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On Some *k*- Oresme Polynomials with Negative Indices

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Abstract
In this study, *k*- Oresme polynomials with negative indices, which are the generalization of Oresme polynomials, were examined and defined. By examining the algebraic properties of recently defined polynomial sequences, some important identities were given. The matrices of negative indices *k*- Oresme polynomials was defined. Some sum formulas were given according to this definition.

Keywords: Matrices, Recurrences, Special sequences and polynomials
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1. Introduction

In [1], Alwyn Francis Horadam defined the well-known number sequence called Horadam numbers denoted by second order linear recurrence relation. The author examined the principle properties of an arbitrary generalized integer sequence and studied particular cases of this sequence [1]-[3]. The sequence studied by Horadam is re-examined by various authors and several applications of this sequence are included in [4]-[7].
For nonzero integers p and q, Horadam sequence is given by the recurrence relation

$$w_{n+2} = pw_{n+1} - qw_n, n \geq 0, \tag{1.1}$$

where $w_n = w_n(w_0, w_1; p, q)$ is the general term. Nicole Oresme, one of the scientists in the 14th century, investigated the sum of the sequences of rational numbers and the properties of this sum [8]. Later in 1974, this author expanded and defined a new integer sequence denoted by $\{\mathbf{O}_n\}$ and this defined sequence is known in the literature as the Oresme sequence [9]. Different sequences are obtained by customizing the coefficients p, q in the Horadam sequence, which has been studied by many authors. The Oresme sequences we are working with here is the version of the coefficients p, q obtained by taking special numbers.The recurrence relation of this sequence is as follows.

$$\mathbf{O}_n = \mathbf{O}_{n-1} - \frac{1}{4}\mathbf{O}_{n-2}; \mathbf{O}_0 = 0, \mathbf{O}_1 = \frac{1}{2}. \tag{1.2}$$

Horadam examined these numbers in more detail and obtained both linear and non-linear relations involving these numbers and gave the generating functions for them. Cook [6] generalized the these numbers as k- Oresme numbers denoted by $\mathbf{O}_n^{(k)}$ and defined by, for $k>2$,

$$\mathbf{O}_n^{(k)}=\mathbf{O}_{n-1}^{(k)}-\frac{1}{k^2}\mathbf{O}_{n-2}^{(k)}, \tag{1.3}$$

in here the initial conditions are $\mathbf{O}_0^{(k)}=0$ and $\mathbf{O}_1^{(k)}=\frac{1}{k}$.
It can be noticed that these numbers are reduced to standard Oresme numbers by taking $k=2$. In [6], for $k^2-4>0$, the closed formula of *k*- Oresme numbers is given by

$$\mathbf{O}_n^{(k)}=\frac{\alpha^n-\beta^n}{\sqrt{k^2-4}}. \tag{1.4}$$

In the last equation $\alpha=\frac{k+\sqrt{k^2-4}}{2k}$ and $\beta=\frac{k-\sqrt{k^2-4}}{2k}$. Some identities and sum formulas for this number sequence are studied in [6], [10]. Moreover, see [6], [10]-[13] for recent studies. In [14], Halici et al. generalized the *k*- Oresme numbers as *k*- Oresme polynomials denoted by $\mathbf{O}_n^{(k)}(x)$. The recurrence relation of nth *k*- Oresme polynomials is as follows.

$$\mathbf{O}_{n+2}^{(k)}(x)=\mathbf{O}_{n+1}^{(k)}(x)-\frac{1}{k^2x^2}\mathbf{O}_n^{(k)}(x),\mathbf{O}_0^{(k)}(x)=0,\mathbf{O}_1^{(k)}(x)=\frac{1}{kx}, \tag{1.5}$$

where $x\in\mathbb{R}$ and $n\in\mathbb{N}$. Taking $k=1$ and $x=1$ in (1.5) respectively, one can get Oresme polynomials and *k*- Oresme numbers. In [12], *k*- Oresme numbers are extended to negative indices and gave the following recurrence relation

$$\mathbf{O}_{-n}^{(k)}=k^2\left(\mathbf{O}_{-n+1}^{(k)}-\mathbf{O}_{-n+2}^{(k)}\right), \tag{1.6}$$

where $\mathbf{O}_{-1}^{(k)}=-k$ and $\mathbf{O}_0^{(k)}=0$ are the initial conditions. The nth term of this sequence is defined by

$$\mathbf{O}_{-n}^{(k)}=-k^{2n}\frac{(\alpha^n-\beta^n)}{\sqrt{k^2-4}}. \tag{1.7}$$

The values α and β are as in the equation (1.4).
Also, the authors in [15] worked on *k*- Oresme polynomials and derivatives. Some results obtained about these polynomials are given below.

$$i)\sum_{i=1}^{n}\mathbf{O}_i^{(k)}(x)=k^2x^2\left(\frac{1}{kx}-\mathbf{O}_{n+2}(x)\right). \tag{1.8}$$

$$ii)\sum_{i=1}^{n}(-1)^i\mathbf{O}_i^{(k)}(x)=\frac{k^2x^2}{2k^2x^2+1}\left(\frac{1}{kx}+(-1)^{n+1}\left(\mathbf{O}_{n+2}^{(k)}(x)-2\mathbf{O}_{n+1}^{(k)}(x)\right)\right). \tag{1.9}$$

$$iii)\sum_{i=1}^{n}\mathbf{O}_{2i+1}^{(k)}(x)=\frac{k^2x^2}{2k^2x^2+1}\left(\frac{k^2x^2}{kx+1}+\frac{k^2x^2}{k^2x^2+1}\mathbf{O}_{2n+1}^{(k)}(x)-k^2x^2\mathbf{O}_{2n+2}^{(k)}(x)\right). \tag{1.10}$$

$$iv)\sum_{i=1}^{n}\mathbf{O}_{2i}^{(k)}(x)=\frac{k^2x^2}{2k^2x^2+1}\left(kx-\left(k^2x^2+1\right)\mathbf{O}_{2n+2}^{(k)}(x)+\mathbf{O}_{2n+1}^{(k)}(x)\right). \tag{1.11}$$

In [16], Soykan studied a different generalization of Oresme sequences.
In this study, we examined the corresponding generation matrix for the polynomial sequence we define in this paper. We gave some combinatorial equations for this new sequence studied with the help of basic matrix calculations. Also, we derived new identities by using the concepts of trace and determinant of a matrix. We also calculated sum formulas for the elements of this sequence.

2. Main Results

Definition 2.1. *For $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, k- Oresme polynomial with negative indices is denoted by $\mathbf{O}_{-n}^{(k)}(x)$ and defined by the recurrence relation*

$$\mathbf{O}_{-n}^{(k)}(x) = (kx)^2 \left(\mathbf{O}_{-n+1}^{(k)}(x) - \mathbf{O}_{-n+2}^{(k)}(x)\right), \tag{2.1}$$

with initial conditions $\mathbf{O}_{-1}^{(k)}(x) = -kx$ and $\mathbf{O}_{0}^{(k)}(x) = 0$.

Some terms of this sequence are

$$\left\{\mathbf{O}_{-n}^{(k)}(x)\right\}_{n\geq 0} = \left\{0, -kx, -(kx)^3, (kx)^3 - (kx)^5, \ldots\right\}.$$

In the case of $k = 2$ and $x = 1$, the recurrence relation (2.1) is reduced to the equation (1.6). If the equation (2.1) is solved, the roots of this equation are

$$\alpha = \frac{kx + \sqrt{(kx)^2 - 4}}{2kx} \text{ and } \beta = \frac{kx - \sqrt{(kx)^2 - 4}}{2kx}, \tag{2.2}$$

respectively.

Corollary 2.2. *The Binet formula for the sequence $\left\{O_{-n}^{(k)}(x)\right\}_{n\geq 0}$ is*

$$\mathbf{O}_{-n}^{(k)}(x) = -(kx)^{2n} \frac{(\alpha^n - \beta^n)}{\sqrt{(kx)^2 - 4}}. \tag{2.3}$$

Proof. For the k- Oresme polynomials with negative indices, let us substitute the closed formula for the k- Oresme numbers with negative indices in equation (1.7).

$$\mathbf{O}_{-n}^{(k)}(x) = \frac{1}{\sqrt{(kx)^2 - 4}} \left(\frac{1}{\alpha^n} - \frac{1}{\beta^n}\right),$$

$$\mathbf{O}_{-n}^{(k)}(x) = -\frac{1}{\sqrt{(kx)^2 - 4}} \left(\frac{\alpha^n - \beta^n}{(\alpha\beta)^n}\right),$$

which implies

$$\mathbf{O}_{-n}^{(k)}(x) = -(kx)^{2n} \frac{1}{\sqrt{(kx)^2 - 4}} \left(\left(\frac{kx + \sqrt{(kx)^2 - 4}}{2kx}\right)^n - \left(\frac{k - \sqrt{(kx)^2 - 4}}{2kx}\right)^n\right).$$

By some elementary operations, the following equation is obtained

$$\mathbf{O}_{-n}^{(k)}(x) = -(kx)^{2n} \frac{(\alpha^n - \beta^n)}{\sqrt{(kx)^2 - 4}}.$$

This proves the corollary. □

Using the terms of the sequence $\left\{\mathbf{O}_{-n}^{(k)}(x)\right\}_{n\geq 0}$, the generating matrix corresponds to these polynomials with negative indices is defined as

$$\mathbb{O} = \frac{1}{kx} \begin{bmatrix} (kx)^2 \mathbf{O}_{0}^{(k)}(x) & -\mathbf{O}_{-1}^{(k)}(x) \\ (kx)^2 \mathbf{O}_{-1}^{(k)}(x) & -\mathbf{O}_{-2}^{(k)}(x) \end{bmatrix}. \tag{2.4}$$

In the following Theorems some fundamental identities for the polynomials mentioned above are deduced by using the matrices $\mathbb{O}$.

Theorem 2.3. *For the matrix $\mathbb{O}$, the following equation is true.*

$$\mathbb{O}^n=\begin{bmatrix} kx\mathbf{O}^{(k)}_{-n+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n}(x) \\ kx\mathbf{O}^{(k)}_{-n}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x) \end{bmatrix}. \tag{2.5}$$

Proof. To prove by induction observe that for $n=1$, then the equation (2.5) is true. Using the fact that $\mathbb{O}^{n+1}=\mathbb{O}^n\mathbb{O}$, we have

$$\mathbb{O}^{n+1}=\begin{bmatrix} kx\mathbf{O}^{(k)}_{-n}(x) & kx\mathbf{O}^{(k)}_{-n+1}(x)-kx\mathbf{O}^{(k)}_{-n}(x) \\ kx\mathbf{O}^{(k)}_{-n-1}(x) & kx\mathbf{O}^{(k)}_{-n}(x)-kx\mathbf{O}^{(k)}_{-n-1}(x) \end{bmatrix}$$

and when the necessary procedures and arrangements are made

$$\mathbb{O}^{n+1}=\begin{bmatrix} kx\mathbf{O}^{(k)}_{-(n+1)+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-(n+1)}(x) \\ kx\mathbf{O}^{(k)}_{-(n+1)}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-(n+1)-1}(x) \end{bmatrix}$$

is obtained. Thus, the proof is completed. □

In the following theorem, we give the generating function of $\left\{\mathbf{O}^{(k)}_{-n}(x)\right\}_{n\geq 0}$.

Theorem 2.4. *The generating function for these polynomials is derived below:*

$$\sum_{i=1}^{\infty}\mathbf{O}^{(k)}_{-n}(x)z^i=-\frac{-kxz}{1-z(kx)^2+z^2(kx)^2}, \tag{2.6}$$

where $x\in\mathbb{R}$.

Proof. Using the definition of generating number function and some elementary operations, we have following equations.

$$f(z)=\mathbf{O}^{(k)}_{0}(x)+z\mathbf{O}^{(k)}_{-1}(x)+z^2\mathbf{O}^{(k)}_{-2}(x)+z^3\mathbf{O}^{(k)}_{-3}(x)\cdots.$$

$$-z(kx)^2f(z)=-z(kx)^2\mathbf{O}^{(k)}_{0}(x)-z^2(kx)^2\mathbf{O}^{(k)}_{-1}(x)-z^3(kx)^2\mathbf{O}^{(k)}_{-2}(x)-z^4(kx)^2\mathbf{O}^{(k)}_{-3}(x)\cdots.$$

$$z^2(kx)^2f(z)=z^2(kx)^2\mathbf{O}^{(k)}_{0}(x)+z^3(kx)^2\mathbf{O}^{(k)}_{-1}(x)+z^4(kx)^2\mathbf{O}^{(k)}_{-2}(x)+z^5(kx)^2\mathbf{O}^{(k)}_{-3}(x)\cdots.$$

From this, the following equation is obtained:

$$f(z)-z(kx)^2f(z)-z^2(kx)^2f(z)=\mathbf{O}^{(k)}_{0}(x)+z\left(\mathbf{O}^{(k)}_{-1}(x)-(kx)^2\mathbf{O}^{(k)}_{0}(x)\right)+z^2\left(\mathbf{O}^{(k)}_{-2}(x)-(kx)^2\mathbf{O}^{(k)}_{-1}(x)+(kx)^2\mathbf{O}^{(k)}_{0}(x)\right)\cdots.$$

By using the relation (2.1), it is obviously seen that

$$f(z)-z(kx)^2f(z)+z^2(kx)^2f(z)=-kxz.$$

Which implies

$$f(z)=\frac{-kxz}{1-z(kx)^2+z^2(kx)^2}.$$

This completes the proof. □

The well-known Catalan and Cassini identities for the sequence $\left\{\mathbf{O}^{(k)}_{-n}(x)\right\}_{n\geq 0}$ are given in the following two Theorems.

Theorem 2.5. *For $n\geq 0$, we have*

$$\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-n-1}(x)-\left(\mathbf{O}^{(k)}_{-n}(x)\right)^2=-(kx)^{2n}. \tag{2.7}$$

Proof. By using the matrix $\mathbb{O}$ given in the equation (2.5) and the fact that $(det(\mathbb{O}))^n = det(\mathbf{O}^n)$, we can write

$$det\begin{bmatrix} kx\mathbf{O}^{(k)}_{-n+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n}(x) \\ kx\mathbf{O}^{(k)}_{-n}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x) \end{bmatrix} = det\begin{bmatrix} 0 & 1 \\ (kx)^2 & (kx)^2 \end{bmatrix}^n.$$

Hence, we have

$$(det(\mathbb{O}))^n = -\mathbf{O}^{(k)}_{n+1}(x)\mathbf{O}^{(k)}_{n-1}(x) + \left(\mathbf{O}^{(k)}_{n}(x)\right)^2 = -(kx)^{2n}.$$

Thus, the desired result is obtained. □

We have given an important identity provided by the elements of this polynomial sequence in the Theorem below.

Theorem 2.6. *For $n \geq r$, the following equality is true.*

$$\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r}(x) - \left(\mathbf{O}^{(k)}_{-n}(x)\right)^2 = -(kx)^{2n-2r}\left(\mathbf{O}^{(k)}_{-r}(x)\right)^2. \tag{2.8}$$

Proof. By substituting the equation (2.5) into the left-hand side of the equation (2.8), we get

$$LHS = \begin{bmatrix} kx\mathbf{O}^{(k)}_{-n+r+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n+r}(x) \\ kx\mathbf{O}^{(k)}_{-n+r}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n+r-1}(x) \end{bmatrix}\begin{bmatrix} kx\mathbf{O}^{(k)}_{-n-r+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-r}(x) \\ kx\mathbf{O}^{(k)}_{-n-r}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-r-1}(x) \end{bmatrix} - \begin{bmatrix} kx\mathbf{O}^{(k)}_{-n+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n}(x) \\ kx\mathbf{O}^{(k)}_{-n}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x) \end{bmatrix}^2.$$

By the matrix operation, the LHS equals to

$$LHS = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix},$$

where

$$\begin{aligned}
A &= (kx)^2\mathbf{O}^{(k)}_{-n+r+1}(x)\mathbf{O}^{(k)}_{-n-r+1}(x) - \mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r}(x),\\
B &= -\mathbf{O}^{(k)}_{-n+r+1}(x)\mathbf{O}^{(k)}_{-n-r}(x) + \tfrac{1}{(kx)^2}\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r-1}(x),\\
C &= (kx)^2\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r+1}(x) - \mathbf{O}^{(k)}_{-n+r-1}(x)\mathbf{O}^{(k)}_{-n-r}(x),\\
D &= -\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r}(x) + \tfrac{1}{(kx)^2}\mathbf{O}^{(k)}_{-n+r-1}(x)\mathbf{O}^{(k)}_{-n-r-1}(x),\\
A' &= (kx)^2\left(\mathbf{O}^{(k)}_{-n+1}(x)\right)^2 - \left(\mathbf{O}^{(k)}_{-n}(x)\right)^2,\\
B' &= \mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-n}(x) + \tfrac{1}{(kx)^2}\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-n-1}(x),\\
C' &= (kx)^2\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-n}(x) - \mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-n-1}(x)
\end{aligned}$$

and

$$D' = -\left(\mathbf{O}^{(k)}_{-n}(x)\right)^2 + \tfrac{1}{(kx)^2}\left(\mathbf{O}^{(k)}_{-n-1}(x)\right)^2.$$

Hence, we obtain

$$\mathbf{O}^{(k)}_{-n+r}(x)\mathbf{O}^{(k)}_{-n-r}(x) - \left(\mathbf{O}^{(k)}_{-n}(x)\right)^2 = -(kx)^{2n-2r}\left(\mathbf{O}^{(k)}_{-r}(x)\right)^2,$$

which proves the theorem. □

In the case of $r = 1$, one can get the Cassini identity from the equation (2.8).
In the below, we give an important identity for these polynomials we are considering with negative indices is given.

Theorem 2.7. *For $n, m \in \mathbb{Z}^+$, we have*

$$\mathbf{O}^{(k)}_{-(n+m)}(x) = kx\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x) - \frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x)\mathbf{O}^{(k)}_{-m}(x). \tag{2.9}$$

Proof. By using (2.5), we can get

$$\mathbb{O}^{n+m} = \begin{bmatrix} kx\mathbf{O}^{(k)}_{-(n+m)+1}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-(n+m)}(x) \\ kx\mathbf{O}^{(k)}_{-(n+m)}(x) & -\frac{1}{kx}\mathbf{O}^{(k)}_{-(n+m)-1}(x) \end{bmatrix}.$$

Since $\mathbb{O}^{n+m} = \mathbb{O}^{n}\mathbb{O}^{m}$, equating the corresponding elements of the matrices we have

$$kx\mathbf{O}^{(k)}_{-(n+m)}(x) = (kx)^2\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x) - \mathbf{O}^{(k)}_{-n-1}(x)\mathbf{O}^{(k)}_{-m}(x).$$

Hence,

$$\mathbf{O}^{(k)}_{-(n+m)}(x) = kx\mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x) - \frac{1}{kx}\mathbf{O}^{(k)}_{-n-1}(x)\mathbf{O}^{(k)}_{-m}(x).$$

□

The well-known an important identity for these polynomials with negatives indices is deduced in the following Theorem.

Theorem 2.8. *For the positive integers m,n, the following is satisfied.*

$$\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-m}(x) - \mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x) = -(kx)^{2m}\mathbf{O}^{(k)}_{-(n-m)}(x). \tag{2.10}$$

Proof. Using the closed formula, we can write $\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-m}(x) - \mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x)$ as,

$$LHS = \frac{1}{(kx)^2-4}\left[\left(-(kx)^{2n-2}\left(\alpha^{n-1}-\beta^{n-1}\right) - (kx)^{2m}\left(\alpha^{m}-\beta^{m}\right)\right) - \left((kx)^{2n}\left(\alpha^{n}-\beta^{n}\right) - (kx)^{2m-2}\left(\alpha^{m-1}-\beta^{m-1}\right)\right)\right],$$

$$LHS = \frac{1}{(kx)^2-4}\left[(kx)^{2n+2m-2}\left(-\alpha^{n-1}\beta^{m} - \beta^{n-1}\alpha^{m} + \alpha^{n}\beta^{m-1} + \beta^{n}\alpha^{m-1}\right)\right],$$

$$LHS = \frac{1}{(kx)^2-4}\left[(kx)^{2n+2m-2}\left(\alpha^{n}\beta^{m}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right) - \alpha^{m}\beta^{n}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\right)\right],$$

$$LHS = \frac{1}{(kx)^2-4}\left[(kx)^{2n+2m-2}\left(\alpha^{n}\beta^{m} - \alpha^{m}\beta^{n}\right)\frac{\alpha-\beta}{\alpha\beta}\right],$$

where α and β are the roots of equation (2.1). By substituting $\alpha-\beta = \frac{\sqrt{(kx)^2-4}}{kx}$ and $\alpha\beta = \frac{1}{(kx)^2}$ into the last equation, we obtain

$$\mathbf{O}^{(k)}_{-n+1}(x)\mathbf{O}^{(k)}_{-m}(x) - \mathbf{O}^{(k)}_{-n}(x)\mathbf{O}^{(k)}_{-m+1}(x) = \frac{1}{(kx)^2-4}\left[(kx)^{2n+2m-2}\left(\alpha^{n}\beta^{m} - \alpha^{m}\beta^{n}\right)\frac{\alpha-\beta}{\alpha\beta}\right].$$

Making necessary arrangements, we get

$$LHS = -(kx)^{2m}\mathbf{O}^{(k)}_{-(n-m)}(x)$$

which completes the proof. □

Now, we have given some sum formulas of this polynomials with negative indices in the Theorem below.

