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EDITORIAL

On behalf of the Editorial Board and my Associate Editors, it is my greatest honor and pleasure to introduce the new scientific journal titled Natural Sciences and Engineering Bulletin (NASE).

With the decision taken by the Institute Board and the University Senate 6 months ago, it was decided to start publishing NASE as the Journal of Graduate School of Natural and Applied Sciences. Graduate School of Natural and Applied Sciences started education in 1987 with the establishment of Gaziantep University. It is the most advanced graduate school in our region with 17 departments, 160 academicians and nearly 2000 doctoral and master's students.

NASE is an international peer-reviewed online journal that publishes original scientific articles in the fields of engineering and science, published by Gaziantep University. The launching of the journal was the result of intense efforts such as establishing the journal's editorial boards, defining the writing rules, ethical principles and publication policy. It is our great pleasure to welcome the members of the Editorial Board of NASE. We rely on their expertise for reviewing and accepting papers to the journal. The journal is published twice a year, in May and November.

The issue contains five articles from the disciplines of mathematics, food engineering and industrial engineering. All articles have been peer-reviewed using the double-blind review system. As the Editor-in-Chief, I would like to express my gratitude to all authors and reviewers who contributed to this issue. I would like to extend my special thanks to NASE Associate Editors Assoc. Prof. Dr. Mine MENEKŞE YILMAZ and Assoc. Prof. Dr. Tolgay KARA for their valuable efforts. I would also like to thank Prof. Dr. Emrah CİNKARA, NASE Language Editor, and Dr. Esra ÜNLÜ and Sibel TUTAR, Technical and Ethical Editors, for their valuable contributions.

I would like to invite you to submit your original research papers to NASE in the future so that this journal can fulfill its mission and benefit the scientific community. On behalf of the editorial board, authors and reviewers, I would like to welcome you to our first NASE issue.

Prof. Dr. Çiğdem AYKAÇ

Editör-in-Chief

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Advances of Digital Transformation Tools in Food Engineering Research: Process Simulation and Virtual Reality Applications in Production Processes

Ali Coşkun DALGIÇ[1*]

Keywords
Artificial Intelligence, Digital Twin, Process Simulation, Virtual Reality, Internet of Things

Abstract – The food industry, research and education, along with the development of technologies, are faced with very broad and complex application areas. In order to meet the nutritional needs and expectations of the growing world population and to ensure food quality and food safety expectations, there is an increasing need for computer applications in the control, modeling, optimization, and data analysis of production systems. In this context, digital transformation tools have a significant impact on food engineering research and education, as well as industrial applications. The aim of this study is to examine the role of digital transformation tools in food engineering, especially process simulation and virtual reality applications in production processes. Process simulation makes it possible to analyze (technical, economic, and environmental) various scenarios by creating mathematical models of production processes in a digital environment. When process simulations communicate with real-world production systems, digital twins are created. With this application, production efficiency increases, waste is reduced, and quality is improved. Virtual and augmented reality can be used in various areas, such as training, simulation, design, and inspection in production facilities. This technology allows users to simulate real-world scenarios and understand production processes more effectively. In the next part of the study, a framework is proposed for the integration of process simulation and virtual/augmented reality applications with other digital transformation tools. It is concluded that this framework will provide a powerful structure for optimizing and improving production processes in the food industry.

1. Introduction

Nowadays, research in the field of food engineering is becoming increasingly complex and researchers in this field are faced with many problems in the development of products, improvement of production processes, quality control, and ensuring food safety. Digital transformation tools have become an important tool in solving these problems. The use of digital transformation technologies in the food industry aims to increase efficiency and productivity, facilitate decision-making, increase profitability, develop innovative technologies and methods, and facilitate risk management (Konfo et al., 2023). There are various digital transformation tools used in the food industry at many stages, from agricultural production to end-user food safety. Artificial intelligence (AI), the Internet of Things (IoT), blockchain, digital twins, smart sensors, 3D printing, robots, big data, and virtual reality applications are among these tools (Nugroho et al., 2023) (Figure 1).

AI, process simulation, and IoT applications have a significant impact on food engineering research. AI, with its ability to analyze complex data and build predictive models, plays an important role in areas such as product formulation, nutrient composition optimization, and automation of production processes. AI comprises the technologies involved in electronic devices, computer systems, and robots, all designed to enhance and optimize the speed, precision, and effectiveness of user tasks (Esmaeily et al., 2024), (Thapa et al., 2023).

Process simulation is modeling and analyzing a process or system in a virtual environment to analyze, improve, and optimize the performance of an organization, business, or system. Digital twins, one of the digital transformation tools, are the creation of digital copies of real assets in the physical world. Digital replication is

[1]***Corresponding Author.** Gaziantep University, Engineering Faculty, Food Engineering Department, 27310 Gaziantep/Türkiye. E-mail: dalgic@gantep.edu.tr
OrcID: 0000-0001-6806-5917

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an important tool used to understand, control, and analyze the real world using process simulations (Koulouris et al., 2021). Process simulation supports decision-making processes in areas such as the design of production facilities, capacity planning, energy consumption optimization, and risk analysis. The evolution of these technologies in the field of food engineering has emerged as a reflection of the transformation in the industry (Saydam et al., 2020), (Dursun et al., 2020) and (Hassoun et al., 2022).

The IoT refers to a network of interconnected devices that can communicate and exchange data with each other over the internet without any physical intervention. Equipped with sensors, actuators, and connectivity features, these devices can collect data and transmit it to other devices or central systems, where it can be analyzed and acted upon. The majority of IoT applications in the food industry focus on monitoring temperature, traceability, humidity, color and enhancing sustainability performance (Konfo et al., 2023), (Hassoun et al., 2022).

Virtual and augmented realities are worlds created by digital technologies in virtual environments. Virtual reality (VR) creates fully digital environments for users to interact with, while augmented reality (AR) enhances the user's perception of their surroundings by overlaying digital content on top of the real world. VR and AR technologies, with the latest technological developments; are becoming widespread in many areas, especially in medicine and production. There are also many applications in the fields of engineering, education, automotive and tourism. VR and AR technologies in industrial manufacturing, from initial product design and assembly to real-time discussions between multidisciplinary teams around the world, provide fewer design errors throughout the production process, improved business solutions and longer effective working times (Crofton et al., 2019).

This comprehensive study aims to investigate the advantages and current potentials of contemporary technologies such as AI, VR, process simulation, and IoT by examining topics that are a combination of them. This study will also discuss a framework working model that includes the basic principles of these technologies, their use, and their integration with each other.

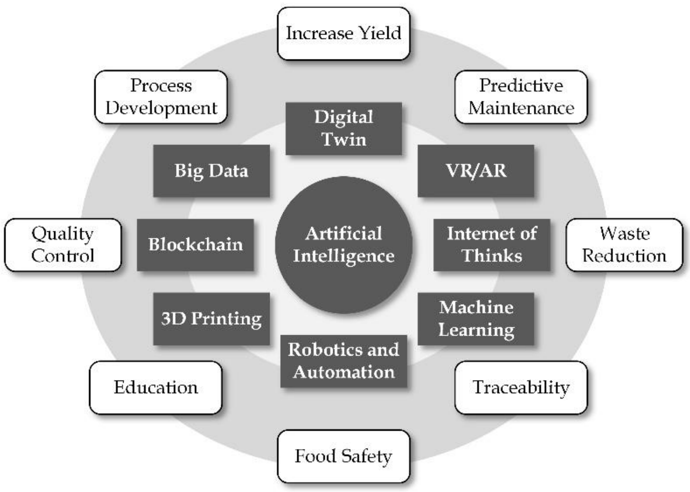

Figure 1. Digital transformation tools and applications in food engineering.

2. Digital Transformation Tools in Food Engineering

2.1. Artificial Intelligence

According to Haenlein et al. (2019), AI is defined as the ability of a system to comprehensively comprehend external data, learn from that data and use that learning in the applications that need it. Therefore, AI takes data from many sources (the IoT, big data sources, and expert systems) and uses knowledge-based rules or machine learning-based patterns to achieve predetermined goals. AI tools can also be classified as machine learning tools, natural language processing tools and image processing tools.

AI applications in the field of food engineering include sensory technology, computer vision systems (CVS), artificial neural networks (ANN) and fuzzy logic (FL) applications (J. Chen et al., 2020) (Table 1). The application of sensor technology is of great significance to AI. Dozens of studies exist that combine various types of sensors and food processing technologies to control key process parameters such as temperature during the drying of fruits and vegetables and pH levels for fermentation (J. Chen et al., 2020). Computer vision is one of the earliest AI technologies applied in food processes and used to measure parameters of physical properties such as size, shape, texture, color and quality of the products in real time (Kakani et al., 2023).

CVSs include tools such as machine learning, graphics, three-dimensional visuals, virtual reality, and augmented reality. There are many studies on this subject. In the review study by J. Chen et al. (2020), CVS applications in food drying systems are included. Among the image processing studies, CVS applications were performed in banana (Jiang et al., 2015), apple (Aghilinategh et al., 2016) and kiwifruit drying processes (Nadian et al., 2016).

ANN, another AI tool, consists of interconnected nodes, or artificial neurons, arranged in layers. Information is processed through the network by propagating signals from input neurons to output neurons through hidden layers. In the ANN, patterns and relationships in these data are first trained using algorithms, and then classification, regression, pattern recognition, and more applications are performed through the network (Barthwal et al., 2024). In many studies, ANNs have been used to modeling and optimization of food processes. Among these studies, ANNs were used in the drying processes of carrot (Erenturk and Erenturk, 2007), banana (Taheri et al., 2018), tomato (Abioye et al., 2024), and green peas (Barzegar et al., 2015) and in the optimization of extraction processes in soursoap fruit (Mahesh et al., 2024).

Fuzzy logic is a field of AI and computing used to address problems related to uncertainty. Unlike systems that work with traditional logic, this method recognizes uncertainty rather than relying on precise truth values (Atthajariyakul and Atthajariyakul, 2006), (Dash et al., 2024), (Li et al., 2021) and (Yousefi-Darani et al., 2019).

Table 1. Application of artificial intelligence in food processing

Products	**Aims**	**AI methods**	**Food Processing**	**References**
Carrot	Mathematical models drying process	ANN	Drying	Erenturk and Erenturk, 2007
Paddy	Optimum drying process conditions	FL	Fluidized bed drying	Atthajariyakul and Atthajariyakul, 2006
Banana	Process model	ANN	Hot air drying	Taheri et al., 2018
Mushrooms	Simulation and optimization of drying process	Genetic algorithm	Drying	Rahman et al., 2014
Banana	Drying uniformity analysis	CVS	Microwave-freeze drying	Jiang et al., 2015
Green peas	Quality optimization	ANN	Hot air infrared-assisted vibratory bed dryer	Barzegar et al., 2015
Apple	Image processing	CVS	Microwave convective drying	Aghilinategh et al., 2016
Kiwifruit	Improvement of kiwifruit drying	CVS	Hybrid hot air-infrared drying	Nadian et al., 2016
Tomato	Modelling of quality attributes	ANN	Convective hot-air drying	Abioye et al., 2024
Soursop fruit	Optimization of extraction techniques	ANN	Solvent extraction	Mahesh et al., 2024
Chili pepper	Improvement of classification method of peppers	ANN	Classification	Cruz et al., 2021
Rice pancake	Rheology and sensory evaluation	FL	Statistical analysis of rheological properties	Dash et al., 2024
Hawthorn	Control of relative humidity	FL	Microwave drying	Li et al., 2021
Dough	Process control	FL	Dough proofing	Yousefi-Darani et al., 2019

2.2. Digital Twins

The concept of a digital twin is a conceptual integrity created using real-world data needed by a simulation created to analyze a system or process (Y. Chen et al., 2020). According to Kritzinger et al. (2018), digital system integration between the physical and digital systems is classified as a digital model, a digital shadow, and a digital twin. A digital model is a digital representation of a physical system without automatic data exchange. A digital shadow is a union in which there is a one-way data flow from the physical system to the digital system created in the digital environment. In the concept of the digital twin, there is a unity including mutual automated data flow between the physical and digital systems (Kritzinger et al., 2018). The actual data transfer between the virtual and real systems is performed by synchronizing and simulating the data obtained from smart devices connected to the physical system through mathematical models.

Process simulation involves using computer-based models to analyze, predict, and optimize the behavior of industrial processes. It entails creating a digital representation of a real-world process, often with specialized software, to simulate how it behaves under different conditions. Process simulators enable mass and energy balances to be realized, equipment specifications to be determined, workforce needs to be estimated, profitability and sensitivity analyses to be made by economic evaluations, and environmental impacts to be investigated with virtual models of production processes created in a computer environment (Koulouris et al., 2021).

There are studies involving digital twins and model studies where high-value-added products are obtained from different food industry wastes (Table 2). The main ones of these wastes are sugar beet/cane molasses (Saydam et al., 2020), (Munagala et al., 2021), (Rathnayake et al., 2018), grain product wastes (Saydam et al., 2020), (Dasgupta et al., 2021), (Koulouris et al., 2021), vegetable oil production wastes (Donaldson et al., 2012), (Innocenzi and Prisciandaro, 2021), (Mabrouki et al., 2015), (Sayar et al., 2018), (Yun et al., 2013), and fruit and vegetable wastes (Lohrasbi et al., 2010), (Martínez-Ruano et al., 2018). Many of these studies involve biotechnological processes and mainly organic acids have been produced (Munagala et al., 2021), (Sayar et al., 2018). These organic acids (lactic acid and acetic acid) have higher economic values than the raw materials used in their production. In addition to organic acids, there are also simulation studies involving pigments (Dursun et al., 2020) and additives. There are also simulation studies involving the production of energy crops such as ethanol (Unrean and Khajeeram, 2016), (Rathnayake et al., 2018), biofuels (Mabrouki et al., 2015), biodiesel (Innocenzi and Prisciandaro, 2021), (Mabrouki et al., 2015), hydrogen gas (Han et al., 2016), and biogas (Mel et al., 2015), (Martínez-Ruano et al., 2018). In Karadeniz et al.'s study, the virtual and physical environment of the ice cream production machine is controlled by the digital twin concept (Karadeniz et al., 2019).

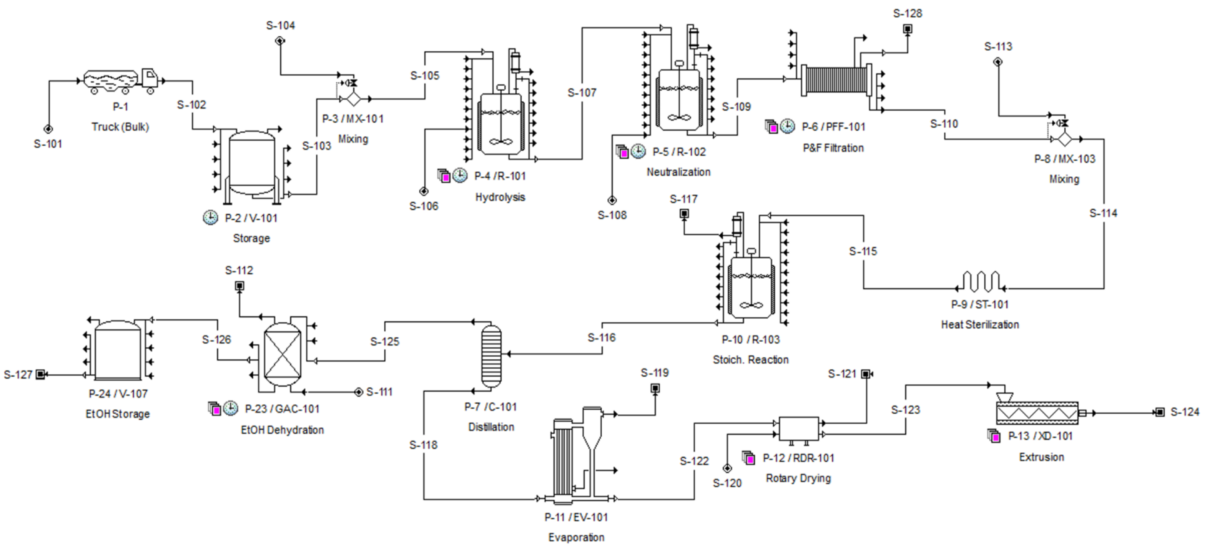

Figure 2. A digital model created by Superpro Designer for the production of bioethanol from sugar beet molasses (Dalgıç, 2018)

In process simulation, analyses can be performed from a single-stage process to a multi-stage process. It enables economic analysis (investment costs, operating costs, and profitability), work-time graphs, mass-energy balances, and sensitivity and uncertainty analysis in batch or continuous systems (Saydam et al., 2020), (Lohrasbi et al., 2010), (Mel et al., 2015), (Martínez-Ruano et al., 2018) and (Sayar et al., 2018). Environmental impact analyses are also performed using mass and energy balances (Dasgupta et al., 2021), (Martínez-Ruano et al., 2018), (Munagala et al., 2021) and (Rathnayake et al., 2018). The process is optimized technically and economically by changing production parameters (capacity, technology, input material characteristics, and economic data) (Saydam et al., 2020), (Unrean and Khajeeram, 2016) and (Yun et al., 2013).

Figure 2 illustrates the digital model for producing bioethanol from sugar beet molasses. This study encompasses alcohol purification following fermentation as well as the drying process for the distillation by-product (shillempe) (Dalgıç, 2018).

Table 2. Application of digital twins and models in food processing

Raw Material	Products	Food Processing	Analysis	DT/DM	References
Sugar beet molasses	Xanthan and sorbitol	Bioprocessing	Techno economic	DM	Saydam et al., 2020
Sugarcane bagasse	Lactic acid	Bioprocessing	Life cycle and economic assessment	DM	Munagala et al., 2021
Cotton stalk	Acetic acid	Bioprocessing	Retro-techno-economic evaluation	DM	Sayar et al., 2018
Agro-industrial wastes	Astaxanthin	Bioprocessing	Techno economic analysis	DM	Dursun et al., 2020
Sunflower seed	Energy and activated carbon	Chemical reaction	Process simulation	DM	Donaldson et al., 2012
Sugar cane bagasses	Ethanol	Fermentation	Optimization and techno-economic assessment	DM	Unrean and Khajeeram, 2016
Citrus waste	Limonene	Extraction	Process design and economic analysis	DM	Lohrasbi et al., 2010
Agricultural biomass	Biogas	Anaerobic digestion	Economic analysis	DM	Mel et al., 2015
Banana peel	Biogas	Anaerobic digestion	Techno- economic and environmental assessment	DM	Martínez-Ruano et al., 2018
Virgin oil and waste cooking oil	Biodiesel	Bioconversion	Technical feasibility	DM	Innocenzi and Prisciandaro, 2021
Palm oil residues	Biofuel	Pyrolysis	Process simulation	DM	Mabrouki et al., 2015
Waste cooking oil	Biodiesel	Bioconversion	Process simulation and energy optimization	DM	Yun et al., 2013
Cassava, cane molasses, and rice straw	Bioethanol	Fermentation	Life cycle assessment	DM	Rathnayake et al., 2018
Corncob	Xylitol	Bioconversion	Energy and life cycle impact assessment	DM	Dasgupta et al., 2021
Food waste	Hydrogen	Bioprocessing	Techno-economic evaluation	DM	Han et al., 2016
Malt	Beer	Fermentation	Production simulation and scheduling	DT	Koulouris et al., 2021
Milk	Ice cream	Ice Cream Machines	Process control	DT	Karadeniz et al., 2019

2.3. Internet of Things

The IoT refers to a network of physical objects or "things" embedded with sensors, software, and other technologies that enable them to communicate and exchange data with other devices and systems over the Internet. The IoT system consists of devices, network structure, software, and security. IoT system devices include all devices implemented in the environment and communication gateways, sensors (e.g., temperature, light, motion, location, etc.), devices that transmit and receive information (e.g., receivers and transmitters), energy supply devices (e.g., batteries, solar panels), and gateways that can manage functions. Devices also include all relevant communication technologies, both wired and wireless, such as Wi-Fi and Bluetooth (Bouzembrak et al., 2019), (Verdouw et al., 2016). The use of IoT in food supply chains has been identified as one of the application areas in food production, processing, storage, distribution, consumption, traceability, visibility, and controllability challenges.

2.4. Virtual and Augmented Reality

VR is defined as a computer-generated digital environment created as if it were real. AR improves the user's perception and interaction with the environment by superimposing digital content on the real world. The main purpose of VR technology is to involve the user in actually experiencing the stimulated environment, and at the same time, the user will use one or more senses to feel like they are in the real world. AR is a field of research that deals with the combination of real-world and computer-generated data. AR is a technology that enables virtual objects to be placed in the real world in real time. The main difference between VR and AR is that VR technology is based on virtual information, while AR technology uses the real environment with additional computer-generated information (Figure 3) (Carrasco and Chen, 2021), (Crofton et al., 2019). Table 3 represents the application of virtual and augmented reality in food processing.

