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CERTAIN RESULTS CONCERNING (p, q)-PARAMETERIZED BETA LOGARITHMIC FUNCTION AND THEIR PROPERTIES

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Abstract. The primary object of this article is to introduce (p, q)-beta logarithmic function with extended beta function by using the logarithmic mean. We evaluate different properties and representations of beta logarithmic function. Further, it is evaluated logarithmic distribution, hypergeometric and confluent hypergeometric functions via logarithmic mean are evaluated and their essential properties are studied. Numerous formulas of (p, q)-beta logarithmic functions such as integral formula, derivative formula, transformation formula and generating function are analyzed.

1. Introduction and Preliminaries

The ordinary hypergeometric functions have been the subject of comprehensive research by various eminent mathematician. These functions play a vital role in different branches of mathematics. Applications of special functions (higher order transcendental functions such as Bessel function, Whittaker function, Wright functions etc.) are found in a broad variety of engineering sub-fields. The Euler beta function plays an important role in special function which introduced by Legendre, Whittaker and Watson etc. Using techniques to unify and generalize specialized

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functions has been an active and interesting area of research. An extension of the Euler beta function was proposed in 1997 by Chaudhry et al. [3] as well as a number of other researchers.

Definition 1. *The beta function (also called the Euler's integral of the first kind) is defined as (see [11, 13]):*

$$B(\xi,\zeta)=\frac{\Gamma(\xi)\ \Gamma(\zeta)}{\Gamma(\xi+\zeta)}=\int_0^1 t^{\xi-1}(1-t)^{\zeta-1}dt,\ \ (Re(\xi)>0,\ \ Re(\zeta)>0) \tag{1}$$

where Γ *(.) is gamma function, the Euler integral of the second kind (commonly used as extension of factorial function to complex numbers defined for all complex numbers except for the non-positive integers).*

As we know that the gamma and beta functions play a crucial role in the development of theory of higher order transcendental functions and their various generalizations are given by the various number of researchers (see [1], [2], [3], [4], [5], [7], [8], [9], [12], [15]).

Gamma function is defined by the convergent improper integral as:

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt,\ \ (Re(x)>0).$$

The underlying extension of Euler's beta function established by Chaudhry et al. [3] is defined as

$$B_p(\xi,\zeta)=\int_0^1 t^{\xi-1}(1-t)^{\zeta-1}\ exp\left[-\frac{p}{t(1-t)}\ \right]dt,\ (Re(p)>0,Re(\xi)>0,Re(\zeta)>0). \tag{2}$$

For $p=0$, the extended beta function reduces to the classical beta function.

In 2004, Chaudhry et al. [4] used new extended beta function $B(\beta,\zeta;\rho)$ to introduced extended Gauss hypergeometric and confluent hypergeometric functions which are defined by their series representation as

$$F_\rho\left(\xi,\zeta;\eta;z\right)=\sum_{n=0}^{\infty}(\xi)_n\ \frac{B_\rho(\zeta+n,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\,\frac{z^n}{n!} \tag{3}$$

$$(\rho\geq 0,\ \ |z|<1,\ \ Re(\eta)>\Re(\zeta)>0),$$

and

$$\Phi_\rho\left(\zeta;\eta;z\right)=\sum_{n=0}^{\infty}\frac{B_\rho(\zeta+n,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\,\frac{z^n}{n!} \tag{4}$$

$$(\rho\geq 0,\ \ |z|<1,\ \ Re(\eta)>Re(\zeta)>0).$$

In 2014, Choi et al. [5] introduced another extension of beta function, denoted by $B_{p,q}(\xi,\zeta)$ and is defined by

$$B_{p,q}(\xi,\zeta)=\int_0^1 t^{\xi-1}(1-t)^{\zeta-1}exp\left[\frac{-p}{t}-\frac{q}{(1-t)}\right]dt, \tag{5}$$

$$(Re(p)>0,\ Re(q)>0),(Re(\xi)>0,\ Re(\zeta)>0).$$

The integral representation for extended Gauss hypergeometric function and extended confluent hypergeometric function are defined as follows :

$$F_{p,q}\left(\xi,\zeta;\eta;z\right)=\frac{1}{B(\zeta,\eta-\zeta)}\int_0^1 t^{\zeta-1}\left(1-t\right)^{\eta-\zeta-1}\left(1-zt\right)^{-\xi}exp\left[\frac{-p}{t}-\frac{q}{(1-t)}\right]dt, \tag{6}$$

$$(p,q\geq 0;\quad |\arg(1-z)|<\pi;\ Re(\eta)>Re(\zeta)>0),$$

and

$$\Phi_{p,q}\left(\zeta;\eta;z\right)=\frac{1}{B(\zeta,\eta-\zeta)}\int_0^1 t^{\zeta-1}\left(1-t\right)^{\eta-\zeta-1}exp\left(zt-\frac{p}{t}-\frac{q}{(1-t)}\right)dt, \tag{7}$$

$$\{p,q\geq 0,\quad Re(\eta)>Re(\zeta)>0\}.$$

Definition 2. *The logarithmic mean for $x,y>0$ (quotient of difference of two non-negative numbers by their logarithmic value) is defined as (see [14])*

$$L(x,y)=\int_0^1 x^{1-t}y^t\ dt=\begin{cases}\frac{x-y}{log(x)-log(y)} & x\neq y,\\ x & x=y.\end{cases} \tag{8}$$

It can be easily seen that the logarithmic mean satisfies the following properties (see [6], [10]):

• The logarithmic mean always lies between the geometric mean and arithmetic mean.

• For $x=y$ all three means that are geometric mean, arithmetic mean and logarithmic mean are same.

• The limiting condition of the logarithmic mean is given as:

$$\lim_{y\to x}L(x,y)=L(x,x)=x.$$

• The logarithmic mean satisfies the following property that is:

$$\frac{1}{L(x,y)}=\int_0^1\frac{dt}{tx+(1-ty)}.$$

• The infinite product of the logarithmic mean of any two positive real numbers are given as:

$$L(x,y)=\prod_{m=1}^{\infty}\left(\frac{x^{2-m}+y^{2-m}}{2}\right).$$

2. Construction of (p, q)- Beta Logarithmic Function

For any fixed $x, y > 0$ the function $x^{1-t}y^t$ is continuous in $[0,1]$ and so it is bounded on $[0,1]$. It means that there exist $c \geq 0$ and for any $x, y, \xi, \zeta > 0$, we have

$$\begin{aligned} 0 &\leq x^{1-t}\, y^t\, t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right] \\ &\leq c\, t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right],\ \forall\ t \in (0,1). \end{aligned}$$

Thus, $x^{1-t}\, y^t\, t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]$ is integrable on $(0,1)$. We introduce the underlying definition that defines the relation between beta function and logarithmic mean.

Definition 3. *For any $x, y, \xi, \zeta \in \mathbb{R}^+$, we define*

$$B^m_{p,q}\ L(x,y;\xi,\zeta) = \int_0^1 x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right]dt, \tag{9}$$

$$(p, q \geq 0,\ Re(\xi) > 0,\ Re(\zeta) > 0),$$

which we call the (p,q) beta-logarithmic function.

Remark 1. *Substituting $x = y = 1$ in (9), we get extended beta function*

$$B^m_{p,q}\ L(1,1;\xi,\zeta) = B^m_{p,q}\ (\xi,\zeta),\ (Re(\xi) > 0,\ Re(\zeta) > 0)$$

where,

$$B^m_{p,q}(\xi,\zeta) = \int_0^1 t^{\xi-1}(1-t)^{\zeta-1} exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right] dt,\ (Re(p) > 0,\ Re(q) > 0). \tag{10}$$

Remark 2. *By setting $x = y = 1$, $p = q = 0$ and $m = 1$ in (9), we get the Euler Beta function (1) (see [11], [13])*

$$B^1_{0,0}\ L(1,1;\xi,\zeta) = B(\xi,\zeta),\ (Re(\xi) > 0,\ Re(\zeta) > 0.$$

Remark 3. *If we take $\xi = \zeta = 1$, $p = q = 0$ and $m = 1$ in (9), we get logarithmic mean (8) (see [14]).*

$$B^1_{0,0}\ L(x,y;1,1) = L(x,y),\ (x, y > 0).$$

3. Properties of (p, q)- Beta Logarithmic Function

In this section, we analyze different properties and representations of a new form of beta function that we call the (p,q) beta logarithmic function. This function is a combined study of a new extended beta function and the logarithmic mean.

Proposition 1. *For $x, y, \xi, \zeta, p, q > 0$, the following assertions hold true:*

$$B^m_{p,q}\ L(x, y; \xi, \zeta) = B^m_{p,q}\ L(y, x; \xi, \zeta), \tag{11}$$

$$B^m_{p,q}\ L(x, x; \xi, \zeta) = x\, B^m_{p,q}\ (\xi, \zeta), \tag{12}$$

and

$$B^m_{p,q}\ L(\delta x, \delta y; \xi, \zeta) = \delta B^m_{p,q}\ L(x, y; \xi, \zeta). \tag{13}$$

Proof. The result (11) may be reached by altering the variable t by $1-u$ in equation (9). The assertions (12) and (13) may be produced by easy computation in equation (9). □

Proposition 2. *For any $x, y, \xi, \zeta, p, q > 0$, the following assertions hold true:*

$$B^m_{p,q}\ L(x, y; \xi + 1, \zeta) + B^m_{p,q}\ L(x, y; \xi, \zeta + 1) = B^m_{p,q}\ L(x, y; \xi, \zeta). \tag{14}$$

Proof. By using the definition (9) to the left side of (14), we get the required assertion (14). □

Corollary 1. *If we set $x = y = 1$ in (14), we obtained the well known result introduced by M. Raïssouli et al. [14]*

$$B^m_{p,q}(\xi + 1, \zeta) + B^m_{p,q}(\xi, \zeta + 1) = B^m_{p,q}(\xi, \zeta). \tag{15}$$

Proposition 3. *For any $x, y, \xi, \zeta > 0$, $p, q \geq 0$, the following assertions hold true:*

$$\begin{aligned} min(x, y)B^m_{p,q}(\xi, \zeta) \leq B^m_{p,q}L(x, y; \xi, \zeta) &\leq xB^m_{p,q}(\xi, \zeta + 1) + yB^m_{p,q}(\xi + 1, \zeta) \\ &\leq max(x, y)B^m_{p,q}(\xi, \zeta). \end{aligned} \tag{16}$$

Proof. From the underlying inequality

$$min(x, y) \leq \sqrt{xy} \leq L(x, y) \leq \left(\frac{x + y}{2}\right) \leq max(x, y) \text{ and } B^m_{p,q}(\xi, \zeta) > 0,$$

we get the following relation

$$min(x, y)B^m_{p,q}(\xi, \zeta) \leq B^m_{p,q}L(x, y; \xi, \zeta). \tag{17}$$

By using the underlying well known Young's inequality

$$x^{1-t}y^t \leq x(1 - t) + yt, \quad \forall\ t \in [0, 1]$$

we get

$$\begin{aligned} B^m_{p,q}\ L(x,y;\xi,\zeta) &\le \int_0^1\ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\Big[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \Big]dt \\ &\le x\left(\int_0^1 t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\Big[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \Big]dt\right) \\ &+ y\left(\int_0^1\ t^{\xi-1}\,(1-t)^{\zeta-1}\, exp\Big[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \Big]dt\right) \\ &\le x\ \big(B^m_{p,q}(\xi,\zeta+1)\big)+\big(y\ B^m_{p,q}(\xi+1,\zeta)\big) \\ &\le\ max(x,y)\ \big(B^m_{p,q}(\xi,\zeta+1)+B^m_{p,q}(\xi+1,\zeta)\big) \end{aligned}$$

by using the relation (15), we achieved the required result. □

Proposition 4. *For any $x, y, \xi, \zeta > 0$, $p, q \ge 0$ the following assertion holds true:*

$$B^m_{p,q}\ L(x,y;\xi,\zeta)=\sum_{n=0}^{\infty} B^m_{p,q}(x,y;\xi+n,\zeta+1). \tag{18}$$

Proof. We have

$$B^m_{p,q}\ L(x,y;\xi,\zeta)=\int_0^1\ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt,$$

By using the series representation $(1-t)^{-1}=\sum_{n=0}^{\infty} t^n$, for $t\in(0,1)$ with the arguments of uniform convergence of this power series, we have

$$\begin{aligned} B^m_{p,q}\ L(x,y;\xi,\zeta) &= \int_0^1\ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta-1}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt \\ &= \int_0^1\ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta}\ (1-t)^{-1}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt \\ &= \sum_{n=0}^{\infty}\int_0^1\ x^{1-t}\ y^t\ t^{\xi-1}\,(1-t)^{\zeta}\ t^n\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt \\ &= \sum_{n=0}^{\infty}\int_0^1\ x^{1-t}\ y^t\ t^{\xi+n-1}\,(1-t)^{\zeta}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\ \right]dt, \end{aligned}$$

using the definition (9) in the above expression, we achieved the desired result. □

Theorem 1. *Let $x, y, \xi, \zeta > 0$, $p, q \ge 0$, the following representation holds true:*

$$B^m_{p,q}\ L(x,y;\xi,\zeta)=\sum_{r,n=0}^{\infty}\frac{B^m_{p,q}(\xi+n,\zeta+r)}{n!r!}(log(x))^r(log(y))^n. \tag{19}$$

Proof. Using the following power series expansion

$$x^{1-t} = \sum_{r=0}^{\infty} \frac{(log(x))^r}{r!}(1-t)^r, \; y^t = \sum_{n=0}^{\infty} \frac{(log(y))^n}{n!}t^n$$

using the above expansion in the result (9), we have

$$B_{p,q}^{m} \; L(x,y;\xi,\zeta) = \int_0^1 \; x^{1-t} \; y^t \; t^{\xi-1} \, (1-t)^{\zeta-1} \exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right] dt$$

$$= \int_0^1 \sum_{r,n=0}^{\infty} \frac{(log(x))^r (log(y))^n}{r!n!} \; t^{\xi+n-1} \, (1-t)^{\zeta+r-1} \exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right] dt$$

$$= \int_0^1 \frac{t^{\xi+n-1} \, (1-t)^{\zeta+r-1}}{r!n!} \; (log(x) \sum_{r,n=0}^{\infty})^r (log(y))^n \exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right] dt,$$

using the definition (10) in the above expression, we achieved the required result (19). □

4. The $(\mathbf{p}, \mathbf{q})$-Beta Logarithmic Random Variable

In this section, we define beta-logarithmic distribution of (9) and obtain its mean, variance and moment generating function.

Definition 4. *For $x, y, \xi, \zeta > 0$, $p, q \geq 0$, the beta-logarithmic distribution is defined as:*

$$f(t) = \begin{cases} \frac{1}{B_{p,q}^{m} \; L(\xi,\zeta)} \; x^{1-t} \; y^t \; t^{\xi-1}(1-t)^{\zeta-1} \exp\left[\frac{-p}{t^m} - \frac{q}{(1-t)^m}\right], & (0 < t < 1), \\ 0, & otherwise. \end{cases} \tag{20}$$

The k^{th}- moment of a random variable X for any real number k is given as:

$$\mathbb{E}(X^k) = \frac{B_{p,q}^{m} \; L(x,y;\xi+k,\zeta)}{B_{p,q}^{m} \; L(x,y;\xi,\zeta)}, \tag{21}$$

$$(\; p,q \geq 0, \;\; x,y,\xi,\zeta > 0).$$

For $k = 1$, we obtain the mean as a particular case of (21) given by

$$\mu = \mathbb{E}(X) = \frac{B_{p,q}^{m} \; L(x,y;\xi+1,\zeta)}{B_{p,q}^{m} \; L(x,y;\xi,\zeta)}. \tag{22}$$

The variance of the distribution is defined as: $\sigma^2 = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2$

$$\sigma^2 = \frac{B_{p,q}^{m} \; L(x,y;\xi,\zeta) \, B_{p,q}^{m} \; L(x,y;\xi+2,\zeta) - \left\{B_{p,q}^{m} \; L(x,y;\xi+1,\zeta)\right\}^2}{\left\{B_{p,q}^{m} \; L(x,y;\xi,\zeta)\right\}^2}. \tag{23}$$

The moment generating function of the distribution is defined as

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathbb{E}(X^n) = \frac{1}{B_{p,q}^m L(x,y;\xi,\zeta)} \sum_{n=0}^{\infty} B_{p,q}^m L(x,y;\xi+n,\zeta)\frac{t^n}{n!}. \tag{24}$$

Here, we recall the following known lemma.

Lemma 1. *Let Y be a random variable with values that exist inside a finite range $[x, y]$. Then, we have for all $\mathcal{E} \in [x, y]$,*

$$\left| P(Y \leq \mathcal{E}) - \frac{y - E(Y)}{y - x} \right| \leq \frac{1}{2} + \frac{|\mathcal{E} - \frac{x+y}{2}|}{y - x}. \tag{25}$$

Proposition 5. *Let X represent a beta-logarithmic random variable with parameters $(x, y; \xi, \zeta)$. Then, for any $k, \mathcal{E} > 0$, the following assumptions are true:*

$$\left| P(X \leq \mathcal{E}) - \frac{B_{p,q}^m\ L(x,y;\xi,\zeta+1)}{B_{p,q}^m\ L(x,y;\xi,\zeta)} \right| \leq \frac{1}{2} + \left| \mathcal{E} - \frac{1}{2} \right| \tag{26}$$

and

$$P(X^k \geq \mathcal{E}) \leq \frac{B_{p,q}^m L(x,y;\xi+k,\zeta)}{\mathcal{E}\ B_{p,q}^m L(x,y;\xi,\zeta)} \tag{27}$$

Proof. With the help of (14) and (22), we have

$$E(X) = 1 - \frac{B_{p,q}^m\ L(x,y;\xi,\zeta+1)}{B_{p,q}^m L(x,y;\xi,\zeta)}, \tag{28}$$

using the above relation in inequality (25), we achieved the desired result (26).

The second inequality (27) can be deduced by using the Markov's inequality

$$P(X^k \geq \mathcal{E}) \leq \frac{E(X^k)}{\mathcal{E}}$$

and the definition of $E(X^k)$, we get the desired result (27). □

5. Hypergeometric and Confluent Hypergeometric Representation by (p, q)-Beta Logarithmic Function

Many researchers gave the extension of hypergeometric and confluent hypergeometric functions (see [4], [5], [12]). Here, we introduce a new hypergeometric and confluent hypergeometric functions in terms of (p,q)-beta logarithmic function.

The (p, q)-beta logarithmic hypergeometric function is defined as:

$$F_{p,q}^m\ L(\xi,\zeta;\eta;z) = \sum_{n=0}^{\infty} (\xi)_n\ \frac{B_{p,q}^m\ L(x,y;\zeta+n,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\ \frac{z^n}{n!}, \tag{29}$$

$$(p, q \geq 0,\ |z| < 1,\ Re(\eta) > Re(\zeta) > 0,\ x, y > 0).$$

The (p,q)-beta logarithmic confluent hypergeometric logarithmic function is defined as:

$$\Phi^{m}_{p,q}\ L\,(\xi;\zeta;z)=\sum_{n=0}^{\infty}\frac{B^{m}_{p,q}\ L(x,y;\xi+n,\eta-\zeta)}{B(\zeta,\eta-\zeta)}\frac{z^n}{n!}, \tag{30}$$

$$(p,q\geq 0,\ x,y,>0,\ Re(\eta)>Re(\zeta)>0, Re(\xi)>0, |z|<1).$$

5.1. **Integral formula.**

Theorem 2. *The following integral formula for the (p,q)-beta logarithmic hypergeometric and (p,q)-beta logarithmic confluent hypergeometric function holds true:*

$$F^{m}_{p,q}\ L\,(\xi,\zeta;\eta;z)=\frac{1}{B(\zeta,\eta-\zeta)}\times\int_0^1 x^{1-t}y^t\ t^{\zeta-1}(1-t)^{\eta-\zeta-1}(1-zt)^{-\xi}\exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]dt, \tag{31}$$

$$(|\arg(1-z)|<\pi;\ p,q\geq 0;\ x,y,\in\mathbb{R}^+;\ Re(\eta)>Re(\zeta)>0),$$

and

$$\phi^{m}_{p,q}\ L\,(\zeta;\eta;z)=\frac{1}{B(\zeta,\eta-\zeta)}\int_0^1 x^{1-t}y^t\ t^{\zeta-1}(1-t)^{\eta-\zeta-1}e^{zt}exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]dt \tag{32}$$

$$(p,q\geq 0;\ x,y,\in\mathbb{R}^+;\ Re(\eta)>Re(\zeta)>0).$$

Proof. By applying the definition of beta logarithmic function (9) into (29) and by rearranging the order of integral and summation, we get

$$F^{m}_{p,q}\ L\,(\xi,\zeta;\eta;z)=\frac{1}{B(\zeta,\eta-\zeta)}\times\int_0^1 x^{1-t}y^t\ t^{\zeta-1}(1-t)^{\eta-\zeta-1}exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]\sum_{n=0}^{\infty}(\xi)_n\frac{(zt)^n}{n!}dt. \tag{33}$$

Applying the binomial theorem in (33), we obtained the desired result (31).

Similarly, we can obtain (32). □

5.2. **Derivative formula.**

Theorem 3. *The following derivative formula for (p,q)-beta logarithmic hypergeometric and (p,q)-beta logarithmic confluent hypergeometric functions hold true:*

$$\frac{d^n}{dz^n}\left\{F^{m}_{p,q}\ L\,(\xi,\zeta;\eta;z)\right\}=\frac{(\xi)_n(\zeta)_n}{(\eta)_n}F^{m}_{p,q}\ L\,(\xi+n,\zeta+n;\eta+n;\ z)\,, \tag{34}$$

and

$$\frac{d^n}{dz^n}\left\{\phi^{m}_{p,q}\ L\,(\zeta;\eta;z)\right\}=\frac{(\zeta)_n}{(\eta)_n}\phi^{m}_{p,q}\ L\,(\zeta+n;\eta+n;\ z)\,, \tag{35}$$

where

$$(p, q \geq 0,\ Re(\eta) > Re(\zeta) > 0);\ n \in \mathbb{N}_0.$$

Proof. We know well known relation of Euler-Beta function,

$$B(\zeta, \eta - \zeta) = \frac{\eta}{\zeta}\ B(\zeta + 1, \eta - \zeta), \tag{36}$$

Differentiating (29) with respect to variable z, we get

$$\frac{d}{dz}\left\{F_{p,q}^{m}\ L(\xi, \zeta; \eta; z)\right\} = \sum_{n=0}^{\infty} (\xi)_n\ \frac{B_{p,q}^{m}\ L(x, y; \zeta + n, \eta - \zeta)}{B(\zeta, \eta - \zeta)}\ \frac{z^{n-1}}{n - 1!}$$

$$= \sum_{n=0}^{\infty} (\xi)_{n+1}\ \frac{B_{p,q}^{m}\ L(x, y; \zeta + n + 1, \eta - \zeta)}{B(\zeta, \eta - \zeta)}\ \frac{z^n}{n!},$$

Using $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ and (36) in the above expression, we obtain

$$\frac{d}{dz}\left\{F_{p,q}^{m}\ L(\xi, \zeta; \eta; z)\right\} = \frac{\xi\zeta}{\eta} \sum_{n=0}^{\infty} (\xi + 1)_n\ \frac{B_{p,q}^{m}\ L(x, y; \zeta + n + 1, \eta - \zeta)}{B(\zeta + 1, \eta - \zeta)}\ \frac{z^n}{n!},$$

where $(\alpha)_n$ is the Pochhammar symbol defined as

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} 1 & (n = 0; \alpha \in \mathbb{C} \setminus \{0\}) \\ \alpha(\alpha + 1)(\alpha + 2) \ldots (\alpha + n - 1) & (n \in \mathbb{N}; \alpha \in \mathbb{C}), \end{cases}$$

Now continuing the same process up-to $(n - 1)$, we get the required result (34). Similarly, by applying the same process on (30), we get the required result (35). □

Remark 4. *If we take p = q = 1 and m =1 in the expression* (34) *and* (35)*, we obtain a similar result in [4].*

5.3. **Transformation formulas.**

Theorem 4. *The following formulae for the hypergeometric logarithmic and confluent hypergeometric logarithmic functions hold true:*

$$F_{p,q}^{m}\ L\,(\xi, \zeta; \eta;\ z) = (1 - z)^{-\xi}\, F_{p,q}^{m}\ L\left(\xi, \eta - \zeta; \eta;\ -\frac{z}{1 - z}\right), \tag{37}$$

$$F_{p,q}^{m}\ L\left(\xi, \zeta; \eta;\ 1 - \frac{1}{z}\right) = z^{\xi}\, F_{p,q}^{m} L\,(\xi, \eta - \zeta; \eta;\ 1 - z)\,, \tag{38}$$

$$F_{p,q}^{m}\ L\left(\xi, \zeta; \eta;\ \frac{z}{1 + z}\right) = (1 + z)^{\xi}\, F_{p,q}^{m}\ L\,(\xi, \eta - \zeta; \eta;\ -z)\,, \tag{39}$$

$$\Phi_{p,q}^{m}\ L\,(\zeta, \eta;\ z) = e^{z}\, \Phi_{p,q}^{m}\ L\,(\eta - \zeta; \eta;\ -z)\,. \tag{40}$$

$$(p, q \geq 0, x, y, \in \mathbb{R}^{+};\ |z| < 1;\ Re(\eta) > Re(\zeta) > 0).$$

Proof. Substituting t by $1-t$ in $(1-zt)^{-\xi}$ and replacing the following equation

$$[1-z(1-t)]^{-\xi} = (1-z)^{-\xi}\left(1+\frac{z}{1-z}t\right)^{-\xi}$$

in (31) we obtain

$$F_{p,q}^{m}\ L\,(\xi,\zeta;\eta;\ z) = \frac{(1-z)^{-\xi}}{B(\zeta,\eta-\zeta)} \times \int_0^1 t^{\zeta-1}\,(1-t)^{\eta-\zeta-1}\left(1+\frac{z}{1-z}t\right)^{-\xi} exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]dt, \tag{41}$$

further, we have

$$F_{p,q}^{m}\ L\,(\xi,\zeta;\eta;\ z) = \frac{(1-z)^{-\xi}}{B(\zeta,\eta-\zeta)} \times \int_0^1 t^{\zeta-1}\,(1-t)^{\eta-\zeta-1}\left(1-\frac{-z}{1-z}t\right)^{-\xi} exp\left[\frac{-p}{t^m}-\frac{q}{(1-t)^m}\right]dt. \tag{42}$$

In view of (31), we get the required result (37).
Substituting z by $1-\frac{1}{z}$ and $\frac{z}{1+z}$ in (37) yield (38) and (39) respectively. □

Similarly applying the same process in (37) by simple calculation, we can establish (40).

Theorem 5. *The following relation holds true:*

$$F_{p,q}^{m}\ L\,(\xi,\zeta;\eta;\ 1) = \frac{B_{p,q}^{m}\ (x,y;\xi,\eta-\xi-\zeta)}{B(\zeta,\eta-\zeta)} \tag{43}$$

$$(p,q\geq 0;\ \ x,y\in\mathbb{R}^{+};\ \ Re(\eta-\xi-\zeta)>0).$$

Proof. Putting $z=1$ in (31) and using the definition (9), we obtain desired result (43). □

6. Generating function of $F_{p,q}^{m}L\,(\xi,\zeta;\eta;\ z)$

Theorem 6. *The generating function for* $F_{p,q}^{m}\ L\,(\xi,\zeta;\eta;\ z)$ *holds the underlying relation*

$$\sum_{k=0}^{\infty}(\xi)_k\ F_{p,q}^{m}\ L(\xi+k,\zeta;\eta;z)\,\frac{t^k}{k!} = (1-z)^{-\xi}\,F_{p,q}^{m}\left(\xi,\zeta;\eta;\ \frac{z}{1-t}\right) \tag{44}$$

$$(p,q\geq 0,|t|<1).$$

Proof. Let left hand side of (44) be denoted by L, then from (29), we have

$$L = \sum_{k=0}^{\infty} (\xi)_k \left(\sum_{n=0}^{\infty} \frac{(\xi + k)_n \, B_{p,q}^m \; L(x, y; \zeta + n, \eta - \zeta)}{B(\zeta, \eta - \zeta)} \frac{z^n}{n!} \right) \frac{t^k}{k!}.$$

Using the identity $(\alpha)_n (\alpha + n)_k = (\alpha)_k (\alpha + k)_n$, we get

$$L = \sum_{n=0}^{\infty} (\xi)_n \frac{B_{p,q}^m \; L(a, b; \zeta + n, \eta - \zeta)}{B(\zeta, \eta - \zeta)} \left(\sum_{k=0}^{\infty} (\xi + n)_k \frac{t^k}{k!} \right) \frac{z^n}{n!}.$$

Since, we know that $\sum_{n=0}^{\infty} (\xi + n)_n \frac{t^n}{n!} = (1 - t)^{-\xi - n}$, we obtain

$$L = \sum_{n=0}^{\infty} (\xi)_n \frac{B_{p,q}^m \; L(x, y; \zeta + n, \eta - \zeta)}{B(\zeta, \eta - \zeta)} (1 - t)^{-\xi - n} \frac{z^n}{n!}$$

$$L = (1 - t)^{-\xi} \sum_{n=0}^{\infty} (\xi)_n \frac{B_{p,q}^m \; L(x, y; \zeta + n, \eta - \zeta)}{B(\zeta, \eta - \zeta)} \left(\frac{z}{1 - t} \right)^n \frac{1}{n!}. \tag{45}$$

Finally by using (29) in (45), we get the desired result (44). □

7. Conclusions

In this article we define a (p, q)-beta logarithmic function which links with logarithmic mean and generalized beta function (see [3], [4]). Here, we analyze yet another extension of the Euler beta function and study a variety of properties, including integral representation, summation formula and derivative formula of the (p, q)-beta logarithmic function. Some analytical properties of this new extended function are developed and discuss its probabilistic concept as an application. Further, we get the beta distribution and the other statistical formula that go along with it. Finally, we expand the definition of hypergeometric and confluent hypergeometric function and explore the different features of the extended definition.

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SEPARATION, COMPACTNESS, AND SOBRIETY IN THE CATEGORY OF CONSTANT LIMIT SPACES

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Abstract. The objective of this article is to characterize each of compact, sober, and T_i for $i = 0, 1, 2$ constant limit spaces as well as to investigate the relationships between them. Finally, we compare our results in some topological categories.

1. Introduction

The lack of natural function spaces in ***Top***, the category of topological spaces and continuous maps which is not cartesian closed has been recognized as an awkward situation for various applications in the field of functional analysis and homotopy theory. The category ***Lim*** of limit spaces and continuous maps which is cartesian closed [17] supercategory of ***Top***. Limit spaces with compatible vector space structures are used to develop a calculus for vector spaces without norm [22].

Baran, in [2], introduced the notion of (strong) closedness in terms of final lifts, initial lifts, and discrete structures which are available in a topological category. He used these notions to generalize each of compact, sober, and T_i, $i = 1, 2, 3, 4$ objects in topological categories in [2, 7, 12].

The sober spaces were introduced in [18] and used in the theory of non-T_2 spaces. In 2022, Baran and Abughalwa [12] gave various forms of sober objects in a topological category and investigated relationships among these various forms.

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The objective of this article is to characterize each of compact, sober, and T_i for $i = 0, 1, 2$ constant limit spaces as well as to investigate the relationships between them. Finally, we compare our results in some topological categories.

2. Preliminaries

Definition 1. *Let $B \neq \emptyset$, $F(B)$ be the set of filters (proper or improper) on B, and the map $K : B \longrightarrow P(F(B))$. We call (B, K) is a constant limit space if K satisfies:*

(i) *$[s] \in K$, $\forall s \in B$, where $[s] = \{U \subset B : s \in U\}$,*

(ii) *if $\alpha \in K$ and $\alpha \subset \beta$, then $\beta \in K$,*

(iii) *if $\alpha, \beta \in K$, then $\alpha \cap \beta \in K$.*

Let (B, K) and (C, L) be constant limit spaces. If $f(\alpha) \in L$ for every $\alpha \in K$, then a map $f : (B, K) \longrightarrow (C, L)$ is called continuous, where $f(\alpha) = \{U \subset C : \exists V \in \alpha$ such that $f(V) \subset U\}$.

We denote ***ConLim*** by the category of constant limit spaces and continuous maps.

Proposition 1. *([5]) **(1)** Let $\{(B_i, K_i), i \in I\}$ in **ConLim**, B be a set, and $\{f_i : B \longrightarrow (B_i, K_i), i \in I\}$ be a source in **Set**. $\{f_i : (B, K) \longrightarrow (B_i, K_i), i \in I\}$ in **ConLim** is an initial lift iff $K = \{\alpha \in F(B) : f_i(\alpha) \in K_i, \forall i \in I\}$.*

(2) *An epi sink $\{f_i : (B_i, K_i) \longrightarrow (B, K)\}$ in **ConLim** is a final lift iff $\alpha \in K$ implies $\bigcap_{i=1}^{n} f(\alpha_i) \subset \alpha$ for some $\alpha_i \in K_i$, $i \in I$.*

(3) *$K = \{\alpha : \alpha = [U], U \subset B$ is finite $\}$ is discrete structure on B, where $[U] = \{V \subset B : U \subset V\}$.*

The constant limit structure on a finite set B is unique. Let $B = \{a_1, a_2, ..., a_n\}$. The discrete structure on B, $K = \{\alpha : \alpha = [U], U \subset B\} = F(B)$, the indiscrete structure on B.

3. Closed Subobjects

Let X be a set, $X^\infty = X \times X \times ...$ be the countable product of X, and $a \in X$. $\bigvee_a^\infty X$ (resp., $X \bigvee_a X$) is formed by taking countably many disjoint (resp., two distinct) copies of X identifying them at the point a.

Definition 2. *([2, 6]) Define $S_a : X \bigvee_a X \longrightarrow X^2$ by*

$$S_a(t_i) = \begin{cases} (t, t) & \text{if } i = 1 \\ (a, t) & \text{if } i = 2 \end{cases},$$

$\triangledown_a : X \bigvee_a X \longrightarrow X$ *by* $\triangledown_a(t_i) = t$,

$A_a^\infty : \bigvee_a^\infty X \longrightarrow X^\infty$ *by* $A_a^\infty(t_i) = (a, a, ..., a, t, a, a, ...)$,

and $\triangledown_a^\infty : \bigvee_a^\infty X \longrightarrow X$ *by* $\triangledown_a^\infty(t_i) = t$ *for each* $i \in I$, *where* I *is the index set* $\{i : t_i$ *is in the i-th component of* $\bigvee_a^\infty X\}$.

Definition 3. *([2]) Let* $\mathcal{U} : \mathcal{E} \longrightarrow$ ***Set*** *be a topological functor [1] and* X *be an object of* $\mathcal{E}$ *with* $\mathcal{U}(X) = B$.

(1) *If the initial lift of the* $\mathcal{U}$*-source* $S_a : B \bigvee_a B \longrightarrow \mathcal{U}(X^2) = B^2$ *and* $\triangledown_a : B \bigvee_a B \longrightarrow \mathcal{UD}((B)) = B$ *is discrete, then* X *is called* T_1 *at* a*, where* $\mathcal{D}$ *is the discrete functor.*

(2) *If the initial lift of the* $\mathcal{U}$*-source*

$$A_a^\infty : \vee_a^\infty B \longrightarrow \mathcal{U}(X^\infty) = B^\infty \quad and \quad \nabla_a^\infty : \vee_a^\infty B \longrightarrow \mathcal{UD}((B)) = B$$

is discrete, then $\{a\}$ *is called closed.*

(3) *If* $\{*\}$ *is closed in* X/M*, then* $M \subset X$ *is called closed, where* X/M *is the final lift of the epi* U*-sink*

$$q : B = U(X) \to B/M = (B \backslash M) \cup \{*\},$$

identifying M *with a point *.*

(4) *If* X/M *is* T_1 *at* $*$*, then* M *is called strongly closed in* X*.*

(5) *If* $B = M = \emptyset$ *iff then* M *is to be (strongly) closed.*

(6) $M \subset X$ *is open (resp., strongly open) iff* M^c *is closed (resp., strongly closed) in* X*.*

Remark 1. (1) *In* ***Top****, by Corollary 2.2.6 of [2],* $M \subset B$ *is closed iff* M *is closed in the usual sense. Moreover, the notion of strong closedness implies closedness and they coincide when a topological space is* T_1 *[4].*

(2) *In an arbitrary topological category, in general, the notions of closedness and strong closedness are independent of each other [4].*

Theorem 1. *Let* $(B, K) \in$ ***ConLim****.* $\emptyset \neq M \subset B$ *is closed (open) iff* $M = B$.

Proof. Suppose $\emptyset \neq M \subset B$ and $M \neq B$. Then $\exists t \in B$ with $t \notin M$. Take $\sigma = \bigcap_{i=1}^\infty [t_i]$ with $t_i \in B/M$. We have $\triangledown_* \sigma = [t]$ and $\pi_j A_*^\infty \sigma = [*] \cap [t] \in K_1$ for all i, where K_1 is the final structure on B/M. Since σ is generated by the infinite set $\{t_1, t_2, ..., t_n, ...\}$, σ does not contain a finite set which contradicts $\{*\}$ is being closed. Hence, $B = M$.

If $M = B$, then $\bigvee_*^\infty(B/M) = \{*\}$ and by Definition 3 (5), $\{*\} = \bigvee_*^\infty(B/M)$ is closed and consequently, M is closed.

The proof for openness follows from Definition 3. □

Theorem 2. *Every subset of constant limit space is both strongly closed and strongly open.*

Proof. Let $(B, K) \in \boldsymbol{ConLim}$ and $M \subset B$. If $M = \emptyset$, then by Definition 3, M is strongly open (strongly closed). Suppose $M \neq \emptyset$ and let K_1 be the quotient structure on B/M induced by $q : (B, K) \longrightarrow (B/M, K_1)$, K_q be the initial structure on $(B/M)\bigvee_*(B/M)$ induced by

$$S_* : (B/M) \vee_* (B/M) \longrightarrow ((B/M)^2, K_1^2)$$

and

$$\triangledown_* : (B/M) \vee_* (B/M) \longrightarrow (B/M, K_d),$$

where K_1^2 is structure on $(B/M)^2$ and K_d is the discrete structure on B/M.

Suppose $\sigma \in K_q$. Then by Proposition 1, $\pi_1 S_*\sigma, \pi_2 S_*\sigma \in K_1$ and $\triangledown_*\sigma \in K_d$. It follows that $\triangledown_*\sigma = [\emptyset]$ or $[U]$, $U \subset B/M$ is finite with $card(U) = m$. If $\triangledown_*\sigma = [\emptyset]$, then $\sigma = [\emptyset]$. If $\triangledown_*\sigma = [U]$, then $\exists V \in \sigma$ such that $U \supset \triangledown_* V$. Since U is finite, $card(V) \leq 2m$ and consequently, V is finite. Hence, by Definition 2, $(B/M, K_1)$ is T_1 at $*$ and M is strongly closed. The proof for strongly open follows from Definition 3. □

Theorem 3. (1) *Let $f : (A, L) \longrightarrow (B, K)$ be in* $\boldsymbol{ConLim}$. *If $M \subset B$ is (strongly) closed, then $f^{-1}(M) \subset A$ is (strongly) closed.*

(2) *Let $(B, K) \in$* $\boldsymbol{ConLim}$. *If $M \subset N$ and $N \subset B$ are (strongly) closed, then $M \subset B$ is (strongly) closed.*

(3) *Let $(B_i, K_i) \in$* $\boldsymbol{ConLim}$ *for $\forall i \in I$ and $M_i \subset B_i$ be (strongly) open (resp., closed) for each $i \in I$. Then $\prod_{i\in I} M_i$ is (strongly) open (resp., closed) in $\prod_{i\in I} B_i$.*

Proof. We get the proof from Theorems 1 and 2. □

Let X be a set and the wedge $X^2 \bigvee_\Delta X^2$ be two distinct copies of X^2 identified along the diagonal Δ [2]. Define $A : X^2 \bigvee_\Delta X^2 \longrightarrow X^3$ by

$$A((s,t)_i) = \begin{cases} (s,t,s) & \text{if } i = 1 \\ (s,s,t) & \text{if } i = 2 \end{cases},$$

$S : X^2 \bigvee_\Delta X^2 \longrightarrow X^3$ by

$$S((s,t)_i) = \begin{cases} (s,t,t) & \text{if } i = 1 \\ (s,s,t) & \text{if } i = 2 \end{cases},$$

and $\nabla : X^2 \bigvee_{\Delta} X^2 \longrightarrow X^2$ by

$$\triangledown((s,t)_i) = (s,t)$$

for $i = 1, 2$.

Definition 4. *([2,5])* **(1)** *If the initial lift of the $\mathcal{U}$-source*

$$A : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3 \quad \text{and} \quad \nabla : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(\mathcal{D}(B^2)) = B^2$$

(resp.,

$$id : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(B^2 \vee_{\triangle} B^2)' = B^2 \vee_{\triangle} B^2 \quad \text{and} \quad \nabla : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(\mathcal{D}(B^2)) = B^2)$$

is discrete, then X is called $\overline{T}_0$ (resp., $T_0^{'}$), where $(B^2 \bigvee_{\triangle} B^2)'$ is the final lift of the U-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \longrightarrow B^2 \vee_{\triangle} B^2\}$ and i_1, i_2 are the canonical injections.

(2) *If X does not contain an indiscrete subspace with (at least) two points, then X is called a T_0 object.*

(3) *If the initial lift of the $\mathcal{U}$-source*

$$S : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3 \quad \text{and} \quad \nabla : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(\mathcal{D}(B^2)) = B^2$$

is discrete, then X is called T_1.

(4) *If the initial lift of the $\mathcal{U}$-sources $A : B^2 \bigvee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3$ and $S : B^2 \bigvee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3$ agree, then X is called pre$\overline{T}_2$.*

(5) *If the initial lift of the $\mathcal{U}$-source $S : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \longrightarrow B^2 \vee_{\triangle} B^2\}$ agree, then X is called pre$T_2^{'}$.*

(6) *X is KT_2 iff X is pre$\overline{T}_2$ and $T_0^{'}$.*

(7) *X is LT_2 iff X is pre$T_2^{'}$ and $\overline{T}_0$.*

(8) *X is NT_2 iff X is pre$\overline{T}_2$ and T_0.*

Remark 2. *In **Top**, by Theorem 2.2.11 of [2] and Remark 1.3 of [6], all of $\overline{T_0}$, T_0' and T_0 (resp., KT_2, NT_2, and LT_2) are equal to T_0 (resp., T_2). In the realm of preT_2 topological spaces, by the Theorem 2.4 of [14], all T_0, T_1, and T_2 spaces are equivalent.*

Theorem 4. *Let $(B, K) \in$ **ConLim**. Then (B, K) is LT_2 iff (B, K) is KT_2.*

Proof. Let (B, K) be KT_2. By Theorem 2.3 of [5], (B, K) is T_0'. Let K_A (resp., K_F) be the initial lift of A (resp., final lift of $\{i_1, i_2 : B^2 \longrightarrow B^2 \vee_\triangle B^2\}$ and $\sigma \in F(B^2 \vee_\triangle B^2))$ with $\sigma \in K_F$. By Proposition 1, $\exists \alpha, \beta \in K^2$ with $\sigma \supset i_1\alpha \cap i_2\beta$, where K^2 is structure on B^2. Hence,

$$\pi_1 A\sigma \supset \pi_1 A(i_1\alpha \cap i_2\beta) = \pi_1\alpha \cap \pi_1\beta,$$

$$\pi_2 A\sigma \supset \pi_2 A(i_1\alpha \cap i_2\beta) = \pi_2\alpha \cap \pi_1\beta,$$

$$\pi_3 A\sigma \supset \pi_3 A(i_1\alpha \cap i_2\beta) = \pi_1\alpha \cap \pi_2\beta.$$

Since K is a constant limit structure on B and $\pi_1\alpha, \pi_2\alpha, \pi_1\beta, \pi_2\beta \in K$, we have $\pi_1\alpha \cap \pi_1\beta$, $\pi_2\alpha \cap \pi_1\beta, \pi_1\alpha \cap \pi_2\beta \in K$, and consequently, $\pi_1 A\sigma, \pi_2 A\sigma, \pi_3 A\sigma \in K$. By Proposition 1, $\sigma \in K_A$. Hence, $K_F \subset K_A$.

Suppose $\sigma \in F(B^2 \vee_\triangle B^2)$ with $\sigma \in K_A$. If $\sigma = [\emptyset]$, then $\sigma \in K_F$. Suppose $\sigma \neq [\emptyset]$. Let $\alpha_{11} = \pi_1 A\sigma, \alpha_{21} = \pi_2 A\sigma$, and $\alpha_{12} = \pi_3 A\sigma$. In case of (1) of Theorem 3.8 of [3], we have $\pi_1 A\sigma = \pi_2 A\sigma$.Let $\sigma_1 = \pi_1^{-1}(\pi_1 A\sigma) \cup \pi_2^{-1}(\pi_3 A\sigma)$. Since $\pi_1 A\sigma_1 = \pi_1 A\sigma = \pi_2 A\sigma \in K$ and $\pi_2 A\sigma_1 = \pi_3 A\sigma \in K$, we get $\sigma_1 \in K^2$.

We now show $i_1\sigma_1 = (\pi_1 A)^{-1}(\pi_1 A\sigma) \cup (\pi_2 A)^{-1}(\pi_2 A\sigma) \cup (\pi_3 A)^{-1}(\pi_3 A\sigma) = \sigma_0$.

If $U \in i_1\sigma$, then $U \supset (U_1 \times U_2)_1$ for some $U_1 \in \pi_1 A\sigma = \pi_2 A\sigma$ and $U_2 \in \pi_3 A\sigma$. Since case 1 of Theorem 3.8 of [3] holds and $\pi_1 A\sigma \cup \pi_3 A\sigma$ is improper, we may assume $U_1 \cap U_2 = \emptyset$.
Note that

$$(\pi_1 A)^{-1}(U_1) \cap (\pi_2 A)^{-1}(U_1) \cap (\pi_3 A)^{-1}(U_2) = (U_1 \times U_2)_1 \in \sigma_0$$

and consequently, $U \in \sigma_0$. Hence, $i_1\sigma_1 \subset \sigma_0$.

If $U \in \sigma_0$, then $U \supset (U_1 \times U_2)_1 \bigvee ((U_1 \cap U_2) \times U_2)_2$ for some $U_1 \in \pi_1 A\sigma = \pi_2 A\sigma$ and $U_2 \in \pi_3 A\sigma$.

Since case (1) of Theorem 3.8 of [3] holds and $\pi_1 A\sigma \cup \pi_3 A\sigma$ is improper, we may assume $U_1 \cap U_2 = \emptyset$. Hence, $U \supset (U_1 \times U_2)_1$ and consequently, $U \in i_1\sigma_1$. Thus, $i_1\sigma_1 = \sigma_0$. By Corollary 3.3 of [3], $i_1\sigma_1 = \sigma_0 \subset \sigma$.

In case (2) of Theorem 3.8 of [3] holds, we have $\pi_1 A\sigma = \pi_3 A\sigma$. Let $\sigma_1 = \pi_1^{-1}(\pi_1 A\sigma) \cup \pi_2^{-1}(\pi_2 A\sigma)$.

Note that

$$\pi_1\sigma_1 = \pi_1 A\sigma \in K,$$

$$\pi_2\sigma_1 = \pi_2 A\sigma \in K.$$

Consequently, $\sigma_1 \in K^2$.

Let $\sigma_0 = (\pi_1 A)^{-1}(\pi_1 A\sigma) \cup (\pi_2 A)^{-1}(\pi_2 A\sigma) \cup (\pi_3 A)^{-1}(\pi_3 A\sigma)$. Since case (2) of Theorem 3.8 of [3] holds, then $i_2\sigma_1 = \sigma_0$ and by Corollary 3.3 of [3], $i_2\sigma_1 \subset \sigma$.

In case (3) of Theorem 3.8 of [3] holds, we have $\pi_3 A\sigma \cap \pi_2 A\sigma \subset \pi_1 A\sigma$.

Let

$$\sigma_1 = \pi_1^{-1}(\pi_3 A\sigma) \cup \pi_2^{-1}(\pi_2 A\sigma)$$

and

$$\sigma_0 = (\pi_1 A)^{-1}(\pi_3 A\sigma) \cup (\pi_2 A)^{-1}((\pi_2 A\sigma) \cap (\pi_3 A\sigma)) \cup (\pi_3 A)^{-1}(\pi_3 A\sigma).$$

By Corollary 3.3 of [3], $\sigma_0 \subset \sigma$, $\pi_1 A\sigma_0 = \pi_3 A\sigma \in K$, $\pi_2 A\sigma_0 = (\pi_2 A\sigma) \cap (\pi_3 A\sigma) \in K$, and $\pi_3 A\sigma_0 = \pi_3 A\sigma \in K$ since K is a constant limit structure on B. We show that $\sigma_0 = i_1\sigma_1 \cap i_2\sigma_1$.

If $U \in \sigma_0$, then $U \supset (U_1 \times (U_2 \cap U_3))_1 \bigvee ((U_1 \cap U_3) \times U_2)_2$ for some $U_1 \in \pi_3 A\sigma$, $U_3 \in (\pi_2 A\sigma) \cap (\pi_3 A\sigma)$, and $U_2 \in \pi_3 A\sigma$.

Note that

$$((U_1 \cap U_3) \times (U_2 \cap U_3)) \in \sigma_1,$$

$$((U_1 \cap U_3) \times (U_3 \cap U_2))_1 \in i_1\sigma_1,$$

$$((U_1 \cap U_3) \times (U_3 \cap U_2))_2 \in i_2\sigma_1,$$

and

$$((U_1 \cap U_3) \times (U_3 \cap U_2))_1 \bigvee ((U_1 \cap U_3) \times (U_3 \cap U_2))_2 \in i_1\sigma_1 \cap i_2\sigma_1.$$

Hence, $U \in i_1\sigma_1 \cap i_2\sigma_1$ and so $\sigma_0 \subset i_1\sigma_1 \cap i_2\sigma_1$.

If $U \in i_1\sigma_1 \cap i_2\sigma_1$, then $U \supset (U_1 \times U_2)_1 \bigvee (U_1 \times U_2)_2$ for some $U_3 \in \pi_2 A\sigma$ and $U_2 \in \pi_3 A\sigma$. Note that

$$U_3 \cup U_2 \in (\pi_2 A\sigma) \cap (\pi_3 A\sigma)$$

and

$$(\pi_1 A)^{-1}(U_3) \cap (\pi_2 A)^{-1}(U_3 \cup U_2) \cap (\pi_3 A)^{-1}(U_3) = (U_3 \times U_2)_1 \bigvee (U_3 \times U_2)_2 \in \sigma_0$$

and consequently, $U \in \sigma_0$. Hence, $\sigma_0 = i_1\sigma_1 \cap i_2\sigma_1 \subset \sigma$. Therefore $K_A \subset K_F$ and consequently, $K_A = K_F$. Since (B, K) is KT_2, by Definition 4, $K_S = K_A$, where K_S is the initial lift of S. Hence, by Definition 4, $K_S = K_F$ and (B, K) is LT_2.

Suppose (B, K) is LT_2. By Theorem 2.3 of [5], (B, K) is T_0' and by Remark 3.6 of [11], (B, K) is $pre\overline{T}_2$. Hence, by Definition 4, (B, K) is KT_2. □

Let $\boldsymbol{T_0'\mathcal{E}}$ (resp., $\boldsymbol{T_0\mathcal{E}}$, $\boldsymbol{\overline{T}_0\mathcal{E}}$, $\boldsymbol{T_1\mathcal{E}}$, $\boldsymbol{KT_2\mathcal{E}}$, $\boldsymbol{LT_2\mathcal{E}}$, and $\boldsymbol{NT_2\mathcal{E}}$) be the subcategory of $\mathcal{E}$ consisting of T_0' (resp., T_0, $\overline{T}_0$, T_1, KT_2, LT_2, and NT_2) objects of $\mathcal{E}$.

Remark 3. **(1)** *By Theorem 2.3 of [5] and Theorem 4, $\overline{T_0}$, T_0' and T_1 constant limit spaces are equivalent. Furthermore, a constant limit space (B, K) is NT_2 iff B is a point or the empty set. Moreover, $NT_2 \Rightarrow KT_2 \iff LT_2$ but the converse is not true, in general. For example, let be $B = \{a, b\}$, and $K = \{[a], [b], [a] \cap [b], [\emptyset]\}$. (B, K) is LT_2 but it is not NT_2.*

(2) *By Theorem 4 and Theorem 2.3 of [5], $\boldsymbol{T_0ConLim}$, $\boldsymbol{\overline{T}_0ConLim}$, $\boldsymbol{T_0'ConLim}$, $\boldsymbol{T_1ConLim}$, $\boldsymbol{KT_2ConLim}$, $\boldsymbol{LT_2ConLim}$, and* ***ConLim*** *are pairwise isomorphic categories. Since* ***ConLim*** *is a cartesian closed, all of these categories are cartesian closed.*

(3) *By Theorems 1 and 4, we have Tietze Extension Theorem for constant limit spaces. If (B, K) is a KT_2 constant limit space and A is non-empty closed subspace of (B, K), then every morphism $f : (A, L) \to (\mathbb{R}, S)$ has an extension morphism $g : (B, K) \to (\mathbb{R}, S)$, where $\mathbb{R}$ is the set of real numbers and S is any constant limit structure on $\mathbb{R}$.*

(4) *By Theorem 1, we have Urysohn's Lemma for constant limit spaces. Suppose (B, K) is a KT_2 constant limit space and M and N are any nonempty disjoint subsets of B. Then there exists a morphism $f : (B, K) \to ([0, 1], L)$, where L is any constant limit structure on $[0, 1]$ with $f(w) = 0$ if $w \in M$ and $f(w) = 1$ if $w \in N$.*

Note that Tietze Extension Theorem and Urysohn's Lemma for constant filter convergence spaces (resp., extended pseudo-quasi-semi metric spaces) are presented in [21, 23, 24].

Definition 5. *Let $(B, K) \in$* ***ConLim*** *and $Z \subset B$.*

$\boldsymbol{scl}(Z) = \bigcap\{H \subset B : Z \subset H \text{ and } H \text{ is strongly closed}\}$ *is said to be the strong closure of Z.*

$\boldsymbol{cl}(Z) = \bigcap\{H \subset B : Z \subset H \text{ and } H \text{ is closed}\}$ *is said to be the closure of Z.*

$\boldsymbol{Q}(Z) = \bigcap\{H \subset B : Z \subset H, H \text{ is closed and open}\}$ *is called the quasi-component closure of Z.*

$\boldsymbol{SQ}(Z) = \bigcap\{H \subset B : Z \subset H, H \text{ is strongly closed and strongly open}\}$ *is said to be the strong quasi-component closure of Z.*

Theorem 5. *$\boldsymbol{cl} = \boldsymbol{\imath} = \boldsymbol{Q}$, the indiscrete closure operator and $\boldsymbol{scl} = \boldsymbol{\delta} = \boldsymbol{SQ}$, the discrete closure operator of* ***ConLim****.*

Proof. Combine Definition 5, Theorems 1, and 5. □

Definition 6. *([19])Let $\boldsymbol{c}$ be a closure operator of $\boldsymbol{\mathcal{E}}$.*

(1) $\boldsymbol{\mathcal{E}}_{0c} = \{W \in \boldsymbol{\mathcal{E}} : s \in \boldsymbol{c}(\{t\})$ *and* $t \in \boldsymbol{c}(\{s\})$ *implies* $s = t$ *with* $s, t \in W\}$,

(2) $\boldsymbol{\mathcal{E}}_{1c} = \{W \in \boldsymbol{\mathcal{E}} : \boldsymbol{c}(\{s\}) = \{s\}, \forall s \in W\}$,

(3) $\boldsymbol{\mathcal{E}}_{2c} = \{W \in \boldsymbol{\mathcal{E}} : \boldsymbol{c}(\triangle) = \triangle$, *the diagonal*$\}$.

Theorem 6. *A constant limit space $(B, K) \in \boldsymbol{ConLim_{icl}}$ for $i = 0, 1, 2$ iff $B = \emptyset$ or $B = \{a\}$, a one point set.*

Proof. We get the proof from Theorem 1. □

Theorem 7. *$\boldsymbol{ConLim_{iscl}}$, $i = 0, 1, 2$ are isomorphic to* ***ConLim****.*

Proof. We get the proof from Theorem 5. □

4. Sober Constant Limit Spaces

In this section, we characterize irreducible, sober, and quasi-sober constant limit spaces.

Definition 7. *([12,16]) Let $\boldsymbol{\mathcal{E}}$ be a topological category and $X \in Ob(\boldsymbol{\mathcal{E}})$.*

(1) *X is called irreducible if Z_1, Z_2 are closed subobjects of X and $X = Z_1 \cup Z_2$, then $X = Z_1$ or $X = Z_2$.*

(2) *X is called quasi-sober if every nonempty irreducible closed subset of X is the closure of a point .*

(3) *X is called $\overline{T_0}$ sober if X is $\overline{T_0}$ and a quasi-sober.*

(4) *X is called T_0' sober if X is T_0' and a quasi-sober.*

(5) *X is called T_0 sober if X is T_0 and a quasi-sober.*

Remark 4. *In* ***Top****, by Remark 3.4 of [12], all of T_0' sober, T_0 sober, and $\overline{T_0}$ sober are equivalent and they reduce to the usual sober. Also, the notion of irreducibility reduces to notion of the usual irreducibility [16].*

Theorem 8. *Let $(B, K) \in$* ***ConLim****.*

(A) *The following are equivalent:*

(1) *A constant limit space* (B, K) *is quasi-sober.*

(2) (B, K) *is* $\overline{T_0}$ *sober.*

(3) (B, K) *is* T_0' *sober.*

(4) (B, K) *is irreducible.*

(B) *The following are equivalent:*

(1) (B, K) *is* T_0*.*

(2) (B, K) *is* T_0 *sober.*

(3) $card(B) \leq 1$*.*

Proof. **(A)** By Theorem 2.4 of [5] and Definition 7, we get (1) $\iff$ (2) $\iff$ (3).

(1) $\implies$ (4): Suppose (B, K) is quasi-sober and $B = B_1 \cup B_2$, where B_1 and B_2 are closed subsets of B. By Theorem 1, $B_1 = B$ or $\emptyset$ and $B_2 = B$ or $\emptyset$. Hence, by Definition 7, (B, K) is irreducible.

(4) $\implies$ (1): Suppose (B, K) is irreducible and $\emptyset \neq B_1 \subset B$ is irreducible closed. Since B_1 is closed, by Theorem 1, $B_1 = B$ and by Theorem 5, $B = B_1 = \boldsymbol{cl}(\{b\})$ for some $b \in B$. Hence, by Definition 7, (B, K) is quasi-sober. Thus, (1) $\iff$ (4).

(B) (1) $\implies$ (2): Suppose (B, K) is T_0 and $\emptyset \neq B_1 \subset B$ is irreducible closed. Since B_1 is closed, by Theorem 1, $B_1 = B$ and hence, by Theorem 5, $B_1 = B = \boldsymbol{cl}(\{b\})$ for some $b \in B$. Hence, consequently, (B, K) is quasi-sober and by Definition 7, (B, K) is T_0 sober.

(2) $\implies$ (3): Suppose (B, K) is T_0 sober and $B \neq \emptyset$ and $B \neq \{a\}$. Then, $\exists s, t \in B$ with $s \neq t$ and $(\{s, t\}, F(\{s, t\}))$ is the indiscrete subspace of (B, K), contradicting to (B, K) is being T_0 sober. Hence, $card(B) \leq 1$.

(3) $\implies$ (1): If $card(B) \leq 1$, then by Definition 4, (B, K) is T_0. □

5. Compact Constant Limit Spaces

Definition 8. *([7]) Let* $\boldsymbol{\mathcal{E}}$ *be a topological category,* $A, B \in Ob(\boldsymbol{\mathcal{E}})$*, and* $f : A \longrightarrow B$ *be a morphism in* $\boldsymbol{\mathcal{E}}$*.*

(1) *If the image of every (strongly) closed subobject of A is a (strongly) closed subobject of B, then f is said to be (strongly) closed.*

(2) *If the projection $\pi_2 : A \times B \longrightarrow B$ is (strongly) closed for every object B in $\mathcal{E}$, then A is called (strongly) compact.*

Remark 5. *In **Top**, by Remark 2.2 of [7], the notion of compactness reduces to usual one, the notion of strong compactness implies compactness and they coincide when a topological space is T_1.*

Theorem 9. *A constant limit space is compact iff it is strongly compact.*

Proof. Suppose (B, K) is a compact constant limit space. We need to show that for each constant limit space (C, L), the projection $\pi_2 : (B, K) \times (C, L) \longrightarrow (C, L)$ is strongly closed. Suppose $M \subset B \times C$ is strongly closed. If $M = \emptyset$, then $\pi_2 M = \emptyset$ is strongly closed. If $M \neq \emptyset$, then by Theorem 2, $\pi_2(M)$ is strongly closed subset of C and hence, by Definition 8, $\pi_2 : (B, K) \times (C, L) \longrightarrow (C, L)$ is strongly closed and consequently, (B, K) is strongly compact.

Suppose (B, K) is a strongly compact constant limit space. We show $\pi_2 : (B, K) \times (C, L) \longrightarrow (C, L)$ is closed for each constant limit space (C, L). Suppose $M \subset B \times C$ is closed. By Theorem 1, $M = \emptyset$ or $M = B \times C$. If $M = \emptyset$, then $\pi_2 M = \emptyset$ is closed in C. If $M = B \times C$, then $C = \pi_2 M$ is closed. By Definition 8, $\pi_2 : (B, K) \times (C, L) \longrightarrow (C, L)$ is closed and hence, (B, K) is compact. □

Theorem 10. *Let $f : (B, K) \longrightarrow (C, L)$ be morphism in **ConLim**.*

(1) *If (B, K) is (strongly) compact, then the subspace $f(B)$ is (strongly) compact.*

(2) *If (B, K) is connected (resp., strongly connected, D-connected, scl-connected, cl-connected), then the subspace $f(B)$ is connected (resp., strongly connected, D-connected, scl-connected, cl-connected).*

(3) *If (B, K) is $\overline{T}_0$ (resp., T_0', T_1, KT_2 or LT_2), then the subspace $f(B)$ is $\overline{T}_0$ (resp., T_0', T_1, KT_2 or LT_2).*

Proof. It follows from Theorems 1, 2, 4, and 9.

□

6. Comparative Evaluation

We compare our findings in some topological categories and we infer:

(1) In ***Top***,

(i) By Theorem 2.2.11 of [2], Remark 1.3 of [6], and Remark 2.6 of [9],

$$\boldsymbol{Top_{2cl}} = \boldsymbol{Top_{2scl}} = \boldsymbol{LT_2Top} = \boldsymbol{NT_2Top} = \boldsymbol{KT_2Top} \subset \boldsymbol{Top_{1cl}}$$

$$= \boldsymbol{Top_{1scl}} \subset \boldsymbol{Top_{0cl}} = \boldsymbol{Top_{0scl}} = \boldsymbol{\overline{T}_0Top} = \boldsymbol{T_0'Top} = \boldsymbol{T_0Top}.$$

and

$$\boldsymbol{Top_{1Q}} = \boldsymbol{Top_{2Q}}$$

(ii) By Remark 3.4 of [12],

$$\boldsymbol{T_0'SobTop} = \boldsymbol{\overline{T}_0SobTop} = \boldsymbol{T_0SobTop}$$

(iii) By Remark 4.4 of [14], there is no implication between $preT_2$ and each of T_0, T_1 and soberity. By Theorem 4.3 of [14], in the realm of $PreT_2$ topological spaces, all T_0 T_1, T_2, and sober spaces are equivalent.

(2) In $\boldsymbol{ConLim}$,

(i) By Theorems 4 and 6,

$$\begin{aligned}\boldsymbol{ConLim_{2cl}} =& \boldsymbol{ConLim_{1cl}} = \boldsymbol{ConLim_{2Q}}\\ =& \boldsymbol{T_0ConLim} \subset \boldsymbol{ConLim_{2scl}}\\ =& \boldsymbol{ConLim_{1scl}} = \boldsymbol{ConLim_{0scl}}\\ =& \boldsymbol{\overline{T}_0ConLim} = \boldsymbol{T_0'ConLim}\\ =& \boldsymbol{T_1ConLim} = \boldsymbol{KT_2ConLim} = \boldsymbol{LT_2ConLim}\end{aligned}$$

(ii) By Theorem 8,

$$\boldsymbol{T_0ConLim} = \boldsymbol{T_0SobConLim}$$

and

$$\boldsymbol{\overline{T}_0SobConLim} = \boldsymbol{T_0'SobConLim} = \boldsymbol{QSobConLim},$$

where $\boldsymbol{QSobConLim}$ is the full subcategory of $\boldsymbol{ConLim}$ consisting of all quasi-sober constant limit spaces.

(iii) By Theorems 8, the categories $\boldsymbol{\overline{T}_0SobConLim}$, $\boldsymbol{T_0'SobConLim}$, and $\boldsymbol{QSobConLim}$ have all limits and colimits.

(iv) By Theorem 8, a T_0 sober constant limit space is T_0' sober, $\overline{T_0}$ sober, a quasi-sober, and irreducible. The constant limit space $(\mathbb{R}, F(\mathbb{R}))$ is quasi-sober, $\overline{T}_0$ sober, and T_0' sober, and irreducible but it is not T_0 sober, where $\mathbb{R}$ is the set of real numbers.

(v) By Theorem 9, a constant limit space (B, K) is compact iff it is strongly compact.

(3) In $\boldsymbol{Lim}$,

(i) By Theorem 2.10 of [9] and Theorem 2.4 of [6],

$$\boldsymbol{Lim_{2scl}} \subset \boldsymbol{LT_2Lim} = \boldsymbol{NT_2Lim} \subset \boldsymbol{KT_2Lim}$$

and

$$\boldsymbol{LT_2Lim} \subset \boldsymbol{Lim_{1cl}} = \boldsymbol{Lim_{1scl}} = \boldsymbol{T_1Lim}$$

$$\subset \boldsymbol{Lim_{0cl}} = \boldsymbol{Lim_{0scl}} = \boldsymbol{\overline{T}_0 Lim} = \boldsymbol{T_0 Lim} = \boldsymbol{T_0' Lim}$$

(4) In $\boldsymbol{ConFCO}$ (the category of constant filter convergence spaces and continuous maps), by Theorems 4.3-4.5 of [20], Theorems 2.1, 2.2, 2.9, and 2.10 of [5],

$$\boldsymbol{LT_2ConFCO} \subset \boldsymbol{NT_2ConFCO} \subset \boldsymbol{KT_2ConFCO} \subset \boldsymbol{ConFCO_{2cl}} = \boldsymbol{ConFCO_{2scl}}$$

$$\subset \boldsymbol{ConFCO_{1cl}} = \boldsymbol{ConFCO_{1scl}} = \boldsymbol{T_0ConFCO} = \boldsymbol{T_1ConFCO}$$

$$= \boldsymbol{\overline{T}_0ConFCO} \subset \boldsymbol{ConFCO_{0cl}} = \boldsymbol{ConFCO_{0scl}} \subset \boldsymbol{T_0'ConFCO}$$

(5) In $\boldsymbol{FCO}$ (the category of filter convergence spaces and continuous maps),
(i) By Theorems 2.9 and 2.11 of [9] and Theorem 4.10 of [11],

$$\boldsymbol{LT_2FCO} \subset \boldsymbol{NT_2FCO} \subset \boldsymbol{KT_2FCO} \subset \boldsymbol{FCO_{2scl}} \subset \boldsymbol{FCO_{2cl}}$$

$$= \boldsymbol{FCO_{1cl}} = \boldsymbol{FCO_{1scl}} = \boldsymbol{T_1FCO} \subset \boldsymbol{FCO_{0cl}}$$

$$= \boldsymbol{FCO_{0scl}} = \boldsymbol{\overline{T}_0FCO} \subset \boldsymbol{T_0FCO} \subset \boldsymbol{T_0'FCO}$$

(ii) By Theorem 6.3 of [10], (B, K) is strongly compact iff every ultrafilter in B converges and every filter convergence space is compact.

(6) In $\boldsymbol{CApp}$ (the category of approach spaces and contraction maps), by Theorems 4.8, 4.9, 4.12, and 4.13 of [26] and Theorems 7, 9, and 10 of [25],

$$\boldsymbol{CApp_{2scl}} \subset \boldsymbol{CApp_{1scl}} \subset \boldsymbol{CApp_{0scl}}$$

and

$$\boldsymbol{CApp_{2cl}} \subset \boldsymbol{CApp_{1cl}} \subset \boldsymbol{CApp_{0cl}} = \boldsymbol{\overline{T}_0CApp} \subset \boldsymbol{T_0CApp} \subset \boldsymbol{T_0'CApp}$$

(7) In $\boldsymbol{psqMet}$ (the category of extended pseudo-quasi-semi metric spaces and non-expansive maps),

(i) By Theorem 6 of [15], Theorems 3.3-3.5 and 3.15 of [23], Theorem 3.10 of [16],

$$\boldsymbol{LT_2pqsMet} = \boldsymbol{KT_2pqsMet} = \boldsymbol{T_1pqsMet} = \boldsymbol{pqsMet_{1SQ}} = \boldsymbol{pqsMet_{1scl}}$$

$$= \boldsymbol{pqsMet_{2scl}} \subset \boldsymbol{pqsMet_{1cl}} = \boldsymbol{pqsMet_{2cl}} = \boldsymbol{pqsMet_{1Q}} = \boldsymbol{\overline{T}_0pqsMet}$$

$$\subset \boldsymbol{T_0pqsMet} \subset \boldsymbol{pqsMet_{0scl}} \subset \boldsymbol{pqsMet_{0cl}} \subset \boldsymbol{T_0'pqsMet}$$

(ii) By Theorem 3.13 of [12], $\{x\}$ is closed for all $x \in X$ and the nonempty proper irreducible closed subsets of X are exactly the one-point subsets iff an extended pseudo-quasi-semi metric space (X,d) is $\overline{T_0}$ sober,

(iii) By Theorem 3.13 of [12], (X,d) is a quasi-sober and an extended quasi-semi metric space iff (X,d) is T_0 sober.

(8) In $\boldsymbol{RRel}$ (the category of reflexive relation spaces and relation preserving functions),

(i) By Theorem 3.7 of [12] and Theorem 3.7 of [13],

$$\boldsymbol{KT_2RRel} \subset \boldsymbol{RRel_{1cl}} = \boldsymbol{T_0RRel} = \boldsymbol{\overline{T}_0RRel}$$

$$\boldsymbol{RRel_{2cl} = RRel_{2scl} = RRel_{1SQ} = RRel_{2SQ} = RRel_{2Q} = LT_2RRel = T_1RRel}$$

(ii) By Theorems 3.8 and 3.9 of [12],

$$\boldsymbol{T_0'SobRRel} = \boldsymbol{QSobRRel},$$

where $\boldsymbol{QSobRRel}$ is the subcategory of $\boldsymbol{RRel}$ consisting of quasi-sober reflexive spaces.

(iii) By Theorems 3.8 and 3.9 of [12],

$$\boldsymbol{T_0SobRRel} = \boldsymbol{\overline{T}_0SobRRel}$$

(iv) By Theorems 3.8 and 3.9 of [12], a reflexive space (B,R) is $\overline{T_0}$ sober iff the nonempty proper irreducible closed subsets of B are exactly the one-point subsets and $\{x\}$ is closed for all $x \in B$ iff (B,R) is T_0 sober.

(v) By Theorems 3.2 and 5.2 of [13], $(B,R) \in \boldsymbol{RRel_{1SQ}}$ iff it is NT_2.

(vi) By Theorem 5.2, Part (1), and Theorem of 3.8 of [12], if $(B,R) \in \boldsymbol{RRel_{1SQ}}$, then it is quasi-sober and T_0 sober.

(vii) By Theorem 5.3 of [13], $\boldsymbol{RRel_{1SQ}} \subset \boldsymbol{RRel_{1Q}}$ and also by Theorem 5.2 of [13], if $(B,R) \in KT_2$, then $(B,R) \in \boldsymbol{RRel_{1SQ}}$ iff $(B,R) \in \boldsymbol{RRel_{1Q}}$.

(viii) By Theorem 3.4 of [14], a reflexive space (A,R) is compact iff for every $x \in A$ there exist $a,b \in A$ with xRa and bRx.

(9) In $\boldsymbol{Rel}$ (the category of relation spaces and relation preserving functions),

(i) By Theorem 3.3 of [14],

$$\boldsymbol{Rel_{1cl} = Rel_{2cl} = Rel_{1Q} = Rel_{2Q} = Rel_{1SQ} = Rel_{2SQ}}$$

(ii) By Theorem 4.5 of [14],

$$\boldsymbol{LT_2Rel \subset NT_2Rel \subset KT_2Rel = \overline{T}_2Rel = pre\overline{T}_2Rel}$$

$$\boldsymbol{\subset Rel_{1Q} = T_1Rel = T_0'Rel = \overline{T}_0Rel = Rel}$$

(iii) By Theorem 3.3 of [14],

$$\boldsymbol{\overline{T}_0SobRel = T_0'SobRel = QSobRel},$$

where $\boldsymbol{QSobRel}$ is the full subcategory of $\boldsymbol{Rel}$ consisting of all quasi-sober relation spaces.

(iv) By Theorem 3.3 of [14], every relation space is compact.

(10) In any topological category,

(i) By Theorem 2.7 of [6], $\overline{T_0}$ implies T_0' but the converse is not true, in general and by Theorem 3.1 of [8], $preT_2'$ implies $pre\overline{T}_2$. Furthermore, there is no relationship between $\overline{T_0}$ and T_0. Also, by Theorem 3.1 of [8], LT_2 implies KT_2 but the converse is not true, in general. Moreover, by Remark 2.8 (7) of [6], notions of KT_2 and NT_2 are independent of each other.

By Theorem 3.5 of [11], in the realm of $pre\overline{T}_2$ objects, $\overline{T_0}$, T_1, and $\overline{T}_2$ objects are equivalent.

(ii) By Theorems 3.5, 3.13 and Parts (2) and (3) of [12], every $\overline{T_0}$ sober object is T_0' sober. Also, there is no implication between T_0 sober and $\overline{T_0}$ sober.

(iii) By Remark 6.2 of [10] the notions of compactness and strongly compactness are different from each other, in general.

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ROBUST REGRESSION TYPE ESTIMATORS FOR BODY MASS INDEX UNDER EXTREME RANKED SET AND QUARTILE RANKED SET SAMPLING

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Abstract. Robust regression-type estimators of population mean that use auxiliary variable information are proposed by considering robust methods under extreme ranked set sampling (ERSS) and quartile ranked set sampling (QRSS). We have used the data concerning body mass index (BMI) for 800 people in Turkey in 2014. The real data example is applied to see efficiency of the estimators in ERSS and QRSS designs and it is found that the proposed estimators are better in these designs than the classical ranked set sampling (RSS) design. In addition, mean square error (MSE) and percent relative efficiency (PRE) are used to compare the performance of the adapted and proposed estimators.

1. Introduction

In sampling survey, the supplementary information is mostly used to enhance accuracy of the estimators due to the correlation between auxiliary and study variables. Auxiliary information has a major role according to the sampling theory. Because of improving the precision of estimates, making use of convenient auxiliary information such as mean, total population, skewness, attribute and correlation is pretty significant. Auxiliary information has been used in ratio, product and exponential type estimators to acquire effective estimators under distinct sampling designs.

RSS is an alternative sampling design to simple random sampling (SRS) for drawing a sample of observations from a population. It is intended for situations where the certain measurement of sample units is hard but they can be readily

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ranked without real measurement. The ranking is done either through nominative judgment or via the use of an accompanying variable, and need not to be precise. This situation is named defective ranking. If the ranking process is correct, it will be referred to as excellent ranking structure. RSS design was first proposed by McIntyre [6]. Many authors such as Samawi and Muttlak [9], Bouza [2], Mehta and Mandowara [7] used judge mental RSS where ranking is done with respect to auxiliary variable. Later, the authors suggested new sampling designs based on ranking and used auxiliary information to get efficiency. Muttlak [8] proposed QRSS. Taking into account ranking error, Samawi et al. [10] suggested ERSS for estimating a population mean. Long et al. [5] suggested ratio estimators of population mean that used either the first or third quartiles of the auxiliary variable under RSS and ERSS. Koyuncu [3] studied regression type estimators (RTE) under different ranked set sampling. Shahzad et al. [11] suggested RTE for mean estimation under RSS besides the sensitivity issue.

Lately, robust tools are used in estimators under different sampling designs. Zaman and Bulut [14] are proposed new ratio type estimators using LTS, Huber MM, LMS, Tukey-M, LAD and Hampel M robust methods in SRS. Ali et al. [1] generalized estimators of Zaman and Bulut [14]. Subzar et al. [13] adapted the diverse robust regression methods to the ratio estimators. Shahzad et al. [12] identified the class of RTE utilizing robust regression tools. Recently, Koyuncu and Al-Omari [4] proposed generalized robust RTE under RSS and MRSS.

The target of this study is to suggest regression type estimators of the population mean using robust statistics under RSS, ERSS and QRSS. The article is composed as follows: In Section 2, RSS, ERSS and QRSS designs were explained. In Section 3, the recent robust literature were reviewed and adapted robust regression type estimators were given. The proposed exponential robust-RTE estimators in RSS, ERSS and QRSS were introduced in Section 4. In Section 5, a numerical study was conducted using a real data set on BMI. All results that were explained briefly and summarized also in Section 6.

2. RSS, ERSS and QRSS Designs

In this section, RSS, ERSS and QRSS designs are explained.

2.1. **RSS Design.**

The RSS procedure can be created by choosing r random samples of size r units from the population and order the units within each sample according to the variable of interest. Let $(X_1, Y_1), (X_2, Y_2), ..., (X_r, Y_r)$ be a SRS of r, then the measured RSS units are indicated by $\left(Y_{(i)j}, X_{[i]j}\right)$, $i = 1, 2, ..., r$, $j = 1, 2, ..., m$ where $\left(X_{[i]j}, Y_{(i)j}\right)$ is the i^{th} ranked unit from the j^{th} cycle of two auxiliary variables and study variable, respectively. [ ] and ( ) demonstrate the i^{th} perfect ordering in the i^{th} set for auxiliary variable X and the i^{th} judgment ordering in the i^{th} set for study variable Y. One of the most correlated auxiliary variables with study variable was choosed to rank the units. Further let $\bar{x}_{RSS} = \frac{1}{mr} \sum_{j=1}^{m} \sum_{i=1}^{r} X_{[i]j}$,

$\bar{y}_{RSS} = \frac{1}{mr} \sum_{j=1}^{m} \sum_{i=1}^{r} Y_{(i)j}$ are the sample means under RSS and $\bar{Y}, \bar{X}$ are population means, respectively for the study and auxiliary variables.

2.2. **ERSS Design.** ERSS investigated by Samawi et al. [10]. To predict the finite population mean $(\bar{Y})$ using ERSS, the operation can be explained briefly as follows:

(1) The process includes drawing sets of each r units randomly from population for which the mean is to be predicted. The most important assumption is the smallest and the biggest units of the set can be fixed visually or with a little cost.
(2) The lowest ranked unit is determined from the first r unit set. Then, the largest ranked unit is determined from the second r unit set. And the lowest ranked unit is determined from the third set of r units and so on. Thus, the first $(r-1)$ determined units is obtained using the first $(r-1)$ sets. The event of choosing the $r-th$ unit from the $r-th$ (i.e very last) set depends on whether r is odd or even.
(3) When r is even, the measurement value of the largest unit ranked is measured.
(4) Two options exist when r is odd:
    (a) The average of the largest and lowest units in the $r-th$ set is measured for the measure of the $r-th$ unit.

    (b) The measure of the median for the measure of the $r-th$ unit is measured.
(5) This procedure complete one cycle of ERSS. The period may be repeated m times until n elements of desired to obtain.

$$\bar{x}_{ERSS_e} = \frac{1}{2} \left(\bar{X}_{[1]} + \bar{X}_{[r]} \right) \tag{1}$$

where $\bar{X}_{[1]} = \frac{2}{r} \sum_{i=1}^{r/2} X_{2i-1[1]}$. and $\bar{X}_{[r]} = \frac{2}{r} \sum_{i=1}^{r/2} X_{2i[r]}$.

To observe that $X_{1[1]}, X_{3[1]}, ..., X_{r-2[1]}$ and $X_{r[1]}$are identically distributed is easy and so are $X_{2[r]}, X_{4[r]}, ..., X_{r-1[r]}$ and $X_{r(r)}$.

$$\bar{x}_{ERSS_o} = \frac{X_{1[1]}, X_{2[r]}, X_{3[1]}, ..., X_{r-1[r]} + X_{r\left[\frac{r+1}{2}\right]}}{r} \tag{2}$$

2.3. **QRSS Design.** Muttlak [8] suggested QRSS to predict the population mean. The procedure of QRSS can be explained concisely as follows:

(1) Select randomly r^2 bivariate sample units of target population.
(2) If the sample size r is even, choose for measurement from the first $\frac{r}{2}$ samples the $q_1 (r+1)$ th and from the second $\frac{r}{2}$ samples the $q_3 (r+1)$th smallest ranked unit.

(3) If the sample size r is odd, choose for measurement from the first $\frac{(r-1)}{2}$ samples the $q_1(r+1)$th and from the last $\frac{(r-1)}{2}$samples the $q_3(r+1)$ th smallest ranked unit and from the remaining sample the median ranked unit.
(4) The nearest integer of $q_1(r+1)$th and $q_3(r+1)$th where $q_1 = 0.25$ and $q_3 = 0.75$ were always taken.
(5) This procedure complete one cycle of QRSS. The cycle may be repeated m times until $n = mr$ elements of desired to obtain.

Let $X_{i[q_1(r+1)]}$ and $X_{i[q_3(r+1)]}$ denote the $(q_1(r+1))$th and $(q_3(r+1))$th order statistics of the i^{th} sample respectively $(i = 1, 2, .., r)$.

The estimator of the population mean using QRSS with a cycle is given in equations 3 and 4, respectively, in the case of even and odd sample sizes.

$$\bar{x}_{QRSS_e} = \frac{1}{r}\left(\sum_{i=1}^{\frac{r}{2}} X_{i[q_1(r+1)]} + \sum_{i=\frac{r}{2}+1}^{r} X_{i[q_3(r+1)]}\right) \tag{3}$$

$$\bar{x}_{QRSS_o} = \frac{1}{r}\left(\sum_{i=1}^{\frac{r-1}{2}} X_{i[q_1(r+1)]} + \sum_{i=\frac{r+1}{2}}^{r} X_{i[q_3(r+1)]} + X_{i[(r+1)/2]}\right) \tag{4}$$

$X_{i[(r+1)/2]}$ is the median of sample $i = (r+1)/2$. To simplify the notations, let $X_{[i:q]}$ specify the $(q_1(r+1))$th order statistic of i^{th} sample $\left(i = 1, 2, ..., \frac{r}{2}\right)$ and $(q_3(r+1))$th order statistic of i^{th} sample $\left(i = \frac{r}{2}+1, \frac{r}{2}, ..., r\right)$ if the sample size n is even. Also specify the $(q_1(r+1))$th order statistic of i^{th} sample $\left(i = 1, 2, ..., \frac{r-1}{2}\right)$, the median of the i^{th} sample $(i = (r+1)/2)$ and the $(q_3(r+1))$th order statistic of i^{th} sample $\left(i = \frac{r-1}{2}+2, \frac{r-1}{2}+3, ..., r\right)$ if the sample size n is odd. Then the estimator of population mean using QRSS can be written as $\bar{x}_{QRSS} = \frac{1}{r}\sum_i^r X_{[i:q]}$.

3. Adapted Robust Regression Type Estimators

Koyuncu and Al-Omari [4] proposed generalized robust-RTE under SRS, RSS and median ranked set sampling (MRSS).

$$\bar{y}_{N(j)} = \left[\bar{y}_{(j)} + b_{i(j)}\left(\bar{X} - \bar{x}_{[j]}\right)\right]\left(\frac{F\bar{X}+G}{F\bar{x}_{[j]}+G}\right)^{\alpha} \tag{5}$$

where F may represent the coefficient of variation C_x, kurtosis $\beta_{2(x)}$, first and third quarters $q_{1(x)}, q_{3(x)}$ or any known population information of auxiliary variable. (j) represents the SRS, RSS and MRSS sampling designs. $b_{i(j)}$ is regression coefficient calculated from the i robust regression method under (j) design. i represents Huber M, LMS, Huber MM, S, LAD or LTS.

