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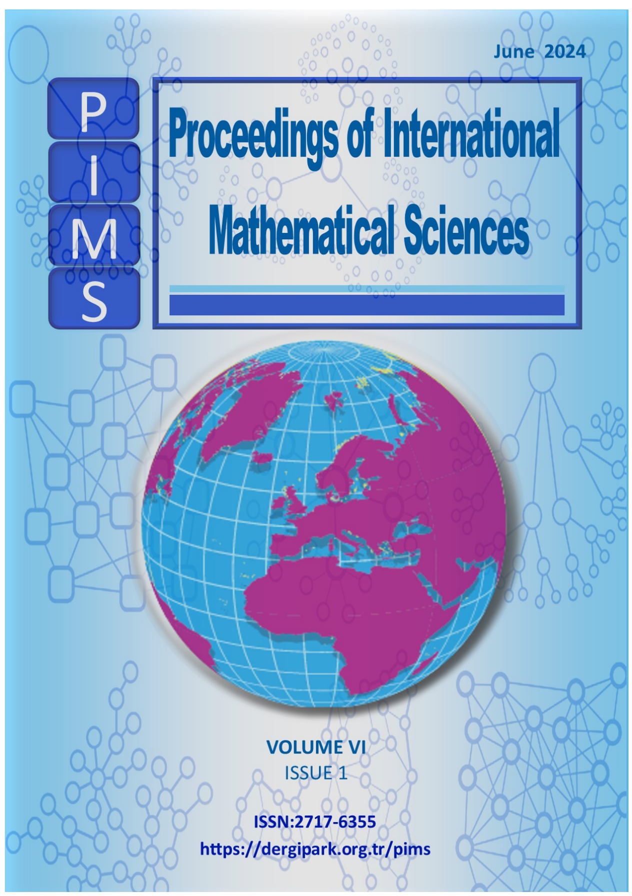

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GENERALIZED KURATOWSKI CLOSURE OPERATORS IN THE BIPOLAR METRIC SETTING

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Abstract. We initiate the investigation of the topological aspects of bipolar metric spaces. In this context, some concepts that generalize open and closed sets, accumulation points, closure and interior operators for bipolar metric spaces, of which little is known about their topological behaviors, are discussed. In addition, some essential properties regarding these notions were obtained, and counterexamples were provided for some expected but not satisfied properties.

1. Introduction

As a natural extension of the concept of length, one of the oldest quantitative concepts, the concept of distance can be thought of as the length of a gap. One of the most celebrated tools that enable the concept of distance to be considered theoretically is metric spaces being neither too restrictive nor too general. Since they are intuitive and simple structures, they have been the subject of many generalizations, abstractions, and variations since the first day, they were defined in Fréchet's doctoral thesis [12].

Although it may be helpful in many situations to define distances between different kinds of objects, substances, people, phenomena, or concepts, it is very recently that bipolar metric spaces have been presented in the literature [32]. While these spaces were mainly studied within the framework of fixed point theory [13, 14, 15, 19, 20, 23, 24, 31, 32, 33, 34, 35, 36, 37, 43], also some applications [21, 26, 27, 28, 29, 39, 44, 46], generalizations [1, 2, 3, 5, 6, 7, 16, 22, 25, 26, 30, 42, 45, 46], and special cases [10] are studied. However, the topological characteristics of bipolar metric spaces constitute a nearly untouched area full of mysteries yet to be explored.

In this study, some topological concepts concerning bipolar metric spaces are examined. Closure operators are one of the significant classes of mappings, studied particularly within the context of the lattice theory from many various perspectives [4, 8, 9, 11, 18, 38, 40, 41]

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and proven to be useful in defining and analyzing generalized topological structures. It turns out that the similar generalized operators in bipolar metric spaces provide less well-behaved properties than their counterparts in metric spaces. Moreover, some expected properties that the aforementioned concepts or operators do not generally provide are explained with illustrative counterexamples, which also contribute to a better understanding of bipolar metric spaces.

2. Preliminaries

In order to fix the notations and inform about the topic, some basic terminology about bipolar metric spaces is given here. Some details on the information presented in this section can be found in [32] and [14].

Definition 2.1. *A bipolar set is a pair (X, Y) of sets. In this case, each of the sets is called a pole. Inspired by the writing order of the sets, the terms "left pole" and "right pole" are used, with their obvious meanings. The intersection of the two poles is referred to as the center of the bipolar set. A point of the left (right) pole is called a left (right) point. For points of the center, the term "central point" is used. Accordingly, points in $X \setminus Y$ are referred as noncentral left points, and points belonging $Y \setminus X$ are called noncentral right points.*

Although bipolar sets seems to be nothing other than ordinary pairs of sets, what makes them more interesting is the category **Bip**. This category has bipolar sets as objects and covariant mappings as morphisms, and more information about it can be found in [14].

Definition 2.2. *Given two bipolar sets (X, Y) and (X', Y'). Then a function $\varphi : X \cup Y \to X' \cup Y'$ is called a covariant mapping (or mapping, for short) from (X, Y) to (X', Y'), provided that $\varphi(X) \subseteq X'$, $\varphi(Y) \subseteq Y'$. This case is written as $\varphi : (X, Y) \rightrightarrows (X', Y')$.*

In addition to mappings, another tool to relate bipolar sets is contravariant mappings, whose usefulness has been tested in many applications, particularly on the fixed point theory.

Definition 2.3. *Let (X, Y) and (X', Y') be bipolar sets. A function $\varphi : X \cup Y \to X' \cup Y'$ is called a contravariant mapping from (X, Y) to (X', Y'), if $\varphi(X) \subseteq Y'$, $\varphi(Y) \subseteq X'$, and this case is denoted by $\varphi : (X, Y) \looparrowright (X', Y')$.*

Since bipolar metric spaces defined on bipolar sets are structures that can be explained with the help of sequences, just like the case of metric spaces, sequences defined on bipolar sets have particular importance. However, since sequences with mixed noncentral left and right points on bipolar metric spaces are essentially useless, the concept of sequence on a bipolar set is defined in terms of arrays consisting of either only left points or only right points. Of course, sequences consisting only of central points are also possible, and these are the most similar ones to sequences in the classical sense.

Definition 2.4. *A right (left) sequence on a bipolar set, is a sequence consisting solely of right (left) points. In the context of bipolar sets, when the generic term "sequence" is used, it is understood that either a right sequence, or a left sequence is meant. If all terms of (u_n) are central points, then it is called a central sequence.*

When a bipolar metric space is given over a bipolar set, there is a notion of convergence for sequences. However, to generalize concepts, such as Cauchy sequences, to bipolar metric spaces, the following additional tool is needed:

Definition 2.5. *A bisequence on a bipolar set (X, Y) is defined to be an ordinary sequence on the product set $X \times Y$.*

Now, as introduced at [32], we present a definition for the notion of a bipolar metric space, which will hereafter be shortly referred as BMS.

Definition 2.6. *Given a bipolar set (X, Y) and let $b : X \times Y \to \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = [0, \infty)$. If b satisfies the following, then it is called a bipolar metric on (X, Y), and in this case (X, Y, b) is called a bipolar metric space.*

(B0) $b(x, y) = 0$ *implies* $x = y$, *for each* $x \in X$ *and* $y \in Y$.
(B1) $x = y$ *implies* $b(x, y) = 0$, *for each* $x \in X$ *and* $y \in Y$.
(B2) $b(u, v) = b(v, u)$, *for each* $u, v \in X \cap Y$.
(B3) $b(x, y) \leq b(x, y') + b(x', y') + b(x', y)$, *for each* $x, x' \in X$ *and* $y, y' \in Y$.

The inequality (B3) is known as the quadrilateral inequality. If (B0) is dropped from the definition, then b is called a bipolar pseudo-metric.

The concept of bipolar metric space was introduced to deal with distances defined between separate kinds of objects frequently occur in both mathematical and applied sciences. Examples include distances between curves and points in $\mathbb{R}^n$, distances between sets and points in a pseudometric space, and distances between arbitrarily chosen points and sites in a Voronoi diagram. Moreover, the value of the characteristic function χ_{A^c} associated with the complement of a crisp or fuzzy set A at a point x also defines a distance between sets and points. Another list of examples includes the distances between branches of a company and delivery addresses, the distance between pairs from sets of stars and planetary bodies based on the observable luminosities, the distances between pairs of some suitable bases and acids based on their reaction rate, and the distance of a group of children and a set of abilities based on test scores. It is possible to examine whether such distances conform more or less to the bipolar metric space structure or their generalizations.

Example 2.1. ***(i)*** *If (M, d) is a metric space, then (M, M, d) is a BMS. Conversely, if (X, X, b) is a BMS, then (X, b) is a metric space.*

(ii) *If (Q, d) is a quasi-metric space [47], and $\tilde{Q} = \{\tilde{q} : q \in Q\}$ be a disjoint copy of Q, that is $\tilde{Q}$ is any set of same cardinality with Q, such that $\tilde{Q} \cap Q = \emptyset$ and the mapping $q \mapsto \tilde{q}$ is a bijection. Then $\left(Q, \tilde{Q}, b\right)$ is a bipolar pseudo-metric space, where b is given by $b(q_1, \tilde{q}_2) := d(q_1, q_2)$, for all $q_1, q_2 \in Q$.*

(iii) *Let (X, δ) be a dislocated metric space [17]. Define the set $U = \{x \in X : \delta(x, x) = 0\}$ and $\tilde{U}^c$ be the disjoint copy of $X \setminus U$. Say $Y = U \cup \tilde{U}^c$. There is a unique function $b : X \times Y \to \mathbb{R}_0^+$ satisfying*

$$\delta(x, y) = \begin{cases} b(x, y), \text{ if } y \in U \\ b(x, \tilde{y}), \text{ if } \tilde{y} \in \tilde{U}^c \text{ for } y \in U^c \end{cases}$$

for each $(x, y) \in X \times Y$. In this case, (X, Y, b) becomes a BMS, and U becomes the center of (X, Y, b)

(iv) *Consider the set C of all functions from $\mathbb{R}$ to the interval $[1, 3]$. Define the function $b : C \times \mathbb{R} \to \mathbb{R}_0^+$ by $b(f, x) = f(x)$. Then $(C, \mathbb{R}, b)$ is a BMS and its center is the empty set.*

Example 2.1 (i) clearly states that BMSs generalize metric spaces. As a result, it can be said that every metric space is a BMS, as formalized in the following proposition. As a principle, each definition given in BMSs must be given in a way that it generalizes its namesake in metric spaces in this context.

Proposition 2.1. *(X, X, b) is a BMS iff (X, b) is a metric space.*

Notation. *Throughout the remainder of this section, (X, Y, b) and (Ξ, Υ, β) will always denote BMSs, while A and B will represent arbitrary subsets of $X \cup Y$. Moreover, for simplicity, we do not distinguish the function b and its restrictions to subsets, in notation.*

Definition 2.7. *If $P \subseteq X$, and $Q \subseteq Y$ are arbitrary subsets, then (P, Q, b) is called a bipolar subspace of (X, Y, b). As a special case, if there exists a set C such that $P = X \cap C$, $Q = Y \cap C$, then (P, Q, b) is called a subspace.*

Clearly, subspaces and bipolar subspaces of a BMS correspond to different concepts. While every subspace is also a bipolar subspace, the converse is not generally true, and subspaces are helpful in most cases because they preserve the balance of the structure to some extent. In contrast, bipolar subspaces can arise in a more chaotic sense. However, they have an instrumental role, especially in constructing examples. For example in the light of Proposition 2.1, a metric space (M, d) can be viewed as a BMS (M, M, d), bipolar subspaces of (M, M, d) will provide plenty of examples of BMS that are not metric spaces. This situation raises the question of whether all BMSs arise this way. Mutlu and Gürdal showed that the answer is negative [32], they nevertheless obtained a partially affirmative result for a generalized type of metric space by utilizing the following tools.

Proposition 2.2. *The function $b_X : X \times X \to \mathbb{R}_0^+$,*

$$b_X(x_1, x_2) = \sup_{y \in Y} |b(x_1, y) - b(x_2, y)|,$$

is a pseudo-metric on X, for every $x_1, x_2 \in X$. Similarly, $b_Y : Y \times Y \to \mathbb{R}_0^+$, defined by

$$b_Y(y_1, y_2) = \sup_{x \in X} |b(x, y_1) - b(x, y_2)|,$$

is a pseudo-metric on Y, for every $y_1, y_2 \in Y$.

The approach in Proposition 2.1, which connects BMSs to classical metric spaces, can be taken one step further with the help of the concept of the center of a BMS consisting of central points, which will naturally be a metric space.

Definition 2.8. *For any BMS (X, Y, b), the metric space $(X \cap Y, b)$ is called the center metric space, and it is denoted by $\mathcal{Z}(X, Y, b)$.*

Definition 2.9. *Let (X, Y, b) be a BMS. The function $\bar{b} : Y \times X \to \mathbb{R}^+$ defined by $\bar{b}(y, x) = b(x, y)$ for every $(y, x) \in Y \times X$, is also a bipolar metric on (Y, X) and $(Y, X, \bar{b})$ is called the opposite of (X, Y, b), denoted by $\overleftrightarrow{(X, Y, b)} = (Y, X, \bar{b})$.*

It is obvious from the definition that, one always has $\overleftrightarrow{\overleftrightarrow{(X, Y, b)}} = (X, Y, b)$.

Definition 2.10. ***(i)*** *A mapping $f : (X, Y, b) \rightrightarrows (Z, W, b')$ is continuous at a left point $x_0 \in X$, if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall y \in Y,\ b(x_0, y) < \delta \Rightarrow b'(f(x_0), f(y)) < \varepsilon,$$

and it is continuous at a right point $y_0 \in Y$, if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in X,\ b(x, y_0) < \delta \Rightarrow b'(f(x), f(y_0)) < \varepsilon.$$

(ii) *Similarly, a contravariant mapping $f : (X, Y, b) \leftrightarrows (Z, W, b')$ is continuous at a left point $x_0 \in X$, if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall y \in Y,\ b(x_0, y) < \delta \Rightarrow b'(f(y), f(x_0)) < \varepsilon,$$

and it is continuous at a right point $y_0 \in Y$, if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in X,\ b(x, y_0) < \delta \Rightarrow b'(f(y_0), f(x)) < \varepsilon.$$

Convergence of sequences on a BMSs is defined as follows.

Definition 2.11. *A left sequence (x_n) converges to a right point $y \in Y$, if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ (which may depend upon ε) such that $b(x_n, y) < \varepsilon$ whenever $n \geq n_0$, and this case is denoted by $(x_n) \rightharpoonup y$ or $\lim_{n\to\infty} x_n = y$.*

A right sequence (y_n) converges to a left point x, if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $n \geq n_0 \implies b(x, y_n) < \varepsilon$, and this is denoted by $(y_n) \rightharpoondown x$ or $\lim_{n\to\infty} y_n = x$.

If a central sequence (u_n) converges to a central point u, such that $(u_n) \rightharpoonup u$ and $(u_n) \rightharpoondown u$, then it is said that (u_n) converges to u and this is denoted by $(u_n) \to u$.

In a BMS, convergence to noncentral left points is not defined for noncentral left sequences, and convergence to noncentral right points is not defined for noncentral right sequences. So, when it is given, for example, that $(u_n) \rightharpoonup v$, then v and (u_n) are readily understood to be a right point and a left sequence, respectively.

Proposition 2.3. *$(x_n) \rightharpoonup y$ on (X, Y, b) iff $(x_n) \rightharpoondown y$ on $\overleftrightarrow{(X, Y, b)}$.*

Remark. *In the light of Proposition 2.3, it is often convenient to consider only left sequences stating and proving general results on convergence in BMSs unless otherwise needed. Similar results for right sequences will readily follow by the duality between a BMS, and its opposite.*

Proposition 2.4. *$(x_n) \rightharpoonup y$ iff $(b(x_n, y)) \to 0$ on $\mathbb{R}$, and $(y_n) \rightharpoondown x$ iff $(b(x, y_n)) \to 0$, on $\mathbb{R}$.*

It is often desirable for convergent sequences to have only one limit. BMSs generally do not have this property, but the uniqueness of the limit can be guaranteed under additional conditions.

Theorem 2.5. [14] *If (X, Y, b) can be embedded as a bipolar subspace into any metric space, then each convergent sequence has a unique limit in (X, Y, b).*

Theorem 2.6. [14] *If a sequence converges to a central point, then this limit is unique.*

Definition 2.12. *A bisequence (x_n, y_n) is called convergent, if there exist points x and y such that $(x_n) \rightharpoonup y$ and $(y_n) \rightharpoondown x$. Moreover, if $x = y$, then (x_n, y_n) is said to be biconvergent to that point.*

Definition 2.13. *A bisequence (x_n, y_n) is called a Cauchy bisequence, if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d(x_n, y_m) < \varepsilon$ whenever $m, n \geq n_0$.*

The following proposition is a concise statement that relates the concepts of Cauchyness, convergence, and biconvergence for bisequences on a BMS.

Proposition 2.7. *Every biconvergent bisequence is Cauchy, and every convergent Cauchy bisequence is biconvergent.*

3. Some Topological Notions on Bipolar Metric Spaces

Throughout the section, except for examples, it has been assumed that a fixed BMS (X, Y, b) is given, and A and B are subsets of $X \cup Y$.

Definition 3.1. *Let $x_0 \in X$, $y_0 \in Y$ and $r > 0$. Then, the set*

$$D_X(x_0; r) = \{y \in Y : b(x_0, y) < r\}$$

is called the left-centric open ball with radius r and center x_0, and the set

$$D_Y(y_0; r) = \{x \in X : b(x, y_0) < r\}$$

is called the right-centric open ball with radius r and center y_0. Similarly, the set

$$\bar{D}_X(x_0; r) = \{y \in Y : b(x_0, y) \leq r\}$$

is called the left-centric closed ball with radius r and center x_0. Similarly, the set

$$\bar{D}_Y(y_0; r) = \{x \in X : b(x, y_0) \leq r\}$$

is called the right-centric closed ball with radius r and center y_0.

An interesting aspect of the definitions given above is the fact that if a ball has a left point as its center, then the ball consists of some right points, and vice versa. Thus, as an extreme case, if x_0 is a noncentral left point, then x_0 will not be an element of the balls accepting it as the center.

Definition 3.2. *A is called a left open set, if for each $y \in A \cap Y$ there exists $r > 0$ such that $D_Y(y; r) \subseteq A$, and A is called a right open set, if for each $x \in A \cap X$ there exists $r > 0$ such that $D_X(x; r) \subseteq A$. If A is both right open and left open, then it is called open.*

By definition, each left-centric open ball is a subset of Y, and each right-centric open ball is a subset of X. Hence, we readily have the subsequent proposition, which explains why the adjectives right and left in the definition of open sets are used in contrast with the types of open balls in the definition.