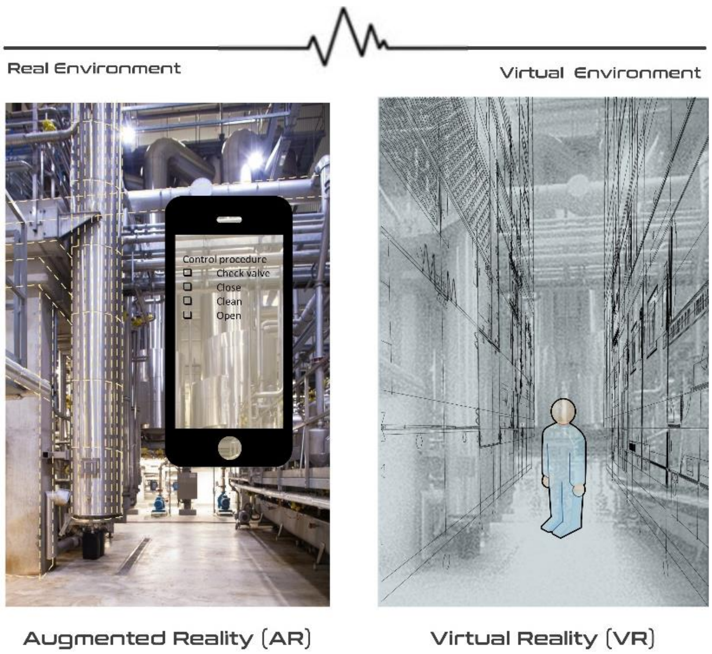

Figure 3. Virtual and real environment difference between VR and AR

2.4.1. Sensory Evaluations

Sensory evaluation is a scientific field that analyzes and measures human responses to the composition and nature of food and drink. It focuses on how we perceive and react to various stimuli using our five senses: sight, smell, touch, taste, and hearing. VR and AR technologies have the potential to revolutionize the way we collect and analyze sensory and consumer data. It is accepted in many studies that consumers' sensory responses are related to the condition of the environments in which food is consumed. It is also stated in many studies that the positive results in laboratories where food products are sensorial analyzed are different from the consumption habits in the real environment. Although there are many factors that may affect these results, one of the main reasons is that sensory analysis laboratories do not represent the real environment (Crofton et al., 2019), (Xu et al., 2021). Some of the sensory analysis studies that created real environment perception using VR and AR include chocolate (Van der Waal et al., 2021), snack foods (Pennanen et al. 2020), fruit juice and cake (Ammann et al., 2020), pastry (Alba-Martínez et al., 2022), and yogurt (Dong et al., 2021).

2.4.2. Food Quality and Safety

Real-time quality assurance and inspection play crucial roles in the food industry, guaranteeing that products adhere to the necessary safety and quality standards. AR technology is now being widely utilized in this field to transform inspection procedures, elevating the accuracy and efficiency of quality control. AR integrates with quality control activities, enabling online data transfer to auditors and the quality control team. These capabilities empower the food industry to continually improve quality standards, reduce waste, make more efficient use of labor, and minimize risks (Liberty et al., 2024).

2.4.3. Food Process Design

It is becoming increasingly difficult for food engineering students to visit large-scale facilities for process and plant design studies. This situation becomes even more difficult to access when the strict measures in occupational health and safety practices are taken into consideration. Apart from educational activities, it may not be possible for any investor to visit a facility in industrial applications in a competitive environment. Virtual reality applications open up opportunities for innovative approaches to provide simulated environments in higher education and in the design phase of industrial investment research (Hungler et al., 2022).

2.4.4. Food Traceability

Traceability, which is one of the most important applications in food safety, enables consumers to access healthy data about the food they consume. Consumers want to access healthy information about where the product they buy comes from and in which process steps it passes. Augmented reality applications offer a powerful interface to access digital information on this subject (Todorović et al., 2019).

Table 3. Application of virtual and augmented reality in food processing

Topics	Applications	VR/AR	References
Sensory evaluation	A research on psychological and physiological effects of chocolate in virtual reality environment	VR	Van der Waal et al., 2021
	Effect of virtual eating environment on consumers' evaluations of healthy and unhealthy snacks	VR	Pennanen et al., 2020
	Color evaluations of fruit juice and cake samples	VR	Ammann et al., 2020
	Investigation the effects of AR environments on the sensory responses of consumers towards different yogurts	AR	Dong et al., 2021
Product development	Product development and visual evaluation of pastry	VR	Alba-Martínez et al., 2022
Process design	Chemical plant design and development	VR	Hungler et al., 2022
Traceability	Food packaging	AR	Todorović et al., 2019

2.5. Blockchain in Food Processing

In food supply chains, ensuring accurate and timely information flow between producers, consumers, and regulatory authorities is an important prerequisite for minimizing food safety problems. Blockchain technology provides transparency in accessing product information within traceability systems. Information at all stages of production, from farm to fork, is recorded. In this case, health risks are reduced and fraud is prevented by eliminating production fraud (Islam and Cullen, 2021).

"Blockchain technology is a database that can be created in an electronic spreadsheet in a short time. The most important feature of these databases is that the information can only be added. Another feature is that each entry in the database (called a block) is cryptographically linked to the last entry" (Yuan and Zhou, 2023).

In the review study many hypotheses were proposed, and these hypotheses were supported by literature studies. One of the proposed hypotheses is that a blockchain-based food traceability system increases consumers' perceptions of product quality and safety, thus increasing their confidence in the product and increasing their willingness to purchase (Tao and Chao, 2024).

Successful traceability implementation requires collaboration and adoption across the entire supply chain, as well as addressing challenges such as interoperability, data privacy, and scalability.

3. Integration of Digital Transformation Tools

Based on the information gathered in the introduction to this review, a framework for how digital transformation tools can be used in food production systems, research, and educational tools is proposed. Figure 4 shows the framework for the integration of digital transformation tools. Each of the digital transformation tools contributes to achieving its goal with many different applications in order to increase efficiency, utilize limited resources, increase profitability, and reduce the environmental impacts of production systems. Therefore, it is imperative that all these tools be used effectively and efficiently.

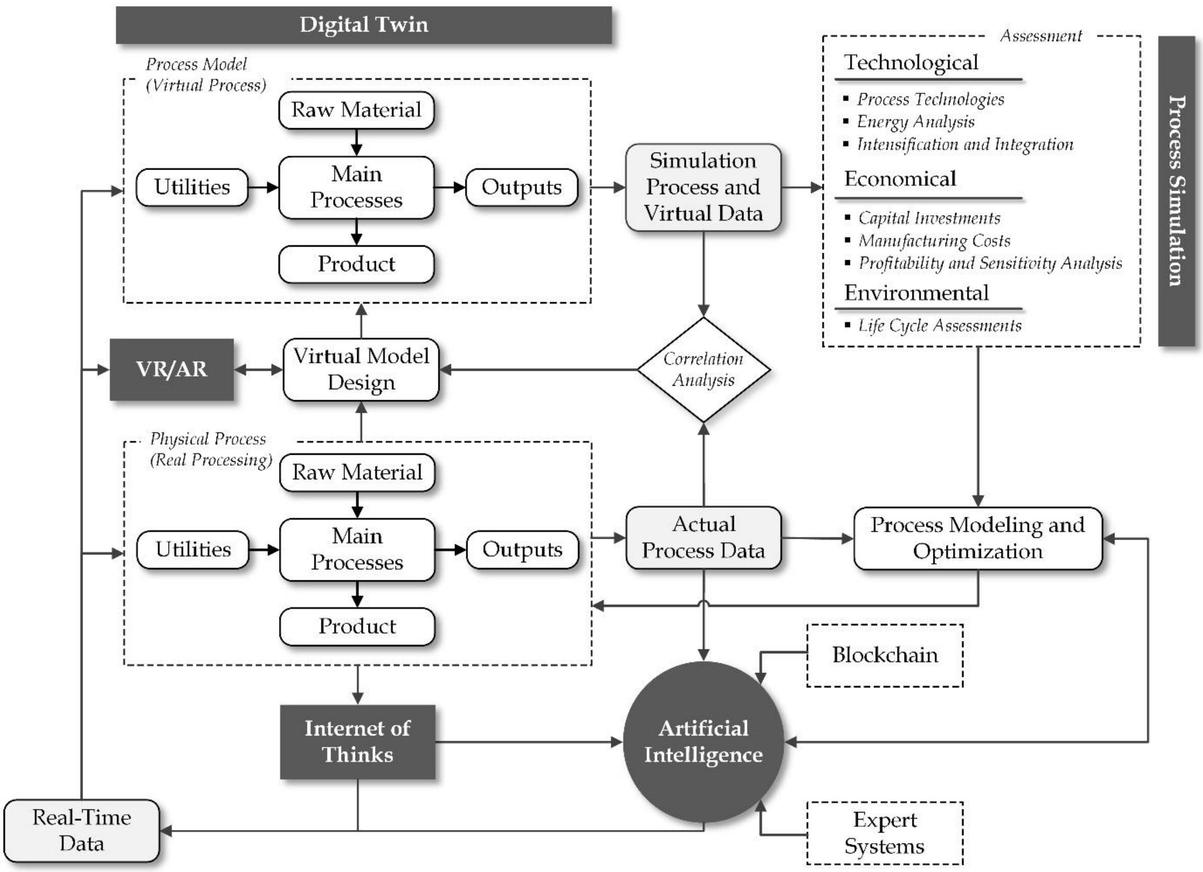

Figure 4. The frame work of the integration of digital transformation tools in food engineering

If we think of production systems as a two-stage system, design and production, we can use process simulations as a feasibility tool at the beginning. Likewise, when we consider VR technologies as a design tool, we can perceptually and technically analyze the production system in a healthier way before implementing it. Other tools can also be utilized during the design phase. AI is, of course, a great support tool for analyzing the designed processes.

In the evaluation of a production stage or equipment that has been digitally modeled in the design phase, many digital analyses are performed. These analyses can be classified as technical, economic, and environmental. As a result of these analyses, decision-making processes on issues such as other investment instruments and legal regulations come into play in the realization of the investment. In the production phase, a connection is created between the physical production and the digital model. This connection can be for data transfer only, or the data transfer between the digital process and the physical process can be for the control of the physical system. This is known as a digital twin. As a result of the evaluation of physical and digital production, the system efficiency is determined by the difference between theoretical and actual. The data generated within the system can be evaluated online. In this case, the IoT is a great help. For the non-system data that the system needs, blockchain, AI, and expert systems can be integrated into the system. AI tools can also be used in the control, monitoring, modeling, analysis, and optimization of the system. The integrated use of process simulations, AI, blockchain, VR/AR, and the IoT has many advantages, including dynamic simulation, optimization, real-time control, predictive maintenance, validation, and calibration of food processes.

4. Potential Challenges and Future Trends in the Food Industry

The application of digital transformation tools in the food industry brings many benefits, such as increased productivity and profitability, but also many implementation challenges. Factors such as high implementation costs in the start-up phase, data security and privacy concerns, the complexity of the supply chain from raw materials to the final product, and the old network structures of small businesses pose obstacles for businesses with limited budgets. In addition, the conservatism of employees in enterprises due to concerns about job loss and the lack of sufficient information infrastructure are also obstacles to the implementation of digital transformation tools.

Addressing data security and privacy concerns; researching and discussing potential risks will facilitate the implementation of digital transformation tools. In addition, risk assessments in terms of cyber security will also be useful in terms of applications. Workforce concerns of employees can be addressed through appropriate and effective training programs for digital transformation tools.

Digital transformation tools are expected to revolutionize the food industry, driving innovation, increasing efficiency, improving product quality and safety, and ultimately meeting the evolving demands of consumers in a rapidly changing market environment. Mainly agricultural practices, food safety systems, innovation of new functional products, and supply chain management will be impacted by AI applications as digital transformation tools for the food industry in the future. The use of virtual and augmented reality applications in food production processes can help optimize training, maintenance and production processes. It is also expected that optimization of production processes, improvement of product quality, product and process design and traceability will become widespread with digital twin applications.

5. Conclusion

In this review, studies involving digital transformation tools in food engineering research, education, and industrial applications were evaluated. As a result of these evaluations, a framework for the integration of digital transformation tools was developed and important findings emerged:

- Digital models of industrial production processes are predominantly created in the studies, and studies on digital twins have just started in food production processes.
- In most of the digital model-process simulation studies, the parameters affecting the processes are evaluated technically, economically, and environmentally.
- The digital twin process can make a great contribution to food engineering studies in real-time control, optimization, data-driven simulation, and predictive maintenance. With the application of the Internet of Things, process simulations can be applied more quickly and efficiently in collecting and monitoring data.
- Virtual and augmented reality applications are mainly used in sensory analysis studies. There are very few food engineering education and process design applications.
- Blockchain applications are mainly involved in traceability studies in food production systems. It is thought that this application can be applied to a wider area regarding food safety.

As a result, it is thought that each of the digital transformation tools can be applied differently in food production processes, and higher-quality and healthier products can be produced in an environmentally friendly and economical way.

Ethics Permissions

This paper does not require ethics committee approval.

Conflict of Interest

Author declare that there is no conflict of interest for this paper.

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A Note on Uniformly Convergence for Positive Linear Operators Involving Euler Type Polynomials

Erkan AGYUZ[1*]

Dedicated to my dear friend Ali Korcan Tan (1986-2023)

Keywords
Positive linear operators, Generating functions, Euler type polynomials, Moment Functions, Korovkin theorem

Abstract – Positive linear operators play a significant role in many domains, particularly numerical and mathematical analysis. Specifically, they are commonly found in a wide variety of methods to resolve optimization and differential equation issues. Basic properties of positive linear operators are linearity, positivity, positive linear and being restrictive. There are various ways to examine the significance of positive linear operators in Approximation Theory. The Convergence Analysis is the most significant of these. In many situations involving numerical analysis and convergence analysis, positive linear operators are essential. Positive linear operators must be able to converge in iterations towards a specific goal, especially in various approximation techniques or iterative solution algorithms. This can be used to solve optimization issues more effectively or to increase the precision of numerical answers. In approximation theory, generating functions are essential. They are specifically used to build algorithms that facilitate the proper approximation to a goal and to examine the approximation in question. The speed at which an approximation converges to a target can also be ascertained via generating functions. An essential tool for evaluating and enhancing the rate of convergence of iterative algorithms is offered by these functions. The aim of this study is to construct a generalized Kantorovich type Szász operators including the generating functions of Euler polynomials with order (-1). Moreover, we derive the moment and central moment functions for these operators. Finally, we show uniformly convergence of operators by using Korovkin theorem.

1. Introduction

An essential component of Approximation Theory is positive linear operators. Finding an approximate solution to a problem is the goal of Approximation Theory as opposed to an exact one. This method makes extensive use of positive linear operators, which are crucial for resolving a wide range of mathematical issues. Positive linear operators are useful tools for addressing mathematical problems and cover a wide range of topics in Approximation Theory. These operators are employed in many engineering and other applied fields, including optimization problems and iterative approaches, because of their properties that guarantee the efficiency and correctness of approximation solutions.

A useful tool in number theory, combinatorics, and other branches of mathematics are generating functions. They encode information about a series of numbers or objects and are simply formal power series. By generating functions, we can simplify issues requiring sequences into problems involving functions, which facilitates their analysis and solution using algebraic and calculus methods (Kilar and Simsek, 2021), (Simsek, 2018) and (Srivastava et. al, 2020).

The generating function of a sequence a_n is defined to be as follows:

[1]***Corresponding Author.** Gaziantep University, Naci Topçuoğlu Vocational School Electronics and Automation Department, Gaziantep Türkiye. E-mail: eagyuz86@gmail.com OrcID: 0000-0003-1110-7578

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$$A_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n,$$

where the a_n are coefficients of generating function for $i = 1,2, ...$ (McBride, 2012).

Concepts of generating functions and positive linear operators are utilized in several branches of mathematical analysis and have some commonalities. Mathematical issues are solved or approximated using both, albeit they are not utilized interchangeably. Nonetheless, generating functions can be used to express positive linear operators in specific situations. Recently, positive linear operators involving generating functions of some special polynomials have been intensively studied by many researchers.

Atakut and Büyükyazıcı constructed a generalization of Kantorovich type operators involving Brenke type polynomials as:

$$L_n^{\alpha_n,\beta_n}(f;x) = \frac{\beta_n}{A(1)B(\alpha_n x)} \sum_{k=0}^{\infty} p_k(\alpha_n x) \int_{k/\beta_n}^{(k+1)/\beta_n} f(t)dt,$$

for α_n, β_n and $p_k(x)$ see reference (Atakut and Büyükyazıcı, 2016).

İçöz *et al.* introduced a new type Szász operators including Appell polynomials at the following equation:

$$M_n(f;x) = \frac{1}{A\big(g(1)\big)B(nxg(1))} \sum_{k=0}^{\infty} p_k(\alpha_n x)\, f\left(\frac{k}{n}\right),$$

for $A, B, g(x)$ and $p_k(x)$ see reference (İçöz et al., 2016).

Menekşe Yılmaz established generalized Kantorovich type operators involving generating functions of Adjoint Bernoulli polynomials as follows:

$$\tilde{A}_n(f;x) = n\frac{e^{-nx}}{e-1} \sum_{k=0}^{n} \frac{\tilde{\beta}_n(nx)}{k!} \int_{k/n}^{(k+1)/n} f(t)dt$$

where $\tilde{\beta}_n(x)$ are called adjoint Bernoulli polynomials (Yılmaz, 2022). For more information on positive linear operators obtained using generating functions of special polynomials, see (Gezer and Yılmaz, 2023), (Özarslan et. al, 2008), (Sofyalıoğlu and Kanat, 2022), (Taşdelen et. al, 2012) and (Yılmaz, 2023).

The moments and central moments functions in Approximation Theory are engaged to define and analyze approximation methods based on orthogonal polynomials. They construct a framework for obtaining effective and attentive approximations of functions over specified intervals, which have applications in various areas of mathematics, engineering, and science. Moment and central moment functions are defined respectively to be as:

$$e_r(t) = t^r, \qquad r = 0,1,2, ...$$

and

$$L_n((e_1 - e_0 x)^r; x) := L_n((t - x)^r; x),$$

where L_n is a positive linear operator (Gupta and Rassias, 2019).

In Approximation Theory, which deals with approximating complicated functions by simpler ones, Korovkin's theorem offers a key result. It provides universal convergence conditions for function sequences, making approximation techniques easier to research and improve.

Theorem 1 (Korovkin Theorem) Let a sequence of linear positive operators $(L_n)_n$, $L_n: V \to \mathcal{F}[a,b]$ where $F[a,b]$ is space of all real-valued functions in the interval $[a,b]$ and V is a linear subspace of $\mathcal{F}[a,b]$. Suppose that $\varphi_0, \varphi_1, \varphi_2 \in V \cap C[a,b]$ forms a Chebychev system on the interval $[a,b]$, if we have

$$\lim_{n\to\infty} L_n(\varphi_j) = \varphi_j$$

uniformly for $j = 0,1,2$, then

$$\lim_{n\to\infty} L_n(f) = f,$$

uniformly, for any $f \in V \cap C[a,b]$ (Paltanea, 2012).

The theorem of Bohman is the particular version of above theorem when $\varphi_j = e_j, j = 0,1,2$. The monomial functions denoted by e_j are defined to be as moment functions.

2. Materials and Methods

In this section we define a new operator. Then we calculate the moment and central moment functions for our operator. We also give some results involving these functions. Finally, by using the moment functions, the uniform convergence of our operator is obtained with the help of Korovkin's theorem.

2.1. Construction of the operator

Let α_n and β_n be strictly increasing sequences such that

$$\lim_{n\to\infty} \frac{1}{\beta_n} = 0\,, \frac{\alpha_n}{\beta_n} = 1 + O\left(\frac{1}{\beta_n}\right),$$

where the O symbol is called big-O.

The negative order Euler polynomials are defined to be as:

$$\sum_{n=0}^{\infty} E_n^{(-k)}(x)\frac{t^n}{n!} = \left(\frac{e^t+1}{2}\right)^k e^{xt},$$

where $k > 0$ and $n > -k$ (Horadam, 1992).

Substituting $k = 1$ into above equation gives the generating function of the Euler polynomials with order (-1) $E_n^{(-1)}$ as follows:

$$\sum_{n=0}^{\infty} E_n^{(-1)}(x)\frac{t^n}{n!} = \left(\frac{e^t+1}{2}\right) e^{xt}.$$

By using above mathematical tools, we obtain a generalized Kantorovich type operators involving Euler polynomials with order (-1), $E_n^{(-1)}(x)$, at the following equation:

$$T_n^{\alpha_n,\beta_n}(f;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\int_{\frac{k}{\beta_n}}^{\frac{k+1}{\beta_n}} f(t)dt \tag{1}$$

where $T_n^{\alpha_n,\beta_n}: L([0,1]) \to C([0,1])$ and $f \in C([0,1])$.

The derivatives of generating functions of Euler polynomials with order (-1) are obtained to be as follows:

$$\sum_{k=0}^{\infty}\frac{kE_n^{(-1)}(x)}{k!}t^{k-1} = \frac{1}{2}e^{\alpha_n x}(e(\alpha_n x+1)+\alpha_n x) \tag{2}$$

$$\sum_{k=0}^{\infty}\frac{k(k-1)E_n^{(-1)}(x)}{k!}t^{k-2} = \frac{1}{2}e^{\alpha_n x}(e(\alpha_n x+1)^2+(\alpha_n x)^2), \tag{3}$$

$$\sum_{k=0}^{\infty}\frac{k(k-1)(k-2)E_n^{(-1)}(x)}{k!}t^{k-3} = \frac{1}{2}e^{\alpha_n x}(e(\alpha_n x+1)^3+(\alpha_n x)^3), \tag{4}$$

$$\sum_{k=0}^{\infty}\frac{k(k-1)(k-2)(k-3)E_n^{(-1)}(x)}{k!}t^{k-4} = \frac{1}{2}e^{\alpha_n x}(e(\alpha_n x+1)^4+(\alpha_n x)^4). \tag{5}$$

2.2. Moments and Central Moments Functions of Operator

The following lemmas are given to prove the uniform convergence of the operator and to obtain some results of the operator.

