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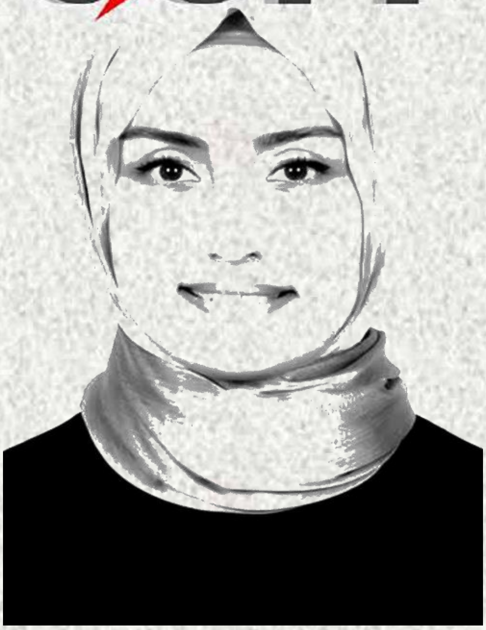

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*To Memory*
*Assoc. Prof. Dr.*
*Zeynep Akdemirci Şanlı*

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Dear Scientists,

Assoc. Prof. Dr. Zeynep ŞANLI was born on August 20, 1988 in Rize, Turkey.

She completed her undergraduate education in 2010 and her master's degree in 2012. In 2019, she completed her thesis titled "Normalization and graphs of Γ2, Γ3 and Hecke groups" and received her doctorate degree.

She worked as a Research Assistant in the Department of Mathematics at Karadeniz Technical University between 2013-2019, as an Assistant Professor in 2021-2024 and as an Associate Professor in the Department of Mathematics at Mersin University Faculty of Science after 2023.

She has many works in the fields of topology, analysis and algebra.

Assoc. Prof. Dr. Zeynep ŞANLI, who passed away on April 6, 2024, was married and had one child.

I would like to thank all the authors who contributed to this special issue of JUM, published in memory of Assoc. Prof. Dr. Zeynep ŞANLI.

Kind regards!

Assoc. Prof. Dr. Gökhan Çuvalcıoğlu
Editor in-Chief

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# SPLIT $(s,t)-$LUCAS QUATERNIONS

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Abstract. In this paper, we introduce a new class of split $(s,t)-$ Lucas quaternions that generalizes the split Lucas quaternions. Additionally, we derive Binet-like formulas, generating functions, binomial sums and Honsberger-like, d'Ocagne-like, Catalan's-like and Cassini's-like identities.

## 1. Introduction

Quaternions are a number system that extends real numbers to one real and three imaginary dimensions. They were first defined by the Irish mathematician Sir William Rowan Hamilton in 1843 and have been applied to mathematics in $3-$dimensional space. Quaternions do not possess the commutative property ($ab = ba$). Nowadays, many researchers are relating quaternions to Fibonacci and other special number sequences. Halıcı investigated the Fibonacci and Lucas quaternions, and gave the generating functions, Binet formulas and some sum formulas for these quaternions [6]. İpek studied on the quaternions of the $(p,q)-$Fibonacci sequence, which are generalizations of the Fibonacci sequence [4]. Likewise, Çimen and İpek also investigated Pell quaternions and Pell-Lucas quaternions [20]. Taşçı defined Padovan and Pell-Padovan quaternions, and gave Binet-like formulas, generating functions, sums formulas and the matrix representation of the Padovan and Pell-Padovan quaternions [21]. Dişkaya and Menken worked on the quaternions of the $(s,t)-$Padovan and $(s,t)-$Perrin sequences, which are generalizations of the Padovan and Perrin sequences [13]. Refer to [5, 9–11, 14] for more details on their research. Split quaternions are a variation of quaternions where the standard basis elements satisfy slightly different multiplication rules. Unlike the quaternion algebra, the split quaternions contain zero divisors, nilpotent elements and nontrival idempotent. The split quaternions were defined by James Cockle in 1849. A split quaternion is defined by

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$$

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where $q_0$, $q_1$, $q_2$ and $q_3$ are real numbers and $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$ are the standart basis in $\mathbb{R}^4$. Then we can write

$$q = S_q + V_q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$$

where $S_q = q_0e_0$ and $V_q = q_1e_1 + q_2e_2 + q_3e_3$. $S_q$ is called the scalar part of the split quaternion q and $V_q$ is called the vector part of the split quaternion q. The split quaternion multipication is defined using the rules;

$$e_0^2 = -1, \quad e_1^2 = e_2^2 = e_3^2 = 1$$

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = -e_1 \quad \text{and} \quad e_3e_1 = -e_1e_3 = e_2.$$

This algebra is associative and non-comutative . Let $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$ and $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3$ be any two split quaternions. Then the addition and subtraction of the split quaternions is

$$q \mp p = (q_0 \mp p_0)e_0 + (q_1 \mp p_1)e_1 + (q_2 \mp p_2)e_2 + (q_3 \mp p_3)e_3$$

and multiplication of the split quaternions is

$$\begin{aligned} qp &= (q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3)(p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3) \\ &= (q_0p_0 - q_1p_1 + q_2p_2 + q_3p_3)e_0 + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)e_1 \\ &+ (q_0p_2 + q_2p_0 - q_1p_3 + q_3p_1)e_2 + (q_0p_3 + q_3p_0 + q_1p_2 - q_2p_1)e_3 \\ &= S_qS_p + < V_q, V_p > + S_qV_p + S_pV_q + V_q\text{x}V_p \end{aligned}$$

where

$$< V_q, V_p >= q_0p_0 - q_1p_1 + q_2p_2 + q_3p_3$$

and

$$V_q\text{x}V_p = \begin{vmatrix} -e_1 & e_2 & e_3 \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix}$$

and for $k \in \mathbb{R}$ the multiplication by scalar is

$$kq = kq_0e_0 + kq_1e_1 + kq_2e_2 + kq_3e_3$$

and conjugate and norm of split quaternion q are

$$\overline{q} = q_0e_0 - q_1e_1 - q_2e_2 - q_3e_3$$

and

$$||q|| = \sqrt{|q\overline{q}|} = \sqrt{q_0^2 + q_1^2 - q_2^2 - q_3^2}$$

The basic operations on the two split quaternions given above can also be seen in [1, 8, 12, 15–19].

In [2], the Lucas sequence $\{L_n\}_{n\geq 0}$ is

(1.1) $$L_0 = 2, \quad L_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n$$

for all $n \geq 2$. Here, $L_n$ is the $n-$th Lucas number. The first few terms of the Lucas numbers are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364.$$

A generalization of the Lucas sequence $\{L_n\}_{n\geq 0}$, which are called the $(s,t)-$Lucas sequence $\{L_{s,t,n}\}_{n\geq 0}$ is defined by the following recurrence relation for $n \geq 0$ and $s,t \geq 1$ such that $s^2+4t>0$;

$$L_{s,t,0}=2, \quad L_{s,t,1}=s \quad \text{and} \quad L_{s,t,n+2}=sL_{s,t,n+1}+tL_{s,t,n} \tag{1.2}$$

$(s,t)-$Lucas sequence refer to reader to [3]. The first few terms of the $(s,t)-$Lucas numbers are

$$2, s, s^2+2t, s^3+3st, s^4+4s^2t+2t^2.$$

To simplify notation, take $L_{s,t,n}=\mathcal{L}_n$. In [3], for every $x \in \mathbb{N}$, one can write the Binet-like formula for the $(s,t)-$Lucas sequence as the form

$$\mathcal{L}_n=\alpha^n+\beta^n \tag{1.3}$$

where $\alpha=\frac{s+\sqrt{s^2+4t}}{2}$ and $\beta=\frac{s-\sqrt{s^2+4t}}{2}$ are the roots of the characteristic equation

$$x^2-sx-t=0 \tag{1.4}$$

associated with the recurrence relation (1.2). Moreover, it can be observed that

$$\alpha^n=\alpha\mathcal{L}_n+t\mathcal{L}_{n-1},$$

$$\beta^n=\beta\mathcal{L}_n+t\mathcal{L}_{n-1},$$

$$\alpha+\beta=s,$$

$$\alpha-\beta=\sqrt{s^2+4t},$$

$$\alpha\beta=-t.$$

The following properties hold [7]:

(1) $\mathcal{L}_m\mathcal{L}_{n+1}+t\mathcal{L}_{m-1}\mathcal{L}_n=\left(s^2+4t\right)\mathcal{F}_{m+n}, \quad m \geq n$
(2) $\mathcal{L}_{n-r}\mathcal{L}_{n+r}-\mathcal{L}_n^2=(-t)^{n-r}\,(s^2+4t), \quad n \geq r$
(3) $\mathcal{L}_{n-1}\mathcal{L}_{n+1}-\mathcal{L}_n^2=(-t)^{n-1}\,(s^2+4t), \quad n \geq 1$

## 2. Split $(p,q)-$Lucas Quaternions

In this section, we define new split quaternions that are split $(p,q)-$Lucas quaternions. Then, we give their Binet-like formula, generating functions, certain binomal sums and Honsberg, d'Ocagne, Catalan's and Cassini's identites.

**Definition 2.1.** The split $(p,q)-$Lucas quaternion $\{\mathcal{QL}_{s,t,n}\}_{n\geq 0}$ is defined by

$$\mathcal{QL}_{s,t,n}=\mathcal{L}_ne_0+\mathcal{L}_{n+1}e_1+\mathcal{L}_{n+2}e_2+\mathcal{L}_{n+3}e_3 \tag{2.1}$$

where $\mathcal{L}_n$ is the $n-$th $(s,t)-$Lucas number.

To simplify notation, take $\mathcal{QL}_{s,t,n}=\mathcal{QL}_n$.

**Theorem 2.2.** *The Binet-like formula for the $n-$th split $(s,t)-$Lucas quaternion is*

$$\mathcal{QL}_n=\hat{\alpha}\alpha^n+\hat{\beta}\beta^n, \qquad n \geq 0 \tag{2.2}$$

*where* $\hat{\alpha}=e_0+\alpha e_1+\alpha^2e_2+\alpha^3e_3$ *and* $\hat{\beta}=e_0+\beta e_1+\beta^2e_2+\beta^3e_3$.

*Proof.* From the definition of $n-$th split $(s,t)-$Lucas quaternion $\mathcal{L}_n$, we obtain

$$\begin{aligned}\mathcal{QL}_n &= \mathcal{L}_n e_0 + \mathcal{L}_{n+1}e_1 + \mathcal{L}_{n+2}e_2 + \mathcal{L}_{n+3}e_3 \\ &= (\alpha^n + \beta^n)\, e_0 + \left(\alpha^{n+1} + \beta^{n+1}\right) e_1 + \left(\alpha^{n+2} + \beta^{n+2}\right) e_2 + \left(\alpha^{n+3} + \beta^{n+3}\right) e_3 \\ &= \alpha^n(e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3) + \beta^n(e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3) \\ &= \hat{\alpha}\alpha^n + \hat{\beta}\beta^n.\end{aligned}$$

Thus, the proof is completed. □

**Theorem 2.3.** *The generating function for the $n-$th split $(s,t)-$Lucas quaternions is*

$$\mathcal{G}_{\mathcal{L}}(x) = \frac{2e_0 + se_1 + (s^2+2t)e_2 + (s^3+3st)e_3 + (-se_0 + 2te_1 + ste_2 + (s^2t + 2t^2)e_3)x}{1 - sx - tx^2}$$

*Proof.* Let

$$\mathcal{G}_{\mathcal{L}}(x) = \sum_{n=0}^{\infty} \mathcal{QL}_n x^n = \mathcal{QL}_0 + \mathcal{QL}_1 x + \mathcal{QL}_2 x^2 + \mathcal{QL}_3 x^3 + \ldots + \mathcal{QL}_n x^n + \ldots$$

be generating function of the split $(p,q)-$Lucas quaternions. This function is multipied every side with $-sx$ such as

$$-sx\mathcal{G}_{\mathcal{L}}(x) = -s\mathcal{QL}_0 x - s\mathcal{QL}_1 x^2 - s\mathcal{QL}_2 x^3 - s\mathcal{QL}_3 x^4 - \ldots - s\mathcal{QL}_n x^{n+1} - \ldots$$

and that is multipied every side with $-tx^2$ such as

$$-tx^2\mathcal{G}_{\mathcal{L}}(x) = -t\mathcal{QL}_0 x^2 - t\mathcal{QL}_1 x^3 - t\mathcal{QL}_2 x^4 - t\mathcal{QL}_3 x^5 - \ldots - t\mathcal{QL}_n x^{n+2} - \ldots$$

Then, we write

$$\begin{aligned}(1 - sx - tx^2)\mathcal{G}_{\mathcal{L}}(x) &= \mathcal{QL}_0 + (\mathcal{QL}_1 - s\mathcal{QL}_0)x + (\mathcal{QL}_2 - s\mathcal{QL}_1 - t\mathcal{QL}_0)x^2 + \ldots \\ &\quad + (\mathcal{QL}_n - s\mathcal{QL}_{n-1} - t\mathcal{QL}_{n-2})x^n\end{aligned}$$

Now using

$$\mathcal{QL}_0 = 2e_0 + se_1 + (s^2+2t)e_2 + (s^3+3st)e_3,$$

$$\mathcal{QL}_1 = se_0 + (s^2+2t)e_1 + (s^3+3st)e_2 + (s^4+4s^2t+2t^2)e_3,$$

$$\mathcal{QL}_2 = (s^2+2t)e_0 + (s^3+3st)e_1 + (s^4+4s^2t+2t^2)e_2 + (s^5+5s^3t+5st^2)e_3$$

and

$$\mathcal{QL}_n - s\mathcal{QL}_{n-1} - t\mathcal{QL}_{n-2} = 0$$

we obtain

$$\mathcal{G}_{\mathcal{L}}(x) = \frac{2e_0 + se_1 + (s^2+2t)e_2 + (s^3+3st)e_3 + (-se_0 + 2te_1 + ste_2 + (s^2t + 2t^2)e_3)x}{1 - sx - tx^2}.$$

Thus, the proof is completed. □

*Remark* 2.4. Let $m$ be a positive integer. Then,

$$(a+b)^m = \sum_{n=0}^{m} \binom{m}{n} a^n b^{m-n} \tag{2.3}$$

where $a$ and $b$ are any real numbers.

**Theorem 2.5.** *Let $m$ be a positive integer. Then,*

$$\sum_{n=0}^{m}\binom{m}{n}s^n t^{m-n}\mathcal{QL}_n=\mathcal{QL}_{2m}.$$

*Proof.* Applying the Binet-like formula (2.2) and combining this with (1.4) and (2.3) we obtain the identity

$$\begin{aligned}\sum_{n=0}^{m}\binom{m}{n}s^n t^{m-n}\mathcal{QF}_n&=\sum_{n=0}^{m}\binom{m}{n}s^n t^{m-n}\left(\hat{\alpha}\alpha^n+\hat{\beta}\beta^n\right)\\&=\sum_{n=0}^{m}\binom{m}{n}\left(\hat{\alpha}(\alpha s)^n t^{m-n}+\hat{\beta}(\beta s)^n t^{m-n}\right)\\&=\hat{\alpha}(s\alpha+t)^m+\hat{\beta}(s\beta+t)^m\\&=\hat{\alpha}\alpha^{2m}+\hat{\beta}\beta^{2m}\end{aligned}$$

Thus, the proof is completed. □

**Theorem 2.6.** *Let $m$ be a positive integer. Then,*

$$\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k\mathcal{QL}_{n-k}=\mathcal{QL}_{n+m}$$

*Proof.* Applying the Binet-like formula (2.2) and combining this with (1.4) and (2.3) we obtain the identity

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k\mathcal{QL}_{n-k}&=\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^k\left(\hat{\alpha}\alpha^{n-k}+\hat{\beta}\beta^{n-k}\right)\\&=\sum_{k=0}^{m}\binom{m}{k}\left(\hat{\alpha}(s\alpha)^{m-k}t^k\alpha^{n-m}+\hat{\beta}(s\beta)^{m-k}t^k\beta^{n-m}\right)\\&=\hat{\alpha}(s\alpha+t)^m\alpha^{n-m}+\hat{\beta}(s\beta+t)^m\beta^{n-m}\\&=\hat{\alpha}\alpha^{n+m}+\hat{\beta}\beta^{n+m}.\end{aligned}$$

Thus, the proof is completed. □

Henceforth, we will get

$$\mathcal{A}_n \text{ instead of } (\hat{\alpha})^2\alpha^n+(\hat{\beta})^2\beta^n,$$
$$\mathcal{B}_n \text{ instead of } \hat{\beta}\hat{\alpha}\alpha^n-\hat{\alpha}\hat{\beta}\beta^n,$$

in the following theorems.

**Theorem 2.7.** ***(Hosberg-like İdentity)*** *Let $\mathcal{QL}_n$ be the split $(s,t)-$Lucas quaternion. The following relations are satisfied*

$$\mathcal{QL}_{n+1}\mathcal{QL}_m+t\mathcal{QL}_n\mathcal{QL}_{m-1}=\mathcal{A}_{n+m}\sqrt{s^2+4t}.$$

*Proof.*

$$
\begin{aligned}
&\mathcal{QL}_{n+1}\mathcal{QL}_m + t\mathcal{QL}_n\mathcal{QL}_{m-1}\\
&= \left(\hat{\alpha}\alpha^{n+1} + \hat{\beta}\beta^{n+1}\right)\left(\hat{\alpha}\alpha^{m} + \hat{\beta}\beta^{m}\right) + t\left(\hat{\alpha}\alpha^{n} + \hat{\beta}\beta^{n}\right)\left(\hat{\alpha}\alpha^{m-1} + \hat{\beta}\beta^{m-1}\right)\\
&= (\hat{\alpha})^2\alpha^{n+m+1} + \hat{\alpha}\hat{\beta}\alpha^{n+1}\beta^{m} + \hat{\beta}\hat{\alpha}\beta^{n+1}\alpha^{m} + (\hat{\beta})^2\beta^{n+m+1}\\
&+ t\left((\hat{\alpha})^2\alpha^{n+m-1} + \hat{\alpha}\hat{\beta}\alpha^{n}\beta^{m-1} + \hat{\beta}\hat{\alpha}\beta^{n}\alpha^{m-1} + (\hat{\beta})^2\beta^{n+m-1}\right)\\
&= (\hat{\alpha})^2\alpha^{n+m+1} + \hat{\alpha}\hat{\beta}\alpha^{n+1}\beta^{m} + \hat{\beta}\hat{\alpha}\beta^{n+1}\alpha^{m} + (\hat{\beta})^2\beta^{n+m+1}\\
&- (\hat{\alpha})^2\alpha^{n+m}\beta - \hat{\alpha}\hat{\beta}\alpha^{n+1}\beta^{m} - \hat{\beta}\hat{\alpha}\beta^{n+1}\alpha^{m} - (\hat{\beta})^2\alpha\beta^{n+m}\\
&= (\hat{\alpha})^2\alpha^{n+m}(\alpha-\beta) + (\hat{\beta})^2\beta^{n+m}(\alpha-\beta)\\
&= \left((\hat{\alpha})^2\alpha^{n+m} + (\hat{\beta})^2\beta^{n+m}\right)(\sqrt{s^2+4t})
\end{aligned}
$$

□

**Theorem 2.8.** ***(d'Ocagne-like Identity)*** *Let $\mathcal{QL}_n$ be the split $(s,t)-$Lucas quaternion. The following relations are satisfied,*

$$\mathcal{QL}_m\mathcal{QL}_{n+1} - \mathcal{QL}_{m+1}\mathcal{QL}_n = (-t)^m\mathcal{B}_{n-m}\ \sqrt{s^2+4t}$$

*Proof.*

$$
\begin{aligned}
&\mathcal{QL}_m\mathcal{QL}_{n+1} - \mathcal{QL}_{m+1}\mathcal{QL}_n\\
&= \left(\hat{\alpha}\alpha^{m} + \hat{\beta}\beta^{m}\right)\left(\hat{\alpha}\alpha^{n+1} + \hat{\beta}\beta^{n+1}\right) - \left(\hat{\alpha}\alpha^{m+1} + \hat{\beta}\beta^{m+1}\right)\left(\hat{\alpha}\alpha^{n} + \hat{\beta}\beta^{n}\right)\\
&= (\hat{\alpha})^2\alpha^{n+m+1} + \hat{\alpha}\hat{\beta}\alpha^{m}\beta^{n+1} + \hat{\beta}\hat{\alpha}\beta^{m}\alpha^{n+1} + (\hat{\beta})^2\beta^{n+m+1}\\
&- (\hat{\alpha})^2\alpha^{n+m+1} - \hat{\alpha}\hat{\beta}\alpha^{m+1}\beta^{n} - \hat{\beta}\hat{\alpha}\beta^{m+1}\alpha^{n} - (\hat{\beta})^2\beta^{n+m+1}\\
&= -\hat{\alpha}\hat{\beta}\alpha^{m}\beta^{n}(\alpha-\beta) + \hat{\beta}\hat{\alpha}\beta^{m}\alpha^{n}(\alpha-\beta)\\
&= (-t)^m\left(\hat{\beta}\hat{\alpha}\alpha^{n-m} - \hat{\alpha}\hat{\beta}\beta^{n-m}\right)\sqrt{s^2+4t}
\end{aligned}
$$

□

**Theorem 2.9.** ***(Catalan's Identity)*** *Let $\mathcal{QL}_n$ be the split $(s,t)-$Lucas quaternion. The following relations are satisfied,*

$$\mathcal{QL}_{n-r}\mathcal{QL}_{n+r} - \mathcal{QL}_n^2 = (-t)^n\mathcal{B}_0F_r\sqrt{s^2+4t} \tag{2.4}$$

*where the Binet formula of the r. Fibonacci number $F_r$ is* $\dfrac{\alpha^r-\beta^r}{\alpha-\beta}$ .

*Proof.*

$$
\begin{aligned}
&\mathcal{QL}_{n-r}\mathcal{QL}_{n+r}-\mathcal{QL}_n^2\\
&=\left(\hat{\alpha}\alpha^{n-r}+\hat{\beta}\beta^{n-r}\right)\left(\hat{\alpha}\alpha^{n+r}+\hat{\beta}\beta^{n+r}\right)-\left(\hat{\alpha}\alpha^{n}+\hat{\beta}\beta^{n}\right)^2\\
&=(\hat{\alpha})^2\alpha^{2n}+\hat{\alpha}\hat{\beta}\alpha^{n-r}\beta^{n+r}+\hat{\beta}\hat{\alpha}\beta^{n-r}\alpha^{n+r}+(\hat{\beta})^2\beta^{2n}\\
&-(\hat{\alpha})^2\alpha^{2n}-\hat{\alpha}\hat{\beta}\alpha^{n}\beta^{n}-\hat{\beta}\hat{\alpha}\beta^{n}\alpha^{n}-(\hat{\beta})^2\beta^{2n}\\
&=\hat{\alpha}\hat{\beta}(\alpha\beta)^n(\beta^r-\alpha^r)+\hat{\beta}\hat{\alpha}(\alpha\beta)^n(\alpha^r-\beta^r)\\
&=(-t)^n\left(\hat{\beta}\hat{\alpha}-\hat{\alpha}\hat{\beta}\right)\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)(\alpha-\beta)\\
&=(-t)^n\mathcal{B}_0F_r\sqrt{s^2+4t}
\end{aligned}
$$

□

**Theorem 2.10.** ***(Cassini's Identity)*** *Let $\mathcal{QF}_n$ be the split $(p,q)-$Lucas quaternion. The following relations are satisfied*

$$\mathcal{QL}_{n-1}\mathcal{QL}_{n+1}-\mathcal{QL}_n^2=(-t)^n\mathcal{B}_0\sqrt{s^2+4t}.$$

*Proof.* We take 1 instead of $r$ in (2.4) to prove the this theorem. □

## 3. Conclusion

In this paper, we introduced a new class of split $(s,t)-$Lucas quaternions, extending the concept of split Lucas quaternions. We derived Binet-like formulas, generating functions, binomial sums, and various identities analogous to those of Honsberger, d'Ocagne, Catalan, and Cassini.

By embedding the $(s,t)-$Lucas sequence within the quaternion algebra, we demonstrated its broader applicability and utility. The derived formulas and identities provide powerful tools for explicit calculations and deeper insights into the sequence's behavior.

Our findings open up new possibilities for future research, particularly in exploring geometric interpretations and potential applications in theoretical physics and computer science. We believe our contributions will stimulate further discovery in this intriguing area of mathematical inquiry.

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**The Declaration of Ethics Committee Approval**

This study does not be necessary ethical committee permission or any special permission.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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# CONFORMABLE EIGENVALUE PROBLEMS WITH TWO PARAMETERS

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Abstract. This study used conformable derivatives to define the eigenvalue problems with two parameters and examined various associated spectral properties. Firstly, the conformable eigenvalue problems with two parameters were reduced to the simpler one parameter problems. Additionally, we focused on the orthogonality properties of eigenfunctions. Secondly, investigating the reality of eigenvalues is important to understand the physical relevance and practical usability of the considered eigenvalue problem. Finally, we examined integral relations, which explain important connections and relationships between different aspects of the problem.

## 1. Introduction

In many important problems in various fields such as basic sciences, natural sciences, finance, and medicine, differential equations are encountered in the mathematical modeling of these problems. The functions that satisfy these equations are also the mathematical solutions to these problems. Therefore, the first step in researching the solutions to any scientific problem is formulating the differential equation. The problems such as the heat flow in a non-uniform rod, the motion of a stretched vibrating string attached at both ends and the computation of the electrostatic field on the surface of a volume are modeled by an eigenvalue problem with an unknown parameter, known in the literature as the Sturm-Liouville differential equation [16, 17, 23]. This equation is considered along with initial or boundary conditions according to the characteristics of the models to be established. The goal here is to determine the unknown parameter and the unknown function that constitutes the problem. The most fundamental properties of Sturm-Liouville problems include the reality of the eigenvalues, the orthogonality of the eigenfunctions, and the completeness of the eigenfunctions [2,9,10]. Sturm-Liouville theory has numerous applications in fields such as physics, mathematics, and engineering [18,22].

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Depending on the nature of the models to be established, the equation may contain more than one parameter [5–8, 11].

Many mathematicians, including Liouville, Riemann, Weyl, Fourier, Laplace, Lagrange, Euler, Abel, Lacroix, Caputo, and Leibniz, have defined fractional-order derivatives in various ways [19]. Fractional differential equations are equations that contain fractional derivatives and fractional integrals [13, 21]. It is a generalization of the classical integer-order derivative concept, allowing the differentiation process to be performed at non-integer (fractional) orders [15]. This concept is widely used in many scientific fields, particularly dynamic systems, control theory, and physics. Because fractional derivatives only satisfy the linearity property of the fundamental characteristics in classical derivatives and are not applicable for all other properties, Khalil and colleagues proposed the definition of the conformable fractional derivative in 2014 to mitigate this complexity [14]. The conformable derivative is a type of fractional derivative that aims to make the concept of fractional derivatives more comprehensible and easier to compute [1]. By preserving some fundamental properties of the classical derivative operator, the conformable derivative allows for broader applicability of fractional derivatives. Therefore, it is increasingly used in scientific research and engineering applications.

Many authors [3, 4, 12, 20] consider the conformable fractional derivative and Sturm-Liouville theory together.

Now, a brief explanation of two-parameter eigenvalue problems is provided. Additionally, we will provide a general overview of some fundamental definitions and accepted results in the field of fractional calculus. This will include a brief introduction to the concepts and notations used in fractional calculus, and it also encompasses a summary of some important results and theorems widely accepted and utilized in the field.

Arscott [7] focused on a series of related eigenvalue problems common to a simple linear homogeneous differential equation dependent on two parameters and stated that the solution must satisfy three limiting conditions:

$$\frac{d^2u}{dz^2} + \{\lambda + \mu f(z) + g(z)\}\, u = 0, \tag{1.1}$$

$$u(a) = u(b) = u(c) = 0, \tag{1.2}$$

where, $f(z)$ and $g(z)$ are functions defined on the interval $[a, c]$ and $\lambda$ and $\mu$ are spectral parameters. Here, this eigenvalue problem with two parameters has been reduced to a one-parameter problem. The spectral properties of the two-parameter eigenvalue operator such as orthogonality, the realness of eigenvalues, and the expansions theorem of eigenfunction have been investigated and some integral relations have been given. Additionally, various integration methods were examined, and results were obtained [5, 6]. In fact, the given problem is the case of a Sturm-Liouville problem with multi-parameters and significant results are obtained [8].

In this study, consider the conformable eigenvalue problems with two parameters

$$D_t^\alpha\left(D_t^\alpha u(t)\right) + \{\lambda + \mu f(t) + g(t)\}\, u = 0, \tag{1.3}$$

$$u(a) = u(b) = u(c) = 0, \tag{1.4}$$

where $b \in [a, c]$, $f(t)$ and $g(t)$ are real-valued continuous functions defined on the interval $[a, c]$ and $\lambda$ and $\mu$ are spectral parameters. This paper aims to reduce two-parameter conformable eigenvalue problems to one-parameter problems; to investigate the orthogonality properties of eigenfunctions and the reality of eigenvalues; and to examine for integral relations that explain important connections and relationships between different aspects of the problem.

1.1. **The conformable fractional derivative.** In this part, we give some basic definitions and properties of the conformable fractional calculus theory [1, 14].

**Definition 1.1.** Consider the function $u : [0,\infty) \to \mathbb{R}$. Then, the "conformable fractional derivative ($\alpha-$ derivative)" of $u$ order $\alpha \in (0,1]$ is defined by:

$$D_t^\alpha u(t) := \lim_{h\to 0} \frac{u(t+ht^{1-\alpha}) - u(t)}{h}.$$

Here, the symbol $D_t^\alpha$ is conformable fractional derivative of $\alpha-$order with respect to $t$.

If $u$ is $\alpha-$differentiable in some $(0,\alpha)$ and $\lim_{t\to 0^+} D_t^\alpha u(t)$ exits, then define

$$D_t^\alpha u(0) = \lim_{t\to 0^+} D_t^\alpha u(t).$$

If $u$ is usual differentiable, then $D_t^\alpha u(t) = t^{1-\alpha}u'(t)$.
One can easily show that $D_t^\alpha$ satisfies all the properties in the following theorem:

**Theorem 1.2.** *Let $\alpha \in (0,1]$ and $u, v$ be $\alpha-$differentiable at a point $t$. Then:*

**i.** $D_t^\alpha (\xi u + \eta v) = \xi D_t^\alpha (u) + \eta D_t^\alpha (v)$, *for all* $\xi, \eta \in \mathbb{R}$.
**ii.** $D_t^\alpha (t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.
**iii.** $D_t^\alpha (uv) = D_t^\alpha (u)\, v + u D_t^\alpha (v)$.
**iv.** $D_t^\alpha \left(\frac{u}{v}\right) = \frac{vD_t^\alpha(u) - uD_t^\alpha(v)}{v^2}$.
**v.** $D_t^\alpha (c) = 0$, *$c$ is a constant.*
**vi.** *If $u$ is usual differentiable, then* $D_t^\alpha (f)(t) = t^{1-\alpha}\frac{du}{dt}$.

**Definition 1.3.** Consider the function $u : [0,\infty) \to \mathbb{R}$. Then, the "conformable fractional integral ($\alpha-$ integral)" of $u$ order $\alpha \in (0,1]$ is defined by:

$$I_\alpha u(t) := \int_0^t u(\xi) d_\alpha \xi = \int_0^t \xi^{\alpha-1} u(\xi) d\xi$$

for $t > 0$. Integral to the right of the last equality is the usual Riemann integral.

**Theorem 1.4.** *Consider two $\alpha-$differentiable functions $u, v : [a,b] \to \mathbb{R}$. Then,*

$$\int_a^b u(t) D_t^\alpha v(t) d_\alpha t = uv\big|_a^b - \int_a^b v(t) D_t^\alpha u(t) d_\alpha t.$$

This formula is called $\alpha-$integration by parts.

## 2. Some Spectral Properties

2.1. **Reduction to one parameter problems.** Let $u(x)$ and $u(y)$ be the solution of (1.3) and

$$U(x,y) = u(x)\,u(y)\,. \tag{2.1}$$

By $\alpha-$ differentiating twice equality (2.1) with respect to $x$ and $y$, we have

$$\begin{aligned} D_x^{\alpha}\left(D_x^{\alpha}U(x,y)\right) =& D_x^{\alpha}\left(D_x^{\alpha}u(x)\right)u(y)\,, \\ D_y^{\alpha}\left(D_y^{\alpha}U(x,y)\right) =& u(x)\,D_y^{\alpha}\left(D_y^{\alpha}u(y)\right). \end{aligned} \tag{2.2}$$

Additionally, since $u(x)$ and $u(y)$ satisfy (1.3), the equations

$$\begin{aligned} D_x^{\alpha}\left(D_x^{\alpha}u(x)\right) + \{\lambda + \mu f(x) + g(x)\}\,u(x) = 0, \\ D_y^{\alpha}\left(D_y^{\alpha}u(y)\right) + \{\lambda + \mu f(y) + g(y)\}\,u(y) = 0 \end{aligned}$$

can be written.

If the equations are multiplied by $u(y)$ and $u(x)$, respectively and after subtracted side by side, from (2.2) we get

$$\begin{aligned} D_x^{\alpha}\left(D_x^{\alpha}U(x,y)\right) - & D_y^{\alpha}\left(D_y^{\alpha}U(x,y)\right) \\ & + \{\mu\left(f(x) - f(y)\right) + g(x) - g(y)\}\,U(x,y) = 0. \end{aligned} \tag{2.3}$$

As a result, the problem (1.3)-(1.4) is reduced to the one parameter eigenvalue problem. Also, when the values $x$ and $y$ are any of the values $a, b, c$, using equality (1.4), it is obtained that the boundary conditions of (2.3) are of the form

$$U(x,y) = 0. \tag{2.4}$$

2.2. **Orthogonality properties.** In this section, the orthogonality property are examined separately for the one parameter and two parameter cases.

Let $\mu$ be a constant in (1.3); $u_1(t)$ and $u_2(t)$ be solutions to the problem (1.3)-(1.4) for different values of $\lambda_1$ and $\lambda_2$, respectively. Then, it can be written as follows

$$D_t^{\alpha}\left(D_t^{\alpha}u_1(t)\right) + \{\lambda_1 + \mu f(t) + g(t)\}\,u_1(t) = 0,$$

$$D_t^{\alpha}\left(D_t^{\alpha}u_2(t)\right) + \{\lambda_2 + \mu f(t) + g(t)\}\,u_2(t) = 0.$$

If above equations are multiplied by $u_2(t)$ and $u_1(t)$, respectively and after subtracted side by side, we obtain

$$D_t^{\alpha}\left(D_t^{\alpha}u_1(t)\,u_2(t) - u_1(t)\,D_t^{\alpha}u_2(t)\right) = \{\lambda_2 - \lambda_1\}\,u_1(t)\,u_2(t)\,.$$

Integrating for $(t_1, t_2) = (a, b)$, $(t_1, t_2) = (a, c)$ or $(t_1, t_2) = (b, c)$ and from boundary conditions (1.4), we get

$$\{\lambda_2 - \lambda_1\}\int_{t_1}^{t_2} u_1(t)\,u_2(t)\,d_{\alpha}t = 0.$$

Considering $\lambda_1 \neq \lambda_2$ yields

$$\int_{t_1}^{t_2} u_1(t)\, u_2(t)\, d_\alpha t = 0. \tag{2.5}$$

The solutions of equation (1.3) corresponding to different values of $\lambda$ and $\mu$ provide a broader orthogonality relation. This concept of orthogonality is called double orthogonality in the classical sense [7]. In conformable fractional calculus, we express the concept of double orthogonality as follows.

**Theorem 2.1.** *Suppose that $v_1(t)$ and $v_2(t)$ are the solutions of the problem (1.3)-(1.4) for different values of $(\lambda_1, \mu_1)$ and $(\lambda_2, \mu_2)$, respectively. Then,*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} v_1(x)\, v_1(y)\, v_2(x)\, v_2(y) \left[f(x) - f(y)\right] d_\alpha y d_\alpha x = 0, \tag{2.6}$$

*where $\lambda_1 \neq \lambda_2$, $\mu_1 \neq \mu_2$ or both; $(x_1, x_2)$ and $(y_1, y_2)$ represent different values of the values-pairs $(a, b), (a, c), (b, c)$.*

*Proof.* Suppose that $v_1(t)$ and $v_2(t)$ are the solutions of the problem (1.3) and

$$V_i := V_i(x, y) = v_i(x)\, v_i(y), i = 1, 2. \tag{2.7}$$

Now, $\alpha-$differentiating twice of the equality (2.7) with respect to $x$ and $y$, we have

$$D_x^\alpha \left(D_x^\alpha V_i(x, y)\right) = D_x^\alpha \left(D_x^\alpha v_i(x)\right) v_i(y),$$
$$D_y^\alpha \left(D_y^\alpha V_i(x, y)\right) = v_i(x)\, D_y^\alpha \left(D_y^\alpha v_i(y)\right)$$

for $i = 1, 2$.

Let us first consider the equality $V_1(x, y) = v_1(x)\, v_1(y)$.

Since $v_1(x)$ and $v_1(y)$ satisfy (1.3), the following equations

$$D_x^\alpha \left(D_x^\alpha v_1(x)\right) + \{\lambda_1 + \mu_1 f(x) + g(x)\}\, v_1(x) = 0,$$
$$D_y^\alpha \left(D_y^\alpha v_1(y)\right) + \{\lambda_1 + \mu_1 f(y) + g(y)\}\, v_1(y) = 0$$

are provided. Multiplying these equations by $v_1(y)$, $v_1(x)$, respectively and subtracting gives

$$\begin{aligned} D_x^\alpha \left(D_x^\alpha V_1(x, y)\right) - D_y^\alpha \left(D_y^\alpha V_1(x, y)\right) \\ + \{\mu_1 (f(x) - f(y)) + g(x) - g(y)\}\, V_1(x, y) = 0. \end{aligned} \tag{2.8}$$

Similarly, we get

$$\begin{aligned} D_x^\alpha \left(D_x^\alpha V_2(x, y)\right) - D_y^\alpha \left(D_y^\alpha V_2(x, y)\right) \\ + \{\mu_2 (f(x) - f(y)) + g(x) - g(y)\}\, V_2(x, y) = 0 \end{aligned} \tag{2.9}$$

for $V_2(x, y) = v_2(x)\, v_2(y)$.

After (2.8) is multiplied by $V_2(x,y)$ and (2.9) is multiplied by $V_1(x,y)$ hence, by subtraction,

$$\begin{aligned}&[V_2 D_x^\alpha (D_x^\alpha V_1) - V_1 D_x^\alpha (D_x^\alpha V_2)] + \left[V_1 D_y^\alpha \left(D_y^\alpha V_2\right) - V_2 D_y^\alpha \left(D_y^\alpha V_1\right)\right] \\ &\qquad + (\mu_2 - \mu_1)(f(x) - f(y)) V_1 V_2 = 0\end{aligned}$$

is obtained. $\alpha-$integrating both sides over the interval $(x_1, x_2)$ and $(y_1, y_2)$ on the last equation gives the following

$$\begin{aligned}&(\mu_2 - \mu_1) \int_{x_1}^{x_2} \int_{y_1}^{y_2} (f(x) - f(y)) V_1 V_2 d_\alpha y d_\alpha x \\ &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} [V_2 D_x^\alpha (D_x^\alpha V_1) - V_1 D_x^\alpha (D_x^\alpha V_2)] d_\alpha x d_\alpha y + \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[V_1 D_y^\alpha \left(D_y^\alpha V_2\right) - V_2 D_y^\alpha \left(D_y^\alpha V_1\right)\right] d_\alpha y d_\alpha x \\ &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} D_x^\alpha [V_2 (D_x^\alpha V_1) - V_1 (D_x^\alpha V_2)] d_\alpha x d_\alpha y + \int_{x_1}^{x_2} \int_{y_1}^{y_2} D_y^\alpha \left[V_1 \left(D_y^\alpha V_2\right) - V_2 \left(D_y^\alpha V_1\right)\right] d_\alpha y d_\alpha x \\ &= \int_{y_1}^{y_2} [V_2 (D_x^\alpha V_1) - V_1 (D_x^\alpha V_2)]_{x_1}^{x_2} d_\alpha y + \int_{x_1}^{x_2} \left[V_1 \left(D_y^\alpha V_2\right) - V_2 \left(D_y^\alpha V_1\right)\right]_{y_1}^{y_2} d_\alpha x.\end{aligned}$$

By virtue of boundary conditions (1.4), we obtain

$$(\mu_2 - \mu_1) \int_{x_1}^{x_2} \int_{y_1}^{y_2} (f(x) - f(y)) V_1 V_2 d_\alpha y d_\alpha x = 0. \tag{2.10}$$

Since $\mu_1 \neq \mu_2$, we reach the result (2.6) from (2.7).

On the other hand, let $\mu_1 = \mu_2$ and $\lambda_1 \neq \lambda_2$. Then, the left side of the equality (2.6) can be arranged as follows:

$$\begin{aligned}&\int_{x_1}^{x_2} f(x) v_1(x) v_2(x) d_\alpha x \int_{y_1}^{y_2} v_1(y) v_2(y) d_\alpha y \\ &- \int_{y_1}^{y_2} f(y) v_1(y) v_2(y) d_\alpha y \int_{x_1}^{x_2} v_1(x) v_2(x) d_\alpha x.\end{aligned} \tag{2.11}$$

From (2.5), the proof is completed. □

### 2.3. **The Reality of Eigenvalues.**

**Theorem 2.2.** *All eigenvalues of problem (1.3)-(1.4) are real.*

*Proof.* The function $u_0(t)$ is an eigenfunction associated with the complex conjugate pair $(\lambda_0, \mu_0)$ and the function $\overline{u_0(t)}$ is an eigenfunction associated with another complex conjugate pair $\left(\overline{\lambda_0}, \overline{\mu_0}\right)$. Thus, if $\mu_1 = \mu_0$, $\mu_2 = \overline{\mu_0}$, $v_1 = u_0, v_2 = \overline{u_0}$ is taken into account on the equality (2.10), the double orthogonality results, we have

$$\left(\overline{\mu_0}-\mu_0\right)\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left(f\left(x\right)-f\left(y\right)\right)u_0\left(x\right)u_0\left(y\right)\overline{v_0}\left(x\right)\overline{v_0}\left(y\right)d_\alpha y d_\alpha x=0.$$

Here, the integrand is not zero; since $\overline{\mu_0}-\mu_0=0$, $\mu_0$ is real.

On the other hand, let $\lambda_0$ and $\overline{\lambda_0}$ be eigenvalues of the same values of $\mu_0$. Similar operations are performed for the $u_0\left(x\right),\overline{u_0}\left(x\right)$ eigenfunctions corresponding to these values. From (2.11), it is obtained that

$$\int_{x_1}^{x_2}\left|u_0\right|^2 d_\alpha x\neq 0$$

and since $\overline{\lambda_0}=\lambda_0$, $\lambda_0$ is real.

Consequently, since $\lambda_0$ and $\mu_0$ are arbitrary, all eigenvalues of the problem (1.3)-(1.4) are real. □

## 3. Some Integral Relations

In this section, two integral relationships are given. These are derived from integral equations satisfied by the solutions of the problem (1.3)-(1.4).

**Theorem 3.1.** *The function*

$$U\left(t\right)=\int_{x_1}^{x_2}G\left(t,x\right)u\left(x\right)d_\alpha x \tag{3.1}$$

*is a solution of the equation (1.3) the following conditions are satisfied*

**i.** *The function $u\left(x\right)$ is a solution of the equation*

$$D_x^\alpha\left(D_x^\alpha u\left(x\right)\right)+\left\{\lambda+\mu f\left(x\right)+g\left(x\right)\right\}u\left(x\right)=0.$$

**ii.** *The function $G\left(t,x\right)$ is a solution of the conformable partial equation*

$$D_t^\alpha\left(D_t^\alpha G\left(t,x\right)\right)-D_x^\alpha\left(D_x^\alpha G\left(t,x\right)\right)+\left\{\mu\left(f\left(t\right)-f\left(x\right)\right)+g\left(t\right)-g\left(x\right)\right\}G=0.$$

**iii.** *The function*

$$G\left(t,x\right)D_x^\alpha\left(u\left(x\right)\right)-u\left(x\right)D_x^\alpha\left(G\left(t,x\right)\right)$$

*has the same value at the endpoints of the intervals $\left(a,b\right),\left(a,c\right),\left(b,c\right)$*

**iv.** *The integral*

$$\int_{x_1}^{x_2}G\left(t,x\right)u\left(x\right)d_\alpha x$$

*exists.*

*Proof.* To complete the proof, we need to show that

$$D_t^\alpha\left(D_t^\alpha U(t)\right)+\{\lambda+\mu f(t)+g(t)\}U(t)=0. \tag{3.2}$$

By the existence of the integral in the condition (iv), the $\alpha-$differential of (3.1) under the integral sign can be taken. Thus,

$$\begin{aligned}&D_t^\alpha\left(D_t^\alpha U(t)\right)+\{\lambda+\mu f(t)+g(t)\}U(t)\\&=\int_{x_1}^{x_2}\left[D_t^\alpha\left(D_t^\alpha G(t,x)\right)+\{\lambda+\mu f(t)+g(t)\}G(t,x)\right]u(x)\,d_\alpha x\end{aligned} \tag{3.3}$$

is obtained. Besides, from the condition (ii), it can be written as

$$\begin{aligned}&D_t^\alpha\left(D_t^\alpha G(t,x)\right)+\{\lambda+\mu f(t)+g(t)\}G(t,x)\\&\qquad=D_x^\alpha\left(D_x^\alpha G(t,x)\right)+\{\lambda+\mu f(x)+g(x)\}G(t,x).\end{aligned} \tag{3.4}$$

The equality (3.4) is also taken into account on the equation (3.3), we have

$$\begin{aligned}&D_t^\alpha\left(D_t^\alpha U(t)\right)+\{\lambda+\mu f(t)+g(t)\}U(t)\\&=\int_{x_1}^{x_2}D_x^\alpha\left(D_x^\alpha G(t,x)\right)u(x)\,d_\alpha x+\int_{x_1}^{x_2}\{\lambda+\mu f(x)+g(x)\}G(t,x)u(x)\,d_\alpha x\end{aligned}$$

When $\alpha-$partial integration is applied to the first integral on the last equality, it is seen to be

$$\begin{aligned}D_t^\alpha\left(D_t^\alpha U(t)\right)&+\{\lambda+\mu f(t)+g(t)\}U(t)\\&=\left[D_x^\alpha\left(G(t,x)\right)u(x)-G(t,x)\,D_x^\alpha\left(u(x)\right)\right]_{x_1}^{x_2}\\&+\int_{x_1}^{x_2}\left[D_x^\alpha\left(D_x^\alpha u(x)\right)+\{\lambda+\mu f(x)+g(x)\}u(x)\right]G(t,x)\,d_\alpha x.\end{aligned}$$

The first term on the last equation and the expression in square brackets in the second term vanishes from conditions (iii) and (i), respectively.

As a result, the proof is completed by obtaining (3.2). □

**Theorem 3.2.** *The function*

$$U(z)=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left[f(x)-f(y)\right]P(x,y,z)\,u(x)\,u(y)\,d_\alpha y d_\alpha x \tag{3.5}$$

*is a solution of the equation (1.3) the following conditions are satisfied*

**i.** *The functions $u(x)$and $u(y)$ are the solutions of the equations*

$$\begin{aligned}D_x^\alpha\left(D_x^\alpha u(x)\right)+\{\lambda+\mu f(x)+g(x)\}u(x)=0,\\D_y^\alpha\left(D_y^\alpha u(y)\right)+\{\lambda+\mu f(y)+g(y)\}u(y)=0,\end{aligned}$$

*respectively.*

**ii.** *$P := P(x,y,z)$ is a solution of following partial equation*

$$\{f(y)-f(z)\} D_x^\alpha (D_x^\alpha P) + \{f(z)-f(x)\} D_y^\alpha \left(D_y^\alpha P\right) + \{f(x)-f(y)\} D_z^\alpha (D_z^\alpha P)$$
$$= -\left[g(x)\{f(y)-f(z)\} + g(y)\{f(z)-f(x)\} + g(z)\{f(x)-f(y)\}\right] P.$$

**iii.** *The equalities*

$$\left[(D_x^\alpha P) u(x) - P(x,y,z) D_x^\alpha u(x)\right]_{x_1}^{x_2} = 0,$$
$$\left[\left(D_y^\alpha P\right) u(y) - P(x,y,z) D_y^\alpha u(y)\right]_{y_1}^{y_2} = 0$$

*are satisfied on the intervals $(a,b), (a,c), (b,c)$.*

**iv.** *The integral*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} \{f(x)-f(y)\} P(x,y,z) u(x) u(y) \, d_\alpha y d_\alpha x$$

*is also exists and convergent.*

*Proof.* To complete the proof, we need to show that

$$D_z^\alpha (D_z^\alpha U(z)) + \{\lambda + \mu f(z) + g(z)\} U(z) = 0. \tag{3.6}$$

By the existence of the integral in the condition (iv), the $\alpha-$differential of (3.5) under the integral sign can be taken. Therefore,

$$\begin{aligned} & D_z^\alpha (D_z^\alpha U(z)) + \{\lambda + \mu f(z) + g(z)\} U(z) \\ & \quad = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \{f(x)-f(y)\} \left[D_z^\alpha (D_z^\alpha P)\right. \\ & \quad \left. + \{\lambda + \mu f(z) + g(z)\} P(x,y,z)\right] u(x) u(y) \, d_\alpha y d_\alpha x \end{aligned} \tag{3.7}$$

is obtained. Besides, with the help of the equality on the condition (ii), the integrand on the equation (3.7) is rearranged as follows

$$\begin{aligned} & \left[\{f(z)-f(y)\} D_x^\alpha (D_x^\alpha P) + \{f(x)-f(z)\} D_y^\alpha \left(D_y^\alpha P\right) - F(x,y,z)\right. \\ & \left. + \{\lambda + \mu f(z) + g(z)\} \{f(x)-f(y)\} P\right] u(x) u(y), \end{aligned} \tag{3.8}$$

where

$$F(x,y,z) = \left[g(x)\{f(y)-f(z)\} + g(y)\{f(z)-f(x)\} + g(z)\{f(x)-f(y)\}\right] P(x,y,z).$$

Then, the function (3.8) is taken into account on the equation (3.7), we have

$$
\begin{aligned}
&D_z^\alpha\left(D_z^\alpha U(z)\right)+\{\lambda+\mu f(z)+g(z)\} U(z)\\
&=\int_{y_1}^{y_2}\{f(z)-f(y)\} u(y) \int_{x_1}^{x_2} D_x^\alpha\left(D_x^\alpha P\right) u(x) d_\alpha x d_\alpha y\\
&+\int_{x_1}^{x_2}\{f(x)-f(z)\} u(x) \int_{y_1}^{y_2} D_y^\alpha\left(D_y^\alpha P\right) u(y) d_\alpha y d_\alpha x \qquad (3.9)\\
&-\int_{x_1}^{x_2}\int_{y_1}^{y_2}[F(x,y,z)\\
&-\{\lambda+\mu f(z)+g(z)\}\{f(x)-f(y)\} P(x,y,z)] u(x) u(y) d_\alpha y d_\alpha x .
\end{aligned}
$$

By applying $\alpha-$partial integration twice to the integrals

$$
\int_{x_1}^{x_2} D_x^\alpha\left(D_x^\alpha P\right) u(x) d_\alpha x, \qquad \int_{y_1}^{y_2} D_y^\alpha\left(D_y^\alpha P\right) u(y) d_\alpha y
$$

on the equality (3.9) respectively,

$$
\begin{aligned}
&\int_{x_1}^{x_2} D_x^\alpha\left(D_x^\alpha P\right) u(x) d_\alpha x=\left[\left(D_x^\alpha P\right) u(x)-P D_x^\alpha u(x)\right]_{x_1}^{x_2}+\int_{x_1}^{x_2} P D_x^\alpha\left(D_x^\alpha u\right)(x) d_\alpha x,\\
&\int_{y_1}^{y_2} D_y^\alpha\left(D_y^\alpha P\right) u(y) d_\alpha y=\left[\left(D_y^\alpha P\right) u(y)-P D_y^\alpha u(y)\right]_{y_1}^{y_2}+\int_{y_1}^{y_2} P D_y^\alpha\left(D_y^\alpha u\right)(y) d_\alpha y
\end{aligned}
$$

are obtained. Here, the first terms on the right-hand side of these equalities vanish on the intervals $(a,b),(a,c),(b,c)$. Then, the equality (3.9) is rearranged that

$$
\begin{aligned}
&D_z^\alpha\left(D_z^\alpha U(z)\right)+\{\lambda+\mu f(z)+g(z)\} U(z)\\
&=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\{f(z)-f(y)\} u(y)\left\{D_x^\alpha\left(D_x^\alpha u\right)(x)+g(x) u(x)\right\} P d_\alpha y d_\alpha x\\
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2}\{f(x)-f(z)\} u(x)\left\{D_y^\alpha\left(D_y^\alpha u\right)(y)+g(y) u(y)\right\} P d_\alpha y d_\alpha x \qquad (3.10)\\
&+\int_{x_1}^{x_2}\int_{y_1}^{y_2}\{\lambda+\mu f(z)\}\{f(x)-f(y)\} u(x) u(y) P d_\alpha y d_\alpha x .
\end{aligned}
$$

On the other hand, the condition (i) can be written as follows

$$
\begin{aligned}
&D_x^\alpha\left(D_x^\alpha u\right)(x)+g(x) u(x)=-\{\lambda+\mu f(x)\} u(x),\\
&D_y^\alpha\left(D_y^\alpha u\right)(y)+g(y) u(y)=-\{\lambda+\mu f(y)\} u(y).
\end{aligned}
$$

If these last representations are taken into consideration in the equation (3.10), we reach

$$\begin{aligned} & D_z^\alpha \left(D_z^\alpha U\left(z\right)\right)+\left\{\lambda+\mu f\left(z\right)+g\left(z\right)\right\}U\left(z\right) \\ & =\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left[\left\{f\left(y\right)-f\left(z\right)\right\}\left\{\lambda+\mu f\left(x\right)\right\}+\left\{f\left(z\right)-f\left(x\right)\right\}\left\{\lambda+\mu f\left(y\right)\right\}\right. \\ & \left.+\left\{f\left(x\right)-f\left(y\right)\right\}\left\{\lambda+\mu f\left(z\right)\right\}\right]u\left(x\right)u\left(y\right)Pd_\alpha y d_\alpha x=0. \end{aligned} \tag{3.11}$$

As a result, the proof is completed by obtaining (3.6). □

## 4. Conclusion

In this study, the focus was initially on classical two-parameter eigenvalue problems. These problems with two parameters have been transformed into one-parameter eigenvalue problems using specific methodologies. Throughout this transformation process, the orthogonal properties of the eigenvalue problems have been emphasized, and certain integral relationships have been established. The recalculation of transitions in the relevant theorems was necessary to obtain the main results. Consequently, the two-parameter eigenvalue problems were formulated using conformable fractional derivatives, and their related properties were examined. Subsequently, these problems were transformed from a two-parameter to a one-parameter format with the help to the properties of conformable fractional derivatives. The research particularly emphasized the reality of the eigenvalues and presented specific integral relationships. And, it was seen to coincide with Arscott's work [7] when the case $\alpha = 1$.

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# ON SOME ASYMPTOTIC EIGENVALUES OF HILL' S EQUATION

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Abstract. In this paper, we deal with Hill' s equation with symmetric single well potential. We find the lower and upper boundaries of the difference between Dirichlet and Neumann eigenvalues of Hill' s equation. We also calculate the eigenvalues of Hill' s equation with two mixed problems, asymptotically.

## 1. Introduction

We consider the following differential equation

$$y''(t) + [\lambda - q(t)]\, y(t) = 0 \tag{1.1}$$

where $\lambda$ is a real parameter and $q(t)$ is a real-valued, continuous and periodic function with period $a$. We also accept that $q(t)$ is a symmetric single well potential with mean value zero. By a symmetric single well potential on $[0, a]$, we mean a continuous function $q(t)$ on $[0, a]$ which is symmetric about $t = \frac{a}{2}$ and non-increasing on $\left[0, \frac{a}{2}\right]$, so we can say that $q(t) = q(a - t)$ and $q'(t)$ exist because of monotony. In literature, a lot of researchers deal with this equation with various boundary conditions, various potentials and they find eigenfunctions, eigenvalues, the expression of Green' s function and instability intervals. Some of those are [1]-[14]. Here we calculate the lower and upper boundaries of the difference between Dirichlet and Neumann eigenvalues of (1.1). We also obtain the eigenvalues of (1.1) with mixed problems.

Let us explain the these problems in the following section (More details can be seen in [11]):

## 2. Preliminaries

We begin with the general second-order equation

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$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0 \tag{2.1}$$

in which the coefficients $a_r(x)$ are complex-valued, piecewise continuous, and periodic, all with the same period $a$. Thus

$$a_r(x + a) = a_r(x) \qquad (0 \leq r \leq 2)$$

where $a$ is a non-zero real constant. It is also assumed that the left and right-hand limits of $a_0(x)$ at every point are non-zero, so that the usual theory of linear differential equations without singular points applies.

The name of Hill' s equation is given to the equation

$$\{P(x)y'(x)\}' + Q(x)y(x) = 0 \tag{2.2}$$

where $P(x)$ and $Q(x)$ are real-valued and have the same period $a$. In addition, it is assumed that $P(x)$ is continuous and nowhere zero and that $P'(x)$ and $Q(x)$ are piecewise continuous. Thus (2.2) is a particular case of (2.1) and it is named after G. W. Hill following his work on it 1877.

When we write $p(x)$ instead of $P(x)$ and $Q(x)$ involves a real parameter $\lambda$ in the form

$$Q(x) = \lambda s(x) - q(x)$$

where $s(x)$ and $q(x)$ are piecewise continuous with period $a$ and there is a constant $s > 0$ such that $s(x) \geq s$. (2.2) is now

$$\{p(x)y'(x)\}' + \{\lambda s(x) - q(x)\}\, y(x) = 0. \tag{2.3}$$

In order to indicate the depence on $\lambda$ which occurs in (2.3), we write $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ for the solutions of (2.3) which satisfy the initial conditions

$$\phi_1(0, \lambda) = 1, \quad \phi_1'(0, \lambda) = 0; \qquad \phi_2(0, \lambda) = 0, \quad \phi_2'(0, \lambda) = 1.$$

Let us define the discriminant as

$$D(\lambda) := \phi_1(a, \lambda) + \phi_2'(a, \lambda). \tag{2.4}$$

Although the parameter $\lambda$ is taken to be real here, it is sometimes necessary to allow it to be complex. Whether $\lambda$ is real or complex, $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$, and their $x-$ derivatives are, for fixed $x$, analytic functions of $\lambda$. Hence, by (2.4), $D(\lambda)$ is an analytic function of $\lambda$. Since, in particular, $D(\lambda)$ is a continuous function of $\lambda$, the values of $\lambda$ for which $|D(\lambda)| < 2$ for an open set on the real $\lambda-$axis. This set, which as we shall see is not empty, can be expressed as the union of a countable collection of disjoint open intervals. (2.3) is stable when $\lambda$ lies in these intervals, and the intervals are therefore called the stability intervals of (2.3). Similarly, the intervals in which $|D(\lambda)| > 2$ are called the instability intervals of (2.3). Finally, the intervals formed by the closures of the stability intervals are, those in which $|D(\lambda)| \leq 2$the conditional stability intervals of are called (2.3). [11] establishes the existence of the stability and instability intervals and gives a precise description of them.

The periodic eigenvalue problem comprises (2.3), considered to hold in $[0, a]$, and the periodic boundary conditions

$$y(a) = y(0), \quad y'(a) = y'(0)$$

and the eigenvalues $\lambda_n$ of this problem satisfy

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots, \text{and} \quad \lambda_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty.$$

Also, $\lambda_n$ are the zeros of the function $D(\lambda) - 2$ and that a given $\lambda_n$ is a double eigenvalue if and only if

$$\phi_2(a, \lambda_n) = \phi_1'(a, \lambda_n) = 0.$$

The semi-periodic (or called as anti-periodic) eigenvalue problem comprises (2.3), considered to hold in $[0, a]$, and the semi-periodic boundary conditions

$$y(a) = -y(0), \quad y'(a) = -y'(0)$$

and the eigenvalues $\mu_n$ of this problem satisfy

$$\mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots, \text{and} \quad \mu_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty.$$

Also, $\mu_n$ are the zeros of the function $D(\lambda) + 2$ and that a given $\mu_n$ is a double eigenvalue if and only if

$$\phi_2(a, \mu_n) = \phi_1'(a, \mu_n) = 0.$$

We also know [11]

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_3 \leq \cdots.$$

We denote also by $\Lambda_n$ and $\nu_n$ respectively the eigenvalues in the two eigenvalue problems which comprise (2.3), considered to hold in $[0, a]$, and the two sets of boundary conditions

$$y(0) = y(a) = 0 \tag{2.5}$$

and

$$y'(0) = y'(a) = 0. \tag{2.6}$$

The equation (2.5) is named as Dirichlet condition, whereas (2.6) is named as Neumann condition. Also from [11], $n = 0, 1, 2, \cdots$

$$\mu_{2n} \leq \Lambda_{2n} \leq \mu_{2n+1}, \qquad \lambda_{2n+1} \leq \Lambda_{2n+1} \leq \lambda_{2n+2}, \tag{2.7}$$

$$\mu_{2n} \leq \nu_{2n+1} \leq \mu_{2n+1}, \qquad \lambda_{2n+1} \leq \nu_{2n+2} \leq \lambda_{2n+2}. \tag{2.8}$$

Let us apply to (2.3) the Liouville transformation

$$t = \int_0^x [s(u)/p(u)]^{1/2}\, du, \quad z(t) = [p(x)s(x)]^{1/4}\, y(x).$$

The transformed equation is

$$z'' + [\lambda - Q(t)z(t)] = 0 \tag{2.9}$$

where

$$Q(t) = q(x) - [p(x)]^{1/4} [s(x)]^{-3/4} \frac{d}{dx} p(x) \frac{d}{dx} [p(x)s(x)]^{-1/4}.$$

It can be seen that the parameter $\lambda$ is unchanged. Also, the periodic and semi-periodic boundary conditions for the $x-$ interval $[0, a]$ are transformed into boundary conditions of the same type for the corresponding $t-$ interval. Hence, the periodic $(\lambda_n)$ and semi-periodic $(\mu_n)$ eigenvalues for (2.9) are the same as for (2.3). We note that $Q(t)$ is $r$ times differentiable if $q^r(x)$, $p^{r+2}(x)$ and $s^{r+2}(x)$ all exist and we can't apply the Liouville transformation if $p''$ and $q''$ do not exist.

## 3. The Results

In this part, we provide our results. Firstly, let us give two mixed problem with Hill' s equation for $t \in [0, a/2]$:

The Mixed Problem 1

$$\begin{aligned} y''(t) + [\lambda - q(t)]\, y(t) &= 0 \\ y'(0) = y(a/2) &= 0, \end{aligned}$$

and its eigenvalue is denoted as $\lambda^{M_1}$;

The Mixed Problem 2

$$\begin{aligned} y''(t) + [\lambda - q(t)]\, y(t) &= 0 \\ y(0) = y'(a/2) &= 0, \end{aligned}$$

and its eigenvalue is denoted as $\lambda^{M_2}$.