They showed that Zaman and Bulut [14] estimators are members of their generalized estimator. Putting suitable values as $\alpha = 1$, $F = 1$, $G = q_{1(x)}, q_{3(x)}$ and j=SRS in the $\bar{y}_{N(j)}$, we can get Zaman and Bulut [14] ratio-RTEs under SRS.

In the same manner, we can extend $\bar{y}_{N(j)}$ estimator to ERSS and QRSS designs putting j=ERSS, j=QRSS respectively. Zaman and Bulut [14] estimators and some members of $\bar{y}_{N(j)}$ can be given as

$$\bar{y}_{EN(j)1} = \frac{\left[\bar{y}_{(j)} + b_{i(j)}\left(\bar{X} - \bar{x}_{[j]}\right)\right]}{\bar{x}_{[j]}}\bar{X} \tag{6}$$

$$\bar{y}_{EN(j)2} = \frac{\left[\bar{y}_{(j)} + b_{i(j)}\left(\bar{X} - \bar{x}_{[j]}\right)\right]}{\bar{x}_{[j]} + C_x}\left(\bar{X} + C_x\right) \tag{7}$$

$$\bar{y}_{EN(j)3} = \frac{\left[\bar{y}_{(j)} + b_{i(j)}\left(\bar{X} - \bar{x}_{[j]}\right)\right]}{\bar{x}_{[j]} + \beta_{2(x)}}\left(\bar{X} + \beta_{2(x)}\right) \tag{8}$$

$$\bar{y}_{EN(j)4} = \frac{\left[\bar{y}_{(j)} + b_{i(j)}\left(\bar{X} - \bar{x}_{[j]}\right)\right]}{\bar{x}_{[j]} + q_{1(x)}}\left(\bar{X} + q_{1(x)}\right) \tag{9}$$

$$\bar{y}_{EN(j)5} = \frac{\left[\bar{y}_{(j)} + b_{i(j)}\left(\bar{X} - \bar{x}_{[j]}\right)\right]}{\bar{x}_{[j]} + q_{3(x)}}\left(\bar{X} + q_{3(x)}\right) \tag{10}$$

To obtain the specific MSE of adapted estimators in equation (5) under (j) design, let us define following notations

$$\vartheta_{o(j)} = \left(\bar{y}_{(j)} - \bar{Y}\right)/\bar{Y}, \vartheta_{1(j)} = \left(\bar{x}_{[j]} - \bar{X}\right)/\bar{X} \quad \vartheta_{0(j)}\vartheta_{1(j)} = \left(\bar{x}_{[j]} - \bar{X}\right)\left(\bar{y}_{(j)} - \bar{Y}\right)/\bar{X}\bar{Y} \tag{11}$$

For the (j) design, expectaions of ϑ terms are given by

$$E\left(\vartheta_{0(j)}^2\right) = V\left(\bar{y}_{(j)}\right)/\bar{Y}^2, E\left(\vartheta_{1(j)}^2\right) = V\left(\bar{x}_{[j]}\right)/\bar{X}^2, E\left(\vartheta_{0(j)}\vartheta_{1(j)}\right) = \text{cov}\left(\bar{x}_{[j]}, \bar{y}_{(j)}\right)/\bar{Y}\bar{X}$$

If (j) design represents SRS, expectaions of ϑ terms are given by

$$E\left(\vartheta_{0(SRS)}^2\right) = \frac{S_y^2}{\bar{Y}^2}, \quad E\left(\vartheta_{1(SRS)}^2\right) = \frac{S_x^2}{\bar{X}^2}, \quad E\left(\vartheta_{0(SRS)}\vartheta_{1(SRS)}\right) = \frac{S_{xy}}{\bar{Y}\bar{X}}$$

If (j) design represents RSS, expectaions of ϑ terms are given by

$$E\left(\vartheta_{0(RSS)}^2\right) = \frac{1}{\bar{Y}^2}\left(\frac{S_y^2}{r} - \frac{1}{r^2}\sum_{i=1}^{r}\left(\mu_{y(j)} - \bar{Y}\right)^2\right)$$

$$E\left(\vartheta_{1(RSS)}^2\right) = \frac{1}{\bar{X}^2}\left(\frac{S_x^2}{r} - \frac{1}{r^2}\sum_{i=1}^{r}\left(\mu_{x[i]} - \bar{X}\right)^2\right),$$

$$E\left(\vartheta_{0(RSS)}\vartheta_{1(RSS)}\right) = \frac{1}{\bar{Y}\bar{X}}\left(\frac{S_{xy}}{r} - \frac{1}{r^2}\sum_{i=1}^{r}\left(\mu_{x[i]} - \bar{X}\right)\left(\mu_{y(i)} - \bar{Y}\right)\right)$$

If the sample size is odd and (j) represents the QRSS design, expectaions of ϑ terms are given by

$$E\left(\vartheta^2_{0(QRSS)_o}\right) = \frac{1}{\bar{Y}^2}\left[\frac{1}{r^2}\left(\frac{(r-1)}{2}\left(S^2_{y\left(\frac{r+1}{4}\right)} + S^2_{y\left(\frac{3(r+1)}{4}\right)}\right) + S^2_{y\left(\frac{r+1}{2}\right)}\right)\right],$$

$$E\left(\vartheta^2_{1(QRSS)_o}\right) = \frac{1}{\bar{X}^2}\left[\frac{1}{r^2}\left(\frac{(r-1)}{2}\left(S^2_{x\left[\frac{r+1}{4}\right]} + S^2_{x\left[\frac{3(r+1)}{4}\right]}\right) + S^2_{x\left[\frac{r+1}{2}\right]}\right)\right],$$

$$E\left(\vartheta_{0(QRSS)_o}\vartheta_{1(QRSS)_o}\right) = \frac{1}{\bar{Y}\bar{X}}\left[\frac{1}{r^2}\left(\frac{(r-1)}{2}\left(S_{xy\left(\frac{r+1}{4}\right)} + S_{xy\left(\frac{3(r+1)}{4}\right)}\right) + S_{xy\left(\frac{r+1}{2}\right)}\right)\right]$$

If the sample size is even and (j) represents the QRSS design, expectaions of ϑ terms are given by

$$E\left(\vartheta^2_{0(QRSS)_e}\right) = \frac{1}{\bar{Y}^2}\left[\frac{1}{2r}\left(S^2_{y\left(\frac{r+1}{4}\right)} + S^2_{y\left(\frac{3(r+1)}{4}\right)}\right)\right],$$

$$E\left(\vartheta^2_{1(QRSS)_e}\right) = \frac{1}{\bar{X}^2}\left[\frac{1}{2r}\left(S^2_{x\left[\frac{r+1}{4}\right]} + S^2_{x\left[\frac{3(r+1)}{4}\right]}\right)\right],$$

$$E\left(\vartheta_{0(QRSS)_e}\vartheta_{1(QRSS)_e}\right) = \frac{1}{\bar{Y}\bar{X}}\left[\frac{1}{2r}\left(S_{xy\left(\frac{r+1}{4}\right)} + S_{xy\left(\frac{3(r+1)}{4}\right)}\right)\right]$$

If the sample size is odd and (j) represents the ERSS design, expectaions of ϑ terms are given by

$$E\left(\vartheta^2_{0(ERSS)_o}\right) = \frac{1}{\bar{Y}^2}\left[\frac{1}{r^2}\left(\frac{(r-1)}{2}\left(S^2_{y(1)} + S^2_{y(r)}\right) + S^2_{y\left(\frac{r+1}{2}\right)}\right)\right],$$

$$E\left(\vartheta^2_{1(ERSS)_o}\right) = \frac{1}{\bar{X}^2}\left[\frac{1}{r^2}\left(\frac{(r-1)}{2}\left(S^2_{x[1]} + S^2_{x[r]}\right) + S^2_{x\left[\frac{r+1}{2}\right]}\right)\right],$$

$$E\left(\vartheta_{0(ERSS)_o}\vartheta_{1(ERSS)_o}\right) = \frac{S_{xy}}{\bar{Y}\bar{X}}\left[\frac{1}{r^2}\left(\frac{(r-1)}{2}\left(S_{xy(1)} + S_{xy(r)}\right) + S_{xy\left(\frac{r+1}{2}\right)}\right)\right]$$

If the sample size is even and (j) represents the ERSS design, expectaions of ϑ terms are given by

$$E\left(\vartheta^2_{0(ERSS)_e}\right) = \frac{1}{\bar{Y}^2}\left[\frac{1}{2r}\left(S^2_{y(1)} + S^2_{y(r)}\right)\right]$$

$$E\left(\vartheta^2_{1(ERSS)_e}\right) = \frac{1}{\bar{X}^2}\left[\frac{1}{2r}\left(S^2_{x[1]} + S^2_{x[r]}\right)\right]$$

$$E\left(\vartheta_{0(ERSS)_e}\vartheta_{1(ERSS)_e}\right) = \frac{1}{\bar{Y}\bar{X}}\left(\frac{1}{2r}\left(S_{xy(1)} + S_{xy(r)}\right)\right)$$

Writing $\bar{y}_{EN(j)}$ given in Equation 5 with ϑ terms, extracting $\bar{Y}$ and squaring both sides we get MSE of generalized estimator $\bar{y}_{EN(j)}$ under (j) design as

$$MSE\left(\bar{y}_{EN(j)i}\right) = E\Big(\bar{Y}^2\vartheta^2_{0(j)} + B_i^2\bar{X}^2\vartheta^2_{1(j)} + \alpha^2\psi^2\bar{Y}^2\vartheta^2_{1(j)} - 2B_i\bar{Y}\bar{X}\vartheta_{0(j)}\vartheta_{1(j)}$$

$$-2\alpha\psi\bar{Y}^2\vartheta_{0(j)}\vartheta_{1(j)} + 2\alpha\psi B_i\bar{Y}\bar{X}\vartheta^2_{1(j)}) \quad (12)$$

where $\psi = \dfrac{F\bar{X}}{F\bar{X}+G}$

$$MSE\left(\bar{y}_{EN(j)i}\right) = V\left(\bar{y}_{(j)}\right) + B_i^2V\left(\bar{x}_{[j]}\right) + \alpha^2R^2_{FG}V\left(\bar{x}_{[j]}\right) - 2B_i\operatorname{cov}\left(\bar{x}_{[j]}, \bar{y}_{(j)}\right)$$

$$-2\alpha R_{FG}\operatorname{cov}\left(\bar{x}_{[j]}, \bar{y}_{(j)}\right) + 2\alpha R_{FG}B_iV\left(\bar{x}_{[j]}\right) \quad (13)$$

where $R_{FG} = \dfrac{F\bar{Y}}{F\bar{X}+G}$ and B_i robust betas calculated with Huber M, LMS, Huber MM, S, LAD or LTS of population.

We can get MSEs of estimators given in Equation6-10 using Equation12 easily putting related expectations and suitable F and G values of each design. The R_{FG}s for the estimators in Equation6-10 can be given as $R_{FG1} = \frac{\bar{Y}}{\bar{X}}$, $R_{FG2} = \frac{\bar{Y}}{\bar{X}+C_x}$, $R_{FG3} = \frac{\bar{Y}}{\bar{X}+\beta_{2(x)}}$, $R_{FG4} = \frac{\bar{Y}}{\bar{X}+q_{1(x)}}$, $R_{FG5} = \frac{\bar{Y}}{\bar{X}+q_{3(x)}}$ respectively.

4. Proposed Robust Regression type Estimators in RSS, ERSS and QRSS

We can define the following estimators for the population mean of the study variable in RSS, ERSS and QRSS design as follows

$$\bar{y}_{E(j)} = \left[\bar{y}_{(j)} + b_{i(j)}\left(\bar{X} - \bar{x}_{[j]}\right)\right]\exp\left(\frac{\bar{X} - \bar{x}_{[j]}}{\bar{X} + 2F + \bar{x}_{[j]}}\right) \quad (14)$$

where F represents the coefficient of variation, kurtosis and quarters $C_x, \beta_{(x)}, q_{1(x)}$, $q_{3(x)}$ or any known population information of auxiliary variable. j represents the sampling design such as RSS, ERSS and QRSS and $b_{i(j)}$ is robust regression coefficient as defined in Section3. For particulars about all these robust regression methods, researchers are referred to Koyuncu and Al-Omari [4]. We have generated some members of $\bar{y}_{E(j)}$ as $\bar{y}_{E(j)1}$-$\bar{y}_{E(j)5}$ setting F=1, C_x,$\beta_{(x)}$, $q_{1(x)}$ and $q_{3(x)}$ respectively in Table2-Table4 under (j) design.

The MSE of $\bar{y}_{E(j)}$ is given by

$$MSE\left(\bar{y}_{E(j)}\right) = V\left(\bar{y}_{(j)}\right) + B_i^2 V\left(\bar{x}_{[j]}\right)$$
$$+\frac{1}{4}R_{Fi}^2 V\left(\bar{x}_{[j]}\right) - 2B_i \operatorname{cov}\left(\bar{x}_{[j]}, \bar{y}_{(j)}\right) - R_{Fi}\operatorname{cov}\left(\bar{x}_{[j]}, \bar{y}_{(j)}\right) + R_{Fi}B_i V\left(\bar{x}_{[j]}\right) \quad (15)$$

where $R_{Fi} = \frac{\bar{Y}}{\bar{X}+Fi}$, B_i is robust regression betas using i^{th} robust method, (j) represents RSS, ERSS and QRSS designs. One can easily obtain the spesific MSE from Eq.11-12 putting expectation terms belong to design.

5. Numerical Study

If a dataset contains outlying observations, classical methods can be affected by outliers. To obtain more reliable results in the estimation, different diagnostic methods and robust tools are used to determine the effect of these observations on the predictions. With robust methods, estimates that are insensitive to the effects of outliers and extreme values, can be obtained with little or no sensitivity. Moving in this direction, in this study, we considered robust methods for the estimation of population mean. To see the performance of robust regression type estimators of the population mean under RSS, ERSS and QRSS sampling designs, a numerical study is considered. A real data is used to observe the performances of the estimators concerning BMI as a study variable and the weight as an auxiliary variable for 800 people in Turkey in 2014. In Table 1, the summary of population information about BMI (Y) and weight (X) variables are given.

Table 1. Population information about Body Mass Index (Y) and Weight (X) variables

N = 800	$\bar{Y} = 23.776$
$\bar{X} = 67.558$	$C_x = 0.2047$
$\rho = 0.8674$	$C_y = 0.1763$
$q_{1(x)} = 56$	$q_{3(x)} = 78$
$\beta_{2(x)} = 0.2318$	$R = 0.3519$
$\S_x^2 = 191.295$	$\S_y^2 = 17.5804$

The scatter plot of BMI data is given in Figure1. As seen in Figure1, the data are not normally distributed and it is observed that some observations in the dataset are outliers. For this reason, the use of robust methods is found appropriate for this dataset. For application we have assumed that r=9 set, m=10 cycle, n=m*r=90 sample size and calculated theoretical MSE for each design using Equations 13 and 15.

The MSE and PRE of Koyuncu and Al-Omari [4] and the proposed estimators have been calculated under RSS and the results are given in Table 2. The MSE and PRE of Koyuncu and Al-Omari [4], Zaman and Bulut [14] adapted estimators and the proposed estimators for ERSS and QRSS designs are given in Table 3 and Table 4, respectively.

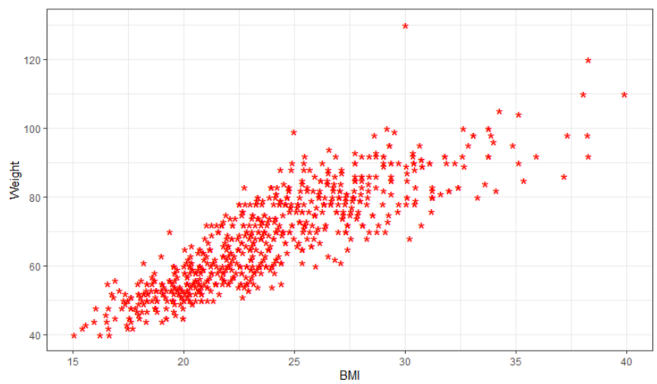

FIGURE 1. Scatter plot of BMI data

The numerical study can be summarized as follows:
The highest PRE values of Koyuncu and Al-Omari [4] and proposed estimators under RSS design are 132.55 and 316.81 respectively (see Table 2). From these values we can say that, for all estimators under RSS design, the best estimator is $\bar{y}_{(RSS)3}$ suggested estimator that used kurtosis of auxiliary variable and LMS robust beta. So, it is concluded that this proposed estimator is approximately three times more effective than other estimators.
The highest PRE values of adapted estimators of Zaman and Bulut [13] and Koyuncu and Al-Omari [4] and proposed estimators under ERSS design are 125.23; 132.47 and 316.70 respectively (see Table 3). From these values we can say that, for all estimators under ERSS design, the best estimator is $\bar{y}_{E(RSS)3}$ suggested estimator that used kurtosis of auxiliary variable and LMS robust beta. So, it is concluded that this proposed estimator is approximately three times more effective than other estimators.
The highest PRE values of adopted estimators of Zaman and Bulut [13] and Koyuncu and Al-Omari [4] and proposed estimators under QRSS design are 125.57; 133.15 and 322.25 respectively (see Table 4). From these values we can say that ,for all estimators under QRSS design, the best estimator is $\bar{y}_{Q(RSS)3}$ suggested estimator that used kurtosis of auxiliary variable and LMS robust beta. So, it was concluded

that this proposed estimator is approximately three times more effective than other estimators. In conclusion, QRSS have the best performance of all proposed estimators in other set sampling designs and LMS have the best performance of all robust methods.

6. Conclusion

We considered robust methods for robust-RTE for mean estimation in RSS, ERSS and QRSS. Firstly, recent proposed robust estimators have been examined. Then, theoretical results for different sampling designs RSS, ERSS and QRSS have been extended. A new exponential-robust- RTE of population mean is proposed and MSEs and PREs of the robust regression type estimators are also obtained for each designs. The existing estimators and proposed estimators have been compared. In conclusion, the suggested estimators perform better than present Zaman and Bulut [14] and Koyuncu and Al-Omari [4] estimators. Also, we demonstrated that the suggested estimator is more effective than adapted estimators of Zaman and Bulut [14] and Koyuncu and Al-Omari [4] in ERSS and QRSS. To see the performance of proposed estimators, we have carried out a numerical study applying on a real data set. When the results of the study are examined, the findings are summarized as follows. The estimators suggested based on the robust methods under RSS designs have better performance over SRS. Also, according to the results obtained from the numerical study, the best method among ranked set sampling methods is QRSS method and it is concluded that the best method among robust methods is LMS. In the light of these results, we desire to develop new estimators in other RSS methods in oncoming studies.

Author Contribution Statements All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Declaration of Competing Interests All authors declare no conflicts of interest in this paper.

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Appendix

TABLE 2. MSE and PRE of Koyuncu and Al-Omari (2020) and proposed estimators under RSS

Estimators	MSE							PRE						
		Robust betas							Robust betas					
Koyuncu and Al-Omari (2020)	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**
$\bar{y}_{N(RSS)1}$	2.454	2.676	2.490	2.502	1.960	2.5986	2.508	100	91.70	98.55	98.09	**125.18***	94.45	97.83
$\bar{y}_{N(RSS)2}$	2.440	2.662	2.476	2.488	1.949	2.584	2.495	100	91.69	98.55	98.08	**125.25***	94.46	97.83
$\bar{y}_{N(RSS)3}$	2.490	2.714	2.527	2.539	1.992	2.636	2.545	100	91.75	98.56	98.10	**125.02***	94.49	97.84
$\bar{y}_{N(RSS)4}$	0.920	1.033	0.938	0.944	0.694	0.9931	0.947	100	89.12	98.09	97.49	**132.55****	92.72	97.16
$\bar{y}_{N(RSS)5}$	0.753	0.845	0.768	0.772	0.576	0.8125	0.775	100	89.06	98.10	97.49	**130.78***	92.71	97.16
Proposed	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**
$\bar{y}_{E(RSS)1}$	0.821	0.922	0.839	0.843	0.623	0.886	0.887	298.71	290.02	297.29	296.30	**314.58***	293.10	282.98
$\bar{y}_{E(RSS)2}$	0.818	0.919	0.834	0.839	0.621	0.883	0.842	298.14	289.53	296.74	296.30	**313.73***	292.55	296.05
$\bar{y}_{E(RSS)3}$	0.829	0.931	0.845	0.851	0.629	0.895	0.854	300.20	291.35	298.77	298.30	**316.81****	294.44	298.06
$\bar{y}_{E(RSS)4}$	0.499	0.545	0.506	0.508	0.434	0.528	0.509	184.53	**189.47***	185.55	185.85	159.92	188.09	186.01
$\bar{y}_{E(RSS)5}$	0.468	0.504	0.473	0.475	0.428	0.490	0.476	160.89	**167.65***	162.21	162.61	134.58	165.58	162.81

*demonstrates the most effective estimators with respect to robust methods

** demonstrates the most effective estimators with respect to all estimators and robust methods

TABLE 3. MSE and PRE of adapted estimators of Zaman and Bulut (2019), Koyuncu and Al-Omari (2020) and proposed estimators under ERSS

Estimators	MSE							PRE						
		Robust betas							Robust betas					
Adapted Estimators of Zaman and Bulut (2019)	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**
$\bar{y}_{EN(ERSS)1}$	2.444	2.666	2.481	2.492	1.952	2.588	2.499	100	91.69	98.54	98.09	**125.23***	94.45	97.83
$\bar{y}_{EN(ERSS)2}$	2.431	2.652	2.467	2.478	1.940	2.574	2.485	100	91.67	98.54	98.08	**125.29***	94.44	97.83
$\bar{y}_{EN(ERSS)3}$	2.481	2.704	2.517	2.529	1.984	2.626	2.535	100	91.74	98.55	98.1	**125.06***	94.48	97.84
Adapted Estimators of Koyuncu and Al-Omari(2020)	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**
$\bar{y}_{EN(ERSS)4}$	0.916	1.028	0.934	0.940	0.692	0.988	0.943	100	89.11	98.09	97.49	**132.47***	92.72	97.15
$\bar{y}_{EN(ERSS)5}$	0.750	0.842	0.764	0.769	0.574	0.809	0.772	100	89.09	98.12	97.5	**130.61***	92.73	97.17
Proposed	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**
$\bar{y}_{E(ERSS)1}$	0.818	0.918	0.834	0.839	0.621	0.882	0.842	287.38	290.37	297.6	297.14	**314.48***	293.4	296.89
$\bar{y}_{E(ERSS)2}$	0.815	0.915	0.831	0.836	0.619	0.879	0.839	286.83	289.86	297.01	296.55	**313.63***	292.87	296.3
$\bar{y}_{an(ERSS)3}$	0.826	0.927	0.842	0.847	0.626	0.891	0.850	288.82	291.69	299.07	298.58	**316.70****	294.78	298.34
$\bar{y}_{E(ERSS)4}$	0.498	0.544	0.505	0.507	0.435	0.527	0.508	180.70	**189.09***	184.98	185.33	158.97	187.6	185.5
$\bar{y}_{E(ERSS)5}$	0.468	0.503	0.473	0.475	0.429	0.490	0.476	158.97	**167.16***	161.58	162.01	133.71	165.07	162.22

*demonstrates the most effective estimators with respect to robust methods

** demonstrates the most effective estimators with respect to all estimators and robust methods

TABLE 4. MSE and PRE of adapted estimators of Zaman and Bulut (2019), Koyuncu and Al-Omari (2020) and proposed estimators under QRSS

	MSE							PRE						
Estimators		Robust betas							Robust betas					
Adapted Estimators of Zaman and Bulut (2019)	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**
$\bar{y}_{EN(QRSS)1}$	2.396	2.616	2.432	2.444	1.910	2.539	2.450	100	91.62	98.53	98.06	**125.50***	94.40	97.81
$\bar{y}_{EN(QRSS)2}$	2.383	2.601	2.419	2.430	1.898	2.525	2.436	100	91.6	98.53	98.07	**125.57***	94.39	97.81
$\bar{y}_{EN(QRSS)3}$	2.432	2.653	2.468	2.480	1.941	2.576	2.486	100	91.67	98.54	98.08	**125.33***	94.43	97.82
Adapted Estimators of Koyuncu and Al-Omari(2020)	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**
$\bar{y}_{EN(QRSS)4}$	0.887	0.997	0.905	0.910	0.666	0.958	0.913	100	98.05	97.44	**133.15***	92.6	97.10	98.05
$\bar{y}_{EN(QRSS)5}$	0.723	0.814	0.738	0.742	0.551	0.781	0.745	100	98.07	97.45	**131.27***	92.6	97.13	98.07
Proposed	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**	**Linear**	**LTS**	**LAD**	**Huber**	**LMS**	**S**	**MM**
$\bar{y}_{E(QRSS)1}$	0.790	0.889	0.806	0.811	0.597	0.854	0.814	303.3	294.22	301.86	301.36	**319.96***	297.43	301.11
$\bar{y}_{E(QRSS)2}$	0.787	0.886	0.803	0.808	0.595	0.850	0.811	302.76	293.71	301.3	300.82	**319.12***	296.87	300.53
$\bar{y}_{E(QRSS)3}$	0.798	0.898	0.814	0.819	0.602	0.862	0.822	304.83	295.53	303.33	302.83	**322.25****	298.79	302.56
$\bar{y}_{E(QRSS)4}$	0.477	0.521	0.484	0.486	0.417	0.505	0.487	185.97	**191.24***	187.07	187.4	159.85	189.78	187.56
$\bar{y}_{E(QRSS)5}$	0.448	0.482	0.453	0.454	0.412	0.469	0.455	161.56	**168.73***	162.91	163.37	133.84	166.55	163.56

*demonstrates the most effective estimators with respect to robust methods

** demonstrates the most effective estimators with respect to all estimators and robust methods

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SOME PROPERTIES OF A CLASS OF GENERALIZED JANOWSKI-TYPE q-STARLIKE FUNCTIONS ASSOCIATED WITH OPOOLA q-DIFFERENTIAL OPERATOR AND q-DIFFERENTIAL SUBORDINATION

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Abstract. Without qualms, studies show that quantum calculus has received great attention in recent times. This can be attributed to its wide range of applications in many science areas. In this exploration, we study a new q-differential operator that generalized many known differential operators. The new q-operator and the concept of subordination were afterwards, used to define a new subclass of analytic-univalent functions that invariably consists of several known and new generalizations of starlike functions. Consequently, some geometric properties of the new class were investigated. The properties include coefficient inequality, growth, distortion and covering properties. In fact, we solved some radii problems for the class and also established its subordinating factor sequence property. Indeed, varying some of the involving parameters in our results led to some existing results.

1. Introduction

Define the set

$$\aleph_\jmath = \{\jmath, \jmath+1, \jmath+2, \dots\}, \ \jmath = 0, 1, 2, \dots .$$

Let $\Xi = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ be the *unit disk* and let

$$\mathfrak{A} = \left\{ f : f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ f(0) = 0, \ f'(0) = 1, \text{ and } z \in \Xi \right\} \tag{1}$$

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be the class of *normalized analytic functions*. Also, let $\mathcal{S}$ which is a subclass of $\mathfrak{A}$ represent the class of functions that are analytic and univalent in $\varXi$. For $\varkappa \in [0,1)$, let $\mathcal{S}^{\star}(\varkappa)$, $\mathcal{C}(\varkappa)$ and $\mathcal{K}(\varkappa)$ represent the classes of starlike functions of order $\varkappa$, convex functions of order $\varkappa$, and close-to-convex functions of order $\varkappa$, respectively. A function f in (1) belongs to the classes $\mathcal{S}^{\star}(\varkappa)$, $\mathcal{C}(\varkappa)$ and $\mathcal{K}(\varkappa)$ if for $z \in \varXi$, $\mathcal{R}e(zf'/f) > \varkappa$, $\mathcal{R}e(z(f''/f') + 1) > \varkappa$ and $\mathcal{R}e(f'/h') > \varkappa$ $(h \in \mathcal{C})$, respectively. We shall let $\mathcal{S}^{\star}(0) = \mathcal{S}^{\star}$, $\mathcal{C}(0) = \mathcal{C}$ and $\mathcal{K}(0) = \mathcal{K}$ simply denote the classes of starlike functions, convex functions and close-to-convex functions, respectively.
Historically, class $\mathcal{S}^{\star}$ of starlike functions was introduced by Alexander [1] and it has been numerously studied in various forms, such as starlike functions of order $\varkappa$, strongly starlike functions, uniformly starlike functions, close-to-starlike functions, bi-starlike functions, Janowski-type starlike functions, Mocanu-type starlike functions, starlike functions of complex order, λ-pseudo-starlike functions, and many more. In deed, an impressive application of starlike functions was demonstrated by Rensaa [32] where the author used starlike functions to solve frequency analysis problem. A frequency analysis problem is the problem of determining unknown frequency f_k $(k \in \aleph_1)$, with its corresponding amplitude a_k $(k \in \aleph_1)$, and of a trigonometric signal $z_k(m)$ where the signal values from k observations are known. We refer readers to [15,25,39] for more information on starlike functions and to [9,41] for some details on its applications.

Suppose $f_1, f_2 \in \mathfrak{A}$, f_1 is said to be subordinate to f_2, notationally expressed as $f_1(z) \prec f_2(z)$ $(z \in \varXi)$, if there exists a Schwarz function: $\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots$ $(|\omega(z)| < 1,\ z \in \varXi)$ such that

$$f_1(z) = f_2(z) \circ \omega(z) = f_2(\omega(z)). \tag{2}$$

In case $f_2(z)$ is univalent in $\varXi$, then $f_1(z) \prec f_2(z) \iff f_1(0) = f_2(0)$ and $f_1(\varXi) \subset f_2(\varXi)$.
Let $\mathcal{P}(\varkappa)$ represent the class of Carathéodory functions of order $\varkappa$ and of the form

$$p_{\varkappa}(z) = 1 + \sum_{k=1}^{\infty}(1-\varkappa)p_k z^k \quad \big(\mathcal{R}e\, p_{\varkappa}(z) > \varkappa \in [0,1),\ p_{\varkappa}(0) = 1,\ z \in \varXi\big). \tag{3}$$

Clearly, $\mathcal{P}(\varkappa) \subseteq \mathcal{P}(0) = \mathcal{P}$, where $\mathcal{P}$ is simply called the class of Carathéodory functions. In 2006, Polatoğlu et al. [30] generalized function is the class $\mathcal{P}$ by introducing the class

$$\mathcal{P}(\lambda;\mathcal{A},\mathcal{B}) := \left\{p(z) \in \mathcal{P} : p(z) \prec (1-\lambda)\frac{1+\mathcal{A}z}{1+\mathcal{B}z} + \lambda \iff p(z) = (1-\lambda)\frac{1+\mathcal{A}\omega(z)}{1+\mathcal{B}\omega(z)} + \lambda\right\} \tag{4}$$

where all parameters are as declared in (8). It is easily seen that $\mathcal{P}(0,1,-1) = \mathcal{P}(1,-1)$ in (3) and $\mathcal{P}(0,\mathcal{A},\mathcal{B}) = \mathcal{P}(\mathcal{A},\mathcal{B})$, the class of Janowski functions introduced in [16], see also [8,39] for more details.

Quantum calculus (simply known as q-calculus) has received a surge in research in recent years, owing to its wide range of applications in mathematics, physics and other sciences. Specifically, its application areas include, for example, quantum physics, operator theory, ordinary fractional calculus, and optimal control problems; see [5,6,17,31,40]. The application of q-calculus (that is, q-differentiation, q-integration and q-analysis,) in the development of Geometric Function Theory (GFT) is particularly noteworthy. Current development in GFT shows that the concept of q-calculus has enticed many geometric function theorists. Since the introduction of the q-derivative and the q-integral by Jackson [13,14], many researchers (see [4,18,21–24,27,28,35,42]) have in diverse ways considered them in the establishment of many properties of the subclasses of $\mathfrak{A}$. In particular, authors in [5,6,17,36] extensively discussed some areas of applications of q-operators, q-functions, q-series and q-analysis in various fields of Pure and Applied Mathematics.

For function $f \in \mathfrak{A}$ of the form (1) and for $q \in (0,1)$, the q-differential operator $\mathcal{D}_q : \mathfrak{A} \longrightarrow \mathfrak{A}$ is define by

$$\left.\begin{array}{l} \mathcal{D}_q f(0) = f'(0) = 1 \quad (z=0) \quad \textit{if it exists,} \\ \mathcal{D}_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)} = 1 + \sum\limits_{k=2}^{\infty} [k]_q a_k z^{k-1} \quad (z \neq 0), \\ f'(z) \text{ as } q \longrightarrow 1, \end{cases} \\ \mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z)) = \sum\limits_{k=2}^{\infty} [k-1]_q [k]_q a_k z^{k-2}, \\ \textit{and} \ \ [k]_q = \frac{1-q^k}{1-q} \ \textit{so that by L'Hŏpital's rule,} \ \lim\limits_{q \uparrow 1} [k]_q = k. \end{array}\right\} \tag{5}$$

Using (5), then the Opoola q-differential operator $D_{q,t}^{n,b,u}$ is defined as follows.

Definition 1. *Let $f \in \mathfrak{A}$, then the Opoola q-differential operator $D_{q,t}^{n,b,u} : \mathfrak{A} \longrightarrow \mathfrak{A}$ $(q \in (0,1),\ n \in \aleph_0)$ is defined by*

$$\left.\begin{array}{l} D_{q,t}^{0,b,u} f(z) = f(z) \\ D_{q,t}^{1,b,u} f(z) = (1+(b-u-1)t)f(z) - zt(b-u) + zt\mathcal{D}_q f(z) = d_{q,t}(f) \\ D_{q,t}^{2,b,u} f(z) = d_{q,t}(D_{q,t}^{1,b,u} f(z)) \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ D_{q,t}^{n,b,u} f(z) = d_{q,t}(D_{q,t}^{n-1,b,u} f(z)) \end{array}\right\} \tag{6}$$

which implies that

$$D_{q,t}^{n,b,u}f(z) = z + \sum_{k=2}^{\infty}(1 + ([k]_q + b - u - 1)t)^n a_k z^k \quad (z \in \Xi) \tag{7}$$

where

$$\left.\begin{array}{c} n \in \aleph_0,\ t \geqq 0,\ b \geqq 0,\ u \in [0,b],\ \lambda \in [0,1),\ -1 \leqq \mathcal{B} < \mathcal{A} \leqq 1, \\ q \in (0,1),\ [k]_q = \frac{1-q^k}{1-q},\ \textit{and}\ \lim\limits_{q\uparrow 1}[k]_q = k. \end{array}\right\} \tag{8}$$

The q-operator in (6) is the q-analogue of the well-known Opoola differential operator introduced in [26]. The following properties hold for the functions in (7).

(1) $\lim\limits_{q\uparrow 1} D_{q,t}^{0,b,u}f(z) = \lim\limits_{q\uparrow 1} D_{q,0}^{n,b,u}f(z) = \lim\limits_{q\uparrow 1} D_{q,0}^{0,b,u}f(z) = f(z) \in \mathfrak{A}$ in (1).

(2) $\lim\limits_{q\uparrow 1} D_{q,1}^{n,b,b}f(z) = \lim\limits_{q\uparrow 1} D_{q,1}^{n,u,u}f(z) = D^n f(z)$, the Sălăgean differential operator introduced in [33].

(3) $\lim\limits_{q\uparrow 1} D_{q,t}^{n,b,b}f(z) = \lim\limits_{q\uparrow 1} D_{q,t}^{n,u,u}f(z) = D_t^n f(z)$, the Al-Oboudi differential operator introduced in [3].

(4) $\lim\limits_{q\uparrow 1} D_{q,t}^{n,b,u}f(z) = D_t^{n,b,u}f(z)$, the Opoola differential operator introduced in [26].

(5) $D_{q,1}^{n,b,b}f(z) = D_{q,1}^{n,u,u}f(z) = D_q^n f(z)$, the Sălăgean q-differential operator introduced by Govindaraj and Sivasubramanian [11].

(6) $D_{q,t}^{n,b,b}f(z) = D_{q,t}^{n,u,u}f(z) = D_{q,t}^n f(z)$ is herein referred to as the Al-Oboudi q-differential operator.

Instances of some recently studied q-operators in GFT can be found in [2,18,20,29].

2. A New Class of q-Starlike Functions

In view of the geometric expression of starlike functions, the Polatoḡlu's function in (4) and the Opoola q-differential operator in Definition 1 , we therefore, present the class $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ as follows.

Definition 2. *A function $f \in \mathfrak{A}$ is said to be a member of the class $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ if it satisfies the q-differential subordination condition*

$$\frac{D_{q,t}^{n+1,b,u}f(z)}{D_{q,t}^{n,b,u}f(z)} \prec (1-\lambda)\frac{1+\mathcal{A}z}{1+\mathcal{B}z} + \lambda \quad (z \in \Xi) \tag{9}$$

where all parameters are as declared in (8).

It can easily be seen that class $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ consists of numerous subclasses of starlike functions when its involving parameters are varied. Some studies on Janowski's q-starlike functions with various definitions can be found in [7,12,19,38].

3. Applicable Lemma

Definition 3 ([43]). (Subordinating Factor Sequence). *The sequence* $\{h_k\}_{k=1}^{\infty}$ *of complex numbers is called a* subordinating factor sequence *if whenever*

$$c(z) = \sum_{k=1}^{\infty} c_k z^k \quad (c_1 = 1,\ z \in \Xi)$$

is analytic-univalently convex in Ξ, $\sum_{k=1}^{\infty} c_k h_k \prec c(z)$.

Lemma 1 ([43]). (Subordinating Factor Sequence). *From Definition 3, the sequence* $\{h_k\}_{k=1}^{\infty}$ *is called a subordinating factor sequence if and only if*

$$\mathcal{R}e\left(1 + 2\sum_{k=1}^{\infty} c_k z^k\right) > 0 \quad (z \in \Xi).$$

4. The Main Results

For brevity and in what follows from (7), let

$$\varDelta_{q,k} = (1 + ([k]_q + b - u - 1)t) \geqq 1, \tag{10}$$

so that

$$D_{q,t}^{n,b,u} f(z) = z + \sum_{k=2}^{\infty} \varDelta_{q,k}^{n} a_k z^k \quad (z \in \Xi), \tag{11}$$

$$\varLambda_q^n(k,\lambda,\mathcal{A},\mathcal{B}) = \varDelta_{q,k}^{n}\Big\{\big(\varDelta_{q,k} - 1\big) + \Big|\varDelta_{q,k}\mathcal{B} - [\mathcal{A} - \lambda(\mathcal{A} - \mathcal{B})]\Big|\Big\}, \tag{12}$$

and henceforth, all parameters shall be as declared in (8).

4.1. Basic Properties.

Theorem 1 (Coefficient Inequality). *Let* $f \in \mathfrak{A}$, *then* $f \in \mathcal{S}_q^{\star}(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{\varLambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A} - \mathcal{B})(1 - \lambda)} |a_k| \leqq 1. \tag{13}$$

All parameters are as declared in (8).

Proof. Suppose inequality (13) holds, then in view of the principle of subordination, we can express (9) as

$$\frac{D_{q,t}^{n+1,b,u} f(z)}{D_{q,t}^{n,b,u} f(z)} = \frac{1 + [\mathcal{A} - \lambda(\mathcal{A} - \mathcal{B})]\omega(z)}{1 + \mathcal{B}\omega(z)} \tag{14}$$

which simplifies to

$$\frac{D_{q,t}^{n+1,b,u} f(z) - D_{q,t}^{n,b,u} f(z)}{[\mathcal{A} - \lambda(\mathcal{A} - \mathcal{B})]D_{q,t}^{n,b,u} f(z) - \mathcal{B}D_{q,t}^{n+1,b,u} f(z)} = \omega(z). \tag{15}$$

Using (11) in (15) leads to

$$\frac{(z+\sum_{k=2}^{\infty}\Delta_{q,k}^{n+1}a_kz^k)-(z+\sum_{k=2}^{\infty}\Delta_{q,k}^{n}a_kz^k)}{[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})](z+\sum_{k=2}^{\infty}\Delta_{q,k}^{n}a_kz^k)-\mathcal{B}(z+\sum_{k=2}^{\infty}\Delta_{q,k}^{n+1}a_kz^k)}$$
$$=\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)a_kz^{k-1}}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}a_kz^{k-1}}=\omega(z). \quad (16)$$

For $|\omega(z)|<1$ and $z\in\Xi$, we have

$$\left|\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)a_kz^{k-1}}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}a_kz^{k-1}}\right|$$
$$\leqq\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)|a_k|}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\Big|\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big||a_k|}\leqq 1.$$

This latter expression on the LHS is bounded above by 1 if

$$\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)|a_k|\leqq(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\Big|\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big||a_k|$$

so that further simplification leads to

$$\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\Big\{\big(\Delta_{q,k}-1\big)+\Big|\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|\Big\}|a_k|\leqq(\mathcal{A}-\mathcal{B})(1-\lambda) \quad (17)$$

and using (12) gives

$$\sum_{k=2}^{\infty}\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})|a_k|\leqq(\mathcal{A}-\mathcal{B})(1-\lambda). \quad (18)$$

Conversely, suppose $f\in\mathcal{S}_q^{\star}(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, then from (16) we have

$$\left|\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)a_kz^{k-1}}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}a_kz^{k-1}}\right|=|\omega(z)|<1$$

and since $\mathcal{R}e\,z\leqq|z|<1$, then it implies that

$$\mathcal{R}e\left\{\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)a_kz^{k-1}}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}a_kz^{k-1}}\right\}<1.$$

Choosing values of z on the real axis of the complex plane and allowing $z\longrightarrow 1$, implies that

$$\frac{\sum_{k=2}^{\infty}\Delta_{q,k}^{n}(\Delta_{q,k}-1)|a_k|}{(\mathcal{A}-\mathcal{B})(1-\lambda)-\sum_{k=2}^{\infty}\Delta_{q,k}^{n}\{\Delta_{q,k}\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\}|a_k|}\leqq 1$$

so that further simplification and using (12) leads to

$$\sum_{k=2}^{\infty} \Lambda_q^n(k, \lambda, \mathcal{A}, \mathcal{B})|a_k| \leqq (\mathcal{A} - \mathcal{B})(1 - \lambda) \tag{19}$$

as asserted. □

Corollary 1. *Observe that from (13), equality occurs for function*

$$f_k(z) = z + \frac{(\mathcal{A} - \mathcal{B})(1 - \lambda)}{\Lambda_q^n(k, \lambda, \mathcal{A}, \mathcal{B})} z^k \quad (k \in \aleph_2, \ z \in \Xi). \tag{20}$$

Corollary 2. *Let $f \in \mathcal{S}_q^\star(n, b, t, u; \lambda, \mathcal{A}, \mathcal{B})$, then*

$$|a_k| \leqq \frac{(\mathcal{A} - \mathcal{B})(1 - \lambda)}{\Lambda_q^n(k, \lambda, \mathcal{A}, \mathcal{B})} \quad (k \in \aleph_2). \tag{21}$$

with extremal function in (20).

Remark 1. *Let $f \in \lim_{q \uparrow 1} \mathcal{S}_q^\star(0, b, 1, b; 0; 1, -1) = \lim_{q \uparrow 1} \mathcal{S}_q^\star(0, u, 1, u; 0; 1, -1) = \mathcal{S}^\star$, then*

$$\sum_{k=2}^{\infty} k|a_k| \leqq 1 \quad (k \in \aleph_2).$$

This is the result of Goodman [10] and Silverman [34].

Theorem 2 (Growth Property)**.** *Let $f \in \mathcal{S}_q^\star(n, b, t, u; \lambda, \mathcal{A}, \mathcal{B})$, then for $r = |z| < 1$,*

$$r - \frac{r^2 \Delta_{q,2}^n (\mathcal{A} - \mathcal{B})(1 - \lambda)}{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} \leqq |D_{q,t}^{n,b,u} f(z)| \leqq r + \frac{r^2 \Delta_{q,2}^n (\mathcal{A} - \mathcal{B})(1 - \lambda)}{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}. \tag{22}$$

Equality occurs for function

$$f_2(z) = z + \frac{(\mathcal{A} - \mathcal{B})(1 - \lambda)}{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})} z^2. \tag{23}$$

Proof. From (13) and for the fact that $\Delta_{q,k}$ is an increasing function of k ($\forall\, k \in \aleph_2$), then

$$\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B}) \sum_{k=2}^{\infty} |a_k| \leqq \sum_{k=2}^{\infty} \Lambda_q^n(k, \lambda, \mathcal{A}, \mathcal{B})|a_k| \leqq (\mathcal{A} - \mathcal{B})(1 - \lambda)$$

which implies that

$$\sum_{k=2}^{\infty} |a_k| \leqq \frac{(\mathcal{A} - \mathcal{B})(1 - \lambda)}{\Lambda_q^n(2, \lambda, \mathcal{A}, \mathcal{B})}. \tag{24}$$

Recall also that for $f \in \mathfrak{A}$ and since $r^k < r = |z| < 1$, then from (11),

$$|D_{q,t}^{n,b,u} f(z)| = \left| z + \sum_{k=2}^{\infty} \Delta_{q,k}^n a_k z^k \right| \leqq r + \sum_{k=2}^{\infty} \Delta_{q,k}^n |a_k| r^k \leqq r + r^2 \Delta_{q,2}^n \sum_{k=2}^{\infty} |a_k| \tag{25}$$

so that putting (24) into (25) leads to

$$|D_{q,t}^{n,b,u}f(z)| \leqq r + \frac{r^2 \Delta_{q,2}^n (\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{26}$$

In the same manner, we can show that

$$|D_{q,t}^{n,b,u}f(z)| \geqq r - \frac{r^2 \Delta_{q,2}^n (\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{27}$$

Putting (26) and (27) together gives (22) as asserted. □

Corollary 3. *Let $f \in \mathcal{S}_q^\star(0,b,t,u;\lambda;\mathcal{A},\mathcal{B})$, then for $r = |z| < 1$,*

$$r - \frac{r^2(\mathcal{A}-\mathcal{B})(1-\lambda)}{\big(\Delta_{q,2}-1\big) + \Big|\Delta_{q,2}\mathcal{B} - [\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|} \leqq |f(z)|$$
$$\leqq r + \frac{r^2(\mathcal{A}-\mathcal{B})(1-\lambda)}{\big(\Delta_{q,2}-1\big) + \Big|\Delta_{q,2}\mathcal{B} - [\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|}.$$

Equality occurs for function

$$f_2(z) = z + \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{\big(\Delta_{q,2}-1\big) + \Big|\Delta_{q,2}\mathcal{B} - [\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|} z^2.$$

Theorem 3 (Distortion Property)**.** *Let $f \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, then for $r = |z| < 1$,*

$$1 - \frac{[2]_q r(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})} \leqq |\mathcal{D}_q(D_{q,t}^{n,b,u}f(z))| \leqq 1 + \frac{[2]_q r(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{28}$$

Equality occurs for the extremal function in (23).

Proof. Recall that for $f \in \mathfrak{A}$, $r^k < r = |z| < 1$, and by using (5) in (11); we have

$$\Big|\mathcal{D}_q(D_{q,t}^{n,b,u}f(z))\Big| = \Big|1 + \sum_{k=2}^{\infty} \Delta_{q,k}^n [k]_q a_k z^{k-1}\Big|$$
$$\leqq 1 + \sum_{k=2}^{\infty} \Delta_{q,k}^n [k]_q |a_k| r^{k-1} \leqq 1 + r[2]_q \Delta_{q,2}^n \sum_{k=2}^{\infty} |a_k| \tag{29}$$

so that using (24) in (29) leads to

$$\Big|\mathcal{D}_q(D_{q,t}^{n,b,u}f(z))\Big| \leqq 1 + \frac{r[2]_q \Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{30}$$

In the same manner, we can show that

$$\Big|\mathcal{D}_q(D_{q,t}^{n,b,u}f(z))\Big| \geqq 1 - \frac{r[2]_q \Delta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{31}$$

Putting (30) and (31) together gives (28) as asserted. □

Corollary 4. *Let $f \in \lim_{q\uparrow 1} \mathcal{S}_q^\star(0,b,t,u;\lambda;\mathcal{A},\mathcal{B})$, then for $r=|z|<1$,*

$$1-\frac{2r(\mathcal{A}-\mathcal{B})(1-\lambda)}{(\nabla-1)+\Big|\nabla\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|} \leqq |f'(z)| \leqq 1+\frac{2r(\mathcal{A}-\mathcal{B})(1-\lambda)}{(\nabla-1)+\Big|\nabla\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|}.$$

where $\nabla = \lim_{q\uparrow 1} \varDelta_{q,2}$. Equality occurs for function

$$f_2(z) = z + \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{(\nabla-1)+\Big|\nabla\mathcal{B}-[\mathcal{A}-\lambda(\mathcal{A}-\mathcal{B})]\Big|}z^2.$$

Theorem 4 (Covering Property)**.** *Let $f \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, then the function $D_{q,t}^{n,b,u}f(z)$ in (11) maps the unit disk $\varXi$ onto a domain that covers the disk*

$$|\varpi| < \frac{\varLambda_q^n(2,\lambda,\mathcal{A},\mathcal{B}) - \varDelta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\varLambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}$$

The result is sharp for the extremal function in (23).

Proof. From (22),

$$|\varpi| = |D_{q,t}^{n,b,u}f(z)| < r - \frac{r^2\varDelta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\varLambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}$$

and observe that as $|z| = r \longrightarrow 1$,

$$|\varpi| < 1 - \frac{\varDelta_{q,2}^n(\mathcal{A}-\mathcal{B})(1-\lambda)}{\varLambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}$$

where some simplifications give the assertion. □

Remark 2. *Let $f \in \lim_{q\uparrow 1} \mathcal{S}^\star(0,b,t,u;0;1,-1) = \mathcal{S}^\star$, then*

$$|\varpi| < \frac{1}{2}.$$

This result agrees with that of Koebe's one-quarter theorem, see [39].

4.2. **Radii Problems.**

Theorem 5 (Radius of Starlikeness)**.** *Let $f \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, then $f \in \mathcal{S}^\star(\varkappa)$ $(\varkappa \in [0,1))$ in the disk*

$$|z| < \mathcal{R}_{\mathcal{S}^\star} := \inf_{k\in\aleph_2}\left\{\frac{\varLambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})(1-\varkappa)}{(\mathcal{A}-\mathcal{B})(1-\lambda)(k-\varkappa)}\right\}^{\frac{1}{k-1}}.$$

The inequality is sharp for the function in (20).

Proof. From the definition of starlikeness, it is sufficient to show that

$$\frac{\frac{zf'(z)}{f(z)} - \varkappa}{1-\varkappa} \prec \frac{1+z}{1-z} \quad (\varkappa \in [0,1)). \tag{32}$$

Using (2) in (32) leads to

$$\frac{zf'(z) - \varkappa f(z)}{(1-\varkappa)f(z)} = \frac{1+\omega(z)}{1-\omega(z)}$$

so that

$$\left|\frac{zf'(z) - f(z)}{zf'(z) + (1-2\varkappa)f(z)}\right| = |\omega(z)| < 1$$

and using (1) leads to

$$\sum_{k=2}^{\infty} \frac{k-\varkappa}{1-\varkappa}|a_k||z|^{k-1} < 1. \tag{33}$$

Note that inequalities (13) and (33) can only be valid if

$$\frac{k-\varkappa}{1-\varkappa}|z|^{k-1} < \frac{\varLambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)}$$

where some simplifications affirm the result. □

Theorem 6 (Radius of Convexity). *Let $f \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, then $f \in \mathcal{C}(\varkappa)$ $(\varkappa \in [0,1))$ in the disk*

$$|z| < \mathcal{R}_{\mathcal{C}} := \inf_{k\in\aleph_2} \left\{\frac{\varLambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})(1-\varkappa)}{(\mathcal{A}-\mathcal{B})(1-\lambda)k(k-\varkappa)}\right\}^{\frac{1}{k-1}}.$$

This inequality is sharp for the function in (20).

Proof. From the definition of convexity, it is sufficient to show that

$$\frac{\frac{zf''(z)}{f'(z)} + 1 - \varkappa}{1-\varkappa} \prec \frac{1+z}{1-z} \quad (\varkappa \in [0,1)). \tag{34}$$

Using (2) in (34) leads to

$$\frac{zf''(z) + (1-\varkappa)f'(z)}{(1-\varkappa)f'(z)} = \frac{1+\omega(z)}{1-\omega(z)}$$

so that

$$\left|\frac{zf''(z)}{zf''(z) + 2(1-\varkappa)f'(z)}\right| = |\omega(z)| < 1$$

and using (1) leads to

$$\sum_{k=2}^{\infty} \frac{k(k-\varkappa)}{1-\varkappa}|a_k||z|^{k-1} < 1. \tag{35}$$

Note that the inequalities (13) and (35) can only be valid if

$$\frac{k(k-\varkappa)}{1-\varkappa}|z|^{k-1} < \frac{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)}$$

where some simplifications affirm the result. □

Theorem 7 (Radius of Close-to-convexity)**.** *Let* $f \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$, *then* $f \in \mathcal{K}(\varkappa)$ $(\varkappa \in [0,1))$ *in the disk*

$$|z| < \mathcal{R}_{\mathcal{K}} := \inf_{k\in\aleph_2} \left\{ \frac{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})(1-\varkappa)}{(\mathcal{A}-\mathcal{B})(1-\lambda)k} \right\}^{\frac{1}{k-1}}.$$

The inequality is sharp for the function in (20).

Proof. From the definition of close-to-convexity, it is sufficient to show that

$$|f'(z)-1| < 1-\varkappa \quad (\varkappa \in [0,1)).$$

Using (1) leads to

$$|f'(z)-1| = \left| \left(1+\sum_{k=2}^{\infty} k a_k z^{k-1}\right) - 1 \right| \leqq \sum_{k=2}^{\infty} k|a_k||z|^{k-1} < 1-\varkappa,$$

that is,

$$\sum_{k=2}^{\infty} \frac{k}{1-\varkappa}|a_k||z|^{k-1} < 1. \tag{36}$$

Note that inequalities (13) and (36) can only be valid if

$$\frac{k}{1-\varkappa}|z|^{k-1} < \frac{\Lambda_q^n(k,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)}$$

where some simplifications affirm the result. □

4.3. **Subordination Property.**

Theorem 8. *Let* $f \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ *and* $c \in \mathcal{C}$, *then*

$$\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}}(f\star c)(z) \prec c(z) \tag{37}$$

and

$$\mathcal{R}e\, f > -\frac{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}. \tag{38}$$

The constant factor

$$\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}} \tag{39}$$

in (37) *cannot be replaced by a larger value. The symbol* $\star$ *is called Hadamard product or convolution.*

The following proof adopts the technique of Srivastava and Attiya [37].

Proof. Let $f \in \mathcal{S}_q^{\star}(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$ and suppose $c(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{C}$, then from (37),

$$\begin{aligned}
&\frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}}(f \star c)(z) \\
&\qquad = \frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}}\left(z + \sum_{k=2}^{\infty} a_k c_k z^k\right) \\
&\qquad\qquad = \sum_{k=1}^{\infty} \frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}} a_k c_k z^k
\end{aligned}$$

and clearly by Definition 3, the subordination result in (37) holds if

$$\left\{\frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}} a_k\right\}_{k=1}^{\infty}$$

is a *subordinating factor sequence* where $a_1 = 1$. Now applying Lemma 1 gives an equivalence inequality

$$\mathcal{R}e\left(1 + \sum_{k=1}^{\infty} \frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})} a_k z^k\right) > 0. \tag{40}$$

Observe that $\mathit{\Lambda}_q^n(k,\lambda,\mathcal{A},\mathcal{B})$ is an increasing function $\forall k \in \aleph_2$, so

$$\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B}) \leqq \mathit{\Lambda}_q^n(k,\lambda,\mathcal{A},\mathcal{B}), \quad \forall k \in \aleph_2.$$

Hence, it follows by using $|z| = r < 1$, triangle inequality and inequality (13) that

$$\begin{aligned}
&\mathcal{R}e\left(1 + \sum_{k=1}^{\infty} \frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})} a_k z^k\right) \\
&\quad = \mathcal{R}e\left(1 + \frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})} \sum_{k=1}^{\infty} a_k z^k\right) \\
&\quad = \mathcal{R}e\left(1 + \frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})} z + \frac{\sum_{k=2}^{\infty} \mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B}) a_k z^k}{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}\right) \\
&\quad \geqq 1 - \frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})} r - \frac{\sum_{k=2}^{\infty} \mathit{\Lambda}_q^n(k,\lambda,\mathcal{A},\mathcal{B}) |a_k|}{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})} r^k \\
&\quad > 1 - \frac{\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})} r - \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{(\mathcal{A}-\mathcal{B})(1-\lambda)+\mathit{\Lambda}_q^n(2,\lambda,\mathcal{A},\mathcal{B})} r \\
&\quad = 1 - r > 0.
\end{aligned}$$

This evidently proves inequality (40) and as well as the subordination result (37). Also, the inequality (38) follows from (37) by taking the convex function

$$c_0(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in \mathcal{C}.$$

To prove the sharpness of the constant (39), consider (see (20)) the function

$$f_2(z) = z + \frac{(\mathcal{A}-\mathcal{B})(1-\lambda)}{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})} z^2 \in \mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$$

so that using (37) leads to

$$\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}} f_2(z) \prec c_0(z) = \frac{z}{1-z}. \tag{41}$$

It can *easily* be verified that for $f_2(z)$,

$$\min_{|z|\leqq r}\left\{\mathcal{R}e\left(\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}} f_2(z)\right)\right\} = -\frac{1}{2} \quad (z \in \Xi)$$

which shows that the constant $\frac{\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})}{2\{(\mathcal{A}-\mathcal{B})(1-\lambda)+\Lambda_q^n(2,\lambda,\mathcal{A},\mathcal{B})\}}$ cannot be replaced by any larger value. □

5. Conclusions

The attention geared towards the study of q-operators by scientists and in particular, by geometric function theorists in recent years is overwhelming. In this study, a new q-differential operator that generalized the famous Sălăgean [33], Al-Oboudi [3] and Opoola differential [26] operators was studied. Subsequently, the q-differential operator and the principle of subordination were used to define a subclass of analytic-univalent functions. This new class was represented by $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$. Further, the geometric properties such as the coefficient inequality, growth, distortion and covering theorems were established for the class $\mathcal{S}_q^\star(n,b,t,u;\lambda,\mathcal{A},\mathcal{B})$. Also, the radii of starlikeness, convexity and close-to-convexity; as well as the subordinating factor sequence problems were solved for the new class. Intermittently, some key corollaries and remarks were given to demonstrate the relationship between this new class (and the new results); and some exiting classes (and their results).

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NUMERICAL RADIUS AND p-SCHATTEN NORM INEQUALITIES FOR POWER SERIES OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on the open disk $D(0,R)$, $f_a(z) := \sum_{k=0}^{\infty} |a_k| z^k$ that has the same radius of convergence R and A, B, $C \in B(H)$ with $\|A\| < R$, then we have the following Schwarz type inequality

$$|\langle C^* A f(A) Bx, y\rangle| \le f_a(\|A\|) \left\langle \left| |A|^{\alpha} B \right|^2 x, x \right\rangle^{1/2} \left\langle \left| |A^*|^{1-\alpha} C \right|^2 y, y \right\rangle^{1/2}$$

for $\alpha \in [0,1]$ and x, $y \in H$. Some natural applications for *numerical radius* and *p-Schatten norm* are also provided.

1. Introduction

The *numerical radius* $\omega(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1}$$

Obviously, by (1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \le \omega(T) \|x\|^2. \tag{2}$$

It is well known that $\omega(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \ge 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;

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(iii) $\omega(T+V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{3}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [7], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [8] improved the inequality (3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{5}$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [5]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{6}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{7}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (6) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{8}$$

and from (7) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{9}$$

For more related results, see the recent books on inequalities for numerical radii [3] and [1].

Let $(H; \langle .,. \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i \rangle < \infty. \tag{10}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{11}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (11) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{12}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{13}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

For a large number of results concerning trace inequalities, see the recent survey paper [4].

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite [12, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} = \left(\sum_{i \in I} \langle |A|^p e_i, e_i \rangle \right)^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{14}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{15}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [12, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{16}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{17}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \ \|BA\|_p \leq \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{18}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{19}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$\left(\left|\operatorname{tr}\left(AB\right)\right| \leq\right) \left\|AB\right\|_1 \leq \left\|A\right\|_p \left\|B\right\|_q, \ A \in \mathcal{B}_p\left(H\right), \ B \in \mathcal{B}_q\left(H\right). \tag{20}$$

For the theory of trace functionals and their applications the reader is referred to [10] and [12].

For $\mathcal{E} := \{e_i\}_{i\in I}$ an orthonormal basis of H we define for $A \in \mathcal{B}_p\left(H\right)$, $p \geq 1$

$$\left\|A\right\|_{\mathcal{E},p} := \left(\sum_{i\in I} \left|\left\langle Ae_i, e_i\right\rangle\right|^p\right)^{1/p}.$$

We observe that $\left\|\cdot\right\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p\left(H\right)$ and

$$\left\|A\right\|_{\mathcal{E},p} \leq \left\|A\right\|_p \text{ for } A \in \mathcal{B}_p\left(H\right).$$

Further, we can take the supremum over all orthonormal basis in H we can also define, for $A \in \mathcal{B}_p\left(H\right)$, that

$$\omega_p\left(A\right) := \sup_{\mathcal{E}} \left\|A\right\|_{\mathcal{E},p} \leq \left\|A\right\|_p,$$

which is a *norm* on $\mathcal{B}_p\left(H\right)$.

It is also known that, if $\mathcal{E} = \{e_i\}_{i\in I}$ and $\mathcal{F} = \{f_i\}_{i\in I}$ are orthonormal basis, then [11]

$$\sup_{\mathcal{E},\mathcal{F}} \sum_{i\in I} \left|\left\langle Te_i, f_i\right\rangle\right|^s = \left\|T\right\|_s^s \text{ for } s \geq 1. \tag{21}$$

2. Vector Inequalities

In 1988 F. Kittaneh [6, Corollary 7] obtained the following Schwarz type inequality for powers of operators:

Lemma 1. *Let $A \in B\left(H\right)$ and $\alpha \in [0,1]$. Then for $n \geq 1$ we have*

$$\left|\left\langle A^n x, y\right\rangle\right|^2 \leq \left\|A\right\|^{2n-2} \left\langle \left|A\right|^{2\alpha} x, x\right\rangle \left\langle \left|A^*\right|^{2(1-\alpha)} y, y\right\rangle \tag{22}$$

for all x, $y \in H$.

We can state the following result as well:

Corollary 1. *Let A, B, $C \in B\left(H\right)$ and $\alpha \in [0,1]$. Then for $n \geq 1$ we have*

$$\left|\left\langle C^* A^n Bx, y\right\rangle\right|^2 \leq \left\|A\right\|^{2n-2} \left\langle \left|\left|A\right|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{23}$$

for all x, $y \in H$.

Proof. If we replace x by Bx and y by Cy in (22), then we get

$$\left|\left\langle C^* A^n Bx, y\right\rangle\right|^2 \leq \left\|A\right\|^{2n-2} \left\langle B^* \left|A\right|^{2\alpha} Bx, x\right\rangle \left\langle C^* \left|A^*\right|^{2(1-\alpha)} Cy, y\right\rangle. \tag{24}$$

Observe that $B^{*}\left|A\right|^{2\alpha}B=\left|\left|A\right|^{\alpha}B\right|^{2}$ and $C^{*}\left|A^{*}\right|^{2(1-\alpha)}C=\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}$, then by (24) we get (23). □

We consider the power series with complex coefficients $f\left(z\right):=\sum_{k=0}^{\infty}a_{k}z^{k}$ with $a_{k}\in\mathbb{C}$ for $k\in\mathbb{N}:=\left\{0,1,...\right\}$. We assume that this power series in convergent on the open disk $D(0,R):=\left\{z\in\mathbb{C}|\ z<R\right\}$. If $R=\infty$ then $D(0,R)=\mathbb{C}$. We define $f_{a}\left(z\right):=\sum_{k=0}^{\infty}\left|a_{k}\right|z^{k}$ which has the same radius of convergence R.

As some natural examples that are useful for applications, we can point out that, if

$$f\left(\lambda\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}}{n}\lambda^{n}=\ln\frac{1}{1+\lambda},\ \ \lambda\in D\left(0,1\right); \tag{25}$$

$$g\left(\lambda\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{\left(2n\right)!}\lambda^{2n}=\cos\lambda,\ \ \lambda\in\mathbb{C};$$

$$h\left(\lambda\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{\left(2n+1\right)!}\lambda^{2n+1}=\sin\lambda,\ \ \lambda\in\mathbb{C};$$

$$l\left(\lambda\right)=\sum_{n=0}^{\infty}\left(-1\right)^{n}\lambda^{n}=\frac{1}{1+\lambda},\ \ \lambda\in D\left(0,1\right);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_{a}\left(\lambda\right)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^{n}=\ln\frac{1}{1-\lambda},\ \ \lambda\in D\left(0,1\right); \tag{26}$$

$$g_{a}\left(\lambda\right)=\sum_{n=0}^{\infty}\frac{1}{\left(2n\right)!}\lambda^{2n}=\cosh\lambda,\ \ \lambda\in\mathbb{C};$$

$$h_{a}\left(\lambda\right)=\sum_{n=0}^{\infty}\frac{1}{\left(2n+1\right)!}\lambda^{2n+1}=\sinh\lambda,\ \ \lambda\in\mathbb{C};$$

$$l_{a}\left(\lambda\right)=\sum_{n=0}^{\infty}\lambda^{n}=\frac{1}{1-\lambda},\ \ \lambda\in D\left(0,1\right).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp\left(\lambda\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{n}\qquad\lambda\in\mathbb{C}, \tag{27}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D\left(0,1\right);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$$ {}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}\lambda^{n}, \alpha,\beta,\gamma > 0, \quad \lambda \in D(0,1);$$

where Γ is *Gamma function.*

The following result is of interest:

Theorem 2. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is convergent on* $D(0,R)$ *and* A, B, $C \in B(H)$ *with* $\|A\| < R$, *then*

$$\left|\langle C^* A f(A) Bx, y\rangle\right|^2 \leq f_a^2(\|A\|) \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{28}$$

for $\alpha \in [0,1]$ *and* x, $y \in H$.

In particular,

$$\left|\langle C^* A f(A) Bx, y\rangle\right|^2 \leq f_a^2(\|A\|) \left\langle \left||A|^{1/2} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1/2} C\right|^2 y, y\right\rangle \tag{29}$$

for x, $y \in H$.

Proof. If we take $n = k+1$, $k \in \mathbb{N}$ in (23) and take the square root, then we get

$$\left|\left\langle C^* A A^k Bx, y\right\rangle\right| \leq \|A\|^k \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle^{1/2}$$

for all x, $y \in H$.

Further, if we multiply by $|a_k| \geq 0$, $k \in \{0, 1, ...\}$ and sum over k from 0 to m, then we get

$$\begin{aligned} &\left|\left\langle C^* A \sum_{k=0}^{m} a_k A^k Bx, y\right\rangle\right| \qquad (30)\\ &= \left|\sum_{k=0}^{m} a_k \left\langle C^* A A^k Bx, y\right\rangle\right| \leq \sum_{k=0}^{m} |a_k| \left|\left\langle C^* A A^k Bx, y\right\rangle\right| \\ &\leq \sum_{k=0}^{m} |a_k| \, \|A\|^k \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle^{1/2} \end{aligned}$$

for all x, $y \in H$.

Since $\|A\| < R$ then series $\sum_{k=0}^{\infty} a_k A^k$ and $\sum_{k=0}^{\infty} |a_k| \|A\|^k$ are convergent and

$$\sum_{k=0}^{\infty} a_k A^k = f(A) \text{ and } \sum_{k=0}^{\infty} |a_k| \|A\|^k = f_a(\|A\|).$$

By taking now the limit over $m \to \infty$ in (30) we deduce the desired result (28). □

Remark 1. *If $A,\ B,\ C \in B(H)$ with $\|A\| < 1$, then for $\alpha \in [0,1]$*

$$\begin{aligned} &\left|\left\langle C^* A (I \pm A)^{-1} Bx, y\right\rangle\right|^2 \\ &\leq (1 - \|A\|)^{-2} \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \end{aligned} \tag{31}$$

and

$$\begin{aligned} &\left|\left\langle C^* A \ln(I \pm A) Bx, y\right\rangle\right|^2 \\ &\leq [\ln(1 - \|A\|)]^2 \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \end{aligned} \tag{32}$$

for all $x,\ y \in H$.

For $\alpha = 1/2$ in (31) and (32) we obtain

$$\left|\left\langle C^* A (I \pm A)^{-1} Bx, y\right\rangle\right|^2 \leq (1 - \|A\|)^{-2} \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \tag{33}$$

and

$$\left|\left\langle C^* A \ln(I \pm A) Bx, y\right\rangle\right|^2 \leq [\ln(1 - \|A\|)]^2 \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \tag{34}$$

for all $x,\ y \in H$.

If $A,\ B,\ C \in B(H)$ and $\alpha \in [0,1]$, then

$$\left|\left\langle C^* A \sin(A) Bx, y\right\rangle\right|^2 \leq [\sinh(\|A\|)]^2 \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{35}$$

and

$$\left|\left\langle C^* A \cos(A) Bx, y\right\rangle\right|^2 \leq [\cosh(\|A\|)]^2 \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{36}$$

for all $x,\ y \in H$.

For $\alpha = 1/2$ in (35) and (36) we obtain

$$\left|\left\langle C^* A \sin(A) Bx, y\right\rangle\right|^2 \leq [\sinh(\|A\|)]^2 \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \tag{37}$$

and

$$\left|\left\langle C^* A \cos(A) Bx, y\right\rangle\right|^2 \leq [\cosh(\|A\|)]^2 \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \tag{38}$$

for all $x,\ y \in H$.

Also, if $A,\ B,\ C \in B(H)$ and $\alpha \in [0,1]$, then

$$\left|\left\langle C^* A \exp(A) Bx, y\right\rangle\right|^2 \leq \exp(2\|A\|) \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle, \tag{39}$$

$$\begin{aligned} &\left|\left\langle C^* A \sinh(A) Bx, y\right\rangle\right|^2 \\ &\leq [\sinh(\|A\|)]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \end{aligned} \tag{40}$$

and

$$\begin{aligned} &\left|\left\langle C^* A \cosh(A) Bx, y\right\rangle\right|^2 \\ &\leq [\cosh(\|A\|)]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \end{aligned} \tag{41}$$

for all x, $y \in H$.

For $\alpha = 1/2$ in (39)-(41) we obtain some simpler inequalities. We omit the details.

3. Norm and Numerical Radius Inequalities

The following vector inequality for positive operators $A \geq 0$, obtained by C. A. McCarthy in [9] is well known,

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \ p \geq 1$$

for $x \in H$, $\|x\| = 1$.

Buzano's inequality [2],

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x, y\rangle|\right] \geq |\langle x, e\rangle \langle e, y\rangle| \tag{42}$$

that holds for any x, y, $e \in H$ with $\|e\| = 1$ will also be used in the sequel.

Our first main result is as follows:

Theorem 3. *Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0, R)$, $\alpha \in [0, 1]$ and A, B, $C \in B(H)$ with $\|A\| < R$, then we have the norm inequality*

$$\|C^* A f(A) B\| \leq f_a(\|A\|) \left\||A|^\alpha B\right\| \left\||A^*|^{1-\alpha} C\right\|. \tag{43}$$

We also have the numerical radius inequalities

$$\omega(C^* A f(A) B) \leq \frac{1}{2} f_a(\|A\|) \left\| \left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right\| \tag{44}$$

and

$$\begin{aligned} &\omega^2(C^* A f(A) B) \\ &\leq \frac{1}{2} f_a^2(\|A\|) \left[\left\||A|^\alpha B\right\|^2 \left\||A^*|^{1-\alpha} C\right\|^2 + \omega\left(\left||A^*|^{1-\alpha} C\right|^2 \left||A|^\alpha B\right|^2\right)\right]. \end{aligned} \tag{45}$$

Proof. We have from (28), by taking the supremum over $\|x\| = \|y\| = 1$, that

$$\|C^* A f(A) B\|^2 = \sup_{\|x\|=\|y\|=1} \left|\left\langle C^* A f(A) Bx, y\right\rangle\right|^2$$

$$\le f_a^2(\|A\|) \sup_{\|x\|=1} \left\langle \left||A|^\alpha B\right|^2 x, x \right\rangle \sup_{\|y\|=1} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y \right\rangle$$
$$= f_a^2(\|A\|) \left\| \left||A|^\alpha B\right|^2 \right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 \right\|$$
$$= f_a^2(\|A\|) \left\||A|^\alpha B\right\|^2 \left\||A^*|^{1-\alpha} C\right\|^2,$$

which gives (43).

From (28) we get, by taking $y = x$, the square root and using the *A-G-mean inequality*, that

$$|\langle C^* A f(A) B x, x\rangle| \tag{46}$$
$$\le f_a(\|A\|) \left\langle \left||A|^\alpha B\right|^2 x, x \right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x \right\rangle^{1/2}$$
$$\le \frac{1}{2} f_a(\|A\|) \left(\left\langle \left||A|^\alpha B\right|^2 x, x \right\rangle + \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x \right\rangle \right)$$
$$= \frac{1}{2} f_a(\|A\|) \left\langle \left(\left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right) x, x \right\rangle$$

for all $x \in H$.

By taking the supremum over $\|x\| = 1$ in (46) we get that

$$\omega(C^* A f(A) B)$$
$$= \sup_{\|x\|=1} |\langle C^* A f(A) B x, x\rangle|$$
$$\le \frac{1}{2} f_a(\|A\|) \sup_{\|x\|=1} \left\langle \left(\left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right) x, x \right\rangle$$
$$= \frac{1}{2} f_a(\|A\|) \left\| \left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right\|,$$

which proves (44).

From (28) for $y = x$ and Buzano's inequality we derive that

$$|\langle C^* A f(A) B x, x\rangle|^2 \tag{47}$$
$$\le f_a^2(\|A\|) \left\langle \left||A|^\alpha B\right|^2 x, x \right\rangle \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x \right\rangle$$
$$\le \frac{1}{2} f_a^2(\|A\|)$$
$$\times \left[\left\| \left||A|^\alpha B\right|^2 x \right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 x \right\| + \left| \left\langle \left||A|^\alpha B\right|^2 x, \left||A^*|^{1-\alpha} C\right|^2 x \right\rangle \right| \right]$$
$$= \frac{1}{2} f_a^2(\|A\|)$$

$$\times \left[\left\| \left| |A|^{\alpha} B \right|^{2} x \right\| \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\| + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right| \right]$$

for all $x \in H, \|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in (47) we get that

$$\begin{aligned}
&\omega^{2} \left(C^{*} A f(A) B \right) \\
&= \sup_{\|x\|=1} \left| \langle C^{*} A f(A) B x, x \rangle \right|^{2} \\
&\leq \frac{1}{2} f_{a}^{2} \left(\|A\| \right) \\
&\times \sup_{\|x\|=1} \left[\left\| \left| |A|^{\alpha} B \right|^{2} x \right\| \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\| + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right| \right] \\
&\leq \frac{1}{2} f_{a}^{2} \left(\|A\| \right) \\
&\times \left[\sup_{\|x\|=1} \left\{ \left\| \left| |A|^{\alpha} B \right|^{2} x \right\| \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\| \right\} \right. \\
&\left. + \sup_{\|x\|=1} \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right| \right] \\
&\leq \frac{1}{2} f_{a}^{2} \left(\|A\| \right) \\
&\times \left[\sup_{\|x\|=1} \left\| \left| |A|^{\alpha} B \right|^{2} x \right\| \sup_{\|x\|=1} \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\| \right. \\
&\left. + \sup_{\|x\|=1} \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right| \right] \\
&= \frac{1}{2} f_{a}^{2} \left(\|A\| \right) \left[\left\| \left| |A|^{\alpha} B \right|^{2} \right\| \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} \right\| + \omega \left(\left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} \right) \right] \\
&= \frac{1}{2} f_{a}^{2} \left(\|A\| \right) \left[\left\| |A|^{\alpha} B \right\|^{2} \left\| |A^{*}|^{1-\alpha} C \right\|^{2} + \omega \left(\left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} \right) \right],
\end{aligned}$$

which proves (45). □

Remark 2. *If we take $\alpha = 1/2$ in Theorem 3, then we get the norm inequality*

$$\left\| C^{*} A f(A) B \right\| \leq f_{a} \left(\|A\| \right) \left\| |A|^{1/2} B \right\| \left\| |A^{*}|^{1/2} C \right\| \tag{48}$$

and the numerical radius inequalities

$$\omega \left(C^{*} A f(A) B \right) \leq \frac{1}{2} f_{a} \left(\|A\| \right) \left\| \left| |A|^{1/2} B \right|^{2} + \left| |A^{*}|^{1/2} C \right|^{2} \right\| \tag{49}$$

and

$$\omega^2 (C^* A f (A) B) \tag{50}$$
$$\leq \frac{1}{2} f_a^2 (\|A\|) \left[\left\| |A|^{1/2} B \right\|^2 \left\| |A^*|^{1/2} C \right\|^2 + \omega \left(\left| |A^*|^{1/2} C \right|^2 \left| |A|^{1/2} B \right|^2 \right) \right].$$

The second main result is as follows:

Theorem 4. *Assume that the conditions of Theorem 3 are satisfied. If $\alpha \in [0,1]$, $r > 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 1$, then*

$$\omega^{2r} (C^* A f (A) B) \leq f_a^{2r} (\|A\|) \left\| \frac{1}{p} \left| |A|^\alpha B \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2rq} \right\|. \tag{51}$$

If $r \geq 1$, then

$$\omega^{2r} (C^* A f (A) B) \leq \frac{1}{2} f_a^{2r} (\|A\|) \left[\| |A|^\alpha B \|^{2r} \left\| |A^*|^{1-\alpha} C \right\|^{2r} \right. \tag{52}$$
$$\left. + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 \right) \right].$$

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then also

$$\omega^{2r} (C^* A f (A) B) \leq \frac{1}{2} f_a^{2r} (\|A\|) \left(\left\| \frac{1}{p} \left| |A|^\alpha B \right|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right\| \right. \tag{53}$$
$$\left. + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 \right) \right).$$

Proof. If we take the power $r > 0$ in (28) written for $y = x$ then we get, by Young and McCarthy inequalities that

$$\begin{aligned}
&|\langle C^* A f (A) B x, x \rangle|^{2r} \\
&\leq f_a^{2r} (\|A\|) \left\langle \left| |A|^\alpha B \right|^2 x, x \right\rangle^r \left\langle \left| |A^*|^{1-\alpha} C \right|^2 x, x \right\rangle^r \\
&\leq f_a^{2r} (\|A\|) \left[\frac{1}{p} \left\langle \left| |A|^\alpha B \right|^2 x, x \right\rangle^{rp} + \frac{1}{q} \left\langle \left| |A^*|^{1-\alpha} C \right|^2 x, x \right\rangle^{rq} \right] \\
&\leq f_a^{2r} (\|A\|) \left[\frac{1}{p} \left\langle \left| |A|^\alpha B \right|^{2rp} x, x \right\rangle + \frac{1}{q} \left\langle \left| |A^*|^{1-\alpha} C \right|^{2rq} x, x \right\rangle \right] \\
&= f_a^{2r} (\|A\|) \left[\left\langle \frac{1}{p} \left| |A|^\alpha B \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2rq} x, x \right\rangle \right]
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$\begin{aligned}
&\omega^{2r} (C^* A f (A) B) \\
&= \sup_{\|x\|=1} |\langle C^* A f (A) B x, x \rangle|^{2r}
\end{aligned}$$

$$\leq f_a^{2r}(\|A\|) \sup_{\|x\|=1} \left[\left\langle \left(\frac{1}{p} \left| |A|^\alpha B \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2rq} \right) x, x \right\rangle \right]$$
$$= f_a^{2r}(\|A\|) \left\| \frac{1}{p} \left| |A|^\alpha B \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2rq} \right\|,$$

which proves (51).

If we take the power $r \geq 1$ in (47) and by using the convexity of the power function, we get

$$\begin{aligned} &|\langle C^* A f(A) Bx, x\rangle|^{2r} \\ &= f_a^{2r}(\|A\|) \\ &\times \left[\frac{\left\| \left| |A|^\alpha B \right|^2 x \right\| \left\| \left| |A^*|^{1-\alpha} C \right|^2 x \right\| + \left| \left\langle \left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 x, x \right\rangle \right|}{2} \right]^r \\ &\leq f_a^{2r}(\|A\|) \\ &\times \frac{\left\| \left| |A|^\alpha B \right|^2 x \right\|^r \left\| \left| |A^*|^{1-\alpha} C \right|^2 x \right\|^r + \left| \left\langle \left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 x, x \right\rangle \right|^r}{2} \end{aligned} \tag{54}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$\begin{aligned} &\omega^{2r}(C^* A f(A) B) \\ &\leq f_a^{2r}(\|A\|) \\ &\times \frac{\left\| \left| |A|^\alpha B \right|^2 \right\|^r \left\| \left| |A^*|^{1-\alpha} C \right|^2 \right\|^r + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 \right)}{2} \\ &= f_a^{2r}(\|A\|) \\ &\times \frac{\left\| |A|^\alpha B \right\|^{2r} \left\| |A^*|^{1-\alpha} C \right\|^{2r} + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 \right)}{2}, \end{aligned}$$

which proves (52).

Also, observe that

$$\begin{aligned} &\left\| \left| |A|^\alpha B \right|^2 x \right\|^r \left\| \left| |A^*|^{1-\alpha} C \right|^2 x \right\|^r \\ &\leq \frac{1}{p} \left\| \left| |A|^\alpha B \right|^2 x \right\|^{pr} + \frac{1}{q} \left\| \left| |A^*|^{1-\alpha} C \right|^2 x \right\|^{qr} \\ &= \frac{1}{p} \left\| \left| |A|^\alpha B \right|^2 x \right\|^{2\frac{pr}{2}} + \frac{1}{q} \left\| \left| |A^*|^{1-\alpha} C \right|^2 x \right\|^{2\frac{qr}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p}\left\langle \left|\left|A\right|^{\alpha} B\right|^{4} x, x\right\rangle^{\frac{pr}{2}} + \frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha} C\right|^{4} x, x\right\rangle^{\frac{qr}{2}} \\
&\leq \frac{1}{p}\left\langle \left|\left|A\right|^{\alpha} B\right|^{2pr} x, x\right\rangle + \frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha} C\right|^{2qr} x, x\right\rangle \\
&= \left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha} B\right|^{2pr} + \frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha} C\right|^{2qr}\right) x, x\right\rangle,
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$. Then

$$
\begin{aligned}
&\frac{\left\|\left|\left|A\right|^{\alpha} B\right|^{2} x\right\|^{r} \left\|\left|\left|A^{*}\right|^{1-\alpha} C\right|^{2} x\right\|^{r} + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha} C\right|^{2} \left|\left|A\right|^{\alpha} B\right|^{2} x, x\right\rangle\right|^{r}}{2} \\
&\leq \frac{1}{2}\left[\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha} B\right|^{2pr} + \frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha} C\right|^{2qr}\right) x, x\right\rangle \right. \\
&\left. + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha} C\right|^{2} \left|\left|A\right|^{\alpha} B\right|^{2} x, x\right\rangle\right|^{r}\right]
\end{aligned}
$$

and by (54)

$$
\begin{aligned}
&\left|\left\langle C^{*} A f\left(A\right) B x, x\right\rangle\right|^{2r} \\
&\leq \frac{1}{2} f_{a}^{2r}\left(\|A\|\right) \left[\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha} B\right|^{2pr} + \frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha} C\right|^{2qr}\right) x, x\right\rangle \right. \\
&\left. + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha} C\right|^{2} \left|\left|A\right|^{\alpha} B\right|^{2} x, x\right\rangle\right|^{r}\right]
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, we derive (53). □

Remark 3. *If we take $r = 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ in (51), then we obtain*

$$
\omega^{2}\left(C^{*} A f\left(A\right) B\right) \leq f_{a}^{2}\left(\|A\|\right) \left\|\frac{1}{p}\left|\left|A\right|^{\alpha} B\right|^{2p} + \frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha} C\right|^{2q}\right\|, \tag{55}
$$

which for $p = q = 2$ gives

$$
\omega^{2}\left(C^{*} A f\left(A\right) B\right) \leq \frac{1}{2} f_{a}^{2}\left(\|A\|\right) \left\|\left|\left|A\right|^{\alpha} B\right|^{4} + \left|\left|A^{*}\right|^{1-\alpha} C\right|^{4}\right\|. \tag{56}
$$

If we take $r = 1$ and $p = q = 2$ in (53), then we get

$$
\begin{aligned}
\omega^{2}\left(C^{*} A f\left(A\right) B\right) \leq \frac{1}{2} f_{a}^{2}\left(\|A\|\right) &\left(\frac{1}{2}\left\|\left|\left|A\right|^{\alpha} B\right|^{4} + \left|\left|A^{*}\right|^{1-\alpha} C\right|^{4}\right\| \right. \\
&\left. + \omega\left(\left|\left|A^{*}\right|^{1-\alpha} C\right|^{2} \left|\left|A\right|^{\alpha} B\right|^{2}\right)\right).
\end{aligned} \tag{57}
$$

If we take $r=2$ *and* $p, q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *in* (53)*, then we get*

$$\omega^4\left(C^*Af(A)B\right)\leq\frac{1}{2}f_a^4\left(\|A\|\right)\left(\left\|\frac{1}{p}\left||A|^{\alpha}B\right|^{4p}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{4q}\right\|+\omega^2\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left||A|^{\alpha}B\right|^2\right)\right). \tag{58}$$

We also have:

Theorem 5. *With the assumptions of Theorem 3, we have for* $r\geq 1$, $\lambda\in[0,1]$ *that*

$$\omega^2\left(C^*Af(A)B\right)\leq f_a^2\left(\|A\|\right)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r}\times\left\|\left|A\right|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^*\right|^{1-\alpha}C\right\|^{2(1-\lambda)} \tag{59}$$

for all $\alpha\in[0,1]$.

