VOLUME VI ISSUE 2 October 2024
ISSN: 2667-7660 http://dergipark.gov.tr/mjm

MALTEPE JOURNAL OF MATHEMATICS

Editorial Board

Contents

Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume VI Issue 2 (2024), Pages 24-49, Doi: https://doi.org/10.47087/mjm.1467436

ON A FAMILY OF DISCRETE ND LADDER-TYPE OPERATORS CONSTRUCTED IN TERMS OF THE HERMITIAN TOEPLITZ COMMUTATOR OPERATOR Z_N

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Abstract. A development of an algebraic system with N-dimensional ladder-type operators associated with the discrete Fourier transform is described, following an analogy with the canonical commutation relations of the continuous case. It is found that a Hermitian Toeplitz matrix Z_N, which plays the role of the identity, is sufficient to satisfy the Jacobi identity and, by solving some compatibility relations, a family of ladder operators with corresponding Hamiltonians can be constructed. The behaviour of the matrix Z_N for large N is elaborated. It is shown that this system can be also realized in terms of the Heun operator W, associated with the discrete Fourier transform, thus providing deeper insight on the underlying algebraic structure.

1. Introduction

The study of discrete structures is significant for the theory of signal processing, entanglement, quantum computation and more [1], and it serves as a source of interesting and surprising considerations worth studying. The problem of the construction of a system of eigenvectors for the discrete Fourier transform (DFT) is still open and has been approached from several directions since J. H. McClellan and T. W. Parks [2]. Recent results of M. K. Atakishiyeva and N. M. Atakishiyev (AA) [3]–[5] aim to enrich the resulting eigensystem with the quality of being canonical. Techniques for associating eigensystems with the DFT include the use of an uncertainty principle associated with cyclic groups of prime order [6], as well as commutative matrices construction for a matrix that commutes with the DFT, thus ensuring that both matrices share the same set of eigenvectors, which provides an orthonormal eigenbasis for the DFT [7]–[9]. This last method is employed in [3],

2020 *Mathematics Subject Classification.* Primary: 47A75 ; Secondaries: 11C20, 39A70, 39B42, 65T50.
Key words and phrases. ladder operators; Hermitian Toeplitz operator; discrete Fourier transform.

Submitted on April 10th, 2024. Accepted on May 24th, 2024.
Communicated by Huseyin CAKALLI and Nazlım Deniz ARAL.

where raising and lowering operators has been found to construct a number operator $\mathcal{N}$ commuting with the DFT, in complete analogy with the linear quantum harmonic oscillator of the continuous case. It is this analogy that motivates the present work targeting the establishment of a broader framework to deal with this problem even more systematically. Can we extend this analogy of the harmonic oscillator for the DFT to mimic the continuous case? To what extent can we make an analogy of the canonical commutation relations (CCR) for finite dimensional Hilbert spaces? This is the common thread that guides us to put forward an algebraic system through some compatibility relations that admit operator solutions of ladder type, beginning with the use of a Hermitian Toeplitz matrix Z_N, which plays the role of the identity. We establish a remarkable relationship between this treatment and the AA-approach, by using the Heun operator W of the latter, thus leading naturally to the proposal of a complete realization of the algebraic system.

The paper is organized as follows: In Section 2, we briefly review some quantum foundations about the CCR. In Section 3, we present the mathematical background necessary for discrete structures in finite dimensional Hilbert space. In Section 4, we establish the discrete commutation relations and show they satisfy the Jacobi identity. In Section 5, we give an explicit matrix representation for Z_N and conduct a brief investigation into the nature of this operator for large N as well. Section 6 is devoted to the main results of this work, namely, the proposal of an algebraic system in terms of the Hermitian Toeplitz operator Z_N and through compatibility relations, whose solutions consist of ladder-type operators. From these operators a family of Hamiltonians $\mathscr{H}$ can be obtained for each N; this solutions, however, fulfill at least two of the four requirements of the proposed algebraic system and not necessarily the other two. In Section 7 we show then how the operators Q and P, which generate Z_N, are related to the raising and lowering operators, as well as the Heun operator W of the AA-approach, through the exponential map. Thus we conclude that this connection could provide a complete realization of the algebraic system; that is, the fulfillment of the four requirements which comprise it. Finally, Section 8 offers concluding remarks on the outstanding issues.

2. Quantum foundations

Canonical commutation relations.

We seek a suitable analogy between continuous and discrete realizations of the canonical commutation relations that underlie the Heisenberg algebra, briefly analyzing the parallels between Classical Mechanics (CM) and Quantum Mechanics (QM)[10].

- Observables in CM are smooth functions on $\mathbb{R}^{2n}$.
- Hermitian operators in QM are regarded as infinitesimal canonical transformations or *infinitesimal automorphisms*, the vector fields are used to obtain (local) canonical transformations by integrating Hamilton's equations. Similarly, Hermitian operators A are employed to derive skew-adjoint operators $2\pi \mathrm{i} A$, which upon the exponentiation yield a one-parameter unitary group.
- The automorphisms of the underlying set $\mathbb{R}^{2n+1}$ are considered, where $\mathbb{R}^{2n+1}$ corresponds to a Lie algebra or a Lie group, depending on whether a bracket operation or a group law is defined.

- The Fourier transform arises naturally from the very wavy nature of QM in an idealized basis of basic plane wave packets (eigenfunctions of the momentum) $e_\xi(x) = e^{2\pi \mathrm{i} x\xi}$, the momentum of which is $h\xi$.
- By constructing the j-th component of momentum through the correspondence of Borel measures with its Hermitian operators, one finds that the Fourier transform intertwines it with the position operator

$$P_j = h\mathcal{F}Q_j\mathcal{F}^{-1} \implies P_j = \frac{h}{2\pi\mathrm{i}}\frac{\partial}{\partial x_j} = hD_j. \tag{2.1}$$

- It has been proven that the action of the exponentials of momentum and position operators is on the functions on L^2 and represents a translation in momentum space and a translation in position space, respectively.
- The basic observables Q_j and P_j satisfy the *canonical cammutation relations* (CCR) (see [10], p.15)

$$[P_j, P_k] = [Q_j, Q_k] = 0, \qquad [P_j, Q_k] = \frac{h\delta_{jk}}{2\pi\mathrm{i}}I. \tag{2.2}$$

Following Seligman's treatment of representation theory [11], the Heinsenberg-Weyl Lie algebra of QM, denoted by $\mathfrak{h}$, with elements Q, P and I over the field of complex numbers is considered. This algebra is defined by the following commutation relations:

$$[Q, P] = \mathrm{i}I, \qquad [Q, I] = 0, \qquad [P, I] = 0. \tag{2.3}$$

The elements Q, P and I form a basis for the algebra $\mathfrak{h}$, so we can express any element E in $\mathfrak{h}$ as:

$$E = xQ + yP + zI, \qquad x, y, z \in \mathbb{C}.$$

Taking Q and P is sufficient for an algebraic basis of $\mathfrak{h}$, as I can be derived from the Lie bracket in the first of equations (2.3). The elements of interest in this abstract scheme are

$$R = \frac{1}{\sqrt{2}}(Q - \mathrm{i}P), \qquad L = \frac{1}{\sqrt{2}}(Q + \mathrm{i}P), \tag{2.4}$$

which satisfy:

$$[L, R] = I, \qquad [L, I] = 0, \qquad [R, I] = 0. \tag{2.5}$$

These operators also form a basis for $\mathfrak{h}$ and thus define $\mathfrak{h}$ as well. The primary goal in this approach is to construct concrete models through representations of $\mathfrak{h}$ with sets of linear operators representable by matrices; this is always feasible since every Lie algebra over $\mathbb{C}$ is isomorphic to some matrix algebra. It may be interesting to note that our search for different bases for representations of the algebra $\mathfrak{h}$ is intimately related with the fact that different bases for the same representations of the Lie group lead to different special functions and provide a group theoretical underpinning for all of these functions (see [12] for a more detailed discussion of this point).

The Heisenberg-Weyl group

For the Heisenberg-Weyl algebra $\mathfrak{h}$ defined in (2.3), we can use its faithful representation and subalgebra of $gl(3, \mathbb{C})$, denoted by $\mathfrak{h}^f$, to build its corresponding Lie Group H^f, which is a subgroup of $GL(3, \mathbb{C})$, through the exponential map

$\exp : \mathfrak{h}^f \to H^f$, such that $\mathfrak{h}^f$ constitutes the tangent space of H^f at the identity. The exponential map

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

exists since the exponential of an arbitrary matrix with finite elements is an absolutely convergent series that yields an invertible matrix; this map sends the zero element in $\mathfrak{h}^f$ to the identity element in H^f. As a manifold, the parameters which form a canonical coordinate system of H^f are given by the Lie group elements of H^f:

$$G(x,y,z) = \exp \mathrm{i}(xQ^f + yP^f + zI^f) = \exp \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ y & \mathrm{i}z & 0 \end{pmatrix}. \tag{2.6}$$

From the properties of the exponential map, one can derive the composition law of the abstract corresponding group H:

$$g(x_1,y_1,z_1)g(x_2,y_2,z_2) = g\Big(x_1+x_2, y_1+y_2, z_1+z_2+\frac{1}{2}[y_1x_2 - x_1y_2]\Big), \tag{2.7}$$

$$e = g(0,0,0), \qquad g(x,y,z)^{-1} = g(-x,-y,-z). \tag{2.8}$$

This group is named the Heisenberg-Weyl (or simply Heisenberg) group. All parameters range over $\mathbb{R}$ so the group manifold is isomorphic to $\mathbb{R}^3$, non-compact and simply connected.

3. Mathematical background for discrete structures

In this section, we explore the discrete structure in finite dimensions, considering the intricacies involved. To address specific nuances, the action of a group G on a set X is represented by a function $f : G \times X \longrightarrow X$ such that for all x in X, $f(e,x) = x$, where e is the identity element of G. An N-dimensional representation of a group G over a field K is a group homomorphism $\phi : G \longrightarrow GL(V)$, where V is an N-dimensional vector space on K, and $GL(V)$ is the group of linear operators on V. If $\phi(g)$ is a unitary operator for every g in G, and its corresponding conjugate transpose satisfies $\phi(g)^\dagger = \phi(g)^{-1}$, we say the representation ϕ is unitary. We consider discrete groups to be Lie groups endowed with the discrete topology; a finite group G acts on itself by automorphisms and can be embedded in some permutation group S_N, which admit a representation on K^N, with K a field.

The Fourier transform arises naturally from the harmonic periodic behaviour of quantum systems, where periodicity is somehow fundamental. Thus, we focus on the finite cyclic abelian group Z_N which is isomorphic to the additive group of integers modulo N, denoted $\mathbb{Z}/N\mathbb{Z}$. We also consider the geometric series:

$$1 + z + z^2 + \ldots + z^{N-1} = \begin{cases} (1-z^N)/(1-z), & \text{if } z \neq 1 \\ N, & \text{if } z = 1 \end{cases}. \tag{3.1}$$

Define $\omega = e^{2\pi \mathrm{i}/N}$, with $\mathrm{i} = \sqrt{-1}$, to be the N-th primitive root of unity, then the set of N-th roots of unity, $\{\omega^k\}$, $k = 0, \ldots, N-1$, is a group and satisfies $1 + \omega + \omega^2 + \ldots + \omega^{N-1} = 0$, since $e^{2\pi \mathrm{i}} - 1 = 0$. Such a group is isomorphic to Z_N and is denoted by C_N. Let's consider $\mathrm{Hom}(\mathbb{Z}/N\mathbb{Z})$ as the set of homomorphisms of $\mathbb{Z}/N\mathbb{Z}$ into C_N. A vector in $\mathbb{C}^N$ is denoted by v and its components by v_j, the canonical basis of $\mathbb{C}^N$ is represented as $e_k = \{(\delta_{k,0}, \ldots, \delta_{k,N-1}) : k = 0, \ldots, N-1\}$, where δ_{kj} is the Kronecker delta function. Operators are denoted by capital letter

T and their matrix entries by T_{lm}. With an abuse of notation, operators and matrix representations of them are denoted with the same letter, unless otherwise specified.

Most of what will be mentioned in this section without proof, can be found in [13].

As discussed in Section 2, the position and momentum operators generate translations in the underlying group, thus we consider a notion of translation in our space of complex-valued functions on $\mathbb{Z}/N\mathbb{Z}$. First we endow it with an inner product to turn it into a Hilbert space $L^2(\mathbb{Z}/N\mathbb{Z})$ by means of

$$\langle f, g\rangle = \sum_{\alpha\in\mathbb{Z}/N\mathbb{Z}} f(\alpha)\overline{g(\alpha)},$$

where $\bar{x}$ denotes the complex conjugate of x in $\mathbb{C}$. A translation operator $T_a : L^2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow L^2(\mathbb{Z}/N\mathbb{Z})$, for every $a \in \mathbb{Z}/N\mathbb{Z}$, is defined by the action of $\mathbb{Z}/N\mathbb{Z}$ on $L^2(\mathbb{Z}/N\mathbb{Z})$ given by

$$T_a f(b) := f(b-a), \quad \forall a, b \in \mathbb{Z}/N\mathbb{Z}.$$

It can be shown that an orthonormal basis for $L^2(\mathbb{Z}/N\mathbb{Z})$ is $\{f_\alpha\}$, $0 \leq \alpha \leq (N-1)$, where

$$f_\alpha(b) = \begin{cases} 1 & \text{if } \alpha = b \\ 0 & \text{if } \alpha \neq b \end{cases} \quad \alpha, b \in \mathbb{Z}/N\mathbb{Z}.$$

If we look for a matrix representation V of T in the $\{f_\alpha\}$ basis, we can take the range of $f \in L^2(\mathbb{Z}/N\mathbb{Z})$, ordered by its argument from $0, \ldots, N-1$, as a vector $(f(0), f(1), \ldots, f(N-1)) \in \mathbb{C}^N$. This means that the set $\{f_\alpha\}$ is represented in $\mathbb{C}^N$ by the canonical basis $\{e_l\}$; therefore the matrix representation V of $T_{a=1} := T$ in the $\{f_\alpha\}$ basis is the $N \times N$ matrix given by

$$V = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix};$$

note that $V = C^{\intercal}$, where C is the circulant permutation matrix with entries $C_{kl} = \delta_{k,l-1}$. Thus the full set of matrices, defined as $V_j = V^j$, are unitary and $V^N = I$, $0 \leq j \leq (N-1)$. These matrices V_j are well known in the literature as the shift matrices and are a basis for the algebra of circulant matrices [14].

On the other hand, the regular representation $\rho : Z_N \to GL(L^2(\mathbb{Z}/N\mathbb{Z}))$ of the cyclic group Z_N is given by $\rho(a_j) = V_j$, $0 \leq j \leq (N-1)$ ([15], p.4), this implies that the matrix representation of the translation operators on the function space $L^2(\mathbb{Z}/N\mathbb{Z})$ is the regular representation of the N-cyclic group, which in turn is completely reducible. Moreover, since $\mathbb{Z}/N\mathbb{Z}$ is abelian, it can be decomposed into a direct sum of one-dimensional irreducible representations. This simple decomposition is the source of a rather intricate structure which gives rise to the Fourier analysis, structure that is employed in this work.

Let C_N^1 be the multiplicative group of complex numbers of absolute value 1, a character on $\mathbb{Z}/N\mathbb{Z}$ is a group homomorphism $\lambda : \mathbb{Z}/N\mathbb{Z} \longrightarrow C_N^1$. Taking characters in $\lambda_1, \lambda_2 \in Hom(\mathbb{Z}/N\mathbb{Z})$ and defining $(\lambda_1+\lambda_2)(a) := \lambda_1(a)\lambda_2(a)\ \ \forall a \in \mathbb{Z}/N\mathbb{Z}$, it can be concluded that $\mathbb{Z}/N\mathbb{Z} \cong Hom(\mathbb{Z}/N\mathbb{Z}) \cong C_N$. It can be shown that $Hom(\mathbb{Z}/N\mathbb{Z})$ consists of elements of the form

$$\lambda_l := \frac{1}{\sqrt{N}}\, e^{-2\pi \mathrm{i} l/N}, \quad l = 0, \ldots, N-1,$$

and the set $\{\lambda_l\}$, $l = 0, \ldots, N-1$, is an orthonormal basis of $L^2(\mathbb{Z}/N\mathbb{Z})$, which is a consequence of the geometric series (3.1). We call this basis $\{\lambda_l\}$, $0 \le l \le (N-1)$, the normalized character basis, NCB, for $L^2(\mathbb{Z}/N\mathbb{Z})$. Because of this, we can expand $f \in L^2(\mathbb{Z}/N\mathbb{Z})$ as a linear combination of the $\{\lambda_l\}$ basis, namely

$$f = \sum_{l=0}^{N-1} \hat{f}_l \lambda_l, \qquad \hat{f}_l \in \mathbb{C};$$

such expansion is the (Nth partial sum of the discrete) Fourier series of f. The coefficients $\hat{f}_l$ can be obtained applying orthonormality of the NCB as usual:

$$\begin{aligned} \langle f, \lambda_m \rangle &= \langle \sum_{l=0}^{N-1} \hat{f}_l \lambda_l, \lambda_m \rangle = \sum_{l=0}^{N-1} \hat{f}_l \langle \lambda_l, \lambda_m \rangle \\ &= \sum_{l=0}^{N-1} \hat{f}_l \delta_{lm} = \hat{f_m}, \quad 0 \le m \le (N-1); \end{aligned}$$

whereby

$$\hat{f_m} = \langle f, \lambda_m \rangle = \sum_{n=0}^{N-1} f(n)\overline{\lambda_m(n)} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{2\pi \mathrm{i} mn/N}, \quad 0 \le m \le (N-1).$$

This is clearly the mth component of a matrix multiplication with the vector $(f(0), f(1), \ldots, f(N-1))$. We give a name to the underlying linear transformation.

Definition 3.1. *Let $\{\lambda_m\}$, $l = 0, \ldots, N-1$, be the normalized character basis and define $\hat{f_m}$ as $(\Phi_N f)(m)$. The discrete Fourier transform (DFT) is the linear operator $\Phi_N : L^2(\mathbb{Z}/N\mathbb{Z}) \to L^2(\mathbb{Z}/N\mathbb{Z})$, defined by*

$$(\Phi_N f)(m) := \langle f, \lambda_m \rangle = \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} f(n) \exp\left(\frac{2\pi \mathrm{i}}{N} mn\right).$$

The DFT is known to satisfy the following properties:

(1) Φ_N is a unitary operator,
(2) $\Phi_N^4 = I$, where I is the identity operator,
(3) the matrix representation of Φ_N in $\{f_\beta\}$ is

$$(\Phi_N)_{mn} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi \mathrm{i}}{N} mn\right), \quad 0 \le m, n \le (N-1);$$

this implies that the columns of the DFT matrix are an orthonormal basis $\{\epsilon_k\}$ for $\mathbb{C}^N$. We call this basis the normalized Fourier basis (NFB).

Next we get back to the shift matrices to connect them with the DFT. Since the V_j are unitary, they must be unitarily similar to a diagonal matrix because of the *spectral theorem*. Thus the DFT plays a fundamental role in this work because the following holds:

Theorem 3.1. *The DFT Φ_N simultaneously diagonalizes the V_j matrices, with eigenvalues*

$$\alpha_{jl} = exp\left(\frac{2\pi\mathrm{i}}{N} jl\right), \qquad 0 \le j, l \le (N-1),$$

and the columns of the DFT as eigenvectors.

Let $\rho_1 : G \longrightarrow U(\mathcal{H}_1)$ and $\rho_2 : G \longrightarrow U(\mathcal{H}_2)$ be unitary representations on G over the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We say that the operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ intertwines ρ_1 and ρ_2 if $A\rho_1(g) = \rho_2(g)A, \ \forall g \in G$; thus ρ_1 and ρ_2 are said to be unitarily equivalent if A is unitary and the operator A is called an intertwining operator. If U is the diagonal operator, obtained through diagonalization of V by the DFT, then U_j and V_j are unitarily equivalent representations of $\mathbb{Z}/N\mathbb{Z}$ with the DFT as intertwining operator. Namely, $V_j = \Phi_N U_j \Phi_N^\dagger, \ j = 0, \ldots, N-1$. With $\omega = \exp(2\pi\mathrm{i}/N)$ the Nth primitive root of unity, $\{\epsilon_k\}$ the NFB and $\{e_k\}$ the canonical basis, the previous theorem implies that

(1) The eigenvectors of U and V satisfy

$$V_j \epsilon_k = \omega^{kj} \epsilon_k \text{ and } U_j e_k = \omega^{kj} e_k,$$

(2) V acts as a shift on the eigenvectors of U and viceversa

$$V_j e_k = e_{k+j} \text{ and } U_j \epsilon_k = \epsilon_{k+j},$$

(3) $U_N = I$;

due to these properties, the matrices U_j are called the clock matrices.

Finally, by applying these properties, $UVe_k = Ue_{k+1} = \omega^{k+1} e_{k+1} = \omega\, \omega^k e_{k+1}$; $VUe_k = V\omega^k e_k = \omega^k V e_k = \omega^k e_{k+1}$. Combining these results, we get $(UV - \omega VU)e_k = 0, \forall k \in \mathbb{Z}/N\mathbb{Z}$, whereof we conclude that the shift and clock matrices V and U satisfy the so-called Weyl commutation relation, namely, $UV = e^{\frac{2\pi\mathrm{i}}{N}} VU$.

4. An algebra with discrete commutation relations

Because of the very definition of translation, the operator V generates translations in the underlying space $L^2(\mathbb{Z}/N\mathbb{Z})$, thus, we look for Hermitian solutions P, whose exponentiation yields the unitary T in complete analogy with eq.(2.6). This request establishes the connection with the quantum picture. This idea is not new; it was previously explored by Santhanam and Tekumalla [16] using a different approach.

Theorem 4.1. *Let j be an element of $\mathbb{Z}/N\mathbb{Z}$. Then a hermitian operator $P_j \in GL(L^2(\mathbb{Z}/N\mathbb{Z}))$, which is a solution of the equation $V_j = exp(i\eta P_j)$, where η is a real parameter, is given by*

$$P_j = \frac{2\pi}{\eta N} \Phi_N \, diag\,(0, j, \ldots, (N-1)j) \Phi_N^{-1}. \tag{4.1}$$

Proof. Due to the unitarity of Φ_N,

$$\begin{aligned} P_j^\dagger &= \frac{2\pi}{\eta N} \left(\Phi_N \mathrm{diag}(0, j, \ldots, (N-1)j) \Phi_N^\dagger\right)^\dagger \\ &= \frac{2\pi}{\eta N} \Phi_N \mathrm{diag}(0, j, \ldots, (N-1)j) \Phi_N^\dagger = P_j, \end{aligned}$$

thus confirming that P_j is Hermitian. Now we show that under exponentiation, we certainly recover V_j. As said before, exponentiation of P_j is well defined since P_j is a matrix with finite elements, it is a series of P_j which converges absolutely and yields a matrix which is invertible; moreover, when multiplied by the imaginary unit i, this matrix becomes a skew-adjoint matrix, which upon exponentiation produces a unitary matrix. Then let $\eta \in \mathbb{R}$ and let's compute the matrix representation of $\exp(\mathrm{i}\eta P_j)$ in the canonical basis $\{e_l\}$; using the unitarity of Φ_N

$$\begin{aligned}\exp(\mathrm{i}\eta P_j) &= \exp\left(\mathrm{i}\eta\frac{2\pi}{\eta N}\Phi_N \mathrm{diag}(0, j, \dots, (N-1)j)\Phi_N^{-1}\right)\\ &= \Phi_N \exp\left(\frac{2\pi \mathrm{i} j}{N}\mathrm{diag}(0, 1, \dots, N-1\right)\Phi_N^{-1},\end{aligned}$$

multiplying by Φ_N on the right and applying e_l, we get

$$\begin{aligned}e^{\mathrm{i}\eta P_j}\Phi_N e_l &= \Phi_N I e_l + \Phi_N \frac{2\pi \mathrm{i} j}{N}\mathrm{diag}(0, 1, \dots, N-1)e_l\\ &\quad + \Phi_N \frac{(2\pi \mathrm{i} j)^2}{2!N^2}\mathrm{diag}^2(0, 1, \dots, N-1)e_l + \cdots\\ &= \Phi_N\left(1 + \frac{2\pi \mathrm{i} j}{N}l + \frac{(2\pi \mathrm{i} j)^2}{2!N^2}l^2 + \cdots\right)e_l\\ &= \Phi_N e^{2\pi \mathrm{i} jl/N}e_l = \Phi_N U_j e_l, \qquad \forall j, l \in \{0, 1, \dots, N-1\}.\end{aligned}$$

Whereby, using the intertwining property of Φ_N, it follows

$$e^{\mathrm{i}\eta P_j}\Phi_N e_l = \Phi_N \Phi_N^{\dagger} V^j \Phi_N e_l = V^j \Phi_N e_l,\ \forall j, l \in \{0, 1, \dots, N-1\}.$$

Thus,

$$\left(e^{\mathrm{i}\eta P_j}\Phi_N - V^j \Phi_N\right)e_l = 0, \quad \forall j, l \in \{0, 1, \dots, N-1\},$$

and consequently

$$e^{\mathrm{i}\eta P_j}\Phi_N = V^j \Phi_N, \quad \forall j \in \{0, 1, \dots, N-1\},$$

from which it follows that

$$e^{\mathrm{i}\eta P_j} = V^j, \quad \forall j \in \{0, 1, \dots, N-1\}.$$

□

Remark. *Clearly, the eigenvalues of P_j are $2\pi jl/(\eta N)$ with the NFB as eigenvectors. Similarly, by exponentiating the U_j, we obtain the same eigenvalues.*

Let the diagonal operators be denoted as

$$Q_j := \frac{2\pi}{\eta N}\mathrm{diag}(0, j, \dots, (N-1)j);$$

thus the DFT intertwines the operators P_j and Q_j as in eq.(2.1). Thereby the DFT intertwines operators at the group level (the V_j's) and at some algebra level as well (the P_j's); this is a direct consequence of the exponential map and the unitarity of the DFT. Thus intertwining is a necessary condition to build possible algebras at the Schrödinger realization level, but not sufficient.

Next, we consider commutators of skew-adjoint operators,

$$\frac{1}{\mathrm{i}}[\mathrm{i}Q, \mathrm{i}P] = \mathrm{i}[Q, P],$$

with

$$Q := \frac{2\pi}{\eta N}\mathrm{diag}(0, 1, \dots, N-1), \qquad P := \Phi_N Q \Phi_N^{-1}. \tag{4.2}$$

Remark. *The real number η is a coupling parameter which will take appropriate values according to the finite discrete (DFT), infinite discrete (Fourier series) or continuous (integral Fourier transform) cases we deal with.*

Since $[Q,P]$ remains constant for the unitary transformation uniparametric group $\{V_j\}_{j=0,\ldots,N-1}$, this suggest the following definition.

Definition 4.1. *Let the commutator of $Q:\mathbb{C}^N \longrightarrow \mathbb{C}^N$ and $P:\mathbb{C}^N \longrightarrow \mathbb{C}^N$ be defined by Z_N through*

$$Z_N := \mathrm{i}[Q,P].$$

Thus we get

Corollary 4.2. *The operators Q, P, Z_N satisfy the Jacobi identity*

$$[Q,[P,Z_N]] + [P,[Z_N,Q]] + [Z_N,[Q,P]] = 0.$$

Proof. Direct computation of the commutators yields

$$[Q,[P,Z_N]] = \mathrm{i}(2(QP)^2 - Q^2P^2 - 2(PQ)^2 + P^2Q^2),$$

$$[P,[Z_N,Q]] = \mathrm{i}(2(PQ)^2 - P^2Q^2 - 2(QP)^2 + Q^2P^2),$$

$$[Z_N,[Q,P] = 0;$$

therefore

$$[Q,[P,Z_N]] + [P,[Z_N,Q]] + [Z_N,[Q,P]] = 0.$$

□

In addition, the following relationships are established.

Corollary 4.3. *The operators Q_j, P_k and Z_N, $j,k = 0,\ldots,N-1$, satisfy the discrete commutation relations*

$$[P_j,P_k] = 0, \quad [Q_j,Q_k] = 0, \quad [P_j,Q_k] = \mathrm{i}jkZ_N.$$

Proof. Since

$$Q = \frac{2\pi}{\eta N}\mathrm{diag}(0,1,\ldots,N-1),$$

then, by Theorem 4.1 and eqs.(4.2), $Q_j = jQ$, thereby $P_j = \Phi_N Q_j \Phi_N^{-1} = \Phi_N jQ\Phi_N^{-1} = jP$, so that $[P_j,Q_k] = P_jQ_k - Q_kP_j = jk(PQ-QP) = -\mathrm{i}^2 jk[P,Q] = -\mathrm{i}jk\mathrm{i}[P,Q] = \mathrm{i}jkZ_N$. The other commutators are trivially satisfied. □

Therefore, Z_N plays the role of the identity in this discrete algebra.

5. About the nature of Z_N

As discussed in the previous section, the operator Z_N plays the role of the identity in this context. In this section we explore more the operator Z_N and suggest that in the limit when $N \to \infty$, $Z_N \to \delta$, where δ is the Dirac delta distribution, which is the identity in distributions under convolution.

5.1. **The explicit form of Z_N.** First we need to know the explicit form of Z_N. So we compute the action of the operators Q_j and P_j on the NFB, to get the explicit form of the matrix entries of $[Q_j, P_k]$. It is clear that P_j is in the canonical representation; since the DFT diagonalizes it, its inverse acts as a transition matrix from $\{e_l\}$ to $\{\epsilon_l\}$, resulting Q_j, which is in the NFB representation (note that the definition we are using for the DFT is with positive sign in the exponent). Since eigenvectors are preserved under the exponential map, the eigenvectors of P_j are the NFB, and because Q_j is diagonal, those of it are the $\{e_l\}$. Also, the corresponding eigenvalues are the exponents of those of V_j, U_j, namely,

$$P_j \epsilon_k = \frac{2\pi jk}{\eta N} \epsilon_k, \qquad Q_j e_k = \frac{2\pi jk}{\eta N} e_k, \tag{5.1}$$

and

$$\epsilon_k = \sum_m \Phi_{mk} e_m, \qquad e_k = \sum_m \Phi^{-1}_{mk} \epsilon_m, \tag{5.2}$$

where Φ_{mk} stands for the matrix entries of Φ_N.

We now compute the transformation of the NFB by the operator $P_k Q_j$ and express it in terms of itself.

Lemma 5.1. *The operator $P_k Q_j$ transforms the NFB as*

$$P_k Q_j \epsilon_m = \frac{4\pi^2 jk}{\eta^2 N^3} \sum_n \sum_{n'} n n' \omega^{n(m-n')} \epsilon_{n'}, \quad 0 \le k, j, m, n, n' \le (N-1).$$

Proof. First we compute $Q_j \epsilon_m$, $0 \le j, m \le (N-1)$ by using the first of eqs.(5.2), to apply the eigenvalue property of Q_j; then the second of eqs.(5.1) to obtain the corresponding eigenvalues; finally the second of eqs.(5.2) to express the result in the NFB basis:

$$\begin{aligned} Q_j \epsilon_m &= Q_j \sum_n \frac{1}{\sqrt{N}} \omega^{nm} e_n = \sum_n \frac{1}{\sqrt{N}} \omega^{nm} Q_j e_n \\ &= \frac{1}{\sqrt{N}} \sum_n \omega^{nm} \frac{2\pi j}{\eta N} n e_n = \frac{2\pi j}{\eta N^2} \sum_n \omega^{nm} n \sum_{n'} \omega^{-nn'} \epsilon_{n'} \\ &= \frac{2\pi j}{\eta N^2} \sum_n \sum_{n'} n \omega^{n(m-n')} \epsilon_{n'}. \end{aligned} \tag{5.3}$$

Next we get

$$P_k Q_j \epsilon_m = \frac{2\pi j}{\eta N^2} \sum_n \sum_{n'} n \omega^{n(m-n')} \epsilon_{n'} \frac{2\pi k n'}{\eta N} \epsilon_{n'} \quad = \quad \frac{4\pi^2 jk}{\eta^2 N^3} \sum_n \sum_{n'} n n' \omega^{n(m-n')} \epsilon_{n'}.$$

□

Therefore the matrix elements of the commutator satisfy the following

Theorem 5.2. *The matrix elements of $[Q_j, P_k]$ in the NFB are given by*

$$[Q_j, P_k]_{lm} = \frac{4\pi^2 jk}{\eta^2 N^3} \sum_n (m-l) n \omega^{n(m-l)},$$

where $j, k, l, m, n = 0, \ldots, N-1$.

Proof. Let $[Q_j, P_k]_{lm}$ be the (l, m) entry of the matrix representation of $[Q_j, P_k]$ in the NFB, then, because of eq.(5.3), Lemma 5.1, and orthonormality, it follows that

$$\begin{aligned}
[Q_j, P_k]_{lm} &= \langle \epsilon_l, [Q_j, P_k]\epsilon_m \rangle = \langle \epsilon_l, \frac{2\pi km}{\eta N} Q_j \epsilon_m - P_k Q_j \epsilon_m \rangle \\
&= \langle \epsilon_l, \frac{2\pi km}{\eta N} \frac{2\pi j}{\eta N^2} \sum_n \sum_{n'} n\omega^{n(m-n')} \epsilon_{n'} \\
&\quad - \frac{4\pi^2 jk}{\eta^2 N^3} \sum_n \sum_{n'} nn'\omega^{n(m-n')} \epsilon_{n'} \rangle \\
&= \frac{4\pi^2 jk}{\eta^2 N^3} \langle \epsilon_l, \sum_n \sum_{n'} (m-n') n\omega^{n(m-n')} \epsilon_{n'} \rangle \\
&= \frac{4\pi^2 jk}{\eta^2 N^3} \sum_n \sum_{n'} (m-n') n\omega^{n(m-n')} \langle \epsilon_l, \epsilon_{n'} \rangle \\
&= \frac{4\pi^2 jk}{\eta^2 N^3} \sum_n (m-l) n\omega^{n(m-l)},
\end{aligned}$$

where bilinearity of $\langle , \rangle$ has been used. □

To estimate the behaviour of Z_N for large N, it is necessary to recenter the matrix elements of $[Q_j, P_k]$ with respect to its entries lm. Set

$$r := \begin{cases} N = 2L+1, & \frac{N-1}{2} = L \\ N = 2M, & \frac{N}{2} = M \end{cases};$$

we define new centered indices l', m', n' the by means of $n' = n - r$, $m' = m - r$, $n' = n - r$, then

$$[Q, P]_{lm} = \frac{4\pi^2}{\eta^2 N^3} \omega^{(m'-l')r} \sum_{n'} (m' - l')(n' - r)\omega^{n'(m'-l')},$$

thereby

$$[Q, P]_{lm} = \omega^{(m-l)r} [Q, P]^C_{l'm'}, \tag{5.4}$$

where $[Q, P]^C_{l'm'}$ is given as in the following definition.

