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Notes on q-Partial Differential Equations for q-Laguerre Polynomials and Little q-Jacobi Polynomials

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Abstract

This article defines two common q-orthogonal polynomials: homogeneous q-Laguerre polynomials and homogeneous little q-Jacobi polynomials. They can be viewed separately as solutions to two q-partial differential equations. Furthermore, an analytic function satisfies a certain system of q-partial differential equations if and only if it can be expanded in terms of homogeneous q-Laguerre polynomials or homogeneous little q-Jacobi polynomials. As applications, several generalized Ramanujan q-beta integrals and Andrews-Askey integrals are obtained.

1. Introduction

The presence of orthogonal polynomials is ubiquitous in various problems encountered in classical mathematical physics. For instance, the Hermite polynomials manifest in the quantum mechanical treatment of harmonic oscillators, while the Laguerre polynomials arise in the propagation of electromagnetic waves. However, the study of q-orthogonal polynomials is also a crucial study topic and can be found in relevant literature [1, 2, 3, 4, 5].

Throughout the paper, it is supposed that $0<|q|<1$ and denote by $\mathbb{N}$ ($\mathbb{C}$) the set of positive integers (complex numbers, respectively). The q-shifted factorials are defined as

$$(a;q)_0=1,\ (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\ (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)$$

and $(a_1,a_2,\cdots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$, where n is a non-negative integer or ∞. The q-derivative of $f(x)$ with respect to x is defined by

$$\mathcal{D}_q\{f(x)\}=\frac{f(x)-f(qx)}{x}.$$

According to the above definition, it is not difficult to verify

$$\mathcal{D}_q\{f(x)g(x)\} \quad=\quad \mathcal{D}_q\{f(x)\}g(x)+f(qx)\mathcal{D}_q\{g(x)\} \tag{1.1}$$

and the Leibniz rule for the product of two functions

$$\mathcal{D}_q^n\{f(x)g(x)\}=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{k(k-n)}\mathcal{D}_q^k\{f(x)\}\mathcal{D}_q^{n-k}\{g(q^kx)\}, \tag{1.2}$$

where

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},\ 0\le k\le n,\ n\in\mathbb{N} \tag{1.3}$$

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is the Gaussian binomial coefficients, also see [6]. For any real number r, the q-shift operator $\eta_{x_i}^r$ is defined by

$$\eta_{x_i}^r\{f(x_1,\cdots,x_n)\} = f(x_1,\cdots,x_{i-1},q^r x_i,x_{i+1},\cdots,x_n).$$

Generalizing Heine's series, or basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} a_1,a_2,\cdots,a_r \\ b_1,b_2,\cdots,b_s \end{matrix};q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \tag{1.4}$$

Here and in what follows, $\binom{n}{k}$ represents the standard combination symbol. The series ${}_r\phi_s$ terminates if one of the numerator parameters is of the form q^{-n}, $n\in\mathbb{N}\cup\{0\}$ and $q\neq 0$. If $0<|q|<1$, the series ${}_r\phi_s$ converges absolutely for all x if $r\leq s$ and for $|x|<1$ if $r=s+1$. The famous q-binomial theorem

$${}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix};q,z\right) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty},\ |z|<1, \tag{1.5}$$

is a q-analogue of Newton's binomial series. This theorem can also derive the following two identities

$$\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},\ |z|<1, \quad \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}z^n = (z;q)_\infty. \tag{1.6}$$

The theory of basic hypergeometric series has been greatly developed for more than a century, and there are many effective ways to study it, such as the Wilf-Zeilberg algorithm, transformation, inversion and operator, for example, see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Ten years ago, Liu first introduced the q-partial differential equation method to study q-series. This innovative approach has attracted the attention of numerous mathematicians, For further details, please refer to [26, 27, 28, 29, 31, 32, 33]. To this end, we initially define the q-partial derivative [28].

Definition 1.1. *A q-partial derivative of a function of several variables is its q-derivative with respect to one of those variables, regarding other variables as constants.*

For convenience, the q-partial derivative of a function f with respect to the variable x is denoted by $\mathcal{D}_{q,x}\{f\}$. In [28], Liu proved the following theorem.

Theorem 1.2. *If $f(x,y)$ is a two-variable analytic function at $(0,0)\in\mathbb{C}^2$, then, f can be expanded in terms of homogeneous Rogers-Szegő polynomials (for definition see (5.1)) if and only if f satisfies the q-partial differential equation $\mathcal{D}_{q,x}\{f\} = \mathcal{D}_{q,y}\{f\}$.*

We should point out that the above theorem has developed a new theory for calculating the q-identity and demonstrated its universality when applied to many types of q-orthogonal polynomials, including Rogers-Szegő polynomials, Hahn polynomials, Stieltjes-Wigert polynomials and Askey-Wilson polynomials, as well as classical orthogonal polynomials such as Hermite polynomials (cf. [30]). Later, some related works by Abdlhusein, Arjika, Aslan, Cao, Jia, Li, Mahaman, Niu and Zhang also fall into Liu's theory. Readers interested can see [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47].
Hahn [48] first discovered the q-Laguerre polynomials, according to Koekoek and Swarttouw [49], they are defined by

$$\mathcal{L}_n^{(\alpha)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix};q,-q^{n+\alpha+1}x\right),\alpha>-1. \tag{1.7}$$

Askey pointed out [50] that the q-Laguerre polynomials converge to the Stieltjes-Wigert polynomials for $\alpha\to\infty$ thus the q-Laguerre polynomials are sometimes called the generalized Stieltjes-Wigert polynomials [49]. The explicit form of q-Laguerre polynomials can write as

$$\mathcal{L}_n^{(\alpha)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_k}(-x)^k. \tag{1.8}$$

To study q-Laguerre polynomials from the perspective of q-partial differential equations following Liu's ideas, it is necessary to introduce homogeneous q-Laguerre polynomials

$$L_n^{(\alpha)}(x,y|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_k}(-x)^k y^{n-k},\alpha>-1. \tag{1.9}$$

Obviously,

$$L_n^{(\alpha)}(x,y|q) = \frac{(q;q)_n}{(q^{\alpha+1};q)_n}y^n\mathcal{L}_n^{(\alpha)}(x/y|q),\ L_n^{(\alpha)}(x,1|q) = \frac{(q;q)_n}{(q^{\alpha+1};q)_n}\mathcal{L}_n^{(\alpha)}(x|q),\ L_n^{(\alpha)}(0,y|q) = y^n.$$

This paper is organized as follows. Section 2 shows that an analytic function satisfies a system of q-partial differential equations, if and only if it can be expanded in terms of homogeneous q-Laguerre polynomials (see Theorem 2.3). Section 3 is an application of Theorem 2.3, where we use the method of q-partial differential equations to prove the generating functions of homogeneous q-Laguerre polynomials with different weights. Section 4 presents that an analytic function can be expanded in terms of homogeneous little q-Jacobi polynomials (see Theorem 4.2) if and only if it satisfies a system of q-partial differential equations. In section 5, we obtain some identities by applying Theorems 2.3 and 4.2, which generalize famous formulas such as Ramanujan q-beta integrals and Andrews-Askey integrals.

2. Homogeneous q-Laguerre polynomials and q-partial differential equations

Firstly, Proposition 2.1 presents an important property of homogeneous q-Laguerre polynomials.

Proposition 2.1. *For $n \in \mathbb{N} \cup \{0\}$, the homogeneous q-Laguerre polynomials satisfy the q-partial differential equation*

$$\mathcal{D}_{q,x}(1-q^{\alpha}\eta_x)\left\{L_n^{(\alpha)}(x,y|q)\right\} = -q^{\alpha+1}\eta_x^2\mathcal{D}_{q,y}\left\{L_n^{(\alpha)}(x,y|q)\right\}, \tag{2.1}$$

namely,

$$\mathcal{D}_{q,x}\left\{L_n^{(\alpha)}(x,y|q) - q^{\alpha}L_n^{(\alpha)}(qx,y|q)\right\} = -q^{\alpha+1}\mathcal{D}_{q,y}\left\{L_n^{(\alpha)}(q^2x,y|q)\right\}.$$

Proof. Let LHS denote the left-hand side of the equation (2.1), and by using the formula $\mathcal{D}_{q,x}\{x^n\} = (1-q^n)x^{n-1}$, we can obtain

$$\text{LHS} = \mathcal{D}_{q,x}\left\{\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_{k-1}} x^k y^{n-k}\right\} = \sum_{k=1}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q (1-q^k)\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_{k-1}} x^{k-1}y^{n-k}.$$

Similarly, use RHS to denote the right-hand side of the equation (2.1). Through simple calculation, we have

$$\begin{aligned} \text{RHS} &= -q^{\alpha+1}\mathcal{D}_{q,y}\left\{\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_k}(q^2x)^k y^{n-k}\right\} \\ &= \sum_{k=0}^{n-1}(-1)^{k+1}\begin{bmatrix} n \\ k \end{bmatrix}_q (1-q^{n-k})\frac{q^{(k+1)^2+(k+1)\alpha}}{(q^{\alpha+1};q)_k} x^k y^{n-k-1} \\ &= \sum_{k=1}^{n}(-1)^k \begin{bmatrix} n \\ k-1 \end{bmatrix}_q (1-q^{n-k+1})\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_{k-1}} x^{k-1}y^{n-k}. \end{aligned}$$

From the definition of the q-binomial coefficients (1.3), it is easy to verify that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q (1-q^k) = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q (1-q^{n-k+1}). \tag{2.2}$$

It follows from (2.2) that LHS $=$ RHS, which completes the proof. □

In order to prove Theorem 2.3, we need the following proposition (for example, see [51, p.5]).

Proposition 2.2. *If $f(x_1,x_2,\cdots,x_k)$ is analytic at the origin $(0,0,\ldots,0) \in \mathbb{C}^k$, then, f can be expanded in an absolutely and uniformly convergent power series,*

$$f(x_1,x_2,\ldots,x_k) = \sum_{n_1,n_2,\ldots,n_k=0}^{\infty}\lambda_{n_1,n_2,\ldots,n_k}x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}.$$

The main result of this section is Theorem 2.3.

Theorem 2.3. *If $f(x_1,y_1,\cdots,x_k,y_k)$ is a $2k$-variable analytic function at $(0,0,\cdots,0) \in \mathbb{C}^{2k}$, then, f can be expanded*

$$\sum_{n_1,\cdots,n_k=0}^{\infty}\lambda_{n_1,\cdots,n_k}L_{n_1}^{(\alpha_1)}(x_1,y_1|q)\cdots L_{n_k}^{(\alpha_k)}(x_k,y_k|q),$$

where $\lambda_{n_1,\cdots,n_k}$ are independent of $x_1,y_1,\cdots,x_k,y_k$, if and only if f satisfies the q-partial differential equations

$$\mathcal{D}_{q,x_j}(1-q^{\alpha_j}\eta_{x_j})\{f\} = -q^{\alpha_j+1}\eta_{x_j}^2\mathcal{D}_{q,y_j}\{f\} \tag{2.3}$$

for $j \in \{1,2,\ldots,k\}$.

Proof. We employ mathematical induction. When $k=1$, it follows from Proposition 2.2 that f can be expanded in an absolutely and uniformly convergent power series in a neighborhood of $(0,0)$. Therefore, there exists a sequence $\{\lambda_{m,n}\}$ independent of x_1 and y_1 for which

$$f(x_1,y_1)=\sum_{m,n=0}^{\infty}\lambda_{m,n}x_1^m y_1^n=\sum_{m=0}^{\infty}x_1^m\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n. \tag{2.4}$$

Substituting the above equation into the following q-partial differential equation

$$\mathcal{D}_{q,x_1}(1-q^{\alpha_1}\eta_{x_1})\{f(x_1,y_1)\}=-q^{\alpha_1+1}\eta_{x_1}^2\mathcal{D}_{q,y_1}\{f(x_1,y_1)\}, \tag{2.5}$$

we obtain

$$\sum_{m=1}^{\infty}(1-q^{\alpha_1+m})(1-q^m)x_1^{m-1}\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=-q^{\alpha_1+1}\sum_{m=0}^{\infty}q^{2m}x_1^m\mathcal{D}_{q,y_1}\left\{\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n\right\}. \tag{2.6}$$

Equating the coefficients of x_1^{m-1} in (2.6), we have

$$\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=\frac{(-q^{\alpha_1+1})q^{2(m-1)}}{(1-q^{\alpha_1+m})(1-q^m)}\mathcal{D}_{q,y_1}\left\{\sum_{n=0}^{\infty}\lambda_{m-1,n}y_1^n\right\}.$$

Iteration $m-1$ times yields

$$\begin{aligned}\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n &= \frac{(-q^{\alpha_1+1})^m q^{m(m-1)}}{(q;q)_m(q^{\alpha_1+1};q)_m}\mathcal{D}_{q,y_1}^m\left\{\sum_{n=0}^{\infty}\lambda_{0,n}y_1^n\right\}\\ &= \frac{(-1)^m q^{m^2+m\alpha_1}}{(q;q)_m(q^{\alpha_1+1};q)_m}\sum_{n=0}^{\infty}\lambda_{0,n}\frac{(q;q)_n}{(q;q)_{n-m}}y_1^{n-m}\\ &= \sum_{n=m}^{\infty}(-1)^m\lambda_{0,n}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{q^{m^2+m\alpha_1}}{(q^{\alpha_1+1};q)_m}y_1^{n-m}.\end{aligned}$$

Noting that the series in (2.4) is an absolutely and uniformly convergent series, substituting the above equation into (2.4) and interchanging the order of the summation, we find

$$\begin{aligned}f(x_1,y_1) &= \sum_{m=0}^{\infty}x_1^m\sum_{n=m}^{\infty}(-1)^m\lambda_{0,n}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{q^{m^2+m\alpha_1}}{(q^{\alpha_1+1};q)_m}y_1^{n-m}\\ &= \sum_{n=0}^{\infty}\lambda_{0,n}\sum_{m=0}^{n}(-1)^m\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{q^{m^2+m\alpha_1}}{(q^{\alpha_1+1};q)_m}x_1^m y_1^{n-m}\\ &= \sum_{n=0}^{\infty}\lambda_{0,n}L_n^{(\alpha_1)}(x_1,y_1|q).\end{aligned}$$

The above calculation shows that the sufficiency of Theorem 2.3 is correct. Conversely, if $f(x_1,y_1)$ can be expanded in terms of $L_n^{(\alpha_1)}(x_1,y_1|q)$, then using Proposition 4.1, we find that $f(x_1,y_1)$ satisfies (2.3). So we can prove the case of $k=1$.
Next, we assume that Theorem 2.3 is true for the case $k-1$. Since f is analytic at $(0,0)$ and satisfies (2.5). Thus, there exists a sequence $\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}$ independent of x_1 and y_1 such that

$$f(x_1,y_1,\ldots,x_k,y_k)=\sum_{n_1=0}^{\infty}c_{n_1}(x_2,y_2,\ldots,x_k,y_k)L_{n_1}^{(\alpha_1)}(x_1,y_1|q). \tag{2.7}$$

Putting $x_1=0$ in (2.7) and using $L_{n_1}^{(\alpha_1)}(0,y_1|q)=y_1^{n_1}$, we obtain

$$f(0,y_1,x_2,y_2,\ldots,x_k,y_k)=\sum_{n_1=0}^{\infty}c_{n_1}(x_2,y_2,\ldots,x_k,y_k)y_1^{n_1}.$$

Using the Maclaurin expansion theorem, we have

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k)=\frac{\partial^{n_1}f(0,y_1,x_2,y_2,\ldots,x_k,y_k)}{n_1!\partial y_1{}^{n_1}}\Big|_{y_1=0}.$$

Since $f(x_1,y_1,\ldots,x_k,y_k)$ is analytic near $(x_1,y_1,\ldots,x_k,y_k)=(0,\ldots,0)\in\mathbb{C}^{2k}$, it follows from the above equation that $c_{n_1}(x_2,y_2,\ldots,x_k,y_k)$ is analytic near $(x_2,y_2,\ldots,x_k,y_k)=(0,\ldots,0)\in\mathbb{C}^{2k-2}$. Substituting (2.7) into (2.3), we find that for $j=2,\ldots,k$,

$$\sum_{n_1=0}^{\infty}\mathcal{D}_{q,x_j}(1-q^{\alpha_j}\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}L_n^{(\alpha_1)}(x_1,y_1|q)=\sum_{n_1=0}^{\infty}(-q^{\alpha_j+1}\eta_{x_j}^2)\mathcal{D}_{q,y_j}\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}L_n^{(\alpha_1)}(x_1,y_1|q).$$

By equating the coefficients of $L_n^{(\alpha_1)}(x_1,y_1|q)$ in the above equation, we obtain

$$\mathcal{D}_{q,x_j}(1-q^{\alpha_j}\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}=-q^{\alpha_j+1}\eta_{x_j}^2\mathcal{D}_{q,y_j}\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}.$$

Therefore, there exists a sequence $\{\lambda_{n_1,n_2,\ldots,n_k}\}$ independent of $x_2,y_2,\ldots,x_k,y_k$ for which

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k)=\sum_{n_2,\ldots,n_k=0}^{\infty}\lambda_{n_1,n_2,\ldots,n_k}L_{n_2}^{(\alpha_2)}(x_2,y_2|q)\cdots L_{n_k}^{(\alpha_k)}(x_k,y_k|q).$$

Then substituting the above equation into (2.7), we proved the sufficiency of Theorem 2.3. Conversely, if f can be expanded in terms of $L_{n_1}^{(\alpha_1)}(x_1,y_1|q)\cdots L_{n_k}^{(\alpha_k)}(x_k,y_k|q)$, it follows from Proposition 2.1 that f satisfies (2.3). This completes the proof. □

Remark 2.4. *Theorem 2.3 implies that all solutions to q-partial differential equation (2.3) can be represented as linear combinations of homogeneous q-Laguerre polynomials. Its applications are discussed in Sections 3 and 5.*

3. Generating functions for homogeneous q-Laguerre polynomials

Since the Stieltjes and Hamburger moment problems corresponding to the q-Laguerre polynomials are indeterminate there exist many different weight functions, see [2, 52, 53, 54] for details. Theorem 3.2 will use Theorem 2.3 to prove the following generating functions of homogeneous q-Laguerre polynomials with different weights. We often refer to the following Hartog's theorem (see [55, p. 28]) to determine if a given function is an analytic function in several complex variables.

Theorem 3.1. *If a complex valued function $f(z_1,z_2,\cdots,z_n)$ is holomorphic (analytic) in each variable separately in a domain $U\in\mathbb{C}^n$, then, it is holomorphic (analytic) in U.*

Theorem 3.2. *(1) We have*

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}}{(q;q)_n}L_n^{(\alpha)}(x,y|q)t^n=(ty;q)_\infty\,{}_0\phi_2\left(\begin{matrix}-\\ q^{\alpha+1},ty\end{matrix};q,-q^{\alpha+1}tx\right). \tag{3.1}$$

(2) For arbitrarily given γ, and for $|ty|<1$, we have

$$\sum_{n=0}^{\infty}\frac{(\gamma;q)_n}{(q;q)_n}L_n^{(\alpha)}(x,y|q)t^n=\frac{(\gamma ty;q)_\infty}{(ty;q)_\infty}\,{}_1\phi_2\left(\begin{matrix}\gamma\\ q^{\alpha+1},\gamma ty\end{matrix};q,-q^{\alpha+1}tx\right). \tag{3.2}$$

Proof. For part (1), denote the right-hand side of (3.1) by $f(x,y)$. It follows from Theorem 3.1 that $f(x,y)$ is an analytic function of x and y. Thus $f(x,y)$ is analytic at $(0,0)\in\mathbb{C}^2$. On the one hand, we have

$$\mathcal{D}_{q,x}(1-q^\alpha\eta_x)\{f(x,y)\}=-tq^{\alpha+1}(ty;q)_\infty\sum_{n=0}^{\infty}\frac{[(-1)^{n+1}q^{\binom{n+1}{2}}]^3}{(q,q^{\alpha+1};q)_n(ty;q)_{n+1}}(-q^{\alpha+1}xt)^n.$$

On the other hand, according to (1.1),

$$\mathcal{D}_{q,y}\{f(x,y)\}=(ty;q)_\infty\sum_{n=0}^{\infty}\frac{-tq^n[(-1)^nq^{\binom{n}{2}}]^3}{(q,q^{\alpha+1};q)_n(ty;q)_{n+1}}(-q^{\alpha+1}xt)^n,$$

from which we obtain

$$-q^{\alpha+1}\eta_x^2\mathcal{D}_{q,y}\{f(x,y)\}=tq^{\alpha+1}(ty;q)_\infty\sum_{n=0}^{\infty}\frac{q^{3n}[(-1)^nq^{\binom{n}{2}}]^3}{(q,q^{\alpha+1};q)_n(ty;q)_{n+1}}(-q^{\alpha+1}xt)^n=\mathcal{D}_{q,x}(1-q^\alpha\eta_x)\{f(x,y)\}.$$

Therefore, by Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$(ty;q)_\infty\,{}_0\phi_2\left(\begin{matrix}-\\ q^{\alpha+1},ty\end{matrix};q,-q^{\alpha+1}xt\right)=\sum_{n=0}^{\infty}\lambda_nL_n^{(\alpha)}(x,y|q). \tag{3.3}$$

Putting $x=0$ in the above equation, using $L_n^{(\alpha)}(0,y|q)=y^n$ and (1.6), we find that

$$\sum_{n=0}^{\infty}\lambda_n y^n=(ty;q)_\infty=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}(ty)^n.$$

Equating the coefficients of y^n in the above equation, we have $\lambda_n=(-1)^n q^{\binom{n}{2}}/[t^n(q;q)_n]$. Then substitute it into (3.3) and equation (3.1) follows.