Lemma 1 (Moment functions) For all $x \in [0,1]$ and $n \in \mathbb{N}$, the $T_n^{\alpha_n,\beta_n}$ satisfy the following equations:

$$T_n^{\alpha_n,\beta_n}(e_0(x);x) = 1, \tag{6}$$

$$T_n^{\alpha_n,\beta_n}(e_1(x);x) = \frac{\alpha_n}{\beta_n}x + \frac{3e+1}{2\beta_n(e+1)}, \tag{7}$$

$$T_n^{\alpha_n,\beta_n}(e_2(x);x) = \frac{\alpha_n^2}{\beta_n^2}x^2 + \frac{\alpha_n}{\beta_n^2}\frac{4e+2}{e+1}x + \frac{10e+1}{3\beta_n^2(e+1)}, \tag{8}$$

where $e_i(x) = x^i \in C([0,1])$ for $i = 0,1,2$.

Proof Let $f(x) = e_0(x) = 1$. By substituting $f(x) = 1$ in (1), the following equation is obtained:

$$T_n^{\alpha_n,\beta_n}(1;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\int_{\frac{k}{\beta_n}}^{\frac{k+1}{\beta_n}} f(t)dt. \tag{9}$$

Using the definition of generating function of $E_n^{(-1)}(x)$ and taking integral in (9), we have

$$T_n^{\alpha_n,\beta_n}(1;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\left(\frac{e+1}{2}\right)e^{\alpha_n x}\frac{1}{\beta_n} = 1, \tag{10}$$

where $t = 1$ and $x \to \alpha_n x$.

Let $f(x) = e_1(x) = x$. By substituting $f(x) = x$ in (1), we give

$$T_n^{\alpha_n,\beta_n}(x;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\int_{k/\beta_n}^{k+1/\beta_n} t dt. \tag{11}$$

Using the definition of generating function of $E_n^{(-1)}(x)$ and taking integral in (11), we obtain

$$T_n^{\alpha_n,\beta_n}(x;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\frac{2k+1}{2{\beta_n}^2}$$

$$= \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{{\beta_n}^2}\sum_{k=0}^{\infty}\frac{kE_n^{(-1)}(\alpha_n x)}{k!} + \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{2{\beta_n}^2}\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}. \tag{12}$$

Applying (2) into (12), we obtain

$$T_n^{\alpha_n,\beta_n}(x;x) = \frac{\alpha_n}{\beta_n}x + \frac{3e+1}{2\beta_n(e+1)}. \tag{13}$$

Let $f(x) = e_2(x) = x^2$. By substituting $f(x) = x^2$ in (1), we give

$$T_n^{\alpha_n,\beta_n}(x^2;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\int_{k/\beta_n}^{k+1/\beta_n} t^2 dt. \tag{14}$$

Using the definition of generating function of $E_n^{(-1)}(x)$ and taking integral in (14), we obtain

$$T_n^{\alpha_n,\beta_n}(x;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\frac{3k^2+3k+1}{3{\beta_n}^3}$$

$$= \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{{\beta_n}^3}\sum_{k=0}^{\infty}\frac{k^2E_n^{(-1)}(\alpha_n x)}{k!} + \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{3{\beta_n}^3}\sum_{k=0}^{\infty}\frac{kE_n^{(-1)}(\alpha_n x)}{k!} +$$
$$\beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{3{\beta_n}^3}\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}. \tag{15}$$

Applying (2) and (3) into (15), we obtain

$$T_n^{\alpha_n,\beta_n}(x^2;x) = \frac{\alpha_n^2}{\beta_n^2}x^2 + \frac{\alpha_n}{\beta_n^2}\frac{4e+2}{e+1}x + \frac{10e+1}{3\beta_n^2(e+1)}. \tag{16}$$

From (10), (13) and (16), the desired results are obtained.

Lemma 2 (Central Moment functions) For all $x \in [0,1]$ and $n \in \mathbb{N}$, the $T_n^{\alpha_n,\beta_n}$ satisfy the following equations:

$$T_n^{\alpha_n,\beta_n}(e_1 - x;x) = \left(\frac{\alpha_n}{\beta_n} - 1\right)x + \frac{3e+1}{2\beta_n(e+1)}, \tag{17}$$

and

$$T_n^{\alpha_n,\beta_n}((e_1 - x)^2;x) = \left(\frac{\alpha_n^2}{\beta_n^2} - \frac{2\alpha_n}{\beta_n} + 1\right)x^2 + \frac{\alpha_n}{\beta_n}\left(\frac{7e+3}{(e+1)\beta_n}\right) + \frac{10e+1}{3{\beta_n}^2(e+1)} \tag{18}$$

Proof Applying the linear property of $T_n^{\alpha_n,\beta_n}$, we have

$$T_n^{\alpha_n,\beta_n}(e_1 - x;x) = T_n^{\alpha_n,\beta_n}(e_1;x) - xT_n^{\alpha_n,\beta_n}(1;x) = \left(\frac{\alpha_n}{\beta_n} - 1\right)x + \frac{3e+1}{2\beta_n(e+1)}, \tag{19}$$

and

$$\begin{aligned} T_n^{\alpha_n,\beta_n}((e_1 - x)^2;x) &= T_n^{\alpha_n,\beta_n}(e_2;x) - 2xT_n^{\alpha_n,\beta_n}(e_1;x) + x^2T_n^{\alpha_n,\beta_n}(1;x) \\ &= \left(\frac{\alpha_n^2}{\beta_n^2} - \frac{2\alpha_n}{\beta_n} + 1\right)x^2 + \frac{\alpha_n}{\beta_n}\left(\frac{7e+3}{(e+1)\beta_n}\right) + \frac{10e+1}{3{\beta_n}^2(e+1)}. \end{aligned} \tag{20}$$

From (19) and (20), the proof is completed.

3. Uniformly Convergence of $T_n^{\alpha_n,\beta_n}$

In this section, we give uniformly convergence property of $T_n^{\alpha_n,\beta_n}$ by using moment functions at the following theorem:

Theorem 2 If $f \in C([0,1])$,

$$\lim_{n\to\infty} T_n^{\alpha_n,\beta_n}(f;x) = f(x) \tag{21}$$

uniformly convergence on $[0,1]$.

Proof We know that,

$$\lim_{n\to\infty} T_n^{\alpha_n,\beta_n}(e_0(x);x) = \lim_{n\to\infty} 1 = 1 \tag{22}$$

$$\lim_{n\to\infty} T_n^{\alpha_n,\beta_n}(x;x) = \lim_{n\to\infty} T_n^{\alpha_n,\beta_n}(e_1(x);x) = \lim_{n\to\infty}\left(\frac{\alpha_n}{\beta_n}x + \frac{3e+1}{2\beta_n(e+1)}\right) = x \tag{23}$$

and

$$\lim_{n\to\infty} T_n^{\alpha_n,\beta_n}(x^2;x) = \lim_{n\to\infty} T_n^{\alpha_n,\beta_n}(e_2(x);x) = \lim_{n\to\infty}\left(\frac{\alpha_n^2}{\beta_n^2}x^2 + \frac{\alpha_n}{\beta_n^2}\frac{4e+2}{e+1}x + \frac{10e+1}{3\beta_n^2(e+1)}\right) = x^2 . \tag{24}$$

Using Theorem 1, the desired result is obtained.

4. Conclusion

In this study, we first introduced a new generalized Kantorovich type Szász operators involving the generating functions of Euler type polynomials with order (-1) and we showed the moment and central moment functions of our operators. The moment and central moment functions are important mathematical tools used to investigate the convergence properties of positive linear operators. With the help of moment functions, we showed that our operator is uniformly convergent.

In future studies, properties of the operator such as convergence speed and convergence error estimation can be investigated.

Ethics Permissions

This paper does not require ethics committee approval.

Conflict of Interest

Author declare that there is no conflict of interest for this paper.

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Fractional Order Mathematical Modeling of COVID-19 Dynamics with Mutant and Quarantined Strategy

İlknur KOCA*

Keywords
Fractional differential equation, Covid-19 model, Existence and Uniqueness

Abstract – Mathematical models provide a common language for communicating ideas, theories, and findings across disciplines. They allow researchers to represent complex concepts in a concise and precise manner, facilitating collaboration and interdisciplinary research. Additionally, visual representations of models help in conveying insights and understanding complex relationships. Mathematical modeling finds applications in various areas across science, engineering, economics, and other fields. Recently disease models have helped us understand how infectious diseases spread within populations. By studying the interactions between susceptible, infected, and recovered individuals, we can identify key factors influencing transmission, such as contact patterns, population density, and intervention strategies. The incorporation of fractional order modeling in studying disease models such as COVID-19 dynamics holds significant importance, offering a more accurate and efficient portrayal of system behavior compared to conventional integer-order derivatives. So in this study, we adopt a fractional operator-based approach to model COVID-19 dynamics. The existence and uniqueness of solutions are crucial properties of mathematical models that ensure their reliability, stability, and relevance for real-world applications. These properties underpin the validity of predictions, the interpretability of results, and the effectiveness of models in informing decision-making processes. Our investigation focuses on positivity of solutions, the existence and uniqueness of solutions within the model equation system, thereby contributing to a deeper understanding of the pandemic's dynamics. Finally, we present a numerical scheme for our model.

1. Introduction

There are various approaches to modeling the spread of infectious diseases, including compartmental models like the SIR (Susceptible-Infectious-Recovered) model, SEIR (Susceptible-Exposed-Infectious-Recovered) model, and their variations (Alkahtani and Koca, 2021), (Anderson and May, 1991), (Kermack and McKendrick, 1927). One of the simplest and most widely used models is the SIR model. It divides the population into three compartments: susceptible, infectious, and recovered. Strengths include simplicity, ease of interpretation, and applicability to large populations. However, it assumes homogeneous mixing, constant parameters and does not consider demographic or spatial heterogeneity. Extending the SIR model by adding an exposed compartment to account for the latent period between infection and becoming infectious, SEIR model better captures the incubation period of the disease.

In contemporary mathematical modeling, there has been a noticeable shift from employing classical derivatives to embracing fractional derivatives. This transition is reflected in recent research, where mathematicians have increasingly incorporated fractional differential operators into their models. These operators, encompassing exponential, Mittag-Leffler kernels, and power-law distributions, offer alternative frameworks for describing diverse phenomena (Atangana and Baleanu, 2016), (Caputo and Fabrizio, 2016), (Podlubny, 1999). Fractional order models are particularly useful for describing systems with long-range interactions or non-local effects. By considering fractional derivatives, these models can more accurately represent the underlying physics or biology (Koca and Ozalp, 2013), (Koca, 2018).

***Corresponding Author.** Department of Economics and Finance, Fethiye Business Faculty, Mugla Sıtkı Kocman University, 48300, Mugla, Türkiye. E-mail: ilknurkoca@mu.edu.tr OrcID: https://orcid.org/0000-0003-4393-1588
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Modeling the spread of COVID-19 has been a critical area of research since the pandemic began. Since the beginning of the COVID-19 pandemic, there has been a significant surge in research and publications related to various aspects of the disease (Dokuyucu and Celik, 2021). Researchers from diverse fields including epidemiology, virology, public health, medicine, mathematics, computer science, and social sciences have contributed to the growing body of knowledge on COVID-19.

In a recent publication (Yu et al., 2024), researchers introduce a novel nonlinear dynamics model called SEIMQR (Susceptible-Exposed-Infected-Mutant-Quarantined-Recovered), designed specifically to delve into the intricacies of COVID-19 transmission dynamics and to forecast its future trends with greater precision. In this paper, different from publication (Yu et al., 2024), we consider their model as fractional order to describe system behavior with greater accuracy and efficiency compared to traditional integer-order derivatives.

2. Preliminaries

Our focus in this section is to provide clear and concise definitions of non-integer fractional derivatives and integrals (Kilbas et al., 2006).

Definition 1 Riemann-Liouville definition of fractional order derivative:

$$ {}_{a}^{RL}D_t^{\varepsilon} f(t) = \frac{1}{\Gamma(n-\varepsilon)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\varepsilon-1} f(\tau) d\tau, \tag{1} $$

where

$$ n-1 < \varepsilon \leqslant n, n \in \mathbb{N} \tag{2} $$

and $\varepsilon \in R$ is a fractional order of the differ-integral of the function $f(t)$.

Definition 2 Caputo's definition of fractional order derivative:

$$ {}_{a}^{C}D_t^{\varepsilon} f(t) = \frac{1}{\Gamma(n-\varepsilon)} \int_a^t (t-\tau)^{n-\varepsilon-1} f^n(\tau) d\tau. \tag{3} $$

Here, $n-1 < \varepsilon \leqslant n$, $n \in \mathbb{N}$, $\varepsilon \in \mathbb{R}$ is a fractional order of the derivative of the function $f(t)$.

Definition 3 According to Riemann-Liouville's perspective, the Riemann-Liouville fractional integral of order $\varepsilon > 0$ for a function $f: (0, \infty) \to R$ is defined as the antiderivative of f with respect to a fractional exponent :

$$ I_t^{\varepsilon} f(t) = \frac{1}{\Gamma(\varepsilon)} \int_0^t (t-\tau)^{\varepsilon-1} f(\tau) d\tau. \tag{4} $$

3. Model Derivation

Herein, we undertake the examination of a Covid-19 model, incorporating a standard incidence specified as follows:

$$ \begin{aligned} \frac{ds(t)}{dt} &= k_1 - k_2 s(t)(i(t)+e(t)) - k_3 s(t) m(t) - k_4 s(t), \\ \frac{de(t)}{dt} &= k_2 s(t)(i(t)+e(t)) + k_3 s(t) m(t) - k_5 e(t) - k_6 e(t) - k_4 e(t), \\ \frac{di(t)}{dt} &= \frac{1}{2} k_5 e(t) - k_7 i(t) - k_6 i(t) - k_8 i(t) - k_1 i(t), \\ \frac{dm(t)}{dt} &= \frac{1}{2} k_5 e(t) + k_7 i(t) - k_8 m(t) - k_6 m(t) - k_4 m(t), \\ \frac{dq(t)}{dt} &= k_6 (e(t)+i(t)+m(t)) - k_1 q(t), \end{aligned} \tag{5} $$

where s, e, i, m, q and $n = s+e+i+m+q$ is the number of total population individuals.

$$ s(t_0) = s_0, e(t_0) = e_0, i(t_0) = i_0, m(t_0) = m_0 \text{ and } q(t_0) = q_0. $$

The entire population within the SEIMQ (Susceptible-Exposed-Infected-Mutant-Quarantined) model can be classified into six distinct groups, each with its respective characteristics outlined as susceptible (s) denotes individuals who have not contracted the virus but are at risk of infection when in contact with carriers, exposed (e) refers to individuals who have been infected with the virus but have yet to display symptoms, infected (i) represents those who have contracted the virus and are exhibiting symptoms, mutant (m) is attributed to individuals infected with a variant strain of the virus and quarantined (q) is designated for individuals isolated to prevent viral transmission to the broader community. The parameters within the system (5) are characterized by positive constants.

The field of calculus, encompassing fractional derivatives and integrals, has garnered growing attention from researchers. Fractional operators have been recognized for their superior ability to depict system behavior compared to integer-order derivatives. Given the significant advantage in memory properties, we propose enhancing the aforementioned system by substituting the integer-order time derivative with the Caputo fractional derivative as presented below:

$$\begin{aligned} {}_0^C D_t^\alpha s(t) &= k_1 - k_2 s(t)(i(t)+e(t)) - k_3 s(t)m(t) - k_4 s(t), \\ {}_0^C D_t^\alpha e(t) &= k_2 s(t)(i(t)+e(t)) + k_3 s(t)m(t) - k_5 e(t) - k_6 e(t) - k_4 e(t), \\ {}_0^C D_t^\alpha i(t) &= \frac{1}{2}k_5 e(t) - k_7 i(t) - k_6 i(t) - k_8 i(t) - k_1 i(t), \\ {}_0^C D_t^\alpha m(t) &= \frac{1}{2}k_5 e(t) + k_7 i(t) - k_8 m(t) - k_6 m(t) - k_4 m(t), \\ {}_0^C D_t^\alpha q(t) &= k_6(e(t)+i(t)+m(t)) - k_1 q(t), \end{aligned} \tag{6}$$

with the initial conditions

$$s(t_0) = s_0, e(t_0) = e_0, i(t_0) = i_0, m(t_0) = m_0 \text{ and } q(t_0) = q_0. \tag{7}$$

3.1. The positivity and boundedness of solutions

The aim of this section is to illustrate the positivity of the solutions of the system concerning with $\forall t \geqslant 0$, we define the norm

$$\|\Phi\|_\infty = \sup_{t\in[0,T]} |\Phi(t)|. \tag{8}$$

We begin by defining the system and then proceed to address the first equation:

$$\begin{aligned} \frac{ds(t)}{dt} &= k_1 - k_2 s(t)(i(t)+e(t)) - k_3 s(t)m(t) - k_4 s(t), \forall t \geqslant 0, \\ &\geq -k_2(i(t)+e(t))s(t) - k_3 m(t)s(t) - k_4 s(t), \forall t \geqslant 0, \\ &\geq \left(-k_2\left(\sup_{t\in[0,T]}|i(t)| + \sup_{t\in[0,T]}|e(t)|\right) - k_3 \sup_{t\in[0,T]}|m(t)| - k_4\right) s(t), \forall t \geqslant 0, \\ &\geq -(k_2(\|i\|_\infty + \|e\|_\infty) + k_3\|m\|_\infty + k_4)s(t), \forall t \geqslant 0. \end{aligned} \tag{9}$$

Then this provides that

$$s(t) \geq s_0 e^{-(k_2(\|i\|_\infty+\|e\|_\infty)+k_3\|m\|_\infty+k_4)t}, \forall t \geqslant 0. \tag{10}$$

Secondly for the function $e(t)$, we obtain

$$\begin{aligned} \frac{de(t)}{dt} &= k_2 s(t)(i(t)+e(t)) + k_3 s(t)m(t) - k_5 e(t) - k_6 e(t) - k_4 e(t), \forall t \geqslant 0, \\ &\geq -(k_5+k_6+k_4)e(t), \forall t \geqslant 0. \end{aligned}$$

So this dictates that

$$e(t) \geq e_0 e^{-(k_5+k_6+k_4)t}, \forall t \geqslant 0. \tag{11}$$

Here we assume that $s(t)$ and $m(t)$ are nonnegative solutions. For equation $i(t)$, we obtain

$$\frac{di(t)}{dt} = \frac{1}{2}k_5 e(t) - k_7 i(t) - k_6 i(t) - k_8 i(t) - k_1 i(t), \forall t \geqslant 0, \tag{12}$$

$$\geq -(k_7 + k_6 + k_8 + k_1) i(t), \forall t \geqslant 0.$$

This dictates that

$$i(t) \geq i_0 e^{-(k_7+k_6+k_8+k_1)t}, \forall t \geqslant 0. \tag{13}$$

Now let us check for the fourth equation is given by

$$\frac{dm(t)}{dt} = \frac{1}{2}k_5 e(t) + k_7 i(t) - k_8 m(t) - k_6 m(t) - k_4 m(t), \forall t \geqslant 0 \tag{14}$$

$$\geq -(k_8 + k_6 + k_4) m(t), \forall t \geqslant 0.$$

So we have

$$m(t) \geq m_0 e^{-(k_8+k_6+k_4)t}, \forall t \geqslant 0. \tag{15}$$

For final equation of model we get

$$\frac{dq(t)}{dt} = k_6 (e(t) + i(t) + m(t)) - k_1 q(t), \forall t \geqslant 0 \tag{16}$$

$$\geq -k_1 q(t), \forall t \geqslant 0.$$

So we have

$$q(t) \geq q_0 e^{-k_1 t}, \forall t \geqslant 0. \tag{17}$$

4. Existence and Uniqueness

Existence and uniqueness conditions are fundamental for establishing the mathematical validity, predictability, and stability of solutions to ordinary differential equations, thereby enabling their application in various scientific and engineering domains. The importance of existence and uniqueness for ordinary differential equations lies in their fundamental role in ensuring the well-posedness of mathematical models and the predictability of solutions. In this section, we provide an in-depth examination of the existence and uniqueness of the equation system. To accomplish this objective, we verify the following theorem (Atangana, 2021).

