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LP-Kenmotsu Manifolds Admitting Bach Almost Solitons

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Abstract

For a Lorentzian para-Kenmotsu manifold of dimension m (briefly, $(LPK)_m$) admitting Bach almost soliton (g, ζ, λ), we explored the characteristics of the norm of Ricci operator. Besides, we gave the necessary condition for $(LPK)_m$ $(m \geq 4)$ admitting Bach almost soliton to be an η-Einstein manifold. Afterwards, we proved that Bach almost solitons are always steady when a Lorentzian para-Kenmotsu manifold of dimension three has Bach almost soliton.

1. Introduction

In 1976, the concept of almost paracontact manifolds was proposed by Sato [1]. An almost paracontact structure on a semi-Riemannian manifold $\mathscr{M}$ was established by Kaneyuki and Kozai in [2]. They created almost paracomplex shape on $\mathscr{M} \times R$. According to Kaneyuki et al. [3], the key variation among an almost paracontact manifold is the signature of metric. In 1995, the authors Sinha and Prasad described para-Kenmotsu as well as special para-Kenmotsu manifolds and found significant properties of para-Kenmotsu manifolds [4]. Afterwards, para-Kenmotsu manifolds drew huge attention and a number of mathematicians brought forward the significant characteristics of such manifolds [5–9].

Semi-Riemannian geometry, used in the relativity theory, was studied in [10]. About four decades ago, Kaigorodov has explored the curvature structure of the spacetime [11]. Raychaudhuri et al. [12] extended the above concepts of the general theory of spacetime. Recently, Haseeb and Rajendra introduced and studied the Lorentzian para-Kenmotsu manifolds [13, 14].

1921 was the year, when Bach initiated Bach tensor [15] to explore conformal geometry. He proved that the Bach tensor is a rank 2 trace-free tensor and is conformally invariant in dimension 4. So, in lieu of Hilbert-Einstein functional, the functional is taken in the following way

$$\mathscr{W}(g) = \int_{\mathscr{M}} \|\mathscr{C}\|_g^2 d\nu_g,$$

where, $\mathscr{M}$ is a manifold of dimension-four and $\mathscr{C}$ repersents the Weyl tensor of type $(1,3)$ given by

$$\begin{aligned}\mathscr{C}(U,\mathscr{V})\mathscr{W} &= \mathscr{R}(U,\mathscr{V})\mathscr{W} + \frac{1}{m-2}[\mathscr{S}(U,\mathscr{W})\mathscr{V} - \mathscr{S}(\mathscr{V},\mathscr{W})U + g(U,\mathscr{W})\mathscr{Q}\mathscr{V} - g(\mathscr{V},\mathscr{W})\mathscr{Q}U] \\ &\quad - \frac{r}{(m-1)(m-2)}[g(U,\mathscr{W})\mathscr{V} - g(\mathscr{V},\mathscr{W})U],\end{aligned} \tag{1.1}$$

here, $\mathscr{R}$ represents the Riemannian curvature tensor, $\mathscr{Q}$ is the Ricci operator and $\mathscr{S}$ denotes the Ricci tensor, such that, $g(\mathscr{Q}U,\mathscr{V}) = \mathscr{S}(U,\mathscr{V})$, $\forall$ differentiable vector fields $U, \mathscr{V}, \mathscr{W}$. Bach tensor of type $(0,2)$ on a semi-Riemannian manifold $(\mathscr{M}^m, g)$ of dimension $m(\geq 3)$ is given by

$$\begin{aligned}\mathscr{B}(U,\mathscr{V}) &= \frac{1}{(m-3)} \sum_{i\in\{1,\ldots,m\}} \sum_{j\in\{1,\ldots,m\}} \varepsilon_i\varepsilon_j (\nabla_{\mathscr{E}_i}\nabla_{\mathscr{E}_j}\mathscr{C}')(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V}) \\ &\quad + \frac{1}{(m-2)} \sum_{i\in\{1,\ldots,m\}} \sum_{j\in\{1,\ldots,m\}} \varepsilon_i\varepsilon_j \mathscr{S}(\mathscr{E}_i,\mathscr{E}_j)\mathscr{C}'(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V}),\end{aligned} \tag{1.2}$$

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here, $g(\mathscr{E}_i,\mathscr{E}_i)=\varepsilon_i$, $g(\mathscr{C}(U,\mathscr{V})\mathscr{W},\mathscr{Y})=\mathscr{C}'(U,\mathscr{V},\mathscr{W},\mathscr{Y})$ and $\{\{\mathscr{E}_i\}_{i=1}^{m-1},\mathscr{E}_m=\zeta\}$ is a local orthonormal frame at each point p of $T_p\mathscr{M}$. Relation (1.1), together with contracting Bianchi second identity, we obtain

$$div\mathscr{C}=\frac{(m-3)}{(m-2)}C_0, \tag{1.3}$$

where, C_0 is Cotton tensor [16] given by

$$C_0(U,\mathscr{V})\mathscr{W}=-(\nabla_{\mathscr{V}}\mathscr{S})(U,\mathscr{W})+(\nabla_U\mathscr{S})(\mathscr{V},\mathscr{W})+\frac{1}{2(m-1)}[(\mathscr{V}r)g(U,\mathscr{W})-(Ur)g(\mathscr{V},\mathscr{W})]. \tag{1.4}$$

In view of equation (1.3), together with equation (1.2), the Bach tensor takes the form,

$$\mathscr{B}(U,\mathscr{V})=\frac{1}{(m-2)}[\sum_{i\in\{1,\dots,m\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\mathscr{V}+\sum_{i\in\{1,\dots,m\}}\sum_{j\in\{1,\dots,m\}}\varepsilon_i\varepsilon_j\mathscr{S}(\mathscr{E}_i,\mathscr{E}_j)\mathscr{C}'(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V})], \tag{1.5}$$

$\forall$ differentiable vector fields $U,\mathscr{V}$. For dimension three, the Weyl tensor vanishes. Therefore, Bach tensor given in equation (1.5) reduces to

$$\mathscr{B}(U,\mathscr{V})=\sum_{i\in\{1,2,3\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\mathscr{V}. \tag{1.6}$$

For further study, the references [17–24] may be seen.
In 2012, Das and Kar [25] studied different characteristics of Bach flow on product manifolds and analysed their outcomes with the Ricci flow. Bach flow is suggested in [26] to specify the Harava-Lifschitz gravity in general relativity. In 2011, Bahuaud and Helliwell in [27] studied the presence of Bach flow for short time. Cao and Chen, in the year 2013, explored Bach flat Ricci solitons [28]. Subsequently, Ho [29] worked comprehensively on the solitons of Bach flow. He also studied the Bach flows on Lie group of dimension 4. In 2020, Helliwell specified Bach flow of dimension 4 on locally homogeneous product manifolds [30]. In recent times, Ghosh [31] investigated the Bach almost solitons (g,ζ,λ) in semi-Riemannian geometry and is given by

$$(\pounds_{\mathscr{X}}g+2\mathscr{B}-2\lambda g)(U,\mathscr{V})=0, \tag{1.7}$$

here, $\pounds_{\mathscr{X}}$ is the Lie derivative operator along $\mathscr{X}$; $\mathscr{X}$ is a potential vector field and $\lambda\in C^\infty(\mathscr{M}^m)$. The Bach almost solitons (g,ζ,λ) is said to be expanding, steady and shrinking according to $\lambda<0$, $\lambda=0$ and $\lambda>0$, respectively.
This article is organized in the following manner: Section 1 contains introduction, based on development of almost paracontact manifold and other concepts. Preliminaries are given in Section 2, based on $(LPK)_m$. Section 3 contains the work on (g,ζ,λ) in $(LPK)_m$. In Section 4, we examine $(LPK)_m$ of dimension 3, which admits Bach almost solitons.

2. Preliminaries

An m-dimensional smooth manifold $\mathscr{M}^m$ is called Lorentzian almost paracontact manifold, if it is equipped with a (1,1)-tensor field ϕ, a contravariant vector field ζ, a 1-form η and a Lorentzian metric g of type (0, 2). The following relations for an m-dimensional Lorentzian metric manifold hold [32],

$$\phi^2(U)=U+\eta(U)\zeta,\ \ \eta(\zeta)+1=0, \tag{2.1}$$

$$g(U,\zeta)=\eta(U),\ \ g(\phi U,\phi\mathscr{V})=\eta(U)\eta(\mathscr{V})+g(U,\mathscr{V}), \tag{2.2}$$

$\forall\, U,\mathscr{V}$ on $\mathscr{M}^m$, and the structure (ϕ,ζ,η,g) is named the Lorentzian almost paracontact structure. An $\mathscr{M}^m$ endowed with (ϕ,ζ,η,g) is known as Lorentzian almost paracontact manifold and holding the following results:

$$\phi\zeta=0,\ \ \eta(\phi U)=0,\ \ \Omega(U,\mathscr{V})=\Omega(\mathscr{V},U), \tag{2.3}$$

here, $\Omega(U,\mathscr{V})=g(U,\varphi\mathscr{V})$.

Definition 2.1. *A Lorentzian almost paracontact manifold $\mathscr{M}^m$ is known as $(LPK)_m$ if*

$$(\nabla_U\phi)(\mathscr{V})=-\eta(\mathscr{V})\phi U-g(\phi U,\mathscr{V})\zeta,$$

$\forall\, U$ and $\mathscr{V}$ on $\mathscr{M}^m$.

Further, for $(LPK)_m$, following results hold good:

$$\nabla_U\zeta+U+\eta(U)\zeta=0, \tag{2.4}$$

$$(\nabla_U\eta)(\mathscr{V})+g(U,\mathscr{V})+\eta(U)\eta(\mathscr{V})=0, \tag{2.5}$$

$$\mathscr{R}(U,\mathscr{V})\zeta=\eta(\mathscr{V})U-\eta(U)\mathscr{V}, \tag{2.6}$$

$$\mathscr{R}(\zeta,\mathscr{V})U = g(U,\mathscr{V})\zeta - \eta(U)\mathscr{V}, \tag{2.7}$$

$$\mathscr{R}(\zeta,U)\zeta = U + \eta(U)\zeta, \tag{2.8}$$

$$\mathscr{S}(U,\zeta) = (m-1)\eta(U), \tag{2.9}$$

$$\mathscr{Q}\zeta = (m-1)\zeta, \tag{2.10}$$

$$\mathscr{S}(\phi\mathscr{V},\phi U) = \mathscr{S}(\mathscr{V},U) + (m-1)\eta(\mathscr{V})\eta(U), \tag{2.11}$$

$\forall\ U,\mathscr{V},\mathscr{W}$ on $(LPK)_m$ [33, 34]. In the above results, ∇ represents the covariant differentiation operator w.r.t. g in semi-Riemannian manifolds.

Proposition 2.2. *We assume $\mathscr{M}$ to be an $(LPK)_m$. Subsequently, we have*

$$\mathscr{S}(\phi U,\mathscr{V}) = \mathscr{S}(U,\phi\mathscr{V}), \tag{2.12}$$

$\forall\ U,\ \mathscr{V}$ on $(LPK)_m$.

Proof. Setting ϕU for U in (2.11), we get,

$$\mathscr{S}(\phi^2 U,\phi\mathscr{V}) = \mathscr{S}(\phi U,\mathscr{V}) + (m-1)\eta(\phi U)\eta(\mathscr{V}).$$

Using equations (2.1) and (2.3) in the foregoing equation, we yield

$$\mathscr{S}(U+\eta(U)\zeta,\phi\mathscr{V}) = \mathscr{S}(\phi U,\mathscr{V}). \tag{2.13}$$

From equation (2.13), the Proposition 2.2 follows. □

3. Bach Almost Solitons and $(LPK)_m$

Definition 3.1. *A semi-Riemannnian manifold is called Bach perfect fluid if Bach almost tensor is given by*

$$\mathscr{B}(U,\mathscr{V}) = \beta\eta(U)\eta(\mathscr{V}) + \alpha g(U,\mathscr{V}), \quad \forall\mathscr{V},U,$$

where, α and β are scalars.

Let $(LPK)_m$ admit (g,ζ,λ). Then (1.7) holds and thus, we have

$$(\pounds_\zeta g)(U,\mathscr{V}) + 2\mathscr{B}(U,\mathscr{V}) = 2\lambda g(U,\mathscr{V}). \tag{3.1}$$

As we have

$$(\pounds_\zeta g)(U,\mathscr{V}) = g(\nabla_U\zeta,\mathscr{V}) + g(U,\nabla_{\mathscr{V}}\zeta). \tag{3.2}$$

The result (2.4), together with (3.2) yields

$$(\pounds_\zeta g)(U,\mathscr{V}) + 2[g(U,\mathscr{V}) + \eta(U)\eta(\mathscr{V})] = 0. \tag{3.3}$$

Putting the preceding result (3.3) in (3.1), we lead to

$$\mathscr{B}(\mathscr{V},U) = (1+\lambda)g(\mathscr{V},U) + \eta(\mathscr{V})\eta(U). \tag{3.4}$$

Result (3.4) shows the succeeding proposition:

Proposition 3.2. *An $(LPK)_m$ admitting a Bach almost soliton (g,ζ,λ) is Bach perfect fluid.*

Replacing $\mathscr{W}$ by ζ in (1.1), we have

$$\begin{aligned}\mathscr{C}(U,\mathscr{V})\zeta = {} & \mathscr{R}(U,\mathscr{V})\zeta + \frac{1}{(m-2)}[\mathscr{S}(U,\zeta)\mathscr{V} - \mathscr{S}(\mathscr{V},\zeta)U + g(U,\zeta)\mathscr{Q}\mathscr{V} - g(\mathscr{V},\zeta)\mathscr{Q}U] \\ & - \frac{r}{(m-1)(m-2)}[g(U,\zeta)\mathscr{V} - g(\mathscr{V},\zeta)U],\end{aligned} \tag{3.5}$$

$\forall$ differentiable vector fields $U,\mathscr{V}$. Operating $\mathscr{Q}$ in (3.5) and using relations (2.2), (2.6), (2.7) and (2.10), we get

$$\mathscr{Q}(\mathscr{C}(U,\mathscr{V})\zeta) = \frac{(r-m+1)}{(m-1)(m-2)}[-\eta(U)\mathscr{Q}\mathscr{V} + \eta(\mathscr{V})\mathscr{Q}U] - \frac{1}{(m-2)}[\eta(\mathscr{V})\mathscr{Q}^2 U - \eta(U)\mathscr{Q}^2\mathscr{V}]. \tag{3.6}$$

The inner product of (3.6) with $\mathscr{X}$ leads to

$$g(\mathscr{Q}(\mathscr{C}(U,\mathscr{V})\zeta),\mathscr{X}) = \frac{(r-m+1)}{(m-1)(m-2)}[\eta(\mathscr{V})g(\mathscr{Q}U,\mathscr{X})-\eta(U)g(\mathscr{Q}\mathscr{V},\mathscr{X})] - \frac{1}{(m-2)}[\eta(\mathscr{V})g(\mathscr{Q}^2U,\mathscr{X})-\eta(U)g(\mathscr{Q}^2\mathscr{V},\mathscr{X})]. \tag{3.7}$$

Let $\{\{\mathscr{E}_i\}_{i=1}^{m-1},\mathscr{E}_m=\zeta\}$ be an orthonormal frame at each point p of $T_p\mathscr{M}$. Now, setting $\mathscr{V}=\mathscr{X}=\mathscr{E}_i$ in (3.7) with summation $i=1$ to m and on evaluation, we get

$$\sum_{i\in\{1,..,m\}}\varepsilon_i g(\mathscr{Q}(\mathscr{C}(U,\mathscr{E}_i)\zeta),\mathscr{E}_i) = -\frac{(r-m+1)^2}{(m-1)(m-2)}\eta(U)+\frac{1}{(m-2)}[|\mathscr{Q}|^2-(m-1)^2]\eta(U). \tag{3.8}$$

Setting ζ in place of $\mathscr{W}$ in relation (1.4) gives

$$C_0(U,\mathscr{V})\zeta = g((\nabla_U\mathscr{Q})\mathscr{V},\zeta)-g((\nabla_{\mathscr{V}}\mathscr{Q})U,\zeta)-\frac{1}{2(m-1)}[U(r)\eta(\mathscr{V})-\mathscr{V}(r)\eta(U)]. \tag{3.9}$$

From equation (2.12), we have the relation

$$\phi\mathscr{Q}U=\mathscr{Q}\phi U. \tag{3.10}$$

From the equation (3.10), we also have

$$g((\nabla_U\mathscr{Q})\mathscr{V},\zeta)=g(\mathscr{Q}U,\mathscr{V})-(m-1)g(U,\mathscr{V}). \tag{3.11}$$

Applying above equation (3.11) in (3.9), it gives

$$C_0(U,\mathscr{V})\zeta = -\frac{1}{2(m-1)}[U(r)\eta(\mathscr{V})-\mathscr{V}(r)\eta(U)]. \tag{3.12}$$

After differentiating covariantly the above relation w.r.t. $\mathscr{W}$ and using the relation (2.5), we obtain

$$(\nabla_{\mathscr{W}}C_0)(U,\mathscr{V})\zeta = -(\nabla_{\mathscr{V}}\mathscr{S})(U,\mathscr{W})+(\nabla_U\mathscr{S})(\mathscr{V},\mathscr{W}) - \frac{1}{2(m-1)}[g(\nabla_{\mathscr{W}}\mathscr{D}r,U)\eta(\mathscr{V})-g(\nabla_{\mathscr{W}}\mathscr{D}r,\mathscr{V})\eta(U)], \tag{3.13}$$

here $\mathscr{D}$ represents the gradient operator. Let $\{\{\mathscr{E}_i\}_{i=1}^{m-1},\mathscr{E}_m=\zeta\}$ be the orthonormal frame at each point p of $T_p\mathscr{M}$. Replacing $U=\mathscr{W}=\mathscr{E}_i$ with summation over $i=1$ to m in equation (3.13), this gives

$$\sum_{i\in\{1,...,m\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,\mathscr{V})\zeta = -\frac{1}{2(m-1)}[(div\mathscr{D}r)\eta(\mathscr{V})-g(\nabla_\zeta\mathscr{D}r,\mathscr{V})]-\frac{\mathscr{V}(r)}{2}. \tag{3.14}$$

Now, by rewriting the equation (1.5), we have

$$\mathscr{B}(U,\mathscr{V}) = \frac{1}{(m-2)}[\sum_{i\in\{1,...,m\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\mathscr{V}+\sum_{i\in\{1,...,m\}}\sum_{j\in\{1,...,m\}}\varepsilon_i\varepsilon_j\mathscr{S}(\mathscr{E}_i,\mathscr{E}_j)\mathscr{C}'(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V})]. \tag{3.15}$$

After evaluation, the second term of the above equation takes the form

$$\begin{aligned}\sum_{i\in\{1,...,m\}}\sum_{j\in\{1,...,m\}}\varepsilon_i\varepsilon_j\mathscr{S}(\mathscr{E}_i,\mathscr{E}_j)\mathscr{C}'(U,\mathscr{E}_i,\mathscr{E}_j,\mathscr{V}) &= -\sum_{i\in\{1,...,m\}}\sum_{j\in\{1,...,m\}}\varepsilon_i\varepsilon_j g(\mathscr{Q}\mathscr{E}_i,\mathscr{E}_j)g(\mathscr{C}(U,\mathscr{E}_i)\mathscr{V},\mathscr{E}_j),\\ &= -\sum_{i\in\{1,...,m\}}\varepsilon_i g(\mathscr{Q}(\mathscr{C}(U,\mathscr{E}_i)\mathscr{V}),\mathscr{E}_i).\end{aligned}$$

Taking the above equation and equation (3.15) together, we obtain

$$\mathscr{B}(U,\mathscr{V}) = \frac{1}{(m-2)}[\sum_{i\in\{1,...,m\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\mathscr{V}-\sum_{i\in\{1,...,m\}}\varepsilon_i g(\mathscr{Q}(\mathscr{C}(U,\mathscr{E}_i)\mathscr{V}),\mathscr{E}_i)]. \tag{3.16}$$

Replacing $\mathscr{V}$ for ζ in the above relation (3.16), it gives

$$\mathscr{B}(U,\zeta) = \frac{1}{(m-2)}[\sum_{i\in\{1,...,m\}}\varepsilon_i(\nabla_{\mathscr{E}_i}C_0)(\mathscr{E}_i,U)\zeta-\sum_{i\in\{1,...,m\}}\varepsilon_i g(\mathscr{Q}(\mathscr{C}(U,\mathscr{E}_i)\zeta),\mathscr{E}_i)]. \tag{3.17}$$

Equations (3.8), (3.14) and (3.17) taken together give

$$\mathscr{B}(U,\zeta) = \frac{1}{(m-2)}[-\frac{U(r)}{2}-\frac{1}{2(m-1)}\{(div\mathscr{D}r)\eta(U)-g(\nabla_\zeta\mathscr{D}r,U)\} + \frac{(r-m+1)^2}{(m-1)(m-2)}\eta(U)-\frac{1}{(m-2)}\{|\mathscr{Q}|^2-(m-1)^2\}\eta(U)]. \tag{3.18}$$

Setting $\mathscr{V}$ for ζ in equation (3.4), we get

$$\mathscr{B}(U,\zeta) = \lambda\eta(U). \tag{3.19}$$

Relation (3.18) and (3.19), taken together give

$$\begin{aligned}\lambda\eta(U) = \frac{1}{(m-2)}[-\frac{U(r)}{2} - \frac{1}{2(m-1)}\{(div\mathscr{D}r)\eta(U) - g(\nabla_\zeta\mathscr{D}r,U)\}\\ + \frac{(r-m+1)^2}{(m-1)(m-2)}\eta(U) - \frac{1}{(m-2)}\{|\mathscr{Q}|^2-(m-1)^2\}\eta(U)].\end{aligned} \tag{3.20}$$

Setting U for ϕU in relation (3.20), we obtain

$$\frac{1}{(m-2)}[-\frac{\phi U(r)}{2} + \frac{1}{2(m-1)}g(\nabla_\zeta\mathscr{D}r,\phi U)] = 0.$$

This implies that

$$g(\nabla_\zeta\mathscr{D}r,\phi U) = (m-1)g(\mathscr{D}r,\phi U).$$

This gives

$$\phi\nabla_\zeta\mathscr{D}r = (m-1)\phi\mathscr{D}r. \tag{3.21}$$

Taking covariant differentiation of equation (2.10) w.r.t. U and using the relations (2.3) and (2.4), we get

$$(\nabla_U\mathscr{Q})\zeta = \mathscr{Q}U - (m-1)U. \tag{3.22}$$

Contracting the preceding equation w.r.t. U, we have

$$\sum_{i\in\{1,\dots,m\}}\varepsilon_i g((\nabla_{\mathscr{E}_i}\mathscr{Q})\zeta,\mathscr{E}_i) = \sum_{i=1}^{m}\varepsilon_i[g(\mathscr{Q}\mathscr{E}_i,\mathscr{E}_i)-(m-1)g(\mathscr{E}_i,\mathscr{E}_i)].$$

or,

$$(div\mathscr{Q})\zeta = r-(m-1)m,$$

or,

$$\zeta(r) = 2[r-m(m-1)], \tag{3.23}$$

which can be written as

$$\pounds_\zeta r = 2r-2m(m-1).$$

Applying the exterior derivative in the above relation, we have

$$d\pounds_\zeta r = 2dr.$$

Since, d and the Lie derivative commutes, therefore, we have

$$\pounds_\zeta dr = 2dr.$$

Writing the above relation in the form of gradient operator, we have

$$\pounds_\zeta\mathscr{D}r = 2\mathscr{D}r,$$

or,

$$\nabla_\zeta\mathscr{D}r - \nabla_{\mathscr{D}r}\zeta = 2\mathscr{D}r.$$

Using the relation (2.4) in the above relation, we lead to

$$\nabla_\zeta\mathscr{D}r = \mathscr{D}r - \zeta(r)\zeta. \tag{3.24}$$

Applying ϕ in the above relation (3.24) and using the relations in (2.3) and (3.21), we get

$$\phi\mathscr{D}r = 0.$$

This implies

$$\mathscr{D}r = -\zeta(r)\zeta. \tag{3.25}$$

Differentiating (3.25) covariantly w.r.t. $\mathscr{X}$, it yields

$$\nabla_\mathscr{X}\mathscr{D}r = -[g(\nabla_\mathscr{X}\mathscr{D}r,\zeta)\zeta - g(\mathscr{D}r,\mathscr{X})\zeta - g(\mathscr{D}r,\zeta)\mathscr{X} - 2g(\mathscr{D}r,\zeta)\eta(\mathscr{X})\zeta], \tag{3.26}$$

which by contracting over $\mathscr{X}$ gives

$$(div\mathscr{D}r) = (m-3)\zeta(r). \tag{3.27}$$

Relations (3.24) and (3.25) give

$$\nabla_{\zeta}\mathscr{D}r = -2\zeta(r)\zeta. \tag{3.28}$$

Using relations (3.25), (3.27) and (3.28) in (3.20), we obtain

$$\begin{aligned}\lambda\eta(U) &= \frac{1}{(m-2)}[\frac{\zeta(r)}{2}\eta(U) - \frac{1}{2(m-1)}\{(m-3)\zeta(r)\eta(U)+2\zeta(r)\eta(U)\}\\ &+\frac{(r+1-m)^2}{(m-1)(m-2)}\eta(U) - \frac{1}{(m-2)}\{|\mathscr{Q}|^2-(m-1)^2\}\eta(U)].\end{aligned} \tag{3.29}$$

On simplification, relation (3.29) gives

$$\lambda = \frac{1}{(m-2)^2}[\frac{(r+1-m)^2}{(m-1)} + (m-1)^2 - |\mathscr{Q}|^2]. \tag{3.30}$$

In the light of the relation (3.30), succeeding theorem holds:

Theorem 3.3. *The Bach almost solitons (g,ζ,λ) on an $(LPK)_m$ are expanding, steady and shrinking according as*

$$[\frac{(r+1-m)^2}{(m-1)} + (m-1)^2] > |\mathscr{Q}|^2,\ [\frac{(r+1-m)^2}{(m-1)} + (m-1)^2] = |\mathscr{Q}|^2 \ \ and\ \ [\frac{(r+1-m)^2}{(m-1)} + (m-1)^2] < |\mathscr{Q}|^2.$$

Consider a Lorentzian para-Kenmotsu space form of m-dimension. Then by relation (3.23), we have $r = m(m-1)$. Hence,

$$\lambda = \frac{1}{(m-2)^2}[m(m-1)^2 - |\mathscr{Q}|^2].$$

The above relation leads the following corollary:

Corollary 3.4. *The Bach almost solitons (g,ζ,λ) on an LP-Kenmotsu space form of dimension m is expanding, steady and shrinking according as $m(m-1)^2 > |\mathscr{Q}|^2$, $m(m-1)^2 = |\mathscr{Q}|^2$ and $m(m-1)^2 < |\mathscr{Q}|^2$.*

Definition 3.5. *An $(LPK)_m$ is named η-Einstein if its $\mathscr{S}$ satisfies [35]*

$$\mathscr{S}(\mathscr{V},U) = ag(\mathscr{V},U) + b\eta(\mathscr{V})\eta(U),$$

$\forall\ \mathscr{V},U$, where, a and b are scalars.

Now, replacing $U=\zeta$ in relation (3.13), we have

$$(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = -(\nabla_{\mathscr{V}}\mathscr{S})(\zeta,\mathscr{W}) + (\nabla_{\zeta}\mathscr{S})(\mathscr{V},\mathscr{W}) - \frac{1}{2(m-1)}[g(\nabla_{\mathscr{W}}\mathscr{D}r,\zeta)\eta(\mathscr{V}) - g(\nabla_{\mathscr{W}}\mathscr{D}r,\mathscr{V})\eta(\zeta)]. \tag{3.31}$$

Taking the inner product of relation (3.26) with $\mathscr{V}$ and replacing $\mathscr{X}$ by $\mathscr{W}$, we obtain

$$g(\nabla_{\mathscr{W}}\mathscr{D}r,\mathscr{V}) = -[g(\nabla_{\mathscr{W}}\mathscr{D}r,\zeta)\eta(\mathscr{V}) - g(\mathscr{D}r,\mathscr{W})\eta(\mathscr{V}) - g(\mathscr{D}r,\zeta)g(\mathscr{V},\mathscr{W}) - 2g(\mathscr{D}r,\zeta)\eta(\mathscr{V})\eta(\mathscr{W})]. \tag{3.32}$$

The relations (3.22), (3.31) and (3.32) give

$$(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = g((\nabla_{\zeta}\mathscr{Q})\mathscr{V},\mathscr{W}) - g((\nabla_{\mathscr{V}}\mathscr{Q})\zeta,\mathscr{W}) - \frac{\zeta(r)}{2(m-1)}[g(\mathscr{V},\mathscr{W}) + \eta(\mathscr{V})\eta(\mathscr{W})]. \tag{3.33}$$

In an $(LPK)_m$, the following result holds (for perusal, see [36])

$$(\nabla_{\zeta}\mathscr{Q})\mathscr{V} = 2\mathscr{Q}\mathscr{V} - 2(m-1)\mathscr{V}. \tag{3.34}$$

Applying relations (3.22) and (3.34) into (3.33), it yields

$$(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = g(\mathscr{Q}\mathscr{V},\mathscr{W}) - (m-1)g(\mathscr{V},\mathscr{W}) - \frac{\zeta(r)}{2(m-1)}[g(\mathscr{V},\mathscr{W}) + \eta(\mathscr{V})\eta(\mathscr{W})]. \tag{3.35}$$

If $(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = 0$ and (3.23), then (3.35) leads to

$$\mathscr{S}(\mathscr{V},\mathscr{W}) = (\frac{r}{m-1} - 1)g(\mathscr{V},\mathscr{W}) + (\frac{r}{m-1} - m)\eta(\mathscr{V})\eta(\mathscr{W}). \tag{3.36}$$

The relation (3.36) leads the following theorem:

Theorem 3.6. *An $(LPK)_m$ $(m \geq 4)$ admitting (g,ζ,λ) is an η-Einstein manifold provided $(\nabla_{\mathscr{W}}C_0)(\zeta,\mathscr{V})\zeta = 0$, $\forall\ \mathscr{V},\mathscr{W}$.*

4. 3-Dimensional Bach Perfect Fluid Lorentzian Para-Kenmotsu Manifold

We consider an $(LPK)_3$ admitting (g,ζ,λ). Curvature tensor of Riemannian manifold in dimension 3 states

$$\mathscr{R}(U,\mathscr{V})\mathscr{W} = -\mathscr{S}(U,\mathscr{W})\mathscr{V} + \mathscr{S}(\mathscr{V},\mathscr{W})U - g(U,\mathscr{W})\mathscr{Q}\mathscr{V} + g(\mathscr{V},\mathscr{W})\mathscr{Q}U - \frac{r}{2}[g(U,\mathscr{W})\mathscr{V} - g(\mathscr{V},\mathscr{W})U], \tag{4.1}$$

$\forall$ differentiable vector fields U, $\mathscr{V}$ and $\mathscr{W}$.
Replacing $U = \mathscr{W} = \zeta$ in (4.1) and using (2.1), (2.8), (2.9) and (2.10), we obtain

$$\mathscr{Q}\mathscr{V} = (\frac{r}{2} - 3)\eta(\mathscr{V})\zeta + (\frac{r}{2} - 1)\mathscr{V}. \tag{4.2}$$

The preceding result gives

$$\mathscr{Q}\phi = \phi\mathscr{Q}.$$

The equation (4.2), together with (2.4), gives

$$(\nabla_U \mathscr{Q})\zeta = \mathscr{Q}U - 2U. \tag{4.3}$$

Equation (3.12), together with (4.3) leads to

$$C_0(U,\mathscr{V})\zeta = \frac{1}{4}[\mathscr{V}(r)\eta(U) - U(r)\eta(\mathscr{V})].$$

The covariant differentiation of above result w.r.t. $\mathscr{W}$ yields

$$(\nabla_{\mathscr{W}} C_0)(U,\mathscr{V})\zeta = -(\nabla_{\mathscr{V}}\mathscr{S})(U,\mathscr{W}) - (\nabla_U \mathscr{S})(\mathscr{V},\mathscr{W}) - \frac{1}{4}[g(\nabla_{\mathscr{W}}\mathscr{D}r,U)\eta(\mathscr{V}) - g(\nabla_{\mathscr{W}}\mathscr{D}r,\mathscr{V})\eta(U)].$$

Putting $\mathscr{W} = U = \mathscr{E}_i$ and taking sum over $i = 1,2,3$ in above relation, where $\{\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3 = \zeta\}$ is orthonormal frame at each point p of $T_p\mathscr{M}$, we have

$$\sum_{i\in\{1,2,3\}} \varepsilon_i (\nabla_{\mathscr{E}_i} C_o)(\mathscr{E}_i,\mathscr{V})\zeta = -\frac{\mathscr{V}(r)}{2} - \frac{1}{4}[(div\mathscr{D}r)\eta(\mathscr{V}) - g(\nabla_\zeta \mathscr{D}r,\mathscr{V})]. \tag{4.4}$$

Taking $\mathscr{V} = \zeta$ in (1.6), we have

$$\mathscr{B}(U,\zeta) = \sum_{i\in\{1,2,3\}} \varepsilon_i (\nabla_{\mathscr{E}_i} C_0)(\mathscr{E}_i, U)\zeta. \tag{4.5}$$

Equations (3.4), (4.4) and (4.5) taken together give

$$\lambda\eta(U) = -\frac{1}{2}g(\mathscr{D}r,U) - \frac{1}{4}[(div\mathscr{D}r)\eta(U) - g(\nabla_\zeta \mathscr{D}r,U)]. \tag{4.6}$$

Replacing ϕU for U in (4.6), we get

$$\phi\nabla_\zeta \mathscr{D}r = 2\phi\mathscr{D}r. \tag{4.7}$$

We have the relation (3.23) and (3.24), for $m = 3$, which yields

$$\nabla_\zeta \mathscr{D}r = \mathscr{D}r - 2(r-6)\zeta. \tag{4.8}$$

The relations (4.7) and (4.8) provide

$$\mathscr{D}r = -2(r-6)\zeta. \tag{4.9}$$

By the covariant diffentiation of (4.9) w.r.t. $\mathscr{X}$ yields

$$\nabla_{\mathscr{X}} \mathscr{D}r = -2g(\mathscr{D}r,\mathscr{X})\zeta + 2(r-6)\mathscr{X} + 2(r-6)\eta(\mathscr{X})\zeta. \tag{4.10}$$