Theorem 2.9. *For $n \geq 1$, we have the followings.*

$$i) \sum_{i=1}^{n} \mathbf{O}^{(k)}_{-i}(x) = -kx\left(1 - kx\mathbf{O}_{-n+1}(x)\right). \tag{2.11}$$

$$ii)\sum_{i=1}^{n}(-1)^{i}\mathbf{O}_{-i}^{(k)}(x)=\frac{1}{2(kx)^{2}+1}\left(kx+(-1)^{n}\left((kx)^{2}\mathbf{O}_{-n}^{(k)}(x)+\mathbf{O}_{-n-1}^{(k)}(x)\right)\right). \tag{2.12}$$

$$iii)\sum_{i=1}^{n}\mathbf{O}_{-(2i+1)}^{(k)}(x)=\frac{1}{2}\left(\frac{\mathbf{O}_{-2n}^{(k)}(x)\left(-2(kx)^{3}-kx\right)}{2(kx)^{2}+1}\left(-kx+(kx)^{2}(2\mathbf{O}_{-2n-1}^{(k)}(x)+1)+(kx)^{4}-1\right)\right). \tag{2.13}$$

$$iv)\sum_{i=1}^{n}\mathbf{O}_{-(2i)}^{(k)}(x)=\frac{(kx)^{2}}{2(kx)^{2}+1}\left(kx-\left((kx)^{2}+1\right)\mathbf{O}_{-2n-2}^{(k)}(x)+\mathbf{O}_{-2n-1}^{(k)}(x)\right). \tag{2.14}$$

Proof. *i)* This equation,

$$\sum_{i=1}^{n}\mathbf{O}_{-i}^{(k)}(x)=-kx(1-kx\mathbf{O}_{0}(x))$$

is true for $n=1$. Let us assume that equality is true for $n\leq m$. Then, we get

$$LHS=-kx\left(1-kx\mathbf{O}_{-n+1}^{(k)}(x)\right)+\left(\mathbf{O}_{-n-1}^{(k)}(x)\right),$$

$$LHS=-kx\left(1-kx\mathbf{O}_{-n+1}^{(k)}(x)\right)+(kx)^{2}\left(kx\mathbf{O}_{-n}^{(k)}(x)-\mathbf{O}_{-n+1}^{(k)}(x)\right)$$

and

$$LHS=-kx\left(1-kx\mathbf{O}_{-n}^{(k)}(x)\right).$$

ii) The proof can be done similarly by using induction method.
iii) By observing that

$$\sum_{i=0}^{n}\mathbf{O}_{-2i-1}^{(k)}(x)=\frac{1}{2}\left(\sum_{i=0}^{2n+1}\mathbf{O}_{-i}^{(k)}(x)-\sum_{i=0}^{2n+1}(-1)^{i}\mathbf{O}_{-i}^{(k)}(x)\right),$$

and using *i* and *ii*, the proof is clear.
iv) Similarly, by observing that

$$\sum_{i=0}^{n}\mathbf{O}_{-2i}^{(k)}(x)=\frac{1}{2}\left(\sum_{i=0}^{2n}\mathbf{O}_{-i}^{(k)}(x)-\sum_{i=0}^{2n}(-1)^{i}\mathbf{O}_{-i}^{(k)}(x)\right),$$

the desired equality can be shown. □

In 2004, Laughlin calculated powers of an arbitrary second order matrix A by using the trace and determinant of this matrix. In [4],[5], Halici and Akyuz deduced and gave some combinatorial identities involving Horadam sequence. The help of these studies, we give some important and proper identity for the polynomials we examined with negative indices in the rest of the section. nth power of an arbitrary matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is given by the following formula:

$$A^{n}=z_{n}A-z_{n-1}DI_{2},$$

where

$$z_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}=\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{1}{2^{n-1}}\binom{n}{2m+1}T^{n-2m-1}(T^{2}-4D)$$

and α, β are the roots of the characteristic equation of Horadam sequence. Notice that, T and D denotes the trace and determinant of the matrix A respectively.
The matrix A^n is given by Laughlin as

$$A^n = \begin{bmatrix} y_n - dy_{n-1} & by_{n-1} \\ cy_n & y_n - ay_{n-1} \end{bmatrix},$$

where

$$y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i} D^i.$$

Theorem 2.10. *For $n \geq 1$, we have*

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} \left(\frac{n\left(2-(kx)^2\right) - 2i\left(1-(kx)^2\right)}{n-i} \right) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (kx)^{-i} \left(\sqrt{(kx)^2 - 4} \right). \tag{2.15}$$

Proof. Applying (2.15) to generating matrix $\mathbb{O}$, we can write

$$\mathbb{O}^n = \begin{bmatrix} y_n - (kx)^2 y_{n-1} & y_{n-1} \\ (kx)^2 y_n & y_n \end{bmatrix}.$$

For $k > 2$, notice that trace and determinant of $\mathbb{O}$ are calculated as $T = (kx)^2$ and $D = -(kx)^2$. Hence, we write y_n as

$$y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i}. \tag{2.16}$$

Using the fact that $\lambda_1^n + \lambda_2^n = 2y_n - (kx)^2 y_{n-1}$, we obtain the left-hand side as

$$LHS = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^{n-i} \left(\frac{\sqrt{(kx)^2-4}}{2kx} \right)^i - \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^{n-i} \left(-\frac{\sqrt{(kx)^2-4}}{2kx} \right)^i,$$

$$LHS = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^n} \frac{1}{(kx)^i} \left[\left(\sqrt{(kx)^2-4} \right)^i - \left(-\sqrt{(kx)^2-4} \right)^i \right],$$

$$LHS = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (kx)^{-i} \left(\sqrt{(kx)^2-4} \right)^i.$$

Furthermore, by using equation (2.16), we can write right-hand side as

$$RHS = 2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} - (kx)^2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} (kx)^{2n-2i}$$

which equals to

$$RHS = 2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} - (kx)^2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i}{n-2i} \frac{n-2i}{n-i} (kx)^{2n-2i}.$$

Since $\frac{n-2i}{n-i} = \frac{n-2\lfloor \frac{n}{2} \rfloor}{n-\lfloor \frac{n}{2} \rfloor} = 0$, we get the desired result as:

$$RHS = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} \left(2 - (kx)^2 \frac{n-2i}{n-i} \right),$$

$$RHS = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} \left(\frac{n(2-(kx)^2) - 2i(1-(kx)^2)}{n-i} \right).$$

Equating the left and right hand sides, we get

$$RHS = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (kx)^{2n-2i} \left(\frac{n(2-(kx)^2) - 2i(1-(kx)^2)}{n-i} \right) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (kx)^{-i} \left(\sqrt{(kx)^2-4} \right)^i.$$

□

3. Conclusion

In this study, we define the corresponding generation matrix for the polynomial sequence we define in this work. We obtained some combinatorial equations for this new sequence studied with the help of basic matrix calculations. Moreover, we gave new identities by using the concepts of trace and determinant of a matrix. We also derived sum formulas for the elements of this sequence.

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On Some Properties of Banach Space-Valued Fibonacci Sequence Spaces

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Abstract
In this work, we give some results about the basic properties of the vector-valued Fibonacci sequence spaces. In general, sequence spaces with Banach space-valued cannot have a Schauder Basis unless the terms of the sequences are complex or real terms. Instead, we defined the concept of relative basis in [1] by generalizing the definition of a basis in Banach spaces. Using this definition, we have characterized certain important properties of vector-term Fibonacci sequence spaces, such as separability, Dunford-Pettis Property, approximation property, Radon-Riesz Property and Hahn-Banach extension property.

Keywords: Approximation property, Dunford-Pettis property, Fibonacci sequence spaces, Radon-Riesz property, Vector-Valued sequence spaces
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1. Introduction

Banach spaces with a Schauder basis have many important advantages. The representation of such spaces with the help of the basis and the ability to approximate the element in countable steps with the help of this representation provide the opportunity to solve many structural and numerical problems. But in general, vector-valued sequence and function spaces do not generally have a Schauder basis. The concept of basis, which we defined in [1] tells us that some of these types of spaces have this type of basis and allows us to examine the structural properties of the space.

In this work we examine certain properties of some Banach space-valued Fibonacci sequence spaces. Their scalar-valued versions are defined and investigated in [2]-[6]. Fibonacci numbers have several applications in the field of Science, Engineering and Architecture. Fibonacci sequence is a sequence in which each number is the sum of the two preceding ones. Numbers that are part of the Fibonacci sequence are known as Fibonacci numbers . Starting from 1 and 1, the sequence begins

$$1,1,2,3,5,8,13,21,34,55,89,144,\ldots.$$

The Fibonacci numbers may be defined by the recurrence relation $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. We refer to [3] for detailed studies concerning Fibonacci numbers. Fibonacci matrix is define by the Fibonacci numbers as $\mathscr{F} = (f_{nk})$ such that

$$f_{n,k}=\begin{cases}\frac{-f_{n+1}}{f_n}, & if\ \ k=n-1\\ \frac{f_n}{f_{n+1}}, & if\ \ k=n\\ 0, & otherwise\end{cases}.$$

More explicitly,

$$\mathscr{F}=\begin{bmatrix} f_1/f_2 & 0 & 0 & 0 & \cdots\\ -f_3/f_2 & f_2/f_3 & 0 & 0 & \cdots\\ 0 & -f_4/f_3 & f_3/f_4 & 0 & \cdots\\ 0 & 0 & -f_5/f_4 & f_4/f_5 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}.$$

By using this sub-triangular infinite matrix Kara [4] introduced the sequence spaces $\ell_p(F)$, $1\le p<\infty$, and $\ell_\infty(F)$ such that

$$\ell_p(\mathscr{F})=\left\{u=(u_n)\in w:\sum_{n=0}^{\infty}\left|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right|^p<\infty\right\}$$

and

$$\ell_\infty(\mathscr{F})=\left\{u=(u_n)\in w:\sup_n\left|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right|<\infty\right\}.$$

For any Banach space V, let us define following V-valued Fibonacci sequence spaces

$$\ell_p(\mathscr{F},V)=\left\{u=(u_n)\in w(V):\sum_{n=0}^{\infty}\left\|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right\|_V^p<\infty\right\}$$

and

$$\ell_\infty(\mathscr{F},V)=\left\{u=(u_n)\in w(V):\sup_n\left\|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right\|_V<\infty\right\}$$

For $V=\mathbb{K}$, the real or complex number, then $\ell_p(\mathscr{F},V)=\ell_p(\mathscr{F})$ and $\ell_\infty(\mathscr{F},V)=\ell_\infty(\mathscr{F})$. It is easy to prove that $\ell_p(\mathscr{F},V)$ and $\ell_\infty(\mathscr{F},V)$ are Banach spaces with norms

$$\|u\|_{\ell_p(\mathscr{F},V)}=\left(\sum_{n=0}^{\infty}\left\|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right\|_V^p\right)^{1/p}$$

and

$$\|u\|_{\ell_\infty(\mathscr{F},V)}=\sup_n\left\|\frac{f_n}{f_{n+1}}u_n-\frac{f_{n+1}}{f_n}u_{n-1}\right\|_V,$$

respectively.

$\mathscr{F}$ is an invertible triangle matrix, that is $\mathscr{F}^{-1}$ exists and it defines an isomorphism from $\ell_\infty(V)$ onto $\ell_\infty(\mathscr{F},V)$ and from $\ell_p(V)$ onto $\ell_p(\mathscr{F},V)$.

We will see in the sequel that $\ell_1(\mathscr{F},V)$ has Dunford-Pettis property and moreover will prove that $\ell_p(\mathscr{F},V)$ have the approximation property for $1\le p<\infty$ in some conditions.

Let us give some known required results from Banach space theory.

Suppose that U and V are Banach spaces. A linear operator S from U into V is compact if $S(B)$ is a relatively compact (means $\overline{S(B)}$ is compact) subset of V whenever B is a bounded subset of U. The collection of all compact linear operators from U into V is denoted by $K(U,V)$, or by just $K(U)$ if $U=V$. The range of a compact linear operator from a Banach space into a Banach space is closed if and only if the operator has finite rank, that is, the range of the operator is finite-dimensional [7]. A Banach space U has the *approximation property* if, for every Banach space V, the set of all finite-rank operators in $B(V,U)$ is dense in $K(V,U)$ [8]. The spaces c_0 and ℓ_p, $1\le p<\infty$, have the approximation property [7].

Let us remember that for any sequence (x_n) in a Banach space U converges weakly to U, or briefly $x_n\xrightarrow{w}x$, whenever $f(x_n)\to f(x_n)$ for each $f\in U^*$, the dual of U. We refer the reader to [7] for the definition of weak topology and weak convergence in detail. Suppose that U and V are Banach spaces. A linear operator S from U into V is weakly compact if $S(B)$ is a relatively weakly compact subset of V whenever B is a bounded subset of U.

Proposition 1.1. *[7] Suppose that S is a linear operator from a Banach space U into a Banach space V. Then S is weakly compact if and only if for any bounded sequence (x_n) in U has a subsequence $\left(x_{n_j}\right)_{j=0}^{\infty}$ such that $\left(Sx_{n_j}\right)$ converges weakly.*

Let us give important definitions of D.Hilbert. Suppose that U and V are Banach spaces. A linear operator S from U into V is *completely continuous* or a Dunford-Pettis operator if $S(K)$ is a compact subset of V whenever K is a weakly compact subset of U [9] .

Definition 1.2. *Suppose that U and V are Banach spaces. A Banach space U has the* Dunford-Pettis property *if, for every Banach space V, each weakly compact linear operator from U into V is completely continuous [7].*

Proposition 1.3. *[7] ℓ_1 has the Dunford-Pettis Property.*

Theorem 1.4. *(R.S.Phillips, [10]) Let V be a linear subspace of the Banach space U and suppose $T : V \to \ell_\infty$ is a bounded linear operator. Then T may be extended to a bounded linear operator $S : U \to \ell_\infty$ having the same norm as T.*

In some cites the operator T in the above theorem is known as a Hahn-Banach operator and then it is said that ℓ_∞ has the *Hahn-Banach extension property.*

2. Some Properties of Banach Space-Value Fibonacci Sequences

Definition 2.1. *[1] Let U and V be Banach spaces and $\mathbb{A}$ be a set. A family $\{\eta_a : a \in \mathbb{A}\}$ of continuous linear functions $\eta_a : V \to U$ is called* Y-basis for *U if the following condition is satisfied. There exist a unique family $\{R_a : a \in \mathbb{A}\}$ of linear functions R_a from U onto V and a subset $\mathscr{D}$ of $\mathscr{F}$, directed by some relation $\ll$, such that, for each $x \in U$, the net $(\pi_F(x) : \mathscr{D})$ converges to x in U. Where, for each $F \in \mathscr{D}$,*

$$\pi_F(x) = \sum_{a\in F} (\eta_a \circ R_a)(x),$$

and $\mathscr{F}$ is the family of all finite subsets of the index set $\mathbb{A}$. Furthermore, $\{\eta_a\}$ is called a Y-Schauder basis for *U whenever each R_a is continuous.*

Definition 2.2. *[1] The family $\{R_a : a \in \mathbb{A}\}$ is called* associate family of functions *(A.F.F.) to the V-basis $\{\eta_a : a \in \mathbb{A}\}$.*

Let $\{\eta_a : a \in \mathbb{A}\}$ be a V-basis for U. Clearly, the finite summation $\pi_F(x)$ defines an operator π_F on U for each $F \in \mathscr{D}$. This operator is called *F-projection* on U corresponding V-basis and it is continuous whenever $\{\eta_a\}$ is a V-Schauder basis.

Remark 2.3. *Let V be a Banach space on the field $\mathbb{C}$ possessing a basis $\{x_n\}$ (in the classical manner). Then the sequence $\{\eta_n\}$ of the functions*

$$\eta_n : \mathbb{C} \to V : \eta_n(t) = tx_n$$

is a $\mathbb{C}$-basis for V in the sense of above Definition. Indeed; take $\mathbb{A} = \mathbb{N}$ and

$$\mathscr{D} = \{\{1\}, \{1,2\}, \{1,2,3\}, \ldots\}$$

with the relation inclusion again, and $\{R_n\}$ as the sequence of associate coordinate functionals (g_n) to the basis $\{x_n\}$. Then $(\pi_F(x) : \mathscr{D})$ converges to x in U iff

$$\sum_{k=1}^{n} (\eta_k \circ R_k)(x) = \sum_{k=1}^{n} g_k(x)x_k,$$

converges to $x = \sum_{n=1}^{\infty} g_n(x)x_n$.

Theorem 2.4. *Let V be a Banach space for which a family $\{\eta_a : a \in \mathbb{A}\}$ be a V-basis for some Banach space V. Then, V is separable if $\mathbb{A}$ is countable [1].*

Main Results

Let us give some main results on V-valued Fibonacci sequence spaces in this section

Theorem 2.5. *Let V be a Banach space. Then $\ell_p(\mathscr{F},V)$ has an unconditional V-Schauder basis.*

Proof. Take $\mathbb{A}=\mathbb{N}$ and consider

$$\begin{aligned} I_n &: V\to \ell_p(V) \\ I_n(z) &= (0,...,0,z,0,...) \end{aligned}$$

for and remember the Fibonacci matrix $\mathscr{F}$. Then obviously each $\mathscr{F}I_n$ defines a bounded linear operator from V into $\ell_p(\mathscr{F},V)$. Now the linear operator sequence

$$\{\mathscr{F}I_n : n\in\mathbb{N}\}$$

is a V-Schauder basis for $\ell_p(\mathscr{F},V)$. Let us prove this. First of all consider the sequence of coordinate projections

$$P_n:\ell_p(\mathscr{F},V)\to V;\ P_n(x)=x_n,$$

as $\{R_n:n\in\mathbb{N}\}$ in the basis definition, and take $\mathscr{D}$ as the family of all F finite subsets of $\mathbb{N}$ which is directed by the inclusion relation $\subseteq$. Then we must show that the net $(\pi_F(x):\mathscr{D})$ converges to x in $\ell_p(\mathscr{F},V)$ where

$$\pi_F(x)=\sum_{n\in F}(\mathscr{F}I_nP_n)(x)=\sum_{n\in F}\mathscr{F}I_n(x_n).$$

Obviously convergence of the above net corresponds to the convergence of the partial sums sequence of the series $\sum_{n=0}^{\infty}\mathscr{F}I_n(x_n)$. Now, consider an arbitrary $\varepsilon>0$. We must find a finite subset $F_0=F_0(\varepsilon)\in\mathscr{D}$ such that, for each finite set $F\supseteq F_0$,

$$\|x-\pi_F(x)\|_{\ell_p(\mathscr{F},V)}\leq\varepsilon.$$

Since $x\in\ell_p(\mathscr{F},V)$ there exists an $n_0(\varepsilon)$ such that $\sum_{n>n_0}^{\infty}\|(\mathscr{F}x)_n\|_V^p<\varepsilon$. Now take F_0 as

$$F_0=\left\{n\in\mathbb{N}:\sum_{n>n_0}^{\infty}\|(\mathscr{F}x)_n\|_V^p>\varepsilon\right\},$$

Then we get

$$\|x-\pi_F(x)\|_{\ell_p(\mathscr{F},V)}=\|\{x_n:n\in\mathbb{N}\setminus F\}\|_{\ell_p(\mathscr{F},V)}\leq\varepsilon,$$

for each finite $F\supseteq F_0$.This implies $(\pi_F(x):\mathscr{D})\to x$ in $\ell_p(\mathscr{F},V)$.

Let us show the uniqueness of the sequence $\{P_n\}$. Suppose

$$\sum_{n\in\mathbb{N}}(\mathscr{F}I_nP_n)(x)=\sum_{n\in\mathbb{N}}\left(\mathscr{F}I_nP_n'\right)(x)$$

and write

$$\pi_F^{\circ}(x)=\sum_{n\in\mathbb{N}}\left(\mathscr{F}I_n\left(P_n-P_n'\right)\right)(x),\quad F\in\mathscr{D}.$$

Remember that

$$\|\pi_F^{\circ}(x)\|_{\ell_p(\mathscr{F},V)}=\left(\sum_{n\in F}\left\|\left(\mathscr{F}I_n\left(P_n-P_n'\right)\right)(x)\right\|^p\right)^{1/p}$$

and

$$\|\pi_F^{\circ}(x)\|_{\ell_p(\mathscr{F},V)}\leq\|\pi_G^{\circ}(x)\|_{\ell_p(\mathscr{F},V)}$$

for $F \subseteq G$. Since $(\pi_F(x) : \mathscr{D}) \to x$ in $\ell_p(\mathscr{F}, V)$ we get

$$\lim_{F \in \mathscr{D}} \|\pi_F^{\circ}(x)\|_{\ell_p(\mathscr{F},V)} = 0.$$

By this observation we have $(P_n - P_n')(x) = 0$ for each n and for every $x \in \ell_p(\mathscr{F}, V)$. This implies, $P_n = P_n'$ for each n. This gives the uniqueness of the basis.

Further, each P_n is continuous because $\|x_n\|_V \leq \|x\|_{\ell_p(\mathscr{F},V)}$. This proves that sequence $\{\mathscr{F} I_n : n \in \mathbb{N}\}$ is a V-Schauder basis for $\ell_p(\mathscr{F}, V)$. □

Theorem 2.6. *For $1 \leq p < \infty$, the Banach space $\ell_p(\mathscr{F}, V)$ has the approximation property if and only if V has.*

Proof. Suppose T be a compact linear operator from a Banach space V into $\ell_p(\mathscr{F}, V)$. We will find a sequence (T_n) of bounded linear operators of finite-rank from V into $\ell_p(\mathscr{F}, V)$. For any $x \in V$, $Tx \in \ell_p(\mathscr{F}, V)$ and for any bounded sequence (x_n) in V, the sequence (Tx_n) has a convergent subsequence $\left(Tx_{n_j}\right)_{j=0}^{\infty}$ in $\ell_p(\mathscr{F}, V)$ since T is compact. Hence

$$\begin{aligned} \left\|Tx_{n_i} - Tx_{n_j}\right\|_{\ell_p(\mathscr{F},V)}^p &= \left\|T\left(x_{n_i} - x_{n_j}\right)\right\|_{\ell_p(\mathscr{F},V)}^p \\ &= \sum_{m=0}^{\infty} \left\| \frac{f_m}{f_{m+1}} T\left(x_{n_i} - x_{n_j}\right)_m - \frac{f_{m+1}}{f_m} T\left(x_{n_i} - x_{n_j}\right)_{m-1} \right\|_V^p \end{aligned}$$

If we remember the definition of the space $\ell_p(\mathscr{F}, V)$,

$$\left\|T\left(x_{n_i} - x_{n_j}\right)\right\|_{\ell_p(\mathscr{F},V)}^p = \left\|(\mathscr{F}T)\left(x_{n_i} - x_{n_j}\right)\right\|_{\ell_p(V)}^p.$$

Now V has the approximation property if and only if $\ell_p(V)$ has. Hence

$$\left\|(\mathscr{F}T)\left(x_{n_i} - x_{n_j}\right)\right\|_{\ell_p(V)}^p \to 0 \text{ as } i, j \to \infty.$$

This means the operator $\mathscr{F}T : V \to \ell_p(V)$ is well-defined and compact. The matrix transformation $\mathscr{F}$ is clearly bounded linear and so is $\mathscr{F}T$. Since $\ell_p(V)$ have the approximation property, there exits a sequence $(A_m)_{m=0}^{\infty}$ of bounded linear operators of finite-rank from V to $\ell_p(V)$ such that $\|\mathscr{F}T - A_m\| \to 0$ as $m \to \infty$. Now the sequence $\left(\mathscr{F}^{-1}A_m\right)_{m=0}^{\infty}$ is the desired sequence of finite-rank operators from V to $\ell_p(\mathscr{F}, V)$. Easily we can see that each $\mathscr{F}^{-1}A_m$ is bounded linear and has the finite-rank. Further

$$\begin{aligned} \left\|T - \mathscr{F}^{-1}A_m\right\| &= \sup_{\|x\|=1} \left\|\left(T - \mathscr{F}^{-1}A_m\right)x\right\|_{\ell_p(\mathscr{F},V)} \\ &= \sup_{\|x\|=1} \left\|Tx - \left(\mathscr{F}^{-1}A_m\right)x\right\|_{\ell_p(\mathscr{F},V)}^p \\ &= \sup_{\|x\|=1} \left\|\mathscr{F}Tx - \mathscr{F}\left(\mathscr{F}^{-1}A_m\right)x\right\|_{\ell_p(V)}^p \\ &= \sup_{\|x\|=1} \left\|(\mathscr{F}T - A_m)x\right\|_{\ell_p(V)}^p \\ &\to 0 \text{ as } m \to \infty. \end{aligned}$$

This completes the proof. □

Theorem 2.7. *$\ell_1(\mathscr{F}, V)$ has the Dunford-Pettis Property if and only if V has.*

Proof. Let T be any weakly compact linear operator from $\ell_1(\mathscr{F}, V)$ into V and compose T with $\mathscr{F}^{-1}$. Then $T\mathscr{F}^{-1}$ is obviously a bounded linear operator from $\ell_1(V)$ into V. Further it is weakly compact if and only if V is. Let us prove this: Suppose U is a bounded in $\ell_1(V)$. By the boundedness of the matrix operator $\mathscr{F}^{-1}$ we have $\mathscr{F}^{-1}(U)$ is a bounded subset of $\ell_1(\mathscr{F}, V)$. Therefore

$$T\left(\mathscr{F}^{-1}(U)\right) = \left(T\mathscr{F}^{-1}\right)(U)$$

is a relatively weakly compact set in V. As a result $T\mathscr{F}^{-1} : \ell_1(V) \to V$ is a weakly compact operator if and only if V is. Now, since $\ell_1(V)$ has the Dunford-Pettis Property if and only if V has, we get $T\mathscr{F}^{-1}$ is completely continuous. Let W be a weakly compact subset of $\ell_1(\mathscr{F}, V)$. Then $\mathscr{F}(W)$ is a weakly compact subset of $\ell_1(V)$, and so

$$\left(T\mathscr{F}^{-1}\right)\mathscr{F}(W) = T(W)$$

is a compact subset in V. □

Theorem 2.8. *Let V be a linear subspace of the Banach space U and suppose $T : V \to \ell_\infty(\mathscr{F}, V)$ is a bounded linear operator. Then T may be extended to a bounded linear operator $H : U \to \ell_\infty(\mathscr{F}, V)$ having the same norm as T if V has the Hahn-Banach extension property.*

Proof. Consider any bounded linear operator $T : V \to \ell_\infty(\mathscr{F}, V)$. Now $\mathscr{F}T : V \to \ell_\infty(V)$ is a bounded linear operator since the Fibonacci matrix is. Now $\ell_\infty(V)$ has the Hahn-Banach extension property since V has.