Definition 5.1. *$[Q, P]^C_{l'm'}$ is called the centered version of $[Q, P]_{lm}$, and is given by*

$$[Q, P]^C_{l'm'} = \frac{4\pi^2}{\eta^2 N^3} \sum_{n'} (m' - l')(n' + r)\omega^{n'(m'-l')},$$

where

$$l', m', n' := \begin{cases} -\frac{N-1}{2}, \ldots, \frac{N-1}{2} & \text{if } N \text{ is odd} \\ -\frac{N}{2}, \ldots, \frac{N}{2} - 1 & \text{if } N \text{ is even} \end{cases}.$$

The centered version can be simplified.

Proposition 5.3. *The centered version of $[Q, P]$ satisfies*

$$[Q, P]^C_{l'm'} = \frac{4\pi^2}{\eta^2 N^3} \sum_{n'} (m' - l') n' \omega^{n'(m'-l')}.$$

Proof. We take from Definition 5.1 the sum with factor $1/N$ and split it into two terms; so on account that $m'-l'=m-l$ one can write

$$\frac{1}{N}\sum_{n'}(m'-l')(n'+r)\omega^{n'(m'-l')} = \frac{1}{N}\sum_{n'}(m'-l')n'\omega^{n'(m'-l')} + \frac{1}{N}\sum_{n'}(m'-l')r\omega^{n'(m'-l')},$$

where the second term vanishies,

$$\begin{aligned}\frac{1}{N}\sum_{n'}(m'-l')r\omega^{n'(m'-l')} &= \frac{m-n}{N}r\sum_{n=0}^{N-1}\omega^{(n-r)(m-l)} \\ &= (m-l)r\omega^{-r(m-l)}\frac{1}{N}\sum_{n=0}^{N-1}\omega^{n(m-l)} \\ &= (m-l)r\omega^{-r(m-l)}\delta_{ml}=0,\ \forall\ m,l=0,\dots,N-1.\end{aligned}$$

The Kronecker delta δ_{ml} appears after using the geometric series in eq.(3.1); therefore, putting this in the Definition 5.1, we get the assertion. □

So the centered version is the non-centered times the phase factor $\omega^{-(m-l)r}$ for each matrix entry.

Corollary 5.4. *$Z_N = \mathrm{i}[Q,P]$ is a traceless Hermitian Toeplitz operator.*

Proof. This clearly follows from Theorem 5.2. □

5.2. **On the behaviour of Z_N for large N.** We are now in a position to roughly estimate the behaviour of $Z_N=\mathrm{i}[Q,P]$ for large N, provided that a restriction on the η parameter is given. In this subsection, we relax formality and rigor to gain intuition on Z_N. We are going to deal a little with tempered distributions as continuous linear functionals on the space of Schwartz functions and also with the Fourier transform of distributions. A gentle treatment of this concepts can be found in [17] and a rigorous one in [18].

Let $\mathcal{F}$ be the set of square integrable periodic functions $f:[-\pi,\pi]\longrightarrow\mathbb{C}$ with convergent Fourier series in the basis $\{\lambda_n(t)=e^{-\mathrm{i}nt} : n=0,1,...,N-1\}$. Then

$$f(t)=\sum_{-\infty}^{\infty}\hat{f}_n e^{-\mathrm{i}nt},$$

with

$$\langle f(t),\lambda_m(t)\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{\mathrm{i}nt}dt =: \hat{f}_n; \tag{5.5}$$

with $\hat{f}_n$ the Fourier transform of f defined through the inner product $\langle , \rangle$ on the corresponding Hilbert space $\mathcal{H}$. Then, introducing the Nth partial sum of f,

$$\begin{aligned} f(t) &= \sum_{-\infty}^{\infty} \hat{f}_n e^{-int} = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}_n e^{-int} \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} f(t') e^{int'} dt' e^{-int} = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} f(t') e^{in(t'-t)} dt' \\ &= \lim_{N\to\infty} \int_{-\infty}^{\infty} f(t') \sum_{n=-N}^{N} e^{in(t'-t)} dt = \lim_{N\to\infty} \int_{-\infty}^{\infty} f(t') D_N(t'-t) dt', \quad (5.6) \end{aligned}$$

where D_N is a well known kernel:

Definition 5.2. *The sequence of functions* $\{D_N\}, N \in \mathbb{N}$, *is called Nth Dirichlet kernel,*

$$D_N(t) := \sum_{n=-N}^{N} e^{int}.$$

It is also well known ([19]) that, using the geometric series (3.1), the Dirichlet kernel obeys, after inserting a factor of $2\pi/N$in the variable t

(1) $D_N(t) = \frac{\sin[(N+\frac{1}{2})t]}{\sin(t)}$,
(2) $\int_{-\pi}^{\pi} D_N(t) dt = 1$,
(3) $D_N(0) = 2N+1$.

Due to these facts, the Dirichlet kernel is taken as an approximating sequence of functions for the Dirac delta distribution δ. So in the sense of distributions we can write

$$\lim_{N\to\infty} D_N = \delta. \quad (5.7)$$

Definition 5.3. *If f and g are two integrable 2π periodic functions, then*

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy$$

is the convolution of f and g.

Hence, it inmediately follows that the partial sums in eq.(5.6) satisfy

$$S_N(f)(t) \quad := \quad \sum_{-N}^{N} \hat{f}_n e^{int} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t') D_N(t'-t) dt' = (f * D_N)(t);$$

thus $f(t) = \lim_{N\to\infty}(D_N * f)(t),\ \forall\ t \in (-\pi,\pi),\ \forall\ f \in \mathcal{F}$, so, using eq.(5.7), we can write

$$\delta * f = f \qquad \forall f \in \mathcal{F}, \quad (5.8)$$

because f determines a well defined tempered distribution through its action on test functions ϕ under the integral sign and, with an abuse of notation, it is common to write f instead of its induced distribution T_f. This means we can consider δ as the identity under convolution in the distributional sense.

Unfortunately, the Dirichlet kernel is known to be problematic as a kernel function and is not a good kernel in the sense of [19], pp. 99, 102, since as $N \to \infty$ the area between the curves and the x-axis measured in absolute value diverges. This makes the kernel very difficult to work with and one has to be very careful when dealing with integrals with absolute value because they may not converge. To avoid this difficulty, it is usual to use the so-called Fejér kernel instead, obtained through arithmetic mean of $D_0, D_1, \dots, D_{N-1}$, which is named a *Cesàro mean*. This remedy the problems because averaging frequently make things better behaved; thus, a non-negative good kernel is obtained (cf.[19], Remark 5, p.103). But convergence of the partial sums $S_N(f)(t)$ is guaranteed however, if f is Lipschitz or differentiable, then they converge pointwise everywhere.

Let us take the derivative of D_N:

$$D'_N(t'-t) := \frac{\mathrm{d}D_N(t'-t)}{\mathrm{d}t'} = \sum_{n=-N}^{N} \mathrm{i}ne^{\mathrm{i}n(t'-t)},$$

so that

$$(t'-t)D'_N(t'-t) = \sum_{n=-N}^{N} \mathrm{i}n(t'-t)e^{\mathrm{i}n(t'-t)}.$$

Now we take a sample of N data given by $t' = \frac{2\pi m}{N}, t = \frac{2\pi l}{N} \in [-\pi, \pi]$. When coupling continuous and discrete treatments, l, m and n are taken according to Definition 5.3. This avoids changing the 2π factor in the exponential. Therefore, using Lemma 5.3 and completing necessary factors we get

$$\begin{aligned}(m-l)D'_N(m-l) &= \sum_{n=-N}^{N} n(m-l)\exp\left(\frac{2\pi\mathrm{i}}{N}n(m-l)\right) \\ &= \frac{4\pi^2}{\eta^2 N^3}\frac{N^3\eta^2\mathrm{i}}{4\pi^2}\sum_{n=-N}^{N}(m-l)n\omega^{n(m-l)} = \frac{N^3\eta^2}{4\pi^2}\mathrm{i}[Q,P]_{lm} \\ &= Z_{lm}, \quad \text{provided } \eta = \frac{2\pi}{N^{3/2}};\end{aligned}$$

that is,

$$Z_{lm} = (m-l)D'_N(m-l). \tag{5.9}$$

A refined computation can be made if we consider a continuum of frequencies; then the Fourier series is replaced by the integral Fourier transform and η would take the value $\sqrt{2\pi/N}$.

The eq.(5.9) suggests the following claim: in the limit as $N \to \infty$, it is expected that $Z_{lm} \to (t'-t)\delta'(t'-t)$. Then, since $f(t)\delta'(t) = -f'(t)\delta(t)$, taking $f(t) = t'-t$, it is found that $Z_{lm} \to -(-1)\delta(t'-t) = \delta(t'-t)$ as $N \to \infty$. Thus, in accordance with eq.(5.8), the limit of Z_N is conjectured to be the identity under convolution in the vector space of tempered distributions. In addition, since Z_N is represented by a Toeplitz matrix, it is natural to ask if its entries are the coefficients of the Fourier series of a real valued function, as it happens for Toeplitz matrices when their entries are the Fourier coefficients of an L^1 function or of a Radon function, converging to zero as $N \to \infty$ for the former and remaining bounded for the latter. Thus, as $N \to \infty$ we wonder what is this real valued function whose Fourier transform is the Dirac δ. It is known that such a function is 1; that is to say, to be precise, the Fourier transform of the distribution T_1 induced by 1 is the distribution δ (see [17], p.203). In the study of algebras of Toeplitz operators, such a real valued function is called the *symbol* of the Toeplitz operator; thus, in our context, the symbol of the limit of the Toeplitz Z_N is the constant 1 as a distribution. The issues about boundedness and compactness is analyzed in [20] for symbols as functions and in [21] for symbols as distributions. Finally, the formal analysis about the veracity of these claims requires the rigorous application of distribution theory and it is the topic for future work.

6. An algebraic system associated with the DFT

Now we proceed to the construction of operators with the ladder property, starting from their generic definition and giving to Z_N its role as a form of identity in its own right. We do not use the traditional definitions $(1/\sqrt{2})(Q \pm \mathrm{i}P)$ since there is no a priori reason to assume they should be valid in the context of the discrete commutation relations. This implies relaxing the condition of the "canonical commutation relations (CCR) equal to a constant" ([22], [23]) and give enough freedom to look for solutions associated to the DFT, involving new operators $\mathscr{H}$, L^- and L^+, but taking a sort of compatibility relations as elemental and even lift them to a fundamental postulate in the context we are dealing with. These compatibility relations are also named compatibility of Hamilton's equations with the Heisenberg equations in [24], p.5 and [25], p.3., as well as Heisenberg-Schrödinger consistency relations. A similar study can be found in [26].

6.1. **Discrete algebraic system from Z_N.** To obtain the form of the compatibility relations in the context we are dealing with, let us take for a while the standard definitions $L = (1/\sqrt{2})(Q + \mathrm{i}P)$, $R = (1/\sqrt{2})(Q - \mathrm{i}P)$ and notice that they imply that

$$\mathscr{H} = \frac{1}{2}\{R, L\}, \qquad Z_N = [R, L].$$

From this it follows that the commutation relations

$$[\mathscr{H}, R] = -\frac{1}{2}\{R, Z_N\}, \qquad [\mathscr{H}, L] = \frac{1}{2}\{L, Z_N\},$$

are valid and one gets

$$[\{R, L\}, R] = -\{R, Z_N\}, \qquad [\{R, L\}, L] = \{L, Z_N\}.$$

We call the above relations the discrete compatibility relations (DCR) and refer to the operator $\mathscr{H}$ as the Hamiltonian operator. Now we leave the standard definitions of L and R and at the same time, leverage such relations as a fundamental postulate in the context of the discrete commutation relations, for their continuous counterpart is not guaranteed to be so in quantum mechanics as is mentioned by Wigner in [27]. Further, the fact that $Z_N, N \in \mathbb{N}$, determines a sequence of Hermitian Toeplitz matrices according to Corollary 5.4, allows to naturally connect those relations with the DFT, in the sense that the entries of Z_N are linked to an approximating sequence of a distribution, whose Fourier transform is the Dirac δ, namely, the distribution T_1 induced by the constant 1, as discussed above.

On the other hand, Toeplitz matrices can be split into two parts, so that

$$Z_N = \sum_{k=1}^{N-1} z_{-k}(B^T)^k + \sum_{k=0}^{N-1} z_k B^k,$$

where B is the backward shift matrix satisfying $B^T = KBK$ with K the reversal matrix defined by ones in the antidiagonal and zero otherwise.

Taking the parity operator, used in the AA-approach, as $S = KV$ with V the basic circulant matrix, there seems to be an enriched structure which provides a possible framework that we believe it is worthwhile analyzing.

Summarizing, the following algebraic system is proposed (note that we will also refer to it as an algebraic scheme in this work).

Construct operators $L^- : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ and $L^+ : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ such that

(1) The operator $\mathscr{H} := (1/2)\{L^+, L^-\}$ is Hermitian (semi-)positive definite;
(2) The discrete compatibility relations (DCR) must be satisfied,

$$[\{L^+, L^-\}, L^+] = -\{L^+, Z_N\}, \ [\{L^+, L^-\}, L^-] = \{L^-, Z_N\}; \tag{6.1}$$

(3) Parity–point reflection–condition: to split the eigenvectors in even and odd parts,

$$[\{L^+, L^-\}, S] = 0; \tag{6.2}$$

(4) Commutation with the DFT: to obtain an eigensystem for Φ_N from the Hamiltonian $\mathscr{H}$,

$$[\{L^+, L^-\}, \Phi_N] = 0. \tag{6.3}$$

In this scheme, non-equally spaced eigenvalues of the Hamiltonian are allowed, for this property plays a key role in quantum information processes. To see if there exist models for this system, the general form for a discrete Hermitian operator is used

$$\mathscr{H} = \sum_{n=0}^{N-1} \xi_n \phi_n \langle \phi_n, \cdot \rangle,$$

where ϕ_n is an eigenbasis of $\mathscr{H}$. Then use the ladder-type generic form to construct creation and annihilation operators through

$$L^+ := \sum_{n=0}^{N-1} r_n \phi_{n+1} \langle \phi_n, \cdot \rangle, \qquad L^- := \sum_{n=0}^{N-1} l_n \phi_{n-1} \langle \phi_n, \cdot \rangle.$$

Sufficient conditions to solve this scheme are provided in what follows.

6.2. **Solving the discrete compatibility relations.** Following the algebraic system proposed in the last paragraph, we establish now sufficient criteria to find solutions for the first two requirements, at least. We aim to establish the conditions under which Hamiltonians can be constructed and to understand their relationship with the DFT within the framework of the DCR. In what follows, bra-ket notation is used.

Let $\mathscr{H}$ be a (semi-)positive definite Hermitian operator on $L^2(\mathbb{Z}/N\mathbb{Z})$ and $\{|\xi_j\rangle | j = 0, \ldots, N-1\}$ be a complete set of eigenvectors of $\mathscr{H}$, such that

$$\mathscr{H}|\xi_j\rangle = \xi_j |\xi_j\rangle, \tag{6.4}$$

also

$$\sum_{n=0}^{N-1} |\xi_n\rangle\langle\xi_n| = I,$$

therefore

$$\mathscr{H} = \sum_{n=0}^{N-1} \xi_n |\xi_n\rangle\langle\xi_n|.$$

Definition 6.1. *The creation and annihilation linear operators $L^+, L^- : \mathbb{C}^N \longrightarrow \mathbb{C}^N$, are defined by*

$$L^+ := \sum_{n=0}^{N-1} l_n^+ |\xi_{n+1}\rangle\langle\xi_n|, \qquad L^- := \sum_{n=0}^{N-1} l_n^- |\xi_{n-1}\rangle\langle\xi_n|.$$

We establish some results to determine under which conditions solutions $L^\pm$ can be found, such that

$$[\{L^+, L^-\}, L^+] = -\{L^+, Z_N\}, \quad [\{L^+, L^-\}, L^-] = \{L^-, Z_N\}$$

holds. Thus we have the following

Lemma 6.1. *$|\xi_j\rangle$ is an eigenvector of $\{L^+, L^-\}$ with eigenvalue $l_{j-1}^+ l_j^- + l_j^+ l_{j+1}^-$.*

Proof. We compute directly, using definitions and orthonormality, that

$$\begin{aligned}
\{L^+, L^-\}|\xi_j\rangle &= (L^+L^- + L^-L^+)|\xi_j\rangle \\
&= L^+ \sum_{n=0}^{N-1} l_n^- |\xi_{n-1}\rangle\langle\xi_n|\xi_j\rangle + L^- \sum_{n=0}^{N-1} l_n^+ |\xi_{n+1}\rangle\langle\xi_n|\xi_j\rangle \\
&= L^+ l_j^- |\xi_{j-1}\rangle + L^- l_j^+ |\xi_{j+1}\rangle \\
&= \sum_{n=0}^{N-1} l_n^+ |\xi_{n+1}\rangle\langle\xi_n| l_j^- |\xi_{j-1}\rangle + \sum_{n=0}^{N-1} l_n^- |\xi_{n-1}\rangle\langle\xi_n| l_j^+ |\xi_{j+1}\rangle \\
&= (l_{j-1}^+ l_j^- + l_j^+ l_{j+1}^-)|\xi_j\rangle.
\end{aligned}$$

□

Lemma 6.2. *The matrix representations for $[\{L^+, L^-\}, L^\pm]$ in the $\{|\xi_j\rangle\}$ basis, are given by*

$$[\{L^+, L^-\}, L^-]_{kj}^\xi = (l_{j-2}^+ l_{j-1}^- - l_j^+ l_{j+1}^-) l_j^+ \delta_{k,j-1},$$

$$[\{L^+, L^-\}, L^+]_{kj}^\xi = (l_{j+1}^+ l_{j+2}^- - l_{j-1}^+ l_j^-) l_j^+ \delta_{k,j+1},$$

respectively.

Proof. Computing the commutator with L^- using the previous Lemma 6.1,

$$\begin{aligned}
[\{L^+, L^-\}, L^-]|\xi_j\rangle &= (\{L^+, L^-\}L^- - L^-\{L^+, L^-\})|\xi_j\rangle \\
&= \{L^+, L^-\} \sum_{n=0}^{N-1} l_n^- |\xi_{n-1}\rangle\langle\xi_n|\xi_j\rangle - L^-(l_{j-1}^+ l_j^- + l_j^+ l_{j+1}^-)|\xi_j\rangle \\
&= \{L^+, L^-\} l_j^- |\xi_{j-1}\rangle - (l_{j-1}^+ l_j^- + l_j^+ l_{j+1}^-) l_j^- |\xi_{j-1}\rangle \\
&= (l_{j-2}^+ l_{j-1}^- - l_j^+ l_{j+1}^-) l_j^- |\xi_{j-1}\rangle,
\end{aligned}$$

whereby, the orthonormality of $\{|\xi_j\rangle\}$ implies

$$\begin{aligned}
\langle\xi_k|[\{L^+, L^-\}, L^-]|\xi_j\rangle &= \langle\xi_k|(l_{j-2}^+ l_{j-1}^- - l_j^+ l_{j+1}^-) l_j^- |\xi_j\rangle \\
&= (l_{j-2}^+ l_{j-1}^- - l_j^+ l_{j+1}^-) l_j^- \delta_{k,j-1}.
\end{aligned}$$

The second equation is obtained similarly. □

Next, we obtain the matrix representation of $\{L^{\pm}, Z_N\}$. Let $Z^{\xi}_{\alpha\beta}$ be the matrix representation of Z_N in $\{|\xi_j\rangle\}$ basis and recall that $\{|e_j\rangle\}$ is the canonical basis. It is clear then that, if $C_{\alpha\beta} := \langle\xi_\alpha|e_m\rangle$ represents the transition matrix from $\{|e_j\rangle\}$ to $\{|\xi_j\rangle\}$, we have

$$Z^{\xi}_{\alpha\beta} = \sum_{m,n} C_{\alpha m} Z^{e}_{mn} C^{-1}_{n\beta} = \sum_{m,n} C_{\alpha m} Z^{e}_{mn} \bar{C}_{\beta n} = \sum_{m,n} \langle\xi_\alpha|e_m\rangle Z^{e}_{mn} \langle e_n|\xi_\beta\rangle,$$

since C is a unitary matrix.

Lemma 6.3. *The matrix representations of $\{L^{\pm}, Z_N\}$ in the $\{|\xi_j\rangle\}$ basis, are given by*

$$\{L^{\pm}, Z_N\}^{\xi}_{kj} = l^{\pm}_{k\mp1} Z^{\xi}_{k\mp1,j} + l^{\pm}_{j} Z^{\xi}_{k,j\pm1},\ \ k,j = 0,1,\dots,N-1,$$

where $Z^{\xi}_{\alpha\beta}$ are the entries of the matrix representation of Z_N in the $\{|\xi_j\rangle\}$ basis.

Proof. Since $Z^{\xi}_{\alpha\beta}$ is the representation of Z_N in the $\{|\xi_j\rangle\}$ basis, then the action of Z_N on a basis vector can be expanded as $Z_N|\xi_j\rangle = \sum_\alpha Z^{\xi}_{\alpha j}|\xi_\alpha\rangle$, so that

$$\begin{aligned}\{L^{\pm}, Z_N\}^{\xi}_{kj} &= \langle\xi_k|L^{\pm}Z_N + Z_N L^{\pm}|\xi_j\rangle = \langle\xi_k|L^{\pm}\sum_\alpha Z^{\xi}_{\alpha j}|\xi_\alpha\rangle + \langle\xi_k|Z_N l^{\pm}_j|\xi_{j\pm1}\rangle \\ &= \langle\xi_k|\sum_\alpha Z^{\xi}_{\alpha j} l^{\pm}_\alpha|\xi_{\alpha\pm1}\rangle + l^{\pm}_j Z^{\xi}_{k,j\pm1} = l^{\pm}_{k\mp1} Z^{\xi}_{k\mp1,j} + l^{\pm}_j Z^{\xi}_{k,j\pm1}.\end{aligned}$$

□

Therefore, putting all this together, one can readily see that the DCR are equivalent to

$$(l^{\pm}_{j\pm1} l^{\mp}_{j\pm2} - l^{\pm}_{j\mp1} l^{\mp}_{j}) l^{\pm}_j \delta_{k,j\pm1} = \mp l^{\pm}_{k\mp1} Z^{\xi}_{k\mp1,j} \mp l^{\pm}_j Z^{\xi}_{k,j\pm1},\ \ \forall k,j = 0,1,\dots,N-1. \quad (6.5)$$

Now conjugate transposition between the $L^{\pm}$ operators is imposed, and the Hermiticity of Z_N becomes essential to employ.

Proposition 6.4. *Let's suppose that the DCR hold. If $(L^-)^\dagger = L^+$, then*

(1) *the DCR are equivalent, and*
(2) *they reduce to*

$$(|l^-_{j-1}|^2 - |l^-_{j+1}|^2) l^-_j \delta_{k,j-1} = l^-_{k+1} Z^{\xi}_{k+1,j} + l^-_j Z^{\xi}_{k,j-1},\ \ \forall k,j = 0,1,\dots,N-1. \quad (6.6)$$

Proof. First consider that

$$\begin{aligned}(L^-)^\dagger|\xi_j\rangle &= \left(\sum_{k=0}^{N-1} l^-_k |\xi_{k-1}\rangle\langle\xi_k|\right)^\dagger |\xi_j\rangle = \left(\sum_{k=0}^{N-1} \overline{l^-_k} |\xi_k\rangle\langle\xi_{k-1}|\right)|\xi_j\rangle \\ &= \sum_{k=0}^{N-1} \overline{l^-_k} |\xi_k\rangle \delta_{k-1,j},\ \ k-1 = j, \\ &= \overline{l^-_{j+1}} |\xi_{j+1}\rangle;\end{aligned}$$

on the other hand $L^+|\xi_j\rangle = l^+_j|\xi_{j+1}\rangle$, thus, if $(L^-)^\dagger = L^+$, we necessarily have $(\overline{l^-_{j+1}} - l^+_j)|\xi_{j+1}\rangle = 0,\ \forall\ j = 0,\dots,N-1$, whereby $l^+_j = \overline{l^-}_{j+1}$. Therefore, using this and eq.(6.5) we get for the DCR

$$(|l^-_{j+2}|^2 - |l^-_j|^2)\overline{l^-_{j+1}} \delta_{k,j+1} = -\overline{l^-_k} Z^{\xi}_{k-1,j} - \overline{l^-_{j+1}} Z^{\xi}_{k,j+1},$$

$$(|l_{j-1}^-|^2 - |l_{j+1}^-|^2) l_j^- \delta_{k,j-1} = l_{k+1}^- Z_{k+1,j}^\xi + l_j^- Z_{k,j-1}^\xi .$$

So to probe part 1 of the proposition, it is enough to conjugate transpose anyone of the last equations to obtain the other using the Hermiticity of Z_N (Corollary 5.4) and a change of index. Part 2 clearly follows as a consequence of the equivalence in part 1 picking the second expression. □

Thus, when imposing the conjugate transposition condition, we can deal with only one of the DCR contained in eq.(6.5), namely that of part 2 of the last proposition; it deploys into the following two equations after applying the definition of the Kronecker delta:

$$|l_{j-1}^-|^2 - |l_{j+1}^-|^2 = Z_{jj} + Z_{j-1,j-1}, \ k = j-1, \tag{6.7}$$

$$l_{k+1}^- Z_{k+1,j}^\xi + l_j^- Z_{k,j-1}^\xi = 0, \ k \neq j-1. \tag{6.8}$$

This means that we have a set of N^2 equations, which we would like to solve for $l^- := \{l_0^-, l_1^-, \dots, l_{N-1}^-\}$; or equivalently, to solve $2N-1$ recurrence relations: the first one given by eq.(6.7) with $k = j-1$ and the other remaining ones correspond to $k = j-N+1, j-N, j-N-1, \dots, j-2, j, j+1, j+2, \dots, j+N-2$ given by eq.(6.8). We make the following correspondence about the indices: $N \to 0$ and $-1 \to N-1$, for instance, $l_N = l_0$ and $l_{-1} = l_{N-1}$. Expanding the first recurrence relation, we deploy N of the N^2 equations as

$$\begin{aligned}
|l_{N-1}^-|^2 - |l_1^-|^2 &= Z_{00}^\xi + Z_{N-1,N-1}^\xi, \ j=0, k=N-1 \\
|l_0^-|^2 - |l_2^-|^2 &= Z_{11}^\xi + Z_{00}^\xi, \ j=1, k=0, \\
|l_1^-|^2 - |l_3^-|^2 &= Z_{22}^\xi + Z_{11}^\xi, \ j=2, k=1, \\
&\vdots \\
|l_{N-2}^-|^2 - |l_0^-|^2 &= Z_{N-1,N-1}^\xi + Z_{N-2,N-2}^\xi, \ j=N-1, k=N-2.
\end{aligned} \tag{6.9}$$

To solve this recurrence relation for a given N, we need to express everything in terms of an initial $|l_0^-|^2$; it turns out that this is possible for N odd only, whereas for N even, l_1^- is additionally required, whence we have to treat the even and odd cases separately. The solution is summarized in the following theorem, in which another of the remarkable properties of Z_N is required, namely, its tracelessness.

Theorem 6.5. *The solutions for the recurrence relation (6.7) satisfy the following hyperbolic relations:*

(1) *for N odd,*

$$\begin{aligned}
|l_{N-1}^-|^2 - |l_0^-|^2 &= Z_{N-1,N-1}^\xi, \\
|l_1^-|^2 - |l_0^-|^2 &= -Z_{00}^\xi;
\end{aligned}$$

(2) *for N even,*

$$\begin{aligned}
|l_{N-2}^-|^2 - |l_0^-|^2 &= Z_{N-2,N-2}^\xi + Z_{N-1,N-1}^\xi, \\
|l_{N-1}^-|^2 - |l_1^-|^2 &= Z_{00}^\xi.
\end{aligned}$$

Proof. Note the l_j^- are interrelated by even and odd indices in (6.9), so we separate the equations in sets of even and odd indices, which means we will have one no interrelating equation when N is even. Then we solve the even and odd cases for N separately.

Case 1: N odd. Lets separate the indices of $j = \{0, 1, \dots, N-1\}$ in even and odd integers. Solving eqs.(6.9) for $j = 2m, m = 1, 2, \dots, (N-1)/2$ in terms of l_0^- and substituting recursively,

$$\begin{aligned} |l_2^-|^2 &= |l_0^-|^2 - Z_{00}^{\xi} - Z_{11}^{\xi}, \\ |l_4^-|^2 &= |l_0^-|^2 - Z_{00}^{\xi} - Z_{11}^{\xi} - Z_{22}^{\xi} - Z_{33}^{\xi}, \\ &\vdots \\ |l_{N-1}^-|^2 &= |l_0^-|^2 + Z_{N-1,N-1}^{\xi} - Tr(Z_N). \end{aligned} \tag{6.10}$$

Similarly for $j = 2m - 1$ in reverse order,

$$\begin{aligned} |l_{N-2}^-|^2 &= |l_0^-|^2 + Z_{N-2,N-2}^{\xi} + Z_{N-1,N-1}^{\xi}, \\ |l_{N-4}^-|^2 &= |l_0^-|^2 + Z_{N-4,N-4}^{\xi} + Z_{N-3,N-3}^{\xi} + Z_{N-2,N-2}^{\xi} + Z_{N-1,N-1}^{\xi}, \\ &\vdots \\ |l_1^-|^2 &= |l_0^-|^2 - Z_{00}^{\xi} + Tr(Z_N). \end{aligned} \tag{6.11}$$

Case 2: N even. This case can be handled similarly, just that it is not possible to express everything in terms of only $|l_0^-|^2$, but $|l_1^-|^2$ becomes also necessary.

For $j = 2m$ in (6.9)

$$|l_{N-2}^-|^2 = |l_0^-|^2 - \sum_{m=0}^{N-3} Z_{mm}^{\xi},$$

for $j = 2m - 1$

$$|l_{N-1}^-|^2 = |l_1^-|^2 - \sum_{m=1}^{N-2} Z_{mm}^{\xi};$$

which are equivalent to

$$|l_{N-2}^-|^2 = |l_0^-|^2 + Z_{N-2,N-2}^{\xi} + Z_{N-1,N-1}^{\xi} - \mathrm{Tr}(Z_N),$$

$$|l_{N-1}^-|^2 = |l_1^-|^2 + Z_{00}^{\xi} - \mathrm{Tr}(Z_N).$$

Finally we get the result, stated in this theorem, remembering that Z_N is traceless, in accordance with Corollary 5.4. □

Note that the odd case requires only three parameters l_0^-, l_1^-, l_{N-1}^-, whereas the even case requires four. The fact that only the modules of the l_j^- are involved, anticipates the existence of many operators $L^{\pm}$ and so, many hamiltonians $\mathscr{H}$.

Hitherto two main properties of Z_N have been used: its Hermiticity and its tracelessness; now its diagonalizability is needed.

Corollary 6.6. *An annihilation operator L^- which is a solution of the DCR, exists if the parameters $l_0^-, l_1^-, l_{N-1}^-, l_{N-2}^-$ satisfy the hyperbolic relations (6.5) and $\{|\xi_j\rangle\}$ is a complete basis of eigenvectors of Z_N.*

Proof. Since $l_0^-, l_1^-, l_{N-1}^-, l_{N-2}^-$ satisfies the relations (6.5), then the recurrence (6.7) is fulfilled. As for the remaining $2N-2$ recurrences in (6.8), the fact that $|\xi_j\rangle$ are eigenvectors of Z_N, implies Z_N^{ξ} is diagonal, therefore $Z_{k,j-1}^{\xi} = Z_{k+1,j}^{\xi} = 0 \; \forall \; k = j-N+1, j-N, \dots, j-2, j, j+1, \dots, j+N-2$; that is, the off-diagonal elements of Z_N^{ξ} vanish, thus, eq.(6.8) is satisfied. □

Therefore, the construction of the operators $L^{\pm}$ is recursively given by (6.10) and (6.11) starting with solutions of the hyperbolic relations.

Example 6.1. *We can obtain a numeric example running code to compute* $Z_N = i[Q,P]$ *for* $N = 13$ *in Wolfram-Mathematica, this program yields the following eigenvalues for* Z_{13}

$$Z^{\xi}_{00} = 11.1582, \quad Z^{\xi}_{1212} = -0.371055,$$

whereby the corresponding hyperbolic relations of Theorem 6.5 which solve the recurrence relation are

$$|l^-_{12}|^2 - |l^-_1|^2 = Z^{\xi}_{00} + Z^{\xi}_{1212} = 10.7872,$$
$$|l^-_1|^2 - |l^-_0|^2 = -11.1582.$$

This implies that $|l^-_1|^2 = |l^-_0|^2 - 11.1582$*, and therefore* $|l^-_0|^2 > 11.1582$*. So if we take* $|l^-_0| = 12$*, then the circle* $C_0 = \{l_0 \in \mathbb{C} || l^-_0| = 12\}$ *determines an infinite set of solutions. We choose building on the imaginary axis, so we take* $l^-_0 = 12i$*; hence* $|l^-_1|^2 = 144 - 11.1582, or |l^-_1| = 11.5257$*, thus we take* $l^-_1 = 11.5257i$*; also,* $|l^-_{12}|^2 = |l^-_1|^2 + 10.7872$ *implies that* $l^-_{12} = 11.9845i$*. The remaining* l^-_j *are given recursively by eqs.(6.10) and (6.11).*

Corollary 6.7. *Let* $\mathbf{h}_N$ *be the set of Hamiltonians* $\mathscr{H} = \frac{1}{2}\{L^+, L^-\}$*, such that* $[\{L^+, L^-\}, L^-] = \{L^-, Z_N\}$*, then* $[\mathscr{H}, Z_N] = 0$*,* $\forall\, \mathscr{H} \in \mathbf{h}_N$*.*

Proof. Now this is clear because of construction since we are taking as $\{|\xi_j\rangle\}$ the eigenvectors of Z_N, which in turn are also eigenvectors of $\mathscr{H}$ in accordance with eq. (6.4); this is, they have the same set of eigenvectors, so they must commute. □

Definition 6.2. *The number operator* $\mathcal{N} : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ *is defined as* $\mathcal{N} := L^+L^-$*.*

Corollary 6.8. $[\mathcal{N}, \mathscr{H}] = [\mathcal{N}, Z_N] = 0$.