By contracting the relation (4.10) over $\mathscr{X}$, we get

$$(div\mathscr{D}r) = 0. \tag{4.11}$$

Using relations (4.8), (4.9) and (4.11) in (4.6), it yields

$$\lambda = 0. \tag{4.12}$$

With the help of (4.12), the relation (3.4) reduces to

$$\mathscr{B}(U,\mathscr{V}) = \eta(U)\eta(\mathscr{V}) + g(U,\mathscr{V}).$$

The above results imply the succeeding theorem:

Theorem 4.1. *Let $(LPK)_3$ admit a (g,ζ,λ), then the manifold is a Bach perfect fluid and (g,ζ,λ) is always steady.*

5. Example

We assume a manifold $\mathscr{M}^3 = \{(u_1, v_1, w_1) \in R^3 : w_1 > 0\}$, here (u_1, v_1, w_1) are the general coordinates in R^3. Consider $\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_2, \hat{\mathscr{E}}_3$, the vector fields on $\mathscr{M}^3$ given as

$$\hat{\mathscr{E}}_1 = w_1 \frac{\partial}{\partial u_1}, \qquad \hat{\mathscr{E}}_2 = w_1 \frac{\partial}{\partial v_1}, \qquad \hat{\mathscr{E}}_3 = w_1 \frac{\partial}{\partial w_1} = \zeta$$

and are linearly independent at each point of $\mathscr{M}^3$. This implies

$$g(\hat{\mathscr{E}}_i, \hat{\mathscr{E}}_j) = \begin{cases} 0, & 1 \le i \ne j \le 3, \\ -1, & i = j = 1,2, \\ 0, & otherwise. \end{cases}$$

Suppose that η is 1-form on $\mathscr{M}^3$ given by $\eta(U) = g(U, \hat{\mathscr{E}}_3) = g(U, \zeta)$, $\forall$ U $\in \chi(\mathscr{M}^3)$. Again, assume that ϕ is $(1,1)$ tensor field on $\mathscr{M}^3$ given below:

$$\phi\hat{\mathscr{E}}_1 = -\hat{\mathscr{E}}_2, \qquad \phi\hat{\mathscr{E}}_2 = -\hat{\mathscr{E}}_1, \qquad \phi\hat{\mathscr{E}}_3 = 0.$$

The linear property of g and ϕ give the following relations

$$\eta(\zeta) = g(\zeta,\zeta) = -1, \phi^2 = U + \eta(U)\zeta,\ \ g(U,\zeta) = \eta(U),\ \eta(\phi U) = 0,\ g(\phi U, \phi\mathscr{V}) = \eta(U)\eta(\mathscr{V}) + g(U,\mathscr{V}).$$

Assuming ∇ to be Levi-Civita connection w.r.t. Lorentzian metric g, then

$$[\hat{\mathscr{E}}_2, \hat{\mathscr{E}}_1] = 0,\ [\hat{\mathscr{E}}_3, \hat{\mathscr{E}}_1] = \hat{\mathscr{E}}_1,\ [\hat{\mathscr{E}}_3, \hat{\mathscr{E}}_2] = \hat{\mathscr{E}}_2.$$

Applying Koszul's formula, we can comfortably obtain

$$\nabla_{\hat{\mathscr{E}}_i}\hat{\mathscr{E}}_j = \begin{cases} -\hat{\mathscr{E}}_3, & i = j = 1,2, \\ -\hat{\mathscr{E}}_i, & i = 1,2, j = 3, \\ 0, & otherwise \end{cases} \tag{5.1}$$

Let $U \in \chi(\mathscr{M}^3)$, then the following relations can also be verified

$$\nabla_U \zeta + U + \eta(U)\zeta = 0,\ \ (\nabla_U \phi)\mathscr{V} = -g(\phi U, \mathscr{V})\zeta - \eta(\mathscr{V})\phi(U).$$

For $U, \mathscr{V}, \mathscr{W} \in \chi(\mathscr{M}^3)$.
Equation (5.1) helps to get the following non-vanishing values:

$$\begin{cases} \mathscr{R}(\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_2)\hat{\mathscr{E}}_1 = -\hat{\mathscr{E}}_2,\ \mathscr{R}(\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_3)\hat{\mathscr{E}}_1 = -\hat{\mathscr{E}}_3,\ \mathscr{R}(\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_2)\hat{\mathscr{E}}_2 = \hat{\mathscr{E}}_1, \\ \mathscr{R}(\hat{\mathscr{E}}_2, \hat{\mathscr{E}}_3)\hat{\mathscr{E}}_2 = -\hat{\mathscr{E}}_3, \mathscr{R}(\hat{\mathscr{E}}_2, \hat{\mathscr{E}}_3)\hat{\mathscr{E}}_3 = -\hat{\mathscr{E}}_2. \end{cases}$$

The above results help to verify

$$\mathscr{R}(U,\mathscr{V})\mathscr{W} = -g(U,\mathscr{W})\mathscr{V} + g(\mathscr{V},\mathscr{W})U. \tag{5.2}$$

Hence, $\mathscr{M}^3$ is a Lorentzian para-Kenmotsu manifold of constant curvature. By contracting (5.2) over W, we obtain

$$\mathscr{S}(U,\mathscr{V}) = 2g(\mathscr{V},\mathscr{W}).$$

This implies

$$r = 6.$$

Then, (4.6) provides $\lambda = 0$. Hence, in this manifold, the Bach almost solitons are steady.

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References

[1] I. Sato, *On a structure similar to the almost contact structure*, Tensor (N. S.), **30** (1976), 219-224.
[2] S. Kaneyuki, M. Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math., **8** (1985), 81-98.
[3] S. Kaneyuki, F. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J., **99** (1985), 173-187.
[4] B.B. Sinha, K.L. Sai Prasad, *A class of almost para contact metric manifolds*, Bull. Calcutta Math. Soc., **87** (1995), 307-312.
[5] T. Q. Binh, L. Tamassy, U. C. De, M. Tarafdar, *Some remarks on almost Kenmotsu manifolds*, Maths. Pannonica, **13** (2002), 31-39.
[6] A. M. Blaga, *Conformal and paracontactly geodesic transformations of almost paracontact metric structures*, Facta Univ. Scr. Math. Inform., **35** (2020), 121-130.
[7] A. M. Blaga, M. Crasmareanu, *Statistical structures in almost paracontact geometry*, Bull. Iranian Math. Soc., **44** (2018), 1407-1413.
[8] G. Calvaruso, *Homogeneous paracontact metric three-manifolds*, Illinois J. Math., **55** (2011), 696-718.
[9] S. Dirik, M. Atceken, U. Yildirim, *Anti invariant submanifolds of a normal para contact metric manifolds*, Gulf J. Math., **10** (2021), 38-49.
[10] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Pure and Applied Mathematics, Vol. 103, Academic Press, New York, 1983.
[11] V. R. Kaigorodov, *The curvature structure of spacetime*, Pro. Geom., **14** (1983), 177-202.
[12] A. K. Raychaudhuri, S. Banerji, A. Banerjee, *General Relativity, Astrophysics and Cosmology*, Springer-Verlag, 1992.
[13] A. Haseeb, R. Prasad, *Certain results on Lorentzian para-Kenmotsu manifolds*, Bol. Soc. Paran. Mat., **39** (2021), 201-220.
[14] A. Haseeb, R. Prasad, *Some results on Lorentzian para-Kenmotsu manifolds*, Bulletin of the Transilvania University of Brasov, Series III : Mathematics, Informatics, Physics, **13(62)** (2020), 185-198.
[15] R. Bach, *Zur Weylschen relativitatstheorie and der Weylschen Erweiterung des Krummungstensorbegriffs*, Math. Z., **9** (1921), 110-135.
[16] Y. Wang, *Cotton tensors on almost co-Kähler 3-manifolds*, Ann. Polon. Math., **120** (2017), 135-148.
[17] J. Bergman, *Conformal Einstein spaces and Bach tensor generalization in n-dimensions*, Ph. D. Thesis, Linköping University, 2004.
[18] U. C. De, K. De, *On a class of three-dimensional trans-Sasakian manifolds*, Commun. Korean Math. Soc., **27** (2012), 795-808.
[19] U. C. De, G. Ghosh, J. B. Jun, P. Majhi, *Some results on para Sasakian manifolds*, Bull. Transilv. Univ. Brasov, Series III : Mathematics, Informatics, Physics, **11 (60)** (2018), 49-63.
[20] U. C. De, A. Sardar, *Classification of (k,μ)-almost co-Kähler manifolds with vanishing Bach tensor and divergence free cotton tensor*, Commun. Korean Math. Soc., **35** (2020), 1245-1254.
[21] H.I. Yoldas, *Notes on η-Einstein solitons on para-Kenmotsu manifolds*, Math. Method Appl. Sci., **46** (2023), 17632-17640.
[22] H. Fu, J. Peng, *Rigidity theorems for compact Bach-flat manifolds with positive constant scalar curvature*, Hokkaido Math. J., **47** (2018), 581-605.
[23] A. Ghosh, R. Sharma, *Sasakian manifolds with purely transversal Bach tensor*, J. Math. Phys., **58** (2017), 103502, 6 pp.
[24] P. Szekeres, *Conformal tensors*, Proc. R. Soc. Lond. Ser. A-Contain. Pap. Math. Phys., **304** (1968), 113-122.
[25] S. Das, S. Kar, *Bach flows of product manifolds*, Int. J. Geom. Methods Mod. Phys., **9** (2012), 1250039, 18 pp.
[26] I. Bakas, F. Bourliot, D. Lust, M. Petropoulos, *Geometric flows in Horava-Lifshitz gravity*, J. High Energy Phys., **2010** (2010), 1-58.
[27] E. Bahuaud, D. Helliwell, *Short-time existence for some higher-order geometric flows*, Commun. Partial Differ. Equ., **36** (2011), 2189-2207.
[28] H.-D. Cao, Q. Chen, *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J., **162** (2013), 1149-1169.
[29] P. T. Ho, *Bach flow*, J. Geom. Phys., **133** (2018), 1-9.
[30] D. Helliwell, *Bach flow on homogeneous products*, SIGMA Symmetry Integrability Geom. Methods Appl., **16** (2020), 35 pp.
[31] A. Ghosh, *On Bach almost solitons*, Beitr. Algebra Geom., **63** (2022), 45-54.
[32] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Nat. Sci., **12** (1989), 151-156.
[33] R. Prasad, A. Haseeb, A. Verma, V. S. Yadav, *A study of φ-Ricci symmetric LP-Kenmotsu manifolds*, Int. J. Maps Math., **7** (2024), 33-44.
[34] Pankaj, S. K. Chaubey, R. Prasad, *Three dimensional Lorentzian para-Kenmotsu manifolds and Yamabe soliton*, Honam Math. J., **43** (2021), 613-626.
[35] D. E. Blair, *Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics*, Springer-Verlag, 509, 1976.
[36] Li, Y., Haseeb, A., Ali, M., *LP-Kenmotsu manifolds admitting η-Ricci solitons and spacetime*, J. Math., **2022**, Article ID 6605127, 10 pages.

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Dynamical Analysis and Solutions of Nonlinear Difference Equations of Thirty Order

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Abstract

Discrete-time systems are sometimes used to explain natural phenomena that happen in nonlinear sciences. We study the periodicity, boundedness, oscillation, stability, and certain exact solutions of nonlinear difference equations in this paper. Using the standard iteration method, exact solutions are obtained. Some well-known theorems are used to test the stability of the equilibrium points. Some numerical examples are also provided to confirm the theoretical work's validity. The numerical component is implemented with Wolfram Mathematica. The method presented may be simply applied to other rational recursive issues.
In this paper, we explore the dynamics of adhering to a rational difference formula

$$x_{n+1} = \frac{x_{n-29}}{\pm 1 \pm x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}},$$

where the initials are arbitrary nonzero real numbers.

1. Introduction

A particular natural phenomenon's evolution is frequently explained over a period of time employing differential equations. Nevertheless, in certain instances, numerous real-life issues can be modeled using discrete time intervals, resulting in difference equations. As a result, recursive equations play an influential and potent role in mathematics. They are effectively employed to explore various applications in engineering, physics, biology, economics, and other fields [1–5]. For example, recursive equations have been effectively employed in modeling various natural phenomena, including population size, the Fibonacci sequence, drug concentrations in the bloodstream, information transmission, pricing dynamics of certain commodities, propagation patterns of annual plants, and more [6–12]. Additionally, certain scholars have utilized difference equations to obtain numerical solutions for certain differential equations. In particular, discretizing a given differential equation produces a corresponding difference equation. For example, the Runge-Kutta scheme arises from discretizing a first-order differential equation. This prompts consideration regarding the convergence of the difference scheme to the solution of a differential equation. The study discussed in reference [13] is dedicated to investigating the preservation of a solution bounded on the entire axis during the transition from differential to difference equations and vice versa. In reference [14], analogous inquiries were undertaken to maintain the oscillatory nature of solutions to second-order equations. Advancements in technology have spurred the utilization of recurrence equations as approximations to partial differential equations. It's noteworthy that fractional-order difference equations are frequently employed to study certain real-life phenomena that arise in nonlinear sciences. Almatrafi et. al. in [15] aim to analyzed the asymptotic stability, global stability, periodicity of the solution of an eighth-order difference equation. Sanbo et. al. in [16], discussed the periodicity, stability, and some solutions of a fifth-order recursive equation. Yeniçerioğlu et. al. in [17], examined the behavior of solutions of the neutral functional differential equations. Using a suitable real root of the corresponding characteristic equation, they explained the asymptotic behavior of the solutions and the stability of the trivial solution. Ahmed et al. [18] discovered new solutions and conducted a dynamical analysis for certain nonlinear difference relations of fifteenth order. Berkal et. al. in [19], have derived the forbidden set and

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determined the solutions of the difference equation that contains a quadratic term. Oğul et. al. in [20], examined soluttions of the sixth-order difference equations.
The inspiration behind this article stems from the exploration of eighteenth-order difference equations outlined in [21]. As such, the objective of this study is to analyze various dynamical properties including equilibrium points, local and global behaviors, boundedness, and analytic solutions of the nonlinear recursive equations (1.1).

$$x_{n+1}=\frac{x_{n-29}}{\pm 1\pm x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}}. \tag{1.1}$$

Here, the initial values $x_{-29},x_{-28},x_{-27},\dots,x_{-2},x_{-1},x_0$, are arbitrary non-zero real numbers. In this work, we also illustrate some 2D figures with the help of Wolfram Mathematica to validate the obtained results.
In this study, stability, periodicity and global asymptotic stability definitions and theorems in the [1] source were used.

2. Solution of the Difference Equation $x_{n+1}=\frac{x_{n-29}}{1+x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}}$

In this section, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers.

$$x_{n+1}=\frac{x_{n-29}}{1+x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}}, \tag{2.1}$$

where,

$$\begin{array}{llllllll}
x_{-29}=A_{30}, & x_{-28}=A_{29}, & x_{-27}=A_{28}, & x_{-26}=A_{27}, & x_{-25}=A_{26}, & x_{-24}=A_{25}, & x_{-23}=A_{24}, & x_{-22}=A_{23},\\
x_{-21}=A_{22}, & x_{-20}=A_{21}, & x_{-19}=A_{20}, & x_{-18}=A_{19}, & x_{-17}=A_{18}, & x_{-16}=A_{17}, & x_{-15}=A_{16}, & x_{-14}=A_{15},\\
x_{-13}=A_{14}, & x_{-12}=A_{13}, & x_{-11}=A_{12}, & x_{-10}=A_{1}1, & x_{-9}=A_{10}, & x_{-8}=A_{9}, & x_{-7}=A_{8}, & x_{-6}=A_{7},\\
x_{-5}=A_{6}, & x_{-4}=A_{5}, & x_{-3}=A_{4}, & x_{-2}=A_{3}, & x_{-1}=A_{2}, & x_{0}=A_{1}. & &
\end{array} \tag{2.2}$$

Theorem 2.1. *Let $\{x_n\}_{n=-29}^{\infty}$ be a solution of (2.1). Then,*

$$x_{30n+1}=\frac{A_{30}\prod_{i=0}^{n}(1+5iA_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1+(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}, \qquad x_{30n+2}=\frac{A_{29}\prod_{i=0}^{n}(1+5iA_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+3}=\frac{A_{28}\prod_{i=0}^{n}(1+5iA_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1+(5i+1)A_4A_{10}A_{16}A_{22}A_{28})}, \qquad x_{30n+4}=\frac{A_{27}\prod_{i=0}^{n}(1+5iA_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1+(5i+1)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+5}=\frac{A_{26}\prod_{i=0}^{n}(1+5iA_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1+(5i+1)A_2A_8A_{14}A_{20}A_{26})}, \qquad x_{30n+6}=\frac{A_{25}\prod_{i=0}^{n}(1+5iA_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1+(5i+1)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n+7}=\frac{A_{24}\prod_{i=0}^{n}(1+(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1+(5i+2)A_6A_{12}A_{18}A_{24}A_{30})}, \qquad x_{30n+8}=\frac{A_{23}\prod_{i=0}^{n}(1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1+(5i+2)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+9}=\frac{A_{22}\prod_{i=0}^{n}(1+(5i+1)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1+(5i+2)A_4A_{10}A_{16}A_{22}A_{28})}, \qquad x_{30n+10}=\frac{A_{21}\prod_{i=0}^{n}(1+(5i+1)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1+(5i+2)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+11}=\frac{A_{20}\prod_{i=0}^{n}(1+(5i+1)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1+(5i+2)A_2A_8A_{14}A_{20}A_{26})}, \qquad x_{30n+12}=\frac{A_{19}\prod_{i=0}^{n}(1+(5i+1)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1+(5i+2)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n+13}=\frac{A_{18}\prod_{i=0}^{n}(1+(5i+2)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1+(5i+3)A_6A_{12}A_{18}A_{24}A_{30})}, \qquad x_{30n+14}=\frac{A_{17}\prod_{i=0}^{n}(1+(5i+2)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1+(5i+3)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+15}=\frac{A_{16}\prod_{i=0}^{n}(1+(5i+2)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1+(5i+3)A_4A_{10}A_{16}A_{22}A_{28})}, \qquad x_{30n+16}=\frac{A_{15}\prod_{i=0}^{n}(1+(5i+2)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1+(5i+3)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+17}=\frac{A_{14}\prod_{i=0}^{n}(1+(5i+2)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1+(5i+3)A_2A_8A_{14}A_{20}A_{26})}, \qquad x_{30n+18}=\frac{A_{13}\prod_{i=0}^{n}(1+(5i+2)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1+(5i+3)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n+19}=\frac{A_{12}\prod_{i=0}^{n}(1+(5i+3)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1+(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}, \qquad x_{30n+20}=\frac{A_{11}\prod_{i=0}^{n}(1+(5i+3)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1+(5i+4)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+21}=\frac{A_{10}\prod_{i=0}^{n}(1+(5i+3)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1+(5i+4)A_4A_{10}A_{16}A_{22}A_{28})}, \qquad x_{30n+22}=\frac{A_{9}\prod_{i=0}^{n}(1+(5i+3)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1+(5i+4)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+23}=\frac{A_{8}\prod_{i=0}^{n}(1+(5i+3)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1+(5i+4)A_2A_8A_{14}A_{20}A_{26})}, \qquad x_{30n+24}=\frac{A_{7}\prod_{i=0}^{n}(1+(5i+3)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1+(5i+4)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n+25}=\frac{A_{6}\prod_{i=0}^{n}(1+(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1+(5i+5)A_6A_{12}A_{18}A_{24}A_{30})}, \qquad x_{30n+26}=\frac{A_{5}\prod_{i=0}^{n}(1+(5i+4)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1+(5i+5)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+27}=\frac{A_{4}\prod_{i=0}^{n}(1+(5i+4)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1+(5i+5)A_4A_{10}A_{16}A_{22}A_{28})}, \qquad x_{30n+28}=\frac{A_{3}\prod_{i=0}^{n}(1+(5i+4)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1+(5i+5)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+29}=\frac{A_{2}\prod_{i=0}^{n}(1+(5i+4)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1+(5i+5)A_2A_8A_{14}A_{20}A_{26})}, \qquad x_{30n+30}=\frac{A_{1}\prod_{i=0}^{n}(1+(5i+4)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1+(5i+5)A_1A_7A_{13}A_{19}A_{25})},$$

where, $x_0,\dots,x_{-29}$ defines as in (2.2).

Proof of Theorem 2.1. The proof of each formula are carried out in similar way. So, we will demonstrate proof using one of the formula. We will employ the mathematical induction method. Let's posit that, with n being greater than zero and supposing our assumption is true for $n=1$. That is,

$$x_{30n-29}=\frac{A_{30}\prod_{i=0}^{n-1}(1+5iA_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+1)A_6A_{12}A_{18}A_{24}A_{30})},\qquad x_{30n-28}=\frac{A_{29}\prod_{i=0}^{n-1}(1+5iA_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-27}=\frac{A_{28}\prod_{i=0}^{n-1}(1+5iA_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1+(5i+1)A_4A_{10}A_{16}A_{22}A_{28})},\qquad x_{30n-26}=\frac{A_{27}\prod_{i=0}^{n-1}(1+5iA_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1+(5i+1)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n-25}=\frac{A_{26}\prod_{i=0}^{n-1}(1+5iA_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1+(5i+1)A_2A_8A_{14}A_{20}A_{26})},\qquad x_{30n-24}=\frac{A_{25}\prod_{i=0}^{n-1}(1+5iA_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1+(5i+1)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n-23}=\frac{A_{24}\prod_{i=0}^{n-1}(1+(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+2)A_6A_{12}A_{18}A_{24}A_{30})},\qquad x_{30n-22}=\frac{A_{23}\prod_{i=0}^{n-1}(1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1+(5i+2)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-21}=\frac{A_{22}\prod_{i=0}^{n-1}(1+(5i+1)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1+(5i+2)A_4A_{10}A_{16}A_{22}A_{28})},\qquad x_{30n-20}=\frac{A_{21}\prod_{i=0}^{n-1}(1+(5i+1)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1+(5i+2)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n-19}=\frac{A_{20}\prod_{i=0}^{n-1}(1+(5i+1)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1+(5i+2)A_2A_8A_{14}A_{20}A_{26})},\qquad x_{30n-18}=\frac{A_{19}\prod_{i=0}^{n-1}(1+(5i+1)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1+(5i+2)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n-17}=\frac{A_{18}\prod_{i=0}^{n-1}(1+(5i+2)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+3)A_6A_{12}A_{18}A_{24}A_{30})},\qquad x_{30n-16}=\frac{A_{17}\prod_{i=0}^{n-1}(1+(5i+2)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1+(5i+3)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-15}=\frac{A_{16}\prod_{i=0}^{n-1}(1+(5i+2)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1+(5i+3)A_4A_{10}A_{16}A_{22}A_{28})},\qquad x_{30n-14}=\frac{A_{15}\prod_{i=0}^{n-1}(1+(5i+2)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1+(5i+3)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n-13}=\frac{A_{14}\prod_{i=0}^{n-1}(1+(5i+2)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1+(5i+3)A_2A_8A_{14}A_{20}A_{26})},\qquad x_{30n-12}=\frac{A_{13}\prod_{i=0}^{n-1}(1+(5i+2)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1+(5i+3)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n-11}=\frac{A_{12}\prod_{i=0}^{n-1}(1+(5i+3)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+4)A_6A_{12}A_{18}A_{24}A_{30})},\qquad x_{30n-10}=\frac{A_{11}\prod_{i=0}^{n-1}(1+(5i+3)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1+(5i+4)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-9}=\frac{A_{10}\prod_{i=0}^{n-1}(1+(5i+3)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1+(5i+4)A_4A_{10}A_{16}A_{22}A_{28})},\qquad x_{30n-8}=\frac{A_9\prod_{i=0}^{n-1}(1+(5i+3)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1+(5i+4)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n-7}=\frac{A_8\prod_{i=0}^{n-1}(1+(5i+3)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1+(5i+4)A_2A_8A_{14}A_{20}A_{26})},\qquad x_{30n-6}=\frac{A_7\prod_{i=0}^{n-1}(1+(5i+3)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1+(5i+4)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n-5}=\frac{A_6\prod_{i=0}^{n-1}(1+(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+5)A_6A_{12}A_{18}A_{24}A_{30})},\qquad x_{30n-4}=\frac{A_5\prod_{i=0}^{n-1}(1+(5i+4)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1+(5i+5)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-3}=\frac{A_4\prod_{i=0}^{n-1}(1+(5i+4)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1+(5i+5)A_4A_{10}A_{16}A_{22}A_{28})},\qquad x_{30n-2}=\frac{A_3\prod_{i=0}^{n-1}(1+(5i+4)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1+(5i+5)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n-1}=\frac{A_2\prod_{i=0}^{n-1}(1+(5i+4)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1+(5i+5)A_2A_8A_{14}A_{20}A_{26})},\qquad x_{30n}=\frac{A_1\prod_{i=0}^{n-1}(1+(5i+4)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1+(5i+5)A_1A_7A_{13}A_{19}A_{25})}.$$

Now, using the main (2.1), one has

$$\begin{aligned}
x_{30n+1}&=\frac{x_{30n-29}}{1+x_{30n-5}x_{30n-11}x_{30n-17}x_{30n-23}x_{30n-29}}\\
&=\frac{\frac{A_{30}\prod_{i=0}^{n-1}(1+5iA_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}}{1+A_6A_{12}A_{18}A_{24}A_{30}\frac{\prod_{i=0}^{n-1}(1+5iA_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}\frac{\prod_{i=0}^{n-1}(1+(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+2)A_6A_{12}A_{18}A_{24}A_{30})}\frac{\prod_{i=0}^{n-1}(1+(5i+2)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+3)A_6A_{12}A_{18}A_{24}A_{30})}\frac{\prod_{i=0}^{n-1}(1+(5i+3)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}\frac{\prod_{i=0}^{n-1}(1+(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1+(5i+5)A_6A_{12}A_{18}A_{24}A_{30})}}\\
&=\frac{A_{30}\prod_{i=0}^{n-1}\frac{(1+5iiA_6A_{12}A_{18}A_{24}A_{30})}{(1+(5i+1)iA_6A_{12}A_{18}A_{24}A_{30})}}{1+5iA_6A_{12}A_{18}A_{24}A_{30}\prod_{i=0}^{n-1}\frac{1+5iA_6A_{12}A_{18}A_{24}A_{30}}{1+(5i+5)A_6A_{12}A_{18}A_{24}A_{30}}}=A_{30}\prod_{i=0}^{n-1}\frac{1+5iA_6A_{12}A_{18}A_{24}A_{30}}{1+(5i+1)iA_6A_{12}A_{18}A_{24}A_{30}}\left(\frac{1}{1+\frac{A_6A_{12}A_{18}A_{24}A_{30}}{1+(5i-5)A_6A_{12}A_{18}A_{24}A_{30}}}\right)\\
&=A_{30}\prod_{i=0}^{n-1}\frac{1+5iA_6A_{12}A_{18}A_{24}A_{30}}{1+(5i+1)A_6A_{12}A_{18}A_{24}A_{30}}\left(\frac{1+(5i-5)A_6A_{12}A_{18}A_{24}A_{30}}{1+(5i-4)A_6A_{12}A_{18}A_{24}A_{30}}\right).
\end{aligned}$$

Hence, we have

$$x_{30n+1}=A_{30}\prod_{i=0}^{n}\frac{1+5iA_6A_{12}A_{18}A_{24}A_{30}}{1+(5i+1)A_6A_{12}A_{18}A_{24}A_{30}}.$$

Similarly,

$$
\begin{aligned}
x_{30n+2} &= \frac{x_{30n-28}}{1+x_{30n-4}x_{30n-10}x_{30n-16}x_{30n-22}x_{30n-28}} \\
&= \frac{\frac{A_{29}\prod_{i=0}^{n-1}(1+5iA_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}}{1+\frac{A_5\prod_{i=0}^{n-1}(1+(5i+4)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1+(5i+5)A_5A_{11}A_{17}A_{23}A_{29})}\frac{A_{11}\prod_{i=0}^{n-1}(1+(5i+3)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1+(5i+4)A_5A_{11}A_{17}A_{23}A_{29})}\frac{A_{17}\prod_{i=0}^{n-1}(1+(5i+2)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1+(5i+3)A_5A_{11}A_{17}A_{23}A_{29})}\frac{A_{23}\prod_{i=0}^{n-1}(1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1+(5i+2)A_5A_{11}A_{17}A_{23}A_{29})}\frac{A_{29}\prod_{i=0}^{n-1}(1+5iA_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}} \\
&= \frac{A_{29}\prod_{i=0}^{n-1}\frac{(1+5iA_5A_{11}A_{17}A_{23}A_{29})}{(1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}}{1+5iA_5A_{11}A_{17}A_{23}A_{29}\prod_{i=0}^{n-1}\frac{1+5iA_5A_{11}A_{17}A_{23}A_{29}}{1+(5i+5)A_5A_{11}A_{17}A_{23}A_{29}}} = A_{29}\prod_{i=0}^{n-1}\frac{1+5iA_5A_{11}A_{17}A_{23}A_{29}}{1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29}}\left(\frac{1}{1+\frac{5iA_5A_{11}A_{17}A_{23}A_{29}}{1+(5i-5)A_5A_{11}A_{17}A_{23}A_{29}}}\right) \\
&= A_{29}\prod_{i=0}^{n-1}\frac{1+5iA_5A_{11}A_{17}A_{23}A_{29}}{1+(5i+1)A_5A_{11}A_{17}A_{23}A_{29}}\left(\frac{1+(5i-5)A_5A_{11}A_{17}A_{23}A_{29}}{1+(5i-4)A_5A_{11}A_{17}A_{23}A_{29}}\right).
\end{aligned}
$$

Therefore, we have

$$
x_{30n+2} = A_{29}\prod_{i=0}^{n}\frac{1+5iA_5A_{11}A_{17}A_{23}A_{29}}{1+(5i+1)iA_5A_{11}A_{17}A_{23}A_{29}}.
$$

Additional relationships can be acquired in the same way, thereby completing the proof. □

Theorem 2.2. *The equation (2.1) has a unique equilibrium point which is the number zero and this equilibrium is not locally asymptotically stable. Also $\bar{x}$ is non hyperbolic.*

Proof of Theorem 2.2. For the equilibriums of equation (2.1), we have

$$
\bar{x} = \frac{\bar{x}}{1+\bar{x}^5},
$$

then

$$
\bar{x}+\bar{x}^6 = \bar{x}, \quad \bar{x}^6 = 0.
$$

In consequence, the equilibrium point of (2.1), is $\bar{x} = 0$.
Consider $f : (0,\infty)^5 \to (0,\infty)$ as the function defined by

$$
f(\xi,\nu,\rho,\chi,\kappa) = \frac{\xi}{1+\xi\nu\rho\chi\kappa}.
$$

Therefore, it is deduced that,

$$
\begin{aligned}
&f_\xi(\xi,\nu,\rho,\chi,\kappa) = \frac{1}{(1+\xi\nu\rho\chi\kappa)^2}, \qquad f_\nu(\xi,\nu,\rho,\chi,\kappa) = \frac{-\xi^2\rho\chi\alpha}{(1+\xi\nu\rho\chi\kappa)^2}, \qquad f_\rho(\xi,\nu,\rho,\chi,\kappa) = \frac{-\xi^2\nu\chi\kappa}{(1+\xi\nu\rho\chi\kappa)^2}, \\
&f_\chi(\xi,\nu,\rho,\chi,\kappa) = \frac{-\xi^2\nu\rho\kappa}{(1+\xi\nu\rho\chi\kappa)^2}, \qquad f_\kappa(\xi,\nu,\rho,\chi,\kappa) = \frac{-\xi^2\nu\chi\rho}{(1+\xi\nu\rho\chi\kappa)^2}.
\end{aligned}
$$

We see that,

$$
f_\xi(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 1, \qquad f_\nu(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0, \qquad f_\rho(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0, \qquad f_\chi(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0, \qquad f_\kappa(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0.
$$

The proof now follows by using Theorem 2.1. □

3. Solution of the Difference Equation $x_{n+1} = \frac{x_{n-29}}{1-x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}}$

In this part, we furnish a specific pattern for the solutions of the difference equation given, assuming that the initial conditions are arbitrary real numbers, where, $x_0,\dots,x_{-29}$ defines as in (2.2)

$$
x_{n+1} = \frac{x_{n-29}}{1-x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}}. \tag{3.1}
$$

Theorem 3.1. *Let's $\{x_n\}_{n=-29}^{\infty}$ be a solution of equation (3.1). Accordingly,*

$$x_{30n+1}=\frac{A_{30}\prod_{i=0}^{n}(1-5iA_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1-(5i+1)A_6A_{12}A_{18}A_{24}A_{30})},\quad x_{30n+2}=\frac{A_{29}\prod_{i=0}^{n}(1-5iA_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+3}=\frac{A_{28}\prod_{i=0}^{n}(1-5iA_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1+(5i+1)A_4A_{10}A_{16}A_{22}A_{28})},\quad x_{30n+4}=\frac{A_{27}\prod_{i=0}^{n}(1-5iA_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1+(5i+1)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+5}=\frac{A_{26}\prod_{i=0}^{n}(1-5iA_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1-(5i+1)A_2A_8A_{14}A_{20}A_{26})},\quad x_{30n+6}=\frac{A_{25}\prod_{i=0}^{n}(1-5iA_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1-(5i+1)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n+7}=\frac{A_{24}\prod_{i=0}^{n}(1-(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1-(5i+2)A_6A_{12}A_{18}A_{24}A_{30})},\quad x_{30n+8}=\frac{A_{23}\prod_{i=0}^{n}(1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1-(5i+2)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+9}=\frac{A_{22}\prod_{i=0}^{n}(1-(5i+1)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1-(5i+2)A_4A_{10}A_{16}A_{22}A_{28})},\quad x_{30n+10}=\frac{A_{21}\prod_{i=0}^{n}(1-(5i+1)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1-(5i+2)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+11}=\frac{A_{20}\prod_{i=0}^{n}(1-(5i+1)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1-(5i+2)A_2A_8A_{14}A_{20}A_{26})},\quad x_{30n+12}=\frac{A_{19}\prod_{i=0}^{n}(1-(5i+1)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1-(5i+2)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n+13}=\frac{A_{18}\prod_{i=0}^{n}(1-(5i+2)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1-(5i+3)A_6A_{12}A_{18}A_{24}A_{30})},\quad x_{30n+14}=\frac{A_{17}\prod_{i=0}^{n}(1-(5i+2)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1-(5i+3)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+15}=\frac{A_{16}\prod_{i=0}^{n}(1-(5i+2)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1-(5i+3)A_4A_{10}A_{16}A_{22}A_{28})},\quad x_{30n+16}=\frac{A_{15}\prod_{i=0}^{n}(1-(5i+2)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1-(5i+3)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+17}=\frac{A_{14}\prod_{i=0}^{n}(1-(5i+2)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1-(5i+3)A_2A_8A_{14}A_{20}A_{26})},\quad x_{30n+18}=\frac{A_{13}\prod_{i=0}^{n}(1-(5i+2)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1-(5i+3)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n+19}=\frac{A_{12}\prod_{i=0}^{n}(1-(5i+3)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1-(5i+4)A_6A_{12}A_{18}A_{24}A_{30})},\quad x_{30n+20}=\frac{A_{11}\prod_{i=0}^{n}(1-(5i+3)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1-(5i+4)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+21}=\frac{A_{10}\prod_{i=0}^{n}(1-(5i+3)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1-(5i+4)A_4A_{10}A_{16}A_{22}A_{28})},\quad x_{30n+22}=\frac{A_9\prod_{i=0}^{n}(1-(5i+3)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1-(5i+4)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+23}=\frac{A_8\prod_{i=0}^{n}(1-(5i+3)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1-(5i+4)A_2A_8A_{14}A_{20}A_{26})},\quad x_{30n+24}=\frac{A_7\prod_{i=0}^{n}(1-(5i+3)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1-(5i+4)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n+25}=\frac{A_6\prod_{i=0}^{n}(1-(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1-(5i+5)A_6A_{12}A_{18}A_{24}A_{30})},\quad x_{30n+26}=\frac{A_5\prod_{i=0}^{n}(1-(5i+4)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1-(5i+5)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n+27}=\frac{A_4\prod_{i=0}^{n}(1-(5i+4)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n}(1-(5i+5)A_4A_{10}A_{16}A_{22}A_{28})},\quad x_{30n+28}=\frac{A_3\prod_{i=0}^{n}(1-(5i+4)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n}(1-(5i+5)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n+29}=\frac{A_2\prod_{i=0}^{n}(1-(5i+4)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n}(1-(5i+5)A_2A_8A_{14}A_{20}A_{26})},\quad x_{30n+30}=\frac{A_1\prod_{i=0}^{n}(1-(5i+4)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n}(1-(5i+5)A_1A_7A_{13}A_{19}A_{25})},$$

holds.