For any $x \in V$, $\mathscr{F}Tx \in \ell_\infty(V)$ and

$$\begin{aligned} \mathscr{F}Tx &= ((\mathscr{F}Tx)_1, (\mathscr{F}Tx)_2, \ldots) \\ &= ((P_1\mathscr{F}T)(x), (P_2\mathscr{F}T)(x), \ldots). \end{aligned}$$

Note that each P_n is coordinate projection from $\ell_\infty(V)$ into V such that $P_n(x) = x_n$. By the Hahn-Banach extension property of $\ell_\infty(V)$, the operator $\mathscr{F}T : V \to \ell_\infty(V)$ can be extended the bounded linear operator $S : U \to \ell_\infty(V)$ with the same norm as $\mathscr{F}T$, that is $\|S\| = \|\mathscr{F}T\|$. Let us define the operator H from U into $\ell_\infty(\mathscr{F}, V)$ such that for $x \in U$,

$$Hx = \left(\mathscr{F}^{-1}S\right)(x).$$

H is well-defined and linear since S and $\mathscr{F}^{-1}$ are. Further

$$\begin{aligned} \|Hx\|_{\ell_\infty(\mathscr{F},V)} &= \left\|\mathscr{F}^{-1}(S(x))\right\|_{\ell_\infty(\mathscr{F},V)} \\ &= \left\|\mathscr{F}\left(\mathscr{F}^{-1}(S(x))\right)\right\|_{\ell_\infty(V)} \\ &= \|S(x)\|_{\ell_\infty(V)} \\ &\leq \|S\| . \|x\| \end{aligned}$$

so that H is bounded. Now for $x \in V$,

$$\begin{aligned} \|Hx\|_{\ell_\infty(\mathscr{F},V)} &= \|S(x)\|_{\ell_\infty(V)} \\ &= \|(\mathscr{F}T)(x)\|_{\ell_\infty(V)} \\ &= \|Tx\|_{\ell_\infty(\mathscr{F},V)} \end{aligned}$$

so that H is an extension of T. Finally

$$\begin{aligned} \|H\| &= \sup_{\|x\|_U=1} \|Hx\|_{\ell_\infty(\mathscr{F},V)} \\ &= \sup_{\|x\|_U=1} \left\|\mathscr{F}^{-1}(S(x))\right\|_{\ell_\infty(\mathscr{F},V)} \\ &= \sup_{\|x\|_U=1} \left\|\mathscr{F}\left(\mathscr{F}^{-1}(S(x))\right)\right\|_{\ell_\infty(V)} \\ &= \sup_{\|x\|_U=1} \|S(x)\|_{\ell_\infty(V)} \\ &= \sup_{\|x\|_U=1} \|Tx\|_{\ell_\infty(\mathscr{F},V)} \\ &= \|T\|. \end{aligned}$$

This completes the proof. □

The following property is another desired property of Banach spaces. Now we see that $\ell_2(\mathscr{F}, V)$ has this property whenever V has, which we call it as the Radon-Riesz Property. The Radon-Riesz property is named after J. Radon and F. Riesz proved that the spaces $Lp(\Omega, \Sigma, \mu)$ for $1 < p < \infty$ have this property [11]-[13]. Radon-Riesz Property also known as the Kadets-Klee property since their further investigation and application of this concept [14]-[16].

Definition 2.9. *[7] A normed space has the* Radon-Riesz property *or the* Kadets-Klee property*, and is called a Radon-Riesz space, if it satisfies the following condition: Whenever (x_n) is a sequence in the space and x an element of the space such that $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$.*

The proof of the following lemma is routine.

Lemma 2.10. *Let V is a Hilbert space. Then $\ell_2(\mathscr{F},V)$ is a Hilbert space with the inner-product*

$$\langle u,v\rangle_{\ell_2(\mathscr{F},V)} = \langle \mathscr{F}u, \mathscr{F}v\rangle_{\ell_2(V)} = \sum_{k=1}^{\infty} \langle (\mathscr{F}u)_k, (\mathscr{F}v)_k\rangle_V .$$

Theorem 2.11. *$\ell_2(\mathscr{F},V)$ has the Radon-Riesz Property whenever V is a Hilbert space possessing the Radon-Riesz Property.*

Proof. Let (u_n) be a sequence in $\ell_2(\mathscr{F},V)$ and u be an element of $\ell_2(\mathscr{F},V)$. Assume that $u_n \overset{w}{\to} u$ in $\ell_2(\mathscr{F},V)$ and assume that $\|u_n\|_{\ell_2(\mathscr{F},V)} \to \|u\|_{\ell_2(\mathscr{F},V)}$. We will prove that (u_n) norm convergent to u that is $u_n \to u$ in $\ell_2(\mathscr{F},V)$. Now the assumption $u_n \overset{w}{\to} u$ means $f(u_n) \to f(u_n)$ for each $f \in \ell_2(\mathscr{F},V)^*$. Let us show that $\|u_n - u\|_{\ell_2(\mathscr{F},V)} \to 0$ to complete the proof:

$$\begin{aligned}
\|u_n - u\|^2_{\ell_2(\mathscr{F},V)} &= \|\mathscr{F}u_n - \mathscr{F}u\|^2_{\ell_2(V)} \\
&= \langle \mathscr{F}u_n - \mathscr{F}u, \mathscr{F}u_n - \mathscr{F}u\rangle_{\ell_2(V)} \\
&= \langle \mathscr{F}u_n, \mathscr{F}u_n\rangle_{\ell_2(V)} - \langle \mathscr{F}u_n, \mathscr{F}u\rangle_{\ell_2(V)} \\
&\quad - \langle \mathscr{F}u, \mathscr{F}u_n\rangle_{\ell_2(V)} + \langle \mathscr{F}u, \mathscr{F}u\rangle_{\ell_2(V)} \\
&= \|\mathscr{F}u_n\|^2_{\ell_2(V)} + \|\mathscr{F}u\|^2_{\ell_2(V)} - \langle \mathscr{F}u_n, \mathscr{F}u\rangle_{\ell_2(V)} - \langle \mathscr{F}u, \mathscr{F}u_n\rangle_{\ell_2(V)}
\end{aligned}$$

Let $z = \mathscr{F}u \in \ell_2(V) = \ell_2(V)^*$ and let us consider $z \circ \mathscr{F}$ such that $(z \circ \mathscr{F})u = \langle \mathscr{F}u, \mathscr{F}u\rangle_{\ell_2(V)}$. Then from the properties of the matrix $\mathscr{F}$ and by the Riesz's Theorem (on $\ell_2(V)$) we have $z \circ \mathscr{F}$ is a continuous linear functional on $\ell_2(\mathscr{F},V)$ and

$$(z \circ \mathscr{F})u_n = z(\mathscr{F}u_n) = \langle \mathscr{F}u_n, \mathscr{F}u\rangle_{\ell_2(V)} .$$

By the assumption $u_n \overset{w}{\to} u$ we have

$$\begin{aligned}
(z \circ \mathscr{F})(u_n) &= \langle \mathscr{F}u_n, \mathscr{F}u\rangle_{\ell_2(V)} \\
&\to \langle \mathscr{F}u, \mathscr{F}u\rangle_{\ell_2(V)}, \text{ as } n \to \infty, \\
&= (z \circ \mathscr{F})(u) \\
&= \|\mathscr{F}u\|^2_{\ell_2(V)}
\end{aligned}$$

Dually, let us now take $z_n = \mathscr{F}u_n \in \ell_2(V)^* = \ell_2(V)$, for each n, then

$$(z_n \circ \mathscr{F})u = z_n(\mathscr{F}u) = \langle \mathscr{F}u, \mathscr{F}u_n\rangle_{\ell_2(V)} .$$

Now again each $z_n \circ \mathscr{F}$ is a continuous linear functional on $\ell_2(\mathscr{F},V)$ and again by the assumption $u_n \overset{w}{\to} u$ we have

$$\begin{aligned}
(z_n \circ \mathscr{F})(u) &= z_n(\mathscr{F}u) \\
&= \langle \mathscr{F}u, \mathscr{F}u_n\rangle_{\ell_2(V)} \\
&= \overline{\langle \mathscr{F}u_n, \mathscr{F}u\rangle_{\ell_2(V)}} \\
&\to \overline{\langle \mathscr{F}u, \mathscr{F}u\rangle_{\ell_2(V)}}, \text{ as } n \to \infty, \\
&= \|\mathscr{F}u\|^2_{\ell_2(V)} .
\end{aligned}$$

Eventually, by the assumption $\|u_n\|_{\ell_2(\mathscr{F},V)} \to \|u\|_{\ell_2(\mathscr{F},V)}$, we have

$$\begin{aligned}
\|u_n - u\|^2_{\ell_2(\mathscr{F},V)} &= \|\mathscr{F}u_n\|^2_{\ell_2(V)} + \|\mathscr{F}u\|^2_{\ell_2(V)} - \langle \mathscr{F}u_n, \mathscr{F}u\rangle_{\ell_2(V)} - \langle \mathscr{F}u, \mathscr{F}u_n\rangle_{\ell_2(V)} \\
&\to \|\mathscr{F}u\|^2_{\ell_2(V)} + \|\mathscr{F}u\|^2_{\ell_2(V)} - \|\mathscr{F}u\|^2_{\ell_2(V)} - \|\mathscr{F}u\|^2_{\ell_2(V)} \\
&= 0, \text{ as } n \to \infty.
\end{aligned}$$

□

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A Class of Implicit Fractional ψ-Hilfer Langevin Equation with Time Delay and Impulse in the Weighted Space

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Abstract
In this paper, the Ulam-Hyers-Rassias stability is discussed and the existence and uniqueness of solutions for a class of implicit fractional ψ-Hilfer Langevin equation with impulse and time delay are investigated. A novel form of generalized Gronwall inequality is introduced. Picard operator theory is employed in authour's analysis. An example will be given to support the validity of our findings.

Keywords: Existence and uniqueness, Generalized Gronwall inequality, ψ-Hilfer Langevin equation, Picard operator theory, Ulam-Hyers-Rassias stability
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1. Introduction

The concept of derivatives of arbitrary order, which is essential to fractional calculus and provides a useful tool for characterizing the inherent properties of numerous materials and processes, has maintained its appeal to a large number of scientists in recent years. In [1], Capelas de Oliveira and Sousa presented a generalization concerning these derivatives, in which they combined many formulations, including the traditional Caputo and Riemann-Liouville operators, and proposed a new fractional differential operator, known as the fractional ψ-Hilfer operator.

Parallel to fractional derivation, another theory is growing: fractional differential equations. This theory has numerous applications, especially in the domains of signal processing, biology, physics, engineering, and finance. (Refer to [2, 3]).

One of the best examples of these fractional differential equations is the Langevin equation, which was initially proposed by Paul Langevin in 1908. Its goal is to give descriptions of specific phenomena that physicians, engineers, economists, and other experts may use. The Langevin equation first described the random movement of particles suspended in a liquid, which is commonly referred to as Brownian motion. In addition to being widely applied in all fields, Brownian motion and stochastic differential equations are also commonly used tools in all scientific fields. (see [4]-[13]).

Furthermore, the best modeling method was found to be fractional differential equations with impulse plus delay included.

Since fractional differential equations with impulse are used to simulate evolutionary situations involving fast changes at a finite or infinite number of instants, they represent a fascinating area of study. Similar to this, fractional differential equations with time delay represent real dynamics. They are used in a wide range of fields, including physics, chemistry, biology, road traffic, and medicine. Their goal is to simulate by taking into account the past. Giving someone a drug, for instance, doesn't result in an instant reaction; instead, you must wait a few minutes to see whether the substance has actually had an impact. (Refer to [14, 15]).

The majority of the time, it can be difficult to solve fractional differential equations exactly, and even when it can be done, it can be time-consuming and difficult to compute. Asking whether they are getting close to the exact solutions or if the error we made was not that big makes it simpler to give an appropriate explanation of the approximate solutions.

The concepts of Ulam-Hyers stability, Lyapunov stability, exponential stability, and finite-time stability are employed to evaluate the behavior of solutions to differential equations or dynamical systems under perturbations. Every kind of stability has uses and benefits of its own. Even though Lyapunov, exponential, and finite-time stabilities are important in their own contexts, Ulam-Hyers stability provides a special benefit by emphasizing the robustness of approximations. This makes it particularly desirable for real-world applications where we need to be sure that small deviations won't result in large inaccuracies because exact conditions are rarely realized. It is a useful tool in the stability analysis toolkit due to its adaptability and wide range of applications. (see [16]-[18]).

Abdo et al. [19] have studied the Ulam-Hyers-Mittag-Leffler stability, uniqueness, and existence of a fractional delay differential equation

$$\begin{cases} {}^{H}\mathfrak{D}_{0+,\iota}^{p_1,p_2;\psi}w(\iota)=f(\iota,w_\iota), \quad 0\le\iota\le b,\\ \mathfrak{I}_{0+,\iota}^{1-\gamma;\psi}w\left(0^{+}\right)=w_0\in\mathbb{R},\\ w(\iota)=\varphi(\iota), \quad -\infty<\iota\le 0.\end{cases}$$

Recently, Lima KB et al. [20] investigated the Ulam-Hyers stability, uniqueness, and existence of the following fractional delay impulsive differential equation:

$$\begin{cases} {}^{H}\mathfrak{D}_{0+,\iota}^{p_1,p_2;\psi}w(\iota)=f(\iota,w_\iota), \quad \iota\in(0,b]\setminus\{\iota_1,\iota_2,\ldots,\iota_m\},\\ \Delta w(\iota_i)=I_i\left(w\left(\iota_i^-\right)\right)=w\left(\iota_i^+\right)-w\left(\iota_i^-\right), \quad i=1,2,\ldots,m,\\ \mathfrak{I}_{0+,\iota}^{1-\gamma}w(0)=w_0,\\ w(\iota)=\varphi(\iota), \quad \iota\in[-r,0].\end{cases}$$

Motivated by the latter work, we present in this work a fairly exhaustive study of a novel class of implicit ψ-Hilfer fractional Langevin equation with delays and impulses given by the form:

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=f\left(\iota,w(\iota),w(\sigma(\iota)),{}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)\right), \quad \iota\in(0,b]\setminus\{\iota_1,\iota_2,\ldots,\iota_m\},\\ \Delta w(\iota_i)=I_i\left(w\left(\iota_i^-\right)\right)=w\left(\iota_i^+\right)-w\left(\iota_i^-\right), \quad i=1,2,\ldots,m,\\ \mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0)=w_0, \quad \gamma=q+(p_1+p_2)(1-q),\\ w(\iota)=\varphi(\iota), \quad \iota\in[-r,0], \quad 0\le r<\infty.\end{cases} \tag{1.1}$$

Where $\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}$ and $\mathfrak{D}_{0^+,\iota}^{\vartheta;\psi}$ represent ψ-fractional integrals in order $1-\gamma$ and ψ-Hilfer fractional derivative in order $\vartheta\in\{p_1,p_2\}$ and type q respectively. $0<\vartheta<1$, $0\le q\le 1$. Also, $f:[0,b]\times\Omega\to\mathbb{R}$ a given function, $I_i:\mathbb{R}\to\mathbb{R}$ and $\varphi:[-r,0]\to\mathbb{R}$ continuous functions, $w\left(\iota_i^+\right)=\lim_{\tau\to 0^+}w(\iota_i+\tau), w\left(\iota_i^-\right)=\lim_{\tau\to 0^-}w(\iota_i-\tau), \iota_i$ satisfies $0=\iota_0<\iota_1<\iota_2<\cdots<\iota_m<\iota_{m+1}=b<\infty$ and $\sigma:[0,b]\to[-r,b]$ is a delay function that is continuous and ensures $\sigma(\iota)\le\iota, \iota\in[0,b]$.

Let $J=[0,b]$, and let $C(J,\mathbb{R})$ and $C^n(J,\mathbb{R})$ be the Banach spaces of continuous functions, n-times continuously differentiable functions on J, respectively. Moreover, for any $f\in C(J,\mathbb{R})$, we have $\|f\|_C=\sup\{|f(\iota)|:\iota\in J\}$. On the other hand, we consider the weighted space in [20], defined by

$$C_{1-\gamma;\psi}(J,\mathbb{R})=\left\{w:J\to\mathbb{R}:(\psi(\iota)-\psi(0))^{1-\gamma}w(\iota)\in C(J,\mathbb{R})\right\}, \quad 0<\gamma\le 1.$$

Define the Banach space

$$PC_{1-\gamma;\psi}(J,\mathbb{R})=\left\{\begin{array}{l} w:J\to\mathbb{R}; \quad w\in C_{1-\gamma;\psi}([\iota_i,\iota_{i+1}],\mathbb{R}), i=0,\ldots,m,\\ \text{and there exist } w\left(\iota_i^+\right), w\left(\iota_i^-\right), \text{with } w(\iota_i)=w\left(\iota_i^-\right), i=1,2,\ldots,m\end{array}\right\}, \quad 0<\gamma\le 1,$$

using the norm

$$\|w\|_{PC_{1-\gamma;\psi}} = \sup_{\iota\in J}\left|(\psi(\iota)-\psi(0))^{1-\gamma}w(\iota)\right|.$$

We then specify the space.

$$\Omega_{\gamma;\psi} = \left\{w:[-r,b]\to\mathbb{R} : w\in C([-r,0],\mathbb{R})\cap PC_{1-\gamma;\psi}(J,\mathbb{R})\right\},$$

using the norm $\|w\|_{\Omega_{\gamma;\psi}} = \max\left\{\|w\|_C, \|w\|_{PC_{1-\gamma;\psi}}\right\}$. One can verify that $\left(\Omega_{\gamma;\psi}, \|\cdot\|_{\Omega_{\gamma;\psi}}\right)$ is a Banach space (see [19, 20]).

2. Preliminaries

Definition 2.1. *([1]) For $p>0$, and $\psi\in C^1(J,\mathbb{R})$, the fractional ψ-Riemann-Liouville operator with order p for an integrable function w can be written as*

$$\mathfrak{I}^{p;\psi}_{0^+,\iota}w(\iota) = \frac{1}{\Gamma(p)}\int_0^{\iota}\psi'(s)(\psi(\iota)-\psi(s))^{p-1}w(s)ds, \tag{2.1}$$

in which $\psi'(\iota)>0$, $\forall\iota\in J$.

Definition 2.2. *([1]) For $0<p<1$, $w\in C(J,\mathbb{R})$, $\psi\in C^1(J,\mathbb{R})$ with $\psi'(\iota)>0$, $\forall\iota\in J$, the fractional ψ-Hilfer derivative operator with order p and type $0\le q\le 1$ of w is represented as*

$${}^H\mathfrak{D}^{p,q;\psi}_{0^+,\iota}w(\iota) = \mathfrak{I}^{q(1-p);\psi}_{0^+,\iota}\left(\frac{1}{\psi'(\iota)}\frac{d}{d\iota}\right)\mathfrak{I}^{(1-q)(1-p);\psi}_{0^+,\iota}w(\iota). \tag{2.2}$$

Lemma 2.3. *([1]) Let $0<p<1$, $0\le q\le 1$, $w\in C^1(J,\mathbb{R})$, then*

$$\mathfrak{I}^{p;\psi}_{0^+,\iota}\,{}^H\mathfrak{D}^{p,q;\psi}_{0^+,\iota}w(\iota) = w(\iota) - \frac{(\psi(\iota)-\psi(0))^{\rho-1}}{\Gamma(\rho)}\mathfrak{I}^{1-\rho;\psi}_{0^+,\iota}w(0), \tag{2.3}$$

where $\rho = p+q(1-p)$.

Lemma 2.4. *([1, 21]) Let $p,q>0,\delta>p$ and $w\in C(J,\mathbb{R})$. Following that $\forall\iota\in J$ there are*
(1) $\mathfrak{I}^{p;\psi}_{0^+,\iota}\mathfrak{I}^{q;\psi}_{0^+,\iota}w(\iota) = \mathfrak{I}^{p+q;\psi}_{0^+,\iota}w(\iota)$,
(2) ${}^H\mathfrak{D}^{p,q;\psi}_{0^+,\iota}\mathfrak{I}^{p;\psi}_{0^+,\iota}w(\iota) = w(\iota)$,
(3) $\mathfrak{I}^{p;\psi}_{0^+,\iota}(\psi(\iota)-\psi(0))^{q-1} = \frac{\Gamma(p)}{\Gamma(p+q)}(\psi(\iota)-\psi(0))^{p+q-1}$,
(4) $\mathfrak{I}^{q;\psi}_{0^+,\iota}(1) = \frac{(\psi(\iota)-\psi(0))^q}{\Gamma(q+1)}$,
(5) ${}^H\mathfrak{D}^{p,q;\psi}_{0^+,\iota}(\psi(\iota)-\psi(0))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta-p)}(\psi(\iota)-\psi(0))^{\delta-p-1}$,
(6) ${}^H\mathfrak{D}^{p,q;\psi}_{0^+,\iota}(\psi(\iota)-\psi(0))^{\delta-1} = 0, \quad 0<\delta<1$.

Lemma 2.5. *([1]) Let $0\le\gamma<1$ and $f\in C_{1-\gamma;\psi}[0,b]$. Then*

$$\mathfrak{I}^{p;\psi}_{0^+,\iota}f(0) = \lim_{\iota\to 0+}\mathfrak{I}^{p;\psi}_{0^+,\iota}f(\iota) = 0, \quad 0\le 1-\gamma<p.$$

To show the Ulam-Hyers-Rassias stability for problem (1.1), we generalise the definitions for ψ-Hilfer given by Rizwan et al in [22].

Take $\varepsilon>0,\theta>0$, $\phi\in\Omega_{\gamma;\psi}$, and considering

$$\begin{cases}\left|{}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota) - f\left(\iota,w(\iota),w(\sigma(\iota)),{}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)\right)\right| \le \varepsilon\phi(\iota), \quad \iota\in J,\\ \left|\Delta w(\iota_k) - I_k\left(w\left(\iota_k^-\right)\right)\right| \le \varepsilon\theta, \quad k=1,\dots,m.\end{cases} \tag{2.4}$$

Definition 2.6. *([20]) Problem (1.1) is Ulam-Hyers-Rassias stable in terms of $(\phi(\iota),\theta)$, when a real number $c_{F,m,\phi}>0$ exists in which, for all $\varepsilon>0$ and all $v\in\Omega_{\gamma;\psi}$ solution of (2.4), there is a solution $w\in\Omega_{\gamma;\psi}$ to the problem (1.1) with*

$$\begin{cases}|v(\iota)-w(\iota)| = 0, \quad \iota\in[-r,0],\\ \left|(\psi(\iota)-\psi(0))^{1-\gamma}(v(\iota)-w(\iota))\right| \le c_{F,m,\phi}\,\varepsilon(\phi(\iota)+\theta), \quad \iota\in J.\end{cases}$$

Remark 2.7. *([20]) A continious function $v \in \Omega_{\gamma;\psi}$ is a solution of (2.4) only if $g \in \Omega_{\gamma;\psi}$ a function and g_k , $k = 1,2,\ldots,m$ a sequence (both depends on v) exist in which*
(1) $|g(\iota)| \leq \varepsilon\phi(\iota)$, $\iota \in J$, $|g_k| \leq \varepsilon\theta$, $\quad k = 1,2,\ldots,m$,
(2) ${}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota) = f\left(\iota, w(\iota), w(\sigma(\iota)), {}^H\mathfrak{D}^{p_1,q;\psi}_{0^+,\iota}\left({}^H\mathfrak{D}^{p_2,q;\psi}_{0^+,\iota}+\lambda\right)w(\iota)\right) + g(\iota), \quad \iota \in J,$
(3) $\Delta w(\iota_k) = I_k\left(w\left(\iota_k^-\right)\right) + g_k, \quad k = 1,2,\ldots,m.$

Definition 2.8. *([23]) Consider the metric space $(\mathscr{E}, d)$. If there is a $w^* \in \mathscr{E}$ in which*
1. $\mathscr{F}_{\mathscr{T}} = \{w^*\}$, in which $\mathscr{F}_{\mathscr{T}} = \{w \in \mathscr{E} : \mathscr{T}(w) = w\}$.
2. $\{\mathscr{T}^n(w_0)\}_{n\in\mathbb{N}}$ converges to w^* for each $w_0 \in \mathscr{E}$.
Then the operator $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ is a Picard operator.