Proof. It is enough to probe that the eigenvectors of $\mathcal{N}$ are those of the set $\{|\xi_j\rangle\}$. Using the definitions of L^+ and L^-, we have

$$\begin{aligned} \mathcal{N} &= L^+L^- = \sum_{j=0}^{N-1}\sum_{j'=0}^{N-1} l^+_j |\xi_{j+1}\rangle\langle\xi_j| l^-_{j'} |\xi_{j'-1}\rangle\langle\xi_{j'}| \\ &= \sum_{j=0}^{N-1}\sum_{j'=0}^{N-1} l^+_j l^-_{j'} |\xi_{j+1}\rangle \delta_{j,j'-1} \langle\xi_{j'}| = \sum_{j=0}^{N-1} l^+_j l^-_{j+1} |\xi_{j+1}\rangle\langle\xi_{j+1}| \\ &= \sum_{j=0}^{N-1} |l^-_{j+1}|^2 |\xi_{j+1}\rangle\langle\xi_{j+1}|, \end{aligned}$$

since $l^+_j = \overline{l^-}_{j+1}$ (see the proof of Proposition 6.4); therefore

$$\mathcal{N}|\xi_\alpha\rangle = \sum_{j=0}^{N-1} |l^-_{j+1}|^2 |\xi_{j+1}\rangle\langle\xi_{j+1}|\xi_\alpha\rangle \quad = \quad |l^-_\alpha|^2 |\xi_\alpha\rangle.$$

Thus $\mathcal{N}$ has the same eigenvectors as $\mathscr{H}$, as desired. □

With these results requirements 1 and 2 of the proposed algebraic scheme are satisfied.

7. On the AA-approach

The purpose of this work is to provide a framework for analysing eigensystems, associated with the DFT for arbitrary N, either by using an operator not necessarily commuting with the DFT, as in the realization in terms of the operator Z_N, or by employing an operator W, which does commute with the DFT. In this section we propose a possible realization for the algebraic scheme 6.1, by using the Heun operator W from the Atakishiyeva and Atakishiyev (AA) approach, which could provide the fulfilment of its four requirements. This can be done due to the existence of a significant connection between the operators Q and P and the raising and lowering operators A and $A^\dagger$ of the AA-approach through the exponential map. Thus, we show how these operators are related and establish corresponding DCR which naturally follow in this approach.

The A and $A^\dagger$ operators, also called *intertwining operators*, are linear transformations $A, A^\dagger : \mathbb{C}^N \to \mathbb{C}^N$ such that (see [3, 4] for more on the subject)

$$A = X + \mathrm{i}Y, \quad A^\dagger = X - \mathrm{i}Y,$$

where $X = \mathrm{diag}(S_0, S_1, \ldots, S_{N-1})$, $S_n := 2\sin(2\pi n/N), n \in \mathbb{Z}_N$, and $Y = i(V^\dagger - V)$. Since X and Y are Hermitian they can be identified as position and momentum operators, respectively. Note that the operators A and $A^\dagger$ satisfy the *intertwining relations*

$$A\Phi_N = \mathrm{i}\Phi A, \qquad A^\dagger\Phi = -\mathrm{i}\Phi A^\dagger.$$

By using them it is possible to prove the following important result: if the discrete number operator is defined as $\mathcal{N} := A^\dagger A$, then

$$[\mathcal{N}, \Phi_N] = 0,$$

hence they have the same eigenvectors.

On the other hand, it is not hard to see that the operator X satisfies the relation (see [4], p. 89)

$$X = \frac{1}{2\mathrm{i}}\left(U - U^\dagger\right),$$

from which it follows that

$$A = \frac{1}{4}\sqrt{\frac{N}{\pi}}\left[V^\dagger - V + \mathrm{i}\left(U^\dagger - U\right)\right],$$

$$A^\dagger = \frac{1}{4}\sqrt{\frac{N}{\pi}}\left[V - V^\dagger - \mathrm{i}\left(U - U^\dagger\right)\right],$$

where, as we know from Section 3, the operators U and V are intertwined by the DFT.

In [4], Theorem 3.1, p.86, it was shown that the set of unitary operators defined as

$$\mathfrak{u}(l; m, n) := \sqrt{N} q^l V^n U^m, \quad 0 \le l, m, n \le N-1,$$

form an irreducible unitary representation $\mathcal{U}(N)$ on $\mathbb{C}^N$ of the finite Heisenberg group H. Additionally, as is proved in [15], Theorem 1.5, p.19, the matrix elements of the unitary, irreducible representations of H are a complete orthonormal set for the vector space of the regular representation $\mathbb{C}[H]$ of H, which in turn is obtained by a basis indexed by the elements of H. It is possible to turn $\mathbb{C}[H]$ into an algebra by means of the product $v_g v_h = v_{gh}$, $g, h \in H$ and extending it linearly; the resulting algebra is the group algebra $\mathbb{C}H$ of H on the field $\mathbb{C}$. Therefore, the matrix elements of A and $A^\dagger$ are elements of the group algebra $\mathbb{C}H$. Recall that, by virtue of Theorem 4.1,

$$V^j = \exp(i\eta P_j),$$

which implies that the raising and lowering operators A and $A^\dagger$ can also be interpreted as linear combinations of exponentiations of the operators Q and P, with matrix elements belonging to the group algebra $\mathbb{C}H$. These arguments probe the deep connection between Q, P, A and $A^\dagger$ as desired. Shortly, we can say that the operators Q, P, L^+, L^- and Z_N *lie at the level* of representations of the Heisenberg algebra $\mathfrak{h}$, whereas the operators X, Y, A, $A^\dagger$ and W *lie at the level* of representations of the Heisenberg group H – turned into a group algebra $\mathbb{C}H$ – and connected through the exponential map, with underlying finite-dimensional Hilbert space $L^2(\mathbb{Z}/N\mathbb{Z})$.

Thus, we have gained insight on the mathematical structure underlying the DFT and now we are in a position to lay down a possible realization for 6.1 lying naturally in the AA-approach. It would be desirable, not mandatory though, that $\mathcal{N}$ was Hermitian; this condition is not fulfilled, however, since A and $A^\dagger$ generate a cubic algebra (for more details see [28]). Fortunately, it has been possible to find a transformation T, which turns $\mathcal{N}$ into an Hermitian matrix by means of a symmetrization with respect to the parity operator S extendable to arbitrary N; this implies that $\mathcal{N}$ is diagonalizable and so there exists an orthonormal eigenbasis of $\mathcal{N}$ and it has been already constructed [5] (at least for $N = 5$ up to the writing of this paper). This means we could use such basis to build an associated family of ladder operators apart from the AA's raising and lowering operators. The natural candidate that plays the role of Z_N is the Heun operator

$$W := -2i\,[X, Y] = [A, A^\dagger],$$

defined in [28], p.7, which additionally can be chosen to commute with Φ_N. So despite the fact that W may not have the Toeplitz property, we could remarkably have in exchange, the fulfilment of the whole four requirements of the algebraic scheme in 6.1. The Toeplitz property will only be relevant when we formally study the intrinsic nature of Z_N, so its absence here is not detrimental to the scheme under analysis, for the existence of solutions of the DCR only required hermiticity, tracelessness and diagonalizability, as we learned in the previous section. Therefore it is important to determine whether or not it is possible to construct operators L^+ and L^-, such that

$$[\{L^+, L^-\}, L^+] = -\{L^+, W\}, \quad [\{L^+, L^-\}, L^-] = \{L^-, W\}.$$

If solutions exist for these equations, then the proposed algebraic scheme will have been solved completely, since

$$[W, \Phi_N] = 0.$$

These observations undoubtedly help to deeply understand the mathematical structure underlying the behaviour of Q, P, A and $A^\dagger$, and we think this can be useful for further developments on the subject.

8. Concluding remarks

In this paper we have established a set of discrete commutation relations obtained from a Hermitian Toeplitz operator Z_N, which plays the role of the identity. Thus, by means of the formulation of the discrete compatibility relations, the algebraic system 6.1 was proposed and it has been shown to admit solutions featuring ladder-type operators that lead to the construction of families of Hamiltonians $\mathscr{H}$ for each N. Also, it was established a remarkable relationship between the operators Q and P, and those of the AA-approach, A and $A^\dagger$, in the sense that these are linear combinations of exponentiations of the former, with matrix elements belonging to the group algebra $\mathbb{C}H$. Then we proposed that the condition $[W, \Phi_N] = 0$ can naturally lead to the fulfilment of the four requirements of the algebraic system and thereby provide a complete model of it, obtaining insight on the underlying algebraic structure of the DFT. We believe this can lead to a well formulated framework for systematically studying discrete systems in finite dimensional Hilbert spaces.

Further analysis is needed to formally deal with the conjectured relationship between Z_N and the δ distribution, as well as to determine whether analytic expressions for the eigenvectors of Z_N can be obtained. It is also important the study of the asymptotics –i.e. upper and lower bounds– of the extreme eigenvalues, if any, trying to follow procedures similar to those in [30]. The establishment of three-term recurrence relations for eigenvectors of the DFT, leading to associated polynomials through Rodrigues-type difference formulas, as well as the extension of the results to the multivariate case still remains pending.

The possible recovery of the continuous case withing the scope of the *Limit Central Theorem*, would be desirable and lies on the veracity of the limit of Z_N as the Dirac distribution, and the knowledge of the explicit form of its eigenvectors. If this limit holds, then Z_N has Schwartz distributional behaviour. Therefore one needs to find an appropriate analog of the identity operator in the CCR (2.2), instead of employing the identity operator under convolution in the distributional sense of the discrete conmutation relations (4.3). This is due to the fact that Schwartz distributions are a bit different notion than the usual probability distributions. Nevertheless, this is not detrimental to the algebra of the discrete commutation relations because in our study under discussion we prioritize a *correspondence principle* from Tarasov. This principle states (see [31] p.4) that the correspondence between discrete and continuous quantum theories lies not so much in the limiting agreement when the step of discretization tends to zero, as in the fact that mathematical operations on the two theories obey, in many cases, the same laws. Finally, the problem of the recovery of the continuous case, associated with the Heun operator, or if the limit of Z_N turns out to be a bounded operator, is left open.

Acknowledgments. NMA is grateful to Alexei Zhedanov for illuminating discussions.

References

[1] I. Bengtsson, K. Zyczkowsky, Geometry of Quantum States: An Introduction to Quantum Entaglement, Cambridge University Press, Cambridge, 2017.

[2] J. H. McClellan and T.W.Parks, *Eigenvalue and Eigenvector Decomposition of the Discrete Fourier Transform*, IEEE Transactions on Audio and Electroacoustics, **AU-20** (1972) 66-74.

[3] M. K. Atakishiyeva and N. M. Atakishiyev, *On the raising and lowering difference operators for eigenvectors of the finite Fourier transform*, J. Phys: Conf. Ser., **597** (2015) 012012.

[4] M. K. Atakishiyeva and N. M. Atakishiyev, *On algebraic properties of the discrete raising and lowering operators, associated with the N-dimensional discrete Fourier transform*, Adv. Dyn. Syst. Appl., **11** (2016) 81-92.

[5] N. M. Atakishiyev, *The eigenvalues and eigenvectors of the 5D discrete Fourier transform number operator revisited*, Maltepe J. Math., **4** (2022) 55-65.

[6] A. Kuznetsov and M. Kwaśnicki, *Minimal Hermite-type eigenbasis of the discrete Fourier transform*, Journal of Fourier Anal. Appl., **25** (2019) 1053-1079.
[7] A. N. Karkishchenko and V. B. Mnukhin, *On a system of eigenvectors of the Fourier transform over finite Gaussian fields*, J. Phys.: Conf. Ser., **1096** (2018) 012040.
[8] B. W. Dickinson and K. Steiglitz, *Eigenvectors and functions of the discrete Fourier transform*, IEEE Trans. Acoust. Speech and Signal Proc., **30** (1982) 25-31.
[9] F. A. Grünbaum, *The eigenvectors of the discrete Fourier transform: a version of the Hermite functions*, J. Math. Anal. Appl., **88** (1982) 355-363.
[10] G. B. Folland, Armonic Analysis in Phase Space, Princeton University Press, New Jersey, 1989.
[11] T. H. Seligman, Group Theory and its Applications in Physics, AIP, New York, 1980.
[12] N. M. Atakishiyev and P. Winternitz, *Bases for representations of quantum algebras*, J. Phys. A: Math. Gen., **33** (2000) 5303-5313.
[13] L. Auslander and R. Tolimieri, *Is computing with the finite Fourier transform pure or applied mathematics*, Bull. Amer. Math. Soc.,**1** (1979) 847-897.
[14] I. Kra and S. R. Simanca, *On Circulant Matrices*, Notes of the AMS, **59** (2012) 368-377.
[15] H. Georgi, Lie Algebras in Particle Physics, Westview Press, USA, 1999.
[16] T. S. Santhanam and A. R. Tekumalla, *Quantum Mechanics in Finite Dimensions*, Foundations of Physics, **6** (1975) 583-587.
[17] M. C. Pereyra and L. A. Word, Harmonic analysis: from Fourier to wavelets, AMS, Providence, Rhode Island, 2012.
[18] J. Alvarez, *A Matehmatical Presentation of Laurent Schwartz Distributions*, Surveys in Mathematics and its Applications, **15** (2020) 1-37.
[19] J. Alvarez and M. Guzman-Partida, *Properties of the Dirichlet kernel*, Electronic J. Math. Anal. Appl., **11** (2023) 96-110.
[20] N. L. Vasilevski, Commutative Algebras of Toeplitz Operators on the Bergman Space, Birkhäuser, Basel, 2008.
[21] A. Perälä, J.Taskinen and J.Virtanen, *Toeplitz operators with distributional symbols on Fock spaces*, Functiones et Approximatio Commentarii Mathematici, **44** (2011) 203-213.
[22] A. Ch. Ganchev and T. D. Palev, *A Lie superalgebraic interpretation of the para-Bose statistics*, J. Math. Phys., **21** (1979) 797-799.
[23] H. S. Green, *A Generalized Method of Field Quantization*, Phys. Rev., **90** (1953) 270-273.
[24] E. I. Jafarov, S. Lievens and J. Van der Jeugt, *The Wigner distribution function for the one-dimensional parabose oscillator*, J. Phys A: Math.Theor., **41** (2008) 235301.
[25] E. I. Jafarov, N. I. Stoilova and J. Van der Jeugt *Finite oscillator models: the Hahn oscillator*, J. Phys. A: Math. Theor., **44** (2011) 265203.
[26] M. Arik, N. M. Atakishiyev and K. B. Wolf, *Quantum algebraic structures compatible with the harmonic oscillator Newton equation*, J. Phys. A: Math. Gen., **32** (1999) L371- L376.
[27] E. P. Wigner, *Do the equations of motion determine the quantum mechanical commutation relations?*, Phys. Rev., **77** (1950) 711-712.
[28] M. K. Atakishiyeva, N. M. Atakishiyev and A. Zhedanov, *An algebraic interpretation of the intertwining operators associated with the discrete Fourier transform*, J. Math. Phys., **62** (2021) 101704.
[29] G. L. Naber, Topology, Geometry and Gauge Fields: Interactions, Springer-Verlag, New York, 2000.
[30] J. M. Bogoya, A. Böttcher, S. M. Grudsky and E. A. Maximenko, *Eigenvalues of Hermitian Toeplitz matrices with smooth simple-loop symbols*, J. Math. Anal. Appl., **42** (2015) 1308-1334.
[31] V. E. Tarasov, *Exact Discrete Analogs of Canonical Commutation and Uncertainty Relations*, Mathematics, **4** (2016) 44.

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Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume VI Issue 2 (2024), Pages 50-60, Doi: https://doi.org/10.47087/mjm.1482207

ON A WEAK FORM OF SEMI-OPEN FUNCTION BY NEUTROSOPHICATION

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Abstract. In this study, a new perspective in neutrosophic topology is brought to some open set definitions in general topology and previously interpreted in different ways in some other non-standard topological spaces. Then, some types of functions and continuity are introduced using these new open sets. In addition to these, the relations between the types of open sets, which we have brought different perspectives, with the continuity and function types that we reinterpreted in neutrosophic topology, are examined and these relations are clarified by giving examples and counterexamples.

1. Introduction

The concept of neutrosophic sets was introduced by Smarandache in his classical paper [10]. After the discovery of the neutrosophic subsets, much attention has been paid to generalize the basic concepts of classical topology in neutrosophic setting and thus a modern theory of neutrosophic topology is developed. The notion of neutrosophic subsets naturally plays a significant role in the study of neutrosophic topology which was introduced by Salama and Alblowi [9] in 2012. In 2021, Acikgoz et. all [1], introduced the concepts of neutrosophic quasi-coincidence and neutrosophic q-neighbourhoods. As in [2, 3], these new concepts were used very effectively and gave some mathematicians the opportunity to reconsider some of the cornerstones of the world of topology in neutrosophic setting. In 1985, Rose [8] defined weakly open functions in a topological spaces. In 1997 J.H. Park et. all [7] introduced the notion of weakly open functions for a fuzzy topological space. In [4], Caldas et. all, introduced and discussed the concept of fuzzy weakly semiopen function which is weaker than fuzzy weakly open and fuzzy almost open functions introduced by [7] and Nanda [6] respectively and obtained several properties and characterizations of these functions comparing with the other functions. In this

2020 *Mathematics Subject Classification.* Primary 54A05; Secondary 54C10, 54D30, 54D101.
Key words and phrases. Neutrosophic semiopen set; Neutrosophic regular open set; Neutrosophic β-open set; Neutrosophic weakly open function; Neutrosophic almost open function; Neutrosophic weakly semiopen function; Neutrosophic θ-closure; neutrosophic hyperconnected space.

Submitted on May 11th, 2024. Accepted on October 3rd, 2024.
Communicated by Huseyin CAKALLI and Nazlım Deniz ARAL.

study, using the concepts of neutrosophic quasi-coincidence and neutrosophic q-neighbourhoods, we reinterpreted this function and open set variants and got reality of that neutrosophic semiopenness implied neutrosophic weakly semiopenness but converse was not true. We also showed that the reverse statement was also true when certain conditions were met.

2. Preliminaries

In this section, we present the basic definitions related to neutrosophic set theory.

Definition 2.1. [9]*A neutrosophic set A on the universe set X is defined as:*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where $T,\ I,\ F : X \to]^{-}0, 1^{+}[$ *and* $^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$.

Scientifically, membership functions, indeterminacy functions and non-membership functions of a neutrosophic set take value from real standart or nonstandart subsets of $]^{-}0, 1^{+}[$. However, these subsets are sometimes inconvenient to be used in real life applications such as economical and engineering problems. On account of this fact, we consider the neutrosophic sets, whose membership function, indeterminacy functions and non-membership functions take values from subsets of $[0, 1]$.

Definition 2.2. [5] *Let X be a nonempty set. If r, t, s are real standard or non standard subsets of* $]^{-}0, 1^{+}[$ *then the neutrosophic set* $x_{r,t,s}$ *is called a neutrosophic point in X given by*

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

For $x_p \in X$*, it is called the support of* $x_{r,t,s}$*, where r denotes the degree of membership value, t denotes the degree of indeterminacy and s is the degree of non-membership value of* $x_{r,t,s}$.

Definition 2.3. [8] *Let A be a neutrosophic set over the universe set X. The complement of A is denoted by* A^c *and is defined by:*

$$A^c = \left\{\left\langle x, F_{\tilde{F}(e)}(x), 1 - I_{\tilde{F}(e)}(x), T_{\tilde{F}(e)}(x)\right\rangle : x \in X\right\}.$$

It is obvious that $[A^c]^c = A$.

Definition 2.4. [8] *Let A and B be two neutrosophic sets over the universe set X. A is said to be a neutrosophic subset of B if* $T_A(x) \leq T_B(x), I_A(x) \leq I_B(x), F_A(x) \geq F_B(x)$*, every x in X. It is denoted by* $A \subseteq B$*. A is said to be neutrosophic soft equal to B if* $A \subseteq B$ *and* $B \subseteq A$*. It is denoted by* $A = B$.

Definition 2.5. [8] *Let* F_1 *and* F_2 *be two neutrosophic soft sets over the universe set X. Then their union is denoted by* $F_1 \cup F_2 = F_3$ *is defined by:*

$$F_3 = \{\langle x, T_{F_3}(x), I_{F_3}(x), F_{F_3}(x) : x \in X\rangle\},$$

where

$$T_{F_3(x)} = \max\{T_{F_1(x)}, T_{F_2}(x)\},$$

$$I_{F_3(x)} = \max\{I_{F_1(x)}, I_{F_2}(x)\},$$

$$F_{F_3(x)} = \min\{F_{F_1(x)}, F_{F_2}(x)\}.$$

Definition 2.6. [8] *Let F_1 and F_2 be two neutrosophic soft sets over the universe set X. Then their intersection is denoted by $F_1 \cap F_2 = F_4$ is defined by:*

$$F_4 = \{\langle x, T_{F_4}(x), I_{F_4}(x), F_{F_4}(x) : x \in X \rangle\},$$

where

$$T_{F_4(x)} = \min\{T_{F_1(x)}, T_{F_2}(x)\},$$

$$I_{F_4(x)} = \min\{I_{F_1(x)}, I_{F_2}(x)\},$$

$$F_{F_4(x)} = \max\{F_{F_1(x)}, F_{F_2}(x)\}.$$

Definition 2.7. [8]*A neutrosophic set F over the universe set X is said to be a null neutrosophic set if $T_F(x) = 0$, $I_F(x) = 0$, $F_F(x) = 1$, every $x \in X$. It is denoted by 0_X.*

Definition 2.8. [8] *A neutrosophic set F over the universe set X is said to be an absolute neutrosophic set if $T_F(x) = 1$, $I_F(x) = 1$, $F_F(x) = 0$, every $x \in X$. It is denoted by 1_X.*

Clearly $0^c_X = 1_X$ and $1^c_X = 0_X$.

Definition 2.9. [8] *Let $NS(X)$ be the family of all neutrosophic sets over the universe the set X and $\tau \subset NS(X)$. Then τ is said to be a neutrosophic topology on X if:*

1) 0_X and 1_X belong to τ;

2) The union of any number of neutrosophic soft sets in τ belongs to τ;

3) The intersection of a finite number of neutrosophic soft sets in τ belongs to τ.

Then (X, τ) is said to be a neutrosophic topological space over X. Each member of τ is said to be a neutrosophic open set [8].

Definition 2.10. [8] *Let (X, τ) be a neutrosophic topological space over X and F be a neutrosophic set over X. Then F is said to be a neutrosophic closed set iff its complement is a neutrosophic open set.*

Definition 2.11. [2] *A neutrosophic point $x_{r,t,s}$ is said to be neutrosophic quasi-coincident (neutrosophic q-coincident, for short) with F, denoted by $x_{r,t,s}$ q F if and only if $x_{r,t,s} \nsubseteq F^c$. If $x_{r,t,s}$ is not neutrosophic quasi-coincident with F, we denote by $x_{r,t,s}$ $\tilde{q}$ F.*

Definition 2.12. [2] *A neutrosophic set F in a neutrosophic topological space (X, τ) is said to be a neutrosophic q-neighborhood of a neutrosophic point $x_{r,t,s}$ if and only if there exists a neutrosophic open set G such that*
$x_{r,t,s}$ q $G \subset F$.

Definition 2.13. [2] *A neutrosophic set G is said to be neutrosophic quasi-coincident (neutrosophic q-coincident, for short) with F, denoted by G q F if and only if $G \nsubseteq F^c$. If G is not neutrosophic quasi-coincident with F, we denote by G $\tilde{q}$ F.*

Definition 2.14. [3] *A neutrosophic point $x_{r,t,s}$ is said to be a neutrosophic interior point of a neutrosophic set F if and only if there exists a neutrosophic open q-neighborhood G of $x_{r,t,s}$ such that $G \subset F$. The union of all neutrosophic interior points of F is called the neutrosophic interior of F and denoted by F°.*

Definition 2.15. [2] *A neutrosophic point $x_{r,t,s}$ is said to be a neurosophic cluster point of a neutrosophic set F if and only if every neutrosophic open q-neighborhood G of $x_{r,t,s}$ is q-coincident with F. The union of all neutrosophic cluster points of F is called the neutrosophic closure of F and denoted by $\overline{F}$.*

Definition 2.16. [2] *Let f be a function from X to Y. Let B be a neutrosophic set in Y with members hip function $T_B(y)$, indeterminacy function $I_B(y)$ and non-membership function $F_B(y)$. Then, the inverse image of B under f, written as $f^{-1}(B)$, is a neutrosophic subset of X whose membership function, indeterminacy function and non-membership function are defined as $T_{f^{-1}(B)}(x) = T_B(f(x))$, $I_{f^{-1}(B)}(x) = I_B(f(x))$ and $F_{f^{-1}(B)}(x) = F_B(f(x))$ for all x in X, respectively. Conversely, let A be a neutrosophic set in X with membership function $T_A(x)$, indeterminacy function $I_A(x)$ and non-membership function $F_A(x)$. The image of A under f, written as $f(A)$, is a neutrosophic subset of Y whose membership function, indeterminacy function and non-membership function are defined as*

$$T_{f(A)}(y) = \begin{cases} sup_{z\in f^{-1}(y)}\{T_A(z)\}, & if\ f^{-1}(y)\ is\ not\ empty, \\ 0, & if\ f^{-1}(y)\ is\ empty, \end{cases}$$

$$I_{f(A)}(y) = \begin{cases} sup_{z\in f^{-1}(y)}\{I_A(z)\}, & if\ f^{-1}(y)\ is\ not\ empty, \\ 0, & if\ f^{-1}(y)\ is\ empty, \end{cases}$$

$$F_{f(A)}(y) = \begin{cases} sup_{z\in f^{-1}(y)}\{F_A(z)\}, & if\ f^{-1}(y)\ is\ not\ empty, \\ 0, & if\ f^{-1}(y)\ is\ empty, \end{cases}$$

for all y in Y, where $f^{-1}(y) = \{x : f(x) = y\}$, respectively.

3. Some Definitions

This section provides some new definitions that form the cornerstones of the sections that follow.

Definition 3.1. *A neutrosophic set F in a neutrosophic topological space (X, τ) is said to be*

a) Neutrosophic semiopen, if $F \subseteq \overline{(F^\circ)}$,
b) Neutrosophic semiclosed, if $\left(\overline{F}\right)^\circ \subseteq F$,
c) Neutrosophic preopen, $F \subseteq \left(\overline{F}\right)^\circ$,
d) Neutrosophic regular open, if $F = \left(\overline{F}\right)^\circ$,
e) Neutrosophic α-open, if $F \subseteq \left(\overline{(F^\circ)}\right)^\circ$,
f) Neutrosophic β-open, $F \subseteq \overline{\left(\overline{F}\right)^\circ}$. Equivalently, if there exists a neutrosophic preopen set A such that $A \subseteq F \subseteq \overline{A}$.

Definition 3.2. *If, F be a neutrosophic set in neutrosophic topological space (X, τ) then, $\overline{F}_s = \bigcap\{F : F \subseteq A,\ A\ is\ neutrosophic\ semiclosed\}$ (resp. $F_s^\circ = \bigcup\{F : F \subseteq A,\ A\ is\ neutrosophic\ semiopen\}$) is called a neutrosophic semiclosure of F (resp. neutrosophic semi-interior of F).*

A neutrosophic set F in a neutrosophic topological space (X, τ) is neutrosophic semiclosed (resp. neutrosophic semiopen) if and only if $F = \overline{F}_s$ (resp. $F = F_s^\circ$).

Definition 3.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a neutrosophic topological space (X, τ) into a neutrosophic topological space (Y, σ). The function f is said to be:*

a) Neutrosophic semiopen, if $f(F)$ is a neutrosophic semiopen set of Y for each neutrosophic open set F in X.
b) Neutrosophic weakly open, if $f(F) \subseteq \left(f(\overline{F})\right)^{\circ}$ is for each neutrosophic open set F in X.
c) Neutrosophic almost open, if $f(F)$ is a neutrosophic open set of Y for each neutrosophic regular open set F in X.
d) Neutrosophic β-open, if $f(F)$ is a neutrosophic β-open set of Y for each neutrosophic open set F in X.

Definition 3.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a neutrosophic topological space (X, τ) into a neutrosophic topological space (Y, σ). The function f is said to be neutrosophic semicontinuous, if $f^{-1}(A)$ is a neutrosophic semiopen set of X, for each $A \in \sigma$.*

4. Neutrosophic Weakly Semiopen Functions

Since the concepts of neutrosophic semicontinuity and neutrosophic semiopenness are indispensable for each other, how the concepts of neutrosophic weak semiopenness and the neutrosophic weak semicontinuity is clarified in this study.

Definition 4.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a neutrosophic topological space (X, τ) into a neutrosophic topological space (Y, σ). The function f is said to be neutrosophic weakly semiopen, if $f(A) \subseteq \left(f\left(\overline{A}\right)\right)^{\circ}_{s}$, for each $A \in \sigma$.*

Obviously, every neutrosophic weakly open function is neutrosophic weakly semiopen and every neutrosophic semiopen function is also neutrosophic weakly semiopen.

Example 4.1. *Let $X = x, y, z$, $Y = a, b, c$ and neutrosophic sets λ and μ are defined as: $\lambda = \{\langle x, 0, 0, 1\rangle, \langle y, 0.3, 0.3, 0.7\rangle, \langle z, 0.2, 0.2, 0.8\rangle\}$*
$\mu = \{\langle a, 0, 0, 1\rangle, \langle b, 0.2, 0.2, 0.8\rangle, \langle c, 0.1, 0.1, 0.9\rangle\}$
Let $\tau = \{0, \lambda, 1\}$ and $\sigma = \{0, \mu, 1\}$. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ defined by $f(x) = a$, $f(y) = b$ and $f(z) = c$ is neutrosophic weakly semiopen but neither neutrosophic semiopen nor neutrosophic weakly open.

Definition 4.2. *A neutrosophic point $x_{r,t,s}$ is said to be a neutrosophic θ-cluster point of a neutrosophic set λ, if, for every neutrosophic open q-nbd μ of $x_{r,t,s}$, $\overline{\mu}$ is q-coincident with λ. The set of all neutrosophic θ-cluster points of λ is called the neutrosophic θ-closure of λ and will be denoted by $\overline{\lambda}_{\theta}$. A neutrosophic λ will be called neutrosophic θ-closed if and only if $\lambda = \overline{\lambda}_{\theta}$. The complement of a neutrosophic θ-closed set is called of neutrosophic θ-open and the neutrosophic θ-interior of λ denoted by λ°_{θ} is defined as $\lambda^{\circ}_{\theta} = \{x_{r,t,s} \mid for\ some\ neutrosophic\ open\ q-$ nbd, β of $x_{r,t,s}, \overline{\beta} \subseteq \lambda\}$.*

Lemma 4.1. *Let λ be a neutrosophic set in a neutrosophic topological space X, then:*

1) λ is a neutrosophic θ-open if and only if $\lambda = \lambda^{\circ}_{\theta}$.
2) $\left(\lambda^{\circ}_{\theta}\right)^{c} = \overline{\left(\lambda^{c}\right)}_{\theta}$ and $\left(\lambda^{c}\right)^{\circ}_{\theta} = \left(\overline{\lambda}_{\theta}\right)^{c}$.

3) $\overline{\lambda}_\theta$ is a neutrosophic closed set but not necessarily is a neutrosophic θ-closed set.

Theorem 4.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a neutrosophic topological space (X, τ) into a neutrosophic topological space (Y, σ). Then, following conditions are equivalent:*

(i) f is neutrosophic weakly semiopen;
(ii) $f\big(\lambda^\circ_\theta\big) \subseteq \big(f(\lambda)\big)^\circ_s$ for every neutrosophic subset λ of X;
(iii) $\Bigg(f^{-1}\big(\beta\big)\Bigg)^\circ_\theta \subseteq f^{-1}\big(\beta^\circ_s\big)$ for every neutrosophic subset β of Y;
(iv) $f^{-1}\Bigg(\overline{\beta_s}\Bigg) \subseteq \overline{\Bigg(f^{-1}\big(\beta\big)\Bigg)}_\theta$ for every neutrosophic subset β of Y;
(v) For each neutrosophic θ-open set λ in X, $f(\lambda)$ is neutrosophic semiopen in Y;
(vi) For any neutrosophic set β of Y and any neutrosophic θ-closed set λ in X containing $f^{-1}(\beta)$, where X is a neutrosophic regular space, there exists a neutrosophic semiclosed set δ in Y containing β such that $f^{-1}(\delta) \subseteq \lambda$.