Also, we have

$$\omega^2\left(C^*Af(A)B\right)\leq f_a^2\left(\|A\|\right)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r}\times\left\|\lambda\left||A|^{\alpha}B\right|^{2r}+(1-\lambda)\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r} \tag{60}$$

for all $\alpha\in[0,1]$ *and* $r\geq 1$.

Proof. From the first part of (47) we have

$$\begin{aligned}
&\left|\left\langle C^*Af(A)Bx,x\right\rangle\right|^2\\
&\leq f_a^2\left(\|A\|\right)\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle\\
&=f_a^2\left(\|A\|\right)\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{1-\lambda}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{\lambda}\\
&\times\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{\alpha}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{1-\lambda}\\
&\leq f_a^2\left(\|A\|\right)\left[(1-\lambda)\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle+\lambda\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle\right]\\
&\times\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{\lambda}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{1-\lambda}
\end{aligned}$$

for all $x\in H$, $\|x\|=1$.

If we take the power $r\geq 1$, then we get by the convexity of power r that

$$\left|\left\langle C^*Af(A)Bx,x\right\rangle\right|^{2r} \tag{61}$$

$$\leq f_a^{2r}(\|A\|)\left[(1-\lambda)\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle+\lambda\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle\right]^r$$
$$\times\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^{r\lambda}\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^{r(1-\lambda)}$$
$$\leq f_a^{2r}(\|A\|)\left[(1-\lambda)\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^r+\lambda\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^r\right]$$
$$\times\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^{r\lambda}\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^{r(1-\lambda)}$$

for all $x\in H$, $\|x\|=1$.

If we use McCarthy inequality for power $r\geq 1$, then we get

$$(1-\lambda)\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^r+\lambda\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^r$$
$$\leq(1-\lambda)\left\langle \left||A|^\alpha B\right|^{2r} x,x\right\rangle+\lambda\left\langle x,\left||A^*|^{1-\alpha}C\right|^{2r} x\right\rangle$$
$$=\left\langle\left[(1-\lambda)\left||A|^\alpha B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle$$

and by (61)

$$\begin{aligned}&\left|\langle C^*Af(A)Bx,x\rangle\right|^{2r}\\&\leq f_a^{2r}(\|A\|)\left[\left\langle\left[(1-\lambda)\left||A|^\alpha B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\&\times\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^{r\lambda}\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^{r(1-\lambda)}\end{aligned}\tag{62}$$

for all $x\in H$, $\|x\|=1$.

If we take the supremum over $\|x\|=1$, then we get

$$\omega^{2r}(C^*Af(A)B)$$
$$=\sup_{\|x\|=1}\left|\langle C^*Af(A)Bx,x\rangle\right|^{2r}$$
$$\leq f_a^{2r}(\|A\|)\sup_{\|x\|=1}\left[\left\langle\left[(1-\lambda)\left||A|^\alpha B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]$$
$$\times\sup_{\|x\|=1}\left\langle \left||A|^\alpha B\right|^2 x,x\right\rangle^{r\lambda}\sup_{\|x\|=1}\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^{r(1-\lambda)}$$
$$=f_a^{2r}(\|A\|)\left\|(1-\lambda)\left||A|^\alpha B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right\|$$
$$\times\left\||A|^\alpha B\right\|^{2r\lambda}\left\||A^*|^{1-\alpha}C\right\|^{2r(1-\lambda)},$$

which gives (59).

We also have

$$
\begin{aligned}
&\left|\left\langle C^{*}Af\left(A\right)Bx,x\right\rangle\right|^{2r}\\
&\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left[\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\
&\times\left[\left\langle\left[\lambda\left|\left|A\right|^{\alpha}B\right|^{2r}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]
\end{aligned}
$$

for all $x\in H$, $\|x\|=1$, which proves (60). □

Remark 4. *If we take $r=1$ in Theorem 5, then we get*

$$
\begin{aligned}
\omega^{2}\left(C^{*}Af\left(A\right)B\right)\leq f_{a}^{2}\left(\left\|A\right\|\right)&\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\| \\
&\times\left\|\left|A\right|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2(1-\lambda)}
\end{aligned}
\tag{63}
$$

and

$$
\begin{aligned}
\omega^{2}\left(C^{*}Af\left(A\right)B\right)\leq f_{a}^{2}\left(\left\|A\right\|\right)&\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\| \\
&\times\left\|\lambda\left|\left|A\right|^{\alpha}B\right|^{2}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|
\end{aligned}
\tag{64}
$$

for all $\alpha,\ \lambda\in[0,1]$.

If we take $\lambda=1/2$ in (63), then we obtain

$$
\begin{aligned}
&\omega^{2}\left(C^{*}Af\left(A\right)B\right)\\
&\leq\frac{1}{2}f_{a}^{2}\left(\left\|A\right\|\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\left\|\left|A\right|^{\alpha}B\right\|\left\|\left|A^{*}\right|^{1-\alpha}C\right\|
\end{aligned}
\tag{65}
$$

If we take $r=2$ in Theorem 5, then we get

$$
\begin{aligned}
\omega^{2}\left(C^{*}Af\left(A\right)B\right)\leq f_{a}^{2}\left(\left\|A\right\|\right)&\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{4}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\|^{1/2} \\
&\times\left\|\left|A\right|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2(1-\lambda)}
\end{aligned}
\tag{66}
$$

and

$$
\begin{aligned}
\omega^{2}\left(C^{*}Af\left(A\right)B\right)\leq f_{a}^{2}\left(\left\|A\right\|\right)&\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{4}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\|^{1/2} \\
&\times\left\|\lambda\left|\left|A\right|^{\alpha}B\right|^{4}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\|^{1/2}
\end{aligned}
\tag{67}
$$

for all $\alpha,\ \lambda\in[0,1]$.

If we take $\lambda = 1/2$ *in* (66)*, then we obtain*

$$\omega^2 (C^* A f(A) B) \tag{68}$$
$$\leq \frac{\sqrt{2}}{2} f_a^2 (\|A\|) \left\| \left|\left|A\right|^\alpha B\right|^4 + \left|\left|A^*\right|^{1-\alpha} C\right|^4 \right\|^{1/2} \left\|\left|A\right|^\alpha B\right\| \left\|\left|A^*\right|^{1-\alpha} C\right\|.$$

4. Inequalities for Trace of Operators

We have the following result for trace of operators:

Theorem 6. *Let* $r \geq 1/2$, p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* pr, $qr \geq 1$. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is convergent on* $D(0, R)$ *and* A, B, $C \in B(H)$ *with* $\|A\| < R$. *If* $|A|^\alpha B \in \mathcal{B}_{2pr}(H)$ *and* $|A^*|^{1-\alpha} C \in \mathcal{B}_{2qr}(H)$ *for* $\alpha \in [0, 1]$, *then* $C^* A f(A) B \in \mathcal{B}_{2r}(H)$ *and*

$$\|C^* A f(A) B\|_{2r} \leq f_a (\|A\|) \left\||A|^\alpha B\right\|_{2pr} \left\|\left|A^*\right|^{1-\alpha} C\right\|_{2qr}. \tag{69}$$

In particular,

$$\|C^* A f(A) B\|_{2r} \leq f_a (\|A\|) \left\||A|^{1/2} B\right\|_{2pr} \left\|\left|A^*\right|^{1/2} C\right\|_{2qr} \tag{70}$$

for $|A|^{1/2} B \in \mathcal{B}_{2pr}(H)$ *and* $|A^*|^{1/2} C \in \mathcal{B}_{2qr}(H)$.

Proof. If we take in (28) the power $r > 0$ and $x = e_i$, $y = f_i$ where $\mathcal{E} = \{e_i\}_{i \in I}$ and $\mathcal{F} = \{f_i\}_{i \in I}$ are orthonormal basis and sum, then we get

$$\sum_{i \in I} \left|\langle C^* A f(A) B e_i, f_i \rangle\right|^{2r} \tag{71}$$
$$\leq f_a^{2r} (\|A\|) \sum_{i \in I} \left\langle \left|\left|A\right|^\alpha B\right|^2 e_i, e_i \right\rangle^r \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^r.$$

If we use the Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we get

$$\sum_{i \in I} \left\langle \left|\left|A\right|^\alpha B\right|^2 e_i, e_i \right\rangle^r \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^r \tag{72}$$
$$\leq \left(\sum_{i \in I} \left\langle \left|\left|A\right|^\alpha B\right|^2 e_i, e_i \right\rangle^{pr} \right)^{1/p} \left(\sum_{i \in I} \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^{qr} \right)^{1/q}$$

By the McCarthy inequality for pr, $qr \geq 1$, we have

$$\sum_{i \in I} \left\langle \left|\left|A\right|^\alpha B\right|^2 e_i, e_i \right\rangle^{pr} \leq \sum_{i \in I} \left\langle \left|\left|A\right|^\alpha B\right|^{2pr} e_i, e_i \right\rangle$$

and

$$\sum_{i \in I} \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^{qr} \leq \sum_{i \in I} \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^{2qr} f_i, f_i \right\rangle,$$

therefore

$$\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}$$
$$\leq\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}e_{i},e_{i}\right\rangle\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}$$
$$=\left(\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2pr}\right)^{1/p}\left(\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2qr}\right)^{1/q}=\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}.$$

By (71) and (72) we derive

$$\sum_{i\in I}\left|\left\langle C^{*}Af\left(A\right)Be_{i},f_{i}\right\rangle\right|^{2r}\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}. \tag{73}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (30), then by (21) we get

$$\left\|C^{*}Af\left(A\right)B\right\|_{2r}^{2r}\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}$$

and the inequality (69) is obtained. □

Remark 5. *If we take $r=1/2$ and $p=q=2$, then by (69) we get*

$$\left\|C^{*}Af\left(A\right)B\right\|_{1}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2} \tag{74}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2}\left(H\right)$ for $\alpha\in\left[0,1\right].$

Also, if $r=1$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (69) we get

$$\left\|C^{*}Af\left(A\right)B\right\|_{2}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2p}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2q} \tag{75}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2q}\left(H\right)$ for $\alpha\in\left[0,1\right].$

We also have:

Theorem 7. *Let $r\geq 1/2,\ p,\ q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Assume that the power series with complex coefficients $f\left(z\right):=\sum_{k=0}^{\infty}a_{k}z^{k}$ is convergent on $D(0,R)$ and $A,\ B,\ C\in B\left(H\right)$ with $\left\|A\right\|<R$. If $\left|A\right|^{\alpha}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2q}\left(H\right)$ for $\alpha\in\left[0,1\right],$ then $C^{*}Af\left(A\right)B\in\mathcal{B}_{2r}\left(H\right)$ and*

$$\left\|C^{*}Af\left(A\right)B\right\|_{2r}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2p}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2q}. \tag{76}$$

In particular,

$$\left\|C^{*}Af\left(A\right)B\right\|_{2r}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{1/2}B\right\|_{2p}\left\|\left|A^{*}\right|^{1/2}C\right\|_{2q} \tag{77}$$

for $\left|A\right|^{1/2}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1/2}C\in\mathcal{B}_{2q}\left(H\right).$

Proof. Assume that $\mathcal{E}=\{e_i\}_{i\in I}$ and $\mathcal{F}=\{f_i\}_{i\in I}$ are orthonormal basis in H. Observe that we have $\frac{1}{\frac{p}{r}}+\frac{1}{\frac{q}{r}}=1$ and by Hölder's inequality for $\frac{p}{r}$ and $\frac{q}{r}$ we have

$$\begin{aligned}&\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{r} \qquad (78)\\ &=\sum_{i\in I}\left[\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{p}\right]^{\frac{r}{p}}\left[\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{q}\right]^{\frac{r}{q}}\\ &\le\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{p}\right)^{r/p}\left(\sum_{i\in I}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{q}\right)^{r/q}.\end{aligned}$$

By McCarthy inequality for $p,\ q>1$ we get

$$\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{p}\le\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2p}e_i,e_i\right\rangle$$

and

$$\sum_{i\in I}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{q}\le\sum_{i\in I}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{2q}f_i,f_i\right\rangle$$

and by (78)

$$\begin{aligned}&\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{r} \qquad (79)\\ &\le\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2p}e_i,e_i\right\rangle\right)^{r/p}\left(\sum_{i\in I}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{2q}f_i,f_i\right\rangle\right)^{r/q}\\ &=\left\|\left|A\right|^{\alpha}B\right\|_{2p}^{2r}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{2q}^{2r}.\end{aligned}$$

By (71) and (79) we get

$$\sum_{i\in I}\left|\left\langle C^*Af\left(A\right)Be_i,f_i\right\rangle\right|^{2r}\le f_a^{2r}\left(\|A\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2p}^{2r}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{2q}^{2r}. \qquad (80)$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (80) we get

$$\left\|C^*Af\left(A\right)B\right\|_{2r}^{2r}\le f_a^{2r}\left(\|A\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2p}^{2r}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{2q}^{2r}$$

and the inequality (76) is thus proved. □

Remark 6. *If we take $p=q=2r=s\ge 1$, then by (76) we get*

$$\left\|C^*Af\left(A\right)B\right\|_{s}\le f_a\left(\|A\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2s}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{2s} \qquad (81)$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2s}\left(H\right)$ and $\left|A^*\right|^{1-\alpha}C\in\mathcal{B}_{2s}\left(H\right)$ for $\alpha\in[0,1]$.

For $\alpha = 1/2$ we have

$$\|C^*Af(A)B\|_s \le f_a(\|A\|) \left\| |A|^{1/2} B \right\|_{2s} \left\| |A^*|^{1/2} C \right\|_{2s} \tag{82}$$

provided that $|A|^{1/2} B \in \mathcal{B}_{2s}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2s}(H)$.

If $r = 2$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, then

$$\|C^*Af(A)B\|_4 \le f_a(\|A\|) \||A|^{\alpha} B\|_{2p} \left\| |A^*|^{1-\alpha} C \right\|_{2q} \tag{83}$$

provided that $|A|^{\alpha} B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$ for $\alpha \in [0,1]$.

In particular,

$$\|C^*Af(A)B\|_4 \le f_a(\|A\|) \left\| |A|^{1/2} B \right\|_{2p} \left\| |A^*|^{1/2} C \right\|_{2q} \tag{84}$$

for $|A|^{1/2} B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2q}(H)$.

Theorem 8. *Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0,R)$, $A,\ B,\ C \in B(H)$ with $\|A\| < R$.*

If $r \ge 1/2$, $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $pr,\ qr \ge 1$ and $||A|^{\alpha} B|^{2pr}$, $\left| |A^*|^{1-\alpha} C \right|^{2qr} \in \mathcal{B}_1(H)$, then $C^*Af(A)B \in \mathcal{B}_{2r}(H)$ and

$$\omega_{2r}^{2r}(C^*Af(A)B) \le f_a^{2r}(\|A\|) \operatorname{tr}\left(\frac{1}{p} ||A|^{\alpha} B|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right). \tag{85}$$

If $r \ge 1$ and $|A|^{\alpha} B, |A^*|^{1-\alpha} C \in \mathcal{B}_{4r}(H)$, then $C^*Af(A)B \in \mathcal{B}_{2r}(H)$ and

$$\begin{aligned} &\omega_{2r}^{2r}(C^*Af(A)B) \\ &\le \frac{1}{2} f_a^{2r}(\|A\|) \left(\||A|^{\alpha} B\|_{4r}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{4r}^{2r} + \omega_r^r \left(\left| |A^*|^{1-\alpha} C \right|^2 ||A|^{\alpha} B|^2 \right) \right) \\ &\le \frac{1}{2} f_a^{2r}(\|A\|) \left(\||A|^{\alpha} B\|_{4r}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{4r}^{2r} + \left\| \left| |A^*|^{1-\alpha} C \right|^2 ||A|^{\alpha} B|^2 \right\|_r^r \right). \end{aligned} \tag{86}$$

If $r \ge 1$, $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $pr, qr \ge 2$, then

$$\begin{aligned} \omega_{2r}^{2r}(C^*Af(A)B) &\le \frac{1}{2} f_a^{2r}(\|A\|) \left[\operatorname{tr}\left(\frac{1}{p} ||A|^{\alpha} B|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right) \right. \\ &\quad \left. + \omega_r^r \left(\left| |A^*|^{1-\alpha} C \right|^2 ||A|^{\alpha} B|^2 \right) \right] \\ &\le \frac{1}{2} f_a^{2r}(\|A\|) \left[\operatorname{tr}\left(\frac{1}{p} ||A|^{\alpha} B|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right) \right. \\ &\quad \left. + \left\| \left| |A^*|^{1-\alpha} C \right|^2 ||A|^{\alpha} B|^2 \right\|_r^r \right]. \end{aligned} \tag{87}$$

Proof. From (28) for $y = x$ we have that

$$\left|\left\langle C^{*}Af\left(A\right)Bx,x\right\rangle\right|^{2}\leq f_{a}^{2}\left(\left\|A\right\|\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle \tag{88}$$

for $x\in H$ with $\|x\|=1$.

If we take the power $r>0$, we get, by Young and McCarthy inequalities, that

$$\begin{aligned}
&\left|\left\langle C^{*}Af\left(A\right)Bx,x\right\rangle\right|^{2r}\\
&\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{r}\\
&\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{pr}+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{qr}\right]\\
&\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\right]\\
&=f_{a}^{2r}\left(\left\|A\right\|\right)\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

If $\mathcal{E}=\{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x=e_i$ and summing over $i\in I$ we get

$$\begin{aligned}
&\left\|C^{*}Af\left(A\right)B\right\|_{\mathcal{E},2r}^{2r}\\
&=\sum_{i\in I}\left|\left\langle C^{*}Af\left(A\right)Be_{i},e_{i}\right\rangle\right|^{2r}\\
&\leq f_{a}^{2r}\left(\left\|A\right\|\right)\sum_{i\in I}\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)e_{i},e_{i}\right\rangle\\
&=f_{a}^{2r}\left(\left\|A\right\|\right)\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right),
\end{aligned}$$

which, by taking the supremum over $\mathcal{E}$, proves (85).

By Buzano's inequality we have

$$\begin{aligned}
&\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle\\
&\leq\frac{1}{2}\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle\right|\right]\\
&=\frac{1}{2}\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right]
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

If we take the power $r\geq 1$ and use the convexity of power function, then we get

$$\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r}$$

$$
\begin{aligned}
&\leq \left[\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle \right|}{2}\right]^{r} \\
&\leq \frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r}+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle \right|^{r}}{2} \\
&= \frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{2\frac{r}{2}}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{2\frac{r}{2}}+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle \right|^{r}}{2} \\
&= \frac{\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{r}{2}}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{r}{2}}+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle \right|^{r}}{2}
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$
\begin{aligned}
&\left\|C^{*}Af\left(A\right)B\right\|_{\mathcal{E},2r}^{2r} \qquad (89)\\
&= \sum_{i\in I}\left|\left\langle C^{*}Af\left(A\right)Be_{i},e_{i}\right\rangle\right|^{2r} \\
&\leq f_{a}^{2r}\left(\|A\|\right)\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\left\langle e_{i},\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i}\right\rangle^{r} \\
&\leq \frac{1}{2}f_{a}^{2r}\left(\|A\|\right)\left[\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}}\right. \\
&\left.+\sum_{i\in I}\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle\right|^{r}\right]
\end{aligned}
$$

Using Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}} \\
&\leq \left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}e_{i},e_{i}\right\rangle^{r}\right)^{1/2}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}e_{i},e_{i}\right\rangle^{r}\right)^{1/2} \\
&\leq \left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4r}e_{i},e_{i}\right\rangle\right)^{1/2}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4r}e_{i},e_{i}\right\rangle\right)^{1/2} \\
&= \left\|\left|A\right|^{\alpha}B\right\|_{4r}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{4r}^{2r},
\end{aligned}
$$

where for the last inequality we used McCarthy's result for $r \geq 1$. This proves (86).

Further, if we use Young's inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$ab\le \frac{1}{p}a^{p}+\frac{1}{q}b^{q},\ \ a,b\ge 0,$$

then we get

$$\begin{aligned}\left\| \left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\| \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r} &\le \frac{1}{p}\left\| \left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{pr}+\frac{1}{q}\left\| \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{qr}\\ &=\frac{1}{p}\left\| \left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\| \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{2\frac{qr}{2}}\\ &=\frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{qr}{2}}\\ &\le \frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\\ &=\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

Therefore

$$\begin{aligned}\left\|C^{*}Af\left(A\right)B\right\|_{\mathcal{E},2r}^{2r} &=\sum_{i\in I}\left|\left\langle C^{*}Af\left(A\right)Be_{i},e_{i}\right\rangle\right|^{2r}\\ &\le f_{a}^{2r}\left(\|A\|\right)\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\left\langle e_{i},\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i}\right\rangle^{r}\\ &\le \frac{1}{2}f_{a}^{2r}\left(\|A\|\right)\left[\sum_{i\in I}\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)e_{i},e_{i}\right\rangle\right.\\ &\left.+\sum_{i\in I}\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle\right|^{r}\right]\\ &=\frac{1}{2}f_{a}^{2r}\left(\|A\|\right)\left[\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)\right.\\ &\left.+\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right\|_{\mathcal{E},r}^{r}\right],\end{aligned}$$

which proves, by taking the supremum over $\mathcal{E}$, the desired inequality (87). □

Remark 7. *Let $\alpha\in[0,1]$. If $r=1/2$, p, $q=2$ and $\left|\left|A\right|^{\alpha}B\right|^{2}$, $\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\in\mathcal{B}_{1}\left(H\right)$, then $C^{*}Af\left(A\right)B\in\mathcal{B}_{1}\left(H\right)$ and by (85) we get*

$$\omega_{1}\left(C^{*}Af\left(A\right)B\right)\le\frac{1}{2}f_{a}\left(\|A\|\right)\operatorname{tr}\left(\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right). \tag{90}$$

If $r=1$ *and* $p,\, q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, then by* (85) *we obtain*

$$\omega_2^2\left(C^*Af\left(A\right)B\right)\leq f_a^2\left(\|A\|\right)\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2p}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\right), \tag{91}$$

provided that $\left|\left|A\right|^{\alpha}B\right|^{2p},\ \left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_1\left(H\right).$

If we take $r=1$ *in* (86)*, then we get*

$$\begin{aligned}&\omega_2^2\left(C^*Af\left(A\right)B\right) &(92)\\ &\leq\frac{1}{2}f_a^2\left(\|A\|\right)\left(\left\|\left|A\right|^{\alpha}B\right\|_4^2\left\|\left|A^*\right|^{1-\alpha}C\right\|_4^2+\omega_1\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2\right)\right)\\ &\leq\frac{1}{2}f_a^2\left(\|A\|\right)\left(\left\|\left|A\right|^{\alpha}B\right\|_4^2\left\|\left|A^*\right|^{1-\alpha}C\right\|_4^2+\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2\right\|_1\right),\end{aligned}$$

provided that $\left|A\right|^{\alpha}B,\ \left|A^*\right|^{1-\alpha}C\in\mathcal{B}_4\left(H\right).$

If $r=1$ *and* $p=q=2$ *in* (87)*, then we get for* $\left|\left|A\right|^{\alpha}B\right|^{2p},\ \left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_1\left(H\right)$ *that*

$$\begin{aligned}\omega_2^2\left(C^*Af\left(A\right)B\right)&\leq\frac{1}{4}f_a^2\left(\|A\|\right)\left[\operatorname{tr}\left(\left|\left|A\right|^{\alpha}B\right|^{2p}+\left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\right)\right. &(93)\\ &\quad\left.+\frac{1}{2}f_a^2\left(\|A\|\right)\omega_1\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2\right)\right.\\ &\leq\frac{1}{4}f_a^2\left(\|A\|\right)\operatorname{tr}\left(\left|\left|A\right|^{\alpha}B\right|^{2p}+\left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\right)\\ &\quad+\frac{1}{2}f_a^2\left(\|A\|\right)\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2\right\|_1.\end{aligned}$$

We also have:

Theorem 9. *With the assumptions of Theorem* 8*, we have for* $r\geq 1,\ \lambda\in[0,1]$ *that*

$$\begin{aligned}\omega_{2r}^{2r}\left(C^*Af\left(A\right)B\right)&\leq f_a^{2r}\left(\|A\|\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\| &(94)\\ &\quad\times\left\|\left|A\right|^{\alpha}B\right\|_{2r}^{2r\lambda}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},\end{aligned}$$

provided that $\left|A\right|^{\alpha}B,\ \left|A^*\right|^{1-\alpha}C\in\mathcal{B}_{2r}\left(H\right).$

In particular,

$$\begin{aligned}\omega_{2r}^{2r}\left(C^*Af\left(A\right)B\right)&\leq\frac{1}{2}f_a^{2r}\left(\|A\|\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{2r}+\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\| &(95)\\ &\quad\times\left\|\left|A\right|^{\alpha}B\right\|_{2r}^{r}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{2r}^{r}.\end{aligned}$$

Proof. If $\mathcal{E}=\{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x=e_i$ in (62) and summing over $i\in I$ we get

$$\begin{aligned}&\sum_{i\in I}\left|\langle C^*Af(A)Be_i,e_i\rangle\right|^{2r} \qquad (96)\\ &\le f_a^{2r}(\|A\|)\sum_{i\in I}\left[\left\langle\left[(1-\lambda)\left||A|^\alpha B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right]e_i,e_i\right\rangle\right]\\ &\times\left\langle\left||A|^\alpha B\right|^2e_i,e_i\right\rangle^{r\lambda}\left\langle\left||A^*|^{1-\alpha}C\right|^2e_i,e_i\right\rangle^{r(1-\lambda)}\\ &\le f_a^{2r}(\|A\|)\left\|(1-\lambda)\left||A|^\alpha B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right\|\\ &\times\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^2e_i,e_i\right\rangle^{r\lambda}\left\langle\left||A^*|^{1-\alpha}C\right|^2e_i,e_i\right\rangle^{r(1-\lambda)}.\end{aligned}$$

If we use Hölder's inequality for $p=\frac{1}{\lambda}$, $q=\frac{1}{1-\lambda}$, then we have

$$\begin{aligned}&\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^2e_i,e_i\right\rangle^{r\lambda}\left\langle\left||A^*|^{1-\alpha}C\right|^2e_i,e_i\right\rangle^{r(1-\lambda)}\\ &\le\left(\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^2e_i,e_i\right\rangle^{r}\right)^{\lambda}\left(\sum_{i\in I}\left\langle\left||A^*|^{1-\alpha}C\right|^2e_i,e_i\right\rangle^{r}\right)^{1-\lambda}\\ &\le\left(\sum_{i\in I}\left\langle\left||A|^\alpha B\right|^{2r}e_i,e_i\right\rangle\right)^{\lambda}\left(\sum_{i\in I}\left\langle\left||A^*|^{1-\alpha}C\right|^{2r}e_i,e_i\right\rangle\right)^{1-\lambda}\\ &=\left\||A|^\alpha B\right\|_{2r}^{2r\lambda}\left\||A^*|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},\end{aligned}$$

which proves (94). □

Remark 8. *If we take $r=1$ in Theorem 9, then we get for $\alpha\in[0,1]$ that*

$$\begin{aligned}\omega_2^2(C^*Af(A)B)\le f_a^2(\|A\|)&\left\|(1-\lambda)\left||A|^\alpha B\right|^2+\lambda\left||A^*|^{1-\alpha}C\right|^2\right\| \qquad (97)\\ &\times\left\||A|^\alpha B\right\|_2^{2\lambda}\left\||A^*|^{1-\alpha}C\right\|_2^{2(1-\lambda)},\end{aligned}$$

provided that $|A|^\alpha B,\ |A^*|^{1-\alpha}C\in\mathcal{B}_2(H)$.

In particular,

$$\begin{aligned}\omega_2^2(C^*Af(A)B)\le\frac{1}{2}f_a^2(\|A\|)&\left\|\left||A|^\alpha B\right|^2+\left||A^*|^{1-\alpha}C\right|^2\right\| \qquad (98)\\ &\times\left\||A|^\alpha B\right\|_2\left\||A^*|^{1-\alpha}C\right\|_2.\end{aligned}$$

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GENERAL LOGARITHMIC CONTROL MODULO AND TAUBERIAN REMAINDER THEOREMS

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Abstract. Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_n \to \infty$. A sequence (ξ_n) is called λ-bounded if

$$\lambda_n(\xi_n - \alpha) = O(1)$$

with the limit $\lim_{n\to\infty} \xi_n = \alpha$. In this work, we obtain several Tauberian remainder theorems on λ-bounded sequences for the logarithmic summability method with help of general logarithmic control modulo of the oscillatory behavior. Tauber conditions in our main results are on the generator sequence and the general logarithmic control modulo.

1. Introduction

Let $\xi = (\xi_n)$ be a sequence of real numbers. Throughout this work, the notation of $(\xi_n) = O(1)$ means that the sequence of (ξ_n) is bounded for large enough n. The $(C,1)$ mean of (ξ_n) is defined by $\sigma_n^{(1)}(\xi) = \dfrac{1}{n+1}\sum_{k=0}^{n} \xi_k$ and the logarithmic mean of (ξ_n) is defined by $\ell_n^{(1)}(\xi) = \dfrac{1}{\gamma_n}\sum_{k=0}^{n} \dfrac{\xi_k}{k+1}$, where $\gamma_n = \sum_{k=0}^{n} \dfrac{1}{k+1} \sim \log n$, where for two sequences (u_n) and (v_n) of positive numbers, we write $u_n \sim v_n$ if $\lim_{n\to\infty} \dfrac{u_n}{v_n} = 1$. A sequence (ξ_n) is said to be $(C,1)$ summable to a finite number α if the limit

$$\lim_{n\to\infty} \sigma_n^{(1)}(\xi) = \alpha \tag{1}$$

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exists and we say that a sequence (ξ_n) is logarithmic summable to a finite number α, if

$$\lim_{n\to\infty} \ell_n^{(1)}(\xi) = \alpha \tag{2}$$

[1]. It is well known that if a sequence (ξ_n) is convergent, then (1) and (2) are exist. In other words, these two methods are regular methods. Also the existence of (1) implies the existence of (2). However the converse implications are not always true. For example the sequence $(\xi_n) = (-1)^n(2n+1)$ is neither ordinary convergent nor $(C, 1)$ convergent. But it is logarithmic convergent to 0.

For a sequence (ξ_n), we have the following identity:

$$\xi_n - \ell_n^{(1)}(\xi) = v_n^{(0)}(\Delta\xi), \tag{3}$$

where $v_n^{(0)}(\Delta\xi) = \dfrac{1}{\gamma_n}\sum_{k=1}^{n}\gamma_{k-1}(\Delta\xi_k)$. The identity (3) is called the logarithmic Kronecker identity and the sequence $(v_n^{(0)}(\Delta\xi))$ is called the generator sequence of (ξ_n). For each integer $k \geq 1$, $\ell_n^{(k)}(\xi)$ is defined by

$$\ell_n^{(k)}(\xi) = \frac{1}{\gamma_n}\sum_{t=0}^{n}\frac{\ell_t^{k-1}(\xi)}{t+1}, \tag{4}$$

where $\ell_n^{(0)}(\xi) = \xi_n$ and $\ell_n^{(1)}(\xi) = \ell_n(\xi)$.

If we get the logarithmic mean of the sequence of $(v_n^{(0)}(\Delta\xi))$, then we obtain

$$\ell_n^{(1)}(v^{(0)}(\Delta\xi)) = \frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{v_k^{(0)}(\Delta\xi)}{k+1} = v_n^{(1)}(\Delta\xi).$$

By getting the logarithmic mean of $(v_n^{(1)}(\Delta\xi))$, then we obtain

$$\ell_n(v^{(1)}(\Delta\xi)) = \frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{v_k^{(1)}(\Delta\xi)}{k+1} = v_n^{(2)}(\Delta\xi).$$

Continuing in this way, we obtain the following sequence:

$$\ell_n(v^{(m-1)}(\Delta\xi)) = \frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{v_k^{(m-1)}(\Delta\xi)}{k+1} = v_n^{(m)}(\Delta\xi),$$

for $m \geq 1$. Hence, all these given sequences can be written as follows:

$$v_n^{(m)}(\Delta\xi) = \begin{cases} \frac{1}{\gamma_n}\sum_{k=0}^{n}\dfrac{v_k^{(m-1)}(\Delta\xi)}{k+1}, & m \geq 1 \\ v_n(\Delta\xi), & m = 0. \end{cases}$$

For a sequence (ξ_n), classical logarithmic control modulo is defined by

$$\omega_n^{(0)}(\xi) = (n+1)\gamma_{n-1}\Delta\xi_n. \tag{5}$$

The general logarithmic control modulo of the oscillatory behavior of integer order $m \geq 1$ of a sequence (ξ_n) is defined by

$$\omega_n^{(m)}(\xi) \quad = \quad \omega_n^{(m-1)}(\xi) - \ell_n^{(1)}(\omega^{(m-1)}(\xi)). \tag{6}$$

Assume that $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_n \to \infty$. A sequence (ξ_n) is called bounded with the rapidity (λ_n) if

$$\lambda_n(\xi_n - \alpha) = O(1) \tag{7}$$

with $\lim_{n\to\infty} \xi_n = \alpha$. Shortly, we say that the sequence (ξ_n) is λ-bounded and the set of all λ-bounded sequences is denoted by m^λ.

Also a sequence (ξ_n) is called λ-bounded by logarithmic method of summability if

$$\lambda_n(\ell_n^{(1)}(\xi) - \alpha) = O(1) \tag{8}$$

with $\lim_{n\to\infty} \ell_n^{(1)}(\xi) = \alpha$. The set of all logarithmic λ-bounded sequences is denoted by (ℓ, m^λ).

Tauberian theory for the logarithmic method have been studied by various authors. A number of authors such as Kwee [2] and Ishiguro [3–5] obtained some Tauberian theorems for the logarithmic method and generalized some classical Tauberian theorems to logaritmic method. Móricz [6] presented some classical type Tauberian theorems for logarithmic method of sequences and established some Tauberian theorems by introducing logarithmic summability method of integrals.

Later, Okur and Totur [7,8] introduced general logarithmic control modulo and classical logarithmic control modulo for logarithmic method of integrals. And they extanded Tuaberian theorems which are given for $(C, 1)$ method. Sezer and Çanak [9,10] investigated new Tauberian conditions with help of general logarithmic control modulo for logarithmic method of sequences and proved some Theorems for logarithmic method of power series.

On the other hand many researchers studied Tauberian remainder theorems for some summability methods such as Kangro [11] and Tammeraid [12–14] after Kangro's work [15] in which the author introduced the concepts of Tauberian remiander theorems by using summability with given rapidity λ. Meronen and Tammeraid [16] presented some Tauberian remainder theorems for $(C, 1)$ summability method from a new perspective. In this work, they used the concept of general control modulo which was defined in [17]. Later Sezer and Çanak [18,19] and Totur and Okur [20,21] proved some results for weighted mean, Hölder and (C, α) summability methods. They also benefited from the concept of general control modulo to obtain Tauberian remainder theorems in these studies.

We aim in this paper to prove some Tauberian remainder theorems for the logarithmic summability method. Firstly, we prove 3 lemmas in section 2 and in each lemma, the relationship between the different-order general logarithmic control modulo of a sequence and its different-order logarithmic means is given. After that, the main theorems are presented in the next section. In the main theorems, we obtain λ-boundedness of a sequence from its logarithmic λ-boundedness by using conditions on generator sequence and general logarithmic control modulo of the given sequence.

2. Auxilary Results

For the proofs of our main results, we require the following lemmas.

Lemma 1. *The following equality is valid.*

$$\omega_n^{(1)}(\xi) = \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi). \tag{9}$$

Proof. Taking $m = 1$ in (6) and using (5), we get

$$\begin{aligned} \omega_n^{(1)}(\xi) &= \omega_n^{(0)}(\xi) - \ell_n^{(1)}(\omega^{(0)}(\xi)) \\ &= \omega_n^{(0)}(\xi) - \frac{1}{\gamma_n}\sum_{k=0}^{n} \frac{(k+1)\gamma_{k-1}\Delta\xi_k}{k+1} \\ &= \omega_n^{(0)}(\xi) - v_n^{(0)}(\Delta\xi). \end{aligned}$$

Using (3) in the last equality, we obtain

$$\omega_n^{(1)}(\xi) = \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi).$$

□

Lemma 2. *The following equality is valid.*

$$\omega_n^{(2)}(\xi) = \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi). \tag{10}$$

Proof. If we take $m = 2$ in (6), we obtain

$$\omega_n^{(2)}(\xi) = \omega_n^{(1)}(\xi) - \ell_n^{(1)}(\omega^{(1)}(\xi)).$$

Using (9), we get

$$\begin{aligned} \omega_n^{(2)}(\xi) &= \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi) - \ell_n^{(1)}(\omega^{(0)}(\xi) - \xi + \ell^{(1)}(\xi)) \\ &= \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi) - \frac{1}{\gamma_n}\sum_{k=0}^{n} \frac{1}{k+1}(\omega_k^{(0)}(\xi) - \xi_k + \ell_k^{(1)}(\xi)) \end{aligned}$$

From (4), we get

$$\omega_n^{(2)}(\xi) = \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi) - v_n^{(0)}(\Delta\xi) + \ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi).$$

By (3), we conclude that

$$\omega_n^{(2)}(\xi) \quad = \quad \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi).$$

□

Lemma 3. *The following equality is valid.*

$$\omega_n^{(3)}(\xi) = \omega_n^{(0)}(\xi) - 3\xi_n + 6\ell_n^{(1)}(\xi) - 4\ell_n^{(2)}(\xi) + \ell_n^{(3)}(\xi). \tag{11}$$

Proof. By taking $m = 3$ in (6), we have

$$\omega_n^{(3)}(\xi) \quad = \quad \omega_n^{(2)}(\xi) - \ell_n^{(1)}(\omega^{(2)}(\xi)).$$

From (10), we obtain

$$\begin{aligned}
\omega_n^{(3)}(\xi) \quad &= \quad \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi) \\
&\quad\; -\ell_n^{(1)}(\omega^{(0)}(\xi) - 2\xi + 3\ell^{(1)}(\xi) - \ell^{(2)}(\xi)) \\
&= \quad \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi) \\
&\quad\; -\frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{1}{k+1}(\omega_k^{(0)}(\xi) - 2\xi_k + 3\ell_k^{(1)}(\xi) - \ell_k^{(2)}(\xi)).
\end{aligned}$$

Now, using (4) in the last equality, we get

$$\begin{aligned}
\omega_n^{(3)}(\xi) \quad &= \quad \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi) \\
&\quad\; -v_n^{(0)}(\Delta\xi) + 2\ell_n^{(1)}(\xi) - 3\ell_n^{(2)}(\xi) + \ell_n^{(3)}(\xi).
\end{aligned}$$

Finally, from definition of the logarithmic Kronecker identity, we have

$$\omega_n^{(3)}(\xi) \quad = \quad \omega_n^{(0)}(\xi) - 3\xi_n + 6\ell_n^{(1)}(\xi) - 4\ell_n^{(2)}(\xi) + \ell_n^{(3)}(\xi).$$

□

3. Main Results

Theorem 1. *Let ξ is λ-bounded by the $(\ell, 1)$ method. If*

$$\lambda_n v_n^{(0)}(\Delta\xi) = O(1), \tag{12}$$

then ξ is λ-bounded.

Proof. Because of ξ is λ-bounded by the $(\ell, 1)$ method, we have

$$\lambda_n\left(\ell_n^{(1)}(\xi) - \alpha\right) = O(1). \tag{13}$$

By the equality

$$\lambda_n (\xi_n - \alpha) = \lambda_n \left(\ell_n^{(1)}(\xi) - \alpha + \xi_n - \ell_n^{(1)}(\xi) \right),$$

we obtain

$$\lambda_n (\xi_n - \alpha) = \lambda_n \left(\ell_n^{(1)}(\xi) - \alpha \right) + \lambda_n v_n^{(0)}(\Delta\xi)$$

using (3). By combining (12) and (13) with the last equailty, we get

$$\lambda_n (\xi_n - \alpha) = O(1).$$

So, ξ is λ-bounded and proof is completed. □

Theorem 2. *Let ξ is λ-bounded by the $(\ell, 1)$ method. If*

$$\lambda_n \omega_n^{(0)}(\xi) = O(1) \tag{14}$$

and

$$\lambda_n \omega_n^{(1)}(\xi) = O(1), \tag{15}$$

then ξ is λ-bounded.

Proof. Benefit from Lemma 1, we get the following equality:

$$\lambda_n (\xi_n - \alpha) = \lambda_n \left(\ell_n^{(1)}(\xi) - \alpha - \omega_n^{(1)}(\xi) + \xi_n - \ell_n^{(1)}(\xi) + \omega_n^{(1)}(\xi) \right).$$

So, we conclude that

$$\lambda_n (\xi_n - \alpha) = -\lambda_n \omega_n^{(1)}(\xi) + \lambda_n \left(\ell_n^{(1)}(\xi) - \alpha \right) + \lambda_n \omega_n^{(0)}(\xi).$$

From λ-boundedness by the $(\ell, 1)$ method, we have (13). Taking (14) and (15) into account we obtain

$$\lambda_n (\xi_n - \alpha) = O(1).$$

This result completed the proof. □

Theorem 3. *Let ξ is λ-bounded by the $(\ell, 1)$ method and the condition (14) is satisfied. If*

$$\lambda_n \omega_n^{(2)}(\xi) = O(1) \tag{16}$$

and

$$\lambda_n \left(\ell_n^{(2)}(\xi) - \alpha \right) = O(1), \tag{17}$$

then ξ is λ-bounded.

Proof. Using Lemma 2, we obtain the equality of

$$\lambda_n (\xi_n - \alpha) = \lambda_n \left(-\omega_n^{(2)}(\xi) + \omega_n^{(0)}(\xi) - \alpha - \xi_n - \ell_n^{(2)}(\xi) + 3\ell_n^{(1)}(\xi) \right).$$

Therefore we get the following result:

$$2\lambda_n (\xi_n - \alpha) = -\lambda_n \omega_n^{(2)}(\xi) + \lambda_n \omega_n^{(0)}(\xi) - \lambda_n \left(\ell_n^{(2)}(\xi) - \alpha \right) + 3\lambda_n \left(\ell_n^{(1)}(\xi) - \alpha \right).$$

Using (13), (14), (16) and (17) we get the result of

$$\lambda_n (\xi_n - \alpha) = O(1).$$

It means that ξ is λ-bounded. □

Theorem 4. *Let ξ is λ-bounded by the $(\ell, 1)$ method and the conditions (14) and (17) are satisfied. If*

$$\lambda_n \omega_n^{(3)}(\xi) = O(1) \tag{18}$$

and

$$\lambda_n \left(\ell_n^{(3)}(\xi) - \alpha\right) = O(1), \tag{19}$$

then ξ is λ-bounded.

Proof. With the Lemma 3 we obtain the following equality:

$$\lambda_n (\xi_n - \alpha) = \lambda_n \left(-\omega_n^{(3)}(\xi) + \omega_n^{(0)}(\xi) + \ell_n^{(3)}(\xi) - 4\ell_n^{(2)}(\xi) + 6\ell_n^{(1)}(\xi) - 2\xi_n - \alpha\right).$$

Then it follows that

$$\begin{aligned} 3\lambda_n (\xi_n - \alpha) &= -\lambda_n \omega_n^{(3)}(\xi) + \lambda_n \omega_n^{(0)}(\xi) + \lambda_n \left(\ell_n^{(3)}(\xi) - \alpha\right) \\ &\quad -4\lambda_n \left(\ell_n^{(2)}(\xi) - \alpha\right) + 6\lambda_n \left(\ell_n^{(1)}(\xi) - \alpha\right). \end{aligned}$$

If we combine (13), (14), (17), (18) and (19), we have the equality

$$\lambda_n (\xi_n - \alpha) = O(1).$$

Therefore we obtain that ξ is λ-bounded. □

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SOME INTEGRAL INEQUALITIES THROUGH TEMPERED FRACTIONAL INTEGRAL OPERATOR

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Abstract. In this article, we adopt the tempered fractional integral operators to develop some novel Minkowski and Hermite-Hadamard type integral inequalities. Thus, we give several special cases of the integral inequalities for tempered fractional integrals obtained in the earlier works.

1. Introduction

The theory of convexity plays a vital role in different fields of pure and applied sciences. Consequently, the classical concepts of convex sets and convex functions have been generalized in different directions. The concept of function is one of the basic structures of mathematics, and many researchers have focused on new function classes and have made efforts to classify the space of functions. One of important class of functions defined as a product of this intense effort is the convex function, which has applications in statistics, inequality theory, convex programming, and numerical analysis. This interesting class of functions is defined as follows (mentioned in ([6]).

Definition 1. *Let $\mathcal{H}$ be an interval in $\mathbb{R}$. Then $f : \mathcal{H} \to \mathbb{R}$ is said to be convex if*

$$f(\xi a + (1-\xi) b) \leq \xi f(a) + (1-\xi) f(b)$$

for all $a, b \in \mathcal{H}$ and $\xi \in [0, 1]$.

For more information, see the papers [1-5] and [22]- [24].
Another aspect due to which the convexity theory has attracted many researchers

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is its close relationship with theory of inequalities. Many famous inequalities can be obtained using the concept of convex functions. For details related to convexity, interested readers are referred to [6,7]. Among these inequalities, Hermite–Hadamard inequality, which provides us a necessary and sufficient condition for a convex function, is one of the most studied results. This result of Hermite and Hadamard reads as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a,\ b\ \in\ I$, with $a < b$. The following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f\left(a\right)+f\left(b\right)}{2} \tag{1}$$

This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions.

Definition 2. *([17-18]) Let $f \in \mathcal{L}^1(a,b)$. The Riemann Liouville integrals $I_{a^+}^{\alpha} f$ and $I_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a\ \geq\ 0$ are defined by*

$$I_{a^+}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)}\int_a^x f\left(\xi\right)\left(x-\xi\right)^{\alpha-1} d\xi, \qquad x >\ a \tag{2}$$

and

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma\left(\alpha\right)}\int_x^b f\left(\xi\right)\left(\xi-x\right)^{\alpha-1} d\xi, \qquad b >\ x \tag{3}$$

The tempered fractional integral was first studied by Buschman [8], but Liu et al. [9], Meerschaert et al. [10] and Fernandez et al. [12] have described the associated tempered fractional calculus more explicitly.

Definition 3. *([10]) Let $[a,b]$ be a real interval and $\zeta \geq 0$, $\alpha > 0$. Then, for a function $f \in \mathcal{L}^1[a,b]$, the left and right tempered fractional integral, respectively, defined by*

$${}_{a^+}\mathfrak{J}^{\alpha,\zeta} f\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(x-\xi\right)^{\alpha-1} e^{-\zeta(x-\xi)} f\left(\xi\right) d\xi \tag{4}$$

and

$${}_{b^-}\mathfrak{J}^{\alpha,\zeta} f\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\xi-x\right)^{\alpha-1} e^{-\zeta(\xi-x)} f\left(\xi\right) d\xi, \tag{5}$$

where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{z-1}e^{-t}dt$.

For any $\zeta > 0$, the positive one-sided tempered fractional operator of a suitable function $f(x)$ can be given by;

$${}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_0^x \left(x-\xi\right)^{\alpha-1} e^{-\zeta(x-\xi)} f\left(\xi\right) d\xi.$$

Remark 1. *If we take $\zeta = 0$ in the equations (4) and (5), then we have the left and right R-L operators (2) and (3) respectively.*

First of all, we define the new incomplete Gamma function following definition as in [11].

Definition 4. *For the real numbers , $\alpha > 0$ and , $x, \zeta \geq 0$, we define the ζ-incomplete Gamma function by*

$$I_\alpha(\alpha, b) = \frac{1}{\Gamma(\alpha)} \int_0^b x^{\alpha-1} e^{-\zeta t} dx$$

If $\zeta = 1$, it reduces to the incomplete Gamma function

$$I_\alpha(\alpha, b) = \frac{1}{\Gamma(\alpha)} \int_0^b x^{\alpha-1} e^{-x} dx.$$

Remark 2. *For the real numbers $\alpha > 0$ and $x, \zeta \geq 0$, we have*

a. $I_{\zeta(b-a)}(\alpha, 1) = \int_0^1 x^{\alpha-1} e^{-\zeta(b-a)x} dx = \frac{1}{(b-a)^\alpha} \mathrm{I}_\alpha(\alpha, \mathrm{b-a})$
b. $\int_0^1 \mathrm{I}_{\alpha(\mathrm{b-a})}(\alpha, x)\, \mathrm{dx} = \frac{\mathrm{I}_\alpha(\alpha\ , \mathrm{b-a})}{(b-a)^\alpha} - \frac{\mathrm{I}_\alpha(\alpha+1\ , \mathrm{b-a})}{(b-a)^{\alpha+1}}$

Recently, Nisar et al. [13] established some inequalities via fractional conformable integral operators. In [14,15], various researchers established Minkowski inequalities involving fractional calculus with general analytic kernels and some novel estimations of Hadamard type inequalities for different kinds of convex functions via tempered fractional integral operator, the Hermite–Hadamard type inequalities for k-fractional conformable integrals are found in [16].
This paper is organized in the following way: In Section 2, the main results, the reverse Minkowski and related Hermite-Hadamard integral inequalities, are established using tempered fractional integral operators. The concluding remarks are given in Section 3.

2. Main Results

In this section, the reverse Minkowski and Hermite-Hadamard type integral inequalities are developed using the tempered integral operator.

Theorem 1. *Let $\zeta \geq 0$, $\alpha > 0$, $p \geq 1$ and let there be two positive functions f_1 and f_2 on $[0,\infty)$ such that for all $x >$ a, ${}^\tau\mathfrak{J}_x^{\alpha,\zeta} {f_1}^p(x) < \infty$, ${}^\tau\mathfrak{J}_x^{\alpha,\zeta} {f_2}^p(x) < \infty$. If $0 < \tau_1 \leq \frac{f_1(\xi)}{f_2(\xi)} \leq \tau_2$, holds for $\tau_1, \tau_2 \in \mathbb{R}^+$ and $\xi \in [0,x]$, then we have:*

$$\left({}^\tau\mathfrak{J}_x^{\alpha,\zeta} {f_1}^p(x)\right)^{\frac{1}{p}} + \left({}^\tau\mathfrak{J}_x^{\alpha,\zeta} {f_2}^p(x)\right)^{\frac{1}{p}} \leq \frac{1+\tau_2(\tau_1+2)}{(\tau_1+1)(\tau_2+1)} \left({}^\tau\mathfrak{J}_x^{\alpha,\zeta} (f_1+f_2)^p(x)\right)^{\frac{1}{p}}. \quad (6)$$

Proof. Under the given condition $\frac{f_1(\xi)}{f_2(\xi)}\leq\tau_2$, $\xi\in[0,x]$, it can be written as

$$(\tau_2+1)^p {f_1}^p(\xi) \leq {\tau_2}^p (f_1+f_2)^p(\xi). \tag{7}$$

Multiplying both sides of (7) by $\frac{(x-\xi)^{\alpha-1}}{\Gamma(\alpha)e^{\zeta(x-\xi)}}$, then integrating the resulting inequality with respect to ξ over $[0,x]$, we obtain,

$$\begin{aligned}(\tau_2+1)^p \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} e^{-\zeta(x-\xi)} {f_1}^p(\xi)\, d\xi \\ \leq \tau_2^p \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} e^{-\zeta(x-\xi)} (f_1+f_2)^p(\xi) d\xi.\end{aligned} \tag{8}$$

Consequently, we obtain

$$(\tau_2+1)^p\, {}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} {f_1}^p(x) \leq {\tau_2}^p\, {}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} (f_1+f_2)^p(x). \tag{9}$$

Hence, we can write

$$\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} {f_1}^p(x)\right]^{\frac{1}{p}} \leq \frac{\tau_2}{(\tau_2+1)} \left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} (f_1+f_2)^p(x)\right]^{\frac{1}{p}}. \tag{10}$$

In contrast, as $\tau_1 f_2(\xi)\leq f_1(\xi)$, it follows that

$$\left(1+\frac{1}{\tau_1}\right)^p {f_2}^p(\xi) \leq \frac{1}{{\tau_1}^p} [f_1(\xi)+f_2(\xi)]^p. \tag{11}$$

Again, if we multiplying both sides of (11) by $\frac{(x-\xi)^{\alpha-1}}{\Gamma(\alpha)e^{\zeta\ (x-\xi)}}$, then integrating the resulting inequality with respect to ξ over $[0,x]$, we obtain,

$$\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} {f_2}^p(x)\right]^{\frac{1}{p}} \leq \frac{1}{(\tau_1+1)} \left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} (f_1+f_2)^p(x)\right]^{\frac{1}{p}}. \tag{12}$$

Adding the inequalities (10) and (12) yields the desired inequality. □

Remark 3. *By setting Theorem 1 for* $\alpha=1$, $\zeta=0$ *and for an arbitrary choice of function, we obtain Theorem 1.2 in [20].*

Remark 4. *In Theorem 1, if we choose* $\zeta=0$, *we obtain Theorem 2.1 in [19].*

Inequality (6) is referred to as the reverse Minkowski inequality for the tempered fractional integral operator.

Theorem 2. *Let* $\zeta \geq 0$, $\alpha > 0$, $p\geq1$ *and let there be two positive functions* f_1 *and* f_2 *on* $[0,\infty)$ *such that for all* $x>$ a, ${}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} {f_1}^p(x)<\infty$, ${}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} {f_2}^p(x)<\infty$. *If* $0<\tau_1\leq\frac{f_1(\xi)}{f_2(\xi)}\leq\tau_2$, *holds for* $\tau_1,\tau_2\in\mathbb{R}^+$ *and* $\xi\in[0,x]$, *then we have:*

$$\begin{aligned}\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} {f_1}^p(x)\right)^{\frac{2}{p}} + \left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} {f_2}^p(x)\right)^{\frac{2}{p}} \\ \geq \left(\frac{(1+\tau_2)(\tau_1+1)}{\tau_2} - 2\right) \left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} {f_1}^p(x)\right]^{\frac{1}{p}} \left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} {f_2}^p(x)\right]^{\frac{1}{p}}.\end{aligned} \tag{13}$$

Proof. The product of inequalities (10) and (12) yields

$$\frac{(1+\tau_2)(\tau_1+1)}{\tau_2}\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}f_1{}^p(x)\right]^{\frac{1}{p}}\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}f_2{}^p(x)\right]^{\frac{1}{p}} \leq \left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}(f_1+f_2)^p(x)\right]^{\frac{2}{p}}. \quad (14)$$

Now, utilizing the Minkowski inequality to the right hand side of (14), one obtains

$$\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}(f_1+f_2)^p(x)\right)^{\frac{2}{p}} \leq \left(\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}f_1{}^p(x)\right]^{\frac{1}{p}}+\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}f_2{}^p(x)\right]^{\frac{1}{p}}\right)^2.$$

Then, we have

$$\begin{aligned}\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}(f_1+f_2)^p(x)\right)^{\frac{2}{p}} \leq &\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}f_1{}^p(x)\right]^{\frac{2}{p}}+\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}f_2{}^p(x)\right]^{\frac{2}{p}}\\ &+2\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}f_1{}^p(x)\right]\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}f_2{}^p(x)\right].\end{aligned} \quad (15)$$

Thus, from the above inequalities, we obtain the inequality (13).

□

Remark 5. *By setting Theorem 2 for $\alpha=1$, $\zeta=0$ and for an arbitrary choice of function, we obtain Theorem 2.2 in [21].*

Remark 6. *In Theorem 2, if we choose $\zeta=0$, we obtain Theorem 2.3 in [19].*

Lemma 1. *([19]) Let $\mathcal{G}$ be a concave function on $[a,\ b]$. Then the following double inequality holds:*

$$\mathcal{G}(a)+\mathcal{G}(b) \leq \mathcal{G}(b+a-x)+\mathcal{G}(x) \leq 2\mathcal{G}\left(\frac{a+b}{2}\right). \quad (16)$$

Theorem 3. *Let $\zeta \geq 0, \alpha > 0, p \geq 1$ and let there be two positive functions $\hbar$ and $\mathcal{L}$ on $[0,\infty)$. If $\hbar^p$ and $\mathcal{L}^q$ are two concave functions on $[0,\infty)$, then we have:*

$$\begin{aligned}2^{-p-q}\big(\hbar(0)+\hbar(x)\big)^p\big(\mathcal{L}(0)+\mathcal{L}(x)\big)^q\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}(x^{\alpha-1})\right]^2&\\ \leq {}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right){}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right).&\end{aligned} \quad (17)$$

Proof. Since the $\hbar^p$ and $\mathcal{L}^q$ are two concave functions on $[0,\infty)$, then by Lemma 1, for any $\xi > 0$ we obtain,

$$\hbar^{\ p}(0)+\hbar^p(x) \leq \hbar^p(x-\xi)+\hbar^p(\xi) \leq 2\hbar^p\left(\frac{x}{2}\right), \quad (18)$$

and

$$\mathcal{L}^q(0)+\mathcal{L}^q(x) \leq \mathcal{L}^q(x-\xi)+\mathcal{L}^q(\xi) \leq 2\mathcal{L}^q\left(\frac{x}{2}\right). \quad (19)$$

Multiplying both sides of (18) and (19) by $\frac{(x-\xi)^{\alpha-1}\xi^{\alpha-1}}{\Gamma(\alpha)e^{\zeta\ (x-\xi)}}$, then integrating the resulting inequality with respect to ξ over $[0,x]$, we obtain,

$$\begin{aligned}\frac{\hbar^p(0)+\hbar^p(x)}{\Gamma(\alpha)}&\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}d\xi\\ &\leq\frac{1}{\Gamma(\alpha)}\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\hbar^p(x-\xi)\,d\xi\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\hbar^p(\xi)\,d\xi\\ &\leq\frac{2\hbar^p\left(\frac{x}{2}\right)}{\Gamma(\alpha)}\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}d\xi,\end{aligned}\tag{20}$$

and

$$\begin{aligned}\frac{\mathcal{L}^q(0)+\mathcal{L}^q(x)}{\Gamma(\alpha)}&\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}d\xi\\ &\leq\frac{1}{\Gamma(\alpha)}\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\mathcal{L}^q(x-\xi)\,d\xi\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\mathcal{L}^q(\xi)\,d\xi\\ &\leq\frac{2\mathcal{L}^q\left(\frac{x}{2}\right)}{\Gamma(\alpha)}\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}d\xi.\end{aligned}\tag{21}$$

Using the change of variables $x-\xi=y$, we have

$$\frac{1}{\Gamma(\alpha)}\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\hbar^p(x-\xi)\,d\xi={}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right),\tag{22}$$

and

$$\frac{1}{\Gamma(\alpha)}\int_0^x\frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\mathcal{L}^q(x-\xi)\,d\xi={}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right).\tag{23}$$

Thus, by using (20) and (22) yields,

$$\begin{aligned}\hbar^p(0)+\hbar^p(x)\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right)&\leq 2\,{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right)\\ &\leq\hbar^p\left(\frac{x}{2}\right)\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right),\end{aligned}\tag{24}$$

Similarly, the use of (21) and (23) yields,

$$\begin{aligned}\hbar^q(0)+\mathcal{L}^q(x)\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right)&\leq 2\,{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right)\\ &\leq\mathcal{L}^q\left(\frac{x}{2}\right)\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right).\end{aligned}\tag{25}$$

The inequalities (24) and (25) imply that

$$\begin{aligned}(\hbar^p(0)+\hbar^p(x))(\mathcal{L}^q(0)+\mathcal{L}^q(x))\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right)^2 \\ \leq 4\,{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right){}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right).\end{aligned} \tag{26}$$

Since $\hbar$ and $\mathcal{L}$ are positive functions, therefore for any $x > 0, p \geq 1$, and $q \geq 1$, we have

$$\left(\frac{\hbar^p(0)+\hbar^p(x)}{2}\right)^{\frac{1}{p}} \geq 2^{-1}(\hbar(0)+\hbar(x)),$$

and

$$\left(\frac{\mathcal{L}^q(0)+\hbar^p(x)}{2}\right)^{\frac{1}{p}} \geq 2^{-1}(\mathcal{L}(0)+\mathcal{L}(x)).$$

Hence, it follows that

$$\frac{(\hbar^p(0)+\hbar^p(x))}{2}{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right) \geq 2^{-p}(\hbar(0)+\hbar(x))^{p\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right), \tag{27}$$

$$\frac{(\mathcal{L}^q(0)+\mathcal{L}^q(x))}{2}{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right) \geq 2^{-q}(\mathcal{L}(0)+\mathcal{L}(x))^{q\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right). \tag{28}$$

The inequalities (27) and (28) imply

$$\begin{aligned}\frac{1}{4}(\hbar^p(0)+\hbar^p(x))(\mathcal{L}^q(0)+\mathcal{L}^q(x))\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right]^2 \\ \geq 2^{-p-q}(\hbar(0)+\hbar(x))^p(\mathcal{L}(0)+\mathcal{L}(x))^q\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right]^2.\end{aligned} \tag{29}$$

Thus, by combining (21) and (24), we get the desired result. □

Remark 7. *By considering Theorem 3, for $\alpha = 1$, $\zeta = 0$ and for an arbitrary choice of function, we obtain Theorem 2.3 in [21].*

Remark 8. *In Theorem 3, if we choose $\zeta = 0$, we obtain Theorem 2.5 in [19].*

Theorem 4. *Let $\zeta \geq 0$, $\alpha, \beta > 0$, $p \geq 1$ and let there be two positive functions $\hbar$ and $\mathcal{L}$ on $[0,\infty)$. If $\hbar^p$ and $\mathcal{L}^q$ are two concave functions on $[0,\infty)$, then we have:*

$$\begin{aligned}2^{2-p-q}(\hbar(0)+\hbar(x))^p(\mathcal{L}(0)+\mathcal{L}(x))^q\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\beta-1}\right)\right]^2 \\ \leq \left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right)+{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\beta-1}\hbar^p(x)\right)\right] \\ \times\left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right)+{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\beta-1}\mathcal{L}^q(x)\right)\right].\end{aligned} \tag{30}$$

Proof. Multiplying both sides of (18) and (19) by $\frac{(x-\xi)^{\alpha-1}\xi^{\beta\alpha-1}}{\Gamma(\alpha)e^{\zeta\ (x-\xi)}}$, then integrating the resulting inequality with respect to ξ over $[0,x]$, we obtain

$$\begin{aligned}\frac{\hbar^p(0)+\hbar^p(x)}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}d\xi \\ \leq \frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\hbar^p(x-\xi)\,d\xi \\ +\frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\hbar^p(\xi)\,d\xi \\ \leq \frac{2\hbar^p\left(\frac{x}{2}\right)}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}d\xi,\end{aligned} \tag{31}$$

and

$$\begin{aligned}\frac{\mathcal{L}^q(0)+\mathcal{L}^q(x)}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}d\xi \\ \leq \frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\mathcal{L}^q(x-\xi)\,d\xi \\ +\frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\mathcal{L}^q(\xi)\,d\xi \\ \leq \frac{2\mathcal{L}^q\left(\frac{x}{2}\right)}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}d\xi.\end{aligned} \tag{32}$$

Using the change of variables $x-\xi=y$, we have

$$\begin{aligned}\frac{\Gamma(\beta)}{\Gamma(\beta)\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\hbar^p(x-\xi)\,d\xi \\ =\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right),\end{aligned} \tag{33}$$

and

$$\begin{aligned}\frac{\Gamma(\beta)}{\Gamma(\beta)\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\mathcal{L}^q(x-\xi)\,d\xi \\ =\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right).\end{aligned} \tag{34}$$

Thus, from (31) and (33), we write

$$\begin{aligned}
&\left(\hbar^{p}(0)+\hbar^{p}(x)\right){}^{\tau}\mathfrak{J}_{x}^{\alpha\beta,\zeta}\left(x^{\alpha-1}\right)\\
&\qquad\leq\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^{p}(x)\right)+{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\hbar^{p}(x)\right)\\
&\qquad\leq 2\hbar^{p}\left(\frac{x}{2}\right){}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right),
\end{aligned}\tag{35}$$

and with (32) and (34), we can write,

$$\begin{aligned}
&\left(\mathcal{L}^{q}(0)+\mathcal{L}^{q}(x)\right){}^{\tau}\mathfrak{J}_{x}^{\alpha\beta,\zeta}\left(x^{\alpha-1}\right)\\
&\qquad\leq\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^{q}(x)\right)+{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\mathcal{L}^{q}(x)\right)\\
&\qquad\leq 2\mathcal{L}^{q}\left(\frac{x}{2}\right){}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right).
\end{aligned}\tag{36}$$

From (30) and (31), it follows that

$$\begin{aligned}
&\left(\hbar^{p}(0)+\hbar^{p}(x)\right)\left(\mathcal{L}^{q}(0)+\mathcal{L}^{q}(x)\right)\left[{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\right]^{2}\\
&\qquad\leq\left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^{p}(x)\right)+{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\hbar^{p}(x)\right)\right]\\
&\qquad\times\left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^{q}(x)\right)+{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\mathcal{L}^{q}(x)\right)\right].
\end{aligned}\tag{37}$$

Since $\hbar$ and $\mathcal{L}$ are positive functions, therefore for any $x > 0$, $p \geq 1$, and $q \geq 1$, we have

$$\frac{\left(\hbar^{p}(0)+\hbar^{p}(x)\right)}{2}{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\geq 2^{-p}(\hbar(0)+\hbar(x))^{p\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right),\tag{38}$$

and

$$\frac{\left(\mathcal{L}^{q}(0)+\mathcal{L}^{q}(x)\right)}{2}{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\geq 2^{-q}(\mathcal{L}(0)+\mathcal{L}(x))^{q\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right).\tag{39}$$

Thus from (38) and (39) it follows that

$$\begin{aligned}
&\frac{1}{4}\left(\hbar^{p}(0)+\hbar^{p}(x)\right)\left(\mathcal{L}^{q}(0)+\mathcal{L}^{q}(x)\right)\left[{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\right]^{2}\\
&\qquad\geq 2^{-p-q}(\hbar(0)+\hbar(x))^{p}(\mathcal{L}(0)+\mathcal{L}(x))^{q}\left[{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\right]^{2}.
\end{aligned}\tag{40}$$

Combining inequalities (37) and (40), we get the desired proof.

□

Remark 9. *By considering Theorem 4 for $\alpha = 1$, $\zeta = 0$ and for an arbitrary choice of function, we obtain Theorem 2.4 in [21].*

Remark 10. *In Theorem 4, if we choose , $\zeta = 0$, we obtain Theorem 2.8 in [19].*

Remark 11. *In Theorem 4, if we choose $\alpha = \beta$, we obtain Theorem 2.4.*

3. Conclusion

The Minkowski and Hermite-Hadamard inequalities for the tempered fractional integral operator have been newly established in this paper. Not only do we prove that the results obtained are mathematically more valuable, but similar inequalities can also be constructed, for example with the help of the incomplete Gamma function used in Remark 2. We hope that our results can stimulate further research in various fields of pure and applied science.

Author Contribution Statements Erdal Gül and Abdüllatif Yalçın have contributed to the establishment of the problem and investigation. Also, Erdal Gül has supervised the editing and conceptualization processes. All the authors have read and approved the final form of the manuscript.

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INTUITIONISTIC FINE SPACE

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Abstract. In the exploration of intuitionistic fine spaces, this article introduces a novel concept known as intuitionistic fine open sets (I_fOS). Delving into the properties of these sets, the study analyzes both intuitionistic fine open and closed sets within the context of intuitionistic fine spaces. The article establishes fundamental definitions, accompanied by illustrative real time example, to provide a comprehensive understanding of the newly introduced sets. Furthermore, the exploration extends to defining and examining key concepts such as intuitionistic fine continuity, intuitionistic fine irresoluteness, and intuitionistic fine irresolute homeomorphism. This progression aims to contribute to the broader comprehension and application of intuitionistic fine spaces in topological contexts.

1. Introduction

Intuitionistic topology (IT), fuzzy and intuitinistic fuzzy topology [2,3] plays a vital role in applied sciences such as pattern recognition, optimization technique, medical diagnosis, decision-making etc., [1,8,12,17,18] creates interest in introducing the new set, intuitionistic fine open set (I_fOS) in this article. The classical version of intuitionistic sets, as proposed by Coker [2,3], serves as a foundational framework for understanding certain topological structures. Additionally, the concept of fine sets, pioneered by P.L. Powar and K. Rajak [14–16], adds further depth to the study of these sets. The linkage between fine sets and the newly introduced concept in this article, namely intuitionistic fine open sets, lies in the amalgamation of these two theoretical underpinnings.

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Keywords. I_fOS, intuitionistic fine continuous, intuitionistic fine irresolute, intuitionistic fine irresolute homeomorphism.

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The introduction of intuitionistic fine open sets can be seen as an evolution or extension of these ideas. In essence, the concept of intuitionistic fine open sets builds upon the foundational notions of intuitionistic sets and fine sets, tailoring them to address and explore more specialized properties or relations. The objective of this article is to introduce a new set of concepts termed intuitionistic fine open sets (I_fOS) in intuitionistic fine spaces. As the discourse advances, an in-depth examination of the characteristics of intuitionistic fine open and closed sets in such spaces is undertaken. The foundational definitions are meticulously laid out, accompanied by essential real time example to elucidate the nuances of these sets.

Moreover, the article delves into the exploration and definition of intuitionistic fine continuity, intuitionistic fine irresoluteness, and intuitionistic fine irresolute homeomorphism, thus adding layers of understanding to the intricacies of intuitionistic fine spaces. In particular, there exists applications in image processing predominantly leverage fuzzy topology [12, 17] and intuitionistic fuzzy topology [1, 8, 19], this article motivates us to anticipate and forecast the potential applications of intuitionistic fine space within the same domain. The focus lies on extrapolating the applications based on the unique characteristics and features inherent to intuitionistic fine space. The aim is to project how the distinct attributes of intuitionistic fine space can contribute to and enhance various aspects, thereby expanding the scope and utility of this topological framework in practical applications.

2. Preliminaries

Definition 1. *[1] Suppose X be a non-empty set, an intuitionistic set (IS) C is an element of form $C = \langle X, C_1, C_2\rangle$, C_1 and C_2 are subsets of X holding $C_1 \cap C_2 = \phi$. C_1 is known as the set of members of C, and C_2 is known as the set of non-members of C.*

Definition 2. *[1] An IT on a non-empty set X is a family τ of ISs in X holding:*
(i) $\underset{\sim}{X}, \underset{\sim}{\phi} \in \tau$.
(ii) $C_1 \cap C_2 \in \tau$ for any $C_1, C_2 \in \tau$.
(iii) $\cup C_i \in \tau$ for arbitrary family $\{C_i : i \in L\} \subseteq \tau$
(X,τ) is called intuitionistic topological space (ITS) and IS in τ is called an intuitionistic open set (IOS) in X, the complement of it is said to be intuitionistic closed set (ICS).

Definition 3. *[3] Suppose X be a non empty set, $p \in X$ an element in X. IS $\underset{\sim}{p} = \langle X, \{p\}, \{p\}^c\rangle$ is an intuitionistic point (IP) in X and the IS $(\underset{\approx}{p}) = \langle X, \phi, \{p\}^c\rangle$ is said to be a vanishing intuitionistic point (VIP) in X.*

Definition 4. *[12] Suppose (X, τ) be a TS, we define $\tau(C_\alpha) = \tau_\alpha = \{K_\alpha(\neq X) : K_\alpha \cap C_\alpha \neq \phi$, for $C_\alpha \in \tau$ and $C_\alpha \neq \phi$, X for some $\alpha \in I$, I an index set$\}$. We define $\tau_f = \{\phi, X\} \bigcup_\alpha \{\tau_\alpha\}$. τ_f of subsets of X is said to be fine collection of subsets of X, (X, τ, τ_f) is known as fine space X generated by topology τ on X.*

Definition 5. *[12] A subset O of (X, τ, τ_f) is called fine open in X, if $O \in$ to τ_f. Its complement is fine-closed set.*

Example 1. *[12] Suppose $X = \{p, q, r\}$ &*
$\tau = \{X, \phi, \{p\}, \{q\}, \{p, q\}\}$. *Let* $A_1 = \{p\}, A_2 = \{q\}, A_3 = \{p, q\}$ *then*
$\tau_1 = \tau(A_1) = \tau\{p\} = \{\{p\}, \{p, q\}, \{p, r\}\}$,
$\tau_2 = \tau(A_2) = \tau\{q\} = \{\{q\}, \{p, q\}, \{q, r\}\}$,
$\tau_3 = \tau(A_3) = \tau\{p, q\} = \{\{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}\}$
$\tau_f = \{\phi, X\} \cup_\alpha \{\tau_\alpha\}$
$\tau_f = \{\phi, X, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}\}$.

Definition 6. *[12]*
(i) The largest fine open set $\subseteq C$ is fine interior of C denoted as $fint(C)$.
(ii) The smallest fine closed set $\supseteq C$ is fine closure of C denoted as $fcl(C)$.

Definition 7. *[12] Suppose (X, τ, τ_f) and $x \in X$, then a fine open set O of $X \in x$ is said to be a fine neighborhood of X.*

Definition 8. *[4, 8] A subset C of (X, τ) is:*
(i) intuitionistic α-open if $C \subseteq Iint(Icl(Iint(C)))$.
(ii) intuitionistic semi-open set if $C \subseteq Icl(Iint(C))$.
(iii) intuitionistic pre-open if $C \subseteq Iint(Icl(C))$.
(iv) intuitionistic β-open if $C \subseteq Icl(Iint(Icl(C)))$.
(v) intuitionistic regular-open if $C = Iint(Icl(C))$.

Definition 9. *[12] Map $g : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f)$ is known as fine continuous if $g^{-1}(V)$ is open in X for every fine open set V of Y.*

Definition 10. *[12] Map $g : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f)$ is known as fine irresolute (or f-irresolute) if $g^{-1}(V)$ is fine-open in X for every fine-open set V of Y.*

Definition 11. *[12] Map $g : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f)$ is f-irresolute homeomorphism if*
(i) g is 1-1 and onto.
(ii) Both maps g and inverse map $g^{-1} : (Y, \tau', \tau'_f) \to (X, \tau, \tau_f)$ are f-irresolute.

3. Intuitionistic Fine Open Sets

Definition 12. *Suppose (X, τ) be an ITS, we define*
$\tau(C_\alpha) = \widehat{\tau}_\alpha = \{K_\alpha (\neq \underset{\sim}{X}) : K_\alpha \cap C_\alpha \neq \underset{\sim}{\phi}$, *for* $C_\alpha \in \tau$ *and* $C_\alpha \neq \underset{\sim}{\phi}, \underset{\sim}{X}$ *for some* $\alpha \in I$, *I an indexed set*$\}$. *We define* $\widehat{\tau_f} = \left\{\underset{\sim}{\phi}, \underset{\sim}{X}\right\} \cup_\alpha \{\widehat{\tau}_\alpha\}$. $\widehat{\tau}_f$ *of subsets of X is known as intuitionistic fine collection of subsets of X &* $(X, \tau, \widehat{\tau}_f)$ *is known as an intuitionistic fine space* $(I_f S)$ *X generated by* τ *on X.*

Definition 13. *A subset O of $I_f S$ X is known as intuitionistic fine open sets $(I_f OS)$ if $O \in \widehat{\tau}_f$. Complement of $(I_f OS)$ is intuitionistic fine closed set $(I_f CS)$.*

Example 2. *Consider* $X = \{p, q, r\}$ *and* $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2\}$ *where*
$A_1 = \langle X, \{r\}, \{p, q\}\rangle$ *and* $A_2 = \langle X, \{r\}, \{p\}\rangle$.
Let $A_\alpha = A_1$ *and* A_2.
$\tau(A_\alpha) = \widehat{\tau}_\alpha = \{\langle X, \{p\}, \{\phi\}\rangle, \langle X, \{q\}, \{\phi\}\rangle, \langle X, \{r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p\}\rangle,$
$\langle X, \{\phi\}, \{q\}\rangle, \langle X, \{\phi\}, \{r\}\rangle, \langle X, \{p\}, \{q\}\rangle, \langle X, \{q\}, \{r\}\rangle,$
$\langle X, \{r\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle, \langle X, \{p\}, \{r\}\rangle, \langle X, \{p, q\}, \{\phi\}\rangle,$
$\langle X, \{q, r\}, \{\phi\}\rangle, \langle X, \{p, r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p, q\}\rangle, \langle X, \{\phi\}, \{p, r\}\rangle,$
$\langle X, \{q\}, \{p, r\}\rangle, \langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle,$
$\langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle\}$.
$\widehat{\tau_f} = \{\underset{\sim}{X}, \underset{\sim}{\phi}\} \cup \{\widehat{\tau}_\alpha\}$.
$\because \widehat{\tau_f} = I_f\tilde{O}S = \{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{p\}, \{\phi\}\rangle, \langle X, \{q\}, \{\phi\}\rangle, \langle X, \{r\}, \{\phi\}\rangle,$
$\langle X, \{\phi\}, \{p\}\rangle, \langle X, \{\tilde{\phi}\}, \{q\}\rangle, \langle X, \{\phi\}, \{r\}\rangle, \langle X, \{p\}, \{q\}\rangle,$
$\langle X, \{q\}, \{r\}\rangle, \langle X, \{r\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle,$
$\langle X, \{p\}, \{r\}\rangle, \langle X, \{p, q\}, \{\phi\}\rangle, \langle X, \{q, r\}, \{\phi\}\rangle, \langle X, \{p, r\}, \{\phi\}\rangle,$
$\langle X, \{\phi\}, \{p, q\}\rangle, \langle X, \{\phi\}, \{p, r\}\rangle, \langle X, \{q\}, \{p, r\}\rangle, \langle X, \{r\}, \{p, q\}\rangle,$
$\langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle$.

3.1. **Real-Time example: Advantage of Intuitionistic Fine open sets.** Suppose $X = \{a, b, c\}$ represents the set of three participants involving in various activities. All combination of subsets (i.e) the power set (intuitionistic) P(X) consist of 27 intuitionistic subsets like $A_1, A_2, ..., A_{27}$ involving membership and non-membership values, in which ϕ represents a set with no participants, X represents a set with all the participants, $\{a\}$ represents a set with one participant and so on. For example, if the following table (Table 1) illustrates the sets to which team they belong to:

Table 1. Intuitionistic subsets and corresponding teams

$X = \langle X, \{p, q, r\}, \{\phi\}\rangle$	Intuitionistic set representing social activity team
$\phi = \langle X, \{\phi\}, \{p, q, r\}\rangle$	Intuitionistic set representing music team
$A_1 = \langle X, \{p, r\}, \{\phi\}\rangle$	Intuitionistic set representing Project group
$A_2 = \langle X, \{p, q\}, \{r\}\rangle$	Intuitionistic set representing Study group
$A_3 = \langle X, \{p, r\}, \{q\}\rangle$	Intuitionistic set representing Sports team
$A_4 = \langle X, \{p\}, \{q\}\rangle$	Intuitionistic set representing individual activity-1 participant
$A_5 = \langle X, \{r\}, \{p\}\rangle$	Intuitionistic set representing individual activity-2 participant

and so on. Also if $\tau = \{\{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{r\}, \{p, q\}\rangle,$
$\langle X, \{r\}, \{p\}\}\rangle\}$ (Ref.Example 2) associated with X, represents possibilities of collection of intuitionistic sets involving in political activity, then we get the collection of intuitionistic fine open sets, $\widehat{\tau_f}$ (Ref.Example 2) gives a clear picture of the

various combinations (intuitionistic sets) of participants engaged in different activities(teams) also involve in political activity so that their intersection is not empty, with the union of possibilities of no and all participants ($\{\underset{\sim}{X}$ and $\underset{\sim}{\phi}\}$).

Definition 14. *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fS, suppose $p \in X$, then an intuitionistic fine open set O of X containing $\underset{\sim}{p}$ is known as an intuitionistic fine neighborhood.*

Example 3. *Consider $X = \{p, q, r\}$ & $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2\}$ where*
$A_1 = \langle X, \{\phi\}, \{p, q\}\rangle$ *and* $A_2 = \langle X, \{\phi\}, \{p\}\rangle$.
$\widehat{\tau_f} = I_fOS = \{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{p\}, \{\phi\}\rangle, \langle X, \{q\}, \{\phi\}\rangle, \langle X, \{r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p\}\rangle,$
$\langle X, \{\phi\}, \{q\}\rangle, \langle X, \{\phi\}, \{r\}\rangle, \langle X, \{p\}, \{q\}\rangle, \langle X, \{q\}, \{r\}\rangle, \langle X, \{r\}, \{p\}\rangle,$
$\langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle, \langle X, \{p\}, \{r\}\rangle, \langle X, \{p, q\}, \{\phi\}\rangle, \langle X, \{q, r\}, \{\phi\}\rangle,$
$\langle X, \{p, r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p, q\}\rangle, \langle X, \{\phi\}, \{p, r\}\rangle, \langle X, \{q\}, \{p, r\}\rangle,$
$\langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle\}$.
Consider $r \in X$ then the intuitionistic fine neighborhoods of the intuitionistic point $\underset{\sim}{r} = \langle X, \{r\}, \{r\}^c\rangle$ ($\underset{\sim}{r} = \langle X, \{r\}, \{p, q\}\rangle$) in X are
$\{\langle X, X, \phi\rangle, \langle X, \{r\}, \{\phi\}\rangle, \langle X, \{r\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle, \langle X, \{q, r\}, \{\phi\}\rangle,$
$\langle X, \{p, r\}, \{\phi\}\rangle, \langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle\}$.

Definition 15. *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fS and suppose $C = \langle X, C_1, C_2\rangle$ be an IS in X then:*
$Icl_f(C) = \bigcap\{J : J$ *is an* I_fCS *in* X & $C \subseteq J\}$
$Iint_f(C) = \bigcup\{J : J$ *is an* I_fOS *in* X & $C \supseteq J\}$.

Example 4. *Suppose $X = \{p, q\}$ and $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1, A_2\}$ where*
$A_1 = \langle X, \{q\}, \{p\}\rangle$ *and* $A_2 = \langle X, \{\phi\}, \{p\}\rangle$.
$\widehat{\tau_f} = I_fOS = \{\underset{\sim}{\phi}, \underset{\sim}{X}, \langle X, \{\phi\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{p\}, \{\phi\}\rangle,$
$\langle X, \{q\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle\}$.
$I_fCS = \{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{p\}, \{\phi\}\rangle, \langle X, \{p\}, \{q\}\rangle, \langle X, \{\phi\}, \{p\}\rangle,$
$\langle X, \{\phi\}, \{q\}\rangle, \langle X, \{\phi\}, \{\phi\}\rangle$.
$Icl(\langle X, \{p\}, \{q\}\rangle) = \underset{\sim}{X}$.
$I_fcl(\langle X, \{p\}, \{q\}\rangle) = \langle X, \{p\}, \{q\}\rangle$.