Proposition 3.1. *The left pole (the set X) an its all supersets are always left open and the right pole (the set Y) and its supersets are always right open.*

Proof. For all $y \in Y$ and $r > 0$, $D_Y(y; r) \subseteq X$ by the definition of a right-centric open ball. Thus the set X and all larger sets containing X are left open. The case of right openness of Y and its supersets is similar. □

Remark. *Note that the largest set $X \cup Y$ is always open, as a consequence of Proposition 3.1.*

Theorem 3.2. *Every left-centric open ball is right open.*

Proof. Consider a left-centric open ball $D_X(x_0; r)$. To show that $D_X(x_0; r)$ is right open, we take an element $x \in D_X(x_0; r) \cap X$. Then, from the definiton of a left-centric open ball, $D_X(x_0; r) \subseteq Y$ and thus x is a central point. Now, for $\rho = r - b(x_0, x) > 0$, we claim that $D_X(x; \rho) \subseteq D_X(x_0; r)$. Suppose that $y \in D_X(x; \rho)$. By the definition of $D_X(x; \rho)$, $y \in Y$. Then $b(x, y) < \rho = r - b(x_0, x)$. Therefore, $b(x_0, x) + b(x, y) < r$. From the quadrilateral inequality, and since x is a central point,

$$b(x_0, y) \leq b(x_0, x) + b(x, x) + b(x, y) = b(x_0, x) + b(x, y) < r.$$

Thus, $y \in D_X(x_0; r)$ and hence, $D_X(x; \rho) \subseteq D_X(x_0; r)$. □

The following example illustrates, surprisingly, that while a left-centric open ball is always a right open set, it does not need to be left open.

Example 3.1. *Let $X = (-\infty, 1]$, $Y = [-1, \infty)$, and $b(x, y) = |x - y|$. Consider the left-centric open ball*

$$D_X(0; 3) = \{y \in [-1, \infty) : |0 - y| < 3\} = [-1, 3).$$

We show that $[-1, 3)$ is not left open. For $y = -1 \in Y$ and any $\varepsilon > 0$, observe that

$$D_Y(-1; \varepsilon) = \{x \in X : |x + 1| < \varepsilon\} = (-\varepsilon - 1, \varepsilon - 1) \cap (-\infty, 1].$$

So, $D_Y(-1; \varepsilon) \nsubseteq D_X(0; 3)$, and $D_X(0; 3)$ is not left open.

Remark. *By symmetry, similar results are valid for right-centric open balls. That is, a right-centric open ball is always left open, but does not need to be right open.*

Definition 3.3. *A is called left closed, if for each right sequence in A, $(y_n) \rightharpoonup x$ implies $x \in A$; and it is called right closed, if for each left sequence in A, $(x_n) \rightharpoondown y$ implies $y \in A$. If a set is both left and right closed, then it is called closed.*

We already know that the left pole of a BMS is left open, and the right pole is right open. A similar result for closedness is given by the following straightforward proposition.

Proposition 3.3. *The left pole is left closed, and the right pole is right closed.*

Theorem 3.4. *Every left-centric closed ball is right closed.*

Proof. Let $\bar{D}_X(x_0; r)$ be a left-centric closed ball. Consider a left sequence (x_n) in $\bar{D}_X(x_0; r)$ such that $(x_n) \rightharpoondown y \in Y$. We need to see that $y \in \bar{D}_X(x_0; r)$. Given an $\varepsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $b(x_n, y) < \varepsilon$ for $n \geq n_0$. Moreover, since (x_n) is a left sequence in $\bar{D}_X(x_0; r) \subseteq Y$, it is a central sequence, and in particular $x_{n_0} \in X \cap Y$. Also, $b(x_0, x_{n_0}) < r$ as (x_n) is in $\bar{D}_X(x_0; r)$. Therefore

$$b(x_0, y) \leq b(x_0, x_{n_0}) + b(x_{n_0}, x_{n_0}) + b(x_{n_0}, y) < r + \varepsilon.$$

Since ε is arbitrary, $b(x_0, y) \leq r$, and this implies $y \in \bar{D}_X(x_0; r)$. □

Now, we provide an example in which a left-centric closed ball is not left closed.

Example 3.2. *Let $X = [-1, 1]$, $Y = (1, \infty)$ and $b(x, y) = |x^2 - y^2|$. Consider the left-centric closed ball $\bar{D}_X(0; 3)$, which is equal to $(1, 3]$. $(y_n) = (\frac{n+1}{n})$ is a right sequence on $\bar{D}_X(0; 3)$ such that $(y_n) \rightharpoonup 1$, and also $(y_n) \rightharpoonup -1$ at the same time. However $1 \notin \bar{D}_X(0; 3)$ so that $\bar{D}_X(0; 3)$ is not left closed. Note here that, there is no convergent left sequence in $\bar{D}_X(0; 3)$. So, the fact that $\bar{D}_X(0; 3)$ is right closed is a vacuous truth in this case.*

Remark. *As a dual result of Theorem 3.4, every right-centric closed ball is left closed.*

Theorem 3.5. *A is left open, if and only if, A^c is right closed, where the complements are taken over the set $X \cup Y$.*

Proof. Let A be left open. Consider a left sequence (x_n) on A^{c}, and suppose that $(x_n) \rightharpoondown y$. We must show that $y \in A^{\mathsf{c}}$. Assume the contrary that $y \in A$. Since A is left open, there exists $\varepsilon > 0$ such that $D_Y(y; \varepsilon) \subseteq A$, and since $(x_n) \rightharpoondown y$, there is an $n_0 \in \mathbb{N}$, such that $b(x_n, y) < \varepsilon$ for $n \geq n_0$. In particular, $x_{n_0} \in D_Y(y; \varepsilon) \subseteq A$, which contradicts by $(x_n) \in A^{\mathsf{c}}$. Thus, $y \in A$ and A is a right closed set.

Conversely, suppose that A^{c} is a right closed set. To show that A is left open, consider a right point $y \in A$. Assume that there exists no $\varepsilon > 0$ such that $D_Y(y; \varepsilon) \subseteq A$. Then for each $\varepsilon > 0, D_Y(y; \varepsilon) \nsubseteq A$, or equivalently, $D_Y(y; \varepsilon) \cap A^{\mathsf{c}} \neq \varnothing$. For each $n \in \mathbb{N}$, pick an $x_n \in D_Y(y; \frac{1}{n}) \cap A^{\mathsf{c}}$. In this case, (x_n) is a left sequence since $D_Y(y; \frac{1}{n}) \subseteq X$, and $(x_n) \subseteq A^{\mathsf{c}}$. However, $(x_n) \rightharpoondown y$, since $b(x_n, y) < \frac{1}{n} \to 0$. This contradicts by the right closedness of A^{c}. Therefore $y \in A$, so that A is left open. □

We now present the following result on generating topologies from a given BMS.

Theorem 3.6. *Let τ_L be the family of left open subsets. Then*

(i) $\varnothing, X \cup Y \in \tau_L$,

(ii) *For every $i \in I$, $A_i \in \tau_L$ implies* $\bigcup_{i \in I} A_i \in \tau_L$,

(iii) $A, B \in \tau_L$ *implies* $A \cap B \in \tau_L$.

Proof. **(i)** The empty set satisfies the conditions in the definition of left open sets vacuously since it has no points, thus $\emptyset \in \tau_L$. $X \cup Y$ is a left open set, since one always have $D_Y(y; r) \subseteq X \subset X \cup Y$, for every right point $y \in X \cup Y$ and $r > 0$.

(ii) Let $\{A_i \subseteq X : i \in I\}$ be a collection of left open subsets. Take $y \in \bigcup_{i \in I} A_i$. Then, $y \in A_i$ for some $i \in I$. Since $A_i \in \tau_L$, there is an $r > 0$ such that $D_Y(y; r) \subseteq A_i$. Therefore, $D_Y(y; r) \subseteq \bigcup_{i \in I} A_i$, and $\bigcup_{i \in I} A_i \in \tau_L$.

(iii) Let A and B be left open subsets. If $y \in A \cap B$, there are $r_A, r_B > 0$ such that $D_Y(y; r_A) \subseteq A$ and $D_Y(y; r_B) \subseteq B$. Set $r = \min\{r_A, r_B\}$. Then, $D_Y(y; r) \subseteq A \cap B$. In other words, $A \cap B$ is left open. □

Remark. *In addition to the topological space (X, τ_L), there is an accompanying topology τ_R on $X \cup Y$, consisting of all right open subsets.*

For any BMS (X, Y, b), we have an associated metric space $\mathcal{Z}(X, Y, b)$, as described in Definition 2.8. A question arise then: does the topology generated by the center metric space equal to the relative topology on $X \cap Y$, corresponding to $\tau_L \cap \tau_R$? The answer is, in general, no, as will be illustrated below.

Example 3.3. *Consider the bipolar subspace $((-\infty, 1], [-1, \infty), b)$ of the standard metric space on $\mathbb{R}$. The center $[-1, 1]$ of this BMS is not left open as for the right point $-1 \in [-1, 1]$, there is no $r > 0$ such that $D_Y(-1; r) \subseteq [-1, 1]$, since $D_Y(-1; r) = (-r - 1, r - 1) \cap (-\infty, 1]$. Hence, $[-1, 1] \notin \tau_L$ and in particular $\tau_L \cap \tau_R$ cannot be a topology on $[-1, 1]$.*

As a result of Theorems 3.5, and 3.6, we have the following corollary for the family $\mathscr{K}_L$ of all left closed subsets.

Corollary 3.7. *Let $\mathscr{K}_L$ be the family of left closed subsets of $X \cup Y$.*

(i) *The empty set $\emptyset$ and $X \cup Y$ are left closed, i.e. $\emptyset, X \cup Y \in \mathscr{K}_L$.*

(ii) *Arbitrary intersections of left closed sets are left closed, i.e. for all $i \in I$, $A_i \in \mathscr{K}_L$ implies $\bigcap_{i \in I} A_i \in \mathscr{K}_L$.*

(iii) *Union of two left closed sets is left closed, i.e. $A, B \in \mathscr{K}_L$ implies $A \cup B \in \mathscr{K}_L$.*

Definition 3.4. *A left point x is called a left accumulation point of A, if $A \cap (D_X(x; r) - \{x\}) \neq \emptyset$ for every $r > 0$. The set of all left accumulation points of A is denoted by $\mathrm{acc}_X(A)$. Similarly, an $y \in Y$ is called right accumulation point if, for every $r > 0$, one has $A \cap (D_Y(y; r) - \{y\}) \neq \emptyset$ and the set of all such points is denoted by $\mathrm{acc}_Y(A)$.*

Definition 3.5. *A left point x is called a left contact point of A, if for every $r > 0$, $A \cap D_X(x; r) \neq \emptyset$. Similarly, an $y \in Y$ is a right contact point, if for every $r > 0$, $A \cap D_Y(y; r) \neq \emptyset$. The set of left contact points of A is called the left closure of A, and is denoted by $\overleftarrow{A}$. The set of right contact points of A is called the right closure of A, and it is denoted by $\overrightarrow{A}$.*

Although every left accumulation point is a left contact point, the converse is shown not to be true in the following example.

Example 3.4. *Let $\mathbb{R}_0^- = (-\infty, 0]$, $\mathbb{R}_0^+ = [0, \infty)$, and $b : \mathbb{R}_0^- \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be defined by $b(x, y) = \lceil y \rceil - \lfloor x \rfloor$, where $\lceil\ \rceil$ and $\lfloor\ \rfloor$ stand for ceiling and floor functions, respectively. Consider the set $\mathbb{R} = \mathbb{R}_0^+ \cup \mathbb{R}_0^-$. If $x \in \mathbb{R}_0^-$ and $x \neq 0$, then*

$$b(x, y) = \lceil y \rceil - \lfloor x \rfloor \geq \lceil y \rceil - (-1) \geq 1.$$

Therefore, $D_X(x; 1) = \emptyset$, which means that $x \neq 0$ is not a left accumulation point, nor a left contact point of $\mathbb{R}$. For the only remaining left point $x = 0$, we have $D_X(0; r) = \{0\}$, if $r \leq 1$.

In this case, $\mathbb{R} \cap (D_X(0; r) - \{x\}) = \emptyset$, *but* $\mathbb{R} \cap D_X(0; r) = \{x\} \neq \emptyset$. *Thus,* $\mathrm{acc}_X(\mathbb{R}) = \emptyset$, *while* $\overleftarrow{\mathbb{R}} = \{0\}$.

Theorem 3.8. *A is left closed iff* $\mathrm{acc}_X(A) \subseteq A$.

Proof. Suppose that A is left closed, $x \in \mathrm{acc}_X(A)$, and assume that $x \notin A$. Then $x \in A^c$, and by Theorem 3.13, A^{c} is right open. In this case, since $x \in A^{\mathrm{c}}$, there exists $r > 0$ such that $D_X(x; r) \subseteq A^{\mathrm{c}}$. Therefore, $A \cap D_X(x; r) = \emptyset$. Thus, x is not a left contact point of A, so it is not a left accumulation point. However, this contradicts by $x \in \mathrm{acc}_X(A)$. Consequently $x \in A$, and we have $\mathrm{acc}_X(A) \subseteq A$.

Conversely, suppose that $\mathrm{acc}_X(A) \subseteq A$ and take an $x \in X$. Consider a right sequence (y_n) on A such that $(y_n) \rightharpoonup x$. We need to show that $x \in A$. If (y_n) is an ultimately constant sequence, that is, if there is some $n_0 \in \mathbb{N}$, such that $y_n = x$ for $n \geq n_0$, then $x \in A$, since (y_n) is a right sequence on A. Otherwise, if (y_n) is not ultimately constant, then for any $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$, such that $b(x, y_n) < \varepsilon$ for all $n \geq n_\varepsilon$. Now consider the left-centric open ball $D_X(x; r)$. Then, $y_n \in D_X(x; r)$ for $n \geq n_r$, and since (y_n) is not ultimately constant, there is an $n^* \geq n_r$ such that $y_{n^*} \neq x$. Hence, $y_{n^*} \in A \cap (D_X(x; r) - \{x\}) \neq \emptyset$, and therefore $x \in \mathrm{acc}_X(A) \subseteq A$. □

Corollary 3.9. *A is left closed, if and only if,* $\overleftarrow{A} \subseteq A$.

Remark. *In contrast with the case of metric spaces, where a set A is closed iff* $\overline{A} = A$, *a left closed set, in general does not have the property* $\overleftarrow{A} = A$ *in a BMS. For instance, in Examlple 3.4,* $\mathbb{R}$ *is left closed, since for any convergent right sequence* (y_n) *in* $\mathbb{R}$, $(y_n) \rightharpoonup x$ *is possible, only if* (y_n) *is ultimately zero, and* $x = 0$. *However* $\overleftarrow{\mathbb{R}} = \{0\} \neq \mathbb{R}$.

The following propositions are direct consequences of definitions.

Proposition 3.10. $\mathrm{acc}_X(A \cup B) = \mathrm{acc}_X(A) \cup \mathrm{acc}_X(B)$.

Proposition 3.11. *If* $A \subseteq B$, *then* $\mathrm{acc}_X(A) \subseteq \mathrm{acc}_X(B)$.

In classical metric spaces, and more generally in topological spaces, the closure operator satisfies four conditions known as the Kuratowski closure axioms; namely it preserves the empty set ($\overline{\emptyset} = \emptyset$), is extensive ($A \subseteq \overline{A}$), idempotent ($\overline{\overline{A}} = \overline{A}$), and distributes over unions of two sets ($\overline{A \cup B} = \overline{A} \cup \overline{B}$). When idempotency is removed from Kuratowski axioms, the remaining three axioms are called Čech closure axioms. Although the Kuratowski and Čech closure axioms are especially prominent because they provide necessary and sufficient conditions to provide equivalent definitions for pretopological and topological spaces, respectively, there are many more properties satisfied by the closure operator of a metric space. We now investigate the extent to which the left closure operator provides similar properties.

Proposition 3.12. *The following hold.*

1. $\overleftarrow{A} \subseteq X$.
2. $\overleftarrow{A}$ *is left closed.*
3. *If* $A \subseteq X \cap Y$, *then* $A \subseteq \overleftarrow{A}$.
4. *If* $A \subseteq K$ *and* K *is left closed, then* $\overleftarrow{A} \subseteq K$.
5. *A is left closed, if and only if* $\overleftarrow{A} \subseteq A$.
6. *If* $A \subseteq B$, *then* $\overleftarrow{A} \subseteq \overleftarrow{B}$.
7. $\overleftarrow{\overleftarrow{A}} \subseteq \overleftarrow{A}$.

8. $\overleftarrow{\varnothing} = \varnothing$.
9. $\overleftarrow{A \cup B} = \overleftarrow{A} \cup \overleftarrow{B}$.

Proof. 1. It follows from the left closure definition.

2. Let (u_n) be a right sequence on $\overleftarrow{A}$ such that $(u_n) \rightharpoonup x$. Since $\overleftarrow{A} \subseteq X$ by definition, (u_n) is in fact a central sequence on $\overleftarrow{A}$. We also have $A \cap D_X(u_n; \frac{1}{n}) \neq \varnothing$ for all $n \in \mathbb{N}^+$. In particular, since $D_X(u_n; \frac{1}{n}) \subseteq Y$, the sets $A \cap D_X(u_n; \frac{1}{n})$ consist of right points. Form a right sequence (y_n) such that $y_n \in A \cap D_X(u_n; \frac{1}{n})$. In this case (y_n) is a right sequence on A and

$$b(x, y_n) \leq b(x, u_n) + b(u_n, u_n) + b(u_n, y_n) \leq b(x, u_n) + \frac{1}{n}.$$

Taking limits on $\mathbb{R}$ as $n \to \infty$ on both sides, we get $(y_n) \rightharpoonup x$ by Proposition 2.4. Then for any given $\varepsilon > 0$, there exists such an $n_0 \in \mathbb{N}$ that $n \geq n_0$ implies $b(x, y_n) < \varepsilon$ for $n \in \mathbb{N}$. Particularly, $b(x, y_{n_0}) < \varepsilon$, or in other terms, $y_{n_0} \in A \cap D_X(x; \varepsilon) \neq \varnothing$. Hence $x \in \overleftarrow{A}$ and the set $\overleftarrow{A}$ is left closed.

3. Suppose $A \subseteq X \cap Y$. Let $u \in A \cap X = A$. Then $u \in D_X(u; r)$, and $A \cap D_X(u; r) \neq \varnothing$ for all $r > 0$. Therefore $u \in \overleftarrow{A}$.