Proof. (i) $\Rightarrow$ (ii): Let λ be any neutrosophic subset of X and $x_{r,t,s}$ be a neutrosophic point in λ°_θ. Then, there exists a neutrosophic open $q - nbd$ γ of $x_{r,t,s}$ such that $\gamma \subseteq \overline{\gamma} \subseteq \lambda$. Then, $f(\gamma) \subseteq f(\overline{\gamma}) \subseteq f(\lambda)$. Since f is neutrosophic weakly semiopen, $f(\gamma) \subseteq (f(\overline{\gamma}))^\circ_s \subseteq (f(\lambda))^\circ_s$. It implies that $f(x_{r,t,s})$ is a point in $(f(\lambda))^\circ_s$. This shows that $x_{r,t,s} \in f^{-1}((f(\lambda))^\circ_s)$. Thus $\lambda^\circ_\theta \subseteq f^{-1}((f(\lambda))^\circ_s)$, and so $f(\lambda^\circ_\theta) \subseteq (f(\lambda))^\circ_s$.
(ii)$\Rightarrow$ (i): Let μ be a neutrosophic open set in X. As $\mu \subseteq (\overline{\mu})^\circ_\theta$ implies,
$f(\mu) \subseteq f((\overline{\mu})^\circ_\theta) \subseteq (f(\overline{\mu}))^\circ_s$. Hence f is neutrosophic weakly semiopen.
(ii) $\Rightarrow$ (iii): Let β be any neutrosophic subset of Y. Then by (ii), $f((f^{-1}(\beta))^\circ_\theta) \subseteq \beta^\circ_s$. Therefore, $(f^{-1}(\beta))^\circ_\theta \subseteq f^{-1}(\beta^\circ_s)$.
(iii) $\Rightarrow$ (ii): This is obvious.
(iii) $\Rightarrow$ (iv): Let β be any neutrosophic subset of Y. Using (iii), we have

$$\Big[\overline{\big(f^{-1}(\beta)\big)}_\theta\Big]^c = \Big[\big(f^{-1}(\beta)\big)^c\Big]^\circ_\theta = \Big[f^{-1}(\beta^c)\Big]^\circ_\theta \subseteq f^1\Bigg(\big(\beta^c\big)^\circ_s\Bigg) = f^{-1}\Bigg(\overline{(\beta_s)}\Bigg)^c = \Big[f^{-1}\overline{(\beta_s)}\Big]^c.$$

Therefore, we obtain $f^{-1}(\overline{\beta_s}) \subseteq \overline{(f^{-1}(\beta))}_\theta$.
(iv) $\Rightarrow$ (iii): Similary we obtain, $[f^{-1}(\beta^\circ_s)]^c \subseteq [(f^{-1}(\beta))^\circ_\theta]^c$, for every neutrosophic subset β of Y, i.e., $(f^{-1}(\beta))^\circ_\theta \subseteq f^{-1}(\beta^\circ_s)$.
(iv) $\Rightarrow$ (v): Let λ be a neutrosophic θ-open set in X. Then, $(f(\lambda))^c$ is a neutrosophic set in Y and by (iv),

$$f^{-1}\Bigg(\overline{\big((f(\lambda))^c\big)_s}\Bigg) \subseteq \overline{f^{-1}\Bigg(\big(f(\lambda)\big)^c\Bigg)}_\theta.$$

Therefore, $\Bigg[f^{-1}\Bigg(\big(f(\lambda)\big)^\circ_s\Bigg)\Bigg]^c \subseteq \overline{(\lambda^c)}_\theta = \lambda^c$. Then, we have $\lambda \subseteq f^{-1}((f(\lambda))^\circ_s)$ which implies $f(\lambda) \subseteq (f(\lambda))^\circ_s$. Hence $f(\lambda)$ is neutrosophic semiopen in Y .

(v) ⇒ (vi): Let β be any neutrosophic set in Y and λ be a neutrosophic θ-closed set in X such that $f^{-1}(\beta) \subseteq \lambda$. Since λ^c is neutrosophic θ-open in X, by (v), $f(\lambda^c)$ is neutrosophic semiopen in Y. Let $\delta = \left[f(\lambda^c)\right]^c$. Then δ is neutrosophic semiclosed and also $\beta \subseteq \delta$. Now, $f^{-1}(\delta) = f^{-1}([f(\lambda^c)]^c) = [f^{-1}(f(\lambda))]^c \subseteq \lambda$.
(vi) ⇒ (iv): Let β be any neutrosophic set in Y. Then, $\lambda = \overline{f^{-1}(\beta)}_\theta$ is neutrosophic θ-closed set in X and neutrosophic $f^{-1}(\beta) \subseteq \lambda$. Then, there exists a neutrosophic semiclosed set δ in Y containing β such that $f^{-1}(\delta) \subseteq \lambda$. Since δ is neutrosophic semiclosed $f^{-1}(\overline{\beta}_s) \subseteq f^{-1}(\delta) \subseteq \overline{f^{-1}(\beta)}_\theta$. □

Furthermore, we can prove the following.

Theorem 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective function. Then the following statements are equivalent:*

(i) f is neutrosophic weakly semiopen.
(ii) $\overline{f(\lambda)}_s \subseteq f(\overline{\lambda})$ for each neutrosophic open set λ in X.
(iii) $\overline{f(\beta^\circ)}_s \subseteq f(\beta)$ for each neutrosophic closed set β in X.

Proof. (i) ⇒ (iii): Let β be a neutrosophic closed set in X. Then we have

$$f(\beta^c) = \big(f(\beta)\big)^c \subseteq \big(f(\overline{\beta^c})\big)^\circ_s$$

and so $\big(f(\beta)\big)^c \subseteq \big(\overline{f(\beta^\circ)_s}\big)^c$. Hence $\overline{f(\beta^\circ)_s} \subseteq f(\beta)$.
(iii) ⇒ (ii): Let λ be a neutrosophic open set in X. Since $\overline{\lambda}$ is a neutrosophic closed set and $\lambda \subseteq \left(\overline{\lambda}\right)^\circ$ by (iii) we have $\overline{f(\lambda)}_s \subseteq \overline{f\big((\overline{\lambda})^\circ\big)}_s \subseteq f(\overline{\lambda})$.
(ii) ⇒ (iii): Similar to (iii) → (ii).
(iii) ⇒ (i) : Clear. □

For the following theorem, the proof is mostly straightforward and is omitted.

Theorem 4.4. *For a function $f : (X, \tau) \to (Y, \sigma)$ the following conditions are equivalent: We define one additional near neutrosophic semiopen condition. This condition when combined with neutrosophic weak semiopenness imply neutrosophic semiopenness.*

(i) f is neutrosophic weakly semiopen;
(ii) For each neutrosophic closed subset β of X, $f(\beta^\circ) \subseteq (f(\beta))^\circ_s$;
(iii) For each neutrosophic open subset λ of X, $f\big((\overline{\lambda})^\circ\big) \subseteq \big(f(\overline{\lambda})\big)^\circ_s$;
(iv) For each neutrosophic regular open subset λ of X, $f(\lambda) \subseteq \big(f(\overline{\lambda})\big)^\circ_s$;
(v) For every neutrosophic preopen subset λ of X, $f(\lambda) \subseteq \big(f(\overline{\lambda})\big)^\circ_s$;
(vi) For every neutrosophic α-open subset λ of X, $f(\lambda) \subseteq \big(f(\overline{\lambda})\big)^\circ_s$.

We define one additional near neutrosophic semiopen condition. This condition when combined with neutrosophic weak semiopenness imply neutrosophic semiopenness.

Definition 4.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to satisfy the neutrosophic weakly semiopen interiority condition if $\big(f(\overline{\lambda})\big)^\circ_s \subseteq f(\lambda)$ for every neutrosophic open subset λ of X.*

Definition 4.4. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be neutrosophic strongly continuous, if for every neutrosophic subset λ of X, $f(\overline{\lambda}) \subseteq f(\lambda)$.*

Obviously, every neutrosophic strongly continuous function satisfies the neutrosophic weakly semiopen interiority condition but the converse does not hold as is shown by the following example.

Example 4.2. *Let* $X = \{a, b\}$, $Y = \{x, y\}$ *and neutrosophic sets* λ *and* μ *are defined as:* $\lambda = \{\langle a, 0.3, 0.3, 0.7\rangle, \langle b, 0.4, 0.4, 0.6\rangle\}$ $\mu = \{\langle x, 0.7, 0.7, 0.3\rangle, \langle y, 0.8, 0.8, 0.2\rangle\}$. *Let* $\tau = \{0, \lambda, 1\}$ *and* $\sigma = \{0, \mu, 1\}$. *Then the mapping* $f : (X, \tau) \to (Y, \sigma)$ *defined by* $f(a) = x$, $f(b) = y$ *satisfies neutrosophic weakly semiopen interiority but is not neutrosophic strongly continuous.*

Theorem 4.5. *Every function that satisfies the neutrosophic weakly semiopen interiority condition into a neutrosophic discrete topological space is neutrosophic strongly continuous.*

Theorem 4.6. *If* $f : (X, \tau) \to (Y, \sigma)$ *is neutrosophic weakly semiopen and satisfies the neutrosophic weakly semiopen interiority condition, then* f *is neutrosophic semiopen.*

Proof. Let λ be a neutrosophic open subset of X. Since f is neutrosophic weakly semiopen $f(\lambda) \subseteq \big(f(\overline{\lambda})\big)^{\circ}_{s}$. However, because f satisfies the neutrosophic weakly semiopen interiority condition, $f(\lambda) = \big(f(\overline{\lambda})\big)^{\circ}_{s}$ and therefore $f(\lambda)$ is neutrosophic semiopen. □

Corollary 4.7. *If* $f : (X, \tau) \to (Y, \sigma)$ *is neutrosophic weakly semiopen and neutrosophic strongly continuous, then* f *is neutrosophic semiopen.*

The following example shows that neither of this neutrosophic interiority condition yield a decomposition of neutrosophic semiopenness.

Example 4.3. *Let* $X = \{a, b, c\}$, $Y = \{x, y, z\}$. *Define* λ *and* μ *as follows :* $\lambda = \{\langle a, 0, 0, 1\rangle, \langle b, 0.2, 0.2, 0.8\rangle, \langle c, 0.7, 0.7, 0.3\rangle\}$, $\mu = \{\langle x, 0, 0, 1\rangle, \langle y, 0.2, 0.2, 0.8\rangle, \langle z, 0.2, 0.2, 0.8\rangle\}$. *Let* $\tau = \{0, \lambda, 1\}$ *and* $\sigma = \{0, \mu, 1\}$. *Then the mapping* $f : (X, \tau) \to (Y, \sigma)$ *defined by* $f(a) = x$, $f(b) = y$ *and* $f(c) = z$ *is neutrosophic semiopen but not neutrosophic weakly semiopen interiority .*

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be neutrosophic contra-open (resp. neutrosophic contra-closed) if $f(\lambda)$ is a neutrosophic closed set (resp. neutrosophic open set) of Y for each neutrosophic open (resp. neutrosophic closed) set λ in X.

Theorem 4.8. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a neutrosophic function. Then, the following statements hold.*

(i) If $f : (X, \tau) \to (Y, \sigma)$ *is neutrosophic preopen and neutrosophic contra-open, then* f *is a neutrosophic weakly semiopen function.*
(ii) If $f : (X, \tau) \to (Y, \sigma)$ *is neutrosophic contra-closed, then* f *is a neutrosophic weakly semiopen function.*

Proof. (i) Let λ be a neutrosophic open subset of X. Since f is neutrosophic preopen $f(\lambda) \subseteq (\overline{f(\lambda)})^{\circ}$ and since f is neutrosophic contra-open, $f(\lambda)$ is neutrosophic closed. Therefore $f(\lambda) \subseteq (\overline{f(\lambda)})^{\circ} = (f(\lambda))^{\circ} \subseteq (f(\overline{\lambda}))^{\circ}$.
(ii) Let λ be an neutrosophic open subset of X. Then, we have $f(\lambda) \subseteq \overline{f(\lambda)} \subseteq (\overline{f(\lambda)})^{\circ}_{s}$. □

The converse of Theorem 4.6 does not hold. Can be seen in Example 4.1.

Theorem 4.9. *Let X be a neutrosophic regular space. Then, $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic weakly semiopen if and only if f is neutrosophic semiopen.*

Proof. The sufficiency is clear.
Necessity. Let λ be a non-null neutrosophic open subset of X. For each $x_{r,s,t}$ neutrosophic point in λ, let $\mu_{x_{r,s,t}}$ be a neutrosophic open set such that $x_{r,s,t} \in \mu_{x_{r,s,t}} \subseteq \overline{x_{r,s,t}} \subseteq \lambda$. Hence, we obtain that

$$\lambda = \bigcup\{\mu_{x_{r,s,t}} : x_{r,s,t} \in \lambda\} = \bigcup\{\overline{x_{r,s,t}} : x_{r,s,t} \in \lambda\}$$

and

$$f(\lambda) = \bigcup\{f(\mu_{x_{r,s,t}}) : x_{r,s,t} \in \lambda\} \subseteq \bigcup\{(f(\overline{x_{r,s,t}}))^\circ_s : x_{r,s,t} \in \lambda\} \subseteq (f(\bigcup\{\overline{x_{r,s,t}} : x_{r,s,t} \in \lambda\}))^\circ_s = (f(\lambda))^\circ_s.$$

Thus, f is semiopen. □

Note that, $f : (X, \tau) \to (Y, \sigma)$ is said to be neutrosophic contra-pre-semiclosed provide that $f(\lambda)$ is neutrosophic semi-open for each neutrosophic semi-closed subset λ of X.

Theorem 4.10. *If $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic weakly semiopen and Y has the property that union of neutrosophic semi-closed sets is neutrosophic semi-closed and if for each neutrosophic semi-closed subset β of X and each fiber*

$$f^{-1}(y_{r,s,t}) \subseteq \beta^c$$

there exists a neutrosophic open subset μ of X for which $\beta \subseteq \mu$ and $f^{-1}(y_{r,s,t})\tilde{q}\overline{\mu}$, then f is neutrosophic contra-pre-semiclosed.

Proof. Assume β is a neutrosophic semi-closed subset of X and let $y_{r,s,t} \in (f(\beta))^c$. Thus, $f^{-1}(y_{r,s,t}) \subseteq \beta^c$ and hence there exists a neutrosophic open subset μ of X for which $\beta \subseteq \mu$ and $f^{-1}(y_{r,s,t}))\tilde{q}\overline{\mu}$. Therefore, $y_{r,s,t} \in (f(\overline{\mu}))^c \subseteq (f(\beta))^c$. Since f is neutrosophic weakly semiopen $f(\mu) \subseteq (f(\overline{\mu}))^\circ_s$. By complement, we obtain

$$y_{r,s,t} \in \overline{(f(\overline{\mu}))^c}_s \subseteq (f(\beta))^c$$

Let $\delta_{y_{r,s,t}} = \overline{(f(\overline{\mu}))^c}_s$. Then $\delta_{y_{r,s,t}}$ is a neutrosophic semi-closed subset of Y containing $y_{r,s,t}$. Hence $(f(\beta))^c = \bigcup\{\delta_{y_{r,s,t}} : y_{r,s,t} \in (f(\beta))^c\}$ is neutrosophic semi-closed and therefore $f(\beta)$ is neutrosophic semi-open. □

Theorem 4.11. *If $f : (X, \tau) \to (Y, \sigma)$ is an neutrosophic almost open function, then it is neutrosophic weakly semiopen. The converse is not generally true.*

Proof. Let λ be a neutrosophic open set in X. Since f is neutrosophic almost open and $(\overline{\lambda})^\circ$ is neutrosophic regular open, $f((\overline{\lambda})^\circ)$ is neutrosophic open in Y and hence $f(\lambda) \subseteq f((\overline{\lambda})^\circ) \subseteq (f(\overline{\lambda}))^\circ \subseteq (f(\overline{\lambda}))^\circ_s$. This shows that f is neutrosophic weakly semiopen. □

Example 4.4. *The function f defined in Example 4.3 is neutrosophic weakly semiopen but not neutrosophic almost open.*

Lemma 4.12. *If $f : (X, \tau) \to (Y, \sigma)$ is a neutrosophic continuous function, then for any neutrosophic subset λ of X, $f(\overline{\lambda}) \subseteq \overline{(f(\lambda))}$.*

Theorem 4.13. *If $f : (X, \tau) \to (Y, \sigma)$ is a neutrosophic weakly semiopen and neutrosophic continuous function, then f is a neutrosophic β-open function.*

Proof. Let λ be a neutrosophic open set in X. Then, by neutrosophic weak semiopenness of f , $f(\lambda) \subseteq (f(\overline{\lambda}))^{\circ}_{s}$. Since f is neutrosophic continuous $f(\overline{\lambda}) \subseteq \overline{f(\lambda)}$ and since for any neutrosophic subset β of X, $(\overline{\beta})^{\circ}_{s} \subseteq \beta \cap \overline{((\overline{\beta})^{\circ})}$, we obtain that, $f(\lambda) \subseteq (f(\overline{\lambda}))^{\circ}_{s} \subseteq (\overline{(f(\lambda)})^{\circ}_{s} \subseteq \overline{((\overline{f(\lambda)})^{\circ})}$. Thus, $f(\lambda) \subseteq \overline{((\overline{f(\lambda)})^{\circ})}$ which shows that $f(\lambda)$ is a neutrosophic β-open set in Y. Hence, f is a neutrosophic β-open function. □

Corollary 4.14. *If $f : (X, \tau) \to (Y, \sigma)$ is a neutrosophic weakly semiopen and neutrosophic strongly continuous function. Then f is a neutrosophic β-open function.*

Definition 4.5. *A neutrosophic topological space (X, τ) is said to be a neutrosophic connected space, if there don't exist neutrosophic clopen sets λ and β such that $\lambda \tilde{q} \beta$ and $\lambda^c \tilde{q} \beta^c$.*

Definition 4.6. *A neutrosophic topological space (X, τ) is said to be a neutrosophic semiconnected space, if there don't exist neutrosophic semiclopen sets λ and β such that $\lambda \tilde{q} \beta$ and $\lambda^c \tilde{q} \beta^c$.*

Theorem 4.15. *If $f : (X, \tau) \to (Y, \sigma)$ is a neutrosophic weakly semiopen of a space X onto a neutrosophic semiconnected space Y, then X is neutrosophic connected.*

Proof. Let X be not connected. Then there exist neutrosophic open sets β and γ in X such that $\beta \tilde{q} \gamma$ and $\beta^c \tilde{q} \gamma^c$. This implies that $f(\beta) \tilde{q} f(\gamma)$ and $f(\beta^c) \tilde{q} f(\gamma^c)$. Since f is neutrosophic weakly semiopen, we have $f(\beta) \subseteq (f(\overline{\beta}))^{\circ}_{s}$ and $f(\gamma) \subseteq (f(\overline{\gamma}))^{\circ}_{s}$ and since β and γ are neutrosophic open and also neutrosophic closed, we have $f(\overline{\beta}) = f(\beta)$, $f(\overline{\gamma}) = f(\gamma)$. Hence $f(\beta)$ and $f(\gamma)$ are neutrosophic semiopen and semiclosed in (Y, σ) such that $f(\beta) \tilde{q} f(\gamma)$ and $((f\beta))^c \tilde{q} (f(\gamma))^c$. Hence, this contrary to the fact that Y is neutrosophic semi-connected. Thus X is neutrosophic connected. □

Definition 4.7. *A space X is said to be neutrosophic hyperconnected if every non-null neutrosophic open subset of X is neutrosophic dense in X.*

Theorem 4.16. *If X is a neutrosophic hyperconnected space, then a function $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic weakly semiopen if and only if $f(X)$ is neutrosophic semi-open in Y.*

Proof. The necessity is clear.
For the sufficiency observe that for any neutrosophic open subset λ of X,
$f(\lambda) \subseteq f(X) = (f(X))^{\circ}_{s} = (f(\overline{\lambda}))^{\circ}_{s}$. □

5. Conclusion

In this study, after presenting the factors that inspired us to focus on this subject and giving the necessary preliminary information, in the third section, we adapted some open set types previously defined in the general topology to neutrosophic spaces, and then we defined some types of functions and continuity using these open set types. In the fourth section, we continued to present new open set and continuity types and illustrated the relationships between them by enriching them with examples. Additionally, we examined the properties of these new types of continuities and open sets and brought a new perspective to the concept of connected space. Our expectation from this study is that it will be one of the cornerstones of various studies to be carried out in the world of mathematics and that it will encourage scientists to conduct various research related to our study.

Conflict of interests

The authors declare that there is no conflict of interests.

Acknowledgment

The authors would like to thank the referees for the valueble suggestions, which improved the paper.

References

[1] A. Acikgoz, H. Çakalli, F. Esenbel, and Lj. D.R. Kočinac, A quest of G-continuity in neutrosophic spaces, Math Meth Appl Sci. **44** (9) (2021) 7834-7844.

[2] A. Acikgoz and F. Esenbel, Neutrosophic Connected topological spaces, Fasciculi Math. **66** (2023) 5-22.

[3] A. Acikgoz and F. Esenbel, Neutrosophic seperation axioms, Maltepe J. Math. **5** (2) (2023) 32-40.

[4] M. Caldas, G. Navalagi and R. Saraf, On Fuzzy Weakly Open Functions, Proyec. **21** (1) (2002) 51-63.

[5] R. Dhavaseelan, S. Jafari, C. Ozel and M. A Al-Shumrani, Generalized neutrosophic contra-continuity, New Trend Neut. Theory Appl. **II** (2018) 365-380.

[6] S. Nanda, On fuzzy topological spaces, Fuzzy Sets Syst. **19** (1986) 193-197.

[7] J. H. Park, Y. B. Park and J. S. Park, Fuzzy weakly open mappings and fuzzy RS-compact sets, Far East J. Math. Sci. **II** (1997) 201-212.

[8] D. A. Rose, On weak openness and almost openness, Internt. J. Math. I. Math. Sci. **7** (1984) 35-40.

[9] A. A. Salama and S. A. Alblowi, Neutrosophic set and neutrosophic topological spaces, IOSR J. Math. **3** (2012) 31–35.

[10] F. Smarandache, Neutrosophic set, A generalisation of the intuitionistic fuzzy sets, Int J Pure Appl Math. **24** (2005) 287-297.

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Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume VI Issue 2 (2024), Pages 61- 77, Doi: https://doi.org/10.47087/mjm.1477504

FINITE ELEMENT ANALYSIS OF LINEARLY EXTRAPOLATED BLENDED BACKWARD DIFFERENCE FORMULA (BLEBDF) FOR THE NATURAL CONVECTION FLOWS

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Abstract. In this paper, we study the stability and convergence of fully discrete finite element method with grad-div stabilization for the incompressible non-isothermal fluid flows. The proposed scheme uses finite element discretization in space and linearly extrapolated blended Backward Differentiation Formula (BLEBDF) in time. We prove the unconditional stability over finite time interval and optimally convergence of the scheme. We also present numerical experiments to verify our theoretical convergence rates and show the reliability of the scheme.

1. Introduction

Most of practical engineering problems including insulating in windows, solar collectors, cooling in electronics are modelled by natural convection flows. In the dimensionless form, the equations governed by these flows are given on the domain $\Omega \subset \mathbb{R}^d (d = 2 \text{ or } 3)$ and a time interval $(0, t^*]$, $t^* < \infty$, as follows

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = RiT\boldsymbol{\xi} + \mathbf{f}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

$$T_t + (\mathbf{u} \cdot \nabla)T - \kappa\Delta T = g, \tag{1.3}$$

where $\mathbf{u}$ is the velocity filed, p the pressure, T the temperature and $\mathbf{f}$ and g are the external forcing and thermal source. $\boldsymbol{\xi}$ is the unit vector in the direction of the gravitational acceleration, ν is the dimensionless kinematic viscosity which is inversely proportional to the Reynolds number, i.e. $\nu = Re^{-1}$, κ the thermal conductivity defined as $\kappa = Re^{-1}Pr^{-1}$ where Pr is the Prandtl number and Ri the Richardson number, and Rayleigh number is defined by $Ra = RiRe^2Pr$. The system is complemented with the appropriate initial and boundary conditions.

2020 *Mathematics Subject Classification.* Primary: 65M60 ; Secondaries: 76M10 .
Key words and phrases. Finite Element Method; Grad-div stabilization; Natural Convection.

Submitted on May 2nd, 2024. Accepted on May 27th, 2024.
Communicated by Hacer ŞENGÜL KANDEMIR and Nazlım Deniz ARAL.

This system is well posed under some restriction on the Rayleigh and Prandtl numbers [21]. Simulations with standard Galerkin finite element method of (1.1)-(1.3) for high Rayleigh number leads to severe computational problems and can exhibit global spurious oscillations, [21, 6, 18]. One remedy to overcome this issue is to use the grad-div stabilization. This type of stabilization adds the penalization term $\gamma\nabla(\nabla\cdot\mathbf{u})$ to the momentum equation which leads to $\gamma(\nabla\cdot\mathbf{u}_h, \nabla\cdot\mathbf{v}_h)$ in the discretization. It was originally proposed in [3] and since then it has studied from both theoretical and computational points of view. The studies on grad-div stabilization show that this stabilization improves mass conservation, leads to much more accurate approximate solutions for the Stokes/Navier-Stokes and related coupled multiphysics flow problems, [5, 12, 14, 15, 19, 20].
The aim of this study is to use this advantage of grad-div stabilized finite element for approximating of the natural convection flows. For the temporal discretization, a new second order time stepping scheme called an blended three step Backward Differentiation Formula (BDF) is used. This selection is due to the fact that such scheme is of second order accuracy with a smaller constant in truncation error term, A-stable and is more accurate than two-step BDF scheme, [22, 16, 2, 10].
The remaining of the paper is organized as follows. Section 2 presents some mathematical preliminaries necessary for the finite element analysis. Section 3 introduces the numerical scheme. Section 4 and 5 provides theoretical results of the stability and convergence. We show that approximate solutions are unconditionally stable over finite time interval and converge both in time and space quadratically. Section 6 provides two numerical experiments. The first one verifies the second order convergence in space and time. The second one, on the other hand, tests the reliability and efficiency of the algorithm. For this aim, we compare the solutions of the scheme with BLEBDF (without the stabilization) on Marsigli's experiment. The results show that our method captures very well the flow pattern at each time level.

2. Mathematical Preliminaries

We consider the domain $\Omega \subset \mathbb{R}^d, d = 2,3$ to be a convex polygon or polyhedra. The L^2-inner product and its induced norm will be denoted as $(\cdot,\cdot)$ and $\|\cdot\|$, the H^k-norm by $\|\cdot\|_k$, and the L^∞-norm by $\|\cdot\|_{L^\infty}$, [1]. Continuous velocity, pressure and temperature spaces are given by respectively:

$$\begin{aligned}
\mathbf{X} &:= \mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla\mathbf{v} \in L^2(\Omega)^{d\times d},\ \mathbf{v} = 0 \ on\ \partial\Omega\},\\
Q &:= L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\, d\mathbf{x} = 0\},\\
Y &:= H_0^1(\Omega).
\end{aligned}$$

Further, we define the space $\mathbf{V} \subset \mathbf{X}$ to be the divergence free subset of $\mathbf{X}$. The dual space of $\mathbf{X}$ is denoted by $\boldsymbol{H}^{-1}$ with the norm

$$\|\mathbf{f}\|_{-1} := \sup_{0\neq\mathbf{v}\in\mathbf{X}} \frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla\mathbf{v}\|}.$$

We frequently use the Poincaré-Friedrich's inequality, [11]: there exists a constant C_P such that

$$\|\mathbf{v}\| \leq C_P\|\nabla\mathbf{v}\|,\ \ \forall\mathbf{v} \in \mathbf{X}.$$

Define the skew symmetrized trilinear form for the non-linear terms to ensure stability of the numerical method

$$b_1(\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}\left[(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - (\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})\right], \quad \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X},$$
$$b_2(\mathbf{u},\theta,\Phi) := \frac{1}{2}\left[(\mathbf{u}\cdot\nabla\theta,\Phi) - (\mathbf{u}\cdot\nabla\Phi,\theta)\right], \quad \mathbf{u}\in\mathbf{X},\theta,\Phi\in Y.$$

Lemma 2.1. *For* $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{X}$ *and* $\mathbf{v},\nabla\mathbf{v}\in L^\infty$*, the skew symmetrized trilinear form* $b(\cdot,\cdot,\cdot)$ *is bounded as follows,* [13]

$$b_1(\mathbf{u},\mathbf{v},\mathbf{w}) \le C\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|, \tag{2.1}$$
$$b_1(\mathbf{u},\mathbf{v},\mathbf{w}) \le C\|\mathbf{u}\|\left(\|\nabla\mathbf{v}\|_{L^3} + \|\mathbf{v}\|_{L^\infty}\right)\|\nabla\mathbf{w}\|. \tag{2.2}$$

We assume that τ_h is a regular, conforming mesh with a maximum diameter h, and $\mathbf{X}_h\subset\mathbf{X}$, $Q_h\subset Q$, $Y_h\subset Y$ be conforming finite element spaces which satisfy approximation properties of piece-wise polynomials of local degree $k, k-1$, and k with $k\ge 1$ respectively, [7]

$$\inf_{\mathbf{v}_h\in\mathbf{X}_h}\{\|\mathbf{u}-\mathbf{v}_h\| + h\|\nabla(\mathbf{u}-\mathbf{v}_h)\|\} \le Ch^{k+1}\|\mathbf{u}\|_{k+1},$$
$$\inf_{q_h\in Q_h}\|p-q_h\| \le Ch^k\|p\|_k.$$
$$\inf_{T_h\in Y_h}\{\|T-T_h\| + h\|\nabla(T-T_h)\|\} \le Ch^{k+1}\|T\|_{k+1}.$$

We also assume that the velocity-pressure finite element pair, $(\mathbf{X}_h, Q_h)$, satisfy the discrete inf-sup condition:

$$\inf_{q_h\in Q_h}\sup_{\mathbf{v}_h\in\mathbf{X}_h}\frac{(q_h,\,\nabla\cdot\mathbf{v}_h)}{||\nabla\mathbf{v}_h\,||\,||\,q_h\,||} \ge \beta > 0.$$

We denote the discretely divergence-free space by $\mathbf{V}_h$ and defined as

$$\mathbf{V}_h := \{\mathbf{v}_h\in\mathbf{X}_h:\ (\nabla\cdot\mathbf{v}_h, q_h) = 0,\ \forall q_h\in Q_h\}.$$

We also introduce the following norms on time interval $[0,t^*]$: $1\le p<\infty$

$$\|\phi\|_{p,k} := \left(\int_0^{t^*}\|\phi(t,\cdot)\|_k^p dt\right)^{1/p}, \quad \|\phi\|_{\infty,k} := \sup_{0\le t\le t^*}\|\phi(t,\cdot)\|_\infty$$

and discrete norms

$$|||\phi|||_{p,k} := \left(\Delta t\sum_{n=0}^{N-1}\|\phi(t^n,\cdot)\|_k^p\right)^{1/p}, \quad |||\phi|||_{\infty,k} := \max_{0\le n\le N}\|\phi(t^n,\cdot)\|_k,$$

where $t^n = n\Delta t$, $n = 0,1,2,...,N = t^*/\Delta t$.
We also introduce the following notations for the stability and convergence analysis

$$\delta[\phi^{n+1}] := \frac{5}{3}\phi^{n+1} - \frac{5}{2}\phi^n + \phi^{n-1} - \frac{1}{6}\phi^{n-2}, \tag{2.3}$$
$$E[\phi^{n+1}] := \phi^{n+1} - 3\phi^n + 3\phi^{n-1} - \phi^{n-2}, \tag{2.4}$$

and estimates which are the conclusion of the use of Taylor's Theorem with integral remainder term, [10]:

$$\|\phi_t^{n+1} - \frac{\delta[\phi^{n+1}]}{\Delta t}\|^2 \le \frac{7}{3}(\Delta t)^3 \int_{t^{n-2}}^{t^{n+1}} \|\phi_{ttt}\|^2 dt, \tag{2.5}$$

$$\|E[\phi^{n+1}]\|^2 \le 9\Delta t^5 \int_{t^{n-2}}^{t^{n+1}} \|\phi_{ttt}\|^2 dt. \tag{2.6}$$

To simplify our finite element analysis, we use the G-stability framework as in [8]. For third order backward differentiation, the positive definite matrix G-matrix and the associated norm are given as

$$G = \frac{1}{12}\begin{pmatrix} 19 & -12 & 3 \\ -12 & 10 & -3 \\ 3 & -3 & 1 \end{pmatrix}, \ \|\mathcal{U}\|_G^2 = (\mathcal{U}, G\mathcal{U}), \ \mathcal{U} \in L^2(\Omega).$$

For $\mathcal{U}^{n+1} := \begin{bmatrix} u^{n+1} & u^n & u^{n-1} \end{bmatrix}^T, \ \forall u^i \in L^2(\Omega)$, the following identity holds, [2]

$$\left(\delta[u^{n+1}], u^{n+1}\right) = \|\mathcal{U}^{n+1}\|_G^2 - \|\mathcal{U}^n\|_G^2 + \frac{1}{12}\|E[u^{n+1}]\|^2. \tag{2.7}$$

The G- and L^2-norms are equivalent in the sense that: there exist $C_l, C_u > 0$ positive constants such that

$$C_l\|\mathcal{U}\|_G^2 \le \|\mathcal{U}\|^2 \le C_u\|\mathcal{U}\|_G^2. \tag{2.8}$$

We also use Young's inequality and the discrete version of Gronwall Lemma.

Lemma 2.2. *Let a, b be non-negative real numbers. Then for any $\varepsilon > 0$,*

$$ab \le \frac{\varepsilon}{p}a^p + \frac{\varepsilon^{-q/p}}{q}b^q; \quad \frac{1}{p} + \frac{1}{q} = 1, \ \text{and} \ \ 1 \le p, q \le \infty.$$

Lemma 2.3 (Gronwall Lemma)**.** *Let $\Delta t, H$ and a_n, b_n, c_n, d_n be non-negative numbers such that*

$$a_N + \Delta t \sum_{n=0}^{N} b_n \le \Delta t \sum_{n=0}^{N-1} d_n a_n + \Delta t \sum_{n=0}^{N-1} c_n + H, \ \text{for } N \ge 0.$$

Then for all $\Delta t > 0$

$$a_N + \Delta t \sum_{n=0}^{N} b_n \le \exp\left(\Delta t \sum_{n=0}^{N-1} d_n\right)\left(\Delta t \sum_{n=0}^{N-1} c_n + H\right).$$

3. Numerical Scheme

The proposed numerical scheme uses three-step backward differentiation in time and finite element in space.