Lemma 2.9. *([24]) Take $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ an increasing Picard operator with $\mathscr{F}_{\mathscr{T}} = \left\{w^*_{\mathscr{T}}\right\}$, and take $(\mathscr{E}, d, \leq)$ an ordered metric space. Then, for each $w \in \mathscr{E}, w \leq \mathscr{T}(w)$ shows $w \leq w^*_{\mathscr{T}}$.*

Lemma 2.10. *([25]) Take w, v be two functions on J that are integrable. Consider $\psi \in C^1(J,\mathbb{R})$ is an increasing function in which $\psi'(\iota) \neq 0, \forall \iota \in J$. Suppose that*
(i) Both w and v are positive.
(ii) For any J, $(g_i)_{i=1,..n}$ are bounded and monotonic increasing functions.
(iii) $p_i > 0 (i = 1,2,\ldots,n)$. If

$$w(\iota) \leq v(\iota) + \sum_{i=1}^{n} g_i(\iota)\int_a^\iota \psi'(s)(\psi(\iota)-\psi(s))^{p_i-1}w(s)\mathrm{d}s,$$

then

$$w(\iota) \leq v(\iota) + \sum_{k=1}^{\infty}\left(\sum_{1',2',3',\ldots k'=1}^{n} \frac{\prod_{i=1}^{k}(g_{i'}(\iota)\Gamma(p_{i'}))}{\Gamma\left(\sum_{i=1}^{k} p_{i'}\right)}\int_a^\iota \left[\psi'(s)(\psi(\iota)-\psi(s))^{\sum_{i=1}^{k} p_{i'}-1}\right]v(s)\mathrm{d}s\right).$$

Assume further that $v(\iota)$ is a nondecreasing function on J. Next, the inequality given by [25, Corollary 2.1], for $n = 2$, gives us

$$w(\iota) \leq v(\iota)\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right) + E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right],$$

where $\psi_g^p(\iota,0) := g(\iota)\Gamma(p)(\psi(\iota)-\psi(0))^p$, and E_p is the Mittag-Leffler function defined in [2] by

$$E_p(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(np+1)}, z \in \mathbb{C}, \mathrm{Re}(p) > 0.$$

Lemma 2.11. *For $n = 2$. Let $w \in \Omega_{\gamma;\psi}$ satisfying the following inequality*

$$w(\iota) \leq v(\iota) + \sum_{l=1}^{2} g_l(\iota)\int_0^\iota \psi'(\tau)(\psi(\iota)-\psi(\tau))^{p_l-1}w(s)ds + \sum_{0<\iota_k<\iota}\beta_k w(\iota_k), \quad \iota \geq 0, \tag{2.5}$$

where $\beta_k > 0, k = 1,\ldots,m$ is a nonnegative constant and $v \in \Omega_{\gamma;\psi}$ is nonnegative too. Following that

$$w(\iota) \leq v(\iota)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right]\right)^k\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right], \quad \iota \in (\iota_k, \iota_{k+1}], \tag{2.6}$$

where $\beta = \max\{\beta_k : k = 1,2,\ldots,m\}$.

Proof. For $n = 2$, and by lemma 2.10, we derive

$$w(\iota) \leq v(\iota)\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right], \quad \iota \in [0,\iota_1], \tag{2.7}$$

$$w(\iota) \leq \left[v(\iota) + \sum_{j=0}^{k}\beta_j w(\iota_j)\right]\left[E_{p_1}\left(\psi_g^{p_1}(\iota,0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota,0)\right)\right], \quad \iota \in [\iota_k, \iota_{k+1}]. \tag{2.8}$$

By induction, for $k=0$, inequality (2.6) holds by (2.7). Suppose that for $k=j<m$, (2.6) holds. After that, using (2.8) and the nondecreasing nature of v and E_p, we obtain for $\iota \in \left(\iota_{j+1}, \iota_{j+2}\right]$,

$$\begin{aligned}
w(\iota) &\leq \left[v(\iota)+\sum_{i=0}^{j+1} \beta_i w\left(\iota_i\right)\right]\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right] \\
&\leq \left[v(\iota)+\sum_{i=1}^{j+1} \beta_i v(\iota_i)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota_i, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota_i, 0)\right)\right]\right)^{i-1}\left[E_{p_1}\left(\psi_g^{p_1}(\iota_i, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota_i, 0)\right)\right]\right] \\
&\quad \times\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right] \\
&\leq \left[v(\iota)+\beta \sum_{i=1}^{j+1} v(\iota)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right]\right)^{i-1}\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right]\right] \\
&\quad \times\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right],
\end{aligned}$$

then

$$\begin{aligned}
w(\iota) &\leq \left[v(\iota)+\beta v(\iota) \frac{1-\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right]\right)^{j+1}}{1-\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right]\right)}\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right]\right] \\
&\quad \times\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right] \\
&= \left[v(\iota)+v(\iota)\left(\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right]\right)^{j+1}-1\right)\right]\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right] \\
&\leq \left[v(\iota)+v(\iota)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right]\right)^{j+1}-v(\iota)\right]\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right] \\
&= v(\iota)\left(1+\beta\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right]\right)^{j+1}\left[E_{p_1}\left(\psi_g^{p_1}(\iota, 0)\right)+E_{p_2}\left(\psi_g^{p_2}(\iota, 0)\right)\right].
\end{aligned}$$

This finishes the proof. □

3. Formula of Solutions

Lemma 3.1. *Let $0<p_1, p_2<1, \quad 0 \leq q \leq 1$, and $h: J \rightarrow \mathbb{R}$ be continuous. A function $w \in \Omega_{\gamma ; \psi}$ given by*

$$w(\iota)=\left[\frac{w_0-\mathfrak{I}_{0^+, \iota}^{p_1+p_2 ; \psi} h^a+\lambda \mathfrak{I}_{0^+, \iota}^{p_2 ; \psi} w(a)}{(\psi(a)-\psi(0))^{\gamma-1}}\right](\psi(\iota)-\psi(0))^{\gamma-1}+\mathfrak{I}_{0^+, \iota}^{p_1+p_2 ; \psi} h(\iota)-\lambda \mathfrak{I}_{0^+, \iota}^{p_2 ; \psi} w(\iota) \tag{3.1}$$

is the unique solution for the problem that follows

$$\begin{cases}{ }^H \mathfrak{D}_{0^+, \iota}^{p_1, q ; \psi}\left({ }^H \mathfrak{D}_{0^+, \iota}^{p_2, q ; \psi}+\lambda\right) w(\iota)=h(\iota), & \iota \in J, \\ w(a)=w_0, \qquad a>0,\end{cases} \tag{3.2}$$

in which $\mathfrak{I}_{0^+, \iota}^{p_1+p_2 ; \psi} h^a=\left.\mathfrak{I}_{0^+, \iota}^{p_1+p_2 ; \psi} h(\iota)\right|_{\iota=a}$.

Proof. Taking the fractional ψ-integral operator of order p_1+p_2 on each side of (3.2). Then utilizing Lemma 2.3, we arrive at

$$w(\iota)-e_1(\psi(\iota)-\psi(0))^{\gamma-1}+\lambda \mathfrak{I}_{0^+, \iota}^{p_2 ; \psi}\left(w(\iota)-\frac{(\psi(\iota)-\psi(0))^{\gamma_1-1}}{\Gamma(\gamma_1)} \mathfrak{I}_{0^+, \iota}^{1-\gamma_1 ; \psi} w(0)\right)=\mathfrak{I}_{0^+, \iota}^{p_1+p_2 ; \psi} h(\iota), \tag{3.3}$$

where $\gamma_1=q+p_1(1-q)$, and e_1 is an arbitrary constant.
Since $1-\gamma<1-\gamma_1$, lemma 2.5 implies that $\mathfrak{I}_{0^+, \iota}^{1-\gamma_1 ; \psi} w(0)=0$.
Hence (3.3) reduces to

$$w(\iota)=e_1(\psi(\iota)-\psi(0))^{\gamma-1}+\mathfrak{I}_{0^+, \iota}^{p_1+p_2 ; \psi} h(\iota)-\lambda \mathfrak{I}_{0^+, \iota}^{p_2 ; \psi} w(\iota). \tag{3.4}$$

In (3.4), the boundary condition $w(a)=w_0$ leads to $e_1=\frac{w_0-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}h^a+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(a)}{(\psi(a)-\psi(0))^{\gamma-1}}$. We substitute e_1 in (3.4), we obtain (3.1).

On the other hand, suppose w can be the unique solution satisfying (3.1), taking the fractional ψ-Hilfer derivative ${}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}$ on either side of (3.1), and using lemma 2.4, we can obtain

$$ {}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}w(\iota)=\mathfrak{I}_{0^+,\iota}^{p_1;\psi}h(\iota)-\lambda w(\iota), $$

then taking fractional ${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}$ again, it follows

$$ {}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=h(\iota). $$

Hence, the proof is complete.

□

We obtain the following result from 3.1, which is useful in what follows.

Lemma 3.2. *A function $w\in\Omega_{\gamma;\psi}$ has a solution of (1.1) if and only if $w\in\Omega_{\gamma;\psi}$ is a solution of the given fractional integral equation*

$$ w(\iota)=\begin{cases}\varphi(\iota), & \iota\in[-r,0],\\ \left[\frac{w_0}{\Gamma(\gamma)}+\sum_{i=1}^{m}\frac{I_i\left(w(\iota_i^-)\right)}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota), & \iota\in J,\end{cases} \tag{3.5} $$

in which $F_w(\iota)=f(\iota,w(\iota),w(\sigma(\iota)),F_w(\iota))$, and $\mathfrak{R}_\psi^\gamma(\iota,0)=(\psi(\iota)-\psi(0))^{\gamma-1}$.

Proof. Assume that w satisfies (1.1), then w satisfies

$$ {}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=f\left(\iota,w(\iota),w(\sigma(\iota)),{}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)\right). $$

Take ${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=F_w(\iota)$. It follows that $F_w(\iota)=f(\iota,w(\iota),w(\sigma(\iota)),F_w(\iota))$. Then, we have

$$ {}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=F_w(\iota). $$

If $\iota\in[-r,0]$, clearly that $w(\iota)=\varphi(\iota)$. For $\iota\in[0,\iota_1]$, ${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=F_w(\iota)$ can be writen as

$$ {}^H\mathfrak{D}_{0^+,\iota}^{p_1+p_2,q;\psi}w(\iota)+\lambda\,{}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}w(\iota)=F_w(\iota). \tag{3.6} $$

Taking the fractional ψ-integral operator of order p_1+p_2 on each side of (3.6). Then utilizing Lemma 2.3, we arrive at

$$ w(\iota)-\frac{(\psi(\iota)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0)+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}\left(\mathfrak{I}_{0^+,\iota}^{p_1;\psi H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}w(\iota)\right)=\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota). \tag{3.7} $$

Utilizing again Lemma 2.3, we can get

$$ w(\iota)-\frac{(\psi(\iota)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0)+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}\left(w(\iota)-\frac{(\psi(\iota)-\psi(0))^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathfrak{I}_{0^+,\iota}^{1-\gamma_1;\psi}w(0)\right)=\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota), \tag{3.8} $$

where $\gamma_1=q+p_1(1-q)$.
Since $1-\gamma<1-\gamma_1$, lemma 2.5 implies that $\mathfrak{I}_{0^+,\iota}^{1-\gamma_1;\psi}w(0)=0$.
Hence by $\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0)=w_0$ and $\mathfrak{R}_\psi^\gamma(\iota,0)=(\psi(\iota)-\psi(0))^{\gamma-1}$, equation (3.8) reduces to

$$ w(\iota)=\frac{w_0}{\Gamma(\gamma)}\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota). \tag{3.9} $$

If $\iota\in[\iota_1,\iota_2]$ then ${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=F_w(\iota)$ with $w(\iota_1^+)=w(\iota_1^-)+I_1\left(w(\iota_1^-)\right)$ By lemma 3.1, one obtain

$$w(\iota)=\left[\frac{w\left(\iota_1^+\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_1}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_1)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{w\left(\iota_1^-\right)+I_1\left(w\left(\iota_1^-\right)\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_1}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_1)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{\frac{w_0}{\Gamma(\gamma)}\mathfrak{R}_\psi^\gamma(\iota_1,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_1}-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_1)+I_1\left(w\left(\iota_1^-\right)\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_1}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_1)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)$$
$$-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{\frac{w_0}{\Gamma(\gamma)}\mathfrak{R}_\psi^\gamma(\iota_1,0)+I_1\left(w\left(\iota_1^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{w_0}{\Gamma(\gamma)}+\frac{I_1\left(w\left(\iota_1^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota).$$

If $\iota\in[\iota_2,\iota_3]$ then again by lemma 3.1

$$w(\iota)=\left[\frac{w\left(\iota_2^+\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_2}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_2)}{\mathfrak{R}_\psi^\gamma(\iota_2,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{w\left(\iota_2^-\right)+I_2\left(w\left(\iota_2^-\right)\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_2}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_2)}{\mathfrak{R}_\psi^\gamma(\iota_2,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{\left[\frac{w_0}{\Gamma(\gamma)}+\frac{I_1\left(w\left(\iota_1^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}\right]\mathfrak{R}_\psi^\gamma(\iota_2,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_2}-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_2)+I_2\left(w\left(\iota_2^-\right)\right)-\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w^{\iota_2}+\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota_2)}{\mathfrak{R}_\psi^\gamma(\iota_2,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)$$
$$+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota)$$
$$=\left[\frac{w_0}{\Gamma(\gamma)}+\frac{I_1\left(w\left(\iota_1^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_1,0)}+\frac{I_2\left(w\left(\iota_2^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_2,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota).$$

Repeating the same fashion in this way for $\iota\in[\iota_k,\iota_{k+1}]$, we get
$w(\iota)=\left[\frac{w_0}{\Gamma(\gamma)}+\sum_{i=1}^k\frac{I_i\left(w\left(\iota_i^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota),\quad k=1,2,\ldots,m.$

In contrast, Suppose w can be the unique solution satisfying (3.5). If $\iota\in[-r,0]$, clearly that $w(\iota)=\varphi(\iota)$. If $\iota\in[0,\iota_1]$, taking the fractional ψ-Hilfer derivative ${}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}$ on either side of

$$w(\iota)=\frac{w_0}{\Gamma(\gamma)}\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota),\tag{3.10}$$

using lemma 2.4, we can obtain

$${}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}w(\iota)=\mathfrak{I}_{0^+,\iota}^{p_1;\psi}F_w(\iota)-\lambda w(\iota).$$

Then taking fractional ${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}$ again, it follows

$${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=F_w(\iota).$$

Now we show that the initial condition $\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0)=w_0$ also holds. We apply $\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}$ on both sides of (3.10), then lemma 2.4 implies that

$$\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(\iota)=w_0+\mathfrak{I}_{0^+,\iota}^{1-\gamma+p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{1-\gamma+p_2;\psi}w(\iota).$$

Since $1-\gamma<1-\gamma+p_1+p_2$ and $1-\gamma<1-\gamma+p_2$, lemma 2.5 implies that

$$\mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0)=w_0.$$

If $\iota\in[\iota_k,\iota_{k+1}],\quad k=1,2,\ldots,m$. Using again lemma 2.4, we obtain

$${}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=F_w(\iota) \text{ and } w\left(\iota_k^+\right)-w\left(\iota_k^-\right)=I_k\left(w\left(\iota_k^-\right)\right).$$

Hence, the proof is complete.

□

4. Existence, Uniqueness and Stability

We present the following hypothesis in order to show the existence, uniqueness, and stability of the solution.

H1: $f:J\times\mathbb{R}^3\to\mathbb{R}$ is continuous and $\mathscr{L}>0$, $0<\mathscr{L}_f<1$ are constants satisfy :

$$|f(\iota,w_1,w_2,w_3)-f(\iota,v_1,v_2,v_3)|\le\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}\{|w_1-v_1|+|w_2-v_2|\}+\mathscr{L}_f|w_3-v_3|,$$

$\iota\in J$ and $w_1,v_1,w_2,v_2,w_3,v_3\in\mathbb{R}$.

H2: $I_i:\mathbb{R}\to\mathbb{R},(i=1,\ldots,m)$ satisfy :

$$\left|I_i\left(w\left(\iota_i^-\right)\right)-I_i\left(v\left(\iota_i^-\right)\right)\right|\le\mathscr{L}_{I_i}\left|w\left(\iota_i^-\right)-v\left(\iota_i^-\right)\right|,$$

where $w,v\in\Omega_{\gamma;\psi}$ and $\mathscr{L}_{I_i}>0$.

H3: The inequality

$$\mathscr{K}:=m\mathscr{L}_I+\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}+\frac{|\lambda|\Gamma(\gamma)(\psi(b)-\psi(0))^{p_2}}{\Gamma(p_2+\gamma)}<1,$$

where $\mathscr{L}_I=\max\{\mathscr{L}_{I_i}:i=1,2,\ldots,m\}$.

H4: A non-decreasing function ϕ, bounded in J, and a constant $\lambda_\phi>0$ exist in which, for each $\iota\in J$,

$$\mathfrak{I}_{0+,\iota}^{p_1+p_2;\psi}\phi(\iota)\le\lambda_\phi\phi(\iota).$$

Theorem 4.1. *Suppose that H1–H4 are true. Then*
(i). There is a unique solution to problem (1.1) in the space $\Omega_{\gamma;\psi}$.
(ii). Problem (1.1) is Ulam-Hyers-Rassias stable.

Proof. Part 1: In this part we will prove the existence as well as the uniqueness of solutions to problem (1.1). Considering Lemma 3.2, we set the operator $\mathscr{A}:\Omega_{\gamma;\psi}\longrightarrow\Omega_{\gamma;\psi}$

$$(\mathscr{A}w)(\iota)==\begin{cases}\varphi(\iota), & \iota\in[-r,0],\\ \left[\frac{w_0}{\Gamma(\gamma)}+\sum_{i=1}^m\frac{I_i\left(w\left(\iota_i^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0)+\mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota)-\lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota), & \iota\in J,\end{cases}\tag{4.1}$$

where $F_w(\iota)=f(\iota,w(\iota),w(\sigma(\iota)),F_w(\iota))$.

As we can see, the solution to problem (1.1) will be the fixed point of $\mathscr{A}$.
We demonstrate that on $\Omega_{\gamma;\psi}$, $\mathscr{A}$ is a contraction map. Let, $w,v\in\Omega_{\gamma;\psi}$. Then for $\iota\in[-r,0]$, we have:

$$\|\mathscr{A}w-\mathscr{A}v\|_C=0.\tag{4.2}$$

Further, for any $\iota\in J$, we have

$$\begin{aligned}|(\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota)|\le&\mathfrak{R}_\psi^\gamma(\iota,0)\left[\sum_{i=1}^m\frac{\left|I_i\left(w\left(\iota_i^-\right)\right)-I_i\left(v\left(\iota_i^-\right)\right)\right|}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right]+\frac{1}{\Gamma(p_1+p_2)}\int_0^\iota\mathscr{N}_\psi^{p_1+p_2}(\iota,s)|F_w(s)-F_v(s)|ds\\&+\frac{|\lambda|}{\Gamma(p_2)}\int_0^\iota\mathscr{N}_\psi^{p_2}(\iota,s)|w(s)-v(s)|ds,\end{aligned}$$

where $\mathscr{N}_{\psi}^{p}(\iota,s)=\psi'(s)(\psi(\iota)-\psi(s))^{p}, \quad p=p_2,p_1+p_2$.
Using (H1), (H2), and

$$|F_w(\iota)-F_v(\iota)|\leq \mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}\{|w(\iota)-v(\iota)|+|w(\sigma(\iota))-v(\sigma(\iota))|\}+\mathscr{L}_f\,|F_w(\iota)-F_v(\iota)|.$$

It follows that

$$|F_w(\iota)-F_v(\iota)|\leq \frac{\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{1-\mathscr{L}_f}\{|w(\iota)-v(\iota)|+|w(\sigma(\iota))-v(\sigma(\iota))|\}.$$

Therefore, we have

$$\begin{aligned}
&|(\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota)|\\
&\leq \Re_{\psi}^{\gamma}(\iota,0)\left[\sum_{i=1}^{m}\mathscr{L}_{I_i}(\psi(\iota_i)-\psi(0))^{1-\gamma}\left|w\left(\iota_i^-\right)-v\left(\iota_i^-\right)\right|\right]\\
&+\frac{\mathscr{L}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)(\psi(s)-\psi(0))^{1-\gamma}\times\{|w(s)-v(s)|+|w(\sigma(s))-v(\sigma(s))|\}\,ds\\
&+\frac{|\lambda|}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)\,|w(s)-v(s)|\,ds.
\end{aligned}$$

Then

$$\begin{aligned}
|(\psi(\iota)-\psi(0))^{1-\gamma}((\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota))|\leq&\sum_{i=1}^{m}\mathscr{L}_{I_i}(\psi(\iota_i)-\psi(0))^{1-\gamma}\left|w\left(\iota_i^-\right)-v\left(\iota_i^-\right)\right|\\
&+\frac{\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)(\psi(s)-\psi(0))^{1-\gamma}\\
&\times\{|w(s)-v(s)|+|w(\sigma(s))-v(\sigma(s))|\}\,ds\\
&+\frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\\
&\times(\psi(s)-\psi(0))^{1-\gamma}|w(s)-v(s)|\,ds.
\end{aligned}$$

Then

$$\begin{aligned}
&|(\psi(\iota)-\psi(0))^{1-\gamma}((\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota))|\\
&\leq m\mathscr{L}_I\|w-v\|_{PC_{1-\gamma;\psi}}\\
&+\frac{2\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\|w-v\|_{PC_{1-\gamma;\psi}}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)ds\\
&+\frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\|w-v\|_{PC_{1-\gamma;\psi}}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}ds\\
&\leq m\mathscr{L}_I\|w-v\|_{PC_{1-\gamma;\psi}}+\frac{2\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}\|w-v\|_{PC_{1-\gamma;\psi}}\times(\psi(\iota)-\psi(0))^{p_1+p_2}\\
&+\frac{\Gamma(\gamma)|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2+\gamma)}\|w-v\|_{PC_{1-\gamma;\psi}}\times(\psi(\iota)-\psi(0))^{p_2+\gamma-1}\\
&\leq\left[m\mathscr{L}_I+\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}+\frac{\Gamma(\gamma)|\lambda|(\psi(b)-\psi(0))^{p_2}}{\Gamma(p_2+\gamma)}\right]\|w-v\|_{PC_{1-\gamma;\psi}}.
\end{aligned}$$

Therefore,

$$\|\mathscr{A}w-\mathscr{A}v\|_{PC_{1-\gamma;\psi}}=\sup_{\iota\in J}\left|(\psi(\iota)-\psi(0))^{1-\gamma}((\mathscr{A}w)(\iota)-(\mathscr{A}v)(\iota))\right|\leq\mathscr{K}\,\|w-v\|_{PC_{1-\gamma;\psi}}. \tag{4.3}$$

From (4.2) and (4.3), we have

$$\begin{aligned}
\|\mathscr{A}w-\mathscr{A}v\|_{\Omega_{\gamma;\psi}}&=\max\left\{\|\mathscr{A}w-\mathscr{A}v\|_{C},\|\mathscr{A}w-\mathscr{A}v\|_{PC_{1-\gamma;\psi}}\right\}\\
&\leq\mathscr{K}\max\left\{0,\|w-v\|_{PC_{1-\gamma;\psi}}\right\}\\
&\leq\mathscr{K}\,\|w-v\|_{\Omega_{\gamma;\psi}}.
\end{aligned}$$

As $\mathscr{K} < 1$, Banach's fixed-point theorem shows that the operator $\mathscr{A}$ has a fixed point, which is the unique solution to problem (1.1).

Part 2: Now, let us discuss the Ulam-Hyers-Rassias stability.
Take $v \in \Omega_{\gamma;\psi}$ as the solution to (2.4) and $w \in \Omega_{\gamma;\psi}$ as the unique solution to the problem:

$$\begin{cases} {}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota) = f\left(\iota, w(\iota), w(\sigma(\iota)), {}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)\right), \quad \iota \in (0,b]\setminus\{\iota_1,\iota_2,\ldots,\iota_m\}, \\ \Delta w(\iota_i) = w\left(\iota_i^+\right) - w\left(\iota_i^-\right) = I_i\left(w\left(\iota_i^-\right)\right), \quad i = 1,2,\ldots,m, \\ \mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi} w(0) = w_0, \quad \gamma = q + (p_1+p_2)(1-q), \\ w(\iota) = \varphi(\iota), \quad \iota \in [-r,0]. \end{cases} \tag{4.4}$$

According to Lemma 3.2, we have

$$w(\iota) = \begin{cases} \varphi(\iota), & \iota \in [-r,0], \\ \left[\frac{w_0}{\Gamma(\gamma)} + \sum_{i=1}^m \frac{I_i\left(w\left(\iota_i^-\right)\right)}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0) + \mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_w(\iota) - \lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}w(\iota), & \iota \in J. \end{cases} \tag{4.5}$$

By assuming that v is a solution of (2.4). Hence, based on Remark 2.7, the solution of

$$\begin{cases} {}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)v(\iota) = f\left(\iota, v(\iota), v(\sigma(\iota)), {}^H\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^H\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)\right) + g(\iota), \quad \iota \in (0,b]\setminus\{\iota_1,\iota_2,\ldots,\iota_m\}, \\ \Delta v(\iota_i) = I_i\left(v\left(\iota_i^-\right)\right) + g_i, \quad i = 1,2,\ldots,m, \\ \mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi} v(0) = w_0, \quad \gamma = q + (p_1+p_2)(1-q), \\ v(\iota) = \varphi(\iota), \quad \iota \in [-r,0]. \end{cases}$$

It can be formulated as follows:

$$v(\iota) = \begin{cases} \varphi(\iota), & \iota \in [-r,0], \\ \left[\frac{w_0}{\Gamma(\gamma)} + \sum_{i=1}^m \frac{I_i\left(v\left(\iota_i^-\right)\right)+g_i}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right]\mathfrak{R}_\psi^\gamma(\iota,0) + \mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}F_v(\iota) - \lambda\mathfrak{I}_{0^+,\iota}^{p_2;\psi}v(\iota) + \mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}g(\iota), & \iota \in J, \end{cases} \tag{4.6}$$

where, $F_v(\iota) = f(\iota, v(\iota), v(\sigma(\iota)), F_v(\iota))$.