**Theorem 3.1.** *The lower and upper boundaries of the difference between Dirichlet and Neumann eigenvalues of (1.1) on $[0, a]$ satisfy, as $n \to \infty$*

*i)*

$$\begin{aligned} &-\frac{a}{(2n+1)^2 \pi^2} \left| \int_0^{a/2} q'(t) \sin\left(\frac{2(2n+1)\pi}{a} t\right) dt \right| + o\left(n^{-3}\right) \\ &\leq \Lambda_{2n} - \nu_{2n+1} \\ &\leq \frac{a}{(2n+1)^2 \pi^2} \left| \int_0^{a/2} q'(t) \sin\left(\frac{2(2n+1)\pi}{a} t\right) dt \right| + o\left(n^{-3}\right), \end{aligned}$$

*ii)*

$$-\frac{a}{8\left(n+1\right)^{2}\pi^{2}}\left|\int_{0}^{a/2}q'\left(t\right)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right|+o\left(n^{-3}\right)$$
$$\leq\Lambda_{2n+1}-\nu_{2n+2}$$
$$\leq\frac{a}{8\left(n+1\right)^{2}\pi^{2}}\left|\int_{0}^{a/2}q'\left(t\right)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right|+o\left(n^{-3}\right).$$

*Proof.* If we subtract (2.8) from (2.7), we reach that

$$\mu_{2n}-\mu_{2n+1}\leq\Lambda_{2n}-\nu_{2n+1}\leq\mu_{2n+1}-\mu_{2n}, \tag{3.1}$$

$$\lambda_{2n+1}-\lambda_{2n+2}\leq\Lambda_{2n+1}-\nu_{2n+2}\leq\lambda_{2n+2}-\lambda_{2n+1}. \tag{3.2}$$

We also have from [1] that the periodic and semi-periodic eigenvalues of (1.1) on $[0,a]$ satisfy, as $n\to\infty$

$$\begin{array}{c}\lambda_{2n+1}^{1/2}\\ \lambda_{2n+2}^{1/2}\end{array}=\frac{2\left(n+1\right)\pi}{a}\mp\frac{a}{8\left(n+1\right)^{2}\pi^{2}}\left|\int_{0}^{a/2}q'\left(t\right)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right|$$
$$-\frac{a^{2}}{64\left(n+1\right)^{3}\pi^{3}}$$
$$\times\left[aq^{2}\left(a\right)+2a\int_{0}^{a/2}q\left(t\right)q'\left(t\right)dt-4\int_{0}^{a/2}tq\left(t\right)q'\left(t\right)dt\right]+o\left(n^{-3}\right)$$

and

$$\begin{array}{c}\mu_{2n}^{1/2}\\ \mu_{2n+1}^{1/2}\end{array}=\frac{\left(2n+1\right)\pi}{a}\mp\frac{a}{2\left(2n+1\right)^{2}\pi^{2}}\left|\int_{0}^{a/2}q'\left(t\right)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right|$$
$$-\frac{a^{2}}{8\left(2n+1\right)^{3}\pi^{3}}$$
$$\times\left[aq^{2}\left(a\right)+2a\int_{0}^{a/2}q\left(t\right)q'\left(t\right)dt-4\int_{0}^{a/2}tq\left(t\right)q'\left(t\right)dt\right]+o\left(n^{-3}\right).$$

If we use this results and equations (3.1) and (3.2), we prove the theorem.

Notice that, the problems are on $[0,a]$, but we can write our solutions on $[0,a/2]$ because of symmetric single well potential $q$.

□

**Theorem 3.2.** *The eigenvalues of the Mixed Problem 1 and the Mixed Problem 2 satisfy, as $n\to\infty$*

*i)*

$$\begin{aligned}
\left[\lambda_{2n}^{M_1}\right]^{1/2} = \left[\lambda_{2n}^{M_2}\right]^{1/2} &= \frac{(2n+1)\pi}{a} \\
&- \frac{a}{2(2n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right| \\
&- \frac{a^2}{8(2n+1)^3\pi^3} \\
&\times \left[aq^2(a) + 2a\int_0^{a/2} q(t)\,q'(t)\,dt - 4\int_0^{a/2} tq(t)\,q'(t)\,dt\right] \\
&+ o\left(n^{-3}\right),
\end{aligned}$$

*ii)*

$$\begin{aligned}
\left[\lambda_{2n+1}^{M_1}\right]^{1/2} = \left[\lambda_{2n+1}^{M_2}\right]^{1/2} &= \frac{(2n+1)\pi}{a} \\
&+ \frac{a}{2(2n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right| \\
&- \frac{a^2}{8(2n+1)^3\pi^3} \\
&\times \left[aq^2(a) + 2a\int_0^{a/2} q(t)\,q'(t)\,dt - 4\int_0^{a/2} tq(t)\,q'(t)\,dt\right] \\
&+ o\left(n^{-3}\right).
\end{aligned}$$

*Proof.* Firstly, it can be note that $\mu$ is the eigenvalues of Hill's equation on the interval $[0, a]$ but the eigenvalues of mixed problems is for Hill' s equation on the interval $[0, a/2]$. [4] doesn' t entirely give mixed eigenvalues but it gives some properties for the mixed eigenvalues and proves that, if you have a symmetric potential

$$\lambda_k^{M_1} = \lambda_k^{M_2} = \mu_k$$

is satisfied. Our potential is symmetric single well, so we can use this equality. From this equality and $\mu_{2n}^{1/2}$ and $\mu_{2n+1}^{1/2}$ (above given from [1]), we prove the theorem and hence, we can give asymptotic eigenvalues. □

## 4. Conclusions

In this study, we find some asymptotic eigenvalues of Hill's equation. Our potential is symmetric single well, so we show that we can write our results on the half interval, we don't need to give asymptotic eigenvalues on the whole interval of the problem, the half interval is enough.

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# SOME CONVERGENCE TYPES OF FUNCTION SEQUENCES AND THEIR RELATIONS

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Abstract. In this paper, we investigate various types of convergence for sequences of functions and examine the relationships among these types. Our findings contribute to a deeper understanding of the structural properties of function sequences and their convergence behaviors.

## 1. Introduction

The study of convergence for sequences of functions is a fundamental topic in mathematical analysis, with significant implications in various branches of mathematics and applied sciences. Convergence types such as pointwise convergence, uniform convergence, and $\alpha-$convergence provide different lenses through which we can understand the behavior of function sequences under various conditions. These convergence concepts are crucial for solving differential equations, analyzing function spaces, and understanding the limits of functions in mathematical modeling.

In this paper, we aim to systematically investigate the different types of convergence for sequences of functions and elucidate the relationships between them. We begin by defining the basic concepts and notations used throughout the paper.

## 2. Preliminaries

Let $X$ be a non-empty subset of $\mathbb{R}$. The set of real-valued functions defined on $X$ is denoted by $F(X,\mathbb{R})$:

$$F := F(X,\mathbb{R}) = \{h \mid h : X \to \mathbb{R}\}.$$

The family of function sequences defined over $X$ is denoted by $FS(X)$. For simplicity, if the domain of functions is known, $FS(X)$ is abbreviated to $FS$.

$$FS = \{(h_n) : \forall n \in \mathbb{N},\ h_n \in F\}.$$

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**Definition 2.1.** ([10]) Let $(h_n) \in FS$ be given. If there exists a function $C \in F$ such that for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq C(t)$ holds, then the function sequence $(h_n)$ is said to be pointwise bounded on $X$.

The family of pointwise bounded function sequences is denoted by $PBFS(X)$. For simplicity, if the domain of functions is known, $PBFS(X)$ is abbreviated to $PBFS$.

$$PBFS = \{(h_n) \in FS : \exists C \in F : \forall n \in \mathbb{N},\, |h_n| \leq C\}.$$

**Example 2.2.** On the interval $X = (0, 1)$, the sequence of functions $(h_n)$ defined by $h_n(t) = n/(nt + 1)$ for each $n \in \mathbb{N}$ is pointwise bounded by $C(t) = 1/t$.

**Proposition 2.3.** The necessary and sufficient condition for a function sequence $(h_n) \in FS$ to be pointwise bounded is that $\sup_{n\in\mathbb{N}} |h_n(t)| < \infty$ for all $t \in X$.

*Proof.* Let $(h_n) \in PBFS$ and let $t \in X$ be given. In this case, there exists a function $C \in F$ such that for every $n \in \mathbb{N}$, $|h_n(t)| \leq C(t)$. Taking the supremum, we have $\sup_{n\in\mathbb{N}} |h_n(t)| < C(t) < \infty$. On the other hand, suppose $\sup_{n\in\mathbb{N}} |h_n(t)| < \infty$ for every $t \in X$. Then, for each $t \in X$, define $C(t) := \sup_{n\in\mathbb{N}} |h_n(t)|$. Obviously, $C \in F$, and for every $n \in \mathbb{N}$, $|h_n| \leq C$ holds. Thus, $(h_n)$ is pointwise bounded. □

**Definition 2.4.** ([10]) Let $(h_n) \in FS$ be given. If there exists a number $M > 0$ such that for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq M$ holds, then the function sequence $(h_n)$ is said to be uniformly bounded on $X$.

The family of function sequences that are uniformly bounded on $X$ is denoted by $UBFS(X)$. For simplicity, if the domain of functions is known, $UBFS(X)$ is abbreviated to $UBFS$.

$$UBFS = \{(h_n) \in FS : \exists M > 0 : \forall n \in \mathbb{N},\, |h_n| \leq M.\}$$

**Example 2.5.** On $X = [0, 1]$, the sequence of functions $(h_n)$ defined by $h_n(t) = t/n$ for each $n \in \mathbb{N}$ is uniformly bounded with $K = 1$.

**Proposition 2.6.** A function sequence $(h_n) \in FS$ is uniformly bounded if and only if $\sup_{n\in\mathbb{N}} \sup_{t\in X} |h_n(t)| < \infty$.

*Proof.* Let $(h_n)$ be a sequence on set $X$ that is uniformly bounded. In other words, there exists $K > 0$ such that for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq K$. Then, taking the supremum over $n$ and $t$, we have $\sup_{n\in\mathbb{N}} \sup_{t\in X} |h_n(t)| < K < \infty$, satisfying the desired condition.

Conversely, if $\sup_{n\in\mathbb{N}} \sup_{t\in X} |h_n(t)| = K < \infty$, then for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq K$. This shows that the sequence $(h_n)$ is uniformly bounded on $X$. □

*Remark* 2.7. In Definition 2.1, if we choose the function $C$ to be the number $M$ mentioned in Definition 2.4, such that for every $t \in X$, $C(x) = K$, then it can be seen that every uniformly bounded function sequence is pointwise bounded: $PBFS \subset UBFS$. Nevertheless, the reverse of this does not hold. The function sequence given in Example 2.2 is pointwise bounded on $X$ but not uniformly bounded.

What conditions need to be imposed on the set $X$ for $UBFS = PBFS$ to hold? The answer is provided in the proposition below.

**Proposition 2.8.** If $X$ is a finite set, then $UBFS = PBFS$.

*Proof.* Let $X$ be a finite set. It's clear that $UBFS \subset PBFS$. Let $(h_n) \in PBFS$ be arbitrary. Then there exists a function $C : X \to \mathbb{R}^+$ such that for every $t \in X$ and every $n \in \mathbb{N}$, $|h_n(t)| \leq C(t)$. Since $X$ is finite, we can choose $K = \max_{t \in X} C(t)$, and for every $n \in \mathbb{N}$ and every $t \in X$, $|h_n(t)| \leq K$. Thus, $PBFS \subset UBFS$. □

If $X$ is not finite, can Proposition 2.8 still hold true? Is it possible to eliminate the condition that the set $X$ is finite? The answer to these questions is provided in the remark below.

*Remark* 2.9. If $X$ is infinite, then it has a countable subset $S = \{t_1, t_2, \cdots\}$. Without loss of generality, we can choose $(t_n)$ to be a monotone sequence. This leads to two cases:

(i) $\lim |t_n| = \infty$
(ii) $\lim |t_n| = a < \infty$

In (i)

$$h_n(t) = \begin{cases} \dfrac{n|t|}{n+1}, & x \in S \\ 0, & t \in X \setminus S \end{cases}$$

In (ii)

$$h_n(t) = \begin{cases} \dfrac{n-1}{n(|t|-a)}, & x \in S \\ 0, & t \in X \setminus S \end{cases}$$

is pointwise bounded but not uniformly bounded. Thus, the condition that $X$ must be finite cannot be removed for the equality $UBFS = PBFS$.

## 3. Some Convergence Types of Function Sequences

### 3.1. Pointiwse Convergence.

**Definition 3.1.** ([10]) Let $(h_n) \in FS$ and $h \in F$ be given. If for every $t \in X$, $\lim_{n\to\infty} h_n(t) = h(t)$ holds, then the sequence $(h_n)$ is said to converge pointwise to the function $h$ on the set $X$, and it is denoted by $h_n \xrightarrow{X} h$.

The above definition can also be expressed as follows: The sequence $(h_n)$ is said to converge pointwise to the function $h$ on the set $X$ if for every $\varepsilon > 0$ and every $t \in X$, there exists a natural number $n_0$ such that for every $n \geq n_0$, $|h_n(t) - h(t)| < \varepsilon$ holds.

If the convergence set is specified, for simplicity, $h_n \xrightarrow{X} h$ is replaced with $h_n \to h$. The family of pointwise convergent function sequences on set $X$ is denoted by $PCFS(X)$. If the domain set is specified, for simplicity, $PCFS(X)$ is replaced with $PCFS$.

$$PCFS = \Big\{(h_n) \in FS : \exists h \in F : \forall t \in X, \lim_{n\to\infty} h_n(t) = h(t)\Big\}$$

**Example 3.2.** $h_n : [0,1] \to \mathbb{R}$ defined by $h_n(t) = t^n$, the sequence $(h_n)$ of functions is pointwise convergent on $[0,1]$ to the function

$$h(t) = \begin{cases} 1, & t = 1 \\ 0, & t \in [0,1). \end{cases}$$

**Proposition 3.3.** $PCFS \subset PBFS$.

*Proof.* Let $(h_n)$ be a sequence in $PCFS$ and let $t \in X$. Then, there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $|h_n(t) - h(t)| < 1$. From this, we have

Let $C(t) := \max_{t\in X} \{|h_1(x)|, |h_2(t)|, \ldots, |h_{n_0-1}(t)|, |h(t)| + 1\}$. Performing this operation for every $t \in X$, we define the function $C$ on $X$ such that for every $n \in \mathbb{N}$ and every $t \in X$,

$$|h_n(t)| \leq C(t)$$

Thus, $(h_n)$ is pointwise bounded on $X$. □

3.2. **Uniform Convergence.**

**Definition 3.4.** [10] Let $(h_n) \in FS$ and $h \in F$ be given. If for every $\varepsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $t \in X$, $|h_n(t) - h(t)| < \varepsilon$, then the sequence $(h_n)$ is said to converge uniformly to the function $h$ on the set $X$, and it is denoted by $h_n \overset{X}{\rightrightarrows} h$.

If the convergence set is specified, for simplicity, $h_n \overset{X}{\rightrightarrows} h$ is replaced with $h_n \rightrightarrows h$. The family of uniformly convergent function sequences on set $X$ is denoted by $UCFS(X)$. If the domain set is specified, for simplicity, $UCFS(X)$ is replaced with $UCFS$.

$$UCFS = \{(h_n) \in FS : \forall\varepsilon > 0, \exists n_0 : \forall t \in X, \forall n \geq n_0, |h_n(t) - h(t)| < \varepsilon\}$$

**Example 3.5.** The sequence of functions $(h_n)$ defined by $h_n(t) = t/n$ for every $n \in \mathbb{N}$ is uniformly convergent to the function $h(t) = 0$ on the interval $[0, 1]$.

*Remark* 3.6. It is evident that $UCFS$ is a subset of $PCFS$, Nevertheless, the reverse of this does not hold. Even though the function sequence given in Example 3.2 is pointwise convergent, it is not uniformly convergent.

Under what conditions on the set $X$ does $UCFS = PCFS$ hold? The answer is provided in the proposition below.

**Proposition 3.7.** If $X$ is a finite set, then $UCFS = PCFS$ holds.

*Proof.* Let $X$ be a finite set. It is obvious that $PCFS \subset UCFS$. Let $h_n \to h$ for functions $h$ and $h_n$ defined on the finite set $X = \{t_1, \cdots, t_k\}$, and let $\varepsilon > 0$ be arbitrary. Then, there exist $n_1, \cdots, n_k \in \mathbb{N}$ such that for every $n \geq n_i$, $|h_n(t_i) - h(t_i)| < \varepsilon$ for $i = \overline{1, k}$. If we choose $N = \max_{1\leq i\leq k} n_i$, then for every $n \geq N$, we have $\sup_{t\in X} |h_n(t) - h(t)| < \varepsilon$. Thus, $UCFS \subset PCFS$ is satisfied. □

If $X$ is not finite, can Proposition 3.7 still hold true? Can the requirement that the set $X$ be finite be removed? The answer to these questions is provided in the remark below.

*Remark* 3.8. If $X$ is infinite, then it has a countable subset $S = \{t_1, t_2, \cdots\}$. Without loss of generality, we can choose $(t_n)$ to be a monotone sequence. This leads to two cases:

(i) $\lim |t_n| = \infty$
(ii) $\lim |t_n| = a < \infty$

In (i)

$$h_n(t) = \begin{cases} \arctan \dfrac{|t|}{n}, & t \in S \\ 0, & t \in X \setminus S \end{cases}$$

In (ii)

$$h_n(t) = \begin{cases} \left(\dfrac{|t|}{a}\right)^n, & t \in S \\ 0, & t \in X \setminus S \end{cases}$$

is pointwise convergent but not uniformly convergent. Thus, the condition that $X$ must be finite cannot be removed for the equality $UBFS = PBFS$.

3.3. **Relationships Between Types of Boundedness and Convergence on Certain Sets.** Let a sequence $(h_n) \in FS$ be given. Here, given $S$ as any finite set, some examples of the relationships between types of boundedness and convergence according to the set $X$ can be provided as follows: For each $n \in \mathbb{N}$,

(1) Let $X \in \{[0,1], (0,1), S, \mathbb{N}, \mathbb{R}\}$ and $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = n|t| + n$. Then the sequence of functions $(h_n)$ is not in $PBFS$.
(2) Let $X \in \{[0,1], (0,1), \mathbb{R}\}$ and $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = ((-1)^n nt)/(1+nt^2)$. The sequence of functions $(h_n)$ is in $PBFS$, but it is not in $PCFS$.
(3) Let $h_n : \mathbb{N} \to \mathbb{R}$ be defined as $h_n(t) = (-1)^n t$. The sequence of functions $(h_n)$ is in $PBFS$, but it is not in both $PCFS$ and $UBFS$.
(4) For $X \in \{[0,1], (0,1), S, \mathbb{N}, \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = (-1)^n \sin t$. The sequence of functions $(h_n)$ is in $UBFS$, but it is not in $PCFS$.
(5) For $X \in \{[0,1], (0,1)\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = t^n$ for each $n \in \mathbb{N}$. The sequence of functions $(h_n)$ is in both $UBFS$ and $PCFS$, but it is not in $UCFS$.
(6) For $X \in \{\mathbb{N}, \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = \arctan(t/n)$. The sequence of functions $(h_n)$ is in both $UBFS$ and $PCFS$, but it is not in $UCFS$.
(7) For $X \in \{[0,1], (0,1), S, \mathbb{N}, \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = \sin t/n$. Then the sequence of functions $(h_n)$ is in both $UBFS$ and $UCFS$.
(8) Let $h_n : [0,1] \to \mathbb{R}$ be defined as

$$h_n(t) = \begin{cases} \dfrac{n-t}{nt}, & t \in (0,1] \\ 0, & t = 0. \end{cases}$$

The sequence of functions $(h_n)$ is in $UCFS$, but it is not in $UBFS$.
(9) Let $h_n : (0,1) \to \mathbb{R}$ be defined as $h_n(t) = (n-t)/(nt)$. The sequence of functions $(h_n)$ is in $UCFS$, but it is not in $UBFS$.
(10) For $X \in \{\mathbb{N}, \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = t + 1/n$. The sequence of functions $(h_n)$ is in $UCFS$, but it is not in $UBFS$.
(11) For $X \in \{[0,1], (0,1), \mathbb{R}\}$, let $h_n : X \to \mathbb{R}$ be defined as $h_n(t) = (nt)/(1+nt^2)$. The sequence of functions $(h_n)$ is in $PCFS$, but it is not in both $UBFS$ and $UCFS$.
(12) Let $h_n : \mathbb{N} \to \mathbb{R}$ be defined as $h_n(t) = t^2/n + t$. The sequence of functions $(h_n)$ is in $PCFS$, but it is not in both $UBFS$ and $UCFS$.

| X | $[0,1]$ | $(0,1)$ | $S$ | $\mathbb{N}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|
| | (1) | (1) | (1) | (1) | (1) |
| | (2) | (2) | (Prop. 2.8) $\emptyset$ | (3) | (2) |
| | (4) | (4) | (4) | (4) | (4) |
| | (5) | (5) | (Prop. 3.7) $\emptyset$ | (6) | (6) |
| | (7) | (7) | (7) | (7) | (7) |
| | (8) | (9) | (Prop. 2.8) $\emptyset$ | (10) | (10) |
| | (11) | (11) | (Prop. 3.7) $\emptyset$ | (12) | (11) |

TABLE 1. Relations between $FS$-$PBFS$-$UBFS$-$PCFS$-$UCFS$.

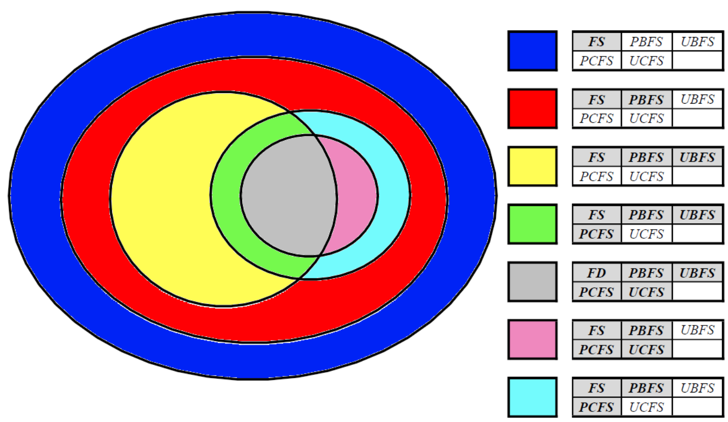


FIGURE 1. Relations between $FS$-$PBFS$-$UBFS$-$PCFS$-$UCFS$ on $[0,1],(0,1),\mathbb{N}$ and $\mathbb{R}$.

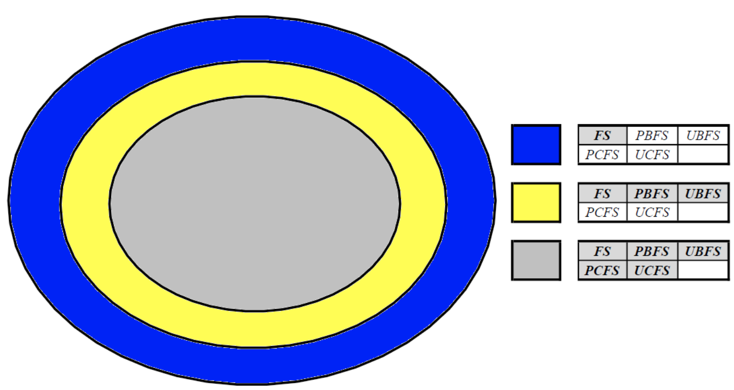


FIGURE 2. Relations Between $FS$-$PBFS$-$UBFS$-$PCFS$-$UCFS$ on Finite Set $S$.

3.4. $\alpha-$**convergence.**

**Definition 3.9.** ([4]) Let $(h_n) \in FS$ and $h \in F$ be given. If for every $t \in X$ and for every sequence $(t_n)$ in $X$ such that $t_n \to t$, we have $h_n(t_n) \to h(t)$, then the sequence $(h_n)$ is said to be $\alpha$-convergent to the function $h$ on the set $X$, and it is denoted by $h_n \overset{X}{\to}_\alpha h$.

If the convergence set is specified, for simplicity, $h_n \overset{X}{\to}_\alpha h$ is replaced with $h_n \to_\alpha h$. The family of $\alpha$-convergent function sequences on set $X$ is denoted by $\alpha CFS\,(X)$. If the domain set is specified, for simplicity, $\alpha CFS\,(X)$ is replaced with $\alpha CFS$.

$$\alpha CFS = \{(h_n) \in FS : \forall t \in X, \forall (t_n)(t_n \to t) \implies h_n(t_n) \to h(t)\}.$$

In [2], Athanassiadou et al. (2015) have proven the proposition equivalent to the definition of $\alpha$-convergence.

**Proposition 3.10.** ([2]) A necessary and sufficient condition for $h_n \overset{X}{\to}_\alpha h$ is that for every given $\varepsilon > 0$ and every $t_0 \in X$, there exist $\delta(\varepsilon, t_0) > 0$ and $n_0(\varepsilon, t_0) \in \mathbb{N}$ such that for every $t \in X$ satisfying $|t - t_0| < \delta$, we have $|h_n(t) - h(t_0)| < \varepsilon$.

From here, the following question can be asked: Is there a relationship between $\alpha$-convergence and other types of convergence, as there is between uniform and pointwise convergence? The answer to this question is provided in the following examples.

**Example 3.11.** For every $n \in \mathbb{N}$, let $h_n : [0,1] \to \mathbb{R}$ be defined as $h_n(t) = t^n$. The sequence $(h_n)$ is pointwise convergent to the function

$$h(t) = \begin{cases} 1, & t = 1 \\ 0, & 0 \le t < 1 \end{cases}$$

while it is not $\alpha$-convergent.

**Example 3.12.** Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous function, and consider an arbitrary sequence $(t_n)$ converging to $t$ such that $h_n(t_n) \not\to h(t)$. If we choose $h_n$ such that $h_n \equiv h$ for every $n \in \mathbb{N}$, then $h_n \not\to_\alpha h$, although $h_n \rightrightarrows h$.

**Example 3.13.** For every $n \in \mathbb{N}$, let $h_n : (0,1] \to \mathbb{R}$ be defined as follow,

$$h_n(t) = \begin{cases} 1 - nt, & 0 < t \le 1/n \\ 0, & 1/n < t \le 1. \end{cases}$$

Then, $h_n \not\rightrightarrows 0$, but $h_n \to_\alpha 0$.

One might wonder: Under what conditions can connections between different types of convergence be made? Below are some of the situations where such connections occur.

**Proposition 3.14.** ([7]) $X \subset \mathbb{R}$ and let $h, h_n : X \to \mathbb{R}$ for every $n \in \mathbb{N}$. In this case,

(a) If $h_n \to_\alpha h$, then $h_n \to h$.
(b) If $h_n \rightrightarrows h$ and $h$ is continuous, then $h_n \to_\alpha h$.
(c) If $X$ is a compact set and $h_n \to_\alpha h$, then $h_n \rightrightarrows h$.
(d) $X$ is compact if and only if $h_n \to_\alpha h$ implies $h_n \rightrightarrows h$.

Which properties need to be imposed on the set $X$ for $\alpha CFS = PCFS = UCFS$? The answer to this question is provided in the proposition below.

**Proposition 3.15.** If $X$ is a finite set then $\alpha CFS = PCFS = UCFS$.

*Proof.* Let $X = \{t_1, t_2, \cdots, t_k\}$ be a finite set. According to Proposition 3.7, $UCFS = PCFS$ on $X$. It suffices to show that $PCFS \subset \alpha CFS$. Let $(t_n) \subset X$ be an arbitrary sequence converging to $t_i$ ($t_i \in X$) with $\lim_{n\to\infty} h_n = h$. Then, there exists an $n_i \in \mathbb{N}$ such that for all $n \geq n_i$, $|h_n(t_i) - h(t_i)| < \varepsilon$. Since $X$ is a finite set, for any convergent sequence defined on this set, after a certain index, every term will be the same. Therefore, there exists $\overline{n}_i \in \mathbb{N}$ such that for all $n \geq \overline{n}_i$, $|t_n - t_i| = 0$. Choosing $\delta$ as any positive number and $n_0 = \max\{n_i, \overline{n}_i\}$, for all $n \geq n_0$,

$$|h_n(t_n) - h(t_i)| = |h_n(t_n) - h_n(t_i)| + |h_n(t_i) - h(t_i)| < 0 + \varepsilon = \varepsilon$$

Therefore, $(h_n) \in \alpha CFS$, and $PCFS \subset \alpha CFS$ is established. □

Proposition 3.15 can also be proven using Proposition 3.14 (b) and (d) aimed at showing $\alpha CFS = UCFS$.

**Proposition 3.16.** Let $X$ be a countable set without accumulation points, and let $(h_n) \in FS$ and $h \in F$ be given. Then, $h_n \to_\alpha h \iff h_n \to h$.

*Proof.* The necessary condition is clear from Proposition 3.14 (a). On the other hand, suppose $h_n \to h$ for any arbitrary $t \in X$ and $\varepsilon > 0$. Then, there exists an $n_t \in \mathbb{N}$ such that for every $n \geq n_t$, $|h_n(t) - h(t)| < \varepsilon$. Now, choosing $\delta < \inf\{|t_i - t_j| : t_i, t_j \in X, i \neq j\}$ and $\overline{n}_t = n_t$, for every $y \in X$ such that $|y - t| < \delta$ and every $n \geq \overline{n}_t$, we have $|h_n(y) - h(t)| = |h_n(t) - h(t)| < \varepsilon$. □

**Proposition 3.17.** If $K \subset \mathbb{R}$ is a compact set and $h_n \xrightarrow{K}_\alpha h$, then $(h_n) \in UBFS$.

*Proof.* Let's assume that $h_n \xrightarrow{K} \alpha h$, and for each $i \in I$, $t_i \in K$ is given. For $\varepsilon = 1$, there exist $\delta_i > 0$ and $n_i \in \mathbb{N}$ such that for all $t \in K$ satisfying $|t - t_i| < \delta_i$ and all $n \geq n_i$, $|h_n(t_n) - h(t_i)| < 1$. Since $h_n \xrightarrow{K}_\alpha h$, $h$ is continuous and attains its maximum value $||h||_\infty = \max_{t\in K} |h(t)|$ on the compact set $K$. Hence,

$$|h_n(t)| - |h(t_i)| \leq |h_n(t) - h(t_i)| < 1 \implies |h_n(t)| \leq 1 + |h(t_i)| \leq 1 + ||h||_\infty = M_0.$$

Here, $B = \bigcup_{t_i\in K} B(t_i, \delta_i)$ is an open cover of the compact set $K$, which has a finite subcover. Without loss of generality, let's consider this subcover as $\bigcup_{i=1}^k B(t_i, \delta_i)$. By choosing $M_i = \max\{|h_1(t_i)|, |h_2(t_i)|, \cdots, |h_{n_0-1}(t_i)|, M_0\}$ for each $i$, and then selecting $M = \max_{1\leq i\leq k} M_i$, we ensure that for all $t \in K$ and all $n \in \mathbb{N}$, $|h_n(t)| \leq M$. Therefore, the sequence $(h_n)$ is uniformly bounded on $X$. □

3.5. **Relationships Between $\alpha-$ Convergence, Boundedness, and Other Types of Convergence On Some Sets.** Let a sequence $(h_n) \in FS$ be given. Here, some examples of the relationships between types of boundedness and convergence according to the structure of the set $X$, with $S$ denoting any finite set, can be given as follows: For each $n \in \mathbb{N}$

(13) Let $h_n : (0,1) \to \mathbb{R}$, $h_n(t) = nt^n(1-t)$. The sequence of functions $(h_n)$ belongs to both PCFS and UBFS. However, the sequence $(h_n)$ does not belong to either $\alpha CFS$ or $UCFS$.

(14) Let $h_n : \mathbb{R} \to \mathbb{R}$, $h_n(t) = (nt)/(1+n^2t^2)$. The sequence of functions $(h_n)$ belongs to both $PCFS$ and $UBFS$. However, the sequence $(h_n)$ does not belong to either $UCFS$ or $\alpha CFS$.

(15) For $X \in \{[0,1],(0,1)\}$, let $h_n : X \to \mathbb{R}$,

$$h_n(t) = \begin{cases} 1, & t \leq \frac{1}{2} \\ t/n, & t > \frac{1}{2} \end{cases}$$

The sequence of functions $(h_n)$ belongs to both $UCFS$ and $UBFS$. However, the sequence $(h_n)$ does not belong to $\alpha CFS$.

(16) Let $h_n : \mathbb{R} \to \mathbb{R}$,

$$h_n(t) = \begin{cases} \dfrac{n+\sin t}{n}, & t > 0 \\ \\ \dfrac{-n\sin t}{n}, & t \leq 0 \end{cases}$$

The sequence of functions $(h_n)$ belongs to both $UCFS$ and $UBFS$. However, the sequence $(h_n)$ does not belong to $\alpha CFS$.

(17) Let $h_n : (0,1) \to \mathbb{R}$,

$$h_n(t) = \begin{cases} t\left(t+\frac{3}{4}\right)^n, & 0 < t < \frac{1}{4} \\ \left(\frac{1}{2}-t\right)\left(\frac{5}{4}-t\right)^n, & \frac{1}{4} \leq t < \frac{1}{2} \\ 0, & \frac{1}{2} \leq t \leq 1-\frac{1}{2n} \\ n+2n^2(t-1), & 1- < \frac{1}{2n} \leq t < 1 \end{cases}$$

The sequence of functions $(h_n)$ belongs to PCFS, but it does not belong to $UBFS$, $UCFS$, or $\alpha CFS$.

(18) For $X \in \{[0,1],(0,1),\mathbb{R}\}$ and let $h_n : X \to \mathbb{R}$.

$$h_n(t) = \begin{cases} \dfrac{n-t}{nt}, & 0 < t < \frac{1}{2} \\ 1, & \text{otherwise} \end{cases}$$

The sequence of functions $(h_n)$ belongs to $UBFS$, $UCFS$, and $\alpha CFS$.

(19) Let $h_n : (0,1) \to \mathbb{R}$,

$$h_n(t) = \begin{cases} 1-nt, & 0 < t \leq \frac{1}{n} \\ 0, & \frac{1}{n} < t < 1 \end{cases}$$

The sequence of functions $(h_n)$ is in $UBFS$ and $\alpha CFS$, but it is not in $UCFS$.

(20) Let $h_n : \mathbb{N} \to \mathbb{R}$,

$$h_n(t) = \begin{cases} 0, & t > n \\ 1, & t \leq n \end{cases}$$

The sequence of functions $(h_n)$ is in $UBFS$ and $\alpha CFS$, but it is not in $UCFS$.

(21) For $X \in \{[0,1],(0,1),S,N,\mathbb{N},\mathbb{R}\}$, let $h_n : X \to \mathbb{R}$, $h_n(t) = 1/(n|t|+n)$. The sequence of functions $(h_n)$ is in $UBFS$, $UCFS$, and $\alpha CFS$.

(22) Let $h_n : (0,1) \to \mathbb{R}$, $h_n(t) = n/(nt+1)$. The sequence of functions $(h_n)$ is in $\alpha CFS$, but it is not in $UBFS$ or $UCFS$.

| | $[0,1]$ | $(0,1)$ | $S$ | $\mathbb{N}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|
| ■ | (1) | (1) | (1) | (1) | (1) |
| ■ | (2) | (2) | (Prop. 2.8) $\emptyset$ | (3) | (2) |
| ■ | (4) | (4) | (4) | (4) | (4) |
| ■ | (5) | (13) | (Prop. 3.15) $\emptyset$ | (Prop. 3.16) $\emptyset$ | (14) |
| ■ | (15) | (15) | (Prop. 3.15) $\emptyset$ | (Prop. 3.16) $\emptyset$ | (16) |
| ■ | (11) | (17) | (Prop.3.15) $\emptyset$ | (Prop. 3.16) $\emptyset$ | (11) |
| ■ | (18) | (18) | (Prop. 3.15) $\emptyset$ | (Prop. 3.16) $\emptyset$ | (18) |
| ■ | (Prop. 3.14 (c)) $\emptyset$ | (19) | (Prop. 3.15) $\emptyset$ | (20) | (6) |
| □ | (21) | (21) | (21) | (21) | (21) |
| ■ | (Prop. 3.17) $\emptyset$ | (9) | (Prop. 2.8) $\emptyset$ | (10) | (10) |
| ■ | (Prop. 3.17) $\emptyset$ | (22) | (Prop. 3.15) $\emptyset$ | (12) | (12) |

TABLE 2. Relations Between $FS$-$PBFS$-$UBFS$-$PCFS$-$UCFS$-$\alpha CFS$.

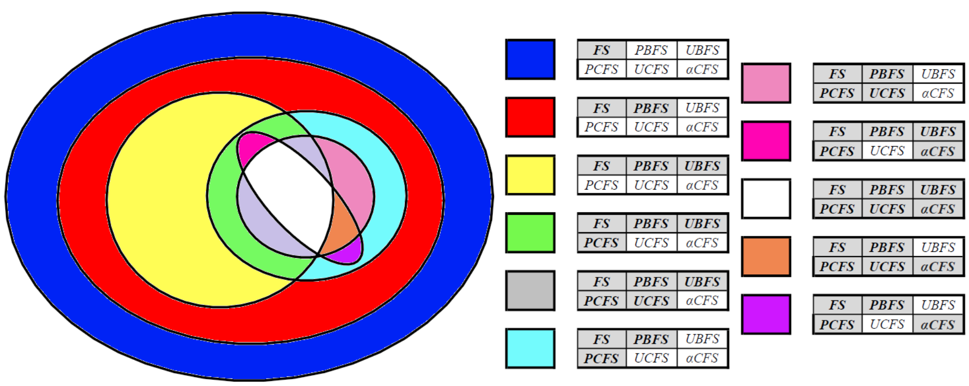


FIGURE 3. Relations Between $FS$-$PBFS$-$UBFS$-$PCFS$-$UCFS$-$\alpha CFS$ on $(0,1)$ and $\mathbb{R}$.

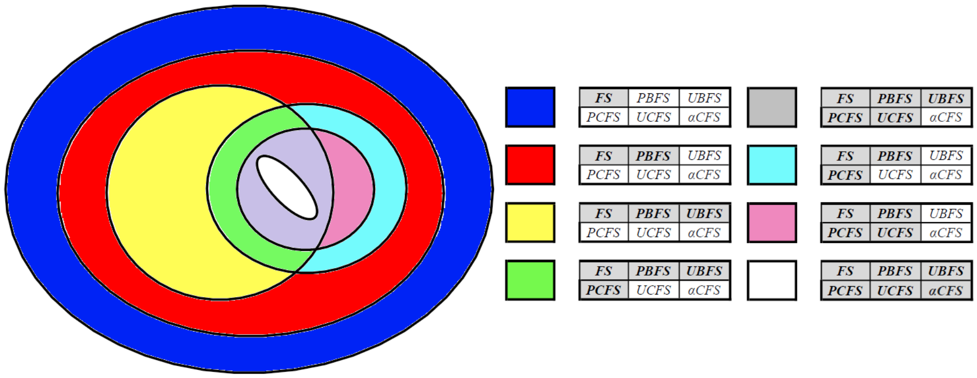


FIGURE 4. Relations Between $FS$-$PBFS$-$UBFS$-$PCFS$-$UCFS$-$\alpha CFS$ on $[0,1]$.

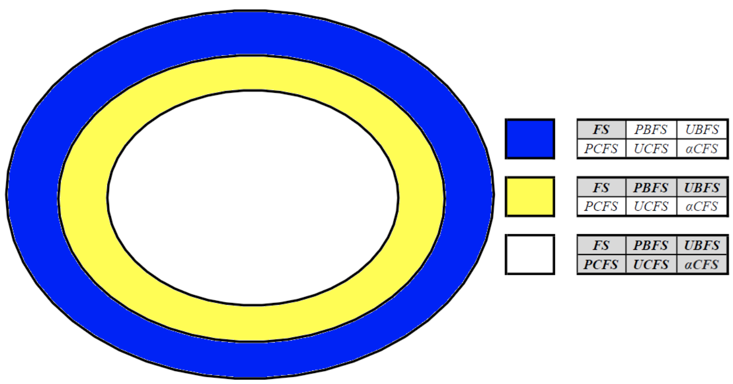


FIGURE 5. Relations Between $FS$-$PBFS$-$UBFS$-$PCFS$-$UCFS$-$\alpha CFS$ on Finite Set.

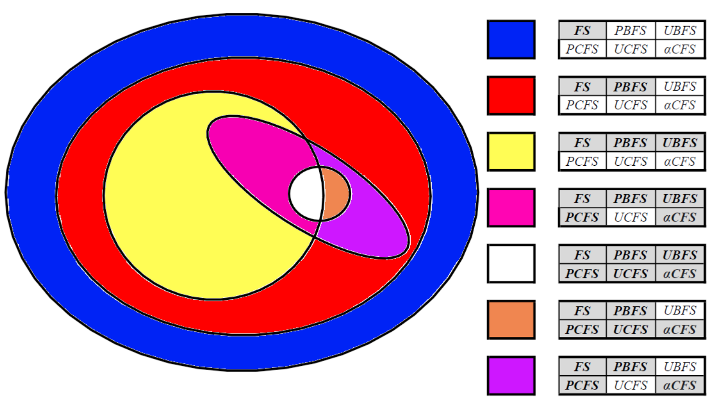


FIGURE 6. Relations Between $FS$-$PBFS$-$UBFS$-$PCFS$-$UCFS$-$\alpha CFS$ on $\mathbb{N}$.

## 4. Conclusion

In conclusion we systematically studied various convergence types of function sequences, including pointwise, uniform, and $\alpha-$convergence, along with their relationships. By examining the conditions under which these convergence types are equivalent or not, we provide examples into the structural properties of function sequences. Notably, our results highlight the critical role of the underlying set's structure (e.g., finite, countable, or compact) in determining these relationships.

The theoretical results presented here contribute to an extended understanding of convergence behaviors for further research in function analysis and its applications. Future work could extend these results to explore additional convergence concepts or their implications in applied sciences.


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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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# TEMPERATURE INDICES OF WELL KNOWN DENDRIMER NETWORKS

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Abstract. Chemical graph theory is a branch of graph theory. In this field, molecules are modeled using graph theory and a mathematical approach is obtained. Thus, predictions can be made about the physical, chemical and bioactivity properties of molecules. In this study, temperature indices depending on the vertex degree are considered. Temperature indices of Polyamidoamine (PAMAM) dendrimers, Porphyrin core dendrimers are obtained and the results are compared numerically using the MATLAB program.

## 1. Introduction

Graph theory is used to solve problems in daily life by expressing objects and the relationships between them with vertices and edges, respectively [1]. Due to the changing living conditions and increasing diseases, it is necessary to discover new chemicals and drugs quickly and inexpensively. With the help of graph indices in chemical graph theory, it has been possible to predict the physical, chemical and bioactivity properties of molecules [2].

Graph indices are the numerical values of chemical networks. Chemical networks are obtained by representing chemical structures with vertices and edges. The atoms of a chemical structure are represented by vertices and the relationships between the chemical structures are represented by edges. The numerical results obtained by using graph indices of chemical networks are used in programs such as SPSS to obtain equations. These equations are used to predict the properties of new chemical structures [2].

Throughout this article, let $\wedge$ be a graph with $V(\wedge), E(\wedge)$. The element numbers of the vertex (edge) set are defined by $|V(\wedge)|(|E(\wedge)|)$. The degree of a vertex $\upsilon$ is defined by $d_\upsilon$ [1].

Fajtlowicz [3] defined the temperature of a vertex $\upsilon$ as

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$$T_v = \frac{d_v}{|V(\wedge)| - d_v}. \tag{1.1}$$

With this motivation, Kulli [4] defined the general first and second temperature indices as follows:

$$GT_1^k(\wedge) = \sum_{\tau v \in E(\wedge)} (T_\tau + T_v)^k \tag{1.2}$$

$$GT_2^k(\wedge) = \sum_{\tau v \in E(\wedge)} (T_\tau T_v)^k . \tag{1.3}$$

Kansal et al. studied temperature indices for predict the physical properties of COVID-19 drugs [5]. Nabeel et al. found temperature indices of some nanotubes [6]. Khan et al. investigated various graph indices based temperate of silicates moleculas [7]. Kulli obtained results about temperatures indices of important nanostructures [8].

Dendrimers have a symmetric core and radially symmetric molecules. Therefore, dendrimer networks are similar to tree graphs. Dendrimers are often used in drug discovery due to their favorable chemical properties. Therefore, dendrimers are widely studied [9]. Zhao et al. studied irregularity indices of some dendrimers [10]. Hasani and Gods investigated M-polynomials of porphrin dendrimers [11]. Sarkar et al. studied generalized Zagreb indices of some reguler dendrimers [12]. Khalaf et al. calculated the degree based indices of four layered prophyrin core dendrimers [13].

In this study, the polyamidoamine and porphyrin core dendrimers are studied. The temperature indices of these dendrimer networks are obtained and the results are compared.

## 2. Preliminaries

In this section,informations about Polyamidoamine dendrimers and Porphyrin cored dendrimers are given.

**PAMAM DENDRIMERS**

Polyamidoamine (PAMAM) dendrimers have an ethylenediamine core, a repetitive branching amidoamine internal structure and a primary amine terminal surface [14]. Figure 1 shows the structure of the PAMAM [14]. Let $PAMAM[r]$ be molecular graph of the polyamidoamine (PAMAM) dendrimers. The PAMAM network have $|V(PAMAM[r])| = 12 \times 2^{r+2} - 23$ and $|E(PAMAM[r])| = 12 \times 2^{r+2} - 24$.

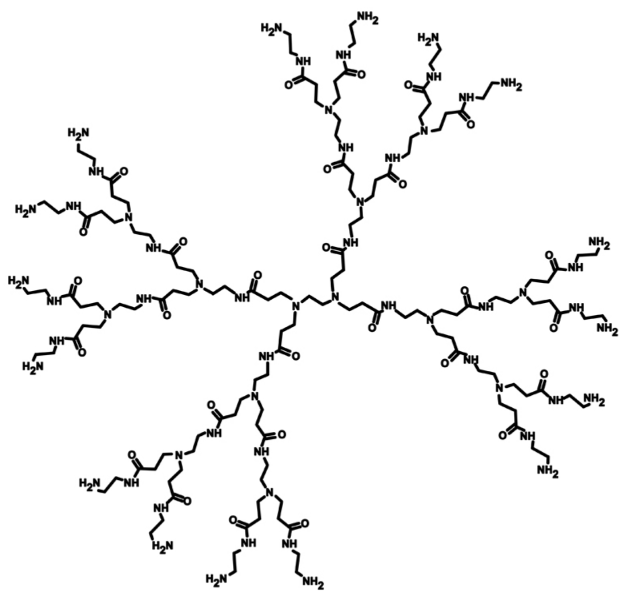

Figure 1. Chemical Structure of PAMAM.

## PORPHYRIN CORED DENDRIMERS

Porphyrin cored dendrimers have a central core and at least 2 branches[15]. In this study, 3 porphyrin cored dendrimers will be examined.

### Porphyrin Cored Dendrimers-2

Porphyrin cored dendrimers-2 have one central core and 4 branches. Figure 2 shows the structure of Porphyrin cored dendrimers-2 [15]. Let $PC2[n]$ be molecular graphs of porphyrin cored dendrimers-2. Then, $|V(PC2[n])| = 8 \times 2^n + 21$ and $|E(PC2[n])| = 8 \times 2^n + 28$ by calculated.

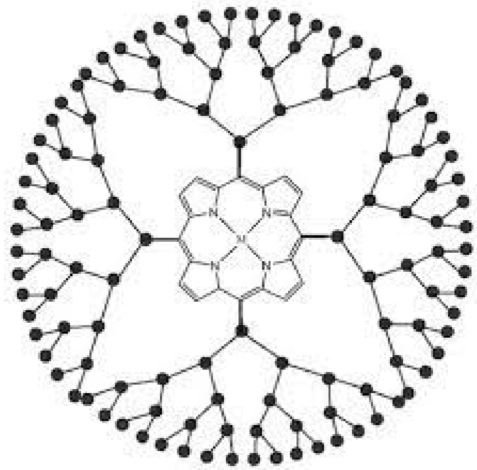

FIGURE 2. Structure of Porphyrin cored dendrimers-2 .

**Porphyrin Cored Dendrimers-3**

Porphyrin cored dendrimers-3 have one central core and 8 branches. Figure 3 shows the structure of Porphyrin cored dendrimers-3 [15]. Let $PC3[n]$ be molecular graphs of porphyrin cored dendrimers-3. Then, $|V(PC3[n])| = 16 \times 2^n + 17$ and $|E(PC3[n])| = 16 \times 2^n + 24$ by calculated.

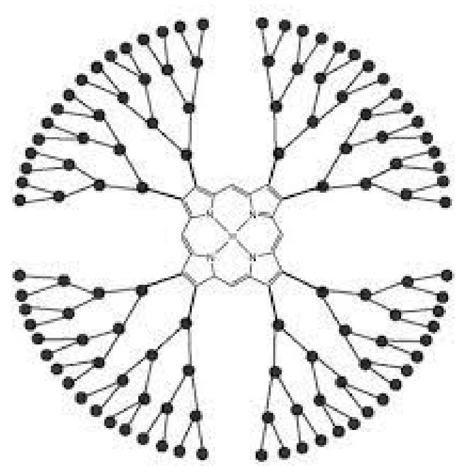

FIGURE 3. Structure of Porphyrin cored dendrimers-3 .

**Porphyrin Cored Dendrimers-4**

Porphyrin cored dendrimers-4 have a central core and 12 branches. Figure 4 shows the structure of Porphyrin cored dendrimers-4 [15]. Let $PC4[n]$ be molecular graphs of porphyrin cored dendrimers-4. Then,$|V(PC4[n])| = 24 \times 2^n + 13$ and $|E(PC3[n])| = 24 \times 2^n + 20$ by calculated.

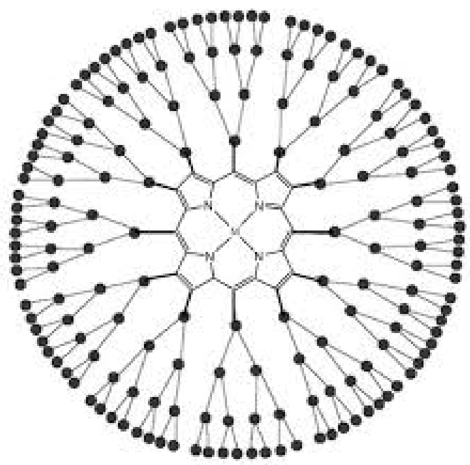

FIGURE 4. Structure of Porphyrin cored dendrimers-4 .

## 3. MAIN RESULTS

In this section, the values of the general first temperature and general second temperature indices of the dendrimer networks mentioned above are calculated. Numerical comparisons of the general first and second temperature indices of each dendrimer network introduced above are made using the MATLAB program.

**Theorem 3.1.** *i. Let the general first temperature index of $PAMAM[r]$ be $GT_1^k(PAMAM[r])$. Then*

$$\begin{aligned} GT_1^k(PAMAM[r]) &= (3 \times 2^r)\left(\frac{1}{12 \times 2^{r+2} - 24} + \frac{2}{12 \times 2^{r+2} - 25}\right)^k \\ &+ (6 \times 2^r - 3)\left(\frac{1}{12 \times 2^{r+2} - 24} + \frac{3}{12 \times 2^{r+2} - 26}\right)^k \\ &+ (18 \times 2^r - 9)\left(\frac{2}{12 \times 2^{r+2} - 25} + \frac{2}{12 \times 2^{r+2} - 25}\right)^k \\ &+ (21 \times 2^r - 12)\left(\frac{2}{12 \times 2^{r+2} - 25} + \frac{3}{12 \times 2^{r+2} - 26}\right)^k . \end{aligned}$$

*ii. Let general second temperature index of $PAMAM[r]$ be $GT_2^k(PAMAM[r])$. Then,*

$$\begin{aligned} GT_2^k(PAMAM[r]) &= (3\times 2^r)\left(\frac{1}{12\times 2^{r+2}-24}\times\frac{2}{12\times 2^{r+2}-25}\right)^k \\ &+ (6\times 2^r-3)\left(\frac{1}{12\times 2^{r+2}-24}\times\frac{3}{12\times 2^{r+2}-26}\right)^k \\ &+ (18\times 2^r-9)\left(\frac{2}{12\times 2^{r+2}-25}\times\frac{2}{12\times 2^{r+2}-25}\right)^k \\ &+ (21\times 2^r-12)\left(\frac{2}{12\times 2^{r+2}-25}\times\frac{3}{12\times 2^{r+2}-26}\right)^k. \end{aligned}$$

*Proof.* The edge partitions of PAMAM dendrimer networks are shown Table 1.

TABLE 1. The edge partitions of PAMAM networks.

| $(d_\tau, d_\upsilon)$ **for** $E(PAMAM[r])$ | $(T_\tau + T_\upsilon)$ **for** $E(PAMAM[r])$ | **The number of edge** |
|---|---|---|
| (1,2) | $(\frac{1}{12\times 2^{r+2}-24},\frac{1}{12\times 2^{r+2}-25})$ | $3\times 2^r$ |
| (1,3) | $(\frac{1}{12\times 2^{r+2}-24},\frac{1}{12\times 2^{r+2}-26})$ | $6\times 2^r-3$ |
| (2,2) | $(\frac{2}{12\times 2^{r+2}-25},\frac{2}{12\times 2^{r+2}-25})$ | $18\times 2^r-9$ |
| (2,3) | $(\frac{2}{12\times 2^{r+2}-25},\frac{3}{12\times 2^{r+2}-26})$ | $21\times 2^r-12$ |

Let $E_{d_\tau,d_\upsilon}$ be $(d_\tau, d_\upsilon)$ for $E(PAMAM[r])$. From Table 1, it is written:

$$TI(PAMAM[r]) = \sum_{\tau\upsilon\in E_{1,2}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{1,3}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{2,2}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{2,3}} W_{\tau\upsilon} \tag{3.1}$$

i. If $W_{\tau\upsilon} = (T_\tau + T_\upsilon)^k$ in Eq.(3.1), then

$$\begin{aligned} GT_1^k(PAMAM[r]) &= (3\times 2^r)\left(\frac{1}{12\times 2^{r+2}-24}+\frac{2}{12\times 2^{r+2}-25}\right)^k \\ &+ (6\times 2^r-3)\left(\frac{1}{12\times 2^{r+2}-24}+\frac{3}{12\times 2^{r+2}-26}\right)^k \\ &+ (18\times 2^r-9)\left(\frac{2}{12\times 2^{r+2}-25}+\frac{2}{12\times 2^{r+2}-25}\right)^k \\ &+ (21\times 2^r-12)\left(\frac{2}{12\times 2^{r+2}-25}+\frac{3}{12\times 2^{r+2}-26}\right)^k. \end{aligned}$$

ii. If $W_{\tau\upsilon} = (T_\tau \times T_\upsilon)^k$ in Eq.(3.1), then

$$\begin{aligned} GT_2^k(PAMAM[r]) &= (3\times 2^r)\left(\frac{1}{12\times 2^{r+2}-24}\times\frac{2}{12\times 2^{r+2}-25}\right)^k \\ &+ (6\times 2^r-3)\left(\frac{1}{12\times 2^{r+2}-24}\times\frac{3}{12\times 2^{r+2}-26}\right)^k \\ &+ (18\times 2^r-9)\left(\frac{2}{12\times 2^{r+2}-25}\times\frac{2}{12\times 2^{r+2}-25}\right)^k \\ &+ (21\times 2^r-12)\left(\frac{2}{12\times 2^{r+2}-25}\times\frac{3}{12\times 2^{r+2}-26}\right)^k. \end{aligned}$$

□

Figure 5 shows plots a) $GT_1^k$ and b) $GT_2^k$ of $PAMAM[r]$. These plots show that the general first temperature index of the $PAMAM[r]$ network grows faster than the general second temperature index.

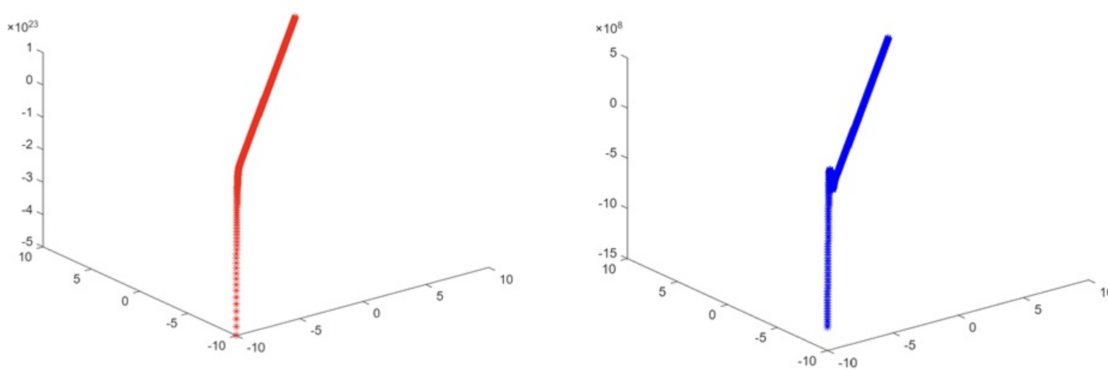


FIGURE 5. The plots of a) $GT_1^k$ and b) $GT_2^k$ of $PAMAM[r]$.

**Theorem 3.2.** *i.* $GT_1^k(PC2[n])$ *is equal to following equation:*

$$\begin{aligned} GT_1^k(PC2[n]) &= (4\times 2^n)\left(\frac{1}{8\times 2^n+20}+\frac{3}{8\times 2^n+18}\right)^k \\ &+ 4\left(\frac{2}{8\times 2^n+19}+\frac{2}{8\times 2^n+19}\right)^k \\ &+ (4\times 2^n+12)\left(\frac{2}{8\times 2^n+19}+\frac{3}{8\times 2^n+18}\right)^k \\ &+ 4\left(\frac{3}{8\times 2^n+18}+\frac{4}{8\times 2^n+17}\right)^k. \end{aligned}$$

*ii.* $GT_2^k(PC2[n])$ *is equal to*

$$\begin{aligned} GT_2^k(PC2[n]) &= (4\times 2^n)\left(\frac{1}{8\times 2^n+20}\times\frac{3}{8\times 2^n+18}\right)^k \\ &+ 4\left(\frac{2}{8\times 2^n+19}\times\frac{2}{8\times 2^n+19}\right)^k \\ &+ (4\times 2^n+12)\left(\frac{2}{8\times 2^n+19}\times\frac{3}{8\times 2^n+18}\right)^k \\ &+ 4\left(\frac{3}{8\times 2^n+18}\times\frac{4}{8\times 2^n+17}\right)^k. \end{aligned}$$

*Proof.* The edge partitons of $PC2[n]$ are shown in the following table (Table 2).

TABLE 2. The edge partitons of $PC2[n]$.

| $(d_\tau, d_\upsilon)$ **for** $E(PC2[n])$ | $(T_\tau + T_\upsilon)$ **for** $E(PC2[n])$ | **The number of edge** |
|---|---|---|
| (1,3) | $(\frac{1}{8\times2^n+20}, \frac{3}{8\times2^n+18})$ | $4\times 2^n$ |
| (2,2) | $(\frac{2}{8\times2^n+19}, \frac{2}{8\times2^n+19})$ | 4 |
| (2,3) | $(\frac{2}{8\times2^n+19}, \frac{2}{8\times2^n+18})$ | 8 |
| (3,3) | $(\frac{3}{8\times2^n+18}, \frac{3}{8\times2^n+18})$ | $4\times 2^n+12$ |
| (3,4) | $(\frac{3}{8\times2^n+18}, \frac{4}{8\times2^n+17})$ | 4 |

Using Table 2, it can be written:

(3.2)

$$TI(PC2[n]) = \sum_{\tau\upsilon\in E_{1,2}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{2,2}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{2,3}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{3,3}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{3,4}} W_{\tau\upsilon}$$

i. If $W_{\tau\upsilon} = (T_\tau + T_\upsilon)^k$ in Eq.(3.2), then

$$\begin{aligned} GT_1^k(PC2[n]) &= (4\times 2^n)\left(\frac{1}{8\times 2^n+20}+\frac{3}{8\times 2^n+18}\right)^k \\ &+ 4\left(\frac{2}{8\times 2^n+19}+\frac{2}{8\times 2^n+19}\right)^k \\ &+ 8\left(\frac{2}{8\times 2^n+19}+\frac{2}{8\times 2^n+18}\right)^k \\ &+ (4\times 2^n+12)\left(\frac{3}{8\times 2^n+18}+\frac{3}{8\times 2^n+18}\right)^k \\ &+ 4\left(\frac{3}{8\times 2^n+18}+\frac{4}{8\times 2^n+17}\right). \end{aligned}$$

ii. If $W_{\tau\upsilon} = (T_\tau \times T_\upsilon)^k$ in Eq.(3.2), then

$$\begin{aligned} GT_2^k(PC2[n]) &= (4\times 2^n)\left(\frac{1}{8\times 2^n+20}\times\frac{3}{8\times 2^n+18}\right)^k \\ &+ 4\left(\frac{2}{8\times 2^n+19}\times\frac{2}{8\times 2^n+19}\right)^k \\ &+ 8\left(\frac{2}{8\times 2^n+19}\times\frac{2}{8\times 2^n+18}\right)^k \\ &+ (4\times 2^n+12)\left(\frac{3}{8\times 2^n+18}\times\frac{3}{8\times 2^n+18}\right)^k \\ &+ 4\left(\frac{3}{8\times 2^n+18}\times\frac{4}{8\times 2^n+17}\right). \end{aligned}$$

□

Figure 6 shows plots a) $GT_1^k$ and b) $GT_2^k$ of $PC2[n]$. This plot gives $GT_1^k(PC2[n])$ growing faster than $GT_2^k(PC2[n])$.

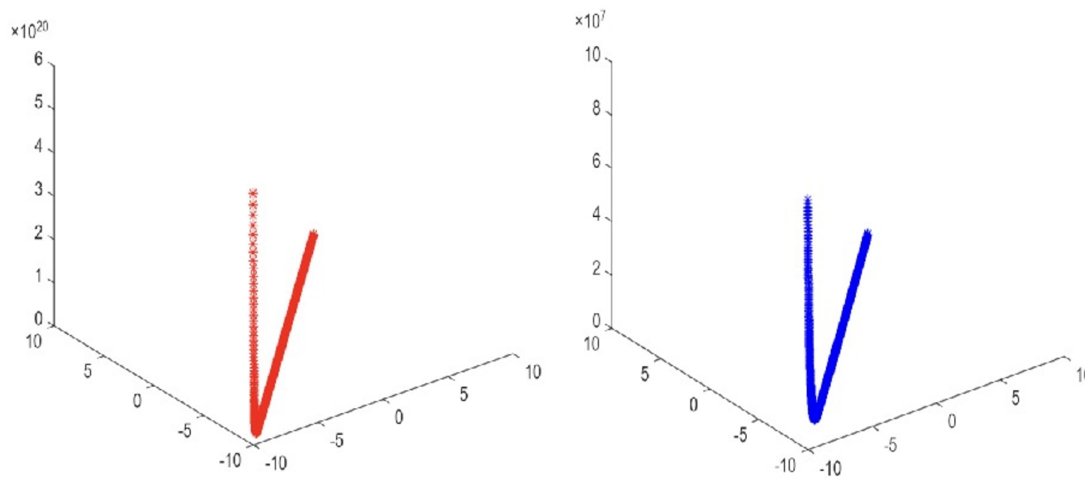

FIGURE 6. The plots of a) $GT_1^k$ and b) $GT_2^k$ of $PC2[n]$.