Proof of Theorem 3.1. Let's suppose that n is greater than 0, and our assumption remains valid for n=1. That is,

$$x_{30n-29}=\frac{A_{30}\prod_{i=0}^{n-1}(1-5iA_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+1)A_6A_{12}A_{18}A_{24}A_{30})},\quad x_{30n-28}=\frac{A_{29}\prod_{i=0}^{n-1}(1-5iA_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-27}=\frac{A_{28}\prod_{i=0}^{n-1}(1-5iA_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1-(5i+1)A_4A_{10}A_{16}A_{22}A_{28})},\quad x_{30n-26}=\frac{A_{27}\prod_{i=0}^{n-1}(1-5iA_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1-(5i+1)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n-25}=\frac{A_{26}\prod_{i=0}^{n-1}(1-5iA_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1-(5i+1)A_2A_8A_{14}A_{20}A_{26})},\quad x_{30n-24}=\frac{A_{25}\prod_{i=0}^{n-1}(1-5iA_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1-(5i+1)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n-23}=\frac{A_{24}\prod_{i=0}^{n-1}(1-(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+2)A_6A_{12}A_{18}A_{24}A_{30})},\quad x_{30n-22}=\frac{A_{23}\prod_{i=0}^{n-1}(1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1-(5i+2)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-21}=\frac{A_{22}\prod_{i=0}^{n-1}(1-(5i+1)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1-(5i+2)A_4A_{10}A_{16}A_{22}A_{28})},\quad x_{30n-20}=\frac{A_{21}\prod_{i=0}^{n-1}(1-(5i+1)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1-(5i+2)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n-19}=\frac{A_{20}\prod_{i=0}^{n-1}(1-(5i+1)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1-(5i+2)A_2A_8A_{14}A_{20}A_{26})},\quad x_{30n-18}=\frac{A_{19}\prod_{i=0}^{n-1}(1-(5i+1)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1-(5i+2)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n-17}=\frac{A_{18}\prod_{i=0}^{n-1}(1-(5i+2)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+3)A_6A_{12}A_{18}A_{24}A_{30})},\quad x_{30n-16}=\frac{A_{17}\prod_{i=0}^{n-1}(1-(5i+2)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1-(5i+3)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-15}=\frac{A_{16}\prod_{i=0}^{n-1}(1-(5i+2)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1-(5i+3)A_4A_{10}A_{16}A_{22}A_{28})},\quad x_{30n-14}=\frac{A_{15}\prod_{i=0}^{n-1}(1-(5i+2)A_3A_9A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1-(5i+3)A_3A_9A_{15}A_{21}A_{27})},$$

$$x_{30n-13}=\frac{A_{14}\prod_{i=0}^{n-1}(1-(5i+2)A_2A_8A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1-(5i+3)A_2A_8A_{14}A_{20}A_{26})},\quad x_{30n-12}=\frac{A_{13}\prod_{i=0}^{n-1}(1-(5i+2)A_1A_7A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1-(5i+3)A_1A_7A_{13}A_{19}A_{25})},$$

$$x_{30n-11} = \frac{A_{12}\prod_{i=0}^{n-1}(1-(5i+3)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}, \qquad x_{30n-10} = \frac{A_{11}\prod_{i=0}^{n-1}(1-(5i+3)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1-(5i+4)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-9} = \frac{A_{10}\prod_{i=0}^{n-1}(1-(5i+3)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1-(5i+4)A_4A_{10}A_{16}A_{22}A_{28})}, \qquad x_{30n-8} = \frac{A_{9}\prod_{i=0}^{n-1}(1-(5i+3)A_3A_{9}A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1-(5i+4)A_3A_{9}A_{15}A_{21}A_{27})},$$

$$x_{30n-7} = \frac{A_{8}\prod_{i=0}^{n-1}(1-(5i+3)A_2A_{8}A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1-(5i+4)A_2A_{8}A_{14}A_{20}A_{26})}, \qquad x_{30n-6} = \frac{A_{7}\prod_{i=0}^{n-1}(1-(5i+3)A_1A_{7}A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1-(5i+4)A_1A_{7}A_{13}A_{19}A_{25})},$$

$$x_{30n-5} = \frac{A_{6}\prod_{i=0}^{n-1}(1-(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+5)A_6A_{12}A_{18}A_{24}A_{30})}, \qquad x_{30n-4} = \frac{A_{5}\prod_{i=0}^{n-1}(1-(5i+4)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1-(5i+5)A_5A_{11}A_{17}A_{23}A_{29})},$$

$$x_{30n-3} = \frac{A_{4}\prod_{i=0}^{n-1}(1-(5i+4)A_4A_{10}A_{16}A_{22}A_{28})}{\prod_{i=0}^{n-1}(1-(5i+5)A_4A_{10}A_{16}A_{22}A_{28})}, \qquad x_{30n-2} = \frac{A_{3}\prod_{i=0}^{n-1}(1-(5i+4)A_3A_{9}A_{15}A_{21}A_{27})}{\prod_{i=0}^{n-1}(1-(5i+5)A_3A_{9}A_{15}A_{21}A_{27})},$$

$$x_{30n-1} = \frac{A_{2}\prod_{i=0}^{n-1}(1-(5i+4)A_2A_{8}A_{14}A_{20}A_{26})}{\prod_{i=0}^{n-1}(1-(5i+5)A_2A_{8}A_{14}A_{20}A_{26})}, \qquad x_{30n} = \frac{A_{1}\prod_{i=0}^{n-1}(1-(5i+4)A_1A_{7}A_{13}A_{19}A_{25})}{\prod_{i=0}^{n-1}(1-(5i+5)A_1A_{7}A_{13}A_{19}A_{25})}.$$

Now, using the main equation (3.1), one has

$$\begin{aligned}
x_{30n+1} &= \frac{x_{30n-29}}{1-x_{30n-5}x_{30n-11}x_{30n-17}x_{30n-23}x_{30n-29}} \\
&= \frac{\frac{A_{30}\prod_{i=0}^{n-1}(1+5iA_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}}{1+A_6A_{12}A_{18}A_{24}A_{30}\frac{\prod_{i=0}^{n-1}(1-5iA_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}\frac{\prod_{i=0}^{n-1}(1-(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+2)A_6A_{12}A_{18}A_{24}A_{30})}\frac{\prod_{i=0}^{n-1}(1-(5i+2)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+3)A_6A_{12}A_{18}A_{24}A_{30})}\frac{\prod_{i=0}^{n-1}(1-(5i+3)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}\frac{\prod_{i=0}^{n-1}(1-(5i+4)A_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n-1}(1-(5i+5)A_6A_{12}A_{18}A_{24}A_{30})}} \\
&= \frac{A_{30}\prod_{i=0}^{n-1}\frac{(1-5iA_6A_{12}A_{18}A_{24}A_{30})}{(1-(5i+1)iA_6A_{12}A_{18}A_{24}A_{30})}}{1+iA_6A_{12}A_{18}A_{24}A_{30}\prod_{i=0}^{n-1}\frac{1-5iA_6A_{12}A_{18}A_{24}A_{30}}{1-(5i+5)A_6A_{12}A_{18}A_{24}A_{30}}} = A_{30}\prod_{i=0}^{n-1}\frac{1-5iA_6A_{12}A_{18}A_{24}A_{30}}{1-(5i+1)iA_6A_{12}A_{18}A_{24}A_{30}}\left(\frac{1}{1-\frac{A_6A_{12}A_{18}A_{24}A_{30}}{1-(5i-5)A_6A_{12}A_{18}A_{24}A_{30}}}\right) \\
&= A_{30}\prod_{i=0}^{n-1}\frac{1-5iA_6A_{12}A_{18}A_{24}A_{30}}{1-(5i+1)A_6A_{12}A_{18}A_{24}A_{30}}\left(\frac{1-(5i-5)A_6A_{12}A_{18}A_{24}A_{30}}{1-(5i-4)A_6A_{12}A_{18}A_{24}A_{30}}\right).
\end{aligned}$$

Hence, we have

$$x_{30n+1} = \frac{A_{30}\prod_{i=0}^{n}(1-5iA_6A_{12}A_{18}A_{24}A_{30})}{\prod_{i=0}^{n}(1-(5i+1)A_6A_{12}A_{18}A_{24}A_{30})}.$$

Similarly,

$$\begin{aligned}
x_{30n+2} &= \frac{x_{30n-28}}{1-x_{30n-4}x_{30n-10}x_{30n-16}x_{30n-22}x_{30n-28}} \\
&= \frac{\frac{A_{29}\prod_{i=0}^{n-1}(1-5iA_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}}{1+\frac{A_5\prod_{i=0}^{n-1}(1-(5i+4)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1-(5i+5)A_5A_{11}A_{17}A_{23}A_{29})}\frac{A_{11}\prod_{i=0}^{n-1}(1-(5i+3)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1-(5i+4)A_5A_{11}A_{17}A_{23}A_{29})}\frac{A_{17}\prod_{i=0}^{n-1}(1-(5i+2)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1-(5i+3)A_5A_{11}A_{17}A_{23}A_{29})}\frac{A_{23}\prod_{i=0}^{n-1}(1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n-1}(1-(5i+2)A_5A_{11}A_{17}A_{23}A_{29})}\frac{A_{29}\prod_{i=0}^{n-1}(1-5iA_5A_{11}A_{17}A_{23}A_{29})}{\prod_{i=0}^{n}(1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}} \\
&= \frac{A_{29}\prod_{i=0}^{n-1}\frac{(1-5iA_5A_{11}A_{17}A_{23}A_{29})}{(1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29})}}{1+5iA_5A_{11}A_{17}A_{23}A_{29}\prod_{i=0}^{n-1}\frac{1-5iA_5A_{11}A_{17}A_{23}A_{29}}{1-(5i+5)A_5A_{11}A_{17}A_{23}A_{29}}} = A_{29}\prod_{i=0}^{n-1}\frac{1-5iA_5A_{11}A_{17}A_{23}A_{29}}{1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29}}\left(\frac{1}{1-\frac{5iA_5A_{11}A_{17}A_{23}A_{29}}{1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29}}}\right) \\
&= A_{29}\prod_{i=0}^{n-1}\frac{1-5iA_5A_{11}A_{17}A_{23}A_{29}}{1-(5i+1)A_5A_{11}A_{17}A_{23}A_{29}}\left(\frac{1-(5i-5)A_5A_{11}A_{17}A_{23}A_{29}}{1-(5i-4)A_5A_{11}A_{17}A_{23}A_{29}}\right).
\end{aligned}$$

Therefore, we have

$$x_{30n+2} = A_{29}\prod_{i=0}^{n}\frac{1-5iA_5A_{11}A_{17}A_{23}A_{29}}{1-(5i+1)iA_5A_{11}A_{17}A_{23}A_{29}}.$$

In a similar way, it is readily achieved in extra relationships. □

Theorem 3.2. *In (3.1) there is a unique equilibrium point located at $\bar{x}=0$, yet it does not fulfill the criteria for local asymptotic stability.*

Proof of Theorem 3.2. The proof follows the same procedure as the proof of Theorem 2.2, thus it is not detailed. □

4. Solution of the Difference Equation $x_{n+1} = \frac{x_{n-29}}{-1+x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}}$

In this case, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers,

$$x_{n+1} = \frac{x_{n-29}}{-1+x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}}, \tag{4.1}$$

where, $x_0,\ldots,x_{-29}$ defines as in (2.2) with $x_{-5}x_{-11}x_{-17}x_{-23}x_{-29} \neq 1$, $x_{-4}x_{-10}x_{-16}x_{-22}x_{-28} \neq 1$, $x_{-3}x_{-9}x_{-15}x_{-21}x_{-27} \neq 1$, $x_{-2}x_{-8}x_{-14}x_{-20}x_{-26} \neq 1$, $x_{-1}x_{-7}x_{-13}x_{-19}x_{-25} \neq 1$, $x_0x_{-6}x_{-12}x_{-18}x_{-24} \neq 1$.

Theorem 4.1. *Each solution $\{x_n\}_{n=-29}^{\infty}$ of equation (4.1) recurs every sixty units and has the structure,*

$$x_{60n+1}=\frac{A_{30}}{-1+A_6A_{12}A_{18}A_{24}A_{30}},\quad x_{60n+2}=\frac{A_{29}}{-1+A_5A_{11}A_{17}A_{23}A_{29}},\quad x_{60n+3}=\frac{A_{28}}{-1+A_4A_{10}A_{16}A_{22}A_{28}},$$
$$x_{60n+4}=\frac{A_{27}}{-1+A_3A_9A_{15}A_{21}A_{27}},\quad x_{60n+5}=\frac{A_{26}}{-1+A_2A_8A_{14}A_{20}A_{26}},\quad x_{60n+6}=\frac{A_{25}}{-1+A_1A_7A_{13}A_{19}A_{25}},$$
$$x_{60n+7}=A_{24}(-1+A_6A_{12}A_{18}A_{24}A_{30}),\quad x_{60n+8}=A_{23}(-1+A_5A_{11}A_{17}A_{23}A_{29}),\quad x_{60n+9}=A_{22}(-1+A_4A_{10}A_{16}A_{22}A_{28}),$$
$$x_{60n+10}=A_{21}(-1+A_3A_9A_{15}A_{21}A_{27}),\quad x_{60n+11}=A_{20}(-1+A_2A_8A_{14}A_{20}A_{26}),\quad x_{60n+12}=A_{19}(-1+A_1A_7A_{13}A_{19}A_{25}),$$
$$x_{60n+13}=\frac{A_{18}}{-1+A_6A_{12}A_{18}A_{24}A_{30}},\quad x_{60n+14}=\frac{A_{17}}{-1+A_5A_{11}A_{17}A_{23}A_{29}},\quad x_{60n+15}=\frac{A_{16}}{-1+A_4A_{10}A_{16}A_{22}A_{28}},$$
$$x_{60n+16}=\frac{A_{15}}{-1+A_3A_9A_{15}A_{21}A_{27}},\quad x_{60n+17}=\frac{A_{14}}{-1+A_2A_8A_{14}A_{20}A_{26}},\quad x_{60n+18}=\frac{A_{13}}{-1+A_1A_7A_{13}A_{19}A_{25}},$$
$$x_{60n+19}=A_{12}(-1+A_6A_{12}A_{18}A_{24}A_{30}),\quad x_{60n+20}=A_{11}(-1+A_5A_{11}A_{17}A_{23}A_{29}),\quad x_{60n+21}=A_{10}(-1+A_4A_{10}A_{16}A_{22}A_{28}),$$
$$x_{60n+22}=A_9(-1+A_3A_9A_{15}A_{21}A_{27}),\quad x_{60n+23}=A_8(-1+A_2A_8A_{14}A_{20}A_{26}),\quad x_{60n+24}=A_7(-1+A_1A_7A_{13}A_{19}A_{25}),$$
$$x_{60n+25}=\frac{A_6}{-1+A_6A_{12}A_{18}A_{24}A_{30}},\quad x_{60n+26}=\frac{A_5}{-1+A_5A_{11}A_{17}A_{23}A_{29}},\quad x_{60n+27}=\frac{A_4}{-1+A_4A_{10}A_{16}A_{22}A_{28}},$$
$$x_{60n+28}=\frac{A_3}{-1+A_3A_9A_{15}A_{21}A_{27}},\quad x_{60n+29}=\frac{A_2}{-1+A_2A_8A_{14}A_{20}A_{26}},\quad x_{60n+30}=\frac{A_1}{-1+A_1A_7A_{13}A_{19}A_{25}},$$

$$x_{60n+31}=A_{30},\quad x_{60n+32}=A_{29},\quad x_{60n+33}=A_{28},\quad x_{60n+34}=A_{27},\quad x_{60n+35}=A_{26},\quad x_{60n+36}=A_{25},\quad x_{60n+37}=A_{24},$$
$$x_{60n+38}=A_{23},\quad x_{60n+39}=A_{22},\quad x_{60n+40}=A_{21},\quad x_{60n+41}=A_{20},\quad x_{60n+42}=A_{19},\quad x_{60n+43}=A_{18},\quad x_{60n+44}=A_{17},$$
$$x_{60n+45}=A_{16},\quad x_{60n+46}=A_{15},\quad x_{60n+47}=A_{14},\quad x_{60n+48}=A_{13},\quad x_{60n+49}=A_{12},\quad x_{60n+50}=A_{11},\quad x_{60n+51}=A_{10},$$
$$x_{60n+52}=A_9,\quad x_{60n+53}=A_8,\quad x_{60n+54}=A_7,\quad x_{60n+55}=A_6,\quad x_{60n+56}=A_5,\quad x_{60n+57}=A_4,\quad x_{60n+58}=A_3,$$
$$x_{60n+59}=A_2,\quad x_{60n+60}=A_1.$$

The solutions consist of 60 periods.

Proof of Theorem 4.1. Suppose,

$$x_{60n-59}=\frac{A_{30}}{-1+A_6A_{12}A_{18}A_{24}A_{30}},\quad x_{60n-58}=\frac{A_{29}}{-1+A_5A_{11}A_{17}A_{23}A_{29}},\quad x_{60n-57}=\frac{A_{28}}{-1+A_4A_{10}A_{16}A_{22}A_{28}},$$
$$x_{60n-56}=\frac{A_{27}}{-1+A_3A_9A_{15}A_{21}A_{27}},\quad x_{60n-55}=\frac{A_{26}}{-1+A_2A_8A_{14}A_{20}A_{26}},\quad x_{60n-54}=\frac{A_{25}}{-1+A_1A_7A_{13}A_{19}A_{25}},$$
$$x_{60n-53}=A_{24}(-1+A_6A_{12}A_{18}A_{24}A_{30}),\quad x_{60n-52}=A_{23}(-1+A_5A_{11}A_{17}A_{23}A_{29}),\quad x_{60n-51}=A_{22}(-1+A_4A_{10}A_{16}A_{22}A_{28}),$$
$$x_{60n-50}=A_{21}(-1+A_3A_9A_{15}A_{21}A_{27}),\quad x_{60n-49}=A_{20}(-1+A_2A_8A_{14}A_{20}A_{26}),\quad x_{60n-48}=A_{19}(-1+A_1A_7A_{13}A_{19}A_{25}),$$
$$x_{60n-47}=\frac{A_{18}}{-1+A_6A_{12}A_{18}A_{24}A_{30}},\quad x_{60n-46}=\frac{A_{17}}{-1+A_5A_{11}A_{17}A_{23}A_{29}},\quad x_{60n-45}=\frac{A_{16}}{-1+A_4A_{10}A_{16}A_{22}A_{28}},$$
$$x_{60n-44}=\frac{A_{15}}{-1+A_3A_9A_{15}A_{21}A_{27}},\quad x_{60n-43}=\frac{A_{14}}{-1+A_2A_8A_{14}A_{20}A_{26}},\quad x_{60n-42}=\frac{A_{13}}{-1+A_1A_7A_{13}A_{19}A_{25}},$$
$$x_{60n-41}=A_{12}(-1+A_6A_{12}A_{18}A_{24}A_{30}),\quad x_{60n-40}=A_{11}(-1+A_5A_{11}A_{17}A_{23}A_{29}),\quad x_{60n-39}=A_{10}(-1+A_4A_{10}A_{16}A_{22}A_{28}),$$
$$x_{60n-38}=A_9(-1+A_3A_9A_{15}A_{21}A_{27}),\quad x_{60n-37}=A_8(-1+A_2A_8A_{14}A_{20}A_{26}),\quad x_{60n-36}=A_7(-1+A_1A_7A_{13}A_{19}A_{25}),$$
$$x_{60n-35}=\frac{A_6}{-1+A_6A_{12}A_{18}A_{24}A_{30}},\quad x_{60n-34}=\frac{A_5}{-1+A_5A_{11}A_{17}A_{23}A_{29}},\quad x_{60n-33}=\frac{A_4}{-1+A_4A_{10}A_{16}A_{22}A_{28}},$$
$$x_{60n-32}=\frac{A_3}{-1+A_3A_9A_{15}A_{21}A_{27}},\quad x_{60n-31}=\frac{A_2}{-1+A_2A_8A_{14}A_{20}A_{26}},\quad x_{60n-30}=\frac{A_1}{-1+A_1A_7A_{13}A_{19}A_{25}},$$

$$x_{60n-29}=A_{30},\quad x_{60n-28}=A_{29},\quad x_{60n-27}=A_{28},\quad x_{60n-26}=A_{27},\quad x_{60n-25}=A_{26},\quad x_{60n-24}=A_{25},\quad x_{60n-23}=A_{24},$$
$$x_{60n-22}=A_{23},\quad x_{60n-21}=A_{22},\quad x_{60n-20}=A_{21},\quad x_{60n-19}=A_{20},\quad x_{60n-18}=A_{19},\quad x_{60n-17}=A_{18},\quad x_{60n-16}=A_{17},$$
$$x_{60n-15}=A_{16},\quad x_{60n-14}=A_{15},\quad x_{60n-13}=A_{14},\quad x_{60n-12}=A_{13},\quad x_{60n-11}=A_{12},\quad x_{60n-10}=A_{11},\quad x_{60n-9}=A_{10},$$
$$x_{60n-8}=A_9,\quad x_{60n-7}=A_8,\quad x_{60n-6}=A_7,\quad x_{60n-5}=A_6,\quad x_{60n-4}=A_5,\quad x_{60n-3}=A_4,\quad x_{60n-2}=A_3,$$
$$x_{60n-1}=A_2,\quad x_{60n}=A_1.$$

Now, it follows from equation (4.1) that

$$x_{60n+1}=\frac{x_{60n-29}}{-1+x_{60n-5}x_{60n-11}x_{60n-17}x_{60n-23}x_{60n-29}}=\frac{A_{30}}{-1+A_6A_{12}A_{18}A_{24}A_{30}}.$$

Then, we have

$$x_{60n+1}=\frac{A_{30}}{-1+A_6A_{12}A_{18}A_{24}A_{30}}.$$

Other relation can be given by the same way. □

Theorem 4.2. *Equation (4.1) has three equilibrium points which $0,\pm\sqrt[5]{2}$, and these equilibrium points aren't locally asymptotically stable.*

Proof of Theorem 4. The proof follows the same procedure as the proof of Theorem 2.2, thus it is not detailed. □

5. Solution of the Difference Equation $x_{n+1}=\frac{x_{n-29}}{-1-x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}}$

In this case, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers,

$$x_{n+1}=\frac{x_{n-29}}{-1-x_{n-5}x_{n-11}x_{n-17}x_{n-23}x_{n-29}}, \tag{5.1}$$

where, $x_0,\ldots,x_{-29}$ defines as in (2.2) with $x_{-5}x_{-11}x_{-17}x_{-23}x_{-29}\neq -1$, $x_{-4}x_{-10}x_{-16}x_{-22}x_{-28}\neq 1$, $x_{-3}x_{-9}x_{-15}x_{-21}x_{-27}\neq -1$, $x_{-2}x_{-8}x_{-14}x_{-20}x_{-26}\neq -1$, $x_{-1}x_{-7}x_{-13}x_{-19}x_{-25}\neq -1$, $x_0x_{-6}x_{-12}x_{-18}x_{-24}\neq -1$.

Theorem 5.1. *Each solution $\{x_n\}_{n=-29}^{\infty}$ of equation (4.1) is periodic with period sixty and is of the form,*

$$x_{60n+1}=\frac{A_{30}}{-1-A_6A_{12}A_{18}A_{24}A_{30}},\quad x_{60n+2}=\frac{A_{29}}{-1-A_5A_{11}A_{17}A_{23}A_{29}},\quad x_{60n+3}=\frac{A_{28}}{-1-A_4A_{10}A_{16}A_{22}A_{28}},$$
$$x_{60n+4}=\frac{A_{27}}{-1-A_3A_9A_{15}A_{21}A_{27}},\quad x_{60n+5}=\frac{A_{26}}{-1-A_2A_8A_{14}A_{20}A_{26}},\quad x_{60n+6}=\frac{A_{25}}{-1-A_1A_7A_{13}A_{19}A_{25}},$$
$$x_{60n+7}=A_{24}(-1-A_6A_{12}A_{18}A_{24}A_{30}),\quad x_{60n+8}=A_{23}(-1-A_5A_{11}A_{17}A_{23}A_{29}),\quad x_{60n+9}=A_{22}(-1-A_4A_{10}A_{16}A_{22}A_{28}),$$
$$x_{60n+10}=A_{21}(-1-A_3A_9A_{15}A_{21}A_{27}),\quad x_{60n+11}=A_{20}(-1-A_2A_8A_{14}A_{20}A_{26}),\quad x_{60n+12}=A_{19}(-1-A_1A_7A_{13}A_{19}A_{25}),$$
$$x_{60n+13}=\frac{A_{18}}{-1-A_6A_{12}A_{18}A_{24}A_{30}},\quad x_{60n+14}=\frac{A_{17}}{-1-A_5A_{11}A_{17}A_{23}A_{29}},\quad x_{60n+15}=\frac{A_{16}}{-1-A_4A_{10}A_{16}A_{22}A_{28}},$$
$$x_{60n+16}=\frac{A_{15}}{-1-A_3A_9A_{15}A_{21}A_{27}},\quad x_{60n+17}=\frac{A_{14}}{-1-A_2A_8A_{14}A_{20}A_{26}},\quad x_{60n+18}=\frac{A_{13}}{-1-A_1A_7A_{13}A_{19}A_{25}},$$
$$x_{60n+19}=A_{12}(-1-A_6A_{12}A_{18}A_{24}A_{30}),\quad x_{60n+20}=A_{11}(-1-A_5A_{11}A_{17}A_{23}A_{29}),\quad x_{60n+21}=A_{10}(-1-A_4A_{10}A_{16}A_{22}A_{28}),$$
$$x_{60n+22}=A_9(-1-A_3A_9A_{15}A_{21}A_{27}),\quad x_{60n+23}=A_8(-1-A_2A_8A_{14}A_{20}A_{26}),\quad x_{60n+24}=A_7(-1-A_1A_7A_{13}A_{19}A_{25}),$$
$$x_{60n+25}=\frac{A_6}{-1-A_6A_{12}A_{18}A_{24}A_{30}},\quad x_{60n+26}=\frac{A_5}{-1-A_5A_{11}A_{17}A_{23}A_{29}},\quad x_{60n+27}=\frac{A_4}{-1-A_4A_{10}A_{16}A_{22}A_{28}},$$
$$x_{60n+28}=\frac{A_3}{-1-A_3A_9A_{15}A_{21}A_{27}},\quad x_{60n+29}=\frac{A_2}{-1-A_2A_8A_{14}A_{20}A_{26}},\quad x_{60n+30}=\frac{A_1}{-1-A_1A_7A_{13}A_{19}A_{25}},$$

$$x_{60n+31}=A_{30},\quad x_{60n+32}=A_{29},\quad x_{60n+33}=A_{28},\quad x_{60n+34}=A_{27},\quad x_{60n+35}=A_{26},\quad x_{60n+36}=A_{25},\quad x_{60n+37}=A_{24},$$
$$x_{60n+38}=A_{23},\quad x_{60n+39}=A_{22},\quad x_{60n+40}=A_{21},\quad x_{60n+41}=A_{20},\quad x_{60n+42}=A_{19},\quad x_{60n+43}=A_{18},\quad x_{60n+44}=A_{17},$$
$$x_{60n+45}=A_{16},\quad x_{60n+46}=A_{15},\quad x_{60n+47}=A_{14},\quad x_{60n+48}=A_{13},\quad x_{60n+49}=A_{12},\quad x_{60n+50}=A_{11},\quad x_{60n+51}=A_{10},$$
$$x_{60n+52}=A_9,\quad x_{60n+53}=A_8,\quad x_{60n+54}=A_7,\quad x_{60n+55}=A_6,\quad x_{60n+56}=A_5,\quad x_{60n+57}=A_4,\quad x_{60n+58}=A_3,$$
$$x_{60n+59}=A_2,\quad x_{60n+60}=A_1.$$

The solutions consist of 60 periods.

Proof. The proof mirrors the proof of Theorem 4.1, and hence, it is not elaborated upon. □

Theorem 5.2. *Equation (5.1) has three equilibrium points which $0,\pm\sqrt[5]{-2}$, and these equilibrium points are not locally asymptotically stable.*

Proof. The proof follows the same procedure as the proof of Theorem 2.2, thus it is not detailed. □

6. Numerical Investigation

We devote this section to verifying the theoretical work obtained in this article.

Example 6.1. *For Eq. 2.1 and 3.1 we consider following initial conditions.*

$$\begin{array}{llllll}
x_{-29}=3.2, & x_{-28}=3.3, & x_{-27}=3.4, & x_{-26}=3.5, & x_{-25}=3.6, & x_{-24}=3.7,\\
x_{-23}=3.8, & x_{-22}=3.9, & x_{-21}=4, & x_{-20}=4.1, & x_{-19}=4.2, & x_{-18}=4.3,\\
x_{-17}=4.4, & x_{-16}=4.5, & x_{-15}=4.6, & x_{-14}=4.7, & x_{-13}=4.8, & x_{-12}=4.9,\\
x_{-11}=5, & x_{-10}=5.1, & x_{-9}=5.2, & x_{-8}=5.3, & x_{-7}=5.4, & x_{-6}=5.5,\\
x_{-5}=5.6, & x_{-4}=5.7, & x_{-3}=5.8, & x_{-2}=5.9, & x_{-1}=6.1, & x_{0}=6.
\end{array}$$

Example 6.2. *For Eq. 4.1 and 5.1 we consider following initial conditions.*

$$\begin{array}{llllll}
x_{-29}=0.32, & x_{-28}=0.33, & x_{-27}=0.34, & x_{-26}=0.35, & x_{-25}=0.36, & x_{-24}=0.37,\\
x_{-23}=0.38, & x_{-22}=0.39, & x_{-21}=0.4, & x_{-20}=0.41, & x_{-19}=0.42, & x_{-18}=0.43,\\
x_{-17}=0.44, & x_{-16}=0.45, & x_{-15}=0.46, & x_{-14}=0.47, & x_{-13}=0.48, & x_{-12}=0.49,\\
x_{-11}=0.5, & x_{-10}=0.51, & x_{-9}=0.52, & x_{-8}=0.53 & x_{-7}=0.54, & x_{-6}=0.55,\\
x_{-5}=0.56, & x_{-4}=0.57, & x_{-3}=0.58, & x_{-2}=0.59, & x_{-1}=0.61, & x_{0}=0.6.
\end{array}$$

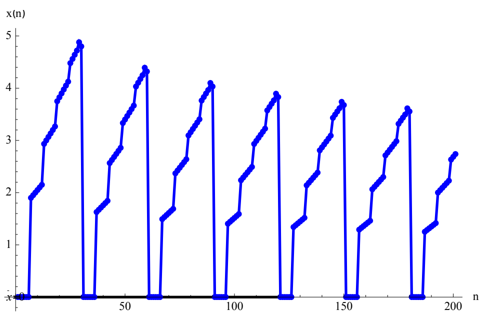

Figure 6.1: Plot illustrates the stability of Eq. 2.1

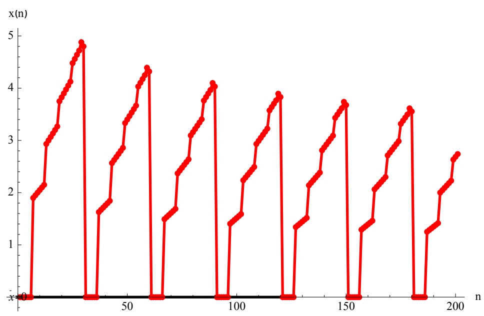

Figure 6.2: Plot illustrates the stability of Eq. 3.1

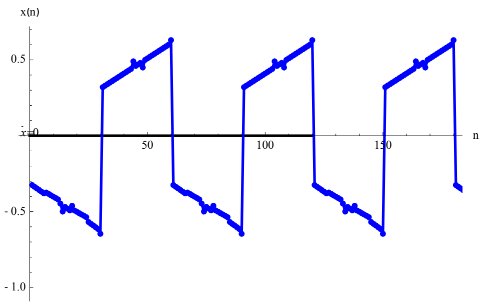

Figure 6.3: Plot illustrates the stability of Eq. 4.1

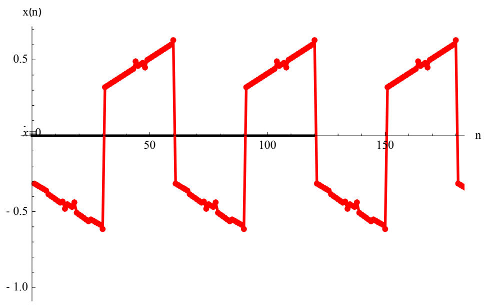

Figure 6.4: Plot illustrates the stability of Eq. 5.1

7. Conclusion

This article extensively explores the qualitative behaviors of difference equations. It effectively examines local stability, periodicity, oscillation, and solutions. Traditional iteration methods are employed to derive exact solutions for the relevant equations.