Theorem 1 With the presence of positive constants p_i and $\overline{p}_i$ satisfying the following:

(i) $\forall i \in \{1,2,3,4,5\}$,

$$|\Phi_i(t, x_i) - \Phi_i(t, x_i',)|^2 \leq p_i |x_i - x_i'|^2. \tag{18}$$

(ii) $\forall (x, t) \in R \times [0, T]$,

$$|\Phi_i(t, x_i)|^2 \leq \overline{p}_i (1 + |x_i|^2). \tag{19}$$

Then the system of equations has a unique system of solutions. Let us revisit our model with taking right side of model as follows:

$$\frac{ds(t)}{dt} = \Phi_1(t, s), \tag{20}$$

$$\frac{de(t)}{dt} = \Phi_2(t, e),$$

$$\frac{di(t)}{dt} = \Phi_3(t, i),$$

$$\frac{dm(t)}{dt} = \Phi_4(t, m),$$

$$\frac{dq(t)}{dt} = \Phi_5(t, q).$$

Here we consider

$$\Phi_1(t, s) = k_1 - k_2 s(t)(i(t) + e(t)) - k_3 s(t) m(t) - k_4 s(t), \tag{21}$$

$$\Phi_2(t,e) = k_2 s(t)(i(t)+e(t)) + k_3 s(t)m(t) - k_5 e(t) - k_6 e(t) - k_4 e(t),$$

$$\Phi_3(t,i) = \frac{1}{2}k_5 e(t) - k_7 i(t) - k_6 i(t) - k_8 i(t) - k_1 i(t),$$

$$\Phi_4(t,m) = \frac{1}{2}k_5 e(t) + k_7 i(t) - k_8 m(t) - k_6 m(t) - k_4 m(t),$$

$$\Phi_5(t,q) = k_6(e(t)+i(t)+m(t)) - k_1 q(t).$$

We commence by examining the function $\Phi_1(t,s)$. Subsequently, we will illustrate that

$$|\Phi_1(t,s) - \Phi_1(t,s_1)|^2 \le p_1 |s - s_1|^2. \tag{22}$$

Afterwards, we express

$$|\Phi_1(t,s) - \Phi_1(t,s_1)|^2 = \left|\begin{matrix} -k_2(i(t)+e(t))(s(t)-s_1(t)) \\ -k_3 m(t)(s(t)-s_1(t)) - k_4(s(t)-s_1(t)) \end{matrix}\right|^2, \tag{23}$$

$$= |(-k_2(i(t)+e(t)) - k_3 m(t) - k_4)(s(t)-s_1(t))|^2,$$

$$\le \{3k_2^2(|i(t)|^2 + |e(t)|^2) + 3k_3^2|m(t)|^2 + 3k_4^2\}|s(t)-s_1(t)|^2,$$

$$\le \begin{Bmatrix} 3k_2^2(\sup\limits_{t\in[0,T]}|i(t)|^2 + \sup\limits_{t\in[0,T]}|e(t)|^2) \\ +3k_3^2 \sup\limits_{t\in[0,T]}|m(t)|^2 + 3k_4^2 \end{Bmatrix} |s(t)-s_1(t)|^2$$

$$\le \{3k_2^2(\|i\|_\infty^2 + \|e\|_\infty^2) + 3k_3^2\|m\|_\infty^2 + 3k_4^2\}|s(t)-s_1(t)|^2,$$

$$\le p_1|s(t)-s_1(t)|^2,$$

where

$$p_1 = \{3k_2^2(\|i\|_\infty^2 + \|e\|_\infty^2) + 3k_3^2\|m\|_\infty^2 + 3k_4^2\}. \tag{24}$$

Proceeding further with the function $\Phi_2(t,e)$, we obtain

$$|\Phi_2(t,e) - \Phi_2(t,e_1)|^2 = |k_2 s(t)(e(t)-e_1(t)) - (k_5+k_6+k_4)(e(t)-e_1(t))|^2 \tag{25}$$

$$\le \{2k_2^2|s(t)|^2 + 2(k_5+k_6+k_4)^2\}|(e(t)-e_1(t))|^2$$

$$\le \left\{2k_2^2 \sup_{t\in[0,T]}|s(t)|^2 + 2(k_5+k_6+k_4)^2\right\}|(e(t)-e_1(t))|^2$$

$$\le \{2k_2^2\|s\|_\infty^2 + 2(k_5+k_6+k_4)^2\}|(e(t)-e_1(t))|^2$$

$$\le p_2|(e(t)-e_1(t))|^2$$

where

$$p_2 = \{2k_2^2\|s\|_\infty^2 + 2(k_5+k_6+k_4)^2\}. \tag{26}$$

Similary we get,

$$|\Phi_3(t,i) - \Phi_3(t,i_1)|^2 = |(-k_7 - k_6 - k_8 - k_1)(i(t)-i_1(t))|^2, \tag{27}$$

$$\le \{(k_7+k_6+k_8+k_1)^2\}|(i(t)-i_1(t))|^2,$$

$$\le p_3|(i(t)-i_1(t))|^2$$

where

$$p_3 = \{k_7+k_6+k_8+k_1\}^2. \tag{28}$$

Similary we get,

$$|\Phi_4(t,m) - \Phi_4(t,m_1)|^2 = |(-k_8 - k_6 - k_4)(m(t)-m_1(t))|^2, \tag{29}$$

$$= \{(k_8+k_6+k_4)^2\}|(m(t)-m_1(t))|^2,$$

$$\leq p_4|(m(t)-m_1(t))|^2$$

where

$$p_4=\{k_8+k_6+k_4\}^2. \tag{30}$$

Finally we get,

$$|\Phi_5(t,q)-\Phi_5(t,q_1)|^2=|-k_1(q(t)-q_1(t))|^2, \tag{31}$$
$$=\{k_1^2\}|(q(t)-q_1(t))|^2,$$
$$\leq p_5|(q(t)-q_1(t))|^2$$

where

$$p_5=\{k_1^2\}. \tag{32}$$

Now that we have checked the first condition for all functions, we move on to verifying the second condition for our model.

$$|\Phi_1(t,s)|^2=|k_1-k_2s(t)(i(t)+e(t))-k_3s(t)m(t)-k_4s(t)|^2, \tag{33}$$
$$=|k_1-(k_2(i(t)+e(t))+k_3m(t)+k_4)s(t)|^2,$$
$$\leq 2k_1^2+2(k_2^2(|i(t)|^2+|e(t)|^2)+k_3^2|m(t)|^2+k_4^2)|s(t)|^2,$$
$$\leq 2k_1^2+2\begin{pmatrix} k_2^2\left(\sup_{t\in[0,T]}|i(t)|^2+\sup_{t\in[0,T]}|e(t)|^2\right) \\ +k_3^2\sup_{t\in[0,T]}|m(t)|^2+k_4^2 \end{pmatrix}|s(t)|^2,$$
$$\leq 2k_1^2+2(k_2^2(\|i\|_\infty^2+\|e\|_\infty^2)+k_3^2\|m\|_\infty^2+k_4^2)|s(t)|^2$$
$$\leq 2k_1^2\left(1+\frac{2\left(k_2^2(\|i\|_\infty^2+\|e\|_\infty^2)+k_3^2\|m\|_\infty^2+k_4^2\right)}{2k_1^2}|s(t)|^2\right)$$
$$\leq \overline{p}_1(1+|s(t)|^2)$$

under the condition

$$\frac{k_2^2(\|i\|_\infty^2+\|e\|_\infty^2)+k_3^2\|m\|_\infty^2+k_4^2}{k_1^2}<1. \tag{34}$$

$$|\Phi_2(t,e)|^2=\left|\begin{matrix} k_2s(t)i(t)+k_3s(t)m(t) \\ +(k_2s(t)-(k_5+k_6+k_4))e(t) \end{matrix}\right|^2, \tag{35}$$
$$\leq 3k_2^2|s(t)|^2|i(t)|^2+3k_3^2|s(t)|^2|m(t)|^2$$
$$+3(k_2^2|s(t)|^2+(k_5+k_6+k_4)^2)e(t),$$
$$\leq 3k_2^2\sup_{t\in[0,T]}|s(t)|^2\sup_{t\in[0,T]}|i(t)|^2+3k_3^2\sup_{t\in[0,T]}|s(t)|^2\sup_{t\in[0,T]}|m(t)|^2$$
$$+3\left(k_2^2\sup_{t\in[0,T]}|s(t)|^2+(k_5+k_6+k_4)^2\right)e(t)$$
$$\leq 3k_2^2\|s\|_\infty^2\|i\|_\infty^2+3k_3^2\|s\|_\infty^2\|m\|_\infty^2$$
$$+3(k_2^2\|s\|_\infty^2+(k_5+k_6+k_4)^2)e(t)$$
$$\leq (3k_2^2\|s\|_\infty^2\|i\|_\infty^2+3k_3^2\|s\|_\infty^2\|m\|_\infty^2)\left(1+\frac{3\left(k_2^2\|s\|_\infty^2+(k_5+k_6+k_4)^2\right)}{3k_2^2\|s\|_\infty^2\|i\|_\infty^2+3k_3^2\|s\|_\infty^2\|m\|_\infty^2}e(t)\right),$$
$$\leq \overline{p}_2(1+|e(t)|^2)$$

under the condition

$$\frac{k_2^2\|s\|_\infty^2+(k_5+k_6+k_4)^2}{k_2^2\|s\|_\infty^2\|i\|_\infty^2+k_3^2\|s\|_\infty^2\|m\|_\infty^2}<1. \tag{36}$$

$$|\Phi_3(t,i)|^2=\left|\frac{1}{2}k_5e(t)-(k_7+k_6+k_8+k_1)i(t)\right|^2, \tag{37}$$

$$\leq \frac{1}{2}k_5^2|e(t)|^2+2(k_7+k_6+k_8+k_1)^2|i(t)|^2,$$

$$\leq \frac{1}{2}k_5^2\sup_{t\in[0,T]}|e(t)|^2+2(k_7+k_6+k_8+k_1)^2|i(t)|^2,$$

$$\leq \frac{1}{2}k_5^2\|e\|_\infty^2+2(k_7+k_6+k_8+k_1)^2|i(t)|^2,$$

$$\leq \left(\frac{1}{2}k_5^2\|e\|_\infty^2\right)\left(1+\frac{2(k_7+k_6+k_8+k_1)^2}{\frac{1}{2}k_5^2\|e\|_\infty^2}|i(t)|^2\right),$$

$$\leq \overline{p}_3(1+|i(t)|^2)$$

under the condition

$$\frac{4(k_7+k_6+k_8+k_1)^2}{k_5^2\|e\|_\infty^2}<1. \tag{38}$$

$$|\Phi_4(t,m)|^2=\left|\frac{1}{2}k_5e(t)+k_7i(t)-(k_8+k_6+k_4)m(t)\right|^2, \tag{39}$$

$$\leq \frac{3}{4}k_5^2|e(t)|^2+3k_7^2|i(t)|^2+3(k_7+k_8+k_6+k_4)^2|m(t)|^2,$$

$$\leq \frac{3}{4}k_5^2\sup_{t\in[0,T]}|e(t)|^2+3k_7^2\sup_{t\in[0,T]}|i(t)|^2$$

$$+3(k_7+k_8+k_6+k_4)^2|m(t)|^2,$$

$$\leq \frac{3}{4}k_5^2\|e\|_\infty^2+3k_7^2\|i\|_\infty^2+3(k_7+k_8+k_6+k_4)^2|m(t)|^2,$$

$$\leq \left(\frac{3}{4}k_5^2\|e\|_\infty^2+3k_7^2\|i\|_\infty^2\right)\left(1+\frac{3(k_7+k_8+k_6+k_4)^2}{\frac{3}{4}k_5^2\|e\|_\infty^2+3k_7^2\|i\|_\infty^2}|m(t)|^2\right)$$

$$\leq \overline{p}_4(1+|m(t)|^2)$$

under the condition

$$\frac{4(k_7+k_8+k_6+k_4)^2}{k_5^2\|e\|_\infty^2+3k_7^2\|i\|_\infty^2}<1. \tag{40}$$

$$|\Phi_5(t,q)|^2=|k_6(e(t)+i(t)+m(t))-k_1q(t)|^2, \tag{41}$$

$$\leq 2k_6^2(|e(t)|^2+|i(t)|^2+|m(t)|^2)+2k_1^2|q(t)|^2,$$

$$\leq 2k_6^2\left(\sup_{t\in[0,T]}|e(t)|^2+\sup_{t\in[0,T]}|i(t)|^2+\sup_{t\in[0,T]}|m(t)|^2\right)+2k_1^2|q(t)|^2,$$

$$\leq 2k_6^2(\|e\|_\infty^2+\|i\|_\infty^2+\|m\|_\infty^2)+2k_1^2|q(t)|^2$$

$$\leq 2k_6^2(\|e\|_\infty^2+\|i\|_\infty^2+\|m\|_\infty^2)\left(1+\frac{2k_1^2}{2k_6^2(\|e\|_\infty^2+\|i\|_\infty^2+\|m\|_\infty^2)}|q(t)|^2\right)$$

$$\leq \overline{p}_5(1+|q(t)|^2)$$

under the condition

$$\frac{k_1^2}{k_6^2(\|e\|_\infty^2+\|i\|_\infty^2+\|m\|_\infty^2)}<1. \tag{42}$$

Provided that the condition for linear growth is satisfied, such that

$$\max\left\{\begin{matrix}\frac{k_2^2(\|i\|_\infty^2+\|e\|_\infty^2)+k_3^2\|m\|_\infty^2+k_4^2}{k_1^2}\\ \frac{k_2^2\|s\|_\infty^2+(k_5+k_6+k_4)^2}{k_2^2\|s\|_\infty^2\|i\|_\infty^2+3k_3^2\|s\|_\infty^2\|m\|_\infty^2}\\ \frac{4(k_7+k_6+k_8+k_1)^2}{k_5^2\|e\|_\infty^2}\\ \frac{4(k_7+k_8+k_6+k_4)^2}{k_5^2\|e\|_\infty^2+3k_7^2\|i\|_\infty^2}\\ \frac{k_1^2}{k_6^2(\|e\|_\infty^2+\|i\|_\infty^2+\|m\|_\infty^2)}\end{matrix}\right\}<1, \tag{43}$$

there exists only one solution set for the system of equations.

5. Numerical Scheme For Model With Riemann-Liouville Derivative

In the forthcoming section, we offer an analysis of the model under consideration within the realm of fractional calculus employing the Riemann-Liouville derivative. When implementing the numerical scheme, we employ the Atangana-Toufik numerical rules (Toufik and Atangana, 2017). Let us now express the model utilizing the Riemann-Liouville derivative as follows:

$$\begin{aligned}&{}_0^{RL}D_t^\alpha s(t)=\Phi_1(t,s),\\&{}_0^{RL}D_t^\alpha e(t)=\Phi_2(t,e),\\&{}_0^{RL}D_t^\alpha i(t)=\Phi_3(t,i),\\&{}_0^{RL}D_t^\alpha m(t)=\Phi_4(t,m),\\&{}_0^{RL}D_t^\alpha q(t)=\Phi_5(t,q),\end{aligned} \tag{44}$$

$s(t_0)=s_0$, $e(t_0)=e_0$, $i(t_0)=i_0$, $m(t_0)=m_0$, and $q(t_0)=q_0$.

Here

$$\begin{aligned}&\Phi_1(t,s)=k_1-k_2s(t)(i(t)+e(t))-k_3s(t)m(t)-k_4s(t),\\&\Phi_2(t,e)=k_2s(t)(i(t)+e(t))+k_3s(t)m(t)-k_5e(t)-k_6e(t)-k_4e(t),\\&\Phi_3(t,i)=\frac{1}{2}k_5e(t)-k_7i(t)-k_6i(t)-k_8i(t)-k_1i(t),\\&\Phi_4(t,m)=\frac{1}{2}k_5e(t)+k_7i(t)-k_8m(t)-k_6m(t)-k_4m(t),\\&\Phi_5(t,q)=k_6(e(t)+i(t)+m(t))-k_1q(t).\end{aligned} \tag{45}$$

We transform the aforementioned system into its numerical counterpart using Lagrange polynomial interpolation.

$$\begin{aligned}&s_{v+1}=\frac{\alpha(\Delta t)^\alpha}{\Gamma(\alpha+2)}\sum_{k=2}^{v}\Phi_1(t_k,s_k).\begin{bmatrix}(v-k+1)^\alpha(v-k+2+\alpha)\\-(v-k)^\alpha(v-k+2+2\alpha)\end{bmatrix}\\&-\frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)}\sum_{k=2}^{v}\Phi_1(t_{k-1},s_{k-1}).\begin{bmatrix}(v-k+1)^{\alpha+1}\\-(v-k)^\alpha(v-k+1+\alpha)\end{bmatrix},\\&e_{v+1}=\frac{\alpha(\Delta t)^\alpha}{\Gamma(\alpha+2)}\sum_{k=2}^{v}\Phi_2(t_k,e_k).\begin{bmatrix}(v-k+1)^\alpha(v-k+2+\alpha)\\-(v-k)^\alpha(v-k+2+2\alpha)\end{bmatrix}\\&-\frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)}\sum_{k=2}^{v}\Phi_2(t_{k-1},e_{k-1}).\begin{bmatrix}(v-k+1)^{\alpha+1}\\-(v-k)^\alpha(v-k+1+\alpha)\end{bmatrix},\\&i_{v+1}=\frac{\alpha(\Delta t)^\alpha}{\Gamma(\alpha+2)}\sum_{k=2}^{v}\Phi_3(t_k,i_k).\begin{bmatrix}(v-k+1)^\alpha(v-k+2+\alpha)\\-(v-k)^\alpha(v-k+2+2\alpha)\end{bmatrix}\\&-\frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)}\sum_{k=2}^{v}\Phi_3(t_{k-1},i_{k-1}).\begin{bmatrix}(v-k+1)^{\alpha+1}\\-(v-k)^\alpha(v-k+1+\alpha)\end{bmatrix}.\end{aligned} \tag{46}$$

$$m_{v+1} = \frac{\alpha(\Delta t)^{\alpha}}{\Gamma(\alpha+2)} \sum_{k=2}^{v} \Phi_2(t_k, m_k). \begin{bmatrix} (v-k+1)^{\alpha}(v-k+2+\alpha) \\ -(v-k)^{\alpha}(v-k+2+2\alpha) \end{bmatrix}$$

$$-\frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+2)} \sum_{k=2}^{v} \Phi_2(t_{k-1}, m_{k-1}). \begin{bmatrix} (v-k+1)^{\alpha+1} \\ -(v-k)^{\alpha}(v-k+1+\alpha) \end{bmatrix},$$

$$q_{v+1} = \frac{\alpha(\Delta t)^{\alpha}}{\Gamma(\alpha+2)} \sum_{k=2}^{v} \Phi_5(t_k, q_k). \begin{bmatrix} (v-k+1)^{\alpha}(v-k+2+\alpha) \\ -(v-k)^{\alpha}(v-k+2+2\alpha) \end{bmatrix}$$

$$-\frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+2)} \sum_{k=2}^{v} \Phi_5(t_{k-1}, q_{k-1}). \begin{bmatrix} (v-k+1)^{\alpha+1} \\ -(v-k)^{\alpha}(v-k+1+\alpha) \end{bmatrix}.$$

4. Conclusion

In this study, we explore a COVID-19 model alongside its fractional order counterpart. We examine the model under linear growth and Lipschitz rules, deriving conditions for the existence and uniqueness of system solutions. Ultimately, we provide numerical approximations to demonstrate the effectiveness of our method.

Ethics Permissions

This paper does not require ethics committee approval.

Conflict of Interest

Author declares that there is no conflict of interest for this paper.

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A Blockchain-Based System for Managing Payments in The Construction Supply Chain

Samer HAFFAR[1], Eren ÖZCEYLAN[2*]

Keywords
Blockchain,
Construction industry,
Payments,
Supply chain,
Ethereum

Abstract – The global construction sector, employing 7% of the worldwide workforce and contributing 13% to global gross domestic product (GDP), is among the largest industries. It encompasses various stakeholders, including owners, contractors, subcontractors, and suppliers. However, the management of payments within this supply chain encounters numerous hurdles, such as delays, rework, errors, late payments, and inadequate supervision and financial oversight. This paper presents a blockchain-based payment handling system for construction supply chains. The system leverages blockchain's features of data transparency and sharing, and decentralization and immutability to provide a secure and trusted tool for handling payments. The system enables tracking work progress, the payment of installments for completed work, and provides a facility for resolving disputes between buyers and sellers on-chain. The system ensures smooth execution and commitments by all parties with blockchain data transparency, escrow payments and independent risk assessment. The paper provides a detailed description of the system design and results of function tests.

1. Introduction

The construction supply chain consists of the activities that lead to starting and completing a construction project of one or more buildings. These activities include identifying the demand for the buildings, construction activities (such as laying the foundation, concreting, welding, plastering, plumbing, etc.), maintaining the building during use, and the demolition of the building. The construction industry is a $10 trillion industry, accounting for 13% of global GDP. A construction supply chain, thus, employs a lot of stakeholders that contribute to its activities, including the owners of the construction project, contractors, suppliers of raw materials as well as engineers and architects (Studer and De Brito Mello, 2021). Construction supply chains are characterized by the flow of materials, information, and finance across all stakeholders to ensure the smooth implementation of construction projects (Xue et al., 2007).

A smooth flow of funds, however, is a rare occurrence in the construction supply chain due to high number of stakeholders involved in construction projects as well as influences and disruptions in the flows of materials and information. Nanayakkara et al. (2021) reported several issues and problems that occur in the construction supply chain when handling payments among stakeholders which include: delays in completing work due to supply chain issues, going over budget, rework and errors, late payments, improper supervision and financial controls, improper withholding of payments, lack of trust between stakeholders. Ramachandra and Rotimi (2015) identified several factors causing delays in payments in the New Zealand construction industry which include: cash flow difficulties due to non-payment on other projects, disputes over payment claims, and dishonesty of the payers.