**Theorem 3.3.** *i.* $GT_1^k(PC3[n])$ *is*

$$\begin{aligned} GT_1^k(PC3[n]) &= 8\times 2^n\left(\frac{1}{16\times 2^n+16}+\frac{3}{16\times 2^n+14}\right)^k \\ &+ 8\left(\frac{2}{16\times 2^n+15}+\frac{3}{16\times 2^n+14}\right)^k \\ &+ (8\times 2^n+12)\left(\frac{3}{16\times 2^n+14}+\frac{3}{16\times 2^n+14}\right)^k \\ &+ 4\left(\frac{3}{16\times 2^n+14}+\frac{4}{16\times 2^n+13}\right)^k. \end{aligned}$$

*ii.* $GT_2^k(PC3[n])$ *is*

$$\begin{aligned} GT_2^k(PC3[n]) &= 8\times 2^n\left(\frac{1}{16\times 2^n+16}\times\frac{3}{16\times 2^n+14}\right)^k \\ &+ 8\left(\frac{2}{16\times 2^n+15}\times\frac{3}{16\times 2^n+14}\right)^k \\ &+ (8\times 2^n+12)\left(\frac{3}{16\times 2^n+14}\times\frac{3}{16\times 2^n+14}\right)^k \\ &+ 4\left(\frac{3}{16\times 2^n+14}\times\frac{4}{16\times 2^n+13}\right)^k. \end{aligned}$$

*Proof.* The edge partitions $PC3[n]$ are given in Table 3.

TABLE 3. The edge partitions of $PC3[n]$.

| $(d_\tau, d_\upsilon)$ **for** $E(PC3[n])$ | $(T_\tau + T_\upsilon)$ **for** $E(PC3[n])$ | **The number of edge** |
|---|---|---|
| (1,3) | $(\frac{1}{16\times2^n+16}, \frac{3}{16\times2^n+14})$ | $8\times 2^n$ |
| (2,3) | $(\frac{2}{16\times2^n+15}, \frac{3}{16\times2^n+14})$ | 8 |
| (3,3) | $(\frac{3}{16\times2^n+14}, \frac{3}{16\times2^n+14})$ | $8\times 2^n+12$ |
| (3,4) | $(\frac{3}{16\times2^n+14}, \frac{4}{16\times2^n+13})$ | 4 |

From Table 3, the following equation can be written:

$$TI(PC3[n]) = \sum_{\tau\upsilon\in E_{1,3}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{2,3}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{3,3}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{3,4}} W_{\tau\upsilon} \tag{3.3}$$

i. If $W_{\tau\upsilon} = (T_\tau + T_\upsilon)^k$ in Eq.3.3, then the proof (i) is completed from Table .
ii. If $W_{\tau\upsilon} = (T_\tau \times T_\upsilon)^k$ in Eq.(3.3), then the proof (ii) is completed with some calculated from Table 3. □

The plots $GT_1^k$ and $GT_2^k$ of $PC3[n]$ are given below (see Figure 7). it is seen that $GT_1^k$ index of $PC3[n]$ grows faster than $GT_2^k$ index of $PC3[n]$ from Figure 7.

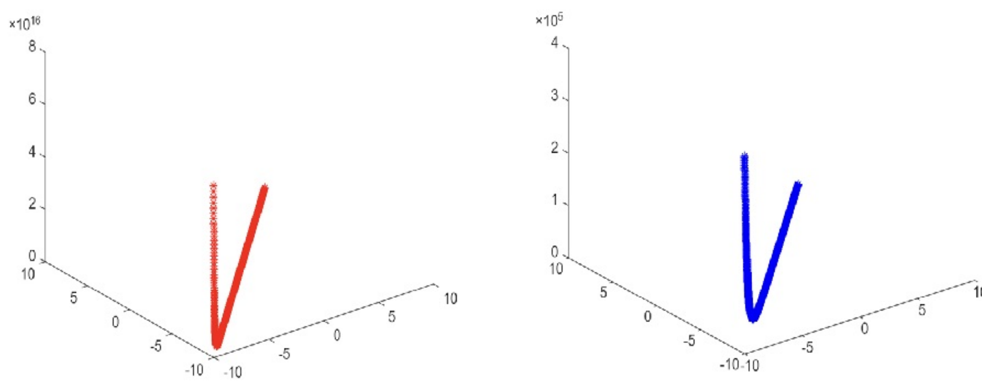


FIGURE 7. The plots of a) $GT_1^k$ and b) $GT_2^k$ of $PC3[n]$.

**Theorem 3.4.** *i. The general first temperature index of $PC4[n]$ is*

$$\begin{aligned} GT_1^k(PC4[n]) &= 12\times 2^n\left(\frac{1}{24\times 2^n+12}+\frac{3}{24\times 2^n+10}\right)^k \\ &+ 8\left(\frac{2}{24\times 2^n+11}+\frac{3}{24\times 2^n+10}\right)^k \\ &+ (8\times 2^n+12)\left(\frac{3}{24\times 2^n+10}+\frac{3}{24\times 2^n+10}\right)^k \\ &+ 4\left(\frac{3}{24\times 2^n+10}+\frac{4}{24\times 2^n+9}\right)^k. \end{aligned}$$

*ii. $GT_2^k(PC4[n])$ is*

$$\begin{aligned} GT_2^k(PC4[n]) &= 12\times 2^n\left(\frac{1}{24\times 2^n+12}\times\frac{3}{24\times 2^n+10}\right)^k \\ &+ 8\left(\frac{2}{24\times 2^n+11}\times\frac{3}{24\times 2^n+10}\right)^k \\ &+ (8\times 2^n+12)\left(\frac{3}{24\times 2^n+10}\times\frac{3}{24\times 2^n+10}\right)^k \\ &+ 4\left(\frac{3}{24\times 2^n+10}\times\frac{4}{24\times 2^n+9}\right)^k. \end{aligned}$$

*Proof.* The edge partitions $PC4[n]$ are given in Table 4.

TABLE 4. The edge partitions of $PC4[n]$.

| $(d_\tau, d_\upsilon)$ **for** $E(PC3[n])$ | $(T_\tau + T_\upsilon)$ **for** $E(PC3[n])$ | **The number of edge** |
|---|---|---|
| (1,3) | $(\frac{1}{24\times2^n+12}, \frac{3}{24\times2^n+10})$ | $12\times 2^n$ |
| (2,3) | $(\frac{2}{24\times2^n+11}, \frac{3}{24\times2^n+10})$ | 8 |
| (3,3) | $(\frac{3}{24\times2^n+10}, \frac{3}{24\times2^n+10})$ | $8\times 2^n+12$ |
| (3,4) | $(\frac{3}{24\times2^n+10}, \frac{4}{24\times2^n+9})$ | 4 |

From Table 4, the following equation is written:

$$TI(PC4[n]) = \sum_{\tau\upsilon\in E_{1,3}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{2,3}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{3,3}} W_{\tau\upsilon} + \sum_{\tau\upsilon\in E_{3,4}} W_{\tau\upsilon} \qquad (3.4)$$

i. If $W_{\tau\upsilon} = (T_\tau + T_\upsilon)^k$ in Eq.3.4, then the proof (i) is completed from Table .
ii. If $W_{\tau\upsilon} = (T_\tau \times T_\upsilon)^k$ in Eq.(3.4), then the proof (ii) is completed with some calculated from Table. □

Figure 8 shows plots a) $GT_1^k$ and b) $GT_2^k$ of $PC4[n]$. This plot shows that $GT_1^k(PC4[n])$ grows faster than $GT_2^k(PC4[n])$.

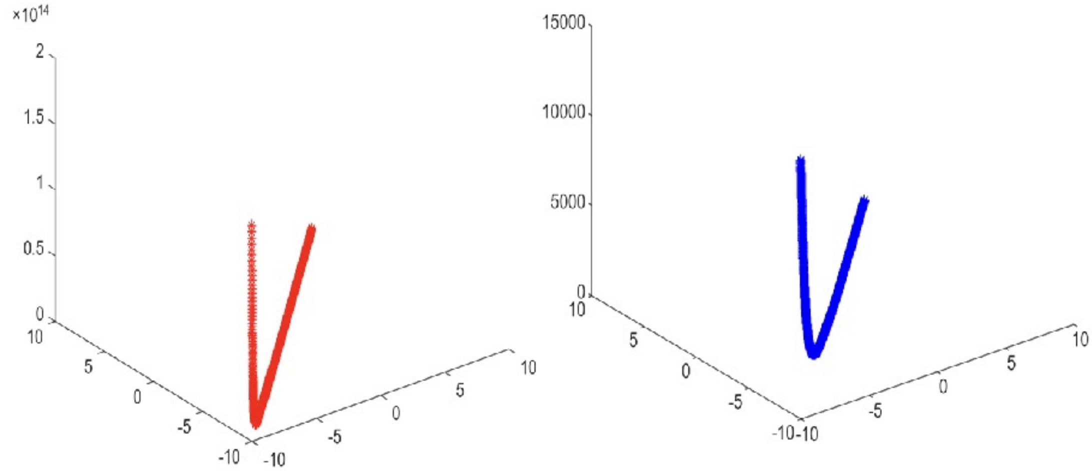


FIGURE 8. The plots of a) $GT_1^k$ and b) $GT_2^k$ of $PC4[n]$.

## 4. Conclusion

Since dendrimers are frequently used in drug discovery, dendrimer networks were considered in this study. Four important dendrimers were studied with temperature indices, which are among the graph indices that have attracted attention recently.

As a result, it was seen that the general first temperature index grew faster than the general second temperature index for PAMAM dendrimer and 3 porphyrin cored dendrimers. The results of this study will shed light on the field of chemical graph theory and fast and cost-effective drug discovery.

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This study does not be necessary ethical committee permission or any special permission.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s)

declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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# BINORMAL CURVES

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Abstract. In this paper, a new class of curves is called binormal curves are introduced. Here,we calculate the Frenet vector fields of binormal curves and using them give the curvature and torsion of such curves. Also, we provide some examples of binormal curves in Euclidean space $R^3$.

## 1. Introduction

Curves are the most important tools for elementary differential geometry. In the study of fundamental theory and characterization of space curves, the calculation of the curvature function, the torsion function and Frenet frame are very interesting and important problem in three dimensional space. So there are many articles on curves in the literature. In [1], the authors investigate the quaternionic Bertrand curves in Euclidean $3-$space. In the papers [2, 8], the curve theory in Galilean space is investigated. Further studies about curves and its applications are found in [3, 4, 5, 9]. In addition to these, the motion of parallel curves in Euclidean $3-$space is given in [7]. Gözütok, Çoban and Sağıroğlu [6] are study the classical differential geometry of curves with respect to conformable fractional derivative. In [7], Aldossary and Gazwani defined the notion of parallel curve based on binormal vector and calculated the Frenet frame of such a curve. Then, in [11] Sağıroğlu and Köse defined the notion of normal curve based on normal vector and gave some properties. Inspried by [11], in this paper we introduce a new class of curves which is called binormal curves. Here we study the Frenet frame of such curves and then calculate the curvature and torsion of binormal curves in Euclidean space $R^3$. Also, we provide some examples of binormal curves. The definitions of curvature, torsion and Frenet frame of a curve is given in [10].

## 2. Preliminaries

Let $\alpha : I \to R^3$ be a unit speed curve , so $\|\alpha'(s)\| = 1$ for each $s$ in $I$. Then

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$T = \alpha'$ is called the unit tangent vector field on $\alpha$. Since $T$ has constant lenght 1, its derivative $T' = \alpha''$ measures the way the curve is turning in $R^3$. Differention of $T.T = 1$ gives $T'$ always orthogonal to $T$. The lenght of $T'$ gives a numerical measurement of the turning of $\alpha$. The function $\kappa(s) = \|T'(s)\|$ for all $s$ in $I$ is called the curvature function of $\alpha$. The unit vector field $N = \frac{T'}{\kappa}$ on $\alpha$ is called the principal normal vector field on $\alpha$. The vector field $B = T \times N$ is called the binormal vector field of $\alpha$. The vector fields $T, N, B$ are called the Frenet frame field of $\alpha$. Also, it is clear that the torsion function $\tau$ of $\alpha$ satisfies $B' = -\tau N$ such that the torsion function $\tau$ measures twisting of $\alpha$.

**Theorem 2.1.** [10] *If $\alpha : I \to R^3$ is a unit speed curve with curvature $\kappa > 0$ and torsion $\tau$, then*

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N. \end{aligned} \tag{2.1}$$

**Theorem 2.2.** [10] *Let $\alpha$ be a regular curve in $R^3$. Then*

$$\begin{aligned} T = \frac{\alpha'}{\|\alpha'\|}, B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, N = B \times T \\ \kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\alpha' \times \alpha''.\alpha'''}{\|\alpha' \times \alpha''\|^2}. \end{aligned} \tag{2.2}$$

## 3. The Curvature Function, The Torsion Function and Frenet Frame of a Binormal Curve

**Definition 3.1.** Let $\alpha$ be a unit speed curve in $R^3$ and $t(s), n(s), b(s)$ be its Franet frame in a point $\alpha(s)$. Including $k$ to represent a real number constant, binormal curve of the curve $\alpha$ is defined as

$$\alpha_b(s) = \alpha(s) + kb(s). \tag{3.1}$$

Now let us consider the binormal curve of a curve $\alpha$ having the equation (3.1). Let be $s_3$ be the arc lenght function of this curve. So, the unit tangent vector of the curve $\alpha_b(s)$ has the form;

(3.2)
$$t_b\left(s_3\right) = \frac{d\alpha_b}{ds}\left(s_3\right).\frac{ds}{ds_3} = \left(t(s) + kb'(s)\right)(s_3).\frac{ds}{ds_3} = \left(t(s) - k\tau(s)n(s)\right)(s_3).\frac{ds}{ds_3}.$$

If we take the dot product with $t$ of both sides;

$$t_b.t = \frac{ds}{ds_3} = cos\theta \tag{3.3}$$

is obtained, where $\theta$ is the angle between the vectors $t_b$ and $t$. If it is multiplied both sides of this equation by the vector $b$, we get the equation

$$t_b.b = 0. \tag{3.4}$$

This shows that the vectors $t_b$ and $b$ are orthogonal. Multiplying both sides by the vector $n$

$$t_b.n = -k\tau(s)\frac{ds}{ds_3} \tag{3.5}$$

is found. If the norm of both sides of the expression $t_b(s_3)$ is taken, we get

$$(3.6)\quad \begin{aligned} \|t_b(s_3)\| &= \left\|(t(s) - k\tau(s)n(s))\frac{ds}{ds_3}\right\| = \|(t(s) - k\tau(s)n(s))\|\frac{ds}{ds_3} \\ &= \frac{ds}{ds_3}\left[\left[(t(s) - k\tau(s)n(s))\right]^{\frac{1}{2}} = \frac{ds}{ds_3}\left[1 + k^2\tau^2(s)\right]^{\frac{1}{2}} \end{aligned}$$

Then since $\|t_b(s_3)\| = 1$,

$$(3.7)\quad \frac{ds_3}{ds} = \sqrt{1 + k^2\tau^2(s)}$$

and from here

$$(3.8)\quad \frac{ds}{ds_3} = \frac{1}{\sqrt{1 + k^2\tau^2(s)}}$$

is obtained.
If it is taken as $t_b = n$, from the derivative of the vector $t_b(s_3)$ with respect to $s_3$, then in according to Frenet formulas

$$(3.9)\quad \kappa_b n_b = (-\kappa(s)t(s) + \tau(s)b(s)).\frac{ds}{ds_3}$$

is found. If we substitute the expression for $\frac{ds}{ds_3}$ in here, we get

$$(3.10)\quad \kappa_b n_b = \frac{-\kappa(s)t(s) + \tau(s)b(s)}{\sqrt{1 + k^2\tau^2(s)}}.$$

If we take the dot product of this vector with itself,
(3.11)

$$\begin{aligned} \kappa_b n_b.\kappa_b n_b = \kappa_b^2 &= \frac{1}{1 + k^2\tau^2(s)}\left(-\kappa(s)t(s) + \tau(s)b(s)\right).\left(-\kappa(s)t(s) + \tau(s)b(s)\right) \\ &= \frac{\kappa^2(s) + \tau^2(s)}{1 + k^2\tau^2(s)} \end{aligned}$$

is obtained. Principal normal vector and binormal vector of binormal curve of the curve $\alpha$ are

$$(3.12)\quad n_b = \frac{-\kappa(s)t(s) + \tau(s)b(s)}{\kappa_b\sqrt{1 + k^2\tau^2(s)}}$$

and

$$(3.13)\quad \begin{aligned} b_b = t_b \times n_b &= \left(\frac{t(s) - k\tau(s)n(s)}{\sqrt{1 + k^2\tau^2(s)}}\right) \times \left(\frac{-\kappa(s)t(s) + \tau(s)b(s)}{\kappa_b\sqrt{1 + k^2\tau^2(s)}}\right) \\ &= \frac{1}{\kappa_b\left(1 + k^2\tau^2(s)\right)}\left(-k\tau^2(s)t(s) - \tau(s)n(s) - k\kappa(s)\tau(s)b(s)\right) \end{aligned}$$

respectively.
Now let us determine the Frenet frame of the curve $\alpha_b(s)$ in terms of the Frenet frame of $\alpha(s)$ in the general case. We know that

$$(3.14)\quad \frac{d\alpha_b(s)}{ds} = \frac{d\alpha(s)}{ds} + k\frac{db(s)}{ds} = t(s) + k(-\tau(s)n(s)) = t(s) - k\tau(s)n(s).$$

If we take the norm of this equation;

$$(3.15)\quad \left\|\frac{d\alpha_b(s)}{ds}\right\| = \sqrt{1 + k^2\tau^2(s)} = K(s)$$

is obtained. Then, the unit tangent vector $t_b(s)$ is as;

$$t_b(s) = \frac{\frac{d\alpha_b(s)}{ds}}{\left\|\frac{d\alpha_b(s)}{ds}\right\|} = \frac{t(s) - k\tau(s)n(s)}{\sqrt{1+k^2\tau^2(s)}}. \tag{3.16}$$

Moreover;

$$\begin{aligned} \frac{d^2\alpha_b(s)}{ds^2} &= \frac{dt(s)}{ds} - k\tau'(s)n(s) - k\tau(s)\frac{dn(s)}{ds} \\ &= k\kappa(s)\tau(s)t(s) + (\kappa(s) - k\tau'(s))n(s) - k\tau^2(s)b(s) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\frac{d\alpha_b(s)}{ds} \times \frac{d^2\alpha_b(s)}{ds^2} = [t(s) - k\tau(s)n(s)] \times \left[k\kappa(s)\tau(s)t(s) + (\kappa(s) - k\tau'(s))n(s) - k\tau^2(s)b(s)\right] \\ &= (\kappa(s) - k\tau'(s))t(s) \times n(s) - k\tau^2(s)t(s) \times b(s) - k^2\kappa(s)\tau^2(s)n(s) \times t(s) + k^2\tau^3(s)n(s) \times b(s) \\ &= k^2\tau^3(s)t(s) + k\tau^2(s)n(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))b(s) \end{aligned} \tag{3.18}$$

are obtained. The norm of this expression is found as;

$$\left\|\frac{d\alpha_b(s)}{ds} \times \frac{d^2\alpha_b(s)}{ds^2}\right\| = \sqrt{k^4\tau^6(s) + k^2\tau^4(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))^2} = L(s) \tag{3.19}$$

The third derivative of the curve $\alpha_b(s)$ with respect to the $s$ parameter is;

$$\begin{aligned} \frac{d^3\alpha_b(s)}{ds^3} &= k\kappa'(s)\tau(s)t(s) + k\kappa(s)\tau'(s)t(s) + k\kappa(s)\tau(s)\frac{dt(s)}{ds} + (\kappa'(s) - k\tau''(s))n(s) \\ &\quad + (\kappa(s) - k\tau'(s))\frac{dn(s)}{ds} - 2k\tau(s)\tau'(s)b(s) - k\tau^2(s)\frac{db(s)}{ds}. \\ &= (k\kappa'(s)\tau(s) + k\kappa(s)\tau'(s) - \kappa^2(s) + k\kappa(s)\tau'(s))t(s) + (k\kappa^2(s)\tau(s) + \kappa'(s) - k\tau''(s) + k\tau^3(s))n(s) \\ &\quad + (\kappa(s)\tau(s) - k\tau(s)\tau'(s) - 2k\tau(s)\tau'(s))b(s) \\ &= (k\kappa'(s)\tau(s) - \kappa^2(s) + 2k\kappa(s)\tau'(s))t(s) + (k\kappa^2(s)\tau(s) + \kappa'(s) - k\tau''(s) + k\tau^3(s))n(s) \\ &\quad + (\kappa(s)\tau(s) - 3k\tau(s)\tau'(s))b(s). \end{aligned} \tag{3.20}$$

Then we get

$$\begin{aligned} \left(\frac{d\alpha_b(s)}{ds} \times \frac{d^2\alpha_b(s)}{ds^2}\right).\frac{d^3\alpha_b(s)}{ds^3} &= k^2\tau^3(s).(k\kappa'(s)\tau(s) - \kappa^2(s) + 2k\kappa(s)\tau'(s)) \\ &\quad + k\tau^2(s).(k\kappa^2(s)\tau(s) + \kappa'(s) - k\tau''(s) + k\tau^3(s)) \\ &\quad + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau(s)).(\kappa(s)\tau(s) - 3k\tau(s)\tau'(s)) \\ &= k^3\kappa'(s)\tau^4(s) - k^2\kappa^2(s)\tau^3(s) + 2k^3\kappa(s)\tau'(s)\tau^3(s) + k^2\kappa^2(s)\tau^3(s) + k\kappa'(s)\tau^2(s) \\ &\quad - k^2\tau^2(s)\tau''(s) + k^2\tau^5(s) + \kappa^2(s)\tau(s) - 3k\kappa(s)\tau(s)\tau'(s) \\ &\quad - k\kappa(s)\tau(s)\tau'(s) + 3k^2\tau(s)(\tau'(s))^2 + k^2\kappa^2(s)\tau^3(s) - 3k^3\kappa(s)\tau^3(s)\tau'(s) \\ &= k^3\kappa'(s)\tau^4(s) + k^2\kappa^2(s)\tau^3(s) - k^3\kappa(s)\tau'(s)\tau^3(s) + k\kappa'(s)\tau^2(s) - k^2\tau^2(s)\tau''(s) \\ &\quad + k^2\tau^5(s) + \kappa^2(s)\tau(s) - 4k\kappa(s)\tau(s)\tau'(s) + 3k^2\tau(s)(\tau'(s))^2. \end{aligned} \tag{3.21}$$

In this case, the expressions of the curvature and torsion of the curve $\alpha_b(s)$ in terms of the curvature and torsion of the curve $\alpha(s)$ are;

(3.22)

$$\kappa_b(s) = \frac{\|\alpha_b'(s) \times \alpha_b''(s)\|}{\|\alpha_b'(s)\|^3} = \frac{\sqrt{k^4\tau^6(s) + k^2\tau^4(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))^2}}{(1 + k^2\tau^2(s))^{\frac{3}{2}}}$$

and

(3.23)

$$\tau_b(s) = \frac{(\alpha_b'(s) \times \alpha_b''(s)).\alpha_b'''(s)}{\|\alpha_b'(s) \times \alpha_b''(s)\|^2}$$

$$= \frac{k^3\kappa'\tau^4 + k^2\kappa^2\tau^3 - k^3\kappa\tau'\tau^3 + k\kappa'\tau^2 - k^2\tau^2\tau'' + k^2\tau^5 + \kappa^2\tau - 4k\kappa\tau\tau' + 3k^2\tau(\tau')^2}{k^4\tau^6 + k^2\tau^4 + (\kappa - k\tau' + k^2\kappa\tau^2)^2}$$

We also obtained the unit tangent vector of the curve $\alpha_b(s)$ as;

$$t_b(s) = \frac{\frac{d\alpha_b(s)}{ds}}{\left\|\frac{d\alpha_b(s)}{ds}\right\|} = \frac{t(s) - k\tau(s)n(s)}{\sqrt{1 + k^2\tau^2(s)}} \tag{3.24}$$

The other Frenet frame elements of this curve;

$$b_b(s) = \frac{\alpha_b'(s) \times \alpha_b''(s)}{|\alpha_b'(s) \times \alpha_b''(s)\|} = \frac{k^2\tau^3 t + k\tau^2 n + (\kappa - k\tau' + k^2\kappa\tau^2)b}{\sqrt{k^4\tau^6 + k^2\tau^4 + (\kappa - k\tau' + k^2\kappa\tau^2)^2}} \tag{3.25}$$

and

(3.26)

$$n_b(s) = b_b(s) \times t_b(s) = \frac{1}{K(s).L(s)}[k^2\tau^3 t + k\tau^2 n + (\kappa - k\tau' + k^2\kappa\tau^2)b] \times [t - k\tau n]$$

$$= \frac{1}{K(s).L(s)}[-k^3\tau^4 t \times n + k\tau^2 n \times t + (\kappa - k\tau' + k^2\kappa^2\tau^2)b \times t + (-k\kappa\tau + k^2\tau\tau' - k^3\kappa\tau^3)b \times n]$$

$$= \frac{1}{K(s).L(s)}[(k\kappa\tau - k^2\tau\tau' + k^3\kappa\tau^3)t + (\kappa - k\tau' + k^2\kappa^2\tau^2)n + (-k^3\tau^4 - k\tau^2)b].$$

**Theorem 3.2.** *The expression of the Frenet frame of the binormal curve $\alpha_b(s)$ in terms of the Frenet frame of the curve $\alpha(s)$ is of the form;*

$$\begin{aligned} t_b(s) &= \frac{t(s) - k\tau(s)n(s)}{\sqrt{1 + k^2\tau^2(s)}} \\ b_b(s) &= \frac{k^2\tau^3(s)t(s) + k\tau^2(s)n(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))b(s)}{\sqrt{k^4\tau^6(s) + k^2\tau^4(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))^2}} \\ n_b(s) &= \frac{1}{K(s).L(s)}[(k\kappa(s)\tau(s) - k^2\tau(s)\tau'(s) + k^3\kappa(s)\tau^3(s))t(s) \\ &+(\kappa(s) - k\tau'(s) + k^2\kappa^2(s)\tau^2(s))n(s) + (-k^3\tau^4(s) - k\tau^2(s))b(s)]. \end{aligned} \tag{3.27}$$

**Theorem 3.3.** *The expression of the curvature and torsion functions of the binormal curve $\alpha_b(s)$ in terms of the curvature and torsion functions of the curve $\alpha(s)$*

*is obtained as;*

$$\kappa_b(s) = \frac{\sqrt{k^4\tau^6 + k^2\tau^4 + (\kappa - k\tau' + k^2\kappa\tau^2)^2}}{(1 + k^2\tau^2)^{\frac{3}{2}}} \tag{3.28}$$

$$\tau_b(s) = \frac{k^3\kappa'\tau^4 + k^2\kappa^2\tau^3 - k^3\kappa\tau'\tau^3 + k\kappa'\tau^2 - k^2\tau^2\tau'' + k^2\tau^5 + \kappa^2\tau - 4k\kappa\tau\tau' + 3k^2\tau(\tau')^2}{k^4\tau^6 + k^2\tau^4 + (\kappa - k\tau' + k^2\kappa\tau^2)^2}$$

**Example 3.4.** Let the curve

$$\alpha(s) = (acos(\frac{s}{a}), asin(\frac{s}{a}), 0), a > 0 \tag{3.29}$$

be given. Then, let us calculate the Frenet apparatus of binormal curve. The definition of binormal curve was $\alpha_b(s) = \alpha(s) + kb(s)$. Let us compute the binormal vector $b(s)$ of the curve $\alpha(s)$. Since the curve $\alpha(s)$ is unit speed curve, we get

$$b(s) = t(s) \times n(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -sin(\frac{s}{a}) & cos(\frac{s}{a}) & 0 \\ -cos(\frac{s}{a}) & -sin(\frac{s}{a}) & 0 \end{vmatrix} = (0, 0, 1). \tag{3.30}$$

Then the equation of the curve $\alpha_b(s)$ is,

$$\alpha_b(s) = (acos(\frac{s}{a}), asin(\frac{s}{a}), 0) + k(0, 0, 1) = (acos(\frac{s}{a}), asin(\frac{s}{a}), k). \tag{3.31}$$

From here;

$$\alpha_b'(s) = (-sin(\frac{s}{a}), cos(\frac{s}{a}), 0) \tag{3.32}$$

and

$$\|\alpha_b'(s)\| = \Big((-sin(\frac{s}{a}))^2 + (cos(\frac{s}{a}))^2\Big)^{\frac{1}{2}} = 1 \tag{3.33}$$

are obtained. Therefore the binormal curve $\alpha_b(s)$ is the unit speed curve. If we take the necessary calculations;

$$\begin{aligned} \alpha_b''(s) &= (-\frac{1}{a}cos(\frac{s}{a}), -\frac{1}{a}sin(\frac{s}{a}), 0) \\ \alpha_b'''(s) &= (\frac{1}{a^2}sin(\frac{s}{a}), -\frac{1}{a^2}cos(\frac{s}{a}), 0) \\ \alpha_b'(s) \times \alpha_b''(s) &= \begin{vmatrix} U_1 & U_2 & U_3 \\ -sin(\frac{s}{a}) & cos(\frac{s}{a}) & 0 \\ -\frac{1}{a}cos(\frac{s}{a}) & -\frac{1}{a}sin(\frac{s}{a}) & 0 \end{vmatrix} = (0, 0, \frac{1}{a}) \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \|\alpha_b'(s) \times \alpha_b''(s)\| &= \sqrt{0^2 + 0^2 + (\frac{1}{a})^2} = \frac{1}{a} \\ \alpha_b'(s) \times \alpha_b''(s).\alpha_b''' &= 0 \end{aligned} \tag{3.35}$$

are found. So the Frenet frame of the binormal curve is;

$$(3.36)\quad \begin{aligned} t_b(s) &= \frac{\alpha_b'(s)}{\|\alpha_b'(s)\|} = (-sin(\frac{s}{a}), cos(\frac{s}{a}), 0) \\ b_b(s) &= \frac{\alpha_b'(s)\times\alpha_b''(s)}{\|\alpha_b'(s)\times\alpha_b''(s)\|} = a(0,0,\frac{1}{a}) = (0,0,1) \\ n_b(s) = b_b(s)\times t_b(s) &= \begin{vmatrix} U_1 & U_2 & U_3 \\ 0 & 0 & 1 \\ -sin(\frac{s}{a}) & cos(\frac{s}{a}) & 0 \end{vmatrix} = (-cos(\frac{s}{a}), -sin(\frac{s}{a}), 0). \end{aligned}$$

The curvature and the torsion functions of this binormal curve are;

$$(3.37)\quad \begin{aligned} \kappa_b(s) &= \frac{\|\alpha_b'(s)\times\alpha_b''(s)\|}{\|\alpha_b'(s)\|^3} = \frac{\frac{1}{a}}{1} = \frac{1}{a} \\ \tau_b(s) &= \frac{\alpha_b'(s)\times\alpha_b''(s).\alpha_b'''(s)}{\|\alpha_b'(s)\times\alpha_b''(s)\|^2} = \frac{0}{(\frac{1}{a})^2} = 0. \end{aligned}$$

Hence the binormal curve of the curve $\alpha(s)$ is a circle.

**Example 3.5.** Let the helix $\alpha(s) = (cos(\frac{s}{\sqrt{2}}), sin(\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}})$ be given. Then let us calculate the Frenet apparatus of the binormal curve. Since $\alpha(s)$ is a unit speed curve, we get

$$(3.38)\quad b(s) = t(s)\times n(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) & \frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) & \frac{1}{\sqrt{2}} \\ -cos(\frac{s}{\sqrt{2}}) & -sin(\frac{s}{\sqrt{2}}) & 0 \end{vmatrix} = (\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}), -\frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}).$$

Then the equation of the binormal curve is,

$$(3.39)\quad \alpha_b(s) = \left(cos(\frac{s}{\sqrt{2}}) + \frac{k}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}), sin(\frac{s}{\sqrt{2}}) - \frac{k}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}} + \frac{k}{\sqrt{2}}\right)$$

Hence

$$(3.40)\quad \alpha_b'(s) = \left(-\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2}cos(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) + \frac{k}{2}sin(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right)$$

and

$$(3.41)\quad \begin{aligned} \|\alpha_b'(s)\| &= \left(\left(-\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2}cos(\frac{s}{\sqrt{2}})\right)^2 + \left(\frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) + \frac{k}{2}sin(\frac{s}{\sqrt{2}})\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{1}{2}} \\ &= \left(1 + \frac{k^2}{4}\right)^{\frac{1}{2}} = \frac{\sqrt{4+k^2}}{2} \end{aligned}$$

are found. Therefore, the binormal curve is not unit speed curve. If we make necessary calculations on arbitrary speed curves;

(3.42)

$$\alpha_b''(s) = \left(-\frac{1}{2}cos(\frac{s}{\sqrt{2}}) - \frac{k}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}), -\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}), 0\right)$$

$$\alpha_b'''(s) = \left(\frac{1}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) - \frac{k}{4}cos(\frac{s}{\sqrt{2}}), -\frac{1}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4}sin(\frac{s}{\sqrt{2}}), 0\right)$$

$$\alpha_b'(s) \times \alpha_b''(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2}cos(\frac{s}{\sqrt{2}}) & \frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) + \frac{k}{2}sin(\frac{s}{\sqrt{2}}) & \frac{1}{\sqrt{2}} \\ -\frac{1}{2}cos(\frac{s}{\sqrt{2}}) - \frac{k}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) & -\frac{1}{2}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) & 0 \end{vmatrix}$$

$$= \left(\frac{1}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) - \frac{k}{4}cos(\frac{s}{\sqrt{2}}), -\frac{1}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4}sin(\frac{s}{\sqrt{2}}), \frac{2+k^2}{4\sqrt{2}}\right)$$

and

(3.43)

$$\|\alpha_b'(s) \times \alpha_b''(s)\|$$

$$= \sqrt{\left(\frac{1}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) - \frac{k}{4}cos(\frac{s}{\sqrt{2}})\right)^2 + \left(-\frac{1}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4}sin(\frac{s}{\sqrt{2}})\right)^2 + \left(\frac{2+k^2}{4\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{k^4+6k^2+8}{32}}$$

$$\alpha_b'(s) \times \alpha_b''(s).\alpha_b''' = \frac{2+k^2}{16}$$

are obtained. Then Frenet frame of the binormal curve is;

(3.44)

$$t_b(s) = \frac{2}{\sqrt{4+k^2}}\left(-\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2}cos(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) + \frac{k}{2}sin(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right)$$

$$b_b(s) = \sqrt{\frac{32}{k^4+6k^2+8}}\left(\frac{1}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) - \frac{k}{4}cos(\frac{s}{\sqrt{2}}), -\frac{1}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4}sin(\frac{s}{\sqrt{2}}), \frac{2+k^2}{4\sqrt{2}}\right)$$

$$n_b(s) = \sqrt{\frac{32}{k^4+6k^2+8}}.\frac{2}{\sqrt{4+k^2}}.(-\frac{1}{4}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4\sqrt{2}}sin(\frac{s}{\sqrt{2}})$$
$$-(\frac{2+k^2}{8})cos(\frac{s}{\sqrt{2}}) - (\frac{2k+k^3}{8\sqrt{2}}sin(\frac{s}{\sqrt{2}}), -\frac{2+k^2}{8}sin(\frac{s}{\sqrt{2}})$$
$$+(\frac{2k+k^3}{8\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{1}{4}sin(\frac{s}{\sqrt{2}}) + \frac{k}{4\sqrt{2}}cos(\frac{s}{\sqrt{2}}), 0).$$

The curvature and the torsion function of the curve are;

$$\kappa_b(s) = \frac{\sqrt{2}.\sqrt{k^2+3}}{4+k^2} \tag{3.45}$$

and

$$\tau_b(s) = \frac{2}{4+k^2}. \tag{3.46}$$

Hence binormal curve is also a helix.

## 4. Conclusions

The definition of binormal curves is given by $\alpha_b(s) = \alpha(s) + kb(s)$ using the unit speed curve $\alpha(s)$. These curves are called parallel curves in the literature. The main goal of this paper is to investigate parallel curves using binormal vector and to study the associated geometry of these curves. We give a similar definition of parallel curves using the normal vector. The aim of this study is contribution to the literature on the theory of parallel curves based on binormal vector in three-dimensional space. In addition, the studies discussed here will later be expanded to surfaces and their geometric properties will be examined. Also, the instrinsic geometric formulas will be derived from the curvatures. This study was conducted at the Karadeniz Technical University in the Department of Mathematics and presented as a master's thesis.

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# SILVER STRUCTURES ON THE RIEMANN EXTENSIONS

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Abstract. In the present paper we deal with an $n-$dimensional differentiable manifold $M$ with a torsion-free linear connection $\nabla$. Here we study some properties of a silver structure on the cotangent bundle $T^*M$ equipped with the Riemannian extension ${}^R\nabla$ and obtain a necessary condition for which the silver semi-Riemannian manifold $\left(T^*M, {}^R\nabla, S\right)$ to be locally decomposable.

## 1. Introduction

The notion of metallic structure on Riemannian manifolds has been studied intesively recently. One of the most studied structure on Riemannian manifolds is silver structure. As a mathematical point of view, the positive solution of the equation

$$x^2 - px - q = 0,$$

for some positive integers $p$ and $q$ is called a $(p, q)-$ structure number which has the form

$$\mu_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

In particular case $p = 2$ and $q = 1$, we note that the last equality gives a silver ratio. In the recent years, the silver sturucture on the differentiable manifolds has been studied intensively in [4, 8, 9].

On the other hand, the cotangent bundle is the dual space of tangent bundle for a differentiable manifold which is very popular topic in Differential Geometry and Mathematical Physics. There are many different types of metrics on the cotangent bundle to study the geometric of such a bundle, for instance, Sasaki metric, Cheeger-Gromoll metric, general natural metrics, Oproius metrics, and etc. One of the most interesting metric is the Riemann extension which is defined by Patterson and Walker in [10]. Then, the notion of Riemann extension has been extensively studied by several authors on different smooth manifolds, for more [2, 3, 5-7, 12, 16].

In the present paper, we study some properties of a silver structure on the

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cotangent bundle equipped with the Riemannian extension. In Sect. 2, we recall some preliminaries on the details concerning the cotangent bundle. In Sect. 3, considering a silver structure on the cotangent bundle $T^*M$ , we give some necessary conditions for which the triple $\left(T^*M, {}^R\nabla, S\right)$ is a locally decomposable silver semi-Riemannian manifold.

## 2. Preliminaries

In this section, we recall some basic notations about the cotangent bundle of [16].

Let $(M, g)$ be a $n-$dimensional differentiable manifold whose cotangent bundle is denoted by $T^*M$. The bundle projection is given as $\pi : T^*M \to M$ and the local coordinates $\left(U, x^j\right),\ j = 1, ..., n$ on $M$ induces a system of local coordinates $\left(\pi^{-1}(U),\ x^j,\ x^{\bar{j}} = p_j\right),\ \bar{j} = n + j = n + 1, ..., 2n$ on $T^*M$, where $x^{\bar{j}} = p_j$ are the components of the covector $p$ in each cotangent space $T^*_x M, x \in U$ with respect to the natural coframe $\left\{dx^j\right\}$.

Also, the set $(r, s)-$type of all tensor fields is denoted by $\Im^r_s(M)$ and $\Im^r_s(T^*M)$ on $M$ and $T^*M$, respectively. Suppose that the vector and a covector (1-form) field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$ have the local expression $X = X^j \frac{\partial}{\partial x^j}$ and $\omega = \omega_j dx^j$ in $U \subset M$, respectively. Then, the horizontal lift ${}^H X \in \Im^1_0(T^*M)$ of $X \in \Im^1_0(M)$ and the vertical lift ${}^V\omega \in \Im^1_0(T^*M)$ of $\omega \in \Im^0_1(M)$ are given, respectively, by

$$\begin{aligned} &{}^H X = X^j \frac{\partial}{\partial x^j} + \sum_j p_h \Gamma^h_{ji} X^i \frac{\partial}{\partial x^{\bar{j}}}, \\ &{}^V\omega = \sum_j \omega_j \frac{\partial}{\partial x^{\bar{j}}} \end{aligned} \tag{2.1}$$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^{\bar{j}}}\right\}$, where $\Gamma^h_{ji}$ are the components of the Levi-Civita connection $\nabla_g$ on $M$.

Moreover, on the cotangent bundle $T^*M$, the Lie bracket satisfies the following relations:

$$\begin{aligned} &\text{i) } \left[{}^H X, {}^H Y\right] = {}^H [X, Y] + \gamma R(X, Y) = {}^H [X, Y] + {}^V (pR(X, Y)), \\ &\text{ii) } \left[{}^H X, {}^V\omega\right] = {}^V (\nabla_X \omega), \qquad iii) \left[{}^V\omega, {}^V\theta\right] = 0, \\ &\text{iv) } {}^V\omega {}^V f = 0, \qquad v) {}^H X {}^V f = {}^V (Xf) \end{aligned} \tag{2.2}$$

for any $X, Y \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$, R denoted the curvature tensor of $\nabla$.

On the other hand, the Riemann extension ${}^R\nabla$ as a semi-Riemannian metric is defined by

$$\begin{aligned} &{}^R\nabla\left({}^V\omega, {}^V\theta\right) = {}^R\nabla\left({}^H X, {}^H Y\right) = 0, \\ &{}^R\nabla\left({}^V\omega, {}^H Y\right) = {}^V(\omega(X)) = \omega(X) \circ \pi \end{aligned} \tag{2.3}$$

for any $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$ on $T^*M$ [2, 16].

## 3. Silver Structure

Let $P \in \Im^1_1(M)$ be an almost product structure on $M$ and $g$ be a (semi-) Riemannian metric such as

$$P^2 = I,\ g(PX, Y) = g(X, PY) \tag{3.1}$$

for any $X, Y \in \Im^1_0(M)$. Then, we call that the pair $(M, g, P)$ is a (semi-)Riemannian almost product manifold [1, 11, 17]. Such metrics in the second equation of (3.1) are said to be pure with respect to $P$ [12, 14].

A necessary and sufficient condition for the almost product structure $P$ to be integrable is that $\nabla P = 0$, where $\nabla$ is Levi-Civita connection of $g$. An almost product manifold with an integrable product structure $P$ is called locally product Riemannian manifold. We know that the locally product Riemannian manifold with structure tensor $P$ is locally decomposable if and only if $P$ is covariantly constant with respect to the Levi–Civita connection $\nabla$. Note that the condition $\nabla P = 0$ is equivalent to $\phi_P g = 0$ where $\phi$ is the Tachibana operator and

$$(3.2)\quad \begin{aligned}(\phi_P g)(X,Y,Z) &= (PX)(g(Y,Z)) - X(g(PY,Z)) + g((L_Y P)X,Z) \\ &+ g(Y,(L_Z P)X)\end{aligned}$$

for all $X,Y,Z \in \Im_0^1(M)$ [12, 15].

**Definition 3.1.** (see [8]) Let $M$ be a $C^\infty$ differentiable manifold. A (1, 1)-type tensor field $S$ on $M$ is called a silver structure on $M$ if

$$(3.3)\quad S^2 = 2S + I$$

is satisfied, where I is the identity map on $M$.

A Riemnnian manifold $(M,g)$ with a silver structure $S$ is said to be Silver Riemannian manifold if the Riemannian metric $g$ is pure with respect to $S$.

The next theorem gives the relationship between the Riemannian silver and almost product structures as follows:

**Theorem 3.2.** *(see[8]) Let $M$ be a Riemannian manifold. If $S$ is a silver structure on $M$, then*

$$P = \frac{1}{\sqrt{2}}(S - I)$$

*is an almost product structure on $M$. Conversely, any almost product structure $P$ on $M$ yields a silver structure on $M$ as follows:*

$$S = I + \sqrt{2}P.$$

**Theorem 3.3.** *(see [4]). Let $(M,g,S)$ be a silver Riemannian manifold, where $S$ is the silver structure and $g$ is the Riemannian metric. Then the followings are satisified:*
*a ) $S$ is integrable if $\phi_S g = 0$,*
*b ) The condition $\phi_S g = 0$ is equivalent to $\nabla S = 0$, where $\nabla$ is the Riemannian connection of $g$,*
*where $\phi_S$ denotes the Tacibana operator and $\nabla$ is the Riemannian connection of $g$.*

In [13], Salimov and Agca presented an almost product structure on $T^*M$ by

$$(3.4)\quad \begin{aligned}P^H X &= {}^V\tilde{X}, \\ P^V\omega &= {}^H\tilde{\omega}.\end{aligned}$$

for any $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, where $\tilde{X} = g \circ X \in \Im_1^0(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M)$ and $P^2 = I$. Applying Theorem 3.2 and (3.4), we find the following silver structure $S$:

$$(3.5)\quad \begin{aligned}S^H X &= {}^H X + \sqrt{2}{}^V\tilde{X}, \\ S^V\omega &= {}^V\omega + \sqrt{2}{}^H\tilde{\omega}.\end{aligned}$$

This silver structure defined by (3.5) is used for Sasaki metric on $T^*M$ in [4].

Now we consider the Riemannian extension ${}^R\nabla$ and the silver structure $S$ on the cotangent bundle $T^*M$. Then, using the Eqs. (3.1) and (3.5), we have the following theorem:

**Theorem 3.4.** *Let $M$ be semi-Riemannian manifold and $T^*M$ be a cotangent bundle of $M$. If $T^*M$ is endowed with a Riemann extension ${}^R\nabla$ and silver structure $S$, then the triple $\left(T^*M, {}^R\nabla, S\right)$ is a silver semi-Riemannian manifold.*

*Proof.* Using (3.1), we write

$$Q\left(\tilde{X}, \tilde{Y}\right) = {}^R\nabla\left(S\tilde{X}, \tilde{Y}\right) - {}^R\nabla\left(\tilde{X}, S\tilde{Y}\right)$$

for any $\tilde{X}, \tilde{Y} \in \Im_0^1\left(T^*M\right)$. From (2.1), (2.3) and (3.5), we have

$$\begin{aligned}
Q\left({}^HX, {}^HY\right) &= {}^R\nabla\left(S^HX, {}^HY\right) - {}^R\nabla\left({}^HX, S^HY\right) \\
&= {}^R\nabla\left({}^HX + \sqrt{2}^V\tilde{X}, {}^HY\right) - {}^R\nabla\left({}^HX, {}^HY + \sqrt{2}^V\tilde{Y}\right) \\
&= \left({}^V\left(\tilde{X}\left(Y\right)\right) - \left(\tilde{Y}\left(X\right)\right)\right) = \sqrt{2}\left(\tilde{X}_iY^i - \tilde{Y}_iX^i\right) \\
&= \sqrt{2}\left(g_{ki}X^kY^i - g_{ki}Y^kX^i\right) = 0, \\
Q\left({}^HX, {}^V\omega\right) &= -Q\left({}^HY, {}^V\omega\right) = {}^R\nabla\left(S^HX, {}^V\omega\right) - {}^R\nabla\left({}^HX, S^V\omega\right) \\
&= {}^R\nabla\left({}^HX + \sqrt{2}^V\tilde{X}, {}^V\omega\right) - {}^R\nabla\left({}^HX, {}^V\omega + \sqrt{2}^H\tilde{\omega}\right) \\
&= {}^V\left(\omega\left(X\right) - \omega\left(X\right)\right) = 0\ , \\
Q\left({}^V\omega, {}^V\theta\right) &= {}^R\nabla\left(S^V\omega, {}^V\theta\right) - {}^R\nabla\left({}^V\omega, S^V\theta\right) \\
&= {}^R\nabla\left({}^V\omega + \sqrt{2}^H\tilde{\omega}, {}^V\theta\right) - {}^R\nabla\left({}^V\omega, {}^V\theta + \sqrt{2}^H\tilde{\theta}\right) \\
&= \sqrt{2}^V\left(\theta\left(\tilde{\omega}\right) - \omega\left(\tilde{\theta}\right)\right) = 0\ ,
\end{aligned}$$

i.e. ${}^R\nabla$ is pure with respect to $S$, which completes the proof. □

Using the Eqs.(2.2), (2.3), (3.2) and (3.5), we obtain the following:

**Lemma 3.5.** *Let $\left(T^*M, {}^R\nabla, S\right)$ be a silver semi-Riemannian manifold. Then, the following component for the Tachibana operator with respect to the silver structure $S$ defined by (3.5) is given by*

$$\begin{aligned}
\left(\phi_S{}^R\nabla\right)\left({}^HX, {}^HY, {}^V\omega\right) &= \left(S^HX\right)\left({}^R\nabla\left({}^HY, {}^V\omega\right)\right) - {}^HX\left({}^R\nabla\left(S^HY, {}^V\omega\right)\right) \\
&\quad + {}^R\nabla\left(\left(L_{{}^HY}S\right){}^HX, {}^V\omega\right) + {}^R\nabla\left({}^HY, \left(L_{{}^V\omega}S\right){}^HX\right) \\
&= -\left({}^R\nabla\left({}^V\omega, \sqrt{2}^H\left(g^{-1}\circ pR\left(Y,X\right)\right)\right)\right) \\
&= -\sqrt{2}^V\left(\omega\left(g^{-1}\circ pR\left(Y,X\right)\right)\right) = -\sqrt{2}^V\left(g^{-1}\left(pR\left(Y,X\right), \omega\right)\right) \\
&= \sqrt{2}^V\left(pR\left(X,Y\right)\tilde{\omega}\right), \\
\left(\phi_F{}^R\nabla\right)\left({}^V\omega, {}^HY, {}^HZ\right) &= \sqrt{2}\left({}^V\left(pR\left(Y,\tilde{\omega}\right)Z + pR\left(Z,\tilde{\omega}\right)Y\right)\right), \\
\left(\phi_F{}^R\nabla\right)\left({}^HX, {}^V\omega, {}^HY\right) &= \sqrt{2}^V\left(pR\left(X,Y\right)\tilde{\omega}\right)
\end{aligned}$$

*Here, we note that the other components are zero.*

Using above Lemma 3.5, we have the following:

**Theorem 3.6.** *The silver semi-Riemannian manifold $\left(T^*M, {}^R\nabla, S\right)$ is a locally decomposable if and only if $M$ is flat.*

**Example 3.7.** Consider the $n$-dimensional Euclidean space $\mathbb{E}^n$ with the Riemannian metric $g_{ij} = \delta_j^i$. It is clear that the Christoffel symbols induced by the Levi-Civita connection $\nabla$ on $\mathbb{E}^n$ are zero.

Let $P$ be an almost product structure on $T^*\mathbb{E}^n$ is given by

$$P = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix},$$

such that $P^2 = I_n$, where $I_n$ denotes the identity matrix of order $n$. Using Theorem 3.2, the almost product structure $P$ on $T^*\mathbb{E}^n$ gives

$$S^H X = {}^H X + \sqrt{2}^H X, \quad S^V \omega = {}^V \omega + \sqrt{2}^V \omega, \tag{3.6}$$

such that the equalities (3.6) are silver structure. Then, one can see that ${}^R\nabla$ is pure with respect to $S$ and the triple $\left(T^*\mathbb{E}^n, {}^R\nabla, S\right)$ becomes a silver semi-Riemannian manifold.

On the other hand, using the Eq.(3.2) and the Tachibana operator with respect to the silver structure defined by (3.6), one has

$$\left(\phi_S {}^R\nabla\right)(X, Y, Z) = 0$$

for any $X, Y, Z \in \Im_0^1(M)$. Then, we obtain that the silver semi-Riemannian manifold $\left(T^*\mathbb{E}^n, {}^R\nabla, S\right)$ is a locally decomposable.

## 4. Conclusion

In this study, a semi-Riemannian manifold $M$ and its cotangent bundle $T^*M$ is considered. Then, by considering the Riemann extension ${}^R\nabla$ and silver structure $S$ on $T^*M$, the components of Tachibana operators are calculated and using them, this characterization is obtained: M is flat if and only if the silver semi-Riemannian manifold $\left(T^*M, {}^R\nabla, S\right)$ is a locally decomposable.

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# A STUDY ON NON-HOMOGENEOUS MULTIPLICATIVE BOUNDARY VALUE PROBLEMS

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Abstract. The paper deals with the non-homogeneous boundary value problems established in the multiplicative calculus. For this problem, the multiplicative meanings of concepts such as adjoint operator, self-adjoint operator, Lagrange identity, Green's formula are obtained. Then, a criterion, called multiplicative Fredholm alternative, is given for the existence of solutions to non-homogeneous multiplicative boundary value problems.

## 1. Introduction

The concept of the classical calculus was developed independently by Newton and Leibniz in the late 17th century when modern science was born [3, 8]. Subsequent studies, including the creation of the idea of limits, placed these developments on a more solid conceptual basis. The classical calculus is widely used today in engineering, science and social sciences [19]. Since the operations of addition and subtraction are the basis of all concepts defined and all theorems established in classical calculus, this calculus is also called additive calculus or Newtonian calculus.

The roles of the operations used between the elements are important in establishing different calculus. Therefore, the various alternative calculus to the classical calculus have been defined with the help of different arithmetic operations. These calculus are defined as Non-Newtonian calculus by Grossman and Katz [15, 16]. Geometric calculus, bigeometric calculus and anaquadratic calculus can be given as examples of these calculus. The concept of geometric calculus was studied widely and related to multiplication, is called multiplicative calculus [9, 24]. In different areas such as biomedical applications [13], differential and integral equations [2,25–31] , finance [10–12], geometry [1,17,18], machine learning [7], numerical applications [21,22,32,35], social sciences [5] and spectral theory [14,23,33,34], significant contributions have been made by multiplicative calculus [4, 6, 24].

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In the literature, some calculus are called multiplicative calculus because the elements in the domain and/or range sets are defined by exponential function arithmetic. Moreover, multiplicative derivatives and integrals are defined in this calculus. Due to the existence of various application areas mentioned above and the existence of provable properties, the multiplicative calculus shown with the help of the multiplicative derivative, whose definition is given in the next section, is preferred.

## 2. Preliminaries

**Definition 2.1.** [4, 24] Suppose that $A \subset \mathbb{R}$, $f : A \to \mathbb{R}^+$. *The multiplicative derivative* of the function $f$ at $x$ is given by

$$f^*(x) = \lim_{\varepsilon \to 0} \left[ \frac{f(x+\varepsilon)}{f(x)} \right]^{\frac{1}{\varepsilon}}, \tag{2.1}$$

if the limit exists and positive.

Suppose that $A \subset \mathbb{R}$, $f : A \to \mathbb{R}^+$ be differentiable in usual case. Then, there is a relation between classical and derivatives as follows:

$$f^*(x) = e^{(\ln of)'(x)}.$$

Repeating this procedure $n$ times, it can be obtained the relation between the $n-$th order classical derivative and $n-$th $^*$derivative as

$$f^{*(n)}(x) = e^{(\ln of)^{(n)}(x)}.$$

**Theorem 2.2.** *[4, 24] Suppose that $f, g$ be multiplicative differentiable and $h$ be usual differentiable at $x$. If $c \in \mathbb{R}^+$ is an arbitrary constant, then the functions $cf$, $fg$, $\frac{f}{g}$, $f^h$, $foh$ and $f+g$ have multiplicative derivatives given by*

**i.** $(cf)^*(x) = f^*(x)$,
**ii.** $(fg)^*(x) = f^*(x)g^*(x)$,
**iii.** $\left(\frac{f}{g}\right)^*(x) = \frac{f^*(x)}{g^*(x)}$,
**iv.** $\left(f^h\right)^*(x) = f^*(x)^{h(x)} f(x)^{h'(x)}$,
**v.** $(foh)^*(x) = f^*(h(x))^{h'(x)}$,
**vi.** $(f+g)^*(x) = f^*(x)^{\frac{f(x)}{f(x)+g(x)}} g^*(x)^{\frac{g(x)}{f(x)+g(x)}}$.

**Definition 2.3.** [4, 24] Suppose that $A \subset \mathbb{R}$, $f : A \to \mathbb{R}^+$ bounded on $[a, b]$. *The multiplicative integral* of the function $f$ on $[a, b]$ is given by $\int_a^b f(x)^{dx}$.

If $f$ is positive and Riemann integrable on $[a, b]$, then it is also multiplicative integrable on $[a, b]$. Additionally, the following equality is satisfied.

$$\int_a^b f(x)^{dx} = e^{\int_a^b (\ln of)(x)dx}.$$

On the contrary,

$$\int_a^b f(x)dx = \ln \int_a^b \left(e^{f(x)}\right)^{dx}$$

if the function $f$ is multiplicative integrable on $[a, b]$.

**Theorem 2.4.** *[4, 24] Suppose that $f, g$ be bounded functions on $[a,b]$. If $f, g$ are $^*$integrable on $[a,b]$, then the functions $f^c$, $fg$, $\frac{f}{g}$ have multiplicative integrals on $[a,b]$ given by*

**i.** $\int_a^b [f(x)^c]^{dx} = \left[\int_a^b f(x)^{dx}\right]^c, c \in \mathbb{R},$

**ii.** $\int_a^b [f(x)g(x)]^{dx} = \int_a^b f(x)^{dx} \int_a^b g(x)^{dx},$

**iii.** $\int_a^b \left[\frac{f(x)}{g(x)}\right]^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}},$

**iv.** $\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx} \int_c^b f(x)^{dx}, a \le c \le b,$

**v.**

$$\int_a^b \left[f^*(x)^{g(x)}\right]^{dx} = f(x)^{g(x)}\Big|_a^b \frac{1}{\int_a^b \left[f(x)^{g'(x)}\right]^{dx}},$$

*where,* $f(x)^{g(x)}\Big|_a^b = \frac{f(b)^{g(b)}}{f(a)^{g(a)}}$.

This equality (v) is known as *multiplicative integration by parts method.*

**Lemma 2.5.** *[14, 20]* $L_2^*[a,b] = \left\{ f : \int_a^b \left[f(x)^{f(x)}\right]^{dx} < \infty \right\}$ *is an multiplicative inner product space with*

$$<,>_*: L_2^*[a,b] \times L_2^*[a,b] \to \mathbb{R}^+, < f, g >_* = \int_a^b \left[f(x)^{g(x)}\right]^{dx},$$

*where $f, g \in L_2^*[a,b]$ are positive functions.*

**Definition 2.6.** [6, 27–30] $n-$th order linear multiplicative differential equation is defined by

$$\left(y^{*(n)}\right)^{a_n(x)} \left(y^{*(n-1)}\right)^{a_{n-1}(x)} \cdots (y^{**})^{a_2(x)} (y^*)^{a_1(x)} y^{a_0(x)} = \phi(x), \tag{2.2}$$

where $\phi(x) > 0$ and $a_k(x),\ k = 0, 1, 2, \cdots, n-1, n$ are functions of $x$.

In equation (2.2), when $\phi(x) = 1$, this equation is called homogeneous multiplicative linear differential equation, otherwise it is called non-homogeneous multiplicative linear differential equation.

## 3. Main Results

In spectral theory, the Lagrange identity is a foundational result that connects a differential operator with its formal adjoint. Through integration, this leads directly to Green's formula. Green's formula helps determine when an operator is self-adjoint by examining the boundary terms. Here, self-adjointness is crucial in spectral theory for several reasons such as all eigenvalues are real, eigenfunctions corresponding to different eigenvalues are orthogonal, and the spectrum is bounded below.

3.1. **Non-Homogeneous Multiplicative Boundary Value Problems.** In this section, non-homogeneous multiplicative boundary value problems are discussed. The multiplicative counterparts of concepts such as the adjoint operator, self-adjoint operator, Lagrange identity and Green's formula are defined for this problem.

Consider the non-homogeneous multiplicative boundary value problem

$$T[y](x) = \{y^{**}\}^{a_0(x)}\{y^*\}^{a_1(x)}y^{a_2(x)} = h(x), \tag{3.1}$$

$$\begin{aligned} B_1[y] &= y(a)^{a_{11}}y^*(a)^{a_{12}}y(b)^{b_{11}}y^*(b)^{b_{12}} = 1,\\ B_2[y] &= y(a)^{a_{21}}y^*(a)^{a_{22}}y(b)^{b_{21}}y^*(b)^{b_{22}} = 1 \end{aligned} \tag{3.2}$$

where the functions $a_i := a_i\,(x)$ for $i = 0, 1, 2$ on the exponentials are continuous real valued functions on the interval $[a, b]$; $a_{ij}, b_{ij}$, for $i, j = 1, 2$ are real constants and $a_0\,(x) \neq 0$ for all $x \in [a, b]$.

**Definition 3.1.** (Multiplicative Formal Adjoint Operator) Let the operator $T$ be the multiplicative differential operator defined by the left-hand side of equation (3.1). In other words, let it be

$$T[y] = \{y^{**}\}^{a_0}\{y^*\}^{a_1}y^{a_2}. \tag{3.3}$$

Then, the multiplicative differential operator $\widetilde{T}$ defined in the form

$$\widetilde{T}\,[y] = (y^{a_0})^{**}((y^{a_1})^*)^{-1}y^{a_2} \tag{3.4}$$

is called *the multiplicative formal adjoint operator* of the operator $T$.

**Example 3.2.** The multiplicative formal adjoint operator of the multiplicative differential operator $T[y] = \{y^{**}\}\,\{y^*\}^6 y^{10}$ is of the form

$$\widetilde{T}\,[y] = \{y^{**}\}\,\{y^*\}^{-6}y^{10}$$

for $a_0 = 1, a_1 = 6, a_2 = 10$ according to formula (3.4).

**Example 3.3.** The multiplicative formal adjoint operator of the multiplicative differential operator $T[y] = \{y^{**}\}^{2x^2}\{y^*\}^{7x}y^{-2}$ is of the form

$$\begin{aligned} \widetilde{T}\,[y] &= (y^{2x^2})^{**}((y^{7x})^*)^{-1}y^{-2}\\ &= [(y^*)^{2x^2}y^{4x}]^*[(y^*)^{7x}y^7]^{-1}y^{-2}\\ &= (y^{**})^{2x^2}(y^*)^x y^{-5} \end{aligned}$$

for $a_0 = 2x^2, a_1 = 7x, a_2 = -2$ according to formula (3.4).

The relationship between the multiplicative differential operator $T$ and its multiplicative formal adjoint operator $\widetilde{T}$ are defined as the multiplicative Lagrange identity.

**Theorem 3.4.** *Suppose that the multiplicative differential operator $T$ is the operator defined by equation (3.1), and the multiplicative differential operator $\widetilde{T}$ is the* *multiplicative formal adjoint of $T$. Therefore, the multiplicative Lagrange identity is of the form*

$$\left\{\{T[u]\odot v\}\ominus\left\{u\odot\widetilde{T}\,[v]\right\}\right\}=\frac{d^*}{dx}[P\,(u,v)], \tag{3.5}$$

*where,*

$$P\,(u,v)=\frac{\{u^*\}^{\ln v^{a_0}}\{u\}^{\ln v^{a_1}}}{\{u\}^{\ln\,(v^{a_0})^*}}. \tag{3.6}$$

*Proof.* From Lemma 2.5, formulas (3.3) and (3.4), the proof can be easily shown similarly to that in [14]. □

**Theorem 3.5.** *(Multiplicative Green's Formula) The multiplicative differential operator $T$ is the operator defined by equation (3.1), and the multiplicative differential operator $\widetilde{T}$ is the multiplicative formal adjoint of $T$. Therefore, the multiplicative Green's formula is of the form*

$$\int_a^b\left\{\left(\{T\,[u]\odot v\}\ominus\left\{u\odot \widetilde{T}[v]\right\}\right)(x)\right\}^{dx}=P\,\,(u,v)\,(x)|_a^b\,, \tag{3.7}$$

*where,* $P\,\,(u,v)\,(x)|_a^b=\frac{P\,(u,v)(b)}{P\,(u,v)(a)}$.