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References

[1] V.L. Kocic, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Mathematics and Its Applications, Kluwer Academic Publishers Group, Dordrecht (1993).

[2] M.R.S Kulenovic, G. Ladas, G., W.S. Sizer, *On the recursive sequence* $\frac{\alpha x_n+\beta x_{n-1}}{\chi x_n+\beta x_{n-1}}$, Math. Sci. Res.Hot-Line, **2**(5) (1998), 1-16.

[3] R. DeVault, G. Ladas, S.W. Schultz, *On the recursive sequence* $x_{n+1}=\frac{A}{x_n}+\frac{1}{x_{n-2}}$, Proc.Amer. Math. Soc., **126**(11) (1998), 3257-3261.

[4] A.M. Amleh, G.A. Grove, G. Ladas, Georgiou, D.A. *On the recursive sequence* $x_{n+1}=\alpha+\frac{x_{n-1}}{x_n}$, J. of Math. Anal. App., *233* (1999), 790-798.

[5] C.H. Gibbons, M.R.S. Kulenovic, G. Ladas, *On the recursive sequence* $\frac{\alpha+\beta xn-1}{\chi+\beta xn-1}$, Math. Sci. Res. Hot-Line, **4**(2) (2000), 1-11.

[6] S. Elaydi, An Introduction to Difference Equations, 3rd Ed., Springer, USA, (2005).

[7] R. Karatas, C. Cinar, D. Simsek, *On positive solutions of the difference equation* $x_{n+1}=\frac{x_{n-5}}{1+x_{n-2}x_{n-5}}$, Int. J. Contemp. Math. Sci, **10**(1) (2006), 495-500 .

[8] M. Aloqeili, *Dynamics of a rational difference equation*, Appl. Math. Comput., **176**(2) (2006), 768-774.

[9] S. Stevic, *A note on periodic character of a higher order difference equation*, Rostock. Math. Kolloq., **61** (2006), 2-30.

[10] E.M. Elsayed, *On the difference equation* $x_{n+1}=\frac{x_{n-5}}{-1+x_{n-2}x_{n-5}}$, Inter. J. Contemp. Math. Sci., **3**(33) (2008), 1657-1664.

[11] R.P. Agarwal, E.M. Elsayed, *On the solution of fourth-order rational recursive sequence*, Adv. Stud. Contemp. Math. , **20**(2) (2010), 525-545.

[12] S. Stevic, B. Iricanin, Z. Smarda, *On a product-type system of difference equations of second order solvable in closed form*, J. Inequal. Appl., **2015**(1) (2012), 327-334.

[13] O. Karpenko, O. Stanzhytskyi, *The relation between the existence of bounded solutions of differential equations and the corresponding difference equations*, J. Difference Equ. Appl., **19**(12) (2013), 1967-1982. https://doi.org/10.1080/10236198.2013.794795

[14] M. Bohner, O. Karpenko, O. Stanzhytskyi, *Oscillation of solutions of second-order linear differential equations and corresponding difference equations*, J. Difference Equ. Appl., **20**(7) (2014), 1112-1126. https://doi.org/10.1080/10236198.2014.893297

[15] M.B. Almatrafi, M.M. Alzubaidi, *Analysis of the qualitative behaviour of an eighth-order fractional difference equation*, Open J. Discrete Appl. Math., **2**(1), (2019), 41-47.

[16] A. Sanbo, A. E.M. Elsayed, *Some properties of the solutions of the difference equation* $x_{n+1}=\alpha x_n+(bx_nx_{n-4})/(cx_{n-3}+dx_{n-4})$, Open J. Discrete Appl. Math., **2**(2) (2019), 31–47.

[17] A. F. Yeniçerioğlu, C. Yazıcı, & V. Yazıcı, *Stability behaviour in functional differential equations of the neutral type*, Univers. J. Math. Appl., **4**(1) (2021), 33-40. https://doi.org/10.32323/ujma.711881

[18] A.M. Ahmed, A.M. Samir, and L.S. Aljoufi, *Expressions and dynamical behavior of solutions of a class of rational difference equations of fifteenth-order*, J. Math. Computer Sci., **25** (2022), 10-22.

[19] M. Berkal, R. Abo-zeid, *On a rational (P+1)th order difference equation with quadratic term*, Univers. J. Math. Appl., **5**(4) (2022), 136-144. https://doi.org/10.32323/ujma.1198471

[20] B. Oğul, D. Şimşek, T.F. Ibrahim, *A qualitative investigation of the solution of the difference equation* $\Psi_{m+1}=\frac{\Psi_{m-3}\Psi_{m-5}}{\Psi_{m-1}(\pm 1\pm\Psi_{m-3}\Psi_{m-5})}$, Commun. Adv. Math. Sci., **6**(2) (2023), 78-85.

[21] B. Oğul, D. Şimşek, H. Öğünmez, A.S. Kurbanlı, *Dynamical behavior of rational difference equation* $x_{n+1}=\frac{x_{n-17}}{\pm 1\pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}$, Bol. Soc. Mat. Mex., **27**(2) (2021), 1-20.

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Some Results on Composition of Analytic Functions in a Unit Polydisc

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Abstract

The manuscript is an attempt to consider all methods which are applicable to investigation a directional index for composition of an analytic function in some domain and an entire function. The approaches are applied to find sufficient conditions of the L-index boundedness in a direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, where the continuous function L satisfies some growth condition and the condition of positivity in the unit polydisc. The investigation is based on a counterpart of the Hayman Theorem for the class of analytic functions in the polydisc and a counterpart of logarithmic criterion describing local conduct of logarithmic derivative modulus outside some neighborhoods of zeros. The established results are new advances for the functions analytic in the polydisc and in multidimensional value distribution theory.

1. Main Definitions and Notations

We will use notations from [1,2]. Let $\mathbb{C}^n$ be an n-dimensional complex vector space, $\mathbf{0} = (0,\dots,0)$, and $\mathbf{b} = (b_1,\dots,b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a fixed direction. Other denotations are the following: $\mathbb{R}_+ = (0,+\infty)$, the unit polydisc $\mathbb{D}^n$ is the Cartesian products of the discs with radius 1, i.e. $\mathbb{D}^n = \{z \in \mathbb{C}^n : |z_j| < 1 \text{ for every } j \in \{1,2,\dots,n\}\}$. A continuous function $L\colon\ \mathbb{D}^n \to \mathbb{R}_+$ is such that for any $z = (z_1,z_2,\dots,z_n) \in \mathbb{D}^n$

$$L(z) > \beta \max_{1\le j\le n} \frac{|b_j|}{1-|z_j|}, \quad \beta = \text{const} > 1. \tag{1.1}$$

Recently, Salo T. with her co-authors [1] introduced a notion of the directional L-index for functions analytic in the polydisc. They proved many criteria belonging functions to the class. They describe the local behavior of the function and its directional derivative and its value distribution on all slices generated by the vector $\mathbf{b}$ and give estimates of logarithmic derivative modulus in the same vector. Now we justify some application of the results to related topics. In particular, we will examine some compositions of a function analytic in $\mathbb{C}^n$ and a function analytic in the $\mathbb{D}^n$, and will present sufficient conditions of boundedness of the L-index in direction for such a composition. Note there are results [3,4] on the finiteness of the index for analytic functions of single variable for which multidimensional analogs are still unknown.
The notation $\mathscr{A}(\mathbb{D}^n)$ we use for the class of functions which are analytic in $\mathbb{D}^n$. Similarly, $\mathscr{A}(\mathbb{C}^n)$ means the class of entire functions of n complex variables.
Let us remind the main definition from [1]. A function $F \in \mathscr{A}(\mathbb{D}^n)$ is said to be of *bounded L-index in a direction* $\mathbf{b}$, if it is possible to find $m_0 \in \mathbb{Z}_+$ such that for every non-negative integer m and for any point z from the polydisc one has

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!L^m(z)} \le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)}\colon \text{ for every } k \in \{0,1,\dots,m_0\}\right\}, \tag{1.2}$$

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where $\partial_{\mathbf{b}}^{0}F(z)$ matches with the function F, $\partial_{\mathbf{b}}F(z)$ is the dot product of the gradient of the function F and the conjugate of the vector $\mathbf{b}$, $\partial_{\mathbf{b}}^{k}F(z)=\partial_{\mathbf{b}}\left(\partial_{\mathbf{b}}^{k-1}F(z)\right)$, $k\geq 2$. The definition firstly appeared for entire functions of single variable in the paper of B. Lepson [5] if $L\equiv 1$, $\mathbf{b}=1$ and in paper [6] if L is an arbitrary positive continuous function and $\mathbf{b}=1$. If such least integer $m_0=m_0(\mathbf{b})$ exists then it is is called the L-index in the direction $\mathbf{b}$ of F. The value m_0 will be denoted by $N_{\mathbf{b}}(F,L)$.
For a fixed point $z^*=(z_1^*,\dots,z_n^*)$ from the polydisc by D_z we denote an intersection of the $\mathbb{D}^n$ and a complex line crossing the point in a given direction $\mathbf{b}$, i.e. $D_{z^*}=\{t\in\mathbb{C}:(z_1^*+tb_1,\dots,z_n^*+tb_n)\in\mathbb{D}^n\}$. In other words, $D_z=\{t\in\mathbb{C}:|t|<\min_{1\leq j\leq n}\frac{1-|z_j|}{|b_j|}\}$. Here if $b_j=0$ then we suppose $\frac{1-|z_j|}{|b_j|}=+\infty$. Denote

$$\lambda_{\mathbf{b}}(\zeta)=\sup_{w\in\mathbb{D}^n}\sup_{s_1,s_2\in D_w}\left\{\frac{L(w+s_2\mathbf{b})}{L(w+s_1\mathbf{b})}:|s_1-s_2|\leq\frac{\zeta}{\min\{L(z+s_2\mathbf{b}),L(z+s_1\mathbf{b})\}}\right\}.$$

As in [1] the $Q_{\mathbf{b}}(\mathbb{D}^n)$ denotes a class of continuous functions $L:\mathbb{D}^n\to\mathbb{R}_+$, which satisfy (1.1) and for each ζ from the segment $[0,\beta]$ the quantity $\lambda_{\mathbf{b}}(\zeta)$ is finite (the parameter β is defined in condition (1.1)).

2. Boundedness of L-index in Direction for Composition of Analytic Functions in the Polydisc

For simplicity, we suppose that for $\Psi\in\mathscr{A}(\mathbb{D}^n)$ there exist $\kappa>0$ and natural p such that for all $z\in\mathbb{D}^n$ and for all integer $m\in\{0,1,\dots,p\}$ next inequality is fulfilled

$$|\partial_{\mathbf{b}}^{m}\Psi(z)|\leq\kappa|\partial_{\mathbf{b}}\Psi(z)|^m. \tag{2.1}$$

For functions $h:\mathbb{C}^m\to\mathbb{C}$ (or $\mathbb{R}$ instead of $\mathbb{C}$) and $g:\mathbb{D}^n\to\mathbb{C}$ by $h_m^{\circ}g$ we denote such a composition $h(\underbrace{g(z),\dots,g(z)}_{m\text{ times}})$ The following proposition was early deduced for the unit ball [7] and n-dimensional complex space [8]. Now we formulate it for the class $\mathscr{A}(\mathbb{D}^n)$.

Theorem 2.1. *Let $\mathbf{b}$ be non-zero n-dimensional complex vector, $f\in\mathscr{A}(\mathbb{C}^m)$, $\Psi\in\mathscr{A}(\mathbb{D}^n)$ and its derivative in the direction $\mathbf{b}$ has empty zero set. Suppose that function l belongs to the class $Q_{\mathbf{1}}^m$ and its values are not lesser than 1, and the function L is defined as $L(z)=\left|\partial_{\mathbf{b}}\Psi(z)\right|l_m^{\circ}\Psi(z)$ and it belongs to the class $Q_{\mathbf{b}}(\mathbb{D}^n)$.*
If the l-index in the direction $\mathbf{1}$ of the function $f\in\mathscr{A}(\mathbb{C}^m)$ is finite and the function Ψ satisfies (2.1) with $N_{\mathbf{1}}(f,l)$ instead of p then the L-index in the direction $\mathbf{b}$ of the function $F(z)=f_m^{\circ}\Psi(z)$ is also finite.
And if the function $F(z)=f_m^{\circ}\Psi(z)$ has finite $N_{\mathbf{b}}(F,L)$ and inequality (2.1) is fulfilled for the function Ψ and $p=N_{\mathbf{b}}(F,L)$ then $N_{\mathbf{1}}(f,l)$ is finite.

Let us formulate some auxiliary propositions. They are counterparts the Hayman Theorem for the class $\mathscr{A}(\mathbb{C}^n)$ [9] and the class $\mathscr{A}(\mathbb{D}^n)$ [1], It was firstly proved by W. Hayman [10] for entire functions of one variable having bounded index.

Theorem 2.2 ([9]). *Let $L\in Q_{\mathbf{b}}^n$. A function $F\in\mathscr{A}(\mathbb{C}^n)$ is of bounded L-index in the direction $\mathbf{b}$ if and only if there exist numbers $p\in\mathbb{Z}_+$, $R>0$ and $C>0$ such that for every $z\in\mathbb{C}^n$ outside the disc of radii R one has*

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)}\leq C\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^k(z)}:\ k\in\{0,\dots,p\}\right\}. \tag{2.2}$$

Theorem 2.3 ([1]). *Let $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$. A function $F\in\mathscr{A}(\mathbb{D}^n)$ has finite $N_{\mathbf{b}}(F,L)$ if and only if for some positive integer p and positive real C, and for every z belonging the polydisc inequality (2.2) holds.*

Proof of Theorem 2.1. Denote $\nabla f=\partial_{\mathbf{1}}f=\sum_{j=1}^{m}\frac{\partial f}{\partial z_j}$, $\nabla^k f\equiv\partial_{\mathbf{1}}^k f$ for $k\geq 2$. Firstly, we present two following formulas from [7,8,11]

$$\partial_{\mathbf{b}}^{k}F(z)=\nabla^k f_m^{\circ}\Psi(z)\left(\partial_{\mathbf{b}}\Psi(z)\right)^k+\sum_{j=1}^{k-1}\nabla^j f_m^{\circ}\Psi(z)Q_{j,k}(z), \tag{2.3}$$

where

$$Q_{j,k}(z)=\sum_{\substack{n_1+2n_2+\dots+kn_k=k\\0\leq n_1\leq j-1}}c_{j,k,n_1,\dots,n_k}\left(\partial_{\mathbf{b}}\Psi(z)\right)^{n_1}\left(\partial_{\mathbf{b}}^2\Psi(z)\right)^{n_2}\dots\left(\partial_{\mathbf{b}}^k\Psi(z)\right)^{n_k},$$

$c_{j,k,n_1,\dots,n_k}$ are non-negative integer numbers, and

$$\nabla^k f_m^{\circ}\Psi(z)=\partial_{\mathbf{b}}^kF(z)\left(\partial_{\mathbf{b}}\Psi(z)\right)^{-k}+\left(\partial_{\mathbf{b}}\Psi(z)\right)^{-2k}\sum_{j=1}^{k-1}\partial_{\mathbf{b}}^jF(z)\left(\partial_{\mathbf{b}}\Psi(z)\right)^jQ^*(z;j,k), \tag{2.4}$$

with

$$Q^*(z;j,k)=\sum_{m_1+2m_2+\dots+km_k=2(k-j)}b_{j,k,m_1,\dots,m_k}\left(\partial_{\mathbf{b}}\Psi(z)\right)^{m_1}\left(\partial_{\mathbf{b}}^2\Psi(z)\right)^{m_2}\dots\left(\partial_{\mathbf{b}}^k\Psi(z)\right)^{m_k},$$

$b_{j,k,m_1,\dots,m_k}$ are some integer coefficients. Their detailed proofs were presented in [7] for the unit ball and use the mathematical induction method. Obviously, their proofs for the polydisc is the same, so we omit them.

Suppose that $N_{\mathbf{1}}(f,l)$ is finite and f belongs to the class $\mathscr{A}(\mathbb{C}^m)$. By Theorem 2.2 inequality (2.2) holds for $n=m$, $F=f$, $L=l$, $\mathbf{b}=\mathbf{1}$. Taking into account (2.1) and (2.3), for $k=p+1$ we obtain

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)}\le\frac{|\nabla^{p+1}f_m^{\circ}\Psi(z)|}{L^{p+1}(z)}|\partial_{\mathbf{b}}\Psi(z)|^{p+1}+\sum_{j=1}^{p}\frac{|\nabla^j f_m^{\circ}\Psi(z)||Q_{j,p+1}(z)|}{L^{p+1}(z)}\le$$
$$\le\max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}:k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\frac{|Q_{j,p+1}(z)|}{(l_m^{\circ}\Psi(z))^{p+1-j}|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right)\le\max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}:k\in\{0,\dots,p\}\right\}\times$$
$$\times\left(C+\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\0\le n_1\le j-1}}c_{j,p+1,n_1,\dots,n_{p+1}}\frac{|(\partial_{\mathbf{b}}\Psi(z))^{n_1}(\partial_{\mathbf{b}}^2\Psi(z))^{n_2}\dots\left(\partial_{\mathbf{b}}^{p+1}\Psi(z)\right)^{n_{p+1}}|}{(l_m^{\circ}\Psi(z))^{p+1-j}|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right)\le$$
$$\le\max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}:k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\0\le n_1\le j-1}}\frac{c_{j,p+1,n_1,\dots,n_{p+1}}\kappa^{p+1}}{(l_m^{\circ}\Psi(z))^{p+1-j}}\right)\le C_1\max_{k\in\{0,\dots,p\}}\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}.$$

Now we substitute the right-hand side of (2.4) instead of $\nabla^k f_m^{\circ}\Psi(z)$ and perform some algebraic transformations:

$$\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}\le\frac{|\partial_{\mathbf{b}}^kF(z)|}{(l_m^{\circ}\Psi(z))^k|\partial_{\mathbf{b}}\Psi(z)|^k}+\sum_{j=1}^{k-1}\frac{|\partial_{\mathbf{b}}^jF(z)||Q^*(z;j,k)|}{(l_m^{\circ}\Psi(z))^k|\partial_{\mathbf{b}}\Psi(z)|^{2k-j}}\le\max_{1\le j\le k}\frac{|\partial_{\mathbf{b}}^jF(z)|}{L^j(z)}\left(1+\sum_{j=1}^{k-1}\frac{|Q^*(z;j,k)|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\right)\le$$
$$\le\max_{j\in\{1,2,\dots,k\}}\left\{L^{-j}(z)\left|\partial_{\mathbf{b}}^jF(z)\right|\right\}\left(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\dots,m_k}|\frac{|(\partial_{\mathbf{b}}\Psi(z))^{m_1}(\partial_{\mathbf{b}}^2\Psi(z))^{m_2}\dots(\partial_{\mathbf{b}}^k\Psi(z))^{m_k}|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\right)\le$$
$$\le\max\left\{\frac{|\partial_{\mathbf{b}}^jF(z)|}{L^j(z)}:1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}\frac{|b_{j,k,m_1,\dots,m_k}|\kappa^k}{(l_m^{\circ}\Psi(z))^{k-j}}\right)\le C_2\max_{1\le j\le k}\frac{|\partial_{\mathbf{b}}^jF(z)|}{L^j(z)}.$$

Hence, it follows that

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)}\le C_1C_2\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}:\ k\in\{0,\dots,p\}\right\}.$$

The last inequality is the same as (2.2) in Theorem 2.3. It means that the theorem is applicable. Hence, we conclude that the directional L-index of the function F is bounded. The first part is proved.
Now we will start coinsiderations vice versa. Assume that the L-index in the direction $\mathbf{b}$ of the function F is bounded. In view of Hayman's Theorem the function must satisfies (2.2). Using (2.1) and (2.4), we will estimate

$$\frac{|\nabla^{p+1}f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^{p+1}}\le\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{(l_m^{\circ}\Psi(z))^{p+1}|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}+\sum_{j=1}^{p}\frac{|\partial_{\mathbf{b}}^jF(z)||Q^*(z;j,p+1)|}{(l_m^{\circ}\Psi(z))^{p+1}|\partial_{\mathbf{b}}\Psi(z)|^{2p+2-j}}\le$$
$$\le\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}:k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\frac{|Q^*(z;j,p+1)|}{(l_m^{\circ}\Psi(z))^{p+1-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(p+1-j)}}\right)\le$$
$$\le\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}:k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\=2(p+1-j)}}|b_{j,p+1,m_1,\dots,m_{p+1}}|\frac{|(\partial_{\mathbf{b}}\Psi(z))^{m_1}(\partial_{\mathbf{b}}^2\Psi(z))^{m_2}\dots\left(\partial_{\mathbf{b}}^{p+1}\Psi(z)\right)^{m_{p+1}}|}{(l_m^{\circ}\Psi(z))^{p+1-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(p+1-j)}}\right)\le$$
$$\le\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}:k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\=2(p+1-j)}}\frac{|b_{j,p+1,m_1,\dots,m_{p+1}}|\kappa^{2p+2-2j}}{l^{p+1-j}(\Psi(z))}\right)\le C_3\max_{k\in\{0,\dots,p\}}\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}.$$

Instead $\partial_{\mathbf{b}}^kF(z)$ in previous expression we substitute (2.3) and again deduce

$$\frac{|\partial_{\mathbf{b}}^kF(z)|}{L^k(z)}\le\frac{|\nabla^k f_m^{\circ}\Psi(z)||\partial_{\mathbf{b}}\Psi(z)|^k}{L^k(z)}+\sum_{j=1}^{k-1}\frac{|\nabla^j f_m^{\circ}\Psi(z)||Q_{j,k}(z)|}{L^k(z)}\le$$
$$\le\max\left\{\frac{|\nabla^j f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j}:1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\frac{|Q_{j,k}(z)|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^k}\right)\le C_4\max\left\{\frac{|\nabla^j f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j}:j\in\{1,2,\dots,k\}\right\}.$$

It implies that

$$\frac{|\nabla^{p+1}f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^{p+1}}\le C_3C_4\max\left\{\frac{|\nabla^j f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j}:j\in\{0,\dots,p\}\right\}.$$

Application of Theorem 2.2 for such values $n=m$, $F=f$, $L=l$, $\mathbf{b}=\mathbf{1}$ give us finiteness of the l-index in the direction $\mathbf{b}$. □

Theorem 2.4. *Let* $\mathbf{b}$ *be a fixed n-dimensional non-zero complex direction, the functions* l, f, Ψ *belong to the classes* $Q_{\mathbf{1}}^m$, $\mathscr{A}(\mathbb{C}^m)$, $\mathscr{A}(\mathbb{D}^n)$, *respectively. For each* $w \in \mathbb{C}^m$ *the values of* $l(w)$ *are not lesser than* 1, *and the* l*-index in the direction* $\mathbf{1}$ *of the function* f *is bounded. Suppose that the function* $L(z) = \max\{1, |\partial_{\mathbf{b}}\Psi(z)|\} l_m^{\circ}\Psi(z)$ *belongs to the class* $Q_{\mathbf{b}}(\mathbb{D}^n)$ *and for every point* z *from the polydisc* $\mathbb{D}^n$ *and for each* $k \in \{1, 2, \ldots, N_{\mathbf{1}}(f,l)+1\}$ *the function* Ψ *satisfies*

$$|\partial_{\mathbf{b}}^k \Psi(z)| \le \kappa (l_m^{\circ}\Psi(z))^{1/(N_{\mathbf{1}}(f,l)+1)} |\partial_{\mathbf{b}}\Psi(z)|^k, \ \ (1 \le kappa \equiv const). \tag{2.5}$$

Then the function $F(z) = f_m^{\circ}\Psi(z)$ *belongs to the function class having bounded* L*-index in the direction* $\mathbf{b}$.

Proof of Theorem 2.4. As above, we will merge methods from appropriate statements in [7,8].
Denote $L_0(z) = l_m^{\circ}\Psi(z)|\partial_{\mathbf{b}}\Psi(z)|$. We estimate Equation (2.3) with L_0 instead of L by modulus and substitute $l_m^{\circ}\Psi(z)|\partial_{\mathbf{b}}\Psi(z)|$ instead of the function L_0, for $k = p+1$ we conclude

$$\begin{aligned} &|\partial_{\mathbf{b}}^{p+1}F(z)|L_0^{-p-1}(z) \le |\nabla^{p+1} f_m^{\circ}\Psi(z)|L_0^{-p-1}(z)\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1} + \sum_{j=1}^{p} |\nabla^j f_m^{\circ}\Psi(z)||Q_{j,p+1}(z)|L_0^{-p-1}(z) \le \\ &\le \frac{|\nabla^{p+1} f_m^{\circ}\Psi(z)|\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}{(l_m^{\circ}\Psi(z))^{p+1}\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1}} + \sum_{j=1}^{p} \frac{|\nabla^j f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j} \cdot \frac{|Q_{j,p+1}(z)|(l_m^{\circ}\Psi(z))^j}{|\partial_{\mathbf{b}}\Psi(z)|^{p+1}\,(l_m^{\circ}\Psi(z))^{p+1}}. \end{aligned} \tag{2.6}$$

Let us remind that $f \in \mathscr{A}(\mathbb{C}^m)$ has finite $N_{\mathbf{b}}(f,l)$ (by hypothesis of the assertion). Theorem 2.2 yields validity of inequality (2.2) in this form

$$(\forall \tau \in \mathbb{C}^m): \quad \frac{|\nabla^{p+1} f(\tau)|}{l^{p+1}(\tau)} \le C \max\left\{\frac{|\nabla^k f(\tau)|}{l^k(\tau)} : \ k \in \{0, \ldots, p\}\right\}$$

for such values of parameters $n = m$, $F = f$, $L = l$, $\mathbf{b} = \mathbf{1}$ and $p = N_{\mathbf{1}}(f,l)$. We enhance (2.6), if we substitute previous inequality with $\tau = (\underbrace{\Psi(z), \ldots, \Psi(z)}_{m \text{ times}})$

$$\begin{aligned} &\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le \max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k} : \ k \in \{0, \ldots, p\}\right\}\left(C + \sum_{j=1}^{p} \frac{|Q_{j,p+1}(z)|(l_m^{\circ}\Psi(z))^{j-p-1}}{|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right) \le \\ &\le \max_{k \in \{0,\ldots,p\}} \frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}\left(C + \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1 \\ 0 \le n_1 \le j-1}} c_{j,p+1,n_1,\ldots,n_{p+1}} \frac{|(\partial_{\mathbf{b}}\Psi(z))^{n_1}\,(\partial_{\mathbf{b}}^2\Psi(z))^{n_2}\ldots\left(\partial_{\mathbf{b}}^{p+1}\Psi(z)\right)^{n_{p+1}}|}{(l_m^{\circ}\Psi(z))^{p+1-j}\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right). \end{aligned} \tag{2.7}$$

Now we use condition (2.5) for the function Ψ. Then inequality (2.7) transforms in the following

$$\begin{aligned} &\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le \max\left\{\frac{|\nabla^k f(\Psi(z))|}{(l_m^{\circ}\Psi(z))^k} : k \in \{0,1,\ldots,p\}\right\}\times \\ &\times\left(C + \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1 \\ 0 \le n_1 \le j-1}} \frac{c_{j,p+1,n_1,\ldots,n_{p+1}}\kappa^{p+1} l(\Psi(z),\ldots,\Psi(z))|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}{(l_m^{\circ}\Psi(z))^{p+1-j}\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right) \le \\ &\le \max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k} : k \in \{0,1,2,\ldots,p\}\right\}\left(C + \sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1 \\ 0 \le n_1 \le j-1}} \frac{c_{j,p+1,n_1,\ldots,n_{p+1}}\kappa^{p+1}}{(l_m^{\circ}\Psi(z))^{p-j}}\right). \end{aligned} \tag{2.8}$$

Since the values of the function l are not lesser than 1, the composition $l_m^{\circ}\Psi(z)$ is also not lesser than 1. We substitute it in (2.8)

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le C_1 \max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k} : k \in \{0, \ldots, p\}\right\}, \tag{2.9}$$

with $C_1 = C + \kappa^{p+1}\sum_{j=1}^{p} \sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1 \\ 0 \le n_1 \le j-1}} c_{j,p+1,n_1,\ldots,n_{p+1}}$. To estimate the fraction $\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}$, we find the modulus of equality (2.4)

$$\begin{aligned} &\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k} \le \frac{|\partial_{\mathbf{b}}^k F(z)|}{(l_m^{\circ}\Psi(z))^k|\partial_{\mathbf{b}}\Psi(z)|^k} + \sum_{j=1}^{k-1} \frac{|\partial_{\mathbf{b}}^j F(z)||Q^*(z;j,k)|}{(l_m^{\circ}\Psi(z))^k|\partial_{\mathbf{b}}\Psi(z)|^{2k-j}} \le \\ &\le \max_{1\le j\le k}\left\{\frac{|\partial_{\mathbf{b}}^j\Psi(z)|}{(l_m^{\circ}\Psi(z))^j|\partial_{\mathbf{b}}\Psi(z)|^j}\right\}\left(1 + \sum_{j=1}^{k-1} \frac{|Q^*(z;j,k)|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\right) \le \\ &\le \max_{1\le j\le k} \frac{|\partial_{\mathbf{b}}^j\Psi(z)|}{(l_m^{\circ}\Psi(z))^j|\partial_{\mathbf{b}}\Psi(z)|^j}\left(1 + \sum_{j=1}^{k-1} \sum_{m_1+2m_2+\ldots+km_k=2(k-j)} |b_{j,k,m_1,\ldots,m_k}| \frac{|(\partial_{\mathbf{b}}\Psi(z))^{m_1}(\partial_{\mathbf{b}}^2\Psi(z))^{m_2}\ldots(\partial_{\mathbf{b}}^k\Psi(z))^{m_k}|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\right). \end{aligned} \tag{2.10}$$

Since $l(w)\geq 1$ and for $s\in\{1,2,\ldots,N_{\mathbf{1}}(f,l)+1\}$ and $N_{\mathbf{1}}(f,l)\geq 1$ one has $s/2\geq 1/(N_{\mathbf{1}}(f,l)+1)$, inequality (2.5) can be reinforced $|\partial_{\mathbf{b}}^{s}\Psi(z)|\leq \kappa l^{s/2}(\Psi(z))|\partial_{\mathbf{b}}\Psi(z)|^{s}$. Applying this inequality to (2.10), we deduce

$$\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}\leq \max_{1\leq j\leq k}\frac{|\partial_{\mathbf{b}}^{j}F(z)|}{(l_m^{\circ}\Psi(z))^j|\partial_{\mathbf{b}}\Psi(z)|^j}\Bigg(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|\kappa^{m_1+m_2+\ldots+m_k}\times$$
$$\times\frac{(l_m^{\circ}\Psi(z))^{(m_1+2m_2+\ldots+km_k)/2}|\partial_{\mathbf{b}}\Psi(z)|^{m_1+2m_2+\ldots+km_k}}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\Bigg)\leq C_2\max\left\{\frac{|\partial_{\mathbf{b}}^{j}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j|\partial_{\mathbf{b}}\Psi(z)|^j}:\ j\in\{1,2,\ldots,k\}\right\},$$

with

$$C_2=1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|\kappa^{m_1+m_2+\ldots+m_k}.$$

Then from inequality (2.9) we get

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)}\leq C_1\max_{k\in\{0,\ldots,p\}}\frac{|f^{(k)}(\Psi(z),\ldots,\Psi(z))|}{(l_m^{\circ}\Psi(z))^k}\leq C_1C_2\max\left\{\frac{|\partial_{\mathbf{b}}^{j}F(z)|}{L_0^{j}(z)}:\ j\in\{0,\ldots,p\}\right\},\tag{2.11}$$

$p=N_{\mathbf{1}}(f,l)$. Remind that inequality (2.11) is proved for all z outside zero set of the function $\partial_{\mathbf{b}}\Phi$ and with usage the condtion $N_{\mathbf{1}}(f,l)\geq 1$. If $N_{\mathbf{1}}(f,l)=0$ then the parameter p also equals zero and estimate (2.9) yields

$$\frac{|\partial_{\mathbf{b}}F(z)|}{L_0(z)}\leq C_1|f_m^{\circ}\Psi(z)|=C_1|F(z)|.$$

Thus, (2.11) is proved for all possible finite values of the directional l-index for the function f.
Since $L(z)=(l_m^{\circ}\Psi(z)\max\{1,|\partial_{\mathbf{b}}\Psi(z)|\}$, we can rewrite inequality (2.11):

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\cdot\frac{L^{p+1}(z)}{L_0^{p+1}(z)}\leq C_1C_2\max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}\frac{L^{k}(z)}{L_0^{k}(z)}:\ k\in\{0,\ldots,p\}\right\}.$$

Then

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\leq C_1C_2\frac{L_0^{p+1}(z)}{L^{p+1}(z)}\max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}\frac{L^{k}(z)}{L_0^{k}(z)}:\ k\in\{0,\ldots,p\}\right\}\leq$$
$$\leq C_1C_2\frac{L_0^{p+1}(z)}{L^{p+1}(z)}\max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}:k\in\{0,\ldots,p\}\right\}\max\left\{\frac{L^{k}(z)}{L_0^{k}(z)}:k\in\{0,\ldots,p\}\right\}=\frac{C_1C_2(L_0(z)/L(z))^{p+1}}{\min\limits_{k\in\{0,\ldots,p\}}(L_0(z)/L(z))^{k}}\max_{k\in\{0,\ldots,p\}}\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}.\tag{2.12}$$

Let $t_0=t(z)=L_0(z)/L(z)$ and $k_0\leq p$ ($k_0\in\mathbb{Z}_+$) be such that $(t_0)^{k_0}=\min_{k\in\{0,\ldots,p\}}t_0^k$. One should observe that $t_0\in(0,1]$ and $p+1-k_0\geq 1$. Hence, $\frac{t_0^{p+1}}{t_0^{k_0}}=t_0^{p+1-k_0}\leq t_0\leq 1$. Therefore, $\frac{(L_0(z)/L(z))^{p+1}}{\min_{k\in\{0,\ldots,p\}}(L_0(z)/L(z))^{k}}=t_0^{p+1-k_0}\leq t_0\leq 1$. Thus, from inequality (2.12) we get

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\leq C_1C_2\max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}:\ k\in\{0,\ldots,p\}\right\}\tag{2.13}$$

for all z outside zero set of the **b**-directional derivative of the function Ψ.
If for some point z from the polydisc $\mathbb{D}^n$ the **b**-directional derivative of the function Ψ vanishes then for any natural value of k does not exceeding $N(f,l)+1$ condition (2.5) means that k-th order **b**-directional derivative of the function Ψ also vanishes at this same point. Substituting this point in (2.3) we conclude that k-th order **b**-directional derivative of the function Ψ also vanishes at this same point for each natural $1\leq k\leq N(f,l)+1$. Hence, for all points z belonging zero set of the **b**-directional derivative of the function Ψ inequality (2.13) is true.
Applying Theorem 2.3 we establish that the function F belong to the class of functions with bounded L-index in the direction **b**. $\square$

3. Application of Logarithmic Criterion to Composition

In this section, we consider an application of the logarithmic criterion to investigation of the index boundedness for a composition of functions from the classes $\mathscr{A}(\mathbb{D}^n)$ and $\mathscr{A}(\mathbb{C}^m)$. Another applications of the statement in function theory of bounded index are decribed in [12–15].
Let us introduce the slice function as $g_z(t):=F(z+t\mathbf{b})$ $(z\in\mathbb{D}^n)$. If one has for some z from the unit polydisc the slice function $g_z(t)$ has empty zero set, then we put $G_r^{\mathbf{b}}(F,z):=\varnothing$; otherwise if $g_z(t)$ identically equals zero then we put $G_r^{\mathbf{b}}(F,z):=\{z+t\mathbf{b}\colon\ |t|\leq\min\limits_{j\in\{1,\ldots,n\}}\frac{1-|z_j|}{|b_j|}\}$.
And last possible case is if $g_z(t)\not\equiv 0$ and $a_{k,z}$ are zeros of $g_z(t)$, then we denote $G_r^{\mathbf{b}}(F,z):=\bigcup_k\left\{z+t\mathbf{b}\colon\ |t-a_{k,z}|\leq\frac{r}{L(z+a_{k,z}\mathbf{b})}\right\}$, $r>0$.
Let $G_r^{\mathbf{b}}(F)=\bigcup_{z^0\in\mathbb{D}^n}G_r^{\mathbf{b}}(F,z^0)$, $n\big(r,z^0,1/F\big)=\sum_{|a_k^0|\leq r}1$ is the counting function of zeros (a_k^0) of the function $F(z^0+t\mathbf{b})$ in the disk $\{t\in\mathbb{C}\colon\ |t|\leq r\}$. Below we formulate two auxiliary propositions proved in [1]. The first of them is the logarithmic criterion analog, and the second of them is weaker sufficient conditions for functions belonging to the class $\mathscr{A}(\mathbb{D}^n)$.