4. Let $A \subseteq K$ and K be a left closed set. Given $x \in \overleftarrow{A}$. Then $A \cap D_X(x, \frac{1}{n}) \neq \varnothing$ for all $n \in \mathbb{N}^+$. Form a right sequence (y_n) such that $y_n \in A \cap D_X(x; \frac{1}{n}) \subseteq K \cap D_X(x; \frac{1}{n})$. (y_n) is a right sequence on K, and $(y_n) \rightharpoonup x$, as $(b(x, y_n)) \to 0$ on $\mathbb{R}$. Since K is left closed, $x \in K$, and $\overleftarrow{A} \subseteq K$.

5. This is Corollary 3.9.

6. Follows immediately from the definitions.

7. A direct consequence of (2) and (5).

8. It is clear, as $\varnothing \cap D_X(x; r)$ will always be empty, for any $x \in X$.

9. By (6), $A, B \subseteq A \cup B$ implies $\overleftarrow{A} \subseteq \overleftarrow{A \cup B}$ and $\overleftarrow{B} \subseteq \overleftarrow{A \cup B}$, so that

$$\overleftarrow{A} \cup \overleftarrow{B} \subseteq \overleftarrow{A \cup B}.$$

On the other hand, if $x \in \overleftarrow{A \cup B}$, then $(A \cup B) \cap D_X(x; r) \neq \varnothing$, so that either $A \cap D_X(x; r) \neq \varnothing$ or $B \cap D_X(x; r) \neq \varnothing$, that is $x \in \overleftarrow{A} \cup \overleftarrow{B}$. □

As can be understood from the proposition above, the left closure operator of a BMS, in general, does not satisfy two of the Kuratowski closure axioms, namely $A \subseteq \bar{A}$ and $\bar{\bar{A}} = \bar{A}$. The following example illustrates that, both $A \subseteq \overleftarrow{A}$ and $\overleftarrow{\overleftarrow{A}} = \overleftarrow{A}$ are in fact not required in BMSs.

Example 3.5. *Let $X = \{(x, y) \in \mathbb{R}^2 : y \geq x\}$ and $Y = \{(x, y) \in \mathbb{R}^2 : y < x\}$. Let $b : X \times Y \to \mathbb{R}_0^+$ be the restriction of Euclidean metric on $\mathbb{R}^2$. Consider the unit disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Observe that $\overleftarrow{D} = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1,\ y = x\}$, and $\overleftarrow{\overleftarrow{D}} = \varnothing$. Thus, neither $D \subseteq \overleftarrow{D}$, nor $\overleftarrow{\overleftarrow{D}} = \overleftarrow{D}$.*

By Proposition 3.12 (2), $\overleftarrow{A}$ is always a left closed set. The following example illustrates that $\overleftarrow{A}$ does not have to be right closed.

Example 3.6. *Let $X = (-\infty, 5)$, $Y = (-5, \infty)$, $A = (-10, 10)$, and $b(x, y) = |x - y|$. Then $\overleftarrow{A} = [-5, 5)$. (x_n) is a left sequence on $\overleftarrow{A}$, where $x_n = \frac{5n}{n+1}$, but $(x_n) \rightharpoonup 5 \notin \overleftarrow{A}$. So, $\overleftarrow{A}$ is not right closed.*

Theorem 3.13. *In a BMS, every singleton is closed.*

Proof. Let $x_0 \in X$ and $C = \{x_0\}$. We must show that C is both left and right closed.

Suppose that (y_n) is a right sequence from C, and $(y_n) \rightharpoonup x$. Since $C = \{x_0\}$, $(y_n) = (x_0, x_0, x_0, \ldots)$ is a constant sequence. In this case, x_0 is both a left and a right point and (y_n) is a central sequence. Then clearly $(y_n) \rightharpoonup x_0$ as $b(x_0, x_0)$ is defined, and x_0 is the only limit by Theorem 2.6. So $x = x_0 \in C$, and C is left closed.

On the other hand, if (x_n) is a left sequence from C, then $(x_n) = (x_0, x_0, x_0, \ldots)$, and only limit of (x_n) is x_0 (if x_0 is a central point), or (x_n) is not convergent (if x_0 is noncentral left point). In both cases, C is right closed. □

Definition 3.6. *The left interior of A is given by*

$$A^{\triangleleft} = \{x \in X : D_X(x; r) \subseteq A, \textit{ for some } r > 0\},$$

and the right interior of A is the set

$$A^{\triangleright} = \{y \in Y : D_Y(y; r) \subseteq A, \textit{ for some } r > 0\}.$$

The points of $A^{\triangleleft}$ are called left interior points, and the points of $A^{\triangleright}$ are called right interior points.

Proposition 3.14. *A set $A \subseteq X \cup Y$ is left open in a BMS (X, Y, b), if and only if, all of its right points are right interior points, that is $A \cap Y \subseteq A^{\triangleright}$.*

Proof. It is a direct consequence of Definitions 3.2 and 3.6. □

Now we give an analog of Proposition 3.12 for left interiors.

Proposition 3.15. *Then the following holds.*

1. *$A^{\triangleleft} \subseteq X$.*
2. *$A^{\triangleleft}$ is left open.*
3. *If $A^{c} \subseteq Y$, then $A^{\triangleleft} \subseteq A$, where the complement is taken in $X \cup Y$.*
4. *If $B \subseteq A$ and B is right open, then $B \cap X \subseteq A^{\triangleleft}$.*
5. *A is right open if and only if $A \cap X \subseteq A^{\triangleleft}$.*
6. *If $A \subseteq B$, then $A^{\triangleleft} \subseteq B^{\triangleleft}$.*
7. *$A^{\triangleleft\triangleleft} \subseteq A^{\triangleleft}$.*
8. *$Y^{\triangleleft} = X$.*
9. *$(A \cap B)^{\triangleleft} = A^{\triangleleft} \cap B^{\triangleleft}$.*

Proof. 1. $A^{\triangleleft} \subseteq X$ by the definition.

2. Let $u \in A^{\triangleleft}$ be a right point. Since $A^{\triangleleft} \subseteq X$, u is a central point. By $u \in A^{\triangleleft}$, there is some $r > 0$, such that $D_X(u; r) \subseteq A$. To show that $A^{\triangleleft}$ is left open, we must find a right-centric open ball with center u contained in $A^{\triangleleft}$.

Consider the right-centric open ball $D_Y(u; \frac{r}{2}) \subseteq X$. To see that $D_Y(u; \frac{r}{2}) \subseteq A^{\triangleleft}$, we must verify that for each $x \in D_Y(u; \frac{r}{2})$, there exists $\varepsilon > 0$ such that $D_X(x; \varepsilon) \subseteq A$. We set $\varepsilon = \frac{r}{2}$. In this case, if $y \in D_X(x; \frac{r}{2})$, then $b(x, y) < \frac{r}{2}$. On the other hand by $x \in D_Y(u; \frac{r}{2})$, we have $b(x, u) < \frac{r}{2}$. Combining these yields,

$$b(u, y) \leq b(u, u) + b(x, u) + b(x, y) < 0 + \frac{r}{2} + \frac{r}{2} = r.$$

Hence $y \in D_X(u; r) \subseteq A$, that is $D_X(x; \frac{r}{2}) \subseteq A$, and $x \in A^{\triangleleft}$. This means that $D_Y(u; \frac{r}{2}) \subseteq A^{\triangleleft}$, and $A^{\triangleleft}$ is left open.

3. Suppose that $A^{c} \subseteq Y$. We show that $A^{c} \subseteq (A^{\triangleleft})^{c}$. Let $y \in A^{c}$. Assume that $y \in A^{\triangleleft}$. Then y is a central point, and $D_X(y; r) \subseteq A$. But by centrality of y, $b(y, y) = 0$ and $y \in D_X(y; r) \subseteq A$, which contradicts by $y \in A^{c}$.

4. Given $x \in B \cap X$. Since B is right open, there is some $r > 0$ such that $D_X(x;r) \subseteq B$. By $B \subseteq A$, we also have $D_X(x;r) \subseteq A$, which means that $x \in A^{\lhd}$.

5. Suppose that $A \cap X \subseteq A^{\lhd}$. If x is a left open in A, then $x \in A \cap X \subseteq A^{\lhd}$, and thus there exist an $r > 0$, such that $D_X(x;r) \subseteq A$. Hence A is right open. Conversely, if A is right open and $x \in A \cap X$, then there exists some $r > 0$, such that $D_X(x;r) \subseteq A$, and this gives $x \in A^{\lhd}$.

6. It is clear from the definition.

7. We know that $A^{\lhd}, A^{\lhd\lhd} \subseteq X$. Suppose for a left point x that $x \notin A^{\lhd}$. Then for each $r > 0$, one have $D_X(x;r) \nsubseteq A$, so that there exists at least a $y_r \in D_X(x;r)$, such that $y_r \notin A$. Now we claim that also $y_r \notin A^{\lhd}$. Assume the contrary that $y_r \in A^{\lhd}$. Since $A^{\lhd} \subseteq X$, y_r is a central point. By $y_r \in A^{\lhd}$, there is an $\varepsilon > 0$ such that $D_X(y_r;\varepsilon) \subseteq A$. However, $y_r \in D_X(y_r;\varepsilon)$, by centrality of y_r. Thus $y_r \in A$, and this is a contradiction. Consequently, our assumption $y_r \in A^{\lhd}$ is false, and $y_r \notin A^{\lhd}$. Therefore $D_X(x;r) \nsubseteq A^{\lhd}$. Since $r > 0$ is arbitrary chosen, x is not an interior point of $A^{\lhd}$, that is $x \notin A^{\lhd\lhd}$.

8. By definition $Y^{\lhd} \subseteq X$, whereas for any $x \in X$, $D_X(x;r) \subseteq Y$ for all $r > 0$, by the definition of a left-centric open ball. Hence $x \in Y^{\lhd}$.

9. By (6), $A \cap B \subseteq A$ and $A \cap B \subseteq B$ imply $(A \cap B)^{\lhd} \subseteq A^{\lhd}$, $(A \cap B)^{\lhd} \subseteq B^{\lhd}$, and $(A \cap B)^{\lhd} \subseteq A^{\lhd} \cap B^{\lhd}$. On the other side, if $x \in A^{\lhd} \cap B^{\lhd}$, then there are $r, s > 0$ such that $D_X(x;r) \subseteq A$, and $D_X(x;s) \subseteq B$. If we set $\varepsilon = \min\{r,s\}$, then $D_X(x;\varepsilon) \subseteq A \cap B$, so that $x \in (A \cap B)^{\lhd}$. □

There is a well-known duality between inteiror and closure operators on metric, and more generally topological spaces, namely $A^{\circ} = (\overline{A^{\mathsf{c}}})^{\mathsf{c}}$. A weaker analog of this for left interior and left closure operations on a BMS, is stated below.

Theorem 3.16. $A^{\lhd} \subseteq \left(\overleftarrow{A^{\mathsf{c}}}\right)^{\mathsf{c}}$.

Proof. Let $x \in A^{\lhd}$. Then $A^{\lhd} \subseteq X$, and there exists $r > 0$ with $D_X(x;r) \subseteq A$, or equivalently $D_X(x;r) \cap A^{\mathsf{c}} = \varnothing$. This means that $x \notin \overleftarrow{A^{\mathsf{c}}}$, hence $x \in (\overleftarrow{A^{\mathsf{c}}})^{\mathsf{c}}$. □

Applying the dual result to Theorem 3.16 for the set A^{c}, we immediately have the following result.

Corollary 3.17. $\overleftarrow{A} \subseteq ((A^{\mathsf{c}})^{\lhd})^{\mathsf{c}}$.

Now we provide a counterexample on falsity of some expectable properties of left interiors.

Example 3.7. *Let* $X = [0,3]$, $Y = [1,4]$, $A = (1,4)$, *and* $b(x,y) = |x-y|$. *Observe that* $X^{\lhd} = [0,3)$, $A^{\lhd} = [0,1) \cup (1,3]$, $A^{\lhd\lhd} = [0,1) \cup (1,3)$, $A^{\mathsf{c}} = [0,1] \cup \{4\}$, $\overleftarrow{A^{\mathsf{c}}} = \{1\}$, *and* $(\overleftarrow{A^{\mathsf{c}}})^{\mathsf{c}} = [0,1) \cup (1,4]$. *Then, one have* $X^{\lhd} \subsetneq X$, $A^{\lhd\lhd} \subsetneq A^{\lhd}$, *and* $A^{\lhd} \subsetneq (\overleftarrow{A^{\mathsf{c}}})^{\mathsf{c}}$.

In Theorem 3.16 and Corollary 3.17, if one takes some complements in X, instead of $X \cup Y$, then also the equalities are satisfied. The key here is to prevent noncentral right points from falling into the right hand sets by restricting only the final complements.

Theorem 3.18. $A^{\lhd} = X \setminus \overleftarrow{A^{\mathsf{c}}}$ *and* $\overleftarrow{A} = X \setminus (A^{\mathsf{c}})^{\lhd}$.

Proof. $A^{\lhd} \subseteq X \setminus \overleftarrow{A^{\mathsf{c}}}$ follows from Proposition 3.12 (1) and Theorem 3.17. On the other side if $x \in X \setminus \overleftarrow{A^{\mathsf{c}}}$, then $x \in X$, but $x \notin \overleftarrow{A^{\mathsf{c}}}$. Therefore $D_X(x;r) \cap A^{\mathsf{c}} = \varnothing$ for some $r > 0$, which impilies $D_X(x;r) \subseteq A$, thence $x \in A^{\lhd}$. The other equality is similarly shown. □

Remark. *It worths noting that the left and right closure operators coincide if $X = Y$, and in this case they are equal to the closure operator of metric space $(X, b) = (Y, b)$. Thus, properties that are not satisfied for the closure operator, are also not available for left closures. For example, in general $\overleftarrow{A \cap B} \neq \overleftarrow{A} \cap \overleftarrow{B}$, and similarly $(A \cup B)^{\triangleleft} \neq A^{\triangleleft} \cup B^{\triangleleft}$.*

Maybe the most conspicuous topological defect of BMSs is the presence of null interior points and missing contanct points. More specifically, it is possible that $\varnothing^{\triangleleft} \neq \varnothing$, $\varnothing^{\triangleright} \neq \varnothing$, $\overleftarrow{X} \subsetneq X$, and $\overrightarrow{Y} \subsetneq Y$. The following theorem provides more information on these two cases.

Theorem 3.19. $\varnothing^{\triangleleft} = X \setminus \overleftarrow{X} = \left\{x \in X : \inf_{y \in Y} b(x, y) \neq 0\right\}$ *and* $\varnothing^{\triangleright} = Y \setminus \overrightarrow{Y} = \left\{y \in Y : \inf_{x \in X} b(x, y) \neq 0\right\}$.

Proof. We only show the equality for the left interior and closure. Other results follows from the duality. For $x \in X$,

$$\begin{aligned} x \in \varnothing^{\triangleleft} &\iff \exists r > 0,\ D_X(x; r) \subseteq \varnothing \\ &\iff \exists r > 0,\ \forall y \in Y,\ b(x, y) \geq r \\ &\iff \inf_{y \in Y} b(x, y) = 0, \end{aligned}$$

and as we have both $\varnothing^{\triangleleft}, \overleftarrow{X} \subseteq X$

$$\begin{aligned} x \notin \overleftarrow{X} &\iff \exists r > 0,\ D_X(x, r) \cap X = \varnothing \\ &\iff \exists r > 0,\ D_X(x, r) = \varnothing \\ &\iff \exists r > 0,\ \forall y \in Y,\ b(x, y) \geq r. \end{aligned}$$

Hence, $x \in \varnothing^{\triangleleft}$ iff $x \notin \overleftarrow{X}$ □

In this context, we now define a better-behaved subclass of BMSs.

Definition 3.7. *A BMS is called nondegenerate, if $\varnothing^{\triangleleft} = \varnothing = \varnothing^{\triangleright}$. Otherwise it is called degenerate.*

Example 3.8. *The BMSs in Example 3.1 and Example 3.3 are degenerate with $\varnothing^{\triangleleft} = (-\infty, -1)$ and $\varnothing^{\triangleright} = (1, \infty)$, while Example 3.6 has a degeneracy with $\varnothing^{\triangleleft} = (-\infty, -5)$ and $\varnothing^{\triangleright} = (5, \infty)$. The space in Example 3.2 is degenerate with $\varnothing^{\triangleleft} = (-1, 1)$ and $\varnothing^{\triangleright} = (1, \infty)$. The space in Example 3.4 is degenerate with $\varnothing^{\triangleleft} = (-\infty, 0)$ and $\varnothing^{\triangleright} = (0, \infty)$. The space in Example 3.5 is degenerate with $\varnothing^{\triangleleft} = \{(x, y) \in \mathbb{R}^2 : y > x\}$ and $\varnothing^{\triangleright} = \{(x, y) \in \mathbb{R}^2 : y < x\}$. And similarly, the space in Example 3.7 is degenerate with $\varnothing^{\triangleleft} = [0, 1)$ and $\varnothing^{\triangleright} = (3, 4]$. On the other hand every metric space is a nondegenerate BMS. However, the class of nondegenerate BMSs is properly larger than the class of metric spaces. An example of a nondegenerate BMS is (X, Y, b), where $X = \mathbb{Q}^2$, $Y = S^1$, the unit circle, and b is the restriction of the Euclidean metric. Another example is the BMS $(\mathbb{Q}, \mathbb{Q}^{\mathsf{c}}, b)$, where $b(x, y) = |x^2 - y^2|$.*

Having both left and right closure and interior operators, it is also possible to talk about left and right boundaries and exteriors on a BMS.