For part (2), denote the right-hand side of (3.2) by $f(x,y)$. It follows from Theorem 3.1 that $f(x,y)$ is an analytic function of x and y for $|ty|<1$. Thus $f(x,y)$ is analytic at $(0,0)\in\mathbb{C}^2$. On the one hand, we have

$$\begin{aligned}\mathcal{D}_{q,x}(1-q^\alpha\eta_x)\{f(x,y)\} &= \mathcal{D}_{q,x}\left\{\frac{(\gamma ty;q)_\infty}{(ty;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_n[(-1)^n q^{\binom{n}{2}}]^2}{(q^{\alpha+1};q)_{n-1}(q,\gamma ty;q)_n}(-q^{\alpha+1}xt)^n\right\}\\ &= \frac{-tq^{\alpha+1}(\gamma ty;q)_\infty}{(ty;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_{n+1}[(-1)^n q^{\binom{n}{2}}]^2 q^{2n}}{(q,q^{\alpha+1};q)_n(\gamma ty;q)_{n+1}}(-q^{\alpha+1}xt)^n.\end{aligned}$$

On the other hand, according to (1.1),

$$\mathcal{D}_{q,y}\{f(x,y)\}=\frac{t(\gamma ty;q)_\infty}{(ty;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_{n+1}[(-1)^n q^{\binom{n}{2}}]^2}{(q,q^{\alpha+1};q)_n(\gamma ty;q)_{n+1}}(-q^{\alpha+1}xt)^n,$$

from which we obtain

$$-q^{\alpha+1}\eta_x^2\mathcal{D}_{q,y}\{f(x,y)\}=\frac{-tq^{\alpha+1}(\gamma ty;q)_\infty}{(ty;q)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma;q)_{n+1}[(-1)^n q^{\binom{n}{2}}]^2 q^{2n}}{(q,q^{\alpha+1};q)_n(\gamma ty;q)_{n+1}}(-q^{\alpha+1}xt)^n=\mathcal{D}_{q,x}(1-q^\alpha\eta_x)\{f(x,y)\}.$$

Hence, by Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$\frac{(\gamma ty;q)_\infty}{(ty;q)_\infty}{}_1\phi_2\left(\begin{matrix}\gamma\\ q^{\alpha+1},\gamma ty\end{matrix};q,-q^{\alpha+1}xt\right)=\sum_{n=0}^{\infty}\lambda_n L_n^{(\alpha)}(x,y|q). \tag{3.4}$$

Putting $x=0$ in the above equation, using $L_n^{(\alpha)}(0,y|q)=y^n$ and (1.5), we find that

$$\sum_{n=0}^{\infty}\lambda_n y^n=\frac{(\gamma ty;q)_\infty}{(ty;q)_\infty}=\sum_{n=0}^{\infty}\frac{(\gamma;q)_n}{(q;q)_n}(ty)^n.$$

Equating the coefficients of y^n in the above equation, we obtain $\lambda_n=t^n(\gamma;q)_n/(q;q)_n$. Then substitute it into (3.4), which completes the proof of (3.2). □

Remark 3.3. *(1) Taking $y=1$, Theorem 3.2 degenerates into generating functions of q-Laguerre polynomials [49, p.109].*
(2) Taking $\gamma=0$ in (3.2), we can obtain a simpler generating function for $L_n^{(\alpha)}(x,y|q)$:

$$\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x,y|q)}{(q;q)_n}t^n=\frac{1}{(ty;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}xt\right). \tag{3.5}$$

4. Homogeneous little q-Jacobi polynomials and q-partial differential equations

A q-analogue of Jacobi polynomials was introduced by Hahn [48] and later studied by Andrews and Askey [56, 57], and named by them as little q-Jacobi polynomials:

$$\mathcal{P}_n^{(\alpha,\beta)}(x|q)={}_2\phi_1\left(\begin{matrix}q^{-n},\alpha\beta q^{n+1}\\ \alpha q\end{matrix};q,qx\right). \tag{4.1}$$

As $q\to 1$, the little q-Jacobi polynomials tend to a multiple of Jacobi polynomials. The little q-Jacobi polynomials with $\beta=0$ are q-analogs of Laguerre polynomials and are orthogonal with respect to a discrete measure on a countable set, called little q-Laguerre (or Wall) polynomials. Moreover, the little q-Legendre polynomials are little q-Jacobi polynomials with $\alpha=\beta=1$. If we set $\beta\to-\alpha^{-1}q^{-1}\beta$, in the little q-Jacobi polynomials and then take the limit $\alpha\to 0$ we obtain the alternative q-Charlier polynomials. For more details about q-Jacobi polynomials, see [49].

To establish the relationship between little q-Jacobi polynomials and q-partial differential equations, similar to Section 2, we naturally introduce homogeneous little q-Jacobi polynomials

$$p_n^{(\alpha,\beta)}(x,y|q)=\sum_{k=0}^{n}q^{k(k+1-2n)/2}\begin{bmatrix}n\\ k\end{bmatrix}_q\frac{(\alpha\beta q^{n+1};q)_k}{(\alpha q;q)_k}(-x)^k y^{n-k}. \tag{4.2}$$

Evidently,

$$p_n^{(\alpha,\beta)}(x,y|q)=y^n\mathcal{P}_n^{(\alpha,\beta)}(x/y|q),\ p_n^{(\alpha,\beta)}(x,1|q)=\mathcal{P}_n^{(\alpha,\beta)}(x|q),\ p_n^{(\alpha,\beta)}(0,y|q)=y^n. \tag{4.3}$$

Firstly, Proposition 4.1 shows an important property of homogeneous little q-Jacobi polynomials.

Proposition 4.1. *The homogeneous little q-Jacobi polynomials satisfy the q-partial differential equation*

$$\mathcal{D}_{q,x}(1-\alpha\eta_x)\left\{p_n^{(\alpha,\beta)}(x,y|q)\right\}=-q\mathcal{D}_{q,y}(\eta_y^{-1}-q\alpha\beta\eta_x^2)\left\{p_n^{(\alpha,\beta)}(x,y|q)\right\}, \tag{4.4}$$

namely,

$$\mathcal{D}_{q,x}\left\{p_n^{(\alpha,\beta)}(x,y|q)-\alpha p_n^{(\alpha,\beta)}(qx,y|q)\right\}=-q\mathcal{D}_{q,y}\left\{p_n^{(\alpha,\beta)}(x,y/q|q)-q\alpha\beta p_n^{(\alpha,\beta)}(q^2x,y|q)\right\}.$$

Proof. If we use LHS to denote the left-hand side of the equation (4.4), we have

$$\begin{aligned}\text{LHS} &= \mathcal{D}_{q,x}\left\{\sum_{k=0}^{n}(-1)^kq^{k(k+1-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(\alpha\beta q^{n+1};q)_k}{(\alpha q;q)_{k-1}}x^ky^{n-k}\right\}\\ &= \sum_{k=1}^{n}(-1)^kq^{k(k+1-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(1-q^k)(\alpha\beta q^{n+1};q)_k}{(\alpha q;q)_{k-1}}x^{k-1}y^{n-k}.\end{aligned}$$

Similarly, use RHS to denote the right-hand side of the equation (4.4). By simple calculation, we obatin

$$\begin{aligned}\text{RHS} &= \mathcal{D}_{q,y}\left\{\sum_{k=0}^{n}(-1)^{k+1}q^{(k+1)(k+2-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(\alpha\beta q^{n+1};q)_{k+1}}{(\alpha q;q)_k}x^ky^{n-k}\right\}\\ &= \sum_{k=0}^{n-1}(-1)^{k+1}q^{(k+1)(k+2-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(1-q^{n-k})(\alpha\beta q^{n+1};q)_{k+1}}{(\alpha q;q)_k}x^ky^{n-k-1}\\ &= \sum_{k=1}^{n}(-1)^kq^{k(k+1-2n)/2}\begin{bmatrix}n\\k-1\end{bmatrix}_q\frac{(1-q^{n-k+1})(\alpha\beta q^{n+1};q)_k}{(\alpha q;q)_{k-1}}x^{k-1}y^{n-k}.\end{aligned}$$

It follows from (2.2) that LHS $=$ RHS. □

The main result of this section is Theorem 4.2.

Theorem 4.2. *If $f(x_1,y_1,\cdots,x_k,y_k)$ is a $2k$-variable analytic function at $(0,0,\cdots,0)\in\mathbb{C}^{2k}$, then, f can be expanded*

$$\sum_{n_1,\cdots,n_k=0}^{\infty}\lambda_{n_1,\cdots,n_k}p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q)\cdots p_{n_k}^{(\alpha_k,\beta_k)}(x_k,y_k|q),$$

where $\lambda_{n_1,\cdots,n_k}$ are independent of $x_1,y_1,\cdots,x_k,y_k$, if and only if f satisfies the q-partial differential equations

$$\mathcal{D}_{q,x_j}(1-\alpha_j\eta_{x_j})\{f\}=-q\mathcal{D}_{q,y_j}(\eta_{y_j}^{-1}-q\alpha_j\beta_j\eta_{x_j}^2)\{f\} \tag{4.5}$$

for $j\in\{1,2,\ldots,k\}$.

Proof. We use mathematical induction. When $k=1$, it follows from Proposition 2.2 that f can be expanded in an absolutely and uniformly convergent power series in a neighborhood of $(0,0)$. Therefore, there exists a sequence $\{\lambda_{m,n}\}$ independent of x_1 and y_1 for which

$$f(x_1,y_1)=\sum_{m,n=0}^{\infty}\lambda_{m,n}x_1^my_1^n=\sum_{m=0}^{\infty}x_1^m\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n. \tag{4.6}$$

Substituting the above equation into following q-partial differential equation

$$\mathcal{D}_{q,x_1}(1-\alpha_1\eta_{x_1})\{f(x_1,y_1)\}=-q\mathcal{D}_{q,y_1}(\eta_{y_1}^{-1}-q\alpha_1\beta_1\eta_{x_1}^2)\{f(x_1,y_1)\}. \tag{4.7}$$

The left-hand side of (4.7) can be written as

$$\sum_{m=1}^{\infty}(1-\alpha_1q^m)(1-q^m)x_1^{m-1}\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=\sum_{m=0}^{\infty}(1-\alpha_1q^{m+1})(1-q^{m+1})x_1^m\sum_{n=0}^{\infty}\lambda_{m+1,n}y_1^n,$$

and right-hand side of (4.7) can be expressed as

$$\mathcal{D}_{q,y_1}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-q)(q^{-n}-\alpha_1\beta_1 q^{2m+1})\lambda_{m,n}x_1^m y_1^n\right\}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-q)(1-q^n)(q^{-n}-\alpha_1\beta_1 q^{2m+1})\lambda_{m,n}x_1^m y_1^{n-1}.$$

Therefore, we obtain

$$\sum_{m=0}^{\infty}(1-\alpha_1 q^{m+1})(1-q^{m+1})x_1^m\sum_{n=0}^{\infty}\lambda_{m+1,n}y_1^n=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-q)(1-q^n)(q^{-n}-\alpha_1\beta_1 q^{2m+1})\lambda_{m,n}x_1^m y_1^{n-1}. \tag{4.8}$$

Equating the coefficients of x_1^m in (4.8), we can easily see that

$$(1-q^m)(1-\alpha_1 q^m)\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n=-q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-1)+1})\lambda_{m-1,n+1}y_1^n.$$

From the recurrence relation of the above equation, we can derive

$$(1-q^{m-1})(1-\alpha_1 q^{m-1})\sum_{n=0}^{\infty}\lambda_{m-1,n}y_1^n = -q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-2)+1})\lambda_{m-2,n+1}y_1^n, \tag{4.9}$$

$$(1-q^{m-2})(1-\alpha_1 q^{m-2})\sum_{n=0}^{\infty}\lambda_{m-2,n}y_1^n = -q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-3)+1})\lambda_{m-3,n+1}y_1^n, \tag{4.10}$$

$$\vdots$$

$$(1-q^2)(1-\alpha_1 q^2)\sum_{n=0}^{\infty}\lambda_{2,n}y_1^n = -q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2\cdot 1+1})\lambda_{1,n+1}y_1^n, \tag{4.11}$$

$$(1-q)(1-\alpha_1 q)\sum_{n=0}^{\infty}\lambda_{1,n}y_1^n = -q\sum_{n=0}^{\infty}(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2\cdot 0+1})\lambda_{0,n+1}y_1^n. \tag{4.12}$$

By equating the coefficients of y_1^n on both sides of (4.9)-(4.12), we easily deduce that

$$\begin{aligned}
\lambda_{m,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-1)+1})}{(1-q^m)(1-\alpha_1 q^m)}\lambda_{m-1,n+1},\\
\lambda_{m-1,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-2)+1})}{(1-q^{m-1})(1-\alpha_1 q^{m-1})}\lambda_{m-2,n+1},\\
&\vdots\\
\lambda_{2,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2\cdot 1+1})}{(1-q^2)(1-\alpha_1 q^2)}\lambda_{1,n+1},\\
\lambda_{1,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2\cdot 0+1})}{(1-q)(1-\alpha_1 q)}\lambda_{0,n+1}.
\end{aligned}$$

By iterating the above equations $m-1$ times, we can deduce that

$$\begin{aligned}
\lambda_{m,n} &= \frac{-q(1-q^{n+1})(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-1)+1})}{(1-q^m)(1-\alpha_1 q^m)}\times\frac{-q(1-q^{n+2})(q^{-(n+2)}-\alpha_1\beta_1 q^{2(m-2)+1})}{(1-q^{m-1})(1-\alpha_1 q^{m-1})}\cdots\\
&\quad\times\frac{-q(1-q^{n+m-1})(q^{-(n+m-1)}-\alpha_1\beta_1 q^{2\cdot 1+1})}{(1-q^2)(1-\alpha_1 q^2)}\times\frac{-q(1-q^{n+m})(q^{-(n+m)}-\alpha_1\beta_1 q^{2\cdot 0+1})}{(1-q)(1-\alpha_1 q)}\lambda_{0,n+m}\\
&= \frac{(-q)^m(q^{n+1};q)_m}{(q;q)_m(\alpha_1 q;q)_m}(q^{-(n+1)}-\alpha_1\beta_1 q^{2(m-1)+1})\cdots(q^{-(n+m)}-\alpha_1\beta_1 q^{2\cdot 0+1})\lambda_{0,n+m}\\
&= q^{m(1-2n-m)/2}\frac{\lambda_{0,n+m}(-1)^m(q;q)_{m+n}}{(q;q)_m(q;q)_n(\alpha_1 q;q)_m}(1-\alpha_1\beta_1 q^{m+n+1})\cdots(1-\alpha_1\beta_1 q^{2m+n})\\
&= \lambda_{0,n+m}(-1)^m q^{m(1-2n-m)/2}\begin{bmatrix} n+m\\ m\end{bmatrix}_q\frac{(\alpha_1\beta_1 q^{m+n+1};q)_m}{(\alpha_1 q;q)_m}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty}\lambda_{m,n}y_1^n &= \sum_{n=0}^{\infty}(-1)^m\lambda_{0,n+m}q^{m(1-2n-m)/2}\begin{bmatrix} n+m\\ m\end{bmatrix}_q\frac{(\alpha_1\beta_1 q^{m+n+1};q)_m}{(\alpha_1 q;q)_m}y_1^n\\
&= \sum_{n=m}^{\infty}(-1)^m\lambda_{0,n}q^{m(1-2n+m)/2}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{(\alpha_1\beta_1 q^{n+1};q)_m}{(\alpha_1 q;q)_m}y_1^{n-m}.
\end{aligned}$$

Noting that the series in (4.6) is an absolutely and uniformly convergent series, substituting the above equation into (4.6) and interchanging the order of the summation, we obtain

$$\begin{aligned} f(x_1,y_1) &= \sum_{m=0}^{\infty} x_1^m \sum_{n=m}^{\infty} \lambda_{0,n}(-1)^m q^{m(1-2n+m)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha_1\beta_1 q^{n+1};q)_m}{(\alpha_1 q;q)_m} y_1^{n-m} \\ &= \sum_{n=0}^{\infty} \lambda_{0,n} \sum_{m=0}^{n} (-1)^m q^{m(1-2n+m)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha_1\beta_1 q^{n+1};q)_m}{(\alpha_1 q;q)_m} x_1^m y_1^{n-m} \\ &= \sum_{n=0}^{\infty} \lambda_{0,n}\, p_n^{(\alpha_1,\beta_1)}(x_1,y_1|q). \end{aligned}$$

The above calculation shows that the sufficiency of Theorem 4.2 is correct. Conversely, if $f(x_1,y_1)$ can be expanded in terms of $p_n^{(\alpha_1,\beta_1)}(x_1,y_1|q)$, then using Proposition 4.1, we find that $f(x_1,y_1)$ satisfies (4.7). So we can prove the case of $k=1$.
Next, we assume that Theorem 4.2 is true for the case $k-1$. Since f is analytic at $(0,0)$. Thus, there exists a sequence $\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}$ independent of x_1 and y_1 such that

$$f(x_1,y_1,\ldots,x_k,y_k) = \sum_{n_1=0}^{\infty} c_{n_1}(x_2,y_2,\ldots,x_k,y_k) p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q). \tag{4.13}$$

Putting $x_1=0$ in (4.13) and using $p_{n_1}^{(\alpha_1,\beta_1)}(0,y_1|q) = y_1^{n_1}$, we obtain

$$f(0,y_1,x_2,y_2,\ldots,x_k,y_k) = \sum_{n_1=0}^{\infty} c_{n_1}(x_2,y_2,\ldots,x_k,y_k) y_1^{n_1}.$$

Using the Maclaurin expansion theorem, we have

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k) = \frac{\partial^{n_1} f(0,y_1,x_2,y_2,\ldots,x_k,y_k)}{n_1!\partial {y_1}^{n_1}}\Big|_{y_1=0}.$$

Since $f(x_1,y_1,\ldots,x_k,y_k)$ is analytic near $(x_1,y_1,\ldots,x_k,y_k)=(0,\ldots,0)\in\mathbb{C}^{2k}$, it follows from the above equation that $c_{n_1}(x_2,y_2,\ldots,x_k,y_k)$ is analytic near $(x_2,y_2,\ldots,x_k,y_k)=(0,\ldots,0)\in\mathbb{C}^{2k-2}$. Substituting (4.13) into (4.5), we find that for $j=2,\ldots,k$,

$$\begin{aligned} &\sum_{n_1=0}^{\infty} \mathcal{D}_{q,x_j}(1-\alpha\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}\, p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q) \\ =&\sum_{n_1=0}^{\infty} (-q)\mathcal{D}_{q,y_j}(\eta_{y_j}^{-1} - q\alpha\beta\eta_{x_j}^2)\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}\, p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q). \end{aligned}$$

By equating the coefficients of $p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q)$ in the above equation, we obtain

$$\mathcal{D}_{q,x_j}(1-\alpha\eta_{x_j})\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\} = -q\mathcal{D}_{q,y_j}(\eta_{y_j}^{-1} - q\alpha\beta\eta_{x_j}^2)\{c_{n_1}(x_2,y_2,\ldots,x_k,y_k)\}.$$

Therefore, by the inductive hypothesis, there exists a sequence $\{\lambda_{n_1,n_2,\ldots,n_k}\}$ independent of x_2, y_2, …, x_k, y_k such that

$$c_{n_1}(x_2,y_2,\ldots,x_k,y_k) = \sum_{n_2,\ldots,n_k=0}^{\infty} \lambda_{n_1,n_2,\ldots,n_k} p_{n_2}^{(\alpha_2,\beta_2)}(x_2,y_2|q)\cdots p_{n_k}^{(\alpha_k,\beta_k)}(x_k,y_k|q).$$

Substituting this equation into (4.13), we proved the sufficiency of the theorem. Conversely, if f can be expanded in terms of $p_{n_1}^{(\alpha_1,\beta_1)}(x_1,y_1|q)\cdots p_{n_k}^{(\alpha_k,\beta_k)}(x_k,y_k|q)$, it follows from (4.4) that f satisfies (4.5). This completes the proof of Theorem 4.2. □

Remark 4.3. *Theorem 4.2 implies that all solutions to q-partial differential equation (4.5) can be represented as linear combinations of homogeneous little q-Jacobi polynomials. See Section 5 for the application of this theorem.*

At the end of this section, we will present the generating function of homogeneous little q-Jacobi polynomials.

Proposition 4.4. *Generating function for homogeneous little q-Jacobi polynomials:*

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}t^n}{(q,\beta q;q)_n} p_n^{(\alpha,\beta)}(a,b|q) = {}_0\phi_1\left(\begin{matrix} - \\ \alpha q \end{matrix}; q, -\alpha q a t\right) {}_2\phi_1\left(\begin{matrix} b/a, - \\ \beta q \end{matrix}; q, -at\right).$$

Proof. It follows from [49] that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}t^n}{(q,\beta q;q)_n}\mathcal{P}_n^{(\alpha,\beta)}(a|q)={}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,\alpha qat\right){}_2\phi_1\left(\begin{matrix}1/a,-\\ \beta q\end{matrix};q,at\right).$$

If a is replaced by a/b in the above equation, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}t^n}{(q,\beta q;q)_n}\mathcal{P}_n^{(\alpha,\beta)}(a/b|q)={}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,\alpha qat/b\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,at/b\right).$$

Letting further $t\to -tb$ in the above equation gives

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}t^n}{(q,\beta q;q)_n}b^n\mathcal{P}_n^{(\alpha,\beta)}(a/b|q)={}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qat\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-at\right).$$

Finally, we can deduce the conclusion by combining the above equation with (4.3). □

By using Proposition 4.1, we can determine that the right-hand side of the equation in Proposition 4.4 satisfies the q-partial differential equation (4.4). Hence, we have the following Corollary 4.5, which will be applied in Section 5.

Corollary 4.5. *We have*

$$\begin{aligned}&\mathcal{D}_{q,a}(1-\alpha\eta_a)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qat\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-at\right)\right\}\\ =\;&-q\mathcal{D}_{q,b}(\eta_b^{-1}-q\alpha\beta\eta_a^2)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qat\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-at\right)\right\}.\end{aligned}$$

5. Applications of Theorems 2.3 and 4.2

The Rogers-Szegő polynomials are famous q-polynomials which play an essential role in the theory of orthogonal polynomials. Liu [28] studied the homogeneous Rogers-Szegő polynomials from the perspective of q-partial differential equations, which are defined as

$$h_n(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q x^k y^{n-k}. \tag{5.1}$$

Further, the homogeneous Hahn polynomials

$$\Psi_n^{(a)}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q (a;q)_k x^k y^{n-k} \tag{5.2}$$

are a generalization of homogeneous Rogers-Szegő polynomials. They were first studied by Hahn [48], and then by Al-Salam and Carlitz [1]. So they are also called Al-Salam-Carlitz polynomials. The following generating functions will be frequently used (cf. [1, 29])

$$\sum_{n=0}^{\infty}\frac{\Psi_n^{(a)}(x,y|q)}{(q;q)_n}t^n=\frac{(axt;q)_\infty}{(xt,yt;q)_\infty},\ \max\{|xt|,|yt|\}<1. \tag{5.3}$$

When $a=0$, (5.3) degenerates into the generating function of homogeneous Rogers-Szegő polynomials

$$\sum_{n=0}^{\infty}\frac{h_n(x,y|q)}{(q;q)_n}t^n=\frac{1}{(xt,yt;q)_\infty},\ \max\{|xt|,|yt|\}<1. \tag{5.4}$$

We present two famous Ramanujan q-beta integrals [58, 59].