*Proof.* The proof is easily obtained by multiplicative integrating equality (3.5) from $a$ to $b$ similarly to that in [14]. □

**Corollary 3.6.** From Lemma 2.5, the multiplicative Green's formula can be also expressed in the form

$$\langle T\,[u]\,,v\rangle_*=P\,\,(u,v)\,(x)|_a^b\left\langle u,\widetilde{T}\,\,[v]\right\rangle_*. \tag{3.8}$$

**Example 3.7.** Suppose that $T$ is the multiplicative differential operator defined by equation (3.1), and $\widetilde{T}$ is the multiplicative formal adjoint of $T$. Therefore,

$$\begin{aligned}\int_a^b\{T\,[u]\odot v\}^{dx}&=\int_a^b\left\{(T\,[u])^{\ln v}\right\}^{dx}=\int_a^b\left\{(\{u^{**}\}^{a_0}\{u^*\}^{a_1}u^{a_2})^{\ln v}\right\}^{dx}\\&=\int_a^b\left\{\{u^{**}\}^{\ln v^{a_0}}\right\}^{dx}\int_a^b\left\{\{u^*\}^{\ln v^{a_1}}\right\}^{dx}\int_a^b\left\{\{u\}^{\ln v^{a_2}}\right\}^{dx}.\end{aligned} \tag{3.9}$$

If multiplicative integration by parts method is applied twice to the first integral and once to the second integral on the right of (3.9), the following integrals are obtained

$$\int_a^b\left\{\{u^{**}\}^{\ln v^{a_0}}\right\}^{dx}=\left.\frac{\{u^*\}^{\ln v^{a_0}}}{\{u\}^{\ln(v^{a_0})^*}}\right|_a^b\int_a^b\left\{\{u\}^{(\ln v^{a_0})^{**}}\right\}^{dx},$$

$$\int_a^b\left\{\{u^*\}^{\ln v^{a_1}}\right\}^{dx}=\{u\}^{\ln v^{a_1}}\Big|_a^b\int_a^b\left\{\{u\}^{-\ln\,(v^{a_1})^*}\right\}^{dx}.$$

If these expressions are substituted into (3.9),

$$\int_a^b \{T[u] \odot v\}^{dx} = \frac{\{u^*\}^{\ln v^{a_0}} \{u\}^{\ln v^{a_1}}}{\{u\}^{\ln (v^{a_0})^*}}\Bigg|_a^b \int_a^b \left\{u^{\ln\left[(v^{a_0})^{**}((v^{a_1})^*)^{-1} v^{a_2}\right]}\right\}^{dx}$$

$$= P\ (u,v)|_a^b \int_a^b u^{\ln \widetilde{T}\,[v]}\,dx = P\ (u,v)|_a^b \int_a^b \left\{u \odot \widetilde{T}\ [v]\right\}^{dx}$$

is found. Consequently, equality (3.8) is obtained.

Now, let $T$ and $\widetilde{T}$ be two multiplicative differential operators defined on $C^{2,*}[a,b]$. Here, $C^{2,*}[a,b]$ is the space of functions defined on the interval $[a,b]$ whose multiplicative derivatives up to the second order are continuous.

Suppose that $D(T), D\left(\widetilde{T}\right) \subset C^{2,*}[a,b]$ and $D(T)$ be the set of functions that belong to $C^{2,*}[a,b]$ and satisfy the boundary conditions given in (3.2). This is the domain of the multiplicative differential operator $T$. The domain $D\left(\widetilde{T}\right)$ of the multiplicative formal adjoint $\widetilde{T}$ is the set of all functions $v$ for which

$$\langle L[u], v\rangle_* = \left\langle u, \widetilde{T}\ [v]\right\rangle_* \tag{3.10}$$

holds for all $u \in D(T)$.

If the multiplicative Green's formula (3.7) is considered along with (3.10), it is seen that the domain of $\widetilde{T}$ consists of the functions $v$ for which

$$P\ (u,v)(x)|_a^b = 1 \tag{3.11}$$

holds for all $u \in D(L)$.

**Definition 3.8.** (Multiplicative Adjoint Operator) *The multiplicative adjoint operator* of $T$ is an operator $\widetilde{T}$ with domain $D\left(\widetilde{T}\right)$.

**Example 3.9.** The multiplicative adjoint operator of $T[y] = \{y^{**}\}\{y^*\}^6 y^{10}$ with a domain

$$D(T) = \left\{y : y \in C^{2,*}[0,\pi]; y(0) = y(\pi) = 1\right\}$$

is calculated as follows:

From Example 3.2, the multiplicative formal adjoint operator of $T$ is $\widetilde{T}[y] = \{y^{**}\}\{y^*\}^{-6} y^{10}$. To complete the solution, we need to determine the domain $D\left(\widetilde{T}\right)$ of the operator $\widetilde{T}$.

Let $v \in C^{2,*}[0,\pi]$. Also,

$$P(u,v)(x)|_0^\pi = \frac{\{u^*\}^{\ln v}\{u\}^{\ln v^6}}{\{u\}^{\ln v^*}}(x)\Bigg|_0^\pi = 1$$

must be satisfied for all $u \in D(T)$ and $v \in D\left(\widetilde{T}\right)$.

Considering the domain of the operator $T$, where $u(0) = u(\pi) = 1$, then equality can be easily written as

$$\frac{\{u^*(\pi)\}^{\ln v(\pi)}}{\{u^*(0)\}^{\ln v(0)}} = 1. \tag{3.12}$$

The goal is to find a condition for the function $v$. Therefore, in order for equality (3.12) to hold for all $u \in D(T)$, it must be that $v(0) = v(\pi) = 1$.

As a result, since $v$ is an arbitrary variable, the domain of the multiplicative adjoint operator $\widetilde{T}$ is

$$D\left(\widetilde{T}\right) = \left\{y : y \in C^{2,*}[0,\pi]; y(0) = y(\pi) = 1\right\}.$$

**Definition 3.10.** (Multiplicative Self-Adjoint Operator) Let $T = \widetilde{T}$ and $D(T) = D\left(\widetilde{T}\right)$, then the operator $T$ is called *the multiplicative self-adjoint operator.*

**Definition 3.11.** (Multiplicative Adjoint Boundary Value Problem) Let $B[u] = 1$ denote the boundary conditions for the operator $T$, and $\widetilde{B}[u] = 1$ indicate the boundary conditions for the adjoint operator $\widetilde{T}$. Then, the boundary value problem

$$\widetilde{T}[u] = 1; \widetilde{B}[u] = 1$$

is called *the multiplicative adjoint* of the boundary value problem

$$T[u] = 1; B[u] = 1.$$

**Example 3.12.** It can be seen from Example 3.2 that the boundary value problem

$$\{y^{**}\}\{y^{*}\}^{-6}y^{10} = 1,$$
$$y(0) = y(\pi) = 1$$

is the multiplicative adjoint of the boundary value problem

$$\{y^{**}\}\{y^{*}\}^{6}y^{10} = 1,$$
$$y(0) = y(\pi) = 1.$$

**Example 3.13.** The multiplicative adjoint of the boundary value problem

$$y^{**}y = 1,$$
$$y^{*}(0)y^{*}(\pi) = 1, \ y(0) = 1$$

is calculated as follows:

Here, the multiplicative adjoint of the operator $T[y] = y^{**}y$ is given by $\widetilde{T}[y] = y^{**}y$.

The domain of $T$ is the form

$$D(T) = \left\{u : u \in C^{2,*}[0,\pi]; u(0) = u(\pi) = 1, u^{*}(0) = 1\right\}.$$

The domain of the multiplicative adjoint operator $\widetilde{T}$ consists of the functions $v$ that satisfy the equality

$$P(u,v)(x)\Big|_0^\pi = \left.\frac{\{u^{*}\}^{\ln v}}{\{u\}^{\ln v^{*}}}\right|_0^\pi = 1$$

for all $u \in D(T)$.

Considering the domain of the operator $T$, where $u^{*}(\pi) = u^{*}(0)^{-1}$ and $u(0) = 1$, then the equality

$$\frac{\{u^{*}(0)\}^{-\ln v(0)v(\pi)}}{\{u(\pi)\}^{\ln v^{*}(\pi)}} = 1 \tag{3.13}$$

is obtained. Therefore, in order for equality (3.13) to hold for all $u \in D(T)$, it must be that

$$v(0)\,v(\pi) = 1,\ \ v^*(\pi) = 1.$$

This implies that the domain of the multiplicative adjoint operator $\widetilde{T}$ is

$$D\left(\widetilde{T}\right) = \left\{v : v \in C^{2,*}[0,\pi];\ v^*(\pi) = 1, v(0)\,v(\pi) = 1\right\}.$$

Consequently, since $u$ and $v$ are arbitrary variables, the multiplicative adjoint of given multiplicative boundary value problem takes the form

$$y^{**}y = 1,$$
$$y(0)y(\pi) = 1,\ \ y^*(\pi) = 1.$$

**Example 3.14.** The multiplicative adjoint of the boundary value problem

$$\{y^{**}\}^{2x^2}\{y^*\}^{7x}y^{-2} = 1,$$
$$y(1) = y(\pi) = 1$$

is calculated as follows:

From Example 3.3, the multiplicative adjoint of the operator

$$T[y] = \{y^{**}\}^{2x^2}\{y^*\}^{7x}y^{-2}$$

is given by

$$\widetilde{T}[y] = (y^{**})^{2x^2}(y^*)^{x}y^{-5}.$$

The domain of $T$ is the form

$$D(T) = \left\{u : u \in C^{2,*}[1,\pi];\ u(1) = u(\pi) = 1\right\}.$$

The domain of the multiplicative adjoint operator $\widetilde{T}$ consists of the functions $v$ that satisfy the equality

$$P(u,v)(x)|_1^\pi = \left.\frac{\{u^*\}^{\ln v^{2x^2}}\{u\}^{\ln v^{7x}}}{\{u\}^{\ln\left(\{v^*\}^{2x^2}v^{4x}\right)}}(x)\right|_1^\pi = 1.$$

Considering the domain of the operator $T$, where $u(1) = u(\pi) = 1$, then the equality

$$\left\{\frac{\{u^*(\pi)\}^{2\pi^2\ln v(\pi)}}{\{u^*(1)\}^{2\ln v(1)}}\right\} = 1 \tag{3.14}$$

is obtained. Therefore, in order for equality (3.14) to hold for all $u \in D(T)$, it must be that

$$v(1) = 1,\ \ v(\pi) = 1.$$

This implies that the domain of the multiplicative adjoint operator $\widetilde{T}$ is

$$D\left(\widetilde{T}\right) = \left\{v : v \in C^{2,*}[0,\pi];\ v(1) = 1, v(\pi) = 1\right\}.$$

Consequently, since $u$ and $v$ are arbitrary variables, the multiplicative adjoint of given multiplicative boundary value problem takes the form

$$\{y^{**}\}^{2x^2}\{y^*\}^{x}y^{-5} = 1,$$
$$y(0) = y(\pi) = 1.$$

3.2. **Multiplicative Fredholm Alternative.** In this section, a criterion will be provided for determining whether a solution exists for a non-homogeneous multiplicative boundary value problem.

**Theorem 3.15.** *(Multiplicative Fredholm Alternative) Let $T$ and $B$ denote a multiplicative operator and a set of boundary conditions, respectively. Then, the necessary condition for the non-homogeneous multiplicative boundary value problem*

$$T[y](x) = h(x), \; x \in (a,b), \; h(x) > 0$$
$$B[y] = 1$$

*to have a solution is that*

$$\int_a^b \{h(x) \odot z(x)\}^{dx} = 1 \tag{3.15}$$

*for every solution $z$ of the homogeneous multiplicative adjoint boundary value problem*

$$\widetilde{T}[z](x) = 1, \; x \in (a,b)$$
$$\widetilde{B}[z] = 1.$$

*Proof.* Let us consider in turn the following non-homogeneous multiplicative boundary value problem and its multiplicative adjoint boundary value problem

$$T[y] = h, \; B[y] = 1, \tag{3.16}$$

$$\widetilde{T}[y] = 1, \; \widetilde{B}[y] = 1, \tag{3.17}$$

respectively.

If $u$ is a solution to the multiplicative boundary value problem (3.16), then equality

$$\langle T[u], v\rangle_* = \langle h, v\rangle_*$$

is satisfied for any function $v$.

Similarly, if $v$ is a solution to the multiplicative adjoint boundary value problem (3.17), then

$$\left\langle u, \widetilde{T}[v]\right\rangle_* = \langle u, 1\rangle_* = 1$$

is satisfied for any function $u$.

According to the last two equalities, the multiplicative Green's formula (3.8) is taken into account and

$$\langle T[u], v\rangle_* = \langle T[u], v\rangle_*$$

or

$$\langle h, v\rangle_* = 1 \tag{3.18}$$

is obtained. That is, equality (3.15) holds.

As a result, since $u$ and $v$ are arbitrary variables, the proof is completed. □

**Corollary 3.16.** A necessary condition for the non-homogeneous multiplicative boundary value problem (3.16) to have a solution is that equation (3.18) holds for all solutions $v$ of the homogeneous multiplicative adjoint boundary value problem (3.17). That is, the function $h$ must be multiplicatively orthogonal to all solutions of the homogeneous multiplicative adjoint boundary value problem (3.17) within the problem's domain.

*Remark* 3.17. If the homogeneous multiplicative adjoint boundary value problem (3.17) has only the multiplicative trivial solution, then for any continuous arbitrary function $h > 0$, the non-homogeneous multiplicative boundary value problem (3.16) has always a solution. In this case, if the boundary conditions are multiplicatively separable or periodic, the solution is also unique.

**Example 3.18.** Determine the conditions that the function $h$ must satisfy for the non-homogeneous multiplicative boundary value problem

$$\begin{gathered} \{y^{**}\}\{y^{*}\}^{6}y^{10} = h\left(x\right), \\ y(0) = y(\pi) = 1 \end{gathered} \tag{3.19}$$

to have a solution.

From Example 3.2, the homogeneous multiplicative adjoint boundary value problem corresponding to (3.19) is

$$\begin{gathered} \{y^{**}\}\{y^{*}\}^{-6}y^{10} = 1, \\ y(0) = y(\pi) = 1. \end{gathered} \tag{3.20}$$

Therefore, the general solution of the problem (3.20) is computed as

$$y = e^{e^{3x}(c_1 \cos x + c_2 \sin x)}$$

with the help of techniques in [28].

Considering the boundary conditions in (3.20),

$$y\left(0\right) = e^{c_1},\ y\left(\pi\right) = e^{-c_1 e^{3\pi}} = 1$$

is obtained. Thus, $c_1 = 0$, and the solution of the problem (3.20) is

$$y = e^{ke^{3x}\sin x}$$

for any arbitrary constant $c_2 = k$.

As a result, for the non-homogeneous multiplicative boundary value problem (3.19) to have a solution, the condition

$$\int_0^{\pi} \left\{ h\left(x\right) \odot e^{ke^{3x}\sin x} \right\}^{dx} = \int_0^{\pi} \left\{ h(x)^{ke^{3x}\sin x} \right\}^{dx} = 1$$

should be held according to the multiplicative Fredholm alternative.

*Remark* 3.19. If we note that $h\left(x\right) = e^{e^{-3x}\cos x}$ in Example 3.18 then

$$\int_0^{\pi} \left\{ e^{\frac{\cos 2x}{2}} \right\}^{dx} = 1$$

or

$$\int_0^{\pi} \frac{\cos 2x}{2} dx = 0$$

holds, and the condition (3.15) is satisfied. That is, for this choice of $h$, there is a solution of the problem (3.19). However, the solution is not unique because the solution of the problem (3.20) according to the problem (3.19) depends on a parameter $k$.

In fact, for $h(x) = e^{e^{-3x}\cos x}$ in Example 3.18, the solution of the problem (3.19)

$$y = e^{ke^{-3x}\sin x + \frac{x}{2}e^{-3x}\sin x}$$

with the help of techniques in [29].

**Example 3.20.** Determine the conditions that the function $h$ must satisfy for the non-homogeneous multiplicative boundary value problem

$$\begin{gathered} \{y^{**}\}^{2x^2}\{y^{*}\}^{7x}y^{-2} = h(x), \\ y(1) = y(\pi) = 1 \end{gathered} \tag{3.21}$$

to have a solution.

From Example 3.3, the homogeneous multiplicative adjoint boundary value problem corresponding to (3.21) is

$$\begin{gathered} \{y^{**}\}^{2x^2}\{y^{*}\}^{x}y^{-5} = 1, \\ y(1) = y(\pi) = 1. \end{gathered} \tag{3.22}$$

The multiplicative differential equation in (3.22) is a multiplicative Cauchy-Euler equation and its general solution is

$$y = e^{c_1 x^{\frac{1-\sqrt{41}}{4}} + c_2 x^{\frac{1+\sqrt{41}}{4}}}$$

with the help of techniques in [28].

Considering the boundary conditions in (3.22), $c_1 = c_2 = 0$ and the multiplicative trivial solution $y = 1$ is obtained. According to Remark 3.17, since the problem (3.22) has only the trivial solution, for any continuous arbitrary function $h > 0$, the problem (3.21) has always a solution.

## 4. Conclusion

We consider non-homogeneous boundary value problems that are redefined with multiplicative calculus techniques. The concepts of adjoint operator, self-adjoint operator, Lagrange identity, Green's formula given in the multiplicative sense for this problem are expressed, and the detailed examples are provided to emphasize their importance. In conclusion, a criterion is provided for determining whether a solution exists for a non-homogeneous multiplicative boundary value problem, and also some examples are given.

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# GRÖBNER-SHIRSHOV BASES AND NORMAL FORMS FOR THE INFINITE COXETER GROUPS OF TYPES $\widetilde{B}_n$ AND $\widetilde{D}_n$

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Abstract. In this paper, we will investigate the infinite Coxeter groups $\widetilde{B}_n$ and $\widetilde{D}_n$. Their Gröbner-Shirshov bases and classifications of normal forms are achieved by leveraging results from the infinite Coxeter groups of types $\widetilde{C}_n$. Additionally, new algorithms are presented for obtaining normal forms of elements within these groups.

## 1. Introduction

To begin with, we revisit certain ideas related to the Gröbner-Shirshov basis theory. Let $S$ represent a set, and $S^*$ denote the free monoid of strings formed by $S$. We refer to the empty string as $e$. A well-ordering $<$ on $S^*$ is referred to as a monomial order if $x < y$ implies $axb < ayb$ for all $a, b \in S^*$. Let $\langle S \rangle$ denote the free associative algebra generated by $S$ over a field $k$. Given $0 \neq f \in \langle S \rangle$, we denote by $\overline{f}$ the leading word of $f$ concerning a specified monomial order. For two monic polynomials $f$ and $g$, $f\langle S \rangle$ if there exists a word $w$ such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in S^*$. The intersection composition of $f$ and $g$ is defined by $\langle f, g \rangle_w = fb - ag$. If $\overline{f} = a\overline{g}b$ for some $a, b \in S^*$, the inclusion composition is defined as $\langle f, g \rangle = f - agb$. In this scenario, the transformation $f \to f - agb$ is known as the elimination of the leading word (ELW) of $f$ in $g$. Let $R \subseteq \langle S \rangle$ be a collection of monic polynomials, and let $f$ be another monic polynomial. We say that $f$ is reduced to $h$ modulo $R$ if $f$ is derived from a sequence of ELWs involving elements of $R$, and no further ELWs of $r$ are possible. A set $R \subseteq \langle S \rangle$ is termed a Gröbner-Shirshov basis, denoted by GSB if every composition of polynomials from $R$ is reduced to zero modulo $R$. A GSB $R$ is considered minimal if there are no inclusion compositions within $R$. If $R \subseteq \langle S \rangle$ is not a GSB, take a composition of intersections of polynomials from $R$ and reduce it modulo $R$. If this reduction results in a non-zero polynomial $r$, add $r$ to the set $R$. Continue this process for each composition of polynomials from $R$ until no further enlargements are required. The final set obtained will be a GSB.

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This procedure is referred to as the Shirshov algorithm. The Composition Diamond Lemma ([13]) is valuable for finding the normal form of a group through its GSB.

When a group $G$ is defined by generators $S$ and relations $R$, each relation $x = y$ in $R$ can be associated with a polynomial $x - y$. Thus, the set of relations can be viewed as a subset of $\urcorner\langle S\rangle$. Consequently, a GSB of $R$, referred to as a GSB for the group $G$, can be found. It's worth noting that $R$ comprises "biwords," essentially differences of words. The Shirshov algorithm maintains this property throughout the computation. Therefore, a GSB of a group can be considered a unique set of relations for that group. Furthermore, the set

$$Red(R) = \{w \in S^* | w \neq x\bar{s}y, x, y \in S^*, s \in R\}$$

constitutes the set of all normal forms of $G$, as established by the Composition Diamond lemma.

Coxeter groups, known as Weyl groups, represent one of the most significant examples of groups defined by generators and defining relations. Consequently, the pursuit of finding GSB for these groups has attracted considerable attention from researchers. GSB for finite Coxeter groups can be found in [1]. For the finite exceptional Coxeter group of type $E_8$, a GSB has been established in [3], while for the finite exceptional Coxeter groups of type $E_6$ and $E_7$, a GSB can be found in [4]. The method of GSB bases introduces a new algorithm for deriving normal forms of elements in groups, monoids, and semigroups, providing a fresh approach to solving the word problem in these algebraic structures. The word problem for a finitely generated group $G$ involves the algorithmic challenge of determining whether two words formed by the generators represent the same element. A novel algorithm for obtaining normal forms and addressing the word problem for Extended Modular, Extended Hecke, and Picard groups through their GSB is explored in [5]. Comparable findings for the singular part of the Brauer semigroup and braid groups via the complex reflection group $G_{12}$ are presented in [11] and [14], respectively. In [6], the authors establish a connection between graph theory and GSB of groups. This article aims to pave the way for further research in this area. GSB for infinite Coxeter groups of type $\widetilde{A}_n$, $\widetilde{C}_n$, as well as for finite Coxeter groups of type $A_n$, $B_n$, and $D_n$, have been obtained in [8], [15], and [12], respectively. Additionally, for the infinite exceptional Weyl group of type $F_4$, a GSB has been constructed in [10]. The author worked on GSB bases for infinite Coxeter group of type $\widetilde{A}_n$ in [7] and the results in this article were obtained from [13].

The primary objective of this article is to derive GSB and normal forms for infinite Coxeter groups of types $\widetilde{B}_n$ and $\widetilde{D}_n$.

## 2. Gröbner-Shirshov Bases

This section focuses on the discussion of GSB for the infinite Coxeter groups of Types $\widetilde{B}_n$ and $\widetilde{D}_n$.

### 2.1. GSB for $\widetilde{B}_n$.

**Definition 2.1.** The presentation of the infinite Coxeter group of type $\widetilde{B}_n$ includes generators $S = \{s_0, s_1, \ldots, s_n\}$ for a positive integer $n \geq 2$ and the following defining relations:

$(RB_1)$ $s_a s_a = e$ for $0 \leq a \leq n$,

$(RB_2)$ $s_a s_b = s_b s_a$ for $0 \leq a < b - 1 < n$ but $(a, b) \neq (0, 2)$,

$(RB_3)$ $s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1}$ for $1 \leq a < n-1$,
$(RB_4)$ $s_0 s_1 = s_1 s_0$,
$(RB_5)$ $s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}$,
$(RB_6)$ $s_0 s_2 s_0 = s_2 s_0 s_2$.

where $e$ represents the identity element of the group.

For the sake of convenience, let us assume that

$$s_{ab} = \begin{cases} s_a s_{a+1} \cdots s_b, & \text{if } 1 \leq a \leq b < n\ ; \\ s_a s_{a+1} \cdots s_n s_{n-1} \cdots s_c, & \text{if } 1 \leq a \leq c = 2n - j \leq n; \\ e, & \text{if } 0 \leq b = a - 1 < n. \end{cases}$$

and

$$s_{ab}^{-1} = \begin{cases} s_b s_{b-1} \cdots s_a, & \text{if } 1 \leq a \leq b < n\ ; \\ s_c s_{c+1} \cdots s_n s_{n-1} \cdots s_a, & \text{if } 1 \leq a \leq c = 2n - j \leq n; \\ e, & \text{if } 0 \leq b = a - 1 < n. \end{cases}$$

It is important to note that $s_{ab}^{-1}$ is, in fact, the inverse of $s_{ab}$ since $s_a s_a = e$ for each $a$.

**Lemma 2.2.** *Assume that $<$ denotes the degree lexicographic order on $S^*$. A GSB for the infinite Coxeter group of type $\widetilde{B}_n$ with respect to $<$ includes the following polynomials:*

- $f_1^{(a)} = s_a s_a - 1$ if $0 \leq a \leq n$,
- $f_2^{(a,b)} = s_a s_b - s_b s_a$ if $0 \leq a < b - 1 < n$ but $(a, b) \neq (0, 2)$,
- $f_3^{(a,b)} = s_{ab} s_a - s_{a+1} s_{ab}$ if $1 \leq a \leq n-2$ and $a < b < 2n - a - 1$,
- $f_4^{(a)} = s_{a,2n-a} s_{a+1} - s_{a+1} s_{a,2n-a}$ if $1 \leq a \leq n-1$,
- $f_5^{(a)} = s_0 s_{2a} s_{1a} - s_1 s_0 s_{2a} s_{1,a-1}$ if $1 \leq a \leq n-1$,
- $f_6^{(a)} = s_0 s_{2,2n-a} s_{1,2n-a+1} - s_1 s_0 s_{2,2n-a} s_{1,2n-a}$ if $2 \leq a \leq n$,
- $f_7^{(a,b)} = s_0 s_{2a} s_{1b} s_0 - s_2 s_0 s_{2a} s_{1b}$ if $2 \leq a \leq 2n-3$ and $0 \leq b \leq 1$,
- $f_8^{(a,b)} = s_0 s_{2a} s_{1b} s_0 s_{2b} - s_2 s_0 s_{2a} s_{1b} s_0 s_{2,a-1}$ if $2 \leq b < a \leq n$,
- $f_9^{(a,b)} = s_0 s_{2,2n-a} s_{1b} s_0 s_{2b} - s_2 s_0 s_{2,2n-a} s_{1a} s_0 s_{2,b-1}$ if $3 \leq a \leq n-1$ and $2 \leq b \leq n-1$,
- $f_{10}^{(a,b)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2b} s_1 - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2b}$ if $1 \leq a \leq 2$ and $2 \leq b \leq 2n-3$,
- $f_{11}^{(a,b)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2b} s_{1,a-1} - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2b} s_{1,a-2}$ if $3 \leq a \leq n-1$, $3 \leq b \leq n$ and $a \leq b$,
- $f_{12} = s_0 s_{2,2n-2} s_0 s_2 - s_2 s_0 s_{2,2n-2} s_0$,
- $f_{13}^{(a,b)} = s_0 s_{2,2n-2} s_{1a} s_0 s_{2,2n-b} s_{1a} - s_2 s_0 s_{2,2n-2} s_{1a} s_0 s_{2,2n-b} s_{1,a-1}$ if $2 \leq b \leq a \leq n-1$,
- $f_{14}^{(a,b)} = s_0 s_{2,2n-2} s_{1,2n-a-1} s_0 s_{2,2n-b} s_{1,2n-a}$ $- s_2 s_0 s_{2,2n-2} s_{1,2n-a-1} s_0 s_{2,2n-b} s_{1,2n-a-1}$ if $2 \leq b \leq a \leq n-1$,
- $f_{15}^{(a,b)} = s_0 s_{2,2n-a} s_{1,2n-b-1} s_0 s_{2,2n-b}$ $- s_2 s_0 s_{2,2n-a} s_{1,2n-b-1} s_0 s_{2,2n-b-1}$ if $2 \leq a - 1 \leq b \leq n-1$,
- $f_{16} = s_0 s_{2,2n-2} s_1 s_0 s_{2,2n-2} s_1 s_2 - s_2 s_0 s_{2,2n-2} s_1 s_0 s_0 s_{2,2n-2} s_1$,
- $f_{17}^{(a,b)} = s_0 s_{2,2n-2} s_{1b} s_0 s_{2,2n-a} s_{1,b-1} - s_2 s_0 s_{2,2n-2} s_{1b} s_0 s_{2,2n-a} s_{1,b-2}$ if $3 \leq b < a \leq n-1$.

*Proof.* The proof is conducted using the Shirshov algorithm.
$< f_{12}, f_5^{(2)} >= f_{10}^{(1,2)} - s_2s_0s_{2,2n-2}f_5^{(1)}s_2$,
$< f_{10}^{(1,b)}, f_7^{(b,1)} >= f_{10}^{(2,b)} - s_2s_0s_{2,2n-2}s_1f_7^{(b,0)}$ if $2 \le b \le n$,

Similarly, other elements can also be found.For a detailed proof, you can refer to the thesis [13] □

Let $R^B$ denote the set of polynomials as outlined in Lemma 2.2. Currently, we are unable to demonstrate that the provided polynomials in the lemma form a GSB for the infinite Coxeter group of type $\widetilde{B}_n$. Verifying this would involve intricate computations to confirm that the remaining compositions in $R^B$ reduce to zero modulo $R^B$. Instead, we will utilize the Composition Diamond lemma to establish that $R$ serves as a GSB for the infinite Coxeter group of type $\widetilde{B}_n$.

## 2.2. GSB for $\widetilde{D}_n$.

**Definition 2.3.** The presentation of the infinite Coxeter group of type $\widetilde{D}_n$ includes generators $S = \{s_0, s_1, \dots, s_n\}$ for a positive integer $n \ge 4$ and the following defining relations:

$(RD_1)$ $s_as_a = 1$ for $0 \le a \le n$,
$(RD_2)$ $s_as_b = s_bs_a$ for $0 < a < b-1 < n$ but $(a,b) \ne (0,2)$ and $(a,b) \ne (n-2,n)$,
$(RD_3)$ $s_as_{a+1}s_a = s_{a+1}s_as_{a+1}$ where $1 \le a < n-1$,
$(RD_4)$ $s_{n-2}s_ns_{n-2} = s_ns_{n-2}s_n$,
$(RD_5)$ $s_0s_2s_0 = s_2s_0s_2$.

For the sake of convenience, let us assume that

$$s_{ij} = \begin{cases} s_as_{a+1}\cdots s_b, & \text{if } 1 \le a < b < n \text{ ;} \\ s_as_{a+1}\cdots s_{n-2}s_ns_{n-1}\cdots s_{2n-b}, & \text{if } 1 \le a \le n-1 < b \le 2n-a; \\ s_a, & \text{if } b = a; \\ 1, & \text{if } b = a-1. \end{cases}$$

From this point forward, we will refrain from using superscripts unless it becomes necessary to distinguish between the groups $\widetilde{B}_n$ and $\widetilde{D}_n$.

**Lemma 2.4.** *Assume that $<$ denotes the degree lexicographic order on $S^*$. A GSB for the infinite Coxeter group of type $\widetilde{D}_n$ with respect to $<$ includes the following polynomials:*

- $g_1^{(a)} = s_as_a - 1$ if $0 \le a \le n$,
- $g_2^{(a,b)} = s_as_b - s_bs_a$ if $1 < b - a$ but $(a,b) \ne (0,2)$ and $(a,b) \ne (n-2,n)$,
- $g_3^{(a)} = s_{a,a+1} - s_{a+1}s_a$ if $a = 0, n-1$,
- $g_4 = s_{n-2,n}s_{n-2} - s_ns_{n-2,n}$,
- $g_5^{(a,b)} = s_{ab}s_a - s_{a+1}s_{ab}$ if $(1 \le a < b \le n-1)$ or $(1 \le a < n-2$ and $n \le b \le 2n-3$ and $2n-b-1 > 1)$,
- $g_6^{(a)} = s_{a,2n-a}s_{a+1} - s_{a+1}s_{a,2n-a}$ if $1 \le a \le n-3$,
- $g_7 = s_{n-2,n+2}s_n - s_{n-1}s_{n-2,n+2}$,
- $g_8 = s_{n-2,n+2}s_{n-1} - s_ns_{n-2,n+2}$,
- $g_9^{(a,b)} = s_0s_{2a}s_{1b}s_0 - s_2s_0s_{2a}s_{1b}$ if $0 \le b \le 1$ and $2 \le a \le 2n-3$,
- $g_{10}^{(a)} = s_0s_{2a}s_{1a} - s_1s_0s_{2a}s_{1,a-1}$ if $2 \le a \le n-1$,

- $g_{11} = s_0s_{2n}s_{1n} - s_1s_0s_{2n}s_{1,n-2}$,
- $g_{12} = s_0s_{2,n-1}s_{1,n+1} - s_1s_0s_{2,n-1}s_{1n}$,
- $g_{13}^{(a)} = s_0s_{2,2n-a}s_{1,2n-a+1} - s_1s_0s_{2,2n-a}s_{1,2n-a}$ if $2 \le a < n$,
- $g_{14}^{(a,b)} = s_0s_{2a}s_{1b}s_0s_{2b} - s_2s_0s_{2a}s_{1b}s_0s_{2,b-1}$ if $(2 \le b \le n-1$ and $n \le a \le 2n-3)$ or $(2 \le b < n-1$ and $3 \le a \le n-1$ and $b < a)$,
- $g_{15} = s_0s_{2,n-1}s_{1n}s_0s_{2n} - s_2s_0s_{2,n-1}s_{1n}s_0s_{2,n-2}$,
- $g_{16}^{(a,b)} = s_0s_{2,2n-2}s_{1a}s_0s_{2b}s_{1,a-1} - s_2s_0s_{2,2n-2}s_{1a}s_0s_{2b}s_{1,a-2}$ if $2 \le a \le b \le n-1$,
- $g_{17} = s_0s_{2,2n-2}s_0s_2 - s_2s_0s_{2,2n-2}s_0$,
- $g_{18}^{(a,b)} = s_0s_{2,2n-2}s_{1a}s_0s_{2b}s_1 - s_2s_0s_{2,2n-2}s_{1a}s_0s_{2b}$ if $(a = 1$ and $2 \le b \le n-1)$ or $(1 \le a \le 2$ and $n \le b \le 2n-3)$,
- $g_{19}^{(a)} = s_0s_{2,2n-a}s_{1,n-1}s_0s_{2,n+1} - s_2s_0s_{2,2n-a}s_{1,n-1}s_0s_{2n}$ if $3 \le a \le n$,
- $g_{20}^{(a)} = s_0s_{2,2n-a}s_{1n}s_0s_{2n} - s_2s_0s_{2,2n-a}s_{1n}s_0s_{2,n-2}$ if $3 \le a \le n-1$,
- $g_{21}^{(a)} = s_0s_{2,2n-2}s_{1a}s_0s_{2,2n-2}s_{12} - s_2s_0s_{2,2n-2}s_{1a}s_0s_{2,2n-2}s_1$ if $1 \le a \le 2$,
- $g_{22} = s_0s_{2,2n-2}s_{1n}s_0s_{2n}s_{1,n-2} - s_2s_0s_{2,2n-2}s_{1n}s_0s_{2n}s_{1,n-3}$,
- $g_{23}^{(a)} = s_0s_{2,2n-2}s_{1,n-1}s_0s_{2,2n-a}s_{1n} - s_2s_0s_{2,2n-2}s_{1,n-1}s_0s_{2,2n-a}s_{1,n-2}$ if $2 \le a \le n-1$,
- $g_{24}^{(a)} = s_0s_{2,2n-2}s_{1n}s_0s_{2,2n-a}s_{1,n-1} - s_2s_0s_{2,2n-2}s_{1n}s_0s_{2,2n-a}s_{1,n-2}$ if $2 \le a \le n-1$,
- $g_{25}^{(a,b)} = s_0s_{2,2n-2}s_{1,2n-a}s_0s_{2,2n-b}s_{1,2n-a+1}$
  $- s_2s_0s_{2,2n-2}s_{1,2n-a}s_0s_{2,2n-b}s_{1,2n-a}$ if $2 \le b < a \le n$,
- $g_{26}^{(a,b)} = s_0s_{2,2n-a}s_{1,2n-b}s_0s_{2,2n-b+1} - s_2s_0s_{2,2n-a}s_{1,2n-b}s_0s_{2,2n-b}$ if $3 \le a \le b \le n-1$,
- $g_{27}^{(a,b,c)} = s_0s_{2,2n-2}s_{1a}s_0s_{2,2n-b}s_{1c} - s_2s_0s_{2,2n-2}s_{1a}s_0s_{2,2n-b}s_{1,c-1}$ if $(c = a$ and $2 \le b \le n-2$ and $3 \le a \le n-2)$ or $(c = a-1$ and $3 \le a \le n-2$ and $a < b \le n)$,

*Proof.* As in the case of $\widetilde{B}_n$, the proof is established using the Shirshov algorithm. For a detailed proof, you can refer to the thesis [13].

□

At this stage, we are unable to demonstrate that the polynomials provided in the lemma form GSB for the infinite Coxeter group of type $\widetilde{D}_n$. We will demonstrate that the set of polynomials found for $\widetilde{B}_n$ and $\widetilde{D}_n$ indeed forms GSB for the infinite Coxeter group of types $\widetilde{B}_n$ and $\widetilde{D}_n$ by examining their normal forms, respectively.

## 3. Normal Forms

The necessary definitions and properties for the normal forms of $\widetilde{C}_n$ are provided in [13] and [15].

3.1. **Normal Forms for $\widetilde{B}_n$.** For $v \in \widetilde{S_n^C}$, let us define $v[a,b] = |\{t \in \mathbb{Z} : t \le a, v(t) \ge b\}|$ for all $a, b \in \mathbb{Z}$. Now, consider $\widetilde{S}_n^B = \{u \in \widetilde{S}_n^C : u[n, n+1] \equiv 0 \bmod 2\}$ which is a subgroup of $\widetilde{S}_n^C$ consisting of elements in the form $\{u \in S_n^C : u[n, n+1] \equiv 0 \bmod 2\}$. It is clear that $\widetilde{S}_n^B$ is a subgroup of $\widetilde{S}_n^C$ with an index of 2. Moreover, for any $u \in \widetilde{S}_n^B$, we can represent it as $u = (s_{nb_n}^C s_{n-1,b_{n-1}}^C \cdots s_{1b_1}^C)(s_0^C s_{1,2n-1}^C)^{\alpha_{2n-1}} \cdots (s_0^C s_1^C)^{\alpha_1}(s_0^C)^{\alpha_0}$ where $\sum_{t=0}^{2n-1} \alpha_t$ is an even number. The following proposition affirms that $\widetilde{S}_n^B$ is indeed the infinite Coxeter group of type $\widetilde{B}_n$.

**Proposition 3.1.** ([2], Proposition 8.5.3)
The group $\widetilde{S}_n^B$ with generating set $\{s_0^B, s_1^B, \ldots, s_n^B\}$ is the infinite Coxeter group of type $\widetilde{B}_n$ where $s_a^B = s_i^C$ for $a = 1, 2, \ldots, n$ and $s_0^B = [2n-1, 2n, 3, \ldots, n]$.

First of all, we give some relations between words in $\widetilde{B}_n$ and words in $\widetilde{C}_n$.

**Lemma 3.2.** *The following statements are equivalent.*
(i) $s_0^C s_1^C s_0^C = s_0^B$,
(ii) $(s_0^C s_{1a}^C)(s_0^C s_{1b}^C) = s_0^B s_{2a}^B s_{1b}^B$ *for* $0 \le a \leqslant b \le 2n-2$.

*Proof.* (i) $s_0^C s_1^C s_0^C = [2n, 2, \ldots, n][2, 1, 3, \ldots, n][2n, 2, \ldots, n] = s_0^B$.
(ii) $s_0^B s_{2a}^B s_{1b}^B = s_0^C s_1^C s_0^C s_{2a}^C s_{1b}^C = s_0^C s_{1a}^C s_0^C s_{1b}^C$ by a series of ELW in $f_2^{(0,c)}$. □

It's worth mentioning that the length of a word in $\widetilde{C}_n$ is two greater than the length of the corresponding word in $\widetilde{B}_n$.

**Lemma 3.3.** *In the context of the infinite Coxeter group of type $\widetilde{C}_n$, the following relation is valid:*

$$(s_0^C s_{1,2n-2}^C)(s_0^C s_{1b}^C)(s_0^C s_{1a}^C) = \begin{cases} (s_0^C s_{1,2n-1}^C)(s_0^C s_{1a}^C)(s_0^C s_{1,b-1}^C), & \text{if } a+b < 2n, \\ (s_0^C s_{1,2n-1}^C)(s_0^C s_{1,a-1}^C)(s_0^C s_{1b}^C), & \text{if } a+b \ge 2n. \end{cases}$$

*This equation is applicable for* $1 \le a, b \le 2n-1$ *with the condition that* $b \le a$ *when* $a < n$ *or* $a < b$ *when* $a \ge n$.

*Proof.* In the scenario where $a + b < 2n$, there are two distinct cases to consider:
(i) $1 \le b \le a < n$,
(ii) $1 \le b < n \le a < 2n - b$.

In both of these cases, the following relationships hold:
$(s_0^C s_{1,2n-2}^C)(s_0^C s_{1b}^C)(s_0^C s_{1a}^C) = (s_0^C s_{1,2n-1}^C)(s_0^C s_{1b}^C)(s_0^C s_{1b-1}^C)(s_0^C s_{b+1,a}^C)$ applying by an ELW in $f_5^{(b)}$.
$(s_0^C s_{1,2n-1}^C)(s_0^C s_{1b}^C)(s_0^C s_{1b-1}^C)(s_0^C s_{b+1,a}^C) = (s_0^C s_{1,2n-1}^C)(s_0^C s_{1a}^C)(s_0^C s_{1,b-1}^C)$ applying by a series of ELW in $f_2$.
In the case where $2n \le a + b$, we have $n \le b < a \le 2n - 2$. Let $a = 2n - c$ and $j = 2n - d$. Therefore;

$$(s_0^C s_{1,2n-2}^C)(s_0^C s_{1b}^C)(s_0^C s_{1a}^C) = (s_0^C s_{1,2n-1}^C)(s_0^C s_{1b}^C)(s_0^C s_{1b}^C) s_{d-2} s_{d-3} \cdots s_c$$

due to an ELW in $f_6^{(b)}$. Furthermore; $(s_0^C s_{1a}^C) s_t = (s_0^c s_{1,t-1}^C) s_{t+1}^C s_{tb}^C$ by an ELW in $f_3^{(t,b)}$. $(s_0^c s_{1,t-1}^C) s_{t+1}^C s_{tb}^C = s_{t+1}^C s_{1b}^C$ by a series of ELW in $f_2$. This results in the desired equality. □

**Corollary 3.4.**

$$(s_0^C s_{1,2n-1}^C)(s_0^C s_{1a}^C)(s_0^C s_{1b}^C) = \begin{cases} (s_0^B s_{2,2n-2}^B s_{1,b+1}^B)(s_0^C s_{1a}^C), & a+b < 2n-1, \\ (s_0^B s_{2,2n-2}^B s_{1b}^B)(s_0^C s_{1,a+1}^C), & a+b \ge 2n-1. \end{cases}$$

This equation holds for $1 \le b \leqslant a \le 2n-2$.

**Lemma 3.5.** *Let $m \geq 1$.*

(i) $(s_0^C s_{1,2n-1}^C)^{2m} = (s_0^B s_{2,2n-2}^B s_1^B)^{2m}$,

(ii) $(s_0^C s_{1,2n-1}^C)^{2m-1}(s_0^C s_{1b}^C) = (s_0^B s_{2,2n-2}^B s_1^B)^{2m-1}(s_0^B s_{2b}^B)$ *for* $2 \leq b \leq 2n-2$,

(iii) $(s_0^C s_{1,2n-1}^C)^{2m-1} s_0^C = (s_0^B s_{2,2n-2}^B s_1^B)^{2(m-1)}(s_0^B s_{2,2n-2}^B)(s_0^B)$,

*Proof.*

(i) We will utilize induction with respect to $m$. $(s_0^B s_{2,2n-2}^B s_1^B)(s_0^B s_{2,2n-2}^B s_1^B) = (s_0^C s_{1,2n-2}^C s_0^C s_1^C)(s_0^C s_{1,2n-2}^C s_0^C s_1^C) = (s_0^C s_{1,2n-2}^C)(s_1^C s_0^C s_1^C s_0^C)(s_{2,2n-2}^C s_0^C s_1^C) = (s_0^C s_{1,2n-1}^C)(s_0^C s_{1,2n-2}^C)(s_0^C s_0^C s_1^C) = (s_0^C s_{1,2n-1}^C)^2$. The first equality is derived from Lemma 3.2, and the second and the third equalities stem from ELW in $f_5^{(1)}$ and $f_2^{(0,c)}$, respectively. Assume that $(s_0^B s_{2,2n-2}^B s_1^B)^{2c} = (s_0^C s_{1,2n-1}^C)^{2c}$ for a positive integer $c$. Consequently, $(s_0^B s_{2,2n-2}^B s_1^B)^{2(c+1)} = (s_0^C s_{1,2n-1}^C)^{2c}(s_0^B s_{2,2n-2}^B s_1^B)^2 = (s_0^C s_{1,2n-1}^C)^{2(c+1)}$.

(ii) $(s_0^B s_{2,2n-2}^B s_1^B)^{2m+1}(s_0^B s_{2b}^B) = (s_0^C s_{1,2n-1}^C)^{2m}(s_0^C s_{1,2n-2}^C s_0^C s_1^C)(s_0^C s_{1b}^C s_0^C)$ by Lemma 3.2.
$(s_0^C s_{1,2n-1}^C)^{2m}(s_0^C s_{1,2n-2}^C s_0^C s_1^C)(s_0^C s_{1b}^C s_0^C) = (s_0^C s_{1,2n-1}^C)^{2m} s_0^C s_{1,2n-2}^C s_1^C s_0^C s_1^C s_0^C s_{2b}^C s_0^C$ by ELW in $f_5^{(1)}$.
$(s_0^C s_{1,2n-1}^C)^{2m} s_0^C s_{1,2n-2}^C s_1^C s_0^C s_1^C s_0^C s_{2b}^C s_0^C = (s_0^C s_{1,2n-1}^C)^{2m+1} s_0^C s_{1b}^C s_0^C s_0^C$ by a series of ELW in $f_2^{(0,c)}$.
$(s_0^C s_{1,2n-1}^C)^{2m+1} s_0^C s_{1b}^C s_0^C s_0^C = (s_0^C s_{1,2n-1}^C)^{2m+1} s_0^C s_{1b}^C$ by ELW in $f_1^{(0)}$.

(iii) $(s_0^B s_{2,2n-1}^B)(s_0^B) = (s_0^C s_{1,2n-2}^C s_0^C)(s_0^C s_1^C s_0^C)$ by Lemma 3.2.
$(s_0^C s_{1,2n-2}^C s_0^C)(s_0^C s_1^C s_0^C) = s_0^C s_{1,2n-1}^C s_0^C$.

The remaining part follows as a straightforward consequence of part (i). □

It should be noted that the length of word in $\widetilde{C}_n$ is $2m$ greater than the length of the corresponding word in $\widetilde{B}_n$.

**Definition 3.6.** The following words are defined in $\widetilde{B}_n$ :

(i) $w_0 = s_{nd_n}^B \cdots s_{ad_a}^B \cdots s_{1d_1}^B$ for $a-1 \leq d_a \leq 2n-a$ and $a = 1, \ldots, n$.

(ii) $w_1 = \prod_{i=1}^{t}(s_0^B s_{2,2n-2}^B s_{1a_i}^B)$ for $t \geq 0$ and $1 \leq a_i \lessdot a_{i-1} \leq 2n-2$.

(iii) $w_2 = \prod_{i=1}^{s}(s_0^B s_{2,b_{2i-1}}^B s_{1b_{2i}}^B)$ for $s \geq 0$ and $0 \leq b_i \lessdot b_{i-1} \leq 2n-3$.

(iv) $$w_3 = \begin{cases} (s_0^B s_{2n-2}^B s_1^B)^{2m}, \\ (s_0^B s_{2n-2}^B s_1^B)^{2m-1}(s_0^B s_{2b}^B), & \text{for } m \geq 0 \text{ and } 1 \leq b \leq 2n-2, \\ (s_0^B s_{2n-2}^B s_1^B)^{2(m-1)}(s_0^B s_{2,2n-2}^B) s_0^B. \end{cases}$$

(v) $w_4 = w_0 w_1 w_2$ where $a_t \geq 2$ and either $b_1 \lessdot a_t$ or $b_1 \not\lessdot a_t$ but
$$\begin{cases} b_2 \lessdot a_t, & a_t + b_1 \geq 2n; \\ b_2 + 1 < a_t, & a_t + b_1 < 2n. \end{cases},$$

(vi) $w_5 = w_0 w_1 w_3$.

Let $W_B = \{w_4, w_5\}$.

**Theorem 3.7.** *Any word $w \in W_C$ in which number of appearance of $s_0$ is even can be converted into a word in $W_B$.*

*Proof.* Since $s_a^B = s_a^C$ for $a = 1, \dots, n$, we focus on words of the form $w = (s_0^C s_{1,2n-1}^C)^m \prod_{i=1}^{t}(s_0^C s_{1b_i}^C)$ where $m+t$ is even $0 \le b_i \leqslant b_{i-1} \le 2n-2$.

If $m = 0$, then according to Lemma 3.2,we can write $w = \prod_{i=1}^{\frac{t}{2}}(s_0^B s_{2,b_{2i-1}}^B s_{b_{2i}}^B)$. As a result, $w$ belongs to $W_B$.

Suppose that $m \ge 1$ and $2n-2 = b_1 = b_2 = \cdots = b_d > b_{d+1}$. Then $w = \left(\prod_{i=1}^{\lfloor \frac{d+1}{2} \rfloor}(s_0^B s_{2,b_{2i-1}}^B s_{1,b_{2i}}^B)\right)w'$ where $w' = (s_0^C s_{1,2n-1}^C)^m \prod_{i=2\lfloor \frac{d+1}{2} \rfloor+1}^{t}(s_0^C s_{1b_i}^C)$ by repeated applications of Corollary 3.4 and Lemma 3.2. Let us rewrite $w'$ as follows, $w' = (s_0^C s_{1,2n-1}^C)^m (s_0^C s_{1a}^C)\prod_{i=0}^{p}(s_0^C s_{1a_i}^C)$. Assume that $a+i+a_i \ge 2n-1$ for $0 \le i \le q \le p$ and $a+q+1+a_i < 2n-1$ for $q+1 \le i \le p$. Let $x = (2n-2)-a$. Now we investigate each case separately. There are 6 cases.

Case (i): $q \ge x-1$ and $m > x$. Corollary 3.4 and Lemma 3.2 imply that $w' = \prod_{i=0}^{x}(s_0^B s_{2,2n-2}^B s_{1a_i}^B)w''$ where $w'' = (s_0^C s_{1,2n-1}^C)^{m-x}\prod_{i=x+1}^{p}(s_0^C s_{1,x_i}^C)$. Now same process can be applied to $w''$. This should be repeated until one of the conditions is not met. Therefore we can assume that $w'$ does not satisfy one of the conditions without loss of generality.

Case (ii): $q \ge x-1$ and $m = x$. Corollary 3.4 and Lemma 3.2 suggest that $w' = \prod_{i=0}^{m}(s_0^B s_{2,2n-2}^B s_{1a_i}^B)\prod_{i=\frac{m+2}{2}}^{\frac{p}{2}}(s_0^B s_{2,a_{2i-1}}^B s_{1,a_{2i}})$ because of $a_x \leqslant a_{x+1}$, $w' \in W_B$ and so is $w$.

Case (iii): $q \ge x-1$ and $m < x$. Corollary 3.4 and Lemma 3.2 suggest that $w' = \left(\prod_{i=0}^{m-1}(s_0^B s_{2,2n-2}^B s_{1a_i}^B)\right)(s_0^B s_{2,a+m}^B s_{1a_m}^B)\prod_{i=\frac{m+2}{2}}^{\frac{p}{2}}(s_0^B s_{2,a_{2i-1}}^B s_{1,a_{2i}})$. If $a+m \leqslant a_{m-1}$, then clearly $w' \in W_B$ which implies $w \in W_B$. Suppose $a+m \nleqslant a_{m-1}$. Since $a_{m-1}+m+a \ge 2n$ and $a_m \leqslant a_{m-1}$, $w' \in W_B$ and so is $w$.

Case (iv): $q < x-1$ and $m \le q$. Similar to the scenario in case (iii).

Case (v): $q < x-1$ and $q < m \le p$. $w'$ equals

$$\left(\prod_{i=0}^{q}(s_0^B s_{2,2n-2} s_{1a_i}^B)\right)\left(\prod_{i=q+1}^{m-1}(s_0^B s_{2,2n-2}^B s_{1a_i+1}^B)\right)(s_0^B s_{2,a+q+1}^B s_{1a_m}^B)\prod_{i=\frac{m+2}{2}}^{\frac{p}{2}}(s_0^B s_{2,a_{2i-1}}^B s_{1,a_{2i}})$$

by Corollary 3.4 and Lemma 3.2. We can observe that $a_q > a_{q+1}$. If $a+q+1 \leqslant a_{m-1}+1$, then it is evident that $w' \in W_B$ which consequently implies that $w \in W_B$. Now consider the scenario where $a+q+1 \nleqslant a_{m-1}+1$. In this case, $a_{m-1}+1 \le a+q+1$ and $a_m+a+q+1 < 2n-1$. It follows that $a_m < n$ and consequently $a_m+1 < a_{m-1}+1$. Therefore, we can conclude that $w' \in W_B$ and hence $w$ is also an element of $W_B$.

Case (vi): Applying Corollary 3.4 and Lemma 3.2 repeatedly provides the following, $w' = \left(\prod_{i=0}^{q}(s_0^B s_{2,2n-2} s_{1a_i}^B)\right)\left(\prod_{i=q+1}^{p}(s_0^B s_{2,2n-2}^B s_{1a_i+1}^B)\right)w''$ where

$$w'' = \begin{cases} (s_0^B s_{2,2n-2}^B s_1^B)^{m-p}, & a+q+1 = 2n-2 \\ (s_0^B s_{2,2n-2}^B s_1^B)^{m-p-1}(s_0^B s_{2,a+q+1}^B), & 1 \le a+q+1 \le 2n-3 \\ (s_0^B s_{2,2n-2}^B s_1^B)^{m-p-2}(s_0^B s_{2,2n-2}^B)(s_0^B), & a+q+1 = 0 \end{cases}$$

by Lemma 3.5. Thus, it is evident that $w^{'} \in W_B$ and consequently, $w$ is also an element of $W_B$.

□

**Lemma 3.8.** *The generating function for words in $W_B$ is given by the expression:*

$$\prod_{a=1}^{n}(1+y+\cdots+y^{2a-1})\frac{1+y^a}{1-y^{n+a}}.$$

*Proof.* We have established a one to one correspondence between words in $W_B$ and words in $W_C$ with the even number of occurrence of $s_0$. Consider a word in $W_C$ of the form $w = (s^C_{nd_n}s^C_{n-1,d_{n-1}}\cdots s^C_{1d_1})\prod_{i=1}^{t}(s^C_0 s^C_{1b_i})$ where $t$ is even and $0 \leq b_i \leqslant b_{i-1} \leq 2n-1$. Since $s^C_a = s^B_a$ for $a = 1, \ldots, n$, $s^C_{nd_n}s^C_{n-1,d_{n-1}}\cdots s^C_{1d_1} = s^B_{nd_n}s^B_{n-1,d_{n-1}}\cdots s^B_{1d_1}$, we can express this word in $W_B$ as $s^B_{nd_n}s^B_{n-1,d_{n-1}}\cdots s^B_{1d_1}$. The generating function for this form of word in $W_B$ is $\prod_{a=1}^{n}(1+y+\cdots+y^{2a-1})$. When converting the $\prod_{i=1}^{t}(s^C_0 s^C_{1b_i})$ part into a word in $W_B$, the corresponding word losses length by the number of occurrences of $s_0$. The generating function for the words in the form $\prod_{i=1}^{t}(s^C_0 s^C_{1b_i})$ where $t \geq 0$ in $W_C$ is $\prod_{a=1}^{n}\frac{1+y^a}{1-y^{n+a}}$. It is important to note we consider all words of the form $\prod_{i=1}^{t}(s^C_0 s^C_{1b_i})$ where $t \geq 0$, and we can add or remove $s^C_0$ from the end of the word if the number of occurrences of $s^C_0$ is odd, without affecting the result. □

Consider the generating function for the infinite Coxeter group of type $\widetilde{B}_n$

$$\prod_{a=1}^{n}\frac{1+y+\cdots+y^{2a-1}}{1-y^{2a-1}}.$$

Using Section 7.1 in [2] we can express this as:

$$\prod_{a=1}^{n}(1+y+\cdots+y^{2a-1})(\frac{1+y^a}{1-y^{n+a}}) = \prod_{a=1}^{n}\frac{1+y+\cdots+y^{2a-1}}{1-y^{2a-1}}$$

which corresponds to the generating function for words in $W_B$.

With t is understanding in place, we can now proceed to unveil the main result about a GSB for the infinite Coxeter group of type $\widetilde{B}_n$..

**Theorem 3.9.** *Let $R^B$ represent the set of all polynomials as described in Lemma 2.2. Then,*

(i) $W_B = Red(R^B)$.

(ii) $R^B$ *serves as a GSB for the infinite Coxeter group of type* $\widetilde{B}_n$.

*Proof.* (i) It is evident that any word in $W_B$ is $R^B$-reduced. Thus, we have $W_B \subseteq \text{Red}(R^B)$. Conversely, if $w \in \text{Red}(R^B)$, then $w$ can be expressed as a permutation in $\widetilde{S}^B_n$. According to Theorem 3.7, this permutation corresponds to a word in $W_B$. Consequently, we obtain $\text{Red}(R^B) \subseteq W_B$.

(ii) We know that any polynomial in $R^B$ is part of a GSB for the infinite Coxeter group of type $\widetilde{B}_n$. If $R^B$ were not a GSB, then, by the Composition Diamond lemma, $\text{Red}(R^B)$ should be a proper subset of the set of normal forms in the infinite Coxeter group of type $\widetilde{B}_n$. This would contradict the fact that $W_B$ and the normal forms of the infinite Coxeter group of type $\widetilde{B}_n$ share the same generating functions.

□

3.2. **Normal Forms for $\widetilde{D}_n$.** Define $\widetilde{S}_n^D$ as a subgroup of $\widetilde{S}_n^B$ consisting of those elements in $\widetilde{S}_n^B$ which, in their complete notation, exhibit an even number of negative entries to the right of 0. $\widetilde{S}_n^D = \{u \in \widetilde{S}_n^B \ : \ u[0,1] \equiv 0 \ (mod 2)\}$. Hence, it follows that $\widetilde{S}_n^D$ is a subgroup of $\widetilde{S}_n^B$ with an index of 2.

**Proposition 3.10.** ([2], Proposition 8.6.3)

The group $\widetilde{S}_n^D$ generated by $\{s_0^D, s_1^D, \ldots, s_n^D\}$, constitutes the infinite Coxeter group of type $\widetilde{D}_n$. In this group, $s_a^D = s_a^B$ for $a = 0, 1, 2, \ldots, n-1$ and $s_n^D = [(n-1 \ -n)]$.

Now, let's attempt to find normal form representations of elements in $\widetilde{D}_n$ with respect to these generators. First and foremost, we'll present some relations between words in $\widetilde{D}_n$ and words in $\widetilde{B}_n$.

**Lemma 3.11.** (i) $s_n^B s_{n-1}^B = s_n^D s_n^B$,
(ii) $s_n^B s_{n-1}^B s_n^B = s_n^D$,
(iii) $s_{n-1}^B s_n^B s_{n-1}^B = s_n^D s_{n-1}^D s_n^B$,
(iv) $s_n^B s_{n-1}^B s_n^B s_{n-1}^B = s_n^D s_{n-1}^D$.

*Proof.* (i) $s_n^B s_{n-1}^B = [(n \ -n)][(n-1 \ n)] = [(n-1 \ -n)][(n \ -n)] = s_n^D s_n^B$.
(ii) $s_n^B s_{n-1}^B s_n^B = s_n^D s_n^B s_n^B = s_n^D$ by part (i).
(iii) $s_{n-1}^B s_n^B s_{n-1}^B = s_n^D s_{n-1}^D s_n^B$ by applying part (i) and ELW in $g_3^{(n-1)}$, respectively.
(iv) $s_n^B s_{n-1}^B s_n^B s_{n-1}^B = s_n^D s_{n-1}^D$ by part (ii). □

**Lemma 3.12.** *For* $1 \leq a \leq n-2$

$$s_{ab_a}^B = \begin{cases} s_{ab_a}^D, & b_a < n; \\ s_{a,n-1}^D s_n^B, & b_a = n; \\ s_{ab_i}^D s_n^B, & b_a > n. \end{cases}$$

*Proof.* Since $s_a^B = s_a^D$ for $1 \leq a \leq n-1$, we also have $s_{ab_a}^B = s_{ab_a}^D$ for $b_a < n$. Similarly $s_{an}^B = s_{a,n-1}^D s_n^B$. Now, let us consider the case where $b_a > n$ and $a \leq n-2$. Then, if part (ii) of Lemma 3.11, ELW's in $f_2^{(i,n)}$ where $i = 2n - b_a, \ldots, n-2$ and ELW's in $f_4^{(n-1)}$ are applied, respectively, then $s_{ab_a}^D s_n^B = s_{ab_a}^B$ will be obtained. □

**Lemma 3.13.** *For* $1 \leq a \leq n-2$

$$s_n^B s_{ab_a}^B = \begin{cases} s_{ab_a}^D s_n^B, & b_a \leq n-2; \\ s_{an}^D s_n^B, & b_a = n-1; \\ s_{an}^D, & b_a = n; \\ s_{ab_a}^D, & b_a > n. \end{cases}$$

*Proof.* (i) $s_n^B s_{ab_a}^B = [(n \ -n)][(a \ a+1 \ \cdots \ b_a+1)] = [(a \ a+1 \ \cdots \ b_a+1)][(n \ -n)] = s_{ab_a}^B s_n^B$ because $b_a + 1 < n$. $s_{ab_a}^B s_n^B = s_{ab_a}^D s_n^B$ because $s_{ab_a}^B = s_{ab_a}^D$.
(ii) $s_{an}^D s_n^B = s_{a,n-2}^B s_n^B s_{n-1}^B s_n^B s_n^B$. If ELW's in $f_1^{(n)}$ and ELW's in $f_2^{(i,n)}$ for $i = n-2, \ldots, a$ are applied, respectively, then $s_{a,n-2}^B s_n^B s_{n-1}^B s_n^B s_n^B = s_n^B s_{a,n-1}^B$.
(iii) $s_n^B s_{an}^B = s_n^B s_{a,n-1}^B s_n^B$. Using part (ii), then $s_n^B s_{a,n-1}^B s_n^B = s_{a,n}^D s_n^B s_n^B = s_{a,n}^D$.

(iv) $s_n^B s_{ab_a}^B = s_n^B s_{a,n}^B s_{n-1}^B \cdots s_{2n-b_a}^B$. Using part (ii), then $s_n^B s_{a,n}^B s_{n-1}^B \cdots s_{2n-b_a}^B = s_{an}^D s_{n-1}^B \cdots s_{2n-b_a}^B = s_{ab_a}^D$ since $s_i^B = s_i^D$ for $i \neq n$.

$\square$

**Definition 3.14.** Let us consider a word $w$ of the form $s_{nj_n}^B s_{n-1,b_{n-1}}^B \cdots s_{ab_a}^B \cdots s_{1b_1}^B$ where each $b_a$ satisfies $a-1 \leq b_a \leq 2n-a$ for $1 \leq a \leq n$. We will define a function $n(w)$, which counts the number of occurrences of $s_n$ in the word $w$.

The following corollary is a result of the equalities $s_n^B s_0^B = s_0^B s_n^B$, $s_0^B = s_0^D$ and the lemmas discussed above.