Blockchain is a distributed tamper-proof record of ordered transactions that is secured cryptographically. The blockchain ledger is stored on a network of computers, each computer having an identical copy of the entire ledger. The tamper-proof nature of the blockchain is ensured by the collective effort of all computers on the network (Levis et al., 2021). The Ethereum blockchain was released in 2015 and brought the concept of smart contracts to blockchains. Smart contracts allow writing and executing software applications on the blockchain (Zhou et al., 2022). Thus, all blockchains share several characteristics which include "immutability", which

[1] Gaziantep University, Natural and Applied Sciences, 27410 Gaziantep, Türkiye. E-mail: sam.hafar@gmail.com OrcID: 0000-0002-9317-0257
[2*] **Corresponding Author.** Gaziantep University, Industrial Engineering Department, 27410 Gaziantep, Türkiye. E-mail: erenozceylan@gmail.com OrcID: 0000-0002-5213-6335
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means it's impossible to make any changes to the data that is stored on a blockchain; "transparency", all data stored on the blockchain is publicly available and accessible.

In this paper, we propose a blockchain-based system that aims to manage and handle the processing of payments in the construction supply chain. The purpose of this system is to provide a solution that addresses the issues with construction supply chain payments reported in the literature. The system has the following features:

- **Tracking Work and Installments**. The system allows the supply chain stakeholders to track work on projects and manage the payment of installments through the supply chain.
- **Dispute Handling**. If any of the stakeholders fails to comply with the agreement, the system can handle initiating and resolving disputes.
- **Risk Mitigation**. The system mitigates the risks associated with payments by utilizing escrow accounts and credit risk scores for all stakeholders, thus creating a motivation for them to commit to their agreements and comply with their obligations.

To the knowledge of the authors, this is the first system in the literature that combines the features of work tracking, payment and installment handling, dispute handling and risk mitigation in one blockchain-based system, which is the main contribution of this study.

1.1. Literature Review

Researchers in the construction supply chain have explored the potential of blockchain technology to tackle various challenges. In their study, Shemov et al. (2020) investigated the application of blockchain as a digital platform to address common issues within the construction supply chain. Despite acknowledging security concerns associated with blockchain use, the authors concluded that the technology offers a viable solution to many challenges, particularly those related to trust and project delays. In another study, Lu, Wu, et al. (2021) developed a blockchain-based model for government supervision of construction work (GSCW) that incorporates an incentive mechanism. Their model facilitates information sharing, preserves privacy, and seamlessly integrates into existing GSCW teams' workflows without disruption. Lu, Li, et al. (2021) introduced a novel solution utilizing smart construction objects (SCOs) to address the challenge of bridging blockchain systems with real-life construction projects. The research demonstrates the effectiveness of this approach in ensuring data accuracy and recording reputation scores. Zhang et al. (2023) explored the potential of blockchain technology in addressing flaws within construction contract management (CCM), such as information sharing and payment processes.

Blockchain technology is not only used for the construction supply chain, but also in other supply chains/supply chain applications as well. The exploration of blockchain technology's application in supply chains is an extensively studied subject in academic literature, driven by its attributes including traceability, transparency, decentralization, immutability, and automation. According to Han and Fang (2024), blockchain technology is used in a variety of supply chain functions, including logistics traceability, supply chain finance, supply chain collaboration, sustainable management, and risk management. Ioannou and Demirel (2022) reported that blockchain can be harnessed in supply chain finance to tackle issues such as limited visibility within the supply chain, cumbersome manual paperwork processes, and the risk of fraud. Archa et al. (2018) presented a system based on blockchain to tackle the problem of drug counterfeiting within the pharmaceutical supply chain. This system monitors the quantities of drugs held by each party and traces the movement of drugs across all parties involved in the supply chain. Rogerson and Parry (2020) outlined several instances where blockchain is employed to combat counterfeiting and enhance trust and visibility within the food supply chain. Kumar et al. (2021) proposed a holistic framework based on permissioned Blockchain technology to address challenges in international trade practices. The framework aims to enhance supply chain and logistics operations by addressing issues such as traceability, data integrity, and decentralized decision-making.

Ensuring the uninterrupted flow of funds within a construction supply chain is crucial for averting issues and delays (Studer and De Brito Mello, 2021). However, due to the rarity of a seamless fund flow, numerous studies have focused on pinpointing the challenges and obstacles encountered by stakeholders in managing payments. Some of these studies aimed to compile a spectrum of issues, as demonstrated by the works of Ramachandra and Rotimi (2015) and Swai et al. (2020). Ramachandra and Rotimi (2015) delved into the root causes of payment problems in the New Zealand construction sector, identifying factors such as cash flow constraints stemming from delays and non-payments on other projects, disputes over payment claims, prevalent payment culture, payer dishonesty, inadequate supervision, financial control, and cost overruns. On the other hand, Swai et al. (2020) explored the factors contributing to unfair payment practices within the UK construction industry. Their findings implied late payments to contractors, conditional payment practices, downward pressure on contractor rates, and retention payments as among the leading factors. Other studies have proposed targeted solutions to specific challenges. For instance, Xie et al. (2019) investigated the impact of payment cycles at two critical junctures in the construction supply chain: from owner to general contractor and from general contractor to subcontractor. Their research revealed that shortening the payment periods at these junctures could expedite fund flow and facilitate the provision of advance funds, thereby ensuring smoother progress on construction projects.

Researchers utilize blockchain technology to automate payments and overcome challenges within supply chains. Kaid and Eljazzar (2018) presented a blockchain-based system aimed at resolving trust and visibility concerns among stakeholders, incorporating a straightforward mechanism for managing payment installments within the supply chain parties. Their approach involves buyers and sellers agreeing on a rule, such as withholding payment until 50% of the contracted services are fulfilled, and subsequently sharing service-related information on the blockchain. In a blockchain-based scheme for supply chain finance proposed by Tsai (2023), three key actors are involved: a large enterprise acting as a buyer, an small and medium enterprise (SME) acting as a seller, and a financial institution. The process begins with the buyer creating a purchase order, followed by the seller shipping the products. Subsequent steps, such as invoicing and receipt confirmation, are then completed to finalize the purchase operation. Financial institutions then issue a loan to the seller, with the buyer responsible for repaying the seller's loan. Omar et al. (2021) proposed a blockchain-based solution to enhance the efficiency of Group Purchasing Organization (GPO) contracts witin the healthcare supply chain (HCSC), addressing current inefficiencies in procurement processes. By integrating blockchain technology and decentralized storage, the solution aims to streamline communication among stakeholders, minimize procurement timelines, and ensure transparency, thus potentially reducing pricing discrepancies and inaccuracies. Alnıpak and Toraman (2024) explored the adoption of Blockchain technology for payment transactions in Turkey's maritime industry, aiming to measure stakeholders' intentions through Technology Acceptance Model (TAM). Findings reveal strong positive relationships between usage intention and perceived usefulness, as well as perceived ease of use.

Researchers explored the application of blockchain technology to address payment-related challenges in the construction supply chain. In their work, Motawa and Kaka (2009) introduced an IT system capable of modeling various payment systems, allowing stakeholders within the construction supply chain to collectively determine the most suitable payment mechanism. This approach aims to ensure smooth cash flow and safeguard the supply chain against potential disruptions, thereby satisfying all involved parties. Meanwhile, Hamledari and Fischer (2021) examined the potential of blockchain-based smart contract solutions to automate progress payment tracking in the construction supply chain. Additionally, Das et al. (2020) presented a decentralized blockchain-based model for managing interim payments in construction projects. Notably, this model eliminates the need for trust among stakeholders and can autonomously enforce the terms and conditions of interim payments. Moreover, it facilitates the confidential sharing of sensitive financial information among stakeholders. Sigalov et al. (2021) introduced the implementation of smart contracts in construction projects to address issues with complex contract structures, delayed payments, and lack of transparency. By integrating Building Information Modeling (BIM) with blockchain-based smart contracts, automated and transparent payment processing is achieved. Their solution facilitate automatic payments upon acceptance of construction work, enhancing efficiency and trust in the construction industry.

2. Materials and Methods

In this section, we provide a detailed description of the proposed payment system with its features and capabilities. We propose a payment process workflow that is adopted in the payment system, so we provide a description of that process. Lastly, we provide a description of the system components, and the blockchain smart contract that was built to implement the proposed system functions.

2.1. System Features

2.1.1. Actors

There are four actors (or users) that have access to the system, each user plays a role in carrying out the payment process, namely: the buyer, the seller, the auditor, and the credit scoring agency. The **buyer** is the supply chain stakeholder that's buying goods and services and wants to handle the payment of the price of these goods and services on the system. The **seller** is the individual or entity selling those goods and services to the buyer**.** The **auditor** is a 3rd party entity that has two tasks: verifying the work that was completed by the seller and confirming its compliance with the terms and conditions of the agreement between the buyer and the seller. The other task of the auditor is resolving disputes between the buyer and the seller. The **credit scoring agency** is an institution that evaluates the financial situation of the buyer and the seller and gives them a credit score.

2.1.2. Work and Installment Tracking

The proposed system relies on the concept of "progress payments", which associates work with payments. This is the reason why there are two components of the system, one is for tracking work, and the other is for tracking installments. These components are independent of each other (i.e., an installment can be paid by the buyer without necessarily having the work associated with it completed). This way, if the buyer is satisfied with the work being done by the seller and they trust them enough, they can pay the installments without having to track the work associated with it on the system. **Work Tracking** consists of three stages: first, the seller announces that they completed a portion of the work as agreed, then the auditor verifies the completed work to make sure it complies with the terms and conditions of the agreement, then the buyer confirms work completion. All that progress is committed to the blockchain in the form of "events". **Installment Tracking** works by defining the installments that need to be paid by the buyer; the buyer pays the installments one after the other to the system, and the seller then withdraws those installments from the system. Thus, the system acts as a "trusted middleman" between the buyer and the seller.

2.1.3. Credit Risk and Escrows

The system requires the buyer and the seller to declare their credit score. Credit risk assessment serves as a valuable tool for both buyers and sellers in evaluating each other's financial stability. This assessment informs decisions regarding various installment payment terms, including the escrow amount, down payment, and installment count. Financial information provided by both parties is evaluated by a credit assessment agency to determine credit risk. Subsequently, this information is shared with both the buyer and seller as part of the payment process. Credit assessment is based on several indicators of a business. For example, the information taken into account when evaluating the credit risk of SMEs may include short term debt/equity, cash/assets, EBITDA/assets, retained earnings/assets and EBITDA/interest expenses (Altman and Sabato, 2007).

Based on the credit risk information, the buyer, and the seller both agree on an "escrow" amount that each of them pays as a guarantee for the smooth implementation and compliance with the terms and conditions of the agreement. Escrow and payment retention are common practices within the construction industry aimed at safeguarding the interests of owners. In the study by Antipin and Trufanova (2021), escrow accounts serve to protect shareholders of a construction project from developers with uncertain financial standing. Funds from shareholders are deposited into the escrow account, which is then utilized by a bank to finance project operations. The developer only receives funds from the escrow account upon project completion and commissioning. Further, as discussed in Swai et al. (2020), payment retention ensures that contractors fulfill their contractual obligations, thereby safeguarding the interests of owners. The buyer and the seller deposit their escrow payments in the system, and then they can withdraw their escrows once the project is over.

2.1.4. Dispute Handling

If one party (buyer or seller) fails to comply with their obligations as defined in the terms and conditions of their contract, the other party can dispute that noncompliance from within the system. If a dispute is initiated, all further work on the system is frozen, including deposits and withdrawals, until the dispute is resolved. There are two possible resolutions for a dispute; either both parties agree to proceed with the agreement as described or they terminate the agreement. If the buyer and seller agree to terminate the agreement, a settlement is made where the buyer and the seller are refunded a portion of whatever amount that is left in the system on their agreement's balance (which consists of the escrows that have been deposited as well as any installments that are still not withdrawn). The auditor is in charge of handling dispute resolution as well as settling balances.

2.1.5. Blockchain

The system is implemented on top of blockchain technology. The use of blockchain technology has several benefits for payments handling which include:

- **Immutability**: All information about a deal is tamper-proof and cannot be changed by anyone outside the functions that the system offers. This makes the data extremely reliable and trustworthy.
- **Transparency**: All information and data, including those of previous deals performed by the buyer and/or the seller, are publicly available and accessible. This allows both the buyer and the seller to learn about the reputation, credibility, and history of dealings (work completion, disputes, delays, etc.) of the other party and ensure that they're a reliable entity to work with.
- **Information Sharing**: Blockchain is a great platform for information sharing. Once a piece of information becomes available on the blockchain, it can be accessible by all parties in real-time all over the world.

2.2. Payment Process Workflow

The system's workflow consists of five stages, namely: Deal Creation, Depositing the Escrow, Work Tracking and Installments, Disputes, and Completion. Figure 1 is a flow chart of the proposed payment process stages. Below is a description of each stage:

- **Deal Creation**: In this stage, a contract (in the system, known as a "Deal") that defines the terms of the service is established between the buyer and the seller. The system is designed to have the seller create the contract and define its terms and buyer to approve. The buyer's approval of the deal is indicated by their depositing the escrow amount in the next stage. When creating the deal, the seller specifies the following: a link to the full contract document, the buyer, the auditor, the credit scoring agency, the number of installments and their amounts, the escrow amount of the buyer and the escrow amount of the seller.
- **Deal Guarantees**: In this stage, the buyer and the seller have to deposit an amount of money in the smart contract that serves as a guarantee against failure of compliance with the terms and conditions of the contract. This stage consists of two steps, credit score evaluation and escrow deposit.
    - **Credit Score**: The independent scoring agency provides the credit score of both the buyer and the seller directly to the smart contract.
    - **Escrow Deposit**: In this stage, both the buyer and the seller deposit their escrow amounts as defined in the deal. This stage concludes when both escrow amounts are deposited. The system prevents any further steps until escrows are deposited. However, the buyer can still deposit installments even if they have not deposited their escrow yet.
- **Work, Installments**: At this stage, the seller starts working on delivering the services as agreed in the contract that is linked to by the deal in the system. This stage has two activities that run concurrently, namely: tracking work and paying installments.
    - **Work Tracking**: Whenever the seller completes a portion of the service, they can announce that to the buyer and the auditor through the system. The auditor then verifies the completed work, then announces that they approve it. Lastly, the buyer reviews the completed work and approves it.
    - **Paying Installments**: If the buyer is satisfied with the service, they deposit the installment that corresponds to the portion of work that is completed. The buyer can choose to pay the installment at any time, even if work is still in progress or has not started yet. The buyer can, thus, pay all installments even before the project starts.
- **Disputes**: This stage is optional and can only be entered if the buyer or the seller decides so. If the buyer or the seller are not satisfied with the other party's compliance with the deal terms and conditions, they can initiate a dispute. When initiating a dispute, the system allows the user to mention the reasons they think the other party is noncompliant with the terms of the deal. When a dispute is initiated, the

system freezes all further activities until that dispute is resolved. The auditor then reviews the dispute and resolves it as per the terms of the contract. The result of the dispute is either “dispute resolution” or “deal termination”.

  - **Dispute resolution**: If the dispute is resolved, work resumes from the point it halted due to the dispute and continues according with the previous stage.
  - **Deal termination**: If the deal is terminated, the auditor then “settles the balances” by refunding the balance of the deal (which consists of the escrows and any unwithdrawn installments) back to the buyer and the seller as per the contract terms and conditions.

- **Completion**: Once all installments are deposited by the buyer, work on the deal officially concludes. At this stage, two main activities take place as follows:
  - **Installment Withdrawal**: The seller continues to withdraw whatever amounts from the deposited installments that are still not withdrawn. Then, once all amounts are withdrawn, the system marks the deal as completed.
  - **Escrow Withdrawal**: Once the deal is marked as completed, the system then allows the buyer and the seller to withdraw the escrows that are deposited at the beginning of the deal.

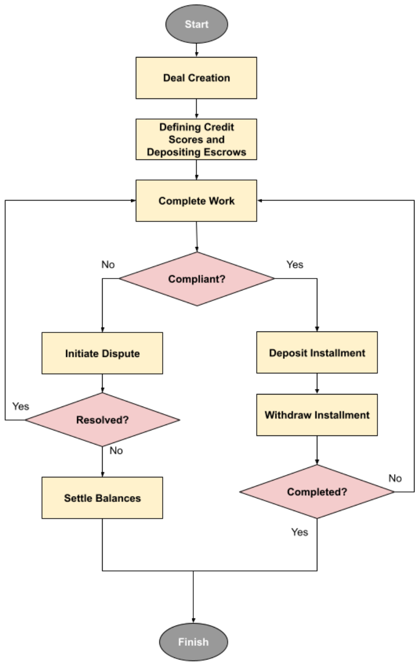

Figure 1. Flow of the payment process.

2.3. The Deals Smart Contract

The system is designed to manage the payments between two parties, one is a buyer buying goods and/or services, a seller selling these goods and services, and an auditor who’s in charge of ensuring the smooth delivery of goods and services and dispute resolution. The system also allows a credit scoring agency to provide the credit scores of the buyer and seller so that escrow amounts are determined in accordance with the risk involved in the deal. Thus, all stakeholders can use the system to handle the flow of goods and services and funds among them, two stakeholders at a time. The system consists of a single smart contract, called Deals. The smart contract offers several functions to the system users that allows them to carry out all actions relevant to the payment handling workflow explained above, which are: create deals, deposit, and withdraw escrows, monitor work, deposit and withdraw installments, initiate and resolve disputes and settle balances. The contract specifications document can be stored on an off-chain file storage service such as interplanetary file system (IPFS) or a cloud storage service. The smart contract was written in the Solidity programming language and can be deployed on the Ethereum blockchain or any blockchain that has an Ethereum virtual machine (EVM). Figure 2 illustrates the relationships between the various system components and its users.

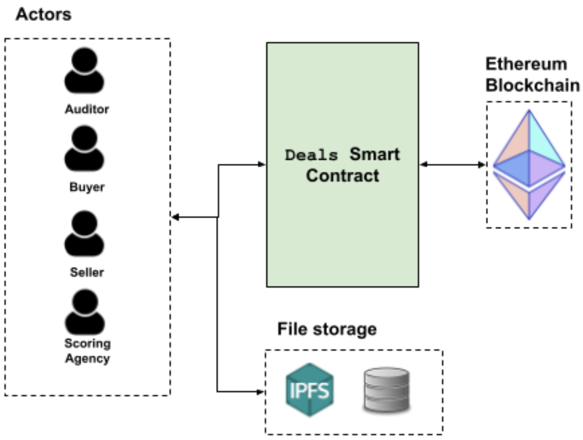

Figure 2. Components of the proposed payment system and their relationship with its actors/users.

2.3.1. Deal Structure

The information and data of every deal managed by the system is stored in a data structure called Deal. All Deal objects are stored in a *mapping* called *deals* and the unique identifiers of the stored deals are stored in an array called *dealIds*. Below are the data members of the Deal data structure and their descriptions:

- **Deal ID**: A unique identifier given to each deal. This identifier is used by the smart contract and the users to refer to the desired deal when executing functions.
- **Specs URI**: A link to the contract specifications document that defines the terms and conditions of the deal between the buyer and seller, which includes, among other terms: a description of the work that needs to be done, the total price, the number of installments and the amount of each installment, as well as the terms and conditions of dispute resolution.
- **Seller**: The seller's account (their address on the blockchain).
- **Buyer**: The buyer's account (their address on the blockchain).
- **Auditor**: The auditor's account (their address on the blockchain).
- **Credit Scoring Agency**: The credit scoring agency's account (their address on the blockchain).
- **Status**: The current status of the deal, which can be one of the statuses in the enum STATUS (Figure 3).
- **Comment**: A comment that is set by the user whenever they execute a function on the smart contract and cause the deal status to change. It is used to share messages among the users.
- **Installments**: An array of all the installments that need to be paid by the buyer to the seller; each element of the array constitutes an installment and specifies its amount in ETH.
- **Current Installment**: The index of the installment the latest installment that is yet to be paid from the Installments array.
- **Balance**: The total amount currently held in the deal by the smart contract, (balance = total deposited escrows + total deposited installments – total withdrawn installments and escrows).
- **Buyer Credit Score**: The credit score of the buyer, which is set by an independent credit scoring agency.
- **Seller Credit Score**: The credit score of the seller, which is set by an independent credit scoring agency.
- **Seller Escrow**: The total amount of escrow required to be deposited by the seller.
- **Buyer Escrow**: The total amount of escrow required to be deposited by the buyer.
- **Seller Escrow Deposited**: True, if the seller deposited their escrow, and false otherwise.
- **Buyer Escrow Deposited**: True, if the buyer deposited their escrow, and false otherwise.

```
enum STATUS {
    NEW,
    CREDIT_SCORES_ADDED, BUYER_ESCROW_DEPOSITED, SELLER_ESCROW_DEPOSITED,
    WORK_FINISHED, WORK_VERIFIED, WORK_CONFIRMED, INSTALLMENT_DEPOSITED,
    DISPUTE_INITIATED, DISPUTE_RESOLVED, DEAL_TERMINATED, BALANCES_SETTLED,
    ALL_INSTALLMENTS_DEPOSITED,DEAL_COMPLETED
}
```

Figure 3. The STATUS *enum* that defines all possible statuses of a deal object.

2.3.2. Events

The smart contract broadcasts a DealStatusChange event every time a function is executed on a deal that causes its status to change. The UpdateStatus function is used by all the other functions in the smart contract to emit the DealStatusChange event. The DealStatusChange event commits the following information to the blockchain:

- **Changed By**: The account address of the user that executed the function.
- **From**: The deal status before the function was executed.
- **To**: The deal status after the function execution.
- **Comment**: The new value of the comment after the status change.
- **Deal Balance**: The balance of the deal whose status was updated.
- **Contract Balance**: The balance of the smart contract, which is the sum of the balances of all deals managed by the smart contract.

