sum_{l,m=0}^{r,k} q^{l+m} \right| = \delta
\end{aligned}$$

and thus $\mathcal{M}_u \subset \tilde{\mathcal{M}}_u^q$. □

3. Dual spaces

In this section, the α-, $\beta(\mathcal{P})$-, $\beta(b\mathcal{P})$-and γ-duals of $\tilde{\mathcal{M}}_u^q$ are determined. For $\Psi, \Lambda \in \Omega$, the set $D(\Psi : \Lambda)$ is defined by

$$D(\Psi : \Lambda) = \left\{ \tau = (\tau_{rk}) \in \Omega : \tau u = (\tau_{rk} u_{rk}) \in \Lambda \quad \text{for all} \quad (u_{rk}) \in \Psi \right\}.$$

Then, α-, $\beta(\vartheta)$- and γ-duals of Ψ are defined as

$$\Psi^{\alpha} = D(\Psi : \mathcal{L}_u), \quad \Psi^{\beta(\vartheta)} = D(\Psi : \mathcal{CS}_{\vartheta}) \quad \text{and} \quad \Psi^{\gamma} = D(\Psi : \mathcal{BS}).$$

Theorem 3.1. $\left[\tilde{\mathcal{M}}_u^q\right]^{\alpha} = \mathcal{L}_u.$

Proof. Consider the sequences $u = (u_{lm}) \in \tilde{\mathcal{M}}_u^q$ with $\nu = (\nu_{lm}) \in \mathcal{M}_u$ and $\tau = (\tau_{lm}) \in \mathcal{L}_u$. In that case, $|\nu_{lm}| < N < \infty$ for at least $N > 0$ for all $l, m \in \mathbb{N}$.

By using the equality (2.7), it is obtained the inequality

$$
\begin{aligned}
\sum_{l,m}|\tau_{lm}u_{lm}| &= \sum_{l,m}\left|\tau_{lm}\sum_{i=l-1}^{l}\sum_{j=m-1}^{m}\frac{(-1)^{l+m-(i+j)}}{q^{l+m}}[i+1]_q[j+1]_q\nu_{ij}\right| \\
&\leq N\sum_{l,m}|\tau_{lm}|\left|\sum_{i=l-1}^{l}\sum_{j=m-1}^{m}\frac{(-1)^{l+m-(i+j)}}{q^{l+m}}[i+1]_q[j+1]_q\right| \\
&= N\sum_{l,m}|\tau_{lm}|<\infty
\end{aligned}
$$

which gives that $\tau\in\left[\tilde{\mathcal{M}}_u^q\right]^\alpha$ and thus $\mathcal{L}_u\subset\left[\tilde{\mathcal{M}}_u^q\right]^\alpha$.

On the other hand, consider that $\tau\in\left[\tilde{\mathcal{M}}_u^q\right]^\alpha\backslash\mathcal{L}_u$. In that case, $\sum_{l,m}|\tau_{lm}u_{lm}|<\infty$ for all $u=(u_{lm})\in\tilde{\mathcal{M}}_u^q$. For choosing $e\in\tilde{\mathcal{M}}_u^q$, since $\tau e=\tau\notin\mathcal{L}_u$, it is reached the contradiction $\tau\notin\left[\tilde{\mathcal{M}}_u^q\right]^\alpha$. Thus, it should be $\tau\in\mathcal{L}_u$. □

We can express the necessary conditions for the matrix class characterizations that will be used in this and the next section and the matrix classes with the help of a lemma as follows:

$$\sup_{r,k\in\mathbb{N}}\sum_{l,m}|b_{rklm}|<\infty, \tag{3.1}$$

$$\exists a_{lm}\in\mathbb{C}\ni\vartheta-\lim_{r,k\to\infty}b_{rklm}=a_{lm}\quad\text{subsists}, \tag{3.2}$$

$$\forall l\in\mathbb{N},\quad\exists m_0\ni b_{rklm}=0,\quad\forall m>m_0, \tag{3.3}$$

$$\forall m\in\mathbb{N},\quad\exists l_0\ni b_{rklm}=0,\quad\forall l>l_0, \tag{3.4}$$

$$\sup_{r,k,l,m\in\mathbb{N}}|b_{rklm}|<\infty, \tag{3.5}$$

$$\sup_{r,k\in\mathbb{N}}\sum_{l,m}|b_{rklm}|^{p'}<\infty, \tag{3.6}$$

$$\exists a_{lm}\in\mathbb{C}\ni\quad bp-\lim_{r,k\to\infty}\sum_{l,m}|b_{rklm}-a_{lm}|=0, \tag{3.7}$$

$$bp-\lim_{r,k\to\infty}\sum_{l=0}^{r}b_{rklm}\quad\text{subsists},\quad\forall m\in\mathbb{N}, \tag{3.8}$$

$$bp-\lim_{r,k\to\infty}\sum_{m=0}^{k}b_{rklm}\quad\text{subsists},\quad\forall l\in\mathbb{N}, \tag{3.9}$$

$$\sum_{l,m}|b_{rklm}|\quad\text{converges}, \tag{3.10}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Lemma 3.1. *[6, 7, 38] For $B=(b_{zntk})\in\Omega$, the following statements hold:*