Proposition 5.1. *For* $m\in\mathbb{R}$, $0<q=e^{-2k^2}<1$, *supposing that* $|yzq|<1$, *we have*

$$\int_{-\infty}^{+\infty}\frac{e^{-\theta^2+2m\theta}}{(yq^{1/2}e^{2ki\theta};q)_\infty(zq^{1/2}e^{-2ik\theta};q)_\infty}d\theta=\sqrt{\pi}e^{m^2}\frac{(-yqe^{2mki};q)_\infty(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty}. \tag{5.5}$$

Supposing that $\max\{|yq^{1/2}e^{2mk}|,|zq^{1/2}e^{-2mk}|\}<1$, *we have*

$$\int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(-yqe^{2k\theta};q)_\infty(zqe^{-2k\theta};q)_\infty d\theta=\sqrt{\pi}e^{m^2}\frac{(yzq;q)_\infty}{(yq^{1/2}e^{2mk};q)_\infty(zq^{1/2}e^{-2mk};q)_\infty}. \tag{5.6}$$

The following Theorem 5.2 is a generalization of the Proposition 5.1.

Theorem 5.2. *For $m \in \mathbb{R}$ and $\alpha > -1$, $0 < q = e^{-2k^2} < 1$, supposing that $|yzq| < 1$, we have*

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{e^{-\theta^2+2m\theta}}{(zq^{1/2}e^{-2ik\theta};q)_\infty} \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x,y|q)h_n(-qe^{2mki},q^{1/2}e^{2ki\theta}|q)}{(q;q)_n} d\theta \\ = \;& \sqrt{\pi}e^{m^2} \frac{(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty} {}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix}; q, -q^{\alpha+2}xz\right). \end{aligned} \tag{5.7}$$

Supposing that $\max\{|yq^{1/2}e^{2mk}|, |zq^{1/2}e^{-2mk}|\} < 1$, *we have*

$$\begin{aligned}&\int_{-\infty}^{+\infty} e^{-\theta^2+2m\theta}(zqe^{-2k\theta};q)_\infty \sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} \frac{L_n^{(\alpha)}(x,y|q)g_n(-qe^{2k\theta},q^{1/2}e^{2mk}|q)}{(q;q)_n} d\theta \\ = \;& \sqrt{\pi}e^{m^2} \frac{(yzq;q)_\infty}{(zq^{1/2}e^{-2mk};q)_\infty} {}_0\phi_2\left[\begin{matrix}-\\ q^{\alpha+1}, zqy\end{matrix}; q, -q^{\alpha+2}zx\right], \end{aligned} \tag{5.8}$$

where $g_n(x,y|q)$ represent the homogeneous Stieltjes-Wigert polynomials:

$$g_n(x,y|q) = h_n(x,y|q^{-1}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)}x^k y^{n-k}.$$

Proof. (1) We use $f(x,y)$ to represent the right-hand side of (5.7). Obviously, $f(x,y)$ is analytic near $(0,0) \in \mathbb{C}^2$. It is evident from (3.5) that $f(x,y)$ satisfies

$$\mathcal{D}_{q,x}(1-q^\alpha \eta_x)\{f(x,y)\} = -q^{\alpha+1}\eta_x^2 \mathcal{D}_{q,y}\{f(x,y)\}.$$

According to Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$\sqrt{\pi}e^{m^2} \frac{(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty} {}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix}; q, -q^{\alpha+2}xz\right) = \sum_{n=0}^{\infty} \lambda_n L_n^{(\alpha)}(x,y|q). \tag{5.9}$$

By letting $x = 0$ in the above equation and using $L_n^{(\alpha)}(0,y|q) = y^n$, we can derive that

$$\sqrt{\pi}e^{m^2} \frac{(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty} = \sum_{n=0}^{\infty} \lambda_n y^n. \tag{5.10}$$

Next, by using equations (5.4) and (5.5),

$$\begin{aligned}\sqrt{\pi}e^{m^2} \frac{(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty} &= \frac{1}{(-yqe^{2mki};q)_\infty} \int_{-\infty}^{+\infty} \frac{e^{-\theta^2+2m\theta}}{(yq^{1/2}e^{2ik\theta}, zq^{1/2}e^{-2ik\theta};q)_\infty} d\theta \\ &= \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \frac{e^{-\theta^2+2m\theta}}{(zq^{1/2}e^{-2ik\theta};q)_\infty} \frac{h_n(-qe^{2mki},q^{1/2}e^{2ki\theta}|q)}{(q;q)_n} d\theta y^n. \end{aligned} \tag{5.11}$$

Then comparing the y^n coefficients of (5.10) and (5.11), we can obtain

$$\lambda_n = \int_{-\infty}^{+\infty} \frac{e^{-\theta^2+2m\theta}}{(zq^{1/2}e^{-2ik\theta};q)_\infty} \frac{h_n(-qe^{2mki},q^{1/2}e^{2ki\theta}|q)}{(q;q)_n} d\theta.$$

Finally, substitute the above equation into (5.9) to complete the proof.
(2) Similarly, we use $f(x,y)$ to represent the right-hand side of (5.8). Obviously, $f(x,y)$ is analytic near $(0,0) \in \mathbb{C}^2$. It is evident from (3.1) that $f(x,y)$ satisfies

$$\mathcal{D}_{q,x}(1-q^\alpha \eta_x)\{f(x,y)\} = -q^{\alpha+1}\eta_x^2 \mathcal{D}_{q,y}\{f(x,y)\}.$$

According to Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of x and y such that

$$\frac{\sqrt{\pi}e^{m^2}(yzq;q)_\infty}{(zq^{1/2}e^{-2mk};q)_\infty} {}_0\phi_2\left[\begin{matrix}-\\ q^{\alpha+1}, zqy\end{matrix}; q, -q^{\alpha+2}zx\right] = \sum_{n=0}^{\infty} \lambda_n L_n^{(\alpha)}(x,y|q). \tag{5.12}$$

By letting $x=0$ in the above equation and using $L_n^{(\alpha)}(0,y|q)=y^n$, we can derive that

$$\sqrt{\pi}e^{m^2}\frac{(yzq;q)_\infty}{(zq^{1/2}e^{-2mk};q)_\infty}=\sum_{n=0}^{\infty}\lambda_n y^n. \tag{5.13}$$

Next, by using equations (5.6) and [28, Theorem 3.1]:

$$(sy,ty;q)_\infty=\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}g_n(s,t|q)\frac{y^n}{(q;q)_n}.$$

So the left-side of (5.13) can be rewritten as

$$\begin{aligned}
\sqrt{\pi}e^{m^2}\frac{(yzq;q)_\infty}{(zq^{1/2}e^{-2mk};q)_\infty} &= (yq^{1/2}e^{2mk};q)_\infty\int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(-yqe^{2k\theta};q)_\infty(zqe^{-2k\theta};q)_\infty d\theta\\
&= \int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(zqe^{-2k\theta};q)_\infty(yq^{1/2}e^{2mk};q)_\infty(-yqe^{2k\theta};q)_\infty d\theta\\
&= \int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(zqe^{-2k\theta};q)_\infty\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{g_n(q^{1/2}e^{2mk},-qe^{2k\theta}|q)y^n}{(q;q)_n}d\theta\\
&= \sum_{n=0}^{\infty}\int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(zqe^{-2k\theta};q)_\infty(-1)^n q^{\binom{n}{2}}\frac{g_n(q^{1/2}e^{2mk},-qe^{2k\theta}|q)}{(q;q)_n}d\theta y^n.
\end{aligned}$$

Then comparing the y^n coefficients of (5.13) and the above equation, we can obtain

$$\lambda_n=\int_{-\infty}^{+\infty}e^{-\theta^2+2m\theta}(zqe^{-2k\theta};q)_\infty(-1)^n q^{\binom{n}{2}}\frac{g_n(q^{1/2}e^{2mk},-qe^{2k\theta}|q)}{(q;q)_n}d\theta.$$

Finally, substitute the above equation into (5.12) to complete the proof. □

Remark 5.3. *When $x=0$, (5.7) and (5.8) degenerate to (5.5) and (5.6), respectively. In the later Theorem 6.4, we will provide an equivalent form of (5.7), and (5.8) is similar, which we leave for interested readers.*

Now, we will present some applications of Theorems 2.3 and 4.2 in q-integral. The Jackson q-integral of the function $f(x)$ from a to b is defined as

$$\int_a^b f(x)d_qx=(1-q)\sum_{n=0}^{\infty}[bf(bq^n)-af(aq^n)]q^n. \tag{5.14}$$

If f is continuous on (a,b), then it is easily seen that

$$\lim_{q\to 1^-}\int_a^b f(x)d_qx=\int_a^b f(x)dx.$$

The famous Andrews-Askey integral formula [60, Theorem 1] can be stated in the following proposition.

Proposition 5.4. *For* $\max\{|bu|,|bv|,|cu|,|cv|\}<1$, *we have*

$$\int_u^v\frac{(qx/u,qx/v;q)_\infty}{(bx,cx;q)_\infty}d_qx=\frac{(1-q)v(q,u/v,qv/u,bcuv;q)_\infty}{(bu,bv,cu,cv;q)_\infty}.$$

In [29, Theorem 4.4], Liu extended Proposition 5.4 and proved the following q-integral formula.

Proposition 5.5. *If there are no zero factors in the denominator of the integral, we have*

$$\int_u^v\frac{(qx/u,qx/v,\beta ax;q)_\infty}{(ax,bx,cx,dx;q)_\infty}d_qx=\frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\sum_{n=0}^{\infty}\frac{W_n(c,d,u,v|q)\Psi_n^{(\beta)}(a,b|q)}{(q;q)_n}$$

with

$$W_n(a,b,u,v|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(av,bv;q)_k}{(abuv;q)_k}u^kv^{n-k}. \tag{5.15}$$

The main results of this section is the following Theorems 5.6 and 5.9.

Theorem 5.6. *For* $\max\{|cu|,|cv|,|du|,|dv|,|bzu|,|bzv|\}<1$, *we have*

$$\begin{aligned}&\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\mathbb{T}(\beta,y;\mathcal{D}_{q,z})\left\{\frac{1}{(bzx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}azx\right)\right\}d_qx\\ =&\ \frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\sum_{n=0}^{\infty}\frac{W_n(c,d,u,v|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}L_n^{(\alpha)}(a,b|q)\end{aligned}$$

with

$$\mathbb{T}(\beta,y;\mathcal{D}_{q,z})=\sum_{k=0}^{\infty}\frac{(\beta;q)_k}{(q;q)_k}(y\mathcal{D}_{q,z})^k,$$

it is called the Cauchy augmentation operator [61, (1.2)].

Proof. We use $I(a,b)$ to represent the left-hand side of the equation in Theorem 5.3, then we have

$$I(a,b)=\mathbb{T}(\beta,y;\mathcal{D}_{q,z})\left\{\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx,bzx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}azx\right)d_qx\right\}. \tag{5.16}$$

It is evident that the function in braces in (5.16) is analytic near $(0,0)\in\mathbb{C}^2$ for $\max\{|cu|,|cv|,|du|,|dv|,|bzu|,|bzv|\}<1$, therefore $I(a,b)$ is also analytic. By using

$$\mathbb{T}(\beta,y;\mathcal{D}_{q,z})\{z^n\}=\sum_{k=0}^{\infty}\frac{(\beta;q)_k}{(q;q)_k}y^k\mathcal{D}_{q,z}^k\{z^n\}=\Psi_n^{(\beta)}(y,z|q)$$

and (3.5), then (5.16) can be rewritten as

$$\begin{aligned}I(a,b)&=\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\mathbb{T}(\beta,y;\mathcal{D}_{q,z})\left\{\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)}{(q;q)_n}(xz)^n\right\}d_qx\\ &=\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}x^nd_qx.\end{aligned}\tag{5.17}$$

According to the definition of q-integral, it can be seen that (5.17) is a linear combination of $L_n^{(\alpha)}(a,b|q)$, namely,

$$\begin{aligned}I(a,b)=&\ (1-q)\sum_{m=0}^{\infty}\left[\frac{vq^m(vq^{m+1}/u,q^{m+1};q)_\infty}{(cvq^m,dvq^m;q)_\infty}\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}(vq^m)^n\right.\\ &\left.-\frac{uq^m(q^{m+1},uq^{m+1}/v;q)_\infty}{(cuq^m,duq^m;q)_\infty}\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}(uq^m)^n\right].\end{aligned}$$

Since $\mathcal{D}_q$ is a difference operator, it follows from the above equation and Proposition 2.1 that

$$\mathcal{D}_{q,a}(1-q^{\alpha}\eta_a)\{I(a,b)\}=-q^{\alpha+1}\eta_a^2\mathcal{D}_{q,b}\{I(a,b)\}.$$

Then by Theorem 2.3, there exists a sequence $\{\lambda_n\}$ independent of a and b such that

$$\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(a,b|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}x^nd_qx=\sum_{n=0}^{\infty}\lambda_nL_n^{(\alpha)}(a,b|q). \tag{5.18}$$

Putting $a=0$ in the above equation, using $L_n^{(\alpha)}(0,b|q)=b^n$ and (5.3), we find that

$$I(0,b)=\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\sum_{n=0}^{\infty}\frac{\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}(bx)^nd_qx=\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}\frac{(\beta ybx;q)_\infty}{(ybx,zbx;q)_\infty}d_qx=\sum_{n=0}^{\infty}\lambda_nb^n. \tag{5.19}$$

Substituting $a\to yb$ and $b\to zb$ in Proposition 5.5 yields the following result

$$\int_u^v \frac{(qx/u,qx/v,\beta ybx;q)_\infty}{(ybx,zbx,cx,dx;q)_\infty}d_qx=\frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\sum_{n=0}^{\infty}\frac{W_n(c,d,u,v|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}b^n.$$

By combining the above q-integral with (5.19) and equating the coefficients of b^n, we can obtain

$$\lambda_n=\frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\frac{W_n(c,d,u,v|q)\Psi_n^{(\beta)}(y,z|q)}{(q;q)_n}.$$

Substituting the above equation into (5.18), Theorem 5.6 follows. □

Remark 5.7. *(1) When $a=b=y=z=0$, Theorem 5.6 immediately reduces to the Proposition 5.4, so Theorem 5.6 is really an extension of the Andrews-Askey integral.*
(2) When $a=0$ and $b=1$, Theorem 5.6 becomes Proposition 5.5.
(3) When $y=0$, $z=1$ and combining (3.5), we obtain

$$\begin{aligned}&\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(bx,cx,dx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}ax\right)d_qx\\ =&\ \frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\sum_{n=0}^{\infty}\frac{W_n(c,d,u,v|q)}{(q;q)_n}L_n^{(\alpha)}(a,b|q).\end{aligned}\tag{5.20}$$

(4) Setting $d=0$ in (5.20) and noticing that $W_n(c,0,u,v|q)=\Psi_n^{(cv)}(u,v|q)$. We immediately obtain following corollary.

Corollary 5.8. *For $\max\{|cu|,|cv|,|bu|,|bv|\}<1$, we have*

$$\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(bx,cx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}ax\right)d_qx=\frac{(1-q)v(q,u/v,qv/u;q)_\infty}{(cu,cv;q)_\infty}\sum_{n=0}^{\infty}\frac{\Psi_n^{(cv)}(u,v|q)L_n^{(\alpha)}(a,b|q)}{(q;q)_n}.$$

Theorem 5.9. *For $\max\{|au|,|av|,|cu|,|cv|,|du|,|dv|,|\alpha q|,|\beta q|\}<1$, we have*

$$\begin{aligned}&\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qax\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-ax\right)d_qx\\ =&\ \frac{(1-q)v(q,u/v,qv/u,dcuv;q)_\infty}{(du,dv,cu,cv;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}W_n(d,c,u,v|q)}{(q,\beta q;q)_n}p_n^{(\alpha,\beta)}(a,b|q).\end{aligned}$$

Proof. We use $I(a,b)$ to represent the left-hand side of the equation in Theorem 5.9. Clearly, $I(a,b)$ is analytic near $(0,0)\in\mathbb{C}^2$. According to the definition of q-integral, we have

$$\begin{aligned}I(a,b)\ =\ &(1-q)\sum_{n=0}^{\infty}\left[\frac{vq^n(vq^{n+1}/u,q^{n+1};q)_\infty}{(cvq^n,dvq^n;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha avq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-avq^n\right)\right.\\ &\left.-\frac{uq^n(uq^{n+1}/v,q^{n+1};q)_\infty}{(cuq^n,duq^n;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha auq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-auq^n\right)\right].\end{aligned}\tag{5.21}$$

By setting $t=vq^n$ in Corollary 4.5, we obtain

$$\begin{aligned}&\mathcal{D}_{q,a}(1-\alpha\eta_a)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha avq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-avq^n\right)\right\}\\ =&\ -q\mathcal{D}_{q,b}(\eta_b^{-1}-q\alpha\beta\eta_a^2)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha avq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-avq^n\right)\right\}.\end{aligned}\tag{5.22}$$

Similarly,

$$\begin{aligned}&\mathcal{D}_{q,a}(1-\alpha\eta_a)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha auq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-auq^n\right)\right\}\\ =&\ -q\mathcal{D}_{q,b}(\eta_b^{-1}-q\alpha\beta\eta_a^2)\left\{{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha auq^{n+1}\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-auq^n\right)\right\}.\end{aligned}\tag{5.23}$$

Since $\mathcal{D}_q$ is a difference operator, it follows from equations (5.21)-(5.23) that

$$\mathcal{D}_{q,a}(1-\alpha\eta_a)\{I(a,b)\}=-q\mathcal{D}_{q,b}(\eta_b^{-1}-q\alpha\beta\eta_a^2)\{I(a,b)\}.$$

By Theorem 4.2, there exists a sequence $\{\lambda_n\}$ independent of a and b such that

$$\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qax\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-ax\right)d_qx=\sum_{n=0}^{\infty}\lambda_np_n^{(\alpha,\beta)}(a,b|q).\tag{5.24}$$

Letting $a=0$ into (5.24) and using $p_n^{(\alpha,\beta)}(0,b|q)=b^n$, we can find

$$I(0,b)=\int_u^v\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(bx)^n}{(q,\beta q;q)_n}\frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}d_qx=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}b^n}{(q,\beta q;q)_n}\int_u^v\frac{x^n(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}d_qx=\sum_{n=0}^{\infty}\lambda_nb^n.\tag{5.25}$$

We note that interchange the order of summation and the q-integral in (5.25) is reasonable, since

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}b^n}{(q,\beta q;q)_n}\quad\text{and}\quad\int_u^v\frac{x^n(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}d_qx$$

can easily infer that they are converges absolutely and uniformly by using the ratio test. Then by q-integral [17, (3.4)]:

$$\int_u^v\frac{x^n(qx/u,qx/v;q)_\infty}{(dx,cx;q)_\infty}d_qx=\frac{(1-q)v(q,u/v,qv/u,dcuv;q)_\infty}{(du,dv,cu,cv;q)_\infty}W_n(d,c,u,v|q).$$

Substituting the above equation into (5.25), we have

$$I(0,b)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}b^n}{(q,\beta q;q)_n}\frac{(1-q)v(q,u/v,qv/u,dcuv;q)_\infty}{(du,dv,cu,cv;q)_\infty}W_n(d,c,u,v|q)=\sum_{n=0}^{\infty}\lambda_nb^n.$$

Equating the coefficients of b^n on both sides of the above equation, we obtain

$$\lambda_n=\frac{(1-q)v(q,u/v,qv/u,dcuv;q)_\infty}{(du,dv,cu,cv;q)_\infty}\frac{q^{n(n-1)/2}W_n(d,c,u,v|q)}{(q,\beta q;q)_n}.$$

Finally, substituting the above equation into (5.24) and Theorem 5.9 follows. □

Remark 5.10. *(1) When $a=b=0$, Theorem 5.9 immediately reduces to the Andrews-Askey integral.*
(2) Setting $d=0$ in Theorem 5.9, we immediately obtain the following corollary.

Corollary 5.11. *For $\max\{|cu|,|cv|,|\alpha q|,|\beta q|\}<1$, we have*

$$\int_u^v\frac{(qx/u,qx/v;q)_\infty}{(cx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\ \alpha q\end{matrix};q,-\alpha qax\right){}_2\phi_1\left(\begin{matrix}b/a,-\\ \beta q\end{matrix};q,-ax\right)d_qx$$
$$=\frac{(1-q)v(q,u/v,qv/u;q)_\infty}{(cu,cv;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(\beta q;q)_n(q;q)_n}\Psi_n^{(cv)}(u,v|q)p_n^{(\alpha,\beta)}(a,b|q).$$

6. Concluding remark

1. This article interprets homogeneous q-Laguerre polynomials and homogeneous little q-Jacobi polynomials mainly from the perspective of q-partial differential equations, providing a new method for studying these two q-orthogonal polynomials. This research method also belongs to Liu's theory of q-partial differential equations.
2. We notice that homogeneous q-Laguerre polynomials and homogeneous Hahn polynomials appear in the Corollary 5.8. To calculate their generating function, we introduce the general double basic hypergeometric series is defined as follows [6, p. 282]

$$\Phi_{D:E;F}^{A:B;C}\left[\begin{matrix}a_A:b_B;c_C\\ d_D:e_E;f_F\end{matrix};q;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_A;q)_{m+n}(b_B;q)_m(c_C;q)_n}{(d_D;q)_{m+n}(q,e_E;q)_m(q,f_F;q)_n}$$
$$\times\left[(-1)^{m+n}q^{\binom{m+n}{2}}\right]^{D-A}\left[(-1)^mq^{\binom{m}{2}}\right]^{1+E-B}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+F-C}x^my^n, \tag{6.1}$$

where a_A abbreviates the array of A parameters $a_1,a_2,\cdots,a_A$, etc, and $q\neq0$ when $\min\{D-A,1+E-B,1+F-C\}<0$. The series (6.1) converges absolutely for $|x|,|y|<1$ when $\min\{D-A,1+E-B,1+F-C\}\geq0$ and $|q|<1$. The series (6.1) is called the q-Kampé de Fériet series when $B=C$ and $E=F$.