**Corollary 3.15.** Let $1 \leq b \lessdot a \leq 2n-2$.

$$s_0^B s_{2a}^B s_{1b}^B = \begin{cases} s_0^D s_{2a}^D s_{1b}^D, & a \leq n-1 \text{ or } b > n \\ s_0^D s_{2,n-1}^D s_{1b}^D s_n^B, & a = n \text{ and } b < n-1 \\ s_0^D s_{2,n-1}^D s_{1n}^D s_n^B, & a = n \text{ and } b = n-1 \\ s_0^D s_{2,n-1}^D s_{1n}^D, & a = n \text{ and } b = n \\ s_0^D s_{2a}^D s_{1b}^D s_n^B, & a > n \text{ and } b < n-1 \\ s_0^D s_{2a}^D s_{1n}^D s_n^B, & a > n \text{ and } b = n-1 \\ s_0^D s_{2a}^D s_{1n}^D, & a > n \text{ and } a = n \end{cases}$$

**Corollary 3.16.** Let $1 \leq b \lessdot a \leq 2n-2$.

$$s_n^B s_0^B s_{2a}^B s_{1b}^B = \begin{cases} s_0^D s_{2a}^D s_{1b}^D s_n^B, & a \leq n-1 \text{ or } b > n \\ s_0^D s_{2n}^D s_{1b}^D s_n^B, & a = n-1 \\ s_0^D s_{2a}^D s_{1b}^D, & a \geq n \text{ and } b < n \\ s_0^D s_{2a}^D s_{1,n-1}^D s_n^B, & a \geq n \text{ and } b = n \end{cases}$$

**Definition 3.17.**

$$a \lessdot b = \begin{cases} a \leq b, & \text{if } a \geq n+1; \\ b = n-1 \text{ or } b \geq n+1, & \text{if } a = n; \\ a < b, & \text{if } a \leq n-1. \end{cases}$$

It is clear that $n$ and $n-1$ are not directly comparable. However, we can say that $n \lessdot n-1$ and $n-1 \lessdot n$.

**Definition 3.18.**

$$a \lesssim b = \begin{cases} a \leq b, & \text{if } a \geq n; \\ b = n-1 \text{ or } b \geq n+1, & \text{if } a = n-1; \\ a < b, & \text{if } a < n-1. \end{cases}$$

Indeed, it is important to note that $n$ and $n-1$ are not directly comparable to each other.

**Definition 3.19.** We define the following words in $\widetilde{D}_n$,

(i) $w_0 = s_{nd_n}^D \cdots s_{ad_a}^D \cdots s_{1d_1}^D$ where $a-1 \leq d_a \leq 2n-a$ for $a = 1, \ldots, n$ except $n-2 \leq d_{n-1} \leq n-1$.

(ii) $w_1 = \prod_{i=1}^{t} (s_0^D s_{2,2n-2}^D s_{1,a_i}^D)$ for $t \geq 0$, $1 \leq a_i \lesssim a_{i-1} \leq 2n-2$.

(iii) $w_2 = \prod_{i=1}^{s} (s_0^D s_{2,b_{2i-1}}^D s_{1,b_{2i}}^D)$ for $s \geq 0$, $1 \leq b_i \lessdot b_{i-1} \leq 2n-3$.

(iv) $w_3 = \begin{cases} (s_0^D s_{2,2n-2}^D s_{1,b_{2i}}^D)^{2m}, \\ (s_0^D s_{2,2n-2}^D s_{1,b_{2i}}^D)^{2m-1} (s_0^D s_{2b}^D), \\ (s_0^D s_{2,2n-2}^D s_{1,b_{2i}}^D)^{2(m-1)} (s_0^D s_{2,2n-2}^D) s_0^D, \end{cases}$ for $m \geq 0$ and $1 \leq b \leq 2n-2$.

(v) $w_4 = w_0 w_1 w_2$ where $a_t \geq 2$ and either $b_1 \lessdot a_t$ or $b_1 \not\lessdot a_t$

but $\begin{cases} b_2 \lesssim a_t, & a_t + b_1 \geq 2n; \\ b_2 + 1 < a_t, & a_t + b_1 < 2n. \end{cases}$

(vi) $w_5 = w_0 w_1 w_3$

Let $W_D = \{w_4, w_5\}$.

**Theorem 3.20.** *Any word $w \in W_B$ where $n(w)$ is even can be transformed into a word in $W_D$.*

*Proof.* Let $w_0 = s^B_{nb_n} s^B_{n-1,b_{n-1}} \cdots s^B_{ab_a} \cdots s^B_{1b_1}$ where $a - 1 \leq b_a \leq 2n - a$ for $1 \leq a \leq n$. Let $t_a = n(s^B_{nb_n} \cdots s^B_{a+1,b_{a+1}})$. Then

$$w_0 = \begin{cases} (s^D_{n,d_n} \cdots s^D_{ad_a} \cdots s^D_{1,d_1}), & n(w) \text{ is even;} \\ (s^D_{n,d_n} \cdots s^D_{ad_a} \cdots s^D_{1,d_1}) s^B_n, & n(w) \text{ is odd.} \end{cases}$$

where

$$d_n = \begin{cases} n, & b_n = n \text{ or } b_{n-1} = n+1; \\ n-1, & \text{otherwise.} \end{cases},$$

$$d_{n-1} = \begin{cases} n-1, & b_{n-1} = n-1 \text{ or } b_{n-1} = n; \text{ and } b_n = n-1; \\ n-1, & b_{n-1} = n+1; \\ n-2, & \text{otherwise.} \end{cases}$$

and

$$d_i = \begin{cases} b_a, & b_a \neq n-1, n; \\ n-1, & b_a = n-1 \text{ or } b_a = n; \text{ and } t_a \text{ is even;} \\ n, & b_a = n-1 \text{ or } b_a = n; \text{ and } t_a \text{ is odd.} \end{cases}$$

for $a = n-2, n-3, \ldots, 1$.

The values of $d_n$ and $d_{n-1}$ can be easily determined using Lemma 3.11. To find the values of other $d_a$, apply recursively either Lemma 3.12 or Lemma 3.13 for $a = n-2, n-3, \ldots, 1$ while using the fact that $s^B_n s^B_n = 1$.

Consider $w_1 = \prod_{i=1}^t (s^B_0 s^B_{2,2n-2} s^B_{a_i})$ for $t \geq 0$ and $1 \leq a_i \lessdot a_{i-1} \leq 2n-2$ and let $\zeta$ be the count of $a_i$'s that are less than or equal to $n-1$ in $w_1$. Through multiple applications of Corollary 3.15 and Corollary 3.16 imply that

$$w_1 = \begin{cases} \prod_{i=1}^t (s^D_0 s^D_{2,2n-2} s^D_{a_i}), & \zeta \text{ is even;} \\ (\prod_{i=1}^t (s^D_0 s^D_{2,2n-2} s^D_{a_i})) s^B_0, & \zeta \text{ is odd.} \end{cases}$$

where $b_i = a_i$ if $a_i \neq n-1$ and $b_i = n$ if $a_i = n-1$.

Now consider $\bar{w}_1 = s^B_n w_1$. Similarly

$$\bar{w}_1 = \begin{cases} \prod_{i=1}^t (s^D_0 s^D_{2,2n-2} s^D_{a_i}), & \zeta \text{ is odd;} \\ (\prod_{i=1}^t (s^D_0 s^D_{2,2n-2} s^D_{a_i})) s^B_0, & \zeta \text{ is even.} \end{cases}$$

where $b_i = a_i$ if $a_i \neq n$ and $b_i = n-1$ if $a_i = n$.

Hence both $w_1$ and $\bar{w}_1$ can be transformed one of the following

$$\begin{cases} \prod_{i=1}^t (s^D_0 s^D_{2,2n-2} s^D_{a_i}), \\ (\prod_{i=1}^t (s^D_0 s^D_{2,2n-2} s^D_{a_i})) s^B_0, \end{cases}$$

where for $t \geq 0$, $1 \leq a_i \lesssim a_{i-1} \leq 2n-2$.

∎

**Lemma 3.21.**

$$\prod_{a=1}^{n-1}[(1+y+y^2+\ldots+y^a)(1+y^a)] = (1+y+y^2+\ldots+y^{n-1})\prod_{a=1}^{n-1}(1+y+y^2+\ldots+y^{2a-1})$$

*Proof.* If $n$ is odd, then

$$\begin{aligned}
\prod_{i=1}^{\frac{n-3}{2}}(1+y+y^2+\cdots+y^{2k})\prod_{t=1}^{n-1}(1+y^t) &= \prod_{i=1}^{\frac{n-3}{2}}(\frac{1-y^{2i+1}}{1-y})\prod_{t=1}^{n-1}(\frac{1-y^{2t}}{1-y^t}) \\
&= \frac{(1-y^{n+1})(1-y^{n+3})\cdots(1-y^{2n-2})}{(1-y)^{\frac{n-1}{2}}} \\
&= \frac{1-y^{n+1}}{1-y}\frac{1-y^{n+3}}{1-y}\cdots\frac{1-y^{2n-2}}{1-y} \\
&= \prod_{m=0}^{\frac{n-3}{2}}(1+y+y^2+\cdots+y^{n+2m}).
\end{aligned}$$

If $n$ is even, then

$$\begin{aligned}
\prod_{i=1}^{\frac{n-2}{2}}(1+y+y^2+\cdots+y^{2k})\prod_{t=1}^{n-1}(1+y^t) &= \prod_{i=1}^{\frac{n-2}{2}}(\frac{1-y^{2i+1}}{1-x^i})\prod_{t=1}^{n-1}(\frac{1-y^{2t}}{1-y^t}) \\
&= \frac{(1-y^{n})(1-y^{n+2})\cdots(1-y^{2n-2})}{(1-y)^{\frac{n}{2}}} \\
&= \frac{1-y^{n}}{1-y}\frac{1-y^{n+2}}{1-y}\cdots\frac{1-y^{2n-2}}{1-y} \\
&= \prod_{m=0}^{\frac{n-1}{2}}(1+y+y^2+\cdots+y^{n+2m-2}).
\end{aligned}$$

□

**Lemma 3.22.** *The generating function for word in $W_D$ is given by:*

$$\frac{1+y+\cdots+y^{n-1}}{1-y^{n-1}}\prod_{a=1}^{n-1}\frac{1+y^a}{1-y^{n-1+a}}.$$

*Proof.* We have established one to one correspondence between words in $W_D$ and the words in $W_C$ where the numbers of occurrences of both $s_0$ and $s_n$ are even. Let us consider a word $w$ of the form:

$$w = (s^C_{nd_n}s^C_{n-1,d_{n-1}}\cdots s^C_{1,d_1})\prod_{i=1}^{t}(s^C_0 s^C_{1b_i}).$$

Here, $t$ is even, $n(w)$ is even and $0 \le b_i \leqslant b_{i-1} \le 2n-1$. First, we examiner the part of the word $s^C_{nd_n}s^C_{n-1,d_{n-1}}\cdots s^C_{1,d_1}$, which corresponds to $s^B_{nd_n}s^B_{n-1,d_{n-1}}\cdots s^B_{1,d_1}$ in $W_B$. According to Theorem 3.20, the corresponding word in $W_D$ has

$$s^D_n s^D_{n-1}s^D_{n-2,b_{n-2}}\cdots s^D_{1b_1}$$

where $a-1 \le b_a \le 2n-a$. The generating function for these words is $(1+y)^2\prod_{a=2}^{n-1}(1+y+y^2+\cdots+y^{a-1}+2y^a+y^{a+1}+\cdots+y^{2a}) = \prod_{a=1}^{n-1}(1+y+\cdots+y^i)(1+y^i) = (1+y+y^2+\cdots+y^{n-1})\prod_{a=1}^{n-1}(1+y+y^2+\cdots+y^{2a-1})$ as given by

Lemma 3.21. Now, let us analyze the word $\overline{w} = \prod_{i=1}^{t}(s_0^C s_{1b_i}^C)$, where $t$ is even, and $n(\overline{w})$ is even. We are assuming that $n(s_{nb_n}^B s_{n-1,b_{n-1}}^B \cdots s_{1,b_1}^B)$ is even; otherwise, we would consider the word $s_n^B \overline{w}$. When converting the word $\prod_{i=1}^{t}(s_0^C s_{1b_i}^C)$ into a word in $W_D$, the resulting word loses its length due to the number of occurrences of both $s_0$ and $s_n$. The generating function for words in the form $\prod_{i=1}^{t}(s_0^C s_{1b_i}^C)$ in $W_C$ is given by $\prod_{a=1}^{n} \frac{1+y^a}{1-y^{n+a}}$. Hence, the generating function for the corresponding words in $W_D$ is $\frac{(1+y)(1+y^2)\cdots(1+y^{n-1})}{(1-y^{n-1})(1-y^n)\cdots(1-y^{2n-2})} = \frac{1}{y^{n-1}} \prod_{a=1}^{n-1} \frac{1+y^a}{1-y^{n-1+a}}$. □

We've established that the generating function for the infinite Coxeter group of type $\widetilde{D}_n$ can be expressed as:

$$\frac{1+y+\cdots+y^{n-1}}{1-y^{n-1}} \prod_{a=1}^{n-1} \frac{1+y+\ldots+y^{2a-1}}{1-y^{2a-1}}.$$

Using Section 7.1 in [2], we can simplify this expression to

$$\frac{1+y+\cdots+y^{n-1}}{1-y^{n-1}} \prod_{a=1}^{n-1} \frac{1+y+\ldots+y^{2a-1}}{1-y^{2a-1}} = (\prod_{a=1}^{n-1}(1+y+\ldots+y^{2a-1})(\frac{1+y^a}{1-y^{n-1+a}})).$$

This result matches the generating function of words in $W_D$.

Now, we are ready to present the main result about a GSB for the infinite Coxeter group of type $\widetilde{D}_n$.

**Theorem 3.23.** *Let $R^D$ be the set of all polynomials as provided in Lemma 2.4. Then*

(i) $W_D = Red(R^D)$.
(ii) *$R^D$ is a GSB for the infinite Coxeter group of type $\widetilde{D}_n$.*

*Proof.* (i) It is evident that any word in $W_D$ is $R^D$-reduced. Therefore, we have $W_D \subseteq \text{Red}(R^D)$. Conversely, if $w \in \text{Red}(R^D)$, then $w$ can be expressed as a permutation in $\widetilde{S}_n^D$, and this permutation corresponds to a word in $W_D$ according to Theorem 3.20. Hence, we have $\text{Red}(R^D) \subseteq W_D$.

(ii) We understand that any polynomial in $R^D$ forms part of a GSB of the infinite Coxeter group of type $\widetilde{D}_n$. If, hypothetically, $R^D$ were not a GSB, then according to Composition Diamond lemma, $\text{Red}(R^D) = W_B$ would be a proper subset of the set of normal forms of the infinite Coxeter group of type $\widetilde{D}_n$. This would contradict to the fact that $W_D$ and normal forms of the infinite Coxeter group of type $\widetilde{D}_n$ share same generating functions.

□

## 4. Conclusion

The main purpose of this article is to derive the GSB and normal forms for infinite Coxeter groups of type $\widetilde{B}_n$ and $\widetilde{D}_n$. Similar to many previously mentioned papers, we use the Shirshov algorithm to obtain a set of $R$ relations. We used it partially. We then asserted that $\text{Red}(R)$ is equal to the set of normal forms of infinite Coxeter groups of type $\widetilde{B}_n$ and $\widetilde{D}_n$. Then, by applying the Composition Diamond lemma, we find that $R$ forms a GSB. At this stage, we took advantage of the combinatorial properties of infinite Coxeter groups of type $\widetilde{B}_n$ and $\widetilde{D}_n$ as presented in [2]. Using this information, we determined a set of normal forms for this group and designed a method to determine the normal form of each element of the group when provided in permutation form. As a result, we have determined the normal form of the product of two normal forms. As a result, the group is completely characterized in terms of these normal forms.

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# HEMI-SLANT SUBMANIFOLDS OF LORENTZIAN KENMOTSU SPACE FORMS

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Abstract. In this paper, we study curvature properties of hemi-slant submanifolds of Lorentzian Kenmotsu space forms. We define Lorentzian Kenmotsu space forms and study their curvature properties. We give an example for hemi-slant submanifold of Lorentzian Kenmotsu space forms. Finally, the curvature properties of distributions are analyzed and the conditions for Einstein are investigated.

## 1. Introduction

Bishop and O'neill investigated negative curvature manifolds [3]. They studied these manifolds using warped product. From the second half of the twentieth century, the warped product began to be used in contact manifolds. Kenmotsu investigated a different class of an almost contact manifold. He defined new conditions by

$$(\nabla_X\varphi)Y = -\eta(Y)\varphi X - g(X,\varphi Y)\xi \qquad (1.1)$$
$$\nabla_X\xi = X - \eta(X)\xi$$

He showed that the contact manifold satisfying these two conditions is normal. But this manifold was not Sasakian [7]. A differentiable manifold called Lorentzian manifold with a Lorentzian metric of index 1. A Lorentzian manifold has lightlike, timelike and spacelike vector fields. Therefore, the Lorentzian metric can also be used on odd dimensional manifolds. So we can study Lorentzian contact manifolds. Firstly, Takahashi defined and studied Lorentzian Sasakian manifolds using the Lorentzian metric on Sasakian manifold [13]. After, Duggal has investigated the space time manifolds [6]. From all these studied, Rosça investigated Lorentzian Kenmotsu manifolds [9]. Many authors have been studied on Lorentzian Kenmotsu manifolds [2, 4, 5, 8, 14, 15].

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In this paper, we are studied curvature properties of hemi-slant submanifolds of Lorentzian Kenmotsu space form. Firstly, we are defined Lorentzian Kenmostsu space forms and study their curvature properties. After, the definition of a hemi-slant submanifold of an Lorentzian Kenmotsu space form is given and an example is presented. Finally, the curvature properties of distributions are analyzed and the conditions for Einstein are investigated.

## 2. Lorentzian Kenmotsu Manifolds

Let $B$ be almost contact manifold with an almost contact structure $(\varphi, \eta, \xi)$, where $\xi$ is a vector field on $B$, $\eta$ is a $1-$form and $\varphi$ is a tensor field of type $(1,1)$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1.$$

If a semi-Riemannian metric $g$ on almost contact manifold $B$ by

$$g(\varphi X, \varphi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \qquad g(\xi, \xi) = \epsilon = -1$$

therefore $(B, \varphi, \eta, \xi, g)$ is called a Lorentzian almost contact manifold. Then we get $\eta(X) = \epsilon g(X, \xi)$. Moreover, $\xi$ is never a spacelike vector field and a lightlike vector field on $B$. We consider a local basis $\{e_1, ..., e_{2n}, \xi\}$ in $TB$ i.e.

$$g(e_i, e_j) = \delta_{ij} \text{ and } g(\xi, \xi) = -1$$

that is $e_1, ..., e_{2n}$ are spacelike vector fields, and $\xi$ is timelike.

We not that, for all $X, Y \in \Gamma(TB)$, if $\Phi(X, Y) = g(X, \varphi Y)$, $\Phi$ is said to be fundamental $2-$form.

On the other hand, manifold is normal if

$$N = [\varphi, \varphi] + 2d\eta \otimes \xi = 0$$

where $[\varphi, \varphi]$ is Nijenhuis tensor field of $\varphi$.

**Definition 2.1.** Let $B$ be a Lorentzian almost contact manifold. $B$ is called a Lorentzian Kenmotsu manifold if normal and $d\eta = 0$ and $d\Phi = 2\epsilon\eta \wedge \Phi$.

**Theorem 2.2.** [10] *Let $B$ be a Lorentzian contact manifold. Therefore for all $X, Y \in \Gamma(TB)$, $B$ is a Lorentzian Kenmotsu manifold if and only if*

$$\left(\overline{\nabla}_X \varphi\right) Y = \epsilon\{g(Y, \varphi X)\xi - \eta(Y)\varphi X\}. \tag{2.1}$$

.

**Corollary 2.3.** *Let $B$ be a Lorentzian Kenmotsu manifold. Therefore we get*

$$\overline{\nabla}_X \xi = \epsilon\varphi^2 X \tag{2.2}$$

*for all $X, Y \in \Gamma(TB)$.*

## 3. Lorentzian Kenmotsu Space Forms

Let Lorentzian Kenmotsu manifold $B$ has constant $\varphi-$holomophic section curvature k. Therefore it is called Lorentzian Kenmotsu-space form. If constant $\varphi-$holomophic section curvature is $k$, manifold $B$ is denoted by $B(k)$. Therefore, curvature tensor satisfied,

$$
\begin{aligned}
R(X,Y,Z,W) \quad = \quad & \frac{k+3}{4}\{g(Z,Y)g(W,X)-g(W,Y)g(Z,X)\} \\
& +\frac{k-1}{4}\{g(Z,\varphi Y)g(W,\varphi X)-g(W,\varphi Y)g(Z,\varphi X) \\
& -2g(W,\varphi Z)g(Y,\varphi X)+g(Z,X)\eta(W)\eta(Y) \\
& -g(Z,Y)\eta(W)\eta(X)+g(W,Y)\eta(Z)\eta(X)\}.
\end{aligned} \tag{3.1}
$$

**Theorem 3.1.** *Let $B$ be a Lorentzian Kenmotsu manifold. If $B$ have constat $\varphi-$ holomophic sectional curvature, therefore the Ricci tensor is not parallel.*

*Proof.* We using (3.1). For all $X, Y \in \Gamma(TB)$, we get

$$
S(X,Y) = \frac{(k-1)+(k+3)n}{2} g(\varphi Y, \varphi X) - 2n\eta(Y)\eta(X)
$$

which proves the assertion. □

**Corollary 3.2.** *Let $B$ be a Lorentzian Kenmotsu manifold. Therefore we get*

$$
\tau = \frac{((k-3)n-2)(2n+1)}{4}
$$

*where $\tau$ is the scalar curvature.*

## 4. Hemi-Slant Submanifolds of Lorentzian Kenmotsu Space Forms

Let $B$ be a submanifold of a Lorentzian Kenmotsu manifold $\overline{B}$ and $\nabla$ be the Levi-Civita connection of $B$. For all $X, Y \in \Gamma(TB)$ and $N \in \Gamma(TB)^{\perp}$, we have

$$
\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{4.1}
$$

$$
\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N. \tag{4.2}
$$

This equations is called Gauss and Weingarten formulas, respectively. Moreover, from (4.1) and (4.2), we get

$$
g(A_N X, Y) = g(h(X,Y), N). \tag{4.3}
$$

For any $X \in \Gamma(TB)$, we give

$$
\varphi X = TX + NX
$$

where $NX$ and $TX$ is the normal and tangential components, respectively.

For any $V \in \Gamma(T^{\perp}B)$, we have

$$
\varphi V = tV + nV
$$

where $nV$ and $tV$ is the normal and tangential components, respectively [12].

**Lemma 4.1.** *Let $B$ be a submanifold of a Lorentzian Kenmotsu manifold $\overline{B}$. Therefore, for all $K, L \in \Gamma(TB)$*

$$
(\nabla_K T)L = A_{NL}K + th(K,L) + \epsilon\{g(TK,L)\xi - \eta(L)TK\} \tag{4.4}
$$

$$
(\nabla_K N)L = nh(K,L) - h(K,TL) - \epsilon\eta(L)NK. \tag{4.5}
$$

From now on, we accept that the $\xi$ is tangent to the submanifold $B$. Therfore, we can consider the orthogonal direct decomposition

$$TB = D \oplus \xi,$$

where $D$ is the orthogonal distribution to $\xi$.

**Definition 4.2.** Let $B$ be a submanifold of Lorentzian Kenmotsu manifold $\overline{B}$ . Therefore $B$ is called anti-invariant if and only if $\varphi(T_x B) \subset T_x^{\perp} B$ for all $x \in B$.

**Definition 4.3.** Let $B$ be a submanifold of a Lorentzian Kenmotsu manifold $\overline{B}$ . If angle betwen $\varphi B$ and $TB$ is a constant, submanifold $B$ is called slant submanifold.

In [1], $Sp\{\xi\}$ defines the timelike vector field distribution. Let $W$ is a spacelike vector field. If vector field $W$ is orthogonal to $\xi$, we get

$$g(\varphi W, \varphi W) = g(W, W) \geq 0.$$

For spacelike vector fields the Cauchy-Schawrz inequality

$$g(W, W) \leq \|W\| \, \|W\|$$

is verified.

Then we have

$$\cos\theta = \frac{g(\varphi W, TW)}{\|\varphi W\| \, \|TW\|}.$$

**Definition 4.4.** Let $B$ be submanifold of of a Lorentzian Kenmotsu manifold $\overline{B}$. Therefore $B$ is called a hemi-slant submanifold which $D_1$ and $D_2$ two orthogonal spacelike distributions such that

$(i)$ $TB = D_1 \oplus D_2 \oplus sp\{\xi\}$
$(ii)$ $D_1$ is anti-invariant.
$(iii)$ $D_2$ is slant with angle $\theta \neq 0$.

Therefore, the angle $\theta$ is called the slant angle of a submanifold $B$.

On the other hand, let $d_i$ be dimension of the distribution $D_i$ for $i = 1, 2$. Therefore we have the following cases:

If $d_2 = 0$, therefore $B$ is an anti-invariant submanifold.
If $d_1 = 0$ and $\theta = 0$, therefore $B$ is an invariant submanifold.
If $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, therefore $B$ is a proper slant timelike submanifold.
If $d_1 d_2 \neq 0$ and $\theta \neq \frac{\pi}{2}$, $B$ is a proper hemi-slant timelike submanifold.
For a local orthonormal frame $\{e_1, ..., e_{2p}, e_{2p+1}, ..., e_{2p+2q}, \xi\}$,

$$D_1 = sp\{e_1, ..., e_{2p}\}, D_2 = sp\{e_{2p+1}, ..., e_{2p+2q}\}$$

where $dimD_1 = 2p$ and $dimD_2 = 2q$.

**Example 4.5.** In what follows, $\mathbb{R}^{2m+1}$ with Lorentzian Kenmotsu structure given by

$$\varphi(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x_i} - X_i\frac{\partial}{\partial y_i}) + Y_i y_i\frac{\partial}{\partial z}$$

$$g = e^{-2z}(\sum_{i=1}^{n} dx_i \otimes dx_i + dy_i \otimes dy_i) - \epsilon\eta \otimes \eta$$

$$\xi = \frac{\partial}{\partial z}, \qquad \eta = dz$$

where $(x_1, ..., x_n, y_1, ..., y_n, z)$ are Cartesian coordinates on $\mathbb{R}^{2m+1}$.

Now, a submanifold $B$ of $\mathbb{R}^7$ defined by

$$B = F(s,l,k,u,t) = (s,0,k,l,u,0,t).$$

Therefore local frame of $TB$

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x_1}, \qquad e_2 = \frac{\partial}{\partial y_1}, \qquad e_3 = \frac{\partial}{\partial x_3}, \\ e_4 &= \frac{\partial}{\partial y_2}, \qquad e_5 = \frac{\partial}{\partial z} = \xi \end{aligned}$$

and

$$e_1^* = \frac{\partial}{\partial x_2}, \qquad e_2^* = \frac{\partial}{\partial y_3}$$

from a basis of $T^{\perp}B$.

We choose

$$D_1 = sp\{e_1, e_2\}$$

and

$$D_2 = sp\{e_3, e_4\},$$

then $D_1$, $D_2$ are anti-invariant and slant distribution. Thus

$$TB = D_1 \oplus D_2 \oplus sp\{\xi\}$$

$B$ is a hemi-slant submanifold of $\mathbb{R}^7$.

## 5. Curvature Properties of Distributions

[11], Let $B$ be a hemi-slant submanifold of a Lorentzian Kenmotsu manifold $\overline{B}$ . From (3.1) and (4.1), a hemi-slant submanifold $B$ has constat $\varphi$-sectional curvatre $k$ if and only if the Riemanian curvatre tensor $R$ satisfied

$$\begin{aligned} R(X,Y,Z,W) &= \frac{k+3}{4}\{g(Z,Y)g(W,X) - g(W,Y)g(Z,X)\} \\ &+\frac{k-1}{4}\{g(\varphi Y,Z)g(\varphi X,W) - g(\varphi Y,W)g(\varphi X,Z) \\ &-2g(\varphi Z,W)g(\varphi X,Y) + g(Z,X)\eta(W)\eta(Y) \\ &-g(Z,Y)\eta(W)\eta(X) + g(W,Y)\eta(Z)\eta(X)\} \\ &+g(h(Z,Y),h(W,X)) - g(h(Z,X),h(W,Y)). \end{aligned} \tag{5.1}$$

**Proposition 1.** Let $B$ be hemi-slant submanifold of Lorentzian Kenmotsu space form $\overline{B}(k)$.Therefore we get

$$\begin{aligned} R(X,Y,Z,W) &= \frac{k+3}{4}\{g(Z,Y)g(W,X) - g(W,Y)g(Z,X)\} \\ &+g(h(Z,Y),h(W,X)) - g(h(Z,X),h(W,Y)) \end{aligned} \tag{5.2}$$

for all $X,Y,Z,W \in \Gamma(D_1)$.

*Proof.* The proof follows from (5.1). □

**Corollary 5.1.** *Let $B$ be hemi-slant submanifold of Lorentzian Kenmotsu space form $\overline{B}(k)$ and anti-invariant distribution $D_1$ is totally geodesic. Therefore $D_1$ is flat if and only if $k = -3$.*

**Theorem 5.2.** *Let $B$ be hemi-slant submanifold of Lorentzian Kenmotsu space form $\overline{B}(k)$.If anti-invariant distribution $D_1$ is totally geodesic, therefore it is Einstein.*

*Proof.* Let $D_1$ is totally geodesic. For all $X, Y \in \Gamma(D_1)$ using (5.2), we have Ricci curvature by

$$S_1(X,Y) = \sum_{i=1}^{2p} \frac{k+3}{4}\{g(X,Y)g(E_i,E_i) - g(X,E_i)g(E_i,Y)\}.$$

Then, by elementary calculations, we get

$$S_1(X,Y) = \frac{(k+3)(2p-1)}{4} g(X,Y)$$

which proves the assertion. □

**Corollary 5.3.** *Let $B$ be hemi-slant submanifold of Lorentzian Kenmotsu space form $\overline{B}(k)$. If $D_1$ is totally geodesic, scalar curvature of $D_1$ given by*

$$\tau_{D_1} = p(p-1)\frac{k+3}{4}$$

**Theorem 5.4.** *Let $B$ be hemi-slant submanifold of Lorenzian Kenmotsu space form $\overline{B}(k)$. Therefore the scalar curvatre of $D_2$ is given by*

$$\tau_{D_2} = q\frac{(k+3)(2q-1)+3(k-1)}{2}.$$

*Proof.* For all $U, V \in \Gamma(D_2)$, from (5.2), Ricci curvature of $D_2$ is given by

$$S_2(U,V) = \frac{3(k-1)(k+3)+(2q-1)}{4} g(U,V)$$

which proves the assertion. □

## 6. Conclusion

Lorentzian manifolds have potential for applications in many fields of mathematics and physics. In particular it is applicable to the theory of relativity, theory of spacetimes. Researchers have increased studies on this field from different areas in recent years. After the defination of Lorentzian Kenmotsu manifold, hemi-slant submanifolds were studied. In this paper, the idea of examining curvature of hemi-slant submanifold are emphasized. The works on this subject will be useful tools for the applications of hemi-slant submanifold with different manifolds.

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The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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# $\mathbf{I}^*$-CONVERGENCE OF FUNCTION SEQUENCES IN ASYMMETRIC METRIC SPACE

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Abstract. In this paper, by considering ideal which is special subfamily of power set of natural numbers $\mathbf{I}^*$-convergence of sequence of functions in asymmetric metric spaces is defined and some results about new concept are given. Obtained results is supported some examples to show differences by the classical ones.

## 1. Introduction

The definition of statistical convergence by using asymptotic density was first introduced by Fast [6] and Steinhaus [17] in the same year 1951, independently. Although, it looks a simple generalization of classical convergence, this definition gave a new perspective to the researchers.

In [8], Freedman A.R. and Sember J. J. introduced a general concept of density and studied the relationship between densities and strong convergence areas of different summability methods. In [2], It has been demonstrated that if a sequence is strongly $\mathfrak{p}$-Cesaro summable or $w_{\mathfrak{p}}$ convergent then the sequence must be statistically convergent for $0 < \mathfrak{p} < \infty$ Furthermore a bounded statistically convergent sequence must be $w_{\mathfrak{p}}$ convergent for any $\mathfrak{p}$, $0 < \mathfrak{p} < \infty$.

Di Maio G. and Kocinac L. D. R. introduced and examined statistical convergence in topological and uniform spaces in [3]. They demonstrated the applicability of this convergence to the theory of choice principles, function spaces, and hyperspaces.

Some years later in the paper [12], Ilkhan and Kara obtained some results about completeness, compactness and pre-compactness by using statistically Cauchy sequences in a quasi metric spaces.

Later, based on the idea of this definition, P. Kostyrko, T. Salat, W. Wilczynki [13] gave the concepts ideal convergence by characterizing the small sets of a space in different ways.

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In article [7], the Bolzano-Weierstrass theorem is generalized by using ideal convergence. The authors of the paper [7] provided instances of ideals possessing and lacking the Bolzano-Weierstrass property, and examined the BW property in relation to submeasures and its extendibility to a maximal **P**-ideal. Apart from these, The study [18] examined the completion of a linear $n$-normed space regarding ideal convergence by introducing the notion of uniform continuous $n$-norm.

B. K. Lahiri and P. Das [14] carried out studies on **I**-convergence and $\mathbf{I}^*$ convergence and obtained important results. These concepts were studied in arbitrary metric spaces or arbitrary topological spaces.

In [11], Argha Gosh discussed and examined the concepts of $\mathbf{I}^*(\alpha)$- convergence and $\mathbf{I}^*$-exhaustiveness of metric function sequences and explained the relationship between these two concepts.

In asymmetric metric spaces (or quasi metric spaces in some sources) which is a larger structure than metric spaces, some properties of quasi metric spaces were given by Otafudu O. O. in [15], Reilly et all. in [16], Doitchinov D. in [4] and Dutta R. in [5], where sequence and function sequence convergence and fixed point results were given. Then, Ghosh A. (in his paper [10]) investigated the convergence of sequences of functions in asymmetric metric spaces with the help of ideals.

In this paper, our aim to give new kind definitions of left (right) $\mathbf{I}^*(\alpha)$-convergence, left (right) $\mathbf{I}^*$- Alexandroff convergence, left (right) $\mathbf{I}^*$-uniformly convergence for function sequences in an asymmetric metric space and some relations between them will be investigated.

## 2. Preliminaries and new results

In this part, we will present several new definitions along with corresponding results related to them.Throughout the text, we are going to use $\mathbf{Y}^{\mathbf{X}}$ to indicate the set of all maps from the asymmetric metric spaces $(\mathbf{X}, \mathfrak{q})$ to $(\mathbf{Y}, \mathfrak{p})$

**Definition 2.1.** Let $\mathbf{X} \neq \emptyset$ be a set and $\mathfrak{q} : \mathbf{X} \times \mathbf{X} \to [0, \infty)$ be a function. The function $\mathfrak{q}$ is defined as an asymmetric metric on $\mathbf{X}$ if it meets the following criteria: $(i)$ $\mathfrak{q}(x, y) \geq 0$, for all $x, y \in \mathbf{X}$; $(ii)$ $\mathfrak{q}(x, y) = 0$ if and only if $x = y$ and $(iii)$ $\mathfrak{q}(x, z) \leq \mathfrak{q}(x, y) + \mathfrak{q}(y, z)$ holds for all $x, y, z \in \mathbf{X}$.

Then, the pair $(\mathbf{X}, \mathfrak{q})$ is referred to as an asymmetric metric space and in addition to this if $\mathfrak{q}$ possesses the property of symmetry, it is classified as a metric and $(\mathbf{X}, \mathfrak{q})$ is termed a metric space.

**Definition 2.2.** Let $(\mathbf{X}, \mathfrak{q})$ be an asymmetric metric space. A left(right) topology $\tau^-(\tau^+)$ induced by $\mathfrak{q}$ is generated by the collection of left(right) open balls

$$\mathbb{B}^-(x, r) := \{y \in \mathbf{X} : \mathfrak{q}(y, x) < r\} \, (\mathbb{B}^+(x, r) := \{y \in \mathbf{X} : \mathfrak{q}(x, y) < r\})$$

for all $x \in \mathbf{X}$ and positive reals $r > 0$, respectively.

A sequence $\tilde{x} = (x_n)$ is said left(right) convergent to a point $x^*$, if for every $\varepsilon > 0$ there exists $n_* = n_*(\varepsilon) \in \mathbb{N}$ such that $x_n \in \mathbb{B}^-(x^*, r)\, (x_n \in \mathbb{B}^+(x^*, r))$ holds for all $n > n_*$.

One of the important problems that arise as a result of the lack of symmetry property is that left(right) limit of a sequence is not unique, in generally. Let's give an example to see this defect of asymmetric metric space:

**Example 2.3.** Let us consider a real valued sequence $\tilde{x} = (x_n)$ as

$$x_n := \begin{cases} \frac{1}{2^n}, & n \text{ is odd}, \\ \frac{1}{3^n}, & n \text{ is even}, \end{cases}$$

and asymmetric metric as

$$\mathfrak{q}(a, b) := \begin{cases} 0, & a \leq b, \\ 1, & a > b. \end{cases}$$

Hence, it is evident that every point of $(-\infty, 0)$ serves as a left limit point of the sequence.

**Definition 2.4.** [9] A subset $\mathfrak{B}$ of $\mathbb{N}$ is considered to have natural density natural density (or asymptotic density) denoted by $d(\mathfrak{B})$ if following limit exists

$$d(\mathfrak{B}) := \lim_{n\to\infty} \frac{|\mathfrak{B}(n)|}{n},$$

where $\mathfrak{B}(n) := \{j \in \mathfrak{B} : j \leq n\}$ and the symbol $|.|$ denotes the cardinality of the inside set.

**Definition 2.5.** [13] Let $\mathbf{X}$ be a non-empty set. A family $\mathbf{I} \subset 2^{\mathbf{X}}$ is termed an ideal on $\mathbf{X}$ if $(i)$ $U \cup V \in \mathbf{I}$ holds for all $U, V \in \mathbf{I}$ and $(ii)$ $U \in \mathbf{I}$ and $V \subset U$, then $V \in \mathbf{I}$ holds.

An ideal $\mathbf{I}$ is is referred to as non-trivial if $\mathbf{I}$ is not equal to $\emptyset$ and $\mathbf{X}$ is not an element of $\mathbf{I}$. A non-trivial ideal is termed admissible if it includes the singleton set $\{x\}$ for every $x \in \mathbf{X}$.

**Definition 2.6.** [13] Let $\mathbf{X} \neq \emptyset$. Then, $\mathfrak{F} \subset 2^{\mathbf{X}}$ is defined as a filter on $\mathbf{X}$ if it meets these criteria: $(i)$ $U \cap V \in \mathfrak{F}$ for all $U, V \in \mathfrak{F}$ and $(ii)$ $U \in \mathfrak{F}$ and $U \subset V$ implies that $V \in \mathfrak{F}$ holds.

For any non-trivial ideal $\mathbf{I} \subset 2^{\mathbf{X}}$ it can be defined a filter as follows

$$\mathfrak{F}(\mathbf{I}) := \{U \subset \mathbf{X} : U^c \in \mathbf{I}\}$$

and it is called a filter associated with $\mathbf{I}$. Following families

$$\mathbf{I}_d = \{U \subset \mathbb{N} : d(U) = 0\};\ \mathfrak{F}(\mathbf{I}_d) := \{U \subset \mathbb{N} : d(U) = 1\}$$

are well known nontrivial admissible ideal and filter.

**Definition 2.7.** [10] Let $\mathbf{I}$ be an admissible ideal.It is referred to as Good, for any sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets where $A_n \notin \mathbf{I}$ for all $n \in \mathbb{N}$, if there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of mutually disjoint sets such that $B_n \subset A_n$, $B_n \in$ Ł and $\bigcup_{n=1}^{\infty} B_n \notin \mathbf{I}$ hold.

A condition equivalent to this definition will be given in the following lemma:

**Lemma 2.8.** *An ideal* $\mathbf{I}$ *is Good iff for every* $\{D_n\}_{n\in\mathbb{N}} \notin \mathfrak{F}(\mathbf{I})$ *there exists pairwise disjoint sets* $\{P_n\}_{n\in\mathbb{N}} \subset \mathfrak{F}(\mathbf{I})$ *such that* $P_n \supset D_n$ *and* $\bigcap_{n=1}^{\infty} P_n \notin \mathfrak{F}(\mathbf{I})$ *hold.*

*Proof.* Assume $\mathbf{I}$ is a Good ideal and consider $\{D_n\}_{n\in\mathbb{N}} \notin \mathfrak{F}(\mathbf{I})$. Then, $\mathbb{N} \setminus D_n \notin \mathbf{I}$. Since $\mathbf{I}$ is Good ideal, there exists $A_n \subset \mathbb{N} \setminus D_n$ such that $A_n \in \mathbf{I}$ and $\bigcup_{n=1}^{\infty} A_n \notin \mathbf{I}$. If $P_n$ is chosen as such $P_n := \mathbb{N} \setminus A_n \in \mathfrak{F}(\mathbf{I})$, then

$$\bigcap_{n=1}^{\infty} P_n = \bigcap_{n=1}^{\infty} \mathbb{N} \setminus A_n = \mathbb{N} \setminus \bigcup_{n=1}^{\infty} A_n \notin \mathfrak{F}(\mathbf{I}).$$

□

**Definition 2.9.** [13] A sequence $\{x_k\}_{k\in\mathbb{N}} \subset (\mathbf{X},\mathfrak{q})$ is described as left(right) **I**-convergent to $x_* \in \mathbf{X}$ if

$$\{k \in \mathbb{N} : \mathfrak{q}\,(x_k, x_*) \geq \varepsilon\} \in \mathbf{I}; (\{k \in \mathbb{N} : \mathfrak{q}\,(x_*, x_k) \geq \varepsilon\} \in \mathbf{I})$$

holds for every $\varepsilon > 0$, respectively.

In this case,it is denoted by symbolically $x_k \overset{\mathbf{I}^-}{\to} x_*$ and $x_k \overset{\mathbf{I}^+}{\to} x_*$, respectively.

**Definition 2.10.** [10] Function sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is defined as left(right) pointwise **I**-convergent to a function $f \in \mathbf{Y}^{\mathbf{X}}$ if $f_k\,(x) \overset{\mathbf{I}^-}{\to} f(x)$ $(f_k\,(x) \overset{\mathbf{I}^+}{\to} f(x))$ holds for each $x \in \mathbf{X}$.

**Definition 2.11.** [10] Function sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is called left(right) **I**-convergent uniformly to $f \in \mathbf{Y}^{\mathbf{X}}$ if for each $\varepsilon > 0$ there exists $A \in \mathfrak{F}(\mathbf{I})$ such that $\mathfrak{p}\,(f_k(x), f(x)) < \varepsilon$ $(\mathfrak{p}\,(f(x), f_k(x)) < \varepsilon)$ holds for all $k \in A$ and $x \in \mathbf{X}$.

**Definition 2.12.** [10] A function $f \in \mathbf{Y}^{\mathbf{X}}$ is referred to as left continuous ($f^-$-continuous) at a point $\xi \in \mathbf{X}$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathfrak{p}\,(f(y), f(\xi)) < \varepsilon$ satisfies for all $y \in \mathbb{B}^-\,(\xi, \delta)$.

Similarly, right continuous ($f^+$-continuous) at $\xi \in \mathbf{X}$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathfrak{p}\,(f(\xi), f(y)) < \varepsilon$ satisfies for all $y \in \mathbb{B}^+\,(\xi, \delta)$.

**Definition 2.13. (Sequential continuity at a point)** A function $f \in \mathbf{Y}^{\mathbf{X}}$ is said to be $(i)$ $f^{-,-}$ continuous at $x^* \in \mathbf{X}$, if whenever a sequence $\{x_k\}_{k\in\mathbb{N}}$ left converges to $x^*$ in $(\mathbf{X},\mathfrak{q})$, then corresponding sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ left converges to $f(x^*)$ in $(\mathbf{Y},\mathfrak{p})$;

$(ii)$ $f^{+,+}$ continuous at a point $x_* \in \mathbf{X}$, if whenever a sequence $\{x_k\}_{k\in\mathbb{N}}$ right converges to $x_*$ in $(\mathbf{X},\mathfrak{q})$, then corresponding sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ right converges to $f(x_*)$ in $(\mathbf{Y},\mathfrak{p})$.

**Definition 2.14.** Let $(\mathbf{X},\mathfrak{q})$ be an asymmetric metric space, $\{x_n\} \subset \mathbf{X}$ be a sequence and $a^* \in \mathbf{X}$. A sequence $\{x_n\}$ is said to be left (right) $\mathbf{I}^*$-convergent to $a^*$, if there exists $K = \{m_1 < m_2 < ... < m_n < ...\}$ such that

$$\lim_{n\to\infty} \mathfrak{q}\,(x_{m_n}, a^*) = 0 \; (\lim_{n\to\infty} \mathfrak{q}(a^*, x_{m_n}) = 0)$$

holds.

It is denoted by symbolically $x_n \overset{\mathbf{I}^{*-}}{\to} a^*$ $(x_n \overset{\mathbf{I}^{*+}}{\to} a^*)$, respectively.

**Definition 2.15.** A sequence of function $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is called left (right) $\mathbf{I}^*\,(\alpha)$ convergent to $f \in \mathbf{Y}^{\mathbf{X}}$ if for any sequence $\{x_k\}$ that left(right) $\mathbf{I}^*$ converges to point $x$ in $\mathbf{I}$, the sequence $(f_k\,\{x_k\})$ is also left $\mathbf{I}^*$-convergence to $f(x)$.

**Definition 2.16.** A sequence of function $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is called left(right) $\mathbf{I}^*$-exhaustive at a point $\acute{a} \in \mathbf{X}$ if there exists $\mathrm{A} = \mathrm{A}(\acute{a}) \in \mathbf{I}$ such that for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, \acute{a}) > 0$ and $n_0 = n_0(\varepsilon, \acute{a}) \in \mathbb{N}$ such that $\mathfrak{q}(\acute{a}, x) < \delta$ $(\mathfrak{q}(x, \acute{a}) < \delta)$ implies $\mathfrak{p}\,(f_n(\acute{a}), f_n(x)) < \varepsilon$ $(\mathfrak{p}\,(f_n(x), f_n(\acute{a})) < \varepsilon)$ for all $n \in \mathbb{N} \setminus \mathrm{A}$ and $n \geq n_0$.

**Definition 2.17.** A function sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is called left(right) pointwise $\mathbf{I}^*$-convergent to a function $f \in \mathbf{Y}^{\mathbf{X}}$ if $f_k\,(\acute{x}) \overset{\mathbf{I}^{*-}}{\to} f(\acute{x})$ ( $f_k\,(\acute{x}) \overset{\mathbf{I}^{*+}}{\to} f(\acute{x})$) satisfies for all $\acute{x} \in \mathbf{X}$.

**Theorem 2.18.** *Let $x \in \mathbf{X}$ and $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to $f$ at point $x \in \mathbf{X}$. If $\{f_k\}_{k\in\mathbb{N}}$ is right pointwise $\mathbf{I}^*$-convergent to $f$ at every point $z \in \mathbf{X} \setminus \{x\}$ and $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x \in \mathbf{X}$, then $f$ is $f^-$-continuous*

*Proof.* Owing to the fact that $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$-exhaustive at $x \in \mathbf{X}$, then there exists $\mathrm{A} = \mathrm{A}(x) \in \mathbf{I}$ such that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ and $\exists n_0 = n_0(\varepsilon, x) \in \mathbb{N}$ such that for every $\mathfrak{q}(y, x) < \delta$ implies

$$\mathfrak{p}(f_n(y), f_n(x)) < \varepsilon$$

for all $n \geq n_0$ and $n \in \mathbb{N} \setminus \mathrm{A}$.

Let $y \in B^-(x, \delta) \setminus \{x\}$. Since, $\{f_k\}$ is right pointwise $\mathbf{I}^*$-convergent to $f$, then we have $f_k(x) \overset{\mathbf{I}^{*+}}{\to} f(x)$ for all $y \in \mathbf{X}$.

So, there exists $K = \{k_1 < k_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ such that $\lim_{n\to\infty} \mathrm{p}(f(y), f_{k_n}(y)) = 0$. Then, for all $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that $\mathfrak{p}(f_n(y), f_n(x)) < \frac{\varepsilon}{3}$ holds for every $k_n \geq n_1$.

Since, $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to $f$ at point $x \in \mathbf{X}$, then there exists $M_2 \in \mathfrak{F}(\mathbf{I})$ such that $\lim_{m\to\infty} \mathrm{p}(f_{k_m}(x), f(x)) < \frac{\varepsilon}{3}$ holds.

Now, $K_1 \cap K_2 \cap (\mathbb{N} \setminus \mathrm{A}) \in \mathfrak{F}(\mathbf{I})$ and this implies that $K_1 \cap K_2 \cap (\mathbb{N} \setminus \mathrm{A}) \neq \emptyset$.

Hence, we can choose $j \in K_1 \cap K_2 \cap (\mathbb{N} \setminus \mathrm{A})$. Then, for all $y \in B^-(x, \delta) \setminus \{x\}$ we have

$$\mathrm{p}(f(y), f(x)) \leq \mathrm{p}(f(y), f_j(y)) + \mathrm{p}(f_j(y), f_j(x)) + \mathrm{p}(f_j(x), f(x)) < \varepsilon.$$

Therefore, $f$ is left continuous. □

**Theorem 2.19.** *If $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to $f$ at point $x \in \mathbf{X}$ and $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x \in \mathbf{X}$, then $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*(\alpha)$ convergent to $f \in \mathbf{Y}^{\mathbf{X}}$ at $x \in \mathbf{X}$.*

*Proof.* For the reason that $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to $f$ at point $x \in \mathbf{X}$, then $f_k(x) \overset{\mathbf{I}^{*-}}{\to} f(x)$. So, it can be find a set $K = \{k_1 < k_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ such that

$$\lim_{m\to\infty} \mathfrak{p}(f_{k_m}(x), f(x)) = 0$$

holds. Hence, for all $\varepsilon > 0$ there exists natural number $n_0$ such that $\mathfrak{p}(f_{k_m}(x), f(x)) < \frac{\varepsilon}{2}$ holds for every $k_m \geq n_0$. Given that $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x \in \mathbf{X}$, then there exists $K' = K'(x) \in \mathbf{I}$ such that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ and $n_1 = n_1(\varepsilon, x) \in \mathbb{N} \ni$ for every $\mathfrak{q}(y, x) < \delta$ implies $\mathfrak{p}(f_n(y), f_n(x)) < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N} \setminus K'$ and $\forall n \geq n_1$.

Let $x_n \overset{\mathbf{I}^{*-}}{\to} x, n \to \infty$. We must show that $f_n(x_n) \overset{\mathbf{I}^{*-}}{\to} f(x), n \to \infty$. Since $x_n \overset{\mathbf{I}^{*-}}{\to} x, n \to \infty$, then there exists

$$K'' = \{\mathrm{m}_1 < \mathrm{m}_2 < ... < \mathrm{m}_\mathrm{k} < ...\} \in \mathfrak{F}(\mathbf{I})$$

such that $\lim_{k\to\infty} \mathfrak{q}(x_{m_k}, x) = 0$.

So, for all $\delta > 0$ there exists $n_1(\delta) \in \mathbb{N}$ such that $\mathfrak{q}(x_{m_k}, x) < \delta$ holds for all $m_k \geq n_1$.

Let us take $K^* := K' \cap K'' \in \mathfrak{F}(\mathbf{I})$ and $n^* := max\{n_0, n_1\} \in \mathbb{N}$. Thus, we have $\mathfrak{p}(f_n(x_n), f_n(x)) < \frac{\varepsilon}{2}$ for all $n \geq n^*$ where $n \in K^*$. Also, for any $j \in K^*$ following inequality

$$\mathfrak{p}(f_j(x_j), f(x)) < \mathfrak{p}(f_j(x_j), f_j(x)) + \mathfrak{p}(f_j(x), f(x)) < \varepsilon$$

holds.

This gives left pointwise $\mathbf{I}^*$-convergence of $\{f_k\}$. So, proof is ended. □

**Theorem 2.20.** *Assume that left* $\mathbf{I}^*$*-convergence signifies right* $\mathbf{I}^*$*-convergence in* $\mathbf{Y}$*. If* $\mathbf{I}$ *is Good and* $\{f_k\}_{k\in\mathbb{N}}$ *is left* $\mathbf{I}^*(\alpha)$ *convergent to* $f\in\mathbf{Y}^{\mathbf{X}}$ *at* $x\in\mathbf{X}$*, then* $\{f_k\}_{k\in\mathbb{N}}\subset\mathbf{Y}^{\mathbf{X}}$ *is left pointwise* $\mathbf{I}^*$*-convergent to* $f$ *at point* $x\in\mathbf{X}$ *and* $\{f_k\}_{k\in\mathbb{N}}$ *is left* $\mathbf{I}^*$*- exhaustive at* $x\in\mathbf{X}$*.*

*Proof.* Obviously, $\{f_k\}_{k\in\mathbb{N}}\subset\mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to $f$ at point $x\in\mathbf{X}$. Assume that $\{f_k\}$ does not left $\mathbf{I}^*$- exhaustive at a point $x$. Then, for every $A=A(x)\in\mathfrak{F}(\mathbf{I})$ there exists an $\varepsilon^{'}>0$ such that for all $\delta=\delta(\varepsilon^{'},x)>0$ and $n_0=n_0(\varepsilon,x)\in\mathbb{N}$ there exists $k\in A$ $(k\geq n_0)$ such that $\mathfrak{q}(z,x)<\delta$ implies

$$\mathfrak{p}(f_k(z),f_k(x))\geq\varepsilon^{'}.$$

Especially, let us choose $A=\mathbb{N}$ and $\delta=\frac{1}{k}$. Then, there exists $n_k\in\mathbb{N}$ such that for some $x_k\in\mathbb{B}^{-}(x,\frac{1}{k})$, implies

$$\mathfrak{p}(f_{n_k}(x_k),f_{n_k}(x))\geq\varepsilon^{'}.$$

We consider only one such $x_k$ corresponding to each such $n_k$. Let $A_k$ denote all such $n_k\in\mathbb{N}$ satisfying the above inequality and $B_k$ denote the collection of corresponding unique $x_k^{'}s$. We claim that $\mathbb{N}\setminus\{A_k\}\notin\mathfrak{F}(\mathbf{I})$. Suppose $\mathbb{N}\setminus\{A_k\}\in\mathfrak{F}(\mathbf{I})$. Then, $\{A_k\}\in\mathbf{I}$. Thus, there exists $n_0^k\in A_k$ such that

$$\mathfrak{p}(f_{n_0^k}(x_0^k),f_{n_0^k}(x))\geq\varepsilon^{'}$$

for some $x_0^k\in\mathbb{B}^{-}(x,\frac{1}{k})$, which is inconsistent with the definition of $A_k$.

Thus, $\mathbb{N}\setminus\{A_k\}\notin\mathfrak{F}(\mathbf{I})$. Since $\mathbf{I}$ is Good ideal, then from Lemma 2.8 for every $\mathbb{N}\setminus\{A_k\}\notin\mathfrak{F}(\mathbf{I})$ there exist $P_k\supset\mathbb{N}\setminus\{A_k\}$ pairwise distinct sets such that $\mathbb{N}\setminus P_k\in\mathfrak{F}(\mathbf{I})$ for every $k\in\mathbb{N}$ and $\bigcap_{k=1}^{\infty}\mathbb{N}\setminus P_k\notin\mathfrak{F}(\mathbf{I})$.

Now, let $P_k=\left\{p_1^k<p_2^k<...\right\}$. Examine a sequence $\{z_n\}$ as follows:

$$\{z_n\}:=\begin{cases}x, & n\notin\bigcap_{k=1}^{\infty}\mathbb{N}\setminus P_k,\\ x_j^k, & n\in P_k,\end{cases}$$

and $n=p_j^k$, $x_j^k\in B_k$ corresponds to the natural number $p_j^k\in A_k$. Let $\varepsilon>0$ be given. As a result, there is a least $k_0\in\mathbb{N}$ such that $\frac{1}{k_0}<\varepsilon$. Now,

$$\{n\in\mathbb{N}:\mathfrak{q}(z_n,x)\geq\varepsilon\}\subset\bigcup_{k=1}^{k_0-1}\mathbb{N}\setminus P_k\in\mathbf{I}$$

Thus, $z_n\xrightarrow{\mathbf{I}^{*-}}x, n\to\infty$. On the flip side,

$$\left\{n\in\mathbb{N}:\mathfrak{p}(f_n(z_n),f_n(x))\geq\varepsilon^{'}\right\}=\mathbb{N}\setminus P_k\in\mathfrak{F}(\mathbf{I})$$

holds which is a contradiction. Hence, $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x\in\mathbf{X}$. □

**Definition 2.21.** A sequence of functions $\{f_k\}_{k\in\mathbb{N}}\subset\mathbf{Y}^{\mathbf{X}}$ is described as left(right) uniformly $\mathbf{I}^*$- convergent to a function $f$ if for every $\varepsilon>0$ and for all $x\in\mathbf{X}$ there exists $K\notin\mathbf{I}$ with $n_0=n_0(\varepsilon)\in K$ such that $\mathfrak{p}(f_n(x),f(x))<\varepsilon$ ( $\mathfrak{p}(f(x),f_n(x))<\varepsilon$) holds for all $n\geq n_0$ and $n\in K$.

Similarly, right uniformly $\mathbf{I}^*$- convergence can also be defined.

**Theorem 2.22.** *Assume that left* $\mathbf{I}^*$*-convergence implies right* $\mathbf{I}^*$*-convergence in* $\mathbf{Y}$ *and* $x\in\mathbf{X}$*. If for every* $\varepsilon>0$ *there exists* $\delta>0$ *and* $K=\{k_1<k_2<...\}\in\mathfrak{F}(\mathbf{I})$ *such that for all* $y\in\mathbb{B}^{-}(x,\delta)$ *we have*

$$\mathfrak{p}(f_{k_n}(y),f_{k_n}(x))<\varepsilon$$

*then,* $\mathfrak{p}(f_{k_n}(x), f_{k_n}(y)) < \varepsilon$ *holds for all* $y \in \mathbb{B}^-(x, \delta)$.

*Proof.* The proof is clear. So, it is omitted here. □

**Definition 2.23.** Let $(\mathbf{X}, \mathfrak{q})$ be an asymmetric metric space and $K \subset \mathbf{X}$ be a set. Then, $K$ is said to be left(right) compact if every open cover of $K$ in left(right) topology has a finite sub-cover.

**Theorem 2.24.** *Assume that left* $\mathbf{I}^*$*-convergence implies right* $\mathbf{I}^*$*-convergence in* $\mathbf{Y}$*. If sequence of functions* $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ *is left pointwise* $\mathbf{I}^*$*-convergent to* $f$ *and* $\{f_k\}_{k\in\mathbb{N}}$ *is left* $\mathbf{I}^*-$*exhaustive on* $\mathbf{X}$*, then* $f$ *is left continuous on* $\mathbf{X}$ *and* $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ *is left uniformly* $\mathbf{I}^*-$*convergent to the function* $f$ *on every left compact subset of* $\mathbf{X}$.

*Proof.* Initially, we will establish that $f$ is left continuous on $\mathbf{X}$. Let $x \in \mathbf{X}$ be an arbitrary element. Since $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x$, then there exists $\mathrm{A} = \mathrm{A}(x) \in \mathfrak{F}(\mathbf{I})$ such that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ and there exists $n^{'} = n^{'}(\varepsilon, x) \in \mathbb{N}$ such that for every $\mathfrak{q}(y, x) < \delta$ implies $\mathfrak{p}(f_k(y), f_k(x)) < \varepsilon$ for all $n \in \mathrm{A}$ and $n \geq n^{'}$.

Postulate that $f$ is not left continuous function. Then, when $\{x_k\}$ is left $\mathbf{I}^*$-convergent to $x$, the sequence $\{f(x_k)\}$ is not left $\mathbf{I}^*$-convergent to $f(x)$. So, there exists $K = \{k_1 < k_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ such that $\lim_{n\to\infty} \mathfrak{q}(x_{k_n}, x) = 0$ holds but $\lim_{n\to\infty} \mathfrak{p}(f(x_{k_n}), f(x)) = 0$. Then, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{q}(x_{k_n}, x) < \varepsilon$ holds for all $n \geq n_0$ but there exists $n_1 \in \mathbb{N}$ such that $\mathfrak{p}\left(f(x_{k_n}), f(x)\right) \geq \varepsilon$ holds for all $n \geq n_1$. This is a conflict with the definition of being left $\mathbf{I}^*$-exhaustive and therefore $f$ is left continuous.

Let $K$ be a left compact subset of $\mathbf{X}$, $\varepsilon > 0$ and $x \in K$. Then, $f$ is left continuous at $\mathbf{x}$. Therefore, there exists $\delta > 0$ such that we have $\mathfrak{p}(f(y), f(x)) < \frac{\varepsilon}{3}$ for $y \in \mathbb{B}^-(x, \delta)$. Since, left $\mathbf{I}^*-$convergence implies right $\mathbf{I}^*$- convergence in $\mathbf{Y}$. Then, we have $\mathfrak{p}(f(y), f(x)) < \frac{\varepsilon}{3}$. Since $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$-exhaustive at $\mathbf{x}$, then there exists $\mathrm{A} = \mathrm{A}(x) \in \mathfrak{F}(\mathbf{I})$ such that for all $\varepsilon > 0$ there are $\delta = \delta(\varepsilon, x) > 0$ and $n^{'} = n^{'}(\varepsilon, x) \in \mathbb{N}$ such that for every $\mathfrak{q}(y, x) < \delta$ implies $\mathfrak{p}(f_k(y), f_k(x)) < \varepsilon$ for all $n \in \mathrm{A}, n \geq n^{'}$.

Now, $K \subset \bigcup_{x\in K} \mathbb{B}^-(x, \delta_x)$ and $K$ is left compact. Then, there exists finite number points

$$x_1, x_2, ..., x_m \in K$$

such that $K \subset \bigcup_{i=1}^m \mathbb{B}^-(x_i, \delta_{x_i})$ holds. Since $\{f_k\}_{k\in\mathbb{N}}$ is left pointwise $\mathbf{I}^*$-convergent to $f$ for each $i$ there are $A_i \in \mathfrak{F}(\mathbf{I})$ such that

$$\mathfrak{p}(f_k(x_i), f(x_i)) < \frac{\varepsilon}{3}$$

holds for each $k \in A_i$. Now, let us consider $B := \bigcap_{i=1}^m A_i \cap A_{x_i}$. Then, $B \in \mathfrak{F}(\mathbf{I})$. If $z \in K$, then there exists $i \in \{1, 2, ..., m\}$ such that $\mathfrak{q}(z, x_i) < \delta x_i < \delta$ implies that

$$\mathfrak{p}(f(x_i), f(z)) < \frac{\varepsilon}{3}$$

and

$$\mathfrak{p}(f_k(z), f_k(x_i)) < \frac{\varepsilon}{3}$$

hold for all $k \in B$ and $z \in \mathbb{B}^-(x_i, \delta_{x_i})$. Hence, we obtain

$$\mathfrak{p}(f_k(z), f(z)) < \mathfrak{p}(f_k(z), f_k(x_i)) + \mathfrak{p}(f_k(x_i), f(x_i)) + \mathfrak{p}(f(x_i), f(z)) < \varepsilon.$$

So, we arrived the proof. □

**Definition 2.25.** A sequence of left(right) continuous function $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is said to be left(right) $\mathbf{I}^*$- Alexandroff convergent to the function $f$ if, $\{f_k\}_{k\in\mathbb{N}}$ is left(right) pointwise $\mathbf{I}^*$-convergent to $f$, for all $\varepsilon > 0$ and $A \in \mathfrak{F}(\mathbf{I})$ there exists $M_A = \{m_1 < m_2 < ...\} \subset A$ and an open cover $U = \{U_k : k \in A\}$ in the left (right) topology of $\mathbf{I}$ such that for every $x \in U_k$ we have $\mathfrak{p}\left(f_{m_k}(x), f(x)\right) < \varepsilon$ $\left(\mathfrak{p}\left(f(x), f_{m_k}(x)\right) < \varepsilon\right)$.