Now let, $w, v \in \Omega_{\gamma;\psi}$. Then for $\iota \in [-r,0]$, we have:

$$\|w - v\|_C = 0. \tag{4.7}$$

Further, for any $\iota \in J$, we have

$$\begin{aligned} |w(\iota) - v(\iota)| \leq& \mathfrak{R}_\psi^\gamma(\iota,0)\left[\sum_{i=1}^m \frac{\left|I_i\left(w\left(\iota_i^-\right)\right) - I_i\left(v\left(\iota_i^-\right)\right)\right|}{\mathfrak{R}_\psi^\gamma(\iota_i,0)}\right] + \mathfrak{R}_\psi^\gamma(\iota,0)\sum_{i=1}^m \frac{|g_i|}{\mathfrak{R}_\psi^\gamma(\iota_i,0)} + \mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}|g(\iota)| \\ &+ \frac{1}{\Gamma(p_1+p_2)}\int_0^\iota \mathscr{N}_\psi^{p_1+p_2}(\iota,s)\left|F_w(s) - F_v(s)\right| ds + \frac{|\lambda|}{\Gamma(p_2)}\int_0^\iota \mathscr{N}_\psi^{p_2}(\iota,s)\left|w(s) - v(s)\right| ds. \end{aligned}$$

Using (H1), (H2), and remark 2.7, we've obtained

$$\begin{aligned} \left|(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))\right| \leq& \sum_{i=1}^m \mathscr{L}_{I_i}(\psi(\iota_i)-\psi(0))^{1-\gamma}\left|w\left(\iota_i^-\right) - v\left(\iota_i^-\right)\right| + \sum_{i=1}^m \varepsilon\theta(\psi(\iota_i)-\psi(0))^{1-\gamma} \\ &+ \frac{\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^\iota \mathscr{N}_\psi^{p_1+p_2}(\iota,s)(\psi(s)-\psi(0))^{1-\gamma} \\ &\times\{|w(s)-v(s)| + |w(\sigma(s)) - v(\sigma(s))|\}\, ds \\ &+ \frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathscr{N}_\psi^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1} \\ &\times(\psi(s)-\psi(0))^{1-\gamma}|w(s)-v(s)|\, ds + (\psi(\iota)-\psi(0))^{1-\gamma}\varepsilon\lambda_\phi\phi(\iota). \end{aligned}$$

Then, if $M=\max\left\{m(\psi(b)-\psi(0))^{1-\gamma},\lambda_\phi(\psi(b)-\psi(0))^{1-\gamma}\right\}$, we get

$$\begin{aligned}|(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))| \leq & M\varepsilon(\theta+\phi(\iota))+\sum_{i=1}^{m}\mathscr{L}_{I_i}(\psi(\iota_i)-\psi(0))^{1-\gamma}\left|w\left(\iota_i^-\right)-v\left(\iota_i^-\right)\right| \\ & +\frac{\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_\psi^{p_1+p_2}(\iota,s)(\psi(s)-\psi(0))^{1-\gamma} \\ & \times\{|w(s)-v(s)|+|w(\sigma(s))-v(\sigma(s))|\}\,ds \\ & +\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_\psi^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1} \\ & \times(\psi(s)-\psi(0))^{1-\gamma}|w(s)-v(s)|\,ds.\end{aligned} \tag{4.8}$$

And now for every $z\in C([-r,b],\mathbb{R}_+)$, we define $\mathscr{T}:C([-r,b],\mathbb{R}_+)\to C([-r,b],\mathbb{R}_+)$ as

$$(\mathscr{T}z)(\iota)=\begin{cases}0, & \iota\in[-r,0],\\ M\varepsilon(\theta+\phi(\iota))+\sum_{i=1}^{m}\mathscr{L}_{I_i}(z(\iota_i)) & \\ +\frac{\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{(1-\mathscr{L}_f)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_\psi^{p_1+p_2}(\iota,s)\left(z(s)+z(\sigma(s))\right)ds & \\ +\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_\psi^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}z(s)ds, & \iota\in J.\end{cases} \tag{4.9}$$

We show that $\mathscr{T}$ is a Picard operator. Let $z,w\in C([-r,b],\mathbb{R}_+)$. Then,

$$\|\mathscr{T}z-\mathscr{T}w\|_C=0. \tag{4.10}$$

Further, for any $\iota\in J$, we have

$$\begin{aligned}|(\mathscr{T}z)(\iota)-(\mathscr{T}w)(\iota)| \leq & \sum_{i=1}^{m}\mathscr{L}_{I_i}\left|z\left(\iota_i^-\right)-w\left(\iota_i^-\right)\right| \\ & +\frac{\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_\psi^{p_1+p_2}(\iota,s)\times\{|z(s)-w(s)|+|z(\sigma(s))-w(\sigma(s))|\}\,ds \\ & +\frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_\psi^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times|z(s)-w(s)|\,ds.\end{aligned}$$

Then

$$\begin{aligned}|(\mathscr{T}z)(\iota)-(\mathscr{T}w)(\iota)| \leq & m\mathscr{L}_I\|z-w\|_c \\ & +\frac{2\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\|z-w\|_c\int_0^{\iota}\mathscr{N}_\psi^{p_1+p_2}(\iota,s)ds \\ & +\frac{|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\|z-w\|_c\int_0^{\iota}\mathscr{N}_\psi^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}ds \\ \leq & m\mathscr{L}_I\|z-w\|_c+\frac{2\mathscr{L}(\psi(\iota)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}\|z-w\|_c\times(\psi(\iota)-\psi(0))^{p_1+p_2} \\ & +\frac{\Gamma(\gamma)|\lambda|(\psi(\iota)-\psi(0))^{1-\gamma}}{\Gamma(p_2+\gamma)}\|z-w\|_c\times(\psi(\iota)-\psi(0))^{p_2+\gamma-1} \\ \leq & \left[m\mathscr{L}_I+\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}+\frac{\Gamma(\gamma)|\lambda|(\psi(b)-\psi(0))^{p_2}}{\Gamma(p_2+\gamma)}\right]\|z-w\|_C.\end{aligned}$$

Therefore,

$$\|\mathscr{T}z-\mathscr{T}w\|_C=\sup_{\iota\in J}|(\mathscr{T}z)(\iota)-(\mathscr{T}w)(\iota)|\leq\mathscr{K}\|z-w\|_C.$$

By $\mathscr{K} < 1$, the operator $\mathscr{T}$ is a contraction mapping. According to [26, Theorem 2.1], we obtain that $\mathscr{T}$ is Picard operator and $\mathscr{F}_{\mathscr{T}} = z^*$. Then for all $\iota \in [-r, b]$,

$$\begin{aligned} z^*(\iota) =& M\varepsilon\left(\theta+\phi(\iota)\right)+\sum_{0<\iota_i<\iota}\mathscr{L}_{I_i}z^*(\iota_i) \\ &+\frac{\mathscr{L}\left(\psi(b)-\psi(0)\right)^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)\times\left(z^*(s)+z^*(\sigma(s)\right)ds \\ &+\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times z^*(s)ds. \end{aligned} \tag{4.11}$$

Next, we prove that z^* is increasing. Take $\iota_1, \iota_2 \in [-r, b]$ with $\iota_1 < \iota_2$. Then for $\iota_1, \iota_2 \in [-r, 0]$, we have $z^*(\iota_2) - z^*(\iota_1) = 0$. Further, for $0 < \iota_1 < \iota_2 \le b$. Define $N_1 = \min_{s\in[0,b]}\left(z^*(s) + z^*(\sigma(s))\right)$ and $N_2 = \min_{s\in[0,b]} z^*(s)$, we have

$$\begin{aligned} z^*(\iota_2)-z^*(\iota_1) =& M\varepsilon\left(\theta+\phi(\iota_2)\right)-M\varepsilon\left(\theta+\phi(\iota_1)\right)+\sum_{0<\iota_i<\iota_2}\mathscr{L}_{I_i}z^*(\iota_i)-\sum_{0<\iota_i<\iota_1}\mathscr{L}_{I_i}z^*(\iota_i) \\ &+\frac{\mathscr{L}\left(\psi(b)-\psi(0)\right)^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\left(\int_0^{\iota_2}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)\left(z^*(s)+z^*(\sigma(s))\right)ds\right. \\ &\left.-\int_0^{\iota_1}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)\left(z^*(s)+z^*(\sigma(s))\right)ds\right) \\ &+\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\left(\int_0^{\iota_2}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times z^*(s)ds\right. \\ &\left.-\int_0^{\iota_1}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times z^*(s)ds\right) \\ \ge& M\varepsilon\left(\phi(\iota_2)-\phi(\iota_1)\right)+\sum_{0<\iota_i<\iota_2-\iota_1}\mathscr{L}_{I_i}z^*(\iota_i)+\frac{N_1\mathscr{L}\left(\psi(b)-\psi(0)\right)^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\left(\int_0^{\iota_2}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)ds\right. \\ &\left.-\int_0^{\iota_1}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)ds\right) \\ &+\frac{N_2|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\left(\int_0^{\iota_2}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}ds\right. \\ &\left.-\int_0^{\iota_1}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}ds\right) \\ \ge& \sum_{0<\iota_i<\iota_2-\iota_1}\mathscr{L}_{I_i}z^*(\iota_i)+\frac{N_1\mathscr{L}\left(\psi(b)-\psi(0)\right)^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2+1)}\left(\left(\psi(\iota_2)-\psi(0)\right)^{p_1+p_2}-\left(\psi(\iota_1)-\psi(0)\right)^{p_1+p_2}\right) \\ &+\frac{N_2|\Gamma(\gamma)\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2+\gamma)}\left(\left(\psi(\iota_2)-\psi(0)\right)^{p_2+\gamma-1}-\left(\psi(\iota_1)-\psi(0)\right)^{p_2+\gamma-1}\right) \\ >& 0. \end{aligned}$$

Therefore, The operator z^* is increasing. Since $\sigma(\iota) \le \iota, z^*(\sigma(\iota)) \le z^*(\iota), \iota \in J$. By (4.11), we get

$$\begin{aligned} z^*(\iota) \le& M\varepsilon\left(\theta+\phi(\iota)\right)+\sum_{0<\iota_i<\iota}\mathscr{L}_{I_i}z^*(\iota_i) \\ &+\frac{2\mathscr{L}\left(\psi(b)-\psi(0)\right)^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_1+p_2}(\iota,s)z^*(s)ds \\ &+\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^{\iota}\mathscr{N}_{\psi}^{p_2}(\iota,s)(\psi(s)-\psi(0))^{\gamma-1}\times z^*(s)ds. \end{aligned} \tag{4.12}$$

As $0<\gamma<1$, then $(\psi(s)-\psi(0))^{\gamma-1}<1$. So, (4.12) reduce to

$$\begin{aligned} z^*(\iota) \leq & M\varepsilon\left(\theta+\phi(\iota)\right)+\sum_{0<\iota_i<\iota}\mathscr{L}_{I_i}z^*(\iota_i) \\ &+\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}\int_0^\iota \mathscr{N}_\psi^{p_1+p_2}(\iota,s)z^*(s)ds \\ &+\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}\int_0^\iota \mathscr{N}_\psi^{p_2}(\iota,s)z^*(s)ds. \end{aligned} \tag{4.13}$$

Using lemma 2.11, with

$$\begin{aligned} &w(\iota)=z^*(\iota), \quad v(\iota)=M\varepsilon\left(\theta+\phi(\iota)\right) \\ &g_1(\iota)=\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)\Gamma(p_1+p_2)}, \quad g_2(\iota)=\frac{|\lambda|(\psi(b)-\psi(0))^{1-\gamma}}{\Gamma(p_2)}, \quad \beta=\mathscr{L}_I, \end{aligned}$$

we obtain

$$\begin{aligned} & z^*(\iota)\leq M\varepsilon\left(\theta+\phi(\iota)\right) \\ & \times\left(1+\mathscr{L}_I\left[E_{p_1+p_2}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)}(\psi(\iota)-\psi(0))^{p_1+p_2}\right)+E_{p_2}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma}(\psi(\iota)-\psi(0))^{p_2}\right)\right]\right)^k \\ & \times\left[E_{p_1+p_2}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma}}{\left(1-\mathscr{L}_f\right)}(\psi(\iota)-\psi(0))^{p_1+p_2}\right)+E_{p_2}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma}(\psi(\iota)-\psi(0))^{p_2}\right)\right] \\ \leq & M\varepsilon\left(\theta+\phi(\iota)\right)\times\left(1+\mathscr{L}_I\left[E_{p_1+p_2}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)}\right)+E_{p_2}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma+p_2}\right)\right]\right)^k \\ & \times\left[E_{p_1+p_2}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)}\right)+E_{p_2}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma+p_2}\right)\right] \\ \leq & c_{f,m,\phi}\varepsilon\left(\theta+\phi(\iota)\right), \quad \iota\in J, \end{aligned}$$

where

$$\begin{aligned} c_{f,m,\phi}= & M\left(1+\mathscr{L}_I\left[E_{p_1+p_2,\psi}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)}\right)+E_{p_2,\psi}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma+p_2}\right)\right]\right)^k \\ & \times\left[E_{p_1+p_2,\psi}\left(\frac{2\mathscr{L}(\psi(b)-\psi(0))^{1-\gamma+p_1+p_2}}{\left(1-\mathscr{L}_f\right)}\right)+E_{p_2,\psi}\left(|\lambda|(\psi(b)-\psi(0))^{1-\gamma+p_2}\right)\right]. \end{aligned}$$

Specifically, when $z(\iota)=(\psi(\iota)-\psi(0))^{1-\gamma}|w(\iota)-v(\iota)|$, using (4.8), we obtain $z\leq\mathscr{T}(z)$, where the Picard operator $\mathscr{T}$ is increasing. Next, applying Lemma 2.9, we get to $z\leq z^*$.Therefore, it follows

$$\left|(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))\right|\leq c_{f,m,\phi}\varepsilon(\phi(\iota)+\theta), \quad \iota\in J. \tag{4.14}$$

Thus, problem (1.1) is Ulam-Hyers-Rassias stable. □

Remark 4.2. *As a consequences of theorem 4.1, we can obtain the Ulam–Hyers stability (U-H). While ϕ is increasing function for any $\iota\in J$, the inequality (4.14) reduce to*

$$\left|(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))\right|\leq c_{f,m,\phi}\varepsilon(\phi(b)+\theta), \quad \iota\in J.$$

Therefore

$$\left|(\psi(\iota)-\psi(0))^{1-\gamma}(w(\iota)-v(\iota))\right|\leq c_f\varepsilon, \quad \iota\in J,$$

where $c_f=c_{f,m,\phi}(\phi(b)+\theta)$, and problem (1.1) is Ulam-Hyers stable.

5. Example

Example 5.1. *Taking the following problem:*

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)=\dfrac{(\psi(\iota)-\psi(0))^2\left(1+|w(\iota)|+|w(\iota-\frac{1}{2})|\right)}{30e^{(\psi(\iota)-\psi(0))^2+2}(|w(\iota)|+|w(\iota-\frac{1}{2})|)} \\ +\dfrac{1}{43\left(1+\left|{}^{H}\mathfrak{D}_{0^+,\iota}^{p_1,q;\psi}\left({}^{H}\mathfrak{D}_{0^+,\iota}^{p_2,q;\psi}+\lambda\right)w(\iota)\right|\right)}, \quad \iota\in(0,1]\backslash\left\{\dfrac{1}{2}\right\}, \\ I_1\left(w\left(\dfrac{1}{2}^-\right)\right)=\dfrac{1+|w\left(\frac{1}{2}^-\right)|}{11|w\left(\frac{1}{2}^-\right)|}, \\ \mathfrak{I}_{0^+,\iota}^{1-\gamma;\psi}w(0)=1, \quad \gamma=\beta+(p_1+p_2)(1-q), \\ w(\iota)=0, \quad \iota\in[-1,0]. \end{cases} \tag{5.1}$$

Define $f:(0,1]\times\mathbb{R}^3\to\mathbb{R}$ by

$$f(\iota,w,v,u)=\frac{(\psi(\iota)-\psi(0))^2(1+|w|+|v|)}{30e^{(\psi(\iota)-\psi(0))^2+2}(|w|+|v|)}+\frac{1}{43(1+|u|)},$$

and $I_1:\mathbb{R}\to\mathbb{R}$ by

$$I_1(u)=\frac{1+|u|}{11|u|}.$$

For $\iota\in(0,1]$, we have

$$\begin{aligned} &|f(\iota,w_1,v_1,u_1)-f(\iota,w_2,v_2,u_2)| \\ &\le\frac{(\psi(\iota)-\psi(0))^2}{30e^{(\psi(\iota)-\psi(0))^2+2}}\left|\frac{1+|w_1|+|v_1|}{|w_1|+|v_1|}-\frac{1+|w_2|+|v_2|}{|w_2|+|v_2|}\right|+\frac{1}{43}\left|\frac{1}{1+|u_1|}-\frac{1}{1+|u_2|}\right| \\ &\le\frac{(\psi(\iota)-\psi(0))^2}{30e^2}(|w_1-w_2|+|v_1-v_2|)+\frac{1}{43}|u_1-u_2| \\ &\le\frac{(\psi(\iota)-\psi(0))^{\gamma+1}}{30e^2}(\psi(\iota)-\psi(0))^{1-\gamma}(|w_1-w_2|+|v_1-v_2|)+\frac{1}{43}|u_1-u_2| \\ &\le\frac{(\psi(1)-\psi(0))^{\gamma+1}}{30e^2}(\psi(\iota)-\psi(0))^{1-\gamma}(|w_1-w_2|+|v_1-v_2|)+\frac{1}{43}|u_1-u_2|. \end{aligned}$$

This implies that f satisfies (H_1) with $\mathscr{L}=\frac{(\psi(1)-\psi(0))^{\gamma+1}}{30e^2}$, and $\mathscr{L}_f=\frac{1}{43}$.
Also,

$$|I_1(w)-I_1(v)|=\frac{1}{11}\left(\left|\frac{1+|w|}{|w|}-\frac{1+|v|}{|v|}\right|\right)\le\frac{1}{11}|w-v|.$$

Therfore, (H_2) is satisfied with $\mathscr{L}_{I_1}=\mathscr{L}_I=\frac{1}{11}$.
Now, take $p_1=\frac{1}{2},p_2=\frac{1}{4},q=1,\lambda=\frac{1}{2}$ and $\psi(\iota)=\iota^2$. Then $\gamma=1$, and $\mathscr{L}=\frac{1}{30e^2}$.
As $m=1$, we have $\mathscr{K}:=\frac{1}{11}+\frac{2\times\frac{1}{30e^2}}{(1-\frac{1}{43})\Gamma(\frac{3}{4}+1)}+\frac{\frac{1}{2}}{\Gamma(\frac{1}{4}+1)}=0,652593<1$ and (H_3) is satisfied.

Furthermore, by selecting $\phi(\iota)=\iota^2$, for any $\iota\in(0,1]$, we have

$$I_{0+,\iota}^{p_1+p_2;\psi}\phi(\iota)=I_{0+,\iota}^{\frac{3}{4};\iota^2}\phi(\iota)=\frac{16}{21\Gamma(\frac{3}{4})}\iota^{\frac{7}{2}}=\frac{16}{21\Gamma(\frac{3}{4})}\iota^{\frac{3}{2}}\phi(\iota)\le\frac{16}{21\Gamma(\frac{3}{4})}\phi(\iota).$$

By setting $\lambda_\phi=\frac{16}{21\Gamma(\frac{3}{4})}$, we get (H_4). So all conditions of theorem 4.1 are satisfied. Hence, (5.1) has a unique solution and is Ulam-Hyers-Rassias stable.

6. Conclusion

During this paper, we have examined the existence and uniqueness of a class of fractional implicit ψ-Hilfer Langevin equations with time delay and impulsive. The obtained results are proven using Banach's fixed-point theorem. Additionally, the Ulam-Heyrs-Rassias stability for problem (1.1) is considered via a novel form of generalized Gronwall inequality and Picard operator theory. Finally, we provide an example to show how the theoretical findings stated previously are valid.

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An Improved Quantitative Optional Randomised Response Technique with Additive Scrambling using Two Questions Approach

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Abstract
In this paper, an improved two-stage and three-stage optional randomized response (ORR) models for quantitative variables that make the use of additive scrambling was proposed. These two-stage and three-stage models achieve efficient estimation of the mean and sensitivity level simultaneously in the single sample by using two questions. It is found that the proposed models perform better than the existing ORR models in terms of estimating sensitive attribute and sensitivity level simultaneously. It is found that the proposed three stage ORR model provides better estimates than the two-stage and one-stage ORR models and offers more privacy to the respondents with suitable choice of design parameters. The properties of the proposed models are demonstrated with the help of a numerical study.

Keywords: Optional randomised response, Sensitive surveys, Sensitivity level, Two-questions approach
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1. Introduction

In the realm of research, assessing the sensitive characteristics poses a unique challenge. Questions about illegal activities, stigmatized behaviours, or private beliefs often fall prey to biased responses, fuelled by factors like social desirability or fear of judgment. This is where the randomized response technique (RRT) emerges as a beacon of hope, offering a robust method to collect accurate data while preserving the privacy of the respondent. First proposed by Warner [1], RRT is a statistical method designed to introduce controlled randomness into the response of sensitive questions. Different modifications of Warner's [1] original method have been developed and empirically applied to different situations concerning the sensitive data. While RRT has revolutionized data collection in sensitive surveys, limitations arise when quantitative information is required. To handle quantitative data, Warner [2] proposed randomised response (RR) model under additive scrambling in which a random number is added to the response and the response so obtained is known as scrambled response. Pollock and Bek [3] investigated and compared additive and multiplicative RR models including Greenberg et al. [4] model. Multiplicative RR model was discussed in detail by Eichhorn and Hayre [5] which was earlier briefly discussed by Pollock and Bek [3]. This contributed to understanding the statistical implications of applying RRT to quantitative data.

Gupta et al. [6] introduced an optional randomized response technique (ORRT) where the respondents decide themselves whether they want to tell the truth (or scramble their true response) depending upon whether the question being asked is perceived by them as nonsensitive (or sensitive). The proportion of respondents who consider the question sensitive is called the sensitivity level (usually denoted by ω) of the question. The Gupta et al. [6] model used multiplicative scrambling as proposed by Eichhorn and Hayre [5] to estimate the mean of a sensitive variable. Gupta et al. [7] introduced additive scrambling in optional randomised response (ORR) model. Gupta et al. [8] and Mehta et al. [9] further improved the one stage additive scrambling ORR models by extending it to the two stage and three stage models respectively. Huang [10] used linear combination of scrambling variables to scramble the response under ORR model. Gupta et al. [11] observed that the ORR model under additive scrambling performs better than the ORR model under linear combination scrambling. In an ORR model, there are two parameters of interest: the sensitivity level of the question and prevalence of the sensitive characteristic in the population. In the above discussed ORR models, split sample approach is used to estimate the mean of sensitive variable and sensitivity level of the research question. However, the split sample approach requires a larger total sample size for estimation.

Gupta et al. [12] estimated the finite population mean and sensitivity level using ORR model in the presence of nonsensitive auxiliary information from a single sample. Tiwari and Mehta [13] proposed an improved methodology for ORR models in which the sensitivity level (ω) was considered to be known and the RRT was applied only for those respondents who considered the particular question a sensitive one. Tiwari and Mehta [14] also proposed an improved ORRT for quantitative variable.

In ORRT, the approach to estimate the unknown sensitivity level of the main research question by means of using RRT is called two-questions approach [15]. In this approach, all respondents are asked two separate questions. The question about sensitivity level of the sensitive question is asked first via randomization device. In this randomization process, the question is “Is the main research question sensitive?” This question can be denoted by Question no. 1. It can be asked along with an unrelated innocuous question. The underlying sensitivity level and its variance can be estimated from the sample by using any binary RRT. The main research question is denoted by Question no. 2, where respondent answer the question using second randomisation device. The two-questions approach eliminates the need of a split sample to estimate the mean and sensitivity level separately. The two-questions approach also increases the precision of the estimate of sensitivity level.

Sihm et al. [16] used two-questions approach to estimate the sensitivity level when using unrelated ORRT. Chhabra et al. [17] extended this method to the multi stage unrelated ORRT. In a similar way, Kalucha et al. [18] used two-questions approach to estimate sensitivity level and estimated the mean of the sensitive variable by using one stage additive scrambling ORRT propounded by Gupta et al. [7].