Theorem 6.1. *If $\max\{|uyt|,|vyt|\}<1$, then, we have*

$$\sum_{n=0}^{\infty}\frac{\Psi_n^{(\beta)}(u,v|q)L_n^{(\alpha)}(x,y|q)}{(q;q)_n}t^n=\frac{(\beta uyt;q)_\infty}{(uyt,vyt;q)_\infty}\Phi_{2:1;0}^{0:2;1}\left[\begin{matrix}-:\beta,vyt;0\\ 0,q^{\alpha+1}:\beta uyt;-\end{matrix};q;-xutq^{\alpha+1},-xvtq^{\alpha+1}\right].$$

Proof. Firstly, applying the q-partial derivative operator $\mathcal{D}_{q,t}^k$ to act both sides of the equation (5.3), and then using the formula (1.2), we deduce that

$$\sum_{n=0}^{\infty}\frac{\Psi_{n+k}^{(\beta)}(u,v|q)}{(q;q)_n}t^n=\frac{(\beta ut;q)_\infty}{(ut,vt;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}_q\frac{(\beta,vt;q)_j}{(\beta ut;q)_j}u^jv^{k-j}. \tag{6.2}$$

Let LHS to denote the left-hand side of the equation in Theorem 6.1, we have

$$\begin{aligned}\text{LHS} &= \sum_{n=0}^{\infty}\frac{\Psi_n^{(\beta)}(u,v|q)}{(q;q)_n}t^n\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_k}x^ky^{n-k}\\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^kt^nq^{k^2+k\alpha}\Psi_n^{(\beta)}(u,v|q)}{(q;q)_k(q;q)_{n-k}(q^{\alpha+1};q)_k}x^ky^{n-k}\\ &= \sum_{k=0}^{\infty}\frac{(-xt)^kq^{k^2+k\alpha}}{(q,q^{\alpha+1};q)_k}\sum_{n=0}^{\infty}\frac{\Psi_{n+k}^{(\beta)}(u,v|q)}{(q;q)_n}(yt)^n.\end{aligned}$$

Letting $t\to yt$ in (6.2), then substituting it into the above equation yields

$$\begin{aligned}\text{LHS} &= \sum_{k=0}^{\infty}\frac{(-xt)^kq^{k^2+k\alpha}}{(q,q^{\alpha+1};q)_k}\frac{(\beta uyt;q)_\infty}{(uyt,vyt;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q\frac{(\beta,yvt;q)_j}{(\beta yut;q)_j}u^jv^{k-j}\\ &= \frac{(\beta uyt;q)_\infty}{(uyt,vyt;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{(k+j)^2+(k+j)\alpha}(\beta,yvt;q)_j(-xtu)^j(-xtv)^k}{(q^{\alpha+1};q)_{k+j}(q;q)_j(q;q)_k(\beta uyt;q)_j},\end{aligned}$$

which is equivalent to the right-hand side of the equation in Theorem 6.1. □

Remark 6.2. *(1) Letting $t\to 1$, $x\to a$, $y\to b$ and $\beta\to cv$ in Theorem 6.1, and then substituting that into the equation in Corollary 5.8, we obtain*

$$\begin{aligned}&\int_u^v\frac{(qx/u,qx/v;q)_\infty}{(bx,cx;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\q^{\alpha+1}\end{matrix};q,-q^{\alpha+1}ax\right)d_qx\\ &= \frac{(1-q)v(q,u/v,qv/u,bcuv;q)_\infty}{(bu,bv,cu,cv;q)_\infty}\Phi^{0:2;1}_{2:1;0}\left[\begin{matrix}-:cv,bv;0\\0,q^{\alpha+1}:bcuv;-\end{matrix};q;-auq^{\alpha+1},-avq^{\alpha+1}\right].\end{aligned}$$

(2) Letting $\beta=0$ in Theorem 6.1, and we immediately obtain the following corollary.

Corollary 6.3. *If $\max\{|uyt|,|vyt|\}<1$, then, we have*

$$\sum_{n=0}^{\infty}\frac{h_n(u,v|q)L_n^{(\alpha)}(x,y|q)}{(q;q)_n}t^n=\frac{1}{(uyt,vyt;q)_\infty}\Phi^{0:1;1}_{2:0;0}\left[\begin{matrix}-:vyt;0\\0,q^{\alpha+1}:-;-\end{matrix};q;-xutq^{\alpha+1},-xvtq^{\alpha+1}\right].$$

Applying Corollary 6.3 to (5.7), we immediately arrive at the following theorem. The proof will be omitted.

Theorem 6.4. *For $m\in\mathbb{R}$ and $\alpha>-1$, $0<q=e^{-2k^2}<1$ and $|yzq|<1$, we have*

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{e^{-\theta^2+2m\theta}}{(yq^{1/2}e^{2ki\theta};q)_\infty(zq^{1/2}e^{-2ik\theta};q)_\infty}\times\Phi^{0:1;1}_{2:0;0}\left[\begin{matrix}-:yq^{1/2}e^{2ki\theta};0\\0,q^{\alpha+1}:-;-\end{matrix};q;xe^{2mki}q^{\alpha+2},-xve^{2ki\theta}q^{\alpha+3/2}\right]d\theta\\ &= \sqrt{\pi}e^{m^2}\frac{(-yqe^{2mki};q)_\infty(-zqe^{-2mki};q)_\infty}{(yzq;q)_\infty}{}_0\phi_1\left(\begin{matrix}-\\q^{\alpha+1}\end{matrix};q,-q^{\alpha+2}xz\right).\end{aligned}$$

Declarations

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New Weighted Inequalities for Functions Whose Higher-Order Partial Derivatives Are Co-Ordinated Convex

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Abstract

The purpose of this study is to establish recent inequalities based on double integrals of mappings whose higher-order partial derivatives in absolute value are convex on the co-ordinates on rectangle from the plane. Also, some special cases of results improved in this study are examined.

1. Introduction

In the past century, Many scholars have been interested in Hermite-Hadamard inequalities Hermite-Hadamard inequalities have attracted the interest of a good many researchers because of wide application fields in numerical analysis and in the theory of some special means. A large number of researchers have worked on new results related to Hermite-Hadamard inequalities for various function classes. One of them is co-ordinated convex functions, and we examine generalizations of these types results for co-ordinated convex functions in this work.
We define a bidimensional interval $\Delta =: [a_1,a_2]\times[b_1,b_2]$ in $\mathbb{R}^2$ with $a_1<a_2$ and $b_1<b_2$. If the inequality

$$\varphi(t\varkappa+(1-t)z, t\tau+(1-t)w) \le t\varphi(\varkappa,\tau)+(1-t)\varphi(z,w)$$

holds, $\varphi:\Delta\to\mathbb{R}$ is said to be convex on Δ, for all $(\varkappa,\tau),(z,w)\in\Delta$ and $t\in[0,1]$. If the partial functions $\varphi_\tau:[a_1,a_2]\to\mathbb{R}$, $\varphi_\tau(u)=\varphi(u,\tau)$ and $\varphi_\varkappa:[b_1,b_2]\to\mathbb{R}$, $\varphi_\varkappa(v)=\varphi(\varkappa,v)$ are convex for all $\varkappa\in[a_1,a_2]$ and $\tau\in[b_1,b_2]$, then $\varphi:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ (see, [1]).
In this case, the definition of co-ordinated convex function can be given as follows.

Definition 1.1. *Let $t,s\in[0,1]$ and $(\varkappa,u),(\tau,v)\in\Delta =: [a_1,a_2]\times[b_1,b_2]$. If the inequality*

$$\varphi(t\varkappa+(1-t)\tau, su+(1-s)v)\le ts\varphi(\varkappa,u)+s(1-t)\varphi(\tau,u)+t(1-s)\varphi(\varkappa,v)+(1-t)(1-s)\varphi(\tau,v)$$

holds, then $\varphi:\Delta\to\mathbb{R}$ will be called co-ordinated convex on Δ.

It is clearly seen that every convex mapping is co-ordinated convex. Also, A coordinated convex function that is not convex does exist (see, [1]).
Furthermore, in [1], Hermite-Hadamard type inequalities for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$ were established by Dragomir.

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Theorem 1.2. *Let $\varphi:\Delta\to\mathbb{R}$ be a co-ordinated convex mapping on Δ. Then we possess the inequalities:*

$$\begin{aligned}\varphi\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\varphi\left(\varkappa,\frac{b_1+b_2}{2}\right)d\varkappa+\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\varphi\left(\frac{a_1+a_2}{2},\tau\right)d\tau\right] \qquad (1.1)\\ &\leq \frac{1}{(a_2-a_1)(b_2-b_1)}\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(\varkappa,\tau)\,d\tau d\varkappa\\ &\leq \frac{1}{4}\left[\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\varphi(\varkappa,b_1)\,d\varkappa+\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\varphi(\varkappa,b_2)\,d\varkappa\right.\\ &+\left.\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\varphi(a_1,\tau)\,d\tau+\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\varphi(a_2,\tau)\,d\tau\right]\\ &\leq \frac{\varphi(a_1,b_1)+\varphi(a_1,b_2)+\varphi(a_2,b_1)+\varphi(a_2,b_2)}{4}.\end{aligned}$$

The above inequalities are sharp.

During the past several years, some mathematicians have worked on double integral inequalities for co-ordinated convex functions. For illustrate, Hadamard's type inequalities including Riemann-Liouville fractional integrals for convex and s-convex functions on the co-ordinates by some authors in [2] and [3]. Latif and Dragomir provided recent double integral inequalities based on the left side of Hermite- Hadamard type inequality by using co-ordinated convex functions in two variables in [4]. Novel weighted integral inequalities for functions whose partial derivatives in absolute value are convex on the co- ordinates on a rectangle from the plane are attained by Erden and Sarıkaya in [5] and [6]. some researchers derived Hermite-Hadamard type results based on the deference between the middle and the rightmost terms in (1.1) by using the derivatives of co-ordinated convex functions in [7]. Also, some mathematicians found out recent inequalities for co-ordinated convex functions in [8], [9], [10], and [11]. In [12], [13], and [14], some Hermite-Hadamard type results for different classes of co-ordinated convex mappings are developed.

On the other side, a large number of researchers have focused on inequalities involving higher-order differentiable functions. To illustrate, some integral inequalities for n-times differentiable functions are established in [15], [16] and [17]. In addition, Erden et al. gave weighted inequalities for $n-$times differentiable functions in [18]. Some mathematicians also focused on double integral inequalities including higher-order partial derivatives for two-dimensional functions in [19], [20] and [21].

In this work, we first establish a novel double integral equality based on higher-order partial derivatives. After that, recent inequalities for convex functions on the co-ordinates on the rectangle from the plane are provided. What is more, we observe relations between results in this work and inequalities presented in the earlier studies.

2. Integral identity

Before we can prove our primary findings, we establish the following equality involving mappings whose partial derivatives are continuous.

Lemma 2.1. *Assuming that $\varphi:[a_1,a_2]\times[b_1,b_2]=:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ is a continuous function such that the partial derivatives $\dfrac{\partial^{k+l}\varphi(t,s)}{\partial t^k\partial s^l}$, $k=0,1,2,...,n-1$, $l=0,1,2,...,m-1$ exists and are continuous on Δ, and suppose that the functions $g:[a_1,a_2]\to[0,\infty)$ and $h:[b_1,b_2]\to[0,\infty)$ are integrable. Additionally, $P_{n-1}(\varkappa,t)$ and $Q_{m-1}(\tau,s)$ are defined by*

$$P_{n-1}(\varkappa,t):=\begin{cases}\frac{1}{(n-1)!}\int\limits_{a_1}^{t}(u-t)^{n-1}g(u)\,du, & a_1\leq t<\varkappa\\[2ex] \frac{1}{(n-1)!}\int\limits_{a_2}^{t}(u-t)^{n-1}g(u)\,du, & \varkappa\leq t\leq a_2\end{cases}$$

and

$$Q_{m-1}(\tau,s):=\begin{cases}\frac{1}{(m-1)!}\int\limits_{b_1}^{s}(u-s)^{m-1}h(u)\,dv, & b_1\leq s<\tau\\[2ex] \frac{1}{(m-1)!}\int\limits_{b_2}^{s}(u-s)^{m-1}h(u)\,dv, & \tau\leq s\leq b_2\end{cases}$$

where $n,m \in \mathbb{N}\backslash\{0\}$. Then, for all $(\varkappa,\tau) \in [a_1,a_2]\times[b_1,b_2]$, we have the identity

$$\begin{aligned}\int_{a_1}^{a_2}\int_{b_1}^{b_2} P_{n-1}(\varkappa,t)\,Q_{m-1}(\tau,s)\,\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}dsdt \quad &= \quad \sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt \\ &- \quad \sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt \qquad (2.1)\end{aligned}$$

where $M_k(\varkappa)$ and $M_l(\tau)$ are defined by

$$M_k(\varkappa)=\int_{a_1}^{a_2}(u-\varkappa)^k g(u)\,du, \quad k=0,1,2,...$$

$$M_l(\tau)=\int_{b_1}^{b_2}(u-\tau)^l h(u)\,du, \quad l=0,1,2,...$$

Proof. Applying integration by parts for partial derivatives given in the lemma, via fundamental analysis operations, the desired identity (2.1) can be obtained. □

3. Some inequalities for co-ordinated convex mappings

For convenience, we give the following notations used to simplify the details of some results given in this section;

$$A_n(\varkappa)=(a_2-a_1)\frac{(\varkappa-a_1)^{n+1}}{n+1}+\frac{(a_2-\varkappa)^{n+2}-(\varkappa-a_1)^{n+2}}{n+2},$$

$$B_n(\varkappa)=(a_2-a_1)\frac{(a_2-\varkappa)^{n+1}}{n+1}+\frac{(\varkappa-a_1)^{n+2}-(a_2-\varkappa)^{n+2}}{n+2},$$

$$C_m(\tau)=(b_2-b_1)\frac{(\tau-b_1)^{m+1}}{m+1}+\frac{(b_2-\tau)^{m+2}-(\tau-b_1)^{m+2}}{m+2}$$

and

$$D_m(\tau)=(b_2-b_1)\frac{(b_2-y)^{m+1}}{m+1}+\frac{(\tau-b_1)^{m+2}-(b_2-\tau)^{m+2}}{m+2}.$$

We start with the following result.

Theorem 3.1. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|\frac{\partial^{n+m}\varphi}{\partial t^n\partial s^m}\right|$ is a convex function on the co-ordinates on Δ, then the following inequality holds:*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right. \qquad (3.1)\\ -&\left.\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right| \\ \leq&\frac{\|g\|_{[a_1,a_2],\infty}}{(a_2-a_1)n!}\frac{\|h\|_{[b_1,b_2]\infty}}{(b_2-b_1)m!}\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|A_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|A_n(\varkappa)D_m(\tau)\right.\\ +&\left.\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|B_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|B_n(\varkappa)D_m(\tau)\right\}\end{aligned}$$

for all $(\varkappa,\tau)\in[a_1,a_2]\times[b_1,b_2]$, where $\|g\|_{[a_1,a_2],\infty}=\sup_{u\in[a_1,a_2]}|g(u)|$ and $\|h\|_{[b_1,b_2],\infty}=\sup_{u\in[b_1,b_2]}|h(u)|$.

Proof. If we take absolute value of both sides of the equality (2.1), we find that

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right|\leq\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|dsdt.$$

Since $\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|$ is a convex function on the co-ordinates on Δ, we have

$$\begin{aligned}\left|\frac{\partial^{n+m}}{\partial t^n\partial s^m}\varphi\left(\frac{a_2-t}{a_2-a_1}a_1+\frac{t-a_1}{a_2-a_1}a_2,\frac{b_2-s}{b_2-b_1}b_1+\frac{s-b_1}{b_2-b_1}b_2\right)\right| &\leq \frac{a_2-t}{a_2-a_1}\frac{b_2-s}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right| \\ &+ \frac{a_2-t}{a_2-a_1}\frac{s-b_1}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right| \\ &+ \frac{t-a_1}{a_2-a_1}\frac{b_2-s}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right| \\ &+ \frac{t-a_1}{a_2-a_1}\frac{s-b_1}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|.\end{aligned} \tag{3.2}$$

Utilizing the inequality (3.2), we can write

$$\begin{aligned}\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|dsdt &\leq \frac{1}{(a_2-a_1)(b_2-b_1)} \\ &\times\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|\int_{a_1}^{a_2}\int_{b_1}^{b_2}(a_2-t)|P_{n-1}(\varkappa,t)|(b_2-s)|Q_{m-1}(\tau,s)|dsdt\right. \\ &+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|\int_{a_1}^{a_2}\int_{b_1}^{b_2}(a_2-t)|P_{n-1}(\varkappa,t)|(s-b_1)|Q_{m-1}(\tau,s)|dsdt \\ &+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|\int_{a_1}^{a_2}\int_{b_1}^{b_2}(t-a_1)|P_{n-1}(\varkappa,t)|(b_2-s)|Q_{m-1}(\tau,s)|dsdt \\ &+\left.\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|\int_{a_1}^{a_2}\int_{b_1}^{b_2}(t-a_1)|P_{n-1}(\varkappa,t)|(s-b_1)|Q_{m-1}(\tau,s)|dsdt\right\}.\end{aligned}$$

If we calculate the above four double inetgrals and also substitute the results in (3.3), because of $\|g\|_{[a_1,\varkappa],\infty}, \|g\|_{[\varkappa,a_2],\infty}\leq\|g\|_{[a_1,a_2],\infty}$ and $\|h\|_{[b_1,\tau]\infty}, \|h\|_{[\tau,b_2]\infty}\leq\|h\|_{[b_1,b_2]\infty}$, we obtain required inequality (3.1) which completes the proof. □

Remark 3.2. *Under the same assumptions of Theorem 3.1 with $n=m=1$, then the following inequality holds:*

$$\left|M_0(\varkappa)M_0(\tau)\varphi(\varkappa,\tau)-M_0(\tau)\int_{a_1}^{a_2}g(t)\varphi(t,\tau)dt-M_0(\varkappa)\int_{b_1}^{b_2}h(s)\varphi(\varkappa,s)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(t)h(s)\varphi(t,s)dsdt\right| \tag{3.3}$$
$$\leq\frac{\|g\|_{[a_1,a_2],\infty}}{(a_2-a_1)}\frac{\|h\|_{[b_1,b_2]\infty}}{(b_2-b_1)}\times\left\{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|A_1(\varkappa)C_1(\tau)+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|A_1(\varkappa)D_1(\tau)\right.$$
$$+\left.\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|B_1(\varkappa)C_1(\tau)+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|B_1(\varkappa)D_1(\tau)\right\}$$

which was given by Erden and Sarikaya in [22] (in case of $\lambda=0$).

Remark 3.3. *If we take $g(u)=h(u)=1$ in (3.3), then we get*

$$\left|(a_2-a_1)(b_2-b_1)\varphi(\varkappa,\tau)-(b_2-b_1)\int_{a_1}^{a_2}\varphi(t,\tau)dt\ -(a_2-a_1)\int_{b_1}^{b_2}\varphi(\varkappa,s)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right| \tag{3.4}$$
$$\leq \frac{1}{(a_2-a_1)(b_2-b_1)}\left\{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|A_1(\varkappa)C_1(\tau)+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|A_1(\varkappa)D_1(\tau)\right.$$
$$\left.+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|B_1(\varkappa)C_1(\tau)+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|B_1(\varkappa)D_1(\tau)\right\}$$

which was given by Erden and Sarıkaya in [6].

Remark 3.4. *Taking $\varkappa=\dfrac{a_1+a_2}{2}$ and $\tau=\dfrac{b_1+b_2}{2}$ in (3.4), it is found that*

$$\left|(a_2-a_1)(b_2-b_1)\varphi\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right)-(b_2-b_1)\int_{a_1}^{a_2}\varphi\left(t,\frac{b_1+b_2}{2}\right)dt\right.$$
$$\left.-\ (a_2-a_1)\int_{b_1}^{b_2}\varphi\left(\frac{a_1+a_2}{2},s\right)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right|$$
$$\leq \frac{(a_2-a_1)^2(b_2-b_1)^2}{16}\left\{\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|}{4}\right\}$$

which was given by Latif and Dragomir in [4].

Corollary 3.5. *Under the same assumptions of Theorem 3.1 with $g(u)=h(u)=1$, then we have the inequality*

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{X_k(\varkappa)}{k!}\frac{Y_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{Y_l(\tau)}{l!}\int_{a_1}^{a_2}\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\ -\sum_{k=0}^{n-1}\frac{X_k(\varkappa)}{k!}\int_{b_1}^{b_2}\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right| \tag{3.5}$$
$$\leq \frac{1}{n!(a_2-a_1)}\frac{1}{m!(b_2-b_1)}\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|A_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|A_n(\varkappa)D_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|B_n(\varkappa)C_m(\tau)\right.$$
$$\left.+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|B_n(\varkappa)D_m(\tau)\right\}$$

where $X_k(\varkappa)$ and $Y_l(\tau)$ are defined by

$$X_k(x)=\frac{(a_2-\varkappa)^{k+1}+(-1)^k(\varkappa-a_1)^{k+1}}{(k+1)} \tag{3.6}$$

and

$$Y_l(y)=\frac{(b_2-\tau)^{l+1}+(-1)^l(\tau-b_1)^{l+1}}{(l+1)}, \tag{3.7}$$

respectively. This result is a Ostrowski type inequality for mappings whose absolute value of heigher degree partial derivatives are co-ordinated convex.