**Definition 2.26.** [10] An asymmetric metric $\mathfrak{q}$ defined on $\mathbf{I}$ is said to have satisfy approximate metric axiom (AMA) if there exists a map $c : \mathbf{X} \times \mathbf{X} \to [0, \infty)$ such that $\mathfrak{q}(y, z) \leq c(z, y).\mathfrak{q}(z, y)$ holds for every $z, y \in \mathbf{X}$ where $c$ meets the condition described as: For all $z$, there exists $\delta_z > 0$ such that for all $y \in \mathbb{B}^+(z, \delta_z)$ implies that $c(z, y) \leq C(z)$ holds, where $C(z) > 0$ is a real number.

**Theorem 2.27.** $(\mathbf{X}, \mathfrak{q})$ *and* $(\mathbf{Y}, \mathfrak{p})$ *be asymmetric spaces. Suppose that* $(\mathbf{Y}, \mathfrak{p})$ *provides the property (AMA) and corresponding map* $C$ *is bounded. If* $\{f_k\}_{k\in\mathbb{N}}$ *is left* $\mathbf{I}^*-$ *Alexandroff convergent to the function* $f$ *then* $f$ *is left continuous.*

*Proof.* Assume that $\{f_k\}_{k\in\mathbb{N}}$ be left $\mathbf{I}^*$- Alexandroff convergent to the function $f$. Then, $\{f_k\}$ is left continuous map, $\{f_k\}$ is left pointwise $\mathbf{I}^*$-convergent to $f$ and for all $\varepsilon > 0, A \in \mathfrak{F}(\mathbf{I})$ there exists

$$M_A = \{m_1 < m_2 < ...\} \subset A$$

and open cover

$$V = \{V_k : k \in A\}$$

in the left topology of $\mathbf{X}$ such that every $x \in V_k$ we have $\mathfrak{p}(f_{m_k}(x), f(x)) < \varepsilon$.

Let $x \in \mathbf{X}$ and $\{x_n\}$ is left $\mathbf{I}^*$-convergent to $x$. Since $\{f_k\}_{k\in\mathbb{N}}$ is left pointwise $\mathbf{I}^*$- convergent to $f$, there exist $K = \{m_1 < m_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ and $n_0(\varepsilon, x) \in \mathbb{N}$ such that

$$\mathfrak{p}(f_{m_k}(x), f(x) < \frac{\varepsilon}{3r}$$

for all $m_n, n_0$. Since the function corresponding to $C$ is bounded, there exist $r > 0$ such that $C(z) < r$ holds for all $z \in \mathbf{X}$. Let $K \in \mathfrak{F}(\mathbf{I})$.

Then, there exists $M_k = \{m_1 < m_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ and open cover $V = \{V_k : k \in A\}$ such that $\mathfrak{p}(f_{m_k}(x), f(x)) < \frac{\varepsilon}{3}$ for every $x \in V_k$.

Since $V = \{V_k : k \in A\}$ is open cover, then we can choose a $k \in \mathbb{N}$ such that $x \in V_k$. Because of $f_{m_k}$ is left continuous at $\mathbf{X}$ and $\{x_n\}$ is left $\mathcal{I}^*$-convergent to $\mathbf{X}$, there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$ when $\{x_n\} \in V_k$ , $\mathfrak{p}(f_{m_k}(x_n), f_{m_k}(x)) < \frac{\varepsilon}{3}$. Since $(\mathbf{Y}, \mathfrak{p})$ satisfies the property $(AMA)$, we can see

$$\begin{aligned}\mathfrak{p}(f(x_n), f(x)) &< \mathfrak{p}(f(x_n), f_{m_k}(x_n)) + \mathfrak{p}(f_{m_k}(x_n), f_{m_k}(x)) + \mathfrak{p}(f_{m_k}(x), f(x)) \\ &< C(f_{m_k}(x_n))\mathfrak{p}(f_{m_k}(x_n), f(x_n)) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.\end{aligned}$$

□

## 3. CONCLUSION

In this work, information about $\mathbf{I}^*$-convergence in asymmetric metric spaces is given. Similar results can be generalized to $\mathbf{I}^{\mathbf{K}}$ convergence, where $\mathbf{I}$ and $\mathbf{K}$ are admissible ideals.

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# HYPERBOLIC HORADAM SPINORS

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Abstract. This study, introduces a new definition of hyperbolic spinors through a transformation from the Horadam split quaternion, which holds significant importance in mathematics and physics. Subsequently, fundamental concepts such as conjugate and norm are elucidated. Leveraging the defined hyperbolic spinor and the recurrence relation of the Horadam sequence, a novel sequence is delineated, and its foundational equations, akin to the generator function and Binet formula, are expressed through theorems.

## 1. Introduction

Number sequences are of significant interest in mathematics. Among these, the number of sequences attributed to Leonardo Fibonacci (1170–1250) stand out prominently. However, numerous sequences exist akin to the Fibonacci sequence, where each term after the second is the sum of the preceding two terms, albeit with different initial values. Notable examples include the Lucas, Pell, modified Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas numbers, each defined with distinct starting points [10].

Mathematically, quaternions represent a number system that extends beyond complex numbers, thus enriching the domain of normed division algebra. This algebraic hierarchy comprises the real numbers R, complex numbers C, quaternions H, and octonions O, marking a significant milestone in modern algebra following their discovery in 1843 by Hamilton [8]. Hamilton's seminal work has resonated across diverse disciplines, spanning from quantum physics to computer science [6, 7, 13]. Within algebraic realms, split quaternions or coquaternions emerge as elements within a 4-dimensional associative algebra initially introduced by James Cockle [5]. Diverging from the quaternion algebra established by Hamilton, which delineates a 4-dimensional real vector space equipped with a multiplicative operation, split quaternions exhibit distinctive attributes. They encompass zero divisors, nilpotent elements, and nontrivial idempotent elements, distinguishing themselves from conventional quaternionic structures.

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Spinor is a significant concept in quantum mechanics, particularly in areas such as spacetime geometry and particle physics. Spinors are used to represent a property called spin, which is the rotation of a particle around its axis, determining its magnetic moment and response to magnetic fields. Spinors are mainly employed in defining fermions (particles with spin) in quantum mechanics, particularly in describing the properties of fundamental particles such as electrons, protons, and neutrons. Consequently, spinors are crucial for understanding and predicting the behavior of fundamental particles [3]. However, spinors are not limited to quantum mechanics alone. Mathematically, spinors are also utilized in general relativity and spacetime geometry. Spinors are particularly used to describe the behavior of particles subject to Lorentz transformations (operations rotating and changing the direction of spacetime). This is particularly important for understanding topics such as spacetime curvature and time dilation within the framework of Einstein's general theory of relativity. Spinors represent an essential concept with broad applications in physics and mathematics, utilized in various fields ranging from theoretical physics to practical applications such as magnetic resonance imaging [12].

## 2. Preliminaries

The recurrence relation defines Horadam number sequence

$$W_{n+2} = pW_{n+1} + qW_n$$

with initial conditions $W_0 = a$, $W_1 = b$, for $n \geq 0$. The characteristic equation of the recurrence relation of this sequence is

$$x^2 - px - q = 0$$

the roots of the equation are

$$\alpha = \frac{1+\sqrt{d}}{2}, \beta = \frac{1-\sqrt{d}}{2}, d = p^2 + 4q.$$

The recurrence relation of the $(p, q)$-Fibonacci number sequence derived from the Horadam number sequence with initial conditions $a = 0$ and $b = 1$ is

$$U_{n+2} = pU_{n+1} + qU_n.$$

where $U_n$ is $n$th $(p, q)$-Fibonacci number, for $n \geq 0$ [4].

Ipek has formulated the recurrence relation for $(p, q)$-Fibonacci quaternions, represented by the equation

$$QU_{n+2} = pQU_{n+1} + qQU_n, n \geq 0.$$

and has subsequently derived various identities. These include the Binet formula, generating functions, and specific binomial sums incorporating $(p, q)$-Fibonacci quaternions.

The recurrence relation defines $(p, q)$- Lucas sequence

$$V_{n+2} = pV_{n+1} + qV_n$$

with initial conditions $V_0 = 2$, $V_1 = b$, for $n \geq 0$ [9].

Patel and Ray introduced the $(p, q)$- Lucas quaternion and they define this quaternion as follows [11].

The $(p, q)$- Lucas quaternion is defined recursively by

$$QV_{n+2} = pQV_{n+1} + qQV_n, n \geq 0.$$

Let's give some information about split quaternions, which play an essential role in our paper. You can find more detailed information in [1].

A split quaternion is defined with $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and the quaternion basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is given such that

$$\mathbf{i}^2 = -1, \mathbf{j}^2 = \mathbf{k}^2 = 1, \quad \mathbf{ijk} = 1, \quad \mathrm{ij} = -\mathrm{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathrm{kj} = \mathrm{i}, \quad \mathbf{ki} = -\mathrm{ik} = \mathbf{j}$$

Let $q_0 = S_q$ and $\mathbf{V}_q = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ be scalar and vectorial parts of the quaternion $q$. So, we can write the quaternion $q$ as $q = S_q + \mathbf{V}_q$. The set of these quaternions is $\mathbb{K}$. Let $p = S_p + \mathbf{V}_p, q = S_q + \mathbf{V}_q \in \mathbb{K}$ be two real quaternions.

$\overline{q}$ is the conjugate of the quaternion $q$ is equal to $\overline{q} = S_q - \boldsymbol{V}_q$ and it is

$$\overline{q} = \mathbf{q}_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3$$

In addition, the norm of a split quaternion

$$N(q) = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

For $n \geq 0$, define the split Horadam quaternion $H_n$ by

$$H_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}$$

where, $W_n$ is the $n$th Horadam number and i,j,k are split quaternionic units [2].

On the other hand, let us consider the vector $(\alpha_1, \alpha_2, \alpha_3)$ with ${\alpha_1}^2 + {\alpha_2}^2 + {\alpha_3}^2 = 0$ in the complex vector space $\mathbb{C}^3$. These vectors form a two-dimensional surface in the two-dimensional $\mathbb{C}^2$ subspace of $\mathbb{C}^3$ . If the parameters of this two-dimensional surface are taken as $\varphi_1$ and $\varphi_2$ , the following equations can be written

$$\begin{aligned} \alpha_1 &= {\varphi_1}^2 - {\varphi_2}^2 \\ \alpha_2 &= i\left({\varphi_1}^2 + {\varphi_2}^2\right) \\ \alpha_3 &= -2\varphi_1\varphi_2 \end{aligned}$$

Thus, each isotropic vector in $\mathbb{C}^3$ corresponds to a vector in $\mathbb{C}^2$ and vice versa. The vector $\varphi = (\varphi_1, \varphi_2) \cong \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ obtained in this way is called a spinor [12].

## 3. Main Theorems and Proofs

Hyperbolic spinors corresponding to the Horadam split quaternion were defined using the transformations provided in the preceding section. Their conjugates, norms and fundamental properties were examined in this section. Additionally, important equalities and theorems, such as the Binet formula and the generating function were proven. Consequently, by determining the initial conditions of the Horadam sequence, special cases of hyperbolic spinous, namely $(p, q)$- Fibonacci and $(p, q)$- Lucas hyperbolic spinors, were introduced and fundamental equations for both were derived.

**Definition 3.1.** *Let $H_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}$ be nth split Horadam quaternion where $W_n$ is nth Horadam number for $n \geq 0$. $H_w$ is the set of split Horadam quaternions. Therefore, we give the linear transformation between the hyperbolic spinors and split quaternions as follows:*

$$\varphi_w : H_w \longrightarrow S$$

$$H_w \longrightarrow \varphi_w\left(W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}\right) = \begin{bmatrix} W_n + jW_{n+3} \\ -W_{n+1} + jW_{n+2} \end{bmatrix}$$

*Furthermore, a new hyperbolic Horadam spinor sequence can be introduced using the spinor defined. The recurrence relation of this sequence is as follows.*

$$SH_{n+1} = pSH_n + qSH_{n-1}$$

*where $p$ and $q$ are real numbers,*

$$SH_0 = \begin{bmatrix} a + j\left(p^2b + qpa + qb\right) \\ -b + j(pb + qa) \end{bmatrix}, SH_1 = \begin{bmatrix} b + j\left(p^3b + p^2qa + 2pqb + qbpa\right) \\ -pb - qa + j\left(p^2b + qpa + qb\right. \end{bmatrix}$$

are initial conditions.

The set of hyperbolic Horadam spinor sequences is

$$\{SH_n\}_{n\in\mathrm{N}}^{\infty}$$
$$= \left\{ \begin{bmatrix} a + j(p^2b + qpa + qb) \\ -b + j(pb + qa) \end{bmatrix}, \begin{bmatrix} b + j(p^3b + p^2qa + 2qbp + q^2bpa) \\ -pb - qa + j(p^2b + qpa + qb) \end{bmatrix}, \ldots, \begin{bmatrix} W_n + jW_{n+3} \\ -W_{n+1} + jW_{n+2} \end{bmatrix}, \ldots \right\}$$

where $SH_n = \begin{bmatrix} W_n + jW_{n+3} \\ -W_{n+1} + jW_{n+2} \end{bmatrix}$ is $n$th hyperbolic Horadam spinor and $W_n$ is nth Horadam number.

**Definition 3.2.** *Let $n \geq 0, n \in \mathbb{Z}$ and the $n$th hyperbolic $(p, q)$- Lucas spinor is $SV_n$. Then, the recurrence relation of hyperbolic $(p, q)$- Lucas spinors is as follows:*

$$SV_{n+2} = pSV_{n+1} + qSV_n$$

*with initial conditions*

$$SV_0 = \begin{bmatrix} 2 + j\left(p^2b + 2pq + pqb\right. \\ -b + j(pb + 2q) \end{bmatrix}$$
$$SV_1 = \begin{bmatrix} b + j\left(p^3b + 2qp^2 + p^2qb + pbq + 2q^2\right) \\ -(pb + 2q) + j\left(p^2b + 2qb + pbq\right) \end{bmatrix}.$$

Similar to number sequences, here, by keeping the coefficients constant in the recurrence relation of the hyperbolic Horadam spinor sequence and changing the initial conditions to $a = 0, b = 1$, the hyperbolic $(p, q)$- Fibonacci spinor sequence can be obtained. Similarly, when $a = 2, b = b$ is taken, the hyperbolic $(p, q)$-Lucas spinor sequence can be obtained as follows:

**Definition 3.3.** *Hyperbolic $(p, q)$- Fibonacci spinor sequence is defined with*

$$SU_{n+2} = pSU_{n+1} + qSU_n$$

*recurrence relation for $n \geq 0$. The initial conditions of this sequence are*

$$SU_0 = \begin{bmatrix} j\left(p^2 + q\right) \\ -1 + jp \end{bmatrix}, SU_1 = \begin{bmatrix} 1 + j\left(p^3 + 2pq\right) \\ -p + j\left(p^2 + q\right) \end{bmatrix}.$$

The terms for this two hyperbolic spinor, defined with $a = 0, b = 1$, have been obtained. In hyperbolic $(p, q)$- Fibonacci spinors, taking $p = 1, q = 1$ results in the recurrence relation of the Fibonacci sequence. Therefore, similar properties provided for hyperbolic Horadam spinors can readily be derived for hyperbolic Fibonacci spinors. A parallel situation also holds for hyperbolic $(p, q)$-Lucas spinors. By classifying hyperbolic Horadam spinors, such as the Fibonacci, Pell, Pell-Lucas, Jacobsthal, Jacobsthal Lucas sequences obtained through the classification of the coefficients and initial conditions of the Horadam sequence, the following table can be derived.

| $p$ | $q$ | $a$ | $b$ | Hyperbolic Horadam spinor |
|---|---|---|---|---|
| $p$ | $q$ | 0 | 1 | Hyperbolic ( $p, q$ )-Fibonacci spinor |
| $p$ | $q$ | 2 | $p$ | Hyperbolic ( $p, q$ )-Lucas spinor |
| 2 | 1 | 0 | 1 | Hyperbolic Pell spinor |
| 1 | 2 | 0 | 1 | Hyperbolic Jacobsthal spinor |
| 1 | 1 | 2 | 1 | Hyperbolic Lucas spinor |
| 2 | 1 | 2 | 2 | Hyperbolic Pell Lucas spinor |
| 2 | 1 | 2 | 1 | Hyperbolic Jacobsthal Lucas spinor |

TABLE 1. Various hyperbolic spinor types.

For example, let's construct the hyperbolic Lucas spinor sequence using the numerical values specified in the table. Let the general term of the sequence be denoted as $SHL_n$. Then, the initial conditions $SHL_0$ and $SHL_1$ are as follows.

$$SHL_0 = \begin{bmatrix} 2+4j \\ -1+3j \end{bmatrix},$$

$$SHL_1 = \begin{bmatrix} 1+7j \\ -3+4j \end{bmatrix}.$$

Since the recurrence relation of the Lucas sequence is satisfied, the other terms of the sequence are obtained using the relation

$$SHL_{n+1} = SHL_n + SHL_{n-1}.$$

**Definition 3.4.** *Let the conjugate of the split Horadam quaternion* $\overline{H_n} = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}$. *The hyperbolic Horadam spinor* $\overline{SH_n}$ *corresponding to the conjugate of the split Horadam quaternion is written by*

$$\left(\overline{SH_n}\right) = \begin{bmatrix} W_n - jW_{n+3} \\ -W_{n+1} - jW_{n+2} \end{bmatrix}$$

*Furthermore, by utilizing conjugate definitions, we can obtain the following. The hyperbolic conjugate of hyperbolic Horadam spinor* $SH_n$ *is*

$$SH_n^* = \begin{bmatrix} W_n - jW_{n+3} \\ W_{n+1} - jW_{n+2} \end{bmatrix}.$$

*Hyperbolic Horadam spinor conjugate* $S\tilde{H}_n = jC\overline{SH_n}$ *is*

$$S\tilde{H}_n = \begin{bmatrix} -W_{n+2} - jW_{n+1} \\ W_{n+3} - jW_n \end{bmatrix}$$

*The mate of hyperbolic Horadam spinor* $S\breve{H}_n = -C\overline{SH_n}$ *is*

$$S\breve{H}_n = \begin{bmatrix} W_{n+1} + jW_{n+2} \\ W_n - jW_{n+3} \end{bmatrix}$$

*where*

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let's give an example for each of the numerical counterparts of the given conjugate definitions.

For $n = 0$ hyperbolic Horadam spinor $SH_0$, the conjugatesof this spinor are as follows.

$$(\overline{SH_0}) = \begin{bmatrix} a - j(pc+qb) \\ -b - jc \end{bmatrix},$$

$$SH_0^* = \begin{bmatrix} a - j(pc+qb) \\ b - jc \end{bmatrix},$$

$$S\tilde{H}_0 = \begin{bmatrix} -a - j(pc+qb) \\ b - jc \end{bmatrix},$$

$$S\check{H}_0 = \begin{bmatrix} a + j(pc+qb) \\ b - jc \end{bmatrix}.$$

**Corollary 3.5.** *For the hyperbolic Horadam spinor $SH_n$ and its conjugates, the following equalities are valid.*

$$\cdot\, CS\hat{H}_n = \overline{SH_n},$$
$$\cdot\, j\tilde{H}_n = -S\hat{H}_n,$$
$$\cdot\, jCS\tilde{H}_n = -\overline{SH_n}.$$

**Proposition 3.6.** *Let the nth hyperbolic Horadam spinor* $\mathrm{SH}_n$ *be the spinor corresponding to the nth split Horadam quaternion $W_n$. In this case, the hyperbolic Horadam spin or representation of the split quaternion norm is as follows:*

$$N(H_n) = (SH_n^*)^\top SH_n.$$

*Proof.* Assume that $n$th hyperbolic Horadam spinor $SH_n$ corresponds to the $n$th split Horadam quaternion $W_n$. Then, the following equation can be obtained as:

$$(SH_n^*)^\top SH_n = \begin{bmatrix} W_n - jW_{n+3} & W_{n+1} - jW_{n+2} \end{bmatrix} \begin{bmatrix} W_n + jW_{n+3} \\ -W_{n+1} + jW_{n+2} \end{bmatrix}$$

We can associate to the product of two Horadam split quaternions with a hyperbolic Horadam spinor matrix product as follows:

$$qw \to \hat{q}w \longrightarrow \hat{Q}SH$$

where $\hat{Q}$ is the hyperbolic, unitary, square matrix defined by

$$\hat{Q} = \begin{bmatrix} W_0 + jW_3 & W_1 + jW_2 \\ -W_1 + jW_2 & W_0 - jW_3 \end{bmatrix}.$$

□

We present fundamental equations, such as the Binet formula, generating function.

**Theorem 3.7.** *The Binet formula for hyperbolic Horadam spinor is as follows.*

$$SH_n = \frac{1}{2\sqrt{d}}\left(\begin{bmatrix} r + \sqrt{d}(a + j(pc+qb)) \\ s + \sqrt{d}(-b + jc) \end{bmatrix}\alpha^n - \begin{bmatrix} r - \sqrt{d}(a + j(pc+qb)) \\ s - \sqrt{d}(-b + jc) \end{bmatrix}\beta^n\right)$$

*where* $r = 2b - pa + j\left(p^2c + pqb + 2qc\right), s = pb - 2c + j(pc + 2qb),$

$$c = pb + qa, \alpha = \frac{1+\sqrt{d}}{2}, \beta = \frac{1-\sqrt{d}}{2}, d = p^2 + 4q.$$

*Proof.* The characteristic equation of the recurrence relation of hyperbolic Horadam spinor sequence is $x^2 - px - q = 0$. The discriminant of this equation is $d = p^2 + 49$ and the roots of $\alpha$ and $\beta$ are $\alpha = \frac{p+\sqrt{d}}{2}, \beta = \frac{p-\sqrt{d}}{2}$. The Bines formula for hyperbolic Horadam spinor sequence is $SH_n = A\alpha^n + B\beta^n$ where $A = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, B = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$ are $2 \times 1$ matrices. When substituted for $n = 0$ and $n = 1$, after performing the necessary operations, the desired result is obtained. □

As a result, the Binet formulas for hyperbolic $(p, q)$- Fibonacci and hyperbolic $(p, q)$- Lucas spinors can be expressed as follows.

**Corollary 3.8.** *The Binet formulas for hyperbolic* $(p, q)$*-Fibonacci and hyperbolic* $(p, q)$*- Lucas spinor are as follows, respectively.*

$$SU_n = \frac{1}{2\sqrt{d}} \left( \begin{bmatrix} 2 + j\left(p^3 + 3pq\right) + \sqrt{d}\left(jp^2 + jq\right) \\ -p + j\left(p^2 + 2q\right) + \sqrt{d}(-1 + jq) \end{bmatrix} \alpha^n - \begin{bmatrix} 2 + j\left(p^3 + 3pq\right) - \sqrt{d}\left(jp^2 + jq\right) \\ -p + j\left(p^2 + 2q\right) - \sqrt{d}(-1 + jq) \end{bmatrix} \beta^n. \right.$$

$$SV_n = \frac{1}{2\sqrt{d}} \left( \begin{matrix} 2b - 2p + j\left(p^3b + p^2bq + 4qp^2 + 4q^2\right) + \sqrt{d}\left(4 + 2p^2bj + 4qbj + 2pqbj\right) \\ -pb - 4q + j\left(p^2b + 2pq + 2pqb\right) + \sqrt{d}(-2b + 2pb + 4q) \end{matrix} \right] \alpha^n -$$
$$\begin{bmatrix} 2b - 2p + j\left(p^3b + p^2qb + 4qp^2 + 4q^2\right) - \sqrt{d}\left(4 + 2p^2bj + 4qbj + 2pbqj\right) \\ -pb - 4q + j\left(p^2b + 2pq + 2pbq\right) - \sqrt{d}(-2b + 2pbj + 4qj) \end{bmatrix} \beta^n.$$

**Theorem 3.9.** *The generating function for the nth hyperbolic Horadam spinor is obtained as follows:*

$$G_w(x) = \frac{1}{1 - px - qx^2} \left(SH_0(1 - px) + SH_1\right),$$

*where*

$$SH_0 = \begin{bmatrix} a + j(pc + qb) \\ -b + jc \end{bmatrix}$$
$$SH_1 = \begin{bmatrix} b + j\left(p^2c + qbp + qc\right) \\ -c + j(pc + qb) \end{bmatrix}, \quad c = pb + qa.$$

*Proof.* Assume that $SH_n$ is the $n$th hyperbolic Horadam spinor and the generating function of the hyperbolic Horadam spinor is

$$G_w(x) = \sum_{n=0}^{\infty} SH_n x^n.$$

First, the function can be written from the recurrence relation of the hyperbolic Horadam spinor sequence as follows:

$$\sum_{n=0}^{\infty} SH_{n+2}x^n = p\sum_{n=0}^{\infty} SH_{n+1}x^n + q\sum_{n=0}^{\infty} SH_n x^n,$$
$$\sum_{n=2}^{\infty} SH_n x^{n-2} = p\sum_{n=1}^{\infty} SH_n x^{n-1} + q\sum_{n=0}^{\infty} SH_n x^n.$$

Then, the following equation can be obtained

$$\frac{1}{x^2}\left[-SH_0 - SH_1 + G_w(x)\right] = p\frac{1}{x}\left[-SH_0 + G_w(x)\right] + qG_w(x)$$

Consequently, for the hyperbolic Horadam spinors, the generating function is obtained as follows.

$$G_w(x) = \frac{1}{1-px-qx^2}\left(SH_0(1-px)+SH_1\right).$$

□

**Corollary 3.10.** *The generating functions for hyperbolic $(p,q)$- Fibonacci spinors and hyperbolic $(p,q)$-Lucas spinor are as follows, respectively.*

$$G_u(x) = \frac{1}{1-px-qx^2}\begin{bmatrix} 1+j\left(p^3+2pq+p^2+q-p^3x-qpx\right) \\ -1-p+j\left(p-p^2x+p^2+q\right) \end{bmatrix},$$

$$G_v(x) = \frac{1}{1-px-qx^2}\begin{bmatrix} 2+b-2px+j\left((1-x)\left(p^3b+2qp^2+p^2bq\right)+2pq+2pqb+2q^2\right) \\ -b-pb-2q+pbx+j\left((1-x)\left(p^2b+2pq\right)+pbq+2q\right) \end{bmatrix}.$$

## 4. Conclusion

This study, defined hyperbolic Horadam spinor sequences using the most general form of number sequences, namely Horadam sequences and split Horadam quaternions. Additionally, $(p,q)$-Fibonacci and $(p,q)$-Lucas hyperbolic spinor sequences were defined using the general forms of Fibonacci and Lucas sequences with parameters $p$ and $q$. The relationships between these newly defined sequences, as well as their internal relationships, were demonstrated through provided equalities. As a result, a new hyperbolic spinor sequence was defined based on the properties of number sequences and the definitions of split quaternions and spinors.

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# ON THE LAGRANGE INTERPOLATIONS OF THE JACOBSTHAL AND JACOBSTHAL-LUCAS SEQUENCES

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Abstract. This study explores the formation of polynomials of at most degree $n$ using the first $n+1$ terms of the Jacobsthal and Jacobsthal-Lucas sequences through Lagrange interpolation. The paper provides a detailed examination of the recurrence relations and various identities associated with the Jacobsthal and Jacobsthal-Lucas Lagrange Interpolation Polynomials.

## 1. Introduction

As is well known, Fibonacci numbers have been prominently featured in applied sciences. There have been many studies on Fibonacci numbers and their generalizations over the centuries. Lucas numbers, which share the same recurrence relation but have different initial conditions from Fibonacci numbers, have many relationships with the Fibonacci numbers. Both Fibonacci and Lucas numbers are sequences of second-order recurrence relations. There are many sequences of the same order, some of which include Pell, Jacobsthal, Pell-Lucas, and Jacobsthal-Lucas sequences. Among the generalizations of the Fibonacci sequence, the Tribonacci sequence has a third-order recurrence relation. Some sequences with a third-order recurrence relation are the Narayana, Perrin, and Padovan sequences. The purpose of this study is to establish a relationship between the Lagrange interpolation with the Jacobsthal sequences.
Jacobsthal numbers have attracted a lot of interest due to their intriguing characteristics. Jacobsthal and Jacobsthal-Lucas numbers appear respectively as the integer sequences A001045 and A014551 from [21, 22]. The Jacobsthal sequence $\{J_n\}_{n\geq 0}$ is

$$J_{n+2} = J_{n+1} + 2J_n. \tag{1.1}$$

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with the initial elements $J_0 = 1$ and $J_1 = 1$. First few terms of the sequence $\{J_n\}$ are

$$1, 1, 3, 5, 11, 21, 43, 85, 171, 341.$$

The Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ is

$$j_{n+2} = j_{n+1} + 2j_n. \tag{1.2}$$

with the initial elements $j_0 = 2$ and $j_1 = 1$. First few terms of the sequence $\{j_n\}$ are

$$2, 1, 5, 7, 17, 31, 65, 127, 257, 511.$$

Some studies related to the sequences $\{J_n\}$ and $\{j_n\}$ can be found in [1–12, 14–16, 18, 19, 23]. The characteristic equation of the recurrences $\{J_n\}$ and $\{j_n\}$ is

$$x^2 - x - 2 = 0 \tag{1.3}$$

where the roots of the equation (1.3) are

$$x_1 = 2 \quad \text{and} \quad x_2 = -1$$

in order for,

$$x_1 + x_2 = 1, \quad x_1 - x_2 = 3 \quad \text{and} \quad x_1 x_2 = -2.$$

The Binet-like formulas of the sequences $\{J_n\}$ and $\{j_n\}$ are

$$J_n = \frac{2^n - (-1)^n}{3} \tag{1.4}$$

and

$$j_n = 2^n + (-1)^n, \tag{1.5}$$

respectively. Some interrelationships are

$$J_n + j_n = 2J_{n+1} \tag{1.6}$$

$$3J_n + j_n = 2^{n+1} \tag{1.7}$$

The Lagrange interpolating polynomial is essentially a rephrased version of the Newton polynomial that eliminates the need to calculate divided differences. Lagrange interpolation is beneficial because it is effective for data points that are unevenly spaced along the independent variable. In the realm of numerical analysis, interpolation refers to the method of identifying the most suitable function based on certain given points. The most basic form of interpolation uses a polynomial. This implies that for a set of specified points, there is a polynomial that intersects all these points. This polynomial approximates the underlying function closely. One technique for polynomial interpolation is the Lagrange interpolation method [20].

Let $P_n$ be the set of all real-valued polynomials of degree at most $n$ defined over the set $\mathbb{R}$ of real numbers, given that $n$ is a nonnegative integer. The basic interpolation problem is as follows: identify a polynomial $p_0 \in P_0$ such that $p_0(x_0) = y_0$, given $x_0$ and $y_0$ in $\mathbb{R}$. This can be solved by using the formula $p_0(x) \equiv y_0$. Examining the subsequent, more general problem is the primary purpose [13].

Let $n \geq 1$, and assume that $x_i$ for $i = 0, 1, \ldots, n$ are distinct real numbers (i.e., $x_i \neq x_j$ for $i \neq j$), and $y_i$ for $i = 0, 1, \ldots, n$ are real numbers. We want to find $p_n \in \mathcal{P}_n$ such that $p_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$.

There exist polynomials $L_k \in \mathcal{P}_n$ for $k = 0, 1, \ldots, n$, such that

$$L_k(x_i) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

for all $i, k = 0, 1, \ldots, n$. Furthermore,

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$$

satisfies the interpolation conditions mentioned above. In other words, $p_n \in \mathcal{P}_n$ and $p_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$. For each fixed $k$, $0 \leq k \leq n$, $L_k$ is required to have $n$ zeros at $x_i$ for $i = 0, 1, \ldots, n$ and $i \neq k$. Thus, $L_k(x)$ is of the form

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

Once these basis polynomials are constructed, the Lagrange interpolation polynomial can be expressed as follows:

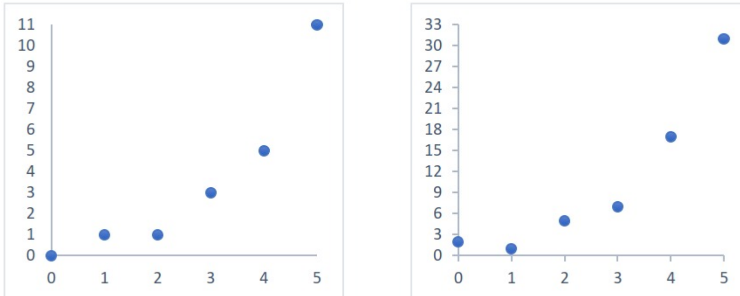
$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k = \sum_{k=0}^{n} \left( \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} \right) y_k. \tag{1.8}$$

Based on these statements, Mufid and at al. showed that a polynomial of degree $n$ at most can be created from the first $n+1$ terms of the Fibonacci sequence using Lagrange interpolation, and that this Fibonacci Lagrange Interpolation Polynomial (FLIP) can be obtained both recursively and implicitly [17].

In this study, we first investigate the formation of polynomials of degree at most $n$ using the first $n+1$ terms of the sequences $\{J_n\}$ and $\{j_n\}$es via Lagrange interpolation. Then, we present a detailed examination of the recurrence relations and various identities associated with the Jacobsthal and Jacobsthal-Lucas Lagrange Interpolation Polynomials.

## 2. The Lagrange interpolation of the Jacobsthal sequences

Before we commence the interpolations of the sequences $\{J_n\}$ and $\{j_n\}$, we will plot these sequences on the $xy-$coordinate system. Let's denote the Jacobsthal point as $p_n = (n, J_n)$ and the Jacobsthal-Lucas point as $q_n = (n, j_n)$, representing the points associated with the $n-$th terms of the sequences $\{J_n\}$ and $\{j_n\}$, respectively. For illustration, the points of the sequences $\{J_n\}$ and $\{j_n\}$ from $n = 0$ to $n = 5$ are depicted in Figure 1.




Jacobsthal points    Jacobsthal-Lucas points

FIGURE 1. The points of the sequences $\{J_n\}$ and $\{j_n\}$

We define $\mathbb{J}_n(x)$ as the polynomial constructed using the Lagrange interpolation from the points $j_k$ for $k \in \{0, 1, \ldots, n\}$. Specifically, we interpolate using the points $(x_k, y_k) = (k, J_k)$. Accordingly, with $x_k = k$ and $y_k = J_k$ in equation (1.8), we write

$$\mathbb{J}_n(x) = \sum_{k=0}^{n} \left( \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x-i}{k-i} \right) J_k. \tag{2.1}$$

The factors $(k - i)$ in equation (2.1) can be simplified as follows.

$$\prod_{\substack{i=0 \\ i \neq k}}^{n} (k-i) = (-1)^{n-k}(n-k)!k! = (-1)^{n-k}\binom{n}{k} n! \tag{2.2}$$

Upon incorporating equation (2.2) into equation (2.1), we obtain:

$$\mathbb{J}_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \left( (-1)^{n-k} \binom{n}{k} \prod_{\substack{i=0 \\ i \neq k}}^{n} (x-i) \right) J_k. \tag{2.3}$$

For instance, we can obtain $\mathbb{J}_n(x)$ for $n = 1, 2, 3, 4$ using equation (2.3) as follows:

$$\mathbb{J}_1(x) = 1,$$

$$\mathbb{J}_2(x) = x^2 - x + 1,$$

$$\mathbb{J}_3(x) = -\frac{1}{6}\left(2x^3 - 12x^2 + 10x - 6\right),$$

$$\mathbb{J}_4(x) = \frac{1}{24}\left(6x^4 - 44x^3 + 114x^2 - 76x + 24\right).$$

In Figure 2, the graphs of the above polynomials are displayed. Recall that $\mathbb{J}_n(k) = J_k$ for $k \in \{0, 1, \ldots, n\}$.

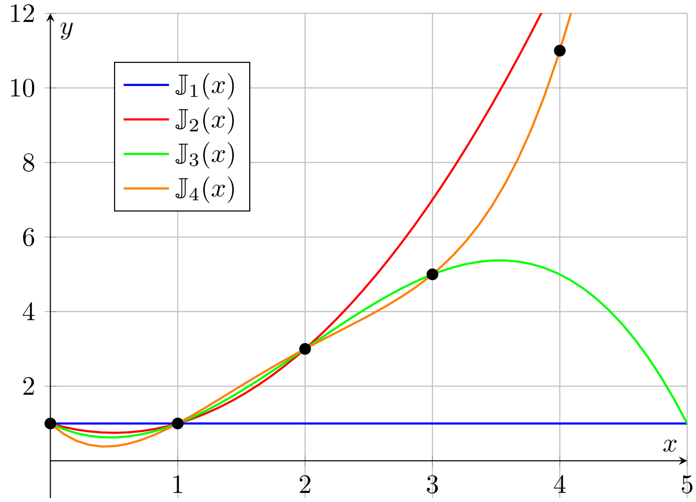



FIGURE 2. Graphs of $\mathbb{J}_1(x)$, $\mathbb{J}_2(x)$, $\mathbb{J}_3(x)$, and $\mathbb{J}_4(x)$

Similarly, the interpolation of the Jacobsthal-Lucas sequence is expressed as:

$$\gimel_n(x) = \frac{1}{n!}\sum_{k=0}^{n}\left((-1)^{n-k}\binom{n}{k}\prod_{\substack{i=0\\ i\neq k}}^{n}(x-i)\right) j_k. \tag{2.4}$$

For instance, we can obtain $\gimel_n(x)$ for $n = 1, 2, 3, 4$ using equation (2.4) as follows:

$$\gimel_1(x) = -x + 2,$$

$$\gimel_2(x) = \frac{1}{2}\left(5x^2 - 7x + 4\right),$$

$$\gimel_3(x) = -\frac{1}{6}\left(7x^3 - 36x^2 + 35x - 12\right),$$

$$\gimel_4(x) = \frac{1}{24}\left(17x^4 - 130x^3 + 331x^2 - 242x + 48\right).$$

In Figure 3, the graphs of the above polynomials are displayed. Recall that $\gimel_n(k) = j_k$ for $k \in \{0, 1, \ldots, n\}$.

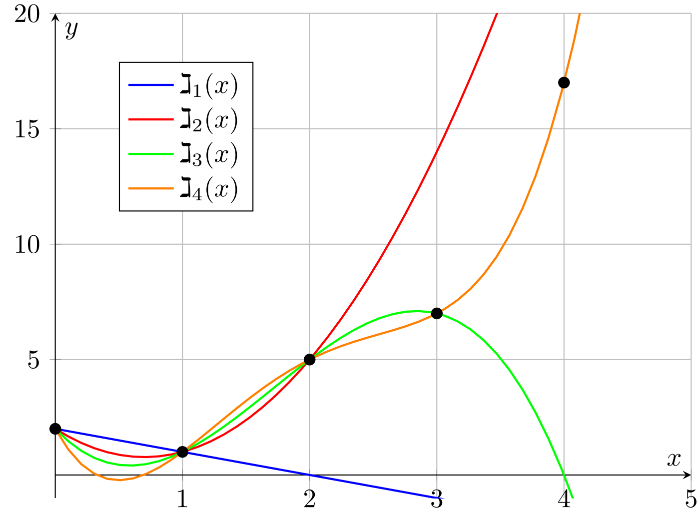



FIGURE 3. Graphs of $\gimel_1(x)$, $\gimel_2(x)$, $\gimel_3(x)$, and $\gimel_4(x)$

We shall establish a leading coefficient theorem for $\mathbb{J}_n(x)$, before obtaining more formulas for it.

**Theorem 2.1.** *The leading coefficients of $\mathbb{J}_n(x)$ and $\gimel_n(x)$ are*

$$\frac{(-1)^{n+1}J_n}{n!}$$

*and*

$$\frac{(-1)^{n+1}j_n}{n!},$$

*respectively.*

*Proof.* The leading coefficient of $\mathbb{J}_n(x)$ is equal to $\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}J_k\binom{n}{k}$. Thus, we just have to demonstrate that

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{n-k}J_k\binom{n}{k} &= \frac{1}{3}\sum_{k=0}^{n}(-1)^{n-k}\left(2^k-(-1)^k\right)\binom{n}{k} \\
&= \frac{1}{3}\left[\sum_{k=0}^{n}(-1)^{n-k}2^k\binom{n}{k} - \sum_{k=0}^{n}(-1)^{n-k}(-1)^k\binom{n}{k}\right] \\
&= \frac{1}{3}\left[(2-1)^n-(-1-1)^n\right] \quad \text{(by identity (1))} \\
&= \frac{(-1)^{n+1}}{3}\left[2^n-(-1)^n\right] \\
&= (-1)^{n+1}\left[\frac{2^n-(-1)^n}{3}\right] \\
&= (-1)^{n+1}J_n
\end{aligned}$$

Similarly, the leading coefficient of $\gimel n(x)$ is found to be $\dfrac{(-1)^{n+1}j_n}{n!}$. □

## 3. Recurrence relations of the $\mathbb{J}_n(x)$ and $\gimel_n(x)$

In this part, we will derive the additional formulas for $\mathbb{J}_n(x)$ and $\gimel_n(x)$. It's remarkable that $\mathbb{J}_n(x)$ and $\gimel_n(x)$ can be constructed recurrence relations, much like the sequences $\{J_n\}$ and $\{j_n\}$ respectively.

Consider the polynomials $\mathbb{J}_{n+1}(x)$ and $\mathbb{J}_n(x)$. For each $i \in \{0, 1, 2, \ldots, n\}$, we have $\mathbb{J}_{n+1}(i) = \mathbb{J}_n(i)$. In other words, the polynomials $\mathbb{J}_{n+1}(x)$ and $\mathbb{J}_n(x)$ intersect at $n+1$ points. Consequently, we can express the relationship as follows:

$$\mathbb{J}_{n+1}(x) - \mathbb{J}_n(x) = a \cdot x(x-1)\cdots(x-n),$$

where $a$ denotes the leading coefficient of $\mathbb{J}_{n+1}(x)$. As a results, when $P_n(x) = x(x-1)\cdots(x-n)$,

$$\mathbb{J}_{n+1}(x) = \mathbb{J}_n(x) + \frac{(-1)^n J_{n+1}}{(n+1)!} P_n(x)$$

is a recursive formula.

By successively applying the recursive formula for $\mathbb{J}_n(x)$, $\mathbb{J}_{n-1}(x)$, ..., $\mathbb{J}_2(x)$, we derive the following implicit formula:

$$\mathbb{J}_{n+1}(x) = \mathbb{J}_1(x) + \sum_{i=1}^{n} \frac{(-1)^i J_{i+1}}{(i+1)!} P_i(x)$$

which simplifies to

$$\mathbb{J}_n(x) = \sum_{i=1}^{n} \frac{(-1)^{i-1} J_i}{i!} P_{i-1}(x). \tag{3.1}$$

Similarly, the recurrence relation of the $\gimel_n(x)$ is derived as:

$$\gimel_n(x) = -2x + 2 + \sum_{i=1}^{n} \frac{(-1)^{i-1} j_i}{i!} P_{i-1}(x). \tag{3.2}$$

Some relationships between recurrence relations (3.1) and (3.2) are as follows:

1.
$$\mathbb{J}_n(x) + \gimel_n(x) = -2x + 2 + 2\sum_{i=1}^{n} \frac{(-1)^{i-1} J_{i+1}}{i!} P_{i-1}(x)$$

2.
$$3\mathbb{J}_n(x) + \gimel_n(x) = -2x + 2 - \sum_{i=1}^{n} \frac{(-2)^i}{i!} P_{i-1}(x)$$

**Theorem 3.1.** *The Binet-like formulas of the recurrence relations of the $\mathbb{J}_n(x)$ and $\gimel_n(x)$ are, respectively,*

$$\mathbb{J}_n(x) = \sum_{i=1}^{n} \frac{1 - (-2)^i}{3i!} P_{i-1}(x)$$

*and*

$$\gimel_n(x) = -2x + 2 - \sum_{i=1}^{n} \frac{(-2)^i + 1}{i!} P_{i-1}(x).$$

*Proof.* It is easily proven using the equalities of (1.4) and (1.5). □

## 4. Conclusion

This study investigated the formation of polynomials of degree at most $n$ using the first $n+1$ terms of the sequences $\{J_n\}$ and $\{j_n\}$ through Lagrange interpolation. The article provided a detailed examination of recurrence relations and various identities associated with Jacobsthal and Jacobsthal-Lucas Lagrange Interpolation Polynomials.

The interpolation polynomials of the sequences $\{J_n\}$ and $\{j_n\}$ offer valuable tools that reflect the properties and structure of these sequences. These polynomials can be used to determine the value of the independent variable x corresponding to a given function value, even when the parameters are not evenly spaced.

These results lay a foundation for future research, opening new avenues to explore the applicability of these important number sequences and their interpolation polynomials in broader fields. Particularly, there is potential for further use and development of these polynomials in areas such as image processing, numerical analysis, and other engineering applications.

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The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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# 1-GUARDABLE SUBGRAPHS OF GRAPHS

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Abstract. Cops and Robbers game is played on a graph. There are two players in the game consisting of set of cops and only one robber. They play in respectively; on each player's turn, the player may either move to an adjacent vertex or stay in its vertex. If one of the cops comes into the vertex with the robber then the robber is captured. Therefore, game ends and cop wins the game. In this study, 1-guardable subgraphs of graphs in the game of Cops and Robbers is considered. It is mentioned about some special subgraphs and their relations. It is known that if the subgraph is 1-guardable then it must be isometric but the converse of this argument may not be true. We show that for the inverse to be true, some conditions must be added.

## 1. Introduction

The game of Cops and Robbers on graphs is presented by Quilliot firstly and advanced by Nowakowski and Winkler [1]. The game is played with two players named cop and robber. Initially, each player chose a vertex respectively on a graph. Players move to adjacent vertices or can stay their location. After some finite moves, cop wins if he comes to the same vertex with the robber. The robber wins if he can avoid the cop forever. The minimum number of cops needed to catch the robber is called cop number and denoted by $c(G)$. If cop number is 1 then graph is called cop-win graph. A graph which is $c(G) > 1$ is sometimes called robber-win. Introductory work about the cop number came in 1984 with Aigner and Fromme [2, 3]. Bonato and Nowakowski have written the book that gives the most detailed information [4].

To determine the cop number is a hard problem on the game of Cops and Robbers. If $G$ is a graph of order $n$, then $c(G) = O(\sqrt{n})$ is known as Meyniel's conjecture [5]. This bound has been tried to improve and Frank showed that

$$c(n) \leq (1 + o(1))n\frac{\log\log(n)}{\log(n)}$$

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for the cop number [5]. Chinifooroshan proved that

$$c(n) = O\left(\frac{n}{\log(n)}\right)$$

for the cop number of any graph $G$ [6]. The best known upper bound for the cop number is given by Lu and Peg

$$c(n) = O\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}\right)$$

by using the probabilistic method [7].

An induced subgraph $H$ of $G$ is said to be $k$-guardable if, after finitely many moves, $k$ cops can move only in the vertices of $H$ in such a way that if the robber moves into $H$ at round $t$, then he will be captured at round $t+1$ for a constant integer $k \geq 1$. It was proved that the isometric path is 1-guardable [2].

In [8], it is shown that the cop number of generalized Petersen graph is at most 4 and also proved that finite isometric subtree of a graph is 1-guardable.

Quilliot was studied on retractions on graphs and he proved several theorems in [9] before the game is introduced. Retracts have an important role in the game. Let $H$ be an induced subgraph of $G$. If there is a homomorphism $f$ from $G$ onto $H$ that is identity on $H$ which means $f(x) = x$ for every vertex $x \in H$. Then this map is said to be retraction and $H$ is called retract of $G$. In [3], it was shown that retract of cop-win graph is also cop-win.

The number of moves is considered on cop-win graphs and it is called capture time. For a given graph with $n$ vertices the capture time is bounded by $(n-3)$ [10]. In another paper, this bound is improved and presented new upper bound as $(n-4)$ for $n \geq 7$ and $\lfloor \frac{n}{2} \rfloor$ for $n \leq 7$ [11].

In this paper, we try to determine under what conditions the subgraph of a graph can be 1-guardable.

## 2. Some Special Subgraphs of Graph

For the given graph $G = (V, E)$ and $H = (W, F)$, $H$ is said to be *subgraph* of $G$ if $W \subset V$ and $F \subset E$. Let $G = (V, E)$ be the any graph and $S \subset V$ be any subset of vertices of $G$. Then the *induced subgraph* $G[S]$ is the graph whose vertex set is $S$ and whose edge set consists of all of the edges in $E$ that have both endpoints in $S$. That is, for any two vertices $u, v \in S$, $u$ and $v$ are adjacent in $G[S]$ if and only if they are adjacent in $G$.

$H$ is an *isometric subgraph* of $G$ if $d_G(u, v) = d_H(u, v)$ for every vertices $u, v$ in $H$. A *convex subgraph* $H$ of an undirected graph $G$ is a subgraph that includes every shortest path in $G$ between its two vertices.

**Proposition 1.** The convex subgraph of a graph is an isometric subgraph.

*Proof.* Let $G$ be any graph and $H$ be a convex subgraph of $G$. Then $H$ includes every shortest path in $G$ between its two vertices. Hence, $d_G(u, v) = d_H(u, v)$ for $u, v \in G$. □

Note that the converse of Proposition 1 may not be true. Although $C_5$ is isometric subgraph of $W_6$ but it is not convex subgraph (see Figure 1).

**Proposition 2.** The isometric subgraph of a graph is an induced subgraph.

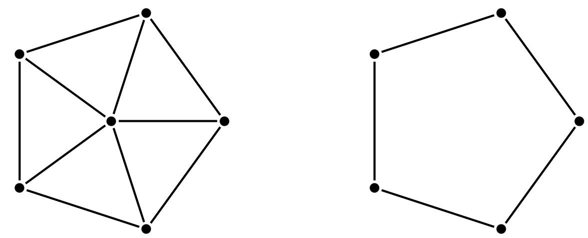

FIGURE 1. $C_5$ is an isometric subgraph of $W_6$ but not convex.

*Proof.* Let $G$ be a graph and $H$ be an isometric subgraph of $G$. Then assume that $H$ is not induced subgraph of $G$. Then there exist $x, y \in V(H)$ such that the edge $xy \notin E(H)$. In this case, $d_G(x, y) = 1$ but $d_H(x, y) \neq 1$. This contradicts with $H$ is isometric. If $H$ is isometric subgraph of $G$ then $H$ must be induced subgraph of $G$. □

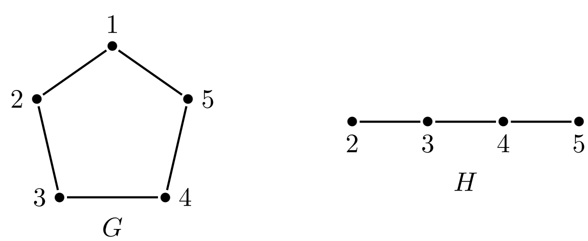


FIGURE 2. $H$ is induced subgraph of $G$ but not isometric.

If $H$ is an induced subgraph of $G$ then $H$ may not be an isometric subgraph of $G$. It is seen easily in Figure 2. While $d_H(2, 5) = 3$, $d_G(2, 3) = 2$.

**Corollary 2.1.** *The convex subgraph of a graph is an induced subgraph.*

*Proof.* It is the conclusion of Proposition 1 and Proposition 2. □

**Proposition 3.** The convex subgraph of a graph is a retract.

*Proof.* Let $G$ be a graph and $H$ be a convex subgraph of $G$. Thus, $H$ is an induced subgraph of $G$. It can be defined $f$ as a graph homomorphism such that

$$f : V(G) \to V(H), \qquad f(v) = \begin{cases} \{u : d(u, v) = d(u, H)\} & , \text{ if } v \notin V(H) \\ v & , \text{ if } v \in V(H) \end{cases}$$

If the set $\{u : d(u, v) = d(u, H)\}$ has more than one element, only one element can be chosen from this set. □

There is an example of homomorphism mentioned in the above proof in Figure 3. If homomorphism is defined as

$$f(v_1) = v_1, f(v_2) = v_2, f(v_3) = v_3, f(v_4) = v4, f(v_5) = v_3, f(v_6) = v_2, f(v_7) = v_2$$

then $f$ is a retraction from $G$ onto $H$. It can be chosen $f(v_6) = v_3$ then there is a different retraction with the same graphs.

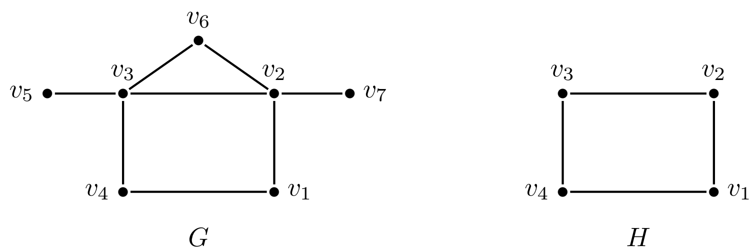


FIGURE 3. $H$ is a convex subgraph which is a retract of $G$.

**Corollary 2.2.** *The convex subgraph of the cop-win graph is cop-win.*

*Proof.* Let $G$ be a cop-win graph and $H$ be a convex subgraph of $G$. Then, by Proposition 3 $H$ is retract. It is known that if $G$ is cop-win, then each retract of $G$ is also cop-win. Hence convex subgraph of the cop-win graph is cop-win. □

**Proposition 4.** The retract of a graph is an isometric subgraph.

*Proof.* Let $G$ be a graph and $H$ be retract of graph $G$ so by definition, there exists a homomorphism $f : V(G) \to V(H)$ such that $f(x) = x$ for every $x \in V(H)$ . In this case for any $x, y \in V(H)$, $f(x) = x$ and $f(y) = y$. Since $f$ is the identity on $V(H)$ we get

$$d_H(f(x), f(y)) = d_H(x, y)$$

and by the definition of homomorphism

$$d_G(x, y) \geq d_H(f(x), f(y)) = d_H(x, y).$$

On the other hand, the shortest path between the vertices $x$ and $y$ can pass through the vertex $u$ such that $u \in V(G) - V(H)$. So, this path does not stay entirely in $H$. Hence $d_G(x, y) \leq d_H(x, y)$ so equality $d_G(x, y) = d_H(x, y)$ is provided. Then, $H$ is an isometric subgraph of $G$. □

## 3. 1-Guardable Subgraphs of Graph

Suppose that H is an induced subgraph of G and assume that $k$ cops guard the subgraph $H$. Then after finitely many moves, for a fixed integer $k \geq 1$, $H$ is said to be $k$-guardable if the robber moves into the subgraph $H$ at round $t$ then he is captured at round $t + 1$. Note that if one cop can guard a subgraph $H$ in $G$, then $H$ must be isometric in $G$. If it is not, then there are two vertices $u, v \in V(H)$ such that $d_H(u, v) > d_G(u, v)$. The robber can travel between $u$ and $v$ infinitely many times without being caught by the cop that guards $H$.

Note that there can also be subgraphs in $G$, not 1-guardable, although they are both isometric and cop-win (see Figure 4 and also see [12]).

Connected graph $G$ is given and $H$ is a subgraph of $G$. Let the set $A$ be a set of access points to subgraph $H$ and define as

$$A = \{v \in V(G) \backslash V(H) | N(v) \cap H \neq \emptyset\},$$

and for $v \in A$, define the set $B_v$ such that

$$B_v = \{u \in V(H) | N_H(v) \subset N_H(u)\}$$

as the set of points in $H$ that watch the points in set $A$. Let the set $B = \bigcup_{v \in A} B_v$ be (see Figure 5). Following conclusions can be given according to these notations.

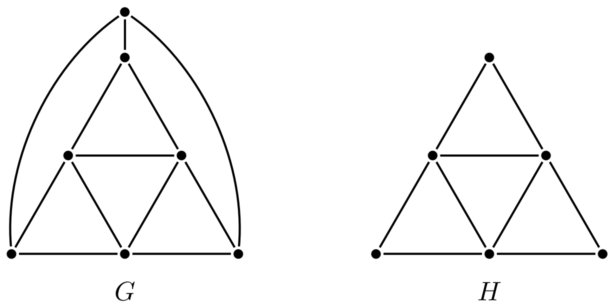


FIGURE 4. An isometric subgraph $H$ of $G$ is cop-win but $H$ is not 1-guardable in $G$.

**Lemma 3.1.** *If $B_v \neq \emptyset$ for all $v \in A$ and*

$$d_G(v_1, v_2) \geq \max\{d_H(x, y) : x, y \in B\},$$

*for every $v_1, v_2 \in A$ then $H$ is an isometric subgraph of $G$.*

*Proof.* Let $x$ and $y$ be any vertices of subgraph $H$. There are three cases:

(1) Let $x, y \in B$. If $x$ and $y$ are adjacent or the same vertices, there is nothing to prove. If they are not adjacent, by the definition of sets $A$ and $B$ and assumption $d_H(x, y)$ can not be different from $d_G(x, y)$.

(2) Let $x \in B$ and $y \notin B$. So the shortest path between $x$ and $y$ must be in $H$. Otherwise, let the path $P$ be the shortest path between $x$ and $y$ but not in $H$ so there exists a vertex $z \in A$ on the path $P$. Since $z \in A$ there is a vertex $z' \in B$ watching $z$. Hence there is another path for $u_1, u_2, \ldots, u_k \in B$ and $w \in H$ such that

$$P' : y \to \cdots \to z \to w \to z' \to u_1 \to u_2 \to \cdots \to u_k \to x.$$

By the inequality given by assumption, the path $P'$ is shorter than the path $P$. This contradicts with $P$ being the shortest path. Hence shortest path $P$ between $x$ and $y$ must be in $H$.

(3) Let $x, y \notin B$. Proof is similar to the previous case. Thus $H$ is an isometric subgraph.

□

**Lemma 3.2.** *If $B_v \neq \emptyset$ for all $v \in A$ and induced subgraph $G[B]$ is a complete graph then $H$ is 1-guardable.*

*Proof.* Let cop be positioned at one of the vertices of set $B$ and let the robber be positioned at one of the vertices $r_i$ which is seen in Figure 5. The cop is watching the robber at one of the vertices $u_i \in B$ and let the next move of robber go through one of the vertices $v_i$. When the robber moves over from vertices $v_i$ to subgraph $H$, cop comes to one of the vertices $u_i \in B_{v_i}$ corresponding to vertex $v_i$. Since $G[B]$ is a complete graph then the robber is captured on the next move. Hence $H$ is 1-guardable. □

If the set $G[B]$ is not complete then $H$ may not be 1-guardable. Let's explain briefly: If $B$ is not complete then cop can not be moved between vertices in $B$. So, the robber can not be captured when he comes into subgraph $H$.

It is a strict condition that the induced subgraph $G[B]$ is a complete graph. If the robber can enter the subgraph $H$ from only one vertex, this condition can be loosened.

**Theorem 3.3.** *Let $G$ be a graph, $H$ is a subgraph of $G$. If there exists a vertex $x \in V(G) \setminus V(H)$ such that $|N_s(x) \cap A| > 1$ for $s > 0$ then let $k = \min\{s : |N_s(x) \cap A| > 1, x \in V(G) \setminus V(H)\}$, otherwise we set $k = d(R, A)$. If for all $v \in A$, $B_v \neq \emptyset$ and the inequalities*

$$\max\{d(x, y) : x, y \in B\} \leq k$$

*and*

$$\min\{d(v_1, v_2) : v_1, v_2 \in A\} \geq \max\{d(x, y) : x, y \in B\}$$

*are valid then $H$ is 1-guardable.*

*Proof.* There are two cases:

(1) Let $w$ be a vertex satisfying $k = \min\{s : |N_s(x) \cap A| > 1, x \in V(G) \setminus V(H)\}$. When the robber comes to vertex $w$, then the cop moves towards to vertices that watch the adjacent vertices of the vertex $w$ in the set $A$. Because of the inequality $\max\{d(x, y) : x, y \in B\} \leq k$ given with statement of theorem, cop can make these moves. Therefore, the cop waits the robber on one of the vertices $u_i \in B_{v_i}$. So the robber is captured in one move when he comes to subgraph $H$.

(2) Suppose that there is no vertex $w$ satisfying the conditions given in theorem. It can be mentioned about two cases:
   (a) There are adjacent vertices in set $A$. In this case, because of the inequality $\min\{d(v_1, v_2) : v_1, v_2 \in A\} \geq \max\{d(x, y) : x, y \in B\}$, induced subgraph $G[B]$ must be complete graph. Then cop can move to any vertex in $B$ that he wants. Therefore the robber is captured in one move when he comes to subgraph $H$.
   (b) If there is no adjacent vertices in set $A$. Then there is a path between $R$ and set $A$ and the robber moves on this path. Because of the both inequalities given in theorem, the cop can comes to vertex $u_i \in B_{v_i}$. Thus the robber is captured in one move when he comes to subgraph $H$.

As a result, subgraph $H$ is 1-guardable. □

**Example 3.4.** Let $G$ be a graph given in Figure 6 and $H = G[\{i, j, k, l\}]$ induced subgraph of $G$. Then we get $A = \{e, f, g, h\}$ and $B = i, j, k, l$. Since

$$|N_2(a) \cap A| = |N_2(b) \cap A| = |N_2(c) \cap A| = |N_2(d) \cap A| = 2$$

we find $k = 2$. Besides, for all $v \in A$ $B_v \neq \emptyset$ and the inequalities

$$2 = \max\{d(x, y) : x, y \in B\} \leq 2 = k$$

and

$$3 = \min\{d(v_1, v_2) : v_1, v_2 \in A\} \geq \max\{d(x, y) : x, y \in B\} = 2$$

are valid. Hence, by Theorem 3.3 we obtain that $H$ is 1-guardable.

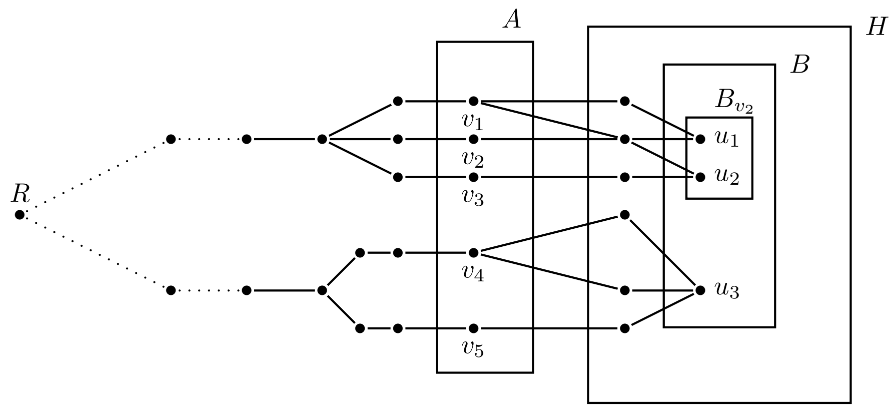


FIGURE 5. Figure that is given in proof of Teorem 3.3

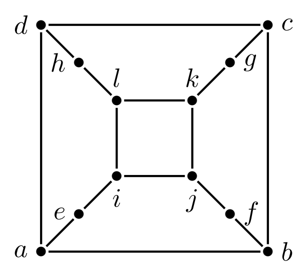


FIGURE 6. Graph $G$ in Example 3.4

## 4. Conclusion

In this paper, we try to determine under what conditions the subgraph of a graph can be 1-guardable. Some special subgraphs of a graph was defined at first then was given the relations between these subgraphs. Some conditions were given with the help of lemmas and theorem in order for the subgraph to be 1-guardable. Finally, an example of a graph that verifies the theorem is given.