Narjis and Shabir [15] proposed three unrelated ORR models under two-questions approach to simultaneously estimate the proportion of sensitive attribute and sensitivity level. In addition, Narjis and Shabir [19] also proposed a multi-question approach to estimate the proportion of sensitive attribute and sensitivity level when an unrelated innocuous attribute is unknown. Recently, Gupta et al. [20] addressed lack of trust in RRTs by proposing an optional enhanced trust (OET) model for quantitative RRT. In OET model, respondents can choose between revealing their true answer or using a scrambling technique either proposed by Warner [2] or Diana and Perry [21] based on their trust in the respective RR model. The OET model introduces three unknowns: mean of sensitive variable, sensitivity level (ω), and trust level (A).

Azeem et al. [22] simplify the OET model by assuming sensitivity level and trust level to be known. However, it is imperative to acknowledge the inherent limitation of this assumption as the sensitivity level and trust level are rarely, if ever, truly known. For a comprehensive summary of RRT, one may refer to Fox and Tracy [23], Chaudhury et al. [24], and Le et al. [25].

There are several models to scramble the quantitative response in RRT such as additive scrambling, multiplicative scrambling, linear combination of scrambling variables etc. However, in this paper we restrict ourselves to additive scrambling in ORRT. In ORR models under additive scrambling in two-stage [8] and three-stage [9], variance of the estimate of sensitivity level inflates as the second stage and third stage probability increases. In addition, these two models used split-sample approach to estimate the prevalence of sensitive characteristics and the sensitivity level. This negatively impacts the estimation of sensitivity level and requires a larger total sample size.

To overcome these limitations in two-stage and three-stage ORR models, in this paper, we propose two improved two-stage and three-stage ORR models under additive scrambling. The proposed models estimate the prevalence of the sensitive characteristic and the sensitivity level of the main research question from two different sets of responses from the same sample. The proposed models are compared with the existing ORR models using additive scrambling. In Section 2, the ORR models using split sample approach discussed by Gupta et al. [8] and Mehta et al. [9] and ORR model using two-questions approach given by Kalucha et al. [18] are discussed in brief. Section 3 deals with the proposed improved two-stage and three stage ORR models using additive scrambling. In Section 4, the efficiency of the proposed ORR models is compared with the existing ORR models using a numerical study. Privacy protection of the proposed ORR models is discussed in Section 5, followed by the conclusion of the study in Section 6.

2. Brief Description of Quantitative ORRT

Let μ and σ^2 be the unknown mean and variance of the sensitive variable X and S is a scrambling variable (independent of X) with known mean θ and known variance σ_s^2. Let ω be the unknown sensitivity level of the survey question in the population. Under these assumptions, a brief discussion of Gupta et al. [8], Mehta et al. [9] and Kalucha et al. [18] models are as follows:

2.1 The split sample spproach - Gupta et al. and Mehta et al. models

In Gupta et al. [8] two stage ORR model, a known proportion (T) of the respondents provide truthful response to the sensitive question. From the remaining known proportion of respondents $(1-T)$, an unknown proportion (ω) provides scrambled responses and the rest unknown proportion $(1-\omega)$ provide truthful responses to the question. To estimate the mean of sensitive variable (X) and sensitivity level (ω), the sample size n is split into two sub-samples with sizes n_1 and n_2, respectively. Under this model, reported responses $(Z_i;\ i=1,2)$ in the two sub-samples are given by,

$$Z_i = \begin{cases} X \text{ with probability } T+(1-T)(1-\omega), \\ (X+S_i) \text{ with probability } (1-T)\omega, \end{cases} \qquad i=1,2$$

Here, $S_i,\ i=1,\ 2$, are independent scrambling variables, independent of X. The unbiased estimators of the mean of sensitive variable and sensitivity level respectively from the sub-samples are given by,

$$\widehat{\mu_G} = \frac{\theta_1\bar{z}_2-\theta_2\bar{z}_1}{\theta_1-\theta_2} \quad \text{and} \quad \widehat{\omega_G} = \frac{\bar{z}_1-\bar{z}_2}{(1-T)(\theta_1-\theta_2)}, \quad \theta_1 \neq \theta_2.$$

Here, $\bar{z}_1$ and $\bar{z}_2$ respectively are the sample mean of reported responses in the two sub-samples. The variances of these estimators are given by,

$$Var\left(\widehat{\mu_G}\right) = \frac{1}{(\theta_1-\theta_2)^2}\left(\theta_2^2\frac{\sigma_{Z_1}^2}{n_1}+\theta_1^2\frac{\sigma_{Z_2}^2}{n_2}\right) \text{and } Var\ \left(\widehat{\omega_G}\right) = \frac{1}{(1-T)^2(\theta_1-\theta_2)^2}\left(\frac{\sigma_{Z_1}^2}{n_1}+\frac{\sigma_{Z_2}^2}{n_2}\right),$$

where $\sigma_{Z_i}^2 = \sigma^2+(1-T)\,\omega\sigma_{S_i}^2+(1-T)\,\omega\,\{1-(1-T)\,\omega\}\,\theta_i^2;\ i=1,\ 2.$

Mehta et al. [9] extended the two-stage ORR model of Gupta et al. [8] to three stages. In three-stage ORRT, in each sub sample a fixed predetermined proportion (T) of respondents is instructed to tell the truth and a fixed predetermined proportion (F) of respondents is instructed to scramble their response. The remaining proportion $(1-T-F)$ of respondents have an option of scrambling their responses additively if they consider the question to be sensitive, or else they can report their true response X. For $F=0$, the model is same as the Gupta et al. [8] model. For both T and F equal to zero, the model is same as the Gupta et al. [7] model. The reported response in the sub-samples is given by,

$$Z_i = \begin{cases} X \text{ with probability } T+(1-T-F)(1-\omega), \\ (X+S_i) \text{ with probability } F+(1-T-F)\,\omega, \end{cases} \qquad i=1,2.$$

The unbiased estimators of the mean of sensitive variable and sensitivity level from the sub-samples are given by,

$$\widehat{\mu_M} = \frac{\theta_1\bar{z}_2-\theta_2\bar{z}_1}{\theta_1-\theta_2} \text{ and } \widehat{\omega_M} = \frac{1}{(1-T-F)}\left(\frac{\bar{z}_1-\bar{z}_2}{(\theta_1-\theta_2)}-F\right),\ \theta_1\neq\theta_2,\ T+F\neq 1.$$

Here, $\bar{z}_1$ and $\bar{z}_2$, respectively are the sample mean of reported responses in the two sub-samples. The variances of these estimators are given by,

$$Var\left(\widehat{\mu_M}\right) = \frac{1}{(\theta_1-\theta_2)^2}\left(\theta_2^2\frac{\sigma_{Z_1}^2}{n_1}+\theta_1^2\frac{\sigma_{Z_2}^2}{n_2}\right)$$

and

$$Var\ \left(\widehat{\omega_M}\right) = \frac{1}{(1-T-F)^2(\theta_1-\theta_2)^2}\left(\frac{\sigma_{Z_1}^2}{n_1}+\frac{\sigma_{Z_2}^2}{n_2}\right).$$

Here for $i=1,\ 2$, $\sigma_{Z_i}^2 = \sigma^2+\{F+(1-T-F)\,\omega\}\,\sigma_{S_i}^2+\{F+(1-T-F)\,\omega\}\,[1-\{F+(1-T-F)\,\omega\}]\,\theta_i^2.$

2.2 The two-questions approach- Kalucha et al. model

In this model, from sample of size n, a proportion $(1-\omega)$ of respondents truthfully answer the sensitive question (main research question) directly while the remaining proportion (ω) scramble their responses additively. Here, the underlying sensitivity level ω and its variance are estimated by using the Greenberg et al. [26] model. Hence, the reported quantitative response is given by,

$$Z = \begin{cases} X \text{ with probability } (1-\omega), \\ (X+S) \text{ with probability } \omega, \end{cases}$$

Here, S is scrambling variable independent of X. The unbiased estimators of the mean of sensitive variable and sensitivity level respectively from the sample are given by,

$$\widehat{\mu_K} = \bar{z} - \widehat{\omega_K}\theta \text{ and } \widehat{\omega_K} = \frac{\widehat{P_y} - (1-P)\pi}{P}.$$

Here, $\bar{z}$ is the sample mean of reported quantitative responses in the sample obtained by asking main research question, $\widehat{P_y}$ is the proportion of 'yes' responses in the sample from first question (viz., Is the main research question sensitive?), P and π respectively are design parameters of Greenberg et al. [26] model to estimate sensitivity level ω. The variances of estimators proposed by Kalucha et al. [18] are given by,

$$Var\left(\widehat{\mu_K}\right) = \frac{\sigma_Z^2}{n} + \theta^2 \frac{P_y(1-P_y)}{nP^2} \text{ and } Var\left(\widehat{\omega_K}\right) = \frac{P_y(1-P_y)}{nP^2},$$

where $\sigma_Z^2 = \sigma^2 + \omega\sigma_S^2 + \omega(1-\omega)\theta^2$ and $P_y = P\omega + (1-P)\pi$.

3. The Proposed Two-Stage and Three-Stage Improved ORR Models

In the proposed two-stage and three-stage models, all respondents are asked two separate questions. The question to estimate the sensitivity level is asked first via randomization device 1. In this randomization process, the question is "Is the main research question sensitive?" It is asked along with an unrelated innocuous question. The underlying sensitivity level ω and its variance are estimated by using the Greenberg et al. [26] model. Let π be the known probability of the binary innocuous unrelated question and P be the known probability of the respondent selecting Question no. 1. The probability of getting "yes" response to the Question no. 1 is $P_y = P\omega + (1-P)\pi$. Solving for ω, we get,

$$\omega = \frac{P_y - (1-P)\pi}{P}.$$

Thus, the unbiased estimate of ω, as per the Greenberg et al. [26] model is given by,

$$\hat{\omega} = \frac{\widehat{P_y} - (1-P)\pi}{P}, \tag{3.1}$$

where $\widehat{P_y}$ is the proportion of 'yes' responses in the sample. The variance of the estimator is given by,

$$Var\ (\hat{\omega}) = \frac{P_y(1-P_y)}{nP^2}.$$

3.1 Two stage improved ORR model

In the same sample, to answer Question no. 2 (main research question), a known proportion (T) of the respondents provide truthful response. From the remaining known proportion of respondents $(1-T)$, an unknown proportion (ω) provides scrambled responses and the rest unknown proportion $(1-\omega)$ provide truthful responses to the main research question. Therefore, the reported quantitative response Z to the main research question according to two-stage ORR model is given by,

$$Z = \begin{cases} X \text{ with probability } T + (1-T)(1-\omega), \\ (X+S) \text{ with probability } (1-T)\omega, \end{cases}$$

The mean and variance of Z respectively are given by,

$$E(Z) = \{T + (1-T)\ (1-\omega)\}E(X) + \{(1-T)\ \omega\}E(X+S)$$
$$E(Z) = E(X) + \{(1-T)\ \omega\}E(S)$$

$$E(Z) = \mu + \omega(1-T)\theta \tag{3.2}$$

and

$$Var\ (Z) = \sigma_Z^2 = E\left(Z^2\right) - \{E(Z)\}^2$$
$$= \{T + (1-T)\ (1-\omega)\}E\left(X^2\right) + \{(1-T)\omega\}E(X+S)^2 - \{E(Z)\}^2$$
$$Var\ (Z) = \sigma^2 + (1-T)\omega\sigma_S^2 + (1-T)\omega\{1-(1-T)\omega\}\theta^2.$$

From equation (3.2), the unbiased estimator of μ under this model (denoted by $\widehat{\mu_1}$) is given by,

$$\widehat{\mu_1} = \bar{z} - (1-T)\theta\hat{\omega},$$

here $\bar{z}$ is sample mean of reported responses and $\hat{\omega}$ is an unbiased estimator of ω given in equation (3.1). The variance of the estimator $\widehat{\mu_1}$ is given by,

$$Var\ \left(\widehat{\mu_1}\right) = Var\ (\bar{z}) + (1-T)^2\theta^2 Var(\hat{\omega})$$
$$Var\ \left(\widehat{\mu_1}\right) = \frac{\sigma_Z^2}{n} + \theta^2(1-T)^2\frac{P_y(1-P_y)}{nP^2}$$

3.2 Three stage improved ORR model

In three stage model, to answer Question no. 2 (main research question) in the same sample, a fixed predetermined proportion (T) of respondents is instructed to tell the truth and a fixed predetermined proportion (F) of respondents is instructed to scramble their response. The remaining proportion $(1-T-F)$ of respondents have an option of scrambling their responses additively if they consider the question to be sensitive, else they can report their true response X. Thus, the reported quantitative response Z to the main research question according to three-stage ORR model is given by,

$$Z = \begin{cases} X \text{ with probability } T + (1-T-F)(1-\omega), \\ (X+S) \text{ with probability } F + (1-T-F)\omega, \end{cases}$$

The mean and variance of Z are given by,

$$E(Z) = \{T + (1-T-F)(1-\omega)\}E(X) + \{F + (1-T-F)\omega\}E(X+S)$$
$$E(Z) = E(X) + \{F + (1-T-F)\omega\}E(S)$$

$$E(Z) = \mu + \{F + (1-T-F)\omega\}\theta \tag{3.3}$$

and

$$Var\ (Z) = \sigma_Z^2 = \{T + (1-T-F)(1-\omega)\}E\left(X^2\right) + \{F + (1-T-F)\omega\}E(X+S)^2 - \{E(Z)\}^2$$
$$\sigma_Z^2 = \sigma^2 + \{F + (1-T-F)\omega\}\sigma_s^2 + \{F + (1-T-F)\omega\}[1 - \{F + (1-T-F)\omega\}]\theta^2.$$

From equation (3.3), the unbiased estimator of μ under this model (denoted by $\widehat{\mu_2}$) is given by,

$$\widehat{\mu_2} = \bar{z} - (1-T-F)\theta\hat{\omega} - F\theta,$$

here $\bar{z}$ is sample mean of reported responses and $\hat{\omega}$ is an unbiased estimator of ω given in equation (3.1). The variance of the estimator $\widehat{\mu_2}$ is given by,

$$Var\ \left(\widehat{\mu_2}\right) = Var\ (\bar{z}) + (1-T-F)^2\theta^2 Var(\hat{\omega})$$
$$Var\ \left(\widehat{\mu_2}\right) = \frac{\sigma_Z^2}{n} + \theta^2(1-T-F)^2\frac{P_y(1-P_y)}{nP^2}$$

Table 1. PRE of proposed estimators of mean (μ) w.r.t Gupta et al. [8][G], Mehta et al. [9][M] and Kalucha et al. [18][K] estimators. $(n = 1000,\ n_1 = n_2 = 500,\ X \sim Poisson\ (4),\ S_1 \sim Poisson\ (2),\ S_2 \sim Poisson\ (5),\ P = 0.70,\ \pi = 0.25)$.

ω	T	F	$PRE\,(\hat{\mu_1},\hat{\mu_G})$	$PRE\,(\hat{\mu_1},\hat{\mu_K})$	$PRE\,(\hat{\mu_2},\hat{\mu_G})$	$PRE\,(\hat{\mu_2},\hat{\mu_M})$	$PRE\,(\hat{\mu_2},\hat{\mu_K})$
0.70	0.55	0.30	682.58	139.78	692.00	735.31	141.71
0.70	0.45	0.40	668.72	130.40	698.43	738.94	136.19
0.70	0.35	0.50	650.07	122.18	706.43	737.56	132.77
0.70	0.25	0.60	627.47	114.91	716.20	731.47	131.16
0.70	0.15	0.70	601.61	108.45	728.04	720.62	131.25
0.70	0.05	0.80	573.01	102.67	742.43	704.57	133.03
0.80	0.55	0.30	690.78	134.98	710.57	736.52	138.84
0.80	0.45	0.40	676.82	126.31	716.55	739.36	133.73
0.80	0.35	0.50	657.52	118.93	722.25	737.25	130.64
0.80	0.25	0.60	633.67	112.57	727.91	730.45	129.31
0.80	0.15	0.70	605.82	107.04	733.73	718.85	129.64
0.80	0.05	0.80	574.41	102.20	739.95	701.96	131.65
0.90	0.55	0.30	699.41	128.44	726.29	737.83	133.38
0.90	0.45	0.40	685.60	120.82	730.55	739.87	128.75
0.90	0.35	0.50	665.87	114.57	732.41	737.04	126.02
0.90	0.25	0.60	640.83	109.40	732.08	729.53	124.98
0.90	0.15	0.70	610.88	105.11	729.60	717.17	125.54
0.90	0.05	0.80	576.21	101.55	724.85	699.42	127.74

4. Efficiency Comparison

The efficiency of the proposed estimators with respect to the estimators suggested by Gupta et al. [8] [G], Mehta et al. [9] [M] and Kalucha et al. [18] [K] is numerically established using the following formula of percent relative efficiency:

$$PRE\,(\widehat{\tau}_i,\widehat{\tau}_j) = \frac{Var\,(\widehat{\tau}_j)}{Var\,(\widehat{\tau}_i)} \times 100;\ \tau = \hat{\mu},\ \hat{\omega};\ i = 1,2;\ j = G,M,K$$

The PRE of the proposed estimators has been computed at various values of model parameters. The results of the numerical study are illustrated in the Table 1 and Table 2. For the numerical analysis, we used distribution of sensitive variable and scrambling variable similar to what is used in Mehta et al. [9]. The distribution of S_1 is used for single sample in proposed ORR models. Table 1 illustrates the PRE of proposed estimator $\hat{\mu}_i$, $(i = 1,2)$ of mean (μ) w.r.t the estimators suggested by Gupta et al. [8], Mehta et al. [9] and Kalucha et al. [18].

It is observed from Table 1 that all the PREs corresponding to the proposed estimators for the mean of sensitive variable, under two-stage and three-stage ORR models are greater than 100. The results indicate that the proposed two-stage and three stage ORR models under two-questions approach are more efficient than ORR models of Gupta et al. [8] and Mehta et al. [9] under split sample approach. Moreover, the proposed two-stage and three-stage ORR models under two questions approach performs better than the one stage ORR model under two-questions approach as suggested by Kalucha et al. [18] for all values of the model parameters. Moreover, it is seen from Table 1 that the proposed three-stage ORR model performs much better than the two-stage and one-stage ORR models under two-questions approach for estimating the mean of sensitive variable. The main focus of the present study is to check whether the estimation of sensitivity level improved under proposed models or not. In this regard, the results concerning the PREs of estimators under proposed models for sensitivity level are demonstrated in Table 2.

It is observed from Table 2 that while comparing with the Gupta et al. [8] and Mehta et al. [9] models, the PRE for the estimators of sensitivity level for the proposed two-stage and three-stage ORR models are significantly higher than 100. A large gain in PRE is observed for proposed ORR models under two-questions approach in comparison to the two-stage and three-stage ORR models under split-sample approach due to the reason that the variance of estimate of sensitivity level inflates as the second stage and third stage probability increases in split samples. However, the two-questions approach used in Kalucha et al. [18] ORR model and in proposed two-stage and three-stage ORR models yields same precision to estimate the sensitivity level but the proposed two-stage and three-stage ORR models also improved the precision of estimate of the mean of sensitive variable under various practical situations (see, Table 1). Hence, the proposed two-stage and three-stage ORR models under two questions approach outperforms the Gupta et al. [8] and Mehta et al. [9] ORR models under split sample approach and Kalucha et al. [18] ORR model under two-questions approach.

Table 2. PRE of proposed estimators of sensitivity level (ω) w.r.t Gupta et al. [8][G], Mehta et al. [9][M] and Kalucha et al. [18][K] estimators. ($n=1000,\ n_1=n_2=500,\ X\sim Poisson\ (4),\ S_1\sim Poisson\ (2),\ S_2\sim Poisson\ (5),\ P=0.70,\ \pi=0.25$).

ω	T	F	$PRE\,(\hat{\omega},\hat{\omega_G})$	$PRE\,(\hat{\omega},\hat{\omega_M})$	$PRE\,(\hat{\omega},\hat{\omega_K})$
0.70	0.55	0.30	3601.77	32415.93	100.00
0.70	0.45	0.40	2572.06	34579.95	100.00
0.70	0.35	0.50	1926.98	36184.35	100.00
0.70	0.25	0.60	1489.17	37229.15	100.00
0.70	0.15	0.70	1174.49	37714.33	100.00
0.80	0.55	0.30	3990.81	35917.27	100.00
0.80	0.45	0.40	2830.57	38055.40	100.00
0.80	0.35	0.50	2099.21	39418.46	100.00
0.80	0.25	0.60	1600.26	40006.45	100.00
0.80	0.15	0.70	1240.05	39819.36	100.00
0.90	0.55	0.30	4608.24	41474.14	100.00
0.90	0.45	0.40	3239.07	43547.47	100.00
0.90	0.35	0.50	2371.29	44527.59	100.00
0.90	0.25	0.60	1776.58	44414.50	100.00
0.90	0.15	0.70	1345.58	43208.20	100.00

5. Privacy Protection

The aspect of privacy protection of respondents is an integral part of the RRT. We examine this aspect for the proposed ORR models. Lanke [27], Yan et al. [28] and Giordano and Perri [29] have discussed this issue in detail. Lanke [27] and Giordano and Perri [29] devised a privacy measure to assess the privacy protection of binary RRTs while Yan et al. [28] derived a privacy measure for the quantitative RRTs. Yan et al. [28] defined the measure of privacy protection as $\nabla=E\,(Z-X)^2$ where X is the true response of the sensitive variable and Z is the reported response. For a given model, the larger the value of ∇, the larger the privacy provided by the model.

The privacy measure for one-stage additive scrambling ORR model is given by $\nabla_1=\left(\theta^2+\sigma_S^2\right)\omega$ and for two-stage additive scrambling ORR model it is given by $\nabla_2=\left(\theta^2+\sigma_S^2\right)\omega\,(1-T)$. Thus, comparing ∇_1 and ∇_2, it is observed that for same precision Gupta et al. [7] model (which is used by Kalucha et al. [18] under two questions approach) is more protective than Gupta et al. [8] model (proposed two-stage improved ORR model). This shows that the proposed two stage ORR model with two-questions approach may be made more protective as compared to Kalucha et al. [18] model with two questions approach, but at the cost of precision. In fact, it is a trade-off between the efficiency and privacy protection. That is, we can have highly efficient estimator by compromising on privacy. Similarly, we can build a more protective model by compromising on the efficiency.

Hussain and Al-Zehrani [30] discussed that Gupta et al. [7] one-stage model is more protective compared to Gupta et al. [8] two-stage model and they argued that Gupta et al. [7] model remain more protective among all other existing ORR models under additive scrambling. This argument is not true in case of Mehta et al. [9] three-stage ORR model. In case of three-stage ORR model due to Mehta et al. [9], the reported response is given by,

$$Z=\begin{cases}X \text{ with probability } T+(1-T-F)(1-\omega),\\ (X+S) \text{ with probability } F+(1-T-F)\omega,\end{cases}$$

then

$$Z-X=\begin{cases}0 \text{ with probability } T+(1-T-F)(1-\omega),\\ S \text{ with probability } F+(1-T-F)\omega,\end{cases}$$

Therefore, the privacy measure in case of the Mehta et al. [9] model is given by,

$$\nabla_3=E\,(Z-X)^2=\{F+(1-T-F)\,\omega\}E(S^2)=\{F+(1-T-F)\,\omega\}\left(\theta^2+\sigma_S^2\right)$$

The privacy comparison of the proposed three-stage ORR model with one-stage and two-stage ORR models can be summarised in the following theorems.

Theorem 5.1. *Three stage ORRT with two questions approach offers more privacy than one stage ORRT with two-questions approach if,* $F\,(1-\omega)>T\omega$.