Corollary 3.6. *Under the same assumptions of Theorem 3.1 with $\varkappa=\frac{a_1+a_2}{2}$ and $\tau=\frac{b_1+b_2}{2}$, then we have the inequality*

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k\left(\frac{a_1+a_2}{2}\right)}{k!}\frac{M_l\left(\frac{b_1+b_2}{2}\right)}{l!}\frac{\partial^{k+l}\varphi\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l\left(\frac{b_1+b_2}{2}\right)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi\left(t,\frac{b_1+b_2}{2}\right)}{\partial\tau^l}dt\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{M_k\left(\frac{a_1+a_2}{2}\right)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi\left(\frac{a_1+a_2}{2},s\right)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right|$$
$$\leq \frac{\|g\|_{[a_1,a_2],\infty}}{(n+1)!}\frac{\|h\|_{[b_1,b_2]\infty}}{(m+1)!}\frac{((a_2-a_1)^{n+1}}{2^{n+1}}\frac{(b_2-b_1)^{m+1}}{2^{m+1}}$$
$$\times\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|\right\}$$

*which is "**weighted mid-point**" inequality for functions whoose absolute value of heigher degree partial derivatives are co-ordinated convex.*

We establish some weighted integral inequalities by using convexity of $\left|\frac{\partial^{n+m}\varphi}{\partial t^n \partial s^m}\right|^q$.

Theorem 3.7. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|\frac{\partial^{n+m}\varphi}{\partial t^n \partial s^m}\right|^q$ is a convex function on the co-ordinates on Δ, $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$, then the following inequality holds:*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt-\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds\right. \\ &+\left.\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right| \\ \leq\ & \frac{\|g\|_{[a_1,a_2],\infty}}{n!\,(np+1)^{\frac{1}{p}}}\frac{\|h\|_{[b_1,b_2],\infty}}{m!\,(mp+1)^{\frac{1}{p}}}(a_2-a_1)^{\frac{1}{q}}(b_2-b_1)^{\frac{1}{q}}\times\left[(\varkappa-a_1)^{np+1}+(a_2-\varkappa)^{np+1}\right]^{\frac{1}{p}}\left[(\tau-b_1)^{mp+1}+(b_2-\tau)^{mp+1}\right]^{\frac{1}{p}} \\ \times\ & \left[\frac{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q}{4}\right]^{\frac{1}{q}} \end{aligned} \tag{3.8}$$

for all $(\varkappa,\tau)\in[a_1,a_2]\times[b_1,b_2]$, where $\|g\|_{[a_1,a_2],\infty}=\sup_{u\in[a_1,a_2]}|g(u)|$ and $\|h\|_{[b_1,b_2],\infty}=\sup_{u\in[b_1,b_2]}|h(u)|$.

Proof. Taking absolute value of (2.1), from Hölder's inequality, it follows that

$$\begin{aligned} &\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right. \\ -\ &\left.\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right| \\ \leq\ & \left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)|^p|Q_{m-1}(\tau,s)|^p dsdt\right)^{\frac{1}{p}}\left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q dsdt\right)^{\frac{1}{q}}. \end{aligned} \tag{3.9}$$

By utulizing the definition of $P_{n-1}(\varkappa,t)$ and $Q_{m-1}(\tau,s)$, we find that

$$\begin{aligned} \left(\int_a^b\int_c^d|P_{n-1}(\varkappa,t)|^p|Q_{m-1}(\tau,s)|^p dsdt\right)^{\frac{1}{p}} &\leq \frac{\|g\|_{[a_1,a_2],\infty}}{n!\,(np+1)^{\frac{1}{p}}}\frac{\|h\|_{[b_1,b_2],\infty}}{m!\,(mp+1)^{\frac{1}{p}}} \\ &\times \left[(\varkappa-a_1)^{np+1}+(a_2-\varkappa)^{np+1}\right]^{\frac{1}{p}}\left[(\tau-b_1)^{mp+1}+(b_2-\tau)^{mp+1}\right]^{\frac{1}{p}}. \end{aligned} \tag{3.10}$$

Since $\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q$ is a convex function on the co-ordinates on Δ, we also have

$$\begin{aligned} \left|\frac{\partial^{n+m}}{\partial t^n\partial s^m}\varphi\left(\frac{a_2-t}{a_2-a_1}a_1+\frac{t-a_1}{a_2-a_1}a_2,\frac{b_2-s}{b_2-b_1}b_1+\frac{s-b_1}{b_2-b_1}b_2\right)\right|^q &\leq \frac{a_2-t}{a_2-a_1}\frac{b_2-s}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q \\ &+ +\frac{a_2-t}{a_2-a_1}\frac{s-b_1}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q \\ &+ \frac{t-a_1}{a_2-a_1}\frac{b_2-s}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q \\ &+ +\frac{t-a_1}{a_2-a_1}\frac{s-b_1}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q. \end{aligned} \tag{3.11}$$

Using the inequality (3.11), it follows that

$$\left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q dsdt\right)^{\frac{1}{q}} \leq (a_2-a_1)^{\frac{1}{q}}(b_2-b_1)^{\frac{1}{q}} \times \left[\frac{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q}{4}\right]^{\frac{1}{q}}. \tag{3.12}$$

Substituting the inequalities (3.10) and (3.12) in (3.9), we deduce the inequality (3.8). Hence, the proof is completed. □

Remark 3.8. *Under the same assumptions of Theorem 3.7 with $n=m=1$, then the following inequality holds:*

$$\begin{aligned}&\left|M_0(\varkappa)M_0(\tau)\varphi(\varkappa,\tau)-M_0(\tau)\int_{a_1}^{a_2}g(t)\varphi(t,\tau)dt-M_0(\varkappa)\int_{b_1}^{b_2}h(s)\varphi(\varkappa,s)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(t)h(s)\varphi(t,s)dsdt\right| \quad (3.13)\\ \leq\ & \|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}(a_2-a_1)^{\frac{1}{q}}(b_2-b_1)^{\frac{1}{q}}\times\left[\frac{(\varkappa-a_1)^{p+1}+(a_2-\varkappa)^{p+1}}{p+1}\right]^{\frac{1}{p}}\left[\frac{(\tau-b_1)^{p+1}+(b_2-\tau)^{p+1}}{p+1}\right]^{\frac{1}{p}}\\ \times & \left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}\end{aligned}$$

which was given by Erden and Sarikaya in [22] (in case of $\lambda=0$).

Corollary 3.9. *Substituting $(\varkappa,\tau)=(a_1,b_1)$, (a_1,b_2), (a_2,b_1) and (a_2,b_2) in (3.13). Subsequently, if we add the obtained results and use the triangle inequality for the modulus, we get the inequality*

$$\begin{aligned}&\left|M_0(\varkappa)M_0(\tau)\frac{\varphi(a_1,b_1)+\varphi(a_1,b_2)+\varphi(a_2,b_1)+\varphi(a_2,b_2)}{4}\right. \quad (3.14)\\ +&\left.\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(t)h(s)\varphi(t,s)dsdt-\frac{1}{2}M_0(\tau)\int_{a_1}^{a_2}g(t)\left[\varphi(t,b_1)+\varphi(t,b_2)\right]dt-\frac{1}{2}M_0(\varkappa)\int_{b_1}^{b_2}h(s)\left[\varphi(a_1,s)+\varphi(a_2,s)\right]ds\right|\\ \leq\ & \|g\|_{[a_1,a_2],\infty}\|h\|_{[b_1,b_2],\infty}\frac{(a_2-a_1)^2(b_2-b_1)^2}{4(p+1)^{\frac{1}{p}}}\times\left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}\end{aligned}$$

which is a weighted Hermite-Hadamard type inequality for double integrals.

Remark 3.10. *If we take $g(u)=h(u)=1$ in (3.14), then we have*

$$\begin{aligned}&\left|\frac{\varphi(a_1,b_1)+\varphi(a_1,b_2)+\varphi(a_2,b_1)+\varphi(a_2,b_2)}{4}\right.\\ +&\left.\frac{1}{(a_2-a_1)(b_2-b_1)}\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt-\frac{1}{2(a_2-a_1)}\int_{a_1}^{a_2}\left[\varphi(t,b_1)+\varphi(t,b_2)\right]dt-\frac{1}{2(b_2-b_1)}\int_{b_1}^{b_2}\left[\varphi(a_1,s)+\varphi(a_2,s)\right]ds\right|\\ \leq\ & \frac{(a_2-a_1)(b_2-b_1)}{4(p+1)^{\frac{1}{p}}}\left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}\end{aligned}$$

which was deduced by Sarikaya et al. in [7].

Remark 3.11. *If we take $g(u)=h(u)=1$ in (3.13), then we get*

$$\left|(a_2-a_1)(b_2-b_1)\varphi(\varkappa,\tau)-(b_2-b_1)\int_{a_1}^{a_2}\varphi(t,\tau)dt-(a_2-a_1)\int_{b_1}^{b_2}\varphi(\varkappa,s)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right| \tag{3.15}$$
$$\leq (a_2-a_1)^{\frac{1}{q}}(b_2-b_1)^{\frac{1}{q}}\times\left[\frac{(\varkappa-a_1)^{p+1}+(a_2-\varkappa)^{p+1}}{p+1}\right]^{\frac{1}{p}}\left[\frac{(\tau-b_1)^{p+1}+(b_2-\tau)^{p+1}}{p+1}\right]^{\frac{1}{p}}$$
$$\times\left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}$$

which was given by Erden and Sarıkaya in [6].

Remark 3.12. *Taking $\varkappa=\frac{a_1+a_2}{2}$ and $\tau=\frac{b_1+b_2}{2}$ in (3.15), we get*

$$\left|(a_2-a_1)(b_2-b_1)\varphi\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right)-(b_2-b_1)\int_{a_1}^{a_2}\varphi\left(t,\frac{b_1+b_2}{2}\right)dt\right.$$
$$\left.-\quad(a_2-a_1)\int_{b_1}^{b_2}\varphi\left(\frac{a_1+a_2}{2},s\right)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right|$$
$$\leq\frac{(a_2-a_1)^2(b_2-b_1)^2}{4(p+1)^{\frac{2}{p}}}\times\left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}$$

which was given by Latif and Dragomir in [4].

Similarly, the other reults related to Theorem 3.7 can be obtained as in Corollary 3.5 and 3.6.

Theorem 3.13. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|\frac{\partial^{n+m}\varphi}{\partial t^n\partial s^m}\right|^q$ is a convex function on the co-ordinates on Δ, $\frac{1}{p}+\frac{1}{q}=1$ and $q\geq1$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right. \tag{3.16}$$
$$\left.-\quad\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right|$$
$$\leq\frac{1}{[(a_2-a_1)(b_2-b_1)]^{\frac{1}{q}}}\frac{\|g\|_{[a_1,a_2],\infty}}{n!(n+1)^{\frac{1}{p}}}\frac{\|h\|_{[b_1,b_2],\infty}}{m!(m+1)^{\frac{1}{p}}}\times\left[(\varkappa-a_1)^{n+1}+(a_2-\varkappa)^{n+1}\right]^{\frac{1}{p}}\left[(\tau-b_1)^{m+1}+(b_2-\tau)^{m+1}\right]^{\frac{1}{p}}$$
$$\times\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q A_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q A_n(\varkappa)D_m(\tau)\right.$$
$$\left.+\quad\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q B_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q B_n(\varkappa)D_m(\tau)\right\}^{\frac{1}{q}}$$

for all $(\varkappa,\tau)\in[a_1,a_2]\times[b_1,b_2]$, where $\|g\|_{[a_1,a_2],\infty}=\sup_{u\in[a_1,a_2]}|g(u)|$ and $\|h\|_{[b_1,b_2],\infty}=\sup_{u\in[b_1,b_2]}|h(u)|$.

Proof. We take absolute value of (2.1). Because of $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}+\frac{1}{q}$ can be written instead of 1. Using Hölder's inequality, we find that

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right. \tag{3.17}$$
$$\left.-\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right|$$
$$\leq\left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|dsdt\right)^{\frac{1}{p}}\times\left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q dsdt\right)^{\frac{1}{q}}.$$

By simple calculations, we can write

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|dsdt\leq\frac{\|g\|_{[a_1,a_2],\infty}}{(n+1)!}\frac{\|h\|_{[b_1,b_2],\infty}}{(m+1)!}\times\left[(\varkappa-a_1)^{n+1}+(a_2-\varkappa)^{n+1}\right]\left[(\tau-b_1)^{m+1}+(b_2-\tau)^{m+1}\right]. \tag{3.18}$$

By similar methods in the proof of Theorem 3.1, from (3.11), we obtain

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q dsdt \leq \frac{\|g\|_{[a_1,a_2],\infty}}{(a_2-a_1)n!}\frac{\|h\|_{[b_1,b_2],\infty}}{(b_2-b_1)m!} \tag{3.19}$$
$$\times\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q A_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q A_n(\varkappa)D_m(\tau)\right.$$
$$\left.+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q B_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q B_n(\varkappa)D_m(\tau)\right\}.$$

Substituting the inequalities (3.18) and (3.19) in (3.17), we easily deduce the required inequality (3.16) which completes the proof. □

Remark 3.14. *In case $(p,q)=(\infty,1)$, if we take limit as $p\to\infty$ in Theorem 3.13, then the inequality (3.16) becomes the inequality (3.1). Thus, we obtain all of the results which are similar to Theorem 3.1.*

4. Conclusion

In this paper, Ostrowski type inequalities for co-ordinated convex functions are developed. It is also shown that the results provided in this paper are potential generalizations of the existing comparable results in the literature. Infuture directions, one may find similar results through different types of co-ordinated convexity.
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Declarations

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An Exact Multiplicity Result for Singular Subcritical Elliptic Problems

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Abstract

For a bounded and smooth enough domain Ω in $\mathbb{R}^n$, with $n \geq 2$, we consider the problem $-\Delta u = au^{-\beta} + \lambda h(.,u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where $\lambda > 0$, $0 < \beta < 3$, $a \in L^\infty(\Omega)$, $ess\,inf\,(a) > 0$, and with $h = h(x,s) \in C\left(\overline{\Omega} \times [0,\infty)\right)$ positive on $\Omega \times (0,\infty)$ and such that, for any $x \in \Omega$, $h(x,.)$ is strictly convex on $(0,\infty)$, nondecreasing, belongs to $C^2(0,\infty)$, and satisfies, for some $p \in \left(1, \frac{n+2}{n-2}\right)$, that $\lim_{s\to\infty} \frac{h_s(x,s)}{s^p} = 0$ and $\lim_{s\to\infty} \frac{h(x,s)}{s^p} = k(x)$, in both limits uniformly respect to $x \in \overline{\Omega}$, and with $k \in C\left(\overline{\Omega}\right)$ such that $\min_{\overline{\Omega}} k > 0$. Under these assumptions it is known the existence of $\Sigma > 0$ such that for $\lambda = 0$ and $\lambda = \Sigma$ the above problem has exactly a weak solution, whereas for $\lambda \in (0,\Sigma)$ it has at least two weak solutions, and no weak solutions exist if $\lambda > \Sigma$. For such a Σ we prove that for $\lambda \in (0,\Sigma)$ the considered problem has it has exactly two weak solutions.

1. Introduction

Let $n \geq 2$, and let Ω be a C^2 bounded domain in $\mathbb{R}^n$, let $a : \Omega \to \mathbb{R}$, and let $h : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$. For $\lambda \geq 0$ and $\beta > 0$, consider the problem:

$$\begin{cases} -\Delta u = au^{-\beta} + \lambda h(.,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \\ u > 0 \text{ in } \Omega. \end{cases} \tag{1.1}$$

Singular problems like the above appear in many applications to physical and chemical process (cf. [1], [2], [3] and their references). After the pioneers works [4] [1], [3], [5], [6], [7], [2] and [8], singular elliptic problems have received a lot of interest in the literature, and many articles concern them. Let us recall some of these works:
The case when $h = 0$ in (1.1) was studied, under different hypothesis on the function a, in [5], [9], [10], and [11]. In particular, [11] gives, when a is regular enough, accurate asymptotic estimates near the boundary for the solutions. [12] studied (1.1) when $h = 0$ and a is a Radon's measure. Also, [2] studied problem (1.1) when $a = -1$, but with $h(.,u)$ replaced by a suitable positive function $h \in L^1(\Omega)$.
[8] considered the problem $-\Delta u = au^{-\beta} + h(.,\lambda u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, and proved that if $\beta > 0$, $a \in C^1\left(\overline{\Omega}\right)$, $a > 0$ in $\overline{\Omega}$, $h \in C^1\left(\overline{\Omega} \times [0,\infty)\right)$ and if, for some positive constant c, $h(x,s) > c(1+s)$ for all $(x,s) \in \overline{\Omega} \times [0,\infty)$, then there exists $\lambda^* > 0$ such that the studied problem has a positive classical solution $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ for any $\lambda \in [0,\lambda^*)$, and has no positive classical solution if $\lambda > \lambda^*$.
[13] addressed the equation $-\Delta u = au^{-\beta} + \lambda u^p$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, and obtained existence and nonexistence theorems when a is a regular enough function, with indefinite sign, $0 < \beta < 1$, $0 < p < 1$ and $\lambda \geq 0$.
[10] studied existence, nonexistence, uniqueness and stability issues for weak solutions of the problem $-\Delta u = p(x)u^{-\beta}$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, when $\beta > 0$ and $p(x)$ behaves like $d_\Omega^{-\gamma}(x)$ as $x \to \partial\Omega$, with $d_\Omega(x) := dist(x,\partial\Omega)$ and $0 < \gamma < 2$.
[14] investigates equations with singular nonlinearities that involve two bifurcation parameters.
[15] gives existence and nonexistence theorems for equations of the form $-\Delta u = g(x,u) + \lambda f(x,u,|\nabla u|)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω with $g(x,s)$ singular at $s = 0$ and also at $x \in \partial\Omega$, and where $f(x,u,|\nabla u|)$ involves a power of $|\nabla u|$.

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[16] studied the problem

$$-\Delta u = a(x)g(u) + \lambda h(u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,\ u > 0 \text{ in } \Omega, \tag{1.2}$$

where $h(s)$ is nondecreasing, positive on $(0,\infty)$, and such that $s^{-1}h(s)$ is nonincreasing; and with g satisfying $\lim_{s\to 0^+} g(s) = \infty$ but in such a way that, for some $\alpha \in (0,1)$ and $\varepsilon > 0$, $s^{\alpha}g(s)$ is bounded on $(0,\varepsilon)$. There it was introduced the space $E := \left\{v \in C^2(\Omega) \cap C^{1,1-\alpha}\left(\overline{\Omega}\right) : \Delta v \in L^1(\Omega)\right\}$ and, among other results, it was proved that if g and h are regular enough on $(0,\infty)$ and $[0,\infty)$ respectively, and if a is regular enough on $\overline{\Omega}$, then:
i) if $\lim_{s\to\infty} s^{-1}h(s) = 0$, problem (1.2) has a solution in E for any $\lambda \geq 0$.
ii) If $\lim_{s\to\infty} s^{-1}h(s) > 0$ and $\lambda \geq \frac{\lambda_1}{\lim_{s\to\infty} s^{-1}h(s)}$ (where λ_1 is the principal eigenvalue for $-\Delta$ in Ω with Dirichlet boundary condition) then (1.2) has no solutions u in E.
iii) If $\lim_{s\to\infty} s^{-1}h(s) > 0$ and $\min_{\overline{\Omega}} a > 0$ then (1.2) has a unique weak solution in E for any λ such that $0 \leq \lambda < \frac{\lambda_1}{\lim_{s\to\infty} s^{-1}h(s)}$.
[17] studied semilinear elliptic problems with singular nonlocal Neumann boundary conditions, obtaining existence and uniqueness (up to a constant) results.
In [18] existence results were obtained for a one dimensional problem involving the fractional $p-$Laplacian with multipoint boundary conditions.
Concerning multiplicity results [19] studied, for $\beta > 0$ and $1 < p \leq 2$, the problem $-\Delta_p u = g(u) + \lambda h(u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω on a smooth, bounded and strictly convex domain in $\mathbb{R}^n$, , and under suitabe conditions on g and h, there was proved that for some $\varepsilon > 0$ if $0 < \lambda < \varepsilon$ then there exist at least two weak solutions.
[20] addressed existence and multiplicity issues for positive weak solutions of a family of (p,q)-Laplacian systems on an open, bounded, and regular enough domain in $\mathbb{R}^n$. Under suitable assumptions on the problem's data, there it was proved the existence of at least two (weak) positive solutions of the system.
[21] proved that if $B : \overline{\Omega} \to M_n(\mathbb{R})$ satisfies the standard symmetry, ellipticity, and regularity conditions, and if $0 < \beta < 1 < p < \frac{n+2}{n-2}$ then, for λ positive and small enough, the problem $-\operatorname{div}(B(x)\nabla u) = u^{-\beta} + \lambda u^p$ in Ω, $u = 0$ on $\partial\Omega$ has two positive weak solutions in $H_0^1(\Omega)$.
[22] addressed the problem $-\Delta_p u = \lambda u^{-\beta} + u^q$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω under the assumptions that $0 < \beta < 1$, $1 < p < \infty$, $q < \infty$ and $p - 1 < q \leq p^* - 1$, with p^* given by $p^* := \frac{np}{n-p}$ if $p < n$, $p^* = Q$ with $Q > p$ if $p = n$, and $p^* = \infty$ if $p > n$. With these assumptions [22] proved that, for some $\lambda^* \in (0,\infty)$, the problem has a weak solution if $\lambda = \lambda^*$, has no weak solution if $\lambda > \lambda^*$, and has at least two weak solutions if $\lambda \in (0,\lambda^*)$.
[23] studied problems of the form

$$\begin{cases} -\Delta u = \lambda\left(u^{-\delta} + u^q + \rho(u)\right) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded and regular enough domain in $\mathbb{R}^n$ with $n \geq 3$, $\lambda > 0$, $\delta > 0$, $0 < q \leq 2^* - 1$ where $2^* - 1 = \frac{n+2}{n-1}$ and $\rho \in C^1([0,\infty))$ satisfies:
a) $\rho(0) = \rho'(0) = 0$, $\rho(t) + t^q \geq 0$, if $q < 2^* - 1$;
b) There exists $\beta < 2^* - 2$ such that $\lim_{t\to\infty} t^{-\beta}\rho^-(t) = 0$ and $\lim_{t\to\infty} t^{--2^*+1}\rho^+(t) = 0$ if $q = 2^* - 1$.
Under these assumptions there it was proved, for λ positive and small enough, the existence of at least two positive solutions of (1.3).
[24] studied problems of the form

$$\begin{cases} -\Delta u = \lambda\left(u^{-\delta} + h(u)e^{u^{\alpha}}\right) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.4}$$

where $\lambda > 0$, $0 < \delta < 1$, $1 \leq \alpha < 2$, and $h \in C^2[0,\infty)$ satisfies $h(0) = 0$, $s \to s^{-\delta} + h(s)e^{s^{\alpha}}$ is convex, and for any $\varepsilon > 0$, $\lim_{s\to\infty} h(s)e^{-\varepsilon s^{\alpha}} = 0$ and $\lim_{s\to\infty} h(s)e^{\varepsilon s^{\alpha}} = \infty$. Under these asumptions there were proved several existence, multiplicity, and bifurcation results for problem (1.4).
[25] studied the problem $-\Delta_N u = \lambda f(.,u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where Ω is a bounded and regular domain in R^N, Δ_N is the $N-$Laplacian on Ω, and where $f(x,s)$ is a regular enough function which may be singular at $s = 0$ and with exponential growth as $s \to \infty$. Under suitable additional assumptions on f, there it was proved the existence of $\Sigma > 0$ such that: for $0 < \lambda < \Sigma$ the problem has at least two solutions, one solution if $\lambda = \Sigma$, and no solutions when $\lambda > \Sigma$.
We mention also that the Nehari manifold method, adapted to the presence of singular nonlinearities through the study of the associated fibering functions, were used to establish multiplicity results for degenerated elliptic singular nonlinear problems involving either the pLaplacian or the weighted $p-q$ Laplacian in [26], [27], and [28]. For additional works concerning singular elliptic problems see e.g., [29], [30], [31], [32], [33], [34], [35], [36], [37], [38]; and for a systematic treatment of the subject of singular problems, we refer the readers to the research books [39] and [40] and their references.
Our aim in this work is to prove an exact multiplicity result for weak solutions of problem (1.1). By a weak sollution we mean, as usual, the given by following:

Definition 1.1. *If $\rho : \Omega \to \mathbb{R}$ is a a measurable function such that $\rho\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, and if u is a function defined on Ω we say that u is a weak solution of the problem*

$$\begin{cases} -\Delta u = \rho \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

if, and only if, $u \in H_0^1(\Omega)$ and $\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega \rho\varphi$ for all $\varphi \in H_0^1(\Omega)$.
Also, for $u \in H^1(\Omega)$ and ρ as above, we will write $-\Delta u \geq \rho$ in Ω (respectively $-\Delta u \leq \rho$ in Ω) to mean that $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \geq \int_\Omega \rho\varphi$ (resp. $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \leq \int_\Omega \rho\varphi$) for any nonnegative $\varphi \in H_0^1(\Omega)$.