Theorem 3.1. *[1] Let $F :\in \mathscr{A}(\mathbb{D}^n)$, $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and $\mathbb{D}^n \setminus G_{\beta}^{\mathbf{b}}(F) \neq \varnothing$. The function F has finite $N_{\mathbf{b}}(F,L)$ if and only if*

1) *for every radius r belonging to the half-closed interval $(0,\beta]$ there exists a positive real $P = P(r)$ such that for every point $z \in \mathbb{D}^n$ outside the set $G_r^{\mathbf{b}}(F)$ the following directional logarithmic derivative estimate is true*

$$|\partial_{\mathbf{b}} F(z)| \leq PL(z)|F(z)|; \tag{3.1}$$

2) *for every radius r belonging to the segment $[0,\beta]$ and some $\widetilde{n}(r) \in \mathbb{Z}_+$ amount of zeros for the slice function in some circles within the unit polydisc is uniformly bounded, i.e.*

$$n\left(r/L(z^0), z^0, 1/F\right) \leq \widetilde{n}(r). \tag{3.2}$$

for each $z^0 \in \mathbb{D}^n$ with $F(z^0 + t\mathbf{b}) \not\equiv 0$.

Theorem 3.2. *[1] Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $\mathbb{D}^n \setminus G_{\beta}^{\mathbf{b}}(F) \neq \emptyset$, $F : \mathbb{D}^n \to \mathbb{C}$ be an analytic function. If the following conditions are satisfied*

1) *there exists $r_1 \in (0, \beta/2)$ (or there exists $r_1 \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{D}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$) such that $n(r_1) \in [-1;\infty)$;*
2) *there exist $r_2 \in (0,\beta)$, $P > 0$ such that $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$ and for all $z \in \mathbb{D}^n \setminus G_{r_2}(F)$ inequality (3.1) holds,*

then the function F has bounded L-index in the direction $\mathbf{b}$.

Within the notion of bounded index the local properties of analytic solutions of ordinary [5, 16, 17], directional [13] and partial differential equations [18] and their systems [19] are considered in many papers. Moreover, application of the Hayman theorem and its analogs is main method to justify sufficent conditions for boundedness of L-index in direction, if they are applied to composition of entire [4, 8, 15] and analytic functions [2, 7].
Below there are presented other results on functions' composition from the classes $\mathscr{A}(\mathbb{D}^n)$ and $\mathscr{A}(\mathbb{C}^m)$. They are proved with usage of logarithmic criterion analog for the unit polydisc (similar results for the unit ball see in [2]). In this section we suppose that $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^m$.

Proposition 3.3. *Let $\Psi \in \mathscr{A}(\mathbb{D}^n)$, $f \in \mathscr{A}(\mathbb{C}^m)$ with an empty zero set.*
1) Suppose that $l \in Q_{\mathbf{1}}(\mathbb{C}^m)$, $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and for every point z from the unit polydisc the value $L(z)$ is not lesser than $\left|\partial_{\mathbf{b}}\Psi(z)\right| l_m^{\circ}\Psi(z)$. If the $\mathbf{1}$-directional l-index of the function f is finite, then the function $F(z) = f_m^{\circ}\Psi(z)$ has finite $N_{\mathbf{b}}(F,L)$.
2) Suppose that $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, the $\mathbf{b}$-directional derivative of the function Ψ has empty zero set and $l \in Q_{\mathbf{1}}(\mathbb{C}^m)$ and such a function $l_m^{\circ}\Psi(z)$ is not lesser than $L(z)/\left|\partial_{\mathbf{b}}\Psi(z)\right|$ for every point z from the polydisc $\mathbb{D}^n$. And if the function $F(z) = f_m^{\circ}\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$, then the $\mathbf{1}$-directional l-index of the function f is also finite.

Proof. It is not difficult to verify that

$$\partial_{\mathbf{b}} F(z) = \partial_{\mathbf{1}} f_m^{\circ}\Psi(z) \cdot \partial_{\mathbf{b}}\Psi(z). \tag{3.3}$$

Remind that zero set of f is empty. So such a function $f_m^{\circ}\Psi(z)$ has also empty zero set. Then $G_r^{\mathbf{b}}(F) = \varnothing$. Thus, it leaves to validate condition 2) in Theorem 3.2. Indeed, we need to justify inequality (3.1) for every point z belonging the polydisc $\mathbb{D}^n$. Using (3.3) for the directional logarithmic derivative estimate we obtain

$$\left|\partial_{\mathbf{b}} F(z)/F(z)\right| = \left|\partial_{\mathbf{1}} f_m^{\circ}\Psi(z)\right| \cdot |\partial_{\mathbf{b}}\Psi(z)|/|f_m^{\circ}\Psi(z)| \tag{3.4}$$

Let f be of bounded l-index in the direction $\mathbf{1}$. By Theorem 3.1 (see also [20]) for ther multivariate entire functions inequality (3.1) is valid for the function f and for all $w \in \mathbb{C}^m$:

$$|\partial_{\mathbf{1}} f(w)| \leq Pl(w) \cdot |f(w)| \tag{3.5}$$

After substitution $w = (\underbrace{\Psi(z),\dots,\Psi(z)}_{m \text{ times}}$ in (3.5) and usage (3.4) the following estimate become valid

$$|\partial_{\mathbf{b}} F(z)|/|F(z)| = Pl_m^{\circ}\Psi(z) \cdot |\partial_{\mathbf{b}}\Psi(z)| \leq PL(z). \tag{3.6}$$

The function F also does not vanish. Thus, we have proved validity of condition 2) in Theorem 3.1. It means that the function F belongs to the class of functions with bounded L-index in the direction $\mathbf{b}$.
By analogy to the first part of the proof we can justify the second part of the assertion. □

By $\mathbf{1}_j$ we denote m-dimensional complex vector, in which j-th component equals one, other components are zeros.

Proposition 3.4. *Let $\Psi_j \in \mathscr{A}(\mathbb{D}^n)$ and $l \in Q_{\mathbf{1}_j}^m$ for $j \in \{1,\dots,m\}$, $f \in \mathscr{A}(\mathbb{C}^m)$ with empty zero set. Suppose that $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and $L(z) \geq \sum_{j=1}^m \left|\partial_{\mathbf{b}}\Psi_j(z)\right| l(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))$ for every point z within the polydisc $\mathbb{D}^n$. If for every $j \in \{1,\dots,m\}$ the function f is of bounded l-index in the direction $\mathbf{1}_j$, then the composite function $F(z) = f(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))$ is of bounded L-index in the direction* $\mathbf{b}$.

Proof. Using direct calculations it can be substantiated

$$\partial_{\mathbf{b}} F(z) = \sum_{j=1}^m f'_{\Psi_j}(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))\partial_{\mathbf{b}}\Psi_j(z). \tag{3.7}$$

Since f has empty zero set, the composite function $f(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))$ does not vanish for all z from the polydisc $\mathbb{D}^n$ that is $G_r^{\mathbf{b}}(F)=\varnothing$. It leaves to validate inequality (3.1) within the polydisc $\mathbb{D}^n$ because it is equivalent condition 2) in Theorem 3.2. From (3.7) it follows that

$$|\partial_{\mathbf{b}}F(z)|/|F(z)|\le\sum_{j=1}^{m}\left|\frac{f'_{\Psi_j}(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))}{f(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))}\right|\cdot|\partial_{\mathbf{b}}\Psi_j(z)| \tag{3.8}$$

Since f is of bounded l-index in each direction $\mathbf{1}_j$, by analog of Theorem 3.1 for entire functions of m complex variables (see [20]) inequality (3.1) holds for the function f and for all $w\in\mathbb{C}^m$:

$$\frac{|\partial_{\mathbf{1}_j}f(w)|}{|f(w)|}\le Pl(w) \tag{3.9}$$

Replacing w by $(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))$ in (3.9) and using it in (3.8) we establish such a directional logarithmic derivative estimate

$$|\partial_{\mathbf{b}}F(z)|/|F(z)|=Pl(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))\cdot\sum_{j=1}^{m}|\partial_{\mathbf{b}}\Psi_j(z)|\le PL(z). \tag{3.10}$$

Since function F has not zero points as the function f, from (3.10) it follows that by Theorem 3.1 $\mathbf{b}$-directional L-index of the function F is finite. Proposition 3.4 is proved. □

The condition of absence zero points in the function f can be replaced by another condition on the function Ψ generated of the notion of multidimensional directional multivalence.
Let us remind the definition of function having bounded value L-distribution in a direction.
Function $F\in\mathscr{A}(\mathbb{D}^n)$ is called [1] a function of *bounded value L-distribution in the direction* $\mathbf{b}$, if for some natural p and for any complex w and for every point z_0 within the polydisc $\mathbb{D}^n$ such that the slice function $F(z^0+t\mathbf{b})$ does not equal identically w, the inequality holds $n\big(1/L(z^0),z^0,1/(F-w)\big)\le p$, i.e. the equation $F(z^0+t\mathbf{b})=w$ has at most p solutions in the disc $\{t:\ |t|\le 1/L(z^0)\}$. Using the one-dimensional notion of multivalence, we can claim that the slice function $F(z^0+t\mathbf{b})$ is p-valent in every disc $\{t:\ |t|\le 1/L(z^0)\}$ for every point $z^0\in\mathbb{D}^n$. For another classes of multivariate analytic and slice holomorphic functions the notion is considered in [21]. If $n=1$, $\mathbf{b}=1$ and $L\equiv 1$ then the notion matches with a definition of function of bounded value distribution [22–25], and if $n=1$, $\mathbf{b}=1$, $L=l\not\equiv 1$ then it is a definition of function of bounded value l-distribution [6,26]. Another approach to multivalence of bivariate function is considered in [27]. Our main result on this topic is the following

Proposition 3.5. *Let $\Psi\in\mathscr{A}(\mathbb{D}^n)$, $f\in\mathscr{A}(\mathbb{S})$, $F(z)=f\circ\Psi(z)$. $l\in Q$, $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$ be such that $L(z)\ge\big|\partial_{\mathbf{b}}\Psi(z)\big|l\circ\Psi(z)$ for any z with $\mathbb{D}^n$. If these functions satisfy such hypotheses*
1) $N(f,l)$ is finite;
2) the function Ψ has bounded value L-distribution in the direction $\mathbf{b}$*,*
3) for any $r_1\in(0;\beta]$ there exist $r_2>0$ and $r_3>0$ for which the following inclusion $G_{r_2}(f;l)\subset\Psi(G^{\mathbf{b}}_{r_1}(F;L))\subset G_{r_3}(f;l)$ is true, then F is of bounded L-index in the direction $\mathbf{b}$.

Proof. The condition 3) allows us to prove inequality (3.6) by similarity to Proposition 3.3.
Inequality (3.2) is valid for F because equality $F(z^0+t\mathbf{b})=0$ yields the equation $\Psi(z^0+t\mathbf{b})=c_k$, where c_k span whole zero set of the function f, $k\in\mathbb{N}$. Since Ψ has bounded value L-distribution in the direction $\mathbf{b}$, the last equation $\Psi(z^0+t\mathbf{b})=c_k$ has at most $p(r_1)$ solutions for given k at the disc $\{t:\ |t|\le\frac{r_1}{L(z^0)}\}$, if $r_1\in(0;\beta)$. Condition 3) means that the set $\{\Psi(z^0+t\mathbf{b}):|t|\le\frac{r_1}{L(z^0)}\}$ includes at most $n(r_3)$ zeros of f. Thus, such a set $\{z^0+t\mathbf{b}:\ |t|\le\frac{r_1}{L(z^0)}\}$ holds at most $p(r_1)\cdot n(r_3)$ zeros of F. In other words, zeros of the F are uniformly distributed in the sense of validity (3.2). Then by the logarithnmic criterion analog (Theorem 3.2) the function F is of bounded L-index in the direction $\mathbf{b}$. □

It is worth recognizing that Theorems 2.1 and 2.4, Propositions 3.3 and 3.5 are varied assumptions by the outer and inner function of the composition. But their consequence is the similar: a composite function is of bounded L-index in the direction $\mathbf{b}$ with alike functions

$$L(z)=\big|\partial_{\mathbf{b}}\Psi(z)\big|\cdot l_m^{\circ}\Psi(z)\text{ or }L(z)=\max\{1,|\partial_{\mathbf{b}}\Psi(z)|\}\cdot l_m^{\circ}\Psi(z).$$

But there were presented examples of analytic functions in the unit ball which dissatisfy concurrently assumptions of these statements (see examples in [2]).

4. Conclusion

Proposition 3.5 has not an analog for another multidimensional approach — so-called index in joint variables. Recent results for composite entire functions with bounded index in joint variables were deduced in [28]. They are similar to Theorem 2.1 and Theorem 2.4. Proposition 3.5 uses the notion of bounded value distribution in a direction. For multivariate complexvalued entire functions F. Nuray [27] introduced a notion of multivalence and indicated some connection between multivalued functions and functions with finite index in joint variables. The multivalence means bounded value distribution in some sense. But we do not know whether is it possible to deduce analogs of Propositon 3.5 for this class of functions which is intensively examined in papers of F. Nuray and R. Patterson [19,29–31].
Let us present a brief description of possible investigations. Other important meanings of the obtained results is their application to composite differential equations. Changing variables we can reduce such a equation to simpler form and investigate the form by index boundedness of its solution. Further, we perform the inverse changing variables and obtain composition of analytic solutions of simpler equations and a mapping given by the changing variables. Therefore, we can apply the obtained results to such compositions and conclude about L-index boundedness in direction of primary equation for some function L and direction $\mathbf{b}$.

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References

[1] A. Bandura, T. Salo, *Analytic in a unit polydisc functions of bounded L-index in direction*, Mat. Stud., **60**(1) (2023), 55–78.
[2] V. P. Baksa, A. I. Bandura, T. M. Salo, Skaskiv O.B., *Note on boundedness of the L-index in the direction of the composition of slice entire functions*, Mat. Stud., **58** (1) (2022), 58–68.
[3] A. I. Bandura, M. M. Sheremeta, *Bounded l-index and $l-M$-index and compositions of analytic functions*, Mat. Stud., **48**(2) (2017), 180-188.
[4] M. M. Sheremeta, *On the l-index boundedness of some composition of functions*, Mat. Stud., **47**(2) (2017), 207–210.
[5] B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index*, in: Entire Functions and Related Parts of Analysis, J. Korevaar (ed.), Proceedings of Symposia in Pure Math., **11**, Am. Math. Soc., Providence (1968), 298–307.
[6] A. D. Kuzyk, M. M. Sheremeta, *Entire functions of bounded l-distribution of values*, Math. Notes, **39**(1) (1986), 3–8.
[7] A. I. Bandura, *Composition, product and sum of analytic functions of bounded L-index in direction in the unit ball*, Mat. Stud., **50**(2) (2018), 115–134.
[8] A. Bandura, *Composition of entire functions and bounded L-index in direction*, Mat. Stud., **47**(2) (2017), 179–184.
[9] A. I. Bandura, O. B. Skaskiv, *Entire functions of bounded L-index in direction*, Mat. Stud., **27**(1) (2007), 30–52. (in Ukrainian)
[10] W. K. Hayman, *Differential inequalities and local valency,* Pacific J. Math., **44** (1) (1973), 117-137.
[11] A. I. Bandura, O. B. Skaskiv, I. R. Tymkiv, *Composition of entire and analytic functions in the unit ball*, Carpathian Math. Publ., **14** (1) (2022), 95–103.
[12] M. M. Sheremeta, Y.S. Trukhan, *Boundedness of the l-index of the Naftalevich-Tsuji product*, Ukr. Math. J., **56**(2) (2004), 305–317.
[13] A. Bandura, O. Skaskiv, L. Smolovyk, *Slice holomorphic solutions of some directional differential equations with bounded L-index in the same direction*, Demonstr. Math., **52**(1) (2019), 482–489.
[14] A. A. Goldberg, M. N. Sheremeta, *Existence of an entire transcendental function of bounded l-index*, Math. Notes, **57**(1) (1995), 88–90.
[15] I. M. Hural, *About some problem for entire functions of unbounded index in any direction*, Mat. Stud., **51**(1) (2019), 107–110.
[16] M. M. Sheremeta, Y. S. Trukhan, *Properties of analytic solutions of three similar differential equations of the second order*, Carp. Math. Publ., **13**(2) (2021), 413–425.
[17] M. M. Sheremeta, Y. S. Trukhan, *Properties of analytic solutions of a differential equation*, Mat. Stud., **52** (2) (2019), 138–143.
[18] A. Bandura, O. Skaskiv, *Analog of Hayman's Theorem and its Application to Some System of Linear Partial Differential Equations*, J. Math. Phys., Anal., Geom., **15**(2) (2019), 170–191.
[19] F. Nuray, R.F. Patterson, *Vector-valued bivariate entire functions of bounded index satisfying a system of differential equations*, Mat. Stud., **49**(1) (2018), 67–74.
[20] A. I. Bandura, *Some improvements of criteria of L-index boundedness in direction*, Mat. Stud., **47**(1) (2017), 27–32.
[21] A. I. Bandura, *Analytic functions in the unit ball of bounded value L-distribution in a direction*, Mat. Stud., **49** (1) (2018), 75–79.
[22] G. H. Fricke, *A note on bounded index and bounded value distribution*, Indian J. Pure Appl. Math. **11** (4) (1980), 428–432.
[23] S. Shah, *Entire functions of bounded value distribution and gap power series*, In: Studies in Pure Mathematics To the Memory of Paul Turán, (P. Erdős, L. Alpár, G. Halász, A. Sárközy, eds.). Birkhäuser Basel, Basel, 1983. pp. 629-634.
[24] R. Roy, S. M. Shah, *The product of two functions of bounded value distribution*, Indian J. Pure Appl. Math. **17**(5) (1986), 690–693.
[25] R. Roy, S. M. Shah, *Functions of bounded index, bounded value distribution and v-bounded index*, Nonlinear Analysis **11** (1987), 1383–1390.
[26] M. M. Sheremeta, *On the univalence of entire functions of bounded l-index*, Mat. Stud., **43**(2) (2015), 185–188.
[27] F. Nuray, R. F. Patterson, *Multivalence of bivariate functions of bounded index*, Le Matematiche, **70**(2) (2015), 225–233.
[28] A. Bandura, T. Salo, O. Skaskiv, *L-Index in Joint Variables: Sum and Composition of an Entire Function with a Function With a Vanished Gradient*, Fractal and Fractional, **7**(8) (2023), Article ID: 593.
[29] F. Nuray, R. F. Patterson, *Entire bivariate functions of exponential type*, Bull. Math. Sci. **2015**, **5** () (2015), 171–177.
[30] F. Nuray, *Bounded index and four dimensional summability methods*, Novi Sad J. Math., **49**(2) (2019), 73–85.
[31] R. F. Patterson, F. A. Nuray, *A characterization of holomorphic bivariate functions of bounded index*, Math. Slov., **67**(3) (2017), 731–736.

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Multiple Positive Symmetric Solutions for the Fourth-Order Iterative Differential Equations Involving p-Laplacian with Integral Boundary Conditions

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Abstract

The purpose of this paper is to investigate the existence of multiple positive symmetric solutions for fourth order p-Laplacian iterative system with integral boundary conditions. Initially, we establish the existence of at least one and two positive symmetric solutions for the fourth order problem using Krasnosel'skii fixed point theorem. Subsequently, we establish the existence of at least three positive symmetric solutions by applying five-functionals fixed point theorem.

1. Introduction

Boundary value problems (BVPs) associated with ordinary differential equations play a significant role in various fields, including physics, chemistry, engineering, biotechnology, and social sciences. The higher order differential equations with specific types of iterative differential equations are important for analyzing the characteristics like monotonicity, convexity, equivariance, smoothness, and numerical solutions (see [1–5]). It is also worth noting that differential equations with integral boundary conditions are crucial in modeling phenomena such as plasma physics, underground water flow, chemical engineering, heat conduction, and thermo-elasticity.

In the theory of differential equations, one of the most significant operators is one dimensional p-Laplacian operator and is defined as $\phi_{\mathrm{p}}(z)=|z|^{\mathrm{p}-2}z$, where $\mathrm{p}>1$, $\phi_{\mathrm{p}}^{-1}=\phi_{\mathrm{q}}$ and $\frac{1}{\mathrm{p}}+\frac{1}{\mathrm{q}}=1$. Such problems can be found in the mathematical modeling of image processing, heat radiation, glaciology, biophysics, plasma physics, rheology, plastic molding, etc (see [6,7]). In particular, fourth-order BVPs with the p-Laplacian operator, have diverse applications in brain warping, fluids in lungs, ice formation, beam theory, and designing special curves on surfaces. The applications highlight the wide range of uses and significance of the p-Laplacian operator in several fields (see [8–14]). Various approaches, like fixed point theorems, iterative techniques, and shooting methods, are employed to establish the existence of solutions for such problems (see [15–17]). In 2000, Avery and Henderson [18] considered the problem

$$\begin{aligned}&y''(z)+\mathtt{f}(y)=0,\ \ 0\leqslant z\leqslant 1,\\&y(0)=0=y(1),\end{aligned}$$

and established the existence of at least three symmetric positive solutions by using the generalization of Leggett-Williams fixed point theorem. In 2015, Akcan and Hamal [19] established the existence of concave symmetric positive solutions for the BVP

$$\begin{aligned}&y''(z)+\mathtt{f}(z,y(z),y'(z))=0,\ 0<z<1,\\&y(0)=y(1)=\psi\int_0^{\eta}y(\mathrm{x})\mathrm{dx},\end{aligned}$$

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where $\psi, \eta \in (0,1)$ by applying monotone iterative technique. In 2016, [20] Ding established the existence of symmetric positive solutions for the p-Laplacian BVP

$$\begin{aligned}&(\phi_p(y'(z)))' + \mathtt{f}(z, y(z), y'(z)) = 0, \ \ 0 \le z \le 1,\\ &y(0) = y(1) = \int_0^1 y(\mathrm{x})\mathtt{g}(\mathrm{x})\mathrm{dx},\end{aligned}$$

by using the fixed point theorem due to Avery and Peterson. In 2020, [21] Asaduzzamana and Ali established the existence of symmetric positive solutions for the BVP

$$\begin{aligned}&-y^{(4)}(z) = \mathtt{f}(y,v), \ z \in [0,1],\\ &-v^{(4)}(z) = \mathtt{f}(y,v), \ z \in [0,1],\\ &y(z) = y(1-z), \ y'''(0) - y'''(1) = y''(z_1) + y''(z_2),\\ &v(z) = v(1-z), \ y'''(0) - v'''(1) = v''(z_1) + v''(z_2), \ \ 0 < z_1 < z_2 < 1,\end{aligned}$$

by applying Krasnoselskii's fixed point theorem. Following that, the researchers have explored the study of symmetric positive solutions, see [22–30]. Inspired by the works mentioned above, we investigate the existence of multiple positive symmetric solutions for the fourth order p-Laplacian iterative system with integral boundary conditions

$$\left.\begin{aligned}&(\phi_p(v(z)y_n''(z)))'' = w(z)\mathtt{f}_n(z, y_{n+1}(z)), \ \ 1 \le n \le i, \ \ z_1 \le z \le z_2,\\ &y_{i+1}(z) = y_1(z), \ \ z_1 \le z \le z_2,\end{aligned}\right\} \tag{1.1}$$

satisfying boundary conditions

$$\left.\begin{aligned}&y_n(z_1) = \int_{z_1}^{z_2} \mathtt{g}(\mathtt{s})y_n(\mathtt{s})\mathtt{ds}, \ \ y_n(z_2) = \int_{z_1}^{z_2} \mathtt{g}(\mathtt{s})y_n(\mathtt{s})\mathtt{ds}, \ \ 1 \le n \le i,\\ &\phi_p(v(z_1)y_n''(z_1)) = \int_{z_1}^{z_2} \mathtt{h}(\mathtt{s})\phi_p(v(\mathtt{s})y_n''(\mathtt{s}))\mathtt{ds}, \ \ \phi_p(v(z_2)y_n''(z_2)) = \int_{z_1}^{z_2} \mathtt{h}(\mathtt{s})\phi_p(v(\mathtt{s})y_n''(\mathtt{s}))\mathtt{ds}, \ \ 1 \le n \le i,\end{aligned}\right\} \tag{1.2}$$

where $i \in \mathbb{N}$ with $2z_1 < z_2$, $\phi_p(z) = |z|^{p-2}z$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. The following conditions are presumed to be valid in the entire paper:

(I1) $\mathtt{f}_n : [z_1, z_2] \times [z_1, \infty) \to [z_1, \infty)$ is continuous, $\mathtt{f}_n(z_2 + z_1 - z, y) = \mathtt{f}_n(z, y)$, $1 \leqslant n \leqslant i$ for all $(z, y) \in [z_1, z_2] \times [z_1, \infty)$. (For existence of solution)

(I2) $v(z), w(z) \in L^1[z_1, z_2]$ are positive, symmetric on $[z_1, z_2]$ (i.e., $v(z_2 + z_1 - z) = v(z)$ for $z \in [z_1, z_2]$). (For positive symmetric solution)

(I3) $\mathtt{g}(z), \mathtt{h}(z) \in L^1[z_1, z_2]$ are non-negative, symmetric on $[z_1, z_2]$, and μ_1, $\mu_2 \in (z_1, z_2)$, $\mu_1 = \int_{z_1}^{z_2} \mathtt{g}(\mathtt{s})\mathtt{ds}$, $\mu_2 = \int_{z_1}^{z_2} \mathtt{h}(\mathtt{s})\mathtt{ds}$. (For positive symmetric solution)

The organization of the remaining part of the paper is as follows. In Section 2, we construct Green's function and estimate the bounds for Green's function for the problem (1.1)-(1.2). In section 3, we establish the existence of at least one and two positive symmetric solutions by using Krasnoselskii's fixed point theorem. Using the five-functional fixed point theorem, we establish the existence of at least three positive symmetric solutions. In Section 4, we provide examples to check the validity of the results.

2. Green's Function and Its Bounds

Here, we determine the solution of (1.1)-(1.2) as a solution of the integral equation that includes Green's function. After that, we establish a few characteristics of the Green's function which are useful in establishing our main results.

Lemma 2.1. *Assume that* (I2) − (I3) *hold. Then for any* $u_1(z) \in C([z_1, z_2], \mathbb{R})$*, the BVP*

$$\phi_p(v(z)y_1''(z)) = u_1(z), \ \ z_1 \le z \le z_2, \tag{2.1}$$

$$y_1(z_1) = \int_{z_1}^{z_2} \mathtt{g}(\mathtt{s})y_1(\mathtt{s})\mathtt{ds}, \ \ y_1(z_2) = \int_{z_1}^{z_2} \mathtt{g}(\mathtt{s})y_1(\mathtt{s})\mathtt{ds}, \tag{2.2}$$

has one and only one solution

$$y_1(z) = -\int_{z_1}^{z_2} \mathtt{H}_1(z,t)v^{-1}(t)\phi_q(u_1(t))\mathrm{d}t,$$

where $\mathtt{H}_1(z,t)$ *is the Green's function and is given by*

$$\mathtt{H}_1(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\mu_1}\int_{z_1}^{z_2} \mathrm{G}(\mathtt{s},t)\mathtt{g}(\mathtt{s})\mathtt{ds}, \tag{2.3}$$

in which

$$\mathrm{G}(z,t) = \frac{1}{z_2 - z_1}\begin{cases}(z - z_1)(z_2 - t), & z \leqslant t,\\ (t - z_1)(z_2 - z), & t \leqslant z.\end{cases} \tag{2.4}$$

Proof. Integrating (2.1) twice from z_1 to z, we get

$$y_1(z)=\int_{z_1}^{z}(z-t)v^{-1}(t)\phi_q(u_1(t))dt+c_1(z-z_1)+c_2.$$

By using boundary conditions (2.2), we get

$$c_1=\frac{-1}{z_2-z_1}\int_{z_1}^{z_2}(z_2-t)v^{-1}(t)\phi_q(u_1(t))dt,\ \text{ and }\ c_2=\int_{z_1}^{z_2}g(s)y_1(s)ds.$$

So, we have

$$\begin{aligned}y_1(z)&=\int_{z_1}^{z}(z-t)v^{-1}(t)\phi_q(u_1(t))dt+\frac{-1}{z_2-z_1}\int_{z_1}^{z_2}(z-z_1)(z_2-t)v^{-1}(t)\phi_q(u_1(t))dt+\int_{z_1}^{z_2}g(s)y_1(s)ds\\&=-\int_{z_1}^{z_2}G(z,t)v^{-1}(t)\phi_q(u_1(t))dt+\int_{z_1}^{z_2}g(s)y_1(s)ds.\end{aligned}$$

After certain computations, we obtain

$$\int_{z_1}^{z_2}g(s)y_1(s)ds=\frac{-1}{1-\mu_1}\int_{z_1}^{z_2}\int_{z_1}^{z_2}G(s,t)v^{-1}(t)\phi_q(u_1(t))dtds.$$

Therefore,

$$\begin{aligned}y_1(z)&=-\int_{z_1}^{z_2}G(z,t)v^{-1}(t)\phi_q(u_1(t))dt+\frac{-1}{1-\mu_1}\int_{z_1}^{z_2}\int_{z_1}^{z_2}G(s,t)v^{-1}(t)\phi_q(u_1(t))dtds\\&=-\int_{z_1}^{z_2}\left[G(z,t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}G(s,t)g(s)ds\right]v^{-1}(t)\phi_q(u_1(t))dt\\&=-\int_{z_1}^{z_2}H_1(z,t)v^{-1}(t)\phi_q(u_1(t))dt.\end{aligned}$$

□

Lemma 2.2. *Suppose* (I3) *holds. For* $\lambda\in(z_1,\frac{z_2}{2})$, *let* $\sigma(\lambda)=\frac{\lambda-z_1}{z_2-z_1}$, $\alpha_1=\frac{1}{1-\mu_1}$. *Then* $G(z,t)$, $H_1(z,t)$ *have the following properties:*

(A1) $0\leqslant G(z,t)\leqslant G(t,t)$, $\forall\, z,t\in[z_1,z_2]$,
(A2) $0\leqslant H_1(z,t)\leqslant\alpha_1 G(t,t)$, $\forall\, z,t\in[z_1,z_2]$,
(A3) $G(z,t)\geqslant\sigma(\lambda)G(t,t)$, $\forall\, z\in[\lambda,z_2-\lambda]$ *and* $t\in[z_1,z_2]$,
(A4) $H_1(z,t)\geqslant\sigma(\lambda)\alpha_1 G(t,t)$, $\forall\, z\in[\lambda,z_2-\lambda]$ *and* $t\in[z_1,z_2]$,
(A5) $G(z_2+z_1-z,z_2+z_1-t)=G(z,t)$, $H_1(z_2+z_1-z,z_2+z_1-t)=H_1(z,t)$, $\forall\, z,t\in[z_1,z_2]$.