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**The Declaration of Ethics Committee Approval**
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**The Declaration of Research and Publication Ethics**
The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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# A PHYSICAL APPROACH TO LAGRANGIAN EQUATIONS WITH BUNDLE STRUCTURE FOR MINKOWSKI 3-SPACE

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Abstract. Minkowski 3-space is important because it is the spatial structure in which physical events occur. Therefore, in this study, we examined the time-dependent Lagrangian energy equations in Minkowski 3-space. To facilitate the examination of the necessary mechanical structure and solutions with respect to time, we employed the jet bundle structure. This approach has made the creation of the necessary geometric structures more comprehensible in terms of coordinate basis. To interpret the obtained Lagrangian energy equations and to understand the significance of the time parameter, an example is also provided in our study.

## 1. Introduction

We worked on the Lagrangian energy structure previously studied in Euclidean space by establishing the necessary mechanical structure in Minkowski space. This energy structure had not been obtained before in 3-dimensional Minkowski space.The structure of Minkowski metric is different from Euclid metric. Also for this difference, it can be seen no studies in this subject. Mathematicians, working in Minkowski space, believe that there is natural phenomenon with Minkowski geometry for explaining physical phenomena occuring in 3-dimensional Euclid space. To obtain the time-dependent energy structure, we constructed the jet bundle structure in 3-dimensional Minkowski space and based our study on the coordinate system of this bundle structure.
The jet bundles can be classfied into two manifold:
1) Total complex manifold,
2) Phase manifold.
The inclusion of time-dependent differential coordinates in the jet bundle structure makes it more suitable for the creation of time-dependent mechanical systems. The consraint, real, complex and Para-complex structures on the time-dependent Lagrangian systems can be researched in [2] and [5]. Then found that in the paper

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[1] Lagrangian and Hamiltonian mechanical systems were instructed on the vector bundle structures and jet bundle forms.Lagrangian equations are solved with real bundles by [2] , [3], [6], [5]. As shown in the studies presented in the references, none of them have been conducted in Minkowski space. Therefore, we chose to carry out our work in Minkowski space as a distinctive approach.

Let $TQ$ be the tangent bundle of an m-dimensional configuration manifold $Q$. Given a Lagrangian energy function $L\ : TQ \rightarrow R$ , the vector field $X$ satisfying the energy equation

$$i_{X_L} w_L = dE_L \tag{1}$$

is unique. Here, $w$ is a 2-form on the bundle $T^*Q$ , and $E_L$ represents the energy associated with $L$.[6], [5] If the families of curves that are solutions to the above energy equation on $TQ$ are integral curves of the vector field $X$ , then the vector field $X$ is called a semi-spray.

The triple $(TQ, w_L, L\ )$ is called Lagrangian system on the tangent bundle $TQ$. [6], [5]

Let, $L{:}R{\times}TQ{=}J\left(R,Q\right){\rightarrow}R$ and $TQ{=}\left\{t\right\}\times TQ$ be Lagrangian function. The coordinate system on $TQ$ is $\{q_i, v_i\}$ .

The Poincare cartan 1-form on the $T^*Q$ associated with $L$ Lagrangian energy function is

$$\alpha_L{=}d_J L{+}Ldv_i \ = \frac{\partial L}{\partial v_i}\ dq_i{+}Ldv_i$$

The Poincare cartan 2-form associated with $L$ Lagrangian energy function is

$$\Omega_L{=}dd_j L{+}dL \wedge Ldv_i$$

If the paths of semisprays verify

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2}$$

then this is called as Euler-Lagrange equation.

[1], proved that equation (2) is not changed; but, this study investigates the possible shapes of Euler-Lagrange system (2) on jet bundles.

**Bundles on Minkowski 3-Space**

A bundle is a triple $(E, \pi, M)$ where $E$ and $M$ are manifolds and $\pi{:}E \rightarrow M$ is a surjective submersion. $E$ is called the total space, $\pi$, the projection and $M$ the base space. This bundle denoted by $\pi$ or $E$. The first jet manifold of $\pi$ is the set $\left\{J^1_p\phi{:}p \in M, \phi \in \Gamma_p\left(\pi\right)\right\}$ and denoted by $J^1E$. Here, $\phi$ is a map $\phi{:}M \rightarrow E$ and is called a section of $\pi$. If it is satisfies the condition $\pi \circ \phi{=}id_M$, then the set of all sections of $\pi$ will be denoted $\Gamma\left(\pi\right)$.

Let $(E, \pi, M)$ a bundle and let $(U, u)$ be an adapted coordinate system on $E$, where $u{=}\ (x, y, z, u_\alpha)$. The induced coordinate system $(U^1, u^1)$ on $J^1E$ is defined by

$$U^1{=}\left\{J^1_p\phi{:}\phi(p) \in U\right\}$$

$$u^1{=}\ (x, y, z, u_\alpha, u^i_\alpha)$$

where $x(J^1_p\phi)\ {=}x(p)$,$y(J^1_p\phi)\ {=}y(p)$,$z(J^1_p\phi)\ {=}z(p)$, $u_\alpha(J^1_p\phi)\ {=}u_\alpha(\phi m)$ and are known as derivative coordinates.[1]

In this study, we consider the bundle structure $\left(E^3_1, \pi, R\right)$. The coordinates of the manifold $E^3_1$are $(x, y, z)$, the coordinate of the manifold $R$ is $(t)$. Also, the coordinates of the manifold $J^1E^3_1$ are $(t, x, y, z, x^{\cdot}, y^{\cdot}, z^{\cdot})$.

Here derivative coordinates denoted by

$$\dot{x} = \frac{dx}{dt}$$

$$\dot{y} = \frac{dy}{dt}$$

$$\dot{z} = \frac{dz}{dt}$$

**Lagrangian Mechanical Systems of Minkowski Space with Bundle Structure**

The Minkowski 3-space $E_1^3$ is the Euclidean 3-space $E^3$ provided with the standard flat metric given by

$$g = -\,dx^2 + dy^2 + dz^2$$

where $(x, y, z)$ is a rectangular coordinate system of $E^3$. Let $g$ be the Minkowski metric, and let $v \in E_1^3$ be,

- $g(v, v) > 0$ or $g(v) = 0$, $v$ is spacelike vector
- $g(v, v) < 0$ , $v$ is timelike vector
- $g(v, v) = 0$ and $v \neq 0$ , $v$ is null( lightlike)

A similar analysis can be performed within an $\alpha$ curve on $E_1^3$ .
$\tau$ is the set of all time-like vectors in $E_1^3$. For $\forall u \in \tau$; the set

$$C(\overrightarrow{u}) = \{\overrightarrow{x} \in \tau : \langle \overrightarrow{u}, \overrightarrow{x} \rangle < 0\} = \left\{\overrightarrow{x} \in E_1^3 : g(x-u, x-u) < 0\right\}$$

defined as timecone.

**Theorem 1.** The time-like vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ in Minkowski 3-space $E_1^3$ are in the same timecone,

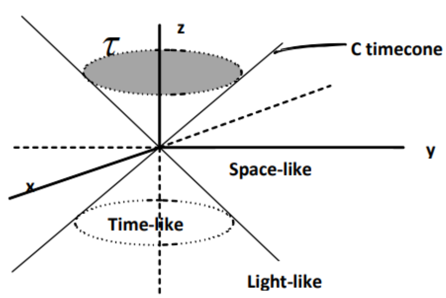


FIGURE 1

$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = -\|\overrightarrow{x}\| \, \|\overrightarrow{y}\| \cosh\theta$ and here $\theta$ is the *Lorentz timelike angle* between $\overrightarrow{x}$ and $\overrightarrow{y}$ vectors.

**Definition 1.1.** A (1,1)- type tensor field $J$ that satisfies the $J^2=0$ condition is approximately called a tangent structure. Here, $J:T\left(J^1E_1^3\right)\longrightarrow T\left(J^1E_1^3\right)$ *is*

$$J\left(\frac{\partial}{\partial x}\right)=-\frac{\partial}{\partial x^{\cdot}},\quad J\left(\frac{\partial}{\partial y}\right)=\frac{\partial}{\partial y^{\cdot}},\quad J\left(\frac{\partial}{\partial z}\right)=\frac{\partial}{\partial z^{\cdot}}=J\left(\frac{\partial}{\partial z^{\cdot}}\right)$$

$$J\left(\frac{\partial}{\partial x^{\cdot}}\right)=J\left(\frac{\partial}{\partial y^{\cdot}}\right)=J\left(\frac{\partial}{\partial z^{\cdot}}\right)=0$$

$$J\left(\frac{\partial}{\partial t}\right)=-\dot{x}\frac{\partial}{\partial x^{\cdot}}+\dot{y}\frac{\partial}{\partial y^{\cdot}}+\dot{z}\frac{\partial}{\partial z^{\cdot}} \tag{3}$$

$J$ can be calculated as a tensor field from (3), as

$$J=\left(-dx-\dot{x}\,dt\right)\times\frac{\partial}{\partial x^{\cdot}}+\left(dy+\dot{y}\,dt\right)\times\frac{\partial}{\partial y^{\cdot}}+\left(dz+\dot{z}\,dt\right)\times\frac{\partial}{\partial z^{\cdot}} \tag{4}$$

A semi-spray is a vector field over $E_1^3$ and defined as below;

$$\varepsilon=\frac{\partial}{\partial t}-\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\dot{z}\frac{\partial}{\partial z}-\varepsilon_1\frac{\partial}{\partial x^{\cdot}}+\varepsilon_2\frac{\partial}{\partial y^{\cdot}}+\varepsilon_3\frac{\partial}{\partial z^{\cdot}} \tag{5}$$

By calculate $J(\varepsilon)$, then equation (6) are found

$$V=J\varepsilon=-2\,\dot{x}\frac{\partial}{\partial x^{\cdot}}+2\,\dot{y}\frac{\partial}{\partial y^{\cdot}}+2\,\dot{z}\frac{\partial}{\partial z^{\cdot}} \tag{6}$$

which is called "*Liouville vector field*"
Moreover, "Poincare-Cartan 1-form" is written as:

$$\alpha L=d_JL+Ldt$$

$$\alpha L=-\dot{x}\frac{\partial L}{\partial x^{\cdot}}dt+\dot{y}\frac{\partial L}{\partial y^{\cdot}}dt+\dot{z}\frac{\partial L}{\partial z^{\cdot}}dt-\frac{\partial L}{\partial x^{\cdot}}dx+\frac{\partial L}{\partial y^{\cdot}}dy+\frac{\partial L}{\partial z^{\cdot}}dz+Ld \tag{7}$$

Then we can write differential operator $d$,

$$d=\frac{\partial}{\partial t}dt-\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz-\frac{\partial}{\partial x^{\cdot}}d\,\dot{x}+\frac{\partial}{\partial y^{\cdot}}d\,\dot{y}+\frac{\partial}{\partial z^{\cdot}}d\,\dot{z} \tag{8}$$

By using the differentiation $d$ (8), then Poincare-Cartan 2-form is obtained.

$$\Omega_L=dd_JL+dL\wedge dt$$

$$\begin{aligned}\Omega_L=(dx\wedge dt)&\left(\frac{\partial^2L}{\partial t\partial\dot{x}}+\dot{x}\frac{\partial^2L}{\partial x\partial\dot{x}}-\dot{y}\frac{\partial^2L}{\partial x\partial\dot{y}}-\dot{z}\frac{\partial^2L}{\partial x\partial\dot{z}}-\frac{\partial L}{\partial x}\right)\\
+(dy\wedge dt)&\left(-\frac{\partial^2L}{\partial t\partial\dot{y}}-\dot{x}\frac{\partial^2L}{\partial y\partial\dot{x}}+\dot{y}\frac{\partial^2L}{\partial y\partial\dot{y}}+\dot{z}\frac{\partial^2L}{\partial y\partial\dot{z}}-\frac{\partial L}{\partial y}\right)\\
+(dz\wedge dt)&\left(-\frac{\partial^2L}{\partial t\partial\dot{z}}-\dot{x}\frac{\partial^2L}{\partial z\partial\dot{x}}+\dot{y}\frac{\partial^2L}{\partial z\partial\dot{y}}+\dot{z}\frac{\partial^2L}{\partial z\partial\dot{z}}-\frac{\partial L}{\partial z}\right)\\
+\left(d\dot{x}\wedge dt\right)&\left(\dot{x}\frac{\partial^2L}{\partial\dot{x}^2}-\dot{y}\frac{\partial^2L}{\partial\dot{x}\partial\dot{y}}-\dot{z}\frac{\partial^2L}{\partial\dot{x}\partial\dot{z}}\right)\\
+\left(d\dot{y}\wedge dt\right)&\left(-\dot{x}\frac{\partial^2L}{\partial\dot{x}\partial\dot{y}}+\dot{y}\frac{\partial^2L}{\partial\dot{y}^2}+\dot{z}\frac{\partial^2L}{\partial\dot{y}\partial\dot{z}}+2\frac{\partial L}{\partial\dot{y}}\right)\\
+\left(d\dot{z}\wedge dt\right)&\left(-\dot{x}\frac{\partial^2L}{\partial\dot{x}\partial\dot{z}}+\dot{y}\frac{\partial^2L}{\partial\dot{y}\partial\dot{z}}+\dot{z}\frac{\partial^2L}{\partial\dot{z}^2}+2\frac{\partial L}{\partial\dot{z}_3}\right)\end{aligned}$$

$$
\begin{aligned}
&+(dx\wedge dy)\left(-\frac{\partial^2 L}{\partial x\partial \dot{y}}+\frac{\partial^2 L}{\partial y\partial \dot{x}}\right)+(dx\wedge dz)\left(-\frac{\partial^2 L}{\partial x\partial \dot{z}}+\frac{\partial^2 L}{\partial z\partial \dot{x}}\right)\\
&+(dy\wedge dz)\left(-\frac{\partial^2 L}{\partial y\partial \dot{z}}+\frac{\partial^2 L}{\partial z\partial \dot{y}}\right)+\left(dx\wedge d\dot{x}\right)\left(-\frac{\partial^2 L}{\partial \dot{x}^2}\right)\\
&+\left(dx\wedge d\dot{y}\right)\left(\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}\right)+\left(dx\wedge d\dot{z}\right)\left(\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}\right)\\
&+\left(dy\wedge d\dot{x}\right)\left(\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}\right)+\left(dy\wedge d\dot{y}\right)\left(-\frac{\partial^2 L}{\partial \dot{y}^2}\right)\\
&+\left(dy\wedge d\dot{z}\right)\left(-\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}}\right)+\left(dz\wedge d\dot{x}\right)\left(\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}\right)\\
&+\left(dz\wedge d\dot{y}\right)\left(-\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}}\right)+\left(dz\wedge d\dot{z}\right)\left(-\frac{\partial^2 L}{\partial \dot{z}^2}\right)
\end{aligned}
$$

(9)

**Definition 1.2.** Solutions of the Euler-Lagrange equation can be found by assuming

$$
\begin{aligned}
&i_\varepsilon\Omega_L=\Omega_L(\varepsilon)=0\\
&i_\varepsilon\Omega_L=\Omega_L(\varepsilon)\\
&=-(\frac{\partial^2 L}{\partial t\partial \dot{x}}+\dot{x}\frac{\partial^2 L}{\partial x\partial \dot{x}}-\dot{y}\frac{\partial^2 L}{\partial x\partial \dot{y}}-\dot{z}\frac{\partial^2 L}{\partial x\partial \dot{z}}-\frac{\partial L}{\partial x}\\
&-\dot{y}\frac{\partial^2 L}{\partial x\partial \dot{y}}+\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{x}}-\dot{z}\frac{\partial^2 L}{\partial x\partial \dot{z}}+\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{x}}\\
&+\varepsilon_1\frac{\partial^2 L}{\partial \dot{x}^2}+\varepsilon_2\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}+\varepsilon_3\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}})dx\\
&-(-\frac{\partial^2 L}{\partial t\partial \dot{y}}-\dot{x}\frac{\partial^2 L}{\partial y\partial \dot{x}}+\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{y}}+\dot{z}\frac{\partial^2 L}{\partial y\partial \dot{z}}+\frac{\partial L}{\partial y}\\
&-\dot{x}\frac{\partial^2 L}{\partial x\partial \dot{y}}+\dot{x}\frac{\partial^2 L}{\partial y\partial \dot{x}}-\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{y}}+\dot{z}\frac{\partial^2 L}{\partial y\partial \dot{z}}\\
&-\varepsilon_1\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}-\varepsilon_2\frac{\partial^2 L}{\partial \dot{y}^2}-\varepsilon_3\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}_3})dy\\
&-(-\frac{\partial^2 L}{\partial t\partial \dot{z}}-\dot{x}\frac{\partial^2 L}{\partial z\partial \dot{x}}+\dot{y}\frac{\partial^2 L}{\partial z\partial \dot{y}}+\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{z}}+\frac{\partial L}{\partial z}\\
&-\dot{x}\frac{\partial^2 L}{\partial x\partial \dot{z}}+\dot{x}\frac{\partial^2 L}{\partial z\partial \dot{x}}-\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{z}}+\dot{y}\frac{\partial^2 L}{\partial z\partial \dot{y}}\\
&-\varepsilon_1\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}-\varepsilon_2\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}}-\varepsilon_3\frac{\partial^2 L}{\partial \dot{z}^2})dz\\
&-(\dot{x}\frac{\partial^2 L}{\partial \dot{x}^2}-\dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}-\dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}\\
&-\dot{x}\frac{\partial^2 L}{\partial \dot{x}^2}-\dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}-\dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}})d\dot{x}\\
&-(-\dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}+\dot{y}\frac{\partial^2 L}{\partial \dot{y}^2}+\dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}}+2\frac{\partial L}{\partial \dot{y}}
\end{aligned}
$$

$$
\begin{aligned}
&+\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{y}}+\dot{y}\frac{\partial^2 L}{\partial\dot{y}^2}+\dot{z}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}})d\dot{y}\\
&-(-\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{z}}+\dot{y}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}}+\dot{z}\frac{\partial^2 L}{\partial\dot{z}^2}+2\frac{\partial L}{\partial\dot{z}}\\
&+\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{z}}+\dot{y}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}}+\dot{z}\frac{\partial^2 L}{\partial\dot{z}^2})d\dot{z}\\
&+(-\dot{x}\frac{\partial^2 L}{\partial t\partial\dot{x}}-\dot{x}^2\frac{\partial^2 L}{\partial x\partial\dot{x}}+\dot{x}\dot{y}\frac{\partial^2 L}{\partial x\partial\dot{y}}+\dot{x}\dot{z}\frac{\partial^2 L}{\partial x\partial\dot{z}}+\dot{x}\frac{\partial L}{\partial x}\\
&-\dot{y}\frac{\partial^2 L}{\partial t\partial\dot{y}}-\dot{x}\dot{y}\frac{\partial^2 L}{\partial y\partial\dot{x}}+\dot{y}^2\frac{\partial^2 L}{\partial y\partial\dot{y}}+\dot{z}\dot{y}\frac{\partial^2 L}{\partial y\partial\dot{z}}+\dot{y}\frac{\partial L}{\partial y}\\
&-\dot{z}\frac{\partial^2 L}{\partial t\partial\dot{z}}-\dot{x}\dot{z}\frac{\partial^2 L}{\partial z\partial\dot{x}}+\dot{y}\dot{z}\frac{\partial^2 L}{\partial z\partial\dot{y}}+\dot{z}^2\frac{\partial^2 L}{\partial z\partial\dot{z}}+\dot{z}\frac{\partial L}{\partial z}\\
&-\varepsilon_1\dot{x}\frac{\partial^2 L}{\partial\dot{x}^2}+\varepsilon_1\dot{y}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{y}}+\varepsilon_1\dot{z}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{z}}\\
&-\varepsilon_2\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{y}}+\varepsilon_2\dot{y}\frac{\partial^2 L}{\partial\dot{y}^2}+\varepsilon_2\dot{z}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}}\\
&-\varepsilon_3\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{z}}+\varepsilon_3\dot{y}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}}+\varepsilon_3\dot{z}\frac{\partial^2 L}{\partial\dot{z}^2}+2\varepsilon_2\frac{\partial L}{\partial\dot{y}}+2\varepsilon_3\frac{\partial L}{\partial\dot{z}})dt
\end{aligned}
$$

(10)

By equalizing equation (10) to zero, then (11) are obtained.

$$
\begin{aligned}
I\quad:\quad 0&=-\frac{\partial^2 L}{\partial t\partial\dot{x}}+\frac{\partial}{\partial x}(-\dot{x}\frac{\partial L}{\partial\dot{x}}+\dot{y}\frac{\partial L}{\partial\dot{y}}+\dot{z}\frac{\partial L}{\partial\dot{z}}+L)\\
&+(\dot{y}\frac{\partial^2 L}{\partial x\partial\dot{y}}-\dot{y}\frac{\partial^2 L}{\partial y\partial\dot{x}}+\dot{z}\frac{\partial^2 L}{\partial x\partial\dot{z}}-\dot{z}\frac{\partial^2 L}{\partial z\partial\dot{x}})\\
&-\frac{\partial}{\partial\dot{x}}(\varepsilon_1\frac{\partial L}{\partial\dot{x}}+\varepsilon_2\frac{\partial L}{\partial\dot{y}}+\varepsilon_3\frac{\partial L}{\partial\dot{z}})
\end{aligned}
$$

$$
\begin{aligned}
II\quad:\quad 0&=\frac{\partial^2 L}{\partial t\partial\dot{y}}-\frac{\partial}{\partial y}(-\dot{x}\frac{\partial L}{\partial\dot{x}}+\dot{y}\frac{\partial L}{\partial\dot{y}}+\dot{z}\frac{\partial L}{\partial\dot{z}}+L)\\
&+(-\dot{x}\frac{\partial^2 L}{\partial y\partial\dot{x}}+\dot{x}\frac{\partial^2 L}{\partial x\partial\dot{y}}-\dot{z}\frac{\partial^2 L}{\partial y\partial\dot{z}}+\dot{z}\frac{\partial^2 L}{\partial z\partial\dot{y}}\\
&+\frac{\partial}{\partial\dot{y}}(\varepsilon_1\frac{\partial L}{\partial\dot{x}}+\varepsilon_2\frac{\partial L}{\partial\dot{y}}+\varepsilon_3\frac{\partial L}{\partial\dot{z}})
\end{aligned}
$$

$$
\begin{aligned}
III\quad:\quad 0&=\frac{\partial^2 L}{\partial t\partial\dot{z}}-\frac{\partial}{\partial z_3}(-\dot{x}\frac{\partial L}{\partial\dot{x}}+\dot{y}\frac{\partial L}{\partial\dot{y}}+\dot{z}\frac{\partial L}{\partial\dot{z}}+L)\\
&+(-\dot{x}\frac{\partial^2 L}{\partial z\partial\dot{x}}+\dot{x}\frac{\partial^2 L}{\partial x\partial\dot{z}}-\dot{y}\frac{\partial^2 L}{\partial z\partial\dot{y}}+\dot{y}\frac{\partial^2 L}{\partial y\partial\dot{z}})\\
&+\frac{\partial}{\partial\dot{z}}(\varepsilon_1\frac{\partial L}{\partial\dot{x}}+\varepsilon_2\frac{\partial L}{\partial\dot{y}}+\varepsilon_3\frac{\partial L}{\partial\dot{z}})
\end{aligned}
$$

$$
\begin{aligned}
IV \quad : \quad & 0 = -\dot{x}\frac{\partial^2 L}{\partial \dot{x}^2} + \dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} + \dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} + \frac{\partial L}{\partial \dot{x}} \\
& + \dot{x}\frac{\partial^2 L}{\partial \dot{x}^2} + \dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} + \dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} - \frac{\partial L}{\partial \dot{x}} \\
& \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}^2} - \dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \frac{\partial L}{\partial \dot{y}}
\end{aligned}
$$

$$
\begin{aligned}
V \quad : \quad & 0 = \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}^2} - \dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \frac{\partial L}{\partial \dot{y}} \\
& - \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}^2} - \dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \frac{\partial L}{\partial \dot{y}}
\end{aligned}
$$

$$
\begin{aligned}
VI \quad : \quad & 0 = \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \dot{z}\frac{\partial^2 L}{\partial \dot{z}^2} - \frac{\partial L}{\partial \dot{z}} \\
& - \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \dot{z}\frac{\partial^2 L}{\partial \dot{z}^2} - \frac{\partial L}{\partial \dot{z}}
\end{aligned}
$$

$$
\begin{aligned}
VII \quad : \quad & 0 = -\dot{x}\frac{\partial^2 L}{\partial t\partial \dot{x}} - \dot{x}^2\frac{\partial^2 L}{\partial x\partial \dot{x}} + \dot{x}\dot{y}\frac{\partial^2 L}{\partial x\partial \dot{y}} + \dot{x}\dot{z}\frac{\partial^2 L}{\partial x\partial \dot{z}} + \dot{x}\frac{\partial L}{\partial x} \\
& - \dot{y}\frac{\partial^2 L}{\partial t\partial \dot{y}} - \dot{x}\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{x}} + \dot{y}^2\frac{\partial^2 L}{\partial y\partial \dot{y}} + \dot{z}\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{z}} + \dot{y}\frac{\partial L}{\partial y} \\
& - \dot{z}\frac{\partial^2 L}{\partial t\partial \dot{z}} - \dot{x}\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{x}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{y}} + \dot{z}^2\frac{\partial^2 L}{\partial z\partial \dot{z}} + \dot{z}\frac{\partial L}{\partial z} \\
& - \varepsilon_1\dot{x}\frac{\partial^2 L}{\partial \dot{x}^2} + \varepsilon_1\dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} + \varepsilon_1\dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} \\
& - \varepsilon_2\dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} + \varepsilon_2\dot{y}\frac{\partial^2 L}{\partial \dot{y}^2} + \varepsilon_2\dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} \\
& - \varepsilon_3\dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} + \varepsilon_3\dot{y}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} + \varepsilon_3\dot{z}\frac{\partial^2 L}{\partial \dot{z}^2} \\
& + 2\varepsilon_2\frac{\partial L}{\partial \dot{y}} + 2\varepsilon_3\frac{\partial L}{\partial \dot{z}}
\end{aligned}
$$

(11)

This equation (11) represents a non-linear equations system. By assuming

(12) $$\varepsilon_1 = -\dot{x}, \quad \varepsilon_2 = \dot{y}, \quad \varepsilon_3 = \dot{z}$$

In this approach, it must be a negative term, because Minkowski metric is negatie definitly. Then following equalities can be written as below;

(13) $$\dot{x}(I) - \dot{y}(II) - \dot{z}(III) + \dot{x}(IV) - \dot{y}(V) - \dot{z}(VI) + (VII) = 0$$

Solving (13) lead to the equation

$$
\begin{aligned}
VIII \quad : \quad 0 = & -\dot{x}\frac{\partial^2 L}{\partial t \partial \dot{x}} - \dot{y}\frac{\partial^2 L}{\partial t \partial \dot{y}} - \dot{z}\frac{\partial^2 L}{\partial t \partial \dot{z}} \\
& -\dot{x}^2\frac{\partial^2 L}{\partial x \partial \dot{x}} + \dot{x}\dot{y}\frac{\partial^2 L}{\partial x \partial \dot{y}} + \dot{x}\dot{z}\frac{\partial^2 L}{\partial x \partial \dot{z}} + \dot{x}\frac{\partial L}{\partial x} \\
& -\dot{x}\dot{y}\frac{\partial^2 L}{\partial y \partial \dot{x}} + \dot{y}^2\frac{\partial^2 L}{\partial y \partial \dot{y}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial y \partial \dot{z}} + \dot{y}\frac{\partial L}{\partial y} \\
& -\dot{x}\dot{z}\frac{\partial^2 L}{\partial z \partial \dot{x}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial z \partial \dot{y}} + \dot{z}^2\frac{\partial^2 L}{\partial z \partial \dot{z}} + \dot{z}\frac{\partial L}{\partial z} \\
& +\dot{x}^2\frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} - \dot{x}\dot{y}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} - \dot{x}\dot{z}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{z}} \\
& +\dot{x}^2\frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} + \dot{x}\dot{y}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} + \dot{x}\dot{z}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{z}} \\
& +\dot{x}\dot{y}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} + \dot{y}^2\frac{\partial^2 L}{\partial \dot{y} \partial \dot{y}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial \dot{y} \partial \dot{z}} \\
& +\dot{x}\dot{z}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{z}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial \dot{y} \partial \dot{z}} + \dot{z}^2\frac{\partial^2 L}{\partial \dot{z} \partial \dot{z}} \\
& -\dot{x}^2\frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} - \dot{x}\dot{y}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} - \dot{x}\dot{z}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{z}} \\
& +2\dot{y}\frac{\partial L}{\partial \dot{y}} + 2\dot{z}\frac{\partial L}{\partial \dot{z}}
\end{aligned}
$$

(14)

We can write this equation in a general form. But for this writing, we can assume a notation for negative terms. We denote this notation as follows,

$$
\delta_i = \begin{cases} -1, & ,i = 1 \\ 1 & ,i = 2,3 \end{cases}
$$

Also we can write

$$
\begin{aligned}
& -\frac{\partial}{\partial t}\left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) - \dot{x}\frac{\partial L}{\partial x}\left(-\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) + \left(\dot{x}\frac{\partial L}{\partial x} + \dot{y}\frac{\partial L}{\partial y} + \dot{z}\frac{\partial L}{\partial z}\right) \\
& +\dot{y}\frac{\partial L}{\partial \dot{y}}\left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) + \left(\dot{x}\frac{\partial L}{\partial x} + \dot{y}\frac{\partial L}{\partial y} + \dot{z}\frac{\partial L}{\partial z}\right) + \dot{z}\frac{\partial L}{\partial \dot{z}}\left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) \\
& + \left(\dot{x}\frac{\partial L}{\partial x} + \dot{y}\frac{\partial L}{\partial y} + \dot{z}\frac{\partial L}{\partial z}\right) + \left(-\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) + \left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) \\
& + \left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) = 0
\end{aligned}
$$

$$\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}} = M$$
$$-\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}} = N$$
$$\dot{x}\frac{\partial L}{\partial x} + \dot{y}\frac{\partial L}{\partial y} + \dot{z}\frac{\partial L}{\partial z} = P$$

$$(15) \quad -\frac{\partial}{\partial t}(M) - \dot{x}\frac{\partial L}{\partial \dot{x}}(N+P) + \dot{y}\frac{\partial}{\partial \dot{y}}(M+P) + \dot{z}\frac{\partial}{\partial \dot{z}}(M+P) + (N+M+P) = 0$$

(15) is the Lagrange equation in Minkowski space . Following examples show an application of equation (14).

**Example** In this example, we will derive the Lagrangian energy equations for a particle moving within a time cone. We will assume that we use the same approach as outlined above for the solution method. First, let us define a helical curve within a time cone and examine the coordinate structure of this curve. In Figure-1, we show the helical curve within the time cone. Since the curve we are working with remains entirely within the time cone, it is also a timelike curve.

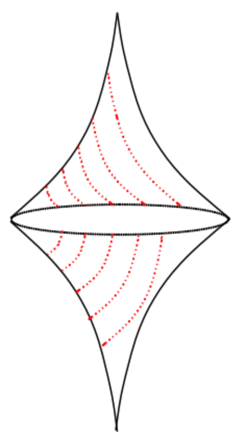

FIGURE 2

This helix is also referred to as a Minkowski helix and is represented as follows.

$$\alpha(\theta) = (r\text{sinh}u\theta, r\text{cosh}u\theta\text{sin}\theta, r\text{cosh}u\theta\text{cos}\theta)$$

Here, $r = r(t)$ is the radius function depending on the time parameter $t$, $\theta$ is fixed angle.

For the reason mentioned above, the $\alpha$ curve is also a timelike curve, and thus $\left\langle \alpha', \alpha' \right\rangle < 0$ .

If $\overrightarrow{x}$ , $\overrightarrow{y}$ is timelike vector,

$$\langle \overrightarrow{x}\ ,\ \overrightarrow{y} \rangle = -\ \| \overrightarrow{x}\ \| \ \| \ \overrightarrow{y} \| \,\text{cosh}\theta$$

The velocity vector for this curve is

$$\alpha'(\theta) = (ur\text{cosh}u\theta, ur\text{sinh}u\theta\text{sin}\theta + r\text{cosh}u\theta\text{cos}\theta, ur\text{sinh}u\theta\text{cos}\theta - r\text{cosh}u\theta\text{sin}\theta)$$

If this curve is time-like, then it can be

(16) $$\left\langle \alpha\,'(\theta), \alpha\,'(\theta) \right\rangle = r^2 \left(\cosh^2 u\theta - u^2\right)$$

Thus, taking into account the condition provided above,

(17) $$-u < \cosh u\theta < u$$

On the other hand, for the helix, the jet bundle coordinate is

$$(t, r\sinh u\theta, r\cosh u\theta\sin\theta, r\cosh u\theta\cos\theta, \dot{r}\sinh u\theta, \dot{r}\cosh u\theta\sin\theta, \dot{r}\cosh u\theta\cos\theta)$$

(18)

When we rearrange equation (14) according to these bundle coordinates,

(19) $$-3\,\dot{r}\,\frac{\partial^2 L}{\partial t \partial r^{\cdot}} + 3\dot{r}^2\,\frac{\partial^2 L}{\partial r \partial r^{\cdot}} + 3\,\dot{r}\,\frac{\partial L}{\partial r} + 5\dot{r}^2\,\frac{\partial^2 L}{\partial r^{\cdot}\,\partial r^{\cdot}} + 4\,\dot{r}\,\frac{\partial L}{\partial r^{\cdot}} = 0$$

we obtain the equation.
Equation (19) is the Lagrangian energy equation for the Minkowski helix.
In this equation, it can be seen that the Lagrangian energy function depends on the parameter $r$ .
Since $r = r(t)$, the energy function $L$ also depends on time.
Also, we consider

$$\frac{dL}{d\,r^{\cdot}} = \lambda \Rightarrow L = \lambda\,\dot{r}$$

With calculation the equation (18), we get solution of Lagrange energy function;

(20) $$L = -\frac{4}{3}\lambda r$$

Furthermore, for radius function

$$-\frac{4}{3}\lambda r = \lambda\,\dot{r}$$

In the solution of the equation, the radius function

$$r = e^{-\frac{4}{3}t}$$

By using this value, the Lagrangian energy is,

(21) $$L = -\frac{4}{3}\lambda e^{-\frac{4}{3}t}$$

From this equation, it can be noticed that radius related by time. Really, with our accaptance, the main parameter is time. The progress of movement is related to time.

## 2. Conclusion

In this paper, unlike in Euclidean space, we have worked in 3-dimensional Minkowski space. Specifically, a jet bundle structure has been established in this space, and all proofs have been examined using the coordinate system of this bundle structure. The advantage of working with the bundle structure is that it allows us to directly construct the time-dependent mechanical system. Lagrangian energy equations are divided into time-dependent and time-independent categories. In applications and physical contexts, it is more practical to work with the time-dependent equation. Using or not using the time parameter in deriving these two equation structures changes and complicates the entire proof in the classical approach. Our method,

which involves using the jet bundle structure, makes the entire analysis more comprehensible by incorporating time within these bundle coordinates.
In our study, we demonstrated the ease of obtaining the energy function using bundle coordinates within the example structure we presented. As shown in the example, due to the differences in the metric structure of Minkowski space, the curves and vectors studied are categorized as timelike, spacelike, or null. Since we worked with a helix lying within a time cone, we dealt with timelike curves and vectors. However, as can be shown with other examples, the obtained equation (14) can also be applied to spacelike curves and vectors.
The system given in equation (11) is a nonlinear system of equations. Its solution is possible under special conditions. When proving this, we considered the general form of the Lagrangian energy structure based on our previous work in Euclidean space and complex space. Accordingly, by experimenting with all the special conditions that would allow us to obtain the Lagrangian energy equation in Minkowski space, we found that the conditions provided in equation (12) are the most suitable and developed the proof based on these. The negativity here is entirely due to the structure of the Minkowski metric.
As a result of energy equations (21), where time provide to in a big-far time interval or partial movement in all large velocity, Lagrange energy is in a develop state case.

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This study does not be necessary ethical committee permission or any special permission.

**The Declaration of Research and Publication Ethics**
The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the authors declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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# GEOMETRIC PHASES, MAGNETIC CURVES FOR DARBOUX FRAMES ON LIGHTLIKE AND TIMELIKE SURFACES

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Abstract. In this paper, we obtain ${}^{D}\mathbf{E}_{\mathcal{T}}$, ${}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}$, ${}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$, ${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$, magnetic curves, Lorentz force equations and geometric phases for Darboux frame of a spacelike curve with non-lightlike principal normal lying on a lightlike surface, null Darboux frame on a timelike surface, 1GDF and 2GDF in the tangential direction. Later, we derive intrinsic directional derivatives in $\widetilde{\mu}$, $\widetilde{U}-lines$ directions for 1GDF on a lightlike surface. Finally, we present geometric phases and magnetic curves in $\widetilde{\mu}$, $\widetilde{U}-lines$ directions for 1GDF on a lightlike surface.

## 1. Introduction

Geometric phase, also known as Berry phase, is occurs when a quantum system undergoes cyclic variation, and the final state of the system depends not only on the initial and final conditions but also on the path taken [1]. The interaction between the electric field and geometric phase has important implications in quantum computing, condensed matter physics, and quantum information processing and optik. This concept has gained crucial interest in last years. The investigation of the electric field change has contributed to the development of materials of science, condensed matter physics and plasma physics [2-9].

In recent times, numerous authors have presented new Darboux frames. Balakrishnan presented certain moving space curves are endowed with a geometric phase for the Darboux frame in Euclidean 3-space [10]. Later, Ertuğ presented the variation of electric field with respect to Darboux triad in Euclidean and Minkowski 3-spaces [11,12]. Alessio et al. [13] have studied null Darboux frame $\{\mathbf{T},\mathbf{V},\mathbf{N}\}$ derivative formulas on a timelike surface:

$$(1.1)\qquad \begin{bmatrix}\mathbf{T}\\ \mathbf{V}\\ \mathbf{N}\end{bmatrix}_{\mathcal{T}} = \begin{bmatrix}\kappa_g^* & 0 & \kappa_n^*\\ 0 & -\kappa_g^* & -\tau_g^*\\ \tau_g^* & -\kappa_n^* & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{V}\\ \mathbf{N}\end{bmatrix}$$

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where the geodesic curvature $\kappa_g^* = \langle \mathbf{T}_\mathcal{T}, \mathbf{V} \rangle$, the geodesic torsion $\tau_g^* = \langle \mathbf{N}_\mathcal{T}, \mathbf{V} \rangle$, the normal curvature $\kappa_n^* = \langle \mathbf{T}_\mathcal{T}, \mathbf{N} \rangle$, tangential direction $\mathcal{T}$ and
$\langle \mathbf{T}, \mathbf{T} \rangle = 0 = \langle \mathbf{V}, \mathbf{V} \rangle = \langle \mathbf{T}, \mathbf{N} \rangle = \langle \mathbf{N}, \mathbf{V} \rangle = \mathbf{0}, \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{T}, \mathbf{V} \rangle = 1$, $\mathbf{T} \times \mathbf{V} = \mathbf{N}$, $\mathbf{V} \times \mathbf{N} = \mathbf{V}$, $\mathbf{N} \times \mathbf{T} = \mathbf{T}$ :

Topbaş et al. [14] introduced Darboux frame $\{\mathbf{T}, \mu, \mathbf{U}\}$ derivative equations of a spacelike curve on null surface in the tangential direction $\mathcal{T}$ in $\mathbb{R}_1^3$ for Darboux frame on lightlike surface [14]:

$$\begin{pmatrix} \mathbf{T} \\ \mu \\ \mathbf{U} \end{pmatrix}_\mathcal{T} = \begin{pmatrix} 0 & \varepsilon_0 \kappa_n & \varepsilon_0 \kappa_g \\ -\kappa_g & \varepsilon_0 \tau_g & 0 \\ -\kappa_n & 0 & -\varepsilon_0 \tau_g \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mu \\ \mathbf{U} \end{pmatrix} \tag{1.2}$$

where the geodesic curvature $\kappa_g = \langle \mu, \mathbf{T}_\mathcal{T} \rangle$, the geodesic torsion $\tau_g = \langle \mathbf{U}, \mu_\mathcal{T} \rangle$ and the normal curvature $\kappa_n = \langle \mathbf{U}, \mathbf{T}_\mathcal{T} \rangle$ and
$\langle \mathbf{T}, \mathbf{T} \rangle = 1$, $\langle \mathbf{U}, \mathbf{U} \rangle = \langle \mu, \mu \rangle = \langle \mathbf{T}, \mu \rangle = \langle \mathbf{T}, \mathbf{U} \rangle = \mathbf{0}$, $\langle \mathbf{U}, \mu \rangle = \varepsilon_0 = \pm 1$ and $\mathbf{T} \times \mu = \varepsilon_0 \mu$, $\mu \times \mathbf{U} = \mathbf{T}$, $\mathbf{U} \times \mathbf{T} = \varepsilon_0 \mathbf{U}$.

Djordjević and Nesovic introduced the first kind generalized Darboux frame (1GDF) derivative formulas of 1GDF on a lightlike surface in the tangential direction $\mathcal{T}$ in $\mathbb{R}_1^3$ as the following [15]:

$$\begin{pmatrix} \widetilde{\mathbf{T}} \\ \widetilde{\mu} \\ \widetilde{\mathbf{U}} \end{pmatrix}_\mathcal{T} = \begin{pmatrix} 0 & \varepsilon_1 \widetilde{\kappa}_n & \varepsilon_1 \widetilde{\kappa}_g \\ -\widetilde{\kappa}_g & \varepsilon_1 \widetilde{\tau}_g & 0 \\ -\widetilde{\kappa}_n & 0 & -\varepsilon_1 \widetilde{\tau}_g \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{T}} \\ \widetilde{\mu} \\ \widetilde{\mathbf{U}} \end{pmatrix} \tag{1.3}$$

where $\left\{ \widetilde{\mathbf{T}} = \mathbf{T} + \varpi \mathbf{U},\ \widetilde{\mu} = -\varepsilon_1 \frac{\varpi}{\zeta} \mathbf{T} + \frac{1}{\zeta} \mu - \varepsilon_1 \frac{\varpi^2}{2\zeta} \mathbf{U},\ \widetilde{\mathbf{U}} = \zeta \mathbf{U} \right\}$, $\varpi \neq 0$, $\zeta \neq 0$ are differentiable functions, generalized geodesic curvature $\widetilde{\kappa}_g = \frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_\mathcal{T}}{\zeta} - \frac{\varpi \tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2 \kappa_n}{2\zeta}$, the generalized geodesic torsion $\widetilde{\tau}_g = \tau_g - \varepsilon_1 \varpi \kappa_n - \varepsilon_1 \frac{\zeta_\mathcal{T}}{\zeta}$, the generalized normal curvature $\widetilde{\kappa}_n = \zeta \kappa_n$,
$\left\langle \widetilde{\mathbf{T}}, \widetilde{\mathbf{T}} \right\rangle = 1$, $\left\langle \widetilde{\mathbf{U}}, \widetilde{\mathbf{U}} \right\rangle = \langle \widetilde{\mu}, \widetilde{\mu} \rangle = \left\langle \widetilde{\mathbf{T}}, \widetilde{\mu} \right\rangle = \left\langle \widetilde{\mathbf{T}}, \widetilde{\mathbf{U}} \right\rangle = \mathbf{0}$, $\left\langle \widetilde{\mathbf{U}}, \widetilde{\mu} \right\rangle = \varepsilon_1 = \pm 1$, $\widetilde{\mathbf{T}} \times \widetilde{\mu} = \varepsilon_1 \widetilde{\mu}$, $\widetilde{\mu} \times \mathbf{U} = \widetilde{\mathbf{T}}$, $\widetilde{\mathbf{U}} \times \widetilde{\mathbf{T}} = \varepsilon_1 \widetilde{\mathbf{U}}$.

Furthermore, Djordjević and Nesovic introduced the second kind generalized Darboux frame $\{\widetilde{\mathbf{T}} = \mathbf{T}^* = \mathbf{T},\ \mu^* = \frac{1}{\zeta} \mu,\ \mathbf{U}^* = \zeta \mathbf{U}\ \}$ derivative formulas of 2GDF on lightlike surface in the tangential direction $\mathcal{T}$ in $\mathbb{R}_1^3$ [15]:

$$\begin{pmatrix} \mathbf{T}^* \\ \mu^* \\ \mathbf{U}^* \end{pmatrix}_\mathcal{T} = \begin{pmatrix} 0 & \varepsilon_1 \zeta \kappa_n & \varepsilon_1 \frac{\kappa_g}{\zeta} \\ -\frac{\kappa_g}{\zeta} & \varepsilon_1 (\tau_g - \varepsilon_1 \frac{\zeta_\mathcal{T}}{\zeta}) & 0 \\ -\zeta \kappa_n & 0 & -\varepsilon_1 (\tau_g - \varepsilon_1 \frac{\zeta_\mathcal{T}}{\zeta}) \end{pmatrix} \begin{pmatrix} \mathbf{T}^* \\ \mu^* \\ \mathbf{U}^* \end{pmatrix} \tag{1.4}$$

where generalized geodesic curvature $\frac{\kappa_g}{\zeta}$, generalized geodesic torsion $\tau_g - \varepsilon_1 \frac{\zeta_\mathcal{T}}{\zeta}$ and generalized normal curvature $\zeta \kappa_n$.
Magnetic curves which a divergence free vector field were studied in [16-20]. Intrinsic directional derivatives have been investigated in [21-28].

In section 1, we give introduction. In Section 2, 3 and 4, we obtain ${}^{D}\mathbf{E}_\mathcal{T}$, ${}^{\mathcal{ND}}\mathbf{E}_\mathcal{T}$, ${}^{1\mathcal{GD}}\mathbf{E}_\mathcal{T}$,

${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$, Lorentz force equations and magnetic curves in the tangential direction. We derive intrinsic directional derivatives in the $\widetilde{\mu}, \widetilde{U}-lines$ directions for 1GDF on a lightlike surface. Later, we present $\widetilde{\mu}, \widetilde{\mathbf{U}}-$magnetic curves and geometric phases in the $\widetilde{\mu}$, $\widetilde{U}-lines$ directions for 1GDF on a lightlike surface .

## 2. ${}^{D}\mathbf{E}_{\mathcal{T}}$, ${}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}$, ${}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ AND ${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$

**${}^{D}\mathbf{E}_{\mathcal{T}}$ for Darboux frame on a lightlike surface in the tangential direction**

In general form, the change of the electric field ${}^{D}\mathbf{E}_{\mathcal{T}}$ for Darboux frame of a spacelike curve with non-lightlike principal normal lying on a lightlike surface can be expressed

$$ {}^{D}\mathbf{E}_{\mathcal{T}} = a_1\mathbf{T} + a_2\mu + a_3\mathbf{U}. \tag{2.1}$$

Assume that

$$ \left\langle {}^{D}\mathbf{E}, \mathbf{T} \right\rangle = 0. \tag{2.2}$$

$$ \left\langle {}^{D}\mathbf{E}, {}^{D}\mathbf{E} \right\rangle = const. \tag{2.3}$$

From Eqs.(1.2), (2.1), (2.2) and (2.3), the followings are obtained:

$$ a_1 = -\varepsilon_0\kappa_n \left\langle {}^{D}\mathbf{E}, \mu \right\rangle - \varepsilon_0\kappa_g \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle \ . \tag{2.4}$$

$$ a_2 = \varsigma_1 \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle, \quad a_3 = -\varsigma_1 \left\langle {}^{D}\mathbf{E}, \mu \right\rangle \tag{2.5}$$

Here, $\varsigma_1$ is a parameter. Assume that, $\left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle \neq 0$, $\left\langle {}^{D}\mathbf{E}, \mu \right\rangle \neq 0$. If Eqs.(2.4) and (2.5) are substituted in Eq.(2.1), then

$$ \begin{aligned} \frac{\delta({}^{D}\mathbf{E})}{\delta\mathcal{T}} &= {}^{D}\mathbf{E}_{\mathcal{T}} = -\varepsilon_0(\kappa_n \left\langle {}^{D}\mathbf{E}, \mu \right\rangle + \kappa_g \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle)\mathbf{T} \\ & +\varsigma_1 \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle \mu - \varsigma_1 \left\langle {}^{D}\mathbf{E}, \mu \right\rangle \mathbf{U} \end{aligned} \tag{2.6}$$

is obtained. $\varsigma_1 \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle \mu - \varsigma_1 \left\langle {}^{D}\mathbf{E}, \mu \right\rangle \mathbf{U}$ denotes the rotation around $\mathbf{T}$ for Darboux frame with a nonnull principal normal lying on a lightlike surface. For $\varsigma_1 = 0$,

$$ {}^{D}\mathbf{E}_{\mathcal{T}} = -\varepsilon_0(\kappa_n \left\langle {}^{D}\mathbf{E}, \mu \right\rangle + \kappa_g \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle)\mathbf{T} \tag{2.7}$$

Lorentz force equation ${}^{D}\Phi$ of the electric field vector ${}^{D}\mathbf{E}$ for Darboux frame of a spacelike curve with a nonlightlike principal normal lying on a lightlike surface is described by

$$ {}^{D}\Phi({}^{D}\mathbf{E}) ={}^{D} \mathbf{E}_{T} = \mathcal{A}_1 \times {}^{D}\mathbf{E}. \tag{2.8}$$

From Eq.(2.8), Lorentz force equations of $\{\mathbf{T}, \mu, \mathbf{U}\}$ are given by

$$ {}^{D}\Phi(\mathbf{T}) = \varepsilon_0\kappa_n\mu + \varepsilon_0\kappa_g\mathbf{U} \tag{2.9}$$

$$ {}^{D}\Phi(\mu) = -\kappa_g\mathbf{T} + \varepsilon_0\varsigma_1\mathbf{U} \tag{2.10}$$

$$ {}^{D}\Phi(\mathbf{U}) = -\kappa_n\mathbf{T} - \varepsilon_0\varsigma_1\mu \tag{2.11}$$

From Eq.(2.9), (2.10), (2.11), the magnetic vector field is obtained:

$$\mathcal{A}_1{=}\varsigma_1\mathbf{T}{-}\kappa_n\mu{+}\kappa_g\mathbf{U}$$

In the general form, it can given by

$$^{D}\mathbf{E}=\varepsilon_0\left\langle E,U\right\rangle\mu{+}\varepsilon_0\left\langle E,\mu\right\rangle\mathbf{U} \tag{2.12}$$

Via Eqs.(2.12), it is derived

$$\begin{aligned} ^{D}\mathbf{E}_{\mathcal{T}} &= \mu(\varepsilon_0\left\langle E,U\right\rangle_{\mathcal{T}}+\tau_g\left\langle E,U\right\rangle) \\ &\quad +\mathbf{U}(\varepsilon_0\left\langle E,\mu\right\rangle_{\mathcal{T}}-\tau_g\left\langle E,\mu\right\rangle) \\ &\quad -\varepsilon_0\mathbf{T}(\kappa_n\left\langle ^{D}\mathbf{E},\mu\right\rangle+\kappa_g\left\langle ^{D}\mathbf{E},\mathbf{U}\right\rangle) \end{aligned} \tag{2.13}$$

Comparing Eqs.(2.7) and (2.13), it can be obtained

$$\left\langle E,\mu\right\rangle_{\mathcal{T}} = \varepsilon_0\tau_g\left\langle E,\mu\right\rangle \tag{2.14}$$

$$\left\langle E,U\right\rangle_{\mathcal{T}} = -\varepsilon_0\tau_g\left\langle E,U\right\rangle \tag{2.15}$$

Geometric phase around $\mathbf{T}$ for the frame $\{\mathbf{T},\mu,\mathbf{U}\}$ on lightlike surface via Eqs.(2.14) and (2.15) is $\varepsilon_0\tau_g$.

**$^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}$ for null Darboux frame on timelike surface in the tangential direction**

The change of electric field $^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}$ for null Darboux frame on a timelike surface in the $\mathcal{T}-lines$ direction is given by

$$\frac{\delta(^{\mathcal{ND}}\mathbf{E})}{\delta\mathcal{T}}={}^{\mathcal{ND}}\,\mathbf{E}_{\mathcal{T}}=b_1\mathbf{T}+b_2\mathbf{V}+b_3\mathbf{N}. \tag{2.16}$$

Assume that

$$\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{N}\right\rangle=0, \tag{2.17}$$

$$\left\langle ^{\mathcal{ND}}\mathbf{E},{}^{\mathcal{ND}}\mathbf{E}\right\rangle=const. \tag{2.18}$$

From Eqs.(1.1), (2.16), (2.17) and (2.18), it can be obtained

$$b_1 = \varsigma_2\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{V}\right\rangle,\ b_2=-\varsigma_2\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{T}\right\rangle \tag{2.19}$$

$$b_3 = \kappa_n^*\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{V}\right\rangle-\tau_g^*\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{T}\right\rangle \tag{2.20}$$

where $\varsigma_2$ is a parameter. Assume that $\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{V}\right\rangle\neq 0$, $\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{T}\right\rangle\neq 0$. If Eqs.(2.19), (2.20) are substituted in Eq.(2.16), then

$$\begin{aligned} ^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}} &= \varsigma_2\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{V}\right\rangle\mathbf{T}-\varsigma_2\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{T}\right\rangle\mathbf{V} \\ &\quad +(\kappa_n^*\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{V}\right\rangle-\tau_g^*\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{T}\right\rangle)\mathbf{N} \end{aligned} \tag{2.21}$$

$\varsigma_2\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{V}\right\rangle\mathbf{T}-\varsigma_2\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{T}\right\rangle\mathbf{V}$ denotes the rotation around $\mathbf{N}$ for null Darboux frame on a timelike surface. For $\varsigma_2=0$,

$$^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}=(\kappa_n^*\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{V}\right\rangle-\tau_g^*\left\langle ^{\mathcal{ND}}\mathbf{E},\mathbf{T}\right\rangle)\mathbf{N} \tag{2.22}$$

Null Darboux Lorentz force equation $^{\mathcal{ND}}\Phi$ of the electric field vector for null Darboux frame on timelike surface is described by

$$ {}^{\mathcal{ND}}\Phi^{(N)}(\mathbf{E}) = {}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}} = \mathcal{A}_2 \times {}^{\mathcal{ND}}\mathbf{E} \tag{2.23} $$

Via Eqs.(2.21) and (2.23), Lorentz force equations of the frame $\{\mathbf{T}, \mathbf{V}, \mathbf{N}\}$

$$ {}^{\mathcal{ND}}\Phi^{(N)}(\mathbf{T}) = \varsigma_2 \mathbf{T} + \kappa_n^* \mathbf{N} \tag{2.24} $$

$$ {}^{\mathcal{ND}}\Phi^{(N)}(\mathbf{V}) = -\tau_g^* \mathbf{N} - \varsigma_2 \mathbf{V} \tag{2.25} $$

$$ {}^{\mathcal{ND}}\Phi^{(N)}(\mathbf{N}) = \tau_g^* \mathbf{T} - \kappa_n^* \mathbf{V} \tag{2.26} $$

Via Eqs.(2.23), (2.24), (2.25) and (2.26), null Darboux magnetic field vector is derived

$$ \mathcal{A}_2 = -\tau_g^* \mathbf{T} + \varsigma_2 \mathbf{N} - \kappa_n^* \mathbf{V} $$

In the general form,

$$ {}^{\mathcal{ND}}\mathbf{E} = \left\langle {}^{\mathcal{ND}}E, V \right\rangle \mathbf{T} + \left\langle {}^{\mathcal{ND}}E, \mathcal{T} \right\rangle \mathbf{V} \tag{2.27} $$

With the aid (2.27), it is obtained

$$ \begin{aligned} {}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}} = \; & \mathbf{T}( \left\langle {}^{\mathcal{ND}}E, V \right\rangle_{\mathcal{T}} + \kappa_g^* \left\langle {}^{\mathcal{ND}}E, V \right\rangle ) \\ & + \mathbf{V}( \left\langle {}^{\mathcal{ND}}E, \mathcal{T} \right\rangle_{\mathcal{T}} - \kappa_g^* \left\langle {}^{\mathcal{ND}}E, \mathcal{T} \right\rangle ) \\ & + \mathbf{N}(\kappa_n^* \left\langle {}^{\mathcal{ND}}E, V \right\rangle - \tau_g^* \left\langle {}^{\mathcal{ND}}E, \mathcal{T} \right\rangle) \end{aligned} \tag{2.28} $$

Comparing Eqs.(2.22) and (2.28), the followings are obtained

$$ \begin{aligned} \left\langle {}^{\mathcal{ND}}E, V \right\rangle_{\mathcal{T}} &= -\kappa_g^* \left\langle {}^{\mathcal{ND}}E, V \right\rangle \\ \left\langle {}^{\mathcal{ND}}E, \mathcal{T} \right\rangle_{\mathcal{T}} &= \kappa_g^* \left\langle {}^{\mathcal{ND}}E, \mathcal{T} \right\rangle \end{aligned} $$

Geometric phase around $\mathbf{N}$ for the frame $\{\mathbf{T}, \mathbf{V}, \mathbf{N}\}$ is $\kappa_g^*$.

### ${}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ for 1GDF on lightlike surface in the tangential direction

The change of electric field ${}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ for 1GDF on lightlike surface in the tangential direction $\mathcal{T}$ is given by

$$ {}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = c_1 \widetilde{\mathbf{T}} + c_2 \widetilde{\mu} + c_3 \widetilde{\mathbf{U}}. \tag{2.29} $$

Consider

$$ \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{T}} \right\rangle = 0 \tag{2.30} $$

$$ \left\langle {}^{1\mathcal{GD}}\mathbf{E}, {}^{1\mathcal{GD}}\mathbf{E} \right\rangle = const. \tag{2.31} $$

From Eqs.(1.3), (2.29), (2.30) and (2.31), the followings are obtained :

$$ c_1 = -\varepsilon_1 \left( \zeta \kappa_n \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + \left( \frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi \tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2 \kappa_n}{2\zeta} \right) \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle \right) \tag{2.32} $$

$$ c_2 = \mathfrak{z}_1^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle, \quad c_3 = -\mathfrak{z}_1^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle \tag{2.33} $$

where $\mathfrak{z}_1^1$ is a parameter. For $\mathfrak{z}_1^1 = 0$, Eqs.(2.32), (2.33) are rewritten in Eq.(2.29),

(2.34)
$$^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = -\varepsilon_1 \left( \zeta\kappa_n \left\langle ^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + (\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2\kappa_n}{2\zeta}) \left\langle ^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle \right) \widetilde{\mathbf{T}}$$

Lorentz force equation of electric field vector for 1GDF on lightlike surface in the tangential direction is described

$$^{1\mathcal{GD}}\Phi^{(\mathcal{T})}(^{1\mathcal{GD}}\mathbf{E}) = ^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = \mathcal{A}_3 \times {}^{1\mathcal{GD}}\mathbf{E} \tag{2.35}$$

$\mathcal{A}_3$ is the magnetic vector field. From Eq.(2.35), Lorentz force equations of 1GDF in the tangential direction can be given by

$$^{1\mathcal{GD}}\Phi^{(\mathcal{T})}(\widetilde{\mathbf{T}}) = \varepsilon_1\zeta\kappa_n\widetilde{\mu} + \varepsilon_1(\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2\kappa_n}{2\zeta})\widetilde{\mathbf{U}} \tag{2.36}$$

$$^{1\mathcal{GD}}\Phi^{(\mathcal{T})}(\widetilde{\mu}) = -(\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2\kappa_n}{2\zeta})\widetilde{\mathbf{T}} + \varepsilon_1\mathfrak{z}_1^1\widetilde{\mathbf{U}} \tag{2.37}$$

$$^{1\mathcal{GD}}\Phi^{(\mathcal{T})}(\widetilde{\mathbf{U}}) = -\zeta\kappa_n\widetilde{\mathbf{T}} - \varepsilon_1\mathfrak{z}_1^1\widetilde{\mu} \tag{2.38}$$

In the general form,

$$^{1\mathcal{GD}}\mathbf{E} = \varepsilon_1 \left\langle ^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle \widetilde{\mu} + \varepsilon_1 \left\langle ^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu} \right\rangle \widetilde{\mathbf{U}} \tag{2.39}$$

With the aid (2.39), it is obtained

$$\begin{aligned} ^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = {} & \widetilde{\mu} \left( \varepsilon_1 \left\langle ^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle_{\mathcal{T}} + (\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1 \frac{\zeta_T}{\zeta}) \left\langle ^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle \right) \\ & + \widetilde{\mathbf{U}} \left( \varepsilon_1 (^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu})_{\mathcal{T}} - (\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1 \frac{\zeta_T}{\zeta}) \left\langle ^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu} \right\rangle \right) \\ & - \varepsilon_1\widetilde{\mathbf{T}} \left( \zeta\kappa_n \left\langle ^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + (\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2\kappa_n}{2\zeta}) \left\langle ^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle \right) \end{aligned} \tag{2.40}$$

Comparing Eqs.(2.34) and (2.40), it can be derived

$$\left\langle ^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu} \right\rangle_{\mathcal{T}} = \varepsilon_1(\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1 \frac{\zeta_T}{\zeta}) \left\langle ^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu} \right\rangle \tag{2.41}$$

$$\left\langle ^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle_{\mathcal{T}} = -\varepsilon_1(\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1 \frac{\zeta_T}{\zeta}) \left\langle ^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle \tag{2.42}$$

From Eqs.(2.41) and (2.42), geometric phase around $\widetilde{\mathbf{T}}$ in the $\mathcal{T} - lines$ direction for 1GDF on lightlike surface is $(\varepsilon_1\tau_g - \varpi\kappa_n - \frac{\zeta_T}{\zeta})$.