Algorithm 3.1. *Let forcing terms $\mathbf{f} \in L^2(0, t^*; \mathbf{H}^{-1}(\Omega))$, $g \in L^2(0, t^*; H^{-1}(\Omega))$ Choose an end time t^* and a time step Δt such that $t^* = N\,\Delta t$. Denote the discrete solutions at time levels $t^n := n\Delta t$ by*

$$\mathbf{u}_h^n := \mathbf{u}_h(t^n), \ \ p_h^n := p_h(t^n), \ \ T_h^n := T_h(t^n), \quad n = 0, 1, 2, ..., N.$$

Let initial conditions $\mathbf{u}_h^{-2}, \mathbf{u}_h^{-1}, \mathbf{u}_h^0 \in \boldsymbol{L}^2(\Omega)$ *and* $T_h^{-2}, T_h^{-1}, T_h^0 \in L^2(\Omega)$ *be given. For* $n = 0, ..., N-1$*, find* $(\mathbf{u}_h^{n+1},\ p_h^{n+1}, T_h^{n+1})$ *such that the equations below are satisfied:* $\forall(\mathbf{v}_h, q_h, \chi_h) \in (\mathbf{X}_h, Q_h, Y_h)$

$$\left(\frac{\delta[\mathbf{u}_h^{n+1}]}{\Delta t}, \mathbf{v}_h\right) + b_1\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}_h^{n+1}, \mathbf{v}_h\right) + \nu\left(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h\right)$$
$$+ \gamma\left(\nabla\cdot\mathbf{u}_h^{n+1}, \nabla\cdot\mathbf{v}_h\right) - \left(p_h^{n+1}, \nabla\cdot\mathbf{v}_h\right) = Ri(\big(3T_h^n - 3T_h^{n-1} + T_h^{n-2}\big)\boldsymbol{\xi}, \mathbf{v}_h\big)$$
$$+ \left(\mathbf{f}^{n+1}, \mathbf{v}_h\right), \qquad (3.1)$$
$$\left(\nabla\cdot\mathbf{u}_h^{n+1},\ q_h\right) = 0, \qquad (3.2)$$
$$\left(\frac{\delta[T_h^{n+1}]}{\Delta t},\ \chi_h\right) + b_2\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, T_h^{n+1}, \chi_h\right) + \kappa\left(\nabla T_h^{n+1}, \nabla\chi_h\right)$$
$$= \left(g^{n+1}, \chi_h\right). \qquad (3.3)$$

4. Stability Analysis

This section is devoted to proving the stability of Algorithm 3.1. We show that the proposed algorithm's solutions are stable over finite time interval without any time step restriction.

Lemma 4.1. *The solutions of Algorithm 3.1 are unconditionally stable over* $(0, t^*)$ *and they satisfy the bound: for any* $\Delta t > 0$

$$\|\mathcal{T}_h^N\|_G^2 + \frac{1}{2}\kappa\Delta t\sum_{n=0}^{N-1}\|T_h^{n+1}\|^2 \le \|\mathcal{T}_h^0\|_G^2 + \frac{1}{2}\kappa^{-1}\|g\|^2_{L^2(0,t^*;H^{-1}(\Omega))} =: K_T, \qquad (4.1)$$

and

$$\|\mathcal{U}_h^N\|_G^2 + \frac{1}{2}\nu\Delta t\sum_{n=0}^{N-1}\|\nabla\mathbf{u}_h^{n+1}\|^2 + \gamma\Delta t\sum_{n=0}^{N-1}\|\nabla\cdot\mathbf{u}_h^{n+1}\|^2$$
$$\le \|\mathcal{U}_h^0\|_G^2 + 27\nu^{-1}Ri^2C_P^2C_uK_T|\boldsymbol{\xi}|^2t^* + \frac{1}{2}\nu^{-1}\|\mathbf{f}\|^2_{L^2(0,t^*;H^{-1}(\Omega))} =: K_{\mathbf{u}}. \qquad (4.2)$$

Proof. We will first obtain the bound on discrete temperature solution. Letting $\chi_h = \Delta t\, T_h^{n+1}$ in (3.3), which vanishes the non-linear term, and then using (2.7) followed by the Cauchy-Schwarz and Young's inequalities we obtain

$$\|\mathcal{T}_h^{n+1}\|_G^2 - \|\mathcal{T}_h^n\|_G^2 + \frac{1}{12}\|E[T_h^{n+1}]\|^2 + \kappa\Delta t\|\nabla T_h^{n+1}\|^2$$
$$= \Delta t\big(g^{n+1}, T_h^{n+1}\big)$$
$$\le \Delta t\|g^{n+1}||_{H^{-1}}\|\nabla T_h^{n+1}\|$$
$$\le \frac{1}{2}\kappa^{-1}\Delta t\|g^{n+1}||^2_{H^{-1}} + \frac{1}{2}\kappa\Delta t\|\nabla T_h^{n+1}||^2. \qquad (4.3)$$

From which, reordering terms gives

$$||\mathcal{T}_h^{n+1}||_G^2 - ||\mathcal{T}_h^n||_G^2 + \frac{1}{12}||E[T_h^{n+1}]||^2 + \frac{1}{2}\kappa\Delta t\|\nabla T_h^{n+1}||^2 \le \frac{1}{2}\kappa^{-1}\Delta t\|g^{n+1}||^2_{H^{-1}}.$$

Dropping the third left hand side term and summing over time steps gives the bound on temperature solution. To get the stability bound on velocity, set $q_h = p_h^{n+1}$ in

(3.2) and $\mathbf{v}_h = \Delta t \mathbf{u}_h^{n+1}$ in (3.1). The pressure and the non-linear terms vanishes, and using the same identity produces

$$\|\mathcal{U}_h^{n+1}\|_G^2 - \|\mathcal{U}_h^n\|_G^2 + \frac{1}{12}\|E[\mathbf{u}_h^{n+1}]\|^2 + \nu\Delta t\|\nabla \mathbf{u}_h^{n+1}\|^2 + \gamma\Delta t\|\nabla\cdot\mathbf{u}_h^{n+1}\|^2$$
$$= \Delta t Ri\big(\big(3T_h^n - 3T_h^{n-1} + T_h^{n-2}\big)\boldsymbol{\xi}, \mathbf{u}_h^{n+1}\big) + \Delta t\big(\mathbf{f}^{n+1}, \mathbf{u}_h^{n+1}\big). \quad (4.4)$$

Using Cauchy-Schwarz, Young's and the Poincaré-Friedrich's inequalities together with (2.8) and (4.1) on the first right hand side term, we have

$$\begin{aligned}
&\Delta t Ri\big(\big(3T_h^n - 3T_h^{n-1} + T_h^{n-2}\big)\boldsymbol{\xi}, \mathbf{u}_h^{n+1}\big)\\
&\le \Delta t Ri\,\left[3\|T_h^n\| + 3\|T_h^{n-1}\| + \|T_h^{n-2}\|\right]|\boldsymbol{\xi}|C_P\|\nabla\mathbf{u}_h^{n+1}\|\\
&\le 27\nu^{-1}\Delta t Ri^2 C_P^2\left[\|T_h^n\|^2 + \|T_h^{n-1}\|^2 + \|T_h^{n-2}\|^2\right]|\boldsymbol{\xi}|^2 + \frac{1}{4}\nu\Delta t\|\nabla\mathbf{u}_h^{n+1}\|^2\\
&\le 27\nu^{-1}\Delta t Ri^2 C_P^2 C_u\|\mathcal{T}_h^n\|_G^2|\boldsymbol{\xi}|^2 + \frac{1}{4}\nu\Delta t\|\nabla\mathbf{u}_h^{n+1}\|^2\\
&\le 27\nu^{-1}\Delta t Ri^2 C_P^2 C_u K_T|\boldsymbol{\xi}|^2 + \frac{1}{4}\nu\Delta t\|\nabla\mathbf{u}_h^{n+1}\|^2,
\end{aligned}$$

and for the forcing term use Cauchy-Schwarz and Young's inequalities to get

$$\Delta t\big(\mathbf{f}^{n+1}, \mathbf{u}_h^{n+1}\big) \le \nu^{-1}\Delta t\|\mathbf{f}^{n+1}\|_{H^{-1}}^2 + \frac{1}{4}\nu\Delta t\|\nabla\mathbf{u}_h^{n+1}\|^2.$$

Combining these estimates together with (4.4) gives

$$\|\mathcal{U}_h^{n+1}\|_G^2 - \|\mathcal{U}_h^n\|_G^2 + \frac{1}{12}\|E[\mathbf{u}_h^{n+1}]\|^2 + \frac{\nu\Delta t}{2}\|\nabla \mathbf{u}_h^{n+1}\|^2 + \gamma\Delta t\|\nabla\cdot\mathbf{u}_h^{n+1}\|^2$$
$$\le 27\nu^{-1}\Delta t Ri^2 C_P^2 C_u K_T|\boldsymbol{\xi}|^2 + \nu^{-1}\Delta t\|\mathbf{f}^{n+1}\|_{H^{-1}}^2. \quad (4.5)$$

Summing over time steps, dropping the third left hand side term gives the desired stability bound on the velocity. □

5. Convergence analysis

This section is devoted to the finite element error analysis of Algorithm 3.1 We will show that the finite element solutions convergences to the true solutions quadratically both in time and space. In the analysis, we will use the following error notations: $\forall n = 0, 1, ..., N$

$$\mathbf{e}_{\mathbf{u}}^n := \mathbf{u}^n - \mathbf{u}_h^n, \quad e_T^n := T^n - T_h^n, \quad (5.1)$$

and error's decomposition

$$\begin{aligned}
\mathbf{e}_{\mathbf{u}}^n &:= \eta_{\mathbf{u}}^n - \phi_{h,\mathbf{u}}^n, \quad \phi_{h,\mathbf{u}}^n := \mathbf{u}_h^n - \tilde{\mathbf{u}}_h^n, \quad \eta_{\mathbf{u}}^n := \mathbf{u}^n - \tilde{\mathbf{u}}_h^n,\\
e_T^n &:= \eta_T^n - \phi_{h,T}^n, \quad \phi_{h,T}^n := T_h^n - \tilde{T}_h^n, \quad \eta_T^n := T^n - \tilde{T}_h^n
\end{aligned} \quad (5.2)$$

True solutions at t^{n+1} satisfies the following equations:

$$\left(\frac{\delta[\mathbf{u}^{n+1}]}{\Delta t}, \mathbf{v}_h\right) + b_1(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, \mathbf{u}^{n+1}, \mathbf{v}_h) + \nu(\nabla \mathbf{u}^{n+1}, \nabla \mathbf{v}_h)$$
$$+ \gamma(\nabla \cdot \mathbf{u}^{n+1}, \nabla \cdot \mathbf{v}_h) - (p^{n+1}, \nabla \cdot \mathbf{v}_h) = Ri((3T^n - 3T^{n-1} + T^{n-2})\boldsymbol{\xi}, \mathbf{v}_h)$$
$$+ (\mathbf{f}^{n+1}, \mathbf{v}_h) + \Lambda_1(\mathbf{u}, T, \mathbf{v}_h), \tag{5.3}$$
$$(\nabla \cdot \mathbf{u}^{n+1}, q_h) = 0, \tag{5.4}$$
$$\left(\frac{\delta[T^{n+1}]}{\Delta t}, \chi_h\right) + b_2(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, T^{n+1}, \chi_h) + \kappa(\nabla T^{n+1}, \nabla \chi_h)$$
$$= (g^{n+1}, \chi_h) + \Lambda_2(\mathbf{u}, T, \chi_h), \tag{5.5}$$

where

$$\Lambda_1(\mathbf{u}, T, \mathbf{v}_h) := \left(\frac{\delta[\mathbf{u}^{n+1}]}{\Delta t} - \mathbf{u}_t^{n+1}, \mathbf{v}_h\right) - b_1(E[\mathbf{u}^{n+1}], \mathbf{u}^{n+1}, \mathbf{v}_h)$$
$$+ Ri(E[T^{n+1}]\boldsymbol{\xi}, \mathbf{v}_h), \tag{5.6}$$
$$\Lambda_2(\mathbf{u}, T, \chi_h) := \left(\frac{\delta[T^{n+1}]}{\Delta t} - T_t^{n+1}, \chi_h\right) - b_2(E[\mathbf{u}^{n+1}], T^{n+1}, \chi_h) \tag{5.7}$$

are consistency errors. We now give the bounds for the consistency errors.

Lemma 5.1.

$$|\Lambda_1(\mathbf{u}, T, \mathbf{v}_h)|$$
$$\leq \frac{9}{\varepsilon}\nu^{-1}\Delta t^5 \left(C\|\nabla \mathbf{u}^{n+1}\|^2 \|\nabla \mathbf{u}_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)} + Ri^2 C_P^2 |\boldsymbol{\xi}|^2 \|T_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)}\right)$$
$$+ \frac{7}{6\varepsilon}\nu^{-1}(\Delta t)^3 C_P^2 \|\mathbf{u}_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)} + \varepsilon\nu\|\nabla \mathbf{v}_h\|^2, \tag{5.8}$$

and

$$|\Lambda_2(\mathbf{u}, T, \chi_h)|$$
$$\leq \frac{9}{2\varepsilon} C\,\kappa^{-1}\,\Delta t^5 \|\nabla T^{n+1}\|^2 \|\nabla \mathbf{u}_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)}$$
$$+ \frac{7}{6\varepsilon}\kappa^{-1}(\Delta t)^3 C_P^2 \|T_{ttt}\|^2_{L^2(t^{n-2},\, t^{n+1};\, L^2)} + \varepsilon\kappa\|\nabla \chi_h\|^2. \tag{5.9}$$

Proof. Using the Cauchy-Schwarz, Young's and Poincaré-Friedrich's inequalities together with (2.1) produces the bounds. □

Theorem 5.2. *Assume that true solutions* $(\mathbf{u}, p, T)$ *satisfies the following regularity conditions*

$$\mathbf{u} \in L^\infty(0, T; \boldsymbol{H}^{k+1}(\Omega)), \quad T \in L^\infty(0, T; H^{k+1}(\Omega)), \quad p \in L^2(0, T; H^{k+1}(\Omega)),$$
$$\mathbf{u}_t \in L^2(0, T; \boldsymbol{H}^1(\Omega)), \quad T_t \in L^2(0, T; H^1(\Omega)), \quad T_{ttt} \in L^2(0, T; L^2(\Omega)),$$
$$\mathbf{u}_{ttt} \in L^2(0, T; \boldsymbol{L}^2(\Omega) \cap \boldsymbol{H}^1(\Omega)).$$

Then the errors defined in (5.1) *satisfy the following bound:*

$$\begin{aligned}&\|\mathbf{e}_{\mathbf{u}}^N\|_G^2 + \|e_T^N\|_G^2 + \frac{1}{12}\sum_{n=0}^{N-1}\left[\|E[\mathbf{e}_{\mathbf{u}}^{n+1}]\|^2 + \|E[e_T^{n+1}]\|^2\right] \\ &+ \frac{\Delta t}{2}\sum_{n=0}^{N-1}\left[\nu\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|^2 + \kappa\|\nabla e_T^{n+1}\|^2\right] + \frac{\gamma\Delta t}{2}\sum_{n=0}^{N-1}\|\nabla\cdot\mathbf{e}_{\mathbf{u}}^{n+1}\|^2 \\ &\leq C(\Delta t^4 + h^{2k}),\end{aligned}$$

where C is the general constant independent of h and Δt.

Proof. We divide the error analysis into thee parts. In the first part, we will give the bounds for the velocity error equation and in the second part, the bounds for the temperature. In the third part, we will apply the Gronwall Lemma and triangle inequality to the error terms to finish the proof.

Step 1 [*The error bound for the velocity*]
Subtracting (3.1)-(3.2) from (5.3)-(5.4) and using error notations given in (5.1) produces: $\forall\, q_h \in Q_h$

$$\begin{aligned}&\left(\frac{\delta[\mathbf{e}_{\mathbf{u}}^{n+1}]}{\Delta t}, \mathbf{v}_h\right) + \nu(\nabla\mathbf{e}_{\mathbf{u}}^{n+1}, \nabla\mathbf{v}_h) + \gamma(\nabla\cdot\mathbf{e}_{\mathbf{u}}^{n+1}, \nabla\cdot\mathbf{v}_h) - (p^{n+1} - q_h, \nabla\cdot\mathbf{v}_h) \\ &\quad + b_1(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, \mathbf{u}^{n+1}, \mathbf{v}_h) - b_1(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) \\ &\qquad = Ri((3e_T^n - 3e_T^{n-1} + e_T^{n-2})\boldsymbol{\xi}, \mathbf{v}_h) + \Lambda_1(\mathbf{u}, T, \mathbf{v}_h), \quad (5.10)\\ &\qquad (\nabla\cdot\mathbf{e}_{\mathbf{u}}^{n+1},\ q_h) = 0, \quad (5.11)\end{aligned}$$

Using error decomposition's in (5.2) and setting $\mathbf{v}_h = \Delta t\phi_{h,\mathbf{u}}^{n+1}$ and applying (2.7) gives

$$\begin{aligned}&\|\phi_{h,\mathbf{u}}^{n+1}\|_G^2 - \|\phi_{h,\mathbf{u}}^n\|_G^2 + \frac{1}{12}\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \nu\Delta t\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \gamma\Delta t\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2 \\ &= \left(\delta[\eta_{\mathbf{u}}^{n+1}], \phi_{h,\mathbf{u}}^{n+1}\right) + \nu\Delta t\left(\nabla\eta_{\mathbf{u}}^{n+1}, \nabla\phi_{h,\mathbf{u}}^{n+1}\right) + \gamma\Delta t\left(\nabla\cdot\eta_{\mathbf{u}}^{n+1}, \nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\right) \\ &\quad - \Delta t\left(p^{n+1} - q_h, \nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\right) + \Delta t\, b_1\left(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}\right) \\ &\quad - \Delta t b_1\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}_h^{n+1}, \phi_{h,\mathbf{u}}^{n+1}\right) \\ &\quad - Ri\,\Delta t\left((3\eta_T^n - 3\eta_T^{n-1} + \eta_T^{n-2})\boldsymbol{\xi},\ \phi_{h,\mathbf{u}}^{n+1}\right) \\ &\quad + Ri\,\Delta t\left((3\phi_{h,T}^n - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2})\boldsymbol{\xi},\ \phi_{h,\mathbf{u}}^{n+1}\right) - \Delta t\,\Lambda_1\left(\mathbf{u}, T, \phi_{h,\mathbf{u}}^{n+1}\right). \qquad (5.12)\end{aligned}$$

We now bound below the right hand side terms of (5.12).The first term is zero due to the L^2-projection. The next three terms are bounded below by using the Cauchy-Schwarz and Young's inequalities as follows:

$$\begin{aligned}\nu\Delta t\left(\nabla\eta_{\mathbf{u}}^{n+1}, \nabla\phi_{h,\mathbf{u}}^{n+1}\right) &\leq \nu\Delta t\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \frac{\nu\Delta t}{4}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2, \\ \gamma\Delta t\left(\nabla\cdot\eta_{\mathbf{u}}^{n+1}, \nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\right) &\leq \gamma\Delta t\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \frac{\gamma\Delta t}{4}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2 \\ \Delta t\left(p^{n+1} - q_h, \nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\right) &\leq \gamma^{-1}\Delta t\inf_{q_h\in Q_h}\|p^{n+1} - q_h\|^2 + \frac{\gamma\Delta t}{4}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2.\end{aligned}$$

For the non-linear terms, we add and subtract the terms below

$$b_1(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1})$$

to get

$$\begin{aligned}
&b_1(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}) - b_1\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \mathbf{u}_h^{n+1}, \phi_{h,\mathbf{u}}^{n+1}\right)\\
&= b_1(3\eta_{\mathbf{u}}^n - 3\eta_{\mathbf{u}}^{n-1} + \eta_{\mathbf{u}}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}) - b_1(3\phi_{h,\mathbf{u}}^n - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1})\\
&+ b_1(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \eta_{\mathbf{u}}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}).
\end{aligned}$$

For the first non-linear term, we apply (2.1) to get

$$\begin{aligned}
&\Delta t\, b_1(3\eta_{\mathbf{u}}^n - 3\eta_{\mathbf{u}}^{n-1} + \eta_{\mathbf{u}}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1})\\
&\quad \le C\,\Delta t\,\left(3\|\nabla\eta_{\mathbf{u}}^n\| + 3\|\nabla\eta_{\mathbf{u}}^{n-1}\| + \|\nabla\eta_{\mathbf{u}}^{n-2}\|\right)\|\nabla\mathbf{u}^{n+1}\|\|\nabla\phi_{\mathbf{u},h}^{n+1}\|\\
&\quad \le 108C\,\Delta t\,\nu^{-1}\left(\|\nabla\eta_{\mathbf{u}}^n\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-2}\|^2\right)\|\nabla\mathbf{u}^{n+1}\|^2 + \frac{\nu\Delta t}{16}\|\nabla\phi_{\mathbf{u},h}^{n+1}\|^2.
\end{aligned}$$

For the second term, use (2.2) together with Young's inequality which leads to

$$\begin{aligned}
&\Delta t\, b_1\left(3\phi_{h,\mathbf{u}}^n - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}, \mathbf{u}^{n+1}, \phi_{h,\mathbf{u}}^{n+1}\right)\\
&\le C\Delta t\|3\phi_{h,\mathbf{u}}^n - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}\|\left(\|\nabla\mathbf{u}^{n+1}\|_{L^3} + \|\mathbf{u}^{n+1}\|_{L^\infty}\right)\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|\\
&\le 16\,\Delta t\,C\,\nu^{-1}\left(\|\nabla\mathbf{u}^{n+1}\|_{L^3}^2 + \|\mathbf{u}^{n+1}\|_{L^\infty}^2\right)\|3\phi_{h,\mathbf{u}}^n - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}\|^2 + \frac{\nu\Delta t}{32}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2.
\end{aligned}$$

For the last non-linear term, we apply (2.1) and Young's inequality to get

$$\begin{aligned}
&\Delta t\, b_1(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, \eta_{\mathbf{u}}^{n+1}, \phi_{h,\mathbf{u}}^{n+1})\\
&\qquad \le C\,\Delta t\,\|\nabla\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\right)\|\|\nabla\eta_{\mathbf{u}}^{n+1}\|\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|\\
&\qquad \le 8\,C\,\nu^{-1}\Delta t\|\nabla\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\right)\|^2\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \frac{\nu\Delta t}{32}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2.
\end{aligned}$$

The next two terms are estimated in a similar way:

$$\begin{aligned}
&Ri\,\Delta t\left((3\phi_{h,T}^n - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2})\boldsymbol{\xi},\ \phi_{h,\mathbf{u}}^{n+1}\right)\\
&\qquad \le Ri\,\Delta t\|3\phi_{h,T}^n - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\||\boldsymbol{\xi}|C_P\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|\\
&\qquad \le 16C_P^2\,Ri^2\nu^{-1}\Delta t\|3\phi_{h,T}^n - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2|\boldsymbol{\xi}|^2 + \frac{\nu\Delta t}{64}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&Ri\,\Delta t\left((3\eta_T^n - 3\eta_T^{n-1} + \eta_T^{n-2})\boldsymbol{\xi},\ \phi_{h,\mathbf{u}}^{n+1}\right)\\
&\qquad \le Ri\,\Delta t\|3\eta_T^n - 3\eta_T^{n-1} + \eta_T^{n-2}\||\boldsymbol{\xi}|C_P\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|\\
&\qquad \le 16C_P^2\,Ri^2\nu^{-1}\Delta t\|3\eta_T^n - 3\eta_T^{n-1} + \eta_T^{n-2}\|^2|\boldsymbol{\xi}|^2 + \frac{\nu\Delta t}{64}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2.
\end{aligned}$$

Now take $\mathbf{v}_h = \phi_{h,\mathbf{u}}^{n+1}$ in (5.8) with $\varepsilon = 3/32$. Then considering the resulting inequality and all these estimates above on the right hand side of (5.12) and combining like terms yields

$$\|\phi_{h,\mathbf{u}}^{n+1}\|_G^2 - \|\phi_{h,\mathbf{u}}^{n}\|_G^2 + \frac{1}{12}\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \frac{\nu\Delta t}{2}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \frac{\gamma\Delta t}{2}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2$$
$$\le (\nu+\gamma)\Delta t\,\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \gamma^{-1}\Delta t \inf_{q_h\in Q_h}\|p^{n+1}-q_h\|^2$$
$$+ 108\,C\,\nu^{-1}\,\Delta t\,\left(\|\nabla\eta_{\mathbf{u}}^{n}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-2}\|^2\right)\|\nabla\mathbf{u}^{n+1}\|^2$$
$$+ 16C\nu^{-1}\left(\|\nabla\mathbf{u}^{n+1}\|_{L^3}^2 + \|\mathbf{u}^{n+1}\|_{L^\infty}^2\right)\|3\phi_{h,\mathbf{u}}^{n} - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}\|^2$$
$$+ 8\,C\,\nu^{-1}\Delta t\|\nabla\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\right)\|^2\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2$$
$$+ 16C_P^2 Ri^2\nu^{-1}\Delta t\|3\phi_{h,T}^{n} - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2|\boldsymbol{\xi}|^2$$
$$+ 16C_P^2\,Ri^2\nu^{-1}\Delta t\|3\eta_T^{n} - 3\eta_T^{n-1} + \eta_T^{n-2}\|^2|\boldsymbol{\xi}|^2$$
$$+ C\nu^{-1}(\Delta t)^4\left(\|T_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|\nabla\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2\right). \tag{5.13}$$

Step 2 [***The error bound for the temperature***] First subtract (3.3) from (5.5) and consider error notations given in (5.1) to get

$$\left(\frac{\delta[e_T^{n+1}]}{\Delta t},\chi_h\right) + \kappa(\nabla e_T^{n+1},\nabla\chi_h) + b_2(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, T^{n+1},\chi_h)$$
$$- b_2(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, T_h^{n+1},\chi_h) = \Lambda_2(\mathbf{u},T,\chi_h). \tag{5.14}$$

Then using error decompositions, setting $\chi_h = \Delta t\phi_{h,T}^{n+1}$ and recalling (2.7) gives

$$\|\phi_{h,T}^{n+1}\|_G^2 - \|\phi_{h,T}^{n}\|_G^2 + \frac{1}{12}\|E[\phi_{h,T}^{n+1}]\|^2 + \kappa\Delta t\|\nabla\phi_{h,T}^{n+1}\|^2$$
$$= \left(\delta[\eta_T^{n+1}],\phi_{h,T}^{n+1}\right) + \kappa\Delta t\left(\nabla\eta_T^{n+1},\nabla\phi_{h,T}^{n+1}\right)$$
$$+ \Delta t b_2(3\mathbf{u}^n - 3\mathbf{u}^{n-1} + \mathbf{u}^{n-2}, T^{n+1},\phi_{h,T}^{n+1})$$
$$- \Delta t b_2(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}, T_h^{n+1},\phi_{h,T}^{n+1}) - \Delta t\Lambda_2\left(\mathbf{u},T,\phi_{h,T}^{n+1}\right). \tag{5.15}$$

Proceeding in a similar way as in Step 1, we can bound the right hand side of (5.15) as follows

$$\|\phi_{h,T}^{n+1}\|_G^2 - \|\phi_{h,T}^{n}\|_G^2 + \frac{1}{12}\|E[\phi_{h,T}^{n+1}]\|^2 + \frac{\kappa\Delta t}{2}\|\nabla\phi_{h,T}^{n+1}\|^2$$
$$\le \kappa\Delta t\,\|\nabla\eta_T^{n+1}\|^2 + 108\,C\,\kappa^{-1}\,\Delta t\,\left(\|\nabla\eta_{\mathbf{u}}^{n}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-2}\|^2\right)\|\nabla T^{n+1}\|^2$$
$$+ 16\,C\kappa^{-1}\left(\|\nabla T^{n+1}\|_{L^3}^2 + \|T^{n+1}\|_{L^\infty}^2\right)\|3\phi_{h,T}^{n} - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2$$
$$+ 8\,C\,\kappa^{-1}\Delta t\|\nabla\left(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\right)\|^2\|\nabla\eta_T^{n+1}\|^2$$
$$+ C\,\kappa^{-1}(\Delta t)^4\left(C_P^2\|T_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|\nabla\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2\right). \tag{5.16}$$

Step 3 [***The application of the Gronwall Lemma***] Add (5.13) to (5.16) to get

$$
\begin{aligned}
&\left(\|\phi_{h,\mathbf{u}}^{n+1}\|_G^2 + \|\phi_{h,T}^{n+1}\|_G^2\right) - \left(\|\phi_{h,\mathbf{u}}^{n}\|_G^2 + \|\phi_{h,T}^{n}\|_G^2\right) + \frac{1}{12}\left(\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \|E[\phi_{h,T}^{n+1}]\|^2\right)\\
&+ \frac{\nu\Delta t}{2}\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \frac{\kappa\Delta t}{2}\|\nabla\phi_{h,T}^{n+1}\|^2 + \frac{\gamma\Delta t}{2}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2\\
&\leq (\nu+\gamma^{-1})\Delta t\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \kappa\Delta t\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \gamma^{-1}\Delta t\inf_{q_h\in Q_h}\|p^{n+1}-q_h\|^2\\
&+ 108\, C\Delta t\left[\nu^{-1}\|\nabla\mathbf{u}^{n+1}\|^2 + \kappa^{-1}\|\nabla T^{n+1}\|^2\right]\left[\|\nabla\eta_{\mathbf{u}}^{n}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-2}\|^2\right]\\
&+ 16C\nu^{-1}\Delta t\left[\|\nabla\mathbf{u}^{n+1}\|_{L^3}^2 + \|\mathbf{u}^{n+1}\|_{L^\infty}^2\right]\|3\phi_{h,\mathbf{u}}^{n} - 3\phi_{h,\mathbf{u}}^{n-1} + 3\phi_{h,\mathbf{u}}^{n-2}\|^2\\
&+ 16C\kappa^{-1}\Delta t\left[\|\nabla T^{n+1}\|_{L^3}^2 + \|T^{n+1}\|_{L^\infty}^2\right]\|3\phi_{h,\mathbf{u}}^{n} - 3\phi_{h,\mathbf{u}}^{n-1} + 3\phi_{h,\mathbf{u}}^{n-2}\|^2\\
&+ 8C\Delta t(\nu^{-1}\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \kappa^{-1}\|\nabla\eta_T^{n+1}\|^2)\|\nabla(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2})\|^2\\
&+ 16C_P^2 Ri^2\nu^{-1}\Delta t\|3\phi_{h,T}^{n} - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2|\boldsymbol{\xi}|^2\\
&+ 16C_P^2\, Ri^2\nu^{-1}\Delta t\|3\eta_T^{n} - 3\eta_T^{n-1} + \eta_T^{n-2}\|^2|\boldsymbol{\xi}|^2\\
&+ C\nu^{-1}(\Delta t)^4\left(\|\nabla\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|T_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2\right)\\
&+ C\kappa^{-1}(\Delta t)^4\left(\|\nabla\mathbf{u}_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2 + \|T_{ttt}\|_{L^2(t^{n-2},\,t^{n+1};\,L^2)}^2\right).
\end{aligned}
$$

Then summing over time steps and using approximating properties produces

$$
\begin{aligned}
&\|\phi_{h,\mathbf{u}}^{N}\|_G^2 + \|\phi_{h,T}^{N}\|_G^2 + \frac{1}{12}\sum_{n=0}^{N-1}\left[\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \|E[\phi_{h,T}^{n+1}]\|^2\right]\\
&+ \frac{\Delta t}{2}\sum_{n=0}^{N-1}\left[\nu\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \kappa\|\nabla\phi_{h,T}^{n+1}\|^2\right] + \frac{\gamma\Delta t}{2}\sum_{n=0}^{N-1}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2\\
&\leq C\Delta t\sum_{n=0}^{N-1}M^{n+1}\left[\|3\phi_{h,\mathbf{u}}^{n} - 3\phi_{h,\mathbf{u}}^{n-1} + \phi_{h,\mathbf{u}}^{n-2}\|^2 + \|3\phi_{h,T}^{n} - 3\phi_{h,T}^{n-1} + \phi_{h,T}^{n-2}\|^2\right]\\
&\quad + \Delta t\sum_{n=0}^{N-1}4C\left[\nu^{-1}\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \kappa^{-1}\|\nabla\eta_T^{n+1}\|^2\right]\|\nabla(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\|^2\\
&\quad + C(\nu^{-1}+\kappa^{-1})|||\eta_{\mathbf{u}}^{n+1}|||_{2,0}^2 + CRi^2\nu^{-1}|||\eta_T^{n+1}|||_{2,0}^2\\
&\quad + C\gamma^{-1}\inf_{q_h\in Q_h}|||p-q_h|||_{2,0}^2\\
&\quad + C\Delta t^4\left[(\nu^{-1}+\kappa^{-1})(\|\nabla\mathbf{u}_{ttt}\|_{2,0}^2 + \|T_{ttt}\|_{2,0}^2) + \nu^{-1}\|\mathbf{u}_{ttt}\|_{2,0}^2\right]\\
&\quad + \|\phi_{h,\mathbf{u}}^0\|_G^2 + \|\phi_{h,T}^0\|_G^2,
\end{aligned}
$$

where

$$
\begin{aligned}
M^{n+1} := \max\{&16C\nu^{-1}[\|\nabla\mathbf{u}^{n+1}\|_{L^3}^2 + \|\mathbf{u}^{n+1}\|_{L^\infty}^2], 16C\kappa^{-1}[\|\nabla T^{n+1}\|_{L^3}^2 + \|T^{n+1}\|_{L^\infty}^2],\\
&16C_P^2 Ri^2\nu^{-1}|\boldsymbol{\xi}|^2\}.
\end{aligned}
$$

Now apply the Gronwall's Lemma to get

$$\|\phi_{h,\mathbf{u}}^N\|_G^2 + \|\phi_{h,T}^N\|_G^2 + \frac{1}{12}\sum_{n=0}^{N-1}\Big[\|E[\phi_{h,\mathbf{u}}^{n+1}]\|^2 + \|E[\phi_{h,T}^{n+1}]\|^2\Big]$$

$$+ \frac{\Delta t}{2}\sum_{n=0}^{N-1}\Big[\nu\|\nabla\phi_{h,\mathbf{u}}^{n+1}\|^2 + \kappa\|\nabla\phi_{h,T}^{n+1}\|^2\Big] + \frac{\gamma\Delta t}{2}\sum_{n=0}^{N-1}\|\nabla\cdot\phi_{h,\mathbf{u}}^{n+1}\|^2$$

$$\leq \exp\left(\Delta t\sum_{n=0}^{N-1}M^{n+1}\right)$$

$$\Bigg(\Delta t\sum_{n=0}^{N-1}C\left(\nu^{-1}+\kappa^{-1}\right)\|\nabla(3\mathbf{u}_h^n - 3\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\|^2 + C(\nu^{-1}+\kappa^{-1})|||\eta_{\mathbf{u}}^{n+1}|||_{2,0}^2$$

$$+ CRi^2\nu^{-1}|||\eta_T^{n+1}|||_{2,0}^2 + C\gamma^{-1}\inf_{q_h\in Q_h}|||p-q_h|||_{2,0}^2$$

$$+ C\Delta t^4\left[(\nu^{-1}+\kappa^{-1})(\|\nabla\mathbf{u}_{ttt}\|_{2,0}^2 + \|T_{ttt}\|_{2,0}^2) + \nu^{-1}\|\mathbf{u}_{ttt}\|_{2,0}^2\right]\Bigg)$$

Using the stability result on the right hand side, drooping the third left hand side term and applying the triangle inequality to the error terms finishes the proof. □

6. Numerical Studies

In this section, we impose two numerical experiments. The first numerical experiment verify our convergence rates obtained in Theorem 5.2 while the second one reveals the effectiveness of the proposed algorithm. The numerical experiment was implemented using the software package FreeFem++, [9].