Proof. From (2.3) and (2.4), it is clear that the properties (A1) and (A2) hold.

For inequality (A3), let $z\in[\lambda,z_2-\lambda]$ and $z\leqslant t$, then

$$\frac{G(z,t)}{G(t,t)}=\frac{(z-z_1)(z_2-t)}{(t-z_1)(z_2-t)}\geqslant\sigma(\lambda),$$

and for $t\leqslant z$,

$$\frac{G(z,t)}{G(t,t)}=\frac{(t-z_1)(z_2-z)}{(t-z_1)(z_2-t)}\geqslant\sigma(\lambda).$$

Hence, the inequality (A3). For the inequality (A4), consider

$$\begin{aligned}H_1(z,t)&=G(z,t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}G(s,t)g(s)ds\\&\geqslant\sigma(\lambda)G(t,t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}\sigma(\lambda)G(t,t)g(s)ds.\end{aligned}$$

Hence, $H_1(z,t)\geqslant\sigma(\lambda)\alpha_1G(t,t)$. For inequality (A5), consider

$$\begin{aligned}G(z_2+z_1-z,z_2+z_1-t)&=\frac{1}{z_2-z_1}\begin{cases}(z_2+z_1-z-z_1)(z_2-(z_2+z_1-t)), & z_2+z_1-z\leqslant z_2+z_1-t,\\(z_2+z_1-t-z_1)(z_2-(z_2+z_1-z)), & z_2+z_1-t\leqslant z_2+z_1-z,\end{cases}\\&=\frac{1}{z_2-z_1}\begin{cases}(z-z_1)(z_2-t), & z\leqslant t,\\(t-z_1)(z_2-z), & t\leqslant z,\end{cases}\\&=G(z,t).\end{aligned}$$

Consider

$$\begin{aligned}
\mathtt{H}_1(z_2+z_1-z,z_2+z_1-t) =&\mathrm{G}(z_2+z_1-z,z_2+z_1-t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}\mathrm{G}(\mathtt{s},z_2+z_1-t)\mathtt{g}(\mathtt{s})\mathtt{ds}\\
=&\mathrm{G}(z,t)+\frac{1}{1-\mu_1}\int_{z_2}^{z_1}\mathrm{G}(z_2+z_1-\mathtt{s},z_2+z_1-t)\mathtt{g}(z_2+z_1-\mathtt{s})\mathtt{d}(z_2+z_1-\mathtt{s})\\
=&\mathrm{G}(z,t)+\frac{1}{1-\mu_1}\int_{z_1}^{z_2}\mathrm{G}(\mathtt{s},t)\mathtt{g}(\mathtt{s})\mathtt{ds}\\
=&\mathtt{H}_1(z,t).
\end{aligned}$$

□

Lemma 2.3. *Assume that* $(\mathtt{I2})-(\mathtt{I3})$ *hold. Then for any* $u_2(z)\in \mathrm{C}([z_1,z_2],\mathbb{R})$*, the BVP*

$$(\phi_{\mathrm{p}}(v(z)y_1''(z)))''=u_2(z),\ \ z_1\le z\le z_2,$$

satisfying boundary conditions

$$y_1(z_1)=\int_{z_1}^{z_2}\mathtt{g}(\mathtt{s})y_1(\mathtt{s})\mathtt{ds},\ \ y_1(z_2)=\int_{z_1}^{z_2}\mathtt{g}(\mathtt{s})y_1(\mathtt{s})\mathtt{ds},$$

$$\phi_{\mathrm{p}}(v(z_1)y_1''(z_1))=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_1''(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(z_2)y_1''(z_2))=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_1''(\mathtt{s}))\mathtt{ds},$$

has a unique solution

$$y_1(z)=\int_{z_1}^{z_2}\mathtt{H}_1(z,t)v^{-1}(t)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t,\mathtt{s})u_2(\mathtt{s})\mathtt{ds}\right]\mathtt{d}t,$$

where $\mathtt{H}_1(z,t)$ *is given in (2.3) and*

$$\mathtt{H}_2(z,t)=\mathrm{G}(z,t)+\frac{1}{1-\ \mu_2}\int_{z_1}^{z_2}\mathrm{G}(\mathtt{s},t)\mathtt{h}(\mathtt{s})\mathtt{ds}.$$

Proof. Let, $u_1(z)=\phi_{\mathrm{p}}(v(z)y_1''(z))$ for $z_1\le z\le z_2$. Then the BVP

$$(\phi_{\mathrm{p}}(v(z)y_1''(z)))''=u_2(z),\ \ z_1\le z\le z_2,$$

$$\phi_{\mathrm{p}}(v(z_1)y_1''(z_1))=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_1''(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(z_2)y_1''(z_2))=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_1''(\mathtt{s}))\mathtt{ds}$$

is equivalent to the problem

$$u_1''(z)=u_2(z),\ \ z_1\le z\le z_2, \tag{2.5}$$

$$u_1(z_1)=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})u_1(\mathtt{s})\mathtt{ds},\ \ u_1(z_2)=\int_{z_1}^{z_2}\mathtt{h}(\mathtt{s})u_1(\mathtt{s})\mathtt{ds}. \tag{2.6}$$

By Lemma 2.1, the BVP (2.5)-(2.6) has unique solution $u_1(z)=-\int_{z_1}^{z_2}\mathtt{H}_2(z,t)u_2(t)\mathtt{d}t$. That is

$$\phi_{\mathrm{p}}(v(z)y_1''(z))=-\int_{z_1}^{z_2}\mathtt{H}_2(z,t)u_2(t)\mathtt{d}t \tag{2.7}$$

Again by Lemma 2.1, the differential equation (2.7) with boundary conditions

$$y_1(z_1)=y_1(z_2)=\int_{z_1}^{z_2}\mathtt{g}(\mathtt{s})y_1(\mathtt{s})\mathtt{ds},$$

has a unique solution

$$y_1(z)=\int_{z_1}^{z_2}\mathtt{H}_1(z,t)v^{-1}(t)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t,\mathtt{s})u_2(\mathtt{s})\mathtt{ds}\right]\mathtt{d}t.$$

This completes the proof. □

Lemma 2.4. *Suppose* $(\mathtt{I3})$ *holds. For* $\lambda\in(z_1,\frac{z_2}{2})$*, let* $\sigma(\lambda)=\frac{\lambda-z_1}{z_2-z_1}$*,* $\alpha_2=\frac{1}{1-\mu_2}$*. Then,* $\mathtt{H}_2(z,t)$ *has the following properties:*

$(\mathtt{A6})$ $0\leqslant \mathtt{H}_2(z,t)\leqslant \alpha_2\mathrm{G}(t,t)$, $\forall\, z,t\in[z_1,z_2]$,
$(\mathtt{A7})$ $\mathtt{H}_2(z,t)\geqslant\sigma(\lambda)\alpha_2\mathrm{G}(t,t)$, $\forall\, z\in[\lambda,z_2-\lambda]$ *and* $t\in[z_1,z_2]$,
$(\mathtt{A8})$ $\mathtt{H}_2(z_2+z_1-z,z_2+z_1-t)=\mathtt{H}_2(z,t)$, $\forall\, z,t\in[z_1,z_2]$.

Note that an i-tuple $\big(y_1(z),y_2(z),\cdots,y_i(z)\big)$ is a solution of (1.1)-(1.2) if and only if

$$y_{\mathtt{n}}(z)=\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_{\mathtt{n}}(t_2,y_{\mathtt{n}+1}(t_2))\mathtt{d}t_2\right]\mathtt{d}t_1,\ \mathtt{n}=1,2,\cdots,i,$$
$$y_{i+1}(z)=y_1(z),\ \ z\in[z_1,z_2],\ \ 1\le\mathtt{n}\le i,$$

i.e.,

$$\begin{aligned}y_1(z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\right.\right.\right.\\&\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\right.\right.\\&\left.\left.\left.\left.\left.\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1.\end{aligned}$$

3. Existence of Positive Symmetric Solutions

Let $\mathrm{B}=\{y:y\in\mathrm{C}([z_1,z_2],\mathbb{R})\}$ be a Banach space with norm $\|y\|=\max_{z\in[z_1,z_2]}|y(z)|$. For $\lambda\in(z_1,\frac{z_2}{2})$, we define the cone $\mathrm{K}\subset\mathrm{B}$ as

$$\mathrm{K}=\big\{y\in\mathrm{B}:y(z)\geqslant0,y(z)\text{ is concave, symmetric on }[z_1,z_2]\text{ and }\min_{z\in[\lambda,z_2-\lambda]}y(z)\geqslant\sigma(\lambda)\|y\|\big\}.$$

Define operator $\mathrm{T}:\mathrm{K}\to\mathrm{B}$ by

$$\begin{aligned}\mathrm{T}y_1(z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\&\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1.\end{aligned}$$

Let,

$$m\leqslant\alpha_1\int_{z_1}^{z_2}\mathrm{G}(t_j,t_j)v^{-1}(t_j)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{f}\mathrm{G}(t_{j+1},t_{j+1})w(t_{j+1})\mathtt{d}t_{j+1}\right]\mathtt{d}t_j,\ j=1,2,\cdots2i-1,$$
$$M\geqslant\sigma(\lambda)\alpha_1\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_j,t_j)v^{-1}(t_j)\phi_{\mathtt{q}}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{j+1},t_{j+1})w(t_{j+1})\mathtt{d}t_{j+1}\right]\mathtt{d}t_j,\ j=1,2,\cdots2i-1.$$

Lemma 3.1. *For each $\lambda\in(z_1,\frac{z_2}{2})$, $\mathrm{T}(\mathrm{K})\subset\mathrm{K}$ and $\mathrm{T}:\mathrm{K}\to\mathrm{K}$ is completely continuous.*

Proof. Since $\mathtt{H}_1(z,t)\geqslant0,\ \mathtt{H}_2(z,t)\geqslant0,\ \forall\,z,t\in[z_1,z_2],\ (\mathrm{T}y_1)(z)\geqslant0$. Let $y_1\in\mathrm{K}$, then consider

$$\begin{aligned}(\mathrm{T}y_1)(z_2+z_1-z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z_2+z_1-z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)\right.\right.\right.\\&w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\right.\right.\\&\left.\left.\left.\left.\left.\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\\=&\int_{z_2}^{z_1}\mathtt{H}_1(z_2+z_1-z,z_2+z_1-t_1)v^{-1}(z_2+z_1-t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(z_2+z_1-t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)\right.\right.\\&\left.\left.\cdots\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\Big)\mathtt{d}t_2\Big]\mathtt{d}(z_2+z_1-t_1)\\=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_2}^{z_1}\mathtt{H}_2(z_2+z_1-t_1,z_2+z_1-t_2)w(z_2+z_1-t_2)\mathtt{f}_1\left(z_2+z_1-t_2,\right.\right.\\&\left.\int_{z_1}^{z_2}\mathtt{H}_1(z_2+z_1-t_2,t_3)\cdots\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\Big]\mathtt{d}t_3\Big)\\&\left.\mathtt{d}(z_2+z_1-t_2)\right]\mathtt{d}t_1\\&\vdots\\=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\\&\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathtt{q}}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\\&\left.\left.\left.\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\\=&(\mathrm{T}y_1)(z).\end{aligned}$$

Hence $\mathrm{T}y_1$ is symmetric on $[z_1,z_2]$. From Lemma 2.2, we get

$$\begin{aligned}
(\mathrm{T}y_1)(z) &= \int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\Bigg(t_{2i-2},\\
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\Bigg]\mathrm{d}t_{2i-1}\Bigg)\cdots\mathrm{d}t_4\Bigg]\mathrm{d}t_3\Bigg)\mathrm{d}t_2\Bigg]\mathrm{d}t_1\\
&\leqslant \alpha_1\int_{z_1}^{z_2}\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\Bigg(t_{2i-2},\\
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\Bigg]\mathrm{d}t_{2i-1}\Bigg)\cdots\mathrm{d}t_4\Bigg]\mathrm{d}t_3\Bigg)\mathrm{d}t_2\Bigg]\mathrm{d}t_1.
\end{aligned}$$

So,

$$\begin{aligned}
\|\mathrm{T}y_1\| \leqslant&\alpha_1\int_{z_1}^{z_2}\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\Bigg(t_{2i-2},\\
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\Bigg]\mathrm{d}t_{2i-1}\Bigg)\cdots\mathrm{d}t_4\Bigg]\mathrm{d}t_3\Bigg)\mathrm{d}t_2\Bigg]\mathrm{d}t_1.
\end{aligned}$$

Again from Lemma 2.2, we get

$$\begin{aligned}
\min_{z\in[\lambda,z_2-\lambda]}\{(\mathrm{T}y_1)(z)\} =& \min_{z\in[\lambda,z_2-\lambda]}\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)\\
&w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\\
&\mathrm{d}t_{2i}\Bigg]\mathrm{d}t_{2i-1}\Bigg)\cdots\mathrm{d}t_4\Bigg]\mathrm{d}t_3\Bigg)\mathrm{d}t_2\Bigg]\mathrm{d}t_1\\
\geqslant&\alpha_1\ \sigma(\lambda)\int_{z_1}^{z_2}\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)\\
&w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\\
&\mathrm{d}t_{2i}\Bigg]\mathrm{d}t_{2i-1}\Bigg)\cdots\mathrm{d}t_4\Bigg]\mathrm{d}t_3\Bigg)\mathrm{d}t_2\Bigg]\mathrm{d}t_1.
\end{aligned}$$

By using above two inequalities one can write

$$\min_{z\in[\lambda,z_2-\lambda]}\{(\mathrm{T}y_1)(z)\}\geqslant\sigma(\lambda)\|\mathrm{T}y_1\|.$$

So, $\mathrm{T}y_1\in\mathrm{K}$ and thus $\mathrm{T}(\mathrm{K})\subset\mathrm{K}$. By using Arzela-Ascoli theorem and standard methods it can be prove T is completely continuous. □

Theorem 3.2. *Let* $(\mathtt{I1})-(\mathtt{I3})$ *hold. Also assume that the following hold,*

$(\mathtt{I4})$ $\displaystyle\lim_{y\to0^+}\frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)}=0,\ \lim_{y\to+\infty}\frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)}=+\infty,\quad 1\le\mathtt{n}\le i$ *for* $z\in[z_1,z_2]$.

Then the BVP (1.1)-(1.2) has at least one positive symmetric solution.

Proof. Since $\displaystyle\lim_{y\to0^+}\frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)}=0$, there exists $l_1>0$ such that

$$\mathtt{f}_{\mathtt{n}}(z,y)\leqslant\eta\phi_{\mathrm{p}}(y),\ 0\leqslant y\leqslant l_1,\ z\in[z_1,z_2],\ \text{where}\ \eta\leqslant\phi_{\mathrm{p}}\left(\frac{1}{m}\right).$$

Let $\Theta_1=\{y\in\mathrm{B}:\|y\|<l_1\}$, if $y_1\in\mathrm{K}\cap\partial\Theta_1$, and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\Bigg]\mathrm{d}t_{2i-1}\\
&\leqslant\int_{z_1}^{z_2}\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\eta\phi_{\mathrm{p}}(y_1(t_{2i}))\mathrm{d}t_{2i}\Bigg]\mathrm{d}t_{2i-1}\\
&\leqslant\phi_{\mathrm{q}}(\eta)\alpha_1 l_1\int_{z_1}^{z_2}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathrm{d}t_{2i}\Bigg]\mathrm{d}t_{2i-1}\\
&\leqslant l_1 m\phi_{\mathrm{q}}(\eta)\leqslant l_1.
\end{aligned}$$

Similarly for $t_{2i-4} \in [z_1, z_2]$

$$\begin{aligned}
&\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-4}, t_{2i-3}) v^{-1}(t_{2i-3}) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-3}, t_{2i-2}) w(t_{2i-2}) \mathtt{f}_{i-1}\left(t_{2i-2}, \int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2}, t_{2i-1}) v^{-1}(t_{2i-1})\right.\right.\\
&\left.\left.\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1}, t_{2i}) w(t_{2i}) \mathtt{f}_i(t_{2i}, y_1(t_{2i})) \mathrm{d}t_{2i}\right] \mathrm{d}t_{2i-1}\right) \mathrm{d}t_{2i-1}\right] \mathrm{d}t_{2i-3}\\
&\leqslant \int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-4}, t_{2i-3}) v^{-1}(t_{2i-3}) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-3}, t_{2i-2}) w(t_{2i-2}) \mathtt{f}_{i-1}\left(t_{2i-2}, l_1\right) \mathrm{d}t_{2i-1}\right] \mathrm{d}t_{2i-3}\\
&\leqslant \int_{z_1}^{z_2} \alpha_1 \mathrm{G}(t_{2i-4}, t_{2i-3}) v^{-1}(t_{2i-3}) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \alpha_2 \mathrm{G}(t_{2i-3}, t_{2i-2}) w(t_{2i-2}) \eta \phi_{\mathrm{p}}(l_1) \mathrm{d}t_{2i-1}\right] \mathrm{d}t_{2i-3}\\
&\leqslant \phi_{\mathrm{q}}(\eta) \alpha_1 l_1 \int_{z_1}^{z_2} \mathrm{G}(t_{2i-3}, t_{2i-3}) v^{-1}(t_{2i-3}) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \alpha_2 \mathrm{G}(t_{2i-2}, t_{2i-2}) w(t_{2i-2}) \mathrm{d}t_{2i-1}\right] \mathrm{d}t_{2i-3}\\
&\leqslant l_1 m \phi_{\mathrm{q}}(\eta) \leqslant l_1.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\mathrm{T}y_1(z) = &\int_{z_1}^{z_2} \mathtt{H}_1(z, t_1) v^{-1}(t_1) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1, t_2) w(t_2) \mathtt{f}_1\left(t_2, \int_{z_1}^{z_2} \mathtt{H}_1(t_2, t_3) v^{-1}(t_3) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3, t_4) w(t_4) \mathtt{f}_2 \cdots \mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\
&\left.\left.\left.\left.\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2}, t_{2i-1}) v^{-1}(t_{2i-1}) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1}, t_{2i}) w(t_{2i}) \mathtt{f}_i(t_{2i}, y_1(t_{2i})) \mathrm{d}t_{2i}\right] \mathrm{d}t_{2i-1}\right) \cdots \mathrm{d}t_4\right] \mathrm{d}t_3\right) \mathrm{d}t_2\right] \mathrm{d}t_1\\
&\leqslant l_1 = \|y_1\|.
\end{aligned}$$

So $\|\mathrm{T}y_1\| \leqslant \|y_1\|$ for all $y_1 \in \mathrm{K} \cap \partial\Theta_1$.
Since $\lim\limits_{y \to +\infty} \dfrac{\mathtt{f}_{\mathtt{n}}(z, y)}{\phi_{\mathrm{p}}(y)} = +\infty$, there exists $\overline{l_2} > 0$ such that

$$\mathtt{f}_{\mathtt{n}}(z, y) \geqslant \zeta \phi_{\mathrm{p}}(y),\ y \geqslant \overline{l_2},\ z \in [z_1, z_2],\ \text{where } \zeta \geqslant \phi_{\mathrm{p}}\left(\frac{1}{M}\right).$$

Let $l_2 = \max\{2l_1, \frac{\overline{l_2}}{\sigma(\lambda)}\}$ and $\Theta_2 = \{y \in \mathrm{B} : \|y\| < l_2\}$. For $y_1 \in \mathrm{K} \cap \partial\Theta_2$, we have

$$\min_{z \in [\lambda, z_2 - \lambda]} y_1(z) \geqslant \sigma(\lambda) \|y_1\| \geqslant \sigma(\lambda) l_2 \geqslant \overline{l_2}.$$

For $t_{2i-2} \in [z_1, z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2}, t_{2i-1}) v^{-1}(t_{2i-1}) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1}, t_{2i}) w(t_{2i}) \mathtt{f}_i(t_{2i}, y_1(t_{2i})) \mathrm{d}t_{2i}\right] \mathrm{d}t_{2i-1}\\
&\geqslant \int_{\lambda}^{z_2 - \lambda} \sigma(\lambda) \alpha_1 \mathrm{G}(t_{2i-1}, t_{2i-1}) v^{-1}(t_{2i-1}) \phi_{\mathrm{q}}\left[\int_{\lambda}^{z_2 - \lambda} \sigma(\lambda) \alpha_2 \mathrm{G}(t_{2i}, t_{2i}) w(t_{2i}) \zeta \phi_{\mathrm{p}}(y_1(t_{2i})) \mathrm{d}t_{2i}\right] \mathrm{d}t_{2i-1}\\
&\geqslant \phi_{\mathrm{q}}(\zeta) \sigma(\lambda) \alpha_1 l_2 \int_{\lambda}^{z_2 - \lambda} \mathrm{G}(t_{2i-1}, t_{2i-1}) v^{-1}(t_{2i-1}) \phi_{\mathrm{q}}\left[\int_{\lambda}^{z_2 - \lambda} \sigma(\lambda) \alpha_2 \mathrm{G}(t_{2i}, t_{2i}) w(t_{2i}) \mathrm{d}t_{2i}\right] \mathrm{d}t_{2i-1}\\
&\geqslant l_2 M \phi_{\mathrm{q}}(\zeta) \geqslant l_2.
\end{aligned}$$

Similarly for $t_{2i-4} \in [z_1, z_2]$

$$\begin{aligned}
&\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-4}, t_{2i-3}) v^{-1}(t_{2i-3}) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-3}, t_{2i-2}) w(t_{2i-2}) \mathtt{f}_{i-1}\left(t_{2i-2}, \int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2}, t_{2i-1}) v^{-1}(t_{2i-1})\right.\right.\\
&\left.\left.\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1}, t_{2i}) w(t_{2i}) \mathtt{f}_i(t_{2i}, y_1(t_{2i})) \mathrm{d}t_{2i}\right] \mathrm{d}t_{2i-1}\right) \mathrm{d}t_{2i-1}\right] \mathrm{d}t_{2i-3}\\
&\geqslant \int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-4}, t_{2i-3}) v^{-1}(t_{2i-3}) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-3}, t_{2i-2}) w(t_{2i-2}) \mathtt{f}_{i-1}\left(t_{2i-2}, l_2\right) \mathrm{d}t_{2i-1}\right] \mathrm{d}t_{2i-3}\\
&\geqslant \int_{\lambda}^{z_2 - \lambda} \sigma(\lambda) \alpha_1 \mathrm{G}(t_{2i-4}, t_{2i-3}) v^{-1}(t_{2i-3}) \phi_{\mathrm{q}}\left[\int_{\lambda}^{z_2 - \lambda} \sigma(\lambda) \alpha_2 \mathrm{G}(t_{2i-3}, t_{2i-2}) w(t_{2i-2}) \zeta \phi_{\mathrm{p}}(l_1) \mathrm{d}t_{2i-1}\right] \mathrm{d}t_{2i-3}\\
&\geqslant \phi_{\mathrm{q}}(\zeta) \sigma(\lambda) \alpha_1 l_2 \int_{\lambda}^{z_2 - \lambda} \mathrm{G}(t_{2i-3}, t_{2i-3}) v^{-1}(t_{2i-3}) \phi_{\mathrm{q}}\left[\int_{\lambda}^{z_2 - \lambda} \sigma(\lambda) \alpha_2 \mathrm{G}(t_{2i-2}, t_{2i-2}) w(t_{2i-2}) \mathrm{d}t_{2i-1}\right] \mathrm{d}t_{2i-3}\\
&\geqslant l_2 M \phi_{\mathrm{q}}(\zeta) \geqslant l_2.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\mathrm{T}y_1(z) = &\int_{z_1}^{z_2} \mathtt{H}_1(z, t_1) v^{-1}(t_1) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1, t_2) w(t_2) \mathtt{f}_1\left(t_2, \int_{z_1}^{z_2} \mathtt{H}_1(t_2, t_3) v^{-1}(t_3) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3, t_4) w(t_4) \mathtt{f}_2 \cdots \mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\
&\left.\left.\left.\left.\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2}, t_{2i-1}) v^{-1}(t_{2i-1}) \phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1}, t_{2i}) w(t_{2i}) \mathtt{f}_i(t_{2i}, y_1(t_{2i})) \mathrm{d}t_{2i}\right] \mathrm{d}t_{2i-1}\right) \cdots \mathrm{d}t_4\right] \mathrm{d}t_3\right) \mathrm{d}t_2\right] \mathrm{d}t_1\\
&\geqslant l_2 = \|y_1\|.
\end{aligned}$$

So $\|\mathrm{T}y_1\| \geqslant \|y_1\|$ for all $y_1 \in \mathrm{K} \cap \partial\Theta_2$. Consequently, Krasnoselskii's fixed point theorem [31, 32] guarantees that T has a fixed point $\mathrm{K} \cap (\overline{\Theta}_2 \setminus \Theta_1)$. □

Theorem 3.3. *Let* $(\mathtt{I1})-(\mathtt{I3})$ *hold. Also assume that the following conditions hold,*

$(\mathtt{I5})$ $\lim_{y\to 0^+}\frac{\mathtt{f}_\mathtt{n}(z,y)}{\phi_\mathrm{p}(y)}=+\infty,\ \lim_{y\to+\infty}\frac{\mathtt{f}_\mathtt{n}(z,y)}{\phi_\mathrm{p}(y)}=0,\ 1\le\mathtt{n}\le i$ *for* $z\in[z_1,z_2]$.

Then the BVP (1.1)-(1.2) has at least one positive symmetric solution.

Proof. We can establish the result by using the previous argument is in Theorem 3.2. □

Theorem 3.4. *Let* $(\mathtt{I1})-(\mathtt{I3})$ *hold. Also assume that the following conditions hold,*

$(\mathtt{I6})$ $\lim_{y\to 0^+}\frac{\mathtt{f}_\mathtt{n}(z,y)}{\phi_\mathrm{p}(y)}=+\infty,\ \lim_{y\to+\infty}\frac{\mathtt{f}_\mathtt{n}(z,y)}{\phi_\mathrm{p}(y)}=+\infty,\ 1\le\mathtt{n}\le i$ *for* $z\in[z_1,z_2]$.

$(\mathtt{I7})$ *There exists a constant* r_1 *such that* $\mathtt{f}_\mathtt{n}(z,y)\leqslant\phi_\mathrm{p}(\frac{r_1}{m})$ *for* $y\in[0,r_1]$, $z\in[z_1,z_2]$.

Then the BVP (1.1)-(1.2) has at least two positive symmetric solutions y_1^* *and* y_1^{**} *such that* $0<\|y_1^*\|<r_1<\|y_1^{**}\|$.

Proof. Since $\lim_{y\to 0^+}\frac{\mathtt{f}_\mathtt{n}(z,y)}{\phi_\mathrm{p}(y)}=+\infty$, there exists $r_*\in(0,r_1)$ such that $\mathtt{f}_\mathtt{n}(z,y)\geqslant\zeta_1\phi_\mathrm{p}(y)$, for $0\leqslant y\leqslant r_*$, $z\in[z_1,z_2]$, where $\zeta_1\geqslant\zeta$; here ζ is given in the proof of Theorem 3.2. Set $\Theta_3=\{y\in\mathrm{B}:\|y\|<r_*\}$. For $y_1\in\mathrm{K}\cap\partial\Theta_3$, and $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathrm{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\
&\geqslant\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathrm{q}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\zeta_1\phi_\mathrm{p}(y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\
&\geqslant\phi_\mathrm{q}(\zeta_1)\sigma(\lambda)\alpha_1 r_*\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathrm{q}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\\
&\geqslant r_*M\phi_\mathrm{q}(\zeta_1)\geqslant r_*.
\end{aligned}$$

Similarly for $t_{2i-4}\in[z_1,z_2]$

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_\mathrm{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-3},t_{2i-2})w(t_{2i-2})\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathrm{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})\right.\right.\right.\\
&\left.\left.\left.w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\mathrm{d}t_{2i-1}\right]\mathrm{d}t_{2i-3}\\
&\geqslant\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_\mathrm{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-3},t_{2i-2})w(t_{2i-2})\mathtt{f}_{i-1}\left(t_{2i-2},r_*\right)\mathrm{d}t_{2i-1}\right]\mathrm{d}t_{2i-3}\\
&\geqslant\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-4},t_{2i-3})v^{-1}(t_{2i-3})\phi_\mathrm{q}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i-3},t_{2i-2})w(t_{2i-2})\zeta_1\phi_\mathrm{p}(r_*)\mathrm{d}t_{2i-1}\right]\mathrm{d}t_{2i-3}\\
&\geqslant\phi_\mathrm{q}(\zeta_1)\sigma(\lambda)\alpha_1 r_*\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_{2i-3},t_{2i-3})v^{-1}(t_{2i-3})\phi_\mathrm{q}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i-2},t_{2i-2})w(t_{2i-2})\mathrm{d}t_{2i-1}\right]\mathrm{d}t_{2i-3}\\
&\geqslant r_*M\phi_\mathrm{q}(\zeta_1)\geqslant r_*.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\mathrm{T}y_1(z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_\mathrm{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_\mathrm{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\
&\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathrm{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathrm{d}t_{2i}\right]\mathrm{d}t_{2i-1}\right)\cdots\mathrm{d}t_4\right]\mathrm{d}t_3\right)\mathrm{d}t_2\right]\mathrm{d}t_1\\
\geqslant& r_*=\|y_1\|.
\end{aligned}$$

So,

$$\|\mathrm{T}y_1\|\geqslant\|y_1\|\text{ for all }y_1\in\mathrm{K}\cap\partial\Theta_3.\tag{3.1}$$

Since $\lim_{y\to+\infty}\frac{\mathtt{f}_\mathtt{n}(z,y)}{\phi_\mathrm{p}(y)}=+\infty$, there exists $r^*>r_1$ such that $\mathtt{f}_\mathtt{n}(z,y)\geqslant\zeta_2\phi_\mathrm{p}(y)$, for $y\geqslant r^*$, $z\in[z_1,z_2]$, where $\zeta_2\geqslant\zeta$; here ζ is given in the proof of Theorem 3.2. Choose $\overline{r^*}>\max\{\frac{r^*}{\sigma(\lambda)},r_1\}$ and set $\Theta_4=\{y\in\mathrm{B}:\|y\|<\overline{r^*}\}$. For any $y_1\in\mathrm{K}\cap\partial\Theta_4$, we get

$$y_1\geqslant\min_{z\in[\lambda,z_2-\lambda]}y_1(z)\geqslant\sigma(\lambda)\|y_1\|\geqslant\sigma(\lambda)\overline{r^*}\geqslant r^*.$$

For $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\geqslant\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\zeta_2\phi_p(y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\geqslant\phi_q(\zeta_2)\sigma(\lambda)\alpha_1\overline{r^*}\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\geqslant\overline{r^*}M\phi_q(\zeta_1)\\
&\geqslant\overline{r^*}.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\mathrm{T}y_1(z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\\
&\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\right.\\
&\left.\left.\left.\left.\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\\
\geqslant&\,\overline{r^*}=\|y_1\|.
\end{aligned}$$

So,

$$\|\mathrm{T}y_1\|\geqslant\|y_1\|\quad\text{for all } y_1\in\mathrm{K}\cap\partial\Theta_4. \tag{3.2}$$

Let $\Theta_5=\{y\in\mathrm{B}:\|y\|<r_1\}$, if $y_1\in\mathrm{K}\cap\partial\Theta_5$, and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\leqslant\int_{z_1}^{z_2}\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\phi_p\left(\frac{r_1}{m}\right)\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\leqslant\frac{r_1}{m}\alpha_1\int_{z_1}^{z_2}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\leqslant r_1.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\mathrm{T}y_1(z)=&\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\mathtt{f}_{i-1}\left(t_{2i-2},\right.\right.\right.\right.\\
&\left.\left.\left.\left.\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_q\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\\
\leqslant&\, r_1=\|y_1\|.
\end{aligned}$$

So,

$$\|\mathrm{T}y_1\|\leqslant\|y_1\|\text{ for all } y_1\in\mathrm{K}\cap\partial\Theta_5. \tag{3.3}$$

Since $r_*\leqslant r_1<r^*$ and from (3.1), (3.2), and (3.3) it follows from Krasnoselskii's fixed point theorem [31, 32] T has a fixed point y_1^* in $\mathrm{K}\cap(\overline{\Theta}_5\setminus\Theta_3)$ and a fixed point y_1^{**} in $\mathrm{K}\cap(\overline{\Theta}_4\setminus\Theta_5)$ such that $0<\|y_1^*\|<r_1<\|y_1^{**}\|$. □

Theorem 3.5. *Let* (I1) – (I3) *hold. Also assume that the following conditions hold,*

(I8) $\lim_{y\to 0^+}\frac{\mathtt{f}_n(z,y)}{\phi_p(y)}=0,\ \lim_{y\to+\infty}\frac{\mathtt{f}_n(z,y)}{\phi_p(y)}=0,\ 1\leq \mathtt{n}\leq i$ *for* $z\in[z_1,z_2]$.

(I9) *There exists a constant* r_2 *such that* $\mathtt{f}_n(z,y)\geqslant\phi_p(\frac{r_2}{M})$ *for* $y\in[\sigma(\lambda)r_2,r_2]$, $z\in[z_1,z_2]$.

Then the BVP (1.1)-(1.2) has at least two positive symmetric solutions y_1^* *and* y_1^{**} *such that* $0<\|y_1^*\|<r_2<\|y_1^{**}\|$.