Theorem 1. *Suppose $(X, \tau, \widehat{\tau}_f)$ be I_fS, then the (arbitrary) union of I_fOS in X is I_fOS in X.*

Proof. Suppose $\{K_\alpha\}_{\alpha \in I}$ be set of I_fOSs of X. Implies $K_\alpha \cap C_\alpha \neq \underset{\sim}{\phi}, \forall \alpha \in I$ and $C_\alpha(\neq \underset{\sim}{\phi}, \underset{\sim}{X}) \in \tau$. We need T.P that $\cup_{\alpha \in I} K_\alpha = K$ is I_fOS. It's enough to S.T $K \cap C_\beta \neq \underset{\sim}{\phi}$ for $C_\beta(\neq \underset{\sim}{\phi}, \underset{\sim}{X}) \in \tau$. Here $(\cup_{\alpha \in I} K_\alpha \cap C_\beta) = (K_\alpha \cap C_\beta) \cup (K_\beta \cap C_\beta)$..... $\Rightarrow$ $\exists$ an index $\beta \in I$ s.t $K_\beta \cap C_\beta \neq \underset{\sim}{\phi}$ ($\because K_\beta \in \widehat{\tau}_f$). Therefore $(\cup K_\alpha) \cap C_\beta \neq \phi \Rightarrow K$ is an I_fOS. □

Remark 1. (1) *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fTS then the union of two I_fCS in X need not be I_fCS in X.*
(2) *Suppose $(X, \tau, \widehat{\tau}_f)$ be an I_fTS then $\cap$ of two I_fOS in X need not be I_fOS in X.*

Theorem 2. *Suppose* $(X, \tau, \widehat{\tau}_f)$ *be an* I_fS*, then the arbitrary intersection of* I_fCSs *in X is* I_fCS *in X.*

Proof. Assuming $\{F_\alpha\}_{\alpha \in I}$ to be set of intuitionistic fine-closed sets of X.
T.P: $\bigcap F_\alpha = F$ is intuitionistic fine closed. It is sufficient T.P F^c is intuitionistic fine-open. Using De Morgan's law to get $F^c = \cup F_\alpha^c$. Using the above remark the union of intuitionistic fine open set implies that $F^c = \cup F_\alpha^c$ is I_fOS. Therefore F is I_fCS. □

Example 5. *Consider Example 3, Suppose* $A = \langle X, \{r\}, \{\phi\} \rangle$ *and*
$B = \langle X, \{p\}, \{r\} \rangle$ *be two* I_fOS *then* $A \cup B = \langle X, \{p, r\}, \{\phi\} \rangle$ *which is an* I_fOS.
Now $C = \langle X, \{q\}, \{r\} \rangle$ *and* $D = \langle X, \{r\}, \{q\} \rangle$ *be* I_fOS
$C \cap D = \langle X, \{\phi\}, \{q, r\} \rangle$ *which is not an* I_fOS.
Here $I_fCS = \{\underset{\sim}{X}, \phi, \langle X, \{\phi\}, \{p\} \rangle, \langle X, \{\phi\}, \{q\} \rangle, \langle X, \{\phi\}, \{r\} \rangle, \langle X, \{p\}, \{\phi\} \rangle,$
$\langle X, \{q\}, \{\phi\} \rangle, \langle X, \{r\}, \{\phi\} \rangle, \langle X, \{q\}, \{p\} \rangle, \langle X, \{r\}, \{q\} \rangle, \langle X, \{p\}, \{r\} \rangle,$
$\langle X, \{p\}, \{q\} \rangle, \langle X, \{q\}, \{r\} \rangle, \langle X, \{r\}, \{p\} \rangle, \langle X, \{\phi\}, \{p, q\} \rangle, \langle X, \{\phi\}, \{q, r\} \rangle,$
$\langle X, \{\phi\}, \{p, r\} \rangle, \langle X, \{p, q\}, \{\phi\} \rangle, \langle X, \{p, r\}, \{\phi\} \rangle, \langle X, \{p, r\}, \{q\} \rangle,$
$\langle X, \{p, q\}, \{r\} \rangle, \langle X, \{p\}, \{q, r\} \rangle, \langle X, \{q\}, \{p, r\} \rangle, \langle X, \{r\}, \{p, q\} \rangle, \langle X, \{\phi\}, \{\phi\} \rangle\}$.
Let $E = \langle X, \{q\}, \{p, r\} \rangle$ *and* $F = \langle X, \{r\}, \{p, q\} \rangle$ *be* I_fCS
$E \cup F = \langle X, \{q, r\}, \{p\} \rangle$ *which is not an* I_fCS.
$E \cap F = \langle X, \{\phi\}, \{X\} \rangle$ *which is an* I_fCS.

Definition 16. *An intuitionistic fine subset C of* $(X, \tau, \widehat{\tau}_f)$ *is:*
(i) an $I_f\alpha OS$ *if* $C \subseteq I_fint(I_fcl(I_fint(C)))$. *(ii) an* I_fSOS *if* $C \subseteq I_fcl(I_fint(C))$.
(iii) an I_fPOS *if* $C \subseteq I_fint(I_fcl(C))$. *(iv) an* $I_f\beta OS$ *if* $C \subseteq I_fcl(I_fint(I_fcl(C)))$.
(v) an I_fROS *if* $C = I_fint(I_fcl(C))$.

Remark 2. *An* I_fOS *C of* $(X, \tau, \widehat{\tau} f)$ *is:*
(i) an intuitionistic fine α*-open set* $(If\alpha OS)$ *if C is an intuitionistic* α *open subset of* (X, τ). *(ii) an intuitionistic fine semi-open set* (I_fSOS) *if C is an intuitionistic semi open subset of* (X, τ). *(iii) an intuitionistic fine pre-open set* (I_fPOS) *if C is an intuitionistic pre open subset of* (X, τ). *(iv) an intuitionistic fine* β*-open set* $(I_f\beta OS)$ *if C is an intuitionistic* β *open subset of* (X, τ). *(v) an intuitionistic fine regular-open* (I_fROS) *if C is an intuitionistic regular open subset of* (X, τ).

Theorem 3. *Suppose* $(X, \tau, \widehat{\tau}_f)$ *be an* I_fS *w.r.t the TS* (X, τ)*, then* $\widehat{\tau}_f \subset$ *every* $ISOS$ *and* $I\alpha OS$.

Proof. Assume, $E \subset \underset{\sim}{X}$ and $E \notin \widehat{\tau}_f$.
T.P: E is not $ISOS$ in $\underset{\sim}{X}$.
$\because E \notin \widehat{\tau}_f \Rightarrow C_\alpha \cap E = \underset{\sim}{\phi}$.
$\forall \alpha \in J \Rightarrow I_fint(E) = \underset{\sim}{\phi} \Rightarrow I_fcl(I_fint(E)) = \underset{\sim}{\phi} \Rightarrow E$ not contained in $I_fcl(I_fint(E))$.
Hence E is not intuitionistic semi open and hence E is not contained in
$I_fint(I_fcl(I_fint(E)))$ and therefore E is not an $I\alpha OS$. □

Theorem 4. *Suppose* $(X, \tau, \widehat{\tau}_f)$ *be the* I_fS *w.r.t the TS* (X, τ)*, then* $\widehat{\tau}_f \subset$ *every* $IPOS$ *and* $I\beta OS$.

Proof. Assume, $E \subset \underset{\sim}{X}$ and $E \notin (X, \tau)$.
T.P: E is not $IPOS$ and not $I\beta OS$ in $\underset{\sim}{X}$
$\because$ it is known that $E \notin \widehat{\tau}_f \Rightarrow C_\alpha \cap E = \phi \forall \alpha \in J \Rightarrow E \subseteq (C_\alpha)^c$ and $I_f cl(E) \subseteq (C_\alpha)^c$ and $(C_\alpha)^c$ is an $I_f CS$ containing E. $\because C_\alpha \cap (C_\alpha)^c = \phi$ and $I_f cl(E) \subseteq (C_\alpha)^c \Rightarrow C_\alpha \cap I_f cl(E) = \underset{\sim}{\phi} \Rightarrow I_f int(I_f cl(E)) = \underset{\sim}{\phi}$ and hence E not contained in $I_f int(I_f cl(E))$. Therefore E is not intuitionistic pre open and thus E is not contained in $I_f cl(I_f int(I_f cl(E)))$. Hence E is not $I\beta OS$. □

Example 6. *Suppose* $X = \{p, q, r\}$ & $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A_1\}$ *where*
$A_1 = \langle X, \{r\}, \{p, q\}\rangle$
$I_f OS = \{\underset{\sim}{X}, \underset{\sim}{\phi}, \langle X, \{\phi\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p\}\rangle, \langle X, \{\phi\}, \{q\}\rangle, \langle X, \{p\}, \{\phi\}\rangle,$
$\langle X, \{q\}, \{\phi\}\rangle, \langle X, \{r\}, \{\phi\}\rangle, \langle X, \{\phi\}, \{p, q\}\rangle, \langle X, \{p, q\}, \{\phi\}\rangle, \langle X, \{q, r\}, \{\phi\}\rangle,$
$\langle X, \{p, r\}, \{\phi\}\rangle, \langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p\}, \{q\}\rangle,$
$\langle X, \{r\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle.$
It is found that $\langle X, \{\phi\}, \{r\}\rangle, \langle X, \{\phi\}, \{q, r\}\rangle, \langle X, \{\phi\}, \{p, r\}\rangle,$
$\langle X, \{p\}, \{q, r\}\rangle, \langle X, \{q\}, \{p, r\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \{q\}, \{r\}\rangle, \langle X, \{p\}, \{r\}\rangle$
are not members of $\widehat{\tau_f}$, *they are not* $I\alpha OS, I\beta OS, ISOS$ *and* $IPOS$ *but they are* $I_f CS$.

Theorem 5. *Suppose* $(X, \tau, \widehat{\tau}_f)$ *be an* $I_f S$ *and C any arbitrary subset of X. Then:*
(i)$Iint(C) \subseteq I_f int(C)$
(ii)$I_f cl(C) \subseteq Icl(C)$.

Proof. By the definitions of interior, closure, intuitionistic fine interior and intuitionistic fine closure $Iint(C) \subseteq I_f int(C)$ and $I_f cl(C) \subseteq Icl(C)$. □

Theorem 6. *Suppose* $(X, \tau, \widehat{\tau}_f)$ *be an* $I_f S$ *and C be any arbitrary subset of X. Then* $x \in I_f cl(C)$ *iff every* $I_f OS$ *'O' containing x intersects C.*

Proof. Assume $(X, \tau, \widehat{\tau}_f)$ to be an $I_f S$ and C be any arbitrary subset of X. Let $x \in I_f cl(C)$, consider every $IfOS$ 'O' containing x by the def. of intuitionistic fine closure we find that every $I_f OS$ 'O' containing x intersects C.
Conversely assume that every intuitionistic fine open set O containing x intersects C then using the definition of intuionistic fine open set we find that $x \in I_f cl(C)$. □

4. Intuitionistic Fine Maps

Definition 17. *Map* $g : (X, \tau, \widehat{\tau}_f) \to (Y, \tau', \widehat{\tau}'_f)$ *is known as intuitionistic fine-continuous if* $g^{-1}(V)$ *is intuitionistic open set (IOS) in* X $\forall$ $I_f OS$ V *of Y.*

Definition 18. *Map* $g : (X, \tau, \widehat{\tau}_f) \to (Y, \tau', \widehat{\tau}'_f)$ *is known as intuitionistic fine-irresolute if* $g^{-1}(V)$ *is intuitionistic fine open in* X $\forall$ $I_f OS$ V *of Y.*

Definition 19. *A map* $g : (X, \tau, \widehat{\tau}_f) \to (Y, \tau', \widehat{\tau}'_f)$ *is known as intuitionistic fine-irresolute homeomorphism if*
(i) g is 1-1 and onto. (ii) Both maps g and inverse map of g are intuitionistic fine irresolute.

Example 7. *Suppose* $X = \{p, q, r\}$ & $\tau = \{\underset{\sim}{X}, \underset{\sim}{\phi}, A\}$ *s.t* $A = \langle X, \{p\}, \{q\}\rangle$ *and suppose* $Y = \{1, 2, 3\}$ *and* $\tau' = \{\underset{\sim}{Y}, \underset{\sim}{\phi}, B\}$ *where* $B = \langle X, \{3\}, \{1, 2\}\rangle$. *Suppose* $g : (X, \tau, \widehat{\tau}_f) \to (Y, \tau', \widehat{\tau}'_f)$ *where g(p)=1, g(q)=2 and g(r)=3.*
Here I_fOS *in X are* $\{\langle X, \phi, X\rangle, \langle X, X, \phi\rangle, \langle X, \phi, \{p\},\rangle, \langle X, \phi, \{q\}\rangle, \langle X, \phi, \{r\}\rangle,$
$\langle X, \{p\}, \phi\rangle, \langle X, \{q\}, \phi\rangle, \langle X, \{r\}, \phi\rangle, \langle X, \phi, \{p, q\}\rangle, \langle X, \phi, \{q, r\}\rangle,$
$\langle X\{p, q\}, \phi\rangle, \langle X, \{q, r\}, \phi\rangle, \langle X, \{p, r\}, \phi\rangle, \langle X, \{p\}, \{q\}\rangle, \langle X, \{q\}, \{r\}\rangle,$
$\langle X, \{r\}, \{p\}\rangle, \langle X, \{q\}, \{p\}\rangle, \langle X, \{r\}, \{q\}\rangle, \langle X, \{p\}, \{r\}\rangle, \langle X, \{p\}, \{q, r\}\rangle,$
$\langle X, \{r\}, \{p, q\}\rangle, \langle X, \{q, r\}, \{p\}\rangle, \langle X, \{p, r\}, \{q\}\rangle, \langle X, \{p, q\}, \{r\}\rangle, \langle X, \phi, \phi\rangle\}$.
I_fOS *in Y are* $\{\langle Y, \phi, Y\rangle, \langle Y, Y, \phi\rangle, \langle Y, \phi, \{1\},\rangle, \langle Y, \phi, \{2\}\rangle, \langle Y, \phi, \{3\}\rangle,$
$\langle Y, \{1\}, \phi\rangle, \langle Y, \{2\}, \phi\rangle, \langle Y, \{3\}, \phi\rangle, \langle Y, \phi, \{1, 2\}\rangle, \langle Y, \phi, \{2, 3\}\rangle,$
$\langle Y\{1, 2\}, \phi\rangle, \langle Y, \{2, 3\}, \phi\rangle, \langle Y, \{1, 3\}, \phi\rangle, \langle Y, \{1\}, \{2\}\rangle, \langle Y, \{2\}, \{3\}\rangle,$
$\langle Y, \{3\}, \{1\}\rangle, \langle Y, \{2\}, \{1\}\rangle, \langle Y, \{3\}, \{2\}\rangle, \langle Y, \{1\}, \{3\}\rangle, \langle Y, \{1\}, \{2, 3\}\rangle,$
$\langle Y, \{3\}, \{1, 2\}\rangle, \langle Y, \{2, 3\}, \{1\}\rangle, \langle Y, \{1, 3\}, \{2\}\rangle, \langle Y, \{1, 2\}, \{3\}\rangle, \langle Y, \phi, \phi\rangle\}$.
Here the given function is not intuitionistic fine continuous but it is intuitionistic fine irresolute and intuitionistic fine irresolute homeomorphism.

Theorem 7. *Suppose* X & Y *be* I_fSs *and suppose* $g : X \to Y$. *The following are equivalent :*
(i) g is intuitionistic fine irresolute.
(ii) For every subset C of X, $g(I_fcl(C)) \subseteq I_fclg(C)$.
(iii) For every I_fCS *D in Y,* $g^{-1}(D)$ *is* I_fCS *in X.*

Proof. $(i) \Rightarrow (ii)$: Suppose that g is intuitionistic fine irresolute. Suppose C be a subset of X. We S.T if $x \in I_fcl(C)$, then $g(x) \in g(I_fcl(C)) \Rightarrow g(x) \in I_fcl(g(C))$. Assume V to be an intuitionistic fine neighbourhood of g(x). Hence $g^{-1}(V)$ is I_fOS of X containing x ; this intersects C in some point y. Then V intersects g(C) in g(y) s.t $g(x) \in I_fcl(g(C))$, which is required.
$(ii) \Rightarrow (iii)$: Suppose D be I_fCS in Y and suppose that $C = g^{-1}(D)$. We have to P.T C is $I_fCS$in X; We P.T $I_fcl(C) = C$. By basic set theory, $g(C) = g(g^{-1}(D)) \subset D$. Hence if $x \in I_fcl(C), g(x) \in g(I_f(C)) \subset I_fcl(C)) \subset I_fcl(D) = D$, $\because$ D is an I_fCS, s.t $x \in g^{-1}(D) = C$. Hence $I_f(C) \subset C$, s.t $I_fcl(C) = C$ as required.
$(iii) \Rightarrow (i)$: Suppose V be an I_fOS of Y. Set $D = Y - V$. Hence $g^{-1}(D) = g^{-1}(Y) - g^{-1}(V) = X - g^{-1}(V)$. Here D is an I_fCS of Y. Hence $g^{-1}(D)$ is I_fCS in X by hypothesis s.t $g^{-1}(V)$ is I_fOS, as required. □

5. Envisaging Applications in Image Processing: The Probable Impacts of Intuitionistic Fine Space

Given that intuitionistic fuzzy sets inherently handle both membership and non-membership values, their significance in image processing is well-established. The similarity between intuitionistic fine space and intuitionistic topological space with fuzzy measures suggests a promising avenue for the application of intuitionistic fine space in the realm of image processing. In the digital plane, each pixel functions as an open set [21]. Notably, intuitionistic sets encompass both membership and non-membership values, with measures for these values defined in intuitionistic fuzzy sets. This framework forms the foundation for various applications such as image extraction, image segmentation, optimization, and more within the domain of image processing. This article posits that the same rationale can be extended to intuitionistic fine open sets, potentially yielding intriguing and valuable results in the context of image processing.

6. Conclusion

The continued advancement of the intuitionistic fine topology, as introduced in this article, into fuzzy, binary space holds the potential to unlock further applications in diverse areas. These areas span from digital lines to computer networking, image processing, and data analysis, among others. Consequently, this article lays the groundwork for future applications of $I_f S$ in both intuitionistic topology and intuitionistic fuzzy topology, paving the way for broader and more varied practical implementations.

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BOUNDS FOR THE MAXIMUM EIGENVALUES OF THE FIBONACCI-FRANK AND LUCAS-FRANK MATRICES

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Abstract. Frank matrix is one of the popular test matrices for eigenvalue routines because it has well-conditioned and poorly conditioned eigenvalues. In this paper, we investigate the bounds for the maximum eigenvalues of the special cases of the generalized Frank matrices which are called Fibonacci-Frank and Lucas-Frank matrices. Then, we obtain the Euclidean norms and the upper bounds for the spectral norms of these matrices.

1. Introduction

The Fibonacci and Lucas number sequences which are the most famous integer sequences, are defined by the recurrence relations ($n \geq 1$) [8]

$$f_{n+1} = f_n + f_{n-1} \quad \text{with} \quad f_0 = 0, f_1 = 1 \tag{1}$$

and

$$l_{n+1} = l_n + l_{n-1} \quad \text{with} \quad l_0 = 2, l_1 = 1. \tag{2}$$

The Binet formulas for the Fibonacci and Lucas number sequences are

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad l_n = \alpha^n + \beta^n, \tag{3}$$

respectively, where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ [8]. Also, there are some summation formulas for these number sequences, for example [8, 22]

$$\sum_{i=1}^{n} f_i = f_{n+2} - 1, \quad \sum_{i=1}^{n} f_i^2 = f_n f_{n+1} \tag{4}$$

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and

$$\sum_{i=1}^{n} l_i = l_{n+2} - 3, \quad \sum_{i=1}^{n} l_i^2 = l_n l_{n+1} - 2. \tag{5}$$

Matrix theory plays an important role in mathematics, engineering and many other sciences because the matrices are very useful tool to solve multidimensional equation systems. A matrix may be assigned numerical items in various ways, for example the determinant, trace, eigenvalues, singular values, spectral radius, matrix norm, etc. Norms for matrices are used to measure the "sizes" of the matrices, have an importance in the matrix theory. Due to the various applications of the Fibonacci and Lucas number sequences, there have been many studies on the norms of the special matrices with entries of the Fibonacci and Lucas numbers [1, 2, 7, 15, 17–20]. The Euclidean (Frobenius) and spectral norm of an $m \times n$ matrix A are defined as

$$\|A\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} \mid a_{ij} \mid^2\right]^{\frac{1}{2}} \quad \text{and} \quad \|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i\left(A^H A\right)}, \tag{6}$$

respectively, where A^H is the conjugate transpose of the matrix A and $\lambda_i\left(A^H A\right)$'s are the eigenvalues of $A^H A$ [6]. The maximum row length norm $r_1(A)$ and the maximum column length norm $c_1(A)$ of any matrix A are defined by

$$r_1(A) = \max_i \sqrt{\sum_j \mid a_{ij} \mid^2} \quad \text{and} \quad c_1(A) = \max_j \sqrt{\sum_i \mid a_{ij} \mid^2}, \tag{7}$$

respectively [6]. Moreover, for any $m \times n$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$, if $A = B \circ C$, then

$$\|A\|_2 \le r_1(B)\, c_1(C), \tag{8}$$

where $B \circ C$ is the Hadamard product of the matrices B and C, which is defined by $B \circ C = [b_{ij}c_{ij}]$ [6].

Frank [3] defined the matrix of order n

$$F_n = \begin{bmatrix} n & n-1 & 0 & 0 & \dots & 0 & 0 \\ n-1 & n-1 & n-2 & 0 & \dots & 0 & 0 \\ n-2 & n-2 & n-2 & n-3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}, \tag{9}$$

which is called Frank matrix. The elements of the Frank matrix $F_n = [g_{ij}]$ are characterized by the formula

$$g_{ij} = \begin{cases} n + 1 - \max(i, j), & i > j - 2 \\ 0, & \text{orherwise.} \end{cases} \tag{10}$$

Hake [5] investigated the determinant, inverse, LU-decomposition and characteristic polynomials of the matrix F_n. Because of its well conditioned and poorly conditioned eigenvalues, the Frank matrix is one of the popular test matrices for eigenvalue routines. As a consequence of Sturm's Theorem, all eigenvalues of the matrix F_n are real and positive [5]. Varah [23] gave a generalization of the Frank matrix and computed its eigensystem. The generalized Frank matrix F_{a_n} is defined as

$$F_{a_n} = \begin{bmatrix} a_n & a_{n-1} & 0 & 0 & \dots & 0 & 0 \\ a_{n-1} & a_{n-1} & a_{n-2} & 0 & \dots & 0 & 0 \\ a_{n-2} & a_{n-2} & a_{n-2} & a_{n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_2 & a_2 & a_2 & \dots & a_2 & a_1 \\ a_1 & a_1 & a_1 & a_1 & \dots & a_1 & a_1 \end{bmatrix}, \tag{11}$$

where $a = (a_1, a_2, a_3, \dots, a_n)$ is a finite sequence with any a_i real numbers [12]. The elements of the generalized Frank matrix $F_{a_n} = \left[(f_a)_{ij}\right]$ are characterized by

$$(f_a)_{ij} = \begin{cases} a_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

It is clear that for $a_i = i, (i = 1, 2, \dots n)$ the generalized Frank matrix turns into the well-known Frank matrix. The authors investigated some properties of the matrix F_{a_n} and presented that the set of all $n \times n$ generalized Frank matrices is an n-dimensional vector space. They obtained the characteristic polynomials of the matrix F_{a_n} as

$$P_n(\lambda) = (\lambda - a_n + a_{n-1}) P_{n-1}(\lambda) - a_{n-1}\lambda P_{n-2}(\lambda), \tag{13}$$

with the initial conditions

$$P_1(\lambda) = \lambda - a_1 \quad \text{and} \quad P_2(\lambda) = \lambda^2 - (a_1 + a_2)\lambda + a_1 a_2 - a_1^2.$$

The Sturm's Theorem gives the exact number of zeros in an interval for any polynomial without multiple zeros, is used for computing the eigenvalues of symmetric or tridiagonal matrices [4,9,16,21,24]. According to the Sturm's Theorem, if the sequence $P_0(x), P_1(x), \dots, P_n(x)$ has the Sturm sequence properties on (a,b) and α, β $(\alpha < \beta)$ are any numbers in (a,b), then $P_n(x)$ has exactly $c(\beta) - c(\alpha)$ different zeros in the interval (α, β), where $c(\alpha)$ denotes the number of changes in sign of consecutive members of the sequence $P_0(\alpha), P_1(\alpha), \dots, P_n(\alpha)$ [4]. Mersin and Bahşi [11] showed that the characteristic polynomial of the generalized Frank matrix F_{a_n} is the form of the Strum sequence for the positive and strictly increasing (or negative and strictly decreasing) sequence $\{a_n\}$. They obtained all eigenvalues of the matrix F_{a_n} are different and positive, also the eigenvalues of the matrices F_{a_i} and $F_{a_{i-1}}$ are interlaced for $1 \le i \le n$, by considering the Sturm's Theorem.

Moreover, as a conclusion of the Sturm's Theorem, for the positive and strictly increasing sequence $\{a_n\}$, the inequalities

$$\lambda_n < a_1 \quad \text{and} \quad a_n < \lambda_1 \tag{14}$$

are hold, where λ_n and λ_1 are the minimum and the maximum eigenvalues of F_{a_n} for $n \geq 2$, respectively [11].

As the special forms of the generalized Frank matrices, Fibonacci-Frank matrix F_{f_n} and Lucas-Frank matrix F_{l_n} are defined as

$$F_{f_n} = \begin{bmatrix} f_n & f_{n-1} & 0 & 0 & \dots & 0 & 0 \\ f_{n-1} & f_{n-1} & f_{n-2} & 0 & \dots & 0 & 0 \\ f_{n-2} & f_{n-2} & f_{n-2} & f_{n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_2 & f_2 & f_2 & f_2 & \dots & f_2 & f_1 \\ f_1 & f_1 & f_1 & f_1 & \dots & f_1 & f_1 \end{bmatrix} \tag{15}$$

and

$$F_{l_n} = \begin{bmatrix} l_n & l_{n-1} & 0 & 0 & \dots & 0 & 0 \\ l_{n-1} & l_{n-1} & l_{n-2} & 0 & \dots & 0 & 0 \\ l_{n-2} & l_{n-2} & l_{n-2} & l_{n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l_2 & l_2 & l_2 & l_2 & \dots & l_2 & l_1 \\ l_1 & l_1 & l_1 & l_1 & \dots & l_1 & l_1 \end{bmatrix}, \tag{16}$$

where f_n and l_n are the ordinary Fibonacci and Lucas numbers [10]. The elements of the matrices $F_{f_n} = [f_{ij}]$ and $F_{l_n} = [l_{ij}]$ are

$$f_{ij} = \begin{cases} f_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad l_{ij} = \begin{cases} l_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Since the determinant of the matrix in equation (15) is zero, Fibonacci-Frank matrix F_{f_n} is used as

$$F_{f_n} = \begin{bmatrix} f_{n+1} & f_n & 0 & 0 & \dots & 0 & 0 \\ f_n & f_n & f_{n-1} & 0 & \dots & 0 & 0 \\ f_{n-1} & f_{n-1} & f_{n-1} & f_{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_3 & f_3 & f_3 & f_3 & \dots & f_3 & f_2 \\ f_2 & f_2 & f_2 & f_2 & \dots & f_2 & f_2 \end{bmatrix} \tag{18}$$

in [10].

We note that F_{f_n} will represent the matrix given by (18), throughout the paper. The determinants, inverses, LU-decompositions and characteristic polynomials of the matrices F_{f_n} and F_{l_n} are examined in [10]. The characteristic polynomial of the Fibonacci-Frank matrix F_{f_n} is obtained from equation (13) as

$$P_n(\lambda) = (f_{n-1} - \lambda) P_{n-1}(\lambda) - f_n \lambda P_{n-2}(\lambda), \tag{19}$$

with the initial conditions $P_0(\lambda) = 1$, $P_1(\lambda) = 1 - \lambda$, $P_2(\lambda) = \lambda^2 - 3\lambda + 1$, and the characteristic polynomial of the Lucas-Frank matrix F_{l_n} is

$$Q_n(\mu) = (l_{n-2} - \mu) Q_{n-1}(\mu) - l_{n-1} \mu Q_{n-2}(\mu), \tag{20}$$

with $Q_0(\mu) = 1$, $Q_1(\mu) = 1 - \mu$, $Q_2(\mu) = \mu^2 - 4\mu + 2$. The characteristic polynomials in equations (19) and (20) have the properties of the Sturm sequences [11].

In this paper, firstly we obtain the number of the eigenvalues of the matrices F_{f_n} and F_{l_n} in the interval $(0,1)$. We examine the bounds for the maximum eigenvalues of the matrices F_{f_n} and F_{l_n}. Then, we present the Euclidean norm and the upper bounds for the spectral norms of these matrices. Additionally, we give an example to illustrate our results.

2. Main Results

Lemma 1. *Let the characteristic polynomials of the Fibonacci-Frank matrix F_{f_n} and Lucas-Frank matrix F_{l_n} be $P_n(\lambda)$ and $Q_n(\mu)$, respectively. Then, for the value of $\lambda = \mu = 1$, we have*

(i) $P_n(1) < 2P_{n-1}(1)$, *for* $n \geq 6$,

(ii) $Q_n(1) > 3Q_{n-1}(1)$, *for* $n \geq 3$.

Proof. (i) We use the induction method on n. Since $P_6(1) = 0 < 2P_5(1) = 12$, the result is true for $n = 6$. Suppose that the result is true for $n = k > 6$. Then,

$$P_k(1) < 2P_{k-1}(1). \tag{21}$$

Hence, we have

$$0 > P_7(1) > P_8(1) > P_9(1) > \ldots > P_{k-1}(1). \tag{22}$$

For $n = k + 1$,

$$\begin{aligned} P_{k+1}(1) - 2P_k(1) &= (f_k - 1) P_k(1) - f_{k+1} P_{k-1}(1) - 2P_k(1) \\ &= f_k P_k(1) - 3P_k(1) - (f_k + f_{k-1}) P_{k-1}(1) \\ &= f_k (P_k(1) - P_{k-1}(1)) - f_{k-1} P_{k-1}(1) - 3P_k(1) \\ &< f_k P_{k-1}(1) - f_{k-1} P_{k-1}(1) - 3P_k(1) \\ &= f_{k-2} P_{k-1}(1) - 3P_k(1) \\ &< f_{k-2} P_{k-1}(1) - 6P_{k-1}(1) \\ &< (f_{k-2} - 6) P_{k-1}(1) \\ &< 0. \end{aligned}$$

Hence,

$$P_{k+1}(1) < 2P_k(1). \tag{23}$$

(ii) The proof is similar to the proof of (i).

□

Theorem 1. *The number of the eigenvalues of the Fibonacci-Frank matrix F_{f_n} in the interval $(0,1)$ is three for $n \geq 7$.*

Proof. Considering the Sturm's Theorem, we must show that $c(1) - c(0) = 3$, where $c(x)$ denotes the number of changes in sign of consecutive members of the sequence in equation (19), for $n \geq 7$.

TABLE 1. The number of sign changes of $P_{i \leq 7}(\lambda)$ for $\lambda = 0$, $\lambda = 1$

Characteristic polynomials $P_i(\lambda)$ for $i \leq 7$	Sign of $P_i(\lambda)$ for $\lambda = 0$	Sign of $P_i(\lambda)$ for $\lambda = 1$
$P_0(\lambda) = 1$	+	+
$P_1(\lambda) = 1 - \lambda$	+	0
$P_2(\lambda) = \lambda^2 - 3\lambda + 1$	+	−
$P_3(\lambda) = -\lambda^3 + 6\lambda^2 - 6\lambda + 1$	+	0
$P_4(\lambda) = \lambda^4 - 11\lambda^3 + 27\lambda^2 - 16\lambda + 2$	+	+
$P_5(\lambda) = -\lambda^5 + 19\lambda^4 - 90\lambda^3 + 127\lambda^2 - 55\lambda + 6$	+	+
$P_6(\lambda) = \lambda^6 - 32\lambda^5 + 273\lambda^4 - 793\lambda^3 + 818\lambda^2 - 297\lambda + 30$	+	0
$P_7(\lambda) = -\lambda^7 + 53\lambda^6 - 776\lambda^5 + 4147\lambda^4 - 8813\lambda^3 + 7756\lambda^2 - 2484\lambda + 240$	+	−
Number of sign changes	$c_7(0) = 0$	$c_7(1) = 3$

From Table 1, we have $c_7(0) = 0$ and $c_7(1) = 3$. Then, $P_7(\lambda)$ has $c_7(1) - c_7(0) = 3$ eigenvalues in the interval $(0,1)$. The eigenvalues of the matrix F_{f_7} are $\lambda_1 = 33.108$, $\lambda_2 = 11.495$, $\lambda_3 = 4.834$, $\lambda_4 = 2.083$, $\lambda_5 = 0.882$, $\lambda_6 = 0.433$, $\lambda_7 = 0.164$. Then, the eigenvalues in the interval $(0,1)$ are λ_5, λ_6 and λ_7, so it is clear that our result is correct for $n = 7$. As it seen in Table 1, $P_7(1) < 0$. From Lemma 1, we have $P_n(1) < 2P_{n-1}(1)$, then $P_n(1) < 0$ for $n > 7$. That is, there is no sign change of $P_n(1)$ for $n > 7$. Hence, $c_n(1) = 3$ is true for $n > 7$. To complete the proof we must show that $c_n(0) = 0$ is true for $n > 7$. From the initial condition of the recurrence relation in equation (19), we have $P_1(0) = 1$. Considering the recurrence relation in equation (19), we have

$$\begin{aligned} P_n(0) &= f_{n-1}P_{n-1}(0) \\ &= f_{n-1}f_{n-2}P_{n-2}(0) \\ &\vdots \\ &= f_{n-1}f_{n-2}f_{n-3}\dots f_1P_1(0) \\ &> 0. \end{aligned}$$

That is, there is no sign change of $P_n(0)$ for any positive integer n. Thus, $c_n(0) = 0$ for $n \geq 1$. Hence, we have $c_n(1) - c_n(0) = 3$ for $n \geq 7$. That is, the number of the eigenvalues of the Fibonacci-Frank matrix F_{f_n} in the interval $(0,1)$ is three for $n \geq 7$, as desired. □

Theorem 2. *The number of the eigenvalues of the Lucas-Frank matrix F_{l_n} in the interval $(0,1)$ is two for $n \geq 4$.*

Proof. The proof is similar to the proof of Theorem 1. □

Lemma 2. *The equalities for the Fibonacci-Frank matrix F_{f_n}*

(i) $trF_{f_n} = f_{n+3} - 2$,

(ii) $trF_{f_n}^2 = 3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2$,

(iii) $\sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2 = 3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2 - \frac{1}{n}\left(f_{n+3} - 2\right)^2$

are valid and also the equalities for the Lucas-Frank matrix F_{l_n}

(i') $trF_{l_n} = l_{n+2} - 3$,

(ii') $trF_{l_n}^2 = 3\left(l_n l_{n+1} - 2\right) - 2l_n^2$,

(iii') $\sum_{i=1}^{n}\left(\mu_i - \frac{trF_{l_n}}{n}\right)^2 = 3\left(l_n l_{n+1} - 2\right) - 2l_n^2 - \frac{1}{n}\left(l_{n+2} - 3\right)^2$

are hold, where λ_i's and μ_i's $(i = 1, 2, \ldots, n)$ are the eigenvalues of the matrices F_{f_n} and F_{l_n}, respectively.

Proof. (i) For the Fibonacci-Frank matrix $F_{f_n} = [f_{ij}]$, we have

$$trF_{f_n} = \sum_{i=2}^{n+1} f_i = \sum_{i=1}^{n} f_i + f_{n+1} - f_1 = f_{n+2} + f_{n+1} - 2 = f_{n+3} - 2.$$

(ii) For the matrix $F_{f_n}^2 = \left[f_{ij}^{(2)}\right]$, we have

$$trF_{f_n}^2 = \sum_{i=1}^{n} f_{ii}^{(2)} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} f_{ik} f_{ki}\right).$$

From the following equalities

$$f_{ik} = \begin{cases} f_{n+2-\max(i,k)}, & i > k-2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_{ki} = \begin{cases} f_{n+2-\max(k,i)}, & k > i-2 \\ 0, & \text{otherwise,} \end{cases}$$

we can say that if $\mid i - k \mid < 2$, then $f_{ik} f_{ki} \neq 0$, otherwise $f_{ik} f_{ki} = 0$. $\mid i - k \mid < 2$ yields $i = k$, $i = k - 1$ and $i = k + 1$ for $1 < i < n$. Since,

${f_{11}}^{(2)} = f_{n+1}^2 + f_n^2$ and ${f_{nn}}^{(2)} = f_2^2 + f_2^2$, we have

$$\begin{aligned}
trF_{f_n}^2 &= \sum_{i=1}^{n} {f_{ii}}^{(2)} = f_{n+1}^2 + f_n^2 + \sum_{i=2}^{n-1}\left(\sum_{k=i-1}^{i+1} f_{ik}f_{ki}\right) + f_2^2 + f_2^2 \\
&= \sum_{i=2}^{n-1}\left(2f_{n+2-i}^2 + f_{n+1-i}^2\right) + f_{n+1}^2 + f_n^2 + 2f_2^2 \\
&= 2\sum_{i=2}^{n-1} f_{n+2-i}^2 + \sum_{i=2}^{n-1} f_{n+1-i}^2 + f_{n+1}^2 + f_n^2 + 2f_2^2 \\
&= 2\left(f_n^2 + f_{n-1}^2 + \ldots + f_3^2\right) + \left(f_{n-1}^2 + f_{n-2}^2 + \ldots + f_3^2 + f_2^2\right) \\
&\qquad + f_n^2 + f_{n+1}^2 + 2f_2^2 \\
&= 2\sum_{i=2}^{n} f_i^2 + \sum_{i=2}^{n} f_i^2 + f_{n+1}^2 \\
&= 3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2.
\end{aligned}$$

(iii) By using (i) and (ii), we get

$$\begin{aligned}
\sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2 &= \sum_{i=1}^{n}\left(\lambda_i\right)^2 - 2\frac{trF_{f_n}}{n}\sum_{i=1}^{n}\lambda_i + \sum_{i=1}^{n}\left(\frac{trF_{f_n}}{n}\right)^2 \\
&= \sum_{i=1}^{n}\left(\lambda_i\right)^2 - 2\frac{\left(trF_{f_n}\right)^2}{n} + n\left(\frac{trF_{f_n}}{n}\right)^2 \\
&= 3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2 - \frac{1}{n}\left(f_{n+3} - 2\right)^2.
\end{aligned}$$

The proofs of (i'), (ii') and (iii') are similar to the proofs of (i), (ii) and (iii), respectively. □

Theorem 3. *There are the following inequalities for the Fibonacci-Frank matrix F_{f_n} and Lucas-Frank matrix F_{l_n} whose eigenvalues are ordered as $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ and $\mu_1 > \mu_2 > \cdots > \mu_n$, respectively*

(i) $$f_{n+1} \le \lambda_1 \le \sqrt{\left(1 - \frac{1}{n}\right)\left(3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2 - \frac{1}{n}\left(f_{n+3} - 2\right)^2\right)} + \frac{1}{n}\left(f_{n+3} - 2\right),$$

(ii) $$l_n \le \mu_1 \le \sqrt{\left(1 - \frac{1}{n}\right)\left(3\left(l_n l_{n+1} - 2\right) - 2{l_n}^2 - \frac{1}{n}\left(l_{n+2} - 3\right)^2\right)} + \frac{1}{n}\left(l_{n+2} - 3\right).$$

Proof. (i) The equation

$$\lambda_1 - \frac{trF_{f_n}}{n} = -\sum_{i=2}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right) \tag{24}$$

is clearly holds for the Fibonacci-Frank matrix F_{f_n}. Then, we can write the inequality

$$|\lambda_1 - \frac{trF_{f_n}}{n}| \leq \sum_{i=2}^{n} |\lambda_i - \frac{trF_{f_n}}{n}|. \tag{25}$$

By means of [13, 14], we have for the sequence of positive real numbers $q = (q_i)$ and the sequences of non-negative real numbers with similar monocity $a = (a_i)$ and $b = (b_i)$, $(i = 1, 2, \dots, m)$

$$\sum_{i=1}^{m} q_i \sum_{i=1}^{m} q_i a_i b_i \geq \sum_{i=1}^{m} q_i a_i \sum_{i=1}^{m} q_i b_i. \tag{26}$$

Moreover, if $a = (a_i)$ and $b = (b_i)$ has opposite monocity, then the sense of the inequality in (26) reverses [13, 14].

If equation (26) is applied to the right hand side of the inequality (25) by using as $a_i = \dfrac{1}{\mid \lambda_i - \frac{trF_{f_n}}{n} \mid}$ and $b_i = q_i = \mid \lambda_i - \frac{trF_{f_n}}{n} \mid$, then we get

$$\mid \lambda_1 - \frac{trF_{f_n}}{n} \mid \leq \sum_{i=2}^{n} \mid \lambda_i - \frac{trF_{f_n}}{n} \mid \leq \sqrt{(n-1) \sum_{i=2}^{n} \mid \lambda_i - \frac{trF_{f_n}}{n} \mid^2}.$$

Hence, we have

$$\begin{aligned}
\left(\lambda_1 - \frac{trF_{f_n}}{n}\right)^2 &\leq (n-1)\left(\sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2 - \left(\lambda_1 - \frac{trF_{f_n}}{n}\right)^2\right), \\
\frac{n}{n-1}\left(\lambda_1 - \frac{trF_{f_n}}{n}\right)^2 &\leq \sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2, \\
\lambda_1 - \frac{trF_{f_n}}{n} &\leq \sqrt{\left(1 - \frac{1}{n}\right)\sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2}.
\end{aligned}$$

Then, by using Lemma 2 (iii),

$$\lambda_1 \leq \sqrt{\left(1 - \frac{1}{n}\right)\left(3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2 - \frac{1}{n}\left(f_{n+3} - 2\right)^2\right)} + \frac{1}{n}\left(f_{n+3} - 2\right)$$

as desired. Considering equation (14), we have $f_{n+1} < \lambda_1$. This completes the proof.

(ii) The proof is similar to the proof of (i).

□

Theorem 4. *The Euclidean (Frobenius) norms of the Fibonacci-Frank matrix F_{f_n} and Lucas-Frank matrix F_{l_n} are*

(i) $\|F_{f_n}\|_F = \sqrt{\frac{1}{5}\left(3l_{2n+1}+l_{2n}\right)+f_{n+1}^2-n-\frac{1}{2}\left(\left(-1\right)^n+5\right)}$,

(ii) $\|F_{l_n}\|_F = \sqrt{2l_{2n-1}+l_{2n}+l_n^2-2n-\frac{1}{2}\left(5\left(-1\right)^n+11\right)}$,

where f_n and l_n are the ordinary Fibonacci and Lucas numbers, respectively.

Proof. (i) By using the Binet formulas for the Fibonacci numbers, we have

$$\begin{aligned}\|F_{f_n}\|_F^2 &= \sum_{i=1}^{n-1}\left(n-i+2\right)f_{i+1}^2+f_{n+1}^2\\ &= \sum_{i=1}^{n-1}\left(n+2\right)f_{i+1}^2-\sum_{i=1}^{n-1}if_{i+1}^2+f_{n+1}^2\\ &= \frac{(n+2)}{5}\sum_{i=1}^{n-1}\left(\alpha^{i+1}-\beta^{i+1}\right)^2-\frac{i}{5}\sum_{i=1}^{n-1}\left(\alpha^{i+1}-\beta^{i+1}\right)^2+f_{n+1}^2\\ &= \frac{n+2}{5}\sum_{i=1}^{n-1}\left(\alpha^2\left(\alpha^2\right)^i+\beta^2\left(\beta^2\right)^i-2\left(-1\right)^{i+1}\right)\\ &\quad -\frac{i}{5}\sum_{i=1}^{n-1}\left(\alpha^2\left(\alpha^2\right)^i+\beta^2\left(\beta^2\right)^i-2\left(-1\right)^{i+1}\right)+f_{n+1}^2.\end{aligned}$$

Using the well known equalities

$$\sum_{i=1}^{n-1}\alpha^i=\frac{\alpha^n-\alpha}{\alpha-1}\quad\text{and}\quad\sum_{i=1}^{n-1}i\alpha^i=\frac{\alpha-n\alpha^n+\left(n-1\right)\alpha^{n+1}}{\left(\alpha-1\right)^2},\tag{27}$$

we have

$$\begin{aligned}\|F_{f_n}\|_F^2 = {}& \frac{n+2}{5}\left(\alpha^2\frac{\left(\alpha^2\right)^n-\alpha^2}{\alpha^2-1}+\beta^2\frac{\left(\beta^2\right)^n-\beta^2}{\beta^2-1}+2\sum_{i=1}^{n-1}\left(-1\right)^i\right)\\ & -\frac{1}{5}\left(\alpha^2\left(\frac{\alpha^2-n\left(\alpha^2\right)^n+\left(n-1\right)\left(\alpha^2\right)^{n+1}}{\left(\alpha^2-1\right)^2}\right)\right.\\ & \left.+\beta^2\left(\frac{\beta^2-n\left(\beta^2\right)^n+\left(n-1\right)\left(\beta^2\right)^{n+1}}{\left(\beta^2-1\right)^2}\right)+2\sum_{i=1}^{n-1}i\left(-1\right)^i\right)+f_{n+1}^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{n+2}{5}\left(\left(\alpha^{2n+1}+\beta^{2n+1}\right)-\left(\alpha^{3}+\beta^{3}\right)-\left((-1)^{n}+1\right)\right)\\
&\quad -\frac{1}{5}\left(\left(\alpha^{2}+\beta^{2}\right)-n\left(\alpha^{2n}+\beta^{2n}\right)+(n-1)\left(\alpha^{2n+2}+\beta^{2n+2}\right)\right.\\
&\qquad \left.-\frac{1+(2n-1)(-1)^{n}}{2}\right)+f_{n+1}^{2}\\
&= \frac{n+2}{5}\left(l_{2n+1}-l_{3}\right)-\frac{1}{5}\left(l_{2}-nl_{2n}+(n-1)\,l_{2n+2}\right)-\frac{(-1)^{n}}{2}\\
&\qquad -\frac{2n+3}{10}+f_{n+1}^{2}\\
&= \frac{1}{5}\left(3l_{2n+1}+l_{2n}\right)+f_{n+1}^{2}-n-\frac{1}{2}\left((-1)^{n}+5\right).
\end{aligned}
$$

Thus, desired result is obtained.

(ii) The proof is similar to the proof of (i).

□

Theorem 5. *There are the following upper bounds for the spectral norms of the Fibonacci-Frank matrix F_{f_n} and Lucas-Frank matrix F_{l_n}*

(i) $\|F_{f_n}\|_2 \le \sqrt{\left(f_{n+1}^2+1\right)\left(f_n^2+n-1\right)}$,

(ii) $\|F_{l_n}\|_2 \le \sqrt{\left(l_n^2+1\right)\left(l_{n-1}^2+n-1\right)}$,

where f_n and l_n are the ordinary Fibonacci and Lucas numbers, respectively.

Proof. (i) By using the Hadamard product, the matrix F_{f_n} can be written as

$$
F_{f_n} = \underbrace{\begin{bmatrix} f_{n+1} & 1 & 0 & 0 & \dots & 0 & 0 \\ f_n & f_n & 1 & 0 & \dots & 0 & 0 \\ f_{n-1} & f_{n-1} & f_{n-1} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_3 & f_3 & f_3 & f_3 & \dots & f_3 & 1 \\ f_2 & f_2 & f_2 & f_2 & \dots & f_2 & f_2 \end{bmatrix}}_{A} \circ \underbrace{\begin{bmatrix} 1 & f_n & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & f_{n-1} & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & f_{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & f_2 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}}_{B}.
$$

Considering the inequality $2f_n^2 < f_{n+1}^2$, which can be proven by the mathematical induction method, we have the maximum row length norm of the matrix $A = [a_{ij}]$ and maximum column length norm of the matrix $B = [b_{ij}]$ as

$$
r_1(A) = \max_i \sqrt{\sum_j \mid a_{ij} \mid^2} = \sqrt{f_{n+1}^2+1}, \tag{28}
$$

$$c_1(B) = \max_j \sqrt{\sum_i |\, b_{ij}\, |^2} = \sqrt{f_n^2 + n - 1}, \tag{29}$$

respectively. Then, by using equation (8), we have

$$\|F_{f_n}\|_2 \le r_1(A)\, c_1(B) = \sqrt{\left(f_{n+1}^2 + 1\right)\left(f_n^2 + n - 1\right)}. \tag{30}$$

(ii) The proof is similar to the proof of (i).

□

Now, we give the following example to illustrate our results:

Example 1. *Consider the Fibonacci-Frank and Lucas-Frank matrices for $n = 8$. Then, the matrices are*

$$F_{f_8} = \begin{bmatrix} 34 & 21 & 0 & 0 & 0 & 0 & 0 & 0 \\ 21 & 21 & 13 & 0 & 0 & 0 & 0 & 0 \\ 13 & 13 & 13 & 8 & 0 & 0 & 0 & 0 \\ 8 & 8 & 8 & 8 & 5 & 0 & 0 & 0 \\ 5 & 5 & 5 & 5 & 5 & 3 & 0 & 0 \\ 3 & 3 & 3 & 3 & 3 & 3 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad F_{l_8} = \begin{bmatrix} 47 & 29 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 29 & 18 & 0 & 0 & 0 & 0 & 0 \\ 18 & 18 & 18 & 11 & 0 & 0 & 0 & 0 \\ 11 & 11 & 11 & 11 & 7 & 0 & 0 & 0 \\ 7 & 7 & 7 & 7 & 7 & 4 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 & 3 & 0 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The characteristic polynomials of the matrices F_{f_8} and F_{l_8} are

$$\begin{aligned} P_8(\lambda) &= \lambda^8 - 87\lambda^7 + 2137\lambda^6 - 19968\lambda^5 + 79377\lambda^4 - 139303\lambda^3 + 106949\lambda^2 \\ &\qquad - 33162\lambda + 3120, \\ Q_8(\mu) &= \mu^8 - 120\mu^7 + 4054\mu^6 - 51792\mu^5 + 278231\mu^4 - 647740\mu^3 \\ &\qquad + 652566\mu^2 - 268188\mu + 33264 \end{aligned}$$

and the eigenvalues of the matrices F_{f_8} and F_{l_8} are

$$\begin{array}{ll} \lambda_1 = 53.563, & \mu_1 = 74.018, \\ \lambda_2 = 18.591, & \mu_2 = 25.689, \\ \lambda_3 = 7.889, & \mu_3 = 10.920, \\ \lambda_4 = 3.753, & \mu_4 = 5.232, \\ \lambda_5 = 1.756, & \mu_5 = 2.300, \\ \lambda_6 = 0.851, & \mu_6 = 1.009, \\ \lambda_7 = 0.432, & \mu_7 = 0.618, \\ \lambda_8 = 0.164, & \mu_8 = 0.214. \end{array}$$

It is clear that λ_6, λ_7 and λ_8 are in the interval $(0,1)$, then the matrix F_{f_8} has three eigenvalues in the interval $(0,1)$. Similarly, since μ_7 and μ_8 are in the interval $(0,1)$, the matrix F_{l_8} has two eigenvalues in the interval $(0,1)$.

There are the following bounds for the maximum eigenvalues of the matrices F_{f_8} *and* F_{l_8} *from Theorem 3*

$$f_9 \leq \lambda_1 \leq \sqrt{\left(1-\frac{1}{8}\right)\left(3\left(f_8 f_9-1\right)+f_9^2-\frac{1}{8}\left(f_{11}-2\right)^2\right)}+\frac{1}{8}\left(f_{11}-2\right),$$

$$34 \leq \lambda_1 = 53.563 \leq 56.210$$

and

$$l_8 \leq \mu_1 \leq \sqrt{\left(1-\frac{1}{8}\right)\left(3\left(l_8 l_9-2\right)-2{l_8}^2-\frac{1}{8}\left(l_{10}-3\right)^2\right)}+\frac{1}{8}\left(l_{10}-3\right),$$

$$47 \leq \mu_1 = 74.018 \leq 77.694.$$

Considering Theorem 4, the Euclidean norms of the matrices F_{f_8} *and* F_{l_8} *are*

$$\|F_{f_8}\|_F = \sqrt{\frac{1}{5}\left(3l_{17}+l_{16}\right)+f_9^2-8-\frac{1}{2}\left(\left(-1\right)^8+5\right)} = 61.065$$

and

$$\|F_{l_8}\|_F = \sqrt{2l_{15}+l_{16}+l_8^2-16-\frac{1}{2}\left(5\left(-1\right)^8+11\right)} = 84.380.$$

By using Theorem 5 we have the following upper bounds for the spectral norms of the matrices F_{f_8} *and* F_{l_8}

$$\|F_{f_8}\|_2 = 56.911 \leq \sqrt{\left(f_9^2+1\right)\left(f_8^2+7\right)} = 719.955$$

and

$$\|F_{l_8}\|_2 = 78.643 \leq \sqrt{\left(l_8^2+1\right)\left(l_7^2+7\right)} = 1368.970.$$

In Example 1, we gave our results for Fibonacci-Frank matrix F_{f_n} and Lucas-Frank matrix F_{l_n} for n=8. The bounds we have obtained for the maximum eigenvalues of these matrices for increasing values of n obtained from Theorem 3 are given in the following tables:

TABLE 2. The bounds for the maximum eigenvalues of the matrix F_{f_n} according to the increasing values of n.

$n=2$	$2 \leq \lambda_1 = 2.618 \leq 2.618$
$n=3$	$3 \leq \lambda_1 = 4.791 \leq 4.828$
$n=4$	$5 \leq \lambda_1 = 7.796 \leq 8.095$
$n=5$	$8 \leq \lambda_1 = 12.654 \leq 13.130$
$n=6$	$13 \leq \lambda_1 = 20.455 \leq 21.337$
$n=7$	$21 \leq \lambda_1 = 33.108 \leq 34.654$
$n=8$	$34 \leq \lambda_1 = 53.563 \leq 56.210$
$n=9$	$55 \leq \lambda_1 = 86.672 \leq 91.153$
$n=10$	$89 \leq \lambda_1 = 140.235 \leq 147.760$
$n=20$	$10946 \leq \lambda_1 = 17247.848 \leq 18340.237$
$n=30$	$1346269 \leq \lambda_1 = 2121345.008 \leq 2262287.634$
$n=40$	$165580141 \leq \lambda_1 = 260908188.115 \leq 278631218.037$
$n=50$	$20365011074 \leq \lambda_1 = 32089585793.157 \leq 34297378604.5$

TABLE 3. The bounds for the maximum eigenvalues of the matrix F_{l_n} according to the increasing values of n.

$n=2$	$3 \leq \mu_1 = 3.414 \leq 3.414$
$n=3$	$4 \leq \mu_1 = 6.702 \leq 6.722$
$n=4$	$7 \leq \mu_1 = 10.761 \leq 11.034$
$n=5$	$11 \leq \mu_1 = 17.512 \leq 18.186$
$n=6$	$18 \leq \mu_1 = 28.258 \leq 29.494$
$n=7$	$29 \leq \mu_1 = 45.762 \leq 47.918$
$n=8$	$47 \leq \mu_1 = 74.018 \leq 77.694$
$n=9$	$76 \leq \mu_1 = 119.780 \leq 125.988$
$n=10$	$123 \leq \mu_1 = 193.798 \leq 204.213$
$n=20$	$15127 \leq \mu_1 = 23835.939 \leq 25345.591$
$n=30$	$1860498 \leq \mu_1 = 2931626.699 \leq 3126404.622$
$n=40$	$228826127 \leq \mu_1 = 360566248.032 \leq 385058873.003$
$n=50$	$28143753123 \leq \mu_1 = 44346716881.237 \leq 47397811506.4$

According to Table 2 and Table 3, the bounds are quite close to the exact values of the maximum eigenvalues of the matrices F_{f_n} and F_{l_n}. Also, the upper bounds are closer to the maximum eigenvalues than the lower bounds. Additionally, we give the following figures to better illustrate this result.

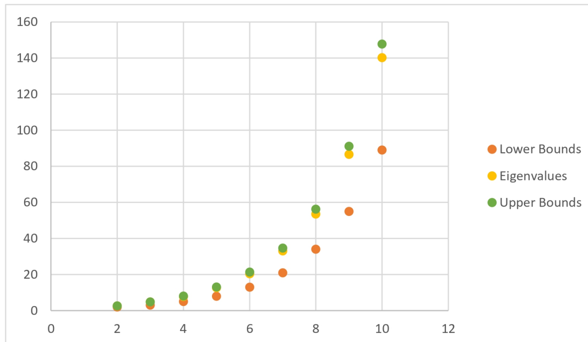

FIGURE 1. The graph of the maximum eigenvalues of the matrix F_{f_n} and their lower and upper bounds for $n = 2, 3, 4, \dots, 10$.

In Figure 1, the horizontal axis contains the values of n from 2 to 10, and the vertical axis contains the maximum eigenvalues of the matrix F_{f_n} corresponding to these values of n, as well as the values of its lower and upper bounds.

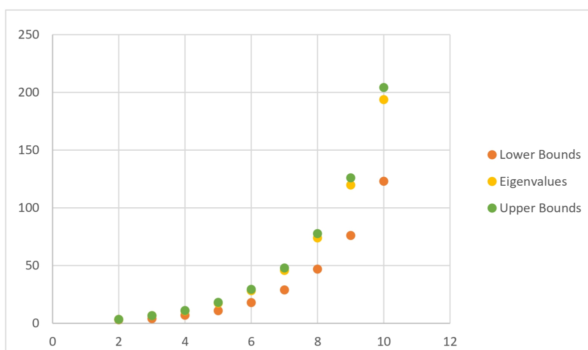

FIGURE 2. The graph of the maximum eigenvalues of the matrix F_{l_n} and their lower and upper bounds for $n = 2, 3, 4, \dots, 10$.

Similarly, in Figure 2, the values of n between 2 and 10 are on the horizontal axis, and the maximum eigenvalue of the matrix F_{l_n} corresponding to these n values and their lower and upper bounds are on the horizontal axis.

As indicated by the graphs in Figures 1 and 2, the lower and upper bounds are very close to the maximum eigenvalues of the matrices F_{f_n} and F_{l_n} for small values of n. As the value of n increases, the distance between these bounds and the maximum eigenvalues widens. In this case, it is observed that the upper bounds remain closer to the maximum eigenvalues than the lower bounds.

Author Contribution Statements The authors confirm sole responsibility for the following concepts involved in this study and design, data collection, analysis and interpretation of results, and manuscript preparation.

Declaration of Competing Interests The authors declare that they have no competing interests.

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$B-$RIESZ POTENTIAL IN $B-$LOCAL MORREY-LORENTZ SPACES

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Abstract. In this paper, the Riesz potential ($B-$Riesz potential) which are generated by the Laplace-Bessel differential operator will be studied. We obtain the necessary and sufficient conditions for the boundedness of the $B-$Riesz potential I_γ^α in the $B-$local Morrey-Lorentz spaces $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ with the use of the rearrangement inequalities and boundedness of the Hardy operators H_ν^β and $\mathcal{H}_\nu^\beta$ with power weights.

1. Introduction

Lorentz spaces, which are very useful in the theory of interpolation, have first been introduced by Lorentz [18]. These spaces are Banach spaces and generalizations of Lebesgue spaces. The Lorentz space $L_{p,q}(\mathbb{R}^n)$, $0 < p, q \leq \infty$, is known as the set of all measurable functions f such that

$$\|f\|_{L_{p,q}(\mathbb{R}^n)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,\infty)} < \infty.$$

Here, by f^* we denote the nonincreasing rearrangement of f and

$$f^*(t) = \inf \{\lambda > 0 : |\{y \in \mathbb{R}^n : |f(y)| > \lambda\}| \leq t\}, \quad t \in (0, \infty).$$

The necessary and sufficient condition for the functional $\|\cdot\|_{L_{p,q}}$ be a norm is $1 \leq q \leq p$ or $p = q = \infty$. If $p = q = \infty$, then $L_{\infty,\infty}(\mathbb{R}^n) \equiv L_\infty(\mathbb{R}^n)$. One can easily observe that $L_{p,p}(\mathbb{R}^n) \equiv L_p(\mathbb{R}^n)$ and $L_{p,\infty}(\mathbb{R}^n) \equiv WL_p(\mathbb{R}^n)$. It is obvious that $L_{p,q} \subset L_p \subset L_{p,r} \subset WL_p$ for $0 < q \leq p \leq q \leq r \leq \infty$. For further details, we refer the interested reader to [5, 18, 19].

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On Lorentz spaces, the boundedness of the Riesz potential and the boundedness of its version related to the Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i} + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad 1 \le k \le n,$$

have been studied by many researchers [3, 4, 10–15, 21]. The Riesz potential connected with the Laplace-Bessel differential operator ($B-$Riesz potential) is generated by generalized shift operator

$$T^y f(x) := C_{\gamma,k} \int_0^{\pi} \dots \int_0^{\pi} f\left[(x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k}, x'' - y''\right] d\gamma(\alpha).$$

Here $C_{\gamma,k} = \pi^{-\frac{k}{2}}\Gamma(\frac{\gamma_i+1}{2})[\Gamma(\frac{\gamma_i}{2})]^{-1}$, $(x_i, y_i)_{\alpha_i} = (x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $1 \le k \le n$ and $d\gamma(\alpha) = \prod_{i=1}^{k} \sin^{\gamma_i - 1}\alpha_i \, d\alpha_i$ [16, 17].

The $B-$convolution operator is defined as:

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^{\gamma} dy.$$

Here, $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0, 1 \le k \le n\}$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$. Let us set $x = (x', x'')$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, and $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$.

The purpose of this paper is to obtain the boundedness of the $B-$Riesz potential operator I_γ^α on $B-$local Morrey-Lorentz spaces with the use of the rearrangement inequalities and the Hardy inequality. Local Morrey-Lorentz spaces $M^{\text{loc}}_{p,q,\lambda}(\mathbb{R}^n)$ which have first been introduced by Aykol et al. [2] and are generalizations of Lorentz spaces. One has $M^{\text{loc}}_{p,q,0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. They have also proved that the Riesz potential operator is bounded in these spaces. In this study, we consider the $B-$Riesz potential by

$$I_\gamma^\alpha f(x) = \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha - Q} f(y) (y')^{\gamma} dy, \quad 0 < \alpha < Q.$$

The maximal operator has a crucial role in the study of the regularity of some partial differential equations and in the study of the boundedness of some singular integrals and on the differentiability properties of functions. For a function $f \in L^{\text{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$, the $B-$maximal operator and $B-$fractional maximal operator are defined by, (see [7]) respectively,

$$M_\gamma f(x) = \sup_{r>0} |B_+(0,r)|_\gamma^{-1} \int_{B_+(0,r)} T^y |f(x)| (y')^{\gamma} dy,$$

$$M_\gamma^\alpha f(x) = \sup_{r>0} |B_+(0,r)|_\gamma^{\frac{\alpha}{Q}-1} \int_{B_+(0,r)} T^y |f(x)| (y')^{\gamma} dy, \quad 0 \le \alpha < Q,$$

where $B_+(x,r) = \{y \in \mathbb{R}^n_{k,+} : |x-y| < r\}$. Let $B_+(0,r) \subset \mathbb{R}^n_{k,+}$ be a measurable set, then

$$|B_+(0,r)|_\gamma = \int_{B_+(0,r)} (x')^\gamma dx = \omega(n,k,\gamma) r^Q,$$

where $\omega(n,k,\gamma) = \dfrac{\pi^{\frac{n-k}{2}}}{2^k} \displaystyle\prod_{i=1}^{k} \dfrac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}$, $Q = n + |\gamma|$. It is easy to observe that $M^0_\gamma f = M_\gamma f$ for $\alpha = 0$ (see [7]). It is well known that the inequality $M^\alpha_\gamma \leq C\, I^\alpha_\gamma$ holds.

On local Morrey-Lorentz space, the necessary and sufficient conditions for the boundedness of the Riesz potential operator are given in [13]. On the other hand, the $B-$ Riesz potential has been investigated in various function spaces by many mathematicians (see, for example [3,10–12]). The above results inspire us to investigate the boundedness of the $B-$Riesz potential defined on $B-$local Morrey-Lorentz spaces.

Throughout the paper, C denotes a positive constant independent of appropriate parameters and not necessary the same at each occurrence.

2. Preliminaries

Given any measurable set E with $|E|_\gamma = \displaystyle\int_E (x')^\gamma dx$ and a measurable function $f : \mathbb{R}^n_{k,+} \to \mathbb{R}$, the $\gamma-$rearrangement of f in decreasing order is defined as

$$f^*_\gamma(t) = \inf\{s > 0 : f_{*,\gamma}(s) \leq t\}, \quad \forall t \in (0,\infty),$$

where $f_{*,\gamma}(s)$ denotes the $\gamma-$distribution function of f given by

$$f_{*,\gamma}(s) = \left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x)| > s\right\}\right|_\gamma .$$

The average function of f^{**}_γ is defined as

$$f^{**}_\gamma(t) = \frac{1}{t}\int_0^t f^*_\gamma(s) ds, \quad t > 0,$$

and the following inequality holds (see [20]):

$$(f+g)^{**}_\gamma(t) \leq f^{**}_\gamma(t) + g^{**}_\gamma(t).$$

Now, we give some characteristics of the $\gamma-$rearrangement of functions:

- if $0 < p < \infty$, then
$$\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx = \int_0^\infty (f^*_\gamma(t))^p dt,$$
- for any $t > 0$,
$$\sup_{|E|_\gamma = t} \int_E |f(x)|(x')^\gamma dx = \int_0^t f^*_\gamma(s) ds, \tag{1}$$

- the following inequality holds:

$$\int_{\mathbb{R}^n_{k,+}} |f(x)g(x)|(x')^\gamma dx \le \int_0^\infty f^*_\gamma(t) g^*_\gamma(t) dt,$$

- the following inequality holds (see [5, 20, 22]):

$$(f+g)^*_\gamma(t) \le f^*_\gamma(t/2) + g^*_\gamma(t/2). \tag{2}$$

Definition 1. *[18] If $0 < p, q \le \infty$, then we define the Lorentz space $L_{p,q,\gamma}(\mathbb{R}^n_{k,+})$ is the set of all measurable functions $f \in \mathbb{R}^n_{k,+}$ such that*

$$\|f\|_{L_{p,q,\gamma}} = \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*_\gamma(t) \right\|_{L_q(0,\infty)} < \infty.$$

If $0 < p \le \infty$, $q = \infty$, then $L_{p,\infty,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\gamma}(\mathbb{R}^n_{k,+})$, where $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ is weak Lebesgue space of all measurable functions f such that

$$\|f\|_{WL_{p,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{t>0} t^{1/p} f^*_\gamma(t) < \infty, \quad 1 \le p < \infty.$$

If $p = q = \infty$ or $1 \le q \le p$, then the functional $\|f\|_{p,q,\gamma}$ is a norm [5, 11, 22]. However if $p = q = \infty$, then $L_{\infty,\infty,\gamma}(\mathbb{R}^n_{k,+}) = L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$.

In case $0 < p, q \le \infty$, a functional $\|\cdot\|^*_{L_{p,q,\gamma}}$ is given by

$$\|f\|^*_{L_{p,q,\gamma}} = \|f\|^*_{L_{p,q,\gamma}(0,\infty)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^{**}_\gamma(t)\|_{L_q(0,\infty)},$$

which is a norm on $L_{p,q,\gamma}(\mathbb{R}^n_{k,+})$ for $1 \le q \le \infty$, $1 < p < \infty$ or $p = q = \infty$.

If $1 < p \le \infty$, $1 \le q \le \infty$, then

$$\|f\|_{p,q,\gamma} \le \|f\|^*_{p,q,\gamma} \le \frac{p}{p-1} \|f\|_{p,q,\gamma},$$

that is, $\|f\|_{p,q,\gamma}$ and $\|f\|^*_{p,q,\gamma}$ are equivalent.

Definition 2. *[8] Let $1 \le p < \infty$, and $0 \le \lambda \le Q$. The $B-$Morrey space $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ is the set of all measurable functions with $f \in L^{\text{loc}}_{p,\gamma}(\mathbb{R}^n_{k,+})$ such that*

$$\|f\|_{L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{x \in \mathbb{R}^n_{k,+}, \rho > 0} \rho^{-\frac{\lambda}{p}} \|f\|_{L_{p,\gamma}(B_+(x,\rho))} < \infty.$$

If $\lambda = 0$, then $L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+})$; if $\lambda > Q$ or $\lambda < 0$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n_{k,+}$. Also, the weak $B-$Morrey space $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ is the set of all functions $f \in WL^{\text{loc}}_{p,\gamma}(\mathbb{R}^n_{k,+})$ with following norm

$$\|f\|_{WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{x \in \mathbb{R}^n_{k,+}, \rho > 0} \rho^{-\frac{\lambda}{p}} \|f\|_{WL_{p,\gamma}(B_+(x,\rho))} < \infty.$$

Definition 3. *[6] Let $0 \le \lambda \le 1$ and $0 \le p < \infty$. The local Morrey space $LM_{p,\lambda} \equiv LM_{p,\lambda}(0,\infty)$ is the set of all functions $f \in L^{\text{loc}}_p(0,\infty)$ such that*

$$\|f\|_{LM_{p,\lambda}(0,\infty)} = \sup_{\rho > 0} \rho^{-\frac{\lambda}{p}} \|f\|_{L_p(0,\rho)} < \infty.$$

Moreover, $WLM_{p,\lambda} \equiv WLM_{p,\lambda}(0,\infty)$ *denotes the weak local Morrey space of all functions* $f \in WL_p^{\rm loc}(0,\infty)$ *such that*

$$\|f\|_{WLM_{p,\lambda}(0,\infty)} = \sup_{\rho>0} \rho^{-\frac{\lambda}{p}} \|f\|_{WL_p(0,\rho)} < \infty.$$

Definition 4. [9] *Given a function* $f \in L_{1,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n)$*, and a ball* $B_+(x,r)$*. By* $f_{B_+}(x)$ *we denote the average of* $T^y f$ *on the ball* B_+*,*

$$f_{B_+}(x) = |B_+|_\gamma^{-1} \int_{B_+} T^y f(x)(y')^\gamma dy.$$

The $BMO-$*Bessel space* $BMO_\gamma(\mathbb{R}_{k,+}^n)$ *is the set of all functions on* $L_{1,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n)$ *with*

$$\|f\|_{*,\gamma} = \sup_{B_+} |B_+|_\gamma^{-1} \int_{B_+} |T^y f(x) - f_{B_+}|(y')^\gamma dy < \infty.$$

Definition 5. *Let* $0 < p,q \leq \infty$ *and* $0 \leq \lambda \leq 1$*. The* $B-$*local Morrey-Lorentz space* $M_{p,q,\lambda,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n)$ *is set of all measurable functions with the quasinorm*

$$\|f\|_{M_{p,q,\lambda,\gamma}^{\rm loc}} = \sup_{\rho>0} \rho^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t)\|_{L_q(0,\rho)} < \infty.$$

If $\lambda > 1$ *or* $\lambda < 0$*, then* $M_{p,q,\lambda,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n) = \Theta$*, where* Θ *is the set of all functions equivalent to* 0 *on* $\mathbb{R}_{k,+}^n$*. Also,*

$$M_{p,q,0,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n) = L_{p,q,\gamma}(\mathbb{R}_{k,+}^n) \quad \textit{and} \quad M_{p,p,\lambda,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n) \equiv M_{p,0,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n).$$

The weak $B-$*local Morrey-Lorentz space* $WM_{p,q,\lambda,\gamma}^{\rm loc}(\mathbb{R}_{k,+}^n)$ *is the set of all measurable functions with the quasinorm*

$$\|f\|_{WM_{p,q,\lambda,\gamma}^{\rm loc}} = \sup_{\rho>0} \rho^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t)\|_{WL_q(0,\rho)} < \infty.$$

We need the boundedness of the Hardy operators which will be used in the proof of our main theorem.

Definition 6. [21] *Let* φ *be a measurable function on* $(0,\infty)$ *and* $\beta \in \mathbb{R}$*. The weighted Hardy operators* H_ν^β *and* $\mathcal{H}_\nu^\beta$ *with power weights are defined as*

$$H_\nu^\beta \varphi(t) = t^{\beta+\nu-1} \int_0^t \frac{\varphi(y)}{y^\nu} dy, \qquad \mathcal{H}_\nu^\beta \varphi(t) = t^{\beta+\nu} \int_t^\infty \frac{\varphi(y)}{y^{\nu+1}} dy.$$

In the following theorem, we state that the Hardy operators are bounded in local Morrey and weak local Morrey spaces.

Theorem 1. [1, 21] *Let* $0 < \lambda < 1$*,* $0 < \beta < 1-\lambda$*,* $1 \leq r < \dfrac{1-\lambda}{\beta}$ *and* $\dfrac{1}{r} - \dfrac{1}{s} = \dfrac{\beta}{1-\lambda}$*.*

i. *If* $\nu < \frac{1}{r'} + \frac{\lambda}{r}$, *then*

$$\left\| H_\nu^\beta \varphi \right\|_{LM_{s,\lambda}(0,\infty)} \le C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)}.$$

ii. *If* $\nu = \frac{1}{r'} + \frac{\lambda}{r}$, *then*

$$\left\| H_\nu^\beta \varphi \right\|_{WLM_{s,\lambda}(0,\infty)} \le C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)}.$$

iii. *If* $\nu > \frac{\lambda - 1}{r}$, *then*

$$\left\| \mathcal{H}_\nu^\beta \varphi \right\|_{LM_{s,\lambda}(0,\infty)} \le C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)}.$$

iv. *If* $\nu = \frac{\lambda - 1}{r}$, *then*

$$\left\| \mathcal{H}_\nu^\beta \varphi \right\|_{WLM_{s,\lambda}(0,\infty)} \le C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)}.$$

3. $B-$Riesz Potential in $B-$Local Morrey-Lorentz Space

This section devoted to obtain the boundedness of the $B-$Riesz potential in $B-$local Morrey-Lorentz and weak $B-$local Morrey-Lorentz space.

For the $B-$Riesz potential, the following inequality

$$(I_\gamma^\alpha f)_\gamma^*(t) \le (I_\gamma^\alpha f)_\gamma^{**}(t) \le C_2 \left(t^{\frac{\alpha}{Q}-1} \int_0^t f_\gamma^*(y)dy + \int_t^\infty y^{\frac{\alpha}{Q}-1} f_\gamma^*(y)dy \right) \qquad (3)$$

holds, where $C_2 = C_{\gamma,k}(Q/\alpha)^2 \omega(n,k,\gamma)^{(Q-\alpha)/Q}$ (see [10]).

Theorem 2. *Let* $0 \le \lambda < 1$, $0 < \alpha < Q$, $1 \le q \le \infty$, $1 \le r \le s \le \infty$, $\frac{r}{r+\lambda} \le p \le \left(\frac{\lambda}{r} + \frac{\alpha}{Q}\right)^{-1}$ *and* $f \in M_{p,r,\lambda,\gamma}^{\text{loc}}$.

(i) *If* $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{Q}\right)^{-1}$, *then* $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{Q}$ *is necessary and sufficient condition for the boundedness of* I_γ^α *from* $M_{p,r,\lambda,\gamma}^{\text{loc}}$ *to* $M_{q,s,\lambda,\gamma}^{\text{loc}}$.

(ii) *If* $p = \frac{r}{r+\lambda}$, *then* $1 - \frac{1}{q} = \frac{\alpha}{Q} - \frac{\lambda}{s}$ *is necessary and sufficient condition for the boundedness of* I_γ^α *from* $M_{p,r,\lambda,\gamma}^{\text{loc}}$ *to* $WM_{q,s,\lambda,\gamma}^{\text{loc}}$.

Proof. (i) *Sufficiency.* Let $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{Q}\right)^{-1}$. From (3), we have

$$\begin{aligned}
\|I_\gamma^\alpha f\|_{M_{q,s,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)} &= \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f)_\gamma^*(t) \right\|_{L_s(0,\rho)} \\
&\le \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f)_\gamma^{**}(t) \right\|_{L_s(0,\rho)} \\
&\le C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}} \left(t^{\frac{\alpha}{Q}-1} \int_0^t f_\gamma^*(y)dy + \int_t^\infty y^{\frac{\alpha}{Q}-1} f_\gamma^*(y)dy \right) \right\|_{L_s(0,\rho)}
\end{aligned}$$

$$\leq C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{Q}-1} \int_0^t f_\gamma^*(y)dy \right\|_{L_s(0,\rho)}$$
$$+ C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}} \int_t^\infty y^{\frac{\alpha}{Q}-1} f_\gamma^*(y)dy \right\|_{L_s(0,\rho)}$$
$$= I_1 + I_2.$$

We take $\nu = \dfrac{1}{p} - \dfrac{1}{r}$ and $\varphi(y) = y^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(y)$. Then, we have

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} - \frac{1}{p} + \frac{\alpha}{Q}.$$

From Theorem 1, we can write $\beta = (1-\lambda)\left(\dfrac{1}{r} - \dfrac{1}{s}\right)$. Then we get $\dfrac{1}{p} - \dfrac{1}{q} = \lambda\left(\dfrac{1}{r} - \dfrac{1}{s}\right) + \dfrac{\alpha}{Q}$. Therefore, again by Theorem 1, we get

$$I_1 = C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{Q}-1} \int_0^t f_\gamma^*(y)dy \right\|_{L_s(0,\rho)}$$
$$= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\beta+\nu-1} \int_0^t \frac{\varphi(y)}{y^\nu} dy \right\|_{L_s(0,\rho)}$$
$$= C \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| H_\nu^\beta \varphi \right\|_{L_s(0,\rho)}$$
$$= C \left\| H_\nu^\beta \varphi \right\|_{LM_{s,\lambda}(0,\infty)}$$
$$\leq C \|\varphi\|_{LM_{r,\lambda}(0,\infty)} = C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \|\varphi\|_{L_r(0,\rho)}$$
$$= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \|t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(y)\|_{L_r(0,\rho)} = C \|f\|_{M^{\text{loc}}_{p,r,\lambda,\gamma}(\mathbb{R}^n_{k,+})}.$$

We now estimate I_2. We take $\nu = \dfrac{1}{p} - \dfrac{1}{r} - \dfrac{\alpha}{Q}$ and $\varphi(y) = y^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(y)$. Then, we get

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} - \frac{1}{p} + \frac{\alpha}{Q}.$$

Therefore, by using Theorem 1, we obtain

$$I_2 = C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}} \int_t^\infty y^{\frac{\alpha}{Q}-1} f_\gamma^*(y)dy \right\|_{L_s(0,\rho)}$$
$$= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\beta+\nu} \int_t^\infty \frac{\varphi(y)}{y^{\nu+1}} dy \right\|_{L_s(0,\rho)}$$
$$= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| \mathcal{H}_\nu^\beta \varphi \right\|_{L_s(0,\rho)}$$

$$= C \left\| \mathcal{H}_\nu^\beta \varphi \right\|_{LM_{s,\lambda}(0,\infty)}$$
$$\leq C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)} = C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \left\| \varphi \right\|_{L_{r,\lambda}(0,\rho)}$$
$$= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \| t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(\tau) \|_{L_r(0,\rho)} = C \left\| f \right\|_{M^{\mathrm{loc}}_{p,r,\lambda,\gamma}(\mathbb{R}^n_{k,+})}.$$

Hence, we obtain that the $B-$Riesz potential I_γ^α bounded from $M^{\mathrm{loc}}_{p,r,\lambda,\gamma}$ to $M^{\mathrm{loc}}_{q,s,\lambda,\gamma}$.

Necessity. Suppose that the $B-$Riesz potential I_γ^α is bounded from $M^{\mathrm{loc}}_{p,r,\lambda,\gamma}$ to $M^{\mathrm{loc}}_{q,s,\lambda,\gamma}$ and $\dfrac{r}{r+\lambda} \leq p \leq \left(\dfrac{\lambda}{r} + \dfrac{\alpha}{Q}\right)^{-1}$. For $\tau > 0$, we define $f_\tau(x) := f(\tau x)$. Then $(f_\tau)_\gamma^*(t) = f_\gamma^*(t\tau^Q)$ and

$$\begin{aligned} \|f_\tau\|_{M^{\mathrm{loc}}_{p,r,\lambda,\gamma}} &= \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \| t^{\frac{1}{p}-\frac{1}{r}} (f_\tau)_\gamma^*(t) \|_{L_r(0,\rho)} \\ &= \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \| t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(t\tau^Q) \|_{L_r(0,\rho)} \\ &= \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \tau^{-\frac{Q}{p}} \| t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(t) \|_{L_r(0,\rho\tau^Q)} \\ &= \tau^{-\frac{Q}{p}+\frac{Q\lambda}{r}} \sup_{\rho>0} (\rho\tau^Q)^{-\frac{\lambda}{r}} \| t^{\frac{1}{p}-\frac{1}{r}} f_\gamma^*(t) \|_{L_r(0,\rho\tau^Q)} \\ &= \tau^{-Q(\frac{1}{p}-\frac{\lambda}{r})} \|f\|_{M^{\mathrm{loc}}_{p,r,\lambda,\gamma}}. \end{aligned}$$

Also, $(I_\gamma^\alpha f_\tau)(x) = \tau^{-\alpha}(I_\gamma^\alpha f)(\tau^Q x)$ and $(I_\gamma^\alpha f_\tau)_\gamma^*(t) = \tau^{-\alpha}(I_\gamma^\alpha f)_\gamma^*(t\tau^Q)$. Then, we get

$$\begin{aligned} \|I_\gamma^\alpha f_\tau\|_{M^{\mathrm{loc}}_{q,s,\lambda,\gamma}} &= \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \| t^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f_\tau)_\gamma^*(t) \|_{L_s(0,\rho)} \\ &= \tau^{-\alpha} \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \| t^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f)_\gamma^*(t\tau^Q) \|_{L_s(0,\rho)} \\ &= \tau^{-\alpha} \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left(\int_0^\infty (t\tau^Q)^{\frac{s}{q}-1} ((I_\gamma^\alpha f)_\gamma^*(t\tau^Q))^s d((t\tau^Q)) \right)^{\frac{1}{s}} \tau^{-\frac{Q}{q}} \\ &= \tau^{-\alpha-\frac{Q}{q}-\frac{Q\lambda}{s}} \sup_{\rho>0} (\rho\tau^Q)^{-\frac{\lambda}{s}} \| t^{\frac{1}{q}-\frac{1}{s}} I_\gamma^\alpha f_\gamma^*(t) \|_{L_s(0,\rho)} \\ &= \tau^{-\alpha-Q(\frac{1}{q}-\frac{\lambda}{s})} \|I_\gamma^\alpha f\|_{M^{\mathrm{loc}}_{q,s,\lambda,\gamma}}. \end{aligned}$$

Since the $B-$Riesz potential I_γ^α is bounded from $M^{\mathrm{loc}}_{p,r,\lambda,\gamma}$ to $M^{\mathrm{loc}}_{q,s,\lambda,\gamma}$, we can write $\|I_\gamma^\alpha f\|_{M^{\mathrm{loc}}_{q,s,\lambda,\gamma}} \leq C\|f\|_{M^{\mathrm{loc}}_{p,r,\lambda,\gamma}}$, where $C > 0$ is a constant. Then

$$\begin{aligned} \|I_\gamma^\alpha f\|_{M^{\mathrm{loc}}_{q,s,\lambda,\gamma}} &= \tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})} \|I_\gamma^\alpha f_\tau\|_{M^{\mathrm{loc}}_{q,s,\lambda,\gamma}} \\ &\leq C\tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})} \|f_\tau\|_{M^{\mathrm{loc}}_{p,r,\lambda,\gamma}} \end{aligned}$$

$$= \tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})}\|f\|_{M^{\rm loc}_{p,r,\lambda,\gamma}}$$
$$= \tau^{\alpha+Q(\frac{1}{q}-\frac{1}{p})+Q\lambda(\frac{1}{r}-\frac{1}{s})}\|f\|_{M^{\rm loc}_{p,r,\lambda,\gamma}}.$$

- If $\frac{1}{p} < \frac{1}{q} + \lambda\left(\frac{1}{r}-\frac{1}{s}\right) + \frac{\alpha}{Q}$, then we have $\|I^\alpha_\gamma f\|_{M^{\rm loc}_{q,s,\lambda,\gamma}} = 0$ as $\tau \to 0$ for all $f \in M^{\rm loc}_{p,r,\lambda,\gamma}$.
- If $\frac{1}{p} > \frac{1}{q} + \lambda\left(\frac{1}{r}-\frac{1}{s}\right) + \frac{\alpha}{Q}$, then we have $\|I^\alpha_\gamma f\|_{M^{\rm loc}_{q,s,\lambda,\gamma}} = 0$ as $\tau \to \infty$ for all $f \in M^{\rm loc}_{p,r,\lambda,\gamma}$.
- If $\frac{1}{p} - \frac{1}{q} \neq \lambda\left(\frac{1}{r}-\frac{1}{s}\right) + \frac{\alpha}{Q}$, then we have $I^\alpha_\gamma f(x) = 0$ for all $f \in M^{\rm loc}_{p,r,\lambda,\gamma}$ and a.e. $x \in \mathbb{R}^n_{k,+}$, which is impossible.