6.1. **Convergence rate verification.** To verify theoretical findings, we pick true velocity, pressure and temperature solutions

$$\mathbf{u}(\mathbf{x},t) = \begin{pmatrix}\cos(\pi(y-t))\\ \sin(\pi(x+t))\end{pmatrix}\exp(t),$$

$$p(\mathbf{x},t) = \sin(x+y)(1+t^2),\quad T(\mathbf{x},t) = \sin(\pi x) + y\exp(t),$$

on region $\Omega = (0,1)\times(0,1)$ with $\nu = \kappa = Ri = \gamma = 1.0$. Forcing terms $\mathbf{f}$ and g are calculated from (1.1)-(1.2). We impose the following boundary conditions:

$$\mathbf{u}_h(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t),\quad T_h(\mathbf{x},t) = T(\mathbf{x},t)\ \text{ on }\ \partial\Omega.$$

To verify the spatial convergence rates, fix end time $t^* = 0.001$ with a time step $\Delta t = t^*/8$, and use $(\mathbf{P}_2, P_1, P_2)$ finite elements Then we run Algorithm 3.1 on successively mesh refinements. The calculated rates show that the spatial convergence for both velocity and temperate are of second order, see Table 1 .

For the temporal convergence rate verification, we fix mesh size $h = 1/128$, and take end time $t^* = 1$. Then we calculate the solutions of Algorithm 3.1 for $\Delta t = 1/4, 1/8, 1/16, 1/32, 1/64$. The errors and rates obtained from these calculations are presented in Table 2 and verify our theoretical findings.

h	$\|\|\|\nabla(\mathbf{u}-\mathbf{u}_h)\|\|\|_{2,0}$	Rate	$\|\|\|\nabla(T-T_h)\|\|\|_{2,0}$	Rate
1/4	8.8479e-5	–	6.2564e-5	–
1/8	2.2657e-5	1.9653	1.6021e-5	1.9978
1/16	5.6982e-6	1.9914	4.0292e-6	1.9654
1/32	1.4267e-6	1.9978	1.0088e-6	1.9978
1/64	3.5681e-7	1.9995	2.52230e-7	1.9994

TABLE 1. Spatial velocity and temporal errors and rates.

Δt	$\|\|\|\nabla(\mathbf{u}-\mathbf{u}_h)\|\|\|_{2,0}$	Rate	$\|\|\|\nabla(T-T_h)\|\|\|_{2,0}$	Rate
1/4	2.0299e-3	–	1.4317e-3	–
1/8	5.1350e-4	1.9829	3.6024e-2	1.9840
1/16	1.2974e-4	1.9845	9.0220e-5	1.9829
1/32	3.2319e-5	2.0052	2.2566e-5	1.9993
1/64	8.0195e-6	2.0108	5.6420e-6	1.9999

TABLE 2. Temporal velocity and temperature errors and rates.

6.2. **Marsigli's experiment.** This numerical experiment tests and aims to show the effectiveness of Algorithm 3.1 on a physical situation which demonstrates that when two fluids with different densities meet, a motion driven by the gravitational force is created: the fluid with higher density rises over the lower one. Since the density differences can be modelled by the temperature differences with the help of the Boussinesq approximation, this physical problem is modelled by the incompressible Boussinesq system (1.1)-(1.3) studied herein. For the experiment's set-up, we follow the paper written by Johnston and co-workers, [17]. The domain is an insulated box, $\Omega = [0,8] \times [0,1]$ and dimensionless flow parameters are set to be

$$Re = 1.000,\ Ri = 4.0,\ Pr = 1.0,\ \gamma = 1.0.$$

We take time step $\Delta t = 0.2$ and use $(\boldsymbol{P}_2, P_1, P_2)$ finite element spaces for the velocity, the pressure and temperature. We run both our algorithm and BLEBDF without grad-div stabilization. The obtained results presented in Figure 1 and Figure 2 indicate that Algorithm 3.1 catches very well the flow pattern at each time level. However, BLEBDF without grad-div stabilization creates very poor solutions and builds significant oscillations as time progresses.

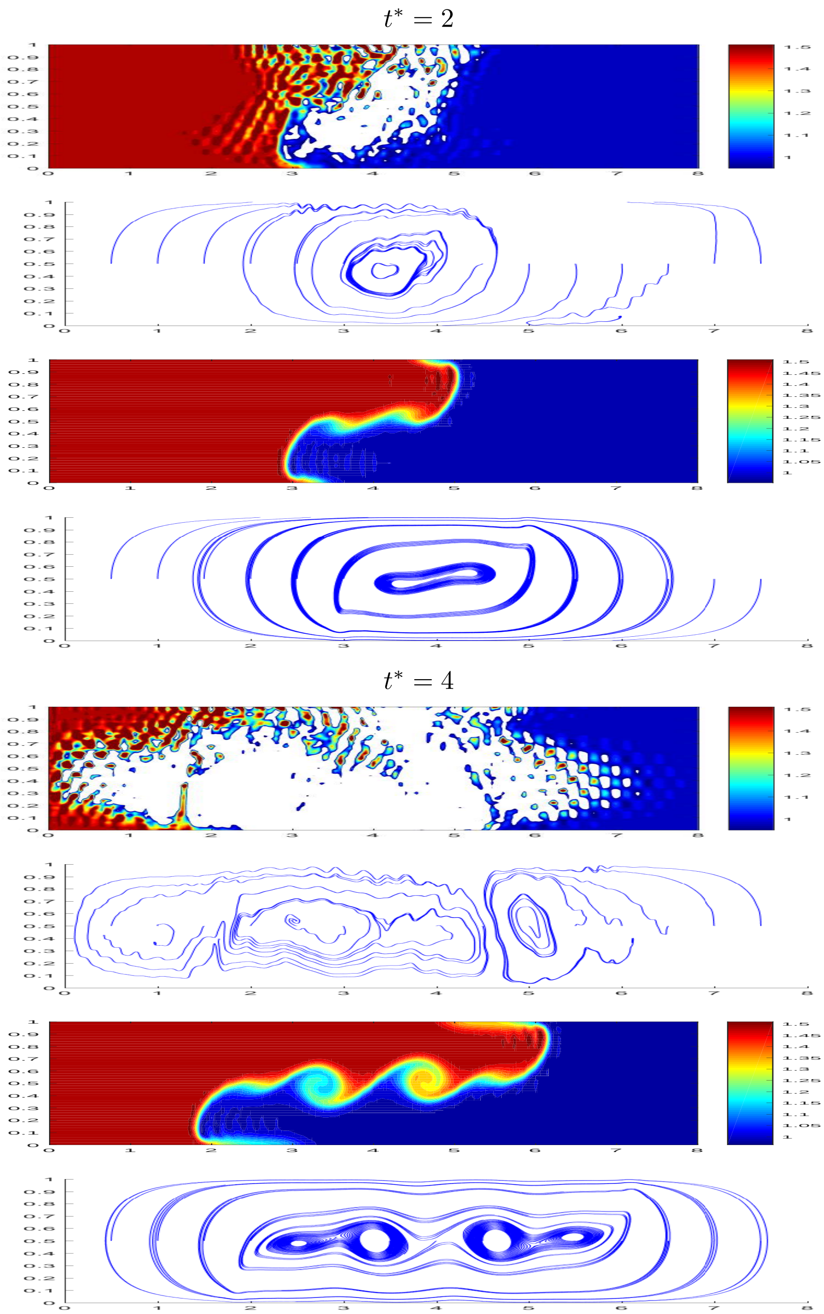

FIGURE 1. The temperature contours and velocity streamlines of BLEBDF without grad-div stabilization and of Algorithm 3.1, respectively, from a coarse mesh computation at $t^* = 2, 4$.

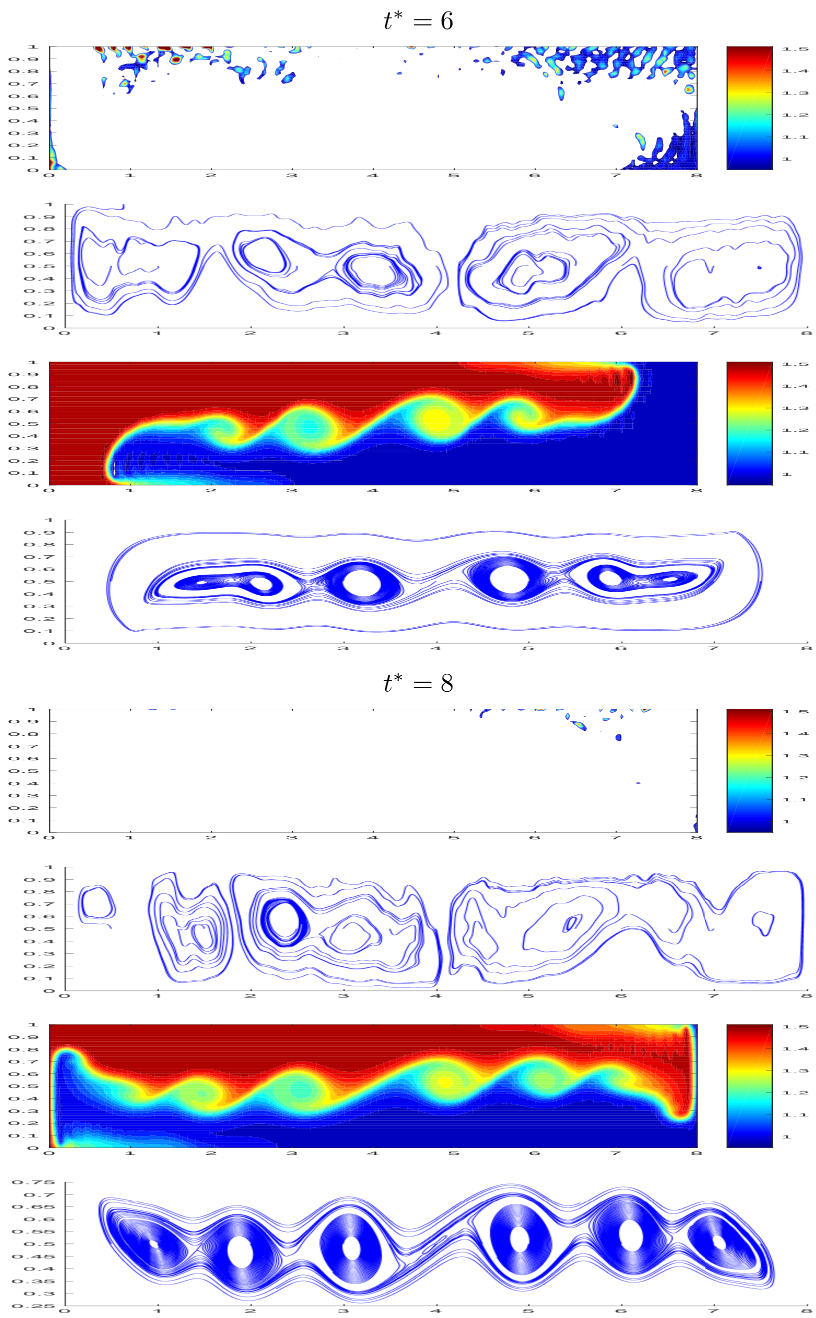

FIGURE 2. The temperature contours and velocity streamlines of BLEBDF without grad-div stabilization and of Algorithm 3.1, respectively, from a coarse mesh computation at $t^* = 6, 8$.

7. Conclusion

In this paper, we used grad-div stabilized finite element method for approximating natural convection flow problems. We applied a new class second order time stepping called linearized blended tree-step BDF in time. The proposed scheme is unconditionally stable over finite time interval and of second order convergent both in time and space. The numerical experiments presented here verifies theoretical convergence rates, and reveals the reliability of the proposed scheme.

Conflict of interests

The authors declare that there is no conflict of interests.

Acknowledgments

The authors would like to thank the referees for the valuable suggestions, which improved the paper.

References

[1] R. A. Adams, Sobolev spaces, Academic Press, New York, (1975).
[2] A. Çıbık, F. G. Eroğlu and S. Kaya, *Long Time Stability of a Linearly Extrapolated Blended BDF Scheme for Multiphysics Flows*, Int. J. Numer. Anal. Mod. **17** (2020) 24-41.
[3] L. P. Franca and T. J. R. Hughes, *Two classes of mixed finite element methods*, Comput. Methods Appl. Mech. Engrg. **69(1)**, (1988) 89-128.
[4] J. de Frutos, B. García-Archilla, V. John and J. Novo, *Analysis of the grad-div stabilization for the time-dependent Navier–Stokes equations with inf-sup stable finite elements*, Adv Comput. Math. **44**, (2018), 195–225.
[5] K. Galvin, A. Linke, L. Rebholz and N. Wilson, *Stabilizing poor mass conservation in incompressible flow problems with large irrotational forcing and application to thermal convection*, Comput. Methods Appl. Mech. Engrg **237–240** (2012) 166–176.
[6] P. M. Gresho, M. Lee, S. T. Chan and R. L. Sani, *Solution of time dependent, incompressible Navier–Stokes and Boussinesq equations using the Galerkin finite element method*, in: Lecture Notes in Math., vol. 771, Springer-Verlag, Berlin/Heidelberg/New York (1980) 203–222.
[7] V. Girault and P. A. Raviart, Finite element approximation of the Navier-Stokes equations, Lecture Notes in Math., Vol. 749, Springer-Verlag, Berlin (1979).
[8] E. Hairer and G. Wanner, Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems, Second edition, Springer-Verlag, Berlin (2002).
[9] F. Hecht. *New development in freefem++*, J. Numer. Math., **20(3-4)** (2012) 251–265.
[10] N. Jiang, *A second-order ensemble method based on a blended backward differentiation formula timestepping scheme for time-dependent Navier–Stokes equations*, Numer. Methods Partial Differ. Equ. **33(1)**, (2017).
[11] V. John, *Finite Element Methods for Incompressible Flow Problems*, Springer Series in Computational Mathematics, Switzerland (2016).
[12] V. John, A. Linke, C. Merdon, M. Neilan and L. Rebholz, *On the divergence constraint in mixed finite element methods for incompressible flows*,SIAM Rev. **59** (2017) 492–544.
[13] W. Layton. An Introduction to the Numerical Analysis of Viscous Incompressible Flows. SIAM, Philadelphia (2008).
[14] A. Linke, G. Matthies and L. Tobiska, *Robust arbitrary order mixed finite element methods for the incompressible Stokes Equations with pressure independent velocity errors*, ESAIM:M2AN. **50** (2016) 289-309.
[15] A. Linke and C. Merdon, *On velocity errors due to irrotational forces in the Navier-Stokes momentum balance*, J. Comput. Phys. **313** (2016) 654-661.
[16] S. Liu, P. Huang and Y. He, *A second-order scheme based on blended BDF for the incompressible MHD system*, Adv. Comput. Math. **49** (2023) 79.
[17] J.-G. Liu, C. Wang and H. Johnston *Fourth order scheme for incompressible Boussinesq equations*, J. Sci. Comput., **18(2)** (2003) 253-285.

[18] H. Melhem, Finite element approximation to heat transfer through combined solid and fluid media, PhD thesis, University of Pittsburgh, (1987).
[19] M. Olshanskii and A. Reusken, *Grad-div stabilization for Stokes equations*, Math. Comp. **73** (2004) 1699-1718.
[20] M. A. Olshanskii, G. Lube, T. Heister and J. Löwe, *Grad-div stabilization and sub-grid pressure models for the incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg. **198** (2009) 3975-3988.
[21] P. H. Rabinowitz, *Existence and nonuniqueness of rectangular solutions of the Benard problem*, Arch. Ration. Mech. Anal. **29**, (1968).
[22] S. S. Ravindran, *An Analysis of the Blended Three-Step Backward Differentiation Formula Time-Stepping Scheme for the Navier-Stokes-Type System Related to Soret Convection*, Numer. Func. Anal. Opt. **36** (2015) 658-686.

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Maltepe Journal of Mathematics
SSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume VI Issue 2 (2024), Pages 78-89, Doi: https://doi.org/10.47087/mjm.1452521

GENERALIZED TENSORIAL SIMPSON TYPE INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACE

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Abstract. Several generalized Simpson tensorial type inequalities for selfadjoint operators have been obtained with variation depending on the conditions imposed on the function $\mathfrak{f}$

$$\Bigg\|\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)$$
$$+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\Bigg\|$$
$$\leq\frac{5\left\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\right\|(2\lambda-1)^2+1)}{144}\left\|\mathfrak{f}'\right\|_{I,+\infty}.$$

1. Introduction and preliminaries

The concept we now call a “tensor” wasn’t originally named that way. When Josiah Willard Gibbs first described the idea in the late 19th century, he used the term “dyadic.” Today, mathematicians define a tensor as the mathematical embodiment of Gibbs’ initial concept. Tensors and inequalities are natural partners, thanks to the widespread use of inequalities in mathematics. These mathematical statements about comparisons have a profound impact on various scientific disciplines. While many types of inequalities exist, some of the most significant ones include Jensen’s, Ostrowski’s, Hermite-Hadamard’s, and Minkowski’s inequalities. For those interested in delving deeper, references [21] and [23] provide more details about inequalities and their fascinating history. Regarding the generalizations of the aforementioned inequalities, numerous studies have been published; for additional information, check the following and the references therein [8, 24, 25, 22, 17, 16, 15, 28, 29, 30, 1, 2, 3, 4, 5, 7, 9, 10].
Classical inequalities of Simpson type have been given by Hezenci et al. [19] and Sarikaya et al. [26]. To enhance the presentation of this work, we will demonstrate

2010 *Mathematics Subject Classification.* Primary: MSC Number ; 26D05; 26D07; Secondaries: MSC Numbers ; 26D20 .

Key words and phrases. Tensorial product, Selfadjoint operators, Convex functions.

Submitted on March 14th, 2024. Accepted on October 11th, 2024.
Communicated by Huseyin CAKALLI and Hacer ŞENGÜL KANDEMIR.

new developments in the theory of inequalities in Hilbert spaces. One such development is the Dragomir's inequality for normal operators given by the following [11]:

Theorem 1.1. *Let $(\mathscr{H};\langle .,.\rangle)$ be a Hilbert space and $\mathfrak{T}:\mathscr{H}\to\mathscr{H}$ a normal linear operator on $\mathscr{H}$. Then*

$$\|\mathfrak{T}x\|^2 \geq \frac{1}{2}\left(\|\mathfrak{T}x\|^2 + |\langle \mathfrak{T}^2 x, x\rangle|\right) \geq |\langle \mathfrak{T}x, x\rangle|^2,$$

for any $x\in H$, $\|x\|=1$. The constant $\frac{1}{2}$ is the best possible.

The Hermite-Hadamard inequality in the selfadjoint operator sense, as provided by Dragomir [12], is another intriguing conclusion.

Theorem 1.2. *Let $\mathfrak{f}: I\to\mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators $\mathfrak{A}$ and $\mathfrak{B}$ with spectra in I we have the inequality*

$$\begin{aligned}\mathfrak{f}\left(\frac{\mathfrak{A}+\mathfrak{B}}{2}\right) &\leq \mathfrak{f}\left(\frac{3\mathfrak{A}+\mathfrak{B}}{4}\right)+\mathfrak{f}\left(\frac{\mathfrak{A}+3\mathfrak{B}}{4}\right)\\ &\leq \int_0^1 \mathfrak{f}((1-t)\mathfrak{A}+t\mathfrak{B})dt\\ &\leq \frac{1}{2}\left[\mathfrak{f}\left(\frac{\mathfrak{A}+\mathfrak{B}}{2}\right)+\frac{\mathfrak{f}(\mathfrak{A})+\mathfrak{f}(\mathfrak{B})}{2}\right] \leq \frac{\mathfrak{f}(\mathfrak{A})+\mathfrak{f}(\mathfrak{B})}{2}.\end{aligned}$$

The first paper related to tensorial inequalities in Hilbert space was written by Dragomir [14]. In the paper, he proved the tensorial version of the Ostrowski type inequality given by the following.

Theorem 1.3. *Assume that $\mathfrak{f}$ is continuously differentiable on I with $\|\mathfrak{f}'\|_{I,+\infty} := \sup_{t\in I}|\mathfrak{f}'(t)| < +\infty$ and $\mathfrak{A},\mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B})\subset I$. Then the following inequality holds:*

$$\begin{aligned}&\left\|\mathfrak{f}((1-\lambda)\mathfrak{A}\otimes 1+\lambda 1\otimes\mathfrak{B}) - \int_0^1 \mathfrak{f}((1-u)\mathfrak{A}\otimes 1 + u1\otimes\mathfrak{B})du\right\| \qquad (1.1)\\ &\leq \|\mathfrak{f}'\|_{I,+\infty}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\end{aligned}$$

for $\lambda\in[0,1]$.

Recently, various inequalities in the same tensorial surrounding have been obtained. The following result of Simpson type was obtained by Stojiljković [31].

Theorem 1.4. *Assume that $\mathfrak{f}$ is continuously differentiable on I and $|\mathfrak{f}''|$ is convex and $\mathfrak{A},\mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B})\subset I$. Then the following inequality holds:*

$$\begin{aligned}&\Bigg\|\frac{1}{6}\left(\mathfrak{f}(\mathfrak{A})\otimes 1 + 4\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1 + 1\otimes\mathfrak{B}}{2}\right)+1\otimes\mathfrak{f}(\mathfrak{B})\right)\\ &-\frac{1}{2}\alpha\Bigg(\int_0^1 \mathfrak{f}\left(\left(\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\left(\frac{1+k}{2}\right)1\otimes\mathfrak{B}\right)k^{\alpha-1}dk\end{aligned}$$

$$+\int_0^1 \mathfrak{f}\left(\left(\left(1-\frac{k}{2}\right)\mathfrak{A}\otimes 1+\frac{k}{2}1\otimes\mathfrak{B}\right)(1-k)^{\alpha-1}dk\right)\Bigg\|$$

$$\leq \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|^2\,\frac{\left(\|\mathfrak{f}''(\mathfrak{A})\|+\|\mathfrak{f}''(\mathfrak{B})\|\right)\left(3\alpha^2+8\alpha+7\right)}{(\alpha+2)(24\alpha+24)}$$

for $\alpha > 0$.

The following inequality has been recently obtained by the same author [32].

Theorem 1.5. *Assume that* $\mathfrak{f}$ *is continuously differentiable on* I *with* $\|\mathfrak{f}'\|_{I,+\infty} := \sup_{t\in I}|\mathfrak{f}'(t)| < +\infty$ *and* $\mathfrak{A},\mathfrak{B}$ *are selfadjoint operators with* $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$. *Then the following inequality holds:*

$$\left\|\int_0^1 \mathfrak{f}((1-\lambda)\mathfrak{A}\otimes 1+\lambda 1\otimes\mathfrak{B})d\lambda-\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}}{2}\right)\right\|$$

$$\leqslant \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|^2\,\frac{\|\mathfrak{f}''\|_{I,+\infty}}{24}.$$

Recently, the following inequality of Ostrowski type was obtained by Stojiljković et al. [33] which generalized the recently obtained results by Dragomir [14].

Theorem 1.6. *The formulation is the same as the one given by Dragomir in his Ostrowski type Theorem given above* (1.1) *with an exception that* $\alpha > 0$, *then*

$$\Bigg\|\Big(\lambda^\alpha+(1-\lambda)^\alpha\Big)\mathfrak{f}((1-\lambda)\mathfrak{A}\otimes 1+\lambda 1\otimes\mathfrak{B})$$

$$-\alpha\Bigg((1-\lambda)^\alpha\int_0^1 \mathfrak{f}((1-\lambda)(1-u)\mathfrak{A}\otimes 1+(u+(1-u)\lambda)1\otimes\mathfrak{B})(1-u)^{\alpha-1}du$$

$$+\lambda^\alpha\int_0^1 u^{\alpha-1}\mathfrak{f}(((1-u)+u(1-\lambda))\mathfrak{A}\otimes 1+u\lambda 1\otimes\mathfrak{B})du\Bigg)\Bigg\|$$

$$\leqslant \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\left(\frac{\lambda^{\alpha+1}}{\alpha+1}+\frac{(1-\lambda)^{\alpha+1}}{\alpha+1}\right)\|\mathfrak{f}'\|_{I,+\infty}.$$

Stojiljković et al., [34] recently obtained a Trapezoid type tensorial inequality which is given by the following.

Theorem 1.7. *Assume that* $\mathfrak{f}$ *is continuously differentiable on* I *with* $\|\mathfrak{f}'\|_{I,+\infty} := \sup_{t\in I}|\mathfrak{f}'(t)| < +\infty$ *and* $\mathfrak{A},\mathfrak{B}$ *are selfadjoint operators with* $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$. *Then the following inequality holds for* $\alpha > 0$*:*

$$\Bigg\|(\mathfrak{f}(\mathfrak{A})\otimes 1+1\otimes\mathfrak{f}(\mathfrak{B})) \tag{1.2}$$

$$-\alpha\left[\int_0^1(1-\lambda)^{\alpha-1}\mathfrak{f}(\lambda 1\otimes\mathfrak{B}+(1-\lambda)\mathfrak{A}\otimes 1)d\lambda+\int_0^1\lambda^{\alpha-1}\mathfrak{f}(\lambda 1\otimes\mathfrak{B}+(1-\lambda)\mathfrak{A}\otimes 1)d\lambda\right]\Bigg\|$$

$$\leq \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\,\frac{1}{1+\alpha}\left(2-2^{1-\alpha}\right)\|\mathfrak{f}'\|_{I,+\infty}.$$

In order to derive similar inequalities of the tensorial type, we need the following introduction and preliminaries.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $\mathfrak{f} : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $\mathfrak{A} = (\mathfrak{A}_1, ..., \mathfrak{A}_k)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of $\mathfrak{A}_i$ is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of $\mathfrak{f}$. If

$$\mathfrak{A}_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of $\mathfrak{A}_i$ for $i = 1, ..., k$ by following , we define

$$\mathfrak{f}(\mathfrak{A}_1, ..., \mathfrak{A}_k) := \int_{I_1} ... \int_{I_k} \mathfrak{f}(\lambda_1, ..., \lambda_k) dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k)$$

as bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [6] extends the definition of Koranyi [20] for functions of two variables and have the property that

$$\mathfrak{f}(\mathfrak{A}_1, ...\mathfrak{A}_k) = \mathfrak{f}_1(\mathfrak{A}_1) \otimes ... \otimes \mathfrak{f}_k(\mathfrak{A}_k),$$

whenever $\mathfrak{f}$ can be separated as a product $\mathfrak{f}(t_1, ..., t_k) = \mathfrak{f}_1(t_1)...\mathfrak{f}_k(t_k)$ of k functions each depending on only one variable.

Recall the following property of the tensorial product

$$(\mathfrak{A}\mathfrak{C}) \otimes (\mathfrak{B} \otimes \mathfrak{D}) = (\mathfrak{A} \otimes \mathfrak{B})(\mathfrak{C} \otimes \mathfrak{D})$$

that holds for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \in B(\mathscr{H})$.

From the property we can deduce easily the following consequences

$$\mathfrak{A}^n \otimes \mathfrak{B}^n = (\mathfrak{A} \otimes \mathfrak{B})^n, n \geqslant 0,$$

$$(\mathfrak{A} \otimes 1)(1 \otimes \mathfrak{B}) = (1 \otimes \mathfrak{B})(\mathfrak{A} \otimes 1) = A \otimes B,$$

which can be extended, for two natural numbers m, n we have

$$(\mathfrak{A} \otimes 1)^n (1 \otimes \mathfrak{B})^m = (1 \otimes \mathfrak{B})^m (\mathfrak{A} \otimes 1)^n = \mathfrak{A}^n \otimes \mathfrak{B}^m.$$

For more information, consult the following book related to tensors [18]. The following Lemma which we require can be found in a paper of Dragomir [13].

Lemma 1.8. *Assume $\mathfrak{A}$ and $\mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}) \subset I, Sp(\mathfrak{B}) \subset J$ and having the spectral resolutions . Let $\mathfrak{f}; \mathfrak{h}$ be continuous on $I, \mathfrak{g}, \mathfrak{k}$ continuous on J and ϕ and ψ continuous on an interval K that contains the sum of the intervals $\mathfrak{f}(I) + \mathfrak{g}(J); \mathfrak{h}(I) + \mathfrak{k}(J)$,then*

$$\begin{aligned} &\phi(\mathfrak{f}(\mathfrak{A}) \otimes 1 + 1 \otimes \mathfrak{g}(\mathfrak{B}))\psi(\mathfrak{h}(\mathfrak{A}) \otimes 1 + 1 \otimes \mathfrak{k}(\mathfrak{B})) \\ &= \int_I \int_J \phi(\mathfrak{f}(t) + \mathfrak{g}(s))\psi(\mathfrak{h}(t) + \mathfrak{k}(s)) dE_t \otimes dF_s. \end{aligned} \tag{1.3}$$

In the paper written by Sarikaya and Bardak [27], the following Lemma is given, which is used to obtain inequalities generalized Simpson type inequalities.

Lemma 1.9. *Let* $\mathfrak{f} : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I°, $a{,}b \in I^\circ$ *with* $a < b$. *If* $\mathfrak{f}' \in L^1[a, b]$, *then the following equality holds:*

$$\frac{(\omega-a)^2}{2(b-a)}\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\frac{1+k}{2}\omega+\frac{1-k}{2}a\right)dk \tag{1.4}$$

$$+\frac{(b-\omega)^2}{2(b-a)}\int_0^1\left(\frac{1}{3}-\frac{k}{2}\right)\mathfrak{f}'\left(\frac{1+k}{2}\omega+\frac{1-k}{2}b\right)dk$$

$$=\frac{1}{6}\mathfrak{f}(\omega)+\frac{1}{3(b-a)}\left[(\omega-a)\mathfrak{f}\left(\frac{a+\omega}{2}\right)+(b-\omega)\mathfrak{f}\left(\frac{\omega+b}{2}\right)\right]-\frac{1}{b-a}\int_{\frac{a+\omega}{2}}^{\frac{\omega+b}{2}}\mathfrak{f}(x)dx,$$

where $\omega = \mu a + (1-\mu)b, \forall\mu \in [0, 1]$.

This paper delves into a novel area of mathematics: tensorial inequalities of the Simpson type for differentiable functions within a tensorial Hilbert space. This field is young and ripe for exploration, and obtaining new bounds for various combinations of convex functions is crucial for its advancement. The paper is structured logically. The "Main Results" section unveils the key findings that contribute to the novelty of this work. Subsequently, the "Examples and Consequences" section showcases practical applications of the obtained results. By choosing specific convex functions, we generate numerous tensorial Simpson-type inequalities and bounds. Finally, the "Conclusion" section summarizes the paper's contributions and highlights its significance for the development of tensorial inequalities. In the following theorem, you'll find a fundamental result that serves as the foundation for deriving further inequalities throughout the paper.

2. Main results

The following Lemma will be crucial in obtaining the inequalities which follow.

Lemma 2.1. *Assume that* $\mathfrak{f}$ *is continuously differentiable on* I, A *and* B *are self-adjoint operators with* $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$, *then*

$$\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right) \tag{2.1}$$

$$+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi$$

$$=\frac{(1-\lambda)^2}{2}(1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1)\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)dk$$

$$+\frac{\lambda^2(1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1)}{2}\int_0^1\left(\frac{1}{3}-\frac{k}{2}\right)\mathfrak{f}'\left(\left(\frac{(1+k)\lambda}{2}\mathfrak{A}\otimes 1\right)+\left(1-\frac{\lambda(1+k)}{2}\right)1\otimes\mathfrak{B}\right)dk.$$

Proof. We will start the proof with Lemma 1.9 (eq. (1.4)). Introducing the substitutions on the right hand side and simplifying the integral, then assuming that $\mathfrak{A}$ and $\mathfrak{B}$ have the spectral resolutions

$$\mathfrak{A}=\int tdE(t) \text{ and } \mathfrak{B}=\int sdF(s).$$

If we take the integral $\int_I\int_I$ over $dE_t \otimes dF_s$, then we get

$$\int_I\int_I\left(\frac{1}{6}\mathfrak{f}(\lambda t+(1-\lambda)s)+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)t+(1-\lambda)s}{2}\right)\right.$$

$$+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda t+(2-\lambda)s}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)t+s\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\Bigg)dE_t\otimes dF_s$$

$$=\int_I\int_I\left(\frac{(1-\lambda)^2}{2}(s-t)\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)dk\right.$$

$$\left.+\frac{\lambda^2(s-t)}{2}\int_0^1\left(\frac{1}{3}-\frac{k}{2}\right)\mathfrak{f}'\left(\left(\frac{(1+k)\lambda}{2}t\right)+\left(1-\frac{\lambda(1+k)}{2}\right)s\right)dk\right)dE_t\otimes dF_s.$$

By utilizing the Fubini's Theorem and Lemma 1.8 (eq. (1.3)) for appropriate choices of the functions involved, we have successively

$$\int_I\int_I\mathfrak{f}(\lambda t+(1-\lambda)s)dE_t\otimes dF_s=\mathfrak{f}\left(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}\right),$$

$$\int_I\int_I\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)t+s\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\Bigg)dE_t\otimes dF_s$$

$$=\int_0^1\int_I\int_I\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)t+s\left(\frac{\phi+(1-\lambda)}{2}\right)dE_t\otimes dF_s d\phi\right.$$

$$=\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+\left(\frac{\phi+(1-\lambda)}{2}1\otimes\mathfrak{B}\right)d\phi,\right.$$

$$\int_I\int_I(s-t)\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)dk dE_t\otimes dF_s$$

$$=\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\int_I\int_I(s-t)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)dE_t\otimes dF_s dk$$

$$=(1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1)\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)dk.$$

Following the same principle for other terms, the equality follows.