Definition 3.8. *The boundary of A is defined to be the set $\partial_L(A) = \overleftarrow{A} \setminus A^{\triangleleft}$, and the right boundary of A is $\partial_R(A) = \overrightarrow{A} \setminus A^{\triangleright}$, the left exterior of A is $\operatorname{ext}_L(A) = (A^{\mathsf{c}})^{\triangleleft}$, and $\operatorname{ext}_R(A) = (A^{\mathsf{c}})^{\triangleright}$.*

Proposition 3.20. *$\partial_L(A)$ is left closed.*

Proof. Let (u_n) be a right sequence on $\partial_L(A) = \overleftarrow{A} \setminus A^{\lhd}$ and $(u_n) \rightharpoonup x \in X$. As $\overleftarrow{A} \subseteq X$, (u_n) is a central sequence, and $x \in \overleftarrow{A}$ by Proposition 3.12 (2). It remains to show that $x \notin A^{\lhd}$. Since $u_n \notin A^{\lhd}$, there is no $r > 0$ such that $D_X(u_n; r) \subseteq A$, so that $D_X(u_n; r) \cap A^{c} \neq \varnothing$, and $u_n \in \overleftarrow{A^{c}}$. Then (u_n) is a right sequence on the left closed set $\overleftarrow{A^{c}}$, and $(u_n) \rightharpoonup x$ yields $x \in \overleftarrow{A^{c}}$. By applying Corollary 3.17 for A^{c}, one have $\overleftarrow{A^{c}} \subseteq (A^{\lhd})^{c}$, and this implies $x \notin A^{\lhd}$ as desired. □

The trichotomy rule $M = L^{\circ} \cup \partial(L) \cup \mathrm{ext}(L)$, which is valid for any subset L in a metric space (M, d), is in general does not work (at least perfectly) for BMSs. By Proposition 3.15 (6), both $A^{\lhd}$ and $\mathrm{ext}_L(A)$ are subsets of $\varnothing^{\lhd}$. Since it is possible that $\varnothing^{\lhd} \neq \varnothing$ on a BMS, $A^{\lhd}$ and $\mathrm{ext}_L(A)$ does not have to be disjoint. Many instances of this case, can be found in Example 3.8. Nevertheless, by removing the requirement for the sets to be pairwise disjoint, the following weaker result can be stated.

Theorem 3.21. *$X = A^{\lhd} \cup \partial_L(A) \cup \mathrm{ext}_L(A)$ and $X = A^{\rhd} \cup \partial_R(A) \cup \mathrm{ext}_R(A)$.*

Proof. By the definitions, Proposition 3.15 (1), and Theorem 3.18,

$$\begin{aligned} A^{\lhd} \cup \partial_L(A) \cup \mathrm{ext}_L(A) &= A^{\lhd} \cup (\overleftarrow{A} \setminus A^{\lhd}) \cup (A^{c})^{\lhd} \\ &= A^{\lhd} \cup \overleftarrow{A} \cup (A^{c})^{\lhd} \\ &= A^{\lhd} \cup (X \setminus (A^{c})^{\lhd}) \cup (A^{c})^{\lhd} \\ &= A \cup X = X. \end{aligned}$$

On the other hand, $X = A^{\rhd} \cup \partial_R(A) \cup \mathrm{ext}_R(A)$ follows from the duality. □

While the class τ_L in Theorem 3.6 is a topology on $X \cup Y$, the left closure operator do not correspond to τ_L. In fact, except for some special cases, it does not correspond to the closure operator of any topology on $X \cup Y$, as it does not satisfy the Kuratowski closure axioms in general. In this context, we finally introduce a modified kind of left and right interior and closure operators on a BMS, which fit better with the topologies τ_L and τ_R.

Definition 3.9. *The set $\overset{\Longleftarrow}{A} := \overleftarrow{A} \cup A$ is called the normalized left closure of A, and the set $\overset{\Longrightarrow}{A} := \overrightarrow{A} \cup A$ is called the normalized right closure of A, the set $A^{\blacktriangleleft} := A \cap (A^{\lhd} \cup X^{c})$ is called the normalized left interior of A, and the set $A^{\blacktriangleright} := A \cap (A^{\rhd} \cup Y^{c})$ is called the normalized right interior of A.*

Theorem 3.22. *A is left closed iff $\overset{\Longleftarrow}{A} = A$, and A is right closed iff $\overset{\Longrightarrow}{A} = A$.*

Proof. By Theorem 3.8, A is left closed iff $\mathrm{acc}_X(A) \subseteq A$, that is $\mathrm{acc}_X(A) \cup A = A$. Comparing definitions 3.4 and 3.5, we have $\mathrm{acc}_X(A) \subseteq \overleftarrow{A}$. If $\overset{\Longleftarrow}{A} = A$, then $\overleftarrow{A} \cup A = A$, and $\overleftarrow{A} \subseteq A$. In this case also $\mathrm{acc}_X(A) \subseteq A$, and A is left closed. On the other hand, if A is left closed, then $\overleftarrow{A} \subseteq A$ by Proposition 3.12 (4), and $\overset{\Longleftarrow}{A} = \overleftarrow{A} \cup A = \overleftarrow{A}$. The result for right closed sets follows from the duality. □

Theorem 3.23. *$A^{\blacktriangleleft} = (\overset{\Longleftarrow}{A^{c}})^{c}$ and $A^{\blacktriangleright} = (\overset{\Longrightarrow}{A^{c}})^{c}$.*

Combining Theorem 3.5, Theorem 3.22, and Theorem 3.23 gives rise to the following corollary.

Corollary 3.24. *A is right open iff $A^{\blacktriangleleft} = A$, and A is left open iff $A^{\blacktriangleright} = A$.*

4. Conclusion

Open sets in metric spaces are studied through two weaker concepts, right open and left open sets, in BMSs. Of course, the same situation applies to the case of closed sets. The duality between left open and right closed sets is particularly interesting. On the other hand, in BMSs, the left closure operator does not satisfy two of the Kuratowski closure axioms, namely extensivity and idempotency. In this respect, left closure operators do not determine a topology on the left pole, nor do they determine a pretopology. Therefore, it is certain that the topology τ_L, and the left closure operator represent different structures. Hence, it is understood that it is not necessary to study topological concepts from only a single point of view in BMSs. Instead, different perspectives can be brought to the structure. Undoubtedly, this attempt, which initiated an independent review of topological concepts in BMSs, is only the beginning, and there is still a long way to advance.

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NOTE ON APPROXIMATION OF TRUNCATED BASKAKOV OPERATORS

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Abstract. In this paper, we firstly introduce nonlinear truncated Baskakov operators on compact intervals and obtain some direct theorems. Also, we give the approximation of fuzzy numbers by truncated nonlinear Baskakov operators.

1. Introduction

Zadeh introduced the ideas of fuzzy sets, in [1]. Fuzzy numbers represented by proper intervals are a significant subject with numerous applications in a wide range of disciplines. Furthermore, it is well recognized that working with fuzzy numbers can be complicated due to the complicated approaches in which their membership function shapes are represented. Therefore, using trapezoidal or triangular fuzzy members to approximate fuzzy numbers, many studies have recently been published (see [13]-[19]).

The core topic of Korovkin type is the approximation of a continuous function by a series of linear positive operators (see [20],[21]). Bede et al. [6] have recently proposed nonlinear positive operators in place of linear positive operators. Although the Korovkin theorem fails for these nonlinear operators, they behave similarly to linear operators in terms of approximation.

The purpose of the paper is to use called max-product Truncated Baskakov operator which is given in the book [6] by applying continuous membership functions to approximate fuzzy integers. These operators additionally maintain the quasi-concavity in a manner analogous to the specific state of the unit interval will be given. These results turn out to be particularly useful for fuzzy numbers since they will enable us to construct fuzzy numbers with the same support in a straightforward manner. Additionally, these operators provide a good order of approximation for the (non-degenerate) segment core.

Baskakov [3] demonstrated the positive and linear operators, which are typically associated to functions that are bounded and uniformly continuous to $\nu \in C\,[0, +\infty]$ and specified

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by

$$V_n(\nu)(\theta) = (1+\theta)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \theta^k (1+\theta)^{-k} \nu\left(\frac{k}{n}\right), \ \forall n \in \mathbb{N}. \tag{1.1}$$

It is known that the pointwise approximation result (see [4]) exists as:

$$|\, V_n(\nu)(\theta) - \nu(\theta) \,| \leq C\omega_2^{\varphi}\left(\nu; \sqrt{\theta(1+\theta)/n}\right), \ \theta \in [0,\infty), n \in \mathbb{N},$$

where $\varphi(\theta) = \sqrt{\theta(1+\theta)}$ and $I = [0,\infty)$. In this case,
$I_h = [h^2/(1-h)^2, +\infty), \ h \leq \delta < 1$. Additionally, $V_n(\nu)$ satisfies the convexity and monotonicity of the function ν on $[0,+\infty)$ (see [5]).

The truncated Baskakov operators are identified by

$$U_n(\nu)(\theta) = (1+\theta)^{-n} \sum_{k=0}^{n} \binom{n+k-1}{k} \theta^k (1+\theta)^{-k} \nu\left(\frac{k}{n}\right), \ \nu \in C[0,1].$$

Truncated Baskakov operator of max product kind $\nu : [0,1] \to \mathbb{R}$ are identified by (see [6])

$$U_t^{(M)}(\nu)(\theta) = \frac{\bigvee_{k=0}^{t} b_{t,k}(\theta)\nu\left(\frac{k}{t}\right)}{\bigvee_{k=0}^{t} b_{t,k}(x)}, \ \theta \in [0,1], t \in \mathbb{N}, \ t \geq 1,$$

where $b_{t,k}(\theta) = \binom{t+k-1}{k}\theta^k(1+\theta)^{-t-k}$, $t \geq 1$, $\theta \in [0,1]$. For any function ν, $U_t^{(M)}(\nu)(\theta)$ is positive, continuous on $[0,1]$ and provides $U_t^{(M)}(\nu)(0) = \nu(0)$ for all $t \in \mathbb{N}, t \geq 2$ (see in [6], Lemma 4.2.1). In [7], authors showed that the uniform approximation order in the entire class $C_+([0,1])$ of positive continuous functions on $[0,1]$ cannot be developed, so there is a function $\nu \in C_+([0,1])$ that the approximation order by the truncated max-product Baskakov operator is $C\omega_1(\nu, 1/\sqrt{t})$. The fundamentally better order of approximation $\omega_1(\nu, 1/t)$ was attained for some functional subclasses, such as the nondecreasing concave functions. Finally, some shape preserving properties were proved. In this work, by demonstrating that the uniform approximation order of the truncated Baskakov operators of max-product kind is the same as in the specific case of the unit interval, we expand their definition to an arbitrary compact interval. Then it is shown that these operators maintain the quasi-concavity. These solutions show out to be particularly suited in the approximation of fuzzy numbers since they enable us to construct fuzzy numbers with the same support in a straightforward manner.

2. Preliminaries

Definition 2.1. *([10],[22])(**fuzzy numbers**) Let u be a fuzzy subset of* $\mathbb{R}$ *with membership function* $\mu_u(\theta) : \mathbb{R} \longrightarrow [0,1]$. *Then u is called a fuzzy number if:*

(1) u is normal, i.e. $\exists\theta_0 \in \mathbb{R}$ *such that* $\mu_u(\theta_0) = 1$;

(2) u is fuzzy convex, i.e. $\mu_u(\lambda\theta + (1-\lambda)\vartheta) \geq \min\{\mu_u(\theta), \mu_u(\vartheta)\}$;

(3) μ_u *is upper semicontinuous; and*

(4) Let (cl) *be the closure operator .i.e.* $supp(u)$ *is compact and,*
$supp(u) = cl\{x \in \mathbb{R} \mid \mu_u(\theta) > 0\}$. *Then* $supp(u)$ *is bounded.*

For any fuzzy number u there is $t_1, t_2, t_3, t_4 \in \mathbb{R}$ and $l_u, r_u : \mathbb{R} \to [0,1]$ such that we describe a membership function μ_u as follows:

$$\mu_u(\theta)=\begin{cases} 0 & if \quad \theta<t_1, \\ l_u(\theta) & if \quad t_1\le\theta\le t_2, \\ 1 & if \quad t_2\le\theta\le t_3, \\ r_u(\theta) & if \quad t_3\le\theta\le t_4, \\ 0 & if \quad t_4<\theta. \end{cases}$$

Here, the left side of a fuzzy number u is $l_u : [t_1,t_2]\to[0,1]$ and, the right side of a fuzzy number u is $r_u : [t_3,t_4]\to[0,1]$, also l_u is nondecreasing, r_u is nonincreasing.

Another type of fuzzy number representation is the α-cut, often known as the LU parametric representation. Using this, we have that the fuzzy number u is given by a pair of functions (u^-,u^+) where $u^-,u^+ : [0,1]\to\mathbb{R}$ provide the next qualifications:

i. u^- is nondecreasing,
ii. u^+ is nonincreasing,
iii. $u^-(1)\le u^+(1)$.

For $u=(u^-,u^+)$, we have $core(u)=[u^-(1),u^+(1)]$ and $supp(u)=[u^-(0),u^+(0)]$. The following well-known relations provide an important connection between a fuzzy number's membership function and its parametric representation:

$$u^-(\alpha)=\inf\{\theta\in\mathbb{R}:u(\theta)\ge\alpha\},$$

$$u^+(\alpha)=\sup\{\theta\in\mathbb{R}:u(\theta)\ge\alpha\},\alpha\in(0,1],$$

and

$$[u^-(0),u^+(0)]=cl(\{\theta\in\mathbb{R}:u(x)>0\}),$$

where cl denotes the closure operator. Additionally, if u is continuous with $supp(u)=[a,b]$ and $core(u)=[c,d]$, then it can be proved that

$$u(u^-(\alpha))=\alpha,\forall\alpha\in[a,c],$$

$$u(u^+(\alpha))=\alpha,\forall\alpha\in[d,b].$$

Therefore, the end points of the intervals

$$[u]_\alpha=\left[u^-(\alpha),u^+(\alpha)\right],\forall\alpha\in[0,1],$$

determine a fuzzy number $u\in\mathbb{R}_F$. Hence, by considering

$$\left\{\left(u^-(\alpha),u^+(\alpha)\right)\mid 0\le\alpha\le1\right\},$$

we can describe a fuzzy number $u\in\mathbb{R}_F$ and write $u=(u^-,u^+)$.

3. Truncated Baskakov Operators defined on compact intervals

Throughout this paper, we indicate the continuous function space identified on interval I by $C(I)$ and the positive continuous function space identified on interval I by $C_+(I)$. From the result of Weierstrass theorem (see [2]), $P(\theta)$ converges to continuous function $\nu(\theta)$ in the interval $[0,1]$, we just have to move functions from $[0,1]$ to an arbitrary interval $[\eta,\zeta]$. In fact, let the continuous function $g:[0,1]\to\mathbb{R}$ and the function $\nu(\theta)$ is continuouson $[\eta,\zeta]$, we put $g(\vartheta)=\nu(\eta+(\zeta-\eta)\vartheta)$.

If u is a continuous fuzzy number with $supp(u)=[\eta,\zeta]$, $a<b$ and $core(u)=[c,d]$, $c<d$ then we identify

$$\widetilde{U}_t(u)(\theta)=\begin{cases} 0, \ \theta\notin[\eta,\zeta] \\ U_t(u;[\eta,\zeta])=\sum_{k=0}^{t}b_{t,k}(\theta)u\left(\eta+(\zeta-\eta)\frac{k}{t}\right),\theta\in[\eta,\zeta]. \end{cases}$$

where $b_{t,k}(\theta)=\binom{t+k-1}{k}\left(\frac{\theta-\eta}{\zeta-\eta}\right)^k(\frac{\zeta-2\eta+\theta}{\zeta-\eta})^{-t-k}$.

$$U_t^{(M)}(\nu;[\eta,\zeta])(\theta)=\frac{\bigvee_{k=0}^{t} b_{t,k}(\theta)\nu\left(\eta+(\zeta-\eta)\frac{k}{t}\right)}{\bigvee_{k=0}^{t} b_{t,k}(\theta)},\ \theta\in[\eta,\zeta]$$

where $b_{t,k}(\theta)=\binom{t+k-1}{k}\left(\frac{\theta-\eta}{\zeta-\eta}\right)^k(\frac{\zeta-2\eta+\theta}{\zeta-\eta})^{-t-k}$.

It is known that $\sum_{k=0}^{t} b_{t,k}(\theta)=1$ for all $\theta\in[\eta,\zeta]$ so we get that $\bigvee_{k=0}^{t} b_{t,k}(\theta)>0$. Hence, we get that $U_t^{(M)}(\nu;[\eta,\zeta])$ is well defined.

We obtain that for each $\nu\in C_+([\eta,\zeta])$, $U_t^{(M)}(\nu;[\eta,\zeta])\in C_+([\eta,\zeta])$, because the continuous function is the maximum of a limited number of continuous functions. Here, we will show that $U_t^{(M)}:C_+([\eta,\zeta])\to C_+([\eta,\zeta])$ retains the quasi-concavity. Also the uniform approximation order of $U_t^{(M)}$ will be the same as that of the linear Baskakov operator. We first require the results and definitions given below.