(i) *$B\in(\mathcal{M}_u:\mathcal{M}_u)$ iff the condition (3.1) holds.*

(ii) *$B\in(\mathcal{M}_u:\mathcal{C}_{\mathcal{P}})$ iff the conditions (3.2), (3.3) and (3.4) hold.*

(iii) *$B\in(\mathcal{M}_u:\mathcal{C}_{b\mathcal{P}})$ iff the conditions (3.1), (3.2), (3.7), (3.8), (3.9) and (3.10) hold.*

(iv) *$B\in(\mathcal{C}_{b\mathcal{P}}:\mathcal{M}_u:)$ iff the condition (3.1) holds.*

(v) *For $0<p\leq 1$, $B\in(\mathcal{L}_p:\mathcal{M}_u)$ iff the condition (3.5) holds.*

(vi) *For $1<p<\infty$, $B\in(\mathcal{L}_p:\mathcal{M}_u)$ iff the condition (3.6) holds.*

It can be given the abbreviations to be used in the next theorem as follows:

$$\begin{aligned}
\Delta_{11}\left(\frac{\tau_{lm}}{q^{l+m}}\right) &= \left(\frac{\tau_{lm}}{q^{l+m}} - \frac{\tau_{l+1,m}+\tau_{l,m+1}}{q^{l+m+1}} + \frac{\tau_{l+1,m+1}}{q^{l+m+2}}\right),\\
\Delta_{10}\left(\frac{\tau_{lk}}{q^{l+k}}\right) &= \left(\frac{\tau_{lk}}{q^{l+k}} - \frac{\tau_{l+1,k}}{q^{l+k+1}}\right),\\
\Delta_{01}\left(\frac{\tau_{rm}}{q^{r+m}}\right) &= \left(\frac{\tau_{rm}}{q^{r+m}} - \frac{\tau_{r,m+1}}{q^{r+m+1}}\right).
\end{aligned} \tag{3.11}$$

Theorem 3.2. *Consider that* $\Psi \subset \Omega$, $\tau = (\tau_{lm}) \in \Omega$ *and the* $4d$ *infinite matrix* $O = (o_{rklm})$ *described by*

$$o_{rklm} := \begin{cases} [l+1]_q[m+1]_q\Delta_{11}\left(\frac{\tau_{lm}}{q^{l+m}}\right) & , \quad 0 \le l \le r-1, \quad 0 \le m \le k-1,\\ [l+1]_q[k+1]_q\Delta_{10}\left(\frac{\tau_{lk}}{q^{l+k}}\right) & , \quad 0 \le l \le r-1, \quad m = k,\\ [r+1]_q[m+1]_q\Delta_{01}\left(\frac{\tau_{rm}}{q^{r+m}}\right) & , \quad 0 \le m \le k-1, \quad l = r,\\ \frac{[r+1]_q[k+1]_q\tau_{rk}}{q^{r+k}} & , \quad m = k, \quad l = r,\\ 0 & , \quad \textit{elsewhere} \end{cases} \tag{3.12}$$

for all $r, k, l, m \in \mathbb{N}$.

In that case;

(i) $\left[\tilde{\mathcal{M}}_u^q\right]^{\beta(\vartheta)} = \{\tau = (\tau_{lm}) : O \in (\mathcal{M}_u : C_\vartheta)\}$, *where* $\vartheta \in \{\mathcal{P}, b\mathcal{P}\}$.

(ii) $\left[\tilde{\mathcal{M}}_u^q\right]^{\gamma} = \{\tau = (\tau_{lm}) : O \in (\mathcal{M}_u : \mathcal{M}_u)\}$.