□

Theorem 2.2. *Assume that* $\mathfrak{f}$ *is continuously differentiable on* I *with* $\|\mathfrak{f}'\|_{I,+\infty}:=\sup_{t\in I}|\mathfrak{f}'(t)|<+\infty$ *and* $\mathfrak{A},\mathfrak{B}$ *are selfadjoint operators with* $Sp(\mathfrak{A}),Sp(\mathfrak{B})\subset I,\lambda\in[0,1]$ *, then*

$$\left\|\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right.\tag{2.2}$$

$$\left.+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\right\|$$

$$\leq\frac{5\left\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\right\|(2\lambda-1)^2+1)}{144}\left\|\mathfrak{f}'\right\|_{I,+\infty}.$$

Proof. If we take the operator norm of the previously obtained Lemma (2.1) and use the triangle inequality, we get

$$\left\|\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right.$$

$$\left.+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\right\|$$

$$\leq \frac{(1-\lambda)^2}{2} \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\| \int_0^1 \left\|\frac{k}{2}-\frac{1}{3}\right\| \left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\| dk$$

$$+\frac{\lambda^2\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|}{2}\int_0^1\left\|\frac{1}{3}-\frac{k}{2}\right\|\left\|\mathfrak{f}'\left(\left(\frac{(1+k)\lambda}{2}\mathfrak{A}\otimes 1\right)+\left(1-\frac{\lambda(1+k)}{2}\right)1\otimes\mathfrak{B}\right)\right\| dk.$$

Realize here that by Lemma 1.8,

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right|$$

$$=\int_I\int_I\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right|dE_t\otimes dF_s.$$

Since

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right|\leqslant\|\mathfrak{f}'\|_{I,+\infty}.$$

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right|$$

$$=\int_I\int_I\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right|dE_t\otimes dF_s$$

$$\leqslant\|\mathfrak{f}'\|_{I,+\infty}\int_I\int_I dE_t\otimes dF_s=\|\mathfrak{f}'\|_{I,+\infty}.$$

From which we get the following,

$$\int_0^1\left\|\frac{1}{3}-\frac{k}{2}\right\|\left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\|dk$$

$$\leqslant\|\mathfrak{f}'\|_{I,+\infty}\int_0^1\left\|\frac{1}{3}-\frac{k}{2}\right\|dk=\frac{5\|\mathfrak{f}'\|_{I,+\infty}}{36}.$$

Evaluation of the second part is analogous, summing everything up we obtain the desired equality. □

Theorem 2.3. *Assume that* $\mathfrak{f}$ *is continuously differentiable on* I *and* $|\mathfrak{f}'|$ *is convex and* $\mathfrak{A},\mathfrak{B}$ *are selfadjoint operators with* $Sp(\mathfrak{A}),Sp(\mathfrak{B})\subset I,\lambda\in[0,1]$, *then*

$$\left\|\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right. \tag{2.3}$$

$$\left.+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\right\|$$

$$\leq\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\frac{\|\mathfrak{f}'(\mathfrak{A})\|(\lambda(\lambda(122\lambda-93)+3)+29)+\|\mathfrak{f}'(\mathfrak{B})\|(\lambda((273-122\lambda)\lambda-183)+61)}{1296}.$$

Proof. Since $|\mathfrak{f}'|$ is convex on I, then we get

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right|\leqslant\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)|\mathfrak{f}'(t)|+\frac{1+k}{2}(1-\lambda)|\mathfrak{f}'(s)|$$

for all $k \in [0,1]$ and $t, s \in I$.
If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\left| \mathfrak{f}' \left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) \mathfrak{A} \otimes 1 + \frac{1+k}{2}(1-\lambda) 1 \otimes \mathfrak{B} \right) \right|$$
$$= \int_I \int_I \left| \mathfrak{f}' \left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) t + \frac{1+k}{2}(1-\lambda)s \right) \right| dE_t \otimes dF_s$$
$$\leqslant \int_I \int_I \left[\left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) |\mathfrak{f}'(t)| + \frac{1+k}{2}(1-\lambda)|\mathfrak{f}'(s)| \right] dE_t \otimes dF_s$$
$$= \left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) |\mathfrak{f}'(\mathfrak{A})| \otimes 1 + \frac{1+k}{2}(1-\lambda) 1 \otimes |\mathfrak{f}'(\mathfrak{B})|$$

for all $k \in [0,1]$.
If we take the norm in the inequality, we get the following

$$\left\| \mathfrak{f}' \left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) \mathfrak{A} \otimes 1 + \frac{1+k}{2}(1-\lambda) 1 \otimes \mathfrak{B} \right) \right\|$$
$$\leqslant \left\| \left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) |\mathfrak{f}'(\mathfrak{A})| \otimes 1 + \frac{1+k}{2}(1-\lambda) 1 \otimes |\mathfrak{f}'(\mathfrak{B})| \right\|$$
$$\leqslant \left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) \| |\mathfrak{f}'(\mathfrak{A})| \otimes 1 \| + \frac{1+k}{2}(1-\lambda) \| 1 \otimes |\mathfrak{f}'(\mathfrak{B})| \|$$
$$= \left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) \| \mathfrak{f}'(\mathfrak{A}) \| + \frac{1+k}{2}(1-\lambda) \| \mathfrak{f}'(\mathfrak{B}) \| .$$

Therefore, we obtain

$$\int_0^1 \left\| \frac{k}{2} - \frac{1}{3} \right\| \left\| \mathfrak{f}' \left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) \mathfrak{A} \otimes 1 + \frac{1+k}{2}(1-\lambda) 1 \otimes \mathfrak{B} \right) \right\| dk$$
$$\leqslant \int_0^1 \left\| \frac{k}{2} - \frac{1}{3} \right\| \left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2} \right) \| \mathfrak{f}'(\mathfrak{A}) \| + \frac{1+k}{2}(1-\lambda) \| \mathfrak{f}'(\mathfrak{B}) \| \right) dk$$
$$= \frac{\| \mathfrak{f}'(\mathfrak{A}) \| (61\lambda + 29) + 61 \| \mathfrak{f}'(\mathfrak{B}) \| (1-\lambda))}{648}.$$

Simplifying the other term and adding them, we obtain the desired inequality.

□

We recall that the function $\mathfrak{f} : I \to \mathbb{R}$ is quasi-convex, if

$$\mathfrak{f}((1-\lambda)t + \lambda s) \leqslant \max(\mathfrak{f}(t), \mathfrak{f}(s)) = \frac{1}{2}(\mathfrak{f}(t) + f(s) + |\mathfrak{f}(s) - \mathfrak{f}(t)|)$$

holds for all $t, s \in I$ and $\lambda \in [0,1]$.

Theorem 2.4. *Assume that $\mathfrak{f}$ is continuously differentiable on I with $|\mathfrak{f}'|$ is quasi-convex on I, $\mathfrak{A}$ and $\mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I, \alpha$, then*

$$\left\| \frac{1}{6}\mathfrak{f}(\lambda \mathfrak{A} \otimes 1 + (1-\lambda) 1 \otimes \mathfrak{B}) + \frac{1-\lambda}{3} \mathfrak{f} \left(\frac{(1+\lambda)\mathfrak{A} \otimes 1 + (1-\lambda) 1 \otimes \mathfrak{B}}{2} \right) \right.$$
$$\left. + \frac{\lambda}{3} \mathfrak{f} \left(\frac{\lambda \mathfrak{A} \otimes 1 + (2-\lambda) 1 \otimes \mathfrak{B}}{2} \right) - \frac{1}{2} \int_0^1 \mathfrak{f} \left(\left(\frac{1+\lambda-\phi}{2} \right) \mathfrak{A} \otimes 1 + 1 \otimes \mathfrak{B} \left(\frac{\phi + (1-\lambda)}{2} \right) \right) d\phi \right\|$$
$$\leq \frac{5 \| 1 \otimes \mathfrak{B} - \mathfrak{A} \otimes 1 \| (2\lambda - 1)^2 + 1)}{288} \| \mathfrak{f}' \|_{I,+\infty} . \tag{2.4}$$

Proof. Since $|\mathfrak{f}'|$ is quasi-convex on I, then we get

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right| \leqslant \frac{1}{2}(|\mathfrak{f}'(t)|+|\mathfrak{f}'(s)|+||\mathfrak{f}'(t)|-|\mathfrak{f}'(s)||)$$

for all $k \in [0,1]$ and $t,s \in I$. If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\begin{aligned}
&\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right| \\
&= \int_I\int_I \left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right| dE_t\otimes dF_s \\
&\leqslant \frac{1}{2}\int_I\int_I (|\mathfrak{f}'(t)|+|\mathfrak{f}'(s)|+||\mathfrak{f}'(t)|-|\mathfrak{f}'(s)||)dE_t\otimes dF_s \\
&= \frac{1}{2}(|\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|+||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})||)
\end{aligned}$$

for all $k \in [0,1]$.
If we take the norm, then we get

$$\begin{aligned}
&\left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\| \\
&\leqslant \left\|\frac{1}{2}(|\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|+||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})||)\right\| \\
&\leqslant \frac{1}{2}\left(\||\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|\|+\||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})|\|\right)
\end{aligned}$$

Which when applied in our case, we get

$$\begin{aligned}
&\int_0^1 \left\|\frac{k}{2}-\frac{1}{3}\right\| \left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\| dk \\
&\leqslant \int_0^1 \left\|\frac{k}{2}-\frac{1}{3}\right\| \left(\frac{1}{2}\left(\||\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|\|+\||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})|\|\right)\right)dk.
\end{aligned}$$

Which when simplified, we obtain the desired inequality. □

3. Some examples and consequences

In the following sequel we provide examples to the obtained Theorems in Main section. Examples consist of taking f to be an exponential operator and applying various conditions as given by the Theorems.

Corollary 3.1. *If $\mathfrak{A}, \mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset [m, M]$ and $1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1$ is invertible, then by* (2.2)*, we get*

$$\begin{aligned}
&\left\|\frac{1}{6}\exp(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\exp\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right. \\
&\left.+\frac{\lambda}{3}\exp\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\exp\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\right\| \\
&\leq \frac{5\,\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\,((2\lambda-1)^2+1)}{144}\exp(M). \qquad (3.1)
\end{aligned}$$

Corollary 3.2. *Since for* $\mathfrak{f}(t) = \exp(t), t \in \mathbb{R}$, $|\mathfrak{f}'|$ *is convex, then by* (2.3)

$$\Bigg\| \frac{1}{6}\exp(\lambda\mathfrak{A}\otimes 1 + (1-\lambda)1\otimes\mathfrak{B}) + \frac{1-\lambda}{3}\exp\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1 + (1-\lambda)1\otimes\mathfrak{B}}{2}\right)$$
$$+\frac{\lambda}{3}\exp\left(\frac{\lambda\mathfrak{A}\otimes 1 + (2-\lambda)1\otimes\mathfrak{B}}{2}\right) - \frac{1}{2}\int_0^1 \exp\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1 + 1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\Bigg\|$$
$$\leq \|1\otimes\mathfrak{B} - \mathfrak{A}\otimes 1\| \frac{\|\exp(\mathfrak{A})\|(\lambda(\lambda(122\lambda-93)+3)+29) + \|\exp(\mathfrak{B})\|(\lambda((273-122\lambda)\lambda-183)+61)}{1296}. \tag{3.2}$$

Setting $\lambda = \frac{1}{2}$, *we obtain*

$$\Bigg\| \frac{1}{6}\exp\left(\frac{\mathfrak{A}\otimes + 1\otimes\mathfrak{B}}{2}\right) + \frac{1}{6}\exp\left(\frac{3\mathfrak{A}\otimes 1 + 1\otimes\mathfrak{B}}{4}\right)$$
$$+\frac{1}{6}\exp\left(\frac{\mathfrak{A}\otimes 1 + 3\cdot 1\otimes\mathfrak{B}}{2}\right) - \frac{1}{2}\int_0^1 \exp\left(\left(\frac{\frac{3}{2}-\phi}{2}\right)\mathfrak{A}\otimes 1 + 1\otimes\mathfrak{B}\left(\frac{\phi+\frac{1}{2})}{2}\right)\right)d\phi\Bigg\|$$
$$\leq \|1\otimes\mathfrak{B} - \mathfrak{A}\otimes 1\| \frac{5(\|\exp(\mathfrak{A})\| + \|\exp(\mathfrak{B})\|}{288}. \tag{3.3}$$

4. Conclusion

Tensors have become important in various fields, for example in physics because they provide a concise mathematical framework for formulating and solving physical problems in fields such as mechanics, electromagnetism, quantum mechanics, and many others. As such inequalities are crucial in numerical aspects. Reflected in this work is the tensorial Sarikaya and Bardak's Lemma, which as a consequence enabled us to obtain Simpson type inequalities in Hilbert space. New Simpson type inequalities are given, examples of specific convex functions and their inequalities using our results are given in the section some examples and consequences. Plans for future research can be reflected in the fact that the obtained inequalities in this work can be sharpened or generalized by using other methods. An interesting perspective can be seen in incorporating other techniques for Hilbert space inequalities with the techniques shown in this paper. One direction is the technique of the Mond-Pecaric inequality, on which we will work on.

References

[1] W. Afzal; M. Abbas; J.E. Macías-Díaz; S. Treanţă. Some H-Godunova–Levin Function Inequalities Using Center Radius (Cr) Order Relation. Fractal Fract. 2022, **6**, 518. https://doi.org/10.3390/fractalfract6090518

[2] W. Afzal.; A.A. Lupas; K. Shabbir. Hermite–Hadamard and Jensen-Type Inequalities for Harmonical (h1, h2)-Godunova–Levin Interval-Valued Functions. Mathematics 2022, **10**, 2970. https://doi.org/10.3390/math10162970

[3] W. Afzal., K. Shabbir, S. Treanţă, K. Nonlaopon. Jensen and Hermite-Hadamard type inclusions for harmonical h-Godunova-Levin functions[J]. Aims Mathematics, 2023, **8**(2): 3303-3321. doi: 10.3934/math.2023170

[4] W. Afzal., K. Shabbir, T. Botmart. Generalized version of Jensen and Hermite-Hadamard inequalities for interval-valued (h_1, h_2)- Godunova-Levin functions[J]. AIMS Mathematics, 2022, **7**(10): 19372-19387. doi: 10.3934/math.20221064

[5] W. Afzal, W. Nazeer, T. Botmart, S. Treanţă. Some properties and inequalities for generalized class of harmonical Godunova-Levin function via center radius order relation[J]. AIMS Mathematics, 2023, **8**(1): 1696-1712. doi: 10.3934/math.2023087

[6] H. Araki and F. Hansen, Jensenís operator inequality for functions of several variables, Proc. Amer. Math. Soc. 128 (2000), No. **7**, 20

[7] S.I Butt; M. Tariq; A. Aslam; H. Ahmad; T.A. Nofal. Hermite–Hadamard type inequalities via generalized harmonic exponential convexity and applications. J. Funct. Spaces 2021, 2021, 5533491.

[8] S.I Butt, I. Javed, P. Agarwal et al. Newton–Simpson-type inequalities via majorization. J Inequal Appl 2023, 16 (2023). https://doi.org/10.1186/s13660-023-02918-0

[9] A. Chandola , R. Agarwal , M. R. Pandey.Some New Hermite–Hadamard, Hermite–Hadamard Fejer and Weighted Hardy Type Inequalities Involving (k-p) Riemann–Liouville Fractional Integral Operator, Appl. Math. Inf. Sci. 16, No. **2**, 287–297 (2022).

[10] H. Chen, U.N. Katugampola. Hermite–Hadamard and Hermite–Hadamard–Fejer type inequalities for generalized fractional integrals. J. Math. Anal. Appl. 2017, **446**, 1274–1291

[11] S.S. Dragomir. Applicable Analysis and Discrete Mathematics 2007 Volume **1**, Issue 1, Pages: 92-110 https://doi.org/10.2298/AADM0701092D

[12] S.S. Dragomir, The Hermite-Hadamard type Inequalities for Operator Convex Functions, Applied Mathematics and Computation Volume 218, Issue **3**, 1 October 2011, Pages 766-772

[13] S.S Dragomir, Tensorial Norm Inequalities For Taylor's Expansions Of Functions Of Selfadjoint Operators In Hilbert Spaces , ResearchGate, November 2022.

[14] S.S Dragomir, An Ostrowski Type Tensorial Norm Inequality For Continuous Functions Of Selfadjoint Operators In Hilbert Spaces, Researchgate, November 2022.

[15] S.S. Dragomir, C.E.M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, in: RGMIA Monographs, Victoria University, 2000.

[16] S.S. Dragomir. A Trapezoid Type Tensorial Norm Inequality for Continuous Functions of Selfadjoint Operators in Hilbert Spaces. Istanbul Journal of Mathematics. 2023;1(2):48-56. https://doi.org/10.26650/ijmath.2023.00006

[17] S.S. Dragomir. Gradient Inequalities for an Integral Transform of Positive Operators in Hilbert Spaces, Annales Mathematicae Silesianae Volume 37 (2023): Issue 2 (September 2023), https://doi.org/10.2478/amsil-2023-0008

[18] H. Guo, What Are Tensors Exactly?, World Scientific, June 2021, https://doi.org/10.1142/12388

[19] F. Hezenci, H. Budak, H. Kara. New version of fractional Simpson type inequalities for twice differentiable functions. Adv Differ Equ 2021, **460** (2021). https://doi.org/10.1186/s13662-021-03615-2

[20] A. Koranyi. On some classes of analytic functions of several variables. Trans. Amer. Math. Soc., **101** (1961), 520-554.

[21] D. S. Mitrinović ,Analytic Inequalities, Springer-Verlag, Berlin, 1970.

[22] J., Nasir, S., Qaisar, S. I., Butt, A. K., Khan, R. S., Mabela, Some Simpson's Riemann–Liouville Fractional Integral Inequalities with Applications to Special Functions, Journal of Function Spaces, 2022, 2113742, 12 pages, 2022. https://doi.org/10.1155/2022/2113742

[23] J. Pečarić , F. Proschan , Y. Tong , Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, INC, United States of America, 1992.

[24] T. Rasheed, S. I. Butt, G. Pečarić, and Đ. Pečarić, New bounds of Popoviciu's difference via weighted Hadamard type inequalities with applications in information theory, Math. Meth. Appl. Sci. 47 (2024), 5750–5763, DOI 10.1002/mma.9889.

[25] T. Rasheed, S.I. Butt, Đ. Pečarić, J. Pečarić, Generalized cyclic Jensen and information inequalities, Chaos, Solitons & Fractals, Volume 163, 2022, 112602, ISSN 0960-0779, https://doi.org/10.1016/j.chaos.2022.112602.

[26] M.Z. Sarikaya, E. Set, M.E. Özdemir, On new inequalities of Simpson's type for convex functions, RGMIA Res. Rep. Coll. **13** (2) (2010) Article2.

[27] M. Z. Sarıkaya and S. Bardak, "Generalized Simpson Type Integral Inequalities", Konuralp J. Math., vol. 7, no. 1, pp. 186–191, 2019.

[28] V. Stojiljkovic, Simpson Type Tensorial Norm Inequalities for Continuous Functions of ´ Selfadjoint Operators in Hilbert Spaces, Creat. Math. Inform., 33 (2024), 105–117. https://doi.org/10.37193/CMI.2024.01.10

[29] V. Stojiljković; R. Ramaswamy; O.A.A. Abdelnaby; S. Radenović. Some Refinements of the Tensorial Inequalities in Hilbert Spaces. Symmetry 2023, **15**, 925. https://doi.org/10.3390/sym15040925

[30] V. Stojiljković.; Hermite–Hadamard–type fractional–integral inequalities for (p, h)-convex fuzzy-interval-valued mappings, Electron. J. Math. **5** (2023) 18–28, DOI: 10.47443/ejm.2023.004

[31] V. Stojiljković. (2023). 'Twice Differentiable Ostrowski Type Tensorial Norm Inequality for Continuous Functions of Selfadjoint Operators in Hilbert Spaces' , Electronic Journal of Mathematical Analysis and Applications, **11**(2), pp. 1-15. doi: 10.21608/ej-maa.2023.199881.1014

[32] V. Stojiljković. Twice Differentiable Ostrowski Type Tensorial Norm Inequality for Continuous Functions of Selfadjoint Operators in Hilbert Spaces. European Journal of Pure and Applied Mathematics, **16**(3), (2023), 1421–1433.

[33] V. Stojiljković., & S.S. Dragomir. (2023). Differentiable Ostrowski type tensorial norm inequality for continuous functions of selfadjoint operators in Hilbert spaces. Gulf Journal of Mathematics, **15**(2), 40-55. https://doi.org/10.56947/gjom.v15i2.1247

[34] V. Stojiljković; N. Mirkov; S. Radenović. Variations in the Tensorial Trapezoid Type Inequalities for Convex Functions of Self-Adjoint Operators in Hilbert Spaces. Symmetry 2024, **16**, 121. https://doi.org/10.3390/sym16010121

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Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume VI Issue 2 (2024), Pages 90-102, https://doi.org/10.47087/mjm.1515500

NEUTROSOPHICATION β-COMPACTNESS

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Abstract. In this study, we first define the concept of neutrosophic β-open set. Then, using this new set definition, we present neutrosophic β-compact and neutrosophic β-closed spaces and examine their properties. Also, we classify these spaces using the concept of neutrosophic filterbase, which is introduced for the first time in this study. And, relationships between these different types and forms of compactness are investigated.

1. Introduction

The concept of compactness is one of the indispensable characters of general topology and other topology forms. In general topology, these concepts of β-open sets and β-continuous functions were first introduced by Abd El-Monsef [1]. Later, these concepts were adapted to different topology forms and some interesting properties of them were investigated in different forms of topology as in [4, 6, 8]. By using these notions, the basic classical results that have been going on in the general topology from the very beginning are generalized. The aim of this paper is to define and investigate the concepts of neutrosophic β-compactness and neutrosophic β-closed spaces. Also, the concept of neutrosophic filterbases is presented and by using this, we characterize these new types of compactness and space. Additionally, the relationship between these new types and some different forms of compactness in neutrosophic topology is clarified and a comparison between them is established.

2. Preliminaries

In this section, we present the basic definitions related to neutrosophic set theory.

2020 *Mathematics Subject Classification.* Primary: 54A05; Secondaries: 54C10 ; 54D30 ; 54D101.
Key words and phrases. Neutrosophic β-open set, Neutrosophic β-compactness Neutrosophic β-closed space, Neutrosophic filterbases.

Submitted on July 12th, 2024. Accepted on September 20th, 2024.
Communicated by Huseyin CAKALLI and Nazlım Deniz ARAL.

Definition 2.1. ([10]) A neutrosophic set A on the universe set X is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where $T,\ I,\ F : X \to]^{-}0, 1^{+}[$ and $^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$.

Scientifically, membership functions, indeterminacy functions and non-membership functions of a neutrosophic set take value from real standart or nonstandart subsets of $]^{-}0, 1^{+}[$. However, these subsets are sometimes inconvenient to be used in real life applications such as economical and engineering problems. On account of this fact, we consider the neutrosophic sets, whose membership function, indeterminacy functions and non-membership functions take values from subsets of $[0, 1]$.

Definition 2.2. ([7]) Let X be a nonempty set. If r, t, s are real standard or non standard subsets of $]^{-}0, 1^{+}[$ then the neutrosophic set $x_{r,t,s}$ is called a neutrosophic point in X given by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

For x_p ∈X, it is called the support of $x_{r,t,s}$, where r denotes the degree of membership value, t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

Definition 2.3. ([9]) Let A be a neutrosophic set over the universe set X. The complement of A is denoted by A^c and is defined by: $A^c = \left\{\left\langle x, F_{\tilde{F}(e)}(x), 1 - I_{\tilde{F}(e)}(x), T_{\tilde{F}(e)}(x)\right\rangle : x \in X\right\}$. It is obvious that $[A^c]^c = A$.

Definition 2.4. ([9]) Let A and B be two neutrosophic sets over the universe set X. A is said to be a neutrosophic subset of B if $T_A(x) \leq T_B(x), I_A(x) \leq I_B(x), F_A(x) \geq F_B(x)$, every $xinX$. It is denoted by $A \subseteq B$. A is said to be neutrosophic soft equal to B if $A \subseteq B$ and $B \subseteq A$. It is denoted by $A = B$.

Definition 2.5. ([9]) Let F_1 and F_2 be two neutrosophic soft sets over the universe set X. Then their union is denoted by $F_1 \cup F_2 = F_3$ is defined by:

$$F_3 = \{\langle x, T_{F_3}(x), I_{F_3}(x), F_{F_3}(x) : x \in X\rangle\},$$

where

$$T_{F_3(x)} = \max\{T_{F_1(x)}, T_{F_2}(x)\},$$

$$I_{F_3(x)} = \max\{I_{F_1(x)}, I_{F_2}(x)\},$$

$$F_{F_3(x)} = \min\{F_{F_1(x)}, F_{F_2}(x)\}.$$

Definition 2.6. ([9]) Let F_1 and F_2 be two neutrosophic soft sets over the universe set X. Then their intersection is denoted by $F_1 \cap F_2 = F_4$ is defined by:

$$F_4 = \{\langle x, T_{F_4}(x), I_{F_4}(x), F_{F_4}(x) : x \in X\rangle\},$$

where

$$T_{F_4(x)} = \min\{T_{F_1(x)}, T_{F_2}(x)\},$$

$$I_{F_4(x)} = \min\{I_{F_1(x)}, I_{F_2}(x)\},$$

$$F_{F_4(x)} = \max\{F_{F_1(x)}, F_{F_2}(x)\}.$$

Definition 2.7. ([9]) A neutrosophic set F over the universe set X is said to be a null neutrosophic set if $T_F(x) = 0$, $I_F(x) = 0$, $F_F(x) = 1$, every $x \in X$. It is denoted by 0_X.

Definition 2.8. ([9]) A neutrosophic set F over the universe set X is said to be an absolute neutrosophic set if $T_F(x) = 1$, $I_F(x) = 1$, $F_F(x) = 0$, every $x \in X$. It is denoted by 1_X.

Clearly $0_X^c = 1_X$ and $1_X^c = 0_X$.

Definition 2.9. ([9]) Let $NS(X)$ be the family of all neutrosophic sets over the universe the set X and $\tau \subset NS(X)$. Then τ is said to be a neutrosophic topology on X if:

1) 0_X and 1_X belong to τ;
2) the union of any number of neutrosophic soft sets in τ belongs to τ;
3) the intersection of a finite number of neutrosophic soft sets in τ belongs to τ.

Then (X, τ) is said to be a neutrosophic topological space over X. Each member of τ is said to be a neutrosophic open set [9].

Definition 2.10. ([9]) Let (X, τ) be a neutrosophic topological space over X and F be a neutrosophic set over X. Then F is said to be a neutrosophic closed set iff its complement is a neutrosophic open set.

Definition 2.11. ([2]) A neutrosophic point $x_{r,t,s}$ is said to be *neutrosophic quasi-coincident* (*neutrosophic q-coincident*, for short) with F, denoted by $x_{r,t,s}\ q\ F$ if and only if $x_{r,t,s} \nsubseteq F^c$. If $x_{r,t,s}$ is not neutrosophic quasi-coincident with F, we denote by $x_{r,t,s}\ \tilde{q}\ F$.

Definition 2.12. ([2]) A neutrosophic set F in a neutrosophic topological space (X, τ) is said to be a *neutrosophic q-neighborhood* of a neutrosophic point $x_{r,t,s}$ if and only if there exists a neutrosophic open set G such that $x_{r,t,s}\ q\ G \subset F$.

Definition 2.13. ([2]) A neutrosophic set G is said to be *neutrosophic quasi-coincident* (*neutrosophic q-coincident*, for short) with F, denoted by $G\ q\ F$ if and only if $G \nsubseteq F^c$. If G is not neutrosophic quasi-coincident with F, we denote by G $\tilde{q}\ F$.

Definition 2.14. ([3]) A neutrosophic point $x_{r,t,s}$ is said to be a neutrosophic interior point of a neutrosophic set F if and only if there exists a neutrosophic open q-neighborhood G of $x_{r,t,s}$ such that $G \subset F$. The union of all neutrosophic interior points of F is called the neutrosophic interior of F and denoted by F°.

Definition 2.15. ([2]) A neutrosophic point $x_{r,t,s}$ is said to be a *neurosophic cluster point* of a neutrosophic set F if and only if every neutrosophic open q-neighborhood G of $x_{r,t,s}$ is q-coincident with F. The union of all neutrosophic cluster points of F is called the *neutrosophic closure* of F and denoted by F^-.

Definition 2.16. ([2]) Let f be a function from X to Y. Let B be a neutrosophic set in Y with members hip function $T_B(y)$, indeterminacy function $I_B(y)$ and non-membership function $F_B(y)$. Then, the inverse image of B under f, written as $f^{-1}(B)$, is a neutrosophic subset of X whose membership function, indeterminacy function and non-membership function are defined as $T_{f^{-1}(B)}(x) = T_B(f(x))$,

$I_{f^{-1}(B)}(x) = I_B(f(x))$ and $F_{f^{-1}(B)}(x) = F_B(f(x))$ for all x in X, respectively. Conversely, let A be a neutrosophic set in X with membership function $T_A(x)$, indeterminacy function $I_A(x)$ and non-membership function $F_A(x)$. The image of A under f, written as $f(A)$, is a neutrosophic subset of Y whose membership function, indeterminacy function and non-membership function are defined as

$$T_{f(A)}(y) = \begin{cases} sup_{z\in f^{-1}(y)}\{T_A(z)\}, & if\ f^{-1}(y)\ is\ not\ empty, \\ 0, & if\ f^{-1}(y)\ is\ empty, \end{cases}$$

$$I_{f(A)}(y) = \begin{cases} sup_{z\in f^{-1}(y)}\{I_A(z)\}, & if\ f^{-1}(y)\ is\ not\ empty, \\ 0, & if\ f^{-1}(y)\ is\ empty, \end{cases}$$

$$F_{f(A)}(y) = \begin{cases} sup_{z\in f^{-1}(y)}\{F_A(z)\}, & if\ f^{-1}(y)\ is\ not\ empty, \\ 0, & if\ f^{-1}(y)\ is\ empty, \end{cases}$$

for all y in Y, where $f^{-1}(y) = \{x : f(x) = y\}$, respectively.

3. Some Definitions

This section provides some new definitions that form the cornerstones of the sections that follow.

Definition 3.1. *A neutrosophic set F in a neutrosophic topological space (X, τ) is said to be*

Neutrosophic semiopen, if $F \subseteq \overline{F^\circ}$,
Neutrosophic preopen, $F \subseteq (\overline{F})^\circ$,
Neutrosophic β-open, $F \subseteq \overline{(\overline{F})^\circ}$. Equivalently, if there exists a neutrosophic preopen set A such that $A \subseteq F \subseteq \overline{A}$.

It is obvious that each neutrosophic semiopen and neutrosophic preopen neutrosophic set implies neutrosophic β-open.

Definition 3.2. *If, F be a neutrosophic set in neutrosophic topological space (X, τ) then, $F_p^- = \bigcap\{F : F \subseteq A,\ A\, is\, neutrosophic\, preclosed\}$ (resp. $F_p^\circ = \bigcup\{F : F \subseteq A,\ A\, is\, neutrosophic\, preopen\}$) is called a neutrosophic preclosure of F (resp. neutrosophic preinterior of F).*

Definition 3.3. *If, F be a neutrosophic set in neutrosophic topological space (X, τ) then, $F_s^- = \bigcap\{F : F \subseteq A,\ A\, is\, neutrosophic\, semiclosed\}$ (resp. $F_s^\circ = \bigcup\{F : F \subseteq A,\ A\, is\, neutrosophic\, semiopen\}$) is called a neutrosophic semiclosure of F (resp. neutrosophic semi-interior of F).*

Definition 3.4. *If, F be a neutrosophic set in neutrosophic topological space (X, τ) then, $F_\beta^- = \bigcap\{F : F \subseteq A,\ A\ is\ neutrosophic\ \beta - closed\}$ (resp. $F_s^\circ = \bigcup\{F : F \subseteq A,\ A\ is\ neutrosophic\ \beta - open\}$) is called a neutrosophic β-closure of F (resp. neutrosophic β-interior of F).*

It is obvious that $(F^c)_\beta^- = (F_\beta^\circ)^c$ and $(F^c)_\beta^\circ = (F_\beta^-)^c$.