Since our results depend largely on those of [35], [36], and [37], let us to briefly review them in the next three remarks:

Remark 1.2. *In [35] and [36], it was considered, for $\beta \in (0,3)$, the problem*

$$\begin{cases} -\Delta u = au^{-\beta} + f(\lambda, ., u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.5}$$

with (1.5) understood in weak sense.
Under suitable assumptions on a and f, ([35] Theorem 1.1) states that there exists $\Sigma > 0$ such that problem (1.5) has (at least) a weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, if and only if, $\lambda \in [0, \Sigma]$
Let us mention also that ([35] Theorems 1.2) says that, for λ positive and small enough, there exist at least two weak solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$. In addition, ([35] Theorem 1.1) says also that any solution u in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.5) belongs to $C(\overline{\Omega})$. In [36] all the hypothesis of [35] were assumed, plus an additional one, and in ([36] Theorem 1.2) it was proved that, for Σ as in [35] and $\lambda \in [0, \Sigma]$, problem (1.5) has a solution $u_\lambda \in H_0^1(\Omega) \cap C(\overline{\Omega})$ which is minimal in the sense that $u_\lambda \leq v$ for all weak solution $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.5). Additionally, ([36] Theorem 1.2) says that $\lambda \to u_\lambda$ is strictly increasing from $[0, \Sigma]$ into $C(\overline{\Omega})$; and ([36] Theorem 1.3) asserts that, for each $\lambda \in (0, \Sigma)$, problem (1.5) has at least two weak solutions $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$.

Remark 1.3. *Problem (1.5) was again considered in [37], where, with further hypothesis added, in ([37], Theorem 1.3) it was proved that the map $\lambda \to u_\lambda$, defined for $\lambda \in [0, \Sigma]$, with Σ and u_λ as in Remark 1.2, is continuous from $[0, \Sigma]$ into $C(\overline{\Omega})$, and belongs to $C^1((0, \Sigma), C(\overline{\Omega}))$.*

Remark 1.4. *Also, again for Σ as in Remark 1.2, ([37], Lemma 5.7) states, for each $\lambda \in [0, \Sigma]$, the existence of a solution $v_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (1.5) which is maximal respect to the partial order $\leq$, that is: v_λ has the property that if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.5) and $u \geq v_\lambda$ a.e. in Ω, then $u = v_\lambda$. We mention also that ([37], Theorem 1.4) states that, for $\lambda = \Sigma$, there exists a unique solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (1.5) (and so, in particular, $u_\Sigma = v_\Sigma$).*

We assume, from now on and without anymore mention, the following conditions H1)-H6) (with the convention that $\frac{n+2}{n-2} = \infty$ if $n = 2$):

H1) $\beta \in (0,3)$.
H2) $a \in L^\infty(\Omega)$ and $ess\,inf\,(a) > 0$.
H3) $h \in C^2(\overline{\Omega} \times [0,\infty))$ and there exists $p \in \left(1, \frac{n+2}{n-2}\right)$ such that $\lim_{s\to\infty} \frac{h(x,s)}{s^p} = k(x)$ uniformly on $x \in \overline{\Omega}$, with $k \in C(\overline{\Omega})$ such that $\min_{\overline{\Omega}} k > 0$.
H4) For all $x \in \Omega$, the function $s \to h(x,s)$ is positive, nondecreasing, strictly convex, and belongs to $C^2(0,\infty)$.
H5) $h_s > 0$ in $\overline{\Omega} \times (0,\infty)$, and $\lim_{s\to\infty} \frac{h_s(x,s)}{s^p} = 0$ uniformly on $x \in \overline{\Omega}$, where h_s denotes the partial derivative of h respect of s.
H6) There exists $q \in [1,\infty)$ and a nonnegative and nonidentically zero function $b \in L^\infty(\Omega)$, such that $h(.,s) \geq bs^q$ *a.e.* in Ω, for any $s \geq 0$.

It is immediate to check that, if β, a, and h, satisfy H1)-H6) and if $f : [0,\infty) \times \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is defined by $f(\lambda, ., s) := \lambda h(., s)$, then β, a, and f satisfy all the conditions required in [37] (and so also all the conditions imposed in [35] and [36] hold), thus all the results in [35], [36], and [37] hold for problem (1.1).

Remark 1.5. *We fix, from now on, Σ as given by Remark 1.2, but taking there $\lambda h(., s)$ instead of $f(\lambda, ., s)$, and for $\lambda \in [0, \Sigma]$, u_λ and v_λ will denote the functions provided by Remarks 1.2 and 1.4, again now with $\lambda h(., s)$ instead of $f(\lambda, ., s)$.*

Our aim in this work is to prove the following

Theorem 1.6. *Let Ω be a C^2 and bounded domain in $\mathbb{R}^n$, $n \geq 2$, and assume the conditions H1)-H6). Let Σ be as in Remark 1.5. Then for $\lambda \in (0, \Sigma)$ problem (1.1) has exactly two weak solutions.*

.
Let us briefly outline the structure of the article. In Section 2 we recall some results of [11] concerning existence, uniqueness, and asymptotic properties near the boundary, for classical solutions of problems of the form $-\Delta u = a^*(x)u^{-\beta}$ in Ω, $u=0$ on $\partial\Omega$, $u>0$ in Ω. Again in Section 2, Lemma 2.10 improves, under the assumptions H1)-H6), the regularity results of [35], [36], and [37]. In fact, it proves that any weak solution of (1.1) belongs to $C^1(\Omega)\cap C(\overline{\Omega})$.
The main objective in Section 3 is to prove that the function v_λ provided by Remark 1.4 is a maximal solution of (1.1), in the sense that $w \le v_\lambda$ for each weak solution of (1.1). After some preliminary lemmas, this is done in Lemma 3.6 by using a sub-supersolution argument. This property of v_λ plays a crucial role in the proof of Theorem 1.6
Section 4 concerns certain principal eigenvalue problems with singular potential needed for the proof of Theorem 1.6.
In Section 5 we prove Theorem 1.6 by a contradiction argument. To do it, we suppose that for some $\lambda \in (0,\Sigma)$ there exists a weak solution w of (1.1) such that $w \neq u_\lambda$ and $w \neq v_\lambda$. We rewrite (1.1) as $S(\lambda,u)=0$, where

$$S(\lambda,u) := u-(-\Delta)^{-1}\left(au^{-\beta}+\lambda h(.,u)\right),$$

and where $(-\Delta)^{-1}$ denotes the solution operator for the problem $-\Delta u = h$ in $\Omega, u=0$ on $\partial\Omega$.
From [37] we know that $S:(0,\infty)\times U_\beta \to Y_\beta$ is a continuously Frechet differentiable operator, where Y_β and U_β are, respectively, a suitable Banach's space and a suitable nonempty open subset of Y_β, with U_β such that any weak solution u of (1.1) belongs to U_β (for the definitions Y_β and U_β, see Definition 2.8 in Section 2).
In Remark 5.2 we observe that, as in [37], if $w \le v_\lambda$ and $w \neq v_\lambda$, then $r_{\lambda,w} > 1$, where $r_{\lambda,w}$ denotes the principal eigenvalue of the operator $-\Delta + \beta a w^{-\beta-1}$ in Ω, with weight function $\lambda h_s(.,w)$, and with homogeneous Dirichlet boundary condition. (notice that the potential $\beta a w^{-\beta-1}$ is singular at $\partial\Omega$).
We observe also in Remark 5.2, Lemma 5.3, and Lemma 5.4 that the condition $r_{\lambda,w} > 1$ allows, as in [37], the use of the implicit function theorem to obtain, for some $\varepsilon > 0$, a local continuously differentiable branch $\xi : (\lambda-\varepsilon,\lambda+\varepsilon) \to U_\beta$ such that $S(\sigma,\xi(\sigma)) = 0$ for all $\sigma \in (\lambda-\varepsilon,\lambda+\varepsilon)$ and $\xi(\lambda)=w$. Then we show that ξ can be extended to a continuously differentiable branch $\Theta : (0,\lambda+\varepsilon) \to U_\beta$ such that $S(\sigma,\Theta(\sigma)) = 0$ for any $\sigma \in (0,\lambda+\varepsilon)$ and $\lim_{\sigma\to 0^+}\Theta = u_0$ with convergence in Y_β, where u_0 is the unique weak solution of (1.1) for $\lambda = 0$.
Next we repeat the same process, but starting with u_λ instead of w, to obtain, for some $\varepsilon' > 0$, a continuously differentiable branch, $\Phi : (0,\lambda+\varepsilon') \to Y_\beta$ such that $S(\sigma,\Phi(\sigma)) = 0$ for $\sigma \in (0,\lambda+\varepsilon')$ and $\lim_{\sigma\to 0^+}\Phi = u_0$ with convergence in Y_β. Our final step within the proof of Theorem 1.6 will be to obtain, for $\sigma \in (0,\lambda)$, an estimate of the norm $\|\Phi(\sigma)-\Theta(\sigma)\|_{H_0^1(\Omega)}$ which, by taking the limit as $\sigma \to 0^+$, will give a contradiction.

2. Preliminaries

Let us introduce some notations we will use: δ_Ω will denote the function defined on $\overline{\Omega}$ by

$$\delta_\Omega(x) := dist(x,\partial\Omega). \tag{2.1}$$

and $(-\Delta)^{-1}$ will denote the inverse of the bijection $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$.
If ξ is a measurable functon defined on Ω we will write $\xi \in H^{-1}(\Omega)$ to mean that the map $\phi \to \int_\Omega \xi\phi$ belongs to $H^{-1}(\Omega)$
If f and g, are two functions defined $a.e.$ in Ω, we will write $f \approx g$ to mean that, for some positive constants c_1 and c_2, $c_1 f \le g \le c_2 f$ in Ω, and we will write $f \lessapprox g$ (respectively $f \gtrapprox g$) to mean that for some positive constant c, $f \le cg$ in Ω (resp. $f \ge cg$ in Ω).
If f and g are functions defined $a.e.$ in Ω, and if no confusion arises, we will write $f=g$ in Ω, $f \le g$ in Ω and $f \ge g$ in Ω to mean that $f=g$ $a.e.$ in Ω, $f \le g$ $a.e.$ in Ω and $f \ge g$ $a.e.$ in Ω respectively.
We will need the following elementary comparison lemma for singular equations:

Lemma 2.1. *i) Let $\beta > 0$, and for $i=1,2$, let $u_i \in H_0^1(\Omega)$, and let $a_i \in L^\infty(\Omega)$ be such that $ess\,inf\,(a_i) > 0$. If $a_2 \ge a_1$ and if u_1 and u_2 satisfy, in weak sense,*

$$\begin{cases} -\Delta u_1 \le a_1 u_1^{-\beta} \text{ in } \Omega, \\ u_1 = 0 \text{ on } \partial\Omega \\ u_1 > 0 \text{ in } \Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 \ge a_2 u_2^{-\beta} \text{ in } \Omega, \\ u_2 = 0 \text{ on } \partial\Omega \\ u_2 > 0 \text{ in } \Omega, \end{cases}$$

then $u_1 \le u_2$ $a.e.$ in Ω.
ii) Let $\beta > 0$, let $a \in L^\infty(\Omega)$ be such that $ess\,inf\,(a) > 0$ and, for $i=1,2$, let $u_i \in H_0^1(\Omega)$ be such that, in weak sense,

$$\begin{cases} -\Delta u_1 \le a u_1^{-\beta} \text{ in } \Omega, \\ u_1 = 0 \text{ on } \partial\Omega \\ u_1 > 0 \text{ in } \Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 \ge a u_2^{-\beta} \text{ in } \Omega, \\ u_2 = 0 \text{ on } \partial\Omega \\ u_2 > 0 \text{ in } \Omega, \end{cases}$$

then $u_1 \le u_2$ $a.e.$ in Ω.

Proof. To see i) observe that, in weak sense,

$$\begin{cases} -\Delta(u_2-u_1) \geq a_2 u_2^{-\beta} - a_1 u_1^{-\beta} \geq a_1\left(u_2^{-\beta} - u_1^{-\beta}\right) \text{ in } \Omega, \\ u_2-u_1 = 0 \text{ on } \partial\Omega, \end{cases}$$

Now we use the test function $\varphi := -(u_2-u_1)^-$ to get

$$\int_\Omega \left\|\nabla\left((u_2-u_1)^-\right)\right\|^2 \leq -\int_\Omega a_1\left(u_2^{-\beta} - u_1^{-\beta}\right)(u_2-u_1)^- \leq 0.$$

Thus, by the Poincaré's inequality, $u_1 \leq u_2$.
The proof of ii) is similar. We have, in weak sense,

$$\begin{cases} -\Delta(u_2-u_1) \geq a\left(u_2^{-\beta} - u_1^{-\beta}\right) \text{ in } \Omega, \\ u_2-u_1 = 0 \text{ on } \partial\Omega, \end{cases}$$

and so, by taking the test function $\varphi := -(u_2-u_1)^-$, we get

$$\int_\Omega \left\|\nabla\left((u_2-u_1)^-\right)\right\|^2 \leq -\int_\Omega a\left(u_2^{-\beta} - u_1^{-\beta}\right)(u_2-u_1)^- \leq 0.$$

which, as before, by the Poincaré's inequality implies $u_1 \leq u_2$ *a.e.* in Ω. □

Remark 2.2. *For $\beta \in (0,3)$ and for $a \in L^\infty(\Omega)$ such that $0 \leq a \not\equiv 0$ it is well known that there exists one and only one weak solution of the problem*

$$\begin{cases} -\Delta w = a w^{-\beta} \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega \\ w > 0 \text{ in } \Omega \end{cases} \tag{2.2}$$

(and, in fact, this follows immediately from Lemma 2.1).
Notice also that, if in addition, $a \in C^\eta_{loc}(\Omega)$ for some $\eta \in (0,1)$ and $a \approx \mathbf{1}$ in Ω then, as a particular case of ([11], Theorem 1), problem (2.2) has a unique classical solution $w \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover, $w \approx \Psi_\beta$ in Ω, with $\Psi_\beta : \overline{\Omega} \to \mathbb{R}$ given by the following definition:

Definition 2.3. *For $\beta \in (0,3)$ let $\Psi_\beta : \overline{\Omega} \to \mathbb{R}$ be defined by*

$$\begin{aligned} \Psi_\beta &:= \delta_\Omega \text{ if } 0 < \beta < 1, \\ \Psi_1 &:= \delta_\Omega\left(\log\left(\frac{\omega_0}{\delta_\Omega}\right)\right)^{\frac{1}{2}} \text{ in } \Omega \text{ and } \Psi_1 := 0 \text{ on } \partial\Omega, \\ \Psi_\beta &:= \delta_\Omega^{\frac{2}{1+\beta}} \text{ if } 1 < \beta < 3, \end{aligned}$$

with ω_0 an arbitrary constant such that $\omega_0 > diam(\Omega)$.

Notice that, in each case, $\Psi_\beta \in C(\overline{\Omega})$. The functions Ψ_β, as well as the estimates from [11] quoted in Remark 2.2 will play a relevant role in our work.

Remark 2.4. *Direct computations using the definitions of the functions Ψ_β show that $\delta_\Omega \Psi_\beta^{-\beta} \in L^2(\Omega)$ and $\Psi_\beta^{1-\beta} \in L^1(\Omega)$ for any $\beta \in (0,3)$.*

Remark 2.5. *If $a \in C^\eta_{loc}(\Omega)$ for some $\eta \in (0,1)$, and $a \approx \mathbf{1}$ in Ω, then the classical solution w of problem 2.2 (given by the result quoted in Remark 2.2) belongs to $H^1_0(\Omega)$ and is a weak solution of (2.2). Indeed, since $w \approx \Psi_\beta$ and since, for $\beta = 1$, $\Psi_\beta \lessapprox d^\gamma_\Omega$ for some $\gamma \in (0,1)$, the assertion follows from ([36], Lemma 3.2), taking there $f(\lambda, ., u) = \lambda h(., u)$ and $\lambda = 0$.*

We recall also the following lemma from [37] concerning the functions Ψ_β :

Lemma 2.6. *(See [37], Lemma 2.9) If $f \in L^\infty(\Omega)$, then $\Psi_\beta^{-\beta} f \in H^{-1}(\Omega)$ and there exists a constant $c > 0$, independent of f, such that $\left\|(-\Delta)^{-1}\left(\Psi_\beta^{-\beta} f\right)\right\|_{H^1_0(\Omega)} \leq c\|f\|_\infty$ and $\left\|\Psi_\beta^{-1}(-\Delta)^{-1}\left(\Psi_\beta^{-\beta} f\right)\right\|_\infty \leq c\|f\|_\infty$.*

Lemma 2.7. $(-\Delta)^{-1}\left(\Psi_\beta^{-\beta}\right) \approx \Psi_\beta$ *in Ω.*

Proof. By Lemma 2.6 $\Psi_\beta^{-\beta} \in H^{-1}(\Omega)$. Let $w \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ be such that

$$\begin{cases} -\Delta w = w^{-\beta} \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w > 0 \text{ in } \Omega, \end{cases} \tag{2.3}$$

(by Remark 2.2 there exists a unique such a w). Then, By Remark 2.5, $w \in H_0^1(\Omega)$ and w is a weak solution of (2.3), and by Remark 2.2, there exist positive constants c_1 and c_2 such that

$$c_1\Psi_\beta \le w \le c_2\Psi_\beta \quad \text{ in } \Omega.$$

Thus $c_1^\beta w^{-\beta} \le \Psi_\beta^{-\beta} \le c_2^\beta w^{-\beta}$ in Ω, and so $c_1^\beta(-\Delta)^{-1}\left(w^{-\beta}\right) \le (-\Delta)^{-1}\left(\Psi_\beta^{-\beta}\right) \le c_2^\beta(-\Delta)^{-1}\left(w^{-\beta}\right)$. Since $(-\Delta)^{-1}\left(w^{-\beta}\right) = w$ and $w \approx \Psi_\beta$, the lemma follows. □

The next definition introduces, for $\beta \in (0,3)$, a Banach space Y_β and an open set U_β in Y_β which will play a significant role in our arguments

Definition 2.8. *For $\beta \in (0,3)$, following [37], we define*

$$\begin{aligned} Y_\beta &:= \left\{u \in H_0^1(\Omega) : \Psi_\beta^{-1}u \in L^\infty(\Omega)\right\}, \\ \|u\|_{Y_\beta} &:= \|\nabla u\|_2 + \left\|\Psi_\beta^{-1}u\right\|_\infty \\ U_\beta &:= \left\{u \in Y_\beta : \inf_\Omega \Psi_\beta^{-1}u > 0\right\}. \end{aligned}$$

As observed in ([37], Lemma 3.2), $\left(Y_\beta, \|.\|_{Y_\beta}\right)$ is a Banach's space, and U_β is a nonempty open set in Y_β.

The next remark recalls a celebrated a-priori estimate for subcritical problems due to Gidas and Spruck. It reads as:

Remark 2.9. *(see [41], Theorem 1.1): Let $g : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ be a nonnegative and continuous function such that $\lim_{s\to\infty} \frac{g(x,s)}{s^p} = k(x)$ uniformly on x, with $p \in \left(1, \frac{n+2}{n-2}\right)$ and with $k \in C\left(\overline{\Omega}\right)$ such that $\min_{\overline{\Omega}} k > 0$. Then there exists $M \in (0,\infty)$ such that $u \le M$ for any solution (in the sense of distributions on Ω) $u \in C^1(\Omega) \cap C\left(\overline{\Omega}\right)$ of the problem*

$$\begin{cases} -\Delta u = g(.,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega \end{cases}$$

Notice that, although the proof of ([41], Theorem 1.1) was written for the case when $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, the proof can be adapted for solutions $u \in C^1(\Omega) \cap C\left(\overline{\Omega}\right)$ (as said at the comments in [41] after the statement of Theorem 1.1).