Proof. **(i)** Consider the sequences $\tau = (\tau_{lm}) \in \Omega$ and $u \in \tilde{\mathcal{M}}_u^q$ with $\nu \in \mathcal{M}_u$ with the relation (2.4). By bearing in mind the equation (2.7), it is reached that

$$\begin{aligned}
\sigma_{rk} &= \sum_{l,m=0}^{r,k} \tau_{lm}u_{lm} = \sum_{l,m=0}^{r,k} \tau_{lm}\left(\frac{1}{q^{l+m}}\sum_{i=l-1}^{l}\sum_{j=m-1}^{m}(-1)^{l+m-(i+j)}[i+1]_q[j+1]_q\nu_{ij}\right)\\
&= \sum_{l=0}^{r-1}[l+1]_q[k+1]_q\Delta_{10}\left(\frac{\tau_{lk}}{q^{l+k}}\right)\nu_{lk} + \sum_{m=0}^{k-1}[r+1]_q[m+1]_q\Delta_{01}\left(\frac{\tau_{rm}}{q^{r+m}}\right)\nu_{rm}\\
&+ \sum_{l=0}^{r-1}\sum_{m=0}^{k-1}[l+1]_q[m+1]_q\Delta_{11}\left(\frac{\tau_{lm}}{q^{l+m}}\right)\nu_{lm} + \frac{[r+1]_q[k+1]_q\tau_{rk}}{q^{r+k}}\nu_{rk} = (O\nu)_{rk}
\end{aligned} \tag{3.13}$$

for $O = (o_{rklm})$ is defined by (3.12). Thus, by using (3.13), we reach that $\tau u = (\tau_{lm}u_{lm}) \in \mathcal{CS}_\vartheta$ whenever $u = (u_{lm}) \in \tilde{\mathcal{M}}_u^q$ iff $\sigma = (\sigma_{rk}) \in \mathcal{C}_\vartheta$ whenever $\nu \in \mathcal{M}_u$. Consequently, $\tau \in \left[\tilde{\mathcal{M}}_u^q\right]^{\beta(\vartheta)}$ iff $O \in (\mathcal{M}_u : \mathcal{C}_\vartheta)$ for $\vartheta \in \{\mathcal{P}, b\mathcal{P}\}$.

(ii) It can be shown to be similar to the first part using the definition of γ-dual. So, it is omitted. □

4. Matrix mappings

This section contains the characterizations of matrix classes $(\tilde{\mathcal{M}}_u^q : \Lambda)$ and $(\Psi : \tilde{\mathcal{M}}_u^q)$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_\mathcal{P}, \mathcal{C}_{b\mathcal{P}}\}$ and $\Psi \in \{\mathcal{M}_u, \mathcal{C}_{b\mathcal{P}}, \mathcal{L}_p\}$ for $0 < p < \infty$.

Theorem 4.1. *Consider that 4d matrices $B = (b_{rklm})$ and $H = (h_{rklm})$ with the equality*

$$h_{rklm} = [l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\frac{b_{rklm}}{q^{r+k}}\right). \tag{4.1}$$

Then, $B \in \left(\tilde{\mathcal{M}}_u^q : \Lambda\right)$ iff $H \in (\mathcal{M}_u : \Lambda)$ and

$$B_{rk} \in \left(\tilde{\mathcal{M}}_u^q\right)^{\beta(\vartheta)}. \tag{4.2}$$

Proof. Suppose that $B \in \left(\tilde{\mathcal{M}}_u^q : \Lambda\right)$. Then, $Bu \in \Lambda$ for all $u \in \tilde{\mathcal{M}}_u^q$ with $\nu = C_{(1,1)}(q)u \in \mathcal{M}_u$. Thus, it is obtained that $B_{rk} \in \left(\tilde{\mathcal{M}}_u^q\right)^{\beta(\vartheta)}$. For the (i,j)th partial sums of the series $\sum_{l,m} b_{rklm}u_{lm}$, it is reached that

$$\begin{aligned}(Bu)_{rk}^{[i,j]} &= \sum_{l,m=0}^{i,j} b_{rklm}u_{lm}\\ &= \sum_{l=0}^{i-1}\sum_{m=0}^{j-1}[l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\frac{b_{rklm}}{q^{l+m}}\right)\nu_{lm} + \sum_{l=0}^{i-1}[l+1]_q[j+1]_q\Delta_{10}^{lj}\left(\frac{b_{rklj}}{q^{l+j}}\right)\\ &+ \sum_{m=0}^{j-1}[i+1]_q[m+1]_q\Delta_{01}^{im}\left(\frac{b_{rkim}}{q^{i+m}}\right) + \frac{[i+1]_q[j+1]_q}{q^{i+j}}b_{rkij}\end{aligned} \tag{4.3}$$

for all $r,k,i,j \in \mathbb{N}$. Let us define the 4d infinite matrix $H_{rk} = (h_{ijlm}^{[r,k]})$ as

$$h_{ijlm}^{[r,k]} := \begin{cases} [l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\dfrac{b_{rklm}}{q^{l+m}}\right) &, \quad 0 \le l \le i-1, \quad 0 \le m \le j-1,\\ [l+1]_q[j+1]_q\Delta_{10}^{lj}\left(\dfrac{b_{rklj}}{q^{l+j}}\right) &, \quad 0 \le l \le i-1, \quad m = j,\\ [i+1]_q[m+1]_q\Delta_{01}^{im}\left(\dfrac{b_{rkim}}{q^{i+m}}\right) &, \quad 0 \le m \le j-1, \quad l = i,\\ \dfrac{[i+1]_q[j+1]_q}{q^{i+j}}b_{rkij} &, \quad m = j, \quad l = i,\\ 0 &, \quad \text{otherwise}\end{cases}$$