Theorem 3.1. i. *([7])Let the function $\nu:[0,1]\to\mathbb{R}_+$ is continuous then we obtain*

$$|\,U_t^{(M)}(\nu;[0,1])(\theta)-\nu(\theta)\,|\le 24\omega_1\left(\nu;\frac{1}{\sqrt{t+1}}\right)_{[0,1]},\ t\in\mathbb{N},\ t\ge 2,\ \theta\in[0,1].$$

ii. *([7]) Let the function $\nu:[0,1]\to\mathbb{R}_+$ is concave then we obtain*

$$|\,U_t^{(M)}(\nu;[0,1])(\theta)-\nu(\theta)\,|\le 2\omega_1\left(\nu;\frac{1}{t}\right)_{[0,1]},\ t\in\mathbb{N},\ \theta\in[0,1].$$

Theorem 3.2. *Let the function $\nu:[0,1]\to\mathbb{R}_+$ and fix $t\in\mathbb{N},\ t\ge 2$. Assume that there exists $c\in[0,1]$ such that ν is nondecreasing on $[0,c]$ and nonincreasing on $[c,1]$. Then, there exists $c'\in[0,1]$ such that $U_t^{(M)}(\nu)$ is nondecreasing on $[0,c']$ and nonincreasing on $[c',1]$. In addition, we have $|\,c-c'\,|\le 1/(t+1)$ and $|\,U_t^{(M)}(\nu)(c)-\nu(c)\,|\le\omega_1(\nu;1/(t+1))$.*

Proof. Let $c\in[0,1]$ and $j_c\in\{0,1,\cdots,t-2\}$ be such that $\left[\frac{j_c}{t-1},\frac{j_c+1}{t-1}\right]$. We investigate the monotonicity on each interval of the form $\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$, $j\in\{0,1,\cdots,t-2\}$. Therefore we can specify the monotoncity of $U_t^{(M)}(\nu)$ on $[0,1]$ using the continuity of $U_t^{(M)}(\nu)$ Let take arbitrary $j\in\{0,1,\cdots,j_c-1\}$ and $\theta\in\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$. By the monotonicity of ν, it follows that $\nu\left(\frac{j}{t}\right)\ge\nu\left(\frac{j-1}{t}\right)\ge\cdots\ge\nu(0)$. From [7](proof of Lemma 3.2) the following claims are true:

$$\text{If } j\le k\le k+1\le t \text{ then} 1\ge m_{k,t,j}(\theta)\ge m_{k+1,t,j}(\theta), \tag{3.1}$$

$$\text{If } 0\le k\le k+1\le j \text{ then} m_{k,t,j}(\theta)\le m_{k+1,t,j}(\theta)\le 1. \tag{3.2}$$

Therefore, it is easily follows that $\nu_{j,t,j}(\theta)\ge\nu_{j-1,t,j}(\theta)\ge\cdots\ge\nu_{0,t,j}(\theta)$. From [7] lemma 3.4, it follows that

$$U_t^{(M)}(\nu)=\bigvee_{k=j}^{t}\nu_{k,t,j}(\theta).$$

Since $U_t^{(M)}(\nu)$ is the maximum of nondecreasing functions, it is nondecreasing on the interval $\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$. $U_t^{(M)}(\nu)$ is continuous, therefore ν is nondecreasing on the interval $\left[0,\frac{j_c}{t-1}\right]$. Let take arbitrary $j\in\{j_c+1,\cdots,t-2\}$ and $\theta\in\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$. By using the monotonicity of ν, it follows that $\nu\left(\frac{i}{t}\right)\ge\nu\left(\frac{i+1}{t}\right)\ge\cdots\ge\nu(1)$. It means that $U_t^{(M)}(\nu)(\theta)=\bigvee_{k=0}^{j}\nu_{k,t,j}(\theta)$ from the assertion (3.1). Thus, it is nonincreasing on $\left[\frac{j}{t-1},\frac{j+1}{t-1}\right]$, because $U_t^{(M)}(\nu)$ is the maximum of nonincreasing functions. Considering the continuity of $U_t^{(M)}(\nu)$, it is immediate that ν is nonincreasing on $\left[\frac{j_c+1}{t-1},1\right]$. Finally let investigate the case when $j=j_c$. If $\frac{j}{t-1}\le c$, then by the monotoncity of ν it follows that $\nu\left(\frac{j_c}{t}\right)\ge\nu\left(\frac{j_c-1}{t}\right)\ge\cdots\ge\nu(0)$. Hence , in the situation we get ν is nondecreasing on $\left[\frac{j_c}{t-1}, \frac{j_c+1}{t-1}\right]$. As a result, ν is nondecreasing on $\left[0, \frac{j_c+1}{t-1}\right]$ and nonincreasing on $\left[\frac{j_c+1}{t-1}, 1\right]$. Also, $c' = \frac{j_c+1}{t-1}$ is the maximum point of $U_t^{(M)}(\nu)$ and It is simple to verify that $\mid c - c' \mid \leq \frac{1}{t-1}$. If $j_c/t \geq c$ then by the monotonicity of ν it follows that$\nu\left(\frac{j_c}{t}\right) \geq \nu\left(\frac{j_c+1}{t}\right) \geq \cdots \geq \nu(1)$. Therefore, we obtain that ν is nonincreasing on $\left[\frac{j_c}{t-1}, \frac{j_c+1}{t-1}\right]$. It follows that ν is nondecreasing on $[0, \frac{j_c}{t-1}]$ and nonincreasing on $[\frac{j_c}{t-1}, 1]$. In addition,$c' = \frac{j_c}{t-1}$ is the maximum point of $U_t^{(M)}(\nu)$ and it is easy to check that $\mid c-c' \mid\leq \frac{1}{t-1}$.

The last part of the theorem is now proved. Let start by noting that for all $\theta \in [0,1]$, $U_t^{(M)}(\nu) \leq \nu(c)$. In fact, the description of $U_t^{(M)}(\nu)$ and the fact that c is global maximum point of ν imply this obvious. It indicates that

$$\begin{aligned} \mid U_t^{(M)}(\nu)(c) - \nu(c) \mid &= \nu(c) - U_t^{(M)}(\nu)(c) = \nu(c) - \bigvee_{k=0}^{t} \nu_{k,t,j_c}(c) \\ &\leq \nu_{j_c,t,j_c}(c) = \nu(c) - \nu\left(\frac{j_c}{t}\right). \end{aligned}$$

Since $c, \frac{j_c}{t} \in \left[\frac{j}{t-1}, \frac{j+1}{t-1}\right]$, we easily get $\nu(c) - \nu\left(\frac{j_c}{t}\right) \leq \omega_1(\nu; 1/t+1)$ and the theorem is proved completely. ◻

Definition 3.1. *([8]) Let $\nu : [\eta,\zeta] \to \mathbb{R}$ be continuous on $[\eta,\zeta]$. The function ν is called:*

i. *quasi-convex if $\nu(\lambda\theta + (1-\lambda)\vartheta) \leq \max\{\nu(\theta), \nu(\vartheta)\}$, for all $\theta, \vartheta \in [\eta,\zeta]$, $\lambda \in [0,1]$,*
ii. *quasi-concave, if $-\nu$ is quasi-convex.*

Remark. *The continuous function ν is quasi-convex on $[a,b]$ according to [9], which is similar to implying there is a point $c \in [\eta, b]$ where ν is nonincreasing on $[\eta, c]$ and nondecreasing on $[c,b]$. It is clear from the definition above that the function ν is quasi-concave on $[\eta,\zeta]$, similarly means that there exists a point $c \in [\eta,\zeta]$ such that ν is nondecreasing on $[\eta, c]$ and nonincreasing on $[c,b]$.*

Now, the main results of this section can be presented.

Theorem 3.3. i. *If $\eta,\zeta \in \mathbb{R}$, $\eta < \zeta$ and $\nu : [\eta,\zeta] \to \mathbb{R}_+$ is continuous then we get*

$$\mid U_t^{(M)}(\nu; [\eta,\zeta])(\theta) - \nu(\theta) \mid\leq 24\left([\zeta-\eta]+1\right)\omega_1\left(\nu; \frac{1}{\sqrt{n+1}}\right)_{[\eta,\zeta]}, \ t \in \mathbb{N},\ t \geq 2,\ \theta \in [\eta,\zeta].$$

ii. *If $\nu : [\eta,\zeta] \to \mathbb{R}_+$ is concave then we get*

$$\mid U_t^{(M)}(\nu; [\eta,\zeta])(\theta) - \nu(\theta) \mid\leq 2\left([\zeta-\eta]+1\right)\omega_1\left(\nu; \frac{1}{t}\right)_{[\eta,\zeta]}, \ t \in \mathbb{N},\ \theta \in [\eta,\zeta].$$

Proof. Consider the continuous function $h(\vartheta)$ on $[0,1]$ as $h(\vartheta) = \nu(\eta + (\zeta-\eta)\vartheta)$. It is simple to verify that $h(\frac{k}{t}) = \nu\left(\eta + k.\frac{(\zeta-\eta)}{t}\right)$ for all $k \in \{0,1,\cdots,t\}$. Let any $\theta \in [\eta,\zeta]$ and $\vartheta \in [0,1]$ such that $\theta = \eta + (\zeta-\eta)\vartheta$. So we have $\vartheta = (\theta-\eta)/(\zeta-\eta)$ and $1+\vartheta = \frac{\zeta+\theta-2\eta}{\zeta-\eta}$. From these equalities and noting the expressions for $h\left(\frac{k}{t}\right)$, we get $U_t^{(M)}(\nu; [\eta,\zeta])(\theta) = U_t^{(M)}(h; [0,1])(\vartheta)$. From Theorem 3.1

$$\mid U_t^{(M)}(\nu; [\eta,\zeta])(\theta) - \nu(\theta) \mid=\mid U_t^{(M)}(h; [0,1])(\vartheta) - h(\vartheta) \mid\leq 24\omega_1\left(h; \frac{1}{\sqrt{t+1}}\right)_{[0,1]}. \quad (3.3)$$

Since $\omega_1\left(h; \frac{1}{\sqrt{t+1}}\right)_{[0,1]} \leq \omega_1\left(\nu; \frac{\zeta-\eta}{\sqrt{t+1}}\right)_{[\eta,\zeta]}$ and the property
$\omega_1(\nu; \lambda\delta)_{[\eta,\zeta]} \leq ([\lambda]+1)\,\omega_1(\nu;\delta)_{[\eta,\zeta]}$, we obtain

$$\omega_1\left(h; \frac{1}{\sqrt{t+1}}\right)_{[0,1]} \leq ([\zeta-\eta]+1)\,\omega_1\left(\nu; \frac{1}{\sqrt{t+1}}\right)_{[\eta,\zeta]}$$

which proves (i).

Using the notation from the above point (i), we get $U_n^{(M)}(\nu;[\eta,\zeta])(\theta) = U_t^{(M)}(h;[0,1])(\vartheta)$, where $h(\vartheta) = \nu(\eta+(\zeta-\eta)\vartheta) = \nu(\theta)$ for all $\vartheta \in [0,1]$. The last equality is equivalent to $\nu(u) = h\left(\frac{u-\eta}{\zeta-\eta}\right)$ for all $u \in [\eta,\zeta]$. By the property of concavity for ν, $\nu(\lambda u_1 + (1-\lambda)u_2) \geq \lambda\nu(u_1) + (1-\lambda)\nu(u_2)$, for all $\lambda \in [0,1]$, $u_1, u_2 \in [\eta,\zeta]$, in terms of h can be written as

$$h\left(\lambda\frac{u_1-\eta}{\zeta-\eta} + (1-\lambda)\frac{u_2-\eta}{\zeta-\eta}\right) \geq \lambda h\left(\frac{u_1-\eta}{\zeta-\eta}\right) + (1-\lambda)h\left(\frac{u_2-\eta}{\zeta-\eta}\right).$$

Denoting $\vartheta_1 = \frac{u_1-\eta}{\zeta-\eta} \in [0,1]$ and $\vartheta_2 = \frac{u_2-\eta}{\zeta-\eta} \in [0,1]$ this immediately implies the concavity of h on $[0,1]$. Then, by Theorem 3.1 (ii), we get

$$\mid U_t^{(M)}(\nu;[\eta,\zeta])(\theta) - \nu(\theta) \mid = \mid U_t^{(M)}(h;[0,1])(\vartheta) - h(\vartheta) \mid \leq 2\omega_1\left(h; \frac{1}{t}\right)_{[0,1]}.$$

□

Theorem 3.4. *Let the function $\nu : [\eta,\zeta] \to \mathbb{R}_+$ and let fix $t \in \mathbb{N}$, $t \geq 1$. Assume that there exists $c \in [\eta,\zeta]$ such that ν is nondecreasing on $[\eta,c]$ and nonincreasing on $[c,\zeta]$. Then, there exists $c' \in [\eta,\zeta]$ such that $U_t^{(M)}(\nu;[\eta,\zeta])$ is nondecreasing on $[\eta,c']$ and nonincreasing on $[c',\zeta]$. Also, we have $\mid c-c' \mid \leq (\zeta-\eta)/(t+1)$ and $\mid U_t^{(M)}(\nu;[\eta,\zeta])(c) - \nu(c) \mid \leq ([\zeta-\eta]+1)\,\omega_1\left(\nu; \frac{1}{\sqrt{t+1}}\right)_{[\eta,\zeta]}$.*

Proof. As in the prior theorem, we establish the function h. Let $c_1 \in [0,1]$ be such that $h(c_1) = c$, we observe that h is nondecreasing on $[0,c_1]$ and nonincreasing on $[c_1,1]$, because it is the composition of ν and the linear nondecreasing function $t \to \eta+(\zeta-\eta)t$. By Theorem 3.2 it results that there exists $c' \in [0,1]$ such that $U_t^{(M)}(h;[0,1])$ is nondecreasing on $[0,c_1']$ and nonincreasing on $[c_1',1]$ and also, we have $\mid U_t^{(M)}(h;[0,1])(c_1) - h(c_1) \mid \leq \omega_1(h; 1/t+1)$ and $\mid c_1 - c_1' \mid \leq 1/(t+1)$. Let $c' = \eta+(\zeta-\eta)c_1'$. If $\theta_1,\theta_2 \in [\eta,c']$ with $\theta_1 \leq \theta_2$ then let $\vartheta_1,\vartheta_2 \in [0,c_1']$ be such that $\theta_1 = \eta+(\zeta-\eta)\vartheta_1$ and $\theta_2 = \eta+(\zeta-\eta)\vartheta_2$. Than it follows that $U_t^{(M)}(\nu;[\eta,\zeta])(\theta_1) = U_t^{(M)}(h;[0,1])(\vartheta_1)$ and $U_t^{(M)}(\nu;[\eta,\zeta])(\theta_2) = U_t^{(M)}(h;[0,1])(\vartheta_2)$. the monotonicity of $U_t^{(M)}(h;[0,1])$ means $U_t^{(M)}(h;[0,1])(\vartheta_1) \leq U_t^{(M)}(h;[0,1])(\vartheta_2)$ that is
$U_t^{(M)}(\nu;[\eta,\zeta])(\theta_1) \leq U_t^{(M)}(\nu;[\eta,\zeta])(\theta_2)$. We obtain that $U_t^{(M)}(\nu;[\eta,\zeta])$ is nondecreasing on $[\eta,c']$. By the same way, we get that $U_t^{(M)}(\nu;[\eta,\zeta])$ is nonincreasing on $[c',\zeta]$. For the remainder of the proof, noting that $\mid c_1-c_1' \mid \leq 1/(t+1)$. we get $\mid c-c' \mid = \mid (\zeta-\eta)(c_1-c_1') \mid \leq (\zeta-\eta)/(t+1)$. In addition, mentioning

$$\mid U_t^{(M)}(h;[0,1])(c_1) - h(c_1) \mid \leq \omega_1\left(h; \frac{1}{t+1}\right)_{[0,1]}$$

and taking into account that $\omega_1\left(h; \frac{1}{t+1}\right)_{[0,1]} \leq ([b-\eta]+1)\,\omega_1\left(\nu; \frac{1}{t+1}\right)_{[\eta,\zeta]}$, we obtain

$$\begin{aligned}\mid U_t^{(M)}(\nu;[\eta,\zeta])(c) - \nu(c) \mid &= \mid U_t^{(M)}(h;[0,1])(c_1) - h(c_1) \mid \\ &\leq \omega_1\left(h; \frac{1}{\sqrt{t+1}}\right)_{[0,1]} \leq ([\zeta-\eta]+1)\,\omega_1\left(\nu; \frac{1}{\sqrt{t+1}}\right)_{[\eta,\zeta]}\end{aligned}$$

and the proof is complete. □

Remark. *The preceding theorem and Remark 3 contribute to the conclusion that* $U_t^{(M)}(\nu)$ *is also quasi-concave if* $\nu : [\eta, \zeta] \to \mathbb{R}_+$ *is continuous and quasi-concave. As we indicated in the Introduction,* $U_t^{(M)}$ *maintains monotonicity and quasi-convexity for functions in the space* $C_+([0,1])$*. It can be demonstrated that these conservation features hold in the general case of the space* $C_+([\eta, \zeta])$ *using reasoning similar to that used in the demonstration of Theorem 3.4.*

4. Applications to fuzzy number approximation

Lemma 4.1. *Let* $\eta, \zeta \in \mathbb{R}$, $\eta < \zeta$*. For* $t \in \mathbb{N}$, $t \geq 2$ *and* $k \in \{0, 1, \cdots, t\}$, $j \in \{0, 1, \cdots, t-2\}$ *and* $\theta \in (\eta + j.(\zeta-\eta)/(t-1), \eta + (j+1).(\zeta-\eta)/(t-1))$.
Let $m_{k,t,j}(\theta) = \frac{b_{t,k}(\theta)}{b_{t,j}(\theta)}$*, where* $b_{t,k}(\theta) = \binom{t+k-1}{k}\left(\frac{\theta-\eta}{\zeta-\eta}\right)^k (\frac{\zeta-2\eta+\theta}{\zeta-\eta})^{-t-k}$*. Then*
$m_{k,t,j}(\theta) \leq 1$.

Proof. We can assume that $\eta = 0$ and $\zeta = 1$, by using the same reasoning as in the proof of Theorems 3.3 and 3.4 we may simply get the conclusion of the lemma in the general case. For fix $\theta \in (j/(t-1), (j+1)/(t-1))$ and from Lemma 3.2 in [7], we obtain

$$m_{0,t,j}(\theta) \leq m_{1,t,j}(\theta) \leq \cdots \leq m_{j,t,j}(\theta),$$

$$m_{j,t,j}(\theta) \geq m_{j+1,t,j}(\theta) \geq \cdots \geq m_{t,t,j}(\theta).$$

From $m_{j,t,j}(\theta) = 1$, it is enough to show that $m_{j+1,t,j}(\theta) < 1$ and $m_{j-1,t,j}(\theta) < 1$. Then, we have

$$\frac{m_{j,t,j}(\theta)}{m_{j+1,t,j}(\theta)} = \frac{j+1}{t+j}\frac{1+\theta}{\theta}.$$

Because the function $g(\vartheta) = (1+\theta)/\theta$ is strictly decreasing on the interval $\left[\frac{j}{t-1}, \frac{(j+1)}{t-1}\right]$, it results that $\frac{1+\theta}{\theta} > \frac{t+j}{j+1}$. Obviously, this suggests $m_{j,t,j}(\theta)/m_{j+1,t,j}(\theta) > 1$ that is $m_{j,t,j}(\theta) < 1$. Similar conclusions lead us to the result that $m_{j-1,t,j}(\theta) < 1$ and we get the proof. □

Now consider a function $\nu \in C_+([\eta, \zeta])$. We may simplify the method to compute $U_t^{(M)}(\nu; [\eta, \zeta])(\theta)$ for some $\theta \in [\eta, \zeta]$ by combining formula(3) with the conclusion of Lemma 4.1. Let take $j \in \{0, 1, ..., t-2\}$ and $\theta \in [\eta+(\zeta-\eta)j/(t-1), \eta+(\zeta-\eta)(j+1)/(t-1)]$. As a result of the features of continuous functions, an immediate result of Lemma 4.1 is that $m_{k,t,j}(\theta) \leq 1$ for all $k \in \{0, 1, ..., t\}$. This means that

$$\bigvee_{k=0}^{t} b_{t,k}(\theta) = b_{t,j}(\theta), \theta \in [\eta + (\zeta-\eta)j/(t-1), \eta + (\zeta-\eta)(j+1)/(t-1)]. \tag{4.1}$$

As a result, for each $k \in \{0, 1, ..., t\}$ and
$\theta \in [\eta + (\zeta-\eta)j/(t-1), \eta + (\zeta-\eta)(j+1)/(t-1)]$

$$\nu_{k,t,j}(\theta) = m_{k,t,j}(\theta).\nu(\eta + (\zeta-\eta)k/t), \tag{4.2}$$

by (3) and (4.1) we obtain

$$U_t^{(M)}(\nu; [\eta, \zeta])(\theta) = \bigvee_{k=0}^{t} \nu_{k,t,j}(\theta),\ \theta \in [\eta + (\zeta-\eta)j/(t-1), \eta + (\zeta-\eta)(j+1)/(t-1)]. \tag{4.3}$$

A similar method from paper [7] that takes into account the particular situation $a = 0, b = 1$ is generalized in the formula above. From Lemma 4.1, for any $k \in \{0, 1, ..., t\}$ and $\theta \in [\eta + (\zeta-\eta)j/(t-1), \eta + (\zeta-\eta)(j+1)/(t-1)]$, we get $\nu_{k,t,j}(\theta) \leq \nu(\eta + (\zeta-\eta)k/t)$.