Hence, we obtain $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r}-\frac{1}{s}\right) + \frac{\alpha}{Q}$.

(ii) *Sufficiency.* Let $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r}+\frac{\alpha}{Q}\right)^{-1}$. From (3), we have

$$\begin{aligned}
\|I^\alpha_\gamma f\|_{WM^{\rm loc}_{q,s,\lambda,\gamma}(\mathbb{R}^n_{k,+})} &= \sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}(I^\alpha_\gamma f)^*_\gamma(t)\right\|_{WL_s(0,\rho)} \\
&\le \sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}(I^\alpha_\gamma f)^{**}_\gamma(t)\right\|_{WL_s(0,\rho)} \\
&\le C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}\left(t^{\frac{\alpha}{Q}-1}\int_0^t f^*_\gamma(y)dy + \int_t^\infty y^{\frac{\alpha}{Q}-1}f^*_\gamma(y)dy\right)\right\|_{WL_s(0,\rho)} \\
&\le C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{Q}-1}\int_0^t f^*_\gamma(y)dy\right\|_{WL_s(0,\rho)} \\
&\quad + C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}}\int_t^\infty y^{\frac{\alpha}{Q}-1}f^*_\gamma(y)dy\right\|_{WL_s(0,\rho)} \\
&= J_1 + J_2.
\end{aligned}$$

We take $\nu = 1 + \dfrac{\lambda-1}{r}$ and $\varphi(y) = y^{1+\frac{\lambda-1}{r}}f^*_\gamma(y)$ in the Hardy operator. Then, we get

$$\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} + \frac{\alpha}{Q} - 1 - \frac{\lambda}{r}.$$

From Theorem 1, we can write $\beta = (1-\lambda)\left(\dfrac{1}{r}-\dfrac{1}{s}\right)$. Then we have $1 - \dfrac{1}{q} = \dfrac{\alpha}{Q} - \dfrac{\lambda}{s}$. Therefore, again by Theorem 1, we get

$$\begin{aligned}
J_1 &= C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\frac{1}{q}-\frac{1}{s}+\frac{\alpha}{Q}-1}\int_0^t f^*_\gamma(y)dy\right\|_{WL_s(0,\rho)} \\
&= C_2\sup_{\rho>0}\rho^{-\frac{\lambda}{s}}\left\|t^{\beta+\nu-1}\int_0^t \frac{\varphi(y)}{y^\nu}dy\right\|_{WL_s(0,\rho)}
\end{aligned}$$

$$
\begin{aligned}
&= C \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| H_\nu^\beta \varphi \right\|_{WL_s(0,\rho)} \\
&= C \left\| H_\nu^\beta \varphi \right\|_{WLM_{s,\lambda}(0,\infty)} \\
&\leq C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)} \\
&= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \left\| \varphi \right\|_{L_r(0,\rho)} \\
&= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \| y^{1+\frac{\lambda-1}{r}} f_\gamma^*(y) \|_{L_r(0,\rho)} \\
&= C \left\| f \right\|_{M^{\mathrm{loc}}_{p,r,\lambda,\gamma}(\mathbb{R}^n_{k,+})}.
\end{aligned}
$$

We now estimate J_2. We take $\nu = 1 + \dfrac{\lambda - 1}{r} - \dfrac{\alpha}{Q}$ and $\varphi(y) = y^{1+\frac{\lambda-1}{r}} f_\gamma^*(y)$ in the Hardy operator. Then, we get

$$
\beta = \frac{1}{q} - \frac{1}{s} + \frac{1}{r} + \frac{\alpha}{Q} - 1 - \frac{\lambda}{r}.
$$

Therefore, by using Theorem 1, we obtain

$$
\begin{aligned}
J_2 &= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\frac{1}{q}-\frac{1}{s}} \int_t^\infty y^{\frac{\alpha}{Q}-1} f_\gamma^*(y) dy \right\|_{WL_s(0,\rho)} \\
&= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| t^{\beta+\nu} \int_t^\infty \frac{\varphi(y)}{y^{\nu+1}} dy \right\|_{WL_s(0,\rho)} \\
&= C_2 \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \left\| \mathcal{H}_\nu^\beta \varphi \right\|_{WL_s(0,\rho)} \\
&= C \left\| \mathcal{H}_\nu^\beta \varphi \right\|_{WLM_{s,\lambda}(0,\infty)} \\
&\leq C \left\| \varphi \right\|_{LM_{r,\lambda}(0,\infty)} \\
&= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \left\| \varphi \right\|_{L_{r,\lambda}(0,\rho)} \\
&= C \sup_{\rho>0} \rho^{-\frac{\lambda}{r}} \| y^{1+\frac{\lambda-1}{r}} f_\gamma^*(y) \|_{L_r(0,\rho)} \\
&= C \left\| f \right\|_{M^{\mathrm{loc}}_{p,r,\lambda,\gamma}(\mathbb{R}^n_{k,+})}.
\end{aligned}
$$

Necessity. Suppose that the $B-$ Riesz potential is I_γ^α bounded from $M^{\mathrm{loc}}_{p,r,\lambda,\gamma}$ to $WM^{\mathrm{loc}}_{q,s,\lambda,\gamma}$ and $p = \dfrac{r}{r+\lambda}$. Again, for $\tau > 0$, we define $f_\tau(x) := f(\tau x)$. Then $\|f_\tau\|_{M^{\mathrm{loc}}_{r/(r+\lambda),r,\lambda,\gamma}} = \tau^{-Q} \|f\|_{M^{\mathrm{loc}}_{r/(r+\lambda),r,\lambda,\gamma}}$ and

$$
\begin{aligned}
\|I_\gamma^\alpha f_\tau\|_{WM^{\mathrm{loc}}_{q,s,\lambda,\gamma}} &= \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \| y^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f_\tau)_\gamma^*(y) \|_{WL_s(0,\rho)} \\
&= \tau^{-\alpha} \sup_{\rho>0} \rho^{-\frac{\lambda}{s}} \| y^{\frac{1}{q}-\frac{1}{s}} (I_\gamma^\alpha f)_\gamma^*(y\tau^Q) \|_{WL_s(0,\rho)}
\end{aligned}
$$

$$= \tau^{-\alpha-\frac{Q}{q}-\frac{Q\lambda}{s}} \sup_{\rho>0}(\rho\tau^{Q})^{-\frac{\lambda}{s}} \|y^{\frac{1}{q}-\frac{1}{s}} I_{\gamma}^{\alpha} f_{\gamma}^{*}(y)\|_{WL_s(0,\rho)}$$
$$= \tau^{-\alpha-Q(\frac{1}{q}-\frac{\lambda}{s})} \|I_{\gamma}^{\alpha} f\|_{WM_{q,s,\lambda,\gamma}^{\mathrm{loc}}}.$$

Since the $B-$Riesz potential I_{γ}^{α} is bounded from $M_{p,r,\lambda,\gamma}^{\mathrm{loc}}$ to $WM_{q,s,\lambda,\gamma}^{\mathrm{loc}}$, we have $\|I_{\gamma}^{\alpha} f\|_{WM_{q,s,\lambda,\gamma}^{\mathrm{loc}}} \leq C\|f\|_{M_{p,r,\lambda,\gamma}^{\mathrm{loc}}}$, where $C > 0$ is a constant. Then we get

$$\begin{aligned}
\|I_{\gamma}^{\alpha} f\|_{WM_{q,s,\lambda,\gamma}^{\mathrm{loc}}} &= \tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})} \|I_{\gamma}^{\alpha} f_{\tau}\|_{WM_{q,s,\lambda,\gamma}^{\mathrm{loc}}} \\
&\leq C\tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})} \|f_{\tau}\|_{M_{p,r,\lambda,\gamma}^{\mathrm{loc}}} \\
&= \tau^{\alpha+Q(\frac{1}{q}-\frac{\lambda}{s})} \|f\|_{M_{p,r,\lambda,\gamma}^{\mathrm{loc}}} \\
&= \tau^{\alpha+Q(\frac{1}{q}-1-\frac{\lambda}{r})+Q\lambda(\frac{1}{r}-\frac{1}{s})} \|f\|_{M_{p,r,\lambda,\gamma}^{\mathrm{loc}}}.
\end{aligned}$$

- If $1 < \frac{1}{q} + \frac{\alpha}{Q} - \frac{\lambda}{s}$, then we have $\|I_{\gamma}^{\alpha} f\|_{WM_{q,s,\lambda,\gamma}^{\mathrm{loc}}} = 0$ as $\tau \to 0$ for all $f \in M_{r/(r+\lambda),r,\lambda,\gamma}^{\mathrm{loc}}$.
- If $1 > \frac{1}{q} + \frac{\alpha}{Q} - \frac{\lambda}{s}$, then we have $\|I_{\gamma}^{\alpha} f\|_{WM_{q,s,\lambda,\gamma}^{\mathrm{loc}}} = 0$ as $\tau \to \infty$ for all $f \in M_{r/(r+\lambda),r,\lambda,\gamma}^{\mathrm{loc}}$.
- If $1 \neq \frac{1}{q} + \frac{\alpha}{Q} - \frac{\lambda}{s}$, then we have $I_{\gamma}^{\alpha} f(x) = 0$ for all $f \in M_{p,r,\lambda,\gamma}^{\mathrm{loc}}$ and a.e. $x \in \mathbb{R}_{k,+}^{n}$, which is impossible.

Hence, we obtain $1 - \frac{1}{q} = \frac{\alpha}{Q} - \frac{\lambda}{s}$. This completes the proof.

□

The following corollary is easily obtained from the inequality $M_{\gamma}^{\alpha} \leq C\, I_{\gamma}^{\alpha}$ and Theorem 2, .

Corollary 1. *Let $0 \leq \lambda < 1$, $0 < \alpha < Q$, $1 \leq q \leq \infty$, $1 \leq r \leq s \leq \infty$, $\frac{r}{r+\lambda} \leq p \leq \left(\frac{\lambda}{r} + \frac{\alpha}{Q}\right)^{-1}$.*

(i) *If $\frac{r}{r+\lambda} < p < \left(\frac{\lambda}{r} + \frac{\alpha}{Q}\right)^{-1}$, then $\frac{1}{p} - \frac{1}{q} = \lambda\left(\frac{1}{r} - \frac{1}{s}\right) + \frac{\alpha}{Q}$ is necessary and sufficient condition for the boundedness of the $B-$fractional maximal operator M_{γ}^{α} from $M_{p,r,\lambda,\gamma}^{\mathrm{loc}}$ to $M_{q,s,\lambda,\gamma}^{\mathrm{loc}}$.*

(ii) *If $p = \frac{r}{r+\lambda}$, then $1 - \frac{1}{q} = \frac{\alpha}{Q} - \frac{\lambda}{s}$ is necessary and sufficient condition for the boundedness of the $B-$fractional maximal operator M_{γ}^{α} from $M_{p,r,\lambda,\gamma}^{\mathrm{loc}}$ to $WM_{q,s,\lambda,\gamma}^{\mathrm{loc}}$.*

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A KIND OF ROTATIONAL SURFACES WITH A LIGHT-LIKE AXIS IN CONFORMALLY FLAT PSEUDO-SPACES OF DIMENSIONAL THREE

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Abstract. In this work, we define the rotational surface with a light-like axis in conformally flat pseudo-spaces $\left(\mathbb{E}_3^1\right)_\lambda$, where λ is a radial-type conformal factor. We relate the principal curvatures of a non-degenerate surface that belongs to conformally equivalent spaces $\left(\mathbb{E}_3^1\right)_\lambda$ and $\mathbb{R}_1^3$, based on the radial conformal factor. Thus, we establish a relationship between the radial conformal factor and the profile curve of the rotational flat surface in $\left(\mathbb{E}_3^1\right)_\lambda$, but also for that of the rotational surface with zero extrinsic curvature.

1. Introduction

The theory of surfaces is one of the significant subfields of study that belong to the field of differential geometry. This theory has a wide variety of applications. For instance, it is used in computer graphics to create 3D models of objects, in physics to describe the behavior of fluids and solids, and in engineering to design structures with optimal shapes [1,2].

In contrast to the creation of a helicoidal surface, which has been differently characterized in a recent publication [3], the formation of a rotational surface is achieved only through the rotation of a curve around an axis. The investigation of rotational surfaces has been the subject of considerable scholarly research. To access studies done in recent years, refer to references [4–6]. The study of special surfaces, such as rotational and helicoidal surfaces, is conducted in the setting of conformally flat spaces. Conformally flat spaces possess distinctive characteristics through the utilization of their conformal factors. The determination of the proper conformal factor is important for undertaking surveys of the aforementioned surfaces in conformally flat spaces. A function f is said to be invariant under a transformation T

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of the space into itself if the condition $f(Tx) = f(x)$ is satisfied for all x. If the conformal factor λ is a function that meets this criterion, it is reasonable to consider such surfaces in conformally flat spaces. An estimation for this type of function can be derived from the Cartesian equation of geometric shapes such as the sphere and the cylinder. In contrast to the cylinder type, which exhibits invariance under both rotational and translational symmetries, the spherical type is only invariant under rotational symmetry. For more on research done in the framework of the spherical type $t := x_1^2 + x_2^2 + x_3^2$, see [7, 8]. For another type, see [9–15]. In the aforementioned studies, the authors consider the various conformal factors, such as $\sqrt{t}$, $\frac{1}{\sqrt{t}}$, and e^{-t}. It is worth noting that the first two factors contribute to the formation of the generic metric, whereas the third factor serves as a metric that is a solution to Einstein's equation.

Yerlikaya [14] introduces the conformally flat pseudo-space of dimensional three, and presents a non-degenerate surface's curvatures for an arbitrary conformal factor. But, this work is based on the utilization of the radial conformal factor as the framework. From this perspective, rotational surfaces in conformally flat pseudo-spaces are analyzed.

2. Basic Notations

Denote the Minkowski space by $\mathbb{R}_1^3$, defined by the Minkowski metric $g(x,y) = -x_1y_1 + x_2y_2 + x_3y_3$ with respect to a cononical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}_1^3$, where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$. Observe that for a pseudo-orthonormal basis $\{\xi_1, \xi_2, \xi_3\}$ of $\mathbb{R}_1^3$, the metric becomes $g(x,y) = x_1y_3 + x_2y_2 + x_3y_1$. In a such basis, the following equalities hold

$$g(\xi_1,\xi_1) = g(\xi_1,\xi_2) = g(\xi_2,\xi_3) = g(\xi_3,\xi_3) = 0, \tag{1}$$

$$g(\xi_1,\xi_3) = g(\xi_2,\xi_2) = 1. \tag{2}$$

For some tools regarding the transition matrix given by

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \tag{3}$$

see [16]. The rotational motion about the null coordinate axis $O\xi_3$ is represented by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to A^{-1}RA \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

i.e.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1+\frac{\theta^2}{2} & -\frac{\theta^2}{2} & \theta \\ \frac{\theta^2}{2} & 1-\frac{\theta^2}{2} & \theta \\ \theta & -\theta & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

or the more useful form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\frac{t^2}{2} & -t & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{4}$$

where $t = -\sqrt{2}\theta$.

Equipped the Minkowski space $\mathbb{R}^3_1$ with a conformally flat pseudo-metric given by the angle-bracket notation

$$\langle w_1, w_2 \rangle_{g_\lambda} = \frac{1}{\lambda^2 (p)} \langle w_1, w_2 \rangle_L, \qquad \forall w_1, w_2 \in T_p\mathbb{R}^3_1,\ \forall p \in \mathbb{R}^3_1,$$

the resulting space is said to be the complete pseudo-Riemannian manifold if the conformal factor λ is bounded. From now on, unless otherwise stated, this pseudo-manifold shall be mentioned as the conformally flat pseudo-space, represented by $\left(\mathbb{E}^1_3\right)_\lambda$. Here, note that the pseudo-metric $\langle , \rangle_L$ is the Minkowski metric whose coefficients are those of Eqs. (1) and (2).

3. Surfaces in a conformally flat pseudo-space with radial conformal metrics $\left(\mathbb{E}^1_3\right)_{\lambda(r)}$

In [14], the author calculates the principal curvatures of a non-degenerate parameterized surface for an arbitrary conformal factor in the conformally flat pseudo-space. Now, we'll modify the process so that it works with the radial conformal factor

$$\lambda = \lambda (r), \quad r = 2x_1x_3 + x_2^2, \tag{5}$$

which implies the spherical type with respect to the pseudo-orthonormal basis of $\mathbb{R}^3_1$. Consider a non-degenerate parametrized surface $M = X(U)$ in the Minkowski space as

$$\begin{aligned} &X : U \subset \mathbb{R}^2 \to \mathbb{R}^3_1 \\ &\quad (s,t) \to X(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t)). \end{aligned}$$

Since this surface also belongs to a pseudo-space that is conformal to the Minkowski space, we can write $\widetilde{N}(s,t) = (\lambda N)(s,t)$ for (s,t) in some planar domain, where N and $\widetilde{N}$ denote the normal vector fields in Minkowski space and conformally flat pseudo-space, respectively. Let $\bar{\nabla}$ be the Levi-Civita connection of $\left(\mathbb{E}^1_3\right)_{\lambda(r)}$. Thus, we get

$$\bar{\nabla}_{X_{,s}} \widetilde{N} = \bar{\nabla}_{X_{,s}} (\lambda N) = X_s(\lambda) N + \lambda \bar{\nabla}_{X_{,s}} N, \tag{6}$$

where $X_{,s}$ denotes the partial derivative of X with respect to the parameter s. Using the properties of the connection $\bar{\nabla}$ and considering N as the linear combination of the pseudo-basis, we write

$$\bar{\nabla}_{X_{,s}} N = N_{,s} + \sum_{i,j,k=1}^{3} X_{,s}^{i} N^{j} \Gamma_{ij}^{k} \xi_k, \tag{7}$$

where Γ_{ij}^{k} denote the Christoffel symbols of the conformal pseudo metric. Note that Eq. (7) holds for the parameter t, as well.

Taking Eq. (5) into account, we have $\frac{\partial \lambda}{\partial x_i} = \frac{\partial \lambda}{\partial r}\frac{\partial r}{\partial x_i}$. From now on, we use the notation $\frac{\partial \lambda}{\partial r} = \dot{\lambda}$. Thus, we can write

$$\Gamma_{ij}^{k} = -\bar{g}_{jk}\frac{\epsilon_j}{\epsilon_k}\frac{\dot{\lambda}(r)}{\lambda}\frac{\partial r}{\partial x_i} - \bar{g}_{ik}\frac{\epsilon_i}{\epsilon_k}\frac{\dot{\lambda}(r)}{\lambda}\frac{\partial r}{\partial x_j} + \bar{g}_{ij}\frac{\epsilon_i}{\epsilon_k}\frac{\dot{\lambda}(r)}{\lambda}\frac{\partial r}{\partial x_k}, \tag{8}$$

where $\epsilon_i = \bar{g}_{ii}$. From Eq. (8) together with Eq. (5), we get

$$\begin{aligned} \Gamma_{11}^{2} = \Gamma_{11}^{3} = \Gamma_{12}^{3} = \Gamma_{13}^{1} = \Gamma_{13}^{3} = \Gamma_{23}^{1} = \Gamma_{33}^{1} = \Gamma_{33}^{2} &= 0, \\ \Gamma_{11}^{1} = 2\Gamma_{12}^{2} = -2\Gamma_{22}^{3} &= -\frac{4x_3\dot{\lambda}(r)}{\lambda} \\ \Gamma_{12}^{1} = -\Gamma_{13}^{2} = \Gamma_{22}^{2} = \Gamma_{23}^{3} &= -\frac{2x_2\dot{\lambda}(r)}{\lambda} \\ -2\Gamma_{22}^{1} = 2\Gamma_{23}^{2} = \Gamma_{33}^{3} &= -\frac{4x_1\dot{\lambda}(r)}{\lambda} \end{aligned} \tag{9}$$

Theorem 1. *Let $X : U \to \mathbb{R}_1^3$ be a non-degenerate surface parametrized as $X(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t))$ in the Minkowski space $\mathbb{R}_1^3$. Consider $X(U)$ as a non-degenerate surface in a conformally flat pseudo-space $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$. Then, the eigenvalues $\widetilde{k}_l$ of X in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$ are calculated as*

$$\widetilde{k}_l = \lambda k_l - 2\dot{\lambda}\left\langle (x_1, x_2, x_3), N\right\rangle, \quad 1 \le l \le 2, \tag{10}$$

where N denotes the normal Gauss mapping of X in $\mathbb{R}_1^3$ and k_l are the eigenvalues of N.

Proof. Let's proceeed with the proof for the parameter s. Putting (9) into Eq. (7), we have

$$\bar{\nabla}_{X_{,s}} N = N_{,s} - \frac{2\dot{\lambda}}{\lambda}\left\langle X, N\right\rangle X_{,s} - \frac{2\dot{\lambda}}{\lambda}\left\langle X_{,s}, X\right\rangle N.$$

Substituting this into Eq. (6), we obtain

$$\bar{\nabla}_{X_{,s}} \widetilde{N} = \lambda N_{,s} - 2\dot{\lambda}\left\langle X, N\right\rangle X_{,s}. \tag{11}$$

Taking $N_{,s} = k_1 X_{,s}$ and $\bar{\nabla}_{X_{,s}} \widetilde{N} = \widetilde{k}_1 X_{,s}$ into account and using Eq. (11), we obtain

$$\widetilde{k}_1 = \lambda k_1 - 2\dot{\lambda} \langle X, N \rangle, \tag{12}$$

which concludes the proof. □

3.1. **Rotational Surfaces with a light-like axis in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$.** We now consider the Gauss and extrinsic curvatures of a non-degenerate rotational surface in conformally flat pseudo-spaces $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$, as it relates to the radial conformal factor. As mentioned in the introduction, helicoidal surfaces are described as the general category to which rotational surfaces belong. For this reason, the ability to define helicoidal surfaces in conformally flat pseudo-spaces, as made possible in [14], also allows for the definition of a new type of rotational surface in these spaces.

Let $\gamma(s) = (s, 0, f(s))$, $s > 0$ be a curve x_1x_3-plane defined on $I \subset$ R, which is called the profile curve. Applying this curve to the rotation in Eq. (4), in the following way:

$$\begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\frac{t^2}{2} & -t & 1 \end{pmatrix} \begin{pmatrix} s \\ 0 \\ f(s) \end{pmatrix},$$

we get a non-degenerate surface given by the parametric form

$$X : I \times \mathbb{R} \to \left(\mathbb{E}_3^1\right)_{\lambda(r)}$$

$$(s,t) \to X(s,t) = \left(s, st, f(s) - \frac{st^2}{2}\right), \tag{13}$$

which implies that it is a rotational surface in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$, where $f(s)$ is a function defined on an open interval I of $\mathbb{R}$.

Lemma 1. *Let $X(s,t) = \left(s, st, f(s) - \frac{st^2}{2}\right)$ be a rotational surface in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$. Thus, the Gaussian curvature of X is computed as*

$$K = \frac{-\epsilon\lambda^2}{s\sqrt{2f'}} \frac{\partial}{\partial s}\left(\frac{\lambda - 2\dot{\lambda} s\,(f + sf'))}{\lambda\sqrt{2f'}}\right), \tag{14}$$

where $\dot{\lambda} = \frac{d\lambda}{dr}$ and $\epsilon = \pm 1$.

Proof. To find the Gaussian curvature of X in the conformally flat pseudo-space $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$, we need to calculate the coefficients of the first fundamental form of X with respect to the conformal metric. Then, it is easily seen that

$$\widetilde{E} = \frac{2f'}{\lambda^2}, \quad \widetilde{F} = 0 \quad \text{and} \quad \widetilde{G} = \frac{s^2}{\lambda^2}. \tag{15}$$

Due to $\widetilde{F}=0$, we have from [17] the knowledge that there is a formula for calculating the Gaussian curvature in the Euclidean version. Based on this knowledge, we modify, in the Minkowskian version, the formula of Gaussian curvature such that

$$K=\frac{-\epsilon}{2\sqrt{\widetilde{E}\widetilde{G}}}\left(\frac{\partial}{\partial t}\left(\frac{\widetilde{E}_t}{\sqrt{\widetilde{E}\widetilde{G}}}\right)+\frac{\partial}{\partial s}\left(\frac{\widetilde{G}_s}{\sqrt{\widetilde{E}\widetilde{G}}}\right)\right). \tag{16}$$

Hence, together with $\widetilde{E}_t=0$ and $\widetilde{G}_s=\frac{2s\lambda^2-4\lambda\dot{\lambda}\left(f+sf'\right)s^2}{\lambda^4}$, using Eq. (16), we get Eq. (14). This concludes the proof.

□

Theorem 2. *Let* $X\left(s,t\right)=\left(s,st,f\left(s\right)-\frac{st^2}{2}\right)$ *be a rotational surface in* $\left(\mathbb{E}^1_3\right)_{\lambda(r)}$. *Thus,* $X\left(s,t\right)$ *is flat in* $\left(\mathbb{E}^1_3\right)_{\lambda(r)}$ *if and only if* $\lambda=\lambda\left(2sf\right)=e^{-\int\frac{c_1\sqrt{2f'-1}}{s}ds}$, $c_1\neq0$.

Proof. It is clear from Eq. (14) that the necessary condition for X to be flat in $\left(\mathbb{E}^1_3\right)_{\lambda(r)}$ have to satisfy the following equation

$$\frac{s\lambda-2s^2\dot{\lambda}\left(f+sf'\right)}{\lambda\sqrt{2s^2f'}}=c_1. \tag{17}$$

Hence, if $c_1=0$, we get a contradiction about the completeness of the metric. If $c_1\neq0$, then Eq. (17) becomes $\frac{\dot{\lambda}}{\lambda}=\frac{c_1\sqrt{2f'-1}}{2s(f+sf')}$. By integrating both sides, we obtain the desired outcome. □

Lemma 2. *Let* $X\left(s,t\right)=\left(s,st,f\left(s\right)-\frac{st^2}{2}\right)$ *be a rotational surface in* $\left(\mathbb{E}^1_3\right)_{\lambda(r)}$. *Thus, the extrinsic curvature of* X *is computed as*

$$\widetilde{K}_E=\frac{-\epsilon}{4sf'^2}\left(\lambda f''-4\dot{\lambda}f'\left(f-sf'\right)\right)\left(\lambda+2s\dot{\lambda}\left(f-sf'\right)\right), \tag{18}$$

where $\epsilon=\pm1$.

Proof. If we proceed through the steps of proving Lemma (1) for the Minkowskian metric, then the coefficients of the first fundamental form are as follows:

$$E=2f',\quad F=0\quad\text{and}\quad G=s^2, \tag{19}$$

and the coefficients of the second fundamental form are calculated as

$$e=-\frac{sf''}{\alpha},\quad f=0\quad\text{and}\quad g=\frac{s^2}{\alpha}, \tag{20}$$

where $\alpha=\sqrt{2s^2f'}$. On the other hand, taking into account the partial derivatives of X, we find

$$\widetilde{k}_i=\lambda k_i-4\dot{\lambda}\frac{sff'\left(1-sf'\right)}{\alpha}. \tag{21}$$

Ultimately, using together Eqs. (19) and (20) with Eq. (21), we get

$$\widetilde{K}_E = \widetilde{k}_1 \widetilde{k}_2 = \frac{-\epsilon}{4sf'^2}\left(\lambda f'' - 4\dot{\lambda} f'\left(f - sf'\right)\right)\left(\lambda + 2s\dot{\lambda}\left(f - sf'\right)\right). \tag{22}$$

□

Theorem 3. *Let* $X(s,t) = \left(s, st, f(s) - \frac{st^2}{2}\right)$ *be a rotational surface in* $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$. *Thus,* $X(s,t)$ *has zero extrinsic curvature in* $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$ *if and only if either one of the next two equations*

$$\lambda = \lambda(2sf) = \frac{c_1\sqrt{f'}}{f - sf'} \text{ or } \lambda = \lambda(2sf) = e^{-\int \frac{f+sf'}{s(f-sf')}ds} \tag{23}$$

are satisfied, where c_1 *is a positive real number.*

Proof. In order for X to have zero extrinsic curvature in $\left(\mathbb{E}_3^1\right)_{\lambda(r)}$, the following equations must be met:

$$\lambda f'' - 4\dot{\lambda} f'\left(f - sf'\right) = 0 \quad \text{or} \quad \lambda + 2s\dot{\lambda}\left(f - sf'\right) = 0.$$

Of these, the first one becomes $\frac{\dot{\lambda}}{\lambda} = \frac{f''}{4f'(f-sf')}$. Using the integration, we get $\lambda = \frac{c_1\sqrt{f'}}{f-sf'}$. As similar to this, we find the other one. The proof concludes here. □

Remark 1. *In the first equality of Eq. (23), for* $\lambda(r) = \frac{1}{\sqrt{r}}$, *rotational surfaces* X *with zero extrinsic curvature are rational kinds. More clearly, from Eq. (18), when* $\lambda(r) = \frac{1}{\sqrt{r}}$, $\widetilde{K}_E = 0$ *if and only if* $sff'' + ff' - sf'^2 = 0$, *whose general solution is* $f(s) = ns^m$, *where* m *is a constant and* n *is a positive real number. Rotational surfaces with zero extrinsic curvature can be determined to be polynomial in character with isothermal parameters by a special solution of the differential equation mentioned above. In the second one, for* $\lambda(r) = e^{-r}$, $\widetilde{K}_E = 0$ *if and only if it satisfies the equation* $2s^2f' - 2sf + 1 = 0$, *which ensures that the general solution is* $f(s) = ms + \frac{1}{4s}$, *where* m *is a real number. By using a special solution of the differential equation, we just talked about above, we can figure out that rotational surfaces with zero extrinsic curvature are of constant Gaussian curvature. Both conformal factors are useful, but in different ways for different models, as was mentioned in the introduction.*

Example 1. *Let's use Theorem 3 to describe a rotational surface with zero extrinsic curvature in* $\left(\mathbb{E}_3^1\right)_{\frac{1}{\sqrt{r}}}$. *From Remark 1, for* $\lambda(r) = \frac{1}{\sqrt{r}}$, *we have the knowledge whose profile curve will be* $f(s) = ns^m$. *Substituting this profil curve into Eq. (13), we get the parametrization of a rotational surface with zero curvature surface as follows:*

$$X(s,t) = \left(s, st, ns^m - \frac{st^2}{2}\right).$$

We now plot it putting for $m = 3$ *and* $n = 2$. *See Fig. (1).*

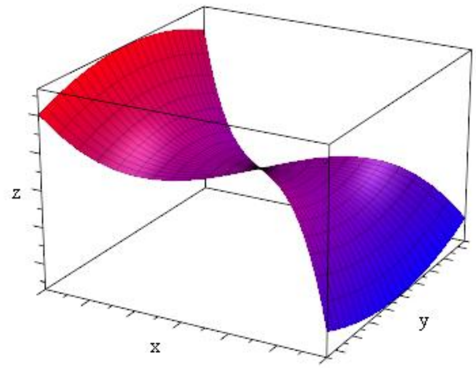

FIGURE 1. The graphic belongs to a rotational surface of rational kind with zero extrinsic curvature in $\left(\mathbb{E}_3^1\right)_{\frac{1}{\sqrt{r}}}$.

We also sketch it out with respect to the constants $m = 3$ *and* $n = \frac{1}{6}$ *that serves as the isothermal parametrization condition. See Fig. (2).*

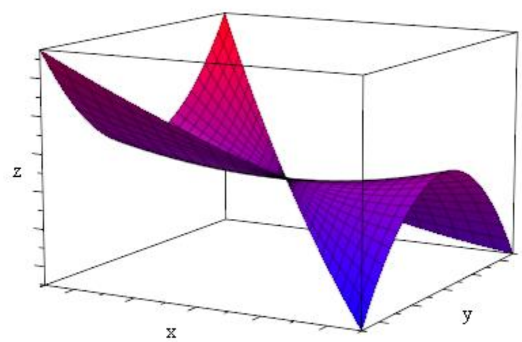

FIGURE 2. The graphic belongs to a rotational surface of rational type with zero extrinsic curvature having the isothermal parameter in $\left(\mathbb{E}_3^1\right)_{\frac{1}{\sqrt{r}}}$.

Example 2. *As similar to Example (1), the profile curve of a rotational surface with zero curvature in* $\left(\mathbb{E}_3^1\right)_{e^{-r}}$ *is* $f(s) = ms + \frac{1}{4s}$. *Applying this to Eq. (13) yields*

$$X(s,t) = \left(s, st, ms + \frac{1}{4s} - \frac{st^2}{2}\right).$$

For $m = 1$, *see Fig. (3).*

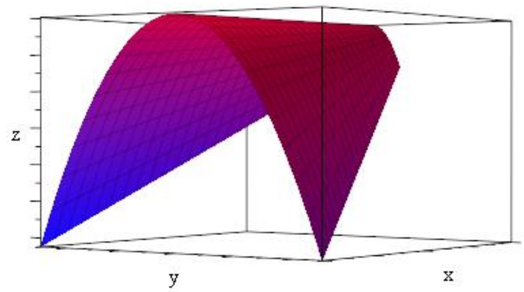

FIGURE 3. The graphic belongs to a rotational surface of with zero extrinsic curvature in $\left(\mathbb{E}_3^1\right)_{e^{-r}}$.

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PARAMETRIC GENERALIZATION OF THE MODIFIED BERNSTEIN-KANTOROVICH OPERATORS

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Abstract. In the current article, a parametrization of the modified Bernstein-Kantorovich operators is studied. Then the Korovkin theorem, approximation properties and central moments of these operators are investigated. The rate of approximation of the operators is obtained by the help of modulus of continuity, functions from Lipschitz class and Peetre-$\mathcal{K}$ functional. Finally, some numerical examples are illustrated to show the effectiveness of the newly defined operators.

1. Introduction

Approximation theory has an important place in studies in the field of mathematics. Let f be a continuous function on the interval $[a, b]$ and then for every $\varepsilon > 0$, there is a polynomial p that satisfies the $\|f(x) - p(x)\| < \varepsilon$ condition. This theorem was given by Weierstrass [19] in 1885. In 1912, Bernstein [3] proved the approximation theorem defined by Weierstrass on the closed interval $[0, 1]$. A generalization of Bernstein operators was made by Chen et al. [7] in 2017. Fuat Usta [18] defined modified Bernstein operators in 2020 as

$$B_\eta^*(g;x) = \frac{1}{\eta}\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-1} g\left(\frac{\zeta}{\eta}\right).$$

By definition of the operator $B_\eta^*(g;x)$, he obtained the following equalities

$$B_\eta^*(1;x) = 1,$$

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$$\begin{aligned} B_{\eta}^{*}(t;x) &= \frac{\eta-2}{\eta}x+\frac{1}{\eta}, \\ B_{\eta}^{*}(t^{2};x) &= \frac{(\eta^{2}-7\eta+6)}{\eta^{2}}x^{2}+\frac{5\eta-6}{\eta^{2}}x+\frac{1}{\eta^{2}}. \end{aligned}$$

Certain examples of articles on parametric generalizations of operators can be found in [2], [4], [5], [6], [8], [7], [9], [10], [12], [13], [14], [16], [17], [20] and [21].

The θ parameterization of modified Bernstein operators were defined for every $g \in C[0,1]$ by Sofyalıoğlu et al. [15] as

$$B_{\eta,\theta}^{*}(g;x)=\sum_{\zeta=0}^{\eta}\rho_{\eta,\zeta}^{(\theta)}(x)g\left(\frac{\zeta}{\eta}\right), \tag{1}$$

where $\eta \geq 1$, $0 \leq \theta \leq 1$, $x \in (0,1)$ and

$$\begin{aligned} \rho_{1,0}^{(\theta)}(x) &= x,\ \rho_{1,1}^{(\theta)}(x)=1-x, \\ \rho_{\eta,\zeta}^{(\theta)}(x) &= \left\{\frac{1}{\eta-1}\begin{pmatrix}\eta-2\\ \zeta\end{pmatrix}(\zeta-(\eta-1)x)^{2}(1-\theta)x\right. \\ &\quad +\frac{1}{\eta-1}\begin{pmatrix}\eta-2\\ \zeta-2\end{pmatrix}(\zeta-1-(\eta-1)x)^{2}(1-\theta)(1-x) \\ &\quad \left.+\frac{1}{\eta}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}(\zeta-\eta x)^{2}\theta x(1-x)\right\}x^{\zeta-2}(1-x)^{\eta-\zeta-2}, \quad \eta \geq 2 \end{aligned}$$

with binomial coefficients

$$\begin{pmatrix}\eta\\ \zeta\end{pmatrix}=\begin{cases}\frac{\eta!}{(\eta-\zeta)!\zeta!} & \text{if } 0\leq\zeta\leq\eta \\ 0 & \text{otherwise}\end{cases}.$$

In this paper, we give the Kantorovich type of parametric generalizations of the modified Bernstein operators created by Sofyalıoğlu et al. [15]. Later, we study approximation properties of the operators. Then we give central moments and rate of convergence.

Now, we define the parametric generalization of the modified Bernstein-Kantorovich operators

$$K_{\eta,\theta}^{*}(g;x)=\sum_{\zeta=0}^{\eta}\rho_{\eta,\zeta}^{(\theta)}(x)\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt, \tag{2}$$

where $\eta \geq 1$, $0 \leq \theta \leq 1$, $x \in (0,1)$ and

$$\begin{aligned} \rho_{1,0}^{(\theta)}(x) &= x,\ \rho_{1,1}^{(\theta)}(x)=1-x, \\ \rho_{\eta,\zeta}^{(\theta)}(x) &= \left\{\frac{\eta}{\eta-1}\begin{pmatrix}\eta-2\\ \zeta\end{pmatrix}(\zeta-(\eta-1)x)^{2}(1-\theta)x\right. \\ &\quad +\frac{\eta}{\eta-1}\begin{pmatrix}\eta-2\\ \zeta-2\end{pmatrix}(\zeta-1-(\eta-1)x)^{2}(1-\theta)(1-x) \end{aligned}$$

$$+\left(\begin{array}{c}\eta\\ \zeta\end{array}\right)(\zeta-\eta x)^2\theta x(1-x)\Big\}x^{\zeta-2}(1-x)^{\eta-\zeta-2},\quad \eta\geq 2$$

with binomial coefficients

$$\left(\begin{array}{c}\eta\\ \zeta\end{array}\right)=\left\{\begin{array}{cc}\frac{\eta!}{(\eta-\zeta)!\zeta!} & \text{if } 0\leq\zeta\leq\eta\\ 0 & \text{otherwise}\end{array}\right. .$$

Choosing $\theta=1$, it is seen that the operators $B^*_{\eta,\theta}(g;x)$ turn into $B^*_{\eta}(g;x)$ given by Usta [18].
The following equalities are going to use in the proof of the next theorem

$$\left(\begin{array}{c}\eta-2\\ \zeta\end{array}\right)=\left(1-\frac{\zeta}{\eta-1}\right)\left(\begin{array}{c}\eta-1\\ \zeta\end{array}\right), \tag{3}$$

$$\left(\begin{array}{c}\eta-2\\ \zeta-1\end{array}\right)=\frac{\zeta}{\eta-1}\left(\begin{array}{c}\eta-1\\ \zeta\end{array}\right). \tag{4}$$

Theorem 1. *The parametric generalization of the modified Bernstein-Kantorovich operators can be expressed as*

$$\begin{aligned}K^*_{\eta,\theta}(g;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt+\frac{\zeta}{\eta-1}\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}}g(t)dt\right]\\ &\times\left(\begin{array}{c}\eta-1\\ \zeta\end{array}\right)(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2}\\ &+\theta\sum_{\zeta=0}^{\eta}\left(\begin{array}{c}\eta\\ \zeta\end{array}\right)(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt.\end{aligned}$$

Proof. We rewrite the Eqn. (2) in more explicit form as

$$\begin{aligned}K^*_{\eta,\theta}(g;x) &= (1-\theta)\left[\sum_{\zeta=0}^{\eta}\frac{\eta}{\eta-1}\left(\begin{array}{c}\eta-2\\ \zeta\end{array}\right)(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2}\right.\\ &\times\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt\\ &+\sum_{\zeta=0}^{\eta}\frac{\eta}{\eta-1}\left(\begin{array}{c}\eta-2\\ \zeta-2\end{array}\right)(\zeta-1-(\eta-1)x)^2x^{\zeta-2}(1-x)^{\eta-\zeta-1}\\ &\left.\times\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt\right]\\ &+\theta\sum_{\zeta=0}^{\eta}\left(\begin{array}{c}\eta\\ \zeta\end{array}\right)(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt.\end{aligned}$$

In other words,

$$\begin{aligned}K^*_{\eta,\theta}(g;x) &= (1-\theta)(\mu_1+\mu_2)+\theta\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1}\\ &\quad\times\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt, \qquad (5)\end{aligned}$$

where μ_1 and μ_2 are

$$\begin{aligned}\mu_1 &= \sum_{\zeta=0}^{\eta-2}\frac{\eta}{\eta-1}\binom{\eta-2}{\zeta}(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt,\\ \mu_2 &= \sum_{\zeta=1}^{\eta}\frac{\eta}{\eta-1}\binom{\eta-2}{\zeta-2}(\zeta-1-(\eta-1)x)^2x^{\zeta-2}(1-x)^{\eta-\zeta-1}\\ &\quad\times\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt.\end{aligned}$$

When we choose the term $\zeta=\eta$ and $\zeta=\eta-1$ respectively, we get $\mu_1=0$.
Similarly, replacing $\zeta=0$ gives $\mu_2=0$.
Therefore, we obtain

$$\mu_2 = \sum_{\zeta=0}^{\eta-2}\frac{\eta}{\eta-1}\binom{\eta-2}{\zeta-1}(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2}\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}}g(t)dt.$$

By using Eqn. (3) and Eqn. (4), we have

$$\begin{aligned}\mu_1+\mu_2 &= \sum_{\zeta=0}^{\eta-2}\frac{\eta}{\eta-1}\left[\binom{\eta-2}{\zeta}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}g(t)dt+\binom{\eta-2}{\zeta-1}\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}}g(t)dt\right]\\ &\quad\times(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2}.\end{aligned}$$

If we rewrite the above equation in (5), we achieve the desired result. □

2. Auxiliary Results

Lemma 1. *For every $x\in(0,1)$, the operator $K^*_{\eta,\theta}(e_m;x)$ has the following identities:*

$$\begin{aligned}K^*_{\eta,\theta}(e_0;x) &= 1,\\ K^*_{\eta,\theta}(e_1;x) &= \frac{\eta-2}{\eta}x+\frac{3}{2\eta},\\ K^*_{\eta,\theta}(e_2;x) &= \frac{(3\eta^3-18\eta^2-3\eta+18)-\theta(6\eta^2-42\eta+36)}{3\eta^2(\eta-1)}x^2\end{aligned}$$

$$+\frac{(18\eta^2-6\eta-24)-\theta(36\eta-48)}{3\eta^2(\eta-1)}x$$
$$+\frac{\left(\eta^2+5\eta+6\right)-12\theta}{3\eta^2(\eta-1)},$$

where $e_m = t^m$ *for* $m = 0, 1, 2$.

Proof. We briefly mention the results of $K^*_{\eta,\theta}(e_m; x)$, where $e_m = t^m$, $m = 0, 1, 2$. For $e_0 = 1$, we write

$$\begin{aligned} K^*_{\eta,\theta}(1;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{1}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}+\frac{\zeta}{\eta-1}\right)\right]\binom{\eta-1}{\zeta} \\ &\quad\times(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\ &\quad+\theta\sum_{\zeta=0}^{\eta}\frac{1}{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1} \\ &= (1-\theta)B^*_\eta(1;x)+\theta B^*_\eta(1;x) \\ &= 1. \end{aligned}$$

For $e_1 = t$, we have

$$\begin{aligned} K^*_{\eta,\theta}(t;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}tdt+\frac{\zeta}{\eta-1}\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}}tdt\right] \\ &\quad\times\binom{\eta-1}{\zeta}(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\ &\quad+\theta\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}tdt. \end{aligned}$$

Since $\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}}tdt=\frac{2\zeta+1}{2\eta^2}$ and $\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}}tdt=\frac{2\zeta+3}{2\eta^2}$,

$$\begin{aligned} K^*_{\eta,\theta}(t;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\left(\frac{2\zeta+1}{2\eta^2}\right)+\frac{\zeta}{\eta-1}\left(\frac{2\zeta+3}{2\eta^2}\right)\right] \\ &\quad\times\binom{\eta-1}{\zeta}(\zeta-(\eta-1)x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\ &\quad+\theta\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2x^{\zeta-1}(1-x)^{\eta-\zeta-1}\left(\frac{2\zeta+1}{2\eta^2}\right) \\ &= (1-\theta)B^*_\eta(t;x)+\frac{1-\theta}{2\eta}+\theta B^*_\eta(t;x)+\frac{\theta}{2\eta} \\ &= \frac{\eta-2}{\eta}x+\frac{3}{2\eta}. \end{aligned}$$

For $e_2 = t^2$, we have

$$\begin{aligned}
K^*_{\eta,\theta}(t^2;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} t^2dt + \frac{\zeta}{\eta-1}\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}} t^2dt\right] \\
&\quad \times \binom{\eta-1}{\zeta}(\zeta-(\eta-1)x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\
&\quad +\theta\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-1}\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} t^2dt.
\end{aligned}$$

Since $\int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} t^2dt = \frac{3\zeta^2+3\zeta+1}{3\eta^3}$ and $\int_{\frac{\zeta+1}{\eta}}^{\frac{\zeta+2}{\eta}} t^2dt = \frac{3\zeta^2+9\zeta+7}{3\eta^3}$,

$$\begin{aligned}
K^*_{\eta,\theta}(t^2;x) &= (1-\theta)\sum_{\zeta=0}^{\eta-1}\frac{\eta}{\eta-1}\left[\left(1-\frac{\zeta}{\eta-1}\right)\left(\frac{3\zeta^2+3\zeta+1}{3\eta^3}\right)\right. \\
&\quad \left.+\frac{\zeta}{\eta-1}\left(\frac{3\zeta^2+9\zeta+7}{3\eta^3}\right)\right] \\
&\quad \times \binom{\eta-1}{\zeta}(\zeta-(\eta-1)x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-2} \\
&\quad +\theta\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-1}\left(\frac{3\zeta^2+3\zeta+1}{3\eta^3}\right) \\
&= \left(\frac{1-\theta}{\eta-1}+\frac{(1-\theta)\,\eta}{\eta-1}+\theta\right)B^*_\eta(t^2;x) \\
&\quad +\left(\frac{1-\theta}{\eta-1}+\frac{1-\theta}{\eta\,(\eta-1)}+\frac{\theta}{\eta}\right)B^*_\eta(t;x)+\frac{1-\theta}{3\eta}+\frac{\theta}{3\eta} \\
&= \frac{(3\eta^3-18\eta^2-3\eta+18)-\theta(6\eta^2-42\eta+36)}{3\eta^2(\eta-1)}x^2 \\
&\quad +\frac{(18\eta^2-6\eta-24)-\theta(36\eta-48)}{3\eta^2(\eta-1)}x \\
&\quad +\frac{\left(\eta^2+5\eta+6\right)-12\theta}{3\eta^2(\eta-1)}.
\end{aligned}$$

□

Lemma 2. *For every $x \in (0,1)$, we have the central moments as*

$$\begin{aligned}
K^*_{\eta,\theta}(t-x;x) &= \frac{-4x+3}{2\eta}, \\
K^*_{\eta,\theta}((t-x)^2;x) &= \frac{1}{3\eta^2(\eta-1)}\left\{18x^2-24x+6+\eta(5+3x-15x^2)\right.
\end{aligned}$$

$$+\eta^2(1+9x-3x^2)$$
$$-\theta[12-48x+36x^2+6\eta^2x^2+\eta(36x-42x^2)]\big\}.$$

Proof. For the sake of brevity, central moments can be expressed as

$$\begin{aligned} K^*_{\eta,\theta}(t-x;x) &= K^*_{\eta,\theta}(e_1;x)-xK^*_{\eta,\theta}(e_0;x), \\ K^*_{\eta,\theta}((t-x)^2;x) &= K^*_{\eta,\theta}(e_2;x)-2xK^*_{\eta,\theta}(e_1;x)+x^2K^*_{\eta,\theta}(e_0;x). \end{aligned}$$

The proof is completed by using these equalities. □

Let $C[0,1]$ be the Banach space of all continuous functions g on $[0,1]$ with the norm

$$\|g\| = \max_{x\in(0,1)} |g(x)|.$$

Theorem 2. *For every $x \in (0,1)$ and $g \in C[0,1]$*

$$\left\|K^*_{\eta,\theta}(g;x)-g(x)\right\| \to 0, \tag{6}$$

uniformly as $\eta \to \infty$.

Proof. In the light of Lemma 1, we have

$$\lim_{\eta\to\infty} K^*_{\eta,\theta}(e_i;x) = t^i, \qquad i=0,1,2.$$

By Korovkin theorem [11] the proof is completed. □

3. Rate of convergence

The modulus of continuity is given by

$$\omega(g,\delta) := \sup_{|t-x|\le\delta}\ \sup_{x\in(0,1)} |g(t)-g(x)|, \quad \delta>0,$$

where $g \in C[0,1]$. Following feature of the modulus of continuity [1]

$$|g(t)-g(x)| \le \left(1+\frac{|t-x|}{\delta}\right)\omega(g,\delta)$$

will be used in the proof of the next theorem.

Theorem 3. *For every $x \in (0,1)$ and $g \in C[0,1]$,*

$$|K^*_{\eta,\theta}(g;x)-g(x)| \le 2\omega(g;\delta_\eta). \tag{7}$$

Here,

$$\begin{aligned} \delta_\eta(x) &= \left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}} \\ &= \left\{\frac{1}{3\eta^2(\eta-1)}\left\{-9\eta^4x+9\eta^3x+18x^2-24x+6+\eta^2(1+18x-3x^2)\right.\right. \\ &\quad \left.\left.+\eta(5-6x-15x^2)-\theta[12-48x+36x^2+6\eta^2x^2+\eta(36x-42x^2)]\right\}\right\}^{1/2}. \end{aligned}$$

Proof. For $K^*_{\eta,\theta}$, we write

$$\begin{aligned}\left|K^*_{\eta,\theta}(g;x)-g(x)\right| &= \left|K^*_{\eta,\theta}(g(t)-g(x);x)\right| \\ &\leq K^*_{\eta,\theta}(\left|g(t)-g(x)\right|;x) \\ &\leq \omega\left(g;\delta\right)\left\{K^*_{\eta,\theta}(1;x)+\frac{1}{\delta}K^*_{\eta,\theta}(\left|t-x\right|;x)\right\} \\ &\leq \omega\left(g;\delta\right)\left\{1+\frac{1}{\delta}\left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}\right\}.\end{aligned}$$

If we select

$$\delta=\delta_\eta=\left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}},$$

then we get

$$\left|K^*_{\eta,\theta}(g;x)-g(x)\right|\leq 2\omega\left(g;\left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}\right),$$

which is the desired result. □

Here, we investigate the rate of convergence of $K^*_{\eta,\theta}(g;x)$ by using functions of Lipschitz class. Let's recall that a function $g\in Lip_M\left(\varsigma\right)$ on $(0,1)$ if the inequality

$$\left|g\left(t\right)-g\left(x\right)\right|\leq M\left|t-x\right|^{\varsigma}\ \ ;\ \forall t,x\in(0,1) \tag{8}$$

holds.

Theorem 4. *Let $x\in(0,1)$, $g\in Lip_M\left(\varsigma\right)$, $0<\varsigma\leq 1$, then we get*

$$\left|K^*_{\eta,\theta}(g;x)-g(x)\right|\leq M\delta^{\varsigma}_{\eta}\left(x\right),$$

where

$$\begin{aligned}\delta_\eta(x) &= \left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}} \\ &= \Bigg\{\frac{1}{3\eta^2(\eta-1)}\left\{-9\eta^4x+9\eta^3x+18x^2-24x+6+\eta^2(1+18x-3x^2)\right. \\ &\quad \left.+\eta(5-6x-15x^2)-\theta[12-48x+36x^2+6\eta^2x^2+\eta(36x-42x^2)]\right\}\Bigg\}^{1/2}.\end{aligned}$$

Proof. Let $x\in(0,1)$, $g\in Lip_M\left(\varsigma\right)$ and $0<\varsigma\leq 1$. From the linearity and monotonicity of the operators $K^*_{\eta,\theta}$, we have

$$\begin{aligned}\left|K^*_{\eta,\theta}(g;x)-g\left(x\right)\right| &\leq K^*_{\eta,\theta}(\left|g\left(t\right)-g\left(x\right)\right|;x) \\ &\leq MK^*_{\eta,\theta}(\left|t-x\right|^{\varsigma};x).\end{aligned}$$

By putting $p=\frac{2}{\varsigma}$, $q=\frac{2}{2-\varsigma}$ in the Hölder inequality, we obtain

$$\begin{aligned}\left|K^*_{\eta,\theta}\left(g;x\right)-g\left(x\right)\right| &\leq M\left[K^*_{\eta,\theta}(\left(t-x\right)^2;x)\right]^{\frac{\varsigma}{2}} \\ &\leq M\delta^{\varsigma}_{\eta}\left(x\right).\end{aligned}$$

By choosing

$$\delta_\eta(x)=\left[K^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}$$

the proof is completed. □

Lastly, we will give the rate of convergence of our operator $K^*_{\eta,\theta}(g;x)$ by means of Peetre-$\mathcal{K}$ functionals. First of all, we give the following lemma.

Lemma 3. *For $x \in (0,1)$ and $g \in C[0,1]$, we get*

$$|K^*_{\eta,\theta}(g;x)| \leq ||g||. \tag{9}$$

Proof. For $K^*_{\eta,\theta}$,

$$\begin{aligned} |K^*_{\eta,\theta}(g;x)| &= \left| \sum_{\zeta=0}^{\eta} \rho^{(\theta)}_{\eta,\zeta}(x) \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} g(t)dt \right| \\ &\leq \sum_{\zeta=0}^{\eta} \rho^{(\theta)}_{\eta,\zeta}(x) \left| \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} g(t)dt \right| \\ &\leq \sum_{\zeta=0}^{\eta} \rho^{(\theta)}_{\eta,\zeta}(x) \int_{\frac{\zeta}{\eta}}^{\frac{\zeta+1}{\eta}} |g(t)|\, dt \\ &\leq ||g|| K^*_{\eta,\theta}(1;x) \\ &= ||g||. \end{aligned}$$

□

$C^2[0,1]$ is the space of the functions g, for which g, g' and g'' are continuous on $[0,1]$. The norm on the space $C^2[0,1]$ is given by

$$\|h\|_{C^2[0,1]} := \|h\|_{C[0,1]} + \|h'\|_{C[0,1]} + \|h''\|_{C[0,1]}.$$

Now, we define classical Peetre-$\mathcal{K}$ functional as follows:

$$\mathcal{K}(g,\lambda) := \inf_{h \in C^2[0,1]} \{\|g-h\| + \lambda \|h''\|\}$$

where $\lambda > 0$.

Theorem 5. *Let $x \in (0,1)$ and $g \in C[0,1]$. Then we have for all $\eta \in \mathbb{N}$,*

$$|K^*_{\eta,\theta}(g;x) - g(x)| \leq 2\mathcal{K}(g;\lambda_\eta(x)),$$

where

$$\begin{aligned} \lambda_\eta(x) &= \frac{1}{6\eta^2(\eta-1)} \left| 10\eta^2 - 4\eta + 6 - 12\theta + \left(-24 + 15\eta - 3\eta^2 - 6\theta(6\eta - 8)\right)x \right. \\ &\quad \left. + \left(18 - 15\eta - 3\eta^2 - 6\theta\left(6 - 7\eta + \eta^2\right)\right)x^2 \right|. \end{aligned}$$

Proof. For a given function $h \in C^2[0,1]$, we have the following Taylor expansion

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-s)h''(s)ds, \;\; t \in (0,1). \tag{10}$$

Applying $K^*_{\eta,\theta}$ operator to the Eqn. (10), we get

$$\begin{aligned}
\left|K^*_{\eta,\theta}(h;x)-h(x)\right| &= \left|K^*_{\eta,\theta}\left((t-x)h'(x);x\right)\right| + \left|K^*_{\eta,\theta}\left(\int_x^t (t-s)h''(s)ds;x\right)\right| \\
&\leq ||h'||\left|K^*_{\eta,\theta}\left(t-x;x\right)\right| + ||h''||\left|K^*_{\eta,\theta}\left(\int_x^t (t-s)ds;x\right)\right| \\
&\leq ||h'||\left|K^*_{\eta,\theta}\left(t-x;x\right)\right| + ||h''||\frac{1}{2}K^*_{\eta,\theta}\left((t-x)^2;x\right).
\end{aligned}$$

So,

$$\left|K^*_{\eta,\theta}(h;x)-h(x)\right| \leq \lambda||h||.$$

Using the above inequality, we get

$$\begin{aligned}
\left|K^*_{\eta,\theta}(g;x)-g(x)\right| &= \left|K^*_{\eta,\theta}(g;x)-g(x)+K^*_{\eta,\theta}(h;x)-K^*_{\eta,\theta}(h;x)+h(x)-h(x)\right| \\
&\leq ||g-h||\left|K^*_{\eta,\theta}(1;x)\right| + ||g-h|| + \left|K^*_{\eta,\theta}(h;x)-h(x)\right| \\
&\leq 2\left(||g-h||+\lambda||h||\right) \\
&= 2\mathcal{K}\left(g;\lambda\right).
\end{aligned}$$

As a result, by choosing

$$\begin{aligned}
\lambda=\lambda_\eta\left(x\right) &= \frac{1}{6\eta^2\left(\eta-1\right)}\left|10\eta^2-4\eta+6-12\theta\right. \\
&\quad +\left(-24+15\eta-3\eta^2-6\theta\left(6\eta-8\right)\right)x \\
&\quad \left.+\left(18-15\eta-3\eta^2-6\theta\left(6-7\eta+\eta^2\right)\right)x^2\right|,
\end{aligned}$$

we obtain

$$|K^*_{\eta,\theta}(g;x)-g(x)| \leq 2\mathcal{K}\left(g;\lambda_\eta\right). \tag{11}$$

Thus, the proof is completed. □

4. Graphical Analysis

In this part, we present some graphics to show the convergence of the operators $K^*_{\eta,\theta}$ to the function g. It is already known that, the operators $K^*_{\eta,\theta}(g;x)$ have been defined for $x \in (0,1)$. For this reason, the closed interval is given by $[0+\epsilon, 1-\epsilon]$, where $\epsilon = 0.0001$.

Example 1. *Let*

$$g(x) = x(x-1)\left(x-\frac{1}{12}\right).$$

Then for $\theta = 0.25$, $\theta = 0.5$ and $\theta = 0.9$, we have plotted the convergence of the new constructed $K^*_{\eta,\theta}$ parametric Bernstein-Kantorovich operators and B^*_η modified Bernstein operators [18] to the function g in Fig. 1 for $\eta = 125$.

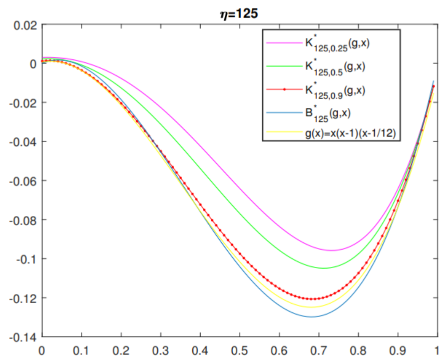


FIGURE 1. Convergence of $K^*_{\eta,\theta}(g;x)$ for different values of θ with fixed $\eta = 125$.

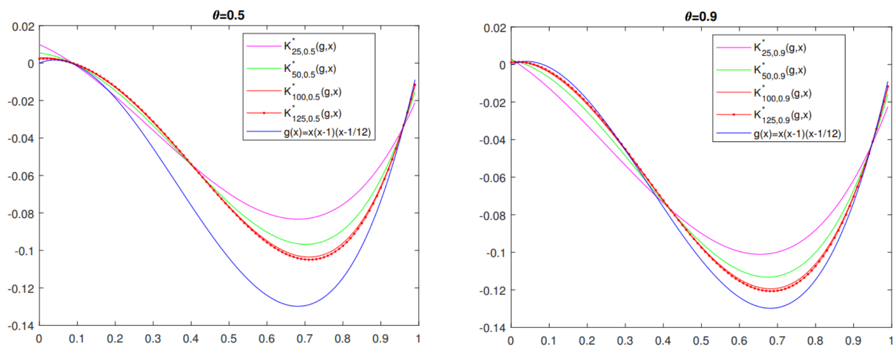


FIGURE 2. Convergence of $K^*_{\eta,\theta}(g;x)$ for different values of η with fixed θ.

In Fig. 2, we have illustrated the convergence of the $K^*_{\eta,\theta}$ operators to the target function $g(x) = x(x-1)\left(x - \frac{1}{12}\right)$ for fixed $\theta = 0.5$ and $\theta = 0.9$, where $\eta \in \{25, 50, 100, 125\}$. The maximum errors for the operators $K^*_{\eta,\theta}$ and B^*_η to the function $g(x) = x(x-1)\left(x - \frac{1}{12}\right)$ are presented in Table 1 for different values of θ and η.

It is obvious from the Table 1 that the best error in the approximation of g by $K^*_{\eta,\theta}$ is achieved when $\theta = 0.999$. Moreover, we note that the error in the approximation of $K^*_{\eta,0.99}(g)$ and $K^*_{\eta,0.999}(g)$ is much smaller than the errors in the approximation $B^*_\eta(g)$, where $\eta \in \{25, 50, 100, 125\}$.

Example 2. *As a second example, we choose*

$$g(x) = xe^{-3x}$$

TABLE 1. Error for approximation of the parametric Bernstein-Kantorovich operators $K^*_{\eta,\theta}$ and modified Bernstein operators B^*_η.

θ	η	$\|\|B^*_\eta(g) - g\|\|$	$\|\|K^*_{\eta,\theta}(g) - g\|\|$
0.99	25	0.0296	0.0262
0.99	50	0.0155	0.0134
0.99	100	0.0079	0.0069
0.99	125	0.0063	0.0056
0.999	25	0.0296	0.0258
0.999	50	0.0155	0.0131
0.999	100	0.0079	0.0066
0.999	125	0.0063	0.0053

and $x \in (0,1)$. *Then for* $\theta = 0.79$, $\theta = 0.89$ *and* $\theta = 0.99$, *we have plotted the convergence of the* $K^*_{\eta,\theta}$ *Bernstein-Kantorovich operators to the function* g *in Fig. 3 for* $\eta = 170$.

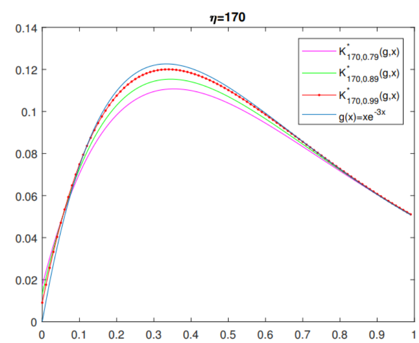
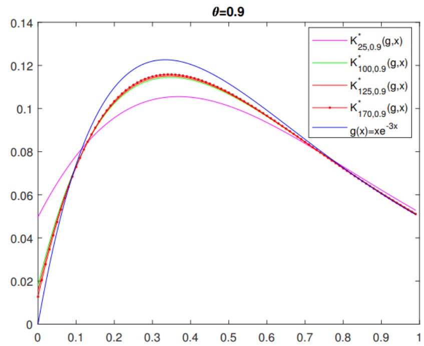


FIGURE 3. Convergence of $K^*_{\eta,\theta}(g;x)$ for $\theta = 0.79$, $\theta = 0.89$ and $\theta = 0.99$.

In Fig. 4, we have presented $K^*_{\eta,\theta}(g;x)$ for fixed $\theta = 0.9$ and $\theta = 0.99$, where $\eta \in \{25, 100, 125, 170\}$.

The error estimation for newly constructed operators $K^*_{\eta,\theta}$ to the function $g(x) = xe^{-3x}$ is presented in Table 2 for different values of θ and η.

It is evident from the Table 2 that the best error in the approximation of g by $K^*_{\eta,\theta}$ is achieved when $\theta = 0.99$ and $\eta = 170$.



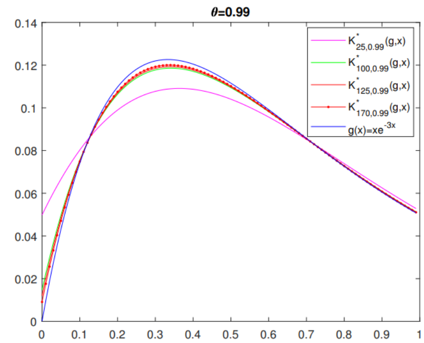


FIGURE 4. Convergence of $K^*_{\eta,\theta}(g;x)$ for $\theta = 0.9$ and $\theta = 0.99$.

TABLE 2. Error for approximation of the $K^*_{\eta,\theta}$ for $\theta = 0.79, 0.89, 0.99$.

θ	η	$\|\|K^*_{\eta,\theta}(g) - g\|\|$
0.79	25	0.0496
0.79	125	0.0194
0.79	170	0.0171
0.89	25	0.0194
0.89	125	0.0157
0.89	170	0.0130
0.99	25	0.0171
0.99	125	0.0119
0.99	170	0.0090

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B-LIFT CURVES AND INVOLUTE CURVES IN LORENTZIAN 3-SPACE

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Abstract. The involute of a curve is often called the perpendicular trajectories of the tangent vectors of a unit speed curve. Furthermore, the B-Lift curve is the curve acquired by combining the endpoints of the binormal vectors of a unit speed curve. In this study, we investigate the correspondences between the Frenet vectors of a curve's B-lift curve and its involute. We also give an illustration of a helix that resembles space in Lorentzian 3-space and show how to visualize these curves by deriving the B-Lift curve and its involute.

1. Introduction

The Lorentz-Minkowski space was expressed in a special metric by the German mathematician Hermann Minkowski in 1907. Unlike the Euclidean space, this space has a temporal dimension. Studies in the Lorentzian space have many physical applications. For example, Lorentzian space is used to formalize Einstein's relativity theory. The character of a vector in Lorentzian space is also defined as spacelike, timelike or lightlike (null).

C. Huygens carried out the curvature of the plane curves at any point in Euclidean space. Sir Isaac Newton defined the curve depending on a parameter and expressed the curvature of the curve. The differential geometry of curves in Euclidean or Lorentzian spaces has been the subject of numerous investigations. [1–9]. Especially at the mutual point of the two curves, new ideas were put forward by establishing connections between Frenet operators. Involute curves and natural lift curves are some of them.

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The involute of a curve is generally referred to as the orthogonal trajectories of the tangent vectors of a unit speed curve. In 1668, the idea of involute curves was first discovered by C. Huygens in optical studies. Afterward, Millman and Parker (1977) [10] and Hacısalihoğlu (1983) [11] clarified the known theorems and results. A basic study on the involute-evolute curves was examined by Çalışkan and Bilici in 2002 [12].They looked into the relationship between the main curve's Frenet operators and its involute curve. They also introduced some important results in 2009 [13], such as curvature and torsion for involute curves, Frenet vectors of non-null curves in Lorentzian space.

By definition, a natural lift curve is created by joining the ends of a unit speed curve's tangent vectors. [14]. The natural lift curve has been investigated by many mathematicians [15–19]. In [18], the authors identified the correlations between the Frenet vectors of the natural lift curve and the main curve. They also gave the characterizations between the natural lift and involute of a curve [19].

In this article, we present the relationships between the B-Lift curve and the involute curve's Frenet vectors in Minkowski 3-space. In this context, the results show that the Frenet vectors of the B-Lift curve and the involute curves are the same; only their signs are different. Additionally, we illustrate our curves and provide an example based on these findings.

2. Preliminaries

The real vector space $\mathbb{R}^3$ that is supplied with a Lorentzian inner product is known as the Lorentzian 3-space $\mathbb{R}^3_1$ and is defined as

$$\langle x, y\rangle_{\mathbb{L}} = -x_1y_1 + x_2y_2 + x_3y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are in $\mathbb{R}^3$ [20].

Let $x = (x_1, x_2, x_3)$ be a vector in $\mathbb{R}^3_1$. Then, x is considered timelike if $\langle$x, x$\rangle < 0$, lightlike if $\langle$x, x$\rangle = 0$ and x $\neq 0$, spacelike if $\langle$x, x$\rangle > 0$ or x=0 [20].

If $\gamma'(s)$ is timelike, lightlike, or spacelike at any $s \in I$, then a curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3_1$ is either timelike, lightlike, or spacelike, respectively. Using the Lorentzian inner product, the norm of the vector $x = (x_1, x_2, x_3)$ is defined as [20]

$$\|x\|_{\mathbb{L}} = \sqrt{|\langle x, x\rangle|}.$$

If $\|x\|_{\mathbb{L}}=1$, the vector x is called a unit vector. The definition of the Lorentzian vector product of the vectors x and y for the vectors x and y in $\mathbb{R}^3_1$ is [21]

$$x \times y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Assume that γ is a unit speed curve. Given by tangent, primary normal, and binormal vectors, respectively, the set $\{T(s), N(s), B(s)\}$ is known as the Frenet frame. For any unit speed curve γ, the Darboux vector represented by W, and we call $W(s) = \tau(s)T(s) + \kappa(s)B(s)$. Let θ be the angle formed by the binormal vector B and the Darboux vector W, then we have

$$\kappa = \|W\|\cos\theta, \tau = \|W\|\sin\theta.$$

We now look at Frenet-Serret formulas based on the curve's Lorentzian characteristics [22]:

i) Suppose that γ is a unit speed spacelike curve and B is a spacelike vector. As a result, N is a timelike vector, while T and B are spacelike vectors. In that condition, we have:

$$N \times B = -T, \quad T \times N = -B, \quad B \times T = -N.$$

The Frenet-Serret formulas follow as

$$\begin{aligned} T^{'} &= \kappa N, \\ N^{'} &= \kappa T + \tau B, \\ B^{'} &= \tau N. \end{aligned}$$

ii) Assume that γ is unit speed spacelike curve and B is a timelike vector. Then, T and N are spacelike vectors, B is a timelike vector. In that case, we can write

$$N \times B = -T, \quad T \times N = B, \quad B \times T = -N.$$

Here are the Frenet-Serret formulas

$$\begin{aligned} T^{'} &= \kappa N, \\ N^{'} &= -\kappa T + \tau B, \\ B^{'} &= \tau N. \end{aligned}$$

iii) Assume that γ is a unit speed timelike curve. Then, N and B are spacelike vectors and T is a timelike vector. In that case, we have

$$N \times B = T, \quad T \times N = -B \quad B \times T = -N.$$

Here are the Frenet-Serret formulas

$$\begin{aligned} T^{'} &= \kappa N, \\ N^{'} &= \kappa T + \tau B, \\ B^{'} &= -\tau N. \end{aligned}$$

Lemma 1 ([23]). *Assume that x and y are linearly independent spacelike vectors that span a spacelike vector subspace in $\mathbb{R}^3_1$. In that case, we get the following inequality:*

$$|\langle x, y\rangle| \leq \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}}.$$

Hence we can write

$$\langle x, y\rangle = \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}} \cos\varphi,$$

where the angle amongst x and y is φ.

Lemma 2 ([23]). *Assume that x and y are linearly independent spacelike vectors that span a timelike vector subspace in $\mathbb{R}_1^3$. Thus we get*

$$|\langle x, y\rangle| > \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}}.$$

Therefore we can write

$$|\langle x, y\rangle| = \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}} \cosh\varphi,$$

where the angle amongst x and y is φ

Lemma 3 ([23]). *Assume that x is a spacelike vector and y is a timelike vector in $\mathbb{R}_1^3$. In that condition, we can write*

$$|\langle x, y\rangle| = \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}} \sinh\varphi,$$

where the angle amongst x and y is φ

Lemma 4 ([23]). *Suppose that x and y are timelike vectors in $\mathbb{R}_1^3$. In that case, we can write*

$$\langle x, y\rangle = -\|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}} \cosh\varphi,$$

where the angle amongst x and y is φ

Definition 1 ([19]). *Let $\gamma = (\gamma(s); T(s), N(s), B(s))$ and $\gamma^* = (\gamma^*(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ are regular curves in $\mathbb{R}_1^3$. $\gamma^*(s^*)$ is called the involute of $\gamma(s)$ ($\gamma(s)$ is called the evolute of $\gamma^*(s^*)$) if $\langle T(s), T^*(s^*)\rangle = 0$. In that case, (γ, γ^*) is called involute-evolute curve couple.*

Proposition 1 ([19]). *Assume that γ is a timelike curve. Then, γ^* is a spacelike curve and B^* is a timelike or spacelike vector. We are aware of the following equations connecting the Frenet frames $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ of curves γ and γ^*:*

i) *Assume that γ is a spacelike curve and B is a spacelike vector.*

a) *If W Darboux vector is timelike, then we can write*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & \cosh\varphi \\ -\cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is spacelike, then we can write*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh\varphi & 0 & \sinh\varphi \\ -\sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Let γ be a spacelike curve and B be a timelike vector.*

a) *If W Darboux vector is timelike, then we can write*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sinh\varphi & 0 & -\cosh\varphi \\ -\cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is spacelike, then we can write*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\varphi & 0 & -\sinh\varphi \\ -\sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 2 ([19]). *Let γ be a spacelike curve and B be spacelike or timelike vector. Then γ^* is a spacelike curve. We know the following equations:*
i) *Let γ be a spacelike curve and B be spacelike vector.*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cos\varphi & 0 & \sin\varphi \\ \sin\varphi & 0 & -\cos\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Let γ be a spacelike curve and B be timelike vector.*
a) *If W Darboux vector is timelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh\varphi & 0 & -\sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is spacelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & -\cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 3 ([19]). *Assume that γ is a spacelike curve and B is a spacelike vector. Then γ^* is a spacelike curve and the following equations are available:*
i) *Let γ^* be a spacelike curve and B^* be a spacelike vector.*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sin\varphi & 0 & -\cos\varphi \\ -\cos\varphi & 0 & -\sin\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Let γ^* be a spacelike curve and B^* be a timelike vector.*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sin\varphi & 0 & -\cos\varphi \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 4 ([19]). *Assume that γ is a spacelike curve and B is a timelike vector. In that case, γ^* is a spacelike curve and the following equations exist:*
i) *Suppose that γ^* is a spacelike curve and B^* is a spacelike vector.*
a) *If W Darboux vector is spacelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sinh\varphi & 0 & \cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\varphi & 0 & \sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Suppose that γ^* is a spacelike curve and B^* is a timelike vector.*

a) *If W Darboux vector is spacelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & -\cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh\varphi & 0 & -\sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Definition 2 ([24]). *If $\gamma : I \to P$ is a unit speed curve, then $\gamma_B : I \to TP$ is known as the B-Lift curve and guarantees the following equation:*

$$\gamma_B(s) = (\gamma(s), B(s)) = B(s)|_{\gamma(s)}, \tag{1}$$

where $P \subset \mathbb{R}^3_1$ is a surface.

3. Involute Curves and B-Lift Curves in Minkowski 3-Space

Proposition 5. *Assume that γ is a timelike curve. Then, γ_B is a spacelike curve and B is spacelike or timelike.*

i) *Suppose that γ_B is a spacelike curve and B_B is timelike vector. The following equations are available:*

a) *If W Darboux vector is spacelike, we can write*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\cosh\varphi & 0 & -\sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, we can write*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\sinh\varphi & 0 & -\cosh\varphi \\ \cosh\varphi & 0 & \sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Assume that γ_B is a spacelike curve and B_B is spacelike vector. We are aware of the following equations connecting the Frenet frames $\{T_B, N_B, B_B\}$ and $\{T, N, B\}$ of curves γ_B and γ:*

a) *If W Darboux vector is spacelike, we know that*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \cosh\varphi & 0 & \sinh\varphi \\ -\sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, we know that*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \sinh\varphi & 0 & \cosh\varphi \\ \cosh\varphi & 0 & \sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 6. *Suppose that γ is a spacelike curve and B is a spacelike vector. Then, γ_B is a timelike curve. We know the following equations:*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \cos\varphi & 0 & \sin\varphi \\ \sin\varphi & 0 & -\cos\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 7. *Suppose that γ is a spacelike curve and B is timelike vector. Then, γ_B is a spacelike curve and B_B is timelike or spacelike vector.*

i) *Let γ_B be a spacelike curve and B_B be a timelike vector.The following equations are available:*

a) *If W Darboux vector is spacelike, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\sinh\varphi & 0 & \cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\cosh\varphi & 0 & \sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Let γ_B be a spacelike curve and B_B be spacelike vector. We have the following equations:*

a) *If W Darboux vector is spacelike, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \sinh\varphi & 0 & -\cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \cosh\varphi & 0 & \sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Corollary 1. *Assume that γ is a timelike curve. Then γ^* is a spacelike curve and B^* is spacelike vector.*

i) *If W Darboux vector is spacelike, then we get*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= N_B, \\ B^* &= B_B. \end{aligned}$$

ii) *If W Darboux vector is timelike, then we get*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= B_B, \\ B^* &= N_B. \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 2. *Assume that γ is a timelike curve. Therefore γ^* is a spacelike curve and B^* is timelike vector.*
i) *If W Darboux vector is spacelike, then we get*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= -B_B, \\ B^* &= N_B. \end{aligned}$$

ii) *If W Darboux vector is timelike, then we get*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= N_B, \\ B^* &= -B_B, \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 3. *Assume that γ is a spacelike curve and B is a spacelike vector. Then γ^* is a timelike curve.*
i) *If W Darboux vector is spacelike, then we have*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= -B_B, \\ B^* &= N_B. \end{aligned}$$

ii) *If W Darboux vector is timelike, then we have*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= N_B, \\ B^* &= -B_B, \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 4. *Assume that γ is a spacelike curve and B is timelike vector. Then γ^* is a timelike curve.*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= N_B, \\ B^* &= -B_B, \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 5. *Assume that γ is a spacelike curve and B is spacelike vector.*
i) *If γ^* is spacelike curve and B^* is spacelike vector, hence we get*

$$\begin{aligned} T^* &= T_B, \\ N^* &= N_B, \\ B^* &= B_B. \end{aligned}$$

ii) *If γ^* is spacelike curve and B^* is timelike vector, hence we get*

$$\begin{aligned} T^* &= T_B, \\ N^* &= -N_B, \\ B^* &= B_B, \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 6. *Assume that γ is a spacelike curve and B is timelike vector.*
i) *If γ and γ^* are spacelike curves with timelike binormal, then we get*

$$\begin{aligned} T^* &= T_B, \\ N^* &= -N_B, \\ B^* &= -B_B. \end{aligned}$$

ii) *If γ^* is spacelike curve and B^* is spacelike vector, then we get*

$$\begin{aligned} T^* &= T_B, \\ N^* &= N_B, \\ B^* &= B_B, \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 7. *Let γ^* and γ_B be involute curve and B-Lift curve of a unit speed curve γ, respectively. Then, the sets $\{T^*, T_B\}$, $\{N^*, N_B\}$ and $\{B^*, B_B\}$ are linearly dependent.*

Example 1. *Suppose that the spacelike circular helix curve is given by*

$$\gamma(s) = (\frac{s}{\sqrt{3}}, 2\cos(\frac{s}{\sqrt{3}}), 2\sin(\frac{s}{\sqrt{3}})).$$

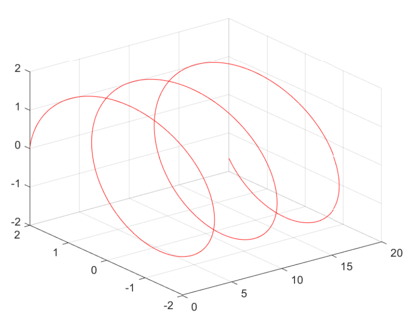

FIGURE 1. The spacelike helix curve $\gamma(s)$

For the spacelike helix curve γ, Frenet frames can be calculated by

$$\begin{aligned} T(s) &= \left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\sin(\frac{s}{\sqrt{3}}), \frac{2}{\sqrt{3}}\cos(\frac{s}{\sqrt{3}})\right), \\ N(s) &= \left(0, -\cos(\frac{s}{\sqrt{3}}), -\sin(\frac{s}{\sqrt{3}})\right), \\ B(s) &= \left(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\sin(\frac{s}{\sqrt{3}}), \frac{1}{\sqrt{3}}\cos(\frac{s}{\sqrt{3}})\right). \end{aligned}$$

Then the B-lift curve is following as

$$\gamma_B(s) = \left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\sin(\frac{s}{\sqrt{3}}), \frac{1}{\sqrt{3}}\cos(\frac{s}{\sqrt{3}})\right).$$

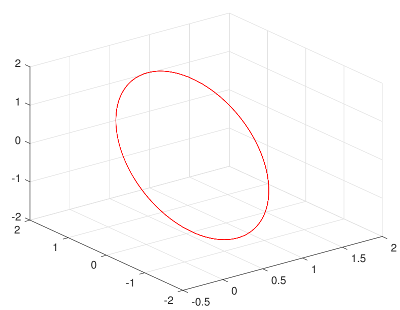

FIGURE 2. B-Lift curve of the curve $\gamma(s)$

For $\lambda=-\sqrt{3}$, the involute of the curve $\gamma(s)$ is given by

$$\begin{aligned} \gamma^*(s) &= \gamma(s) + \lambda.T(s) \\ &= \left(\frac{s}{\sqrt{3}}, 2\cos(\frac{s}{\sqrt{3}}), 2\sin(\frac{s}{\sqrt{3}})\right) + (-\sqrt{3}).\left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\sin(\frac{s}{\sqrt{3}}), \frac{2}{\sqrt{3}}\cos(\frac{s}{\sqrt{3}})\right) \\ &= \left(\frac{s}{\sqrt{3}} - 1, 2(\cos(\frac{s}{\sqrt{3}}) + \sin(\frac{s}{\sqrt{3}})), 2(\sin(\frac{s}{\sqrt{3}}) - \cos(\frac{s}{\sqrt{3}}))\right) \end{aligned}$$

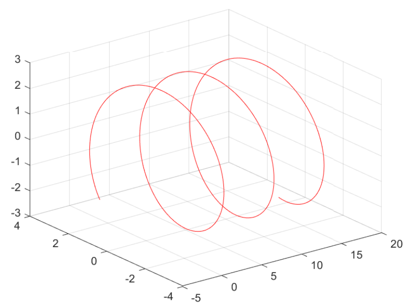

FIGURE 3. Involute curve of the curve $\gamma(s)$

4. Conclusions

In this study, the relations of a spacelike or timelike unit speed curve given in Minkowski-3 space with the B-Lift curve were examined. Furthermore, the equations relating the Frenet operators of the involute curve and the B-Lift curve were discovered. As a consequence, we may summarize the findings of this study as follows:

1. When the Frenet apparatus of the B-Lift curve of a unit speed curve are compared with the Frenet apparatus of the involute curve of a unit speed curve, it is shown that the Frenet vectors are similar; only their signs differ.

2. By giving an example, we obtained the B-Lift curve and the Frenet operators of the involute curve of a given curve and checked the results we found with the help of an example.

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COMPLEX DEFORMABLE CALCULUS

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Abstract. In this paper, we give a new complex deformable derivative and integral of order λ which coincides with the classical derivative and integral for the special values of the parameters. We examine the basic properties of this derivative and integral. We also investigate the basic concepts of complex analysis for the λ-complex deformable derivative. Finally, we give some applications.

1. Introduction

The derivative of a complex-valued function is defined as a certain limit, similar to the derivative of real-valued functions. The official definition is

$$f'(\zeta_0) = \lim_{\varepsilon \to 0} \frac{f(\zeta_0 + \varepsilon) - f(\zeta_0)}{\varepsilon}. \tag{1}$$

For this limit to exist, it must be equal to the same complex number from any direction. Therefore, the differentiability of a complex-valued function at a point is more complex than the differentiability of a real-valued function at a point. See [1,7].