Lemma 2.10. *If u is a weak solution of (1.1) for some $\lambda \ge 0$, then*
i) $u \ge \zeta$, where ζ is the (unique) weak solution of the problem

$$\begin{cases} -\Delta\zeta = a\zeta^{-\beta} \text{ in } \Omega, \\ \zeta = 0 \text{ on } \partial\Omega. \end{cases}$$

ii) There exists a positive constant c, independent of λ and u, such that $u \ge c\Psi_\beta$ in Ω.
iii) $u \in C\left(\overline{\Omega}\right) \cap C^1(\Omega)$.
iv) $u \in U_\beta$.

Proof. i) follows immediately from the equations satisfied by u and ζ and the comparison Lemma 2.1.
To see ii) consider two positive constants k_1 and k_2 such that $k_1 \le a \le k_2$ in Ω. Since u is a weak solution of (1.1) we have, in weak sense,

$$\begin{cases} -\Delta u = au^{-\beta} + \lambda h(.,u) \ge k_1 u^{-\beta} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Let $u_0 \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ be the (unique) solution of (2.3) given by Remark 2.2. By Remark 2.5 $u_0 \in H_0^1(\Omega)$ and u_0 is a weak solution of (2.3) and, by Remark 2.2, $u_0 \ge c\Psi_\beta$ for some constant $c > 0$. Now, in weak sense,

$$\begin{cases} -\Delta\left(k_1^{\frac{1}{1+\beta}}u_0\right) = k_1\left(k_1^{\frac{1}{1+\beta}}u_0\right)^{-\beta} \text{ in } \Omega, \\ u_0 = 0 \text{ on } \partial\Omega. \end{cases}$$

Thus, by (2.4), (2.4), and Lemma 2.1, we have $u \geq k_1^{\frac{1}{1+\beta}} u_0 \geq ck_1^{\frac{1}{1+\beta}} \Psi_\beta$ in Ω, and so ii) holds.
Let us prove iii). We have $0 \leq a \in L^\infty(\Omega)$ and, by ii), $u \geq c\Psi_\beta$ for some constant $c > 0$. Therefore, for some constant $c' > 0$ we have $0 \leq au^{-\beta} \leq c'\Psi_\beta^{-\beta}$. Thus $au^{-\beta} = g\Psi_\beta^{-\beta}$ for some $g \in L^\infty(\Omega)$ and then Lemma 2.6 gives that $au^{-\beta} \in \left(H_0^1(\Omega)\right)'$. Let $z := (-\Delta)^{-1}\left(au^{-\beta}\right)$. Then, for some constant $c'' > 0$,

$$0 \leq z \leq c'(-\Delta)^{-1}\left(\Psi_\beta^{-\beta}\right) \leq c''\Psi_\beta,$$

the last inequality by Lemma 2.7. Thus $z \in L^\infty(\Omega)$. Since $0 \leq au^{-\beta} \leq c'\Psi_\beta^{-\beta}$ we have also $au^{-\beta} \in L^\infty_{loc}(\Omega)$. Thus, by the inner elliptic estimates (see e.g., [44], Theorem 8.24), $z \in C^1(\Omega)$, and since $0 \leq z \leq c''\Psi_\beta$, we have also that z is continuous at $\partial\Omega$. Thus $z \in C\left(\overline{\Omega}\right) \cap C^1(\Omega)$. Now,

$$\begin{cases} -\Delta(u-z) = \lambda h(.,u) \text{ in } \Omega, \\ u - z = 0 \text{ on } \partial\Omega. \end{cases}$$

Let $w := u - z$. Since $-\Delta(u-z) = \lambda h(.,u) \geq 0$ in Ω and $u - z = 0$ on $\partial\Omega$, the weak maximum principle gives that $w \geq 0$ *a.e.* in Ω. Thus $u \geq z$ in Ω. For $(x,s) \in \overline{\Omega} \times [0,\infty)$ let $h^*(x,s) := h(x, s + z(x))$. Then $h^* \in C\left(\overline{\Omega} \times [0,\infty)\right)$ and

$$\begin{cases} -\Delta w = \lambda h^*(.,w) \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w > 0 \text{ in } \Omega, \end{cases} \tag{2.4}$$

Now,

$$\frac{h^*(x,s)}{s^p} = \frac{h(x,s+z(x))}{s^p} = \frac{h(x,s)}{s^p} + \frac{h(x,s+z(x)) - h(x,s)}{s^p}, \tag{2.5}$$

and the mean value theorem gives that, for some $\theta = \theta_x \in (0,1)$,

$$\frac{|h(x,s+z(x)) - h(x,s)|}{s^p} = \frac{h_s(x,s+\theta z(x))z(x)}{s^p} \leq \frac{h_s(x,2s)}{s^p}\|z\|_\infty = 2^p\frac{h_s(x,2s)}{(2s)^p}\|z\|_\infty \text{ for } s \geq \|z\|_\infty$$

and thus

$$\lim_{s\to\infty} \frac{|h(x,s+z(x)) - h(x,s)|}{s^p} = 0$$

uniformly on $x \in \overline{\Omega}$. Let k be as given by H3). Then, by (2.5) $\lim_{s\to\infty} \frac{h^*(x,s)}{s^p} = k(x)$ uniformly on $x \in \overline{\Omega}$, and so, by Remark 2.9 and (2.4), $w \in L^\infty(\Omega)$. Then $\lambda h^*(.,w) \in L^\infty(\Omega)$, and thus, from (2.4), $w \in W^{2,q}(\Omega)$ for any $q \in [1,\infty)$. Then $w \in C\left(\overline{\Omega}\right) \cap C^1(\Omega)$, and thus, since $z \in C\left(\overline{\Omega}\right) \cap C^1(\Omega)$ we get that $u \in C\left(\overline{\Omega}\right) \cap C^1(\Omega)$. Thus iii) holds.
To prove iv) it only remains to see that $u \in Y_\beta$, i.e., to see that, for some positive constant c, $u \leq c\Psi_\beta$ in Ω. By iii), $u \in C\left(\overline{\Omega}\right)$, and then, by our assumptions on h, we have $\lambda h(.,u) \in L^\infty(\Omega)$. Thus, for some positive constant M, $au^{-\beta} + \lambda h(.,u) \leq Mu^{-\beta}$ in Ω. Therefore $-\Delta u \leq Mu^{-\beta}$ and so $-\Delta\left(M^{-\frac{1}{1+\beta}}u\right) \leq \left(M^{-\frac{1}{1+\beta}}u\right)^{-\beta}$ and thus, by Lemma 2.1, $M^{-\frac{1}{1+\beta}}u \leq u_0$ with u_0 as in the proof of i). By Remark 2.2 $u_0 \leq c'\Psi_\beta$ for some positive constant c'. Therefore $u \leq c'M^{\frac{1}{1+\beta}}\Psi_\beta$ in Ω, which concludes the proof of iv). □

Remark 2.11. *Lemma 2.10 says that any weak solution of (1.1) belongs to U_β, and so it improves ([37], Lemma 3.5) which, applied to our actual case, only says that any weak solution in $L^\infty(\Omega)$ belongs also to U_β.*

3. On the maximal solution of problem (1.1)

Let Σ be as in Remark 1.5 and, for each $\lambda \in [0,\Sigma]$, let v_λ as given there, which, we recall, has the property that if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.1) and $u \geq v_\lambda$, then $u = v_\lambda$.
Notice that $u_\lambda \neq v_\lambda$ for any $\lambda \in (0,\Sigma)$. Indeed, if $\lambda \in (0,\Sigma)$, ([36], Theorems 1.2 and 1.3) give two weak solutions of (1.1). Suppose that $u_\lambda = v_\lambda$, and consider any arbitrary weak solution w of (1.1). Since u_λ is minimal we have $u_\lambda \leq w$ and so we would have $v_\lambda \leq w$, which implies $v_\lambda = w$. Then $w = v_\lambda = u_\lambda$, which contradicts existence of two weak solutions of (1.1).
Our main purpose in this section is to prove that $u \leq v_\lambda$ for any weak solution u of problem (1.1). To do it, we will proceed by contradiction, using a sub-supersolutions argument.

Definition 3.1. *Let $\zeta : \Omega \to \mathbb{R}$ be a measurable function such that $\zeta\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$. As usual, a function $u : \Omega \to \mathbb{R}$ is called a weak subsolution of the problem*

$$\begin{cases} -\Delta u = \zeta \textit{ in } \Omega, \\ u = 0 \textit{ on } \partial\Omega \end{cases} \tag{3.1}$$

if $u \in H^1(\Omega)$, $u \leq 0$ *on* $\partial\Omega$, *and*

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle \leq \int_\Omega \zeta \varphi \tag{3.2}$$

for any nonnegative $\varphi \in H_0^1(\Omega)$. *Weak superolutions are similarly defined by reversing the above inequalties. Following [46], we say also that u is a subsolution, in the sense of distributions, of the problem*

$$-\Delta u = \zeta \text{ in } \Omega,$$

if $u \in L^1_{loc}(\Omega)$ *and (3.2) holds for any nonnegative* $\varphi \in C_c^\infty(\Omega)$. *Supersolutions, in the sense of distributions, are similarly defined by reversing the inequality (3.2).*

Proposition 3.2. *For* $\beta \in (0,3)$ *let* U_β *be as given in Definition 2.8. If* $\underline{u} \in U_\beta$ *and* $\overline{u} \in U_\beta$ *are a weak subsolution and a weak supersolution, respectively, of problem (1.1) such that* $\underline{u} \leq \overline{u}$, *then problem (1.1) has a weak solution* u^* *satisfying* $\underline{u} \leq u^* \leq \overline{u}$ *in* Ω.

Proof. Clearly $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution, respectively, in the sense of distributions, of (1.1). Let

$$k(x) := a(x)\underline{u}(x)^{-\beta} + \lambda h(x, \overline{u}(x))$$

Let Ψ_β be as given by Definition 2.8. Since $\underline{u}^{-\beta} \approx \Psi_\beta^{-\beta} \in L^1_{loc}(\Omega)$ and $\overline{u} \approx \Psi_\beta \in L^\infty(\Omega)$ then we have $k \in L^1_{loc}(\Omega)$. Also, $s \to a(x)s^{-\beta}$ is nonincreasing and $s \to \lambda h(x,s)$ is nondecreasing, in both cases for $a.e.\ x \in \Omega$, thus for $a.e.\ x \in \Omega$ it holds that

$$0 \leq a(x)s^{-\beta} + \lambda h(x,s) \leq k(x) \text{ for all } s \in [\underline{u}(x), \overline{u}(x)],$$

then, by ([46], Theorem 2.4) (1.1) has a solution $z \in W^{1,2}_{loc}(\Omega)$, in the sense of distributions, such that $\underline{u} \leq u \leq \overline{u}$ $a.e.$ in Ω. Since $0 \leq \underline{u} \leq \overline{u} \approx \Psi_\beta \in L^\infty(\Omega)$ we have that $u \in W^{1,2}_{loc}(\Omega) \cap L^\infty(\Omega)$ Since $\underline{u} \approx \Psi_\beta$ and $\overline{u} \approx \Psi_\beta$ we have $u \approx \Psi_\beta$. Now:
If $0 < \beta < 1$ we have $\Psi_\beta = \delta_\Omega$ and so $\underline{u} \approx \delta_\Omega$.
If $\beta = 1$ then $\Psi_\beta = \delta_\Omega \left(\log\left(\frac{\omega}{\delta_\Omega}\right)\right)^{\frac{1}{2}}$ and so, for any $\gamma \in (0,1)$, $\delta_\Omega \lessapprox \Psi_\beta \lessapprox d_\Omega^\gamma$ which gives $\delta_\Omega \lessapprox u \lessapprox \delta_\Omega^\gamma$.
If $1 < \beta < 2$ then $\Psi_\beta = \delta_\Omega^{\frac{2}{1+\beta}}$ and then $u \approx \delta_\Omega^{\frac{2}{1+\beta}}$.
Thus, by ([36], Lemma 3.2), $u \in H_0^1(\Omega)$ and u is a weak solution of (1.1). □

Remark 3.3. *(see [42], Proposition 5.9) Let U be a domain in* $\mathbb{R}^n$. *Let* f_1, $f_2 \in L^1(U)$. *If* u_1, $u_2 \in L^1(U)$ *are such that* $\Delta u_1 \geq f_1$ *and* $\Delta u_2 \geq f_2$ *in the sense of distributions in* U, *then*

$$\Delta \max\{u_1, u_2\} \geq \chi_{\{u_1 > u_2\}} f_1 + \chi_{\{u_2 > u_1\}} f_2 + \chi_{\{u_1 = u_2\}} \frac{f_1 + f_2}{2}$$

in the the sense of distributions in U.

Lemma 3.4. *If* u, v *are weak subsolutions (repectively weak supersolutions) of (1.1) then* $w := \max\{u,v\}$ *(resp.* $w := \min\{u,v\}$*) is a weak subsolution (resp. a weak supersolution) of (1.1).*

Proof. Suppose that u, v are weak subsolutions of (1.1) and consider an arbitrary $\varphi \in C_c^\infty(\Omega)$ and an open domain U such that $supp(\varphi) \subset U \subset \overline{U} \subset \Omega$. Since $u, v \in H_0^1(\Omega)$ we have $u, v \in L^1(U)$ and by Lemma 2.10, there exists a positive constant c such that $u \geq c\Psi_\beta$ and $v \geq c\Psi_\beta$ $a.e$ in Ω. Thus $au^{-\beta}$ and $av^{-\beta}$ belong to $L^1(U)$. Also, again by Lemma 2.10, u and v belong to $C(\overline{\Omega})$ and so, since h is nonnegative and $s \to h(x,s)$ is nondecreasing for $a.e.\ x \in \Omega$, we have $0 \leq h(.,u) \leq h(., \|u\|_\infty) \in L^1(U)$ and thus $h(.,u) \in L^1(U)$. Similarly, $h(.,v) \in L^1(U)$ and so $au^{-\beta} + h(.,u)$ and $av^{-\beta} + h(.,v)$ belong to $L^1(U)$. Thus, by Remark 3.3 i) used with $u_1 = u$, $u_2 = v$, $f_1 = au^{-\beta} + h(.,u)$ and $f_2 = av^{-\beta} + h(.,v)$, we have

$$\int_U \langle \nabla w, \nabla \varphi \rangle \leq \int_U \left(aw^{-\beta} + h(.,w)\right) \varphi.$$

Since u, v belong to $C(\overline{\Omega})$ we have $w \in C(\overline{\Omega})$, and so $0 \leq h(.,w) \leq h(., \|w\|_\infty)$ and thus the mapping $\psi \to \int_\Omega h(.,w)\psi$ is continuous on $H_0^1(\Omega)$, and since $w \in H_0^1(\Omega)$, the mapping $\psi \to \int_\Omega \langle \nabla w, \nabla \psi \rangle$ is also continuous on $H_0^1(\Omega)$. On the other hand, since $w \geq c\Psi_\beta$ $a.e$ in Ω, Lemma 2.6 gives the continuity of $\psi \to \int_\Omega aw^{-\beta}\psi$ on $H_0^1(\Omega)$. Thus, by density, (3.3) holds for any $\varphi \in H_0^1(\Omega)$ and so w is a subsolution of (1.1).
The assertion of the lemma in the case when u, v are supersolutions of (1.1) follows from the previous one and from the fact that $\min(u,v) = -\max(-u,-v)$. □

Lemma 3.5. *For any $k>1$ the following two statements are equivalent:*
i) The problem

$$-\Delta u = kau^{-\beta}+\lambda h(.,u) \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega,\ u>0 \text{ in } \Omega, \tag{3.3}$$

has at least a weak solution.
ii) The problem

$$-\Delta u = au^{-\beta}+\lambda h(.,u) \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega,\ u>0 \text{ in } \Omega, \tag{3.4}$$

has at least a weak solution.

Proof. Suppose that i) holds and let z be a solution of problem (3.3). Thus z is a supersolution of problem (3.4). Let u_0 be the (unique) solution of the problem

$$\begin{cases} -\Delta u_0 = au_0^{-\beta} \text{ in } \Omega, \\ u_0 = 0 \text{ on } \partial\Omega, \\ u_0>0 \text{ in } \Omega. \end{cases}$$

Then u_0 is a subsolution of (3.4). Also,

$$-\Delta\left(k^{-\frac{1}{1+\beta}}z\right) = k^{-\frac{1}{1+\beta}}kaz^{-\beta}+k^{-\frac{1}{1+\beta}}h(\lambda,.,z)\geq a\left(k^{-\frac{1}{1+\beta}}z\right)^{-\beta}$$

and so, by Lemma 2.1, $k^{-\frac{1}{1+\beta}}z\geq u_0$ in Ω. Thus $u_0\leq k^{-\frac{1}{1+\beta}}z\leq z$ in Ω. Then, by Proposition 3.2, (3.4) has a solution u such that $u_0\leq u\leq z$ *a.e* in Ω. Thus i) implies ii).
Suppose now that ii) holds, and let u be a solution of (3.4). Then $-\Delta ku = kau^{-\beta}+k\lambda h(.,u)\geq kau^{-\beta}+\lambda h(.,u)$ and so ku is a supersolution of (3.3). Also, $-\Delta\left(\frac{1}{2}u\right)=\frac{1}{2}au^{-\beta}+\frac{1}{2}\lambda h(.,u)\leq kau^{-\beta}+\lambda h(.,u)$ and so $\frac{1}{2}u$ is a subsolution of (3.3) which satisfies $\frac{1}{2}u\leq ku$. Then, by Proposition 3.2, (3.3) has a solution $\widetilde{u}$ such that $\frac{1}{2}u\leq\widetilde{u}\leq ku$ *a.e* in Ω. Thus ii) implies i). □

Lemma 3.6. *Let Σ be as in Remark 1.5 and let $\lambda\in(0,\Sigma)$. Then $w\leq v_\lambda$ for any weak solution w of problem (1.1).*

Proof. We proceed by the way of contradiction. Suppose that w is a weak solution of problem (1.1) such that

$$|\{x\in\Omega : w(x)>v_\lambda(x)\}|>0.$$

For $k>1$, by Remark 3.5, the problem

$$\begin{cases} -\Delta u = kau^{-\beta}+\lambda h(.,u) \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \\ u>0 \text{ in } \Omega. \end{cases}$$

has a weak solution z, and by Lemma 2.10, $z\in U_\beta$. Since $k>1$, $\overline{u}:=z$ is a supersolution of problem (1.1). On the other hand, by Remark 3.3, $\underline{u}:=\max(v_\lambda,w)$ is a subsolution of problem (1.1) and clearly $\underline{u}\geq v_\lambda$ and $\underline{u}\neq v_\lambda$. Observe that, for k large enough,

$$\underline{u}\leq\overline{u} \text{ a.e. in } \Omega. \tag{3.5}$$

Indeed,

$$-\Delta\left(k^{-\frac{1}{1+\beta}}z\right) = k^{-\frac{1}{1+\beta}}kaz^{-\beta}+k^{-\frac{1}{1+\beta}}\lambda h(.,z)\geq a\left(k^{-\frac{1}{1+\beta}}z\right)^{-\beta}.$$

Let u_0 be the (unique) solution of the problem

$$\begin{cases} -\Delta u_0 = au_0^{-\beta} \text{ in } \Omega, \\ u_0 = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, by Lemma 2.1, $k^{-\frac{1}{1+\beta}}z\geq u_0$ in Ω and so $z\geq k^{\frac{1}{1+\beta}}u_0$ in Ω. On the other hand, since (by Lemma 2.10) u_0, w and v_λ belong to U_β, there exist positive constants c_0, c_1 and c_2 such that $u_0\geq c_0\Psi_\beta$, $w\leq c_1\Psi_\beta$ and $v_\lambda\leq c_2\Psi_\beta$. Thus $z\geq k^{\frac{1}{1+\beta}}u_0\geq k^{\frac{1}{1+\beta}}c_0\Psi_\beta\geq k^{\frac{1}{1+\beta}}c_0c_1^{-1}w$ and, similarly, $z\geq k^{\frac{1}{1+\beta}}c_0c_2^{-1}v_\lambda$. Then (3.5) holds for $k>\max\left(1,\left(c_0^{-1}c_1\right)^{1+\beta},\left(c_0^{-1}c_2\right)^{1+\beta}\right)$.
Notice that, by the assumptions on h, $\lambda h(.,s)\in L^2(\Omega)$ for any $s>0$. Thus, by Proposition 3.2, problem (1.1) has a solution u^* such that $\underline{u}\leq u^*\leq\overline{u}$, which, since $\underline{u}\geq v_\lambda$ and $\underline{u}\neq v_\lambda$, contradicts the property of v_λ stated at the beggining of the section. □

4. Some facts about a class of principal eigenvalue problems

In this brief section we recall some facts concerning a class of principal eigenvalue problems with singular potential and weight function, which we will need to prove Theorem 1.6.

Definition 4.1. *Let* $\mathcal{B} := \left\{ b : \Omega \to \mathbb{R} : \delta_\Omega^2 b \in L^\infty(\Omega) \right\}$, *and for* $b \in \mathcal{B}$ *let* $\|b\|_{\mathcal{B}} := \left\| \delta_\Omega^2 b \right\|_\infty$, *and let* $\mathcal{B}^+ := \{ b \in \mathcal{B} : b \geq 0 \text{ in } \Omega \}$ *and* $P := \{ m \in L^\infty(\Omega) : m > 0 \text{ a.e. in } \Omega \}$.

Notice that $\mathcal{B}$ provided with the norm $\|.\|_{\mathcal{B}}$ is a Banach space.