Lemma 4.2. *Let $\eta, b \in \mathbb{R}, \eta < \zeta$. If $\nu : [\eta, \zeta] \to \mathbb{R}_+$ is bounded then we get $U_t^{(M)}(\nu; [\eta, \zeta])(\eta + j(\zeta - \eta)/(t-1)) \geq \nu(\eta + j(\zeta - \eta)/(t-1))$ for all $j \in \{0, 1, \cdots t-2\}$.*

Proof. From Lemma 4.1, since
$\eta + j(\zeta - \eta)/(t-1) \in (\eta + (\zeta - \eta)j/(t-1), \eta + (\zeta - \eta)(j+1)/(t-1))$ and $m_{k,t,j}(\eta + j(\zeta - \eta)/(t-1)) = \frac{b_{t,k}(\eta + j(\zeta-\eta)/(t-1))}{b_{t,j}(\eta + j(\zeta-\eta)/(t-1))}$ for all $k \in \{0, 1, \cdots, t\}$ it means that

$$\bigvee_{k=0}^{t} b_{t,k}(\eta + j(\zeta - \eta)/t) = b_{t,j}(\eta + j(\zeta - \eta)/t).$$

Then we obtain

$$\begin{aligned} U_t^{(M)}(\nu; [\eta, \zeta])(\eta + j(\zeta - \eta)/t) &= \frac{\bigvee_{k=0}^{t} b_{t,k}(\eta + j(\zeta - \eta)/t)\, \nu(\eta + j(\zeta - \eta)/t)}{b_{t,j}(\eta + j(\zeta - \eta)/t)} \\ &\geq \frac{b_{t,j}(\eta + j(\zeta - \eta)/t)\, \nu(\eta + j(\zeta - \eta)/t)}{b_{t,j}(\eta + j(\zeta - \eta)/t)} \\ &= \nu(\eta + j(\zeta - \eta)/t). \end{aligned}$$

and lemma is proved. □

Theorem 4.3. *Let u be a fuzzy number with $supp(u) = [\eta, b]$ and $core(u) = [c, d]$ such that $\eta \leq c < d \leq \zeta$. Then for sufficiently large t, it result that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$ is a fuzzy number such that :*

i. *$supp(u) = supp(\widetilde{U}_t^{(M)}(u; [\eta, \zeta]))$;*
ii. *if $core(\widetilde{U}_t^{(M)}(u; [\eta, \zeta])) = [c_t, d_t]$, then c_t and d_t can be identified precisely and also we get $|c - c_t| \leq (\zeta - \eta)/t$ and $|d - d_t| \leq (\zeta - \eta)/t$;*
iii. *if u is continuous on $[a, b]$, then*

$$\left|\widetilde{U}_t^{(M)}(u; [\eta, \zeta]) - u(\theta)\right| \leq 6([\zeta - \eta] + 1)\, \omega_1\left(u, \frac{1}{\sqrt{t}}\right)_{[\eta, \zeta]},$$

for all $\theta \in \mathbb{R}$.

Proof. Let $t \in \mathbb{N}$, with the inequality $(\zeta - \eta)/t < d - c$. From Theorem 3.4, there is $c' \in [\eta, \zeta]$ such that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$ is nondecreasing on $[\eta, c']$ and nonincreasing on $[c', \zeta]$. Beside, from the description of $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$, it gives us that
By indicating $\| \ . \ \|$ the uniform norm on $B([\eta, \zeta])$ the space of bounded functions on $[\eta, \zeta]$, $\| \ \widetilde{U}_t^{(M)}(u; [\eta, \zeta]) \ \| \leq \| \ u \ \|$ and since $\| \ u \ \| = 1$, it means that $\| \ \widetilde{U}_t^{(M)}(u) \ \| \leq 1$. Consequently, it is sufficient to demonstrate that $\widetilde{U}_t^{(M)}(u)$ is a fuzzy number in order to obtain existence of $\alpha \in [\eta, \zeta]$ such that $\widetilde{U}_t^{(M)}(u)(\alpha) = 1$. Let $\alpha = \eta + j(\zeta - \eta)/n$ where j is choosen with the property that $c < \alpha < d$. Such j exists as $(\zeta - \eta)/t < d - c$. Since $\alpha \in core(u)$, it results $u(\alpha) = 1$. Also,from Lemma 4.2, we can write that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])(\alpha) \geq u(\alpha)$ and obviously this means that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$ is a fuzzy number. Since we have $F_t^{(M)}(u; [\eta, \zeta])(\eta) = u(\eta)$, $U_t^{(M)}(u; [\eta, \zeta])(b) = u(b)$ and the description of u and $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])$, it follows that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])(\theta) = 0$ outside $[\eta, \zeta]$. Now, by $u(\theta) > 0$ and $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])(\theta) = U_t^{(M)}(u; [\eta, \zeta])(\theta)$ for all $\theta \in (\eta, \zeta)$, we can obtain that $\widetilde{U}_t^{(M)}(u; [\eta, \zeta])(\theta) > 0$ for all $\theta \in (\eta, \zeta)$ which proves (i).

Now, let $t \in \mathbb{N}$ with $(\zeta - \eta)/t \leq d - c$. Then take $k(t, c), k(t, d) \in \{1, \cdots, t-1\}$ be with the property that $\eta + (\zeta - \eta)(k(t, d) - 1)/t < c \leq \eta + (\zeta - \eta)k(t, c)/n$ and $\eta + (\zeta - \eta)k(t, c)/t \leq d < \eta + (\zeta - \eta)(k(t, d) + 1)/n$. Since $(\zeta - \eta)/t \leq d - c$ it is obvious that $k(t, c) \leq k(t, d)$. Also, $k(t, c)$ and $k(t, d)$ were chosen, we get that $u(\eta + (\zeta - \eta)k/t) = 1$ for any

$k \in \{k(t,c), \cdots, k(t,d)\}$ and $u(\eta + (\zeta-\eta)k/t) < 1$ for any $k \in \{0, \cdots, t\} \setminus \{k(t,c), \cdots, k(t,d)\}$.
For some $\theta \in [\eta + k(t,c)(\zeta-\eta)/t, \eta + (k(t,c)+1)(\zeta-\eta)/t]$, we have

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) = \bigvee_{k=0}^{n} u_{k,t,k(t,c)}(\theta).$$

We have

$$u_{k(t,c),t,k(t,c)}(\theta) = m_{k(t,c),t,k(t,c)}(\zeta)u(\eta + (\zeta-\eta)k(t,c)/n) = u(\eta + (\zeta-\eta)k(t,c)/t) = 1$$

and from the description of $k(t,c)$ and by Lemma 4.2,so for any $k \in \{0, \cdots, t\}$, we get

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) = u(\eta + (\zeta-\eta)k(t,c)/t) = 1,$$

$\forall\, \theta \in [\eta + k(t,c)(\zeta-\eta)/t, \eta + (k(t,c)+1)(\zeta-\eta)/t]$. Similarly we obtain that

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) = u(\eta + (\zeta-\eta)k(t,d)/t) = 1,$$

$\forall\, \theta \in [\eta + k(t,d)(\zeta-\eta)/t, \eta + (k(t,d)+1)(\zeta-\eta)/t]$. Let take arbitrarily

$$\theta \in (\eta + (k(t,c)-1)(\zeta-\eta)/t, \eta + k(t,c)(\zeta-\eta)/t),$$

we have

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) = \bigvee_{k=0}^{n} u_{k,t,k(t,c)-1}(\theta).$$

If $k \in \{k(t,c), \cdots, k(t,d)\}$, then we get

$$\begin{aligned} u_{k,t,k(t,c)-1}(\theta) =& m_{k,t,k(t,c)-1}(\theta)u(\eta + (\zeta-\eta)k/t) < u(\eta + (\zeta-\eta)k/t) \\ &= u(\eta + (\zeta-\eta)k(t,c)/n) = \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta + (\zeta-\eta)k(t,c)/t). \end{aligned}$$

Let $k \notin \{k(t,c), \cdots, k(t,d)\}$, then we get

$$\begin{aligned} u_{k,t,k(t,c)-1}(\theta) =& m_{k,t,k(t,c)-1}(\theta)u(\eta + (\zeta-\eta)k/t) \le u(\eta + (\zeta-\eta)k/t) \\ &< u(\eta + (\zeta-\eta)k(t,c)/t) = \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta + (\zeta-\eta)k(t,c)/t). \end{aligned}$$

From the propertiy of quasi-concavity of $\widetilde{U}_t^{(M)}(u;[\eta,\zeta])$ on $[\eta,\zeta]$ it easily results that

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) < \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta + (\zeta-\eta)k(t,c)/t),\ \forall\theta \in [\eta, \eta + k(t,c)(\zeta-\eta)/t)].$$

Similarly, we get

$$\widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\theta) < \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta + (\zeta-\eta)(k(t,d)+1)/t),$$

$\forall\, \theta \in [\eta + (k(t,d)+1)(\zeta-\eta)/t), \zeta]$. From the previous inequalities

$$\begin{aligned} \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta + (\zeta-\eta)k(t,c)/t) =& u(\eta + (\zeta-\eta)k(t,c)/t) = u(\eta + (\zeta-\eta)k(t,d)/t) \\ =& \widetilde{U}_t^{(M)}(u;[\eta,\zeta])(\eta + (\zeta-\eta)(k(t,d)+1)/t) = 1, \end{aligned}$$

we get that $\widetilde{U}_t^{(M)}(u;[\eta,\zeta])$ reaches its maximum value only in the range $[\eta+(\zeta-\eta)k(t,c)/t, \eta+(\zeta-\eta)(k(t,d)+1)/t]$ which by the description of $\widetilde{U}_t^{(M)}(u;[\eta,\zeta])$ implies that $core\left(\widetilde{U}_t^{(M)}(u;[\eta,\zeta])\right) = [\eta + (\zeta-\eta)k(t,c)/t, \eta + (\zeta-\eta)(k(t,d)+1)/t]$. Then, indicating $c_t = \eta + (b-\eta)k(t,c)/n$ one can see that both c_t and c belong to the interval $[\eta+(\zeta-\eta)(k(t,c)-1)/t, \eta+(\zeta-\eta)k(t,c)/t]$ of length $(\zeta-\eta)/n$ and so $|c-c_t| \le (\zeta-\eta)/t$. Correlatively, indicating $d_t = \eta + (\zeta-\eta)(k(t,d) + 1)/(t+1)$ we get that $|d - dn| \le (\zeta-\eta)/t$ and the statement (ii) has been proven.

(iii) From Theorem 3.3, the proof is immediate.

□

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FIXED POINT THEOREMS FOR CONTRAVARIANT MAPS IN BIPOLAR $b-$METRIC SPACES WITH INTEGRATION APPLICATION

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ABSTRACT. As a natural extension of the metric and the bipolar metric, this article introduces the new abstract bipolar $b-$ metric. The bipolar $b-$metric is a novel technique addressed in this article; it is explained by combining the well-known $b-$metric in the theory of metric spaces, as defined by Mutlu and Gürdal (2016) [10], with the description of the bipolar metric. In this new definition, well-known mathematical terms such as Cauchy and convergent sequences are utilized. In the bipolar $b-$metric, fundamental topological concepts are also defined to investigate the existence of fixed points implicated in such mappings under different contraction conditions. An example is provided to demonstrate the presented results.

1. INTRODUCTION

Fixed point theory is a fabulous blend of Topology, Analysis, and Geometry. It has been a crucial part of exploring linear and nonlinear phenomena. There are also excellent applications of fixed-point theorems to encourage mathematical inquiry, economics, game theory, computer science, and digital image embedding. Thanks to its application in mathematics and other disciplines, the Banach ([4]) contraction principle has become widely used. Two situations enable this principle to become widespread. Either the mapping space is universalized, or the map's contraction condition is extended.

One of the exciting topics of the last few decades is the theory of fixed points. In particular, the issue of changing the abstract structure of the mapping to form a fixed point has been intensively studied. The concept of a metric space has been variously revised, expanded, and generalized to ensure the existence of a fixed point for certain mappings

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defined in these new constructs. The most exciting and general concept is the $b-$metric space. Several mathematicians have considered it by different names (such as the quasi-metric [5] and the general metric), but it became famous for the publications of Bakhtin [3] and Czerwik [6].

It is exciting to get fixed point theorems for covariant and contravariant maps with different contraction maps in both expanding and non-expanding topological spaces. There are unique bivariate metric spaces like $b-$metric spaces [3, 6], extended $b-$metric spaces, and trivariate metric-type spaces like bipolar metric spaces [10].

In [10] the new distance function, the distance between the members of two different sets is different from the empty set. A successful description of generalized and improved metric spaces is called bipolar metric spaces. This study also validated new versions of Banach and Kannan Caristi's fixed point theorems (see [14]).

Recent articles on bipolar metric space refer to popular theorems of fixed point theory contained in them (see [14] and [15]). In addition, various issues related to this theory are covered (see [1, 2, 7, 11, 12, 8, 13, 16, 17]).

This study aims to combine the bipolar metric space defined in [10] 2016 with the $b-$metric space definition, which is a new approach for general metric spaces. This article discusses the existence of and gives examples of some fixed point theorems in the bipolar $b-$metric.

2. Preliminaries

In this section, the definition and theorem that will be required for the analysis will be reminded again for convenience.

Recall (see, e.g., [3, 6]) that a $b-$metric d on a set X is a generalization of standard metric, where the triangular inequality is replaced by

$$d(x,z) \leq b[d(x,y) + d(y,z)],$$

for all $x, y, z \in X$, for some fixed $b \geq 1$.

Definition 2.1. *([10]). A bipolar metric space is a triple (X, Y, d) such that $X, Y \neq \emptyset$ and $d : X \times Y \longrightarrow \mathbb{R}^+$ is a function satisfying the following conditions:*

(i) $d(x,y) = 0 \Longleftrightarrow x = y$,

(ii) if $x, y \in X \cap Y$, then $d(x,y) = d(y,x)$,

(iii) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$

for all $(x,y), (x_1,y_1), (x_2,y_2) \in X \times Y$. Then d is called a bipolar metric on the pair (X, Y).

Example 2.1. *Let $X = \{(a, 2a) | \ a \in \mathbb{R}\}$, $Y = \{(d, c) | \ d, c \in \mathbb{R}\} = \mathbb{R}^2$ and*

$$d(x,y) = \mid a - d \mid + \mid 2a - c \mid,$$

for every $x = (a, 2a) \in X$ and $y = (d, c) \in Y$. Obviously $X \cap Y = X$ and conditions (i) and (ii) of Definition 2.1 are satisfied.

For each $x = (a, 2a), x' = (a', 2a') \in X$ and $y = (d, c), y' = (d', c') \in Y$, we have

$$\begin{aligned} d(x,y) &= \mid a - d \mid + \mid 2a - c \mid \\ &\leq \mid a - d' \mid + \mid 2a - c' \mid + \mid a' - d' \mid + \mid 2a' - c' \mid + \mid a' - d \mid + \mid 2a' - c \mid \\ &= d(x,y') + d(x',y') + d(x',y). \end{aligned}$$

So, condition (iii) of Definition 2.1 is also satisfied and d is a bipolar metric.

Definition 2.2. *([10]). Let (X_1, Y_1) and (X_2, Y_2) be pairs of sets.*

(a) *Let $f : X_1 \cup Y_1 \longrightarrow X_2 \cup Y_2$ be a given function.*

If $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$, f is said a covariant map from (X_1, Y_1) to (X_2, Y_2) and write $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$.

(b) *$f : X_1 \cup Y_1 \longrightarrow X_2 \cup Y_2$ be a given function.*
If $f(X_1) \subseteq Y_2$, and $f(Y_1) \subseteq X_2$, f is said a contravariant map from (X_1, Y_1) to (X_2, Y_2) and $f : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$ is written in this paper.

Example 2.2. *Let $X = \{(a, 0)|\ a \in \mathbb{R}\}$, $Y = \{(b, c)|\ b, c \in \mathbb{R}\} = \mathbb{R}^2$ and $f : X \cup Y \longrightarrow X \cup Y$ defined by $f(x, y) = (x^2, xy)$ for every $(x, y) \in X \cup Y$. Obviously $f(X) \subseteq X$ and $f(Y) \subseteq Y$. Therefore, f is a covariant map from (X, Y) to (X, Y), that is $f : (X, Y) \rightrightarrows (X, Y)$.*

It is superimposed on the $b-$metric with the bipolar metric, just as different previously defined metrics are combined in one definition. An example of these is the metric structure in defining bipolar and ultrametric,and the description presented is the definition of bipolar $b-$metric.

3. MAIN RESULTS

It is superimposed on the $b-$metric with the bipolar metric, just as different previously defined metrics are combined in one definition. An example of these is the metric structure in defining bipolar and ultrametric [7], and the description presented is the definition of bipolar $b-$metric.

Definition 3.1. *A bipolar $b-$metric space is a triple (X, Y, d) such that $X, Y \neq \emptyset$ and $d : X \times Y \longrightarrow \mathbb{R}^+$ is a function satisfying the following conditions:*
(i) $d(x, y) = 0 \Longleftrightarrow x = y$,
(ii) if $x, y \in X \cap Y$, then $d(x, y) = d(y, x)$,
(iii) $d(x_1, y_2) \leq b[d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)]$ for all $(x, y), (x_1, y_1), (x_2, y_2) \in X \times Y$ and $b \geq 1$. We say d is a bipolar $b-$metric on the pair (X, Y).