The fractional derivative emerged in 1965 when Leibniz asked L'Hospital what does it mean derivative of order 1/2 [3]. Since then, the fractional derivative has attracted the attention of many researchers. Many fractional derivatives have been defined so far. An integral form was generally used in these definitions. Fractional derivatives, introduced by mathematicians such as Caputo, Riemann-Liouville, Hadamard, Riesz, and Grunwald-Letnikov, are the most popular of the fractional derivatives. See [4–6] for more information on the fractional derivative.

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In 2014, Khlalil et al. [2] presented a limit-based definition of the fractional derivative for real-valued functions, similar to the standard derivative definition. Recently, Zulfeqarr et al. [8] introduced a new derivative called the deformable derivative, similar to Khalil's definition of the derivative. The conformable derivative is defined for functions whose domain is zero and positive numbers. Therefore, this derivative definition lacks to include negative numbers. Deformable derivative overcomes this deficiency in the conformable derivative.

The aim of this study is to introduce the λ-complex deformable derivative. In the second section, we give the relationship between λ-complex differentiability and complex differentiability. In the third section, we investigate the fundamental properties of the λ-complex deformable derivative. On the other hand, we examine the fundamental concepts of complex analysis according to this derivative. In the fourth section, we introduce the deformable integral operator for complex functions and give some of its properties. In the last section, we give some applications.

2. Complex Deformable Derivative

We first give the λ-complex deformable derivative definition.

Definition 1. *Let f be a complex-valued function and $0 \leq \lambda \leq 1$. Then the λ-complex deformable derivative is defined by*

$$D^{\lambda} f(z) = \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(z+\varepsilon\lambda) - f(z)}{\varepsilon} \tag{2}$$

where $\lambda + \delta = 1$.

Note that this definition is compatible with $\lambda = 0$ and $\lambda = 1$. Because, if $\lambda = 0$, we get $D^0 f(z) = f(z)$, and if $\lambda = 1$, we get $Df(z) = f'(z)$. In this study, we assume that $0 < \lambda \leq 1$ unless otherwise stated.

The first result implies a relationship between the complex differentiability and the λ-complex deformable differentiability.

Theorem 1. *A complex differentiable f at $\zeta_0 \in \mathbb{C}$ is always λ-complex deformable differentiable at that point for any λ. Moreover, we have*

$$D^{\lambda} f(\zeta_0) = \delta f(\zeta_0) + \lambda D f(\zeta_0), \text{ where } \lambda + \delta = 1. \tag{3}$$

Proof. By definition, we have

$$\begin{aligned} D^{\lambda} f(\zeta_0) &= \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(\zeta_0+\lambda\varepsilon) - f(\zeta_0)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \left(\frac{f(\zeta_0+\lambda\varepsilon) - f(\zeta_0)}{\varepsilon} + \delta f(\zeta_0+\lambda\varepsilon) \right) \\ &= \lambda D f(\zeta_0) + \delta \lim_{\varepsilon \to 0} f(\zeta_0+\lambda\varepsilon). \end{aligned}$$

Since f is differentiable at ζ_0, it is continuous at ζ_0. Hence, $\lim_{\varepsilon \to 0} f(\zeta_0+\lambda\varepsilon)$ exist. Thus, the proof is complete. □

We know that a differentiable function f is continuous. The following result gives a similar result for λ-complex deformable differentiable functions.

Theorem 2. *If f is λ-complex deformable differentiable at a point ζ_0, then f is continuous at ζ_0.*

Proof. By hypothesis, the limits $\lim_{\varepsilon\to 0} \frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}$ and $\lim_{\varepsilon\to 0}\varepsilon$ exist and equal $D^\lambda f(\zeta_0)$ and 0, respectively. Hence, we can write

$$\begin{aligned}\lim_{\varepsilon\to 0}((1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)) &= \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}\varepsilon \\ &= \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}\lim_{\varepsilon\to 0}\varepsilon \\ &= D^\lambda f(\zeta_0).0 = 0.\end{aligned}$$

Since $\lim_{\varepsilon\to 0}(1+\varepsilon\delta)=1$, we get $\lim_{\varepsilon\to 0} f(\zeta_0+\varepsilon\lambda)=f(\zeta_0)$. Thus, f is continuous at ζ_0.

□

Theorem 3. *Let f be λ-complex deformable differentiable function at ζ_0. Then f is differentiable at ζ_0.*

Proof. By the description of complex differentiability

$$\begin{aligned}Df(\zeta_0) &= \frac{1}{\lambda}\lim_{\varepsilon\to 0}\frac{f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon} \\ &= \frac{1}{\lambda}\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)-\varepsilon\delta f(\zeta_0+\varepsilon\lambda)}{\varepsilon} \\ &= \frac{1}{\lambda}\left(\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}-\delta\lim_{\varepsilon\to 0}f(\zeta_0+\varepsilon\lambda)\right).\end{aligned}$$

Since the function f is λ-complex deformable differentiable at point ζ_0, it is continuous at the same point. Thus, the proof is complete. □

Theorem 4. *A function f is λ-complex deformable differentiable at ζ_0 if and only if it is differentiable at ζ_0.*

Definition 2. *Suppose that f is an m-times differentiable at ζ_0. For $\lambda\in(m,m+1]$, λ-complex deformable differentiable at ζ_0 is defined as*

$$D^\lambda f(\zeta_0)=\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\{\delta\})D^m f(\zeta_0+\varepsilon\{\lambda\})-D^m f(\zeta_0)}{\varepsilon}$$

where $\{\lambda\}$ is the fractional part of λ and $\{\lambda\}$+$\{\delta\}$=1.

By the above definition, if f is (m+1)-times differentiable, we get

$$D^\lambda f(\zeta_0)=\{\delta\}D^m f(\zeta_0)+\{\lambda\}D^{m+1}f(\zeta_0).$$

3. Basic Properties of Complex Deformable Derivative

In this part, we investigate certain properties of λ-complex deformable derivative.

Theorem 5. *The operator D^λ provides the following properties:*

(a) $D^\lambda(\alpha f(z)+\beta g(z))=\alpha D^\lambda f(z)+\beta D^\lambda g(z)$ *(Linearity),*
(b) $D^{\lambda_1}.D^{\lambda_2}=D^{\lambda_2}.D^{\lambda_1}$. *(Commutativity),*
(c) *For any constant c, $D^\lambda(c)=\delta c$,*
(d) $D^\lambda(fg)(z)=(D^\lambda f(z))g(z)+\lambda f(z)Dg(z)$.

Theorem 6. *The operator D^λ possesses the following property*

$$D^\lambda\left(\frac{f}{g}\right)(z)=\frac{g(z)D^\lambda(f(z))-\lambda f D(g(z))}{g^2(z)}.$$

Proof. We have

$$\begin{aligned}D^\lambda\left(\frac{f}{g}\right)(z) &= \delta\left(\frac{f(z)}{g(z)}\right)+\lambda D\left(\frac{f(z)}{g(z)}\right)\\ &= \delta\left(\frac{f(z)}{g(z)}\right)+\lambda\left(\frac{(Df(z))g(z)-f(z)(Dg(z))}{g^2(z)}\right)\\ &= \frac{g(z)\left(\lambda Df(z)+\delta f(z)\right)-\lambda f(z)(Dg(z))}{g^2(z)}\\ &= \frac{g(z)D^\lambda f(z)-\lambda f(z)(Dg(z))}{g^2(z)}.\end{aligned}$$

□

The following result gives the chain rule for the λ-complex deformable derivative.

Theorem 7. *Suppose f and g are λ-complex deformable differentiable at ζ_0. Then,*

$$D^\lambda(fog)(\zeta_0)=\delta(fog)(\zeta_0)+\lambda D(fog)(\zeta_0).$$

Proof. Since

$$D^\lambda f(\zeta_0)=\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}=\lim_{\varepsilon\to 0}\left[\frac{f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}+\delta f(\zeta_0+\varepsilon\lambda)\right],$$

we have

$$\begin{aligned}D^\lambda f(g(\zeta_0)) &= \lim_{\varepsilon\to 0}\left[\frac{f(g(\zeta_0+\varepsilon\lambda))-f(g(\zeta_0))}{\varepsilon}+\delta f(g(\zeta_0+\varepsilon\lambda))\right]\\ &= \lim_{\varepsilon\to 0}\left[\frac{f(g(\zeta_0+\varepsilon\lambda))-f(g(\zeta_0))}{g(\zeta_0+\varepsilon\lambda)-g(\zeta_0)}\frac{g(\zeta_0+\varepsilon\lambda)-g(\zeta_0)}{\varepsilon}+\delta f(g(\zeta_0+\varepsilon\lambda))\right]\\ &= \lim_{\varepsilon\to 0}\left[\frac{f(g(\zeta_0)+\varepsilon_0)-f(g(\zeta_0))}{\varepsilon_0}\frac{g(\zeta_0+\varepsilon\lambda)-g(\zeta_0)}{\varepsilon}+\delta f(g(\zeta_0+\varepsilon\lambda)\right],\end{aligned}$$

where $\varepsilon_0 \to 0$ as $\varepsilon \to 0$. We obtain

$$\begin{aligned} D^\lambda f(g(\zeta_0)) &= \lim_{\varepsilon_0 \to 0} \frac{f(g(\zeta_0)+\varepsilon_0) - f(g(\zeta_0))}{\varepsilon_0} \lim_{\varepsilon \to 0} \frac{g(\zeta_0+\varepsilon\lambda) - g(\zeta_0)}{\varepsilon} + \lim_{\varepsilon \to 0} \delta f(g(\zeta_0+\varepsilon\lambda)) \\ &= Df(g(\zeta_0))\lambda Dg(\zeta_0) + \delta f(g(\zeta_0)) \\ &= \lambda D[f(g(\zeta_0))] + \delta f(g(\zeta_0)). \end{aligned}$$

The proof is completed. □

Proposition 1.

(a) $D^\lambda(z^n) = \delta z^n + n\lambda z^{n-1}$, $n \in \mathbb{R}$.
(b) $D^\lambda(e^z) = e^z$.
(c) $D^\lambda(sinz) = \delta sinz + \lambda cosz$.
(d) $D^\lambda(logz) = \delta logz + \frac{\lambda}{z}$.

We now give the notion of real deformable partial derivatives.

Definition 3. *Suppose $f(x_1, x_2, ..., x_j)$ is real function. Then the formula for the partial derivative of f with respect to x_i is given by*

$$\frac{\partial^\lambda}{\partial x_i^\lambda} f(x_1, x_2, ..., x_j) = \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(x_1, ..., x_{i-1}, x_i + \lambda\varepsilon, ..., x_j) - f(x_1, x_2, ..., x_j)}{\varepsilon}. \quad (4)$$

$\frac{\partial^\lambda}{\partial x_i^\lambda} f$ can be also represented $f_{x_i}^{(\lambda)}$.

Theorem 8. *Let $f(z) = \mathfrak{u}(x,y) + i\mathfrak{v}(x,y)$ be an λ-complex deformable differentiable at $\zeta_0 = x_0 + iy_0$. Then the λ-complex deformable derivative of f*

$$D^\lambda f(\zeta_0) = \mathfrak{u}_x^{(\lambda)}(x_0, y_0) + i\mathfrak{v}_x^{(\lambda)}(x_0, y_0) = \mathfrak{v}_y^{(\lambda)}(x_0, y_0) - i\mathfrak{u}_y^{(\lambda)}(x_0, y_0). \quad (5)$$

Proof. Let $\varepsilon = a + ib$. For $b = 0$ and $a \to 0$, we get

$$\begin{aligned} D^\lambda f(\zeta_0) &= \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(\zeta_0 + \varepsilon\lambda) - f(\zeta_0)}{\varepsilon} \\ &= \lim_{a \to 0} \left[\frac{(1+a\delta)\mathfrak{u}(x_0 + a\lambda, y_0) - \mathfrak{u}(x_0, y_0)}{a} + i \frac{(1+a\delta)\mathfrak{v}(x_0 + a\lambda, y_0) - \mathfrak{v}(x_0, y_0)}{a} \right] \\ &= \mathfrak{u}_x^{(\lambda)}(x_0, y_0) + i\mathfrak{v}_x^{(\lambda)}(x_0, y_0). \end{aligned}$$

For $a = 0$ and $b \to 0$, we get

$$\begin{aligned} D^\lambda f(\zeta_0) &= \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(\zeta_0 + \varepsilon\lambda) - f(\zeta_0)}{\varepsilon} \\ &= \lim_{b \to 0} \left[\frac{(1+ib\delta)\mathfrak{u}(x_0, y_0 + ib\lambda) - \mathfrak{u}(x_0, y_0)}{ib} + i \frac{(1+ib\delta)\mathfrak{v}(x_0, y_0 + ib\lambda) - \mathfrak{v}(x_0, y_0)}{ib} \right] \\ &= \mathfrak{v}_y^{(\lambda)}(x_0, y_0) - i\mathfrak{u}_y^{(\lambda)}(x_0, y_0). \end{aligned}$$

Therefore, we have

$$D^\lambda f(\zeta_0) = \mathfrak{u}_x^{(\lambda)}(x_0, y_0) + i\mathfrak{v}_x^{(\lambda)}(x_0, y_0) = \mathfrak{v}_y^{(\lambda)}(x_0, y_0) - i\mathfrak{u}_y^{(\lambda)}(x_0, y_0).$$

□

Corollary 1. *Let* $f(z) = \mathfrak{u}(x,y)+i\mathfrak{v}(x,y)$ *be an* λ*-complex deformable differentiable at* ζ_0*. Then,* $\mathfrak{u}(x,y)$ *and* $\mathfrak{v}(x,y)$ *satisfy the* λ*-deformable Cauchy-Riemann equations as*

$$\mathfrak{u}_x^{(\lambda)} = \mathfrak{v}_y^{(\lambda)} \text{ and } \mathfrak{u}_y^{(\lambda)} = -\mathfrak{v}_x^{(\lambda)}. \tag{6}$$

The conversely of Corollary 1 is not always true. For example, consider the function

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

For $z = x + iy \neq 0$*, we have*

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \left(\frac{x - iy}{x + iy} \right)^2 .$$

This limit is equal to 1 when approaching the origin along the real axis, and -1 when approaching along the $y = x$ *line. Then the function* f *is not differentiable at* $z = 0$*. Therefore,* f *is not* λ*-complex deformable differentiable at* $z = 0$*. On the other hand, since*

$$\mathfrak{u}_x^{(\lambda)}(0,0) = \lambda = \mathfrak{v}_y^{(\lambda)}(0,0)$$

and

$$\mathfrak{u}_y^{(\lambda)}(0,0) = 0 = -\mathfrak{v}_x^{(\lambda)}(0,0)$$

the λ*-deformable Cauchy Riemann equations satisfy at* $z = 0$*.*

Now we give the notion of an λ-deformable analytic function using a complex deformable derivative.

Definition 4. *The function that is* λ*-complex deformable differentiable at every point of an open set* U *is called to be* λ*-deformable analytic in* U*.*

Definition 5. *A function that is analytic at every point in the complex plane is called a* λ*-deformable entire function.*

Definition 6. *A mapping is called conformal at the point* ζ_0 *if it preserves the angles between pairs of regular curves intersecting at* ζ_0*.*

Theorem 9. *Let* f *be* λ*-deformable analytic in* D*. If* $D^\lambda f(\zeta_0) \neq 0$ *at* $\zeta_0 \in D$*, then* f *is conformal at* ζ_0*.*

4. Complex Deformable Integral

In this section, we introduce λ-complex deformable integral and examine some of its basic properties.

Definition 7. *Let C be a smooth curve given by the equation $z(t) = x(t) + iy(t)$, $a \leq t \leq b$. If the function $f(z)$ is defined and continuous on C, then $f(z(t))$ is also continuous and we can set*

$$I_C^{\lambda} f = \frac{1}{\lambda} e^{\frac{-\delta}{\lambda} z} \int_a^b e^{\frac{\delta}{\lambda} z(t)} f(z(t)) z'(t) dt.$$

Definition 8. *Let $f : [a, b] \to \mathbb{C}$ be a continuous function. We define λ-complex deformable integral of f,*

$$I_a^{\lambda} f = \frac{1}{\lambda} e^{\frac{-\delta}{\lambda} z} \int_a^b e^{\frac{\delta}{\lambda} z} f(z) dz. \tag{7}$$

Proposition 2. *The operator $I_a^{\lambda} f$ possesses the following properties:*

(a) $I_a^{\lambda}(bf + cg) = bI_a^{\lambda} f + cI_a^{\lambda} g$,
(b) $I_a^{\lambda_1} I_a^{\lambda_2} = I_a^{\lambda_2} I_a^{\lambda_1}$, *where* $\lambda_i + \delta_i = 1, i = 1, 2$.

Definition 9. *Let f be continuous on D. If there exists a function F such that $D^{\lambda}(F)(z) = f(z)$ for every z in D, then F is called an anti-λ-complex deformable derivative of f.*

We now give another version of the fundamental theorem of calculus.

Theorem 10. *Let a function f be continuous on a domain D. Then $I_a^{\lambda} f$ is λ-complex deformable differentiable in D.*

Proof. If we set $F = I_a^{\lambda} f$ then we have

$$D^{\lambda}(I_a^{\lambda} f(z)) = D^{\lambda}(F(z)) = \lambda DF(z) + \delta F(z).$$

Moreover, a particular solution of the differential equation $\lambda DF + \delta F = f$ is obtained as

$$F(z) = \frac{1}{\lambda} e^{\frac{-\delta}{\lambda} z} \int_a^b e^{\frac{\delta}{\lambda} z} f(z) dz.$$

The proof is completed. □

Theorem 11. *Let a function f be continuous on D and F is a continuous anti-λ-complex deformable derivative of f in D. Then we have*

$$I_a^{\lambda} \left(D^{\lambda} f(z) \right) = I_a^{\lambda} \left(g(z) \right) = f(t) - e^{\frac{\delta}{\lambda}(a-z)} f(a).$$

Proof. Since

$$F(z) = \left(D^{\lambda} f(z) \right) = \lambda Df + \delta f$$

we get

$$\begin{aligned} I_a^{\lambda} F(z) &= \lambda I_a^{\lambda} Df(z) + \delta I_a^{\lambda} f(z) \\ &= e^{\frac{-\delta}{\lambda} z} \int_a^b e^{\frac{\delta}{\lambda} z} Df(z) dz + \delta I_a^{\lambda} f(z) \end{aligned}$$

$$= e^{\frac{-\delta}{\lambda}z}\left(\left[e^{\frac{\delta}{\lambda}z}Df(z)\right]_a^b - \frac{\delta}{\lambda}\int_a^b e^{\frac{\delta}{\lambda}z}f(z)dz\right) + \delta I_a^\lambda f(z)$$
$$= f(z) - e^{\frac{\delta}{\lambda}(a-z)}f(a).$$

The proof is completed. □

5. Applications to Differential Equations

The linear first-order λ-complex differential equation can be expressed in the form

$$D^\lambda w + p(z)w = q(z)$$

where $\omega = f(z)$ be a complex valued function and p(z) is continuous complex valued function. Using expression (3), we get

$$D\omega + \frac{\delta + p(z)}{\lambda}\omega = q(z).$$

Then commonly written as

$$\omega = \frac{1}{\mu(z)}\int \mu(z)q(z)dz + \frac{C}{\mu(z)},$$

with

$$\mu(z) = e^{\int \frac{\delta + p(z)}{\lambda}dz}$$

the integrating factor. Thus, we obtain the general solution of the linear first-order λ-complex deformable differential equation is given by

$$w = e^{\frac{-(\delta + \int P(z)dz)}{\lambda}}\int e^{\frac{(\delta + \int P(z)dz)}{\lambda}}q(z)dz + Ce^{\frac{-(\delta + \int P(z)dz)}{\lambda}} \tag{8}$$

where C is arbitrary complex constant.

Example 1. *Suppose $\omega = f(z)$ is λ-complex deformable differentiable function. Solve*

$$D^\lambda\omega + \omega = 0.$$

Using expression (3), we can write

$$D\omega + \left(\frac{\delta + 1}{\lambda}\right)\omega + \omega = 0.$$

Hence, by using an integrating factor

$$\mu(z) = e^{\int\left(\frac{\delta+1}{\lambda}\right)dz} = e^{\left(\frac{\delta+1}{\lambda}\right)z},$$

we have

$$\omega = Ce^{-\left(\frac{\delta+1}{\lambda}\right)z}$$

where C is an arbitrary complex constant.

Example 2. *Suppose $\omega = f(z)$ is λ-complex deformable differentiable function. Solve*

$$D^{\frac{1}{3}}\omega + \omega = e^{2z}.$$

Using expression (3), we can write

$$\frac{1}{3}D\omega + \frac{2}{3}\omega + \omega = e^{2z},$$

or equally

$$D\omega + 5\omega + \omega = 3e^{2z}.$$

Hence, by using an integrating factor

$$\mu(z) = e^{\int 5dz} = e^{5z},$$

we have

$$\begin{aligned} \omega &= 3e^{-5z}\int e^{7z}dz + Ce^{-5z} \\ &= \frac{3}{7}e^{2z} + Ce^{-5z} \end{aligned}$$

where C is an arbitrary complex constant.

Example 3. *Suppose $\omega = f(z)$ is λ-complex deformable differentiable function. Solve*

$$D^{\frac{1}{2}}\omega + \frac{1}{z}\omega = z^2\omega^2.$$

To solve the $\frac{1}{2}$-complex deformable differential equation, the expansion (3) is first used, and then both sides of the obtained equation are multiplied by ω^{-2}. Then we have

$$\omega^{-2}D\omega + \left(1 + \frac{2}{z}\right)\omega^{-1} = 2z^2.$$

By substituting $\omega^{-1} = \eta$ in the above equation, we get

$$D\eta - \left(1 + \frac{2}{z}\right)\eta = -2z^2.$$

Since the integrating factor of the last differential equation is

$$\mu(z) = e^{-z}z^{-2},$$

we find the general solutions

$$\eta = e^z z^2 \int e^{-z}z^{-2}(-2z^2)dz + Ce^z z^2 = z^2(2 + Ce^z)$$

where C is an arbitrary complex constant. Thus, its general solution may be expressed as

$$\omega = \frac{1}{z^2(2 + Ce^z)}.$$

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MATHEMATICAL MODEL FOR THE DYNAMICS OF ALCOHOL-MARIJUANA CO-ABUSE

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Abstract. A mathematical model for the dynamics of alcohol-marijuana co-abuse is presented in this work. In the past years legalization of recreational marijuana in several states in the United States has added a new layer to alcohol addiction. Much research has been done for alcohol addiction or drug abuse independently, but few include the incidence of marijuana use for alcohol users. A compartmental epidemiological model is used, and results such as the existence and boundedness of solutions, the basic reproduction number using the next-generation method, the disease-free equilibrium, and an analytical expression for the endemic equilibrium are included. Numerical simulations with parameters obtained from data in the United States are performed for different compartments of the population as well as the reproduction number for the alcohol and marijuana sub-models. The model can be adapted for different regions worldwide using appropriate data. This work contributes to understanding the dynamics of the co-abuse of addictive substances. Even though alcohol and marijuana are both legal, they can be of great harm to the brain of the individual when combined, having tremendous consequences for society as a whole. Creating awareness of a public health concern with facts based on scientific research is the ultimate goal of this work.

1. Introduction

Alcohol consumption is a widely accepted social practice between friends and family and sometimes in work environments. Despite its status as an intoxicating substance, classification as a central nervous system depressant [16], and its risk of addiction for one in ten who try alcohol, networking events, business meetings,

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and social gatherings normalize the use of this substance. Since alcohol is socially acceptable, many users disregard the consequences that come with its use [16]. Currently, alcohol use can be placed on a spectrum. Drinking habits can be classified as occasional, such as casual drinking at social events, as moderate, including binge drinking, or as heavy, which is the most dangerous and can lead to many health problems and even death.

The behavior becomes a public health crisis when someone cannot cut back on consumption despite efforts to control consumption habits. This leads to physical health issues, possible driving while intoxicated, and the less quantified suffering that abusers experience with out-of-control behavior when intoxicated [12]. In addition to drinking alcohol, cigarette smoking, marijuana, and other drug use while drinking alcohol is common, as studies show that using one often involves using the other in the same event/occurrence [30].

Studies have shown that peer pressure in adolescence plays a significant role in young adults starting to consume alcohol and/or marijuana [28]. Furthermore, college students are constantly participating in activities that involve the use of alcohol and/or marijuana [20]. Some studies have shown that certain conditions in individuals make them more vulnerable to becoming addicted to alcohol and/or marijuana [14].

Researchers from NIH:NIAAA present a comprehensive study on risk factors periodically, [19], where several studies have shown the severity of alcoholism in modern society. Another study [27] examines alcohol abuse using a mathematical model with recovery and relapse from epidemiology. Neurologists have studied the brain during addictions and concluded that when trauma happens at an early age, the brain structure of an individual may change predisposing someone to be more likely to be addicted than others [14]. Additionally, the use of alcohol and/or marijuana during pregnancy causes “impaired neuro-development” [17].

From well-known studies on rats in the 1950’s where pleasure in the brain was identified when certain areas were electronically stimulated and rats would seek that sensation despite negative consequences, scientists observed something that is now termed “hijacking” of the brain [23]. This is where the brain confuses the pleasurable results of the drug with survival such as eating for nutrients or procreating to perpetuate the species [23].

Similar studies have yet to be done for the use of marijuana and the nature of each differs in that alcohol is a depressant but marijuana has other properties, [11]. In the United States the use of medical and recreational marijuana is fully legal and decriminalized in several states [33]. However not enough research has been done on the effects of the use of marijuana, or the co-abuse of alcohol and marijuana. Many teenagers are engaging in the use of alcohol and marijuana at an early age, without knowing the side effects. Unfortunately for many of them, the recreational use of this combination ends in a disorder they cannot control, affecting their health, their family environment, and their future as productive members of society. In the majority of cases, treatment plans are not affordable for most families, and numbers as few as one out of ten of those who have a substance use disorder attend treatment with approximately 13% attending alcohol treatment, 50% drugs only, and 30% both [22].

There is plenty of literature using mathematical models to analyze the dynamics of alcohol addiction or marijuana addiction individually, see for example [9,13,25,27,29]. However, the literature surrounding the co-abuse of alcohol and marijuana, to the authors' knowledge, is sparse. As the use of marijuana becomes socially acceptable and legalized in many states, related data for the co-abuse is not totally available. Since multiple health organizations show several studies where the co-abuse of alcohol and marijuana is at the top of health concerns in the United States [1,5,6], the necessity of developing mathematical models to contribute to the analysis of alcohol-marijuana co-abuse is imperative.

A mathematical model for the dynamics of the co-abuse of alcohol and marijuana is presented in this work, by using a system of ordinary differential equations under certain assumptions for the whole population. In the first part of this work, the analysis of the model is carried out. The system was divided in two subsystems, one corresponding to the dynamics of alcohol, and the second to the dynamics of marijuana, the evaluation of the basic reproduction number was performed for each sub-system by using the next generation method [31]. The basic reproduction number for the entire system is evaluated in terms of the population parameters. Stability results for the **disease-free** equilibrium are included. Furthermore a section with the analytic solution for the endemic equilibrium is included. The last section of this work includes multiple simulations for different compartments of the population using parameters for the population and simulations for the reproduction number for the sub-systems as well as the entire system. Most of these parameters were gathered from health organizations in the United States, [1,4,6].

By using mathematical modeling, the ultimate goal of this work is to contribute to understanding the co-abuse of alcohol and marijuana as it is now a public concern of the 21st century and to create awareness in teenagers, young-adults, and adults of the consequences of co-abuse. Public health reports indicate that even though both alcohol and marijuana are legal, this does not mean they are good for consumption together. This model can be used for different geographic regions, by changing the parameter values in the simulations.

2. Model Formulation

A compartmental epidemiological model is used to describe the dynamics of the co-abuse of alcohol and marijuana in the population. The total population is divided in 13 compartments, $S(t)$ – Susceptible, $E_a(t)$ – latent alcohol consumers, $E_m(t)$ – latent marijuana consumers, $E_{am}(t)$ – latent alcohol-marijuana consumers, $U_a(t)$ – alcohol users, $U_m(t)$ – marijuana users, $U_{am}(t)$ – alcohol-marijuana users, $T_a(t)$ – alcohol users in treatment, $T_m(t)$ – marijuana users in treatment, T_{am}

– alcohol-marijuana users in treatment, $Q_a(t)$ – alcohol quitters (recovery from alcoholism), $Q_m(t)$ – marijuana quitters (recovery from marijuana abuse), $Q_{am}(t)$ – alcohol-marijuana quitters (recovery from alcohol-marijuana co-abuse). For ease of exposition, we will simply write *compartments*, rather than list each compartment individually. With the above assumptions in place, we have

$$N(t) = S(t) + E_a(t) + E_m(t) + E_{am}(t) + U_a(t) + U_m(t) + U_{am}(t) \\ + T_a(t) + T_m(t) + T_{am}(t) + Q_a(t) + Q_m(t) + Q_{am}(t).$$

Our model draws inspiration from a co-abuse model for alcohol and methamphetamine presented in [26]. We build on this model by introducing a latent compartment E consisting of users who are not yet addicted. It is assumed that susceptible individuals become alcohol/marijuana users after an effective contact with alcohol/marijuana users. In this model, latent classes represent individuals who use alcohol/marijuana moderately, user classes represent individuals who use alcohol/marijuana on a regular basis (these are the infected individuals in general epidemiological terminology, meaning they are alcoholic individuals or marijuana addicted individuals or both). In Table 1, the symbol $*$ indicates that the range for those parameters were estimated for numerical simulation purposes. Those values are still a very good approximation following the literature. Data collection for marijuana use is still in process due to the fact that legalization of recreational marijuana is pretty recent in many states in the United States. The main public health organizations are making a great effort to collect data as mentioned in [18,32].

We assume a homogeneous mixing of populations. A complete analysis for the theory that human social networks may exhibit a "three degrees of influence" property was included in [15], which suggests that individuals acquire habits of alcohol use, marijuana use, or both, based on interactions with different populations. In this model we also assume that individuals who consume alcohol at any level, including during treatment (rehabilitation), contribute to the new alcohol user population. Individuals from T_{am} relapsing during treatment from abusing multiple drugs [4,21] also have the potential to influence susceptible individuals to drink alcohol. Therefore, individuals adopt the habit of alcohol consumption at the rate λ_1 given by the following expression:

$$\lambda_1 = \beta_1\left(\frac{E_a + \theta_1 U_a + \theta_2 T_a + E_{am} + \theta_3 U_{am} + \theta_4 T_{am}}{N}\right),$$

where β_1 denotes the effective contact rate (the contact with an alcoholic drinker that will result in one taking alcohol). Similarly, individuals acquire the habit of smoking marijuana at the rate λ_2 given by

$$\lambda_2 = \beta_2\left(\frac{E_m + \epsilon_1 U_m + \epsilon_2 T_m + E_{am} + \epsilon_3 U_{am} + \epsilon_4 T_{am}}{N}\right),$$

where β_2 denotes the effective contact rate (the contact with a marijuana user that will result in one smoking marijuana).

It is assumed that individuals under alcohol/marijuana treatment tend to have lower recruitment rates relative to alcoholics without treatment or marijuana addicts. Then the following relations hold: $\theta_1 > 1$, $\theta_3 > 1$, $\theta_2 < 1$, $\theta_4 < 1$, $\epsilon_1 > 1$, $\epsilon_3 > 1$, $\epsilon_2 < 1$, $\epsilon_4 < 1$.

Epidemiological models for co-abuse or co-infections are of tremendous interest in recent research [8]. For example in [7] a complete study for the co-infection between HIV and HCV was developed.

The following system of ordinary differential equations captures the dynamics of alcohol-marijuana co-abuse:

$$\frac{dS}{dt} = \Lambda - (\lambda_1 + \lambda_2 + \mu)S \tag{1}$$

$$\frac{dE_a}{dt} = \lambda_1 S + \rho_1 E_{am} - (\eta_a \lambda_2 + \sigma_a + \mu)E_a \tag{2}$$

$$\frac{dE_{am}}{dt} = \eta_a \lambda_2 E_a + \eta_m \lambda_1 E_m - (\rho_1 + \rho_2 + \sigma_{am} + \mu)E_{am} \tag{3}$$

$$\frac{dE_m}{dt} = \lambda_2 S + \rho_2 E_{am} - (\sigma_m + \eta_m \lambda_1 + \mu)E_m \tag{4}$$

$$\frac{dU_a}{dt} = \sigma_a E_a + \rho_3 U_{am} + \psi_a T_a - (\eta_a \lambda_2 + \alpha_a + \xi_a + \delta_a + \mu)U_a \tag{5}$$

$$\begin{aligned}\frac{dU_{am}}{dt} = {} & \sigma_{am} E_{am} + \eta_a \lambda_2 U_a + \eta_m \lambda_1 U_m + \psi_{am} T_{am} \\ & - (\rho_3 + \rho_4 + \alpha_{am} + \xi_{am} + \delta_{am} + \mu)U_{am}\end{aligned} \tag{6}$$

$$\frac{dU_m}{dt} = \sigma_m E_m + \rho_4 U_{am} + \psi_m T_m - (\eta_m \lambda_1 + \alpha_m + \xi_m + \mu)U_m \tag{7}$$

$$\frac{dT_a}{dt} = \alpha_a U_a - (\psi_a + \gamma_a + \mu)T_a \tag{8}$$

$$\frac{dT_{am}}{dt} = \alpha_{am} U_{am} - (\psi_{am} + \gamma_{am} + \mu)T_{am} \tag{9}$$

$$\frac{dT_m}{dt} = \alpha_m U_m - (\psi_m + \gamma_m + \mu)T_m \tag{10}$$

$$\frac{dQ_a}{dt} = \xi_a U_a + \gamma_a T_a - \mu Q_a \tag{11}$$

$$\frac{dQ_{am}}{dt} = \xi_m U_{am} + \gamma_{am} T_{am} - \mu Q_{am} \tag{12}$$

$$\frac{dQ_m}{dt} = \xi_m U_m + \gamma_m T_m - \mu Q_m \tag{13}$$

Figure 1 represents the transition between compartments for the alcohol-marijuana co-abuse model.

In the next section a complete mathematical analysis is developed, positiveness and boundedness of solutions are always fundamental properties for a consistent dynamical system in epidemiology. The basic reproduction number is included. The

TABLE 1. Parameters values[1]

Symbol	Description	Value
Λ	Recruitment rate for susceptible	.0546 [1]
μ	Natural mortality rate	0.001 [2]
β_1	Alcohol transmission rate	$.24 - .27$ [3,10]
β_2	Marijuana transmission rate	0.169 [1]
σ_a	Alcoholism effective rate	0.056 [1]
σ_m	Marijuana users effective rate	0.011 [1]
σ_{am}	Co-abusers effective rate	[0.01-0.06] [1]
α_a	Alcoholism treatment rate	0.131 [1]
α_m	Marijuana users treatment rate	0.09 [1]
α_{am}	Co-abusers treatment rate	0.32 [1]
γ_a	Alcoholism recovery rate after treatment	0.87 [24]
γ_m	Marijuana users recovery rate after treatment	0.45 [1]
γ_{am}	Co-abusers recovery rate after treatment	[0.1-0.4] [1]
θ_1, θ_3	Weight contributions to λ_1 from U_a, U_{am}	[1.01-1.05] [1]
θ_2, θ_4	Weight contributions to λ_1 from T_a, T_{am}	[0.01-0.03] [1]
ϵ_1, ϵ_3	Weight contributions to λ_2 from U_m, U_{am}	[1.01-1.08] [1]
ϵ_2, ϵ_4	Weight contributions to λ_2 from T_m, T_{am}	[0.4-0.7] [1]
η_a	Rate at which alcohol users become marijuana users	[0.5-0.9] [1]
η_m	Rate at which marijuana users become alcohol users	[0.5-0.9] [1]
ψ_a	Relapsing rate from alcoholism	0.13 [24]
ψ_m	Relapsing rate from marijuana use	$[0.4 - 0.6]^*$
ψ_{am}	Relapsing rate from Co-abusers	$[0.4 - 0.6]^*$
ξ_a	Quitting rate from alcohol abusers without treatment	0.36 [1]
ξ_m	Quitting rate from marijuana abusers without treatment	$[0.1 - 0.4]^*$
ξ_{am}	Quitting rate from Co-abusers without treatment	$[0.2 - 0.6]^*$
δ_a	Alcohol-induced mortality rate	.000392 [3]
δ_{am}	Co-abusers mortality rate	$[0.0004 - 0.0007]^*$
ρ_1	Rate at which individuals from E_{am}-class back to E_a-class	$[0.4 - 0.7]^*$
ρ_2	Rate at which individuals from E_{am}-class back to E_m-class	$[0.4 - 0.7]^*$
ρ_3	Rate at which Co-abusers back to U_a-class	$[0.4 - 0.8]^*$
ρ_4	Rate at which Co-abusers back to U_m-class	$[0.2 - 0.7]^*$

[1]The scenarios used to choose most of the parameters were obtained from statistics found in [1,5,6] for the state of Virginia, United States in 2017. For example, σ_a, σ_m, and α_a are taken from the Behavioral Health Barometer for Virginia which can be found in [1]. Other parameters, in particular those for co-abuse such as α_{am}, were estimated using rates for general drug and alcohol co-abuse or by using compartmental rates as bounds. For example, bounds for σ_{am} were assumed based on σ_a and σ_m. The validity of these bounds, such as σ_{am} can be checked using Crosstab, also from [1]. Individual state level data is not available in Crosstab for general public use. In the case of σ_{am}, this Crosstab tells us that almost 3% of Virginians and Marylanders co-abused alcohol and marijuana in 2017.

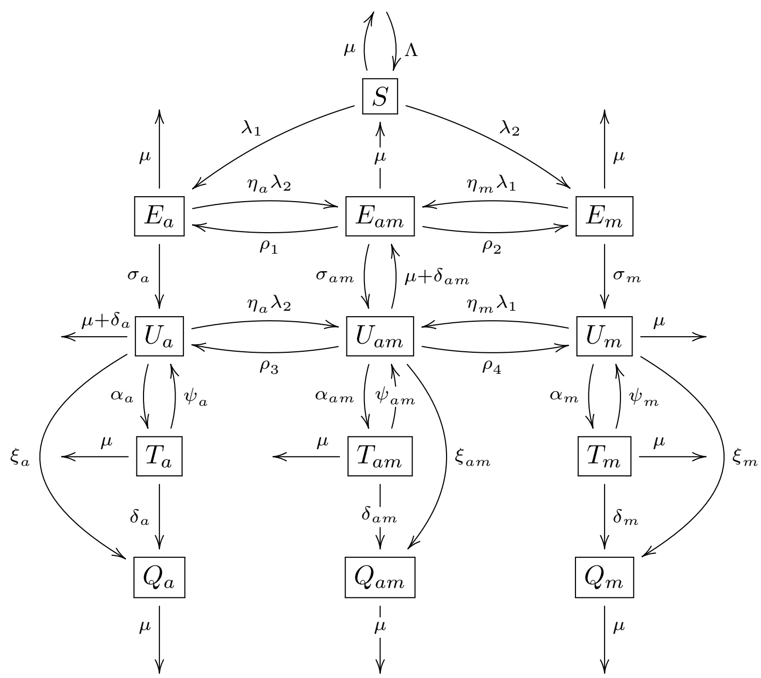

Figure 1. Visual representation of alcohol-marijuana co-abuse model.

free disease equilibrium and an analytical expression for the endemic equilibrium point are included. Some stability results are proven as well.

3. Mathematical Analysis for the Alcohol-Marijuana Co-Abuse Model

Positiveness and long-term behavior for the solutions of System (1)–(13) are established in this section. Assume that the variables and the parameters are all non-negative for all times $t \geq 0$.

Theorem 1. *If each compartment is non-negative at $t = 0$, then each compartment is non-negative for time $t > 0$. Moreover,*

$$\lim_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}.$$

Proof. Assume that T is the maximum time for the epidemic. That is,

$$T = \sup\{S > 0, E_a \geq 0, E_{am} \geq 0, \ldots, Q_m \geq 0\} \in [0, t].$$

Therefore for $T > 0$, from System (1)–(13), equation (1) is equivalent to

$$\frac{dS}{dt} + (\mu + \lambda_1 + \lambda_2)S = \Lambda,$$

from which it holds

$$S(T) \geq S(0) \exp\left\{-\mu T + \int_0^t (\lambda_1(s) + \lambda_2(s))ds\right\}.$$

Hence, $S(T) \geq 0$ for all $T > 0$.

From System (1)–(13) equation (2),

$$\begin{aligned}\frac{dE_a}{dt} &= \lambda_1 S + \rho_1 E_{am} - (\eta_a \lambda_2 + \sigma_a + \mu)E_a \\ &\geq -(\eta_a \lambda_2 + \sigma_a + \mu)E_a.\end{aligned}$$

Then

$$E_a(T) \geq E_{a0} \exp\left\{-\left((\sigma_a + \mu)t + \int_0^t \eta_a \lambda_2(s)ds\right)\right\}.$$

Hence, $E_a(T) \geq 0$ for all $T > 0$. The positiveness of the remaining compartments can be shown in a similar way.

The evolution change in the population is given by

$$\frac{dN}{dt} = \Lambda - \mu N - \delta_a - \delta_{am} U_{am}.$$

Then

$$\frac{dN}{dt} \leq \Lambda - \mu N,$$

from which it holds

$$\frac{dN}{dt} + \mu N \leq \Lambda.$$

Then

$$N(t) \leq \frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu}\right)\exp(-\mu t).$$

Since $(N_0 - \Lambda/\mu)$ is a constant and $\mu > 0$,

$$\frac{\Lambda}{\mu} + \left(N_0 - \frac{\Lambda}{\mu}\right)\exp(-\mu t) \to \frac{\Lambda}{\mu} \text{ as } t \to \infty.$$

So $\lim_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}$ as desired. □

The feasible region D, for System (1)–(13) is therefore

$$D = \left\{(S, E_a, E_{am}, E_m, \ldots, Q_a, Q_{am}, Q_m) \in \mathbb{R}_+^{13} \mid N \leq \frac{\Lambda}{\mu}\right\}.$$

3.1. **Basic reproduction number.** The basic reproduction number is the number of secondary infections. In the context of this model, if an individual is an alcoholic or a marijuana user or both, after an effective contact with susceptible individuals, the basic reproduction number corresponds to how many susceptible individuals become alcoholics or marijuana codependent or alcohol-marijuana co-abusers. The next generation matrix method is used to find the basic reproduction number for the co-abuse model System (1)–(13) [31]. First the basic reproduction number is found for the alcohol model, denoted $\mathscr{R}_{a0}$. Second the basic reproduction number was found for the marijuana model, denoted $\mathscr{R}_{m0}$. The basic reproduction number $\mathcal{R}_{am}$ is the larger of $\mathcal{R}_{a0}$ and $\mathcal{R}_{m0}$. So one only needs to calculate the reproduction number for these individual models to determine the reproduction number of the co-abuse model, see [7] for a detailed calculation of the basic reproduction number for a co-abuse model.

In the next sub-sections, System (1)–(13) is sub-divided into two models, one corresponding to the dynamics of alcohol use, and the other to the marijuana use.

3.2. **Alcohol abuse model.** Taking S together with the first column of Figure 1, one can see that the alcohol abuse model is given by

$$\frac{dS}{dt} = \Lambda - (\mu + \tilde{\lambda}_1)S \tag{14}$$
$$\frac{dE_a}{dt} = \tilde{\lambda}_1 S - (\sigma_a + \mu)E_a \tag{15}$$
$$\frac{dU_a}{dt} = \sigma_a E_a + \psi_a T_a - (\alpha_a + \xi_a + \delta_a + \mu)U_a \tag{16}$$
$$\frac{dT_a}{dt} = \alpha_a U_a - (\psi_a + \gamma_a + \mu)T_a \tag{17}$$
$$\frac{dQ_a}{dt} = \xi_a U_a + \gamma_a T_a - \mu Q_a \tag{18}$$

where

$$\tilde{\lambda}_1 = \beta_1\left(\frac{E_a + \theta_1 U_a + \theta_2 T_a}{N}\right).$$

The corresponding matrices to apply to the next generation method to are

$$\mathscr{F} = \begin{bmatrix} \tilde{\lambda}_1 S \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathscr{V} = \begin{bmatrix} (\sigma_a + \mu)E_a \\ -\sigma_a E_a - \psi_a Ta + (\alpha_a + \xi_a + \delta_a + \mu)U_a \\ -\alpha_a U_a + (\psi_a + \gamma_a + \mu)T_a \\ -\Lambda + (\mu + \tilde{\lambda}_1)S \\ -\xi_a U_a - \gamma_a T_a + \mu Q_a \end{bmatrix}.$$

The alcohol model has a disease-free equilibrium $X_a^0 = (\Lambda/\mu, 0, 0, 0, 0)$. The matrices F and V at the disease-free equilibrium, following the next generation matrix method in [31], are given by:

$$F = \begin{bmatrix} \beta_1 & \beta_1\theta_1 & \beta_1\theta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \sigma_a + \mu & 0 & 0 \\ -\sigma_a & \alpha_a + \xi_a + \delta_a + \mu & -\psi_a \\ 0 & -\alpha_a & \psi_a + \gamma_a + \mu \end{bmatrix},$$

or

$$V = \begin{bmatrix} \sigma_a + \mu & 0 & 0 \\ -\sigma_a & b_1 & -\psi_a \\ 0 & -\alpha_a & b_2 \end{bmatrix},$$

where $b_1 = \alpha_a + \xi_a + \delta_a + \mu$ and $b_2 = \psi_a + \gamma_a + \mu$.

The basic reproduction number $\mathscr{R}_{a0}$ corresponds to the spectral value of the matrix FV^{-1}, so

$$\mathscr{R}_{a0} = \frac{\beta_1}{\sigma_a + \mu} + \frac{\beta_1\theta_1\sigma_a}{b_1(\sigma_a + \mu)(1 - \Phi_a)} + \frac{\beta_1\theta_2\sigma_a\alpha_a}{b_1 b_2(\sigma_a + \mu)(1 - \Phi_a)},$$

where $\Phi_a = \alpha_a\psi_a/b_1 b_2$.

3.3. **Marijuana abuse model.** Taking S together with the right column of Figure 1, one can see that the Marijuana abuse model is given by

$$\frac{dS}{dt} = \Lambda - (\mu + \tilde{\lambda}_2)S \tag{19}$$

$$\frac{dE_m}{dt} = \tilde{\lambda}_2 S - (\sigma_m + \mu)E_m \tag{20}$$

$$\frac{dU_m}{dt} = \sigma_m E_m + \psi_m T_m - (\alpha_m + \delta_m + \mu)U_m \tag{21}$$

$$\frac{dT_m}{dt} = \alpha_m U_m - (\psi_m + \gamma_m + \mu)T_m \tag{22}$$

$$\frac{dQ_m}{dt} = \xi_m U_m + \gamma_m T_m - \mu Q_m \tag{23}$$

where

$$\tilde{\lambda}_2 = \beta_2\left(\frac{E_m + \epsilon_1 U_m + \epsilon_2 T_m}{N}\right).$$

The corresponding matrices to apply the next generation method to are

$$\mathscr{F} = \begin{bmatrix} \tilde{\lambda}_2 S \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathscr{V} = \begin{bmatrix} (\sigma_m + \mu)E_m \\ -\sigma_m E_m - \psi_m Tw + (\alpha_m + \xi_m + \delta_m + \mu)U_m \\ -\alpha_m U_m + (\psi_m + \gamma_m + \mu)T_m \\ -\Lambda + (\mu + \tilde{\lambda}_2)S \\ -\xi_a U_a - \gamma_a T_a + \mu Q_a \end{bmatrix}.$$

Similarly to the alcohol model, the marijuana model has a disease-free equilibrium $X_m^0 = (\Lambda/\mu, 0, 0, 0, 0)$, and the matrices F and V at the marijuana-free equilibrium, following the next generation matrix method, are given by

$$F = \begin{bmatrix} \beta_2 & \beta_2\epsilon_1 & \beta_2\epsilon_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \sigma_m + \mu & 0 & 0 \\ -\sigma_m & \alpha_m + \xi_m + \mu & -\psi_m \\ 0 & -\alpha_m & \psi_m + \gamma_m + \mu \end{bmatrix},$$

or

$$V = \begin{bmatrix} \sigma_a + \mu & 0 & 0 \\ -\sigma_m & c_1 & -\psi_m \\ 0 & -\alpha_m & c_2 \end{bmatrix},$$

where $c_1 = \alpha_m + \xi_m + \mu$ and $c_2 = \psi_m + \gamma_m + \mu$.

The basic reproduction number $\mathscr{R}_{m0}$ is

$$\mathscr{R}_{m0} = \frac{\beta_2}{\sigma_m + \mu} + \frac{\beta_2\epsilon_1\sigma_m}{c_1(\sigma_m + \mu)(1 - \Phi_m)} + \frac{\beta_2\epsilon_2\sigma_m\alpha_m}{c_1c_2(\sigma_m + \mu)(1 - \Phi_m)},$$

where $\Phi_m = \alpha_m\psi_m/c_1c_2$.

Then the basic reproduction number for System (1)–(13) is given by

$$\mathscr{R}_{am} = \max\{\mathscr{R}_{a0}, \mathscr{R}_{m0}\}.$$

Graphs for $\mathscr{R}_{a0}$, $\mathscr{R}_{m0}$, and $\mathscr{R}_{am}$ were obtained using Matlab, the graphs gave us an insight for the behaviour of the basic reproduction number when parameters are varied. Most of the parameters used were found from publicly available data and recent literature.

In Figure 2, notice that $\mathscr{R}_{a0} > 1$ for values of $\sigma_a < 0.3$. From the data $\sigma_a = 0.056 < 0.3$, meaning that alcoholism is not under control, a similar situation is observed for the marijuana model, since $\mathscr{R}_{m0} > 1$ for values of $\sigma_m < 0.2$, and from the data $\sigma_m = 0.0011 < 0.2$. So marijuana use is not under control either. For this range of values, notice that $\mathscr{R}_{am} = \mathscr{R}_{a0}$. Then the alcohol-marijuana co-abuse is an epidemic and can become a pandemic if severe actions are not implemented.

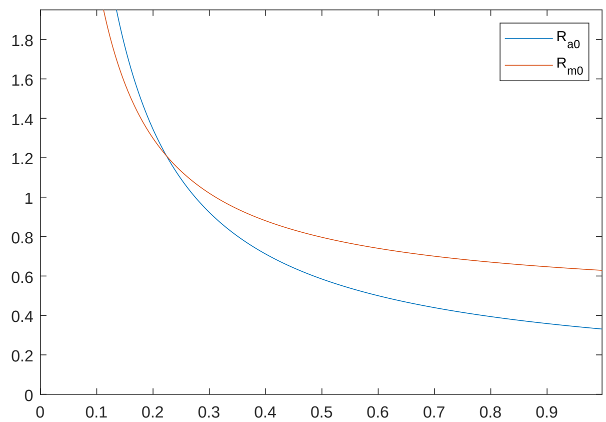

FIGURE 2. $\mathscr{R}_{a0}$ and $\mathscr{R}_{m0}$ with $.1 \leq \sigma_a, \sigma_m \leq .9$, $\theta_1 = 1.01, \epsilon_1 = 1.05$ and $\theta_2 = .01, \epsilon_2 = .7$. On this interval, $\mathscr{R}_{am} = \mathscr{R}_{a0}$.

Theorem 2. *The disease-free equilibrium X_a^0 for the Alcohol abuse model is stable.*

Proof. The Jacobian for the Alcohol abuse model at X_a^0, is given by

$$J_a(X_a^0) = \begin{bmatrix} -(\mu+\tilde{\lambda}_1) & 0 & 0 & 0 & 0 \\ \tilde{\lambda}_1 & -(\sigma_a+\mu) & 0 & 0 & 0 \\ 0 & \sigma_a & -(\alpha_a+\xi_a+\delta_a+\mu) & \psi_a & 0 \\ 0 & 0 & \alpha_a & -(\psi_a+\gamma_a+\mu) & 0 \\ 0 & 0 & \xi_a & \gamma_a & -\mu \end{bmatrix},$$

The eigenvalues of $J_a(X_a^0)$ are given by

$$-\mu, \quad -(\mu+\tilde{\lambda}_1), \quad -(\mu+\sigma_a),$$

$$-\tfrac{1}{2}\alpha_a-\tfrac{1}{2}\delta_a-\tfrac{1}{2}\gamma_a-\mu-\tfrac{1}{2}\psi_a-\tfrac{1}{2}\xi_a-\tfrac{1}{2}\big(\alpha_a^2+2\alpha_a\delta_a-2\alpha_a\gamma_a+2\alpha_a\psi_a+2\alpha_a\xi_a+\delta_a^2-2{*}\delta_a\gamma_a$$
$$-2\delta_a\psi_a+2\delta_a\xi_a+\gamma_a^2+2\gamma_a\psi_a-2\gamma_a\xi_a+\psi_a^2-2\psi_a\xi_a+\xi_a^2\big)^{1/2},$$

and

$$\tfrac{1}{2}\big(\alpha_a^2+2\alpha_a\delta_a-2\alpha_a\gamma_a+2\alpha_a\psi_a+2\alpha_a\xi_a+\delta_a^2-2\delta_a\gamma_a-2\delta_a\psi_a+2\delta_a\xi_a+\gamma_a^2+2\gamma_a\psi_a - 2\gamma_a\xi_a + \psi_a^2 - 2\psi_a\xi_a + \xi_a^2\big)^{1/2} - \tfrac{1}{2}\delta_a - \tfrac{1}{2}\gamma_a - \mu - \tfrac{1}{2}\psi_a - \tfrac{1}{2}\xi_a - \tfrac{1}{2}\alpha_a.$$

notice that the real parts for the five eigenvalues are negative, therefore when t approaches infinity, the solutions approach X_a^0. □

Similarly, it is possible to show results for the marijuana abuse model, and for the alcohol-marijuana co-abuse model.

4. Characterization of the Endemic Equilibrium

Analytic expressions for the endemic equilibrium are presented in this section. Setting equations from System (1)–(13) to zero and performing several calculations, the endemic equilibrium is obtained depending on the force of infectious λ_1^* and λ_2^*, and the parameters for the model. System (1)–(13) was sub-divided to accomplish this task, the first set of equations correspond to the variables S, E_a, E_{am}, E_m, as follows:

$$S^* = \frac{\Lambda}{\mu + \lambda_1^* + \lambda_2^*} \tag{24}$$

$$E_{am}^* = \frac{\lambda_1^*\lambda_2^*[b_5\eta_a + b_4\eta_m]}{b_3b_4b_5[1-\Phi_3]}S^* \tag{25}$$

$$E_a^* = \frac{\rho_1}{b_4}E_{am}^* + \frac{\lambda_1}{b_4}S^* \tag{26}$$

$$E_m^* = \frac{\rho_2}{b_5}E_{am}^* + \frac{\lambda_2}{b_5}S^* \tag{27}$$

where $b_3 = \rho_1 + \rho_2 + \sigma_{am} + \mu$, $b_4 = \eta_a\lambda_2^* + \sigma_a + \mu$, $b_5 = \eta_m\lambda_1^* + \sigma_m + \mu$ and $\Phi_3 = \frac{\eta_a\lambda_2^*\rho_1}{b_3b_4} + \frac{\eta_w\lambda_1^*\rho_2}{b_3b_5}$.

The second set of equations correspond to the variables U_a, U_m, U_{am}, as follows:

$$U_{am}^* = \frac{\sigma_a\eta_a\lambda_2^*}{\Gamma_{am}\Phi_4}E_a^* + \frac{\sigma_m\eta_m\lambda_1^*}{\Gamma_{am}\Phi_5}E_m^* + \frac{\sigma_{am}}{\Gamma_{am}}E_{am}^* \tag{28}$$

$$U_a^* = \frac{\rho_3}{\Phi_4}U_{am}^* + \frac{\sigma_a}{\Phi_4}E_a^* \tag{29}$$

$$U_m^* = \frac{\rho_4}{\Phi_5}U_{am}^* + \frac{\sigma_m}{\Phi_5}E_m^*, \tag{30}$$

where $c_3 = \psi_{am}+\gamma_{am}+\mu$, $c_4 = \rho_3+\rho_4+\alpha_{am}+\xi_m+\delta_{am}+\mu$, $\Phi_4 = \eta_a\lambda_2^*+b_1-\frac{\alpha_a\psi_a}{b_2}$, $\Phi_5 = \eta_m\lambda_1^* + c_1 - \frac{\alpha_m\psi_m}{c_2}$, and $\Gamma_{am} = c_4 - \frac{\alpha_{am}\psi_{am}}{c_3} - \frac{\rho_3\eta_a\lambda_2^*}{\Phi_4} - \frac{\rho_4\eta_m\lambda_1^*}{\Phi_5}$.

The last set of equations correspond to the variables T_{am}^*, T_a^*, T_m^*, Q_{am}^*, Q_a^*, Q_m^*, as follows:

$$T_{am}^* = \frac{\alpha_{am}}{c_3}U_{am}^* \tag{31}$$

$$T_a^* = \frac{\alpha_a}{b_2}U_a^* \tag{32}$$

$$T_m^* = \frac{\alpha_m}{c_2} U_m^*, \tag{33}$$

and

$$Q_{am}^* = \left(\frac{\xi_{am}}{\mu} + \frac{\gamma_{am}\alpha_{am}}{\mu c_3} \right) U_{am}^* \tag{34}$$

$$Q_a^* = \left(\frac{\xi_a}{\mu} + \frac{\gamma_a \alpha_a}{\mu b_2} \right) U_a^* \tag{35}$$

$$Q_m^* = \left(\frac{\xi_m}{\mu} + \frac{\gamma_m \alpha_m}{\mu c_2} \right) U_m^* \tag{36}$$

Reaching an analytic expression for the endemic equilibrium is considered of great value in epidemiological models because the disease free-equilibrium and the endemic equilibrium are the two stages that the population approaches, in the long term. But, in general, for the majority of contagious diseases, the disease remains in the populations, and therefore approaches to the endemic equilibrium.

5. Numerical Simulations

In this section, numerical simulations are presented.

Figure 3 shows the behavior for compartments S, E, U, T, Q, for the sub-model of alcohol abuse. Observe that with a small initial amount of latent population (E), after interactions with susceptible individuals, the number of alcohol users (U) increased, peaking in the fifth year. With treatment programs in effect, the number of alcoholics decreases after the fifth year. Notice that population Q increases, indicating treatment is effective.

The most relevant compartment to observe in this work is the compartment U_a of alcohol-dependent individuals. In Figure 3, it is noticeable that if the alcoholism effective rate increases, the number of alcoholic individuals will increase, having a peak around the fifth year. As treatment plans and programs are implemented, the number of alcohol-dependent users starts decreasing, even though the treatment rate is very low.

For the marijuana abuse model, Figure 4 shows that the population of marijuana users U_m steadily increases during the first five years after 2017. Between five to thirty years after 2017, the user population appears to transition from a slow increase to a slow decrease. After the thirty year mark, the user population decreases more quickly, perhaps due to treatment for marijuana-abuse not being as ubiquitous as treatment for alcohol-abuse.

When simulating System (1)–(13), in Figure 5 notice that the compartment classes E_{am}, U_{am}, T_{am}, and Q_{am} for co-abuse reach their maximum close to the first year, meaning that the use of both alcohol and marijuana lead faster to addiction that the independent use of the substances. Again, if entering treatment happen as soon as the individual detect an addiction, then the probability of recovery is higher than without treatment.

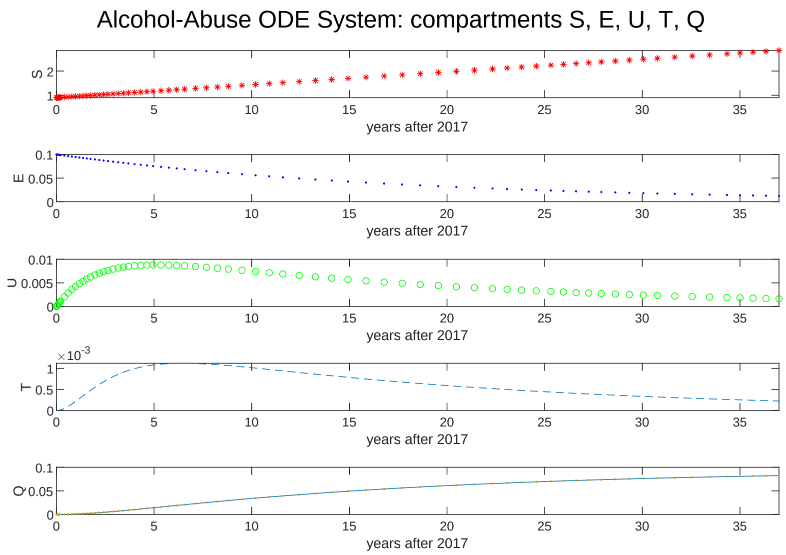

FIGURE 3. Simulation of alcohol-abuse system with $\theta_1 = 1.05$ and $\theta_2 = 0.7$.

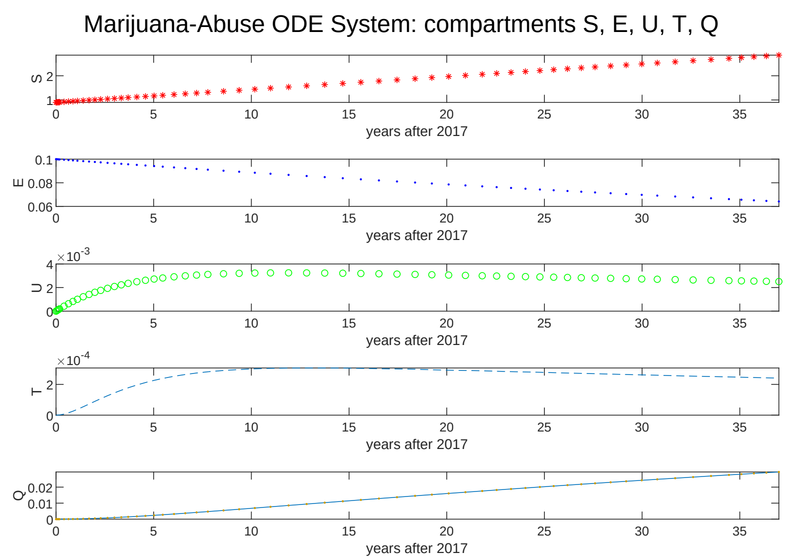

FIGURE 4. Simulation of marijuana-abuse system with $\epsilon_1 = 1.05$ and $\epsilon_2 = 0.7$.

For the alcohol abuse model, in Figure 6, alcohol users U_a increases significantly during the first five years after the year 2017. Notice that the dominant graph of U_a corresponds to $\sigma_a = 0.076$, which is the maximum value for σ_a.

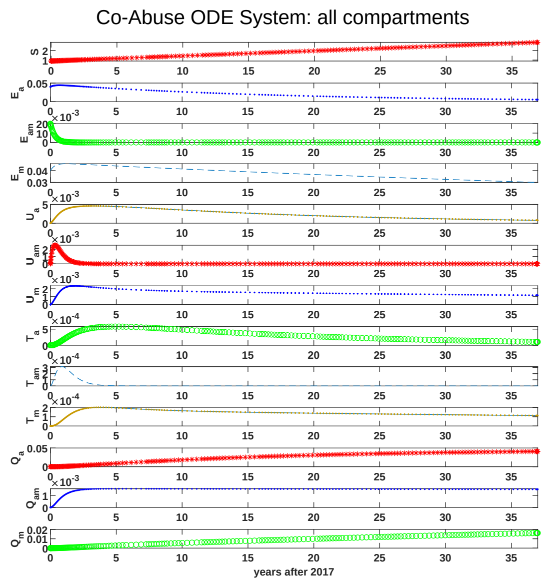

FIGURE 5. Simulation of co-abuse system with $\epsilon_1 = 1.05$ and $\epsilon_2 = 0.7$.

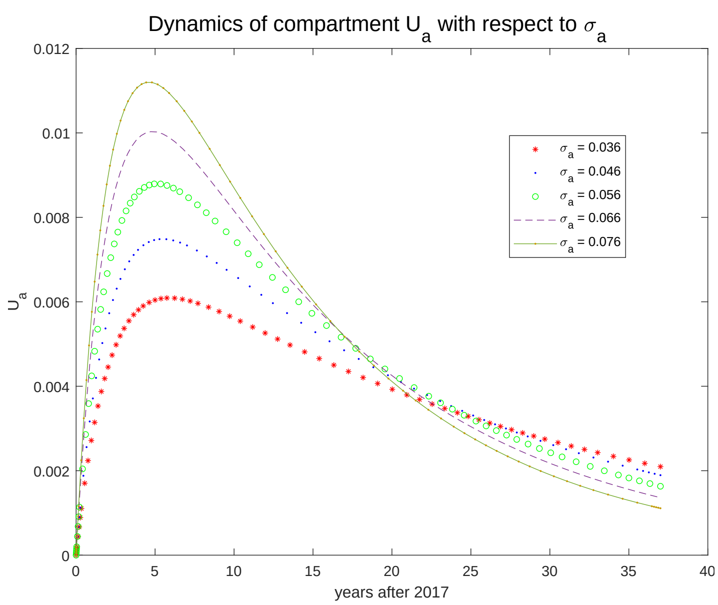

FIGURE 6. U_a with $.25 \leq \sigma_a \leq .9$, $\theta_1 = 0.01$ and $\theta_2 = 1.01$.

For the marijuana abuse model, in Figure 7, data corresponding to marijuana treatment is not available yet because legalization is very recent in many states.

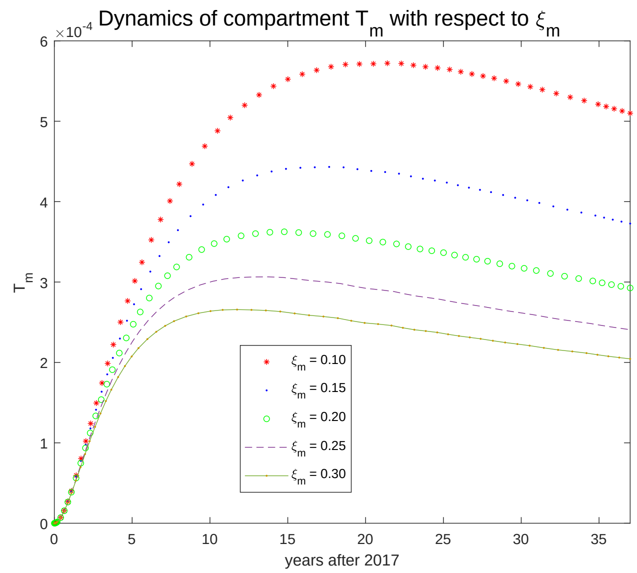

FIGURE 7. T_m with $.1 \leq \xi_m \leq .3$, $\epsilon_1 = 1.01$ and $\epsilon_2 = 1.01$.

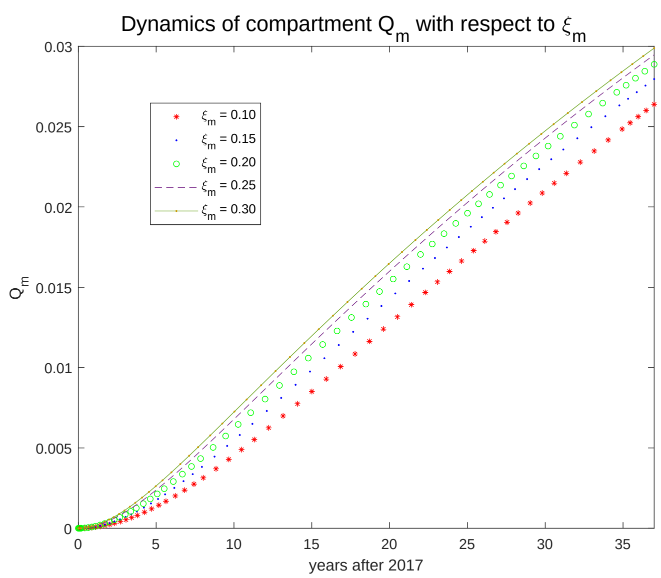

FIGURE 8. Q_m with $.1 \leq \xi_m \leq .3$, $\theta_1 = 0.01$ and $\theta_2 = 1.01$.

The parameters used for this simulations were estimated based on parameters for treatment for other types of drugs.

For the marijuana abuse model, in Figure 8, the parameter in consideration ξ corresponds to the quitting rate of marijuana users without treatment. The reason Q_m increases slowly is because of a lack of treatment programs for marijuana addicts.

6. Conclusions

Even though alcohol-marijuana co-abuse will always be present in modern society, the question of how to prevent co-abuse from becoming a pandemic can be asked. A mathematical model that describes the dynamics of the alcohol-marijuana co-abuse allowed us to identify the most relevant parameters to help control this epidemic. Alcohol and marijuana are two addictive substances that when combined can cause severe damage to young generations. This co-abuse is a social problem that is growing out of control, and if public health entities do not implement prevention programs, susceptible individuals are at high risk of becoming addicted to both substances. Analytical evaluations and numerical simulations show that for some parameters the alcohol-marijuana co-abuse can be controlled under certain constraints. The basic reproduction number for the independent models of alcohol and marijuana, and for the co-abuse model, in terms of the parameters, is a very standard way of defining public policies with the purpose of avoiding pandemics. This work can be implemented for any region by changing the parameters for the model using data values that correspond to each region. The model is well defined since positiveness and boundedness of solutions were shown. Stability for the disease-free equilibrium was attained by evaluation of eigenvalues for the sub-matrix for the newly infectious (latent and alcoholic individuals). Additionally, we were able to find an analytic expression for the endemic equilibrium which will help to identify where the individuals from different compartments will approach in the long term. Simulations for the most relevant parameters were included, showing that it is possible to control the co-abuse by implementing different approaches. These approaches include voluntarily quitting the use of the substances or, for a faster recovery, by entering a treatment program. Unfortunately, if quitting excessive alcohol consumption is not accomplished on time, the consequences of alcoholism can cause irreparable damage in the individual, for example, cardio vascular diseases, diabetes, cirrhosis, and some type of cancers, among others. Similarly, recent research shows that uncontrolled use of marijuana can cause neural damage, mental health disorders such as anxiety, depression, and in some cases paranoia. Also, some studies show that the constant use of marijuana causes digestive problems. When modeling co-abuse, it is expected that health consequences will be worse. Therefore, conduction of more research in this area, and collection of data are very important in order to create awareness and prevention programs for a real problem in society.

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ON UNRESTRICTED DUAL-GENERALIZED COMPLEX HORADAM NUMBERS

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Abstract. This research introduces a novel category of dual-generalized complex numbers, with components represented by unrestricted Horadam numbers. We present various recurrence relations, summation formulas, the Binet formula, and the generating function associated with these numbers. Additionally, a comprehensive bilinear index-reduction formula is derived, which encompasses Vajda's, Catalan's, Cassini's, D'Ocagne's, and Halton's identities as specific cases.

1. Introduction

Hypercomplex numbers have many applications such as in physics, geometry, robotics, and quantum mechanics. There are many studies related to different types of hypercomplex numbers. One among them is dual-generalized complex numbers. They are defined by Gurses et al. [11] as a generalization of dual-complex numbers, hyper-dual numbers, and dual-hyperbolic numbers. The set of dual-generalized complex numbers is defined by

$$\mathbb{DC}_{\mathfrak{p}} = \{a_0 + a_1 J + a_2\varepsilon + a_3 J\varepsilon \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}, \tag{1}$$

where the dual unit ε and the generalized complex unit J adhere to the following rules:

$$J^2 = \mathfrak{p}, -\infty < \mathfrak{p} < \infty, \varepsilon^2 = 0, \varepsilon \neq 0, \varepsilon J = J\varepsilon. \tag{2}$$

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TABLE 1. Multiplication table of units J, ε, $J\varepsilon$.

	1	J	ε	$J\varepsilon$
1	1	J	ε	$J\varepsilon$
J	J	$\mathfrak{p}$	$J\varepsilon$	$\mathfrak{p}\varepsilon$
ε	ε	$J\varepsilon$	0	0
$J\varepsilon$	$J\varepsilon$	$\mathfrak{p}\varepsilon$	0	0

The multiplication scheme for the basis elements of dual generalized complex numbers can also be given in the following table.

Clearly, when the parameter $\mathfrak{p}$ takes the value of -1, the newly introduced commutative number system corresponds into dual-complex numbers. Similarly, for $\mathfrak{p} = 0$, it aligns with hyper-dual numbers, and for $\mathfrak{p} = 1$, it corresponds to dual-hyperbolic numbers. Consequently, an examination of dual-generalized complex numbers allows for the simultaneous understanding of dual-complex numbers, hyper-dual numbers, and dual-hyperbolic numbers. For a more in-depth understanding of dual-generalized complex numbers, one may refer to the relevant literature [5, 6, 9, 11, 17, 18] and the cited references therein.

Extensive research has been conducted on quaternion sequences within specific quaternion algebras. Notably, Horadam [14] explored Fibonacci quaternions within the realm of real quaternion algebra, focusing on quaternion sequences comprising Fibonacci number components. Expanding on the concept of Fibonacci quaternions, Sentürk et al. [19] introduced unrestricted Horadam quaternions within a generalized quaternion algebra by

$$H_n^{(x,y,z)} = w_n + w_{n+x}i + w_{n+y}j + w_{n+z}k,$$

where $\{w_n\}$ is the Horadam sequence [15] defined by

$$w_n = pw_{n-1} + qw_{n-2}, \quad n \geq 2 \tag{3}$$

with the arbitrary initial values w_0, w_1 and nonzero integers p, q. Here the basis $\{1, i, j, k\}$ satisfies the following multiplication rules:

$$\begin{aligned} i^2 &= -\lambda,\ j^2 = -\mu, k^2 = -\lambda\mu, \\ ij &= -ji = k,\ jk = -kj = \mu i,\ ki = -ik = \lambda j, \end{aligned}$$

with $\lambda, \mu \in \mathbb{R}$. For $\lambda = \mu = 1$, it simplifies to the real quaternion algebra, and when $x = 1, y = 2$, and $z = 3$, the unrestricted Horadam quaternions reduce to the Horadam quaternions in [13]. Some matrix representations of Horadam quaternions can be found in [22], and for some recent papers related to special types of quaternions with unrestricted subscripts can be found in [2, 3, 7, 8]. For more on Horadam sequences, see [16, 20].

Several researchers have explored the realm of dual-generalized complex numbers incorporating components resembling Fibonacci sequences. Specifically, Cihan et al. [4] pioneered the study of dual-hyperbolic Fibonacci and Lucas numbers, while

Gungor and Azak [10] established the framework for dual-complex Fibonacci and Lucas numbers. In a similar context, Tan et al. [21] introduced the concept of hyper-dual Horadam quaternions. Furthermore, Gurses et al. [12] innovatively presented the dual-generalized complex Fibonacci quaternions, utilizing dual Fibonacci numbers as coefficients in lieu of real numbers. Recently, Tan and Ocal [23] introduced the dual generalized complex Horadam quaternions.

Inspired by the studies mentioned earlier, we now present the unrestricted dual generalized complex Horadam numbers. We obtain some recurrence relations, the generating function, and the Binet formula of these numbers. We also obtain the general bilinear index-reduction formula of these numbers which reduces to the Vajda's, Halton's, Catalan's, Cassini's, and D'Ocagne's identities as a special case. Moreover, we give summation formulas and a matrix representation of them.

We conclude this section with some preliminaries related to the Horadam sequence.

The Horadam sequence $\{w_n\}$ transforms into the (p,q)-Fibonacci sequence $\{u_n\}$ when $w_0 = 0, w_1 = 1$, and into the (p,q)-Lucas sequence $\{v_n\}$ when $w_0 = 2, w_1 = p$. When $p = q = 1$, these sequences simplify to the traditional Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$, respectively.

The Binet formula of Horadam sequence $\{w_n\}$ is

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{4}$$

where $A := w_1 - w_0\beta, B := w_1 - w_0\alpha$, and α, β are the roots of the characteristic polynomial $x^2 - px - q$, that is; $\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \beta = \frac{p-\sqrt{p^2+4q}}{2}$. Also we have $\alpha\beta = -q, \alpha + \beta = p, \Delta := \alpha - \beta = \sqrt{p^2 + 4q}$ with $p^2 + 4q > 0$.

2. Main Results

In this section, we initially establish the concept of unrestricted dual-generalized complex Horadam numbers, followed by an exploration of some fundamental properties associated with these numbers. Throughout this section, we simply denote the unrestricted dual-generalized complex Horadam numbers as unrestricted DGC Horadam numbers. Let also x, y and z be arbitrary positive integers.

Definition 1. *The nth unrestricted DGC Horadam number is defined as*

$$\widetilde{w}_n^{(x,y,z)} = w_n + w_{n+x}J + w_{n+y}\varepsilon + w_{n+z}J\varepsilon,$$

where w_n is the nth Horadam number, ε is dual unit, and J is generalized complex unit adhering to the multiplication rules in (2).

In the following table, we give some special cases of the unrestricted dual-generalized complex DGC Horadam numbers $\widetilde{w}_n^{(1,2,3)}$. We should note that when

Table 2. Special cases of the unrestricted DGC Horadam numbers.

$\mathfrak{p}$	p	q	w_0	w_1	
$\mathfrak{p}$	1	1	0	1	DGC Fibonacci numbers [12]
$\mathfrak{p}$	1	1	2	1	DGC Lucas numbers [12]
-1	1	1	0	1	Dual-complex Fibonacci numbers [10]
-1	1	1	2	1	Dual-complex Lucas numbers [10]
-1	k	1	0	1	Dual-complex k-Fibonacci numbers [1]
1	1	1	0	1	Dual-hyperbolic Fibonacci numbers [4]
1	1	1	2	1	Dual-hyperbolic Lucas numbers [4]
0	1	1	w_0	w_1	Hyper-dual Fibonacci numbers [22]

$x = 1, y = 2$, and $z = 3$, the unrestricted dual-generalized complex Horadam numbers $\widetilde{w}_n^{(x,y,z)}$ reduce to the conventional dual generalized complex Horadam numbers in [23].

The addition, subtraction, and multiplication of two unrestricted DGC Horadam numbers $\widetilde{w}_n^{(x,y,z)}$ and $\widetilde{w}_m^{(x,y,z)}$ are defined as

$$\begin{aligned}\widetilde{w}_n^{(x,y,z)} \pm \widetilde{w}_m^{(x,y,z)} &= (w_n \pm w_m) + (w_{n+x} \pm w_{m+x})\, J \\ &\quad + (w_{n+y} \pm w_{m+y})\, \varepsilon + (w_{n+z} \pm w_{m+z})\, J\varepsilon\end{aligned}$$

and

$$\begin{aligned}\widetilde{w}_n^{(x,y,z)}\widetilde{w}_m^{(x,y,z)} &= (w_n w_m + \mathfrak{p} w_{n+x} w_{m+x}) + (w_n w_{m+x} + w_{n+x} w_m)\, J \\ &\quad + (w_n w_{m+y} + w_{n+y} w_m + \mathfrak{p} w_{n+x} w_{m+z} + \mathfrak{p} w_{n+z} w_{m+x})\, \varepsilon \\ &\quad + (w_n w_{m+z} + w_{n+x} w_{m+y} + w_{n+y} w_{m+x} + w_{n+z} w_m)\, J\varepsilon,\end{aligned}$$

respectively.

Theorem 1. *The unrestricted DGC Horadam numbers satisfy the following relation:*

$$\widetilde{w}_n^{(x,y,z)} = p\widetilde{w}_{n-1}^{(x,y,z)} + q\widetilde{w}_{n-2}^{(x,y,z)}, \ n \geq 2.$$

Proof. Using the definition of unrestricted DGC Horadam numbers and the definition of classical Horadam numbers, we get

$$\begin{aligned}p\widetilde{w}_{n-1}^{(x,y,z)} + q\widetilde{w}_{n-2}^{(x,y,z)} &= p\,(w_{n-1} + w_{n-1+x} J + w_{n-1+y}\varepsilon + w_{n-1+z} J\varepsilon) \\ &\quad + q\,(w_{n-2} + w_{n-2+x} J + w_{n-2+y}\varepsilon + w_{n-2+z} J\varepsilon) \\ &= (pw_{n-1} + qw_{n-2}) + (pw_{n-1+x} + qw_{n-2+x})\, J \\ &\quad + (pw_{n-1+y} + qw_{n-2+y})\, \varepsilon + (pw_{n-1+z} + qw_{n-2+z})\, J\varepsilon \\ &= w_n + w_{n+x} J + w_{n+y}\varepsilon + w_{n+z} J\varepsilon = \widetilde{w}_n^{(x,y,z)}.\end{aligned}$$

□

In the following Theorem, we give a relation between (p, q)-Fibonacci numbers and the unrestricted DGC Horadam numbers.

Theorem 2. *For $n \geq 1$, we have*

$$\widetilde{w}_n^{(x,y,z)} = u_n \widetilde{w}_1^{(x,y,z)} + q u_{n-1} \widetilde{w}_0^{(x,y,z)}.$$

Proof. From the definition of (p,q)-Fibonacci numbers and the definition of the unrestricted DGC Horadam numbers, we get

$$\begin{aligned}
& u_n \left(w_1 + w_{x+1} J + w_{y+1} \varepsilon + w_{z+1} J\varepsilon \right) + q u_{n-1} \left(w_0 + w_x J + w_y \varepsilon + w_z J \varepsilon \right) \\
= \; & u_n w_1 + q u_{n-1} w_0 \\
& + \left(u_n w_{x+1} + q u_{n-1} w_x \right) J + \left(u_n w_{y+1} + q u_{n-1} w_y \right) \varepsilon + \left(u_n w_{z+1} + q u_{n-1} w_z \right) J\varepsilon \\
= \; & w_n + w_{n+x} J + w_{n+y} \varepsilon + w_{n+z} J \varepsilon \\
= \; & \widetilde{w}_n^{(x,y,z)}.
\end{aligned}$$

□

Theorem 3. *The generating function for unrestricted DGC Horadam numbers is*

$$G(t) = \frac{\widetilde{w}_0^{(x,y,z)} + \left(\widetilde{w}_1^{(x,y,z)} - p \widetilde{w}_0^{(x,y,z)} \right) t}{1 - pt - qt^2}.$$

Proof. Let

$$G(t) := \sum_{n=0}^{\infty} \widetilde{w}_n^{(x,y,z)} t^n = \widetilde{w}_0^{(x,y,z)} + \widetilde{w}_1^{(x,y,z)} t + \sum_{n=2}^{\infty} \widetilde{w}_n^{(x,y,z)} t^n.$$

From Theorem 1, we have

$$\begin{aligned}
& \left(1 - pt - qt^2\right) G(t) \\
= \; & \widetilde{w}_0^{(x,y,z)} + \widetilde{w}_1^{(x,y,z)} t + \sum_{n=2}^{\infty} \widetilde{w}_n^{(x,y,z)} t^n - p \widetilde{w}_0^{(x,y,z)} t - p \sum_{n=2}^{\infty} \widetilde{w}_{n-1}^{(x,y,z)} t^n - q \sum_{n=2}^{\infty} \widetilde{w}_{n-2}^{(x,y,z)} t^n \\
= \; & \widetilde{w}_0^{(x,y,z)} + \widetilde{w}_1^{(x,y,z)} t - p \widetilde{w}_0^{(x,y,z)} t + \sum_{n=2}^{\infty} \left(\widetilde{w}_n^{(x,y,z)} - p \widetilde{w}_{n-1}^{(x,y,z)} - q \widetilde{w}_{n-2}^{(x,y,z)} \right) t^n \\
= \; & \widetilde{w}_0^{(x,y,z)} + \left(\widetilde{w}_1^{(x,y,z)} - p \widetilde{w}_0^{(x,y,z)} \right) t.
\end{aligned}$$

Thus, we get the desired result. □

Theorem 4. *The Binet formula of unrestricted DGC Horadam numbers is*

$$\widetilde{w}_n^{(x,y,z)} = \frac{A \underline{\alpha} \alpha^n - B \underline{\beta} \beta^n}{\alpha - \beta},$$

where $\underline{\alpha} = 1 + \alpha^x J + \alpha^y \varepsilon + \alpha^z J \varepsilon$ *and* $\underline{\beta} = 1 + \beta^x J + \beta^y \varepsilon + \beta^z J \varepsilon$.