2.3.3. Modifiers

The smart contract contains several modifiers that are designed to enforce user access permissions as well as the rules that apply at different stages of the payment process lifecycle as described in the workflow section of this paper. Table 1 lists the available modifiers and what they do when applied to the smart contract functions.

Table 1. A list of the modifiers defined in the smart contract and the behavior changes they make to the functions that they're applied to.

Modifier	Behavior
dealActive(dealId)	Requires the deal to be active to allow a function to proceed with its execution.
dealCompleted(dealId)	Requires the deal to be completed to allow a function to execute.
escrowDeposited(dealId)	Requires that both the buyer's and the seller's deposited to allow function execution.
noDispute(dealId)	Requires that no dispute is active to allow a function to execute.
onlyCreditScoringAgency(dealId)	Requires that the user attempting to execute a function is the credit scoring agency of the deal.
onlyBuyer(dealId)	Requires that the user attempting to execute a function is the buyer of the deal.
onlySeller(dealId)	Requires that the user attempting to execute a function is the seller of the deal.
onlyAuditor(dealId)	Requires that the user attempting to execute a function is the auditor of the deal.
onlyBuyerSeller(dealId)	Requires that the user attempting to execute a function is either the buyer or the seller of the deal.
onlyUser(dealId)	Requires that the user attempting to execute a function is either the buyer, the seller, or the auditor of the deal.
dealExists(dealId)	Requires that a deal with the provided ID exists to allow the function to execute.

2.3.4. Functions/Algorithms

These functions in the smart contract allow its users to modify the deal and add updates to it relevant to the stage of the payment process that it is at. The ability to execute a function is restricted by the rules and behaviors enforced by the modifiers in Table 1 applied to it.

2.3.4.1. Add Deal

This function allows a user to add a new deal and define this information for it: Specs UIR, Buyer, Auditor, Buyer Escrow, Seller Escrow, and Installments. It is assumed that the user that adds the deal is the seller, therefore their address is taken from their transaction information. The function generates a deal ID, creates a Deal object and adds it to the deals mapping. Figure 4 shows the code of the Add Deal function.

```
function addDeal(
    string memory specsURI,
    address buyer, address auditor, address creditScoringAgency,
    uint buyerEscrow, uint sellerEscrow,
    uint[] memory installments) public returns(Deal memory) {

    //create id
    uint dealId = dealIds.length > 0 ? dealIds.length + 1 : 1;
    STATUS status = STATUS.NEW;

    //create deal object
    Deal memory deal = Deal(
        dealId, specsURI,
        msg.sender, buyer, auditor, creditScoringAgency,
        status, "", installments, 0, 0,
        0, 0,
        sellerEscrow, buyerEscrow, false, false);

    //add deal
    deals[dealId] = deal;
    dealIds.push(dealId);

    return deal;

}
```

Figure 4. The Add Deal function code.

2.3.4.2. Set Credit Scores

Allows the credit scoring agency to provide the credit score of both the buyer and the seller. The credit scores must be provided immediately after adding the deal, otherwise, the operation will be reverted. Figure 5 shows the code of the Set Credit Scores function.

```
function setCreditScore(uint dealId, uint buyerCreditScore, uint sellerCreditScore)
    dealExists(dealId) dealActive(dealId) onlyCreditScoringAgency(dealId) public {

    require(deals[dealId].status == STATUS.NEW, "Failed! Credit scores must be set right after creating the deal.");

    deals[dealId].buyerCreditScore = buyerCreditScore;
    deals[dealId].sellerCreditScore = sellerCreditScore;
        updateStatus(dealId, STATUS.CREDIT_SCORES_ADDED, "");

}
```

Figure 5. The Set Credit Score function code.

2.3.4.3. Deposit Escrow

The Deposit Escrow function allows the buyer and the seller to deposit their escrows. Escrows are deposited after credit scores are saved to the deal. The function requires that the full amount of escrow as defined in the deal object to be deposited, otherwise the operation is rejected. Figure 6 shows the code of the Deposit Escrow function.

```
function depositEscrow(uint dealId)
    dealExists(dealId) dealActive(dealId) onlyBuyerSeller(dealId) public payable {

    require(deals[dealId].status == STATUS.CREDIT_SCORES_ADDED,
    "Failed! Cannot deposit escrow before credit scores are provided.");

    if((deals[dealId].seller == msg.sender) && (deals[dealId].sellerEscrow == msg.value)) {
        deals[dealId].balance = deals[dealId].balance + msg.value;
        deals[dealId].sellerEscrowDeposited = true;
        contractBalance = contractBalance + msg.value;
        updateStatus(dealId, STATUS.SELLER_ESCROW_DEPOSITED, "");
    }
    else if((deals[dealId].buyer == msg.sender) && (deals[dealId].buyerEscrow == msg.value)) {
        deals[dealId].balance = deals[dealId].balance + msg.value;
        deals[dealId].buyerEscrowDeposited = true;
        contractBalance = contractBalance + msg.value;
        updateStatus(dealId, STATUS.BUYER_ESCROW_DEPOSITED, "");
    }
    else {
        revert("Please make sure that you deposit the exact escrow amount.");
    }

}
```

Figure 6. The Deposit Escrow function code.

2.3.4.4. Complete Work

The Complete Work function allows the seller to announce the completion of a portion of the work as per the agreement with the buyer. It also allows the auditor to announce their verification of the completed work, and the buyer to confirm that work was completed. Figure 7 shows the code of the Complete Work function.

```
function completeWork(uint dealId, string memory comment)
dealActive(dealId) onlyUser(dealId) escrowDeposited(dealId) noDispute(dealId) public {

    if(deals[dealId].seller == msg.sender) {
        updateStatus(dealId, STATUS.WORK_FINISHED, comment);
    }
    if(deals[dealId].auditor == msg.sender) {
        updateStatus(dealId, STATUS.WORK_VERIFIED, comment);
    }
    if(deals[dealId].buyer == msg.sender) {
        updateStatus(dealId, STATUS.WORK_CONFIRMED, comment);
    }

}
```

Figure 7. The Complete Work function code.

2.3.4.5. Deposit Installment

The Deposit Installment function allows the buyer to deposit the amount of the current installment, which is referenced by the Current Installment field of the Deal object. The buyer is required to deposit the full amount of the installment, otherwise the operation will be rejected. If this was the last installment deposited by the buyer, the system changes the status of the deal to ALL_INSTALLMENTS_DEPOSITED. Figure 8 shows the code of the Deposit Installment function.

```
function depositEscrow(uint dealId)
    dealExists(dealId) dealActive(dealId) onlyBuyerSeller(dealId) public payable {

    require(
            ((deals[dealId].status == STATUS.CREDIT_SCORES_ADDED) && (msg.sender == deals[dealId].buyer))
            ||
            ((deals[dealId].status == STATUS.BUYER_ESCROW_DEPOSITED) && (msg.sender == deals[dealId].seller)),
    "Failed! Buyer must deposit their escrow after credit scores are defined. Seller deposits their escrow after buyer.");

    if((deals[dealId].seller == msg.sender) && (deals[dealId].sellerEscrow == msg.value)) {
        deals[dealId].balance = deals[dealId].balance + msg.value;
        deals[dealId].sellerEscrowDeposited = true;
        contractBalance = contractBalance + msg.value;
        updateStatus(dealId, STATUS.SELLER_ESCROW_DEPOSITED, "");
    }
    else if((deals[dealId].buyer == msg.sender) && (deals[dealId].buyerEscrow == msg.value)) {
        deals[dealId].balance = deals[dealId].balance + msg.value;
        deals[dealId].buyerEscrowDeposited = true;
        contractBalance = contractBalance + msg.value;
        updateStatus(dealId, STATUS.BUYER_ESCROW_DEPOSITED, "");
    }
    else {
        revert("Please make sure that you deposit the exact escrow amount.");
    }

}
```

Figure 8. The Deposit Installment function code.

2.3.4.6. Withdraw Installment

Allows the seller to withdraw the installments that are deposited by the buyer. Once all installments are withdrawn, the system marks the deal as completed by changing its status to DEAL_COMPLETED. Figure 9 shows the code of the Withdraw Installment function.

```
function withdrawInstallment(uint dealId, uint amount) onlySeller(dealId) noDispute(dealId) public payable {

    //ensure that only installments can be withdrawn
    uint maxAmount = deals[dealId].balance - deals[dealId].sellerEscrow - deals[dealId].buyerEscrow;

    if(amount <= maxAmount) {
        address payable withdrawTo = payable(msg.sender);
        uint amountToTransfer = amount;
        withdrawTo.transfer(amountToTransfer);
        deals[dealId].balance = deals[dealId].balance - amountToTransfer;
        contractBalance = contractBalance - amountToTransfer;
        updateStatus(dealId, deals[dealId].status, "AMOUNT_WITHDRAWN");
    } else {
        revert("Cannot withdraw more than the maximum allowed amount.");
    }

    uint totalEscrow =  deals[dealId].sellerEscrow + deals[dealId].buyerEscrow;

    if((deals[dealId].balance == totalEscrow) &&
    (deals[dealId].currentInstallment == deals[dealId].installments.length)) {

        updateStatus(dealId, STATUS.DEAL_COMPLETED, "");
    }
}
```

Figure 9. The Withdraw Installment function code.

2.3.4.7. Withdraw Escrow

Once the deal is completed, this function allows the buyer and the seller to withdraw the escrows that they deposited at the beginning of the deal. Figure 10 shows the code of the Withdraw Escrow function.

```
function withdrawEscrow(uint dealId) dealCompleted(dealId) onlyBuyerSeller(dealId) public payable {

    address payable withdrawTo = payable(msg.sender);

    uint amountToTransfer = msg.sender == deals[dealId].buyer ?
        deals[dealId].buyerEscrow : deals[dealId].sellerEscrow;

    if(amountToTransfer <= deals[dealId].balance) {
        withdrawTo.transfer(amountToTransfer);
        deals[dealId].balance = deals[dealId].balance - amountToTransfer;
        contractBalance = contractBalance - amountToTransfer;
        updateStatus(dealId, deals[dealId].status, "ESCROW_WITHDRAWN");
    } else {
        revert("Transaction failed! Insufficient balance.");
    }
}
```

Figure 10. The Withdraw Escrow function code.

2.3.4.8. Initiate Dispute

The Initiate Dispute function allows the buyer or the seller to halt any further activities on the deal and initiate a dispute with the other party. Figure 11 shows the code of the Initiate Dispute function.

```
function initiateDispute(uint dealId, string memory comment) dealActive(dealId) onlyBuyerSeller(dealId) public  {

    updateStatus(dealId, STATUS.DISPUTE_INITIATED, comment);

}
```

Figure 11. The Initiate Dispute function code.

2.3.4.9. Resolve Dispute

Allows the auditor to resolve an active dispute by either allowing the deal to proceed (by marking the status of the deal as DISPUTE_RESOLVED) or terminating the deal (by marking it as DEAL_TERMINATED). If the deal is terminated, all but the Settle Balances function are deactivated. Figure 12 shows the code of the Resolve Dispute function.

```
function resolveDispute(uint dealId, STATUS status, string memory comment)
dealActive(dealId) onlyAuditor(dealId) public  {

    require((deals[dealId].status == STATUS.DISPUTE_INITIATED), "No dispute was found.");
    require((status == STATUS.DEAL_TERMINATED) || (status == STATUS.DISPUTE_RESOLVED),
    "Deal can only be resolved or terminated.");

    updateStatus(dealId, status, comment);

}
```

Figure 12. The Resolve Dispute function code.

2.3.4.10. Settle Balances

Allows the auditor to refund the remaining balance to the buyer and the seller. The auditor has to refund the entire balance of the deal, otherwise, the operation is reverted. The balance consists of the escrows deposited by the buyer and the seller as well as any deposited installments that are not withdrawn yet by the seller. Figure 13 shows the code of the Settle Balances function. This function can only be executed if the deal was terminated by the Resolve Dispute function.

```
function settleBalances(uint dealId, uint buyerRefund, uint sellerRefund)
dealActive(dealId) onlyAuditor(dealId) public payable {

    require((deals[dealId].status == STATUS.DEAL_TERMINATED), "Action failed! Deal not terminated.");

    uint totalRefund = buyerRefund + sellerRefund;
    require(deals[dealId].balance == totalRefund, "Failed! You must clear out the entire balance.");

    if(deals[dealId].balance == totalRefund) {

        address payable buyer = payable(deals[dealId].buyer);
        buyer.transfer(buyerRefund);
        deals[dealId].balance = deals[dealId].balance - buyerRefund;
        contractBalance = contractBalance - buyerRefund;

        address payable seller = payable(deals[dealId].seller);
        seller.transfer(sellerRefund);
        deals[dealId].balance = deals[dealId].balance - sellerRefund;
        contractBalance = contractBalance - sellerRefund;

        updateStatus(dealId, STATUS.BALANCES_SETTLED, "");

    }

}
```

Figure 13. The Settle Balances function code.

3. Results and Discussion

An experiment of a hypothetical deal was conducted to test the system functions. The cost of invoking each of the smart contract functions is in Table 2. The Remix IDE was used to write and test the code, and a local Hard Hat Node was used to test the smart contract. Results show that the system enables complete oversight and control of construction supply chain deals and the enforcement of their terms and conditions. The test results demonstrate that all system functions are working as they should in managing and handling payments. Further, test results show that the cost of handling payments on top of the system is quite low, despite the record high price of the Ethereum coin of $3,260 at the time of the test, which makes the system an efficient, reliable, and cost-effective alternative to handling payments for construction supply chain stakeholders.

Table 2. Smart contract functions and their cost of execution.

Function	Cost in Gas	Cost in USD[1]
Deploy	3,916,767.00	0.64
Add Deal	380,959.00	0.06
Set Credit Score	86,570.00	0.01
Deposit Escrow	60,624.00	0.01
Complete Work	65,623.00	0.01
Deposit Installment	76,084.00	0.01
Withdraw Installment	65,923.00	0.01
Withdraw Escrow	55,208.00	0.01
Initiate Dispute	107,758.00	0.02
Resolve Dispute	57,571.00	0.01
Settle Balances	65,837.00	0.01

[1]1 ETH = ~$3,260, 1 Gas = 20 GWEI (at the time of writing this paper).

3.1. Functional Test Experiment and Results

The deal is as follows: A contractor wants to hire an architect to create the designs for an upcoming construction project that the contractor plans to start. The contractor and the architect agree on a deal with the total price of $8,900, divided as 20% down payment, then two 30% payments that are associated with agreed upon milestones, then a last payment of 20% upon the delivery of the final deliverable. The contractor and the architect want to use the proposed system to handle the payment process and ensure smooth execution of the project. The deal, thus, becomes as follows:

- For the purposes of the proposed system, the buyer is the contractor, and the seller is the architect.
- The payments are divided into 4 installments, which are $1780, $2670, $2670, and $1780.
- An independent credit scoring agency gave a rating of 1377 to the contractor and 721 to the architect. Therefore, the buyer and the seller agreed that the escrow amounts are $445 for the contractor (buyer), and $1,335 for the seller (architect).
- The buyer and seller agreed that another contractor trusted by both to be the auditor for the deal.

After agreeing on all the deal specifications, the buyer and seller signed a contract detailing all the terms and conditions. Then, the seller uploaded the contract to a cloud storage service and obtained a link to it. The seller then converted the installments and escrows into ETH (Ethereum currency) according to the exchange price of $3,260 and created a Deal on the proposed system using the Add Deal function with the following details:

- The link to the contract.
- The account addresses of the buyer, auditor, and the credit scoring agency.
- The credit scores of both the buyer and the seller.
- The buyer and seller escrows, which are 0.14 ETH and 0.41 ETH, respectively.
- The installments, which are (all in ETH): 0.55, 0.82, 0.82, 0.55.

Then, once created, the system returned a Deal object with the exact same information as well as the deal's unique identifier, confirming the creation of the Deal. Figure 14 shows the result of invoking the Add Deal function. Monetary inputs and outputs are provided in the WEI format, which is a fractional representation of ETH where 1 ETH = 1 * 10-18. Figure 15 shows the result of invoking the Get Deals function, which returns the data of a specific deal object. Whenever a function that returns a structure object (struct object, such as the Deal object in our case), the EVM returns a "tuple" that shows the data types of data structure as well as the values corresponding to each data member. Arrays (such as the installments array in our case) are shown as a comma separated series of values. And that is what is shown in Figure 15.

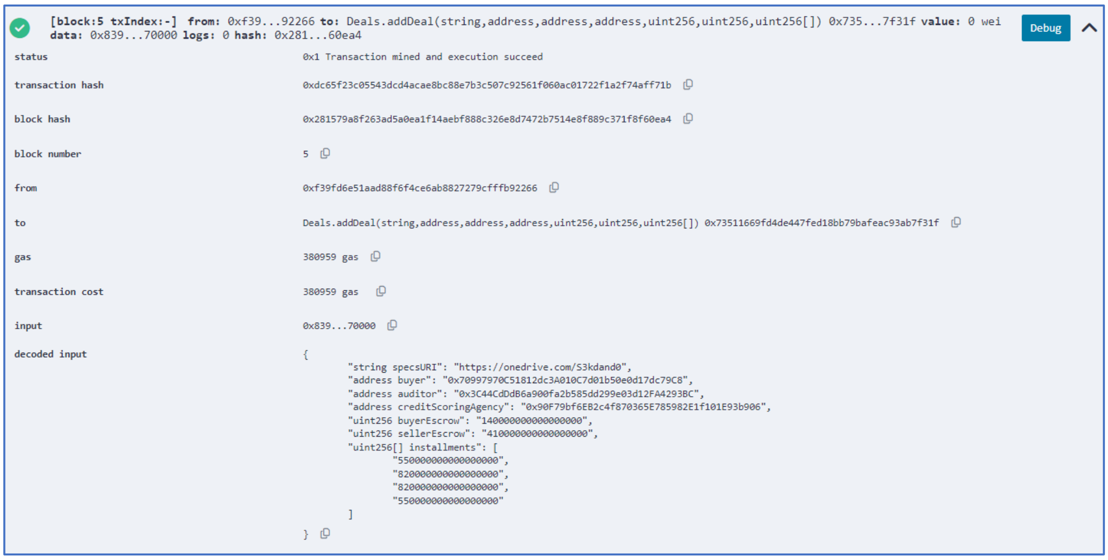

Figure 14. Results of invoking the Add Deal function.

Figure 15. Results of invoking the Get Deal function.

The credit scoring agency then added the credit scores of both the buyer and seller. The buyer then deposited their escrow amount of 0.14 ETH followed by depositing 0.41 ETH by the seller for their escrow amount. The buyer then deposited the down payment of 0.55 ETH. Figure 16 shows the result of invoking the Deposit Installment. The DealStatusChange event shows that the status was changed by the buyer address, the old and new statuses as a number, the total balance of the deal and the smart contract after the successful invocation of the function. The balances are shown in WEI. The total deal balance amount is 1100000000000000000 WEI, which is 1.1 ETH (which equals the 0.14 buyer escrow, the 0.41 seller escrow, and the 0.55 first installment). Since there's no other deals managed by the smart contract at the time of the experiment, the contract balance is the same as the deal balance.

Figure 16. Results of invoking the Deposit Installment function for the down payment.

After paying three of the four installments, the buyer initiated a dispute to complain about a delay caused by the seller in delivering the last deliverable. Figure 17 shows the result of initiating the dispute using the Initiate Dispute function. The auditor investigated the issue and reached an agreement with the seller and the buyer on an updated delivery date. The auditor then invoked the Resolve Dispute function and passed the DISPUTE_RESOLVED status with the comment "Buyer and seller agreed to have the last deliverable ready in 15 days", and that's what is shown as the result of invoking the Resolve Dispute function in Figure 18. The deal then proceeded until the last installment was paid by the buyer and withdrawn by the seller. The buyer then withdrew their escrow followed by the seller's withdrawal of their escrow. Figure 19 shows the result of invoking Withdraw Installment where the seller withdrew the last installment. In that figure, there are logs of two DealStatusChange events; the first event is announcing the withdrawal of the amount while the second is switching the status of the deal from ALL_INSTALLMENTS_DEPOSITED to DEAL_COMPLETED. Figure 20 shows the result invoking the Withdraw Escrow by the seller, which was the last amount to be withdrawn from the Deal, which is why the Deal balance after the withdrawal was 0.

```
decoded input      {
                       "uint256 dealId": "1",
                       "string comment": "The last deliverable has been delayed by 23 days."
                   }

decoded output     -

logs               [
                       {
                           "from": "0x2279b7a0a67db372996a5fab50d91eaa73d2ebe6",
                           "topic": "0xc36e28cb15c475884135730b5cef436b79fff334ed415f85ee7baf92f01ef2d7",
                           "event": "DealStatusChange",
                           "args": {
                               "0": "0x70997970C51812dc3A010C7d01b50e0d17dc79C8",
                               "1": 7,
                               "2": 8,
                               "3": "The last deliverable has been delayed by 23 days.",
                               "4": "2740000000000000000",
                               "5": "2740000000000000000",
                               "changedBy": "0x70997970C51812dc3A010C7d01b50e0d17dc79C8",
                               "from": 7,
                               "to": 8,
                               "comment": "The last deliverable has been delayed by 23 days.",
                               "dealBalance": "2740000000000000000",
                               "contractBalance": "2740000000000000000"
                           }
                       }
                   ]
```


Figure 17. Results of invoking the Initiate Dispute function.


```
decoded input      {
                       "uint256 dealId": "1",
                       "uint8 status": 9,
                       "string comment": "Seller and buyer agreed to extend the deadline by 15 days."
                   }

decoded output     -

logs               [
                       {
                           "from": "0x2279b7a0a67db372996a5fab50d91eaa73d2ebe6",
                           "topic": "0xc36e28cb15c475884135730b5cef436b79fff334ed415f85ee7baf92f01ef2d7",
                           "event": "DealStatusChange",
                           "args": {
                               "0": "0x3C44CdDdB6a900fa2b585dd299e03d12FA4293BC",
                               "1": 8,
                               "2": 9,
                               "3": "Seller and buyer agreed to extend the deadline by 15 days.",
                               "4": "2740000000000000000",
                               "5": "2740000000000000000",
                               "changedBy": "0x3C44CdDdB6a900fa2b585dd299e03d12FA4293BC",
                               "from": 8,
                               "to": 9,
                               "comment": "Seller and buyer agreed to extend the deadline by 15 days.",
                               "dealBalance": "2740000000000000000",
                               "contractBalance": "2740000000000000000"
                           }
                       }
                   ]
```


Figure 18. Results of invoking the Resolve Dispute function.