Proof. Considering the difference of the privacy measures of the models, we observe

$$\begin{aligned}\nabla_3 - \nabla_1 &= \left(\theta^2 + \sigma_S^2\right)\{F + (1-T-F)\,\omega\} - \left(\theta^2 + \sigma_S^2\right)\omega \\ &= \left(\theta^2 + \sigma_S^2\right)(F - F\omega - T\omega) \\ &= \left(\theta^2 + \sigma_S^2\right)\{F(1-\omega) - T\omega\}\end{aligned}$$

For $\nabla_3 - \nabla_1 > 0$, we get $F(1-\omega) > T\omega$. Hence the theorem. □

Theorem 5.2. *Three stage ORRT with two-questions approach always offers more privacy than two stage ORRT with two-questions approach.*

Proof. Considering the difference of the privacy measures of the models, we observe

$$\begin{aligned}\nabla_3 - \nabla_2 &= \left(\theta^2 + \sigma_S^2\right)\{F + (1-T-F)\,\omega\} - \left(\theta^2 + \sigma_S^2\right)\omega(1-T) \\ &= \left(\theta^2 + \sigma_S^2\right)(F - F\omega)\end{aligned}$$

For $\nabla_3 - \nabla_2 > 0$, we get $F > F\omega$ which is always true. This proves the theorem. □

The above results establish the superiority of the proposed three-stage improved ORR model. Hence, our two-questions approach in three-stage ORR model is more protective in comparison to Gupta et al. [8] and Kalucha et al. [18] ORR models. Numerically, from Table 1 it can be observed that for the suitable choice of parameters, the proposed three stage ORR model with two-questions approach performs better in terms of efficiency and at the same time protect the privacy of respondents more than the existing models. For example, when $\omega = 0.80$ taking $T = 0.15$ and $F = 0.70$, the proposed three-stage ORR model has greater precision and the parameter values also satisfy the conditions under Theorem 5.1 and Theorem 5.2. Thus, it may be concluded that if the parameters are chosen carefully, the three-stage quantitative ORR model with two questions approach offers better efficiency and more privacy than the Gupta et al. [8], Mehta et al. [9] and Kalucha et al. [18] ORR models.

6. Conclusion

Using two-questions approach, improved two-stage and three-stage ORR models for quantitative variables have been proposed and their properties are discussed. It is observed from the numerical comparisons that the proposed two-stage and three-stage ORR models using two-questions approach are found to be more efficient as compared to the two-stage and three-stage ORR models using the usual split-sample approach. It is also observed that proposed ORR models can be made more efficient than the existing ORR models by choosing appropriate design parameters. It is also found that the proposed three-stage ORR model under two-questions approach offers more privacy than one-stage and two-stage ORR models.

In addition, the proposed three-stage ORR model using two-questions approach with suitable choices of design parameters performs better than the two-stage and one-stage ORR models and provides more privacy as compared to one stage ORRT using two-questions approach. It is found that there is a significant gain in precision under the two-questions approach to estimate the sensitivity level. Moreover, the precision of estimate of sensitivity level can also be increased by using forced RRT or two-stage RRT.

On the basis of our study, we may conclude that, the proposed three-stage ORR model under the two-questions approach stands out as a particularly valuable tool for surveys that grapple with highly sensitive issues. This model may be highly useful for data collection in areas where respondents might be reluctant to answer truthfully due to fear of judgment or social stigma. This model may be quite relevant in research on illegal activities, stigmatized health conditions, or unpopular opinions. Thus, it may be recommended that to estimate the mean of sensitive variable along with the sensitivity level in a survey concerning sensitive information, the proposed three stage ORR model under two-questions approach is better choice among other existing ORR models.

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A One-Parameter Class of Separable Solutions for An Age-Sex-Structured Population Model with an Infinite Range of Reproductive Ages, A Discrete Set of Offspring, and Maternal Care

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Abstract
A mathematical model based on a discrete newborn set is proposed to describe the evolution of a sex-age-structured population, taking into account the temporary pair of sexes, infinite ranges of reproductive age of sexes, and maternal care of offspring. Pair formation is modeled by a weighted harmonic mean type function. The model is based on the concept of density of families composed of mothers with their newborns. All individuals are divided into the pre-reproductive and reproductive age groups. Individuals of the pre-reproductive class are divided into the newborn and teenager groups. Newborns are under maternal care while the teenagers can live without maternal care but cannot mate. Females of the reproductive age group are divided into singles and those who care for their offspring. The model is composed of a coupled system of integro-partial differential equations. Sufficient conditions for the existence of a one-parameter class of separable solutions of this model are found in the case of stationary vital rates.

Keywords: Age-sex-structured population models, Population models with parental care, Two sex population models
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1. Introduction

The purpose of this work is to analyze a mathematical model for a spatially homogeneous population structured by age and sex taking into account temporary (only for the mating period) pairs of sexes, infinite reproductive age ranges of sexes, a discrete set of offspring, and maternal care of them.

In mathematical biology, the Sharpe-Lotka-McKendrick [1], Fredrickson [2], Hoppensteadt-Staroverov [3], [4], and Hadeler [5] models are well known. The first of them is usually used for the evolution description of the age-structured asexual populations. The model [2] describes two-sex age-structured populations with temporary pairs of sexes. The Hoppensteadt–Staroverov model and its Hadeler modification including a maturation period describe the evolution of age-structured two-sex populations with permanent pairs of sexes. The existence of the separable solutions to model [3], [4], [5] is studied in [6] and [7].

But, all these models do not address the child care phenomenon which is native to many species of mammals and birds. Birds and some species of mammals care for their offspring in pairs. In populations of some species of mammals and fishes only the mother cares for her offspring. Several models (see [8]–[12] and literature there) were proposed for description of child care in two-sex populations with temporary and permanent pairs of sexes. In the first case ([9]), only the mother cares for her offspring. In the second case ([8], [10]-[12]), both parents take care of offspring. Models [8], [10], and[11] are based on the idea of the newborn density which is described by a corresponding PDE. However, a problem arises when describing spatially distributed populations using models of this type, because the equations describing the movement of newborns do not guarantee that they follow the mother or both parents. To overcome this problem, some models have been proposed based on a discrete set of newborns and density of the family (mother-newborns [9] or both parents-newborns [12]). In addition to child care, work [9] takes into account the pregnancy of females. It is also assumed that the reproductive age intervals of males and females in model [9] are finite. To the best of our knowledge, there has been no work in the last decade that has examined the dynamics of the caregiver population.

In the present paper, we revise model [9] by dropping the Environmental pressure and female's pregnancy and contrary to model [9] assume that the age reproductive intervals of both parents are infinite. This is the novelty of the model under consideration. As in [9], all individuals are divided into pre-reproductive and reproductive age groups. Individuals of pre-reproductive class are divided into the newborn and teenager groups. Newborns are under maternal care while the teenagers can live without maternal care but cannot mate. Individuals of the reproductive age class are divided into singles and those who care for their offspring. The goal of this paper is to find sufficient conditions for the existence of separable solutions of the proposed model in the case of stationary vital rates.

The plan of this work is the following: In Section 2, the basic notions are given. In Section 3, we describe the model. Separable solutions are studied in section 4. Some concluding remarks in section 5 conclude the paper.

2. Notations

The following notions are used in this paper:

T, τ_{i*}: child care and maturation period, respectively ($i=1$ for males, $i=2$ for females);

$u_i(t,\tau_i)$: density at time t of individuals of age τ_i ($\tau_i \in (T,\tau_{i*})$ for juveniles, $\tau_i \in (\tau_{i*},\infty)$ for adult individuals, $i=1$ for males, $i=2$ for females);

$u_{2k_1k_2}(t,\tau_1,\tau_2,\tau_3)$: density at time t of females aged τ_2 who take care of k_1 sons and k_2 daughters of age τ_3, born from fathers of age τ_1;

$\nu_i(t,\tau_i)$: mortality at time t of individuals aged τ_i ($i=1$ for males, $i=2$ for females);

$\nu_{2k_1k_2}(t,\tau_1,\tau_2,\tau_3)$: mortality at moment t of mothers aged τ_2 caring for k_1 sons and k_2 daughters of age τ_3, born from fathers of age τ_1;

$\nu_{2k_1k_2;s_1s_2}(t,\tau_1,\tau_2,\tau_3)$: mortality at time t of k_1-s_1 sons and k_2-s_2 daughters of age τ_3, born from fathers of age τ_1 and who are under care of mothers aged τ_2;

$p_i(t,\tau_i)u_i(t,\tau_i)$; density of individuals of age τ_i who wish to mate at time t ($i=1$ for males, $i=2$ for females);

$u_i^0(\tau_i)$: initial density of individuals aged τ_i ($i=1$ for males, $i=2$ for females);

$u^0_{2k_1k_2}(\tau_1,\tau_2,\tau_3)$: initial density of females aged τ_2 who take care of k_1 sons and k_2 daughters aged τ_3;

$|k|=k_1+k_2$, $|s|=s_1+s_2$ with integer valued k_1,k_2,s_1,s_2 where $|k|,|s|=0,1,\dots,n$; $\sum\limits_{|k|=1}^{n} a_{k_1k_2} = \sum\limits_{k_1=0}^{n-1}\sum\limits_{k_2=1}^{n-k_1} a_{k_1k_2}$;

$p(t,\tau_1,\tau_2)\alpha_{2k_1k_2}(t,\tau_1,\tau_2)dt$: probability to produce k_1 sons and k_2 daughters in the time interval $[t,t+dt]$ by a temporal pair formed of a male aged $\tau_1$and female of age τ_2;

$p_{k_1k_2}=p_1\,p_2\,p\,\alpha_{2k_1k_2}$.

$[u_2(t,\tau)]$: jump discontinuity of function u_2 at line $\tau_2=\tau$.

3. The Model

In this section, we present a deterministic model to describe the evolution of a population structured by sex and age. We take into account temporary pairs of sexes, a discrete set of offspring, and maternal care for them. By temporary pairs, we mean pairs that exist during the mating period, duration of which is not taken into account. We use a weighted harmonic mean pair formation function and assume that when a mother dies all offspring under her care die. Using the balance law, we derive the following equations for the dynamic description of a population with a discrete set of offspring:

$$\begin{cases} \partial_t u_1 + \partial_{\tau_1} u_1 + \nu_1 u_1 = 0 \quad \text{in } (0,\infty)\times(T,\infty), \\ u_1|_{\tau_1=T} = \int\limits_{\tau_{1*}}^{\infty} d\tau_1 \int\limits_{\tau_{2*}+T}^{\infty} \sum\limits_{|k|=1}^{n} k_1 u_{2k_1k_2}|_{\tau_3=T}\, d\tau_2 \quad \text{in } [0,\infty), \\ u_1|_{t=0} = u_1^0 \quad \text{in } [T,\infty), \end{cases} \tag{3.1}$$

$$\begin{cases} \partial_t u_2 + \partial_{\tau_2} u_2 + \nu_2 u_2 = S_2^u, \\ u_2|_{\tau_2=T} = \int\limits_{\tau_{1*}}^{\infty} d\tau_1 \int\limits_{\tau_{2*}+T}^{\infty} \sum\limits_{|k|=1}^{n} k_2 u_{2k_1k_2}|_{\tau_3=T}\, d\tau_2 \quad \text{in } [0,\infty), \\ [u_2(t,\tau)] = 0 \quad \text{in } [0,\infty),\ \tau = \tau_{2*}, \tau_{2*}+T, \\ u_2|_{t=0} = u_2^0 \quad \text{in } [T,\infty) \end{cases} \tag{3.2}$$

where

$$S_2^u = \begin{cases} 0 \quad \text{in } (0,\infty)\times(T,\tau_{2*}), \\ \sum\limits_{|k|=0}^{n} \int\limits_{\tau_{1*}}^{\infty} d\tau_1 \left(\int\limits_{0}^{\tau_2-\tau_{2*}} \nu_{2k_1k_2;00} u_{2k_1k_2}\, d\tau_3 - u_{2k_1k_2}|_{\tau_3=0} \right) \quad \text{in } (0,\infty)\times(\tau_{2*},\tau_{2*}+T), \\ \sum\limits_{|k|=0}^{n} \int\limits_{\tau_{1*}}^{\infty} d\tau_1 \left(\int\limits_{0}^{T} \nu_{2k_1k_2;00} u_{2k_1k_2}\, d\tau_3 + u_{2k_1k_2}|_{\tau_3=T} - u_{2k_1k_2}|_{\tau_3=0} \right) \quad \text{in } (0,\infty)\times(\tau_{2*}+T,\infty), \end{cases}$$

$$\begin{cases} \partial_t u_{2k_1k_2} + \sum\limits_{j=1}^{2} \partial_{\tau_j} u_{2k_1k_2} + \left(\nu_{2k_1k_2} + \sum\limits_{|s|=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2} \right) u_{2k_1k_2} = S_{2k_1k_2}^u \quad \text{in } (0,\infty)\times[\tau_{1*},\infty)\times(\tau_{2*}+\tau_3,\infty)\times(0,T), \\ u_{2k_1k_2}|_{\tau_3=0} = \dfrac{p_{k_1k_2} u_1 u_2}{\sum\limits_{j=1}^{2} \int\limits_{\tau_{j*}}^{\infty} p_j u_j\, d\tau_j} \quad \text{in } (0,\infty)\times[\tau_{1*},\infty)\times(\tau_{2*},\infty), \\ u_{2k_1k_2}|_{t=0} = u_{2k_1k_2}^0 \quad \text{in } [\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T] \end{cases} \tag{3.3}$$

where

$$S_{2k_1k_2}^u = \begin{cases} 0, \ |k| = n, \\ \sum\limits_{|s|=|k|+1}^{n} \nu_{2s_1s_2;k_1k_2} u_{2s_1s_2}, \ |k| = n-1, n-2, \dots, 1. \end{cases}$$

We add to this system the following compatibility conditions:

$$u_i^0|_{\tau_i=T} = \int_{\tau_{1*}}^{\infty} d\tau_1 \int_{\tau_{2*}+T}^{\infty} \sum_{|k|=1}^{n} k_i u_{2k_1k_2}^0|_{\tau_3=T}\, d\tau_2, \ i = 1,2,$$

$$u_{2k_1k_2}^0|_{\tau_3=0} = \frac{p_{2k_1k_2}|_{t=0} u_1^0 u_2^0}{\sum\limits_{j=1}^{2} p_j|_{t=0} u_j^0\, d\tau_j} \ \text{in } [\tau_{1*},\infty)\times[\tau_{2*},\infty).$$

4. Separable Solutions

In this section, we study system (3.1)–(3.3) with the vital rates p, p_1, p_2, ν_1, ν_2, $\nu_{2k_1k_2}$, $\nu_{2k_1k_2;s_1s_2}$, $\alpha_{2k_1k_2}$ independent of time t and look for solutions of the form

$$\begin{cases} u_i(t,\tau_i) = \exp\{\lambda t\} w_i(\tau_i), \\ w_i(\tau_i) = a_i v_i(\tau_i),\ v_i(T) = 1, \quad i = 1,2, \\ u_{2k_1k_2} = \exp\{\lambda t\} w_{2k_1k_2}, \\ w_{2k_1k_2} = a_1 a_2 e^{-\lambda\tau_3} v_1(\tau_1) v_2(\tau_2-\tau_3) v_{2k_1k_2}(\tau_1,\tau_2,\tau_3)/\alpha, \quad |k| = 1,2,\dots,n, \\ \quad \alpha = a_1 \int_{\tau_{1*}}^{\infty} p_1 v_1\, d\tau_1 + a_2 \int_{\tau_{2*}}^{\infty} p_2 v_2\, d\tau_2 \end{cases} \tag{4.1}$$

where constants λ, $a_1 = w_1(T)$, $a_2 = w_2(T)$, and functions v_i, $v_{2k_1k_2}$ are to be determined. Set:

$$y_i = a_i/\alpha,\ ||v_i|| = \int_{\tau_{i*}}^{\infty} v_i\,d\tau_i,\ i=1,2,\ \gamma = \sum_{|k|=1}^{n} p_{2k_1k_2},\ \ P = \int_{\tau_{1*}}^{\infty} \gamma v_1\,d\tau_1,$$

$$l_2 = \nu_2 + \lambda + y_1 P,\ l_{2k_1k_2} = \nu_{2k_1k_2} + \sum_{|s|=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2},\ |k| = 1,\dots,n,$$

$$r = \sum_{|k|=1}^{n} v_{2k_1k_2}|_{\tau_3=T},\ R = \int_{\tau_{1*}}^{\infty} r v_1\,d\tau_1,\ q = \sum_{|k|=1}^{n} \nu_{2k_1k_2;00}\, v_{2k_1k_2},\ Q = \int_{\tau_{1*}}^{\infty} q v_1\,d\tau_1,$$

$$\beta_i(x) = \int_{\tau_{1*}}^{\infty} v_1(\tau_1) \sum_{|k|=1}^{n} k_i v_{2k_1k_2}(\tau_1, x+T, T)\,d\tau_1,\ i=1,2.$$

Substituting functions (4.1) into system (3.1)–(3.3) and performing calculations, we get the following equations:

$$\begin{cases} v_1' + (\nu_1+\lambda)v_1 = 0 \text{ in } (T,\infty),\ v_1(T)=1, \\ 1 = y_2 e^{-\lambda T} \int_{\tau_{2*}}^{\infty} v_2(x)\beta_1(x)\,dx, \end{cases} \tag{4.2}$$

$$\begin{cases} v_2' + (\nu_2+\lambda)v_2 = S_2^v,\ v_2(T)=1,\ [u_2(\tau)]=0,\ \tau=\tau_{2*},\tau_{2*}+T, \\ 1 = y_1 e^{-\lambda T} \int_{\tau_{2*}}^{\infty} v_2(x)\beta_2(x)\,dx, \end{cases} \tag{4.3}$$

where

$$S_2^v = \begin{cases} 0 \quad \text{in } (T,\tau_{2*}), \\ y_1\left(\int_0^{\tau_2-\tau_{2*}} v_2(\tau_2-\tau_3)Q(\tau_2,\tau_3)e^{-\lambda\tau_3}\,d\tau_3 - v_2(\tau_2)P(\tau_2)\right) \quad \text{in } (\tau_{2*},\tau_{2*}+T), \\ y_1\left(\int_0^{T} v_2(\tau_2-\tau_3)Q(\tau_2,\tau_3)e^{-\lambda\tau_3}\,d\tau_3 + e^{-\lambda T}v_2(\tau_2-T)R(\tau_2) - v_2(\tau_2)P(\tau_2)\right) \quad \text{in } (\tau_{2*}+T,\infty), \end{cases} \tag{4.4}$$

$$\begin{cases} \sum_{j=2}^{3} \partial_{\tau_j} v_{2k_1k_2} + \left(\nu_{2k_1k_2} + \sum_{|s|=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2}\right) v_{2k_1k_2} = S_{2k_1k_2}^v \quad \text{in } [\tau_{1*},\infty)\times(\tau_{2*}+\tau_3,\infty)\times(0,T) \\ v_{2k_1k_2}|_{\tau_3=0} = p_{2k_1k_2} \quad \text{in } [\tau_{1*},\infty)\times[\tau_{2*},\infty), \end{cases} \tag{4.5}$$

where

$$S_{2k_1k_2}^v = \begin{cases} 0,\ |k| = n, \\ \sum_{|s|=|k|+1}^{n} \nu_{2s_1s_2;k_1k_2} v_{2s_1s_2},\ |k| = n-1, n-2, \dots, 1. \end{cases}$$

We also have the equation for λ,

$$y_1||p_1v_1|| + y_2||p_2v_2|| = 1. \tag{4.6}$$

Integrating Eqs. $(4.2)_1$ and $(4.3)_1$, we get $w_i(\tau_i) = w_i(T)v_i(\tau_i)$ where

$$v_1(\tau_1) = \exp\left\{ -\int_T^{\tau_1} (\nu_1+\lambda)\,ds \right\} \text{ in } [T,\infty),$$

$$v_2(\tau_2) = \exp\left\{ -\int_T^{\tau_2} (\nu_2+\lambda)\,ds \right\} \text{ in } [T,\tau_{2*}].$$

Now we transform Eq. (4.3) with a given positive y_1 into a set of Volterra's type integral equations. To do this, we change variables on the right hand side of Eq. (4.4), then integrate Eq. $(4.3)_1$, and after then change the order of integration. As a result, we have

$$v_2(\tau_2) = f(\tau_2) + \int\limits_{\tau_{2*}+jT}^{\tau_2} G(\tau_2,y)v_2(y)\,dy \text{ in } [\tau_{2*}+jT, \tau_{2*}+(j+1)T] \tag{4.7}$$

with $j = 0,1,2,\ldots$, where

$$G(\tau_2,y) = y_1 \int\limits_y^{\tau_2} \exp\Big\{-\int\limits_z^{\tau_2} l_2(s)\,ds - \lambda(z-y)\Big\} Q(z,z-y)\,dz,$$

$$f(\tau_2) = \begin{cases} v_2(\tau_{2*})\exp\Big\{-\int\limits_{\tau_{2*}}^{\tau_2} l_2\,ds\Big\} \text{ in } [\tau_{2*},\tau_{2*}+T], \\ v_2(\tau_{2*}+jT)\exp\Big\{-\int\limits_{\tau_{2*}+jT}^{\tau_2} l_2\,ds\Big\} + y_1\int\limits_{\tau_{2*}+jT}^{\tau_2} \exp\Big\{-\int\limits_z^{\tau_2} l_2\,ds\Big\}\,dz\Big(\int\limits_{z-T}^{\tau_{2*}+jT} v_2(y)Q(z,z-y)e^{-\lambda(z-y)}\,dy \\ \quad +v_2(z-T)R(z)e^{-\lambda T}\Big) \quad \text{in } [\tau_{2*}+jT,\tau_{2*}+(j+1)T], \end{cases}$$

with $j = 1,2,\ldots$ and

$$v_2(\tau_{2*}) = \exp\Big\{-\int\limits_T^{\tau_{2*}} (\nu_2+\lambda)\,ds\Big\}.$$

Define:

$\nu_{i*} = \inf\limits_{[\tau_{i*},\infty)} \nu_i,\ \nu_i^* = \sup\limits_{[\tau_{i*},\infty)} \nu_i,\ p_{i*} = \inf\limits_{[\tau_{i*},\infty)} p_i, p_i^* = \sup\limits_{[\tau_{i*},\infty)} p_i,$

$\nu_{2k_1k_2*} = \inf\limits_{[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]} \nu_{2k_1k_2},\ p_* = \inf\limits_{[\tau_1*,\infty)\times[\tau_{2*},\infty)} p,$

$\nu_{2k_1k_2}^* = \sup\limits_{[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]} \nu_{2k_1k_2},\ p_* = \sup\limits_{[\tau_1*,\infty)\times[\tau_{2*},\infty)} p,$

$\nu_{2k_1k_2;s_1s_2*} = \inf\limits_{[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]} \nu_{2k_1k_2;s_1s_2},$

$\nu_{2k_1k_2;s_1s_2*} = \sup\limits_{[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]} \nu_{2k_1k_2;s_1s_2},$

$\alpha_{2k_1k_2*} = \inf\limits_{[\tau_{1*},\infty)\times[\tau_2*,\infty)} \alpha_{2k_1k_2},\ \alpha_{2k_1k_2}^* = \sup\limits_{[\tau_{1*},\infty)\times[\tau_2*,\infty)} \alpha_{2k_1k_2},$

$l_{2k_1k_2*} = \nu_{2k_1k_2*} + \sum\limits_{s=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2*},\ l_{2k_1k_2}^* = \nu_{2k_1k_2}^* + \sum\limits_{s=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2}^*,$

$\gamma^* = \sum\limits_{|k|=1}^{n} p_{2k_1k_2}^*,\ \gamma_* = \sum\limits_{|k|=1}^{n} p_{2k_1k_2*},\quad \nu_* = \min(\nu_{1*},\nu_{2*}),$

$l_{2*} = \nu_{2*}+\lambda+\gamma_*||v_1||,\ l_2^* = \nu_2^*+\lambda+\gamma^*||v_1||.$

Consider two functions:

$$\overline{v}_{2k_1k_2}(\tau_3) = \begin{cases} p_{2k_1k_2}^*\exp\{-l_{2k_1k_2*}\tau_3\}, \quad |k| = n, \\ p_{2k_1k_2}^*\exp\{-l_{2k_1k_2*}\tau_3\} + \int\limits_0^{\tau_3}\exp\{-(\tau_3-z)l_{2k_1k_2*}\} \sum\limits_{|s|=|k|+1} \nu_{2s_1s_2;k_1k_2}^*\overline{v}_{2k_1k_2}(z)\,dz, \quad |k| = n-1,n-2,\ldots,1, \end{cases} \tag{4.8}$$

and

$$\underline{v}_{2k_1k_2}(\tau_3) = \begin{cases} p_{2k_1k_2*}\exp\{-l_{2k_1k_2}^*\tau_3\}, \quad |k| = n, \\ p_{2k_1k_2*}\exp\{-l_{2k_1k_2}^*\tau_3\} + \int\limits_0^{\tau_3}\exp\{-(\tau_3-z)l_{2k_1k_2}^*\} \sum\limits_{|s|=|k|+1} \nu_{2s_1s_2;k_1k_2*}\underline{v}_{2k_1k_2}(z)\,dz, \quad |k| = n-1,n-2,\ldots,1, \end{cases}$$

(4.9)

in $[0,T]$. Functions $\bar{v}_{2k_1k_2}$ and $\underline{v}_{2k_1k_2}$ for $|k| = n-1, n-2, \ldots, 1$ can be found recurrently starting from $|k| = n-1$ since $\bar{v}_{2k_1k_2}$ and $\underline{v}_{2k_1k_2}$ for $|k| = n$ are known.