Proof. Using the Binet formula of Horadam numbers in (4), we have

$$\widetilde{w}_n^{(x,y,z)} = w_n + w_{n+x} J + w_{n+y} \varepsilon + w_{n+z} J \varepsilon$$

$$\begin{aligned}
&= \left(\frac{A\alpha^n - B\beta^n}{\alpha-\beta}\right) + \left(\frac{A\alpha^{n+x} - B\beta^{n+x}}{\alpha-\beta}\right) J \\
&\quad + \left(\frac{A\alpha^{n+y} - B\beta^{n+y}}{\alpha-\beta}\right) \varepsilon + \left(\frac{A\alpha^{n+z} - B\beta^{n+z}}{\alpha-\beta}\right) J\varepsilon \\
&= \frac{A\alpha^n}{\alpha-\beta}\left(1+\alpha^x J + \alpha^y \varepsilon + \alpha^z J\varepsilon\right) - \frac{B\beta^n}{\alpha-\beta}\left(1+\beta^x J + \beta^y \varepsilon + \beta^z J\varepsilon\right) \\
&= \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha-\beta}.
\end{aligned}$$

□

From Theorem 4, we derive the Binet formulas of unrestricted DGC (p,q)-Fibonacci and Lucas cases:

$$\widetilde{u}_n^{(x,y,z)} = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha-\beta} \quad \text{and} \quad \widetilde{v}_n^{(x,y,z)} = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n, \tag{5}$$

respectively. By considering (5), the following relation can be easily derived:

$$\widetilde{v}_n^{(x,y,z)} = \widetilde{u}_{n+1}^{(x,y,z)} + q\widetilde{u}_{n-1}^{(x,y,z)}.$$

To establish various properties of unrestricted DGC Horadam numbers, we require the following lemma.

Lemma 1. *Let x, y, z be positive integers with $z > y > x$. Then we have*

$$\underline{\alpha}\underline{\beta} = \widetilde{v}_0^{(x,y,z)} - 1 + (-q)^x \left(\left(1 + v_{z-x}\varepsilon\right)\mathfrak{p} + v_{y-x}J\varepsilon\right).$$

Proof.

$$\begin{aligned}
\underline{\alpha}\underline{\beta} &= \left(1+\alpha^x J + \alpha^y \varepsilon + \alpha^z J\varepsilon\right)\left(1+\beta^x J + \beta^y \varepsilon + \beta^z J\varepsilon\right) \\
&= 1 + \mathfrak{p}\left(\alpha\beta\right)^x \\
&\quad + \left(\alpha^x + \beta^x\right) J \\
&\quad + \left(\alpha^y + \beta^y + \mathfrak{p}\left(\alpha^x\beta^z + \alpha^z\beta^x\right)\right)\varepsilon \\
&\quad + \left(\alpha^z + \beta^z + \alpha^x\beta^y + \alpha^y\beta^x\right) J\varepsilon
\end{aligned}$$

$$\begin{aligned}
\underline{\alpha}\underline{\beta} &= 1 + \mathfrak{p}\left(-q\right)^x + v_x J + v_y \varepsilon + \mathfrak{p}\left(\alpha^x\beta^z + \alpha^z\beta^x\right)\varepsilon + v_z J\varepsilon + \left(\alpha^x\beta^y + \alpha^y\beta^x\right) J\varepsilon \\
&= 1 + v_x J + v_y \varepsilon + v_z J\varepsilon + \mathfrak{p}\left(-q\right)^x + \mathfrak{p}\left(\alpha^x\beta^z + \alpha^z\beta^x\right)\varepsilon + \left(\alpha^x\beta^y + \alpha^y\beta^x\right) J\varepsilon \\
&= \widetilde{v}_0^{(x,y,z)} - 1 + \mathfrak{p}\left(-q\right)^x + \mathfrak{p}\left(\alpha^x\beta^z + \alpha^z\beta^x\right)\varepsilon + \left(\alpha^x\beta^y + \alpha^y\beta^x\right) J\varepsilon \\
&= \widetilde{v}_0^{(x,y,z)} - 1 + \mathfrak{p}\left(-q\right)^x + \mathfrak{p}\left(\left(\alpha\beta\right)^x\left(\alpha^{z-x} + \beta^{z-x}\right)\right)\varepsilon + \left(\left(\alpha\beta\right)^x\left(\alpha^{y-x} + \beta^{y-x}\right)\right) J\varepsilon \\
&= \widetilde{v}_0^{(x,y,z)} - 1 + \left(-q\right)^x\left(\mathfrak{p} + \mathfrak{p}\left(\alpha^{z-x} + \beta^{z-x}\right)\varepsilon + \left(\alpha^{y-x} + \beta^{y-x}\right) J\varepsilon\right) \\
&= \widetilde{v}_0^{(x,y,z)} - 1 + \left(-q\right)^x\left(\left(1 + v_{z-x}\varepsilon\right)\mathfrak{p} + v_{y-x}J\varepsilon\right).
\end{aligned}$$

□

Utilizing the Binet formula for unrestricted DGC Horadam numbers and applying Lemma 1, we derive the following identity.

Theorem 5. *(General bilinear index-reduction formula) For nonnegative integers* a, b, c, d *such that* $a+b=c+d$, $b>a, d>c$, *we have*

$$\widetilde{w}_a^{(x,y,z)}\widetilde{w}_b^{(x,y,z)}-\widetilde{w}_c^{(x,y,z)}\widetilde{w}_d^{(x,y,z)}=-\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left((-q)^a v_{b-a}-(-q)^c v_{d-c}\right).$$

Proof. Let $\Delta=\alpha-\beta$. Using the Binet formula of unrestricted DGC Horadam numbers, we have

$$\begin{aligned}
&(\alpha-\beta)^2\left(\widetilde{w}_a^{(x,y,z)}\widetilde{w}_b^{(x,y,z)}-\widetilde{w}_c^{(x,y,z)}\widetilde{w}_d^{(x,y,z)}\right)\\
&=\left(A\underline{\alpha}\alpha^a-B\underline{\beta}\beta^a\right)\left(A\underline{\alpha}\alpha^b-B\underline{\beta}\beta^b\right)-\left(A\underline{\alpha}\alpha^c-B\underline{\beta}\beta^c\right)\left(A\underline{\alpha}\alpha^d-B\underline{\beta}\beta^d\right)\\
&=A^2\underline{\alpha}^2\alpha^{a+b}-AB\underline{\alpha}\underline{\beta}\alpha^a\beta^b-AB\underline{\beta}\underline{\alpha}\alpha^b\beta^a+B^2\underline{\beta}^2\beta^{a+b}\\
&\quad-A^2\underline{\alpha}^2\alpha^{c+d}+AB\underline{\alpha}\underline{\beta}\alpha^c\beta^d+AB\underline{\beta}\underline{\alpha}\beta^c\alpha^d-B^2\underline{\beta}^2\beta^{c+d}\\
&=A^2\underline{\alpha}^2\left(\alpha^{a+b}-\alpha^{c+d}\right)-AB\underline{\alpha}\underline{\beta}\left(\alpha^a\beta^b-\alpha^c\beta^d+\alpha^b\beta^a-\alpha^d\beta^c\right)+B^2\underline{\beta}^2\left(\beta^{a+b}-\beta^{c+d}\right)
\end{aligned}$$

$$\begin{aligned}
&= -AB\underline{\alpha}\underline{\beta}\left(\alpha^a\beta^b+\alpha^b\beta^a-\alpha^c\beta^d-\alpha^d\beta^c\right)\\
&= -AB\underline{\alpha}\underline{\beta}\left[\left((\alpha\beta)^a\left(\alpha^{b-a}+\beta^{b-a}\right)\right)-\left((\alpha\beta)^c\left(\alpha^{d-c}+\beta^{d-c}\right)\right)\right]\\
&= -AB\underline{\alpha}\underline{\beta}\left((-q)^a v_{b-a}-(-q)^c v_{d-c}\right).
\end{aligned}$$

Thus we get the desired result. □

From Theorem 5, we have the following corollaries.

Corollary 1. *(Vajda's identity) For* $a=m+k, b=n-k, c=m$, *and* $d=n$, *we have*

$$\begin{aligned}
&\widetilde{w}_{m+k}^{(x,y,z)}\widetilde{w}_{n-k}^{(x,y,z)}-\widetilde{w}_m^{(x,y,z)}\widetilde{w}_n^{(x,y,z)}\\
&= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left((-q)^{m+k}v_{n-m-2k}-(-q)^m v_{n-m}\right)\\
&= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}(-q)^m\left((-q)^k v_{n-m-2k}-v_{n-m}\right).
\end{aligned}$$

Since $v_{n-m}-(-q)^k v_{n-m-2k}=\Delta^2 u_k u_{n-m-k}$, we also have

$$\widetilde{w}_{m+k}^{(x,y,z)}\widetilde{w}_{n-k}^{(x,y,z)}-\widetilde{w}_m^{(x,y,z)}\widetilde{w}_n^{(x,y,z)}=AB\underline{\alpha}\underline{\beta}(-q)^m u_k u_{n-m-k}.$$

Corollary 2. *(Catalan's identity) For* $a=n-m, b=n+m$ *and* $c=d=n$, *we have*

$$\begin{aligned}
&\widetilde{w}_{n-m}^{(x,y,z)}\widetilde{w}_{n+m}^{(x,y,z)}-\widetilde{w}_n^{(x,y,z)}\widetilde{w}_n^{(x,y,z)}\\
&= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left((-q)^{n-m}v_{2m}-2(-q)^n\right)
\end{aligned}$$

$$= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}(-q)^{n-m}(v_{2m} - 2(-q)^m).$$

Since $v_{2m} - 2(-q)^m = \Delta^2 u_m^2$, we also have

$$\widetilde{w}_{n-m}^{(x,y,z)}\widetilde{w}_{n+m}^{(x,y,z)} - \widetilde{w}_n^{(x,y,z)}\widetilde{w}_n^{(x,y,z)} = -AB\underline{\alpha}\underline{\beta}(-q)^{n-m}u_m^2.$$

Corollary 3. *(Cassini's identity) For $a = n-1, b = n+1$ and $c = d = n$, we have* $\widetilde{w}_{n-1}^{(x,y,z)}\widetilde{w}_{n+1}^{(x,y,z)} - \widetilde{w}_n^{(x,y,z)}\widetilde{w}_n^{(x,y,z)} = -AB\underline{\alpha}\underline{\beta}(-q)^{n-1}$

By using Lemma 1, we get

$$\widetilde{w}_{n-1}^{(x,y,z)}\widetilde{w}_{n+1}^{(x,y,z)} - \widetilde{w}_n^{(x,y,z)}\widetilde{w}_n^{(x,y,z)}$$
$$= -AB(-q)^{n-1}\left(\widetilde{v}_0^{(x,y,z)} - 1 + (-q)^x((1 + v_{z-x}\varepsilon)\mathfrak{p} + v_{y-x}J\varepsilon)\right).$$

Corollary 4. *(d'Ocagne's identity) For $a = n, b = m+1, c = n+1$, and $d = m$, we have*

$$\widetilde{w}_n^{(x,y,z)}\widetilde{w}_{m+1}^{(x,y,z)} - \widetilde{w}_{n+1}^{(x,y,z)}\widetilde{w}_m^{(x,y,z)} = -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}(-q)^n(v_{m-n+1} + qv_{m-n-1}).$$

Since $v_{m-n+1} + qv_{m-n-1} = -\Delta^2 u_{m-n}$, we also have

$$\widetilde{w}_n^{(x,y,z)}\widetilde{w}_{m+1}^{(x,y,z)} - \widetilde{w}_{n+1}^{(x,y,z)}\widetilde{w}_m^{(x,y,z)} = AB\underline{\alpha}\underline{\beta}(-q)^n u_{m-n}.$$

Corollary 5. *(Halton's identity) For $a = m+k, b = n, c = k$, and $d = m+n$, we have*

$$\begin{aligned}\widetilde{w}_{m+k}^{(x,y,z)}\widetilde{w}_n^{(x,y,z)} - \widetilde{w}_k^{(x,y,z)}\widetilde{w}_{m+n}^{(x,y,z)} &= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}\left((-q)^{m+k}v_{n-m-k} - (-q)^k v_{m+n-k}\right)\\ &= -\frac{AB}{\Delta^2}\underline{\alpha}\underline{\beta}(-q)^k((-q)^m v_{n-k-m} - v_{n-k+m}).\end{aligned}$$

Since $v_{n-k+m} - (-q)^m v_{n-k-m} = \Delta^2 u_m u_{n-k}$, we have

$$\widetilde{w}_{m+k}^{(x,y,z)}\widetilde{w}_n^{(x,y,z)} - \widetilde{w}_k^{(x,y,z)}\widetilde{w}_{m+n}^{(x,y,z)} = AB\underline{\alpha}\underline{\beta}(-q)^k u_m u_{n-k}.$$

Next, we give a relation between the unrestricted DGC (p,q)-Fibonacci numbers and the unrestricted DGC (p,q)-Lucas numbers.

Theorem 6. *For nonnegative integers n and m such that $m \geq n$, we have*

$$\begin{aligned}\widetilde{v}_n^{(x,y,z)}\widetilde{u}_m^{(x,y,z)} - \widetilde{v}_m^{(x,y,z)}\widetilde{u}_n^{(x,y,z)} &= 2(-q)^n u_{m-n}\Big(\widetilde{v}_0^{(x,y,z)} - 1\\ &+ (-q)^x((1 + v_{z-x}\varepsilon)\mathfrak{p} + v_{y-x}J\varepsilon)\Big).\end{aligned}$$

Proof. Using the Binet formula of unrestricted DGC Horadam numbers, we have

$$\begin{aligned}&\Delta\left(\widetilde{v}_n^{(x,y,z)}\widetilde{u}_m^{(x,y,z)} - \widetilde{v}_m^{(x,y,z)}\widetilde{u}_n^{(x,y,z)}\right)\\ =\; &(\underline{\alpha}\alpha^n + \underline{\beta}\beta^n)(\underline{\alpha}\alpha^m - \underline{\beta}\beta^m) - (\underline{\alpha}\alpha^m + \underline{\beta}\beta^m)(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n)\end{aligned}$$

$$\begin{aligned} &= \underline{\alpha}^2\alpha^{n+m} - \underline{\alpha\beta}\alpha^n\beta^m + \underline{\alpha\beta}\alpha^m\beta^n - \underline{\beta}^2\beta^{n+m} \\ &\quad -\underline{\alpha}^2\alpha^{n+m} + \underline{\alpha\beta}\alpha^m\beta^n - \underline{\alpha\beta}\alpha^n\beta^m + \underline{\beta}^2\beta^{n+m} \\ &= 2(\alpha\beta)^n \underline{\alpha\beta}\left(\alpha^{m-n} - \beta^{m-n}\right) \\ &= 2(-q)^n \underline{\alpha\beta}\Delta u_{m-n}. \end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned} \widetilde{v}_n^{(x,y,z)}\widetilde{u}_m^{(x,y,z)} - \widetilde{v}_m^{(x,y,z)}\widetilde{u}_n^{(x,y,z)} &= 2(-q)^n u_{m-n}\Bigg(\widetilde{v}_0^{(x,y,z)} - 1 \\ &\quad + (-q)^x\left((1+v_{z-x}\varepsilon)\mathfrak{p} + v_{y-x}J\varepsilon\right)\Bigg). \end{aligned}$$

□

Presently, we provide a sum formula for unrestricted DGC Horadam numbers.

Theorem 7. *For $n \geq 2$, we have*

$$\sum_{r=1}^{n-1}\widetilde{w}_r^{(x,y,z)} = \frac{\widetilde{w}_n^{(x,y,z)} - \widetilde{w}_1^{(x,y,z)} + q\left(\widetilde{w}_{n-1}^{(x,y,z)} - \widetilde{w}_0^{(x,y,z)}\right)}{p+q-1}.$$

Proof. Using the Binet formula for unrestricted DGC Horadam numbers, we have

$$\begin{aligned} \sum_{r=1}^{n-1}\widetilde{w}_r^{(x,y,z)} &= \sum_{r=1}^{n-1}\frac{A\underline{\alpha}\alpha^r - B\underline{\beta}\beta^r}{\alpha-\beta} \\ &= \frac{A\underline{\alpha}}{\alpha-\beta}\sum_{r=1}^{n-1}\alpha^r - \frac{B\underline{\beta}}{\alpha-\beta}\sum_{r=1}^{n-1}\beta^r \\ &= \frac{A\underline{\alpha}}{\alpha-\beta}\left(\frac{\alpha^n-\alpha}{\alpha-1}\right) - \frac{B\underline{\beta}}{\alpha-\beta}\left(\frac{\beta^n-\beta}{\beta-1}\right) \\ &= \frac{1}{(\alpha-\beta)(1-p-q)}\Big(-\left(A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n\right) - q\left(A\underline{\alpha}\alpha^{n-1} - B\underline{\beta}\beta^{n-1}\right) \\ &\quad + q\left(A\underline{\alpha} - B\underline{\beta}\right) + \left(A\underline{\alpha}\alpha - B\underline{\beta}\beta\right)\Big) \\ &= \frac{-\widetilde{w}_n^{(x,y,z)} - q\widetilde{w}_{n-1}^{(x,y,z)} + q\widetilde{w}_0^{(x,y,z)} + \widetilde{w}_1^{(x,y,z)}}{1-p-q}. \end{aligned}$$

□

Theorem 8. *For nonnegative integers n and r, we have*

$$\sum_{m=0}^{n}\binom{n}{m}q^{n-m}p^m\widetilde{w}_{m+r}^{(x,y,z)} = \widetilde{w}_{2n+r}^{(x,y,z)}.$$

Proof. Using the Binet formula for unrestricted DGC Horadam numbers, we obtain

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}q^{n-m}p^{m}\widetilde{w}_{m+r}^{(x,y,z)} \\
&=\sum_{m=0}^{n}\binom{n}{m}q^{n-m}p^{m}\left(\frac{A\underline{\alpha}\alpha^{m+r}-B\underline{\beta}\beta^{m+r}}{\alpha-\beta}\right) \\
&=\frac{A\underline{\alpha}\alpha^{r}}{\alpha-\beta}\sum_{m=0}^{n}\binom{n}{m}q^{n-m}\left(p\alpha\right)^{m}-\frac{B\underline{\beta}\beta^{r}}{\alpha-\beta}\sum_{m=0}^{n}\binom{n}{m}q^{n-m}\left(p\beta\right)^{m} \\
&=\frac{A\underline{\alpha}\alpha^{r}}{\alpha-\beta}\left(q+p\alpha\right)^{n}-\frac{B\underline{\beta}\beta^{r}}{\alpha-\beta}\left(q+p\beta\right)^{n} \\
&=\frac{A\underline{\alpha}\alpha^{2n+r}-B\underline{\beta}\beta^{2n+r}}{\alpha-\beta}=\widetilde{w}_{2n+r}^{(x,y,z)}.
\end{aligned}$$

□

Ultimately, we present a matrix representation for unrestricted DGC Horadam numbers.

Theorem 9. *For $n \geq 0$, we have*

$$\begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}^{n}\begin{bmatrix} \widetilde{w}_{2}^{(x,y,z)} & \widetilde{w}_{1}^{(x,y,z)} \\ \widetilde{w}_{1}^{(x,y,z)} & \widetilde{w}_{0}^{(x,y,z)} \end{bmatrix}=\begin{bmatrix} \widetilde{w}_{n+2}^{(x,y,z)} & \widetilde{w}_{n+1}^{(x,y,z)} \\ \widetilde{w}_{n+1}^{(x,y,z)} & \widetilde{w}_{n}^{(x,y,z)} \end{bmatrix}.$$

Proof. By using Theorem 2, it can be easily demonstrated through mathematical induction on n. □

By computing the determinant on both sides of the matrix equality mentioned earlier, we derive Cassini's identity for the sequence $\{\widetilde{w}_n\}$ in a straightforward manner as:

$$\widetilde{w}_{n+2}^{(x,y,z)}\widetilde{w}_{n}^{(x,y,z)}-\widetilde{w}_{n+1}^{(x,y,z)}\widetilde{w}_{n+1}^{(x,y,z)}=(-q)^{n}\left(\widetilde{w}_{2}^{(x,y,z)}\widetilde{w}_{0}^{(x,y,z)}-\widetilde{w}_{1}^{(x,y,z)}\widetilde{w}_{1}^{(x,y,z)}\right).$$

3. Conclusion

In this paper we define a novel category of dual-generalized complex numbers, with components represented by unrestricted Horadam numbers. The main advantage to introducing unrestricted dual-generalized complex Horadam numbers is that many unrestricted dual-generalized complex numbers with the well-known numbers such as Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas, Pell, Pell-Lucas can be deduced as particular cases of these unrestricted DGC numbers. We state recurrence relations, summation formulas, Binet formula, and generating function associated with these numbers. In addition, a comprehensive bilinear index-reduction formula is derived, which encompasses Vajda's, Catalan's, Cassini's, D'Ocagne's, and Halton's identities as specific cases. For interested readers, the results of this paper could be applied for any other type of hypercomplex numbers.

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LYAPUNOV-TYPE INEQUALITIES FOR LINEAR HYPERBOLIC AND ELLIPTIC EQUATIONS ON A RECTANGULAR DOMAIN

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Abstract. In the case of oscillatory potential, we present some new Lyapunov -type inequalities for linear hyperbolic and elliptic equations on a rectangular domain in $\mathbb{R}^2$. No sign restriction is imposed on the potential function. As applications of the Lyapunov-type inequalities obtained, we give some estimations for disconjugacy of hyperbolic and elliptic Dirichlet boundary value problems.

1. Introduction

In the paper, we first obtain a Lyapunov-type inequality for the linear hyperbolic equation of the form

$$u_{tt}(x,t) - u_{xx}(x,t) + q(t)u(x,t) = 0, \qquad (x,t) \in \mathcal{R} \tag{1}$$

satisfying the Dirichlet boundary condition

$$u(x,t) = 0, \qquad (x,t) \in \partial\mathcal{R}, \tag{2}$$

where

$$\mathcal{R} = \{(x,t) : x \in [x_1, x_2], t \in [t_1, t_2]\}, \tag{3}$$

and that no sign restriction is imposed on the potential function $q(t) \in \mathrm{L}^1[t_1, t_2]$.

Secondly, we give an analogous result for the linear elliptic equation of the form

$$u_{tt}(x,t) + u_{xx}(x,t) + q(t)u(x,t) = 0, \qquad (x,t) \in \mathcal{R} \tag{4}$$

satisfying the Dirichlet boundary condition (2).

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The well-known Lyapunov inequality [13] for Hill's equation

$$x''(t) + \nu(t)x(t) = 0 \tag{5}$$

states that if t_1 and t_2 $(t_1 < t_2)$ are consecutive zeros of a nontrivial solution $x(t)$ of this equation, then the inequality

$$\int_{t_1}^{t_2} |\nu(t)|\mathrm{d}t > \frac{4}{t_2 - t_1} \tag{6}$$

holds. Inequality (6) was later strengthened by replacement of $|\nu|$ by ν^+, i.e.,

$$\int_{t_1}^{t_2} \nu^+(t)\mathrm{d}t > \frac{4}{t_2 - t_1}, \tag{7}$$

cf. Wintner [17], and thereafter by some other authors, where $\nu^+ = \max\{\nu, 0\}$. Inequality (7) is the best possible in the sense that the constant "4" can not be replaced by any larger constant in (7) due to Hartman [8, Theorem 5.1]. Inequalities (6) and (7) and their several generalizations to Hamiltonian systems, higher order differential equations, nonlinear and half-linear differential equations, difference and dynamic equations, functional and impulsive differential equations, have found many applications in areas like oscillation and Sturmian theory, disconjugacy, asymptotic theory, eigenvalue problems, boundary value problems, and various properties of the solutions of related differential equations, see [5, 12, 16] and their references. We also refer reader to recently published monograph by Agarwal et al. [1] for the historical development of Lyapunov inequalities and its applications.

The classical result of Lyapunov is usually connected with the disconjugacy of Eq. (5), i.e. the inequality

$$\int_{t_1}^{t_2} \nu^+(t)\mathrm{d}t \leq \frac{4}{t_2 - t_1} \tag{8}$$

implies that (5) is disconjugate in $[t_1, t_2]$.

There has been an increasing interest for the Lyapunov-type inequalities for partial differential equations in the last few decades; see for example [2–4, 6, 7, 9, 10, 14, 15] and their references. In 2006, Canada et al. [2] considered the linear partial differential equations

$$\begin{cases} -\Delta u(x) = a(x)u(x), & x \in \Omega \\ \dfrac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded and regular domain and the function $a : \Omega \to \mathbb{R}$. They proved how the relation between the quantity p and $N/2$ play a crucial role by considering the sub-critical $(1 < p < N/2)$, super-critical $(p > N/2)$ and the critical $(p = N/2)$ cases.

In 2016, de Nápoli and Pinasco [7] proved Lyapunov-type inequalities for the p-Laplacian equations

$$\begin{cases} \Delta_p u(x) + w(x)|u(x)|^{p-2}u(x) = 0, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $p > 1$ and the weight function $w \in \mathrm{L}^s$ for some s depending on p and N. They obtained Lyapunov-type inequalities for two separate cases $p < N$ and $p > N$, and the case $p = N$ was given as an open problem for the reader. Recently, Kumar and Tyagi [11] solved this open problem and established Lyapunov-type inequality for a class of the following N-Laplace equations:

$$\begin{cases} \Delta_N v(x) + f(x)|v(x)|^{N-2}v(x) = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega \end{cases}$$

under some conditions on μ, g, R and b, where

$$f(x) = \mu g(x) \left(|x| \log \frac{R}{|x|} \right)^{-N} + b(x).$$

In 2020, Jleli at al. [10] established Lyapunov-type inequalities for the partial differential equations of the form

$$\begin{cases} -G_\gamma u(x,y) = w(x)u(x,y), & (x,y) \in \Gamma \\ u(x,y) = 0, & (x,y) \in \partial\Gamma, \end{cases}$$

where $\Gamma = (a,b) \times \mathcal{O}$; $(a,b) \in \mathbb{R}^2$ and $\mathcal{O}$ is an open bounded subset in $\mathbb{R}^N$ for $N \geq 1$. Here G_γ is the Grushin operator

$$G_\gamma u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + x^{2\gamma}\Delta_y u(x,y), \qquad (x,y) \in \Gamma.$$

When $\gamma = 0$, Grushin operator reduces to the standart Laplace operator but the presence of $x^{2\gamma}$, this operator can not be elliptic on Γ. In 2020, Jleli et al. [10] proved the Lyapunov-type inequality for the Grushin operator via sign change criteria.

In this paper, we obtain a Lyapunov-type inequality for the hyperbolic equation (1) satisfying the Dirichlet boundary condition (2). Moreover, we extend this result to the elliptic equation (4) satisfying the Dirichlet boundary condition (2). To obtain such type of inequalities, we use the separation of variables technique in problems both (1)–(2) and (4)–(2). In Section 3, we present several examples which illustrate how easily the results obtained can be applied to the related equations. At the end of the paper, we impose some open problems.

2. Main Results

Throughout this section, we denote $h^+ = \max\{h, 0\}$ and we shall assume that the potential q is in the set $\mathrm{L}^1[t_1, t_2]$.

Now, let us restate Prb. (1)–(2) as

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + q(t)u(x,t) = 0, & (x,t) \in \mathcal{R}; \\ u(x,t_1) = u(x,t_2) = 0, & x_1 \le x \le x_2; \\ u(x_1,t) = u(x_2,t) = 0, & t_1 \le t \le t_2, \end{cases} \tag{9}$$

where $\mathcal{R}$ is defined in (3).

The first main result of the paper is the following.

Theorem 1 (Lyapunov-type inequality)**.** *If u is a nontrivial solution of Prb.* (9)*, then the inequality*

$$\int_{t_1}^{t_2} \left[(x_2 - x_1)^2 q(t) + \pi^2\right]^+ \mathrm{d}t > \frac{4}{t_2 - t_1}(x_2 - x_1)^2 \tag{10}$$

holds.

Proof. Let u be a nontrivial solution of Prb. (9). The method of separation of variables starts by looking the solutions of Eq. (1) of the form

$$u(x,t) = y(x)z(t), \tag{11}$$

where the variables separate with $y(x) \not\equiv 0$ on (x_1, x_2) and $z(t) \not\equiv 0$ on (t_1, t_2). Substituting (11) in (1), we obtain

$$y(x)z''(t) - y''(x)z(t) + q(t)y(x)z(t) = 0 \tag{12}$$

for $x \in (x_1, x_2)$ and $t \in (t_1, t_2)$. Since $y(x)z(t) \not\equiv 0$, dividing both sides of (12) by it, we separate the variables x and t as

$$\frac{z''(t)}{z(t)} + q(t) = \frac{y''(x)}{y(x)}. \tag{13}$$

The left-hand side of (13) is a function of t only, whereas the right-hand side has just x. But x and t are independent variables so (13) is possible only when both sides of it are constant; that is

$$\frac{z''(t)}{z(t)} + q(t) = \frac{y''(x)}{y(x)} = \lambda \tag{14}$$

for some real number λ. On the other hand, we have boundary conditions to be satisfied. The first boundary conditions in (9) imply that

$$y(x)z(t_1) = 0 \quad \text{and} \quad y(x)z(t_2) = 0 \tag{15}$$

for all $x \in (x_1, x_2)$. Since $y(x) \not\equiv 0$ on (x_1, x_2), (15) is possible only when $z(t_1) = z(t_2) = 0$. Applying the similar argument to the second boundary conditions in (9), we must have $y(x_1) = y(x_2) = 0$. Using these conditions together with (14), we can conclude that $z(t)$ is a nontrivial solution of the boundary value problem

$$\begin{cases} z''(t) + [q(t) - \lambda]z(t) = 0, \\ z(t_1) = z(t_2) = 0 \end{cases} \tag{16}$$

and $y(x)$ is a nontrivial solution of the boundary value problem

$$\begin{cases} y''(x) - \lambda y(x) = 0, \\ y(x_1) = y(x_2) = 0. \end{cases} \tag{17}$$

We note that t_1, t_2 ($t_1 < t_2$) and x_1, x_2 ($x_1 < x_2$) are consecutive zeros of $z(t)$ and $y(x)$, respectively. Now consider the boundary value problem

$$\begin{cases} w''(x) + kw(x) = 0, \\ w(x_1) = w(x_2) = 0, \end{cases} \tag{18}$$

where k is a constant. It is known that the eigenvalues k_n of Prb. (18) are

$$k_n = \frac{n^2\pi^2}{(x_2 - x_1)^2}, \qquad n = 1, 2, \dots,$$

and hence the smallest eigenvalue of it is $k_1 = \pi^2/(x_2 - x_1)^2$. Since Prb. (17) has a nontrivial solution, we take $\lambda = -k_1$. Now replacing λ by $-k_1$ in Prb. (16), it turns out that

$$\begin{cases} z''(t) + Q(t)z(t) = 0, \\ z(t_1) = z(t_2) = 0, \end{cases} \tag{19}$$

where

$$Q(t) = q(t) + \frac{\pi^2}{(x_2 - x_1)^2}.$$

Applying Lyapunov's result to Prb. (19), we see that inequality (10) holds. The proof of Theorem 1 is complete. □

In case

$$q(t) > -\frac{\pi^2}{(x_2 - x_1)^2} \quad \text{for} \quad t \in (t_1, t_1),$$

inequality (10) turns out to be

$$\int_{t_1}^{t_2} \left[(x_2 - x_1)^2 q(t) + \pi^2\right] \mathrm{d}t > \frac{4}{t_2 - t_1}(x_2 - x_1)^2$$

which requires that

$$\int_{t_1}^{t_2} q(t)\mathrm{d}t > \frac{4}{t_2 - t_1} - \frac{t_2 - t_1}{(x_2 - x_1)^2}\pi^2.$$

Corollary 1. *If the inequality*

$$\int_{t_1}^{t_2} \left[(x_2 - x_1)^2 q(t) + \pi^2\right]^+ \mathrm{d}t \le \frac{4}{t_2 - t_1}(x_2 - x_1)^2 \tag{20}$$

holds, then Prb. (9) *has no nontrivial solution.*

Now consider the elliptic equation (4) satisfying the Dirichlet boundary condition (2) by restating it as

$$\begin{cases} u_{tt}(x,t) + u_{xx}(x,t) + q(t)u(x,t) = 0, & (x,t) \in \mathcal{R}; \\ u(x,t_1) = u(x,t_2) = 0, & x_1 \le x \le x_2, \\ u(x_1,t) = u(x_2,t) = 0, & t_1 \le t \le t_2, \end{cases} \tag{21}$$

where $\mathcal{R}$ is defined in (3).

The following is the second main result of the paper.

Theorem 2 (Lyapunov-type inequality)**.** *If u is a nontrivial solution of Prb. (21), then the inequality*

$$\int_{t_1}^{t_2} \left[(x_2 - x_1)^2 q(t) - \pi^2\right]^+ \mathrm{d}t > \frac{4}{t_2 - t_1}(x_2 - x_1)^2 \tag{22}$$

holds.

Proof. The proof Theorem 2 is analogous to that of Theorem 1, and hence it is left to the reader. □

When

$$q(t) > \frac{\pi^2}{(x_2 - x_1)^2} \quad \text{for} \quad t \in (t_1, t_1),$$

inequality (10) turns out to be

$$\int_{t_1}^{t_2} \left[(x_2 - x_1)^2 q(t) - \pi^2\right] \mathrm{d}t > \frac{4}{t_2 - t_1}(x_2 - x_1)^2$$

which requires that

$$\int_{t_1}^{t_2} q(t)\mathrm{d}t > \frac{4}{t_2 - t_1} + \frac{t_2 - t_1}{(x_2 - x_1)^2}\pi^2.$$

Corollary 2. *If the inequality*

$$\int_{t_1}^{t_2} \left[(x_2 - x_1)^2 q(t) - \pi^2\right]^+ \mathrm{d}t \le \frac{4}{t_2 - t_1}(x_2 - x_1)^2 \tag{23}$$

holds, then Prb. (21) has no nontrivial solution.

3. Applications

In this section, we give some disconjugacy estimations for hyperbolic and elliptic Dirichlet boundary value problems, by applying the Lyapunov-type inequalities obtained in Section 2.

Example 1. Consider the hyperbolic boundary value problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + (1 - \pi^2)u(x,t) = 0, & (x,t) \in \mathcal{R}_0, \\ u(x,0) = u(x,\pi) = 0, & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le \pi, \end{cases} \tag{24}$$

where $\mathcal{R}_0$ is the rectangular region

$$\mathcal{R}_0 = \{(x,t) : x \in [0,1], t \in [0,\pi]\}.$$

Substituting $q(t) = 1 - \pi^2$, $x_2 - x_1 = 1$ and $t_2 - t_1 = \pi$ in Lyapunov-type inequality (10), we see that it is satisfied by $\pi^2 > 4$. Note that the solution of Prb. (24) is the function $u(x,t) = \sin(\pi x)\sin t$.

Example 2. Consider the hyperbolic boundary value problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \mu(2 + t - t^2)u(x,t) = 0, & (x,t) \in \mathcal{R}_1, \\ u(x,0) = u(x,1) = 0, & 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, & 0 \le t \le 1, \end{cases} \tag{25}$$

where μ is a positive constant and $\mathcal{R}_1$ is the rectangular region

$$\mathcal{R}_1 = \{(x,t) : x \in [0,\pi], t \in [0,1]\}. \tag{26}$$

In the view of Lyapunov-type inequality (10), the following inequality must be satisfied:

$$\int_0^1 h^+(t;\mu)\mathrm{d}t > 4, \qquad (\mu > 0) \tag{27}$$

where $h(t;\mu) = 2\mu + 1 + \mu t - \mu t^2$. It can be shown that $h(t;\mu) > 0$ for all $t \in [-1,2]$, and hence (27) turns to

$$\int_0^1 h(t;\mu)\mathrm{d}t = \int_0^1 [2\mu + 1 + \mu t - \mu t^2]\mathrm{d}t = \frac{13}{6}\mu + \frac{1}{6} > 4. \tag{28}$$

So Prb. (25) has no nontrivial solution, if $\mu \le 23/13 \approx 1,76923$ by Corollary 1. In particular if $\mu = 4$, then Prb. (25) has a nontrivial solution

$$u(x,t) = t(1-t)e^{t(1-t)}\sin x, \qquad (x,t) \in \mathcal{R}_1.$$

Example 3. Consider the elliptic boundary value problem

$$\begin{cases} u_{tt}(x,t) + u_{xx}(x,t) + \sigma(5/2 + t - t^2)u(x,t) = 0, & (x,t) \in \mathcal{R}_1, \\ u(x,0) = u(x,1) = 0, & 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, & 0 \le t \le 1, \end{cases} \tag{29}$$

where σ is a positive constant and $\mathcal{R}_1$ is the rectangular region defined in (26). In the view of Lyapunov-type inequality (22), the inequality

$$\int_0^1 \nu^+(t;\sigma)\mathrm{d}t > 4 \qquad (\sigma > 0) \tag{30}$$

must be hold, where $\nu(t;\sigma) = 5\sigma/2 - 1 + \sigma t - \sigma t^2$, $\sigma > 0$. It is clear that $\nu(t;\sigma) < 0$ for all $\sigma \in (0, 4/11)$. Moreover, if $\sigma \ge 4/11$, then $\nu(t;\sigma) \ge 0$ for all $t \in [(1 - \sqrt{11})/2, (1 + \sqrt{11})/2]$, and hence (30) turns to

$$\int_0^1 \nu(t;\sigma)\mathrm{d}t = \int_0^1 [5\sigma/2 - 1 + \sigma t - \sigma t^2]\mathrm{d}t = \frac{8}{3}\sigma - 1 > 4. \tag{31}$$

So Prb. (29) has no nontrivial solution, if $\sigma \leq 15/8 \approx 1,875$ by Corollary 2. In particular if $\sigma = 4$, then Prb. (29) has a nontrivial solution

$$u(x,t) = t(1-t)e^{t(1-t)} \sin x, \qquad (x,t) \in \mathcal{R}_1.$$

Finally, we present some open problems concerning possible extensions of Theorem 1 and Theorem 2. It will be of interest to find a Lyapunov-type inequalities for the linear parabolic equation of the form

$$u_t(x,t) - u_{xx}(x,t) + p(t)u(x,t) = 0, \qquad (x,t) \in \mathcal{R} \tag{32}$$

satisfying the Dirichlet boundary condition (2), where $\mathcal{R}$ is defined in (3), and that no sign restriction is imposed on the potential function $p(t) \in \mathrm{L}^1[t_1,t_2]$. In fact, the nonlinear cases

$$u_{tt}(x,t) \pm u_{xx}(x,t) + F(t,u(x,t)) = 0, \qquad (x,t) \in \mathcal{R}$$

and

$$u_t(x,t) - u_{xx}(x,t) + G(t,u(x,t)) = 0, \qquad (x,t) \in \mathcal{R}$$

are of immense interest. Moreover, Lyapunov-type inequalities for elliptic, hyperbolic and parabolic equations of the form

$$u_{tt}(x,t) \pm \Delta u(x,t) + F(t,u(x,t)) = 0, \qquad (x,t) \in \Omega$$

and

$$u_t(x,t) - \Delta u(x,t) + G(t,u(x,t)) = 0, \qquad (x,t) \in \Omega$$

may give remarkable results under some appropriate boundary conditions, where Ω is any closed subset of $\mathbb{R}^n$.

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A NEW APPROACH TO CONSTRUCT AND EXTEND THE SCHUR STABLE MATRIX FAMILIES

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Abstract. In this study, Schur stability, sensitivity and continuity theorems have been mentioned. In addition, matrix families, interval matrix and extend of the intervals also have been mentioned. The $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ intervals of the matrix families have been determined so that the linear sums family $\mathcal{L}$ and convex combination family $\mathcal{C}$ are Schur stable. Samely, the $\mathcal{I}^*_{\mathcal{L}}$ and $\mathcal{I}^*_{\mathcal{C}}$ intervals have been determined and $\mathcal{L}$ and $\mathcal{C}$ are ω^*-Schur stable. Afterwards, the methods which based on continuity theorems and the algorithms which based on the methods have been given. Extended intervals have been obtained with the help of the methods and the algorithms. All definitions are supported by examples.

1. Introduction

One of the real problems of the stability analysis is to determine the stability of the matrix families. In this paper the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ have been presented to make the matrix families $\mathcal{L}$ and $\mathcal{C}$ Schur stable. Here the matrix families $\mathcal{L}$ and $\mathcal{C}$ consist of linear sum and convex combination, respectively. Also, these intervals are extended with the help of continuity theorems and the matrix families are constructed in order to provide Schur stability [13, 16]. There are many studies in the literature specifically related to linear sum and convex combination [5–8, 19, 24]. Unlike studies that control the Schur stability of interval matrices, Schur stable interval matrices are constructed in this study.

In 1892, Lyapunov studied the behavior of solutions of systems and developed the concept of stability (see, for instance, [1, 9, 18]). The stability problem is reduced

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to the problem of the existence of a positive definite solution of the matrix equation known as the Lyapunov equation with this concept for linear systems.

A necessary and sufficient condition for the matrix A to be discrete-asymptotic stable is that the eigenvalues of the matrix A lay in the unit disk, that is, $|\lambda_i(A)| < 1$ for all $i = 1, 2, \ldots, N$, where $\lambda_i (i = 1, 2, \ldots, N)$ are the eigenvalues of the matrix A [1, 18]. On the other hand, this is also known as spectral criterion in the literature. The spectral criterion can also be represented by the spectrum. $\sigma(A) = \{\lambda \mid \lambda = \lambda_i(A)\}$ to be spectrum, the matrix A is said to be Schur stable if it satisfies the condition $\sigma(A) \subset C_s = \{z \mid |z| < 1\}$ [27]. Let's also give the family of Schur stable matrices as follows;

$$S_N = \{A \in M_N(\mathbb{C}) \mid |\lambda_i(A)| < 1 \ \ (i = 1, 2, ..., N)\}.$$

If the locations of these eigenvalues are known approximately, stability analysis of the system can be done with help of many well-known methods. Stability analysis of many control systems is concerned with the region where the eigenvalues of the matrices are located. Gerschgorin and Rouche theorems, which are used in determining the region, can be given as an example to this situation [18,20]. However, it is not easy to determine the eigenvalue in practice. Small changes in the inputs of the matrices lead to the big changes in the eigenvalues, i.e. the eigenvalue problem is an ill-posed problem for the non-symmetric matrices [9,28]. We can give the example of Ostrowski to explain this situation better. $A_\omega = (a_{ij}) \in M_N(\mathbb{R})$; $a_{i,i} = 0.5,\ a_{i,i+1} = 10,\ a_{N,1} = \omega,\ i = 1, 2, ..., N-1$. It is seen that $\|A_{10^{-100}} - A_0\| = 10^{-100}$ and $\lambda_i(A_{10^{-100}}) = 1.5$ so $|\lambda_i(A_{10^{-100}}) - \lambda_i(A_0)| \leq 1$ [1]. As can be seen here, while the matrix A_0 is Schur stable, the matrix $A_{10^{-100}}$ is not Schur stable because of $\lambda_i(A_{10^{-100}}) = 1.5$. Therefore, it is more convenient to use the parameters calculated with the help of the solution of a linear algebraic equation which characterizes the stability for the determination of stability.

Thus, the stability problem is reduced to the problem of the existence of a positive definite solution of the matrix equation given as the Lyapunov equation [1,2,18]. According to Lyapunov's theorem, the Lyapunov matrix equation, which determines the Schur stability of the systems, is given as follow

$$A^* H A - H + I = 0. \tag{1}$$

If this system of equations has a positive definite solution

$$H = \sum_{k=0}^{\infty} (A^*)^k A^k,\ \ H = H^* > 0 \tag{2}$$

then the matrix A is said to be Schur stable [1, 9, 18, 23, 27]. The existence of $H = H^* > 0$ equivalent to having the eigenvalues of the matrix A inside the unit circle.

The parameter $\omega(A) = \|H\| \geq 1$, which determines the quality of the stability, is known as the Schur stability parameter of the matrix A [1,9,11]. Furthermore, ω^* is the practical Schur stability parameter of the matrix A, where $1 < \omega^* \in \mathbb{R}$

and the users choose the value ω^* in view of their problem. If $\omega(A) \leq \omega^*$ then the matrix A is ω^*-Schur stable. Otherwise, the matrix A is ω^*-Schur unstable matrix [1,3,26]. Let's examine the following matrices in order to see the notion of quality of stability more easily.

Let's take $A_k \in S_N$ as follow

$$A_k = \begin{pmatrix} -0.1 & 10^{k-1} - 1 \\ 0 & 0.1 \end{pmatrix}, \ k \in \mathbb{N}.$$

It is clear that, although $\sigma(A_k) = \{-0.1, 0.1\}$ for $k \in \mathbb{N}$, it can be seen from the Table 1 that the values of $\omega(A_k)$ also increase as the values of k increase.

Table 1. The quality of Schur stability of the matrix A_k

k	1	2	3	4	5
$\omega(A_k)$	1.0101	82.0282	9803	998102	9.999e+007

Also the quality of the Schur stability increases as it approaches 1. Especially, in case of $A = 0$, when we substitute it in (2), $H = I$ and $\omega(A) = 1$ are obtained. This state is also known as the perfect state.

In [25], Hurwitz stability intervals for the matrix families were studied. The matrix families were introduced. The intervals were determined to make these families Hurwitz stable. A method and an algorithm were given to extend these intervals.

This study is an analogy of [25]. Here, Schur and ω^*-Schur stability of linear sum and convex combination families are discussed. In Section 2, $\mathcal{L}$ and $\mathcal{C}$ matrix families are introduced, $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ intervals are determined to make these families Schur stable. The illustrative examples related to the subject are given. In Section 3, $\mathcal{I}^*_{\mathcal{L}}$ and $\mathcal{I}^*_{\mathcal{C}}$ intervals are determined to make these families ω^*-Schur stable. Thereafter, the illustrative examples related to the subject are given. In Section 4, a new approach is given for the Schur stability of the matrix families. According to the approach, the methods which based on continuity theorems are given. These theorems shows the sensitivity of Schur stability and ω^*-Schur stability. The algorithms which based on the methods are given. The extended intervals $\mathcal{I}^e_{\mathcal{L}}$, $\mathcal{I}^e_{\mathcal{C}}$, $\mathcal{I}^{*e}_{\mathcal{L}}$ and $\mathcal{I}^{*e}_{\mathcal{C}}$ are obtained with the help of methods and algorithms. End of the paper, examples are given. The numerical results in the article are obtained using the computer dialogue system MVC [10].

2. Schur Stability of the Matrix Families

Let's give the theorems which determining the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ for the matrix families

$$\mathcal{L} = \mathcal{L}(A_1, A_2) = \{A(r) = A_1 + rA_2 \mid A_1, \ A_2 \in M_N(\mathbb{C})\} \tag{3}$$

and

$$\mathcal{C} = \mathcal{C}(A_1, A_2) = \{A(r) = (1-r)A_1 + rA_2 \mid A_1,\ A_2 \in M_N(\mathbb{C})\} \tag{4}$$

to be Schur stable for $A_1 \in S_N$ and $A_2 \in M_N(\mathbb{C})$. Before giving the theorems for these matrix families, let's give the continuity theorem which determines the sensitivity of the stability. We use this theorem for Schur stability. Let's remember the family of Schur stable matrices as follows;

$$S_N = \{A \in M_N(\mathbb{C}) \mid \omega(A) < \infty\}.$$

Theorem 1. *Let $A \in S_N$. If $\|B\| < \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$ then the matrix $A+B \in S_N$ and*

$$\omega(A+B) \leqslant \tfrac{\omega(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)},$$

$$|\omega(A+B) - \omega(A)| \leqslant \tfrac{(2\|A\|+\|B\|)\|B\|\omega^2(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}$$

holds [4, 14].

Theorem 2. *If $A_1 \in S_N,\ A_2 \in M_N(\mathbb{C})$ and $r \in \mathcal{I}_{\mathcal{L}} = [\underline{r}, \overline{r}]$ then $\mathcal{L}(A_1, A_2) \subset S_N$, where $-l = u = -\frac{\|A_1\|}{\|A_2\|} + \frac{1}{\|A_2\|}\sqrt{\|A_1\|^2 + \frac{1}{\omega(A_1)}}$, $l < \underline{r} < \overline{r} < u$.*

Proof. Let us consider the given linear sum as follow

$$A(r) = A_1 + rA_2.$$

If $A_2 = 0$ then $A(r) = A_1$. We know that $A_1 \in S_N$ so $A(r) \in S_N$, too. Let's take $A_2 \neq 0$. If we substitute $A(r)$ in the Lyapunov equation, we get the equation as follow

$$(A_1 + rA_2)^* H (A_1 + rA_2) - H + I = 0$$
$$A_1^* H A_1 - H = -\left(I + rA_1^* H A_2 + rA_2^* H A_1 + r^2 A_2^* H A_2\right).$$

At that rate,

$$C = I + rA_1^* H A_2 + rA_2^* H A_1 + r^2 A_2^* H A_2 > 0$$

$C = C^* > 0$ is available. The obtained result is written as follows

$$\|C\| \le 1 + 2|r| \|A_1\| \|H\| \|A_2\| + r^2 \|A_2\|^2 \|H\|$$

then, if the inequality is substituted in the equation which is the solution of the Lyapunov equation

$$H = \sum_{k=0}^{\infty} (A_1^*)^k C A_1^k$$
$$\|H\| \le \|C\| \omega(A_1)$$
$$\|H\| \le \Big(1 + 2|r| \|A_1\| \|H\| \|A_2\| + r^2 \|A_2\|^2 \|H\|\Big) \omega(A_1)$$
$$\|H\| \le \frac{\omega(A_1)}{1 - 2|r| \|A_1\| \|A_2\| \omega(A_1) - r^2 \|A_2\|^2 \omega(A_1)}$$

is obtained. While $A_1 \in S_N$, the following condition must be verified for $A(r)$ to be Schur stable

$$1 - 2|r| \|A_1\| \|A_2\| \omega(A_1) - r^2 \|A_2\|^2 \omega(A_1) \geq 0.$$

Then if the inequality is arranged with according to r, the Schur stability intervals $[\underline{r}, \overline{r}]$ of the matrix $A(r)$ are obtained, where

$$\underline{r} > l = \frac{\|A_1\| \omega(A_1) - \sqrt{\|A_1\|^2 (\omega(A_1))^2 + \omega(A_1)}}{\|A_2\| \omega(A_1)}$$

and

$$\overline{r} < u = \frac{-\|A_1\| \omega(A_1) + \sqrt{\|A_1\|^2 (\omega(A_1))^2 + \omega(A_1)}}{\|A_2\| \omega(A_1)}.$$

□

Theorem 3. *If $A_1 \in S_N$, $A_2 \in M_N(\mathbb{C})$ and $r \in \mathcal{I}_{\mathcal{C}} = [\underline{r}, \overline{r}]$ then $\mathcal{C}(A_1, A_2) \subset S_N$, where $-l = u = -\frac{\|A_1\|}{\|A_2 - A_1\|} + \frac{1}{\|A_2 - A_1\|}\sqrt{\|A_1\|^2 + \frac{1}{\omega(A_1)}}$, $l < \underline{r} < \overline{r} < u$.*

Proof. If we write $A_2 - A_1$ instead of A_2 in Theorem 2, proof is clear from Theorem 2. □

Here, the equation expressed as a convex combination is shown with $A(r) = (1-r)A_1 + rA_2$ and the values r are examined in such a way that the convex sums of two matrices are Schur stable without the condition $r \in (0,1)$.

Let's examine the values r of the matrix families $\mathcal{L}(A_1, A_2)$ and $\mathcal{C}(A_1, A_2)$, which provide the Schur stability, by using the Schur stability of A_1. During this review, the articles of Duman and Aydın were taken into consideration [14, 15].

It is possible to write the convex combination as a special case of the linear sum. In other words, we can express the convex combination given as $A(r) = (1-r)A_1 + rA_2$ as a linear sum as $A(r) = A_1 + r(A_2 - A_1)$. In order to the matrix $A(r)$ to be Schur stable, let's determine the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ using the Schur stability of the matrix A_1.

Example 1. *For $\alpha \in (-1,1)$, $A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let's examine the interval $\mathcal{I}_{\mathcal{L}}$ which leaves the matrix families $\mathcal{L}(A_1, A_2)$ Schur stable.*

According to the Theorem 2, from $\|A_1\| = |\alpha|$, $\|A_2\| = 1$, $\omega(A_1) = \frac{1}{1-\alpha^2}$, we obtained as follows,

$$l = |\alpha| - 1 \ , \ u = -|\alpha| + 1.$$

$\omega(A(r)) = \frac{1}{1-(\alpha+r)^2}$ is known, then

$$\begin{cases} \alpha < 0 \ , \ \lim_{r \to |\alpha|-1} \left(\frac{1}{1-(\alpha+r)^2} \right) = \infty \\ \\ \alpha > 0 \ , \ \lim_{r \to -|\alpha|+1} \left(\frac{1}{1-(\alpha+r)^2} \right) = \infty \end{cases}$$

is obtained.

Example 2. *Let's examine the following matrices*

$$A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For these matrices we obtained $\|A_1\| = 0.5$, $\|A_2 - A_1\| = 0.5$, $\omega(A_1) = 1.33333$. *So we know that* A_1 *is Schur stable.*

TABLE 2. The effectiveness of the interval $\mathcal{I}_{\mathcal{C}}$

r	-0.9999	-0.99	-0.9	...	0.9	0.99	0.9999
$\omega(A(r))$	10000.3	100.251	10.2564	...	10.2564	100.251	10000.3

According to the Theorem 3, we obtained $l = -1$, $u = 1$. *As can be seen in the Table 2, the condition numbers change according to the values* r *selected from the intervals* $\mathcal{I}_{\mathcal{C}}$. *Also the quality of the stability decrease as the value* r *approaches* -1 *or* 1.

Remark 1. *In particular, if taken* $A_1 = A_2 = 0$, *we get the matrix family* $\mathcal{L}(0,0) = \{0\} \subset S_N$. *Lets take* $A_2 \neq 0$, *the matrix family* $\mathcal{L}(0, A_2)$, $r \in \mathcal{I}_{\mathcal{L}}$ *specified here, which is obtained in the form of* $-l = u = \frac{1}{\|A_2\|}$ *for* $\|A_1\| = 0$ *and* $\omega(A_1) = 1$. *If we call this interval obtained for the* r *value "perfect interval", we can say that the result obtained here is the "perfect state".*

3. ω^*-Schur Stability of the Matrix Families

Let ω^* be the practical Schur stability parameter, where $1 < \omega^* \in \mathbb{R}$ and the users choose the value ω^* in view of their problem. If $\omega(A) \leq \omega^*$ then the matrix A is ω^*-Schur stable matrix. Otherwise, the matrix A is ω^*-Schur unstable matrix [1,3,26].

Although there are theorems known as continuity theorems in the literature that determine the sensitivity of the problem, these theorems show under which conditions the given problems maintain the same property [1,9,11,14,15,17]. Let's give the continuity theorem which determines the sensitivity of the ω^*-Schur stability.

Theorem 4. *Let* A *be a* ω^*-*Schur stable matrix* ($\omega(A) \leq \omega^*$). *If the matrix* B *satisfies* $\|B\| \leq \sqrt{\|A\|^2 + \frac{\omega^* - \omega(A)}{\omega^* \omega(A)}} - \|A\|$, *then* $A + B$ *is* ω^*-*Schur stable [14].*

Let's define the family of ω^*-Schur stable matrices as follows;

$$S_N^* = \{A \in S_N \mid \omega(A) \leq \omega^*\}.$$

Now, considering Theorem 4, let's give the following two theorems.

Theorem 5. *If* $A_1 \in S_N^*$, $A_2 \in M_N(\mathbb{C})$ *and* $r \in \mathcal{I}_{\mathcal{L}}^* = [\underline{r}, \overline{r}]$ *then* $\mathcal{L}(A_1, A_2) \subset S_N^*$, *where* $-l^* = u^* = -\frac{\|A_1\|}{\|A_2\|} + \frac{1}{\|A_2\|}\sqrt{\|A_1\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_1)}}$, $l^* \leq \underline{r} < \overline{r} \leq u^*$.

Proof. If $A_2 = 0$ then $A(r) = A_1$. We know that $A_1 \in S_N^*$ so $A(r) \in S_N^*$ too. Lets take $A_2 \neq 0$. For $r \in \mathcal{I}_{\mathcal{L}}^*$ we can write $l^* \leq r \leq u^*$. Then we get following inequality,

$$r^2 \left\|A_2\right\|^2 \omega\left(A_1\right)\omega^* + 2\left|r\right| \left\|A_1\right\| \left\|A_2\right\| \omega\left(A_1\right)\omega^* - \omega^* + \omega\left(A_1\right) \leq 0.$$

If we arrange above inequality

$$\frac{\omega\left(A_1\right)}{1 - r^2\left\|A_2\right\|^2 \omega\left(A_1\right) - 2\left|r\right| \left\|A_1\right\| \left\|A_2\right\| \omega\left(A_1\right)} \leq \omega^*$$

holds. Since $\omega\left(A_1 + rA_2\right) \leq \frac{\omega(A_1)}{1 - r^2\|A_2\|^2\omega(A_1) - 2|r|\|A_1\|\|A_2\|\omega(A_1)}$ is valid from the Theorem 2, the inequality $\omega\left(A_1 + rA_2\right) \leq \omega^*$ is found. □

Theorem 6. *If $A_1 \in S_N^*,\ A_2 \in M_N\left(\mathbb{C}\right)$ and $r \in \mathcal{I}_{\mathcal{C}}^* = [\underline{r}, \overline{r}]$ then $\mathcal{C}\left(A_1, A_2\right) \subset S_N^*$, where $-l^* = u^* = -\frac{\|A_1\|}{\|A_2 - A_1\|} + \frac{1}{\|A_2 - A_1\|}\sqrt{\left\|A_1\right\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_1)}}$, $l^* \leq \underline{r} < \overline{r} \leq u^*$.*

Proof. It is obvious from the previous proof. □

Now let's give the following illustrative example on this subject.

Example 3. *For $\alpha \in \left(-\sqrt{1 - \frac{1}{\omega^*}}, \sqrt{1 - \frac{1}{\omega^*}}\right) \subset (-1, 1)$, $A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \in S_N^*$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let's examine the interval $\mathcal{I}_{\mathcal{L}}^*$ which leaves the matrix family $\mathcal{L}\left(A_1, A_2\right)$ is ω^*-Schur stable.*

According to the Theorem 5, we obtained,

$$l^* = |\alpha| - \sqrt{1 - \frac{1}{\omega^*}}\ ,\ u^* = -\left|\alpha\right| + \sqrt{1 - \frac{1}{\omega^*}}$$

from $\|A_1\| = |\alpha|$, $\|A_2\| = 1$, $\omega\left(A_1\right) = \frac{1}{1-\alpha^2}$. $\omega\left(A\left(r\right)\right) = \frac{1}{1-(\alpha+r)^2}$ is known, then

$$\begin{cases} \alpha < 0\ ,\ \lim_{r \to |\alpha| - \sqrt{1 - \frac{1}{\omega^*}}} \left(\frac{1}{1-(\alpha+r)^2}\right) = \omega^* \\ \\ \alpha > 0\ ,\ \lim_{r \to -|\alpha| + \sqrt{1 - \frac{1}{\omega^*}}} \left(\frac{1}{1-(\alpha+r)^2}\right) = \omega^* \end{cases}$$

is obtained.

Example 4.

$$A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\ ,\ A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For these matrices we obtained $\omega\left(A_1\right) = 1.33333$, $\|A_1\| = 0.5$, $\|A_2 - A_1\| = 0.5$. So we know that $A_1 \in S_N^*$. If we choose $\omega^* = 10$ then we get $-l = u = 0.897367$. Let's examine the interval $r \in \mathcal{I}_{\mathcal{C}}^*$ which leaves the matrix family $\mathcal{C}\left(A_1, A_2\right)$ is 10$-$Schur stable. According to the Theorem 6, as can be seen in the Table 3, sharp intervals are obtained for the specified $\omega^* = 10$ parameter. It is seen that $\omega^* < \omega(A(r))$ for the r value selected outside these intervals.

TABLE 3. Sharpness of the interval $\mathcal{I}_{\mathcal{C}}^{*}$ of 10−Schur stability

r	-0.897368	l	-0.897366	...	0.897366	u	0.897368
$\omega\left(A(r)\right)$	10.0003	10	9.99937	...	9.99937	10	10.0003

Remark 2. *From the above example, when values of Schur stability parameter $\omega\left(A\left(r\right)\right)$ are checked for the r values, it can be seen clearly that Theorem 2, Theorem 3, Theorem 5 and Theorem 6 gave sharp bounds.*

4. Obtaining the Extended Intervals

The intervals $\mathcal{I}_{\mathcal{L}}, \mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{L}}^{*}$ and $\mathcal{I}_{\mathcal{C}}^{*}$ are given in the Section 3. Although these intervals are found, actually it has been realized that big intervals which preserve Schur stability or $\omega^{*}-$ Schur stability of the matrix families $\mathcal{L}$ and $\mathcal{C}$ can be found. For this reason, the intervals are extended with certain rule in this section. Here, the extended intervals for the matrix families which preserve the Schur stability or $\omega^{*}-$ Schur stability are given. In addition, the extended intervals also allow us to introduce the Schur stable interval matrices or and $\omega^{*}-$ Schur stable interval matrices. To extend the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$, the methods which based on continuity theorems are given and the algorithms which based on the methods are given. Similarly, to extend the intervals $\mathcal{I}_{\mathcal{L}}^{*}$ and $\mathcal{I}_{\mathcal{C}}^{*}$, the methods and the algorithms are given. So it can be obtained bigger intervals which preserve the Schur stability or $\omega^{*}-$ Schur stability of the matrix families $\mathcal{L}$ and $\mathcal{C}$. In this process, the stepsize is determined from the continuity theorems which are Theorem 2, Theorem 3, Theorem 5 and Theorem 6. The extended intervals $\mathcal{I}_{\mathcal{L}}^{e}, \mathcal{I}_{\mathcal{C}}^{e}, \mathcal{I}_{\mathcal{L}}^{*e}$ and $\mathcal{I}_{\mathcal{C}}^{*e}$ are obtained at the end of processing. Let's give the methods and the algorithms as below.

4.1. A method and an algorithm to find the extended interval $\mathcal{I}_{\mathcal{L}}^{e}$.

4.1.1. *A method.* Keeping the Schur stability of the matrix family $\mathcal{L}\left(A_1, B\right)$, a method is given to extend the intervals with the Schur stable matrix A_1 and the matrix B. $\mathcal{I}_{\mathcal{L}} = [\underline{r}, \overline{r}]$ has been chosen with Theorem 2. For $r \in \mathcal{I}_{\mathcal{L}}$, the matrices $A\left(r\right) = A_r = A_1 + rB$ are Schur stable.

i) Defining the stepsize

The stepsize parameter r is used to extend the interval $\mathcal{I}_{\mathcal{L}}$. So, generalizing form the Theorem 2, it is chosen as $r_k \lessapprox -\frac{\|A_k\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_k\|^2 + \frac{1}{\omega(A_k)}}$.

ii) Determining the initial value

From the Theorem 2, the first value of the parameter r_1 is taken as $r_1 \lessapprox u$.

iii) Calculating the upper bound u^e

To extend the upper bound of the intervals $\mathcal{I}_{\mathcal{L}}$, the following steps are done,

$$A_k = A_{k-1} + r_{k-1}B \ , \ r_1 \lessapprox u \ , \ k \geq 2, \tag{5}$$

$$r_k \lessapprox -\frac{\|A_k\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_k\|^2 + \frac{1}{\omega(A_k)}}, \tag{6}$$

$$u_k = u_{k-1} + r_k \ , \ u_1 = r_1. \tag{7}$$

The new matrix A_k in the equality (5) is obtained as Schur stable. r_k in equality (6) is calculated with Theorem 2. u_k in the equality (7) is the upper bound of the extended interval obtained in step k. At the end of this process, the upper bound u^e of the extended interval $\mathcal{I}^e_{\mathcal{L}}$ is obtained.

iv) Calculating the lower bound l^e

Similar to the above application, to extend the lower bound of the intervals $\mathcal{I}_{\mathcal{L}}$, the matrix A_k is taken as $A_k = A_{k-1} - r_{k-1}B$ in the equality (5) and the equality (7) is replaced by the recurrence relation $l_k = l_{k-1} - r_k \ , l_1 = -r_1$. l_k is the lower bound of the extended interval obtained in step k. The result obtained with the new equations, the lower bound l^e of the extended interval $\mathcal{I}^e_{\mathcal{L}}$ is obtained.

Remark 3. *If the method is applied consecutively to get the upper bound, the stepsize r_k is become smaller and the parameter ω continues to grow by increasing. A similar situation is also observed for the lower bound. Because of these reasons, the working with very small numbers is non-practical.*

4.1.2. *An algorithm.* As given in the Remark 3, to stop the calculation, the stopping criterion is given as follow.

Stopping parameter r^*

After a certain step, the new stepsize becomes too small. Calculations with such values are not practical due to some reasons (i.e. floating point arithmetic)(see. [12,16]). r^* is called the practical parameter for the stepsize which chosen by user small enough [21,22]. With this criterion, less processing is needed and the given method run smoothly.

Let's give the algorithm to extend the upper bound of the intervals $\mathcal{I}_{\mathcal{L}}$.

Algorithm 1.1 (for the upper bound u^e)

(1) Input; $A \in S_N,\ B,\ r^*,\ \gamma \lessapprox 1$.
(2) Calculate $\omega(A),\ \|A\|,\ \|B\|$
$\beta = -\frac{\|A\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A\|^2 + \frac{1}{\omega(A)}},\ r_1 = \gamma.\beta.$
(3) Take $k = 1,\ A_1 := A,\ u_1 := r_1$.
(4) If $r_1 < r^*$ then write "The interval cannot be extended based on the available data." and go 7. step.
(5) Calculate;
$A_{k+1} = A_k + r_k B \ , \|A_{k+1}\| \ , \omega(A_{k+1}),$
$\beta_{k+1} = -\frac{\|A_{k+1}\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_{k+1}\|^2 + \frac{1}{\omega(A_{k+1})}},$
$r_{k+1} = \gamma.\beta_{k+1}.$
(6) If $r_{k+1} \geq r^*$ then calculate $u_{k+1} = u_k + r_{k+1}$, take $k := k+1$ and go 5. step.

(7) Write as $M := k$ and the upper bound of interval $u^e = u_M$.

To extend the lower bound of the intervals $\mathcal{I}_{\mathcal{L}}$, steps (5)-(7) in Algorithm 1.1 are taken as follow.

Algorithm 1.2 (for the lower bound l^e)

(5́) Calculate;
$A_{k+1} = A_k - r_k B$, $\|A_{k+1}\|$, $\omega(A_{k+1})$,
$r_{k+1} = \gamma.\beta_{k+1}$.

(6́) If $r_{k+1} \geq r^*$ then calculate $l_{k+1} = l_k - r_{k+1}$ ($l_1 := -r_1$), take $k := k+1$ and go 5́. step.

(7́) Write as $M := k$ and the lower bound of interval $l^e = l_M$.

Finally, the found values u^e and l^e are combined and these values constitute of the Schur stability interval $\mathcal{I}^e_{\mathcal{L}} = [l^e, u^e]$ of the matrix family $\mathcal{L}(A_1, B)$. Here, the interval $\mathcal{I}^e_{\mathcal{L}}$ preserves the Schur stability of the given matrix family.

Theorem 7 (Generalization of the Theorem 2)**.** *If $A_1 \in S_N$, $B \in M_N(\mathbb{C})$ and $r \in \mathcal{I}^e_{\mathcal{L}} = [l^e, u^e]$ then $\mathcal{L}(A_1, B) \subset S_N$, where u^e and l^e are defined as in Algorithm 1.1 and Algorithm 1.2, respectively.*

Proof. It is clear from the Theorem 2, Algorithm 1.1 and Algorithm 1.2. □

4.2. **A method and an algorithm to find the extended interval $\mathcal{I}^{*e}_{\mathcal{L}}$.**

4.2.1. *A method.* Keeping the ω^*-Schur stability of the matrix family $\mathcal{L}(A_1, B)$, a method is given to extend the intervals with the ω^*-Schur stable matrix A_1 and the matrix B. $\mathcal{I}^*_{\mathcal{L}} = [\underline{r}, \overline{r}]$ has been chosen with Theorem 5. For $r \in \mathcal{I}^*_{\mathcal{L}}$, the matrices $A(r) = A_r = A_1 + rB$ are ω^*-Schur stable. The stepsize chosen as $r_k = -\frac{\|A_k\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_k\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_k)}}$, the initial value taken as u^*. To extend the upper bound of the intervals $\mathcal{I}^*_{\mathcal{L}}$, the following steps are done,

$$A_k = A_{k-1} + r_{k-1}B \ , \ r_1 = u^* \ , \ k \geq 2 \tag{8}$$

$$r_k = -\frac{\|A_k\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_k\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_k)}}, \tag{9}$$

$$u_k = u_{k-1} + r_k \ , \ u_1 = r_1. \tag{10}$$

On the other hand, to extend the lower bound of the intervals $\mathcal{I}^*_{\mathcal{L}}$, the matrix A_k is taken as $A_k = A_{k-1} - r_{k-1}B$ in the equality (8) and the equality (10) is replaced by the recurrence relation $l_k = l_{k-1} - r_k$, $l_1 = -r_1$. At the end of this process, the upper bound u^{*e} and lower bound l^{*e} of the extended interval $\mathcal{I}^{*e}_{\mathcal{L}}$.

Remark 4. *Let's take A_1 and B.*

$$A_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix} \ , \ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

From the Theorem 5, it is known that $u = 0.748683$ for $\omega^* = 10$. If the method is applied consecutively to get the upper bound, the stepsize is become smaller and the *parameter ω approaches to 10 as in Table 4. A similar situation is also observed for the lower bound. Because of these reasons, the working with very small numbers is non-practical. For this reason, as in Remark 3 for the Algorithm 1.1 and Algorithm 1.2, the stopping parameter r^* is needed for the algorithm to stop.*

TABLE 4. The values r and $\omega(A_k)$ corresponding to the number of steps k

k	*1*	*50*	*100*	*200*	*300*	*380*
r	*0.748683*	*0.00817741*	*0.00310605*	*0.00079299*	*0.000245453*	*9.98825e-005*
$\omega(A_k)$	*1.66462*	*7.23792*	*8.61111*	*9.57978*	*9.86377*	*9.94385*

4.2.2. *An algorithm.* As given in the Remark 4, to stop the calculation, the stopping criterion r^* is used as follow.

Let's give the algorithm to extend the upper bound of the intervals $\mathcal{I}_{\mathcal{L}}^*$.

Algorithm 2.1 (for the upper bound u^{*e})

(1) Input; $A \in S_N$, B, ω^*, r^*.
(2) Calculate $\omega(A)$.
(3) If $\omega(A) > \omega^*$ then "The matrix A is not ω^*-Schur stable" and finish the algorithm.
(4) Calculate $\|A\|$, $\|B\|$, $u^* = -\frac{\|A\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A)}}$.
(5) Take $k = 1$, $A_1 := A$, $r_1 := u^*$, $u_1 := r_1$.
(6) Calculate;
$$A_{k+1} = A_k + r_k B \ , \ \|A_{k+1}\| \ , \ \omega(A_{k+1}),$$
$$r_{k+1} = -\frac{\|A_{k+1}\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_{k+1}\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_{k+1})}}.$$
(7) If $r_{k+1} \geq r^*$ then calculate $u_{k+1} = u_k + r_{k+1}$, take $k := k+1$ and go 6. step.
(8) Write as $M := k$ and the upper bound of interval $u^{*e} = u_M$.

To extend the lower bound of the intervals $\mathcal{I}_{\mathcal{L}}^*$, steps (6)-(8) in Algorithm 2.1 are taken as follow.

Algorithm 2.2 (for the lower bound l^{*e})

(6́) Calculate;
$$A_{k+1} = A_k - r_k B \ , \ \|A_{k+1}\| \ , \ \omega(A_{k+1}),$$
$$r_{k+1} = -\frac{\|A_{k+1}\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_{k+1}\|^2 - \frac{1}{\omega^*} + \frac{1}{\omega(A_{k+1})}}.$$
(7́) If $r_{k+1} \geq r^*$ then calculate $l_{k+1} = l_k - r_{k+1}$ ($l_1 := -r_1$), take $k := k+1$ and go 6́. step.
(8́) Write as $M := k$ and the lower bound of interval $l^{*e} = l_M$.

Finally, the found values u^{*e} and l^{*e} are combined and these values constitute of the ω^*-Schur stability interval $\mathcal{I}_{\mathcal{L}}^{*e} = [l^{*e}, u^{*e}]$ of the matrix family $\mathcal{L}(A_1, B)$. Here, the interval $\mathcal{I}_{\mathcal{L}}^{*e}$ preserves the ω^*-Schur stability of the given matrix family.

Theorem 8 (Generalization of the Theorem 5)**.** *If $A_1 \in S_N^*$, $B \in M_N(\mathbb{C})$ and $r \in \mathcal{I}_{\mathcal{L}}^{*e} = [l^{*e}, u^{*e}]$ then $\mathcal{L}(A_1, B) \subset S_N^*$, where u^{*e} and l^{*e} are defined as in Algorithm 2.1 and Algorithm 2.2, respectively.*

Proof. It is clear from the Theorem 5, Algorithm 2.1 and Algorithm 2.2. □

Example 5. *Let us consider the matrices A_1 and B as follow,*

$$A_1^1 = \begin{pmatrix} -0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \; A_1^2 = \begin{pmatrix} 0.2 & 1 \\ 0 & 0.1 \end{pmatrix},$$
$$B_1 = E_{11} + E_{22}, \; B_2 = E_{12}, \; B_3 = E_{11} + E_{12} + E_{22}$$

Here E_{ij} is a real matrix which the element in position (i,j) equals 1 and all other elements are 0.

Let's examine the Table 5 (Table 6). The matrices A_1, B and the parameters r^* and ω^* are the input elements, where r^* and ω^* selected by the users. l (l^*) are the lower bounds and u (u^*) are the upper bounds of the interval $\mathcal{I}_{\mathcal{L}}$ ($\mathcal{I}_{\mathcal{L}}^*$) which is calculated with the help of Theorem 2 (Theorem 5). l^e (l^{*e}) are the lower bounds and u^e (u^{*e}) are the upper bounds of the interval $\mathcal{I}_{\mathcal{L}}^e$ ($\mathcal{I}_{\mathcal{L}}^{*e}$) which is the extended interval obtained by the Algorithm 1.1 (Algorithm 2.1) and Algorithm 1.2 (Algorithm 2.2). M indicates how many steps the algorithms stopped.

Table 5. The computed values for the data A_1, B, r^*

A_1	B	γ	r^*	$\overline{r} = \gamma.u$	u^e	M
A_1^1	B_1	0.9	0.01	0.81	0.891	2
			0.001		0.8991	3
		0.95	0.01	0.855	0.89775	2
			0.001		0.899888	3
A_1^2	B_1	0.9	0.1	0.185918	0.320172	2
			0.01		0.638475	12
		0.95	0.1	0.196247	0.334315	2
			0.01		0.635644	11
A_1^2	B_2	0.9	0.1	0.185918	0.429824	3
			0.01		2.35532	88
		0.95	0.1	0.196247	0.448457	3
			0.01		2.42436	90
A_1^2	B_3	0.9	0.01	0.114904	0.486169	14
			0.001		0.667356	77
		0.95	0.01	0.121287	0.494272	14
			0.001		0.66961	75

(a) The computed values $\overline{r}$ and u^e

A_1	B	γ	r^*	$\underline{r} = -\gamma.u$	l^e	M
A_1^1	B_1	0.9	0.01	-0.81	-0.891	2
			0.001		-0.8991	3
		0.95	0.01	-0.855	-0.89775	2
			0.001		-0.899888	3
A_1^2	B_1	0.9	0.1	-0.185918	-0.663416	4
			0.01		-0.932985	13
		0.95	0.1	-0.196247	-0.6847	4
			0.01		-0.940808	13
A_1^2	B_2	0.9	0.1	-0.185918	-2.40175	8
			0.01		-4.36254	94
		0.95	0.1	-0.196247	-2.45135	8
			0.01		-4.42464	95
A_1^2	B_3	0.9	0.01	-0.114904	-1.08201	9
			0.001		-1.09759	14
		0.95	0.01	-0.121287	-1.08676	9
			0.001		-1.0975	13

(b) The computed values $\underline{r}$ and l^e

For example, according to Table 5a (Algorithm 1.1) and Table 5b (Algorithm 1.2), the initial value is obtained as $u = -l = 0.206575$ for the matrices A_1^2, B_2.

- *For $\gamma = 0.9$,*
  - *The extended upper bound is obtained as $u^e = 0.320172$ in 2 steps for $r^* = 0.1$ and $u^e = 0.638475$ in 12 steps for $r^* = 0.01$.*
  - *The extended lower bound is obtained as $l^e = -0.663416$ in 4 steps for $r^* = 0.1$ and $l^e = -0.932985$ in 13 steps for $r^* = 0.01$.*

The extended interval $\mathcal{I}^e_{\mathcal{L}} = [l^e, u^e] = [l_{13}, u_{12}] = [-0.932985, 0.638475]$ *is obtained from the Table 5 for the matrices* A_1^2, B_2 *and the parameter* $r^* = 0.01$ *and* $\gamma = 0.9$.

- *For* $\gamma = 0.95$,
  - *The extended upper bound is obtained as* $u^e = 0.334315$ *in* 2 *steps for* $r^* = 0.1$ *and* $u^e = 0.635644$ *in* 11 *steps for* $r^* = 0.01$.
  - *The extended lower bound is obtained as* $l^e = -0.6847$ *in* 4 *steps for* $r^* = 0.1$ *and* $l^e = -0.940808$ *in* 13 *steps for* $r^* = 0.01$.

The extended interval $\mathcal{I}^e_{\mathcal{L}} = [l^e, u^e] = [l_{13}, u_{11}] = [-0.940808, 0.635644]$ *is obtained from the Table 5 for the matrices* A_1^2, B_2 *and the parameter* $r^* = 0.01$ *and* $\gamma = 0.95$.

TABLE 6. The computed values for the data A_1, B, ω^*, r^*

A_1	B	ω^*	r^*	u^*	u^{*e}	M
A_1^1	B_1	10	0.01	0.848683	0.848683	-
		100	0.01	0.894987	0.894987	-
A_1^2	B_1	10	0.1	0.165268	0.281339	1
			0.01		0.476099	7
		100	0.1	0.202507	0.341518	1
			0.01		0.615616	9
A_1^2	B_2	10	0.05	0.165268	0.627035	6
			0.005		1.51639	69
		100	0.1	0.202507	0.457834	2
			0.01		2.3151	82
A_1^2	B_3	10	0.01	0.102141	0.344321	8
			0.001		0.403404	24
		100	0.01	0.125156	0.479542	12
			0.001		0.612721	54

(a) The computed values u^* and u^{*e}

A_1	B	ω^*	r^*	l^*	l^{*e}	M
A_1^1	B_1	10	0.01	-0.848683	-0.848683	-
		100	0.01	-0.894987	-0.894987	-
A_1^2	B_1	10	0.1	-0.165268	-0.604153	3
			0.01		-0.769007	8
		100	0.1	-0.202507	-0.694796	3
			0.01		-0.921211	11
A_1^2	B_2	10	0.05	-0.165268	-2.60586	11
			0.005		-3.51925	75
		100	0.1	-0.202507	-2.48489	7
			0.01		-4.31782	87
A_1^2	B_3	10	0.01	-0.102141	-1.03721	8
			0.001		-1.04426	10
		100	0.01	-0.125156	-1.07752	7
			0.001		-1.08955	10

(b) The computed values l^* and l^{*e}

On the other hand, according to Table 6a (Algorithm 2.1) and Table 6b (Algorithm 2.2), if the parameter ω^* *is chosen as* 10, *the initial value is obtained as* $u^* = -l^* = 0.165268$ *for the matrices* A_1^2, B_2. *If the stopping parameter* r^* *is chosen as* $r^* = 0.05$ $(r^* = 0.005)$,

- *the extended upper bound is obtained as* $u^{*e} = 0.627035$ $(u^{*e} = 1.51639)$ *in 6 (69) steps.*
- *the extended lower bound is obtained as* $l^{*e} = -2.60586$ $(l^{*e} = -3.51925)$ *in 11 (75) steps.*

The extended interval $\mathcal{I}^{*e}_{\mathcal{L}} = [l^{*e}, u^{*e}] = [-2.60586, 0.627035]$ *is obtained from the Table 6 for the matrices* A_1^2, B_2 *and the parameters* $\omega^* = 10$, $r^* = 0.05$.

According to the Table 5 and the Table 6, let's give the following;

- *The interval* $\mathcal{I}^e_{\mathcal{L}}$ *is bigger than the interval* $\mathcal{I}^{*e}_{\mathcal{L}}$ *with same condition.*
- *The number of steps increases while the stopping parameter decreases.*
- *If the matrices* A_1 *and* B *are taken diagonal, the extended intervals are obtained by the theorems.*
- *If the parameter* ω^* *is chosen bigger, the extended interval* $\mathcal{I}^{*e}_{\mathcal{L}}$ *is obtained bigger in the same conditions.*

4.3. Methods and algorithms to find the extended interval $\mathcal{I}_{\mathcal{C}}^{e}$ and $\mathcal{I}_{\mathcal{C}}^{*e}$.

The methods and the algorithms can be given to extend the intervals $\mathcal{I}_{\mathcal{C}}$ and $\mathcal{I}_{\mathcal{C}}^{*}$ as similar to the methods and algorithms to extend the intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{L}}^{*}$ in Section 4. So, in this paper, the methods and the algorithms to find the intervals $\mathcal{I}_{\mathcal{C}}^{e}$ and $\mathcal{I}_{\mathcal{C}}^{*e}$ won't be given to avoid repeat.

5. Conclusion

In this study, the matrix families $\mathcal{L}$ and $\mathcal{C}$ based on linear sum and convex combination were constructed, respectively. This construction is a new approach that preserves the Schur stability of the matrix families. The intervals $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{I}_{\mathcal{C}}$ that make these matrix families Schur stable were determined in the Theorem 2 and Theorem 3 and supported by the illustrative examples. Here it is seen that the sharp results are obtained from the Theorem 2 and Theorem 3, especially in the Example 1 and Example 2, for the matrix families $\mathcal{L}$ and $\mathcal{C}$. Similarly, the intervals $\mathcal{I}_{\mathcal{L}}^{*}$ and $\mathcal{I}_{\mathcal{C}}^{*}$ that provide $\omega^{*}-$Schur stability of the matrix families $\mathcal{L}$ and $\mathcal{C}$ are determined in the Theorem 5 and Theorem 6 and supported with the numerical examples. It is seen that the Theorem 5 and Theorem 6 give sharp results in the Example 3 and Example 4. At the end, the methods and the algorithms are given to extended the intervals $\mathcal{I}_{\mathcal{L}}, \mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{L}}^{*}$ and $\mathcal{I}_{\mathcal{C}}^{*}$. Here, the methods are based on continuity theorems and the algorithms based on the methods. With the help of these theorems, the obtained intervals are extended and the results are presented with the numerical example.

On the other hand, unlike other studies in the literature, this study shows the importance of continuity theorems which guarantee Schur stability. With the help of these theorems, the matrix families are extended in such a way that their Schur stability is preserved. Also, in many studies, the matrices A_1 and B were taken as Schur stable but in this study there is no need for the matrix B to be Schur stable or ω^{*}-Schur stable.

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MINIMAL TRANSLATION SURFACES IN A STRICT WALKER 3-MANIFOLD

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Abstract. In this paper, we study minimal translations surfaces in a strict Walker 3-manifold. Based on the existence of two isometries, we classify minimal translation surfaces on this class of manifold. Some drawings have been added to illustrate the shape of certain surfaces.

1. Introduction

Minimal surfaces are the most natural objects in differential geometry, and have been studied during the last two and half centuries since J. L. Lagrange. In particular, minimal surfaces have encountered striking applications in other fields, like mathematical physics, conformal geometry, computer aided design, among others. In order to search for more minimal surfaces, some natural geometric assumptions arise. Translation surfaces were studied in the Euclidean 3-dimensional space and they are represented as graphs $z = \alpha(x)+\beta(y)$, where α and β are smooth functions. Scherk [11] proved in 1835 that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right|,$$

where a is a non-zero constant. Since then, minimal translation surfaces were generalized in several directions. For example, the Euclidean space $\mathbb{R}^3$ was replaced with other spaces of dimension 3-usually being 3-dimensional Lie groups and the

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notion of translation was often replaced by using the group operation (see for example [6], [8], [14] and references therein). Another generalizations of Scherk surfaces are: affine translation surfaces in Euclidean 3-space [7], affine translation surfaces in affine 3-dimensional space [12] and translation surfaces in Galilean 3-space [14]. On the other hand, Scherk surfaces were generalized to minimal translation surfaces in Euclidean spaces of arbitrary dimensions(see [5], [9]). In [13], the authors introduce and define the notion of translation surfaces in the Heisenberg group $\mathbb{H}(1;1)$ as the formal analogue to those in the Euclidean 3-space.