Definition 3.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a neutrosophic topological space (X, τ) into a neutrosophic topological space (Y, σ). The function f is said to be neutrosophic β-continuous, if $f^{-1}(A)$ is a neutrosophic β-open set of X, for each $A \in \sigma$.*

Definition 3.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a neutrosophic topological space (X, τ) into a neutrosophic topological space (Y, σ). The function f is said to be neutrosophic $M\beta$-continuous, if $f^{-1}(A)$ is a neutrosophic β-open set of X, for each neutrosophic β-open set A in (Y, σ).*

Lemma 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a neutrosophic topological space (X, τ) into a neutrosophic topological space (Y, σ). Then the following are equivalent:*

a) f is neutrosophic $M\beta$-continuous.
b) $f(F^-_\beta) \subseteq (f(F))^-_\beta$, for every neutrosophic set F in (X, τ):

Proof. *a)* $\implies$ *b)*: Let F be a neutrosophic set in (X, τ), then $f(F))^-_\beta$ is neutrosophic β-closed. By (a), $f^{-1}(f(F))^-_\beta$ is neutrosophic β-closed and so $f^{-1}(f(F))^-_\beta = (f^{-1}(f(F))^-_\beta)^-_\beta$. Since $F \subseteq f^{-1}(f(F))$, $F^-_\beta \subseteq (f^{-1}(f(F)))^-_\beta \subseteq (f^{-1}((f(F))^-_\beta))^-_\beta = f^{-1}(f(F))^-_\beta)$. Hence $f(F^-_\beta) \subseteq (f(F))^-_\beta$.
b) $\implies$ *a)*: Let G be a neutrosophic β-closed in (Y, σ): By (b), if $F = f^{-1}(G)$; then $(f^{-1}(G))^-_\beta \subseteq f^{-1}((f(f^{-1}(G)))^-_\beta) \subseteq f^{-1}(G^-_\beta) = f^{-1}(G)$. Since $f^{-1}(G) \subseteq (f^{-1}(G))^-_\beta$, then $f^{-1}(G) = (f^{-1}(G))^-_\beta$. Hence $f^{-1}(G)$ is a neutrosophic β-closed in set in (X, τ). Hence f is neutrosophic $M\beta$-continuous. □

Lemma 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a neutrosophic topological space (X, τ) into a neutrosophic topological space (Y, σ). Then the following are equivalent:*

a) f is neutrosophic β-continuous.
b) $f(F^-_\beta) \subseteq (f(F))^-$, for every neutrosophic set F in (X, τ):

Proof. Obvious. □

Theorem 3.3. *If $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic open function*[5]*, then $f^{-1}(G^-) \subseteq (f^{-1}(G))^-$ for every neutrosophic set G in (Y, σ).*

Definition 3.7. *A collection of neutrosophic subsets ω of a neutrosophic topological space (X, τ) is said to form a neutrosophic filterbases if and only if, for every finite collection $\{F_i : i = 1, ..., n\}$, $\bigcap_{i=1}^{n} F_i \neq 0_x$.*

Definition 3.8. *A collection φ of neutrosophic sets in a neutrosophic topological space (X, τ) is said to be cover of a neutrosophic set G of X if and only if, $T_{(\cup_{F\in\varphi} F)}(x) = 0$, $I_{(\cup_{F\in\varphi} F)}(x) = 0$ and $F_{(\cup_{F\in\varphi} F)}(x) = 0$, where x is any support in G. A neutrosophic cover φ of a neutrosophic set G in a neutrosophic topological space (X, τ) is said to have a finite subcover if and only if, there exists a finite subcollection $\rho = \{F_1, ..., F_n\}$ of φ such that $T_{(\bigcup_{i=1}^{n} F_i)}(x) \geq T_G(x)$, $I_{(\bigcup_{i=1}^{n} F_i)}(x) \geq I_G(x)$, and $F_{(\bigcup_{i=1}^{n} F_i)}(x) \geq F_G(x)$, where x is any support in G.*

Definition 3.9. *A neutrosophic topological space (X, τ) is said to be neutrosophic strongly compact if and only if, every neutrosophic preopen cover of X has a finite subcover.*

Definition 3.10. *A neutrosophic topological space (X, τ) is said to be neutrosophic semicompact if and only if, every neutrosophic semiopen cover of X has a finite subcover.*

Definition 3.11. *A neutrosophic topological space (X, τ) is said to be neutrosophic almost compact if and only if, every neutrosophic open cover of X has a finite subcollection whose closures cover X .*

Definition 3.12. *A neutrosophic topological space (X, τ) is said to be neutrosophic S-closed if and only if, every neutrosophic semiopen cover of X has a finite subcollection whose closures cover X.*

Definition 3.13. *A neutrosophic topological space (X, τ) is said to be neutrosophic s-closed if and only if, every neutrosophic semiopen cover of X has a finite subcollection whose semiclosures cover X.*

Definition 3.14. *A neutrosophic topological space (X, τ) is said to be neutrosophic P-closed if and only if, every neutrosophic preopen cover of X has a finite subcollection whose preclosures cover X.*

4. Neutrosophic β-Compact Space

In this subheading, we introduce the term neutrosophic β-compactness, which forms the basis of our study. Then, we examine the relationship of this new concept we introduced with other concepts introduced for the first time in this study and conduct an in-depth research on its properties.

Definition 4.1. *A neutrosophic topological space (X, τ) is said to be neutrosophic β-compact if and only if, for every family φ of neutrosophic β-open sets such that $\cup_{G\in\varphi} G = 1_X$, there is a finite subfamily $\omega \subseteq \varphi$ such that $\cup_{G\in\omega} G = 1_X$.*

Definition 4.2. *A neutrosophic set δ in a neutrosophic topological space (X, τ) is said to be neutrosophic β -compact relative to (X, τ) if and only if, for every family φ of neutrosophic β-open sets such that $T_{(\cup_{F\in\varphi} F)}(x) \geq T_\delta(x)$, $I_{(\cup_{F\in\varphi} F)}(x) \geq I_\delta(x)$ and $F_{(\cup_{F\in\varphi} F)}(x) \leq F_\delta(x)$, where x is any support in G, there is a finite subfamily $\rho \subseteq \varphi$ such that $T_{(\cup_{F\in\rho} F)}(x) \geq T_\delta(x)$, $I_{(\cup_{F\in\rho} F)}(x) \geq I_\delta(x)$ and $F_{(\cup_{F\in\rho} F)}(x) \leq F_\delta(x)$, where x is any support in G.*

Remark. *Since each of neutrosophic semiopenness and neutrosophic preopenness implies neutrosophic β-openness, it is clear that every neutrosophic β-compactness implies each of neutrosophic strongly compactness and neutrosophic semicompactness. But the converse need not to be true.*

Theorem 4.1. *A neutrosophic topological space (X, τ) is neutrosophic β-compact if and only if, for every collection $\{F_i : i \in I\}$ of neutrosophic β-closed neutrosophic sets in (X, τ) having the finite intersection property $\bigcap_{i\in I} F_i \neq 0_x$.*

Proof. Let $\{F_i : i \in I\}$ be a collection of neutrosophic β-closed sets with the finite intersection property. Suppose that $\bigcap_{i\in I} F_i = 0_X$. Then, $\bigcup_{i\in I} F_i^c = 1_X$. Since $\{F_i^c : i \in I\}$ is a collection of neutrosophic β-open sets cover of X, then from the neutrosophic β-compactness of X it follows that there exists a finite subset $J \subseteq I$ such that $\bigcup_{j\in J} F_j^c = 1_X$. Then, $\bigcap_{j\in J} F_j = 0_X$, which gives a contradiction and therefore $\bigcap_{i\in I} F_i \neq 0_X$.
Conversely, Let $\{F_i : i \in I\}$ be a collection of neutrosophic β-open sets cover of

X. Suppose that for every finite subset $J \subseteq I$, we have $\bigcup_{j\in J} F_j \neq 1_X$. Then, $\bigcap_{j\in J} F_J^c \neq 0_X$. Hence $\{F_i^c : i \in I\}$ satisfies the finite intersection property. Then, from the hypothesis we have $\bigcap_{i\in I} F_i^c \neq 0_X$, which implies $\bigcup_{j\in J} F_j \neq 1_X$ and this contradicting that $\{F_i : i \in I\}$ is a neutrosophic β-open cover of X. Thus, X is neutrosophic β-compact. □

Now, we give some results of neutrosophic β-compactness in terms of neutrosophic filterbases.

Theorem 4.2. *A neutrosophic topological space (X, τ) is neutrosophic β-compact if and only if, every filterbases μ in (X, τ), $\bigcap_{H\in\mu} H_\beta^- \neq 0_X$.*

Proof. Let φ be a neutrosophic β-open cover of X and φ have no a finite subcover. Then, for every finite subcollection $\{F_1, ..., F_n\}$ of φ, there exists $x \in X$ such that $T_{F_j}(x) < 1$ or $I_{F_j}(x) < 1$ or $F_{F_j}(x) > 0$ for every $j = 1, ..., n$. Then, $T_{F_j^c}(x) > 0$ or $I_{F_j^c}(x) > 0$ or $F_{F_j^c}(x) < 1$. So, $\bigcap_{j\in J} F_j^c \neq 0_X$. $\{F_j^c : F_j \in \varphi\}$ forms a filterbases in (X, τ). Since φ is neutrosophic β-open cover of X , then $T_{F_j}(x) = 1$, $I_{F_j}(x) = 1$ and $F_{F_j}(x) = 0$ for every $x \in X$ and hence $\bigcap_{F_j\in\varphi} (F_j^c)_\beta^- = \bigcap_{F_j\in\varphi} F_j^c = 0_X$ which is a contradiction. Then, every neutrosophic β-open cover of X has a finite subcover and hence X is neutrosophic β-compact.
Conversely, suppose there exists a filterbases μ such that $\bigcap_{H\in\mu} H_\beta^- = 0_X$. So, $\bigcup_{H\in\mu} (H_\beta^-)^c = 1_X$. Hence, $\{(H_\beta^-)^c : H \in \mu\}$ is a neutrosophic β-open cover in (X, τ). Since X is neutrosophic β-compact, then there exists a finite subcover $\{((H_1)_\beta^-)^c, \ldots\ldots\ldots\ldots\ldots, ((H_n)_\beta^-)^c\}$. Then, $\bigcup_{i=1}^n ((H_1)_\beta^-)^c = 1_X$ and hence $\bigcup_{i=1}^n (H_i)^c = 1_X$. So, $\bigcap_{i=1}^n H_i = 0_X$, which is a contradiction, since the H_i are members of filterbases μ. Therefore, $\bigcap_{H\in\mu} H_\beta^- \neq 0_X$ every filterbases μ. □

Theorem 4.3. *A neutrosophic set α in a neutrosophic topological space (X, τ) is neutrosophic β-compact relative to X if and only if, for every filterbase μ such that every finite number of members of μ is neutrosophic quasi-coincident with α,*

$$(\bigcap_{H\in\mu} H_\beta^-) \cap \alpha \neq 0_X.$$

Proof. Suppose that α is not be neutrosophic β-compact relative to X, then there exists a neutrosophic β-open cover φ of α, which has a finite subcover. Then, for every finite subcover δ of φ, $\alpha \nsubseteq \bigcup_{H_i\in\delta} H_i$. So, $\bigcap_{H_i\in\delta} H_i^c \nsubseteq \alpha^c \neq 0_X$. Then, $\mu = \{H_i)^c : H_i \in \varphi\}$ forms a filterbases and $(\bigcap_{H_i\in\delta} H_i^c) q\alpha$. From our assumption, $(\bigcap_{H_i\in\delta} (H_i^c)_\beta^-) \cap \alpha \neq 0_X$. Clearly, $(\bigcap_{H_i\in\delta} H_i^c \cap \alpha \neq 0_X$. Then, $(\bigcap_{H_i\in\delta} H_i^c \neq 0_X$. This implies that $(\bigcup_{H_i\in\delta} H_i \neq 1_X$. From this contradiction, α is neutrosophic β-compact relative to X. Conversely, suppose that there exists a filterbases μ such that every finite number of members of μ is neutrosophic quasi-coincident with α and $(\bigcap_{H\in\mu} H_\beta^-) \cap \alpha = 0_X$.
This implies $\alpha \subseteq (\bigcap_H \in \mu H_\beta^-)^c$. So, $\alpha \subseteq \bigcup_H \in \mu (H_\beta^-)^c$. Then, $\varphi = \{(H_\beta^-)^c : H \in \mu\}$ is neutrosophic β-open cover of α. Since α is neutrosophic β-compact relative to X, then there exists a finite subcover δ of φ, say $\delta = \{((H_i)_\beta^-)^c, \ldots\ldots\ldots\ldots\ldots, ((H_n)_\beta^-)^c\}$, such that $\alpha \subseteq \bigcup_{i=1}^n ((H_i)_\beta^-)^c$. Then, $\bigcap_{i=1}^n (H_i)_\beta^- \subseteq \alpha^c$. This means that $(\bigcap_{i=1}^n (H_i)_\beta^-) \tilde{q}\alpha$. From thi contradiction, it is clear that, for every filterbase μ such that every finite number of members of μ is neutrosophic quasi-coincident with α, $(\bigcap_{H\in\mu} H_\beta^-) \cap \alpha \neq 0_X$. □

Theorem 4.4. *Every neutrosophic β-closed subset of a neutrosophic β-compact space is neutrosophic β-compact relative to X.*

Proof. Let φ be a neutrosophic filterbases in X such that $\alpha q(\bigcap\{H : H \in \delta\})$ holds for every finite subcollection δ of φ and a neutrosophic β-closed set α. Consider $\varphi^* = \{\alpha\} \cup \varphi$. For any finite subcollection δ^* of φ^*, if $\alpha \notin \delta^*$ then $\bigcap \delta^* \neq 0_X$. If $\alpha \in \delta^*$ and since $\alpha q(\bigcap\{H : H \in \delta^* - \{\alpha\}\})$, then $\bigcap \delta^* \neq 0_X$. Hence δ^* is a neutrosophic filterbases in X. Since X is neutrosophic β-compact, then $(\bigcap_H \in \varphi^*(H^{-}_{\beta}) \neq 0_X$. So, $(\bigcap_H \in \varphi(H^{-}_{\beta}) \cap \alpha = (\bigcap_H \in \varphi(H^{-}_{\beta}) \cap \alpha^{-}_{\beta} \neq 0_X$. Hence by Theorem 4.3, we have α is neutrosophic β-compact relative to X. □

Theorem 4.5. *If a function $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic $M\beta$-continuous and α is neutrosophic β-compact relative to X, then so is $f(\alpha)$.*

Proof. Let $\{F_i : i \in I\}$ be a neutrosophic β-open cover of $f(\alpha)$. Since f is neutrosophic $M\beta$-continuous, $\{f^{-1}(F_i) : i \in I\}$ is neutrosophic β-open cover of α. Since α is neutrosophic β-compact relative to X, there is a finite subfamily $\{f^{-1}(F_i) : i = 1, \ldots\ldots, n\}$ such that $\alpha \subseteq \bigcup_{i=1}^{n} f^{-1}(F_i)$. Then, $f(\alpha) \subseteq f(\bigcup_{i=1}^{n} f^{-1}(F_i)) \subseteq \bigcup_{i=1}^{n} F_i$. Therefore, $f(\alpha)$ is neutrosophic β-compact relative to Y. □

Lemma 4.6. *If $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic open and neutrosophic continuous function, then f is neutrosophic $M\beta$-continuous.*

Proof. Let ω be a neutrosophic β-open set in Y; then $\omega \subseteq \overline{(\overline{\varpi})^\circ}$. So, $f^{-1}(\omega) \subseteq f^{-1}(\overline{(\overline{\varpi})^\circ}) \subseteq \overline{f^{-1}((\overline{\varpi})^\circ)}$. Since f is neutrosophic continuous[5], then $f^{-1}((\overline{\varpi})^\circ) = (f^{-1}((\overline{\varpi})^\circ))$. Also by Theorem 3.3, $f^{-1}((\overline{\varpi})^\circ) = (f^{-1}((\overline{\varpi})^\circ))^\circ \subseteq (f^{-1}(\overline{\varpi}))^\circ \subseteq \overline{(f^{-1}(\omega))^\circ}$. Thus, $f^{-1}(\omega) \subseteq \overline{f^{-1}((\overline{\varpi})^\circ)} \subseteq (\overline{(f^{-1}(\omega))})^\circ)$. This implies that $f^{-1}(\omega)$ is neutrosophic β-open. So, f is neutrosophic $M\beta$-continuous. □

Corollary 4.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be neutrosophic open and neutrosophic continuous function. Consider that X is neutrosophic β-compact. Then, $f(X)$ is neutrosophic β-compact .*

Proof. It is follows directly from Lemma 4.6 and Theorem 4.5. □

Definition 4.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be neutrosophic $M\beta$-open if and only if, the image of every neutrosophic β-open set in X is neutrosophic β-open in Y.*

Theorem 4.8. *Let $f : (X, \tau) \to (Y, \sigma)$ be a neutrosophic $M\beta$-open bijective function and Y is neutrosophic β-compact, then X neutrosophic β-compact.*

Proof. Let $\{F_i : i \in I\}$ be a neutrosophic β-open cover of X. Then, $\{f(F_i) : i \in I\}$ is a neutrosophic β-open cover of Y. Since Y is neutrosophic β-compact, there is a finite subset $J \subseteq I$ such that $\{F_j : j \in J\}$ is a neutrosophic β-open cover of Y. So, $1_Y = \bigcup_{j \in J} f(F_i)$. Since f is a neutrosophic $M\beta$-open bijective function, $1_X = f^{-1}(1_Y) = f^{-1}(f(\bigcup_{j \in J} F_j)) = \bigcup_{j \in J} F_j$. Therefore, X is neutrosophic β-compact. □

5. Neutrosophic β-Closed Spaces

Definition 5.1. *A neutrosophic set α in a neutrosophic topological space (X, τ) is said to be a neutrosophic βq-neighborhood of a neutrosophic point $x_{r,t,s}$ in X, if there exists a neutrosophic β-open set $F \subseteq \alpha$ such that $x_{r,t,s} qF$.*

Proof. Let $x_{r,t,s} \in \bar{\alpha_\beta}$ and there exists a neutrosophic βq-neighborhood G of $x_{r,t,s}$, $G\tilde{q}\alpha$. Then there exists a neutrosophic open set $F \subseteq G$ in X such that $x_{r,t,s} qF$, which implies $F\tilde{q}\alpha$ and hence $\alpha \subseteq F^c$. Since F^c neutrosophic β-closed, $\bar{\alpha_\beta} \subseteq F^c$. From $x_{r,t,s} \notin F^c$, we see that $x_{r,t,s} \notin \bar{\alpha_\beta}$. This is a contradiction. Conversely, let $x_{r,t,s} \notin \bar{\alpha_\beta} = \bigcap\{F : F is \beta - closed in X, \alpha \subseteq F\}$. Then, there exists a neutrosophic β-closed set F such that $x_{r,t,s} \notin F$ and $\alpha \subseteq F$. Hence, $x_{r,t,s} qF^c$ and $\alpha\tilde{q}F^c$, where F^c is neutrosophic β-open in X. Then, F^c is a neutrosophic βq-neighborhood of $x_{r,t,s}$ and $\alpha\tilde{q}F^c$. □

Definition 5.2. *A neutrosophic topological space (X, τ) is said to be neutrosophic β-closed if and only if, for every family φ of neutrosophic β-open sets such that $\bigcup_{H\in\varphi} H = 1_X$, there is a finite subfamily $\delta \subseteq \varphi$ such that $\bigcup_{H\in\delta} \bar{H_\beta} = 1_X$.*

Remark. *From the above definition and other types of neutrosophic compactness, one can draw the following diagram:*

$$\begin{array}{lll} Neutrosophic\ semi-compact & \rightarrow Neutrosophic\ s-closed & \rightarrow Neutrosophic\ S-closed \\ \downarrow & \downarrow & \\ Neutrosophic\ \beta-compact & \rightarrow Neutrosophic\ \beta-closed & \\ \downarrow & \downarrow & \\ Neutrosophic\ strongly\ compact & \rightarrow Neutrosophic\ P-closed & \end{array}$$

Example 5.1. *Let (X, τ) be a neutrosophic topological and neutrosphic sets α_n be defined as $\alpha_n = \{\langle x, 1 - \frac{1}{n}, 1 - \frac{1}{n}, \frac{1}{n}\rangle : x \in X\}$ for each $n \in N^+$. Consider a neutrosophic topological space (X, τ) that $\{\alpha_n : n \in N^+\}$ is a neutrosophic base for τ. Then, $\{\alpha_n : n \in N^+\}$ is obviously a neutrosophic β-open cover of X. In (X, τ), $\bar{(\alpha_n)_\beta} = 1_X$ for each $n \geq 3$. Then, X is neutrosophic β-closed but not neutrosophic β-compact.*

Remark. *Example 5.1 also shows that:*

(i) Each of the concepts neutrosphic s-closed, neutrosphic S-closed and neutrosphic P-closed spaces does not imply neutrosphic β-compact.
(ii) Since $\{\alpha_n : n \in N^+\}$ is also neutrosophic semiopen cover of X, then X is also neutrosophic β-closed space but not neutrosophic semi-compact space.
(iii) Since $\{\alpha_n : n \in N^+\}$ is also neutrosophic preopen cover of X, then X is also neutrosophic β-closed space but not neutrosophic strongly compact space.

Example 5.2. *Let $X = [0, 1]$ and consider the following neutrosophic sets $\alpha_1 = \{\langle x, \frac{1,7}{\sqrt{3}}, \frac{1,7}{\sqrt{3}}, 1 - \frac{1,7}{\sqrt{3}}\rangle : x \in X\}$, $\alpha_2 = \{\langle x, \frac{1,73}{\sqrt{3}}, \frac{1,73}{\sqrt{3}}, 1 - \frac{1,73}{\sqrt{3}}\rangle : x \in X\}$, $\alpha_3 = \{\langle x, \frac{1,732}{\sqrt{3}}, \frac{1,732}{\sqrt{3}}, 1 - \frac{1,732}{\sqrt{3}}\rangle : x \in X\}$, $\alpha_4 = \{\langle x, \frac{1,7320}{\sqrt{3}}, \frac{1,7320}{\sqrt{3}}, 1 - \frac{1,7320}{\sqrt{3}}\rangle : x \in X\}$, $\alpha_5 = \{\langle x, \frac{1,73205}{\sqrt{3}}, \frac{1,73205}{\sqrt{3}}, 1 - \frac{1,73205}{\sqrt{3}}\rangle : x \in X\}$, where $\sqrt{3} = 1,7320508075688\ldots\ldots$ Let $\varphi = \{\alpha_i : i \in N^+\} \cup \{0_X, 1_X\}$. It is clear that φ is a neutrosophic topology on*

X. Now, the collection $\{\alpha_i : i \in N^+\}$ is a neutrosophic semiopen (resp. neutrosophic preopen) cover of X but not has a finite subcover. So X is not neutrosophic semicompact space (resp. neutrosophic strongly compact space). Since the neutrosophic semi-closure (resp. neutrosophic pre-closure) of every neutrosophic semiopen (resp. neutrosophic preopen) set of X is 1_X, then X is neutrosophic s-closed (resp. neutrosophic P-closed).

Remark. *Example 5.2 is also shows that each of the concepts neutrosophic S-closed and neutrosophic P-closed spaces does not imply each of neutrosophic semicompact and neutrosophic strongly compact spaces.*

Remark. *From the remark just after, Definition 4.2, Example 5.1, the remark just after Example 5.1, Example 5.2 and the remark just after Example 5.2, it is clear that:*
(i) Neutrosophic S-closed and neutrosophic P-closed spaces are independent notions.
(ii) Neutrosophic S-closed and neutrosophic strongly compact spaces are independent notions.
(iii) Neutrosophic P-closed and neutrosophic semicompact spaces are independent notions.
(iv) Neutrosophic β-compact, neutrosophic semicompact and neutrosophic strongly compact spaces are independent notions.

Theorem 5.1. *A neutrosophic topological space (X, τ) is neutrosophic β-closed if and only if, for every neutrosophic β-open filterbases φ in (X, τ), $\bigcap_{H\in\varphi} H^{-}_{\beta} \neq 0_X$.*

Proof. Let δ be a neutrosophic β-open cover of X and let for every finite subfamily ρ of δ, $\bigcup_{B\in\rho} B^{-}_{\beta} \neq 1_X$. Then, $0_X \subset \bigcap_{B\in\rho}(B^{-}_{\beta})^c$. Thus, $\varphi = \{(B^{-}_{\beta})^c : B \in \delta\}$ forms a neutrosophic β-open filterbases in X. Since δ is a neutrosophic β-open cover of X, $\bigcap_{B\in\delta} B^c = 0_X$. For $B \subseteq B^{-}_{\beta}$, $(B^{-}_{\beta})^c \subseteq B^c$. As B^c is neutrosophic β-closed, $((B^{-}_{\beta})^c)^{-}_{\beta} \subseteq B^c$. So, $\bigcap_{B\in\delta}((B^{-}_{\beta})^c)^{-}_{\beta} = 0_X$. This contradicts with our assumption. This means that every neutrosophic β-open cover of X has a finite subfamily ρ such that $\bigcup_{B\in\rho} B^{-}_{\beta} = 1_X$. Hence, X is neutrosophic β-closed.
Conversely, suppose there exists a neutrosophic β-open filterbases φ in X such that $\bigcap_{H\in\varphi} H^{-}_{\beta} = 0_X$. So, $\bigcup_{H\in\varphi}(H^{-}_{\beta})^c = 1_X$. Then, $\delta = \{(H^{-}_{\beta})^c : B \in \varphi\}$ is a neutrosophic β-open cover of X. Since X is neutrosophic β-closed, then δ has a finite subfamily ρ such that $\bigcup_{H\in\rho}((H^{-}_{\beta})^c)^{-}_{\beta} = 1_X$. So, $\bigcap_{H\in\rho}(((H^{-}_{\beta})^c)^{-}_{\beta})^c = 0_X$. Thus, $\bigcap_{H\in\rho} H = 0_X$. This contradicts with our assumption that φ is a neutrosophic filterbese in (X, τ). □

Definition 5.3. *A neutrosophic set α in a neutrosophic topological space (X, τ) is said to be neutrosophic β-closed relative to X if and only if, for every family δ of neutrosophic β-open sets such that $\bigcup_{B\in\delta} B = \alpha$, there is a finite subfamily $\rho \subseteq \delta$ such that $\alpha \subseteq \bigcup_{B\in\rho} B^{-}_{\beta}$.*

Theorem 5.2. *A neutrosophic set α in a neutrosophic topological space (X, τ) is neutrosophic β-closed relative to X if and only if, every neutrosophic β-open filterbases φ in (X, τ), $(\bigcap_{H\in\varphi} H^{-}_{\beta}) \cap \alpha = 0_X$, there exists a finite subfamily ω of φ such that $(\bigcap_{H\in\omega} H)\tilde{q}\alpha$.*

Proof. Let α be a neutrosophic β-closed relative to X; suppose φ is a neutrosophic β-open filterbases in (X, τ) such that for every finite subfamily ω of φ,

$(\bigcap_{H\in\omega} H)q\alpha$ but $(\bigcap_{H\in\varphi} H^-_\beta)\cap\alpha = 0_X$. For every support x in α, $T_{(\bigcap_{H\in\varphi} H^-_\beta)}(x) = 0$, $I_{(\bigcap_{H\in\varphi} H^-_\beta)}(x) = 0$ and $F_{(\bigcap_{H\in\varphi} H^-_\beta)}(x) = 1$. This implies that, for every support x in α, $T_{(\bigcap_{H\in\varphi}(H^-_\beta)^c)}(x) = 1$, $I_{(\bigcap_{H\in\varphi}(H^-_\beta)^c)}(x) = 1$ and $F_{(\bigcap_{H\in\varphi}(H^-_\beta)^c)}(x) = 0$. Then, $\delta = \{(H^-_\beta)^c) : B \in \varphi\}$ is neutrosophic β-open cover of α and hence there exists a finite subfamily $\omega \subseteq \varphi$ such that $\alpha \subseteq (\bigcup_{H\in\omega}((H^-_\beta)^c)^-_\beta$. So, $\bigcap_{H\in\omega}(((H^-_\beta)^c)^-_\beta)^c \subseteq \alpha^c$. Then, $\bigcap_{H\in\omega}(((H^-_\beta)^\circ_\beta \subseteq \alpha^c$. This means that $\bigcap_{H\in\omega} H \subseteq \alpha^c$. Therefore, $(\bigcap_{H\in\omega} H)\tilde{q}\alpha^c$. This is a contradiction.
Conversely, let α not be a neutrosophic β-closed set relative to X; then there exists a neutrosophic β-open cover δ of α such that every finite subfamily $\rho \subseteq \delta$, $\bigcup_{B\in\rho} B^-_\beta \nsubseteq \alpha$. Then, $\alpha^c \nsubseteq (B^-_\beta)^c$. This implies that $\bigcap_{B\in\rho}(B^-_\beta)^c \neq 0_X$. So, $\varphi = \{(B^-_\beta)^c : B \in \delta\}$ forms a neutrosophic β-open filterbases in (X,τ). Let there exists a finite subfamily $\{(B^-_\beta)^c : B \in \rho\}$ such that $(\bigcap_{B\in\rho}(B^-_\beta)^c)\tilde{q}\alpha$. Then, $\alpha \subseteq \bigcup_{B\in\rho B^-_\beta}$. Hence, exists a finite subfamily $\rho \subseteq \delta$ such that $\alpha \subseteq \bigcup_{B\in\rho B^-_\beta}$, which is a contradiction. Then, for each finite subfamily $\omega = \{(B^-_\beta)^c : B \in \rho\}$ of φ, we have $(\bigcap_{B\in\rho}(B^-_\beta)^c)q\alpha$. From our assumption, $(\bigcap_{B\in\delta}((B^-_\beta)^c)^-_\beta \cap \alpha \neq 0_X$. So, $\bigcap_{B\in\delta}(((B^-_\beta)^c)^-_\beta)^c \cup \alpha^c \neq 1_X$. Clearly, $\bigcap_{B\in\delta}(((B^-_\beta)^c)^-_\beta)^c \cup \alpha^c \neq 1_X$. Then, $\bigcup_{B\in\delta}(((B^-_\beta)^\circ_\beta \neq 1_X$. So, $\bigcup_{B\in\delta} B \neq 1_X$. This contradicts with the fact that δ is a neutrosophic β-open cover of α. Therefore α is neutrosophic β-closed relative to X. □

Definition 5.4. *A neutrosophic set α in a neutrosophic topological space (X,τ) is said to be neutrosophic β-regular, if it is both neutrosophic β-open and neutrosophic β-closed set.*

Proposition 5.3. *If α is neutrosophic β-open set in (X,τ), then α^-_β is neutrosophic β-regular.*

Proof. Since α^-_β is neutrosophic β-closed, we must show that α^-_β is neutrosophic β-open. Since α is neutrosophic β-open in (X,τ), $\vartheta \subseteq \alpha \subseteq \vartheta^-$ holds for some neutrosophic preopen set ϑ in (X,τ). Therefore, we have $\vartheta \subseteq \vartheta^-_\beta \subseteq \alpha^-_\beta \subseteq \vartheta^-$ and hence α^-_β is neutrosophic β-open. □

Theorem 5.4. *For a neutrosophic topological space (X,τ), the following are equivalent:*

(a) (X,τ) is a neutrosophic β-space.
(b) Every neutrosophic β-regular cover of X has a finite subcover.
(c) For every collection $\{B_i : i \in I\}$ of neutrosophic β-regular sets such that $(\bigcap_{i\in I} B_i = 0_X$, there exists a finite subset $G \subseteq I$ such that $(\bigcap_{i\in G} B_i = 0_X$.

Proof. It is obvious from Proposition 5.3 and from the facts that, for every collection $\{B_i : i \in I\}$, $(\bigcup_{i\in I} B_i)^c = \bigcap_{i\in I}(B_i)^c$, $(\bigcap_{i\in I} B_i)^c = \bigcup_{i\in I}(B_i)^c$ and B is neutrosophic β-open set if and only if, B^c is neutrosophic β-closed set. □

Theorem 5.5. *Let $f : (X,\tau) \to (Y,\sigma)$ be a neutrosophic β-continuous surjection function. If (X,τ) is neutrosophic β-closed space, then (Y,σ) is neutrosophic almost compact.*

Proof. Let $\{B_i : i \in I\}$ be a neutrosophic open cover of Y. Then, $\{f^{-1}(B_i) : i \in I\}$ is a neutrosophic β-open cover of X. By hypothesis, there exists a finite subset $G \subseteq I$ such that $(\bigcup_{i\in G}(f^{-1}(B_i))^-_\beta = 1_X$. From the surjectivity of f and by Lemma 3.2,

$1_Y = f(1_X) = f((\bigcup_{i\in G}(f^{-1}(B_i))^{-}_{\beta}) \subseteq \bigcup_{i\in G}(f(f^{-1}(B_i)))^{-}_{\beta} = \bigcup_{i\in G}(B_i)^{-}$. Hence (Y, σ) is neutrosophic almost compact. □

Using Lemma 3.1, we have also the following theorem which can proved similarly to Theorem 5.5.

Theorem 5.6. *If $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic $M\beta$-continuous surjection function and (X, τ) is neutrosophic β-closed space, then (Y, σ) is so.*

6. Conclusion

In our article, first of all, all the factors that make it necessary for us to do this study and the definitions that form the cornerstones of our study are given in the introduction and preliminaries section, respectively. In the third chapter, some types of open sets and continuities, which have already been introduced and whose properties have been examined in detail in general topology, but have never been introduced before in neutrosophic spaces, are given and some properties of these concepts are given. In the fourth and fifth sections, some types of space are given using the new concepts given in the third section, and the relationships between these spaces are shown through diagrams. In order to better demonstrate these relationships, reverse examples were also given.

Conflict of interests

The authors declare that there is no conflict of interests.

Acknowledgment

The authors admit that the statements of some results in this paper were presented at the Seventh International Conference of Mathematical Sciences (ICMS 2023), [4].

References

[1] M.E. Abd El-Monsef, S.N. El-Deeb, and R.A. Mahmoud, β-open sets and β-continuous Mappings, Bull. Fac. Sci. Assiut Univ. **12** (1983) 77-90.

[2] A. Acikgoz, H. Çakalli, F. Esenbel, and Lj. D.R. Kočinac, A quest of G-continuity in neutrosophic spaces, Math. Meth. Appl. Sci. **44** (9) (2021) 7834-7844.

[3] A. Acikgoz, and F. Esenbel, Neutrosophic Connected topological spaces, Fasc. Math. **66** (2023) 5-22.

[4] A. Acikgoz, F. Esenbel and O. Mucuk, Neutrosophication β-Compactness, Seventh International Conference of Mathematical Sciences (ICMS 2023) Abstract Book page 17 Maltepe University Istanbul Turkey.

[5] A. Acikgoz, and F. Esenbel F. Neutrosophic seperation axioms, Maltepe J Math. **5** (2), (2023) 32–40.

[6] B. Ganesan, On fuzzy β-compact spaces and fuzzy β-extremally disconnected spaces, Kybernetika [cybernetics]. **33**(3) (1997) 271-277.

[7] R. Dhavaseelan, S. Jafari, C. Ozel, and C.M. A Al-Shumrani, Generalized neutrosophic contra-continuity, In: F Smarandache, S Pramanik, eds. New Trends in Neutrosophic Theory and Applications Vol. II (2018) 365-380.

[8] I.M. Hanafy, Fuzzy β-Compactness and Fuzzy β-Closed Spaces, Turk J Math. **28** (2004) 281–293.

[9] A.A. Salama, and S.A. Alblowi, Neutrosophic set and neutrosophic topological spaces, IOSR J. Math. **3** (2012) 31-35.

[10] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Int. J. Pure Appl. Math. **24** (2005) 287-297.

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