Example 3.1. *Let $X = \{(a, 0)|\ a \in \mathbb{R}\}$, $Y = \{(d, c)|\ d, c \in \mathbb{R}\} = \mathbb{R}^2$ and*

$$d(x, y) = (a - d)^2 + \mid c \mid,$$

for every $x = (a, 0) \in X$ and $y = (d, c) \in Y$. Obviously $X \cap Y = X$ and conditions (i) and (ii) of Definition 3.1 are satisfied.
For each $x = (a, 0), x' = (a', 0) \in X$ and $y = (d, c), y' = (d', c') \in Y$ and $b = 3$, we have

$$\begin{aligned} d(x, y) &= (a - d)^2 + \mid c \mid = [(a - d') + (d' - a') + (a' - d)]^2 + \mid c \mid \\ &\leq 3(a - d')^2 + \mid c' \mid + 3(a' - d')^2 + \mid c' \mid + 3(a' - d)^2 + \mid c \mid \\ &= 3[d(x, y') + d(x', y') + d(x', y)]. \end{aligned}$$

So, condition (iii) of Definition 3.1 is also satisfied and d is a bipolar $b-$metric for $b = 3$.

It is useful to state the following here. If $b = 1$ in a bipolar $b-$metric, a bipolar metric is obtained (see [10]).
It should be noted that in the preceding example, if (X, Y, d) is a bipolar $b-$metric space, then (X, Y, d) is not necessarily a bipolar metric space, because the triangle inequality does not hold.

Let Ψ denote a family of mappings such that for each $\psi \in \Psi$,
$\psi : [0, \infty) \longrightarrow (0, \infty)$ and

(1) $\psi(t)$ is continuous and it is decreasing for every $t \in [0, \infty)$,

(2) $\int_0^{ms}\psi(t)dt \le m\int_0^s \psi(t)dt$ for every $s>0$ and $m\ge 1$.

For example, if $\psi:[0,\infty)\longrightarrow(0,\infty)$ defined by $\psi(t)=e^{-t}$, $\psi(t)=\frac{1}{1+t}$, then it is easy to see that $\psi\in\Psi$.

Example 3.2. *Let (X,Y,d) be a bipolar $b-$metric space. If it is defined with*

$$\rho(x,y)=\int_0^{d(x,y)}\psi(t)dt,\ \ for\ every\ (x,y)\in X\times Y\ and\ \psi\in\Psi,$$

then (X,Y,ρ) is a bipolar $b-$metric space.

Proof. Obviously conditions (i) and (ii) of Definition 3.1 are satisfied. Now, since d is bipolar $b-$metric hence for all $(x,y),(x_1,y_1)\in X\times Y$ and $b\ge 1$, we have $d(x,y)\le b[d(x,y_1)+d(x_1,y_1)+d(x_1,y)]$. Since ψ is positive we get:

$$\begin{aligned}
\rho(x,y) &= \int_0^{d(x,y)}\psi(t)dt\\
&\le \int_0^{b[d(x,y_1)+d(x_1,y_1)+d(x_1,y)]}\psi(t)dt\\
&= \int_0^{b\,d(x,y_1)}\psi(t)dt+\int_{b\,d(x,y_1)}^{b[d(x,y_1)+d(x_1,y_1)]}\psi(t)dt\\
&+ \int_{b[d(x,y_1)+d(x_1,y_1)]}^{b[d(x,y_1)+d(x_1,y_1)+d(x_1,y)]}\psi(t)dt.
\end{aligned}$$

If set $t=b\,d(x,y_1)+s$, since ψ is decreasing then we get:

$$\begin{aligned}
\int_{b\,d(x,y_1)}^{b[d(x,y_1)+d(x_1,y_1)]}\psi(t)dt &= \int_0^{b\,d(x_1,y_1)}\psi(b\,d(x,y_1)+s)ds \le \int_0^{b\,d(x_1,y_1)}\psi(s)ds\\
&\le b\int_0^{d(x_1,y_1)}\psi(s)ds.
\end{aligned}$$

Similarly, if set $t=b[d(x,y_1)+d(x_1,y_1)]+s$, then

$$\begin{aligned}
\int_{b[d(x,y_1)+d(x_1,y_1)]}^{b[d(x,y_1)+d(x_1,y_1)+d(x_1,y)]}\psi(t)dt &= \int_0^{b\,d(x_1,y)}\psi(b[d(x,y_1)+d(x_1,y_1)]+s)ds\\
&\le \int_0^{b\,d(x_1,y)}\psi(s)ds \le b\int_0^{d(x_1,y)}\psi(s)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho(x,y) &= \int_0^{d(x,y)}\psi(t)dt\\
&\le \int_0^{b\,d(x,y_1)}\psi(t)dt+\int_{b\,d(x,y_1)}^{b[d(x,y_1)+d(x_1,y_1)]}\psi(t)dt\\
&+ \int_{b[d(x,y_1)+d(x_1,y_1)]}^{b[d(x,y_1)+d(x_1,y_1)+d(x_1,y)]}\psi(t)dt\\
&\le b\int_0^{d(x,y_1)}\psi(t)dt+b\int_0^{d(x_1,y_1)}\psi(t)dt+b\int_0^{d(x_1,y)}\psi(t)dt\\
&= b[\rho(x,y_1)+\rho(x_1,y_1)+\rho(x_1,y)].
\end{aligned}$$

So, condition (iii) of Definition 3.1 is also satisfied and ρ is a bipolar $b-$metric. $\square$

Remark. *If d_1, d_2 are bipolar $b-$metrics on (X_1, Y_1) and (X_2, Y_2), respectively, we shall sometimes write $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ and $f : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$.*

Definition 3.2. *Let (X, Y, d) be a bipolar $b-$metric space*

(a) The set X is called the left pole, Y is called the right pole and $X \cap Y$ is called the center of (X, Y, d). Especially, the points in the left pole are called left points, the points in the right pole are called right points, and the points in the center are called central points.

(b) A sequence $\{x_n\} \subseteq X$ is called a left sequence, and a sequence $\{y_n\} \subseteq Y$ is called a right sequence. In a bipolar $b-$metric space, a left or right sequence is simply called a sequence.

(c) A sequence $\{u_n\}$ is said to be convergent to a point u, if and only if $\{u_n\}$ is a left sequence, u is a right point and $\lim\limits_{n\longrightarrow\infty} d(u_n, u) = 0$, or $\{u_n\}$ is a right sequence, u is a left point and $\lim\limits_{n\longrightarrow\infty} d(u, u_n) = 0$.

(d) A bi-sequence $\{(x_n, y_n)\}$ on (X, Y, d) is a sequence on the set $X \times Y$. If the sequence $\{x_n\}$ and $\{y_n\}$ are convergent, then the bi-sequence $\{(x_n, y_n)\}$ is said to be convergent, and if $\{x_n\}$ and $\{y_n\}$ converge to a common fixed point, then $\{(x_n, y_n)\}$ is said to be bi-convergent.

(e) $\{(x_n, y_n)\}$ is called a Cauchy bi-sequence if $\lim\limits_{n,m\longrightarrow\infty} d(x_n, y_m) = 0$.

(f) A bipolar $b-$metric space is called complete, if every Cauchy bi-sequence is convergent, hence bi-convergent.

Definition 3.3. *Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar $b-$metric spaces*

(a) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left-continuous at a point $x_0 \in X_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x_0, y) < \delta$ implies $d_2(fx_0, fy) < \varepsilon$ for all $y \in Y_1$.

(b) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right-continuous at a point $y_0 \in Y_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x, y_0) < \delta$ implies $d_2(fx, fy_0) < \varepsilon$ for all $x \in X_1$.

(c) A map f is called continuous if it is left-continuous at each point $x \in X_1$ and right-continuous at each point $y \in Y_1$.

(d) A contravariant map $f : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$ is continuous if and only if it is continuous as a covariant map $f : (X_1, Y_1) \rightrightarrows (Y_2, X_2)$.

Remark. *([10]). A covariant or contravariant map f from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) is continuous if and only if $\{u_n\} \longrightarrow v$ on (X_1, Y_1, d_1) implies $\{f(u_n)\} \longrightarrow f(v)$ on (X_2, Y_2, d_2).*

In bipolar $b-$metric space we have the following Lemma.

Lemma 3.1. *Let (X, Y, d) be a bipolar $b-$metric space with $b \geq 1$, and suppose that $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ are convergent to y, x respectively, where $y \in Y$ and $x \in X$. Then we have*

$$\frac{1}{b} d(x, y) \leq \liminf_{n\longrightarrow\infty} d(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} d(x_n, y_n) \leq b d(x, y).$$

In particular if $b = 1$, then we have $\lim\limits_{n\longrightarrow\infty} d(x_n, y_n) = d(x, y)$.

Proof. By property (iii) of Definition 3.1, we have

$$d(x, y) \quad \leq \quad b[d(x, y_n) + d(x_n, y_n) + d(x_n, y)].$$

Taking the lower limit as $n \to \infty$ we obtain

$$\frac{1}{b}d(x,y) \leq \liminf_{n\longrightarrow\infty} d(x_n,y_n).$$

On the other hand

$$d(x_n,y_n) \leq b[d(x_n,y)+d(x,y)+d(x,y_n)].$$

And taking the upper limit as $n \to \infty$ we obtain

$$\limsup_{n\longrightarrow\infty} d(x_n,y_n) \leq bd(x,y).$$

So we obtain the desired result. □

Definition 3.4. *Let (X,Y,d) be a bipolar $b-$metric space and assume that $f,g:(X,Y,d) \rightrightarrows (X,Y,d)$. Then the pair $\{f,g\}$ is said to be compatible if and only if $\lim_{n\longrightarrow\infty} d(gfx_n, fgy_n) = 0$, whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in X and Y respectively, such that $\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gy_n = t$ for some $t \in X\cap Y$.*

Lemma 3.2. *Let (X,Y,d) be a bipolar $b-$metric space and $f,g:(X,Y,d) \rightrightarrows (X,Y,d)$ such that the pair $\{f,g\}$ be compatible and g is continuous. Suppose that $\{x_n\}\subseteq X$ and $\{y_n\}\subseteq Y$ such that $\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gy_n = u$ for some $u \in X\cap Y$. Then $\lim_{n\longrightarrow\infty} fgy_n = gu$.*

Proof. Since f and g are compatible, hence $\lim_{n\longrightarrow\infty} d(gfx_n, fgy_n) = 0$. Since g is continuous it follows that

$$\lim_{n\to\infty} gfx_n = gu.$$

By property (iii) of Definition 3.1, we have

$$d(gu, fgy_n) \leq b[d(gu,gu)+d(gfx_n,gu)+d(gfx_n,fgy_n)].$$

Taking the limit as $n \to \infty$ we obtain

$$\lim_{n\longrightarrow\infty} d(gu,fgy_n) \leq b[\lim_{n\longrightarrow\infty} d(gu,gu)+\lim_{n\longrightarrow\infty} d(gfx_n,gu)+\lim_{n\longrightarrow\infty} d(gfx_n,fgy_n)] = 0.$$

Therefore, $\lim_{n\longrightarrow\infty} d(gu,fgy_n) = 0$, so we obtain the desired result. □

In this section, we first express and prove some different extensions and generalizations of the Banach contraction principle [4] on bipolar $b-$metric spaces.

Theorem 3.3. *Let (X,Y,d) be a complete bipolar $b-$metric space and $f,g:(X,Y,d) \rightrightarrows (X,Y,d)$ with:*

(i) *$f(X)\subseteq g(X)$, $f(Y)\subseteq g(Y)$ and g is continuous,*

(ii)

$$d(f(x),f(y)) \leq \frac{\lambda}{b^2} d(g(x),g(y)),$$

for all $(x,y)\in X\times Y$ and $0<\lambda<b$,

(iii) *the pair (f,g) is compatible.*

Then the functions $f,g: X\cup Y \longrightarrow X\cup Y$ have a unique common fixed point in $X\cap Y$. There exists a unique point $u \in X\cap Y$ such that $f(u)=g(u)=u$.

Proof. Let $x_0 \in X$ and $y_0 \in Y$. For each $n \in \mathbb{N}$, define $f(x_n) = g(x_{n+1}) = a_n$ and $f(y_n) = g(y_{n+1}) = b_n$. Then $\{(a_n, b_n)\}$ is a bisequence on (X, Y, d). For each positive integer n, we have

$$\begin{aligned}
d(a_n, b_n) &= d(f(x_n), f(y_n)) \\
&\leq \frac{\lambda}{b^2} d(g(x_n), g(y_n)) = \frac{\lambda}{b^2} d(a_{n-1}, b_{n-1}) = \frac{\lambda}{b^2} d(f(x_{n-1}), f(y_{n-1})) \\
&\leq \frac{\lambda^2}{b^4} d(g(x_{n-1}), g(y_{n-1})) = \frac{\lambda^2}{b^4} d(f(x_{n-2}), f(y_{n-2})) \\
&\vdots \\
&\leq \frac{\lambda^n}{b^{2n}} d(a_0, b_0).
\end{aligned}$$

Also,

$$\begin{aligned}
d(a_n, b_{n+1}) &= d(f(x_n), f(y_{n+1})) \\
&\leq \frac{\lambda}{b^2} d(g(x_n), g(y_{n+1})) = \frac{\lambda}{b^2} d(a_{n-1}, b_n) = \frac{\lambda}{b^2} d(f(x_{n-1}), f(y_n)) \\
&\leq \frac{\lambda^2}{b^4} d(g(x_{n-1}), g(y_n)) = \frac{\lambda^2}{b^4} d(f(x_{n-2}), f(y_{n-1})) \\
&\vdots \\
&\leq \frac{\lambda^n}{b^{2n}} d(a_0, b_1).
\end{aligned}$$

Hence for $m \geq n$ we get

$$\begin{aligned}
d(a_m, b_n) &\leq b[d(a_m, b_{n+1}) + d(a_n, b_{n+1}) + d(a_n, b_n)] \\
&\leq bd(a_m, b_{n+1}) + \frac{\lambda^n}{b^{2n-1}} [d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Also,

$$\begin{aligned}
d(a_m, b_{n+1}) &\leq b[d(a_m, b_{n+2}) + d(a_{n+1}, b_{n+2}) + d(a_{n+1}, b_{n+1})] \\
&\leq bd(a_m, b_{n+2}) + \frac{\lambda^{n+1}}{b^{2n+1}} [d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(a_m, b_{m-1}) &\leq b[d(a_m, b_m) + d(a_{m-1}, b_m) + d(a_{m-1}, b_{m-1})] \\
&\leq bd(a_m, b_m) + \frac{\lambda^{m-1}}{b^{2m-3}} [d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Therefore, if set $d(a_0, b_1) + d(a_0, b_0) = \alpha$ then we have:

$$\begin{aligned}
d(a_m, b_n) &\leq bd(a_m, b_{n+1}) + \frac{\lambda^n}{b^{2n-1}} \alpha \\
&\leq b^2 d(a_m, b_{n+2}) + \frac{\lambda^n}{b^{2n-1}} \alpha + \frac{\lambda^{n+1}}{b^{2n+1}} \alpha \\
&\vdots \\
&\leq b^{m-n} d(a_m, b_m) + \frac{\lambda^n}{b^{2n-1}} \alpha + \frac{\lambda^{n+1}}{b^{2n+1}} \alpha + \cdots + \frac{\lambda^{m-1}}{b^{2m-3}} \alpha \\
&\leq b^{m-n} \frac{\lambda^m}{b^{2m}} \alpha + \frac{\lambda^n}{b^{2n-1}} \alpha + \frac{\lambda^{n+1}}{b^{2n+1}} \alpha + \cdots + \frac{\lambda^{m-1}}{b^{2m-3}} \alpha
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda^n}{b^{2n-1}}\alpha + \frac{\lambda^{n+1}}{b^{2n+1}}\alpha + \cdots + \frac{\lambda^{m-1}}{b^{2m-3}}\alpha + \frac{\lambda^m}{b^{m+n}}\alpha \\
&\leq (\frac{\lambda}{b})^n\alpha + (\frac{\lambda}{b})^{n+1}\alpha + \cdots + (\frac{\lambda}{b})^{m-1}\alpha + (\frac{\lambda}{b})^m\alpha \\
&\leq \frac{(\frac{\lambda}{b})^n\alpha}{1-\frac{\lambda}{b}} \longrightarrow 0.
\end{aligned}$$

Therefore, (a_n, b_n) is a Cauchy bisequence. Since (X, Y, d) is complete, (a_n, b_n) converges, and thus biconverges to a point $u \in X \cap Y$ and

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_{n+1}) = \lim_{n\to\infty} a_n = u,$$

and

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} g(y_{n+1}) = \lim_{n\to\infty} b_n = u.$$

We show that u is a common fixed point of f and g.

Let g be continuous it follows that

$$\lim_{n\to\infty} gf(x_n) = g(u), \qquad \lim_{n\to\infty} gg(x_n) = g(u).$$

Since f and g are compatible, so by Lemma 3.2 $\lim_{n\longrightarrow\infty} fg(y_n) = g(u)$. Putting $x = gx_n$ and $y = u$ in inequality (ii) of Theorem 3.3 we obtain

$$d(fg(x_n), f(u)) \leq \frac{\lambda}{b^2} d(gg(x_n), g(u)). \tag{3.1}$$

Now, by taking the upper limit when $n \to \infty$ in (3.1) and using Lemma 3.1 we get

$$\begin{aligned}
\frac{1}{b}d(g(u), f(u)) &\leq \limsup_{n\longrightarrow\infty} d(fg(x_n), g(u)) \\
&\leq \frac{\lambda}{b^2} \limsup_{n\longrightarrow\infty} d(gg(x_n), g(u)) \\
&= \frac{\lambda}{b^2} bd(g(u), g(u)) = 0.
\end{aligned}$$

Consequently $d(g(u), f(u)) = 0$, it follows that $f(u) = g(u)$. Now, we show that $f(u) = u$. Putting $x = u$ and $y = y_n$ in inequality (ii) of Theorem 3.3 we obtain

$$d(f(u), f(y_n)) \leq \frac{\lambda}{b^2} d(g(u), g(y_n)). \tag{3.2}$$

Similarly by taking the upper limit when $n \to \infty$ in (3.2) and using Lemma 3.1 we obtain

$$\begin{aligned}
\frac{1}{b}d(f(u), u) &\leq \limsup_{n\longrightarrow\infty} d(f(u), f(y_n)) \leq \frac{\lambda}{b^2} \limsup_{n\longrightarrow\infty} d(g(u), g(y_n)) \\
&\leq \frac{\lambda}{b^2} bd(g(u), u)) = \frac{\lambda}{b} d(f(u), u)) \\
&< \frac{1}{b} d(f(u), u),
\end{aligned}$$

it follows that $g(u) = f(u) = u$. If there exists another common fixed point v in $X \cap Y$ of f and g, then

$$\begin{aligned}
d(u, v) &= d(f(u), f(v)) \leq \frac{\lambda}{b^2} d(g(u), g(v)) = \frac{\lambda}{b^2} d(u, v) \\
&< d(u, v),
\end{aligned}$$

which implies that $d(u,v) = 0$ and $u = v$. Thus u is a unique common fixed point of f and g. The proof of the theorem is completed. □

Now we give an example to support our result.