3.2. Management Implications

According to Raj et al. (2022), there are three payment modalities in construction supply chains: cash on delivery, advance payment, and credit payment. Cash on delivery is when the buyer pays the price of the goods once they arrive in their location. Advance payment means that the buyer pays a down payment prior to initiating the service, then pays the rest throughout the service provision period. Credit payment means that the seller sells the goods and services to the seller by extending a line of credit, and then the buyer settles the balance later. The proposed system enables construction supply chain stakeholders to handle payments with all three modalities.

Further, the proposed system creates an environment of mutual accountability where it's in the interest of both the buyer and the seller to be committed to their obligations to each other as per the terms and conditions of the deal. This environment is supported by the transparency, decentralization, immutability, and efficiency that blockchain technology offers. That environment is also supported by the full transparency about the history of deals and financial standing of both parties, and by the risk of losing the escrow money that both parties deposit in the system as a guarantee of smooth execution of the deal. Thus, the system helps construction supply chain stakeholders overcome the many problems and issues with payment handling that are found in the literature (Nanayakkara et al., 2021), (Ramachandra and Rotimi, 2015), (Xie et al., 2019), (Swai et al., 2020), including going over budget, rework and errors, late payments, improper withholding of payments, lack of trust, and disputes over quality of work.

```
logs    [
            {
                "from": "0x2279b7a0a67db372996a5fab50d91eaa73d2ebe6",
                "topic": "0xc36e28cb15c475884135730b5cef436b79fff334ed415f85ee7baf92f01ef2d7",
                "event": "DealStatusChange",
                "args": {
                    "0": "0xf39Fd6e51aad88F6F4ce6aB8827279cffFb92266",
                    "1": 12,
                    "2": 12,
                    "3": "AMOUNT_WITHDRAWN",
                    "4": "550000000000000000",
                    "5": "550000000000000000",
                    "changedBy": "0xf39Fd6e51aad88F6F4ce6aB8827279cffFb92266",
                    "from": 12,
                    "to": 12,
                    "comment": "AMOUNT_WITHDRAWN",
                    "dealBalance": "550000000000000000",
                    "contractBalance": "550000000000000000"
                }
            },
            {
                "from": "0x2279b7a0a67db372996a5fab50d91eaa73d2ebe6",
                "topic": "0xc36e28cb15c475884135730b5cef436b79fff334ed415f85ee7baf92f01ef2d7",
                "event": "DealStatusChange",
                "args": {
                    "0": "0xf39Fd6e51aad88F6F4ce6aB8827279cffFb92266",
                    "1": 12,
                    "2": 13,
                    "3": "",
                    "4": "550000000000000000",
                    "5": "550000000000000000",
                    "changedBy": "0xf39Fd6e51aad88F6F4ce6aB8827279cffFb92266",
                    "from": 12,
                    "to": 13,
                    "comment": "",
                    "dealBalance": "550000000000000000",
                    "contractBalance": "550000000000000000"
                }
            }
        ]
```


Figure 19. Results of invoking the Withdraw Installment function.


```
logs    [
            {
                "from": "0x2279b7a0a67db372996a5fab50d91eaa73d2ebe6",
                "topic": "0xc36e28cb15c475884135730b5cef436b79fff334ed415f85ee7baf92f01ef2d7",
                "event": "DealStatusChange",
                "args": {
                    "0": "0x70997970C51812dc3A010C7d01b50e0d17dc79C8",
                    "1": 13,
                    "2": 13,
                    "3": "ESCROW_WITHDRAWN",
                    "4": "0",
                    "5": "0",
                    "changedBy": "0x70997970C51812dc3A010C7d01b50e0d17dc79C8",
                    "from": 13,
                    "to": 13,
                    "comment": "ESCROW_WITHDRAWN",
                    "dealBalance": "0",
                    "contractBalance": "0"
                }
            }
        ]
```


Figure 20. Results of invoking the Withdraw Escrow function by the buyer.

3.3. Comparison

In this section, we compare our solution with several other solutions from literature, namely, the works proposed by Raj et al. (2022), Tsai (2023), and Wu et al. (2023). Table 3 shows the comparison between our study and these studies.

3.4. Limitations and Future Work

As with any study, ours has some limitations. Firstly, the smart contract in its current design does not support the handling of paying a commission to the auditor to do their job. So, there's always a risk that the auditor may delay the payment process if they don't play their role in a timely manner. By introducing a feature where a commission is paid to the auditor by the smart contract each time, they complete a task, or an entire fee that is paid to the auditor once the deal completes, the auditor will have a motivation to complete their work so they could collect their fees. Secondly, the proposed system does not support the ability to make changes to the deal while in progress; should there be any changes that need to be made to the deal, the buyer and seller have to terminate it and create a new one with the desired changes. Introducing the ability to make changes to the deal makes the process more efficient and less time-consuming. Thirdly, the credit scoring part of the payment process is handled off-chain. A feature can be introduced into the system where information about the buyer's and seller's financial standing can be shared automatically with another smart contract that represents the credit scoring agency, and that smart contract can then calculate the credit score for each of the buyer and seller in real-time and share it with the system; the system then pays the credit scoring smart contract a fee for that service. Lastly, a lot more testing is still needed to test the functionality of the system, specifically the edge cases that involve money transfers. This is to ensure that there's always a reliable way that money can be taken out of the smart contract and doesn't get stuck in it forever; also, to ensure that there are no security loopholes that could enable potential breaches and theft. Future research can focus on introducing the improvements suggested in this section, so the system is more capable of handling payments in a reliable, efficient, and secure manner. Further, a next step based on our study would be to pilot the system where a buyer and a seller in some construction projects manage their payments on the system to examine the system's effectiveness in a real-world scenario and gather feedback of what improvements need to be made.

Table 3. A comparison between the system proposed in our study with other studies in literature.

Criteria	**Our Solution**	(Raj et al., 2022)	(Tsai, 2023)	(Wu et al., 2023)
Sector	Construction	Sector-neutral	Sector-neutral	Construction
Actors	Buyer, seller, auditor, credit scoring agency	Buyer, seller, 3PL[1]	SMEs (seller), financial institutions, core enterprises (buyer)	Client, quality inspector, main contractor, progress inspector, quantity supervisor, 3rd party auditor
Work Tracking	Yes	Yes	Yes	Yes
Installment Tracking	Yes	Yes	Yes	No
Payment Guarantees	Escrow, credit risk, auditor	3PL	Financial institution	Consensus mechanism
Dispute Handling	Yes	No	No	No

[1]3rd party logistics provider.

4. Conclusion

In this paper, we presented a blockchain-based system for handling payments in the construction supply chain. It works by allowing a buyer and a seller to create a deal, define its terms and conditions, track the agreed upon goods and services delivery until completion, and manage the payment of installments that are associated with these goods and services. The system ensures a transparent, secure, reliable, and safe environment for the buyer and the seller by requiring both parties to declare their financial standing and have it reviewed and scored by a separate credit scoring agency. Further, deals are ensured by having both parties deposit an escrow amount that serves as a guarantee that can be used to compensate one party if the other fails to comply with their obligations. The system enables initiating and solving disputes transparently on the blockchain. The system introduced the role of an auditor, which is a 3rd party in charge of verifying completed work and resolving disputes. The system functions were tested with an experiment and their costs were recorded. Test results showed that the system can enable an efficient, reliable and cost-effective environment for handling payments. Future work can focus on introducing the ability to make changes to an ongoing/incomplete deal, automating the credit scoring operation, and introducing a feature for paying the auditor a fee in exchange for their services. The system also requires further testing, ideally in a real-world scenario with a buyer and seller from a construction project to test its effectiveness and get more feedback for improvements.

Ethics Permissions

This paper does not require ethics committee approval.

Author Contributions

SH found the gap, developed the solution, tested the results, and wrote the manuscript. EÖ reviewed the manuscript and scientific proofing.

Conflict of Interest

The authors declare that there's no affiliation or involvement in an organization or entity with a financial or non-financial interest in the subject matter or materials discussed in this paper.

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On $n-\delta$ −semiprimary Ideals in Commutative Rings

Mohammad HAMODA [1], Ece YETKİN ÇELİKEL [2*]

Keywords
Prime ideal,
δ −primary ideal,
δ −semiprimary ideal,
n −semiprimary ideal

Abstract – Let R be a commutative ring with identity and n a positive integer. A generalization of prime ideals is introduced in (Anderson and Badawi, 2021). A proper ideal J of R is said to be an n-semiprimary ideal if whenever $a,b \in R$ with $a^n b^n \in J$, then $a^n \in J$ or $b^n \in J$. Let $\delta: Id(R) \longrightarrow Id(R)$ be an expansion function of ideals of R where $Id(R)$ is the set of all ideals of R. The aim of this paper is to introduce the class of $n-\delta$ −semiprimary ideals generalizing the notion of n −semiprimary ideals. We call a proper ideal J of R an $n-\delta$ −semiprimary ideal if whenever $a^n b^n \in J$ for $a,b \in R$, then $a^n \in \delta(J)$ or $b^n \in \delta(J)$. Several properties and characterizations regarding this class of ideals with many supporting examples are presented. Additionally, we call a proper ideal J of R a strongly $n-\delta$ −semiprimary ideal of R if whenever $K^n L^n \subseteq J$ for proper ideals K and L of R, then $K^n \subseteq \delta(J)$ or $L^n \subseteq \delta(J)$. We investigate the relationship between these two concepts. Moreover, the behaviour of $n-\delta$ −semiprimary ideals under homomorphisms, in localization rings, in division rings, in cartesian product of rings and in idealization rings is investigated.

1. Introduction and Preliminaries

Throughout this article, all rings are assumed to be commutative with identity. By $Id(R)$ and $Id(R)^*$, we denote the set of all ideals and particularly, the set of all proper ideals of a ring R, respectively. Recall from Zhao (2001) that a function $\delta: Id(R) \longrightarrow Id(R)$ providing $J \subseteq \delta(J)$, and whenever $J \subseteq K$ implies $\delta(J) \subseteq \delta(K)$ for all $J, K \in I(R)$ is called an expansion of ideals (in briefly e.f.i). For example, the identity function δ_J, where $\delta_J(J) = J$ for all $J \in Id(R)$, is a trivial e.f.i of R. Also, the function $\delta_{\sqrt{J}}(J) = \sqrt{J}$ for each ideal J of R is an e.f.i of R. Generalizing the concept of prime ideals, in 2001, Zhao introduced the concept of δ-primary ideals. According to Zhao (2001), given an e.f.i δ of ideals of R, $J \in Id(R)^*$ is called a δ-primary ideal in R if $a,b \in R$ with $ab \in J$, then $a \in$ J or $b \in \delta(J)$. After that, Badawi et al. (Badawi et al., 2018) defined the class of δ-semiprimary ideals. $J \in Id(R)^*$ is said to be δ-semiprimary in R if $a,b \in R$ with $ab \in J$ implies $a \in \delta(J)$ or $b \in \delta(J)$. As a different generalization of prime ideals, Anderson and Badawi defined n-semiprimary ideals in their recent research (Anderson and Badawi, 2021). Let $n \geq 1$. Then, $J \in Id(R)^*$ is called an n-semiprimary ideal of R if whenever $a^n b^n \in J$ for $a,b \in R$, then $a^n \in J$ or $b^n \in J$. Clearly, 1-semiprimary ideal is a just prime ideal. For the other extentions of prime and primary ideals, the reader may consult for example (Anderson and Badawi, 2011), (Badawi and Fahid, 2018), (Badawi et al., 2018), (Yetkin Celikel, 2021), (Hamoda, 2023) and (Ulucak et al., 2018).

The motivation of writing this article lies to create new concepts that can be used in many branches in commutative algebra and its applications and to develop related results. In section 2, we present the main results concerning $n-\delta$ −semiprimary ideals with supporting examples and counterexamples. Among many results in this paper, the behavior of this class of ideals under homomorphisms, localizations, cartesian products and idealizations are investigated. We proved that if $J \in Id(R)^*$ and $P^n \subseteq \delta(J)$ for a positive integer n where P be a prime ideal of R including J, then J is $k-\delta$ −semiprimary in R for any integer $k \geq n$.

[1]Department of Mathematics, Faculty of Applied Science, Al-Aqsa University, Gaza, Palestine. E-mail: ma.hmodeh@alaqsa.edu.ps OrcID: 0000-0002-5452-9220
[2]***Corresponding Author.** Department of Basic Sciences, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep, Turkey. E-mail: ece.celikel@hku.edu.tr, yetkinece@gmail.com OrcID: 0000-0001-6194-656X
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2. Properties of $n-\delta-$semiprimary ideals

Our starting point is the following definition. Unless otherwise stated, throughout δ is assumed to be an expansion function of ideals (e.f.i) of a ring R and n ∈ ℕ.

Definition 1 Let δ be an e.f.i of R and $J \in Id(R)^*$. Then, J is said to be $n-\delta-$semiprimary in R if whenever $a^n b^n \in J$ for $a, b \in R$, then $a^n \in \delta(J)$ or $b^n \in \delta(J)$.

It is clear to see that a $1-\delta-$semiprimary ideal coincides with $\delta-$semiprimary ideal. Any $\delta-$semiprimary ideal and n-semiprimary ideal is $n-\delta-$semiprimary. However, the converses of these relationships do not hold in general. The following two examples are presented to justify that there are $n-\delta-$semiprimary ideals of a ring R that are not an $n-$semiprimary ideal.

Example 1 Consider $R=\mathbb{Z}$ and $J=16\mathbb{Z}$. Then $\sqrt{J}=2\mathbb{Z}$ and clearly I is $2-\delta_{\sqrt{J}}-$semiprimary in R. However, J is not a $2-$semiprimary in R as $2^2 2^2 \in J$ and $2^2 \not\in J$.

Example 2 Consider $R=\mathbb{Z}_2[\{X_n\}_{n=1}^{\infty}]$ and the ideal $K=(\{X_n^n\}_{n=1}^{\infty})$ of R. Then K is $n-\delta_{\sqrt{I}}-$semiprimary in R where $\delta_{\sqrt{K}}(K)=\sqrt{K}=(\{X_n\}_{n=1}^{\infty})$. On the other hand, K is not an $n-$semiprimary ideal for each $n \geq 1$ as $X_{2n}^n X_{2n}^n = X_{2n}^{2n} \in K$, but $X_{2n}^n \not\in K$.

Next, we introduce an $n-\delta-$semiprimary ideal of a ring R which is not $\delta-$primary ideal.

Example 3 Let $A=\mathbb{Z}_2[X_1, X_2]$ with indeterminates X_1 and X_2. Take $I=(X_2^2, X_1X_2)A$ and $K=(X_2^2, X_1^2X_2^2)A$ and let $R=A/J$. Then, $\sqrt{I/K}=(X_2, X_1X_2)A/K$. One can check easily that K is $n-\delta_{\sqrt{I}}-$semiprimary in R *for* any $n \geq 1$. However, I/K is not a $\delta_{\sqrt{I}}-$primary in R, since $X_1X_2+K \in I/J$, $X_1+K \not\in \delta_{\sqrt{I}}(I/K)=\sqrt{I/K}$ and $X_2+K \not\in I/K$.

Proposition 1 For $J \in Id(R)^*$, we have the following statements.

1. If J is $n-\delta-$semiprimary in R, then J is $mn-\delta-$semiprimary in R for all $m \in \mathbb{N}$.

2. J is $n-\delta_I-$semiprimary in R if and only if J is $n-$semiprimary in R.

3. Let δ and γ be two e.f.i of R such that $\delta(J) \subseteq \gamma(J)$. If J is $n-\delta-$semiprimary in R, then J is $\gamma-\delta-$semiprimary in R.

4. If $\delta(\delta(J))=\delta(J)$, then $\delta(J)$ is $n-\delta-$semiprimary in R if and only if $\delta(J)$ is $n-$semiprimary in R.

Proof Straightforward.

Proposition 2 Let J ∈ Id(R)*. Then, J is $n-\delta-$semiprimary in R if and only if either $\delta(J)$ is a prime or $\delta(J)$ is $n-\delta-$semiprimary in R providing $\delta(\delta(J))=\delta(J)$.

Proof Assume that $\delta(J)$ is an $n-\delta-$semiprimary in R and let $a^n b^n \in J$ for $a, b \in R$ and $a^n \not\in \delta(J)$. Since $J \subseteq \delta(J)$ and $\delta(J)$ is an $n-\delta-$semiprimary in R, we have $b^n \in \delta(\delta(J))=\delta(J)$, as required. The converse part is clear.

Theorem 1 Let $J \in Id(R)^*$ and $Q^n \subseteq \delta(J)$ for some $n \geq 1$ where Q is prime in of R including J. Then, for all $k \geq n$, J is an $k-\delta-$semiprimary in R.

Proof Let $a^k b^k \in J \subseteq Q$ for $a, b \in R$ and $k \geq n$. Then, $a \in Q$ or $b \in Q$. Hence, $a^k \in Q^k \subseteq \delta(J)$ or $b^k \in Q^k \subseteq \delta(J)$, and therefore J is $k-\delta-$semiprimary in R.

As direct consequences of Theorem 1, we verify the following corollary

Corollary 1 Let R be a Noetherian ring and $J \in Id(R)^*$ such that $\sqrt{J}$ is prime. Then, there is an $n \geq 1$ provided that J is $k-\delta-$semiprimary in R for any $k \geq n$.

Proof Put $Q=\sqrt{J}$. Then there is an $n \geq 1$ satisfying $Q^n \subseteq J \subseteq \delta(J)$ as R is Noetherian. Therefore, from by Theorem 1, J is $k-\delta-$semiprimary for any $k \geq n$.

Corollary 2 For prime ideals $J_1 \subseteq \ldots \subseteq J_k$ of R and positive integers $n_1, \ldots, n_k$, $J = J_1^{n_1} \ldots J_k^{n_k}$ is $m-\delta-$semiprimary where $m \geq n_1 + \cdots + n_k$.

Proof Since $\sqrt{J} = J_1$ is prime and $J_1^n \subseteq J_1^{n_1} \ldots J_k^{n_k} = J$, where $n = n_1 + \cdots + n_k$, Theorem 1 yields that J is $m-\delta-$semiprimary where $m \geq n = n_1 + \cdots + n_k$.

The following example is given to illustrate that the converse of Theorem 1 need not be true.

Example 4 Let $R = \mathbb{Z}_q[X, Y]$, where $q \geq 2$ be a prime integer and let $J = (X^q, Y^q)$. Then $\sqrt{J} = (X, Y)$. $(\sqrt{J})^q \nsubseteq J$ since $YX^{q-1} \notin J$. On the other hand, let $g^q h^q \in \sqrt{J} \subseteq (X, Y)$ for $g, h \in R$. Then, $g \in (X, Y)$ or $h \in (X, Y)$. Thus, $g^q \in J \subseteq \delta(J)$ or $h^q \in J \subseteq \delta(J)$ and hence, J is $q-\delta-$semiprimary in R.

Let $J \in Id(R)^*$. We say that J is $n-\delta-$primary in R if there exists $n \geq 1$ if whenever $ab \in J$ for $a, b \in R$ implies either $a \in \delta(J)$ or $b^n \in \delta(J)$. Next, we show that $n-\delta-$primary ideals are subclass of the class of $n-\delta-$semiprimary ideals.

Proposition 3 Any $n-\delta-$primary ideal is an $n-\delta-$semiprimary ideal.

Proof Let J be $n-\delta-$primary in R. Assume that $a^n b^n \in J$ for $a, b \in R$ and $a^n \notin \delta(J)$. Let k be the minimum positive integer satisfying $a^n b^k \in \delta(J)$. Then, $(a^n b^{k-1})b = a^n b^k \in \delta(J)$. Since $a^n b^{k-1} \notin \delta(J)$ and J is $n-\delta-$primary in R, we have $b^n \in \delta(J)$; so we are done.

Now, we give an example for an $n-\delta-$semiprimary ideal of a ring R which is not $n-\delta-$primary ideal for all n.