### $^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ for 2GDF in the tangential direction on lightlike surface

The change of the electric field $^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ for 2GDF on lightlike surface in the $\mathcal{T} - lines$ direction can be written by

$$^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = d_1\mathbf{T}^* + d_2\mu^* + d_3\mathbf{U}^*. \tag{2.43}$$

Assume that

$$\left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{T}^* \right\rangle = 0 \tag{2.44}$$

$$\left\langle {}^{2\mathcal{GD}}\mathbf{E}, {}^{2\mathcal{GD}}\mathbf{E} \right\rangle = const. \tag{2.45}$$

From Eqs.(1.4), (2.43), (2.44) and (2.45), it can be derived:

$$d_1 = -\varepsilon_1 \zeta \kappa_n \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^* \right\rangle - \varepsilon_1 \frac{\kappa_g}{\zeta} \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^* \right\rangle \tag{2.46}$$

$$d_2 = \varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^* \right\rangle, \quad d_3 = -\varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^* \right\rangle \tag{2.47}$$

$\varsigma_3$ is a parameter. Assume that $\left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^* \right\rangle \neq 0$, $\left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^* \right\rangle \neq 0$. If Eqs.(2.46) and (2.47) are substituted in Eq.(2.43), then

$$\begin{aligned} {}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} &= -\varepsilon_1 (\zeta \kappa_n \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^* \right\rangle + \frac{\kappa_g}{\zeta} \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^* \right\rangle) \mathbf{T}^* \\ &\quad + \varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^* \right\rangle \mu^* - \varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^* \right\rangle \mathbf{U}^* \end{aligned} \tag{2.48}$$

$\varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^* \right\rangle \mu^* - \varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^* \right\rangle \mathbf{U}^*$ denotes the rotation around $\mathbf{T}^*$. For $\varsigma_3 = 0$,

$${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = -\varepsilon_1 (\zeta \kappa_n \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^* \right\rangle + \frac{\kappa_g}{\zeta} \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^* \right\rangle) \mathbf{T}^* \tag{2.49}$$

Lorentz force equation ${}^{2\mathcal{GD}}\Phi^{(\mathcal{T})}$ of the electric field vector ${}^{2\mathcal{GD}}\mathbf{E}$ in the tangential direction can be described by

$${}^{2\mathcal{GD}}\Phi^{(T)}({}^{2\mathcal{GD}}\mathbf{E}) = {}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = \mathcal{A}_4 \times {}^{2\mathcal{GD}}\mathbf{E}_2 \tag{2.50}$$

From Eq.(2.50), Lorentz force equations of 2GDF in the tangential direction are given by:

$${}^{2\mathcal{GD}}\Phi^{(\mathcal{T})}(\mathbf{T}^*) = \varepsilon_1 \zeta \kappa_n \mu^* + \varepsilon_1 \frac{\kappa_g}{\zeta} \mathbf{U}^* \tag{2.51}$$

$${}^{2\mathcal{GD}}\Phi^{(\mathcal{T})}(\mu^*) = -\frac{\kappa_g}{\zeta} \mathbf{T}^* + \varepsilon_1 \varsigma_3 \mathbf{U}^* \tag{2.52}$$

$${}^{2\mathcal{GD}}\Phi^{(\mathcal{T})}(\mathbf{U}^*) = -\zeta \kappa_n \mathbf{T}^* - \varepsilon_1 \varsigma_3 \mu^* \tag{2.53}$$

$$\mathcal{A}_4 = \varepsilon_1 \varsigma_3 \mathbf{T}^* - \zeta \kappa_n \mu^* + \frac{\kappa_g}{\zeta} \mathbf{U}^* \tag{2.54}$$

is the magnetic vector field satisfying Eqs.(2.51), (2.52) and (2.53). Also,

$${}^{2\mathcal{GD}}\mathbf{E} = \varepsilon_1 \left\langle {}^{2\mathcal{GD}}E, U^* \right\rangle \mu^* + \varepsilon_1 \left\langle {}^{2\mathcal{GD}}E, \mu^* \right\rangle \mathbf{U}^* \tag{2.55}$$

Via Eq.(2.55), it can be obtained

$${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = -\varepsilon_1 (\zeta \kappa_n \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^* \right\rangle + \frac{\kappa_g}{\zeta} \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^* \right\rangle) \mathbf{T}^*$$

$+\mu^* \left(\varepsilon_1 \left\langle {}^{2\mathcal{GD}}E, U^* \right\rangle_{\mathcal{T}} + (\tau_g - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta}) \left\langle {}^{2\mathcal{GD}}E, \mu^* \right\rangle\right)$ (2.56)

$+\mathrm{U}^* \left(\varepsilon_1 \left\langle {}^{2\mathcal{GD}}E, \mu^* \right\rangle_{\mathcal{T}} - (\tau_g - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta}) \left\langle {}^{2\mathcal{GD}}E, \mu^* \right\rangle\right)$

Comparing Eqs.(2.49) and(2.56) ,

$$\left\langle {}^{2\mathcal{GD}}E, \mu^* \right\rangle_{\mathcal{T}} = \varepsilon_1(\tau_g - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta}) \left\langle {}^{2\mathcal{GD}}E, \mu^* \right\rangle \tag{2.57}$$

$$\left\langle {}^{2\mathcal{GD}}E, U^* \right\rangle_{\mathcal{T}} = -\varepsilon_1(\tau_g - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta}) \left\langle {}^{2\mathcal{GD}}E, U^* \right\rangle \tag{2.58}$$

From Eqs.(2.57) and (2.58), geometric phase around $\mathbf{T}^*$ for 2GDF in the tangential direction is $\varepsilon_1(\tau_g - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta})$.

## 3. Magnetic curves for Darboux frames

### V−magnetic curves for null Darboux frame in the $\mathcal{T} - lines$ direction

Let $\alpha$ be a distinguished curve on timelike surface with null Darboux frame. $\alpha$ is called **V**-magnetic curve if it satisfied null Darboux Lorentz force equation

$$\mathbf{V}_T = {}^{\mathcal{ND}}\, \Phi^{(V)}(\mathbf{V}) = \mathcal{A}_5 \times \mathbf{V} \tag{3.1}$$

Here, $\mathcal{A}_5$ is magnetic vector field. It can be written by

$$^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T}) = \imath_1 \mathbf{T} + \imath_2 \mathbf{V} + \imath_3 \mathbf{N} \tag{3.2}$$

From Eqs.(1.1), (3.1) and (3.2), it can be derived

$$\left\langle {}^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T}), \mathbf{T} \right\rangle = \imath_2 = 0 \tag{3.3}$$

$$\left\langle {}^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T}), \mathbf{V} \right\rangle = \imath_1 = \kappa_g^* \tag{3.4}$$

$$\left\langle {}^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T}), \mathbf{N} \right\rangle = \imath_3 = \varsigma_4 \tag{3.5}$$

$\varsigma_4$ is a function. If Eqs.(3.3), (3.4) and (3.5) are substituted Eq.(3.2),

$$^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T}) = \kappa_g^* \mathbf{T} + \varsigma_4 \mathbf{N} \tag{3.6}$$

obtained. Also,

$$^{\mathcal{ND}}\Phi^{(V)}(\mathbf{N}) = \tau_g^* \mathbf{T} - \varsigma_4 \mathbf{V} \tag{3.7}$$

$$\mathcal{A}_5 = -\tau_g^* \mathbf{T} - \varsigma_4 \mathbf{V} + \kappa_g^* \mathbf{N}$$

.

### $\widetilde{\mu}$−magnetic curves for 1GDF on lightlike surface in the $\mathcal{T} - lines$ direction

$\alpha$ is called $\widetilde{\mu}$-magnetic curve for 1GDF on lightlike surface if it satisfied the Lorentz force equation

$$\widetilde{\mu}_{\mathcal{T}} = {}^{1\mathcal{GD}}\, \Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mu}) = \widetilde{\mathbf{X}}_1 \times \widetilde{\mu} \tag{3.8}$$

Furthermore, it can be written by

$$^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = j_1 \widetilde{\mathbf{T}} + j_2 \widetilde{\mu} + j_3 \widetilde{\mathbf{U}} \tag{3.9}$$

Via Eqs.(1.3), (3.8) and (3.9), it can be derived

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{T}} \right\rangle = j_1 = 0 \quad (3.10)$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{U}} \right\rangle = \mathfrak{z}_1^2,\ j_2 = \varepsilon_1 \mathfrak{z}_1^2 \quad (3.11)$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mu} \right\rangle = \varepsilon_1 j_3 \ \Rightarrow j_3 = \varepsilon_1 \zeta \kappa_n \quad (3.12)$$

If Eqs.(3.10), (3.11) and (3.12) are rewritten in (3.9),

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = \varepsilon_1 \mathfrak{z}_1^2 \widetilde{\mu} + \varepsilon_1 (\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi \mathcal{T}}{\zeta} - \frac{\varpi \tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2 \kappa_n}{2\zeta})\widetilde{\mathbf{U}}$$

As similar,

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{U}}) = -\mathfrak{z}_1^2 \widetilde{\mathbf{T}} - \varepsilon_1 (\tau_g - \varepsilon_1 \varpi \kappa_n - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta})\widetilde{\mathbf{U}}$$

is derived. The magnetic field vector

$\widetilde{\mathbf{X}}_1 = -\mathfrak{z}_1^2 \widetilde{\mu} + (\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi \mathcal{T}}{\zeta} - \frac{\varpi \tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2 \kappa_n}{2\zeta})\widetilde{\mathbf{U}} + (\tau_g - \varepsilon_1 \varpi \kappa_n - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta})\widetilde{\mathbf{T}}$

is obtained.

**$\widetilde{\mathbf{U}}-$magnetic curves for 1GDF on lightlike surface in the $\mathcal{T}-lines$ direction**

$\alpha$ is called $\widetilde{\mathbf{U}}-$magnetic curve for 1GDF on lightlike surface if it satisfied the Lorentz force equation for 1GDF

$$\widetilde{\mathbf{U}}_{\mathcal{T}} = {}^{1\mathcal{GD}}\ \Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{U}}) = \widetilde{\mathbf{X}}_2 \times \widetilde{\mathbf{U}} \quad (3.13)$$

where $\widetilde{\mathbf{X}}_2$ is magnetic field vector Let $\alpha$ be a spacelike curve 1GDF on lightlike surface. We get,

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}) = v_1 \widetilde{\mathbf{T}} + v_2 \widetilde{\mu} + v_3 \widetilde{\mathbf{U}} \quad (3.14)$$

Via Eqs.(3.13) and (3.14), it can be obtained

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{T}} \right\rangle = v_1 = 0 \quad (3.15)$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{U}} \right\rangle = \varepsilon_1 v_2 \Rightarrow v_2 = \varepsilon_1 \zeta \kappa_n \quad (3.16)$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}), \widetilde{\mu} \right\rangle = \varepsilon_1 v_3 \Rightarrow v_3 = \varepsilon_1 \mathfrak{z}_1^3 \quad (3.17)$$

If Eqs.(3.15), (3.16) and (3.17) are substituted in Eq.(3.14 ), then

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}) = \varepsilon_1 \mathfrak{z}_1^3 \widetilde{\mathbf{U}} + \varepsilon_1 \zeta \kappa_n \widetilde{\mu}$$

As similarly,

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mu}) = -\varepsilon_1 \mathfrak{z}_1^3 \widetilde{\mathbf{T}} + (\tau_g - \varepsilon_1 \varpi \kappa_n - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta})\widetilde{\mu}$$

## 4. Intrinsic directional derivatives for 1GDF on a lightlike surface and magnetic curves

${}^{1\mathcal{GD}}\nabla$ is called the gradient operator for 1GDF on lightlike surface. It can be written by

$$^{1\mathcal{GD}}\nabla = \widetilde{\mathbf{T}}\frac{\delta}{\delta\mathcal{T}} + \varepsilon_1\widetilde{\mu}\frac{\delta}{\delta\widetilde{U}} + \varepsilon_1\widetilde{\mathbf{U}}\frac{\delta}{\delta\widetilde{\mu}}$$

where $\widetilde{\mathcal{T}}$, $\widetilde{\mu}$ and $\widetilde{U}$ are the arc length coordinates on the $\widetilde{\mathcal{T}}$, $\widetilde{\mu}-lines$ and $\widetilde{U}-lines$ for 1GDF. $\frac{\delta}{\delta\widetilde{\mathcal{T}}}$, $\frac{\delta}{\delta\widetilde{\mu}}$ and $\frac{\delta}{\delta\widetilde{U}}$ are the intrinsic directional derivatives in $\widetilde{\mathcal{T}}-lines$, $\widetilde{\mu}-lines$ and $\widetilde{U}-lines$ directions. The divergence vector for 1GDF is

$$^{1\mathcal{GD}}\ div\widetilde{\mathbf{T}} = \left\langle {}^{1\mathcal{GD}}\nabla, \widetilde{\mathbf{T}}\right\rangle = \varepsilon_1\left\langle \frac{\delta\widetilde{\mathbf{T}}}{\delta\widetilde{U}}, \widetilde{\mu}\right\rangle + \varepsilon_1\left\langle \frac{\delta\widetilde{\mathbf{T}}}{\delta\widetilde{\mu}}, \widetilde{\mathbf{U}}\right\rangle$$

where

$$\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})} = \left\langle \frac{\delta\widetilde{\mathbf{T}}}{\delta\widetilde{U}}, \widetilde{\mu}\right\rangle,\ \varphi^{(\widetilde{\mathcal{T}}\widetilde{U})} = \left\langle \frac{\delta\widetilde{\mathbf{T}}}{\delta\widetilde{\mu}}, \widetilde{\mathbf{U}}\right\rangle$$

$$^{1\mathcal{GD}}\ div\widetilde{\mu} = \left\langle {}^{1\mathcal{GD}}\nabla, \widetilde{\mu}\right\rangle = -\widetilde{\kappa}_g + \varepsilon_1\left\langle \frac{\delta\widetilde{\mu}}{\delta\widetilde{\mu}}, \widetilde{\mathbf{U}}\right\rangle \tag{4.1}$$

$$^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}} = \left\langle {}^{1\mathcal{GD}}\nabla, \widetilde{\mathbf{U}}\right\rangle = -\widetilde{\kappa}_n + \varepsilon_1\left\langle \frac{\delta\widetilde{\mathbf{U}}}{\delta\widetilde{U}}, \widetilde{\mu}\right\rangle \tag{4.2}$$

$$\gamma^{(\widetilde{\mu})} = \left\langle {}^{1\mathcal{GD}}\nabla\times\widetilde{\mu}, \widetilde{\mu}\right\rangle = \varepsilon_1\left\langle \frac{\delta\widetilde{\mu}}{\delta\widetilde{\mu}}, \widetilde{\mathbf{T}}\right\rangle$$

$$\gamma^{(\widetilde{U})} = \left\langle {}^{1\mathcal{GD}}\nabla\times\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}\right\rangle = -\varepsilon_1\left\langle \frac{\delta\widetilde{\mathbf{U}}}{\delta\widetilde{U}}, \widetilde{\mathbf{T}}\right\rangle$$

$\gamma^{(T)}$, $\gamma^{(\widetilde{\mu})}$, $\gamma^{(\widetilde{U})}$ are total moments of the $\widetilde{\mathbf{T}}$, $\widetilde{\mu}$, $\widetilde{\mathbf{U}}$ fields of spacelike curve for 1GDF on lightlike surface. Intrinsic directional derivatives in $\widetilde{\mu}$ and $\widetilde{\mathbf{U}}$ lines directions for 1GDF on lightlike surface are obtained

$$\frac{\delta}{\delta\widetilde{\mu}}\begin{bmatrix}\widetilde{\mathbf{T}}\\ \widetilde{\mu}\\ \widetilde{\mathbf{U}}\end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_1\varphi^{(T\widetilde{U})} & -\gamma^{(\widetilde{\mu})}\\ \varepsilon_1\gamma^{(\widetilde{\mu})} & \widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu} & 0\\ -\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})} & 0 & -(\widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu})\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{T}}\\ \widetilde{\mu}\\ \widetilde{\mathbf{U}}\end{bmatrix} \tag{4.3}$$

(4.4)

$$\frac{\delta}{\delta\widetilde{U}}\begin{bmatrix}\widetilde{\mathbf{T}}\\ \widetilde{\mu}\\ \widetilde{\mathbf{U}}\end{bmatrix} = \begin{bmatrix} 0 & \gamma^{(\widetilde{U})} & \varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\\ -\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})} & -(\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}}) & 0\\ -\varepsilon_1\gamma^{(\widetilde{U})} & 0 & (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{T}}\\ \widetilde{\mu}\\ \widetilde{\mathbf{U}}\end{bmatrix}$$

**Geometric phase in $\widetilde{\mu}-lines$ direction for 1GDF on lightlike surface**

In the general form, the change of the electric field $\frac{\delta({}^{1\mathcal{GD}}\mathbf{E}_1)}{\delta\widetilde{\mu}}$ for 1GDF on lightlike surface in $\widetilde{\mu}-lines$ direction is

$$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{\mu}} = {}^{1\mathcal{GD}}\ \mathbf{E}_{\widetilde{\mu}} = l_1\widetilde{\mathbf{T}} + l_2\widetilde{\mu} + l_3\widetilde{\mathbf{U}}. \tag{4.5}$$

Assume that

$$\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{T}} \right\rangle = 0 \tag{4.6}$$

$$\left\langle {}^{1\mathcal{GD}}\mathbf{E}, {}^{1\mathcal{GD}}\ \mathbf{E} \right\rangle = const. \tag{4.7}$$

Here,

$$l_1 = -\varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + \gamma^{(\widetilde{\mu})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle. \tag{4.8}$$

$$l_2 = \mathfrak{z}_2^1\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle, \quad l_3 = -\mathfrak{z}_2^1\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle \tag{4.9}$$

When Eqs.(4.6), (4.7) are written in Eq.(4.5),

$$\begin{aligned} \frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{\mu}} &= (-\varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + \gamma^{(\widetilde{\mu})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle)\widetilde{\mathbf{T}} \\ &\quad + \mathfrak{z}_2^1\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle\widetilde{\mu} - \mathfrak{z}_2^1\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle\widetilde{\mathbf{U}} \end{aligned} \tag{4.10}$$

where $\mathfrak{z}_2^1$ is a parameter. $\mathfrak{z}_2^1({}^{1\mathcal{GD}}\mathbf{E}\times\widetilde{\mathbf{T}})$ is the rotation around $\widetilde{\mathbf{T}}$ for 1GDF on lightlike surface in $\widetilde{\mu} - lines$ direction.

When $\mathfrak{z}_2^1 = 0$, Eq.(4.10) is derived as the following:

$$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{\mu}} = {}^{1\mathcal{GD}}\ \mathbf{E}_{\widetilde{\mu}} = (-\varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + \gamma^{(\widetilde{\mu})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle)\widetilde{\mathbf{T}} \tag{4.11}$$

Lorentz force equation ${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}$ of ${}^{1\mathcal{GD}}\mathbf{E}$ in $\widetilde{\mu} - lines$ direction is described

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}({}^{1\mathcal{GD}}\mathbf{E}) = {}^{1\mathcal{GD}}\ \mathbf{E}_{\widetilde{\mu}} = \widetilde{\mathbf{Y}}_1 \times {}^{1\mathcal{GD}}\mathbf{E} \tag{4.12}$$

With Eqs.(4.11) and (4.12), Lorentz force equations of 1GDF on lightlike surface in the $\widetilde{\mu}-lines$ direction are derived :

$$\begin{aligned} {}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}(\widetilde{\mathbf{T}}) &= \varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\widetilde{\mu} - \gamma^{(\widetilde{\mu})}\widetilde{\mathbf{U}} && (4.13) \\ {}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}(\widetilde{\mu}) &= \varepsilon_1\gamma^{(\widetilde{\mu})}\widetilde{\mathbf{T}} + \varepsilon_1\mathfrak{z}_2^1\widetilde{\mu} && (4.14) \\ {}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}(\widetilde{\mathbf{U}}) &= -\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\widetilde{\mathbf{T}} - \varepsilon_1\mathfrak{z}_2^1\widetilde{\mathbf{U}} && (4.15) \end{aligned}$$

Here

$$\widetilde{\mathbf{Y}}_1 = \mathfrak{z}_2^1\widetilde{\mathbf{T}} - \gamma^{(\widetilde{\mathcal{T}}\widetilde{\mu})})\widetilde{\mu} - \varepsilon_{\mathbf{1}}\varphi^{(\widetilde{\mu}}\widetilde{\mathbf{U}}$$

the magnetic vector field satisfies Eqs.(4.13), (4.14) and (4.15).

**$\widetilde{\mu}-$magnetic curves in the $\widetilde{\mu}-lines$ direction for 1GDF on lightlike surface**

The curve is called $\widetilde{\mu}$-magnetic curve 1GDF on lightlike surface in the $\widetilde{\mu}-lines$ direction if it satisfied Lorentz force equation

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mu}) = \varepsilon_1\gamma^{(\widetilde{\mu})}\widetilde{\mu})\widetilde{\mu} = \widetilde{\mathbf{Y}}_2\times\widetilde{\mu}$$

Consider

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = l_1\widetilde{\mathbf{T}}+l_2\widetilde{\mu}+l_3\widetilde{\mathbf{U}} \tag{4.16}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}),\widetilde{\mathbf{T}}\right\rangle = l_1 = 0 \tag{4.17}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}),\widetilde{\mathbf{U}}\right\rangle = \mathfrak{z}_2^2 \Rightarrow l_2 = \varepsilon_1\mathfrak{z}_2^2 \tag{4.18}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}),\widetilde{\mu}\right\rangle = \varepsilon_1 l_3 \Rightarrow l_3 = -\gamma^{(\widetilde{\mu})} \tag{4.19}$$

$\mathfrak{z}_2^2$ is parameter. If Eqs.(4.17), (4.18) and (4.19) are substituted in Eq.(4.16)

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = \varepsilon_1\mathfrak{z}_2^2\widetilde{\mu} - \gamma^{(\widetilde{\mu})}\widetilde{\mathbf{U}}$$

As similarly,

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{U}}) = -(\widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu})\widetilde{\mathbf{U}} - \mathfrak{z}_2^2\widetilde{\mathbf{T}}$$

Thus,

$$\widetilde{\mathbf{Y}}_2 = -\varepsilon_1(\widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu})\widetilde{\mathbf{T}} - \mathfrak{z}_2^2\widetilde{\mu} + \varepsilon_1\gamma^{(\widetilde{\mu})}\widetilde{\mathbf{U}}$$

### $\widetilde{\mathbf{U}}-$magnetic curves 1GDF on lightlike surface in the $\widetilde{\mu}-lines$ direction

The curve is called $\widetilde{\mathbf{U}}-$magnetic curve if it satisfied the Lorentz force equation 1GDF on lightlike surface in the $\widetilde{\mu}-lines$ direction

$$\widetilde{\mathbf{U}}_{\widetilde{\mu}} = {}^{1\mathcal{GD}}\ \Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{U}}) = \widetilde{\mathbf{Y}}_3\times\widetilde{\mathbf{U}} \tag{4.20}$$

It can be written by

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}) = h_1\widetilde{\mathbf{T}}+h_2\widetilde{\mu}+h_3\widetilde{\mathbf{U}} \tag{4.21}$$

Using Eqs.(4.20) and (4.21), it can be obtained

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}),\widetilde{\mathbf{T}}\right\rangle = h_1 = 0 \tag{4.22}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}),\widetilde{\mathbf{U}}\right\rangle = \varepsilon_1 h_2 \Rightarrow h_2 = \varepsilon_1\varphi^{(\widetilde{T}\widetilde{U})} \tag{4.23}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}),\widetilde{\mu}\right\rangle = \varepsilon_1 h_3 \Rightarrow h_3 = \varepsilon_1\mathfrak{z}_2^3 \tag{4.24}$$

$\mathfrak{z}_2^3$ is a parameter. Via Eqs.(4.22), (4.23) and (4.24), Eq.(4.21) is rewritten by

$$\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}) = \varepsilon_1\mathfrak{z}_2^3\widetilde{\mathbf{U}} + \varepsilon_1\varphi^{(\widetilde{T}\widetilde{U})}\widetilde{\mu}$$

As similar,

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mu}) = -\mathfrak{z}_2^3\widetilde{\mathbf{T}}+(\widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu})\widetilde{\mu}.$$

### $\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}}$ in $\widetilde{\mathbf{U}}-lines$ direction for 1GDF on lightlike surface

$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}}$ for 1GDF in $\widetilde{\mathbf{U}}-lines$ direction in the general form on lightlike surface is given by

$$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}} = {}^{1\mathcal{GD}}\,\mathbf{E}_{\widetilde{U}} = f_1\widetilde{\mathbf{T}} + f_2\widetilde{\mu} + f_3\widetilde{\mathbf{U}}. \tag{4.25}$$

Assume that

$$\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{T}} \right\rangle = 0 \tag{4.26}$$

$$\left\langle {}^{1\mathcal{GD}}\mathbf{E}, {}^{1\mathcal{GD}}\,\mathbf{E} \right\rangle = const. \tag{4.27}$$

Via Eqs.(4.26) and (4.27), it can be obtained:

$$f_1 = -(\varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mathbf{U}} \right\rangle + \gamma^{(\widetilde{U})}\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mu} \right\rangle) \tag{4.28}$$

$$f_2 = \mathfrak{z}_3^1\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mathbf{U}} \right\rangle, \quad f_3 = -\mathfrak{z}_3^1\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mu} \right\rangle \tag{4.29}$$

where $\mathfrak{z}_3^1$ is a parameter. If Eqs.(4.26), (4.27) are written in Eq.(4.25), then

$$\begin{aligned} \frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}} &= {}^{1\mathcal{GD}}\mathbf{E}_{\widetilde{U}} = -(\varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mathbf{U}} \right\rangle + \gamma^{(\widetilde{U})}\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mu} \right\rangle)\widetilde{\mathbf{T}} \\ &\quad + \mathfrak{z}_3^1\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mathbf{U}} \right\rangle\widetilde{\mu} - \mathfrak{z}_3^1\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mu} \right\rangle\widetilde{\mathbf{U}} \end{aligned} \tag{4.30}$$

$\mathfrak{z}_3^1\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mu} \right\rangle - \mathfrak{z}_3^1\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mu} \right\rangle\widetilde{\mathbf{U}}$ gives the rotation around $\widetilde{\mathbf{T}}$ for 1GDF on lightlike surface in the in $\widetilde{U}-lines$ direction. When $\mathfrak{z}_3^1 = 0$, Eq.(4.30) is

$$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}} = {}^{1\mathcal{GD}}\,\mathbf{E}_{\widetilde{U}} = -(\varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mathbf{U}} \right\rangle - \gamma^{(\widetilde{U})}\left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mu} \right\rangle)\widetilde{\mathbf{T}} \tag{4.31}$$

Lorentz force equation ${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}$ of ${}^{1\mathcal{GD}}\mathbf{E}$ in $\widetilde{U}-lines$ direction is given by

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}({}^{1\mathcal{GD}}\mathbf{E}) = {}^{1\mathcal{GD}}\,\mathbf{E}_{\widetilde{U}} = \widetilde{\mathbf{Y}}_3 \times {}^{1\mathcal{GD}}\mathbf{E}_1 \tag{4.32}$$

With Eqs.(4.31) and (4.32), Lorentz force equations of 1GDF on lightlike surface are given by:

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}) = \gamma^{(\widetilde{U})}\widetilde{\mu} + \varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\widetilde{\mathbf{U}} \tag{4.33}$$

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mu}) = -\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\widetilde{\mathbf{T}} + \varepsilon_1\mathfrak{z}_3^2\widetilde{\mu} \tag{4.34}$$

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{U}}) = -\varepsilon_1\gamma^{(\widetilde{U})}\widetilde{\mathbf{T}} - \varepsilon_1\mathfrak{z}_3^2\widetilde{\mathbf{U}} \tag{4.35}$$

Here

$$\widetilde{\mathbf{Y}}_3 = \mathfrak{z}_3^1\widetilde{\mathbf{T}} - \varepsilon_1\gamma^{(\widetilde{U})}\widetilde{\mu} - \varphi^{(\mathcal{T}\widetilde{\mu})}\widetilde{\mathbf{U}}$$

magnetic vector field satisfies Eqs.(4.33), (4.34) and (4.35).

**$\widetilde{\mu}-$magnetic curves for 1GDF on lightlike surface in the $\widetilde{\mathbf{U}}-$ *lines* direction**

If the curve is called $\widetilde{\mu}$-magnetic curve in the $\widetilde{\mathbf{U}}-lines$ direction if it satisfied the Lorentz force equation

$$\begin{aligned} \frac{\delta\widetilde{\mu}}{\delta\widetilde{U}} &= {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mu}) = -\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\widetilde{\mathbf{T}} - (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mu} \\ &= \widetilde{\mathbf{Z}}_1 \times \widetilde{\mu} \end{aligned} \tag{4.36}$$

Consider

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = m_1\widetilde{\mathbf{T}} + m_2\widetilde{\mu} + m_3\widetilde{\mathbf{U}}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{T}} \right\rangle = m_1 = 0 \tag{4.37}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mu} \right\rangle = \mathfrak{z}_3^2 \Rightarrow m_3 = \varphi^{(\mathcal{T}\widetilde{\mu})} \tag{4.38}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{U}} \right\rangle = \varepsilon_1 m_2 \Rightarrow m_2 = \varepsilon_1\mathfrak{z}_3^2 \tag{4.39}$$

Via Eqs.(4.37), (4.38) and (4.39), it is obtained

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = \varepsilon_1\mathfrak{z}_3^2\widetilde{\mu} + \varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\widetilde{\mathbf{U}} \tag{4.40}$$

As similarly,

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{U}}) = -\mathfrak{z}_3^2\widetilde{\mathbf{T}} + (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mathbf{U}} \tag{4.41}$$

Here, the magnetic vector field

$$\widetilde{\mathbf{Z}}_1 = -\varepsilon_1(\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mathbf{T}} - \mathfrak{z}_3^2\widetilde{\mu} + \varphi^{(\mathcal{T}\widetilde{\mu})}\widetilde{\mathbf{U}}$$

satisfies Eqs.(4.36), (4.40) and (4.41).

**$\widetilde{\mathbf{U}}-$magnetic curves for 1GDF on lightlike surface in the $\widetilde{\mathbf{U}}-lines$ direction**

If the curve is called $\widetilde{\mathbf{U}}-$magnetic curve for 1GDF on lightlike surface, if it satisfied the Lorentz force equation in the $\widetilde{U}-lines$ lines

$$\widetilde{\mathbf{U}}_{\widetilde{U}} = {}^{1\mathcal{GD}}\ \Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{U}}) = \widetilde{\mathbf{Z}}_2 \times \widetilde{\mathbf{U}} \tag{4.42}$$

It can be written by

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}) = n_1\widetilde{\mathbf{T}} + n_2\widetilde{\mu} + n_3\widetilde{\mathbf{U}} \tag{4.43}$$

From Eqs.(4.42) and (4.43), it can be obtained

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{T}} \right\rangle = n_1 = 0 \tag{4.44}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}), \widetilde{\mu} \right\rangle = \varepsilon_1 n_3 \Rightarrow n_3 = \varepsilon_1\mathfrak{z}_3^3 \tag{4.45}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{U}} \right\rangle = \varepsilon_1 n_2 \Rightarrow n_2 = \gamma^{(\widetilde{U})} \tag{4.46}$$

Via Eqs.(4.44), (4.45) and (4.46), it is obtained

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}) = \gamma^{(\widetilde{U})}\widetilde{\mu} + \varepsilon_1\mathfrak{z}_3^3\widetilde{\mathbf{U}} \tag{4.47}$$

As similar,

$$ {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mu}) = -\mathfrak{z}_3^3\widetilde{\mathbf{T}} - (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mu} \tag{4.48} $$

Here,

$$ \widetilde{\mathbf{Z}}_2 = (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mathbf{T}} - \varepsilon_1\gamma^{(\widetilde{U})}\widetilde{\mu} + \mathfrak{z}_3^3\widetilde{\mathbf{U}} $$

satisfies Eqs.(4.42), (4.47) and (4.48).

## 5. Conclusion

In this manuscript, we studied variations of electric fields, geometric phases, Lorentz force equations and magnetic curves for Darboux frame of a spacelike curve on null surface, null Darboux frame on timelike surface, first and second kinds generalized Darboux frames on lightlike surface in the tangential direction. Finally, we presented geometric phases, Lorentz force equations and magnetic curves via anholonomic coordinates for the first generalized Darboux frame on lightlike surface.

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The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the authors declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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# SOME CONSTRUCTION METHODS FOR IMPLICATION OPERATORS ON BOUNDED LATTICES

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Abstract. In this paper, we focus the construction methods of implications on bounded lattices. We introduce several methods to obtain implication on bounded lattices. We basically base on implication defined on subinterval such as $[a, 1]$ or $[0, b]$ of the bounded lattice $L$ which has $a, b \in L$ with $a \leq b$ for these construction methods. We also use fuzzy logic operators such as t-norms, t-conorms, negations and implications on $L$ in some construction methods. In addition, we give some remarks and examples to make the new construction methods.

## 1. Introduction

Fuzzy implications generalize the classical implications taking values from $\{0, 1\}$ to the fuzzy logic, where the truth values belong to the unit interval $[0, 1]$. Since fuzzy implications have been used in many areas such as fuzzy control, approximate reasoning, and decision support systems, fuzzy control and etc. [8, 9, 6, 11, 12], the construction methods of these operators are especially important for aplications of them and thus, fuzzy implications construction methods have attracted the attention of researchers. In [8], Baczyński and Jayaram introduced construction methods of implication which are obtained from fuzzy logic operators on unit real interval $[0, 1]$.

In [10], Neres et al. proposed fuzzy implications construction methods, which is called as a new construction technique, from a pair of bivariate aggregation functions and a fuzzy negation on unit interval real $[0, 1]$. In [7] Karaçal et al. introduced two construction methods to built implication operators on bounded lattices by means of t-norms, t-conorms and implications. In [3], Kesicioğlu et al. offered implication construction methods which is called the linear and $g-$convex combination for implications on bounded lattices, where they benefited from fuzzy logic operators. In [4], Karaçal et al. gived many construction methods for

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implications by means of an arbitrary element and basic logic connectives such as t-norms, t-conorms and negations on bounded lattices.

In this paper, our main aim is to obtain an implication from an implication on subinterval $[a,1]$ ($[0,b]$) of the bounded lattice $L$ where $a,b \in L$ with $a \le b$. In addition, we give some other construction methods for implications on bounded lattices via some logic operators besides implications. Firstly, in the section 2 we remind some main definitions and results, which are useful for our paper. In the next section, we give construction methods to built implications on bounded lattices and we add various examples and results from these construction methods. Finally, we finish with concluding remarks .

## 2. Preliminaries

In this section, we list some basic notions and results which will be use in the paper.

**Definition 2.1.** [5] Let $(L, \le, 0, 1)$ be a bounded lattice and $a, b \in L$ with $a \le b$. The subinterval $[a,b]$ is defined as

$$[a,b] = \{x \in L \mid a \le x \le b\}.$$

Similarly, $(a,b] = \{x \in L \mid a < x \le b\}$, $[a,b) = \{x \in L \mid a \le x < b\}$ and $(a,b) = \{x \in L \mid a < x < b\}$ can be defined.

**Definition 2.2.** [1, 2] Let $(L, \le, 0, 1)$ be a bounded lattice. A function $T : L^2 \longrightarrow L$ is a t-norm if it satisfies the following conditions for any $x, y \in L$.

(T1) $T(x,y) = T(y,x)$ (commutavity).
(T2) $T(x,1) = x$ (neutral element).
(T3) If $y \le z$, then $T(x,y) \le T(x,z)$ (monotonicity).
(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity).

**Definition 2.3.** [1, 2] Let $(L, \le, 0, 1)$ be a bounded lattice. A function $S : L^2 \longrightarrow L$ is a t-conorm if it satisfies the following conditions for any $x, y \in L$.

(S1) $S(x,y) = S(y,x)$ (commutavity).
(S2) $S(x,0) = x$ (neutral element).
(S3) If $y \le z$, then $S(x,y) \le S(x,z)$ (monotonicity).
(S4) $S(x,S(y,z)) = S(S(x,y),z)$ (associativity).

**Example 2.4.** Let $(L, \le, 0, 1)$ be a bounded lattice. Two basic t-norms $T_D$ and $T_\wedge$ on a bounded lattice $L$ are respectively given by

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1, \\ x & \text{if } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$T_\wedge(x,y) = x \wedge y.$$

Two basic t-conorms $S_D$ and $S_\vee$ on a bounded lattice $L$ are respectively given as follows:

$$S_D(x,y) = \begin{cases} y & \text{if } x = 0, \\ x & \text{if } y = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$S_\vee(x,y) = x \vee y.$$

**Definition 2.5.** [8, 3, 7] Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$.

**Definition 2.6.** [8, 3, 7] A function $I : L^2 \to L$ on a bounded lattice $(L, \leq, 0, 1)$ is called an implication if it satisfies the following conditions:

(I1) $I$ is a decreasing operation on the first variable, that is, for every $x, z \in L$ with $x \leq z$, $I(z,y) \leq I(x,y)$ for all $y \in L$.

(I2) $I$ is an increasing operation on the second variable, that is, for every $y, z \in L$ with $y \leq z$, $I(x,y) \leq I(x,z)$ for all $x \in L$.

(I3) $I(0,0) = 1$.

(I4) $I(1,1) = 1$.

(I5) $I(1,0) = 0$.

**Theorem 2.7.** [8] *Let $S : [0,1]^2 \to L$ be a t-conorm and $N : [0,1] \to [0,1]$ be a negation. Then the function $I : [0,1]^2 \to [0,1]$ defined by, for all $x, y \in L$,*

$$I(x,y) = S(N(x), y) \tag{2.1}$$

*is an implication.*

**Theorem 2.8.** [4] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L$. Then the function $I_a : L^2 \to L$ defined by, for all $x, y \in L$,*

$$I_a(x,y) = \begin{cases} 1 & x \leq y, \\ 0 & x > y, \\ a & \textit{otherwise,} \end{cases} \tag{2.2}$$

*is an implication.*

**Theorem 2.9.** [4] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm and $N : L \to L$ be a negation. Then the function $I : L^2 \to L$ defined by, for all $x, y \in L$,*

$$I(x,y) = \begin{cases} 1 & x \leq y, \\ y & x > y, \\ S(N(x), y) & \textit{otherwise,} \end{cases} \tag{2.3}$$

*is an implication.*

**Theorem 2.10.** [4] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $N : L \to L$ be a negation and $J_1, J_2, J_3 : L^2 \to L$ be implications. Then the function $I : L^2 \to L$ defined by*

$$I(x,y) = J_3(N(J_1(x,y)), J_2(x,y)) \tag{2.4}$$

*is an implication.*

**Theorem 2.11.** [3] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm, $T : L^2 \to L$ be a t-norm, $I, J : L^2 \to L$ be implications and $a \in L$. The function $TS_a : L^2 \to L$ defined by, for all $x, y \in L$,*

$$TS_a(x, y) = T(S(a, I(x, y)), J(x, y)) \tag{2.5}$$

*is an implication.*

**Theorem 2.12.** [3] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm, $T : L^2 \to L$ be a t-norm, $I, J : L^2 \to L$ be implications and $a \in L$. The function $ST_a : L^2 \to L$ defined by, for all $x, y \in L$,*

$$ST_a(x, y) = S(T(a, I(x, y)), J(x, y)) \tag{2.6}$$

*is an implication.*

**Theorem 2.13.** [7] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm, $T : L^2 \to L$ be a t-norm, $I, J : L^2 \to L$ be implications, $N : L \to L$ be a negation and $a \in L$. The function $K^{I,J}_{a,T,S,N} : L^2 \to L$ defined by, for all $x, y \in L$,*

$$K^{I,J}_{a,T,S,N} = S(T(a, I(x, y)), T(N(a), J(x, y)))) \tag{2.7}$$

*is an implication if and only if $S(a, N(a)) = 1$.*

## 3. Some construction methods of implication on $L$

In this section, we offer many construction methods of implication operators. In Theorem 3.1 (3.4) focus on extension of an implication on the subinterval $[a, 1]$ ($[0, b]$) to bounded lattice $L$, where $a, b \in L$ such as $a \leq b$. In the following construction methods, we give some different construction methods for implications on bounded lattices considering some logic operators such as t-norms, t-conorms, negations as well as implications. Also we illustrate the new construction methods with the several examples.

**Theorem 3.1.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$ and $J : [a, 1]^2 \longrightarrow [a, 1]$ be an implication. Then, the function $I_1 : L^2 \longrightarrow L$ defined by,*

$$I_1(x, y) = \begin{cases} 1 & \text{if } (x = 0 \text{ or } y = 1) \text{or } (x \notin [a, 1] \text{ and } y \in [a, 1]), \\ 0 & \text{if } (x, y) = (1, 0), \\ J(x, y) & \text{if } (x, y) \in [a, 1]^2, \\ a & \text{if } x \in [a, 1] \text{ and } y \notin [a, 1], \\ b & \text{if } x \notin [a, 1] \text{ and } y \notin [a, 1], \end{cases} \tag{3.1}$$

*is an implication on $L$.*

*Proof.* I3, I4 and I5 are obtained directly from the definition of $I_1$.

(I1) Let us show that $I_1$ is a decreasing function on the first variable. Then it should be $I_1(x_2, y) \leq I_1(x_1, y)$ for every elements $x_1, x_2, y \in L$ with $x_1 \leq x_2$. If $x_1 = 0$ or $(x_2, y) = (1, 0)$ or $y = 1$, the proof is trivial. The proof can be split into all possible cases.

1. Let $(x_1, y) \in [a, 1]^2$.

$$I_1(x_2, y) = J(x_2, y) \leq J(x_1, y) = I_1(x_1, y).$$

2. Let $x_1 \in [a,1]$ and $y \notin [a,1]$.

$$I_1(x_2,y) = a \le a = I_1(x_1,y).$$

3. Let $x_1 \notin [a,1]$ and $y \in [a,1]$.
3.1. If $x_2 \in [a,1]$,

$$I_1(x_2,y) = J(x_2,y) \le 1 = I_1(x_1,y).$$

3.2. If $x_2 \notin [a,1]$,

$$I_1(x_2,y) = 1 \le 1 = I_1(x_1,y).$$

4. Let $x_1 \notin [a,1]$ and $y \notin [a,1]$.
4.1. If $x_2 \in [a,1]$,

$$I_1(x_2,y) = a \le b = I_1(x_1,y).$$

4.2. If $x_2 \notin [a,1]$,

$$I_1(x_2,y) = b \le b = I_1(x_1,y).$$

(I2) Let us show that $I_1$ is an increasing function on the second variable. Then it should be $I_1(x,y_1) \le I_1(x,y_2)$ for every elements $x,y_1,y_2 \in L$ with $y_1 \le y_2$. If $x = 0$ or $y_2 = 1$ or $(x,y_1) = (1,0)$, the proof is immediate. The proof can be split into all possible cases.
1. Let $(x,y_1) \in [a,1]^2$.

$$I_1(x,y_1) = J(x,y_1) \le J(x,y_2) = I_1(x,y_2).$$

2. Let $x \in [a,1]$ and $y_1 \notin [a,1]$.
2.1. If $y_2 \in [a,1]$,

$$I_1(x,y_1) = a \le J(x,y_2) = I_1(x,y_2).$$

2.2. If $y_2 \notin [a,1]$,

$$I_1(x,y_1) = a \le a = I_1(x,y_2).$$

3. Let $x \notin [a,1]$ and $y_1 \in [a,1]$.

$$I_1(x,y_1) = 1 \le 1 = I_1(x,y_2).$$

4. Let $x \notin [a,1]$ and $y_1 \notin [a,1]$.
4.1. If $y_2 \in [a,1]$,

$$I_1(x,y_1) = b \le 1 = I_1(x,y_2).$$

4.2. If $y_2 \notin [a,1]$,

$$I_1(x,y_1) = b \le b = I_1(x,y_2).$$

□

*Remark* 3.2. (i) If $b = 1$, then the implication $I_1$ given by the formula (3.1) can be rewritten as follows:

$$I_1(x,y) = \begin{cases} 1 & \text{if } (x = 0 \text{ or } y = 1) \text{or } (x \notin [a,1] \text{ and } y \in [a,1]), \\ 0 & \text{if } (x,y) = (1,0), \\ J(x,y) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } x \in [a,1] \text{ and } y \notin [a,1], \\ 1 & \text{if } x \notin [a,1] \text{ and } y \notin [a,1]. \end{cases} \tag{3.2}$$

(ii) If $a = b$, then the implication $I_1$ given by the formula (3.5) can be rewritten as follows:

$$I_1(x,y) = \begin{cases} 1 & \text{if } (x = 0 \text{ or } y = 1) \text{or } (x \notin [a,1] \text{ and } y \in [a,1]), \\ 0 & \text{if } (x,y) = (1,0), \\ J(x,y) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{otherwise.} \end{cases} \tag{3.3}$$

In order to apply the formula (3.1), we include the following example.

**Example 3.3.** Consider the bounded lattice $(L = \{0, t_1, t_2, t_3, t_4, t_5, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Fig. 1.

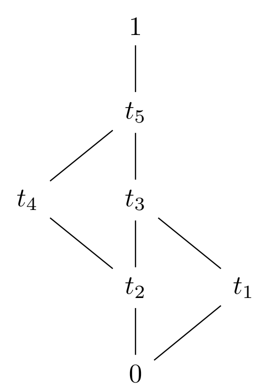


**Figure 1.** The lattice $L$.

Let use take the implication $J : [t_2, 1]^2 \to [t_2, 1]$ as in Table 1:

| $J$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | 1 |
|---|---|---|---|---|---|
| $t_2$ | 1 | 1 | 1 | 1 | 1 |
| $t_3$ | $t_4$ | $t_5$ | $t_4$ | $t_5$ | 1 |
| $t_4$ | $t_3$ | $t_3$ | $t_5$ | $t_5$ | 1 |
| $t_5$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | 1 |
| 1 | $t_2$ | $t_3$ | $t_4$ | $t_5$ | 1 |

**Table 1.** The implication $J$ on $[t_2, 1]$.

By applying the formula (3.1) in Theorem 3.1 with $a = t_2$ and $b = t_3$, the implication $I_1$ can be obtained as in Table 2.

| $I_1$ | $0$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $1$ |
|---|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t_1$ | $t_3$ | $t_3$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t_2$ | $t_2$ | $t_2$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t_3$ | $t_2$ | $t_2$ | $t_4$ | $t_5$ | $t_4$ | $t_5$ | $1$ |
| $t_4$ | $t_2$ | $t_2$ | $t_3$ | $t_3$ | $t_5$ | $t_5$ | $1$ |
| $t_5$ | $t_2$ | $t_2$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $1$ |
| $1$ | $0$ | $t_2$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $1$ |

**Table 2.** The implication $I_1$ on $L$.

**Theorem 3.4.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$ and $J : [0, b]^2 \longrightarrow [0, b]$ be an implication. Then, the function $I_1^* : L^2 \longrightarrow L$ defined by,*

$$I_1^*(x,y) = \begin{cases} 1 & \textit{if } x = 0 \textit{ or } y = 1, \\ 0 & \textit{if } (x,y) = (1,0) \textit{ or } (x \notin [0,b] \textit{ and } y \in [0,b]), \\ J(x,y) & \textit{if } (x,y) \in [0,b]^2, \\ b & \textit{if } x \in [0,b] \textit{ and } y \notin [0,b], \\ a & \textit{if } x \notin [0,b] \textit{ and } y \notin [0,b], \end{cases} \tag{3.4}$$

*is an implication on $L$.*

*Proof.* The proof can be done in a similar fashion as the proof of Theorem 3.1. Therefore, we omit it. □

We present another construction method for implication operators. For this construction method, we use some logic operators on a bounded lattice $L$, an implication on the subinterval $[a, 1]$ of the bounded lattice $L$ and $a, b \in L$.

**Theorem 3.5.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $T : L^2 \to L$ be a t-norm, $N : L \to L$ be a negation and $J : [a, 1]^2 \longrightarrow [a, 1]$ be an implication. Then, the function $I_2 : L^2 \longrightarrow L$ defined by,*

$$I_2(x,y) = \begin{cases} 1 & \textit{if } (x = 0 \textit{ or } y = 1) \textit{or } (x \notin [a,1] \textit{ and } y \in [a,1]), \\ 0 & \textit{if } (x,y) = (1,0), \\ J(x,y) & \textit{if } (x,y) \in [a,1]^2, \\ T(N(x),a) & \textit{if } x \in [a,1] \textit{ and } y \notin [a,1], \\ T(N(x),b) & \textit{if } x \notin [a,1] \textit{ and } y \notin [a,1], \end{cases} \tag{3.5}$$

*is an implication on $L$.*

*Proof.* I3, I4 and I5 are obtained directly from the definition of $I_2$.

(I1) Let us show that $I_2$ is a decreasing function on the first variable. Then it should be $I_2(x_2, y) \leq I_2(x_1, y)$ for every elements $x_1, x_2, y \in L$ with $x_1 \leq x_2$. If $x_1 = 0$ or $(x_2, y) = (1, 0)$ or $y = 1$, the proof is trivial. The proof can be split into all possible cases.

1. Let $(x_1, y) \in [a, 1]^2$.

$$I_2(x_2, y) = J(x_2, y) \le J(x_1, y) = I_2(x_1, y).$$

2. Let $x_1 \in [a, 1]$ and $y \notin [a, 1]$.

$$I_2(x_2, y) = T(N(x_2), a) \le T(N(x_1), a) = I_2(x_1, y).$$

3. Let $x_1 \notin [a, 1]$ and $y \in [a, 1]$.
3.1. If $x_2 \in [a, 1]$,

$$I_2(x_2, y) = J(x_2, y) \le 1 = I_2(x_1, y).$$

3.2. If $x_2 \notin [a, 1]$,

$$I_2(x_2, y) = 1 \le 1 = I_2(x_1, y).$$

4. Let $x_1 \notin [a, 1]$ and $y \notin [a, 1]$
4.1. If $x_2 \in [a, 1]$,

$$I_2(x_2, y) = T(N(x_2), a) \le T(N(x_1), a) \le T(N(x_1), b) = I_2(x_1, y).$$

4.2. If $x_2 \notin [a, 1]$,

$$I_2(x_2, y) = T(N(x_2), b) \le T(N(x_1), b) = I_2(x_1, y).$$

(I2) Let us show that $I_2$ is an increasing function on the second variable. Then it should be $I_2(x, y_1) \le I_2(x, y_2)$ for every elements $x, y_1, y_2 \in L$ with $y_1 \le y_2$. If $x = 0$ or $y_2 = 1$ or $(x, y_1) = (1, 0)$, the proof is immediate. The proof can be split into all possible cases.

1. Let $(x, y_1) \in [a, 1]^2$.

$$I_2(x, y_1) = J(x, y_1) \le J(x, y_2) = I_2(x, y_2).$$

2. Let $x \in [a, 1]$ and $y_1 \notin [a, 1]$.
2.1. If $y_2 \in [a, 1]$,

$$I_2(x, y_1) = T(N(x), a) \le a \le J(x, y_2) = I_2(x, y_2).$$

2.2. If $y_2 \notin [a, 1]$,

$$I_2(x, y_1) = T(N(x), a) \le T(N(x), a) = I_2(x, y_2).$$

3. Let $x \notin [a, 1]$ and $y_1 \in [a, 1]$.

$$I_2(x, y_1) = 1 \le 1 = I_2(x, y_2).$$

4. Let $x \notin [a, 1]$ and $y_1 \notin [a, 1]$.
4.1. If $y_2 \in [a, 1]$,

$$I_2(x, y_1) = T(N(x), b) \le 1 = I_2(x, y_2).$$

4.2. If $y_2 \notin [a, 1]$,

$$I_2(x, y_1) = T(N(x), b) \le T(N(x), b) = I_2(x, y_2).$$

□

*Remark* 3.6. Let $T$ be the t-norm $T_\wedge$ in Theorem 3.5.
(i) The implication $I_2$ given by the formula (3.5) can be rewritten

$$I_2(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ N(x)\wedge a & \text{if } x\in[a,1] \text{ and } y\notin[a,1],\\ N(x)\wedge b & \text{if } x\notin[a,1] \text{ and } y\notin[a,1],\end{cases}\tag{3.6}$$

(ii) If $b=1$, then the implication $I_2$ given by the formula (3.5) can be rewritten as follows:

$$I_2(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ N(x)\wedge a & \text{if } x\in[a,1] \text{ and } y\notin[a,1],\\ N(x) & \text{if } x\notin[a,1] \text{ and } y\notin[a,1],\end{cases}\tag{3.7}$$

(iii) If $a=b$, then the implication $I_2$ given by the formula (3.5) can be rewritten as follows:

$$I_2(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ N(x)\wedge a & \text{otherwise.}\end{cases}\tag{3.8}$$

Now, let us illustrate the application of Theorem 3.5 with the following example.

**Example 3.7.** Consider the lattice $(L=\{0,t_1,t_2,t_3,t_4,t_5,1\},\leq,0,1)$ as given in Fig. 1, the t-norm $T:L^2\to L$ as in $T_\wedge$ and the implication $J:[t_2,1]^2\to[t_2,1]$ as given in Table 1. Let the negation $N:L\to L$ be as in formula 3.9.

$$N(x)=\begin{cases}1 & \text{if } x=0,\\ t_3 & \text{if } x\in\{t_1,t_3\},\\ t_5 & \text{if } x=t_2,\\ t_4 & \text{if } x=t_4,\\ t_2 & \text{if } x=t_5,\\ 0 & \text{if } x=1.\end{cases}\tag{3.9}$$

By applying the formula (3.5) in Theorem 3.5 with $a=t_2$ and $b=t_3$, the implication $I_2$ can be obtained as in Table 3.

| $I_2$ | $0$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $1$ |
|---|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t_1$ | $t_3$ | $t_3$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t_2$ | $t_2$ | $t_2$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t_3$ | $t_2$ | $t_2$ | $t_4$ | $t_5$ | $t_4$ | $t_5$ | $1$ |
| $t_4$ | $t_2$ | $t_2$ | $t_3$ | $t_3$ | $t_5$ | $t_5$ | $1$ |
| $t_5$ | $t_2$ | $t_2$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $1$ |
| $1$ | $0$ | $0$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $1$ |

**Table 3.** The implication $I_2$ on $L$.

**Theorem 3.8.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $S : L^2 \to L$ be a t-conorm, $N : L \to L$ be a negation and $J : [0, b]^2 \longrightarrow [0, b]$ be an implication. Then, the function $I_2^* : L^2 \longrightarrow L$ defined by,*

$$I_2^*(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ 0 & \text{if } (x,y) = (1,0) \text{ or } (x \notin [0,b] \text{ and } y \in [0,b]), \\ J(x,y) & \text{if } (x,y) \in [0,b]^2, \\ S(N(x),b) & \text{if } x \in [0,b] \text{ and } y \notin [0,b], \\ S(N(x),a) & \text{if } x \notin [0,b] \text{ and } y \notin [0,b], \end{cases} \tag{3.10}$$

*is an implication on $L$.*

*Proof.* The proof can be done in a similar fashion as the proof of Theorem 3.1. Therefore, we omit it. □

In the following theorem, we present a method to construct implication operators. To do that, we use a t-norm $T$ on $L$, an implication on a subinterval of $L$ and arbitrary fix elements of $L$.

**Theorem 3.9.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $T : L^2 \to L$ be a t-norm and $J : [a, 1]^2 \longrightarrow [a, 1]$. Then, the function $I_3 : L^2 \longrightarrow L$ defined by,*

$$I_3(x,y) = \begin{cases} 1 & \text{if } (x = 0 \text{ or } y = 1) \text{or } (x \notin [a,1] \text{ and } y \in [a,1]), \\ 0 & \text{if } (x,y) = (1,0), \\ J(x,y) & \text{if } (x,y) \in [a,1]^2, \\ T(a,y) & \text{if } x \in [a,1] \text{ and } y \notin [a,1], \\ T(b,y) & \text{if } x \notin [a,1] \text{ and } y \notin [a,1], \end{cases} \tag{3.11}$$

*is an implication on $L$.*

*Proof.* The proof can be done in a similar fashion as the proof of Theorem 3.1. Therefore, we omit it. □

**Theorem 3.10.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $S : L^2 \to L$ be a t-conorm and $J : [0, b]^2 \longrightarrow [0, b]$. Then, the function $I_3^* : L^2 \longrightarrow L$ defined by,*

$$I_3^*(x,y)=\begin{cases}1 & \text{if } x=0 \text{ or } y=1,\\ 0 & \text{if } (x,y)=(1,0) \text{ or } (x\notin[0,b] \text{ and } y\in[0,b]),\\ J(x,y) & \text{if } (x,y)\in[0,b]^2,\\ S(b,y) & \text{if } x\in[0,b] \text{ and } y\notin[0,b],\\ S(a,y) & \text{if } x\notin[0,b] \text{ and } y\notin[0,b],\end{cases} \tag{3.12}$$

*is an implication on $L$.*

*Proof.* The proof can be done in a similar fashion as the proof of Theorem 3.1. Therefore, we omit it. □

In the following Theorem 3.11, a construction method for implication operators is presented considering some logic operators on a bounded lattice $L$ or on a subinterval of the bounded lattice $L$ and $a,b\in L$.

**Theorem 3.11.** *Let $(L,\leq,0,1)$ be a bounded lattice, $a,b\in L$ with $a\leq b$, $T:L^2\to L$ be a t-norm, $K:L^2\to L$ be an implication and $J:[a,1]^2\longrightarrow[a,1]$ be an implication. Then, the function $I_4:L^2\longrightarrow L$ defined by,*

$$I_4(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1) \text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ T(K(x,y),a) & \text{if } x\in[a,1] \text{ and } y\notin[a,1],\\ T(K(x,y),b) & \text{if } x\notin[a,1] \text{ and } y\notin[a,1],\end{cases} \tag{3.13}$$

*is an implication on $L$.*

*Proof.* I3, I4 and I5 are obtained directly from the definition of $I_4$.

(I1) We need to show that $I_4$ is a decreasing function on the first variable. Then it should be $I_4(x_2,y)\leq I_4(x_1,y)$ for $x_1,x_2,y\in L$ and $x_1\leq x_2$. If $x_1=0$ or $(x_2,y)=(1,0)$ or $y=1$, the proof is trivial. The proof can be split into all possible cases.

1. Let $(x_1,y)\in[a,1]^2$.

$$I_4(x_2,y)=J(x_2,y)\leq J(x_1,y)=I_4(x_1,y).$$

2. Let $x_1\in[a,1]$ and $y\notin[a,1]$.

$$I_4(x_2,y)=T(K(x_2,y),a)\leq T(K(x_1,y),a)=I_4(x_1,y).$$

3. Let $x_1\notin[a,1]$ and $y\in[a,1]$.

3.1. If $x_2\in[a,1]$,

$$I_4(x_2,y)=J(x_2,y)\leq 1=I_4(x_1,y).$$

3.2. If $x_2\notin[a,1]$,

$$I_4(x_2,y)=1\leq 1=I_4(x_1,y).$$

4. Let $x_1\notin[a,1]$ and $y\notin[a,1]$

4.1. If $x_2\in[a,1]$,

$$I_4(x_2,y) = T(K(x_2,y),a) \le T(K(x_1,y),a) \le T(K(x_1,y),b) = I_4(x_1,y).$$

4.2. If $x_2 \notin [a,1]$,

$$I_4(x_2,y) = T(K(x_2,y),b) \le T(K(x_1,y),b) = I_4(x_1,y).$$

(I2) We need to show that $I_4$ is an increasing function on the second variable. Then it should be $I_4(x,y_1) \le I_4(x,y_2)$ for $x,y_1,y_2 \in L$ and $y_1 \le y_2$. If $x = 0$ or $y_2 = 1$ or $(x,y_1) = (1,0)$, the proof is immediate. The proof can be split into all possible cases.

1. Let $(x,y_1) \in [a,1]^2$.

$$I_4(x,y_1) = J(x,y_1) \le J(x,y_2) = I_4(x,y_2).$$

2. Let $x \in [a,1]$ and $y_1 \notin [a,1]$.

2.1. If $y_2 \in [a,1]$,

$$I_4(x,y_1) = T(K(x,y_1),a) \le a \le J(x,y_2) = I_4(x,y_2).$$

2.2. If $y_2 \notin [a,1]$,

$$I_4(x,y_1) = T(K(x,y_1),a) \le T(K(x,y_2),a) = I_4(x,y_2).$$

3. Let $x \notin [a,1]$ and $y_1 \in [a,1]$.

$$I_4(x,y_1) = 1 \le 1 = I_4(x,y_2).$$

4. Let $x \notin [a,1]$ and $y_1 \notin [a,1]$.

4.1. If $y_2 \in [a,1]$,

$$I_4(x,y_1) = T(K(x,y_1),b) \le 1 = I_4(x,y_2).$$

4.2. If $y_2 \notin [a,1]$,

$$I_4(x,y_1) = T(K(x,y_1),b) \le T(K(x,y_2),b) = I_4(x,y_2).$$

□

*Remark* 3.12. Let $T$ be the t-norm $T_\wedge$ in Theorem 3.11.

(i) The implication $I_4$ given by the formula (3.13) can be rewritten as

$$I_4(x,y) = \begin{cases} 1 & \text{if } (x=0 \text{ or } y=1) \text{or } (x \notin [a,1] \text{ and } y \in [a,1]), \\ 0 & \text{if } (x,y) = (1,0), \\ J(x,y) & \text{if } (x,y) \in [a,1]^2, \\ K(x,y) \wedge a & \text{if } x \in [a,1] \text{ and } y \notin [a,1], \\ K(x,y) \wedge b & \text{if } x \notin [a,1] \text{ and } y \notin [a,1]. \end{cases} \tag{3.14}$$

(ii) If $b = 1$, then the implication $I_4$ given by the formula (3.13) can be rewritten as follows:

$$(3.15)\quad I_4(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ K(x,y)\wedge a & \text{if } x\in[a,1] \text{ and } y\notin[a,1],\\ K(x,y) & \text{if } x\notin[a,1] \text{ and } y\notin[a,1].\end{cases}$$

(iii) If $a = b$, then the implication $I_4$ given by the formula (3.13) can be rewritten as follows:

$$(3.16)\quad I_4(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ K(x,y)\wedge a & \text{otherwise.}\end{cases}$$

(iv) If $K(x,y) = N(x) \vee y$, then the implication $I_4$ given by the formula (3.13) can be rewritten as follows:

$$(3.17)\quad I_4(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ (N(x)\vee y)\wedge a & \text{if } x\in[a,1] \text{ and } y\notin[a,1],\\ (N(x)\vee y)\wedge b & \text{if } x\notin[a,1] \text{ and } y\notin[a,1].\end{cases}$$

We illustrate a example for Theorem 3.11.

**Example 3.13.** Consider the lattice $(L = \{0, t_1, t_2, t_3, t_4, t_5, 1\}, \le, 0, 1)$ as given in Fig. 1 and the implication $J : [t_2, 1]^2 \to [t_2, 1]$ as given in Table 1 and the implication $K$ be as in Table 4.

| $K$ | 0 | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | 1 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $t_1$ | 0 | 1 | $t_3$ | 1 | $t_5$ | 1 | 1 |
| $t_2$ | 0 | $t_5$ | 1 | 1 | 1 | 1 | 1 |
| $t_3$ | 0 | $t_1$ | $t_2$ | 1 | $t_5$ | 1 | 1 |
| $t_4$ | 0 | $t_5$ | $t_2$ | $t_5$ | 1 | 1 | 1 |
| $t_5$ | 0 | $t_1$ | $t_2$ | $t_3$ | $t_4$ | 1 | 1 |
| 1 | 0 | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | 1 |

**Table 4.** The implication $K$ on $L$.

By applying the formula (3.13) in Theorem 3.11 with $a = t_2$ and $b = t_3$, the implication $I_4$ can be obtained as in Table 5.

| $I_4$ | $0$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $1$ |
|---|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t_1$ | $0$ | $t_3$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t_2$ | $0$ | $t_2$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t_3$ | $0$ | $0$ | $t_4$ | $t_5$ | $t_4$ | $t_5$ | $1$ |
| $t_4$ | $0$ | $t_2$ | $t_3$ | $t_3$ | $t_5$ | $t_5$ | $1$ |
| $t_5$ | $0$ | $0$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $1$ |
| $1$ | $0$ | $0$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $1$ |

**Table 5.** The implication $I_4$ on $L$.

**Theorem 3.14.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $S : L^2 \to L$ be a t-conorm, $K : L^2 \to L$ be an implication and $J : [0, b]^2 \longrightarrow [0, b]$ be an implication. Then, the function $I_4^* : L^2 \longrightarrow L$ defined by,*

$$
(3.18)\quad I_4^*(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ 0 & \text{if } (x,y) = (1,0) \text{ or } (x \notin [0,b] \text{ and } y \in [0,b]), \\ J(x,y) & \text{if } (x,y) \in [0,b]^2, \\ S(K(x,y),b) & \text{if } x \in [0,b] \text{ and } y \notin [0,b], \\ S(K(x,y),a) & \text{if } x \notin [0,b] \text{ and } y \notin [0,b], \end{cases}
$$

*is an implication on $L$.*

*Proof.* The proof can be done in a similar fashion as the proof of Theorem 3.11. Therefore, we omit it. □

*Remark* 3.15. (i) If we take the restriction of the implication operations in Theorems 3.1, 3.5, 3.9 and 3.11 on $[a, 1]$, it is obtained that $I_1 = I_3 = I_5 = I_7 = J$.
(ii) If we take the restriction of the implication operations in Theorems 3.4, 3.8, 3.10 and 3.14 on $[0, b]$, it is obtained that $I_2 = I_4 = I_6 = I_8 = J$.

## 4. Conclusion

In this study, construction methods for implications on bounded lattices have been investigated by means of a implication operator which is defined on the subinterval $[a, 1]$ ($[0, b]$) of the bounded lattice $L$ having $a, b \in L$ with $a \leq b$. We also have benefited from some fuzzy logic operators in some of the methods. In addition we, the construction methods are clarified with the examples and corollaries.

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**The Declaration of Research and Publication Ethics**

The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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