In this paper, we define and classify minimal translation surfaces in a 3-dimensional strict Walker manifold. The strict Walker manifolds are described in terms of a suitable coordinates (x, y, z) of the manifolds $\mathbb{R}^3$ and their metric depends on an arbitrary function of two variables $f = f(y, z)$ and their metric tensor is given by

$$g_f^\epsilon = \epsilon dy^2 + 2dxdz + fdz^2 \tag{1}$$

where $\epsilon = \pm 1$. These manifolds are denoted by (M, g_f^ϵ). In [4], the authors study a class of minimal surfaces in the three-dimensional Lorentzian Walker manifolds. Their proved the existence of minimal flat and non totally geodesic graphs on three dimensional Lorentizain Walker manifolds. In [2], the authors have found that the strict Walker manifold (M, g_f^ϵ) where f depends only on the variable y are very important. Here we will work with the manifold (M, g_f^ϵ) where f depends only on y and f is not locally a constant.

Three dimensional geometry plays a central role in the investigation of many problem in Riemannian and Lorentzian geometry. The fact that Ricci operator completly determines the curvature tensor is crucial to these investigations, (for detail see [1]).

The paper is organised as follow: in section 2, we recall some preliminaries results for three-dimensional Walker manifold (M, g_f^ϵ) and we give some basic formula for immersed surface in (M, g_f^ϵ). We consider two families of translation surfaces in (M, g_f^ϵ) which are used in the main result. In the last section we classify those which are minimal.

2. Preliminaries

A Walker n-manifold is a pseudo-Riemannian manifold, which admits a field of null parallel r-planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker [15]. Walker has derived adapted coordinates to a parallel plan field. Hence, the metric of a three-dimensional Walker manifold (M, g_f^ϵ) with coordinates (x, y, z) is expressed as

$$g_f^\epsilon = dx \circ dz + \epsilon dy^2 + f(x, y, z)dz^2 \tag{2}$$

and its matrix form as

$$g_f^\epsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & f \end{pmatrix} \quad \text{with inverse} \quad (g_f^\epsilon)^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some function $f(x,y,z)$, where $\epsilon = \pm 1$ and thus $\mathcal{D} = \mathrm{Span}\{\partial_x\}$ as the parallel degenerate line field. Notice that, when $\epsilon = 1$ and $\epsilon = -1$ the Walker manifold has signature $(2,1)$ and $(1,2)$ respectively, and therefore is Lorentzian in both cases. Hence, if (M, g_f^ϵ) is a strict Walker manifolds i.e., $f(x,y,z) = f(y,z)$, then the associated Levi-Civita connection satisfies

$$\nabla_{\partial_y}\partial z = \frac{1}{2} f_y \partial_x, \quad \nabla_{\partial_z}\partial z = \frac{1}{2} f_z \partial_x - \frac{\epsilon}{2} f_y \partial_y. \tag{3}$$

Let now u and v be two vectors in M. Denoted by (e_1, e_2, e_3) the canonical frame in $\mathbb{R}^3$. The vector product of u and v in (M, g_f^ϵ) with respect to the metric g_f^ϵ is the vector denoted by $u \times v$ in M defined by

$$g_f^\epsilon(u \times v, w) = \det(u, v, w) \tag{4}$$

for all vector w in M, where $\det(u,v,w)$ is the determinant function associated to the canonical basis of $\mathbb{R}^3$. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then by using (4), we have:

$$u \times v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) e_1 - \epsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} e_3$$

Let D be an open subset of the plane $\mathbb{R}^2$ satisfying this interval condition: horizontal or vertical lines intersect D in intervals (if at all). A *two-parameter map* is a smooth map $\varphi : D \to M$. Thus φ is composed of two interwoven families of *parameter curves*:

(1) the u-parameter curves $v = v_0$ of φ is $u \mapsto \varphi(u, v_0)$.
(2) the v-parameter curves $u = u_0$ of φ is $v \mapsto \varphi(u_0, v)$.

The partial velocities $\varphi_u = d\varphi(\partial_u)$ and $\varphi_v = d\varphi(\partial_v)$ are vector fields on φ. Evidently $\varphi_u(u_0, v_0)$ is the velocity vector at u_0 of the u-parameter curve $v = v_0$, and symmetrically for $\varphi_v(u_0, v_0)$. If φ lies in the domain of a coordinate system $x^1, \ldots, x^n$, then its coordinate functions $x_i \circ \varphi$ $(1 \le i \le n)$ are real-valued functions on D and

$$\varphi_u = \sum \frac{\partial x^i}{\partial u} \partial_i, \quad \varphi_v = \sum \frac{\partial x^i}{\partial v} \partial_i.$$

So far M could be a smooth manifold: now suppose it is pseudo-Riemannian. If Z is a smooth vector field on φ, its partial covariant derivatives are: $Z_u = \frac{\nabla Z}{\partial u}$, the covariant derivative of Z along u-parameter curves, and $Z_v = \frac{\nabla Z}{\partial v}$, the covariant derivative of Z along v-parameter curves. Explicitly, $Z_u(u_0, v_0)$ is the covariant derivative at u_0 of the vector field $u \mapsto Z(u, v_0)$ on the curve $u \mapsto \varphi(u, v_0)$. In terms of coordinates, $Z = \sum Z^i \partial_i$, where each $Z^i = Z(x^i)$ is a real valued function on D. Then

$$Z_u = \sum_k \Big\{ \frac{\partial Z^k}{\partial u} + \sum_{i,j} \Gamma^k_{ij} Z^i \frac{\partial x^j}{\partial u} \Big\} \partial_k. \tag{5}$$

In the special case $Z = \varphi_u$, the derivative $Z_u = \varphi_{uu}$ gives the accelerations of u-parameter curves, while φ_{vv} gives v-parameter accelerations. With coordinate notation as above, we have:

$$\varphi_{uv} = \sum_k \Big\{ \frac{\partial^2 x^k}{\partial v \partial u} + \sum_{i,j} \Gamma^k_{ij} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \Big\} \partial_k. \tag{6}$$

Now we will assume that φ is an isometric immersion. The first fondamental form of the immersion φ is given by

$$\begin{cases} E = g_f\left(\varphi_*(\partial_u), \varphi_*(\partial_u)\right) \\ F = g_f\left(\varphi_*(\partial_u), \varphi_*(\partial_v)\right) \\ G = g_f\left(\varphi_*(\partial_v), \varphi_*(\partial_v)\right). \end{cases} \tag{7}$$

The coefficients of the second fundamental form of φ are

$$\begin{cases} L &= \varepsilon_1 g_f\left(\varphi_{uu}, \xi\right) \\ M &= \varepsilon_1 g_f\left(\varphi_{uv}, \xi\right) \\ N &= \varepsilon_1 g_f\left(\varphi_{vv}, \xi\right) \end{cases} \tag{8}$$

where $\varepsilon_1 = g^{\epsilon}_f(\xi, \xi)$ the sign of the unit normal ξ along φ.
The mean curvature of φ is given by

$$H = \varepsilon_1 \frac{1}{2} \Big(\frac{LG - 2MF + NE}{EG - F^2} \Big). \tag{9}$$

The idea of translation surface have its origine in the classical text of G. Darboux [3] where the so-called "*surfaces définies par des propriétés cinématiques*" is introduced. A Darboux surface of translation is defined kinematically as the movement of a curve by a uniparameter family of rigid motion of $\mathbb{R}^3$. Hence, such a surface in locally described by $\varphi(s,t) = A(t).\alpha(s) + \beta(t)$ where α and β are parametrized curves in $\mathbb{R}^3$ and $A(t)$ is an orthogonal transformation. $A(t)$ being identity is the case which is most investigated. So a surface S in $\mathbb{R}^3$ is called a translation surface if S can be locally discribed as a sum

$$\varphi(s,t) = \alpha(s) + \beta(t).$$

Next, we consider a three-dimensional strict Walker manifold (M, g^{ε}_f), where f is not locally a constant and depends only on the variable y. For any real number a, the following two maps:

$$\begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (x,y,z+a) \end{array}$$

and

$$\begin{array}{rcl}\mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x,y,z) & \mapsto & (x+a,y,z)\end{array}$$

are isometries of (M, g_f^{ε}). Based in these isometries, we will define two types of translation surfaces.

Definition 1. *A non-degenerate surface S of sign ε_1 in (M, g_f^{ε}) is a translation surface if it can be described locally by an isometric immersion $\varphi : U \subset \mathbb{R}^2 \to (M, g_f^{\epsilon})$ of the form*

$$\varphi(u,v) = (u, v, \alpha(u) + \beta(v)), \qquad \textit{Type I} \tag{10}$$

or

$$\varphi(u,v) = (\alpha(u) + \beta(v), u, v), \qquad \textit{Type II} \tag{11}$$

where α and β are smooth functions on opens of $\mathbb{R}$.

The aim of this work is to classify the minimal translation surfaces in (M, g_f^{ε}) of the Type I and type II as above.

3. Main Results

3.1. **Minimal translation surfaces of Type I.** Let us consider a translation surface of Type I in (M, g_f^{ϵ}) parametrized by $\varphi(u,v) = (u, v, \alpha(u) + \beta(v))$. In this case we have $x = u$, $y = v$ and $z = \alpha(u) + \beta(v)$. For a function g of one variable u (respectively v) we denote $\frac{dg}{du}$ by $\dot{g}$ (respectively $\frac{dg}{dv}$ by g'). The tangent plane of S is spanned by

$$\varphi_u = \partial_x + \dot{\alpha}\partial_z \quad \text{and} \quad \varphi_v = \partial_y + \beta'\partial_z. \tag{12}$$

The unit normal ξ (up to orientation) is given by

$$\xi = \frac{1}{\Delta}\left[(1 + \dot{\alpha}f)\partial_x - \varepsilon\beta'\partial_y - \dot{\alpha}\partial_z\right]. \tag{13}$$

where $\Delta = \|\varphi_u \times \varphi_v\|$. We obtain the coefficients of the first fundamental form of φ as

$$E = 2\dot{\alpha} + \dot{\alpha}^2 f, \quad F = \beta' + \dot{\alpha}\beta' f, \quad G = \varepsilon + \beta'^2 f. \tag{14}$$

And using (6) we have

$$\varphi_{uu} = \begin{pmatrix} 0 \\ -\frac{\varepsilon}{2}\dot{\alpha}^2 f_y \\ \ddot{\alpha} \end{pmatrix}, \quad \varphi_{uv} = \begin{pmatrix} \frac{1}{2}\dot{\alpha} f_y \\ -\frac{\varepsilon}{2}\dot{\alpha}\beta' f_y \\ 0 \end{pmatrix}, \quad \varphi_{vv} = \begin{pmatrix} \beta' f_y \\ -\frac{\varepsilon}{2}\beta'^2 f_y \\ \beta'' \end{pmatrix}. \tag{15}$$

Then the coefficients of the second fundamental form of φ

$$L = \frac{\varepsilon_1}{\Delta}\left[\frac{\varepsilon}{2}\beta'\dot{\alpha}^2 f_y + \ddot{\alpha}\right],$$

$$\begin{aligned} M &= \frac{\varepsilon_1}{\Delta}\left[-\frac{1}{2}\dot{\alpha}^2 f_y + \frac{\varepsilon}{2}\dot{\alpha}\beta'^2 f_y\right], \\ N &= \frac{\varepsilon_1}{\Delta}\left[-\dot{\alpha}\beta' f_y + \frac{\varepsilon}{2}\beta'^3 f_y + \beta''\right]. \end{aligned} \tag{16}$$

Consequently, the minimality condition (9) may be expressed as follows:

$$\ddot{\alpha}(\varepsilon + \beta'^2 f) + \dot{\alpha}^2(-\frac{1}{2}\beta' f_y + f\beta'') + 2\dot{\alpha}\beta'' = 0. \tag{17}$$

Since $y = v$, we can rewrite the minimal condition for Type I in the form

$$\ddot{\alpha}(\varepsilon + \beta'^2 f) + \dot{\alpha}^2(-\frac{1}{2}\beta' f' + f\beta'') + 2\dot{\alpha}\beta'' = 0. \tag{18}$$

We have the following solutions:

(1) **Case 1:** Assume that $\dot{\alpha} = 0$ that is $\alpha = \alpha_0$ (constant). We get the following surface:

$$(s_1): \quad \varphi(u,v) = (u, v, \alpha_0 + \beta(v))$$

for any smooth functions β.

(2) **Case 2:** Assume that $\dot{\alpha} \neq 0$ and $\ddot{\alpha} = 0$. Equation (18) becomes

$$\frac{\ddot{\alpha}}{\dot{\alpha}}(\varepsilon + \beta'^2 f) + \dot{\alpha}(-\frac{1}{2}\beta' f' + f\beta'') + 2\beta'' = 0. \tag{19}$$

Since $\ddot{\alpha} = 0$, from (19) we have:

$$\begin{cases} \alpha(u) = au + b \quad \text{with} \quad a \in \mathbb{R}^*,\ b \in \mathbb{R} \\ (af + 2)\beta'' = \frac{1}{2}af'\beta'. \end{cases} \tag{20}$$

(a) If $\beta' = 0$, then $\beta = \beta_0$ is a constant with $\alpha(u) = au + b$, $a \in \mathbb{R}^*$ satisfy (19) as (18). Thus we have the plan:

$$(s_2): \quad \varphi(u,v) = (u, v, au + \tilde{b}), \quad a \in \mathbb{R}^*,\ \tilde{b} \in \mathbb{R}$$

(b) Now assume $\beta' \neq 0$. An easy integration of the second equation in (20) gives

$$\beta(v) = \tilde{c}\int_{v_*}^{v} \sqrt{|2 + af|}\,dv,$$

where $\tilde{c} \in \mathbb{R}^*$, v_* is a real number such that v and v_* belong to interval on which $(2 + af > 0)$ or $(2 + af < 0)$. So we get the solution

$$(s_3): \quad \varphi(u,v) = \left(u, v, au + b + \tilde{c}\int_{v_*}^{v} \sqrt{|2 + af|}\,dv\right), \quad a, \tilde{c} \in \mathbb{R}^*,\ b \in \mathbb{R}.$$

(3) **Case 3:** Assume that $\dot{\alpha} \neq 0$ and $\ddot{\alpha} \neq 0$. Then equation (18) can be written as (19) anywhere where $\dot{\alpha} \neq 0$. By differentiating the equation (19) with respect to u and v, we get:

$$\frac{d}{du}\left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right)(\varepsilon + \beta'^2 f)' + \ddot{\alpha}(-\frac{1}{2}\beta' f' + f\beta'')' = 0. \tag{21}$$

(a) **Case 3-1:** $(\varepsilon+\beta'^2 f)'=0$. Since $\ddot{\alpha}\neq 0$, the equation (21) gives $(-\frac{1}{2}\beta' f'+f\beta'')'=0$. So we get

$$\begin{cases} \varepsilon+\beta'^2 f &= c_1 \\ -\frac{1}{2}f'\beta'+f\beta'' &= c_2, \end{cases} \tag{22}$$

where $c_1, c_2\in\mathbb{R}$. Then the equation(19) becomes

$$\left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right)c_1+\dot{\alpha}c_2=-2\beta''. \tag{23}$$

Since the left member depends only on u and the right member depends only on v, then there exist a constant c_3 and we have:

$$\begin{cases} \beta' &= -\frac{1}{2}c_3 v+c_4 \\ \left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right)c_1+\dot{\alpha}c_2 &= c_3, \end{cases} \tag{24}$$

where $c_3, c_4\in\mathbb{R}$. If $c_3=0$, then $\beta''=0$ and $\beta'=c_4$. From (22), one gets $\varepsilon+c_4^2 f=c_1$. Then $c_4^2 f'=0$ and $c_4=0$ by the hypothesis on f. So $\beta'=0$ implies $c_2=0$ and $c_1=\varepsilon$. Using this with (24) we get $\ddot{\alpha}=0$ (contradiction with the hypothesis). So $c_3\neq 0$. And then $\beta'\neq 0$ and $\beta''=-\frac{1}{2}c_3\neq 0$. Then (22) becomes

$$\begin{cases} f &= \frac{c_1-\varepsilon}{(-\frac{1}{2}c_3 v+c_4)^2} \\ -\frac{1}{2}f'\beta'+f\beta'' &= c_2. \end{cases} \tag{25}$$

So we get $f'=\frac{c_3(c_1-\varepsilon)}{(-\frac{1}{2}c_3 v+c_4)^3}$. Thus (25) gives $\frac{c_3(c_1-\varepsilon)}{(-\frac{1}{2}c_3 v+c_4)^2}=c_2$, and then we must have $c_2=0$ and $c_1=\varepsilon$. Then we get $f=0$ (a contradiction). So the sub-case $(\varepsilon+\beta'^2 f)'=0$ is not possible.

(b) **Case 3-2:** $(\varepsilon+\beta'^2 f)'\neq 0$. The equation (21) becomes

$$\frac{\frac{d}{du}\left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right)}{\ddot{\alpha}}=-\frac{(-\frac{1}{2}\beta' f'+f\beta'')'}{(\varepsilon+\beta'^2 f)'}. \tag{26}$$

Since the left member depends only on u and the right member depends only on v, its must be constant c. So we get $\frac{d}{du}\left(\frac{\ddot{\alpha}}{\dot{\alpha}}\right)=c\ddot{\alpha}$ and $(-\frac{1}{2}\beta' f'+f\beta'')'=-c(\varepsilon+\beta'^2 f)'$. Then, there exist constants $c_1, c_2\in\mathbb{R}$ such that

$$\frac{\ddot{\alpha}}{\dot{\alpha}}=c\dot{\alpha}+c_1 \quad\text{and}\quad (-\frac{1}{2}\beta' f'+f\beta'')=-c(\varepsilon+\beta'^2 f)+c_2. \tag{27}$$

If we put the equations (27) in (19), we get

$$c_1(\varepsilon+\beta'^2 f)+\dot{\alpha}c_2+2\beta''=0.$$

If we differentiate with respect to u, we obtain $\ddot{\alpha}c_2=0$ i.e., $c_2=0$. So we get:

$$\begin{cases} c_1(\varepsilon+\beta'^2 f)+2\beta'' &= 0 \\ -c(\varepsilon+\beta'^2 f) &= -\frac{1}{2}\beta' f'+f\beta'' \\ \frac{\ddot{\alpha}}{\dot{\alpha}} &= c\dot{\alpha}+c_1 \end{cases} \tag{28}$$

And now we have two possibilities: $c_1 = 0$ or $c_1 \neq 0$.

- **Case 3-2-1:** $c_1 = 0$. We have $c \neq 0$ otherwise $\ddot{\alpha} = 0$. The first equation in (28) gives $\beta'' = 0$, so $\beta' = \beta'_0 \in \mathbb{R}$. And we get

$$-c(\varepsilon + \beta'^2 f) = -\frac{1}{2} f' \beta'_0. \tag{29}$$

If $\beta'_0 = 0$, then by using (29) we get $c\varepsilon = 0$, which is impossible. Therefore $\beta'_0 \neq 0$. An easy integration of (29) gives $f(v) = Ke^{2c\beta'_0 v} - \frac{\varepsilon}{(\beta'_0)^2}$ and $\beta = \beta'_0 v + \beta_0$. The equation $\frac{\ddot{\alpha}}{\dot{\alpha}} = c\dot{\alpha}$ gives $\alpha = -\frac{1}{c}\log|cu + c_1|$, $c \in \mathbb{R}^*$ and $c_1 \in \mathbb{R}$. Then we get solution of the form

$$(s_4): \begin{cases} \varphi(u,v) &= (u, v, -\frac{1}{c}\log|cu + c_1| + \beta'_0 v + \beta_0) \\ f(v) &= Ke^{2c\beta'_0 v} - \frac{\varepsilon}{(\beta'_0)^2} \end{cases}$$

where $K, c, \beta'_0 \in \mathbb{R}^*$ and $c_1, \beta_0 \in \mathbb{R}$.

- **Case 3-2-2:** $c_1 \neq 0$. The first and the second equations in (28) give:

$$\begin{cases} (f - \frac{2c}{c_1})\beta'' &= \frac{1}{2} f' \beta' \\ \beta'^2 f &= -(2\beta'' + \varepsilon c_1). \end{cases}$$

If $\beta' = 0$ then $\beta'' = 0$ and $\varepsilon c_1 = 0$, which is impossible since $c_1 \neq 0$. Therefore we have $\beta' \neq 0$. So we get

$$\begin{cases} f &= -\frac{2\beta'' + \varepsilon c_1}{\beta'^2} \\ \frac{\beta''}{\beta'} &= \frac{1}{2} \frac{f'}{f - \frac{2c}{c_1}}. \end{cases} \tag{30}$$

The second equation of (30) gives

$$\beta' = \pm c_* \sqrt{\left| f - \frac{2c}{c_1} \right|} \quad \text{with} \quad c_* \in \mathbb{R}^*_+.$$

Denoted by $\mu = \operatorname{sign}\left(f - \frac{2c}{c_1}\right)$ and we get:

$$\begin{cases} \beta'^2 &= \mu c_*^2 \left(f - \frac{2c}{c_1}\right) \\ \beta'' &= \pm c_* \frac{\mu f'}{2\sqrt{\mu\left(f - \frac{2c}{c_1}\right)}} \end{cases} \tag{31}$$

The first equation of (31) gives: $\beta = \pm \int_{v_*}^{v} \sqrt{\left| f - \frac{2c}{c_1} \right|} d\tau$ where v_* and v belong to an intervall on which $\left(f - \frac{2c}{c_1}\right) > 0$ or $\left(f - \frac{2c}{c_1}\right) <$

0.
The first equation of (30) gives

$$f = -\frac{\pm c_* \frac{\mu f'}{2\sqrt{\mu\left(f-\frac{2c}{c_1}\right)}} + \varepsilon c_1}{\mu c_*^2\left(f - \frac{2c}{c_1}\right)}.$$

If we put $t = \sqrt{\mu\left(f - \frac{2c}{c_1}\right)}$ then $t^2 = \mu\left(f - \frac{2c}{c_1}\right)$, we have $f = \mu t^2 + \frac{2c}{c_1}$ and t satisfy $-\mu c_*^2(t^2 + \frac{2c}{c_1})t^2 \pm c_* t' = \varepsilon c_1$. We get the solution:

$$(s_5): \quad \varphi(u,v) = (u, v, \alpha(u) + \beta(v))$$

where α and β are given by:

(i) $\alpha(u) = Ae^{c_1 u} + B$ and $\beta(v) = \pm c_* \int_{v_*}^{v} \sqrt{|f|}d\tau$ with $f = \mu t^2$ $(\mu = \pm 1)$ where t is solution of differential equation $-\mu c_*^2 t^4 \pm c_* t' = \varepsilon c_1$;

(ii) $\alpha(u) = \int_{u_*}^{u} \frac{d\tau}{Ke^{-c_1 u} - \frac{c}{c_1}}$ and $\beta(v) = \pm c_* \int_{v_*}^{v} \sqrt{|f - \frac{2c}{c_1}|}d\tau$, where $K, c, c_1 \in \mathbb{R}^*, c_* > 0$ with $f = \mu(t^2 + \frac{2c}{c_1})$ where t is solution of $-\mu c_*^2(t^2 + \frac{2c}{c_1})t^2 \pm t' = \varepsilon c_1$.

We conclude with the following:

Theorem 1. *A translation surface S of Type I in (M, g_f^ϵ) where f depends only on y and not locally a constant, is minimal if and only if there is an interval I $(u \in I)$ and an interval J $(v \in J)$ such that on $I \times J$ the surface take one of the following form*

1) $\varphi(u,v) = (u, v, \alpha_0 + \beta(v))$ *for any smooth functions β where $\alpha_0 \in \mathbb{R}$.*
2) $\varphi(u,v) = (u, v, au + \tilde{b})$, *where $a \in \mathbb{R}^*, \tilde{b} \in \mathbb{R}$.*
3) $\varphi(u,v) = \left(u, v, au + b + \tilde{c}\int^{v} \sqrt{|2 + af|}d\tau\right)$, *where $a, \tilde{c} \in \mathbb{R}^*$, $b \in \mathbb{R}$.*
4) $\varphi(u,v) = (u, v, -\frac{1}{c}\log|cu + c_1| + \beta_0' v + \beta_0)$ *where the function $f(v) = Ke^{2c\beta_0' v} - \frac{\varepsilon}{(\beta_0')^2}$ and $K, c, \beta_0' \in \mathbb{R}^*$ and $c_1, \beta_0 \in \mathbb{R}$.*
5) $\varphi(u,v) = (u, v, \alpha(u) + \beta(v))$ *where α and β are given by*
   (i) $\alpha(u) = Ae^{c_1 u} + B$, $A \in \mathbb{R}^*, B \in \mathbb{R}$ *and* $\beta(v) = \pm c_* \int^{v} \sqrt{|f|}d\tau$, *with $f = \mu t^2$ where $t = t(v)$ is solution of differential equation $\pm c_* t' = \mu c_*^2 t^4 + \varepsilon c_1$;*
   (ii) $\alpha(u) = \int^{u} \frac{d\tau}{Ke^{-c_1 u} - \frac{c}{c_1}}$ *and* $\beta(v) = \pm c_* \int^{v} \sqrt{|f - \frac{2c}{c_1}|}d\tau$; $K, c, c_1 \in \mathbb{R}^*$, *$c_* > 0$ with $f = \mu(t^2 + \frac{2c}{c_1})$ where $t = t(v)$ is solution of $\pm c_* t' = \mu c_*^2(t^2 + \frac{2c}{c_1})t^2 \varepsilon c_1$.*

Example 1. *Let (M, g_f^ϵ) be a Walker manifold where the function $f(y) = y^2$. Let S be a translation surface in M satisfying the condition of the theorem 1. In 3) of the above theorem, if we take $a = 2, b = 0, \tilde{c} = 1$ then the surface S is given by (see figure (A)):*

$$\varphi(u,v) = \left(u, v, 2u + \frac{1}{\sqrt{2}}\ln(v + \sqrt{1+v^2}) + \frac{1}{\sqrt{2}}v\sqrt{1+v^2}\right). \tag{32}$$

In 5)i), if we take $A = 1, B = -3, c_* = 1, c_1 = 1$ then the surface S is given by (see figure (B)):

$$\varphi(u,v) = \left(u, v, e^u + \frac{1}{2}v^2 - 3\right). \tag{33}$$

3.2. **Minimal translation surfaces of Type II.** Let us consider a translation surface S of Type II in (M, g_f^ϵ) parametrized by $\varphi(u,v) = (\alpha(u) + \beta(v), u, v)$. In this case we have $x = \alpha(u) + \beta(v)$, $y = u$ and $z = v$. For a function g of one variable u (respectively v) we denote $\frac{dg}{du}$ by $\dot{g}$ (respectively $\frac{dg}{dv}$ by g'). The tangent plane of S is spanned by

$$\varphi_u = \dot{\alpha}\partial_x + \partial_y \quad \text{and} \quad \varphi_v = \beta'\partial_x + \partial_z, \tag{34}$$

while the unit normal ξ (up to orientation) is given by

$$\xi = \frac{1}{\Delta}\left[(-\beta' - f)\partial_x - \varepsilon\dot{\alpha}\partial_y + \partial_z\right] \tag{35}$$

where $\Delta = \|\varphi_u \times \varphi_v\|$. We obtain the coefficients of the first fundamental form of φ as

$$E = \varepsilon, \quad F = \dot{\alpha}, \quad G = 2\beta' + f. \tag{36}$$

And we have by using (6)

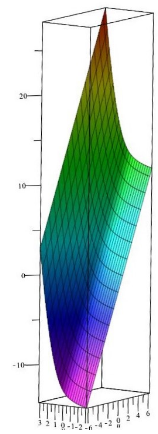
$$\varphi_{uu} = \begin{pmatrix} \ddot{\alpha} \\ 0 \\ 0 \end{pmatrix}, \quad \varphi_{uv} = \begin{pmatrix} \frac{1}{2}f_y \\ 0 \\ 0 \end{pmatrix}, \quad \varphi_{vv} = \begin{pmatrix} \beta'' \\ -\frac{\varepsilon}{2}f_y \\ 0 \end{pmatrix}. \tag{37}$$

Then the coefficients of the second fundamental form of φ

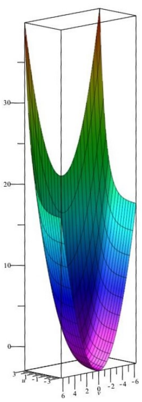
$$\begin{aligned} L &= \frac{\varepsilon_1}{\Delta}\big(\ddot{\alpha}\big), \\ M &= \frac{\varepsilon_1}{\Delta}\Big(\frac{1}{2}f_y\Big), \\ N &= \frac{\varepsilon_1}{\Delta}\Big(\beta'' + \frac{\varepsilon}{2}\dot{\alpha}f_y\Big). \end{aligned} \tag{38}$$

Consequently, the minimality condition (9) may be expressed as follows:

$$\ddot{\alpha}(2\beta' + f) - \frac{1}{2}\dot{\alpha}\dot{f} + \varepsilon\beta'' = 0 \tag{39}$$



(A) $\varphi(u,v) = \left(u, v, 2u + \frac{1}{\sqrt{2}}\ln(v + \sqrt{1+v^2}) + \frac{1}{\sqrt{2}}v\sqrt{1+v^2}\right)$



(B) $\varphi(u,v) = \left(u, v, e^u + \frac{1}{2}v^2 - 3\right)$

FIGURE 1. Figures of the Example 1

Taking the derivatives with respect to v, we get

$$2\ddot{\alpha}\beta'' + \varepsilon\beta''' = 0. \tag{40}$$

We will consider the following cases:

(1) **Case 1:** Assume that $\ddot{\alpha} = 0$. Since (40), we get $\beta'' = \beta_0'' \in \mathbb{R}$ and $\dot{\alpha} = \dot{\alpha}_0 \in \mathbb{R}$. And the equation (39) becomes $-\frac{1}{2}\dot{\alpha}_0\dot{f} + \varepsilon\beta_0'' = 0$. We have the following two subcases:

(a) **Case 1-1:** $\dot{\alpha}_0 = 0$. If $\dot{\alpha}_0 = 0$ then $\beta_0'' = 0$. Thus we get $\alpha = \alpha_0$ and $\beta(v) = av + b$. We get the plane

$$(s_1') : \quad \varphi(u,v) = (a_1 v + a_2, u, v); \quad a_1, a_2 \in \mathbb{R}.$$

(b) **Case 1-2:** $\dot{\alpha} \neq 0$. If $\dot{\alpha} \neq 0$ then $\beta_0'' \neq 0$ and we get $\dot{f} = \frac{2\varepsilon\beta_0''}{\dot{\alpha}}$. We get the solution

$$(s_2') : \begin{cases} \varphi(u,v) &= (a_1 u + a_2 v + a_3, u, v) \\ f(u) &= 2\varepsilon\frac{a_2}{a_1} u + a_4 \end{cases}$$

where $a_1, a_2 \in \mathbb{R}^*$, $a_3, a_4 \in \mathbb{R}$.

(2) **Case 2:** Assume that $\ddot{\alpha} \neq 0$. We will consider the following two sub-cases.

(a) **Case 2-1:** $\beta'' = 0$. If $\beta'' = 0$ then $\beta' = \beta_0' \in \mathbb{R}$. And the equation in (39) becomes

$$\frac{\ddot{\alpha}}{\dot{\alpha}} = \frac{1}{2}\left(\frac{\dot{f}}{2\beta_0' + f}\right),$$

which gives

$$\begin{cases} \alpha(u) &= \tilde{c}\int_{u^*}^{u} \sqrt{|f + 2a|}\,d\tau, \quad a \in \mathbb{R} \\ \beta(v) &= av + d \end{cases}$$

where u^* and u belong to an interval on which $(f+2a > 0)$ or $(f+2a < 0)$. We get the solution

$$(s_3') : \begin{cases} \varphi(u,v) = \left(\tilde{c}\int_{u^*}^{u} \sqrt{|f + 2a|}\,d\tau + av + d, u, v\right) \\ \tilde{c} \in \mathbb{R}^*, \ a, d \in \mathbb{R} \end{cases}$$

(b) **Case 2-2:** $\beta'' \neq 0$. If $\beta'' \neq 0$ then there exist $c \in \mathbb{R}^*$ such that

$$\begin{cases} 2\ddot{\alpha} &= c \\ \frac{\beta'''}{\beta''} &= -c\varepsilon. \end{cases}$$

Thus we have

$$\begin{cases} 2\dot{\alpha} &= cu + c_1 \\ \beta'' &= -c\varepsilon\beta' + c_2 \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$. And the equation in (39) becomes

$$\frac{c}{2}(2\beta' + f) - \frac{1}{4}(cu + c_1)\dot{f} + \varepsilon(-\varepsilon c\beta' + c_2) = 0,$$

that is

$$\frac{c}{2}f = \frac{1}{4}(cu + c_1)\dot{f} + \varepsilon c_2.$$

And then we have the solution

$$(s_4) \quad \varphi(u,v) = \left(\frac{1}{4}cu^2 + \frac{1}{2}c_1 u + \tilde{c}_1 + \frac{\varepsilon c_2}{c}v + K_1 e^{-\varepsilon c v}, u, v\right)$$

with $f(u) = K_2(cu + c_1)^2 + \frac{2c_2\varepsilon}{c}$, where $c_1, \tilde{c}_1, c_2, \tilde{c}_2 \in \mathbb{R}$ and $K_1, K_2 \in \mathbb{R}^*$.

We have the following result:

Theorem 2. *A translation surface S of Type II in (M, g_f^ϵ) where f depends only on y and not locally a constant, is minimal if and only if there is an interval I $(u \in I)$ and an interval J $(v \in J)$ such that on $I \times J$ the surface take one of the following form*

(1) $\varphi(u,v) = (a_1 v + a_2, u, v); \quad a_1, a_2 \in \mathbb{R}.$
(2) $\varphi(u,v) = (a_1 u + a_2 v + a_3, u, v); \quad a_1, a_2 \in \mathbb{R}^*, a_3, a_4 \in \mathbb{R}$ *with* $f(u) = 2\varepsilon\frac{a_2}{a_1}u + a_4.$
(3) $\varphi(u,v) = \left(\tilde{c}\int_{u^*}^{u}\sqrt{|f + 2a|}d\tau + av + d, u, v\right); \quad a, d \in \mathbb{R}.$
(4) $\varphi(u,v) = \left(\frac{1}{4}cu^2 + \frac{1}{2}c_1 u + \tilde{c}_1 + \frac{\varepsilon c_2}{c}v + K_1 e^{-\varepsilon cv}, u, v\right); \quad c_1, \tilde{c}_1, c_2, \tilde{c}_2 \in \mathbb{R},\ c, K_1, K_2 \in \mathbb{R}^*$ *with* $f(u) = K_2(cu + c_1)^2 + \frac{2c_2\varepsilon}{c}.$

Example 2. *Let (M, g_f^ϵ) be a Walker manifold where the function $f(y) = 2y^2$. Let S be a translation surface in M satisfying the condition of the theorem 2. In 3) of the above theorem 2, if we take $a = 1, \tilde{c} = 1, d = 0$ then the surface S is given by (see figure 2a):*

$$\varphi(u,v) = \left(\frac{1}{\sqrt{2}}\ln(u + \sqrt{1+u^2}) + \frac{1}{\sqrt{2}}u\sqrt{1+u^2} + v, u, v\right). \tag{41}$$

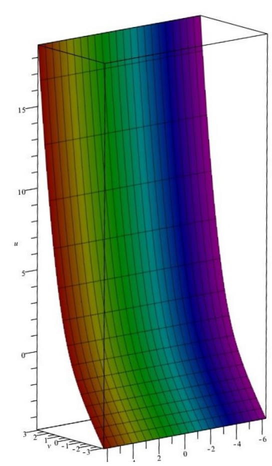


FIGURE 2. $\varphi(u,v) = \left(\frac{1}{\sqrt{2}}\ln(u + \sqrt{1+u^2}) + \frac{1}{\sqrt{2}}u\sqrt{1+u^2} + v, u, v\right)$,
Figure of the Example 2.

4. Conclusion

In this paper we have defined two types of translation surfaces using two kind of isometries in a strict Walker manifold (M, g_f^ϵ). First we have studied and classified the minimality of the translation surface of type I and we draw some examples of these family of surfaces. Secondely, we considered the family of translation surfaces of type II and we studied their minimality. We classify these surfaces and draw some example.

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A SECOND-ORDER NUMERICAL METHOD FOR PSEUDO-PARABOLIC EQUATIONS HAVING BOTH LAYER BEHAVIOR AND DELAY PARAMETER

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Abstract. In this paper, singularly perturbed pseudo-parabolic initial-boundary value problems with time-delay parameter are considered by numerically. Initially, the asymptotic properties of the analytical solution are investigated. Then, a discretization with exponential coefficient is suggested on a uniform mesh. The error approximations and uniform convergence of the presented method are estimated in the discrete energy norm. Finally, some numerical experiments are given to clarify the theory.

1. Introduction

Singularly perturbed problems are defined by a small parameter ε multiplying the highest order derivative term in the differential equation. The solutions of them typically include the boundary or interior layers depending on the situation of the problem. Because of the existence of the layers, the solution shows a multiscale character, i.e., the solution behaves stable and slowly away from the layer region while it behaves unstable and rapidly in the layer region. Therefore, the conventional numerical approaches do not produce the reliable results and ε-uniform computational techniques are required [19, 24, 35, 37, 43, 45, 47, 51] (see, also the references therein). To examine singular perturbation problems and their applications more comprehensively, one may refer in [19, 24, 35, 37, 43, 45, 47, 51].

Intercalarily, many mathematical models of real life situations in science are explained with the singularly perturbed delay differential equations (SPDDEs). Their

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applications can be found in processes for metal plates, spread of HIV and bacterial infections, control theory, population dynamics, neurobiology, thermo elasticity, hydrodynamics of liquid helium, mechanic systems, laser optics and financial mathematics [10, 20, 31, 38, 57, 61] (see, also the references therein). In the literature, SPDDEs have been investigated widely by many authors and different numerical methods have been introduced. These include: Reproducing kernel method [16,28,29], initial value technique [59], numerical integration method [27,50,57], Numerov method [15], the method of hybrid difference schemes [13,14], discontinuous Galerkin method [64], collocation methods [39, 60, 63], Ritz-Galerkin method [34], $hp-$finite element method [46], fitted mesh technique [33, 42], domain decomposition approach [56], cubic spline methods [36], finite difference methods [5,7,9,22,53] and so on [11,25,44,48,49,54].

In this paper, we consider the singularly perturbed linear initial-boundary value pseudo-parapolic problem with time-delay on the domain $\bar{D} = \bar{\Omega} \times [0, T]$; $\bar{\Omega} = [0, l]$, $\Omega = (0, l)$, $D = \Omega \times (0, T]$:

$$Lu \equiv L_1 \left[\frac{\partial u}{\partial t}\right] + L_2 u + c(t)\, u(x, t - r) = f(x, t), \quad (x, t) \in D, \tag{1}$$

$$u(x, t) = \varphi(x, t), \quad (x, t) \in \bar{\Omega} \times [-r, 0], \tag{2}$$

$$u(0, t) = u(l, t) = 0, \ t \in (0, T], \tag{3}$$

where

$$\begin{aligned} L_1 \left[\frac{\partial u}{\partial t}\right] &= -\varepsilon \frac{\partial^3 u}{\partial x^2 \partial t} + a(x) \frac{\partial u}{\partial t}, \\ L_2 [u(x,t)] &= -\varepsilon \frac{\partial^2 u}{\partial x^2} + b(x, t)\, u(x, t), \end{aligned}$$

and $0 < \varepsilon \ll 1$ is the perturbation parameter; the functions a, b, c, f and φ are sufficiently smooth, $r > 0$ is delay parameter and $a(x) \geq \alpha > 0$. The problem (1)-(3) have been studied on Boglaev-type adaptive mesh by conducting linear basis functions and energy inequalities in [32]. Also, G. Amiraliyev and Y. Mamedov [4] have proposed an exponentially difference scheme for solving the problem (1)-(3) without delay parameter.

Pseudo-parabolic or Sobolev type problems have had an important role in the literature. For scientific background and existence-uniqueness results of pseudo-parabolic problems without singular perturbation and the delay parameter, one may refer in [55, 58]. I. Amirali et. al [1] have constructed two-level difference scheme for semilinear pseudo-parabolic initial-boundary value problems with delay parameter (Please, see also a series of the papers [2, 3, 8]). C. Zhang and Z. Tan [65] have used linearized compact finite difference methods for solving nonlinear delay Sobolev partial differential equations. On the other hand, latterly, various numerical schemes have been proposed for parabolic type problems with singular perturbation case. L. Govindarao and J. Mohapatra [30] have suggested a numerical scheme comprised of implicit-trapezoidal scheme on temporal direction and hybrid type scheme on spatial direction for solving singularly perturbed delay parabolic initial-boundary value problems. In the paper [6], a fully discrete scheme has been generated on Shishkin mesh to solve singularly perturbed Sobolev initial-boundary value problem with initial jump. S. Kumar and M. Kumar [40] have discretized singularly perturbed nonlinear delay parabolic type partial differential equations on a generalized Shishkin mesh by using quasilinearization techniques. M. M. Woldaregay et. al [61] have developed a numerical approach by using Crank-Nicolson technique for temporal discretization and exponentially fitted difference scheme for spatial discretization to analyse parabolic convection-diffusion problems with layer behavior. N. A. Mbroh et. al [41] have designed a numerical discretization using fitted operator finite difference method on spatially direction and Crank Nicolson finite difference approach on time direction. S. Yadav and P. Rai [62] have constructed a higher-order difference method consisting of hybrid scheme on Shishkin mesh and implicit Euler method on a uniform mesh to examine singularly perturbed delay parabolic turning point problems of convection-diffusion type. Authors in [10,12] have provided the standard finite difference scheme on piecewise uniform fitted mesh to analyze singularly perturbed delay parabolic initial-boundary value problems. L Govindarao et. al [31] have established a fourth-order numerical scheme on Shishkin-type mesh by using Richardson extrapolation to examine singularly perturbed delay parabolic reaction-diffusion problems. A. B. Chiyaneh and H. Duru [17,18] have formulated difference schemes to resolve singularly perturbed Sobolev initial-boundary value problems with time-delay parameter. S. Elango et. al [23] have provided finite difference scheme on the rectangular piecewise uniform mesh by using trapezoidal rule for solving singularly perturbed partial delay differential equations with integral boundary condition. F. W. Gelu and G. F. Duressa [26] have suggested B-spline collocation technique on Shishkin mesh to obtain a numerical approximation of singularly perturbed delay parabolic problems of reaction-diffusion type. In [21], singularly perturbed Sobolev type initial-boundary value problems with Robin boundary condition have been discretized on a uniform mesh.

Our focus in this study is to present a robust and stable finite difference scheme on a uniform mesh for solving problem (1)-(3). With in this mind, we use the interpolating quadrature rules and exponential basis functions (see [4]).

The rest of this paper is as follows: In Section 2, some priori estimates for the continuous problem are given. The finite difference scheme is constructed on a uniform mesh in Section 3. Section 4 presents the stability and convergence analysis of the proposed scheme in the discrete energy norm. Two numerical examples are solved and the computed results are tabulated in Section 5. Lastly, the paper ends with a brief conclusion.

2. A Priori Bounds

In this section, we give the asymptotic behavior of the analytical solution and its derivatives.

Lemma 1. *The solution $u(x,t)$ of the problem (1)-(3) satisfies that*

$$\varepsilon\left\|\frac{\partial u}{\partial x}\right\|^2+\alpha\left\|u\right\|^2\le\left\{\left[\varepsilon\left\|\frac{\partial\varphi\left(x,0\right)}{\partial x}\right\|^2+\left\|\varphi\left(x,0\right)\right\|^2\right]e^{Ct}\right.$$

$$\left.+\int_0^t c^*\left\|\varphi\left(x,s\right)\right\|^2e^{Ct}ds+\int\limits_0^t\left\|f\right\|^2e^{Ct}ds\right\}$$

where $\|.\|=\|.\|_{L_2(0,l)}$, C is a generic positive constant and $c^*=\max\limits_{t\in[0,T]}|c(t)|$.

Proof. The proof of the lemma can be found in the paper [32]. □

Lemma 2. *Under the assumptions $a\in C^2[0,l]$, $b\in C^2\left(\bar{D}\right)$, $f\in C\left(\bar{D}\right)$ and*

$$|a(0)-b(0,t)|\le C\varepsilon,\quad |a(l)-b(l,t)|\le C\varepsilon, \tag{4}$$

asymptotic expansion of the solution of the problem (1)-(3) can be written in the form

$$\begin{aligned}u\left(x,t\right)&=u_0\left(x,t\right)+\vartheta_0\left(\xi,t\right)+w_0\left(\eta,t\right)\\&+\sqrt{\varepsilon}\left[u_1\left(x,t\right)+\vartheta_1\left(\xi,t\right)+w_1\left(\eta,t\right)\right]+R^*\left(x,t\right),\end{aligned} \tag{5}$$

where the functions $u_0\left(x,t\right)$, $u_1\left(x,t\right)$, $\vartheta_0\left(\xi,t\right)$, $w_0\left(\eta,t\right)$, $\vartheta_1\left(\xi,t\right)$, $w_1\left(\eta,t\right)$ are the solutions of the following problems:

$$\begin{cases}a(x)\frac{\partial u_0}{\partial t}+b(x,t)u_0+c(t)u_0\left(x,t-r\right)=f(x,t),\\u_0(x,t-r)=\varphi(x),\ \ -r\le t\le 0;\end{cases}$$

$$\begin{cases}a\left(x\right)\frac{\partial u_1}{\partial t}+b\left(x,t\right)u_1+c(t)u_1\left(x,t-r\right)=-\sqrt{\varepsilon}\left[\frac{\partial^3u_0}{\partial t\partial x^2}+\frac{\partial^2u_0}{\partial x^2}\right],\\u_1\left(x,t\right)=0,-r\le t\le 0;\end{cases}$$

$$\begin{cases}-\varepsilon\frac{\partial^3\vartheta_0}{\partial t\partial\xi^2}+a\left(0\right)\frac{\partial\vartheta_0}{\partial t}-\varepsilon\frac{\partial^2\vartheta_0}{\partial\xi^2}+a(0)\vartheta_0+c(t)\vartheta_0\left(x,t-r\right)=0,\\\vartheta_0(\xi,t)=0,\ \ -r\le t\le 0;\\\vartheta_0(0,t)=-u_0\left(0,t\right);\ \ \vartheta_0(\frac{l}{\sqrt{\varepsilon}},t)=0,\end{cases}$$

$$\begin{cases}-\varepsilon\frac{\partial^3\vartheta_1}{\partial t\partial\xi^2}+a(0)\frac{\partial\vartheta_1}{\partial t}-\varepsilon\frac{\partial^2\vartheta_1}{\partial\xi^2}+a(0)\vartheta_1+c(t)\vartheta_1\left(x,t-r\right)\\=-\xi\frac{\partial b}{\partial x}(0,t)\vartheta_0-\xi a'(0)\frac{\partial\vartheta_0}{\partial t},\\\vartheta_1(\xi,t)=0,\quad -r\le t\le 0;\\\vartheta_1(0,t)=-u_1\left(0,t\right);\ \vartheta_1(\frac{l}{\sqrt{\varepsilon}},t)=0,\end{cases}$$

$$\begin{cases}-\varepsilon\frac{\partial^3w_0}{\partial t\partial\eta^2}+a(l)\frac{\partial w_0}{\partial t}-\varepsilon\frac{\partial^2w_0}{\partial\eta^2}+a(l)w_0+c(t)w_0\left(x,t-r\right)=0,\\w_0(\eta,t)=0,\quad -r\le t\le 0;\\w_0(\frac{l}{\sqrt{\varepsilon}},t)=0;w_0(0,t)=-u_0\left(l,t\right),\end{cases}$$

$$\begin{cases} -\varepsilon\frac{\partial^3 w_1}{\partial t\partial\eta^2}+a(l)\frac{\partial w_1}{\partial t}-\varepsilon\frac{\partial^2 w_1}{\partial\eta^2}+a(l)w_1+c(t)w_1\,(x,t-r) \\ =-\eta\frac{\partial b}{\partial x}(l,t)\omega_0-\eta a'(l)\frac{\partial^2 w_0}{\partial t^2}, \\ w_1(\eta,t)=0, \quad -r\le t\le 0; \\ w_1(\frac{l}{\sqrt{\varepsilon}},t)=0\,;\;w_1(0,t)=-u_1\,(l,t)\,, \end{cases}$$

where $\xi=\frac{x}{\sqrt{\varepsilon}}$ *and* $\eta=\frac{l-x}{\sqrt{\varepsilon}}$. *Additionally, the remainder term of the asymptotic expansion can be estimated as*

$$\varepsilon^s\left\|\frac{\partial^{k+s}R^*}{\partial t^k\partial x^s}\right\|\le C\varepsilon^{1-s/2}\quad k,\;s=0,1,2.$$

Proof. The proof can be shown by using a similar approach of [4,17,18,21,32]. □

Lemma 3. *Under the conditions of Lemma (2), using*

$$\left|\frac{\partial^{k+s}\vartheta_0}{\partial t^k\partial x^s}\right|\le C\varepsilon^{-s/2}e^{-x\sqrt{a(0)/\varepsilon}}$$

and

$$\left|\frac{\partial^{k+s}w_0}{\partial t^k\partial x^s}\right|\le C\varepsilon^{-s/2}e^{-(l-x)\sqrt{a(l)/\varepsilon}},$$

we have the following bound:

$$\left|\frac{\partial^{k+s}u}{\partial t^k\partial x^s}\right|\le C\left\{1+\varepsilon^{-s/2}\left[e^{-x\sqrt{a(0)/\varepsilon}}+e^{-(l-x)\sqrt{a(l)/\varepsilon}}\right]\right\}, \tag{6}$$

$$(x,t)\in\bar{D},\;k=0,\;1,\;2;\;s=0,\;1,\;2.$$

Proof. The proof of the lemma is similar to those of [4,17,18,21,32]. □

3. Spatial and Temporal Discretization

In this section, we propose the discretization for the problem (1)-(3). Let $\omega_{h\tau}=\omega_h\times\omega_\tau$ denote the mesh on D:

$$\omega_h=\{x_i=ih,\;i=1,2,...,N-1,\;h=l/N\}$$
$$\omega_\tau=\{t_j=j\tau,\;j=1,2,...,M;\tau=T/M\}$$

and

$$\bar{\omega}_h=\omega_h\cup\{x_0=0,x_N=l\}\,,\;\bar{\omega}_\tau=\omega_\tau\cup\{t=0\}\,.$$

For any mesh function $v(x)$ described on $\bar{\omega}_h$, we use the difference formulas in [52]:

$$v_i=v(x_i),\;v_{\bar{x},i}=\frac{v_i-v_{i-1}}{h},$$
$$v_{x,i}=\frac{v_{i+1}-v_i}{h},\;v_{\bar{x}x,i}=\frac{v_{i+1}-2v_i+v_{i-1}}{h^2}.$$

Also, for a function $w\equiv w_i^j\equiv w(x_i,t_j)$ defined on $\bar{\omega}_\tau$, we need (see [52])

$$w_{\bar{t},i}^j=\frac{w_i^j-w_i^{j-1}}{\tau}.$$

Now, we begin to establish the difference scheme according to the space variable. To formulate the difference method, the following integral identity is used:

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} L_1\left[\frac{\partial u}{\partial t}\right]\varphi_i(x)\,dx + h^{-1}\int_{x_{i-1}}^{x_{i+1}} L_2[u]\varphi_i(x)\,dx$$
$$+h^{-1}\int_{x_{i-1}}^{x_{i+1}} c(t)\,u(x,t-r)\,\varphi_i(x)\,dx = h^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x,t)\varphi_i(x)\,dx \tag{7}$$

where the exponential basis function

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) \equiv \frac{\sinh\gamma_{i-0,5}(x-x_{i-1})}{\sinh\gamma_{i-0,5}h}, & x\in[x_{i-1},x_i], \\ \varphi_i^{(2)}(x) \equiv \frac{\sinh\gamma_{i+0,5}(x_{i+1}-x)}{\sinh\gamma_{i+0,5}h}, & x\in[x_i,x_{i+1}], \\ 0, \quad x\notin(x_{i-1},x_{i+1}). \end{cases}$$

Also $\gamma_i = \sqrt{a_i/\varepsilon}$ $(i = 1,2,...,N-1)$, and $a(x_{i\pm 0,5}) = a\left(x_i \pm \frac{h}{2}\right)$. The functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ are the solutions of the following problems, respectively:

$$\begin{cases} -\varepsilon\varphi_i^{(1)''}(x) + a_{i-0.5}\varphi^{(1)}(x) = 0, \quad x_{i-1} < x < x_i, \\ \varphi^{(1)}(x_{i-1}) = 0,\ \varphi^{(1)}(x_i) = 1, \end{cases}$$

$$\begin{cases} -\varepsilon\varphi_i^{(2)''}(x) + a_{i+0.5}\varphi^{(2)}(x) = 0,\ x_i < x < x_{i+1}, \\ \varphi^{(2)}(x_i) = 1, \quad \varphi^{(2)}(x_{i+1}) = 0. \end{cases}$$

For the first term of the equality (7), applying interpolating quadrature rules in [4] and some processes in [17], it is found that

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} L_1\left[\frac{\partial u}{\partial t}\right]\varphi_i(x)\,dx = h^{-1}\int_{x_{i-1}}^{x_{i+1}}\left\{-\varepsilon\frac{\partial^3 u}{\partial x^2\partial t} + a(x)\frac{\partial u}{\partial t}\right\}\varphi_i(x)\,dx$$
$$= h^{-1}\int_{x_{i-1}}^{x_i}\left\{-\varepsilon\frac{\partial^3 u}{\partial x^2\partial t} + a_{i-0.5}\frac{\partial u}{\partial t}\right\}\varphi_i^{(1)}(x)\,dx$$
$$+h^{-1}\int_{x_i}^{x_{i+1}}\left\{-\varepsilon\frac{\partial^3 u}{\partial x^2\partial t} + a_{i+0.5}\frac{\partial u}{\partial t}\right\}\varphi_i^{(2)}(x)\,dx$$
$$+h^{-1}\int_{x_{i-1}}^{x_i}[a(x)-a_{i-0.5}]\frac{\partial u}{\partial t}\varphi_i^{(1)}(x)\,dx + h^{-1}\int_{x_i}^{x_{i+1}}[a(x)-a_{i+0.5}]\frac{\partial u}{\partial t}\varphi_i^{(2)}(x)\,dx.$$

Then, we get

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} L_1\left[\frac{\partial u}{\partial t}\right]\varphi_i(x)\,dx = -\varepsilon\left(\theta_0\left[\frac{\partial u}{\partial t}\right]_{\bar{x}}\right)_{x,i} + A\theta_1\left(\frac{\partial u}{\partial t}\right)_i + R_{1,i}^*(t) \tag{8}$$

where

$$A = \frac{1}{2}(a_{i-0.5} + a_{i+0.5}),$$

$$R_{1,i}^*(t) = h^{-1}\int_{x_{i-1}}^{x_i}[a(x)-a_{i-0.5}]\frac{\partial u}{\partial t}\varphi_i^{(1)}(x)\,dx + h^{-1}\int_{x_i}^{x_{i+1}}[a(x)-a_{i+0.5}]\frac{\partial u}{\partial t}\varphi_i^{(2)}(x)\,dx$$

$$+\left[\left(a_{i-0.5}-A\right)\theta_1^{(1)}+\left(a_{i+0.5}-A\right)\theta_1^{(2)}\right]\left(\frac{\partial u}{\partial t}\right)$$

and

$$\begin{aligned}\theta_0 &\equiv (\theta_0)_i=1+\varepsilon^{-1}a_{i-0.5}\int_{x_{i-1}}^{x_i}(x-x_i)\varphi_i^{(1)}(x)\,dx\\ &= \frac{\rho\sqrt{a_{i-0.5}}}{\sinh\left(\rho\sqrt{a_{i-0.5}}\right)},\left(\rho=h/\sqrt{\varepsilon}\right),\end{aligned}$$

$$\theta_1^{(1)}=h^{-1}\int_{x_{i-1}}^{x_i}\varphi_i^{(1)}(x)\,dx=\frac{1}{\rho\sqrt{a_{i-0.5}}}\tanh\frac{\rho\sqrt{a_{i-0.5}}}{2},$$

$$\theta_1^{(2)}=h^{-1}\int_{x_i}^{x_{i+1}}\varphi_i^{(2)}(x)\,dx=\frac{1}{\rho\sqrt{a_{i+0.5}}}\tanh\frac{\rho\sqrt{a_{i+0.5}}}{2},$$

$$\theta_{1,i}=\theta_{1,i}^{(1)}+\theta_{1,i}^{(2)},\quad A_i=\frac{1}{2}\left(a\left(x_{i-0.5}\right)+a\left(x_{i+0.5}\right)\right).$$

For the second term of the equation (7), we obtain

$$\begin{aligned}h^{-1}\int_{x_{i-1}}^{x_{i+1}}L_2[u]\,\varphi_i(x)\,dx &= h^{-1}\int_{x_{i-1}}^{x_{i+1}}\left(-\varepsilon\frac{\partial^2u}{\partial x^2}+b(x,t)\,u(x,t)\right)\varphi_i(x)\,dx\\ &= -\varepsilon\left(\theta_0u_{\bar{x}}\right)_{x,i}+B\theta_1u(x_i,t)+\theta_1R_{2,i}^*(t)\end{aligned}\tag{9}$$

where

$$B=\frac{1}{2}\left[b(x_{i-0.5},t)+b(x_{i+0.5},t)\right]$$

and

$$R_{2,i}^*(t)=\theta_1^{-1}\left[h^{-1}\int_{x_{i-1}}^{x_{i+1}}L_2[u]\,\varphi_i(x)\,dx+\left(\varepsilon\left(\theta_0u_{\bar{x}}\right)_{x,i}-B\theta_1u(x_i,t)\right)\right].$$

For the third term of the equation (7), it is found that

$$\begin{aligned}h^{-1}\int_{x_{i-1}}^{x_{i+1}}c(t)\,u(x,t-r)\,\varphi_i(x)\,dx &= h^{-1}c(t)\int_{x_{i-1}}^{x_{i+1}}u(x,t-r)\,\varphi_i(x)\,dx\\ &= c(t)\,u(x_i,t-r)\,h^{-1}\int_{x_{i-1}}^{x_{i+1}}\varphi_i(x)\,dx+R_{3,i}^*(t)\\ &= \theta_1c(t)\,u(x_i,t-r)+R_{3,i}^*(t)\end{aligned}\tag{10}$$

where

$$R_{3,i}^*(t)=\theta_1^{-1}h^{-1}c(t)\int_{x_{i-1}}^{x_{i+1}}\left[u(x,t-r)-u(x_i,t-r)\right]\varphi_i(x)\,dx.$$

The term of the right side of the equation (7), we can write:

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}}f(x,t)\varphi_i(x)\,dx=\theta_1F_i-\theta_1R_{4,i}^*(t)\tag{11}$$

where

$$R_{4,i}^{*}(t)=-h^{-1}\theta_1^{-1}\int_{x_{i-1}}^{x_{i+1}}\left[f(x,t)-F\left(x_i,t\right)\right]\varphi_i(x)\,dx$$

and

$$F\left(x_i,t\right)=\frac{1}{2}\left[f(x_{i-0.5},t)+f(x_{i+0.5},t)\right].$$

By combining (8),(9),(10) and (11), we have

$$-\varepsilon\left(\theta_0\left[\frac{\partial u}{\partial t}\right]_{\bar{x}}\right)_{x,i}+A\theta_1\left(\frac{\partial u}{\partial t}\right)-\varepsilon\left(\theta_0 u_{\bar{x}}\right)_{x,i}+B\theta_1 u\left(x_i,t\right)+\theta_1 c(t)u\left(x_i,t-r\right)+\theta_1 R_i^{*}=\theta_1 F_i$$

where

$$R_i^{*}=R_{1,i}^{*}(t)+R_{2,i}^{*}(t)+R_{3,i}^{*}(t)+R_{4,i}^{*}(t).$$

Then, to obtain the discretization for the time variable, we consider the integral equality in the form

$$\tau^{-1}\int_{t_{j-1}}^{t_j}\left[Lu-f\left(x_i,t\right)\right]dt=\tau^{-1}\int_{t_{j-1}}^{t_j}\left\{\varepsilon\left[\theta_0\left(\frac{\partial u}{\partial t}\right)_{\bar{x}}\right]_{x,i}+A\theta_1\left(\frac{\partial u}{\partial t}\right)_i-\varepsilon\left(\theta_0 u_{\bar{x}}\right)_{x,i}\right.$$
$$\left.+B\theta_1 u(x_i,t)+\theta_1 c(t)u(x_i,t-r)-\theta_1 F_i+\theta_1 R_i^{*}\right\}dt \qquad (12)$$

Applying the interpolating quadrature rules [4] to first two terms of the equation (12), we have

$$\tau^{-1}\int_{t_{j-1}}^{t_j}\left[-\varepsilon\left(\theta_0\left(\frac{\partial u}{\partial t}\right)_{\bar{x}}\right)_{x,i}+A\theta_1\left(\frac{\partial u}{\partial t}\right)_i\right]dt=-\varepsilon\left(\theta_0 u_{\bar{t}\bar{x}}\right)_x+\theta_1 A u_{\bar{t}}.$$

For the third and fourth terms of the equation (12), it is obtained that

$$\tau^{-1}\int_{t_{j-1}}^{t_j}\left[-\varepsilon\left(\theta_0 u_{\bar{x}}\right)_x+\theta_1 B_i u_i\right]dt=-\varepsilon\left(\theta_0 u_{\bar{x}}^{(\sigma)}\right)_x+\theta_1 B_i^j u_i^{(\sigma)}+\varepsilon\left(\theta_0 R^{(0)}\right)_{\bar{x}}+R_1^{(1)}$$

where

$$R^{(0)}=u^{(\sigma)}\left(x_i,t_j\right)-\tau^{-1}\int_{t_{j-1}}^{t_j}u\left(x_i,\eta\right)d\eta$$

and

$$R_1^{(1)}=\theta_1 B_i^j\left(\tau^{-1}\int_{t_{j-1}}^{t_j}u(x_i,t)dt-u_i^{(\sigma)}\right).$$

For the term involving the delay parameter, rewriting $\tau=T/M$ and $rM/T=M_0$, we have

$$\theta_1\tau^{-1}\int_{t_{j-1}}^{t_j}c(t)u\left(x_i,t-r\right)dt=\theta_1 c^j u_i^{j-M_0}+R_c^j$$

where

$$R_c^j=\theta_1\left\{\tau^{-1}\int_{t_{j-1}}^{t_j}\left[c(t)-c^j\right]u\left(x_i,t-r\right)dt\right.$$

$$+\tau^{-1}\int_{t_{j-1}}^{t_j} c^j\left(u\left(x_i,t-r\right)-u\left(x_i,t_j-r\right)\right)dt\Bigg\}.$$

For the term of the right side of the equation (12), we find

$$\tau^{-1}\int_{t_{j-1}}^{t_j}\theta_1 F(x_i,t)dt=\theta_1 F_i^j+R_f,$$

where

$$R_f=\tau^{-1}\int_{t_{j-1}}^{t_j}\theta_1 F(x_i,t)dt-\theta_1 F_i^j.$$

Thus, we can suggest the following difference scheme

$$\ell u_i^j:=\ell_1\left(u_{\bar{t},i}^j\right)+\ell_2\left(u_i^j\right)+\theta_1 c^j u_i^{j-M_0}+R_i^j=\theta_1 F_i^j. \tag{13}$$

where

$$\ell_1\left(u_{\bar{t},i}^j\right)=-\varepsilon\left(\theta_0 u_{\bar{t}\bar{x}}^j\right)_{x,i}+\theta_1 A u_{\bar{t}}^j,$$
$$\ell_2\left(u_i^j\right)=-\varepsilon\left(\theta_0 u_{\bar{x}}^{(\sigma)}\right)_{x,i}+\theta_1 B_i^j u_i^{(\sigma)},$$

and the remainder term is denoted by

$$R_i^j=\varepsilon\left(\theta_0 R^{(0)}\right)_x+\theta_1 R^{(1)}+\theta_1 R_{c,j}$$

where

$$R^{(1)}=\tau^{-1}\int_{t_{j-1}}^{t_j}R_i^*(t)dt+R_1^{(1)}-R_f.$$

By omitting the remainder term R_i^j in (13), we can write for the approximate solution

$$-\varepsilon\left(\theta_0 y_{\bar{t}\bar{x}}^j\right)_{x,i}+\theta_1 A y_{\bar{t}}^j-\varepsilon\left(\theta_0 y_{\bar{x}}^{(\sigma)}\right)_{x,i}+\theta_1 B_i^j y_i^{(\sigma)}+\theta_1 c^j y_i^{j-M_0}=\theta_1 F_i^j, \tag{14}$$

$$y\left(x_i,t_j\right)=\varphi\left(x_i,t_j\right),\ -M_0\le j\le 0,\ 0\le i\le N, \tag{15}$$

$$y_0^j=y_N^j=0. \tag{16}$$

4. Error Bounds

Let u_i^j be the solution of the problem (1)-(3) and let y_i^j be the solution of the problem (14)-(16). Then, the error function $z_i^j=y_i^j-u_i^j$ be the solution of the following discrete problem:

$$\ell_1\left(z_{\bar{t},i}^j\right)+\ell_2\left(z_i^j\right)+\theta_1 c^j z_i^{j-M_0}=R_i^j. \tag{17}$$

$$z\left(x_i,t_j\right)=0,\quad 0\le i\le N,\quad -M_0\le j\le 0, \tag{18}$$

$$z_0^j=z_N^j=0,\ t\in\bar{\omega}_\tau, \tag{19}$$

where

$$\ell_1\left(z_{\bar{t},i}^j\right)=-\varepsilon\left(\theta_0 z_{\bar{t}\bar{x}}^j\right)_{x,i}+\theta_1 A z_{\bar{t}}^j$$

and

$$\ell_2\left(z_i^j\right)=-\varepsilon\left(\theta_0 z_{\bar{x}}^{(\sigma)}\right)_{x,i}+\theta_1 B_i^j z_i^{(\sigma)}.$$

Lemma 4. *Under the conditions* $1+\sigma\tau>0$ *and* $A+\sigma\tau B>0$*, the following estimate is satisfied:*

$$\|z\|^2\le C\tau\sum_{k=1}^{j}\left\|\theta_1^{-1}R\right\|_{-*}^2.$$

where

$$\left\|\theta_1^{-1}R\right\|_{-*}^2=(\theta_1)^{-1}\sup_{v}\frac{|(R,v)|^2}{(\theta_0 v_{\bar{x}},v_{\bar{x}})+(\theta_1 v,v)}$$
$$\le(\theta_1)^{-1}\left\{\varepsilon\left(\theta_0 R^{(0)},R^{(0)}\right)+\left(\theta_1 R^{(1)},R^{(1)}\right)+(\theta_1 R_c,R_c)\right\}.$$

Proof. To carry out the error analysis, we use a similar approach in [4,17,32]. First of all, we consider the following equation for the discrete problem (17)-(19):

$$(\ell z,z)=(R,z).$$

From here, we get

$$\left(-\varepsilon\left(\theta_0 z_{\bar{t}\bar{x}}\right)_{x,i},z\right)=\frac{\varepsilon}{2}\left(\theta_0 z_{\bar{t}\bar{x},i},z_{\bar{x},i}\right)=\frac{\varepsilon}{2}\left(\theta_0 z_{\bar{x},i},z_{\bar{x},i}\right)_{\bar{t}}+\frac{\varepsilon\tau}{2}\left(\theta_0 z_{\bar{t}\bar{x},i},z_{\bar{t}\bar{x},i}\right)$$

and

$$(\theta_1 A z_{\bar{t}},z)=\frac{1}{2}(A\theta_1 z,z)_{\bar{t}}+\frac{\tau}{2}\left(A\theta_1 z_{\bar{t}},z_{\bar{t}}\right).$$

Then, to obtain the bound for the term $z^{(\sigma)}(x_i,t_j)$, we take the following equality into account:

$$z^{(\sigma)}=\sigma z+(1-\sigma)\check{z}=\sigma z(x_i,t_j)+(1-\sigma)z(x_i,t_{j-1}) \tag{20}$$

Hence, we acquire as

$$\begin{aligned}\left(-\varepsilon\left(\theta_0 z_{\bar{x}}^{(\sigma)}\right)_{x,i},z\right) &= \varepsilon\left(\theta_0\check{z}_{\bar{x}},\check{z}_{\bar{x}}\right)+\varepsilon\tau\left(\theta_0\check{z}_{\bar{x}},z_{\bar{t}\bar{x}}\right)\\ &\quad+\frac{\varepsilon\sigma\tau}{2}\left(\theta_0 z_{\bar{x}},z_{\bar{x}}\right)_{\bar{t}}+\frac{\varepsilon\sigma\tau^2}{2}\left(\theta_0 z_{\bar{t}\bar{x}},z_{\bar{t}\bar{x}}\right)\end{aligned}$$

and

$$\begin{aligned}\left(\theta_1 B z^{(\sigma)},z\right) &= \left(\theta_1 B\check{z},\check{z}\right)+\tau\left(\theta_1 B\check{z},z_{\bar{t}}\right)\\ &\quad+\frac{\sigma\tau}{2}\left(\theta_1 B z,z\right)_{\bar{t}}+\frac{\sigma\tau^2}{2}\left(\theta_1 B z_{\bar{t}},z_{\bar{t}}\right).\end{aligned}$$

Furthermore, we can write the following estimates:

$$\left|\left(\theta_1 c^j z_i^{j-M_0}, z\right)\right| \le c^* \mu_1 \left(\theta_1 z_i^{j-M_0}, z_i^{j-M_0}\right) + \frac{c^*}{4\mu_1}\left(\theta_1 z, z\right),$$

$$\left|\left(-\varepsilon\left(\theta_0 R^{(0)}\right)_x, z\right)\right| \le \varepsilon\left|\left(\theta_0 R^{(0)}, z_{\bar{x}}\right)\right| \le \varepsilon\mu_2 \left|\left(\theta_0 z_{\bar{x}}, z_{\bar{x}}\right)\right| + \frac{\varepsilon}{4\mu_2}\left(\theta_0 R^{(0)}, R^{(0)}\right),$$

$$\left|\left(\theta_1 R^{(1)}, z\right)\right| \le \mu_3 \left|\left(\theta_1 z, z\right)\right| + \frac{1}{4\mu_3}\left(\theta_1 R^{(1)}, R^{(1)}\right)$$

and

$$\left|\left(\theta_1 R_c, z\right)\right| \le \mu_4 \left|\left(\theta_1 R_c, R_c\right)\right| + \frac{1}{4\mu_4}\left(\theta_1 z, z\right).$$

Taking $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \frac{1}{2}$ and merging these results, we obtain

$$\begin{aligned}
&\frac{1}{2}\left\{\varepsilon\left(1+\sigma\tau\right)\left(\theta_0 z_{\bar{x},i}, z_{\bar{x},i}\right) + \left(A+\sigma\tau B\right)\left(\theta_1 z, z\right)\right\}_{\bar{t}} \\
&+\frac{\tau}{2}\left\{\varepsilon\left(1+\sigma\tau\right)\left(\theta_0 z_{\bar{t}\bar{x},i}, z_{\bar{t}\bar{x},i}\right) + \left(A+\sigma\tau B\right)\left(\theta_1 z_{\bar{t}}, z_{\bar{t}}\right)\right\} \\
&+\varepsilon\left(\theta_0 \check{z}_{\bar{x}}, \check{z}_{\bar{x}}\right) + \left(\theta_1 B\check{z}, \check{z}\right) + \varepsilon\tau\left(\theta_0 \check{z}_{\bar{x}}, z_{\bar{t}\bar{x}}\right) + \tau\left(\theta_1 B\check{z}, z_{\bar{t}}\right) \\
&\le \frac{c^*}{2}\left(\theta_1 z_i^{j-M_0}, z_i^{j-M_0}\right) + \frac{c^*}{2}\left(\theta_1 z, z\right) + \frac{\varepsilon}{2}\left(\theta_0 z_{\bar{x}}, z_{\bar{x}}\right) \\
&+\frac{\varepsilon}{2}\left(\theta_0 R^{(0)}, R^{(0)}\right) + \left(\theta_1 z, z\right) + \frac{1}{2}\left(\theta_1 R^{(1)}, R^{(1)}\right) + \frac{1}{2}\left(\theta_1 R_c, R_c\right)
\end{aligned} \tag{21}$$

which concludes the proof of the lemma. □

Lemma 5. *Under the conditions of the Lemma (3) and rewriting $\sigma = 0.5$ in the relation (20), the remainder term R holds the following estimate:*

$$\|R\| \le C\left(h+\tau^2\right).$$

Proof. The proof of the lemma is similar manner as in [4,17]. □

5. Numerical Results

In this section, the numerical method is tested on two examples to validate the theory. To determine the reliability of the numerical approximation, we take into consideration the elimination method in [52]. Firstly, the difference equation (14) can be written explicitly:

$$\begin{aligned}
-\varepsilon h^{-2}\tau^{-1}&\left[\theta_{0,i+1}\left(y_{i+1}^j - y_{i+1}^{j-1} - y_i^j + y_i^{j-1}\right) - \theta_{0,i}\left(y_i^j - y_i^{j-1} - y_{i-1}^j + y_{i-1}^{j-1}\right)\right] \\
&+\tau^{-1}A\theta_1\left(y_i^j - y_i^{j-1}\right) - \varepsilon h^{-2}\left[\theta_{0,i+1}\left(y_{i+1}^{(\sigma)} - y_i^{(\sigma)} - y_i^{(\sigma)} + y_{i-1}^{(\sigma)}\right)\right] \\
&+\theta_{1,i}By_i^{(\sigma)} + \theta_1 c^j y_i^{j-M_0} = \theta_{1,i}F_i.
\end{aligned} \tag{22}$$

Secondly, we adapt the relation (22) according to the following difference equality:

$$D_i^* y_{i-1}^j - E_i^* y_i^j + G_i^* y_{i+1}^j = -H_i^*,\ i = 2, 3, ..., N-1, \quad j = 2, 3, ..., M-1.$$

Here, to express the term $y_i^{(\sigma)}$ in (22), we use

$$y^{(\sigma)} = \sigma y + (1-\sigma)\check{y} = \sigma y(x_i, t_j) + (1-\sigma) y(x_i, t_{j-1}). \tag{23}$$

Substituting $\sigma = \frac{1}{2}$ in the equation (23), we obtain

$$D_i^* = -\varepsilon h^{-2}\left(\theta_{0,i}\tau^{-1} - \frac{\theta_{0,i+1}}{2}\right),\ G_i^* = -\varepsilon h^{-2}\theta_{0,i+1}\left(\tau^{-1} + \frac{1}{2}\right),$$

$$E_i^* = -\varepsilon h^{-2}\theta_{0,i+1}\left(\tau^{-1} + 1\right) + \theta_{0,i+1}\tau^{-1} - \theta_{1,i}\left(A_i\tau^{-1} + \frac{B_i}{2}\right)$$

and

$$\begin{aligned} H_i^* = -\theta_{1,i}F_i + \theta_{1,i}c^j y_i^{j-M_0} + y_{i-1}^{j-1}\left(-\varepsilon h^{-2}\left(\frac{\theta_{0,i+1}}{2} + \theta_{0,i}\tau^{-1}\right)\right) \\ + y_{i+1}^{j-1}\left(-\varepsilon h^{-2}\theta_{0,i+1}\left(\frac{1}{2} + \tau^{-1}\right)\right) \\ + y_i^{j-1}\left(\theta_{1,i}\frac{B_i}{2} - \varepsilon h^{-2}\left(\theta_{0,i+1} + \tau^{-1}(1 + \theta_{0,i})\right) - \theta_{0,i+1}A_i\tau^{-1}\right). \end{aligned}$$

Thirdly, the coefficients of the elimination method [52] are indicated as

$$\alpha_{i+1} = \frac{G_i^*}{E_i^* - D_i^*\alpha_i}, \quad \beta_{i+1} = \frac{H_i^* + D_i^*\beta_i}{E_i^* - D_i^*\alpha_i},$$

and the output of the computational approach is calculated by

$$y_i^j = \alpha_{i+1} y_{i+1}^j + \beta_{i+1}, \quad i = 1, ..., N-1.$$

In numerical calculations, we use the double-mesh approach [19,24]. The approximate errors and ε-uniform maximum pointwise errors are noted as

$$e_\varepsilon^N = \max_{\omega_h \times \omega_\tau}\left|y_i^{\varepsilon,N} - y_i^{\varepsilon,2N}\right|$$

and

$$e^N = \max_\varepsilon e_\varepsilon^N.$$

Additionally, the order of convergence and ε-uniform rate of convergence are calculated as

$$p_\varepsilon^N = \frac{\ln(e_\varepsilon^N / e_\varepsilon^{2N})}{\ln 2},\ p^N = \frac{\ln(e^N/e^{2N})}{\ln 2}.$$

Example 1. *Consider the following singularly perturbed pseudo-parabolic initial-boundary value problem:*

$$\begin{aligned} -\varepsilon\frac{\partial^3 u}{\partial t \partial x^2} + (x^2(1-x)+1)\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (3 + t\sin(\pi x t))u + (1+t^2)u(x, t-r) \\ = e^{-t}\sin t(x + \sin(\pi x)),\quad (x,t) \in (0,l) \times (0,T] \end{aligned} ,$$

subject to the conditions

$$u(x,t) = \varphi(x,t) = e^{-t}\sin(\pi x),\ (x,t) \in \bar{\Omega} \times [-r,0],$$
$$u(0,t) = u(1,t) = 0,\ t \in (0,T],$$

where $l = 1$, $r = 1$ and $T = 2$. The computed results are shown in Table 1.

Example 2. *Take into account the second test problem:*

$$-\varepsilon\frac{\partial^3 u}{\partial t\partial x^2} + (1 + \frac{x}{2}(1-x))\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (3 + t\cos(\pi x t))u + (1 + \frac{t^2}{2})u(x, t-r)$$
$$= e^{-t}\sin t(x - \cos(\pi x)),\ (x,t) \in (0,l) \times (0,T] \quad ,$$

with

$$u(x,t) = \varphi(x,t) = e^{-t}\sin(2\pi x),\ (x,t) \in \bar{\Omega} \times [-r,0],$$
$$u(0,t) = u(1,t) = 0,\ t \in (0,T],$$

where $l = 1$, $r = 1$ and $T = 2$. The experimental results are presented in Table 2.

In Tables 1-2, the maximum nodal errors and order of convergence are presented for the values $N = 2^n$, $(n = 7, 8, ..., 11)$ and $\varepsilon = 2^{-2w}$, $(w = 1, 2, ..., 8)$. It is concluded that as the value N increases the maximum pointwise errors e^N, e^{2N} are decrease. This implies the reliability of the proposed scheme. Even though the presented numerical algorithm produce stable results, it can be further improved in terms of computational timing.

6. Discussion and Conclusion

In this paper, we have generated a new and efficient numerical scheme to solve initial-boundary value problems of singularly perturbed delay pseudo-parabolic equations. Using the energy estimates and difference analogues of integral inequalities, the error bounds and the parameter-uniform convergence of the proposed scheme have been analyzed. Two test problems have been solved and the experimental results have been reflected in Tables 1-2. From these results, it is observed that the order of convergence p^N is almost 2. In a nutshell, numerical applications agree with the theory. To improve the outlines of this study, the suggested approach can be carried out for more sophisticated problems involving higher dimensional equations, nonlinear functions, fractional derivatives.

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Table 1. Maximum pointwise errors e^N, e^{2N} and the order of convergence p^N on $\omega_h \times \omega_\tau$

ε				N		
		128	256	512	1024	2048
2^{-2}	e^N	0.00002343	0.00000586	0.00000146	0.00000037	0.00000009
	e^{2N}	0.00000586	0.00000146	0.00000037	0.00000009	0.00000002
	p^N	1.9997	2.0003	2.0002	2.0002	2.0000
2^{-4}	e^N	0.00009372	0.00002343	0.00000586	0.00000146	0.00000037
	e^{2N}	0.00002343	0.00000586	0.00000146	0.00000037	0.00000009
	p^N	1.9997	2.0003	2.0002	2.0001	2.0001
2^{-6}	e^N	0.00037489	0.00009374	0.00002346	0.00000586	0.00000146
	e^{2N}	0.00009374	0.00002343	0.00000586	0.00000146	0.00000037
	p^N	1.9998	2.0003	2.0002	2.0001	2.0001
2^{-8}	e^N	0.00150000	0.00037497	0.00009371	0.00002342	0.00000586
	e^{2N}	0.00037497	0.00009371	0.00002342	0.00000586	0.00000146
	p^N	2.0001	2.0004	2.0003	2.0001	2.0000
2^{-10}	e^N	0.00600672	0.00150030	0.00037488	0.00009370	0.00002342
	e^{2N}	0.00150030	0.00037488	0.00009370	0.00002342	0.00000585
	p	2.0013	2.0007	2.0003	2.0001	2.0001
2^{-12}	e^N	0.02413597	0.00600810	0.00149997	0.00037481	0.00009369
	e^{2N}	0.00600810	0.00149997	0.00037481	0.00009369	0.00002342
	p^N	2.0062	2.0019	2.0006	2.0002	2.0001
2^{-14}	e^N	0.09832477	0.02414373	0.00600685	0.00149968	0.00037477
	e^{2N}	0.02414320	0.00600685	0.00149968	0.00037477	0.00009368
	p^N	2.0259	2.0069	2.0019	2.0005	2.0001
2^{-16}	e^N	0.42375898	0.09838510	0.02413962	0.00600574	0.00149951
	e^{2N}	0.09838312	0.02413933	0.00600574	0.00149951	0.00037474
	p^N	2.1067	2.0270	2.0069	2.0018	2.0005

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TABLE 2. Maximum pointwise errors e^N, e^{2N} and the order of convergence p^N on $\omega_h \times \omega_\tau$

ε		N				
		128	256	512	1024	2048
2^{-2}	e^N	0.00002862	0.00000715	0.00000179	0.00000045	0.00000011
	e^{2N}	0.00000715	0.00000179	0.00000045	0.00000011	0.00000003
	p^N	2.0013	2.0008	2.0004	2.0002	2.0001
2^{-4}	e^N	0.00011449	0.00002860	0.00000714	0.00000179	0.00000045
	e^{2N}	0.00002859	0.00000714	0.00000179	0.00000045	0.00000011
	p^N	2.0013	2.0008	2.0004	2.0002	2.0001
2^{-6}	e^N	0.00045795	0.00011438	0.00002858	0.00000714	0.00000179
	e^{2N}	0.00011438	0.00002858	0.00000714	0.00000179	0.00000045
	p^N	2.0013	2.0008	2.0004	2.0002	2.0001
2^{-8}	e^N	0.00183193	0.00045753	0.00011432	0.00002857	0.00000714
	e^{2N}	0.00045752	0.00011432	0.00002857	0.00000714	0.00000179
	p^N	2.0014	2.0008	2.0004	2.0002	2.0002
2^{-10}	e^N	0.00732988	0.00183027	0.00045727	0.00011428	0.00002857
	e^{2N}	0.00183020	0.00045727	0.00011428	0.00002857	0.00000714
	p^N	2.0017	2.0009	2.0004	2.0002	2.0001
2^{-12}	e^N	0.02935407	0.00732329	0.00182923	0.00045713	0.00011426
	e^{2N}	0.00732307	0.00182922	0.00045713	0.00011426	0.00002856
	p^N	2.0030	2.0012	2.0005	2.0002	2.0001
2^{-14}	e^N	0.11797011	0.02932844	0.00731919	0.00182867	0.00045706
	e^{2N}	0.02932840	0.00731914	0.00182866	0.00045706	0.00011425
	p^N	2.0080	2.0025	2.0008	2.0003	2.0001
2^{-16}	e^N	0.48115439	0.11789250	0.02931321	0.00731696	0.00182837
	e^{2N}	0.11789250	0.02931274	0.00731695	0.00182837	0.00045702
	p^N	2.0290	2.0078	2.0022	2.0006	2.0002

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