Example 3.3. *Let* $X = \{(a,0) \mid a \in \mathbb{R}\}$ *and* $Y = \{(d,c) \mid d \in \mathbb{R},\ c \in [0,\infty)\}$ *be endowed with bipolar* $b-$*metric* $d(x,y) = (a-d)^2 + c$, *where* $x = (a,0) \in X$ *and* $y = (d,c) \in Y$. *By Example 3.1,* (X,Y,d) *is a bipolar* $b-$*metric for* $b = 3$. *For every* $(x,y) \in X \cup Y$, *define* $f,g : X \cup Y \longrightarrow X \cup Y$ *by* $f(x,y) = \frac{1}{3}(\sin(x), \ln(1+\frac{y}{3}))$ *and* $g(x,y) = (x, y^2 + y)$. $f,g : (X,Y) \rightrightarrows (X,Y)$, *that is* f *and* g *are two covariant maps from* (X,Y) *to* (X,Y). *It is easy to see that the pairs* $\{f,g\}$ *are compatible mappings.*

Also for each $x \in X$ *and* $y \in Y$ *we have*

$$\begin{aligned} d(fx,fy) &= d(f(a,0), f(d,c)) \\ &= \frac{1}{9}(\sin(a) - \sin(d))^2 + \frac{1}{3}\ln(1+\frac{c}{3}) \\ &\leq \frac{1}{9}(\sin(a) - \sin(d))^2 + \frac{1}{9}c \\ &\leq \frac{1}{9}(a-d)^2 + \frac{1}{9}(c^2 + c) \\ &= \frac{1}{9}d(g(a,0), g(d,c)) = \frac{1}{9}d(gx,gy) \\ &\leq \frac{\lambda}{b^2}d(gx,gy), \end{aligned}$$

where $1 \leq \lambda < 3$ *and* $b = 3$. *Thus* f *and* g *satisfy the conditions given in Theorem 3.3 and* $(0,0) \in X \cap Y$ *is the unique common fixed point of* f *and* g.

Now we get the special cases of Theorem 3.3 as follows:

Corollary 3.4. *Let* (X,Y,d) *be a complete bipolar* $b-$*metric space and* $f : (X,Y,d) \rightrightarrows (X,Y,d)$ *be a mapping such that*

$$d(fx,fy) \leq \frac{\lambda}{b^2}d(x,y), \quad \text{for all } (x,y) \in X \times Y \quad \text{with } 0 < \lambda < b.$$

Then f *has a unique fixed point in* $X \cap Y$.

Proof. If we take g as identity map on X and on Y, then from Theorem 3.3 follows that f has a unique fixed point. □

Corollary 3.5. *Let* (X,Y,d) *be a complete bipolar metric space and* $f,g : (X,Y,d) \rightrightarrows (X,Y,d)$ *with:*

(i) $f(X) \subseteq g(X)$, $f(Y) \subseteq g(Y)$ *and* g *is continuous,*

(ii)

$$d(f(x), f(y)) \leq \lambda d(g(x), g(y)),$$

for all $(x,y) \in X \times Y$ *and* $0 < \lambda < 1$,

(iii) *the pair* (f,g) *is compatible.*

Then the functions $f,g : X \cup Y \longrightarrow X \cup Y$ *have a unique the common fixed point in* $X \cap Y$. *There exists a unique point* $u \in X \cap Y$ *such that* $f(u) = g(u) = u$.

Proof. It is enough to set $b = 1$ in Theorem 3.3. □

The following corollary gives the Theorem of Mutlu, Gürdal [10].

Corollary 3.6. *Let (X, Y, d) be a complete bipolar $b-$metric space and let $f : (X, Y, d) \rightrightarrows (X, Y, d)$ with:*

$$d(f(x), f(y)) \leq \lambda d(x, y), \ \ for\ all\ (x, y) \in X \times Y \ and\ 0 < \lambda < 1.$$

Then the function $f : X \cup Y \longrightarrow X \cup Y$ has a unique fixed point in $X \cap Y$. There exists a unique point $u \in X \cap Y$ such that $f(u) = u$.

Proof. If we take g as identity map on X and on Y, then from Corollary 3.5 follows that f has a unique fixed point. □

4. Fixed point theorem for contravariant maps

Below we prove a similar result for contravariant maps.

Definition 4.1. *Let (X, Y, d) be a bipolar $b-$metric space and $f : (X, Y, d) \rightleftarrows (X, Y, d)$ and $g : (X, Y, d) \rightrightarrows (X, Y, d)$. Then the pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n \longrightarrow \infty} d(gfy_n, fgx_n) = 0$, whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in X and Y respectively, such that $\lim_{n \longrightarrow \infty} fy_n = \lim_{n \longrightarrow \infty} gx_n = t$ for some $t \in X \cap Y$.*

Lemma 4.1. *Let (X, Y, d) be a bipolar $b-$metric space and let $f : (X, Y, d) \rightleftarrows (X, Y, d)$ and $g : (X, Y, d) \rightrightarrows (X, Y, d)$ such that the pair $\{f, g\}$ be compatible and g is continuous. Suppose that $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ such that $\lim_{n \longrightarrow \infty} fy_n = \lim_{n \longrightarrow \infty} gx_n = u$ for some $u \in X \cap Y$. Then $\lim_{n \longrightarrow \infty} fgx_n = gu$.*

Proof. Since f and g are compatible, hence $\lim_{n \longrightarrow \infty} d(gfy_n, fgx_n) = 0$. Since g is continuous it follows that

$$\lim_{n \to \infty} gfy_n = gu.$$

By property (iii) of Definition 3.1, we have

$$d(gu, fgx_n) \leq b[d(gu, gu) + d(gfy_n, gu) + d(gfy_n, fgx_n)].$$

Taking the limit as $n \to \infty$ we obtain

$$\begin{aligned} \lim_{n \longrightarrow \infty} d(gu, fgx_n) &\leq b[\lim_{n \longrightarrow \infty} d(gu, gu) + \lim_{n \longrightarrow \infty} d(gfy_n, gu) \\ &\quad + \lim_{n \longrightarrow \infty} d(gfy_n, fgx_n)] = 0. \end{aligned}$$

Therefore, $\lim_{n \longrightarrow \infty} d(gu, fgx_n) = 0$, so we obtain the desired result. □

Theorem 4.2. *Let (X, Y, d) be a complete bipolar $b-$metric space and $f : (X, Y, d) \rightleftarrows (X, Y, d)$ and $g : (X, Y, d) \rightrightarrows (X, Y, d)$ with:*

(i) *$f(X) \subseteq g(Y)$, $f(Y) \subseteq g(X)$ and g is continuous,*

(ii)

$$d(f(y), f(x)) \leq \frac{\lambda}{b^2} d(g(x), g(y)),$$

for all $(x, y) \in X \times Y$ and $0 < \lambda < b^2$,

(iii) *the pair (f, g) is compatible.*

Then the functions $f, g : X \cup Y \longrightarrow X \cup Y$ have a unique common fixed point in $X \cap Y$. There exists a unique point $u \in X \cap Y$ such that $f(u) = g(u) = u$.

Proof. Let $x_0 \in X$ and $y_0 \in Y$. For each $n \in \mathbb{N}$, define $f(x_n) = g(y_n) = b_n$ and $f(y_n) = g(x_{n+1}) = a_n$. Then $\{(a_n, b_n)\}$ is a bisequence on (X, Y, d). For each positive integer n, we have

$$\begin{aligned}
d(a_n, b_n) &= d(f(y_n), f(x_n)) \\
&\leq \frac{\lambda}{b^2} d(g(x_n), g(y_n)) = \frac{\lambda}{b^2} d(a_{n-1}, b_n) = \frac{\lambda}{b^2} d(f(y_{n-1}), f(x_n)) \\
&\leq \frac{\lambda^2}{b^4} d(g(x_n), g(y_{n-1})) = \frac{\lambda^2}{b^4} d(a_{n-1}, b_{n-1}) \\
&\vdots \\
&\leq \frac{\lambda^{2n}}{b^{4n}} d(a_0, b_0).
\end{aligned}$$

Also,

$$\begin{aligned}
d(a_n, b_{n+1}) &= d(f(y_n), f(x_{n+1})) \\
&\leq \frac{\lambda}{b^2} d(g(x_{n+1}), g(y_n)) = \frac{\lambda}{b^2} d(a_n, b_n) = \frac{\lambda}{b^2} d(f(y_n), f(x_n)) \\
&\leq \frac{\lambda^2}{b^4} d(g(x_n), g(y_n)) = \frac{\lambda^2}{b^4} d(a_{n-1}, b_n) \\
&\vdots \\
&\leq \frac{\lambda^{2n}}{b^{4n}} d(a_0, b_1).
\end{aligned}$$

Hence for $m \geq n$ we get

$$\begin{aligned}
d(a_m, b_n) &\leq b[d(a_m, b_{n+1}) + d(a_n, b_{n+1}) + d(a_n, b_n)] \\
&\leq bd(a_m, b_{n+1}) + \frac{\lambda^{2n}}{b^{4n-1}}[d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Also,

$$\begin{aligned}
d(a_m, b_{n+1}) &\leq b[d(a_m, b_{n+2}) + d(a_{n+1}, b_{n+2}) + d(a_{n+1}, b_{n+1})] \\
&\leq bd(a_m, b_{n+2}) + \frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}}[d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(a_m, b_{m-1}) &\leq b[d(a_m, b_m) + d(a_{m-1}, b_m) + d(a_{m-1}, b_{m-1})] \\
&\leq bd(a_m, b_{m-2}) + \frac{\lambda^{2(m-1)}}{b^{4(m-1)-1}}[d(a_0, b_1) + d(a_0, b_0)].
\end{aligned}$$

Therefore, if set $d(a_0, b_1) + d(a_0, b_0) = \alpha$ then we have:

$$\begin{aligned}
d(a_m, b_n) &\leq bd(a_m, b_{n+1}) + \frac{\lambda^{2n}}{b^{4n-1}}\alpha \\
&\leq b^2 d(a_m, b_{n+2}) + \frac{\lambda^{2n}}{b^{4n-1}}\alpha + \frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}}\alpha \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&\leq b^{m-n}d(a_m,b_m)+\frac{\lambda^{2n}}{b^{4n-1}}\alpha+\frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}}\alpha+\cdots+\frac{\lambda^{2(m-1)}}{b^{4(m-1)-1}}\alpha\\
&\leq b^{m-n}\frac{\lambda^{2m}}{b^{4m}}\alpha+\frac{\lambda^{2n}}{b^{4n-1}}\alpha+\frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}}\alpha+\cdots+\frac{\lambda^{2(m-1)}}{b^{4(m-1)-1}}\alpha\\
&\leq \frac{\lambda^{2n}}{b^{4n-1}}\alpha+\frac{\lambda^{2(n+1)}}{b^{4(n+1)-1}}\alpha+\cdots+\frac{\lambda^{2(m-1)}}{b^{4(m-1)-1}}\alpha+\frac{\lambda^{2m}}{b^{3m+n}}\alpha\\
&\leq \frac{1}{b}[(\frac{\lambda^2}{b^4})^n\alpha+(\frac{\lambda^2}{b^4})^{n+1}\alpha+\cdots+(\frac{\lambda^2}{b^4})^{m-1}\alpha+(\frac{\lambda^2}{b^4})^m\alpha]\\
&\leq \frac{1}{b}[\frac{(\frac{\lambda^2}{b^4})^n\alpha}{1-\frac{\lambda^2}{b^4}}]\longrightarrow 0.
\end{aligned}$$

Therefore, (a_n,b_n) is a Cauchy bisequence. Since (X,Y,d) is complete, (a_n,b_n) converges, and thus biconverges to a point $u\in X\cap Y$ and

$$\lim_{n\to\infty}f(y_n)=\lim_{n\to\infty}g(x_{n+1})=\lim_{n\to\infty}a_n=u,$$

and

$$\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}g(y_n)=\lim_{n\to\infty}b_n=u.$$

We show that u is a common fixed point of f and g.

Since g is continuous it follows that

$$\lim_{n\to\infty}gf(y_n)=g(u),\qquad \lim_{n\to\infty}gg(x_{n+1})=g(u).$$

Since f and g are compatible, so by Lemma 4.1 $\lim_{n\longrightarrow\infty}fg(x_n)=g(u)$. Putting $x=gx_n$ and $y=u$ in inequality (ii) of Theorem 4.2 we obtain

$$d(f(u),fg(x_n))\quad\leq\quad\frac{\lambda}{b^2}\,d(gg(x_n),g(u)). \tag{4.1}$$

Now, by taking the upper limit when $n\to\infty$ in (4.1) and using Lemma 3.1 we get

$$\begin{aligned}
\frac{1}{b}d(f(u),g(u)) &\leq \limsup_{n\longrightarrow\infty}d(f(u),fg(x_n))\\
&\leq \frac{\lambda}{b^2}\limsup_{n\longrightarrow\infty}d(gg(x_n),g(u))\\
&= \frac{\lambda}{b^2}bd(g(u),g(u))=0.
\end{aligned}$$

Consequently $d(f(u),g(u))=0$, it follows that $f(u)=g(u)$. Now, we show that $f(u)=u$. Putting $x=u$ and $y=y_n$ in inequality (ii) of Theorem 4.2 we obtain

$$d(f(y_n),f(u))\quad\leq\quad\frac{\lambda}{b^2}d(g(u),g(y_n)). \tag{4.2}$$

Similarly by taking the upper limit when $n\to\infty$ in (4.2) and using Lemma 3.1 we obtain

$$\begin{aligned}
\frac{1}{b}d(u,f(u)) &\leq \limsup_{n\longrightarrow\infty}d(f(y_n),f(u))\leq\frac{\lambda}{b^2}d(g(u),g(y_n))\\
&\leq \frac{\lambda}{b^2}bd(g(u),u)=\frac{\lambda}{b}d(f(u),u)\\
&< \frac{1}{b}d(u,f(u)),
\end{aligned}$$

it follows that $g(u) = f(u) = u$. If there exists another common fixed point v in $X \cap Y$ of f and g, then

$$\begin{aligned} d(u, v) &= d(f(u), f(v)) \leq \frac{\lambda}{b^2} d(g(v), g(u)) = \frac{\lambda}{b^2} d(u, v) \\ &< d(u, v), \end{aligned}$$

which implies that $d(u, v) = 0$ and $u = v$. Thus u is a unique common fixed point of f and g. The proof of the theorem is completed. □

Example 4.1. *Let*

$$X = \left\{ (x_n) \in \mathbb{R}^\infty \mid x_n \leq 0 \text{ for each } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \sqrt{-x_n} < \infty \right\}$$

and

$$Y = \left\{ (y_n) \in \mathbb{R}^\infty \mid y_n \geq \text{ for each } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \sqrt{y_n} < \infty \right\}.$$

Defined $d : X \times Y \to \mathbb{R}$, *by* $d(x, y) = \left(\sum_{n=1}^{\infty} \sqrt{y_n - x_n} \right)^2$, *where* $x = (x_n) \in X$ *and* $y = (y_n) \in Y$.

Then (X, Y, d) *is a bipolar* $b-$*metric space with the constant* $b = 4$.

$$f : (X, Y, d) \rightleftarrows (X, Y, d), \ f(u_n) = (-u_n)$$
$$g : (X, Y, d) \rightrightarrows (X, Y, d), \ g(u_n) = (u_n^3) + 2u_n$$

These are compatible as $f \circ g = g \circ f$. *Note that*

$$d(f(y_n), f(x_n)) \leq \frac{8}{4^2} d(g(x_n), g(y_n)),$$

and also g *is continuous and* $f(X) \subseteq g(Y)$, $f(Y) \subseteq g(X)$. *Then by Theorem 4.2.* f *and* g *must have a unique common fixed point. Indeed, the only point* $(u_n) = (0, 0, 0, \ldots) \in X \cap Y$ *is a common fixed point of* f *and* g.

Now we get the special cases of Theorem 4.2 as follows:

Corollary 4.3. *Let* (X, Y, d) *be a complete bipolar* $b-$*metric space and* $f : (X, Y, d) \rightleftarrows (X, Y, d)$ *be a mapping such that*

$$d(fy, fx) \leq \frac{\lambda}{b^2} d(x, y), \quad \text{for all} \quad (x, y) \in X \times Y, \quad \text{with} \quad 0 < \lambda < b^2.$$

Then f *has a unique fixed point in* $X \cap Y$.

Proof. If we take g as identity map on X and on Y, then from Theorem 4.2 follows that f has a unique fixed point. □

Corollary 4.4. *Let* (X, Y, d) *be a complete bipolar metric space and* $f : (X, Y, d) \rightleftarrows (X, Y, d)$ *and* $g : (X, Y, d) \rightrightarrows (X, Y, d)$ *with:*

(i) $f(X) \subseteq g(Y)$, $f(Y) \subseteq g(X)$ *and* g *is continuous,*

(ii)

$$d(f(y), f(x)) \leq \lambda d(g(x), g(y)),$$

for all $(x, y) \in X \times Y$ *and* $0 < \lambda < 1$,

(iii) *the pair* (f, g) *is compatible.*

Then the functions $f, g : X \cup Y \longrightarrow X \cup Y$ *have a unique common fixed point in* $X \cap Y$. *There exists a unique point* $u \in X \cap Y$ *such that* $f(u) = g(u) = u$.

Proof. It is enough to set $b = 1$ in Theorem 4.2. □

The following corollary gives the Theorem of Mutlu, Gürdal [10].

Corollary 4.5. *Let (X, Y, d) be a complete bipolar metric space and let $f : (X, Y, d) \rightleftarrows (X, Y, d)$ with:*

$$d(f(y), f(x)) \leq \lambda d(x, y), \ \ for\ all\ (x, y) \in X \times Y\ and\ 0 < \lambda < 1.$$

Then the function $f : X \cup Y \longrightarrow X \cup Y$ has a unique fixed point in $X \cap Y$. There exists a unique point $u \in X \cap Y$ such that $f(u) = u$.

Proof. If we take g as identity map on X and on Y, then from Corollary 4.4 follows that f has a unique fixed point. □

Remark. *In Fixed point theory for readers' interest well known of Meir-Keeler type contraction, Čirič type of quasi contraction, Nadler type of contraction Sehgal-Guseman type of contraction, the completion of bipolar $b-$metric space, Suzuki-Berinde type of contraction, etc., that is these structures can be proven in bipolar $b-$metric space exploration.*

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