Lemma 4.1. *Let $\nu_* + \lambda > 0$ be a given positive constant. Assume that functions $\nu_{2k_1k_2}$ and $\nu_{2k_1k_2;s_1s_2}$ lie in $C^{0,1,1}([\tau_{1*},\infty) \times (\tau_{2*}+\tau_3,\infty) \times (0,T)) \cap C([\tau_{1*},\infty) \times [\tau_{2*}+\tau_3,\infty) \times [0,T])$, $\alpha_{2k_1k_2} \in C^{0,1}([\tau_{1*},\infty) \times (\tau_{2*},\infty)) \cap C([\tau_{1*},\infty) \times]\tau_{2*},\infty))$, $p_{2k_1k_2} \in C^{0,1}([\tau_{1*},\infty) \times (\tau_{2*},\infty)) \cap C([\tau_{1*},\infty) \times [\tau_{2*},\infty))$ and let they be nonnegative bounded functions in domains of their definition. Then problem (4.5) has a unique nonnegative solution $v_{2k_1k_2} \in C^{0,1,1}([\tau_{1*},\infty) \times (\tau_{2*}+\tau_3,\infty) \times (0,T)) \cap C([\tau_{1*},\infty) \times [\tau_{2*}+\tau_3,\infty) \times [0,T])$ such that $\underline{v}_{2k_1k_2} \le v_{2k_1k_2} \le \bar{v}_{2k_1k_2}$ in $[\tau_{1*},\infty) \times [\tau_{2*}+\tau_3,\infty) \times [0,T]$ where $\bar{v}_{2k_1k_2}$ and $\underline{v}_{2k_1k_2} \in C^1([0,T])$ are determined by formulas (4.8) and (4.9), respectively.*

Proof. Conditions of this lemma let us to solve linear equation (4.5) to have

$$v_{2k_1k_2}(\tau_1,\tau_2,\tau_3) = \begin{cases} p_{2k_1k_2}(\tau_1,\tau_{23}) \exp\left\{ -\int\limits_0^{\tau_3} l_{2k_1k_2}(\tau_1, s+\tau_{23}, s)\, ds \right\}, \quad |k| = n, \\ p_{2k_1k_2}(\tau_1,\tau_{23}) \exp\left\{ -\int\limits_0^{\tau_3} l_{2k_1k_2}(\tau_1, s+\tau_{23}, s)\, ds \right\} \\ \quad + \int\limits_0^{\tau_3} \exp\left\{ -\int\limits_z^{\tau_3} l_{2k_1k_2}(\tau_1, s+\tau_{23}, s)\, ds \right\} \sum\limits_{|s|=|k|+1} \nu_{2s_1s_2;k_1k_2}(\tau_1, z+\tau_{22}, z) v_{2k_1k_2}(\tau_1, z+\tau_{23}, z)\, dz, \\ \quad |k| = n-1, n-2, \ldots, 1, \end{cases} \tag{4.10}$$

in $[\tau_{1*},\infty) \times [\tau_{2*}+\tau_3,\infty) \times [0,T]$, where $\tau_{23} = \tau_2 - \tau_3$.

Function (4.10) for $|k| = n-1, n-2, \ldots, 1$ can be found recurrently starting from $|k| = n-1$ since $v_{2k_1k_2}$ for $|k| = n$ is known. Note that function $v_{2k_1k_2}$ is independent of parameters y_1 and λ. Direct comparison of Eq. (4.8) with (4.10) and Eq. (4.9) with (4.10) proves the inequality $\underline{v}_{2k_1k_2} \le v_{2k_1k_2} \le \bar{v}_{2k_1k_2}$ in $[\tau_{1*},\infty) \times [\tau_{2*}+\tau_3,\infty) \times [0,T]$. Differentiability of $\bar{v}_{2k_1k_2}$ and $\underline{v}_{2k_1k_2}$ in $[0,T]$ follows from Eqs.(4.8) with (4.9). □

Let
$q_* = \sum\limits_{k|=1}^{n} \nu_{2k_1k_2,00*} \min\limits_{[0,T]} \underline{v}_{2k_1k_2}$, $q^* = \sum\limits_{k|=1}^{n} \nu^*_{2k_1k_2,00} \max\limits_{[0,T]} \bar{v}_{2k_1k_2}$,
$r^* = \sum\limits_{|k|=1}^{n} \bar{v}_{2k_1k_2}(T)$, $r_* = \sum\limits_{|k|=1}^{n} \underline{v}_{2k_1k_2}(T)$.

Then $\gamma_*||v_1|| \le P \le \gamma^*||v_1||$, $q_*||v_1|| \le Q \le q^*||v_1||$, $r_*||v_1|| \le R \le r^*||v_1||$.

Consider two following systems:

$$\bar{v}_2' + l_{2*}\bar{v}_2 = \begin{cases} y_1||v_1|| \int\limits_0^{\tau_2-\tau_{2*}} \bar{v}_2(\tau_2-\tau_3) e^{-\lambda\tau_3}\, d\tau_3\, q^* \quad \text{in } (\tau_{2*}, \tau_{2*}+T), \quad \bar{v}_2(\tau_{2*}) = \exp\{-(\tau_{2*}-T)(\nu_{2*}+\lambda)\}, \\ y_1||v_1|| \left(\int\limits_0^T \bar{v}_2(\tau_2-\tau_3) e^{-\lambda\tau_3}\, d\tau_3\, q^* + \bar{v}_2(\tau_2-T) e^{-\lambda T} r^* \right) \quad \text{in } (\tau_{2*}+T, \infty), \quad [\bar{v}_2(\tau_{2*}+T)] = 0 \end{cases} \tag{4.11}$$

and

$$\underline{v}_2' + l_{2*}\underline{v}_2 = \begin{cases} y_1||v_1|| \int\limits_0^{\tau_2-\tau_{2*}} \underline{v}_2(\tau_2-\tau_3) e^{-\lambda\tau_3}\, d\tau_3\, q_* \quad \text{in } (\tau_{2*}, \tau_{2*}+T), \quad \underline{v}_2(\tau_{2*}) = \exp\{-(\tau_{2*}-T)(\nu_2^*+\lambda)\}, \\ y_1||v_1|| \left(\int\limits_0^T \underline{v}_2(\tau_2-\tau_3) e^{-\lambda\tau_3}\, d\tau_3\, q_* + \underline{v}_2(\tau_2-T) e^{-\lambda T} r_* \right) \quad \text{in } (\tau_{2*}+T, \infty), \quad [\underline{v}_2(\tau_{2*}+T)] = 0. \end{cases} \tag{4.12}$$

Applying the argument used to construct Eq. (4.7), Eqs. (4.11) and (4.12) on each interval $[\tau_{2*}+jT, \tau_{2*}+(j+1)T]$, $j = 0, 1, \ldots$, can be transformed to Volterra integral equations having unique positive solutions.

Lemma 4.2. *Assume that function $\nu_2 \in C([\tau_{2*},\infty))$ and parameter y_1 are positive and let conditions of Lemma 4.1 be fulfilled. Then Eq. (4.7) has a unique positive solution $v_2 \in C^1((\tau_{2*},\infty)) \cap C([\tau_{1*},\infty))$. Moreover, $\underline{v}_2 \le v_2 \le \bar{v}_2$ in $[\tau_{2*},\infty)$ where $\bar{v}_2$ and $\underline{v}_2$ are unique positive solutions of Eqs. (4.11) and (4.12), respectively.*

Proof. The proof of the existence and uniqueness of the solution is based on the existence and uniqueness theorem of the Volterra linear integral equation. It remains to prove the inequality $\underline{v}_2 \le v_2 \le \bar{v}_2$. Set $Z = \bar{v}_2 - v_2$. Subtracting Eq. (4.3)$_1$ from Eq. (4.11) we get the equation

$$\begin{cases} Z' + l_{2*}Z = y_1 \int\limits_0^{\tau_2-\tau_{2*}} Z(\tau_2-\tau_3)\,d\tau_3 Q(\tau_2,\tau_3)e^{-\lambda\tau_3}\,d\tau_1 + f(\tau_2) \quad \text{in } (\tau_{2*},\tau_{2*}+T), \\ Z(\tau_{2*}) = \exp\Big\{-\int\limits_T^{\tau_{2*}}(v_{2*}+\lambda)\,ds\Big\} - \exp\Big\{-\int\limits_T^{\tau_{2*}}(v_2+\lambda)\,ds\Big\} \end{cases}$$

with a known nonnegative term

$$f(\tau_2) = (l_2 - l_{2*})v_2 + y_1 \int\limits_0^{\tau_2-\tau_{2*}} v_2(\tau_2-\tau_3)\Big(q^*||v_1|| - Q(\tau_2,\tau_3)\Big)e^{-\lambda\tau_3}\,d\tau_3.$$

This equation can be easily transformed into the Volterra integral equation with a nonnegative kernel and nonnegative known term. Hence it has a unique nonnegative solution $\bar{v}_2 - v_2$ in $[\tau_{2*},\tau_{2*}+T]$ and therefore $v_2 \le \bar{v}_2$. Similarly, we prove that $\underline{v}_2 \le v_2$ in $[\tau_{2*},\tau_{2*}+T]$. Subtracting (4.12) from Eq. (4.3)$_1$ and arguing similarly as above, we prove the inequality $\underline{v}_2 \le v_2 \le \bar{v}_2$ in $[\tau_{2*}+T,\infty)$. □

It is well known that a solution to the linear Volterra integral equation with a parameter that has a continuous kernel and a continuous known term with respect to the (argument, parameter) variable is also continuous with respect to the same variable. Hence, functions v_2, $\bar{v}_2$, and $\underline{v}_2$ are continuous with respect to (τ_2, y_1, λ).

Now we prove that $||v_2||$ is continuous with respect to parameters y_1 and λ. We integrate Eq. (4.3)$_1$ to have

$$\bar{v}_2(\tau_2) = \bar{v}_2(\tau_{2*})\exp\Big\{-\int\limits_{\tau_{2*}}^{\tau_2} l_{2*}\,ds\Big\} + \begin{cases} I_1 \text{ in } (\tau_{2*},\tau_{2*}+T), \\ I_2+I_3+I_4 \text{ in } (\tau_{2*}+T,\infty) \end{cases} \tag{4.13}$$

where

$$I_1 = y_1\int\limits_{\tau_{2*}}^{\tau_2}\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\int\limits_0^{z-\tau_{2*}} v_2(z-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q^*||v_1|| = y_1\int\limits_0^{\tau_2-\tau_{2*}} e^{-\lambda\tau_3}\,d\tau_3\int\limits_{\tau_3+\tau_{2*}}^{\tau_2} v_2(z-\tau_3)\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\,q^*||v_1||,$$

$$I_2 = y_1\int\limits_{\tau_{2*}}^{\tau_{2*}+T}\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\int\limits_0^{z-\tau_{2*}} v_2(z-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q^*||v_1|| = y_1\int\limits_0^T e^{-\lambda\tau_3}\,d\tau_3\int\limits_{\tau_3+\tau_{2*}}^{\tau_{2*}+T} v_2(z-\tau_3)\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\,q^*||v_1||,$$

$$I_3 = y_1\int\limits_{\tau_{2*}+T}^{\tau_2}\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\int\limits_0^T v_2(z-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q^*||v_1|| = y_1\int\limits_0^T e^{-\lambda\tau_3}\,d\tau_3\int\limits_{T+\tau_{2*}}^{\tau_2} v_2(z-\tau_3)\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\,q^*||v_1||,$$

$$I_4 = y_1\int\limits_{\tau_{2*}+T}^{\tau_2}\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}e^{-\lambda T}v_2(z-T)\,dz\,r^*||v_1||.$$

Observe that

$$I_2+I_3 = y_1\int\limits_0^T e^{-\lambda\tau_3}d\tau_3\int\limits_{\tau_3+\tau_{2*}}^{\tau_2} v_2(z-\tau_3)\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\,q^*||v_1||.$$

Integrating Eq. (4.13) we find

$$||\bar{v}_2|| = \int\limits_{\tau_{2*}}^{\infty}\bar{v}_2(\tau_{2*})\exp\Big\{-\int\limits_{\tau_{2*}}^{\tau_2} l_{2*}\,ds\Big\}d\tau_2||v_1|| + J_1 + J_2 + J_3,$$

where

$$J_1 = y_1 \int_{\tau_{28}}^{\tau_{2*}+T} d\tau_2 \int_0^{\tau_2-\tau_{2*}} e^{-\lambda\tau_3}\, d\tau_3 \int_{\tau_3+\tau_{2*}}^{\tau_2} \bar{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\, ds\Big\}\, dz\, q^*||v_1||,$$

$$J_2 = y_1 \int_{\tau_{2*}+T}^{\infty} d\tau_2 \int_0^T e^{-\lambda\tau_3} d\tau_3 \int_{\tau_3+\tau_{2*}}^{\tau_2} \bar{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\, ds\Big\} e^{-\lambda\tau_3}\, dz\, q^*||v_1||,$$

$$J_3 = y_1 \int_{\tau_{2*}+T}^{\infty} d\tau_2 \int_{\tau_{2*}+T}^{\tau_2} \exp\Big\{-\int_z^{\tau_2} l_{2*}\, ds\Big\} e^{-\lambda T}\bar{v}_2(z-T)\, dz\, r^*||v_1||.$$

Changing the order of integration we have

$$J_1 = y_1 \int_0^T e^{-\lambda\tau_3}\, d\tau_3 \int_{\tau_{2*}+\tau_3}^{\tau_{2*}+T} d\tau_2 \int_{\tau_3+\tau_{2*}}^{\tau_2} \bar{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\, ds\Big\}\, dz\, q^*||v_1||,$$

$$J_2 = y_1 \int_0^T e^{-\lambda\tau_3}\, d\tau_3 \int_{\tau_{2*}+T}^{\infty} d\tau_2 \int_{\tau_3+\tau_{2*}}^{\tau_2} \bar{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\, ds\Big\} e^{-\lambda\tau_3}\, dz\, q^*||v_1||,$$

$$J_3 = y_1 e^{-\lambda T} \int_{\tau_{2*}}^{\infty} \bar{v}_2(x)\, dx \int_{x+T}^{\infty} \exp\Big\{-\int_{x+T}^{\tau_2} l_{2*}\, ds\Big\}\, d\tau_2\, r^*||v_1||$$

and then

$$J_1+J_2 = y_1 \int_0^T e^{-\lambda\tau_3}\, d\tau_3 \int_{\tau_{2*}+\tau_3}^{\infty} d\tau_2 \int_{\tau_3+\tau_{2*}}^{\tau_2} \bar{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_2^{\lambda}\, ds\Big\}\, dz\, q^*||v_1||.$$

Thus

$$\begin{cases} ||\bar{v}_2|| = \bar{v}_2(\tau_{2*}) \int_{\tau_{2*}}^{\infty} \exp\Big\{-\int_{\tau_{2*}}^{\tau_2} l_{2*}\, ds\Big\}\, d\tau_2 + y_1 \int_0^T e^{-\lambda\tau_3}\, d\tau_3 \int_{\tau_{2*}+\tau_3}^{\infty} d\tau_2 \int_{\tau_3+\tau_{2*}}^{\tau_2} \bar{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\, ds\Big\}\, dz\, q^*||v_1|| \\ \quad + y_1 e^{-\lambda T} \int_{\tau_{2*}}^{\infty} \bar{v}_2(x)\, dx \int_{x+T}^{\infty} \exp\Big\{-\int_{x+T}^{\tau_2} l_{2*}\, ds\Big\}\, d\tau_2\, r^*||v_1||. \end{cases} \tag{4.14}$$

and after changing the order of integration

$$\begin{cases} ||\bar{v}_2|| = \bar{v}_2(\tau_{2*}) \int_{\tau_{2*}}^{\infty} \exp\Big\{-\int_{\tau_{2*}}^{\tau_2} l_{2*}\, ds\Big\}\, d\tau_2 + y_1 \int_0^T e^{-\lambda\tau_3}\, d\tau_3 \int_{\tau_{2*}+\tau_3}^{\infty} \bar{v}_2(z-\tau_3)\, dz \int_z^{\infty} \exp\Big\{-\int_z^{\tau_2} l_{2*}\, ds\Big\}\, d\tau_2\, q^*||v_1|| \\ \quad + y_1 e^{-\lambda T} \int_{\tau_{2*}}^{\infty} \bar{v}_2(x)\, dx \int_{x+T}^{\infty} \exp\Big\{-\int_{x+T}^{\tau_2} l_{2*}\, ds\Big\}\, d\tau_2\, r^*||v_1|| \end{cases} \tag{4.15}$$

If $\nu_* + \lambda > 0$, then

$$||\bar{v}_2|| l_{2*} = \exp\Big\{-(\tau_{2*}-T)(\nu_{2*}+\lambda)\Big\} + \Big(\int_0^T e^{-\lambda\tau_3}\, d\tau_3 q^* + e^{-\lambda T} r^*\Big) ||\bar{v}_2||\,||v_1||y_1.$$

Hence

$$||\bar{v}_2|| := \omega_r(y_1,\lambda) = \frac{\exp\{-(\tau_{2*}-T)(\nu_{2*}+\lambda)\}}{A(y_1,\lambda)} \tag{4.16}$$

provided that

$$A(y_1,\lambda) := \nu_{1*} + \lambda + y_1||v_1||N(\lambda) > 0 \tag{4.17}$$

where

$$N(\lambda) := \gamma_* - \int_0^T e^{-\lambda \tau_3} d\tau_3 \, q^* - e^{-\lambda T} r^*.$$

Observe that condition (4.17) is fulfilled if $N(\lambda) \geq 0$ and $\nu_* + \lambda \geq \varepsilon$, $\varepsilon > 0$. Eq. (4.16) under condition (4.17) shows that $||\bar{v}_2||$ is continuous in (y_1, λ). This, the positivity and continuity of $\bar{v}_2$ with respect to (τ_2, y_1, λ) show that $||\bar{v}_2||$ converges uniformly with respect to $\lambda \in [-\nu_* + \varepsilon, \lambda']$ and $y_1 \in [0, y_1']$ where $\lambda' < \infty$, $y_1' < \infty$. Then Lemma 4.2 shows that $||v_2||$ converges uniformly too and the continuity of v_2 with respect to (τ_2, y_1, λ) proves the continuity of $||v_2||$ with respect to $\lambda \in [-\nu_* + \varepsilon, \lambda']$ and $y_1 \in [0, y_1']$.

Define

$$\tilde{q}_1(y_1, \lambda) = e^{-\lambda T} \int_{\tau_{2*}}^{\infty} v_2(x) \beta_2(x) \, dx,$$

$$\tilde{q}_2(y_1, \lambda) = e^{-\lambda T} \int_{\tau_{2*}}^{\infty} v_2(x) \beta_1(x) \, dx.$$

Using Lemma 4.1, we can prove that functions $\tilde{q}_1(y_1, \lambda)$ and $\tilde{q}_2(y_1, \lambda)$ are continuous in $\lambda \geq -\nu_* + \varepsilon$ and $y_1 > 0$. Eqs. $(4.2)_2$ and $(4.3)_2$ can be rewritten as follows:

$$\begin{cases} y_1 - \dfrac{1}{\tilde{q}_1(y_1, \lambda)} = 0, \\ y_2 - \dfrac{1}{\tilde{q}_2(y_1, \lambda)} = 0. \end{cases} \tag{4.18}$$

Function

$$z(y_1, \lambda) = y_1 - \frac{1}{\tilde{q}_1(y_1, \lambda)}$$

is continuous with respect to (y_1, λ). Obviously, $z|_{y_1 = 0} < 0$. Eq. $(4.3)_1$ shows that

$$v_2(\tau_2) \geq \tilde{v}_2(\tau_2) := \exp\Big\{ -\int_T^{\tau_2} (\nu_2(\tau_2) + \lambda) \, d\tau_2 - (\tau_2 - T) p_2^* p^* \sum_{|k|=1}^{n} \alpha^*_{2k_1k_2} \, d\tau_2 \Big\}$$

for all $y_1 \geq 0$. Define:

$$\widehat{q}_1(\lambda) = e^{-\lambda T} \int_{\tau_{2*}}^{\infty} \tilde{v}_2(x) \beta_2(x) \, dx, \; \widehat{q}_2(\lambda) = e^{-\lambda T} \int_{\tau_{2*}}^{\infty} \tilde{v}_2(x) \beta_1(x) \, dx.$$

Then by definition $\tilde{q}_1(y_1, \lambda) > \widehat{q}_1(\lambda)$ for all $y_1 \geq 0$. Hence, $z|_{y_1 = 1/\widehat{q}_1(\lambda)} > 0$. The continuity of z shows that function $z(y_1, \lambda)$ has at least one positive root $\bar{y}_1(\lambda) \in (0, 1/\widehat{q}_1(\lambda))$, which is continuous in λ. Then Eq. $(4.18)_2$ shows that $y_2(\bar{y}_1(\lambda), \lambda)$, is also continuous with respect to $\lambda \geq -\nu_* + \varepsilon$ with small $\varepsilon > 0$.

Now we find constant λ. Set:

$$\bar{h}_i = \sum_{||k||=1}^{n} k_i \bar{v}_{2k_1k_2}(T), \quad \underline{h}_i = \sum_{|k|=1}^{n} k_i \underline{v}_{2k_1k_2}(T), \quad i = 1, 2, \quad B(\lambda) = \bar{y}_1(\lambda) ||p_1 v_1|| + y_2(\bar{y}_1(\lambda), \lambda) ||p_2 v_2||.$$

It is evident that

$$\widehat{q}_i(\lambda) \geq \underline{h}_i e^{-\lambda T} ||\tilde{v}_2|| ||v_1||, \; i = 1, 2.$$

Then using Eqs. (4.6), (4.16), and (4.18), we get

$$\bar{y}_1(\lambda) ||p_1 v_1|| \leq \frac{p_1^*}{\widehat{q}_2} ||v_1|| \leq \frac{p_1^* e^{\lambda T}}{\underline{h}_2 ||\tilde{v}_2||}, \; \bar{y}_2(\lambda) ||p_2 v_2|| \geq \frac{p_2^* e^{\lambda T}}{\bar{h}_2 ||v_1||},$$

Hence

$$H_l(\lambda) := \frac{p_{2*} e^{\lambda T}}{\bar{h}_2 ||v_1||} \leq B(\lambda) \leq H_r(\lambda) := \frac{p_1^* e^{\lambda T}}{\underline{h}_2 ||\tilde{v}_2||} + \frac{p_2^* e^{\lambda T}}{\underline{h}_1 ||v_1||} \tag{4.19}$$

provided that condition (4.17) with $y_1 = 1/\widehat{q}_1(\lambda)$ (i.e., $A(1/\widehat{q}_1(\lambda), \lambda) > 0$) is satisfied. Analysis of inequalities (4.19) allows us to formulate the following assertion:

Lemma 4.3. *Let conditions of Lemmas 4.1 and 4.2 be satisfied. Assume that* $\lambda_0 \geq -\nu_* + \varepsilon$ *with a small* $\varepsilon > 0$ *and* $\lambda_1 > \lambda_0$ *are such that* $H_r(\lambda_0) < 1$, $H_l(\lambda_1) > 1$, *and* $N(\lambda_0) \geq 0$. *Then function* $B(\lambda) - 1$ *has at least one real root* λ_2.

The proof of lemma is obvious, since H_l, H_r, and N are monotonous functions of λ.

Based on Lemmas 4.1–4.3, we formulate the following proposition:

Theorem 4.1. *Let conditions of Lemmas 4.1–4.3 be satisfied. Then system* (3.1)–(3.3) *has a one-parametric class of separable solutions.*

5. Conclusion

We proposed a deterministic model for two-sex population with a discrete set of offspring and maternal care assuming that pairs of sexes exist only during the period of mating, which is disregarded. The Environmental pressure is also neglected in our model. The reproductive age intervals in model [9] are finite. Contrary to model [9], we let the reproductive age intervals be infinite. The existence of the separable solutions is proved under some conditions on the model data.

To close the paper, we discuss conditions that led to the existence of the solutions to characteristic equation (4.6) of our model and equation (4.9) for exponent λ of the model [9] in the case of the absence of the Environmental pressure. Equation (4.9) of model [9] has at least one real solution without any additional restriction on the model data. As shown in Theorem 1 of our model, the proof of the solvability of characteristic equation (4.6) is based on the proof of the continuity of the norm $||v_2||$ with respect to parameter λ. Knowing this, some robust restrictions on the model data were formulated, that are sufficient for the existence of the solution to characteristic equation (4.6).

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