the relation (4.3) can be restated as

$$(Bu)_{rk}^{[i,j]} = (H_{rk}\nu)_{[i,j]}. \tag{4.4}$$

Moreover, if we take ϑ-limit on $H_{rk} = \left(h_{ijlm}^{[r,k]}\right)$ for $i,j \to \infty$, it is obtained that

$$\vartheta - lim_{i,j\to\infty} h_{ijlm}^{[r,k]} = [l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\frac{b_{rklm}}{q^{l+m}}\right). \tag{4.5}$$

From (4.5), it can be defined the 4d matrix $H = (h_{rklm})$ by

$$h_{rklm} = [l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\frac{b_{rklm}}{q^{l+m}}\right). \tag{4.6}$$

If we take ϑ-limit on (4.4) for $i,j \to \infty$, we see that $Bu = H\nu$. Thus, $H\nu \in \Lambda$ while $\nu \in \mathcal{M}_u$ and $H \in (\mathcal{M}_u : \Lambda)$.

Conversely, suppose that $B_{rk} \in \left(\tilde{\mathcal{M}}_u^q\right)^{\beta(\vartheta)}$ and $H \in (\mathcal{M}_u : \Lambda)$. Let $u \in \tilde{\mathcal{M}}_u^q$ with $\nu = C_{(1,1)}(q)u \in \mathcal{M}_u$. In this case, Bu exists. By the (i,j)th partial sums of $\sum_{t,k} b_{zntk}u_{tk}$, it is obtained the equality

$$\sum_{l,m=0}^{i,j} b_{rklm}u_{lm} = \sum_{l,m=0}^{i,j} h_{ijlm}^{[r,k]}\nu_{lm}$$

for all $r,k,l,m \in \mathbb{N}$. If we take ϑ-limit as $i,j \to \infty$ on the equation above, we reach that $Bu = H\nu$. Consequently, $B \in \left(\tilde{\mathcal{M}}_u^q : \Lambda\right)$. □

Corollary 4.1. *Consider that 4d matrices* $B = (b_{rklm})$ *and* $H = (h_{rklm})$ *with (4.1). Then;*

(i) $B \in \left(\tilde{\mathcal{M}}_u^q : \mathcal{M}_u\right)$ *iff the condition (3.1) holds with* H *in place of* B *and the condition (4.2) holds.*

(ii) $B \in \left(\tilde{\mathcal{M}}_u^q : \mathcal{C}_{\mathcal{P}}\right)$ *iff the conditions (3.2), (3.3) and (3.4) hold with* H *in place of* B *and the condition (4.2) holds.*

(iii) $B \in \left(\tilde{\mathcal{M}}_u^q : \mathcal{C}_{b\mathcal{P}}\right)$ *iff the conditions (3.1), (3.2), (3.7), (3.8), (3.9) and (3.10) hold with* H *in place of* B *and the condition (4.2) holds.*

Lemma 4.1. *[39] Suppose that* $\Psi,\Lambda \subset \Omega$*, a 4d matrix* $B = (b_{rklm})$ *and 4d triangle* $Y = (y_{rklm})$*. Then,* $B \in (\Psi : \Lambda_Y)$ *iff* $YB \in (\Psi : \Lambda)$*.*

Corollary 4.2. *Consider the 4d matrices* $B = (b_{rklm})$ *and* $W = (w_{rklm})$ *with the equality*

$$w_{rklm} = \sum_{l,m=0}^{r,k} c_{rkij}(q) b_{ijlm}.$$

In that case;

(i) $B \in \left(\mathcal{M}_u : \tilde{\mathcal{M}}_u^q\right)$ *iff the condition (3.1) holds with* W *instead of* B*.*

(ii) $B \in \left(\mathcal{C}_{b\mathcal{P}} : \tilde{\mathcal{M}}_u^q :\right)$ *iff the condition (3.1) holds with* W *instead of* B*.*

(iii) *For* $0 < s \leq 1$, $B \in \left(\mathcal{L}_s : \tilde{\mathcal{M}}_u^q\right)$ *iff the condition (3.5) holds with* W *instead of* B*.*

(iv) *For* $1 < s < \infty$, $B \in \left(\mathcal{L}_s : \tilde{\mathcal{M}}_u^q\right)$ *iff the condition (3.6) holds with* W *instead of* B*.*

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