Example 5 Let $R = \mathbb{Z}_2[X_1, X_2]$ with indeterminates X_1 and X_2. For all $n \geq 2$, consider $K = (X_1X_2, X_2^n)$. Then, $Q = \sqrt{K} = (X_2)$ is prime in R and $Q^n \subseteq K$. Define $\delta: Id(R) \longrightarrow Id(R)$ by $\delta(J) = J + M$ for each ideal J of R, where (X_1, X_2) is the unique maximal ideal. Thus, δ is an e.f.i of R. By Theorem 1, K is an $n-\delta-$semiprimary in R. However, $X_2X_1 \in K, X_2 \notin \delta(K)$ and $X_1^m \notin \delta(K)$ for any $m \in \mathbb{N}$. Hence, K is not $m-\delta-$primary in R for all $m \in \mathbb{N}$.

Recall from Zhao (2001) that an e.f.i δ of a ring R is said to be intersection preserving if $\delta(I_1 \cap \ldots \cap I_n) = \delta(I_1) \cap \ldots \cap \delta(I_n)$ for any ideals $I_1, \ldots, I_n$ of R.

Proposition 4 Suppose that δ is intersection preserving and $J_1, \ldots, J_t$ are $n-\delta-$semiprimary ideals of R satisfying $\delta(J_i) = \delta(J_k)$ for all $i,\ k \in \{1,2,\ldots,t\}$. Then, $\bigcap_{i=1}^t J_i$ *is* $n-\delta-$semiprimary in R.

Proof Assume that $a^n b^n \in \bigcap_{i=1}^t J_i$ for $a, b \in R$ and $a^n \notin \delta(\bigcap_{i=1}^t J_i)$. Since $\delta(\bigcap_{i=1}^t J_i) = \bigcap_{i=1}^t \delta(J_i) = \delta(J_i)$, we have $a^n \notin \delta(J_i)$. Since $a^n b^n \in J_i$ for all $i \in \{1,2,\ldots,t\}$ and J_i is $n-\delta-$semiprimary, we have $b^n \in \delta(J_i) = \delta(\bigcap_{i=1}^t J_i)$ for all $i \in \{1,2,\ldots,t\}$, so we are done.

Proposition 5 Let $I_1,\ I_2,\ I_3 \in Id(R)^*$ with the order $I_1 \subseteq I_2 \subseteq I_3$. *If* I_3 is an $n-\delta-$semiprimary ideal of R such that $\delta(I_1) = \delta(I_3)$, then I_2 is an $n-\delta-$semiprimary ideal of R.

Proof Let $a^n b^n \in I_2$ for $a, b \in R$ and $a^n \notin \delta(I_2)$. From our assumptions, we have $\delta(I_1) = \delta(I_2) = \delta(I_3)$. From the inclusion $I_1 \subseteq I_2$, we have $a^n b^n \in I_2$. Since I_3 is an $n-\delta-$semiprimary ideal of R and $a^n \notin \delta(I_3)$, we conclude $b^n \in \delta(I_3) = \delta(I_2)$. Thus, I_2 is an $n-\delta-$semiprimary ideal of R.

Recall from (Ulucak et al., 2018) that if $f: R \longrightarrow S$ is a homomorphism or rings, γ and δ are e.f.i of R and S, respectively, then it is said that f is a $\gamma\delta-$ring homomorphism if $\gamma(f^{-1}(J)) = f^{-1}(\delta(J))$ for all $J \in Id(S)$. In this case, we have $f(\gamma(J)) = \delta(f(J))$ for all $J \in Id(R)$.

Proposition 6 Let γ and δ be e.f.i of R and R', respectively, and $f: R \longrightarrow R'$ be a $\gamma\delta-$ring homomorphism.

1. If J' is $n-\delta-$semiprimary in R', then $f^{-1}(J')$ is $n-\gamma-$semiprimary in R.

2. Suppose that f is onto and $J \in Id(R)^*$containing $ker(f)$. Then J is $n-\gamma-$semiprimary in R if and only if $f(J)$ is $n-\delta-$semiprimary in R'.

Proof 1. Let J' be an $n-\delta-$semiprimary in R' and $a^n b^n \in f^{-1}(J')$ for $a,b \in R$. Then, $f(a^n b^n) = f(a^n)f(b^n) \in J'$ which yields that either $f(a^n) \in \delta(J')$ or $f(b^n) \in \delta(J')$. Hence, $a^n \in f^{-1}(\delta(J'))$ or $b^n \in f^{-1}(\delta(J'))$ and we are done as $f^{-1}(\delta(J')) = \gamma(f^{-1}(J'))$.

2. Let $a,b \in R'$ and $a^n b^n \in f(J)$. Say, $a^n = f(c)^n$ and $b^n = f(d)^n$ for some $c,d \in R$. Then, clearly $f(c)^n f(d)^n = f(c^n d^n) \in f(J)$ and $ker(f) \subseteq J$ imply that $c^n d^n \in J$. Since J is an $n-\gamma-$semiprimary in R, we have either $c \in \gamma(J)$ or $d^n \in \gamma(J)$. Thus, $a^n \in f(\gamma(J))$ or $b^n \in f(\gamma(J))$. The claim follows from $f(\gamma(J)) = \delta(f(J))$.

Recall from Ulucak et al. (2018) that if for an e.f.i δ of R and $J \in Id(R)^*$, the function $\delta_q: R/J \longrightarrow R/J$ defined by $\delta_q(K/J) = \delta(K)/J$ for $K \in Id(R)$ with $J \subseteq K$ is also an e.f.i of R/J. Hence, we conclude the next result for quotient rings.

Corollary 3 Let $I,K \in Id(R)^*$ with the order $I \subseteq K$. Then, K is $n-\delta-$semiprimary in R if and only if K/I is $n-\delta_q-$semiprimary in R/I.

Example 6 Consider the polynomial ring $R[X]$ and its e.f.i. δ. Let $\delta_q: R[X]/(X) \longrightarrow R[X]/(X)$ defined by $\delta_q(K/(X)) = \delta(K)/(X)$ for all ideals $(X) \subseteq K$ of $R[X]$. Then, δ_q is an e.f.i of $R[X]/(X) \approx R$. For any $J \in Id(R)^*$, of R, since $(J,X)/(X) \approx J$, from Corollary 3, (J,X) is $n-\delta-$semiprimary in $R[X]$ if and only if J is $n-\delta_q-$semiprimary in R.

Let S be a multiplicatively closed subset (in briefly, m.c.s) of a ring R and δ be an e.f.i of R. Then, a function δ_S defined by $\delta_S(I_S) = (\delta(I))_S$ is an e.f.i of R_S.

Proposition 7 Let S be a m.c.s of R and $J \in Id(R)^*$. Then we have the following statements.

1. Suppose that J is $n-\delta-$semiprimary in R with $J \cap S = \emptyset$. Then, J_S is $n-\delta_S-$semiprimary in R_S.

2. Suppose that J_S is an $n-\delta_S-$semiprimary ideal of R_S satisfying $Z_{\delta(J)}(R) \cap S = \emptyset$. Then J is $n-\delta-$semiprimary in R.

Proof 1. Let $a,b \in R_S$ and $a^n b^n \in J$. Then, $a = \frac{r_1}{s_1}$ and $b = \frac{r_2}{s_2}$ for some $r_1, r_2 \in R$ and $s_1, s_2 \in S$. Hence, $ur_1^n r_2^n \in J$ for some $u \in S$ and so $(ur_1)^n r_2^n \in J$ yields either $(ur_1)^n \in \delta(J)$ or $r_2^n \in \delta(J)$.Thus, we conclude either $a^n = \frac{u^n r_1^n}{u^n s_1^n} \in \delta(J)_S$ or $b^n = \frac{r_2^n}{s_2^n} \in \delta(I)_S$ and we are done as $\delta_S(J_S) = \delta(J)_S$.

2. Let $a,b \in R$ and $a^n b^n \in J$. Then we have $(\frac{a}{1})^n (\frac{b}{1})^n \in J$ which implies either $(\frac{a}{1})^n\ \delta(J_S)$ or $(\frac{b}{1})^n \in \delta(J_S)$. Since $\delta_S(J_S) = (\delta(J))_S$, there are some $u,u' \in S$ satisfying $ua^n \in \delta(J)$ or $u'b^n \in \delta(J)$. Now, $Z_{\delta(I)}(R) \cap S = \emptyset$ implies that we have either $a^n \in \delta(J)$ or $b^n \in \delta(J)$, as required.

Now, we give the following definition.

Definition 2 Let δ be an e.f.i of a ring $R, J \in Id(R)^*$ and $n \geq 1$. J is said to be strongly $n-\delta-$semiprimary in R if whenever $K^n L^n \subseteq I$ for some $K,L \in Id(R)^*$, then $K^n \subseteq \delta(I)$ or $L^n \subseteq \delta(I)$.

Observe that a strongly $1-\delta-$semiprimary ideal is just a $\delta-$semiprimary ideal. Any strongly $n-\delta-$semiprimary ideal is an $n-\delta-$semiprimary ideal. In the following example, we show that those are distinct concepts.

Example 7 Let δ_I be an e.f.i of the ring $\mathbb{Z}_2[X,Y]$, and let $J = (X^2, Y^2)$. By Example 4, I is $2-\delta_J-$semiprimary in $\mathbb{Z}_2[X,Y]$, with prime ideal $K = J = (X,Y)$. It is clear that $K^2K^2 = K^4 \subseteq J$, but $K^2 \nsubseteq J = \delta_J(J)$. Thus, J is not strongly $2-\delta_J-$semiprimary in $\mathbb{Z}_2[X,Y]$.

We recall from Anderson et al. (1994) that for a $J \in Id(R)^*$, the ideal generated by n th powers of elements of J is denoted by $J_n = (a^n : a \in J)$. Note that $J_n \subseteq J^n \subseteq J$ and the equality holds when $n = 1$. Moreover, it is verified that if $n!$ is unit in R, then $J_n = J^n$. Next, we give a characterization for strongly $n-\delta-$semiprimary ideals of R.

Theorem 2 Let δ be an e.f.i of a ring $R, J \in Id(R)^*$ and $n \geq 1$ such that $n!$ is unit. Then we have the following equivalent three conditions.

1. J is strongly $n-\delta-$semiprimary in R.

2. For each element $a \in R$, any $L \in Id(R)$ with $a^n L^n \subseteq J$ and $a^n \not\subseteq \delta(J)$, we have $L^n \subseteq \delta(J)$.

3. J is $n-\delta-$semiprimary in R.

Proof (1) $\Longrightarrow$ (2) Let $a \in R$, $L \in Id(R)$ with $a^n L^n \subseteq J$ and $a^n \not\subseteq \delta(J)$. Put $K =< a >$. Then $K^n \nsubseteq J$ and this implies that $L^n \subseteq \delta(J)$, as needed.

(2) $\Longrightarrow$ (3) Suppose that $K^n L^n \subseteq J$ for K , $L \in Id(R)^*$ and $K^n \nsubseteq \delta(J)$. Since $n!$ is a unit, we have $J_n = J^n$, and hence $a^n \not\subseteq \delta(J)$ for some $a \in K$. Thus, $L^n \subseteq \delta(J)$ by (ii).

(3) $\Longrightarrow$ (1) Let $a, b \in R$ and $a^n b^n \in J$. Taking $L =< b >$ in (iii), we are done.

Let $R_1, \ldots, R_k$ be commutative rings with identity and $R = R_1 \times \ldots \times R_k$. Recall from Badawi and Fahid (2018) that an ideal of $R = R_1 \times \ldots \times R_k$ has the form $I_1 \times \ldots \times I_k$ for some ideals I_i of R_i for each $i = 1, \ldots, k$. Then, $\delta_\times$ be an e.f.i of R which is defined by $\delta_\times(I_1 \times \ldots \times I_k) = \delta_1(I_1) \times \ldots \times \delta_k(I_k)$ for each ideal I_i of R_i where δ_i is an e.f.i of R_i for each $i \in \{1, \ldots, k\}$. Next, we characterize $n-\delta_\times-$semiprimary ideals of cartesian product of rings.

Theorem 3 Let δ_1 and δ_2 be e.f.i of rings R_1 and R_2, respectively. For $J_1 \times J_2 \in Id(R_1 \times R_2)^*$, the following are equivalent.

1. $J_1 \times J_2$ is an $n-\delta_\times-$semiprimary in $R_1 \times R_2$.

2. J_1 is $n-\delta_1-$semiprimary in R_1 and $\delta_2(J_2) = R_2$ or J_2 is $n-\delta_2-$semiprimary in R_2 and $\delta_1(J_1) = R_1$.

Proof Note that if $\delta_\times(J) = R$, then the claim is clear.

(1) $\Longrightarrow$ (2) Assume that both of $\delta_1(J_1)$ and $\delta_2(J_2)$ are proper. Since $(0,0) = (1,0)^n(0,1)^n \in J$ but neither $(1,0)^n \in \delta_\times(J)$ nor $(0,1)^n \in \delta_\times(J)$, we get a contradiction. Hence, we may assume that $\delta_1(J_1)$ is proper and $\delta_2(J_2) = R_2$. Suppose that $a^n b^n \in J_1$ and $a^n \notin \delta(J_1)$ for some $a, b \in R_1$. Then $(a,0)^n(b,0)^n \in J$ and $(a,0)^n \notin \delta_\times(J)$ which implies $(b,0)^n \in \delta_\times(J)$. Thus, $b^n \in J_1$ and J_1 is $n-\delta_\times-$semiprimary in R_1. In the case of $\delta_1(J_1) = R_1$ and $\delta_2(J_2) = R_2$ is similar.

(2) $\Longrightarrow$ (1) We may suppose that J_1 is $n-\delta_1-$semiprimary in R_1 and $\delta_2(J_2) = R_2$. Let $(a_1, a_2)^n(b_1, b_2)^n \in J = J_1 \times J_2$ such that $(a_1, a_2)^n \not\subseteq \delta_\times(J)$. Then $a_1^n b_1^n \in J_1$ and $a_1^n \notin \delta_1(J_1)$ imlply that $b_1^n \in \delta_1(J_1)$. Hence $(b_1, b_2)^n \in \delta_\times(J)$, so we are done.

In general, we conclude the following result.

Theorem 4 Let $R = R_1 \times \ldots \times R_k$, where $R_1, \ldots, R_k$ are rings for $k \leq 2 < \infty$. Let δ_i be an e.f.i of R_i for each $i \in \{1, \ldots, k\}$. Let $J = J_1 \times \ldots \times J_k \in Id(R)^*$ for some ideals $J_1, \ldots, J_k$ of $R_1, \ldots, R_k$, respectively. Then, we have the following equivalent statements.

1. J is $n-\delta_\times-$semiprimary in R.

2. Either $J = \prod_{r=1}^{k} J_r$ such that for some $t \in \{1, \ldots, k\}$, J_t is an $n-\delta_t-$semiprimary in R_t, and $J_r = R_r$ for every $r \in \{1, \ldots, k\}$ for every $r \in \{1, \ldots, k\} \backslash \{t\}$ or $J = \prod_{r=1}^{k} J_r$ such that for some $t, m \in \{1, \ldots, k\}$.

Proof We use the mathematical induction method. Suppose that $k = 2$. Then the claim holds by Theorem 3. Hence, let $3 \leq k < \infty$. Assume that the claim is true when $A = R_1 \times \ldots \times R_{k-1}$. We verify the claim when $R = A \times R_k$. Since clearly $\delta_A(J_1 \times \ldots \times J_{k-1}) = \delta_1(J_1) \times \ldots \delta_{k-1}(J_{k-1})$, from Theorem 3, J is $n-\delta_\times-$semiprimary in R if and only if either $J = B \times R_k$ for some $n-\delta_A-$semiprimary ideal B of A or $J = A \times B_k$ for some $n-\delta_k-$ semiprimary ideal B_k of R_k . It must be clear that for a $P \in Id(A)^*$ is $n-\delta_A-$ semiprimary in A if and only if $P = \prod_{r=1}^{k-1} J_r$ such that for some $t \in \{1, \ldots, k-1\}$, J_t is $n-\delta_t-$semiprimary in R_t, and $J_r = R_r$ for every $r \in \{1, \ldots, k-1\} \backslash \{t\}$, we are done.

Let δ be an e.f.i of a ring R. For $I \in Id(R)^*$, we define

$D_R(I) = \{n \in \mathbb{N}: I\ is\ an\ n - \delta - semiprimary\ ideal\ of\ R\}$ and $\mu_R(I) = min\ D_R(I)$.

If $D_R(I) = \emptyset$, we define $\mu_R(I) = \infty$.

Theorem 5 Let δ be an e.f.i of a commutative Noetherian integral domain R. If for any $J \in Id(R)$ with $\mu_R(J) = 2$ implies $J = M^2$ for some maximal ideal M of R, then R is a Dedekind domain.

Proof Assume that J is an ideal of R with $M^2 \subseteq J \subset M$ for a maximal ideal M of R. Then, J is $2 - \delta -$semiprimary in R by Theorem 1. Also, J is not prime (maximal). Thus, $\mu_R(J) = 2$. Thus, $J = M^2$ by assumption. Thus, we have no ideal of R satisfying $M^2 \subset J \subset M$ for every maximal ideal M of R and from Theorem 6.20 in Larson and McCarthy (1971), R is a Dedekind domain.

Recall that an integral domain R is said to be a valuation domain if either $x|y$ or $y|x$ $(in\ R)$ for all $0 \neq x, y \in R$. We conclude the following result in valuation domains.

Theorem 6 Let δ be e.f.i of a valuation domain R with $\sqrt{\delta(J)} = \delta(\sqrt{J})$ and $I \in Id(R)^*$ with $K = \sqrt{J}$. *If* K *is* non-idempotent, then J is $n - \delta -$semiprimary in R for some $n \geq 1$.

Proof If $K = \sqrt{J}$ is not idempotent, then $K^n \subseteq J$ for some $n \in \mathbb{N}$, see Theorem 17.1 (5) in Gilmer (1972). Thus, J is $n - \delta -$semiprimary in R by Theorem 1.

Recall from Anderson and Winders (2009) that the idealization ring of an $R -$module M over a ring R defined by $R(+)M = R \times M$ with the following binary operations given by $(x, r) + (y, s) = (x + y, r + s)$ and $(x, r)(y, s) = (xy, yr + xs)$, respectively, and the identity id $(1,0)$. Also, since clearly $(\{0\}(+)M)^2 = \{0\}$, $\{0\}(+)M \subseteq Nil(R(+)M)$.

We define a function $\delta_{(+)}: Id(R(+)M) \longrightarrow Id(R(+)M)$ such that $\delta_{(+)}(J(+)N) = \delta(J)(+)M$ for every ideal $J \in Id(R)$ and every submodule N of M. Then, $\delta_{(+)}$ is an e.f.i of $R(+)M$.

Theorem 7 Let δ be an e.f.i of a ring R, M be an $R -$module, A be a submodule of $M, J \in Id(R)^*$ ideal of R and $n \in \mathbb{N}$. Then, we have the following equivalent conditions.

1. $J(+)A$ is $n - \delta_{(+)} -$semiprimary in $R(+)M$.

2. J is $n - \delta -$semiprimary in R.

Proof (1) $\Rightarrow$ (2) Let $J(+)A$ be $n - \delta_{(+)} -$semiprimary in $R(+)M$, and let $x^n y^n \in J$ for $x, y \in R$. Then, $(x^n, 0)(y^n, 0) = (x^n y^n, 0) \in J(+)A$. It implies that $x^n \in \delta(J)$ or $y^n \in \delta(J)$ since $J(+)A$ is $n - \delta_{(+)} -$semiprimary in $R(+)M$. Therefore, J is $n - \delta -$semiprimary in R.

(2) $\Rightarrow$ (1) Assume that J is $n - \delta -$semiprimary in R, and let $(x, r)^n (y, s)^n = (x^n y^n, z) \in J(+)A$ where $z = n(y^n xr + x^n ys)$ and $(x, r), (y, s) \in R(+)M$. Hence, $x^n y^n \in$ J. Since J is $n - \delta -$semiprimary in R, we have $x^n \in \delta(J)$ or $y^n \in \delta(J)$. Since $\delta_{(+)}(J(+)A) = \delta(J)(+)M$, we conclude either $(x, r)^n \in \delta_{(+)}(J(+)A)$ or $(y, s)^n \in \delta_{(+)}(J(+)A)$. Therefore, $J(+)A$ is $n - \delta_{(+)} -$semiprimary in $R(+)M$.

3. Conclusion

In this study, a generalization of both of $n -$semiprimary and $\delta -$semiprimary ideals which is called $n - \delta -$semiprimary ideals is presented. By this way, we found answers to the following questions. What is the location of the algebraic structure of this class of ideals in the literature? Which properties of it is similar to those of $n -$semiprimary and $\delta -$semiprimary ideals? Is this property stable under localizations, homomorphisms, cartesian products and idealizations? (see: Proposition 6 and 7, Corollary 3, Theorems 2, 3 and 7). Consequently, there are many open questions arising from this study. What if one defines weakly $n - \delta -$semiprimary ideals with the following definition, what will be the differences between these two? For example, a proper ideal I of R a weakly $n - \delta -$semiprimary ideal if whenever $0 \neq a^n b^n \in I$ for $a, b \in R$, then $a^n \in \delta(I)$ or $b^n \in \delta(I)$. On the other hand, extending this algebraic structure in rings to submodules, $n - \delta -$semiprimary submodule of an R-module M can be described.

Ethics Permissions

This paper does not require ethics committee approval.

Author Contributions

Both of the authors conceived of the presented idea. Both of the authors discussed the results and contributed to the final manuscript.

Conflict of Interest

There are no conflicts of interests/competing interests.

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