Proof. We can establish the result by using the previous argument is in Theorem 3.4. □

Next, we establish sufficient conditions for the existence of at least three positive symmetric solutions for the BVP (1.1)-(1.2) by using the five functionals fixed point theorem. For that we define the nonnegative continuous concave functionals ψ_1,ψ_2 and the nonnegative continuous convex functionals $\gamma_1,\gamma_2,\gamma_3$ on K by

$$\psi_1(y)=\min_{z\in I}|y|,\quad \psi_2(y)=\min_{z\in I_1}|y|,\quad \gamma_1(y)=\max_{z\in[z_1,z_2]}|y|,\quad \gamma_2(y)=\max_{z\in I_1}|y|,\quad \gamma_3(y)=\max_{z\in I},|y|,$$

$$\psi_1(y)=\min_{z\in I}|y|\leqslant\max_{z\in I_1}|y|=\gamma_2(y), \tag{3.4}$$

$$\|y\| = \frac{1}{\sigma(\lambda)}\min_{z\in I}|y| \leqslant \frac{1}{\sigma(\lambda)}\max_{z\in[z_1,z_2]}|y| = \frac{1}{\sigma(\lambda)}\gamma_1(y), \tag{3.5}$$

where $I=[\lambda, z_2-\lambda]$, $I_1=[\lambda_1,\lambda_2]$, $\lambda<\lambda_1<\lambda_2<z_2-\lambda$. Then for nonnegative numbers $\mathtt{d}_1,\mathtt{d}_2,\mathtt{d}_3,\mathtt{d}_4,$ and $\mathtt{d}_5$, convex sets are defined as follows

$$\begin{aligned}
&\mathrm{K}(\gamma_1,\mathtt{d}_3)=\{y\in\mathrm{K}:\gamma_1(y)<\mathtt{d}_3\},\\
&\mathrm{K}(\gamma_1,\psi_1,\mathtt{d}_1,\mathtt{d}_3)=\{y\in\mathrm{K}:\mathtt{d}_1\leqslant\psi_1(y);\gamma_1(y)\leqslant\mathtt{d}_3\},\\
&\overline{\mathrm{K}}(\gamma_1,\gamma_2,\mathtt{d}_4,\mathtt{d}_3)=\{y\in\mathrm{K}:\gamma_2(y)\leqslant\mathtt{d}_4;\gamma_1(y)\leqslant\mathtt{d}_3\},\\
&\mathrm{K}(\gamma_1,\gamma_3,\psi_1,\mathtt{d}_1,\mathtt{d}_2,\mathtt{d}_3)=\{y\in\mathrm{K}:\mathtt{d}_1\leqslant\psi_1(y);\gamma_3(y)\leqslant\mathtt{d}_2;\gamma_1(y)\leqslant\mathtt{d}_3\},\text{ and}\\
&\overline{\mathrm{K}}(\gamma_1,\gamma_2,\psi_2,\mathtt{d}_5,\mathtt{d}_4,\mathtt{d}_3)=\{y\in\mathrm{K}:\mathtt{d}_5\leqslant\psi_2(y);\gamma_2(y)\leqslant\mathtt{d}_4;\gamma_1(y)\leqslant\mathtt{d}_3\}.
\end{aligned}$$

Theorem 3.6. *Suppose that* $0<\mathtt{d}_1<\mathtt{d}_2<\frac{\mathtt{d}_2}{\sigma(\lambda)}<\mathtt{d}_3$ *such that* $\mathtt{f}_\mathtt{n}$ *satisfies the following conditions:*

(I10) $\mathtt{f}_\mathtt{n}(z,y)\leqslant\phi_\mathtt{p}(\frac{\mathtt{d}_1}{m})$ *for* $y\in[\sigma(\lambda)\mathtt{d}_1,\mathtt{d}_1]$, $z\in[z_1,z_2]$,
(I11) $\mathtt{f}_\mathtt{n}(z,y)\geqslant\phi_\mathtt{p}(\frac{\mathtt{d}_2}{M})$ *for* $y\in[\mathtt{d}_2,\frac{\mathtt{d}_2}{\sigma(\lambda)}]$, $z\in I$,
(I12) $\mathtt{f}_\mathtt{n}(z,y)\leqslant\phi_\mathtt{p}(\frac{\mathtt{d}_3}{m})$ *for* $y\in[0,\mathtt{d}_3]$, $z\in[z_1,z_2]$,

Then the BVP (1.1)-(1.2) has at least three positive symmetric solutions y_1^*,y_1^{**}, *and* y_1^{***} *such that* $\gamma_2(y_1^*)<\mathtt{d}_1$, $\mathtt{d}_2<\psi_1(y_1^{**})$ *and* $\mathtt{d}_1<\gamma_2(y_1^{***})$ *with* $\psi_1(y_1^{***})<\mathtt{d}_2$.

Proof. From Lemma 3.1 the operator T is completely continuous. From (3.4) and (3.5), for each $y\in\mathrm{K}$, $\psi_1(y)\leqslant\gamma_2(y)$ and $\|y\|\leqslant\frac{1}{\sigma(\lambda)}\gamma_1(y)$. Now to show that $\mathrm{T}:\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}\to\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}$. Let $y\in\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}$, then $0\leqslant|y|\leqslant\mathtt{d}_3$.
By (I12), and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\leqslant\int_{z_1}^{z_2}\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\eta\phi_\mathtt{p}(\frac{\mathtt{d}_3}{m})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\leqslant\frac{\mathtt{d}_3}{m}\alpha_1\int_{z_1}^{z_2}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\leqslant\mathtt{d}_3.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}
\gamma_1(\mathrm{T}y_1(z))=\max_{z\in[z_1,z_2]}&\left[\int_{z_1}^{z_2}\mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2}\mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\
&\mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\
&\left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\
&\leqslant\mathtt{d}_3.
\end{aligned}$$

Therefore $\mathrm{T}:\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}\to\overline{\mathrm{K}(\gamma_1,\mathtt{d}_3)}$. It obvious that

$$\frac{\mathtt{d}_2(\sigma(\lambda)+1)}{\sigma(\lambda)}\in\{y\in\mathrm{K}(\gamma_1,\gamma_3,\psi_1,\mathtt{d}_2,\frac{\mathtt{d}_2}{\sigma(\lambda)},\mathtt{d}_3):\psi_1(y)>\mathtt{d}_2\}\neq\emptyset\quad\text{and}$$

$$\mathtt{d}_1(\sigma(\lambda)+1)\in\{y\in\overline{\mathrm{K}}(\gamma_1,\gamma_2,\psi_2,\sigma(\lambda)\mathtt{d}_1,\mathtt{d}_1,\mathtt{d}_3):\gamma_2(y)<\mathtt{d}_1\}\neq\emptyset.$$

Next, let $y\in\mathrm{K}(\gamma_1,\gamma_3,\psi_1,\mathtt{d}_2,\frac{\mathtt{d}_2}{\sigma(\lambda)},\mathtt{d}_3)$ or $y\in\overline{\mathrm{K}}(\gamma_1,\gamma_2,\psi_2,\sigma(\lambda)\mathtt{d}_1,\mathtt{d}_1,\mathtt{d}_3)$. Then, $\mathtt{d}_2\leqslant|y|\leqslant\frac{\mathtt{d}_2}{\sigma(\lambda)}$ and $\mathtt{d}_1\sigma(\lambda)\leqslant|y|\leqslant\mathtt{d}_1$.
By (I11) and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}
&\int_{z_1}^{z_2}\mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{z_1}^{z_2}\mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\geqslant\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\phi_\mathtt{p}(\frac{\mathtt{d}_2}{M})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\geqslant\frac{\mathtt{d}_2}{M}\sigma(\lambda)\alpha_1\int_{\lambda}^{z_2-\lambda}\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_\mathtt{q}\left[\int_{\lambda}^{z_2-\lambda}\sigma(\lambda)\alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\
&\geqslant\mathtt{d}_2.
\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}\psi_1(\mathrm{T}y_1(z)) = \min_{z\in I}\ & \left[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ \geqslant\ & \mathsf{d}_2.\end{aligned}$$

By $(\mathtt{I10})$, and for $t_{2i-2}\in[z_1,z_2]$, we have

$$\begin{aligned}&\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\ &\leqslant \int_{z_1}^{z_2} \alpha_1\mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\eta\,\phi_{\mathrm{p}}(\frac{\mathsf{d}_1}{m})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\ &\leqslant \frac{\mathsf{d}_1}{m}\alpha_1\int_{z_1}^{z_2} \mathrm{G}(t_{2i-1},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \alpha_2\mathrm{G}(t_{2i},t_{2i})w(t_{2i})\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\\ &\leqslant \mathsf{d}_1.\end{aligned}$$

Continuing in this fashion, we get

$$\begin{aligned}\gamma_2(\mathrm{T}y_1(z)) = \max_{z\in I_1}\ & \left[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ \leqslant\ & \mathsf{d}_1.\end{aligned}$$

Next, let $y\in\mathrm{K}(\gamma_1,\psi_1,\mathsf{d}_2,\mathsf{d}_3)$ with $\gamma_3(\mathrm{T}y_1(z)) > \frac{\mathsf{d}_2}{\sigma(\lambda)}$. Then

$$\begin{aligned}\psi_1(\mathrm{T}y_1(z)) = \min_{z\in I}\ & \left[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1}v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ \geqslant\ & \sigma(\lambda)\left[\int_{z_1}^{z_2} \alpha_1\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ \geqslant\ & \sigma(\lambda)\max_{z\in[z_1,z_2]}\left[\int_{z_1}^{z_2} \alpha_1\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2}\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ \geqslant\ & \sigma(\lambda)\max_{z\in I}\left[\int_{z_1}^{z_2} \alpha_1\mathrm{G}(t_1,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\left(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots\right.\right.\right.\right.\\ & \mathtt{f}_{i-1}\left(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\left[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\right]\mathtt{d}t_{2i-1}\right)\cdots\\ & \left.\left.\left.\left.\mathtt{d}t_4\right]\mathtt{d}t_3\right)\mathtt{d}t_2\right]\mathtt{d}t_1\right]\\ =\ & \sigma(\lambda)\gamma_3(\mathrm{T}y_1(z)) > \mathsf{d}_2.\end{aligned}$$

Let $y\in\overline{\mathrm{K}}(\gamma_1,\gamma_2,\mathsf{d}_1,\mathsf{d}_3)$ with $\psi_2(\mathrm{T}y) < \sigma(\lambda)\mathsf{d}_1$. Then we have

$$
\begin{aligned}
\gamma_2(\mathrm{T}y_1(z)) = \max_{z\in I_1} & \Bigg[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots \\
& \mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\Bigg]\mathtt{d}t_{2i-1}\Bigg)\cdots \\
& \mathtt{d}t_4\Bigg]\mathtt{d}t_3\Bigg)\mathtt{d}t_2\Bigg]\mathtt{d}t_1\Bigg] \\
\leqslant \max_{z\in[z_1,z_2]} & \Bigg[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots \\
& \mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\Bigg]\mathtt{d}t_{2i-1}\Bigg)\cdots \\
& \mathtt{d}t_4\Bigg]\mathtt{d}t_3\Bigg)\mathtt{d}t_2\Bigg]\mathtt{d}t_1\Bigg] \\
\leqslant \frac{1}{\sigma(\lambda)}\min_{z\in I} & \Bigg[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots \\
& \mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\Bigg]\mathtt{d}t_{2i-1}\Bigg)\cdots \\
& \mathtt{d}t_4\Bigg]\mathtt{d}t_3\Bigg)\mathtt{d}t_2\Bigg]\mathtt{d}t_1\Bigg] \\
\leqslant \frac{1}{\sigma(\lambda)}\min_{z\in I_1} & \Bigg[\int_{z_1}^{z_2} \mathtt{H}_1(z,t_1)v^{-1}(t_1)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_1,t_2)w(t_2)\mathtt{f}_1\Bigg(t_2,\int_{z_1}^{z_2} \mathtt{H}_1(t_2,t_3)v^{-1}(t_3)\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_3,t_4)w(t_4)\mathtt{f}_2\cdots \\
& \mathtt{f}_{i-1}\Bigg(t_{2i-2},\int_{z_1}^{z_2} \mathtt{H}_1(t_{2i-2},t_{2i-1})v^{-1}(t_{2i-1})\phi_{\mathrm{q}}\Bigg[\int_{z_1}^{z_2} \mathtt{H}_2(t_{2i-1},t_{2i})w(t_{2i})\ \mathtt{f}_i(t_{2i},y_1(t_{2i}))\mathtt{d}t_{2i}\Bigg]\mathtt{d}t_{2i-1}\Bigg)\cdots \\
& \mathtt{d}t_4\Bigg]\mathtt{d}t_3\Bigg)\mathtt{d}t_2\Bigg]\mathtt{d}t_1\Bigg] \\
= \frac{1}{\sigma(\lambda)} & \psi_2(\mathrm{T}y_1(z)) < \mathtt{d}_1.
\end{aligned}
$$

So, proved all the conditions of the five functionals fixed point theorem [33]. Therefore, the BVP (1.1)-(1.2) has at least three positive symmetric solutions y_1^*, y_1^{**}, and y_1^{***} such that $\gamma_2(y_1^*) < \mathtt{d}_1$, $\mathtt{d}_2 < \psi_1(y_1^{**})$ and $\mathtt{d}_1 < \gamma_2(y_1^{***})$ with $\psi_1(y_1^{***}) < \mathtt{d}_2$. □

4. Examples

In this section, as an application, the results are demonstrated with examples.

Example 4.1.

Consider the following problem

$$
\left.\begin{aligned}
&(\phi_{\mathrm{p}}(v(z)y_{\mathtt{n}}''(z)))'' = w(z)\mathtt{f}_{\mathtt{n}}(z,y_{\mathtt{n}+1}(z)),\ \ 1\le \mathtt{n}\le 2,\ \ 0\le z\le 1,\\
&y_3(z) = y_1(z),
\end{aligned}\right\} \tag{4.1}
$$

satisfying boundary conditions

$$
\left.\begin{aligned}
&y_{\mathtt{n}}(0) = \int_0^1 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\ \ y_{\mathtt{n}}(1) = \int_0^1 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\\
&\phi_{\mathrm{p}}(v(0)y_{\mathtt{n}}''(0)) = \int_0^1 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(1)y_{\mathtt{n}}''(1)) = \int_0^1 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds},
\end{aligned}\right\} \tag{4.2}
$$

where $v(z) = 2+z-z^2$, $w(z) = 10$, $\mathtt{g}(z) = \frac{1}{4}$, $\mathtt{h}(z) = \frac{5}{9}$,

$$
\mathtt{f}_1(z,y) = \mathtt{f}_2(z,y) = \begin{cases} z^2(1-z)^2y^3, & (z,y)\in[0,1]\times(0,6];\\ 6z^2(1-z)^2y^2, & (z,y)\in[0,1]\times[6,\infty). \end{cases}
$$

After algebraic computations, we get $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{5}{9}$, $\alpha_1 = \frac{4}{3}$, $\mathtt{f} = \frac{9}{4}$,

$$
\begin{aligned}
\mathtt{H}_1(z,t) &= \mathrm{G}(z,t) + \frac{1}{1-\mu_1}\int_0^1 \mathrm{G}(\mathtt{s},t)\mathtt{g}(\mathtt{s})\mathtt{ds},\\
\mathtt{H}_2(z,t) &= \mathrm{G}(z,t) + \frac{1}{1-\ \mu_2}\int_0^1 \mathrm{G}(\mathtt{s},t)\mathtt{h}(\mathtt{s})\mathtt{ds},
\end{aligned}
$$

in which

$$
\mathrm{G}(z,t) = \begin{cases} z(1-t), & z\leqslant t,\\ t(1-z), & t\leqslant z. \end{cases}
$$

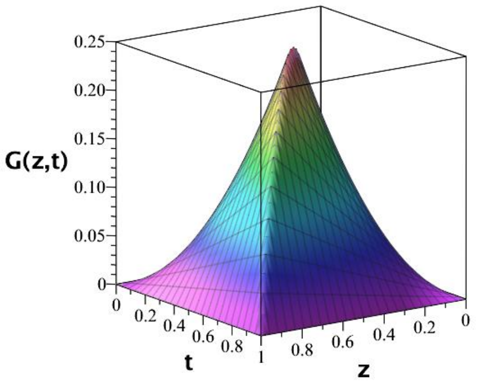

Figure 4.1: Pictorial representation of $G(z,t)$

Let $\lambda = \frac{103}{356}$ then $\sigma(\lambda) = \frac{103}{356}$ and $M = 0.3790187963$, $M = 0.01103127360$

$$\lim_{y\to 0^+} \frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = \lim_{y\to 0^+} \frac{z^2(1-z)^2y^3}{y} = \lim_{y\to 0^+} \left(\frac{1}{2}\right)^4 y^2 = 0,$$

$$\lim_{y\to\infty} \frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = \lim_{y\to\infty} \frac{6z^2(1-z)^2y^2}{y} = \lim_{y\to\infty} 6\times(0.0423)y = \infty,$$

$$\mathtt{f}_{\mathtt{n}}(z,y) \leqslant \eta\phi_{\mathrm{p}}(y) = 2y, \forall z\in[0,1],\ 0\leqslant y\leqslant 5,$$

$$\mathtt{f}_{\mathtt{n}}(z,y) \geqslant \zeta\phi_{\mathrm{p}}(y) = 101y, \forall z\in[0,1],\ y\geqslant 53.$$

Hence by Theorem 3.2, the BVP (4.1)-(4.2) has at least one positive symmetric solution.

Example 4.2.

Consider the following problem

$$\left.\begin{aligned}&(\phi_{\mathrm{p}}(v(z)y''_{\mathtt{n}}(z)))'' = w(z)\mathtt{f}_{\mathtt{n}}(z,y_{\mathtt{n}+1}(z)),\ \ 1\le \mathtt{n}\le 2,\ \ 1\le z\le 3,\\ &y_3(z) = y_1(z),\end{aligned}\right\} \tag{4.3}$$

satisfying boundary conditions

$$\left.\begin{aligned}&y_{\mathtt{n}}(1) = \int_1^3 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\ \ y_{\mathtt{n}}(3) = \int_1^3 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\\ &\phi_{\mathrm{p}}(v(1)y''_{\mathtt{n}}(1)) = \int_1^3 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y''_{\mathtt{n}}(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(3)y''_{\mathtt{n}}(3)) = \int_1^3 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y''_{\mathtt{n}}(\mathtt{s}))\mathtt{ds},\end{aligned}\right\} \tag{4.4}$$

where $v(z) = 2,\ w(z) = z^2(4-z)^2,\ \mathtt{g}(z) = \frac{2}{7},\ \mathtt{h}(z) = \frac{3}{5},$

$$\mathtt{f}_1(z,y) = \mathtt{f}_2(z,y) = \begin{cases} \dfrac{1}{5}z(4-z)y, & (z,y)\in[1,3]\times(0,20];\\ 4z(4-z)+(y-20)e^y, & (z,y)\in[1,3]\times[20,\infty).\end{cases}$$

After algebraic computations, we get $\mu_1 = \frac{2}{7},\ \mu_2 = \frac{3}{5},\ \alpha_1 = \frac{7}{5},\ \mathtt{f} = \frac{5}{2},$

$$\mathtt{H}_1(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\mu_1}\int_1^3 \mathrm{G}(\mathtt{s},t)\mathtt{g}(\mathtt{s})\mathtt{ds},$$

$$\mathtt{H}_2(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\mu_2}\int_1^3 \mathrm{G}(\mathtt{s},t)\mathtt{h}(\mathtt{s})\mathtt{ds},$$

in which

$$\mathrm{G}(z,t) = \frac{1}{2}\begin{cases}(z-1)(3-t), & z\leqslant t,\\ (t-1)(3-z), & t\leqslant z.\end{cases}$$

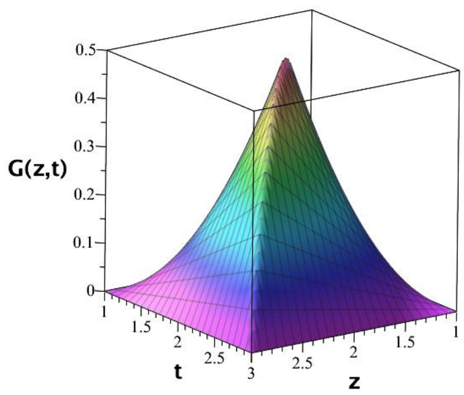

Figure 4.2: Pictorial representation of $G(z,t)$

Let $\lambda = 1.5$ then $\sigma(\lambda) = 0.25$ and $m = 11.276666$,

$$\lim_{y\to 0^+} \frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = \frac{\frac{1}{5}z(4-z)y}{y^2} = +\infty, \ \lim_{y\to +\infty} \frac{\mathtt{f}_{\mathtt{n}}(z,y)}{\phi_{\mathrm{p}}(y)} = \frac{4z(4-z)+(y-20)e^y}{y^2} = +\infty \text{ for } z \in [1,3].$$

Choose a constant $r_1 = 20$ such that

$$\mathtt{f}_{\mathtt{n}}(z,y) \leqslant \phi_{\mathrm{p}}\left(\frac{r_1}{M}\right) = 76.07776843 \text{ for } y \in [0,20],\ z \in [1,3].$$

Hence by Theorem 3.4, the BVP (4.3)-(4.4) has at least two positive symmetric solutions y_1^* and y_1^{**}. such that

$$0 < \|y_1^*\| < 20 < \|y_1^{**}\|.$$

Example 4.3.

Consider the following problem

$$\left.\begin{array}{l} (\phi_{\mathrm{p}}(v(z)y_{\mathtt{n}}''(z)))'' = w(z)\mathtt{f}_{\mathtt{n}}(z,y_{\mathtt{n}+1}(z)),\ \ 1 \le \mathtt{n} \le 2,\ \ 0 \le z \le 1, \\ y_3(z) = y_1(z), \end{array}\right\} \tag{4.5}$$

satisfying boundary conditions

$$\left.\begin{array}{l} y_{\mathtt{n}}(0) = \int_0^1 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds},\ \ y_{\mathtt{n}}(1) = \int_0^1 \mathtt{g}(\mathtt{s})y_{\mathtt{n}}(\mathtt{s})\mathtt{ds}, \\ \phi_{\mathrm{p}}(v(0)y_{\mathtt{n}}''(0)) = \int_0^1 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds},\ \ \phi_{\mathrm{p}}(v(1)y_{\mathtt{n}}''(1)) = \int_0^1 \mathtt{h}(\mathtt{s})\phi_{\mathrm{p}}(v(\mathtt{s})y_{\mathtt{n}}''(\mathtt{s}))\mathtt{ds}, \end{array}\right\} \tag{4.6}$$

where $v(z) = \frac{2}{5}$, $w(z) = \frac{4}{11}$, $\mathtt{g}(z) = \frac{1+z-z^2}{2}$, $\mathtt{h}(z) = \frac{10}{17}$,

$$\mathtt{f}_1(z,y) = \mathtt{f}_2(z,y) = \begin{cases} e^{z(1-z)} + \dfrac{\sin(y)}{4} + \dfrac{6y^4}{7}, & (z,y) \in [0,1] \times (0,5]; \\ e^{z(1-z)} + \dfrac{\sin(y)}{4} + \dfrac{3750}{7}, & (z,y) \in [0,1] \times [5,\infty). \end{cases}$$

After algebraic computations, we get $\mu_1 = \frac{7}{12}$, $\mu_2 = \frac{10}{17}$, $\alpha_1 = \frac{12}{5}$, $\mathtt{f} = \frac{17}{7}$,

$$\mathtt{H}_1(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\mu_1}\int_0^1 \mathrm{G}(\mathtt{s},t)\mathtt{g}(\mathtt{s})\mathtt{ds},$$

$$\mathtt{H}_2(z,t) = \mathrm{G}(z,t) + \frac{1}{1-\ \mu_2}\int_0^1 \mathrm{G}(\mathtt{s},t)\mathtt{h}(\mathtt{s})\mathtt{ds},$$

in which

$$\mathrm{G}(z,t) = \begin{cases} z(1-t), & z \leqslant t, \\ t(1-z), & t \leqslant z. \end{cases}$$

Let $\lambda = \frac{1}{3}$ then $\sigma(\lambda) = \frac{1}{3}$ and $m = .1471861472$, $M = 0.003791260308$. Choose $\mathtt{d}_1 = 1.5$, $\mathtt{d}_2 = 5$, $\mathtt{d}_3 = 100$ then

$$\mathtt{f}_{\mathtt{n}}(z,y) \leqslant \phi_{\mathrm{p}}\left(\frac{\mathtt{d}_1}{m}\right) = 10.19117647 \ \text{ for } \ y \in [0.5,1.5],\ z \in [0,1],$$

$$\mathtt{f}_{\mathtt{n}}(z,y) \geqslant \phi_{\mathrm{p}}\left(\frac{\mathtt{d}_2}{M}\right) = 211.6628959 \ \text{ for } \ y \in [5,15],\ z \in I,$$

$$\mathtt{f}_{\mathtt{n}}(z,y) \leqslant \phi_{\mathrm{p}}\left(\frac{\mathtt{d}_3}{m}\right) = 679.4521 \ \text{ for } \ y \in [0,100],\ z \in [0,1],$$

Hence by Theorem 3.6, the BVP has (4.5)-(4.6) has at least three positive symmetric solutions y_1^*, y_1^{**}, and y_1^{***}. such that $\gamma_2(y_1^*) < 1.5$, $5 < \psi_1(y_1^{**})$ and $1.5 < \gamma_2(y_1^{***})$ with $\psi_1(y_1^{***}) < 5$.

5. Conclusion

The current research work is devoted to establish the presence and characteristics of positive symmetric solutions for iterative system of p-Laplacian problem with integral boundary conditions based on the Krasnoselskii's and five functionals fixed point theorems. We anticipate that our findings will inspire and serve as a reference for future developments in this field.

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References

[1] S. S. Cheng, J. G. Si, X. P. Wang, *An existence theorem for iterative functional differential equations*, Acta Math. Hungar., **94** (2002), 1-17.
[2] E. Eder, *The functional differential equation* $x'(t) = x(x(t))$, J. Differ. Equ., **54**(3) (1984), 390-400.
[3] N. Oprea, *Numerical solutions of first order iterative functional-differential equations by spline functions of even degree*, Sci. Bull. Petru Maior Univ. Tirgu Mures, **6** (2009), 34-37.
[4] J. G. Si, X. P. Wang, S. S. Cheng, *Nondecreasing and convex* C^2*-solutions of an iterative functional differential equation*, Aequationes Math., **60** (2000), 38-56.
[5] D. Yang, W. Zhang, *Solutions of equivariance for iterative differential equations*, Appl. Math. Lett., **17**(7) (2004), 759-765.
[6] J. I. Diaz, F. D. Thelin, *On a nonlinear parabolic problem arising in some models related to turbulent flows*, SIAM J. Math. Anal., **25**(4) (1994), 1085-1111.
[7] R. Glowinski, J. Rappaz, *Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology*, Math. Model. Numer. Anal., **37**(1) (2003), 175-186.
[8] F. Bernis, *Compactness of the support in convex and nonconvex fourth order elasticity problems*, Nonlinear Anal., **6**(11) (1982), 1221-1243.
[9] D. Halpern, O. E. Jensen, J. B. Grotberg, *A theoretic study of surfactant and liquid delivery into the lung*, J. Appl. Physiol., **85** (1998), 333-352.
[10] M. Hofer, H. Pottmann, *Energy-minimizing splines in manifolds*, ACM Trans. Graph., **23**(3) (2004), 284-293.
[11] F. Mémoli, G. Sapiro, P. Thompson, *Implicit brain imaging*, Neuroimage, **23** (2004), 179-188.
[12] T. G. Myers, J. P. F. Charpin, *A mathematical model for atmospheric ice accretion and water flow on a cold surface*, Int. J. Heat Mass Tranf., **47**(25) (2004), 5483-5500.
[13] T. G. Myers, J. P. F. Charpin, S. J. Chapman, *The flow and solidification of thin fluid film on an arbitrary three-dimensional surface*, Phys. Fluids, **14**(8) (2002), 2788-2803.
[14] A. W. Toga, *Brain Warping*, Academic Press, New York, 1998.
[15] X. Han, Y. He, H. Wei, *Existence of positive periodic solutions for a nonlinear system of second-order ordinary differential equations*, Electron. J. Differ. Equ., **2022**(83) (2022), 1-11.
[16] K. R. Prasad, L. T. Wesen, N. Sreedhar, *Existence of positive solutions for second-order undamped Sturm-Liouville boundary value problems*, Asian-Eur. J. Math., **9**(3) (2016), 1-9, Article ID 1650089.
[17] Y. Sun, L. Liu, J. Zhang, R. P. Agarwal, *Positive solutions of singular three-point boundary value problems for second-order differential equations*, J. Comput. Appl. Math., **230**(2) (2009), 738-750.
[18] R. I. Avery, J. Henderson, *Three symmetric positive solutions for a second-order boundary value problem*, Appl. Math. Lett., **13**(3) (2000), 1-7.
[19] U. Akcan, N. A. Hamal, *Existence and monotone iteration of concave positive symmetric solutions for a three-point second-order boundary value problems with integral boundary conditions*, Dyn. Syst. Appl., **24**(2015), 259-270.
[20] Y. Ding, *Existence of positive symmetric solutions for an integral boundary-value problem with* ϕ *Laplacian operator*, Electron. J. Differ. Equ., **2016**(336) (2016), 1-9.
[21] Md. Asaduzzaman, Md. Z. Ali, *On the symmetric positive solutions of nonlinear fourth order ordinary differential equations with four-point boundary value conditions: a fixed point theory approach*, J. Nonlinear Sci. Appl., **13**(6) (2020), 364-377.
[22] U. Akcan, N. A. Hamal, *Existence of concave symmetric positive solutions for a three-point boundary value problems*, Adv. Differ. Equ., **2014** (2014), 1-12, Article ID 313.
[23] M. Feng, *Existence of symmetric positive solutions for a boundary value problem with integral boundary conditions*, Appl. Math. Lett., **24**(8) (2011), 1419-1427.
[24] J. Henderson, H. B. Thompson, *Multiple symmetric positive solutions for a second order boundary value problem*, Proc. Amer. Math. Soc., **128**(8) (2000), 2373-2379.
[25] H. Pang, Y. Tong, *Symmetric positive solutions to a second-order boundary value problem with integral boundary conditions*, Bound. Value Probl., **2013** (2013), 1-9, Article ID 150.
[26] K. R. Prasad, K. Bhushanam, *Denumerably many positive symmetric solutions for integral 2-point iterative systems*, J. Anal., **32**(2) (2024), 823-841.
[27] N. Sreedhar, K. R. Prasad, S. Balakrishna, *Existence of symmetric positive solutions for lidstone type integral boundary value problems*, TWMS J. App. Eng. Math., **8** (2018), 295-305.
[28] Y. Sun, *Existence and multiplicity of symmetric positive solutions for three-point boundary value problem*, J. Math. Anal. Appl., **329**(2) (2007), 998–1009.
[29] Y. Sun, *Three symmetric positive solutions for second-order nonlocal boundary value problems*, Acta Math. Appl. Sin. Engl. Ser., **27**(2) (2011), 233-242.
[30] X. Zhang, W. Ge, *Symmetric positive solutions of boundary value problems with integral boundary conditions*, Appl. Math. Comput., **219**(8) (2012), 3553-3564.
[31] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.
[32] M. A. Krasnoselskii, *Positive Solutions of Operator Equations,* Noordhoff, Groningen, 1964.
[33] R. I. Avery, *A generalization of the Leggett-Williams fixed point theorem*, Math. Sci. Res. Hot-Line, **3**(7) (1999), 9-14.

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Binomial Transforms of the Third-Order Jacobsthal and Modified Third-Order Jacobsthal Polynomials

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Abstract

In this study, we define the binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal polynomials. Further, the generating functions, Binet formulas and summation of these binomial transforms are found by recurrence relations. Also, we establish the relations between these transforms by deriving new formulas. Finally, the Vajda, d'Ocagne, Catalan and Cassini formulas for these transforms are obtained.

1. Introduction

The study of number sequences has been the subject of several studies published in recent decades. Algebraic properties, generating function, Binet's formula and some well-known identities have been studied in this research topic.
In 2013, Cook and Bacon [1] introduced the notion of third-order Jacobsthal numbers $\{J_n^{(3)}\}_{n\geq\mathbb{N}}$ as an extension to the famous properties of the Jacobsthal sequence. The recurrence relation of this number is $J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)}$ for $n \geq 0$, where $J_0^{(3)} = 0$ and $J_1^{(3)} = J_2^{(3)} = 1$. A new study on the modified third-order Jacobsthal numbers $K_{n+2}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 6J_n^{(3)}$ was published in 2020 by Morales [2]. The recurrence relation of this number is $K_{n+3}^{(3)} = K_{n+2}^{(3)} + K_{n+1}^{(3)} + 2K_n^{(3)}$ for $n \geq 0$, where $K_0^{(3)} = 3$, $K_1^{(3)} = 1$ and $K_2^{(3)} = 3$. In addition, Soykan et. al. in [3] studied the binomial transforms of the generalized third-order Jacobsthal numbers.
Some generalizations of third-order Jacobsthal numbers can be obtained in various ways (see, e.g., [4–6]). A natural extension is to consider for $x \in \mathbb{C}$ sequences of third-order Jacobsthal and modified third-order Jacobsthal polynomials $\{J_n^{(3)}(x)\}_{n\geq\mathbb{N}}$ and $\{K_n^{(3)}(x)\}_{n\geq\mathbb{N}}$, respectively. Third-order Jacobsthal and modified third-order Jacobsthal polynomials are defined by the recurrence relations

$$\begin{aligned} &J_{n+3}^{(3)}(x) = (x-1)J_{n+2}^{(3)}(x) + (x-1)J_{n+1}^{(3)}(x) + xJ_n^{(3)}(x), \\ &J_0^{(3)}(x) = 0,\ J_1^{(3)}(x) = 1,\ J_2^{(3)}(x) = x-1 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &K_{n+3}^{(3)}(x) = (x-1)K_{n+2}^{(3)}(x) + (x-1)K_{n+1}^{(3)}(x) + xK_n^{(3)}(x), \\ &K_0^{(3)}(x) = 3,\ K_1^{(3)}(x) = x-1,\ K_2^{(3)}(x) = x^2-1, \end{aligned} \tag{1.2}$$

respectively. For more information, see [7].
On the other hand, some matrix-based transforms can be introduced for a given sequence. The binomial transform is one such transform and there are also other transforms such as rising and falling binomial transforms (see, e.g., [8]). Also, there is an interesting study on watermarking and the binomial transform. In [9], Falcón and Plaza studied the binomial transforms of the k-Fibonacci sequences. In [10], Prodinger gave some information about binomial transform. In [11], a novel Binomial transform based fragile watermarking technique has been proposed for color image authentication. In [12], Yilmaz defined and studied the binomial transforms of the Balancing and

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Lucas-Balancing polynomials. In [13], Özkoç and Gündüz studied the binomial transform for quadra Fibona-Pell sequence and quadra Fibona-Pell quaternion. In [14], Yilmaz and Aktaş studied special transforms of the generalized bivariate Fibonacci and Lucas polynomials. Other examples can be reviewed in [15, 16].
Now we give some preliminaries related our study. Given an integer sequence $\Psi = \{\psi_0, \psi_1, \psi_2, \cdots\}$, the binomial transform $\mathscr{B}$ of the sequence Ψ, $\mathscr{B}(\Psi) = \{\Phi_n\}$, is given by

$$\Phi_n = \sum_{j=0}^{n} \binom{n}{j} \psi_j.$$

Furthermore, in [17], Boyadzhiev studied the following properties of the binomial transform Φ_n:

$$\sum_{j=0}^{n} \binom{n}{j} j\psi_j = n(\Phi_n - \Phi_{n-1})$$

and

$$\sum_{j=1}^{n} \binom{n}{j} \psi_j j^{-1} = \sum_{j=1}^{n} \Phi_j j^{-1}.$$

Motivated essentially by the previous papers, the objective of this study is to apply the binomial transforms to the third-order Jacobsthal $\{J_n^{(3)}(x)\}$ and modified third-order Jacobsthal polynomials $\{K_n^{(3)}(x)\}$ in Eqs. (1.1) and (1.2). Furthermore, the generating functions of binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal polynomials are found by recurrence relations. Also, we describe the Vajda and d'Ocagne formulas and the relations between these transforms by deriving new formulas.

2. Binomial Transforms of Third-Order Jacobsthal Polynomials

In this section, we will mainly focus on binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal polynomials to get some important results. In fact, as a middle step, we will also present the recurrence relations, generating functions and Binet formulas.

Definition 2.1. *Let $\mathscr{J}_n(x)$ and $\mathscr{K}_n(x)$ be the third-order Jacobsthal and modified third-order Jacobsthal polynomials, respectively. The binomial transforms of these polynomials can be expressed as follows:*

1. *the binomial transform of the third-order Jacobsthal polynomial is*

$$\mathscr{J}_n(x) = \sum_{j=0}^{n} \binom{n}{j} J_j^{(3)}(x),$$

2. *the binomial transform of the modified third-order Jacobsthal polynomial is*

$$\mathscr{K}_n(x) = \sum_{j=0}^{n} \binom{n}{j} K_j^{(3)}(x).$$

Before starting the results, it is useful to say $\binom{n}{j} = 0$ for $j > n$.
The following lemma will be key to the proof of the next theorem.

Lemma 2.2. *For $n \geq 0$, the following equalities hold:*

$$\mathscr{J}_{n+1}(x) - \mathscr{J}_n(x) = \sum_{j=o}^{n} \binom{n}{j} J_{j+1}^{(3)}(x), \tag{2.1}$$

$$\mathscr{K}_{n+1}(x) - \mathscr{K}_n(x) = \sum_{j=o}^{n} \binom{n}{j} K_{j+1}^{(3)}(x). \tag{2.2}$$

Proof. We will only prove Eq. (2.1) since the proof of Eq. (2.2) is analogous. By using Definition 2.1 and the well known binomial equality for $1 \leq j \leq n$

$$\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}, \tag{2.3}$$

we obtain

$$\begin{aligned} \mathscr{J}_{n+1}(x) &= \sum_{j=1}^{n+1} \binom{n+1}{j} J_j^{(3)}(x) + J_0^{(3)}(x) \\ &= \sum_{j=0}^{n+1} \binom{n}{j} J_j^{(3)}(x) + \sum_{j=1}^{n+1} \binom{n}{j-1} J_j^{(3)}(x) \\ &= \sum_{j=0}^{n} \binom{n}{j} \left(J_j^{(3)}(x) + J_{j+1}^{(3)}(x) \right), \end{aligned}$$

which is desired result. □

Theorem 2.3. *For $n \geq 0$, we have to*

1. *the recurrence relation of sequences $\{\mathscr{J}_n(x)\}$ is*

$$\mathscr{J}_{n+3}(x) = (x+2)\left[\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right] + (x+1)\mathscr{J}_n(x), \tag{2.4}$$

with initial conditions $\mathscr{J}_0(x) = 0$, $\mathscr{J}_1(x) = 1$ and $\mathscr{J}_2(x) = x+1$.

2. *the recurrence relation of sequences $\{\mathscr{K}_n(x)\}$ is*

$$\mathscr{K}_{n+3}(x) = (x+2)\left[\mathscr{K}_{n+2}(x) - \mathscr{K}_{n+1}(x)\right] + (x+1)\mathscr{K}_n(x), \tag{2.5}$$

with initial conditions $\mathscr{K}_0(x) = 3$, $\mathscr{K}_1(x) = x+2$ and $\mathscr{K}_2(x) = x^2+2x$.

Proof. Similar to the proof of the previous theorem, only the first case (2.4) will be proved. We omit the other cases since the proofs are similar. By considering Definition 2.1 and $J_0^{(3)}(x) = 0$, we obtain

$$\mathscr{J}_{n+3}(x) = \sum_{j=0}^{n+3}\binom{n+3}{j}J_j^{(3)}(x) = \sum_{j=0}^{n+2}\binom{n+3}{j+1}J_{j+1}^{(3)}(x).$$

By taking into account Eq. (2.3), we get

$$\mathscr{J}_{n+3}(x) = \sum_{j=0}^{n+2}\left[\binom{n+1}{j+1} + 2\binom{n+1}{j} + \binom{n+1}{j-1}\right]J_{j+1}^{(3)}(x).$$

By considering recurrence relations of third-order Jacobsthal polynomials

$$J_{j+3}^{(3)}(x) = (x-1)J_{j+2}^{(3)}(x) + (x-1)J_{j+1}^{(3)}(x) + xJ_j^{(3)}(x), \ j \geq 0,$$

and the equality $\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$, we obtain

$$\begin{aligned}
\mathscr{J}_{n+3}(x) &= \sum_{j=0}^{n+2}\binom{n+1}{j+1}J_{j+1}^{(3)}(x) + 2\sum_{j=0}^{n+2}\binom{n+1}{j}J_{j+1}^{(3)}(x) + \sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j+1}^{(3)}(x) \\
&= \sum_{j=0}^{n+1}\binom{n+1}{j}J_j^{(3)}(x) + 2\sum_{j=0}^{n+1}\binom{n+1}{j}J_{j+1}^{(3)}(x) \\
&\quad + \sum_{j=0}^{n+2}\binom{n+1}{j-1}\left[(x-1)J_j^{(3)}(x) + (x-1)J_{j-1}^{(3)}(x) + xJ_{j-2}^{(3)}(x)\right] \\
&= \mathscr{J}_{n+1}(x) + 2\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) \\
&\quad + x\sum_{j=0}^{n+2}\binom{n+1}{j-1}J_j^{(3)}(x) - \sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j-1}^{(3)}(x) \\
&\quad - \sum_{j=0}^{n+2}\binom{n+1}{j-1}J_j^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j-1}^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j-2}^{(3)}(x).
\end{aligned}$$

Using Lemma 2.2 and $\binom{n+1}{j-1} = \binom{n}{j-1} + \binom{n}{j-2}$, we have

$$\begin{aligned}
\mathscr{J}_{n+3}(x) &= \mathscr{J}_{n+1}(x) + 2\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) + x\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) - \mathscr{J}_{n+1}(x) \\
&\quad - \sum_{j=0}^{n+2}\binom{n+1}{j-1}J_j^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n+1}{j-1}J_{j-1}^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n}{j-1}J_{j-2}^{(3)}(x) + x\sum_{j=0}^{n+2}\binom{n}{j-2}J_{j-2}^{(3)}(x) \\
&= \mathscr{J}_{n+1}(x) + 2\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) + x\left(\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right) - \mathscr{J}_{n+1}(x) + (x+1)\mathscr{J}_n(x) \\
&\quad - \sum_{j=0}^{n}\binom{n}{j}\left[J_{j+1}^{(3)}(x) - (x-1)J_j^{(3)}(x) - (x-1)J_{j-1}^{(3)}(x) - xJ_{j-2}^{(3)}(x)\right] \\
&= (x+2)\left[\mathscr{J}_{n+2}(x) - \mathscr{J}_{n+1}(x)\right] + (x+1)\mathscr{J}_n(x)
\end{aligned}$$

which completes the proof in this case. □

Remark 2.4. *For $n \geq 0$ and $x = 2$ in Theorem 2.3, we have to*

1. *the recurrence relation of binomial transform for third-order Jacobsthal numbers $J_n^{(3)}$ is*

$$\mathscr{J}_{n+3} = 4\left[\mathscr{J}_{n+2} - \mathscr{J}_{n+1}\right] + 3\mathscr{J}_n,$$

with initial conditions $\mathscr{J}_0 = 0$, $\mathscr{J}_1 = 1$ and $\mathscr{J}_2 = 3$.

2. *the recurrence relation of binomial transform for modified third-order Jacobsthal numbers $K_n^{(3)}$ is*

$$\mathscr{K}_{n+3} = 4\left[\mathscr{K}_{n+2} - \mathscr{K}_{n+1}\right] + 3\mathscr{K}_n,$$

with initial conditions $\mathscr{K}_0 = 3$, $\mathscr{K}_1 = 4$ and $\mathscr{K}_2 = 8$.

Also, the generating functions for third-order Jacobsthal and modified third-order Jacobsthal polynomials play a vital role in determining some important identities of these new polynomial sequences. In the following theorem, we develop the generating functions for the binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal polynomials.

Theorem 2.5. *The generating functions of the binomial transforms for $\{\mathscr{J}_n(x)\}$ and $\{\mathscr{K}_n(x)\}$ are*

$$g(\mathscr{J}_n(x);\lambda)=\sum_{j=0}^{\infty}\mathscr{J}_j(x)\lambda^j=\frac{\lambda}{1-(x+2)\lambda+(x+2)\lambda^2-(x+1)\lambda^3} \tag{2.6}$$

and

$$g(\mathscr{K}_n(x);\lambda)=\sum_{j=0}^{\infty}\mathscr{K}_j(x)\lambda^j=\frac{3-(2x+4)\lambda+(x+2)\lambda^2}{1-(x+2)\lambda+(x+2)\lambda^2-(x+1)\lambda^3}. \tag{2.7}$$

Proof. We omit the third-order Jacobsthal case in Eq. (2.6) since the proof is similar. For Eq. (2.7), assume that $g(\mathscr{K}_n(x);\lambda)$ is the generating function of the binomial transform for $\{\mathscr{K}_n(x)\}$. The, we obtain

$$g(\mathscr{K}_n(x);\lambda)=\sum_{j=0}^{\infty}\mathscr{K}_j(x)\lambda^j=\mathscr{K}_0(x)+\mathscr{K}_1(x)\lambda+\mathscr{K}_2(x)\lambda^2+\cdots.$$

Using Theorem 2.3, we have

$$\begin{aligned}g(\mathscr{K}_n(x);\lambda)&=\mathscr{K}_0(x)+\mathscr{K}_1(x)\lambda+\mathscr{K}_2(x)\lambda^2+\sum_{j=3}^{\infty}\left((x+2)\left[\mathscr{K}_{j-1}(x)-\mathscr{K}_{j-2}(x)\right]+(x+1)\mathscr{K}_{j-3}(x)\right)\lambda^j\\&=\mathscr{K}_0(x)+(\mathscr{K}_1(x)-(x+2)\mathscr{K}_0(x))\lambda+(\mathscr{K}_2(x)-(x+2)(\mathscr{K}_1(x)-\mathscr{K}_0(x)))\lambda^2\\&\quad+(x+2)\lambda g(\mathscr{K}_n(x);\lambda)-(x+2)\lambda^2 g(\mathscr{K}_n(x);\lambda)-(x+1)\lambda^3 g(\mathscr{K}_n(x);\lambda).\end{aligned}$$

Now rearrangement the equation implies that

$$g(\mathscr{K}_n(x);\lambda)=\frac{\mathscr{K}_0(x)+(\mathscr{K}_1(x)-(x+2)\mathscr{K}_0(x))\lambda+(\mathscr{K}_2(x)-(x+2)(\mathscr{K}_1(x)-\mathscr{K}_0(x)))\lambda^2}{1-(x+2)\lambda+(x+2)\lambda^2-(x+1)\lambda^3},$$

which is equal to $\sum_{j=0}^{\infty}\mathscr{K}_j(x)\lambda^j$ in the theorem. □

Further, we note that $g(\mathscr{J}_n(x);\lambda)$ and $g(\mathscr{K}_n(x);\lambda)$ may be obtained from the generating functions of the third-order Jacobsthal and third-order Jacobsthal polynomials in [7], we have

$$g\left(J_n^{(3)}(x);\lambda\right)=\frac{\lambda}{1-(x-1)\lambda-(x-1)\lambda^2-x\lambda^3}$$

and

$$g\left(K_n^{(3)}(x);\lambda\right)=\frac{3-(x-1)\lambda-(x-1)\lambda^2}{1-(x-1)\lambda-(x-1)\lambda^2-x\lambda^3}.$$

It is seen by using the following result proved by Prodinger in [10]:

$$g(\mathscr{J}_n(x);\lambda)=\frac{1}{1-\lambda}g\left(J_n^{(3)}(x);\frac{\lambda}{1-\lambda}\right)$$

and

$$g(\mathscr{K}_n(x);\lambda)=\frac{1}{1-\lambda}g\left(K_n^{(3)}(x);\frac{\lambda}{1-\lambda}\right).$$

To derive new identities of the binomial transform of third-order Jacobsthal and modified third-order Jacobsthal polynomials, we now present an explicit formula for $\{\mathscr{J}_n(x)\}$ and $\{\mathscr{K}_n(x)\}$ for $n\geq 0$.

Theorem 2.6. *The Binet formulas of sequences $\{\mathscr{J}_n(x)\}$ and $\{\mathscr{K}_n(x)\}$ are*

$$\mathscr{J}_n(x)=\frac{x(x+1)^n}{x^2+x+1}+\frac{\omega_1^{n-1}}{(x+\omega_2)(\omega_1-\omega_2)}-\frac{\omega_2^{n-1}}{(x+\omega_1)(\omega_1-\omega_2)} \tag{2.8}$$

and

$$\mathscr{K}_n(x)=(x+1)^n+\omega_1^n+\omega_2^n,$$

where ω_1 and ω_2 are the conjugate roots of the characteristic equation $\lambda^3-(x+2)\lambda^2+(x+2)\lambda-(x+1)=0$.

Proof. (2.8): From Theorem 2.5 and Eq. (2.6), we have

$$g(\mathscr{J}_n(x);\lambda)=\sum_{j=0}^{\infty}\mathscr{J}_j(x)\lambda^j=\frac{\lambda}{1-(x+2)\lambda+(x+2)\lambda^2-(x+1)\lambda^3}.$$

Using the partial fraction decomposition, $g(\mathscr{J}_n(x);\lambda)$ can be expressed as

$$g(\mathscr{J}_n(x);\lambda)=\frac{1}{\Phi(x)}\left[\frac{x}{1-(x+1)\lambda}+\frac{\omega_2(x+\omega_1)}{i\sqrt{3}(1-\omega_1\lambda)}-\frac{\omega_1(x+\omega_2)}{i\sqrt{3}(1-\omega_2\lambda)}\right],$$

where $\Phi(x)=x^2+x+1$.
However, note that $\omega_1+\omega_2=1$, $\omega_1-\omega_2=i\sqrt{3}$ and $\omega_1\omega_2=1$. Then, we have

$$\begin{aligned}g(\mathscr{J}_n(x);\lambda)&=\frac{1}{\Phi(x)}\left[\frac{x}{1-(x+1)\lambda}+\frac{\omega_2(x+\omega_1)}{i\sqrt{3}(1-\omega_1\lambda)}-\frac{\omega_1(x+\omega_2)}{i\sqrt{3}(1-\omega_2\lambda)}\right]\\&=\frac{1}{\Phi(x)}\sum_{n=0}^{\infty}\left[x(x+1)^n+\frac{\omega_2(x+\omega_1)\omega_1^n}{i\sqrt{3}}-\frac{\omega_1(x+\omega_2)\omega_2^n}{i\sqrt{3}}\right]x^n\\&=\sum_{n=0}^{\infty}\left[\frac{x(x+1)^n}{x^2+x+1}+\frac{\omega_1^{n-1}}{(x+\omega_2)(\omega_1-\omega_2)}-\frac{\omega_2^{n-1}}{(x+\omega_1)(\omega_1-\omega_2)}\right]x^n.\end{aligned}$$

Thus, by the equality of generating function, we get

$$\mathscr{J}_n(x)=\frac{x(x+1)^n}{x^2+x+1}+\frac{\omega_1^{n-1}}{(x+\omega_2)(\omega_1-\omega_2)}-\frac{\omega_2^{n-1}}{(x+\omega_1)(\omega_1-\omega_2)}.$$

The proof of the binomial transform of modified third-order Jacobsthal polynomials $\mathscr{K}_n(x)$ can be seen by taking Theorem 2.5 and Eq. (2.7). □

3. Some Properties of Binomial Transforms of Third-Order Jacobsthal Polynomials

Now, we give the sums of binomial transforms for third-order Jacobsthal and modified third-order Jacobsthal polynomials.

Theorem 3.1. *For $n\geq 3$, sums of sequences $\mathscr{J}_n(x)$ and $\mathscr{K}_n(x)$ are*

$$\sum_{j=0}^{n}\mathscr{J}_j(x)=\frac{1}{x}\left[\mathscr{J}_{n+2}(x)-(x+1)(\mathscr{J}_{n+1}(x)-\mathscr{J}_n(x))\right] \tag{3.1}$$

and

$$\sum_{j=0}^{n}\mathscr{K}_j(x)=\frac{1}{x}\left[\mathscr{K}_{n+2}(x)-(x+1)(\mathscr{K}_{n+1}(x)-\mathscr{K}_n(x))+x-1\right]. \tag{3.2}$$

Proof. (3.1): By considering recurrence relation in Eq. (2.4), we have

$$\begin{aligned}\mathscr{J}_3(x)&=(x+2)\left[\mathscr{J}_2(x)-\mathscr{J}_1(x)\right]+(x+1)\mathscr{J}_0(x)\\\mathscr{J}_4(x)&=(x+2)\left[\mathscr{J}_3(x)-\mathscr{J}_2(x)\right]+(x+1)\mathscr{J}_1(x)\\\mathscr{J}_5(x)&=(x+2)\left[\mathscr{J}_4(x)-\mathscr{J}_3(x)\right]+(x+1)\mathscr{J}_2(x)\\&\cdots\\\mathscr{J}_{n-1}(x)&=(x+2)\left[\mathscr{J}_{n-2}(x)-\mathscr{J}_{n-3}(x)\right]+(x+1)\mathscr{J}_{n-4}(x)\\\mathscr{J}_n(x)&=(x+2)\left[\mathscr{J}_{n-1}(x)-\mathscr{J}_{n-2}(x)\right]+(x+1)\mathscr{J}_{n-3}(x).\end{aligned}$$

Adding these equations, we obtain

$$\begin{aligned}\sum_{j=0}^{n}\mathscr{J}_j(x)&=(x+2)\mathscr{J}_{n-1}(x)+(x+1)\sum_{j=0}^{n-3}\mathscr{J}_j(x)\\&=(x+2)\mathscr{J}_{n-1}(x)+(x+1)\sum_{j=0}^{n}\mathscr{J}_j(x)-(x+1)\left[\mathscr{J}_n(x)+\mathscr{J}_{n-1}(x)+\mathscr{J}_{n-2}(x)\right].\end{aligned}$$

Then, using the relation $\mathscr{J}_{n+3}(x)=(x+2)\left[\mathscr{J}_{n+2}(x)-\mathscr{J}_{n+1}(x)\right]+(x+1)\mathscr{J}_n(x)$, we have

$$\sum_{j=0}^{n}\mathscr{J}_j(x)=\frac{1}{x}\left[(x+1)(\mathscr{J}_n(x)+\mathscr{J}_{n-2}(x))-\mathscr{J}_{n-1}(x)\right]=\frac{1}{x}\left[\mathscr{J}_{n+2}(x)-(x+1)(\mathscr{J}_{n+1}(x)-\mathscr{J}_n(x))\right].$$

Similar to (3.1), by considering equation (2.5), Eq. (3.2) into account similar to the proof of (3.1). □

Now, we give the sums of the first n of binomial transforms for third-order Jacobsthal and modified third-order Jacobsthal polynomials with even subscripts.

Corollary 3.2. *Sums of sequences $\mathscr{J}_n(x)$ and $\mathscr{K}_n(x)$ with even subscripts are*

$$\sum_{j=0}^{n} \mathscr{J}_{2j}(x) = \frac{1}{x}\left[(x+3)\mathscr{J}_{2n+2}(x) - (x^2+3x+3)\mathscr{J}_{2n+1}(x) + (2x^2+5x+3)\mathscr{J}_{2n}(x) - x\right]$$

and

$$\sum_{j=0}^{n} \mathscr{K}_{2j}(x) = \frac{1}{x}\left[(x+3)\mathscr{K}_{2n+2}(x) - (x^2+3x+3)\mathscr{K}_{2n+1}(x) + (2x^2+5x+3)\mathscr{K}_{2n}(x) - (2x^3+7x^2+12x+15)\right].$$

Proof. The proof can be easily established using [18, Theorem 2.1]. □

Now, we give the combinatorial equalities of the binomial transforms for third-order Jacobsthal and modified third-order Jacobsthal polynomials.

Theorem 3.3. *For $n \geq 0$, we have the equalities*

$$\sum_{i=0}^{n}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^j \left(-\frac{x+2}{x+1}\right)^i \mathscr{J}_{i+j}(x) = (x+1)^{-n}\mathscr{J}_{3n}(x) \tag{3.3}$$

and

$$\sum_{i=0}^{n}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^j \left(-\frac{x+2}{x+1}\right)^i \mathscr{K}_{i+j}(x) = (x+1)^{-n}\mathscr{K}_{3n}(x). \tag{3.4}$$

Proof. (3.3): Let λ stand for a root of the characteristic equation of Eq. (2.4). Then, we have $\lambda^3 = (x+2)(\lambda^2-\lambda)+x+1$ and we can write by considering binomial expansion with $x+1 \neq 0$:

$$\begin{aligned}
\left(\frac{\lambda^3}{x+1}\right)^n &= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{\lambda^3}{x+1}-1\right)^i \\
&= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{x+2}{x+1}(\lambda^2-\lambda)\right)^i \\
&= \sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{x+2}{x+1}\lambda^2\right)^j\left(-\frac{x+2}{x+1}\lambda\right)^{i-j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\left(-\frac{x+2}{x+1}\right)^i\lambda^{i+j}.
\end{aligned}$$

If we replace to ω_1 and ω_2 by λ and rearrange, then we obtain

$$\begin{aligned}
\frac{\mathscr{J}_{3n}(x)}{(x+1)^n} &= \frac{1}{(x+1)^n}\left[\frac{x(x+1)^{3n}}{x^2+x+1} + \frac{\omega_1^{3n-1}}{(x+\omega_2)(\omega_1-\omega_2)} - \frac{\omega_2^{3n-1}}{(x+\omega_1)(\omega_1-\omega_2)}\right] \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\left(-\frac{x+2}{x+1}\right)^i\mathscr{J}_{i+j}(x),
\end{aligned}$$

where ω_1 and ω_2 are the roots of the characteristic equation $\lambda^3-(x+2)\lambda^2+(x+2)\lambda-(x+1)=0$. Finally, Eq. (3.4) can be obtained in a similar way. □

For simplicity of notation, let

$$\begin{aligned}
\mathscr{Z}_n(x) &= \frac{(x+\omega_1)\omega_1^{n-1}-(x+\omega_2)\omega_2^{n-1}}{\omega_1-\omega_2} = \frac{A\omega_1^n - B\omega_2^n}{\omega_1-\omega_2}, \\
\mathscr{W}_n &= \omega_1^n+\omega_2^n = \frac{1}{x^2+x+1}\left[(x+2)\mathscr{Z}_{n+1}(x)-(2x+1)\mathscr{Z}_n(x)\right],
\end{aligned} \tag{3.5}$$

where $A = \omega_2 x+1$ and $B = \omega_1 x+1$.
Further, the Binet formula of the binomial transforms for third-order Jacobsthal and modified third-order Jacobsthal polynomials are given by

$$\mathscr{J}_n(x) = \frac{1}{x^2+x+1}\left[x(x+1)^n + \mathscr{Z}_n(x)\right] \tag{3.6}$$

and

$$\mathscr{K}_n(x) = (x+1)^n + \mathscr{W}_n.$$

Note that $\mathscr{Z}_{n+2}(x) = \mathscr{Z}_{n+1}(x) - \mathscr{Z}_{n+2}(x)$, with initial conditions $\mathscr{Z}_0(x) = -x$ and $\mathscr{Z}_1(x) = 1$.
The Vajda's identity for the sequence $\mathscr{Z}_n(x)$ and binomial transform of third-order Jacobsthal polynomials is given in the next theorem.

Theorem 3.4. *Let $n \geq 0$, $p \geq 0$, $q \geq 0$ be integers. Then, we have*

$$\mathscr{Z}_{n+p}(x)\mathscr{Z}_{n+q}(x) - \mathscr{Z}_n(x)\mathscr{Z}_{n+p+q}(x) = (x^2+x+1)\mathscr{A}_p\mathscr{A}_q \tag{3.7}$$

and

$$\begin{aligned} &\mathscr{J}_{n+p}(x)\mathscr{J}_{n+q}(x) - \mathscr{J}_n(x)\mathscr{J}_{n+p+q}(x) \\ &\quad = \frac{1}{(x^2+x+1)^2}\left[(x^2+x+1)\mathscr{A}_p\mathscr{A}_q - x^2(x+1)^n\left(\mathscr{B}_{n+p}(q) - (x+1)^p\mathscr{B}_n(q)\right)\right], \end{aligned} \tag{3.8}$$

where $\mathscr{A}_n = \frac{\omega_1^n-\omega_2^n}{\omega_1-\omega_2}$ and $\mathscr{B}_n(q) = \mathscr{Z}_{n+q}(x) - (x+1)^q\mathscr{Z}_n(x)$.

Proof. (3.7): By using Eq. (3.5), $A = \omega_2 x + 1$ and $B = \omega_1 x + 1$ and $AB = x^2+x+1$, we have

$$\begin{aligned} &\mathscr{Z}_{n+p}(x)\mathscr{Z}_{n+q}(x) - \mathscr{Z}_n(x)\mathscr{Z}_{n+p+q}(x) \\ &\quad = \frac{1}{(\omega_1-\omega_2)^2}\left[\left(A\omega_1^{n+p} - B\omega_2^{n+p}\right)\left(A\omega_1^{n+q} - B\omega_2^{n+q}\right) - (A\omega_1^n - B\omega_2^n)\left(A\omega_1^{n+p+q} - B\omega_2^{n+p+q}\right)\right] \\ &\quad = \frac{1}{(\omega_1-\omega_2)^2}\left[AB(\omega_1^p - \omega_2^p)(\omega_1^q - \omega_2^q)\right] \\ &\quad = (x^2+x+1)\mathscr{A}_p\mathscr{A}_q, \end{aligned}$$

where $\mathscr{A}_n = \frac{\omega_1^n-\omega_2^n}{\omega_1-\omega_2}$ is the n-th companion sequence of $\mathscr{Z}_n(x)$.
(3.8): By formulas (3.5), (3.6) and Eq. (2.4), we get

$$\begin{aligned} &\mathscr{J}_{n+p}(x)\mathscr{J}_{n+q}(x) - \mathscr{J}_n(x)\mathscr{J}_{n+p+q}(x) \\ &= \frac{1}{(x^2+x+1)^2}\left[\left(x(x+1)^{n+p} + \mathscr{Z}_{n+p}(x)\right)\left(x(x+1)^{n+q} + \mathscr{Z}_{n+q}(x)\right) - \left(x(x+1)^n + \mathscr{Z}_n(x)\right)\left(x(x+1)^{n+p+q} + \mathscr{Z}_{n+p+q}(x)\right)\right] \\ &= \frac{1}{(x^2+x+1)^2}\left[\mathscr{Z}_{n+p}(x)\mathscr{Z}_{n+q}(x) - \mathscr{Z}_n(x)\mathscr{Z}_{n+p+q}(x) + x^2(x+1)^{n+p}\mathscr{B}_n(q) - x^2(x+1)^n\mathscr{B}_{n+p}(q)\right] \\ &= \frac{1}{(x^2+x+1)^2}\left[(x^2+x+1)\mathscr{A}_p\mathscr{A}_q - x^2(x+1)^n\left(\mathscr{B}_{n+p}(q) - (x+1)^p\mathscr{B}_n(q)\right)\right], \end{aligned}$$

where $\mathscr{B}_n(q) = \mathscr{Z}_{n+q}(x) - (x+1)^q\mathscr{Z}_n(x)$. □

It is easily seen that for special values of p and q by Theorem 3.4, we get new identities for binomial transform of the third-order Jacobsthal polynomials:

- Catalan's identity: $q = -p$.
- Cassini's identity: $p = 1$, $q = -1$.
- d'Ocagne's identity: $p = 1$, $q = m - n$, with $m \geq n$.

Corollary 3.5. *Catalan identity for binomial transform of the third-order Jacobsthal polynomials. Let $n \geq 0$, $p \geq 0$ be integers such that $n \geq p$. Then*

$$\begin{aligned} &\mathscr{J}_{n+p}(x)\mathscr{J}_{n-p}(x) - (\mathscr{J}_n(x))^2 \\ &\quad = \frac{1}{(x^2+x+1)^2}\left[-(x^2+x+1)\mathscr{A}_p^2 - x^2(x+1)^n\left(\mathscr{B}_{n+p}(-p) - (x+1)^p\mathscr{B}_n(-p)\right)\right]. \end{aligned}$$

Corollary 3.6. *Cassini identity for binomial transform of the third-order Jacobsthal polynomials. Let $n \geq 1$ be an integer. Then*

$$\begin{aligned} &\mathscr{J}_{n+1}(x)\mathscr{J}_{n-1}(x) - (\mathscr{J}_n(x))^2 \\ &\quad = \frac{1}{(x^2+x+1)^2}\left[-(x^2+x+1) - x^2(x+1)^n\left(\mathscr{B}_{n+1}(-1) - (x+1)\mathscr{B}_n(-1)\right)\right]. \end{aligned}$$

Corollary 3.7. *d'Ocagne identity for binomial transform of the third-order Jacobsthal polynomials. Let $n \geq 0$, $m \geq 0$ be integers such that $m \geq n$. Then*

$$\begin{aligned} &\mathscr{J}_{n+1}(x)\mathscr{J}_m(x) - \mathscr{J}_n(x)\mathscr{J}_{m+1}(x) \\ &\quad = \frac{1}{(x^2+x+1)^2}\left[(x^2+x+1)\mathscr{A}_{m-n} - x^2(x+1)^n\left(\mathscr{B}_{n+1}(m-n) - (x+1)\mathscr{B}_n(m-n)\right)\right]. \end{aligned}$$

4. Conclusion

In this paper, we first define the binomial transforms of third-order Jacobsthal polynomials $\mathscr{J}_n(x)$ and give some identities of this new sequence of polynomials. By taking into account these transforms and its properties, identities of the binomial transforms of third-order Jacobsthal and modified third-order Jacobsthal numbers can also be obtained. Furthermore, if we replace $x = 2$ in $\mathscr{J}_n(x)$, we obtain the binomial transform of third-order Jacobsthal numbers and if we replace $x = 2$ in $\mathscr{K}_n(x)$, we obtain the binomial transform of modified third-order Jacobsthal numbers (in the same sense as Soykan in [3]). Finally, we obtained the generating functions, Binet formulas, summations, and relationships for the binomial transforms of the third-order Jacobsthal and modified third-order Jacobsthal polynomials.

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References

[1] C. K. Cook, M. R. Bacon, *Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations*, Ann. Math. Inform., **41** (2013), 27-39.
[2] G. Morales, *A note on modified third-order Jacobsthal numbers*, Proyecciones, **39**(2) (2020), 409-420.
[3] Y. Soykan, E. Taşdemir, M. Göcen, *Binomial transform of the generalized third-order Jacobsthal sequence*, Asian-Eur. J. Math., **15**(12) (2022), 1-12.
[4] G. Morales, *Identities for third order Jacobsthal quaternions*, Adv. Appl. Clifford Algebr., **27**(2) (2017), 1043-1053.
[5] G. Morales, *Some results on dual third-order Jacobsthal quaternions*, Filomat, **33**(7) (2019), 1865-1876.
[6] G. Morales, *Third-order Jacobsthal generalized quaternions*, J. Geom. Symmetry Phys., **50** (2018), 11–27.
[7] G. Morales, *On third-order Jacobsthal polynomials and their properties*, Miskolc Math. Notes, **22**(1) (2021), 123-132.
[8] K. W. Chen, *Identities from the binomial transform*, J. Number Theory, **124** (2007), 142-150.
[9] S. Falcón, A. Plaza, *Binomial transforms of the k-Fibonacci sequences*, Int. J. Nonlinear Sci. Numer. Simul., **10** (2009), 1305-1316.
[10] H. Prodinger, *Some information about the binomial transform*, Fibonacci Q., **32**(5) (1994), 412-415.
[11] S. K. Ghosal, J. K. Mandal, *Binomial transform based fragile watermarking for image authentication*, J. Inf. Secur. Appl., **19** (2014), 272-281.
[12] N. Yilmaz, *Binomial transforms of the Balancing and Lucas-Balancing polynomials*, Contrib. Discrete Math., **15**(3) (2020), 133-144.
[13] A. Özkoç, E. Gündüz, *Binomial transform for quadra Fibona-Pell sequence and quadra Fibona-Pell quaternion*, Univers. J. Math. Appl., **5**(4) (2022), 145-155.
[14] N. Yilmaz, I. Aktaş, *Special transforms of the generalized bivariate Fibonacci and Lucas polynomials*, Hacet. J. Math. Stat., **52**(3) (2023), 640-651.
[15] C. Kızılateş, N. Tuglu, B. Çekim, *Binomial transforms of quadrapell sequences and quadrapell matrix sequences*, J. Sci. Arts, **1**(38) (2017), 69-80.
[16] C. Kızılateş, *On the quadra Lucas-Jacobsthal numbers*, Karaelmas Fen ve Müh. Derg., **7**(2) (2017), 619-621.
[17] K. N. Boyadzhiev, *Notes on the Binomial Transform*, World Scientific, Singapore, 2018.
[18] Y. Soykan, *Summing formulas for generalized Tribonacci numbers*, Univers. J. Math. Appl., **3**(1) (2020), 1-11.