Remark 4.2. *For* $b \in \mathcal{B}^+$ *and* $m \in P$ *consider the principal eigenvalue problem*

$$\begin{cases} -\Delta z + bz = \rho m z \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega, \\ z > 0 \text{ in } \Omega \end{cases} \tag{4.1}$$

where $\rho \in \mathbb{R}$ *and the equation is understood in weak sense, i.e.,* $z \in H_0^1(\Omega)$ *and*

$$\int_\Omega (\langle \nabla z, \nabla \varphi \rangle + bz\varphi) = \rho \int_\Omega m z \varphi$$

for any $\varphi \in H_0^1(\Omega)$. *Notice that, although b may be singular at* $\partial\Omega$ *(for instance* $\delta_\Omega^{-2} \in \mathcal{B}^+$*), by ([43], Theorem 4.1), the principal eigenvalue of (4.1) exists, is unique, positive, and simple. In order to emphasize its dependence on m and b, we will denote such a* ρ *by* $\rho_{m,b}$. *Similarly, we will denote by* $\phi_{m,b}$ *its positive eigenfunction normalized by* $\left\| \phi_{m,b} \right\|_2 = 1$. *In addition, by ([43], Theorem 4.3),* $\rho_{m,b}$ *is given by the usual Rayleigh's variational formula*

$$\rho_{m,b} = \inf_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega \left(|\nabla w|^2 + bw^2 \right)}{\int_\Omega m w^2} \tag{4.2}$$

Remark 4.3. *Let P and* $\mathcal{B}^+$ *be as in Definition 4.1, with P provided with the topology inherited from* $L^\infty(\Omega)$ *and* $\mathcal{B}^+$ *endowed with the topology inherited from the Banach space* $\mathcal{B}$. *Then, by ([43], Theorem 4.5) we have:*
i) The map $(m,b) \to \rho_{m,b}$ *is continuous from* $P \times \mathcal{B}^+$ *into* $\mathbb{R}$.
ii) The map $(m,b) \to \phi_{m,b}$ *is continuous from* $P \times \mathcal{B}^+$ *into* $H_0^1(\Omega)$.

Definition 4.4. *Let* Σ *be as in Remark 1.5 and let* U_β *be as given by Definition 2.8. For* $u \in U_\beta$, $x \in \Omega$, *and* $\sigma \in (0, \Sigma)$, *let*

$$b_u(x) := \beta a(x) u^{-\beta-1}(x)$$

and let

$$N^{\sigma,u}(x) := \sigma h_s(x, u(x))$$

where $h_s(x,t) := \frac{\partial h(x,s)}{\partial s}_{|s=t}$.

Remark 4.5. *Let* Σ *be as in Definition 4.4 and for* $\sigma \in (0,\Sigma)$ *and* $u \in U_\beta$, *consider the principal eigenvalue problem*

$$\begin{cases} -\Delta z + \beta a u^{-\beta-1} z = r\sigma h_s(.,u) z \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega, \\ z > 0 \text{ in } \Omega, \end{cases} \tag{4.3}$$

which is, with the above notations, the problem

$$\begin{cases} -\Delta z + b_u z = r N^{\sigma,u} z \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega, \\ z > 0 \text{ in } \Omega, \end{cases} \tag{4.4}$$

Notice that since $u \in U_\beta$ *and* $\sigma \in (0,\Sigma)$ *then* $b_u \in \mathcal{B}^+$ *and* $N^{\sigma,u} \in P$. *Indeed, since* $0 \leq a \in L^\infty(\Omega)$ *and* $u \in U_\beta$ *there exists a constant* $c > 0$ *such that* $0 \leq b_u \leq c\Psi_\beta^{-\beta-1}$. *Thus* $b_u \in \mathcal{B}^+$. *In fact, let* Ψ_β *be as defined by Definition 2.3 and let* δ_Ω *be defined by (2.1). Then (since* $0 < a \in L^\infty(\Omega)$*):*

i) If $0 < \beta < 1$ *then* $\Psi_\beta = \delta_\Omega$ *and so* $\delta_\Omega^2 b_u \leq c\delta_\Omega^2 \Psi_\beta^{-\beta-1} = c\delta_\Omega^{1-\beta} \in L^\infty(\Omega)$,

ii) If $\beta = 1$ *then* $\Psi_\beta = \delta_\Omega \left(\log\left(\frac{\omega_0}{\delta_\Omega} \right) \right)^{\frac{1}{2}}$ *and so* $\delta_\Omega^2 b_u \leq c\delta_\Omega^2 \Psi_\beta^{-2} = c \left(\log\left(\frac{\omega_0}{\delta_\Omega} \right) \right)^{-1} \in L^\infty(\Omega)$, *iii) If* $1<\beta<3$ *then* $\Psi_\beta=\delta_\Omega^{\frac{2}{1+\beta}}$ *and so* $\delta_\Omega^2 b_u \le c\delta_\Omega^2\Psi_\beta^{-\beta-1}=c\delta_\Omega^2\delta_\Omega^{--(\beta+1)\frac{1}{1+\beta}}\in L^\infty(\Omega)$.

Therefore, for any $\beta\in(0,3)$ *and* $u\in U_\beta$, *we have* $b_u\in\mathcal{B}^+$. *On the other hand,* $N^{\sigma,u}=\sigma h_s(.,u(.))$ *and so, from the assumptions on h stated at the introduction, it is clear that* $N^{\sigma,u}>0$ *in* Ω *and that* $N^{\sigma,u}\in L^\infty(\Omega)$, *and so* $N^{\sigma,u}\in P$.
Then, by Remark 4.2, problem (4.3) has a unique principal eigenvalue $r=\rho_{N^{\sigma,u},b_u}$ *which is unique, positive, simple, and it is given by the corresponding Rayleigh's variational formula.*

Remark 4.6. *In order to simplify the notation the principal eigenvalue of problem (4.3) will be denoted, from now on, by* $r_{\sigma,u}$ *(instead of* $\rho_{N^{\sigma,u},b_u}$*), and its normalized positive principal eigenfunction (normalized by requiring* $\|.\|_2=1$*) will be denoted by* $\phi_{\sigma,u}$ *(instead of* $\phi_{N^{\sigma,u},b_u}$*).*

We will need also the following lemma

Lemma 4.7. *Let* Y_β *and* U_β *be as given in Definition 2.8 and let* $u\in U_\beta$. *Let* Σ *be as in Remark 1.5 and let* $\sigma\in(0,\Sigma)$. *Let* $\{\sigma_j\}_{j\in\mathbb{N}}$ *and* $\{u_j\}_{j\in\mathbb{N}}$ *be sequences in* $(0,\Sigma)$ *and* U_β *respectively, and assume that* $\{\sigma_j\}_{j\in\mathbb{N}}$ *converges to* σ *and that* $\{u_j\}_{j\in\mathbb{N}}$ *converges to u in* Y_β. *Then*
i) $\{b_{u_j}\}_{j\in\mathbb{N}}$ *converges to* b_u *in* $\mathcal{B}$.
ii) $\{N^{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ *converges to* N^{σ_j,u_j} *in* $L^\infty(\Omega)$.
iii) $\{r_{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ *converges to* $r_{\sigma,u}$ *in* $\mathbb{R}$ *and* $\{\phi_{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ *converges to* $\phi_{\sigma,u}$ *in* $H_0^1(\Omega)$.

Proof. Let Ψ_β be as given by Definition 2.3. Since $u\in U_\beta$ there exists $c>0$ such that

$$u\ge c\Psi_\beta \text{ in } \Omega. \tag{4.5}$$

Let Y_β and $\|.\|_{Y_\beta}$ be as given by Definition 2.8, and let $B^{Y_\beta}\left(u,\frac{c}{2}\right)$ be the open ball in Y_β centered at u and with radius $\frac{c}{2}$. Thus for any $z\in B^{Y_\beta}\left(u,\frac{c}{2}\right)$ we have $\left\|\Psi_\beta^{-1}(z-u)\right\|_\infty<\frac{c}{2}$ and so $z>u-\frac{c}{2}\Psi_\beta\ge\left(c-\frac{c}{2}\right)\Psi_\beta=\frac{c}{2}\Psi_\beta$ in Ω. Now, $\{u_j\}_{j\in\mathbb{N}}$ converges to u in Y_β and so there exists $j_0\in\mathbb{N}$ such that $u_j\in B^{Y_\beta}\left(u,\frac{c}{2}\right)$ for any $j\ge j_0$. Then

$$u_j\ge\frac{c}{2}\Psi_\beta \text{ in } \Omega \text{ for any } j\ge j_0. \tag{4.6}$$

Let b_{u_j} and b_u be defined by Definition 4.4. Observe that, for $j\in\mathbb{N}$,

$$\left|\delta_\Omega^2 b_{u_j}-\delta_\Omega^2 b_u\right|=\left|\beta a\delta_\Omega^2\left(\left(u_j^{-\beta-1}-u^{-\beta-1}\right)\right)\right|\le c\left|\delta_\Omega^2\left(u_j^{-\beta-1}-u^{-\beta-1}\right)\right| \text{ in } \Omega, \tag{4.7}$$

where $c=\beta\|a\|_\infty$ is a positive constant independent of j. Now, for $x\in\Omega$, the mean value theorem gives that

$$u_j^{-\beta-1}(x)-u^{-\beta-1}(x)=-(\beta+1)\theta_{j,x}^{-\beta-2}(u_j(x)-u(x)) \tag{4.8}$$

for some number $\theta_{j,x}$ belonging to the open segment with endpoints $u_j(x)$ and $u(x)$, and so, by (4.5) and (4.6),

$$\theta_{j,x}\ge\frac{c}{2}\Psi_\beta(x) \text{ in } \Omega \text{ for any } x\in\Omega \text{ whenever } j\ge j_0. \tag{4.9}$$

Therefore, from (4.7), (4.8), and (4.9), we have, for any $j\ge j_0$,

$$\left|\delta_\Omega^2 b_{u_j}-\delta_\Omega^2 b_u\right|\le\frac{c(\beta+1)\delta_\Omega^2|u_j-u|}{\left(\frac{c}{2}\Psi_\beta\right)^{\beta+2}}=c'\frac{\delta_\Omega^2\Psi_\beta\left|\Psi_\beta^{-1}(u_j-u)\right|}{\Psi_\beta^{\beta+2}}=c'\delta_\Omega^2\Psi_\beta^{-\beta-1}\left|\Psi_\beta^{-1}(u_j-u)\right| \text{ in } \Omega, \tag{4.10}$$

with c' a positive constant independent of j. Direct computations using the definition of the functinos Ψ_β give that

$$\delta_\Omega^2\Psi_\beta^{-\beta-1}\in L^\infty(\Omega). \tag{4.11}$$

Then, by (4.10) and (4.11) we get

$$\left|\delta_\Omega^2 b_{u_j}-\delta_\Omega^2 b_u\right|\le c''\left|\Psi_\beta^{-1}(u_j-u)\right| \text{ in } \Omega,$$

with c'' a positive constant independent of j, and since $\{u_j\}_{j\in\mathbb{N}}$ converges to u in Y_β we have also that $\lim_{j\to\infty}\left\|\Psi_\beta^{-1}(u_j-u)\right\|_\infty=0$ and then

$$\lim_{j\to\infty}\left\|\delta_\Omega^2 b_{u_j}-\delta_\Omega^2 b_u\right\|_\infty=0$$

which gives that $\left\{b_{u_j}\right\}_{j\in\mathbb{N}}$ converges to b_u in $\mathcal{B}$. Thus i) holds

Let us see that $\{N^{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ converges to $N^{\sigma,u}$ in $L^\infty(\Omega)$. Since each Ψ_β is bounded, $\lim_{j\to\infty}\left\|\Psi_\beta^{-1}(u_j-u)\right\|_\infty=0$ implies that

$$\lim_{j\to\infty}\left\|u_j-u\right\|_\infty=0.$$

Also $u\in L^\infty(\Omega)$ (because $\Psi_\beta^{-1}u\in L^\infty(\Omega)$). Then $\left\{u_j\right\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Thus there exists $M>0$ such that $\|u\|_\infty\le M$ and $\left\|u_j\right\|_\infty\le M$ for all $j\in\mathbb{N}$. Then for each j there exists $E_j\subset\Omega$ such that $\left|E_j\right|=0$ and $0\le u_j\le M$ in $\Omega\setminus E_j$, and there exists $E\subset\Omega$ such that $|E|=0$ and $0\le u\le M$ in $\Omega\setminus E$. Let $F:=E\cup\cup_{j\in\mathbb{N}}E_j$. Then $|F|=0$, $0\le u\le M$ in $\Omega\setminus F$ and $0\le u_j\le M$ in $\Omega\setminus F$ for all $j\in\mathbb{N}$. Now, by our assumptions on h stated at the introduction, there exists a constant $M^*>0$ such that $|h_s(x,t)|\le M^*$ and $|h_{ss}(x,t)|\le M^*$ for any $(x,t)\in\Omega\times[0,M]$. Then by the triangle inequality and the mean value theorem we have, for any $x\in\Omega\setminus F$ and for all $j\in\mathbb{N}$,

$$\begin{aligned}
&|N^{\sigma_j,u_j}(x)-N^{\sigma,u}(x)|=\left|\sigma_j h_s(x,u_j(x))-\sigma h_s(x,u(x))\right|\\
&\le\left|(\sigma_j-\sigma)h_s(x,u_j(x))\right|+\left|\sigma h_s(x,u_j(x))-\sigma h_s(x,u(x))\right|\\
&\le\left|(\sigma_j-\sigma)\right|\left|h_s(x,u_j(x))\right|+\sigma\left|h_{ss}(x,\zeta_{j,x})\right|\left|u_j(x)-u(x)\right|
\end{aligned}\tag{4.12}$$

where $\zeta_{j,x}$ is a number belonging to the open segment with endpoints $u_j(x)$ and $u(x)$. Then, for $x\in\Omega\setminus F$ and for all $j\in\mathbb{N}$, $\left|h_{ss}(x,\zeta_{j,x})\right|\le M^*$ and so, for such x and j, (4.12) gives

$$|N^{\sigma_j,u_j}(x)-N^{\sigma,u}(x)|\le M^*\left|\sigma_j-\sigma\right|+\sigma M^*\left|u_j(x)-u(x)\right|$$

which, since $\lim_{j\to\infty}\left\|u_j-u\right\|_\infty=0$ and $\lim_{j\to\infty}\sigma_j=\sigma$, implies that $\{N^{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ converges to $N^{\sigma,u}m$ in $L^\infty(\Omega)$. Thus ii) holds. Now, iii) follows from i), ii), and Remark 4.3. ☐

5. Proof of the main results

We fix, for the whole section, Σ as given by Remark 1.5.

Definition 5.1. *Let Y_β and U_β be as in Definition 2.8, and let $S:(0,\Sigma)\times U_\beta\to Y_\beta$ be defined by*

$$S(\lambda,u):=u-(-\Delta)^{-1}\left(au^{-\beta}+\lambda h(.,u)\right).\tag{5.1}$$

By ([37], Lemma 3.3) we have $au^{-\beta}+\lambda h(.,u)\in H^{-1}(\Omega)$, and $(-\Delta)^{-1}\left(au^{-\beta}+\lambda h(\lambda,u)\right)\in Y_\beta$ for any $(\lambda,u)\in(0,\Sigma)\times U_\beta$, therefore S is well defined. Moreover, by ([37], Lemma 3.7), ([37], Corollary 3.8), and ([37], Lemma 3.9) (all of them applied with $f(\lambda,.,s):=\lambda h(.,s)$) the operator S is continuously Fréchet differentiable in $(0,\Sigma)\times U_\beta$, and its differential at $(\lambda,u)\in(0,\Sigma)\times U_\beta$, denoted by $DS_{(\lambda,u)}$, is given by

$$DS_{(\lambda,u)}(\tau,\psi)=\psi-(-\Delta)^{-1}\left(-\beta a\psi u^{-\beta-1}+\tau h(.,u)+\psi\lambda\tfrac{\partial h}{\partial s}(.,u)\right),\tag{5.2}$$

and its partial derivative $D_2S_{(\lambda,u)}$ at (λ,u) (i.e. the Fréchet differential at u, of the mapping $v\to S(\lambda,v)$) is given by

$$D_2S_{(\lambda,u)}(\psi)=\psi-(-\Delta)^{-1}\left(\left(-\beta au^{-\beta-1}+\lambda\frac{\partial h}{\partial s}(.,u)\right)\psi\right)\tag{5.3}$$

We recall that, as said in Remark 4.6, the principal eigenvalue of a problem of the form (4.3) will be denoted by $r_{\sigma,u}$.

Remark 5.2. *In ([37], Lemma 5.17) it is proved that if $\lambda\in(0,\Sigma)$ and if u_λ is the minimal solution (as provided by Remark 1.2) of (1.1) then,*

$$r_{\lambda,u_\lambda}>1.$$

where r_{λ,u_λ} denotes the principal eigenvalue of problem 4.3, taking there $u=u_\lambda$. By using this fact and a maximum principle with weight function given by ([37], Lemma 4.4), in ([37], Lemma 5.18) it was proved that

$$D_2S_{(\lambda,u_\lambda)}:Y_\beta\to Y_\beta \text{ is bijective.}$$

An inspection of the proofs of lemmas ([37], Lemma 5.17) and ([37], Lemma 5.18) shows that they work also if u_λ is replaced by any weak solution u of (1.1) such that $u\le v_\lambda$ and $u\ne v_\lambda$.

Lemma 5.3. *Let $\lambda\in(0,\Sigma)$, and let u be a weak solution of (1.1) such that $u\ne v_\lambda$, then:*
ii) $r_{\lambda,u}>1$.
ii) $D_2S_{(\lambda,u)}:Y_\beta\to Y_\beta$ is bijective.

Proof. By Lemma 3.6) we actually know that $u \leq v_\lambda$ for any weak solution of (1.1), then the lemma follows from Remark 5.2 □

Now we can prove the following

Lemma 5.4. *Let $\lambda \in (0,\Sigma)$, and let u be a weak solution of (1.1) such that $r_{\lambda,u} > 1$, then there exist $\varepsilon > 0$ and an open set $V \subset Y_\beta$ such that $u \in V \subset U_\beta$ and if $J := (\lambda - \varepsilon, \lambda + \varepsilon)$ then:*
i) $J \subset (0,\Sigma)$ and for any $\sigma \in J$ there exists a unique $\xi(\sigma) \in V$ such that

$$\begin{cases} S(\sigma, \xi(\sigma)) = 0, \\ \xi(\lambda) = u. \end{cases}$$

Moreover, $\xi : J \to Y_\beta$ is continuously differentiable, and its derivative ξ' satisfies, in weak sense, for any $\sigma \in J$,

$$\begin{cases} -\Delta(\xi'(\sigma)) = -\beta a(\xi(\sigma))^{-(1+\beta)} \xi'(\sigma) + h(.,\xi(\sigma)) + \sigma \frac{\partial h}{\partial s}(.,\xi(\sigma)) \xi'(\sigma) \text{ in } \Omega, \\ \xi'(\sigma) = 0 \text{ on } \partial\Omega.. \end{cases} \tag{5.4}$$

ii) $r_{\sigma,\xi(\sigma)} > 1$ for any $\sigma \in J$.
iii) $\sigma \to \xi(\sigma)$ is nondecreasing on J.
iv) $\sigma \to r_{\sigma,\xi(\sigma)}$ is nonincreasing on J.

Proof. The first assertion of i) follows from Lemma 5.3 and the implicit function theorem and, since $S(\sigma,\xi(\sigma)) = 0$ for any $\sigma \in J$, (5.4) follows from (5.2) and the chain rule.
To see ii), observe that, by i), $\sigma \to \xi(\sigma)$ is continuous from J into Y_β. Then, by lemma 4.7, $\sigma \to r_{\sigma,\xi(\sigma)}$ is continuous on J. Thus, since $r_{\lambda,u} > 1$, by diminishing ε if necessary, we get that $r_{\sigma,\xi(\sigma)} > 1$ for all $\sigma \in J$. Thus ii) holds.
Let us see iii). We rewrite (5.4) as

$$\begin{cases} -\Delta(\xi'(\sigma)) + \beta a(\xi(\sigma))^{-\beta-1} \xi'(\sigma) = N^{\sigma,\xi(\sigma)} \xi'(\sigma) + h(.,\xi(\sigma)) \text{ in } \Omega, \\ \xi'(\sigma) = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, since $h(.,\xi(\sigma)) \geq 0$ and $r_{\sigma,\xi(\sigma)} > 1$, the maximum principle with weight stated in ([37] Lemma 4.4 ii)) gives that $\xi'(\sigma) \geq 0$ for any $\sigma \in J$. Thus $\sigma \to \xi(\sigma)$ is nondecreasing on J, and so iii) holds.
To see iv), observe that for σ, $\tau \in J$ such that $\sigma \leq \tau$ we have, by iii), $\xi(\sigma) \leq \xi(\tau)$ in Ω, and so, by the assumptions on h stated at the introduction,

$$N^{\sigma,\xi(\sigma)} = \sigma \frac{\partial h}{\partial s}(.,\xi(\sigma)) \leq \sigma \frac{\partial h}{\partial s}(.,\xi(\tau)) \leq \tau \frac{\partial h}{\partial s}(.,\xi(\tau)) = N^{\tau,\xi(\tau)} \text{ in } \Omega.$$

Then

$$r_{\sigma,\xi(\sigma)} = \inf_{z \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[|\nabla z|^2 + \beta a \xi(\sigma)^{-\beta-1} z^2\right]}{\int_\Omega N^{\sigma,\xi(\sigma)} z^2} \geq \inf_{z \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[|\nabla z|^2 + \beta a \xi(\tau)^{-\beta-1} z^2\right]}{\int_\Omega N^{\tau,\xi(\tau)} z^2.} = r_{\tau,\xi(\tau)},$$

and thus iv) holds.

□

Let us recall the Hardy's inequality (see e.g., [45], p. 313):

$$\sup_{0 \neq \varphi \in H_0^1(\Omega)} \frac{\left\|\varphi \delta_\Omega^{-1}\right\|_{L^2(\Omega)}}{\|\nabla \varphi\|_{L^2(\Omega)}} < \infty$$

Lemma 5.5. *Let Σ be as given by Remark 1.5 and let Y_β and U_β be as given in Definition 2.8. Let $\lambda_0 \in (0,\Sigma)$ and let w_0 be a weak solution of problem (1.1) such that $w_0 \neq v_{\lambda_0}$, where v_{λ_0} is the maximal solution of (1.1) corresponding to $\lambda = \lambda_0$. Then there exists $\varepsilon > 0$ and a function $\Theta : [0, \lambda_0 + \varepsilon) \to U_\beta$ such that*
i) $\Theta(\lambda_0) = w_0$.
ii) $\Theta \in C^1\left((0,\lambda_0+\varepsilon), Y_\beta\right) \cap C\left([0,\lambda_0+\varepsilon), Y_\beta\right)$.
iii) $S(\sigma, \Theta(\sigma)) = 0$ for any $\sigma \in (0, \lambda_0 + \varepsilon)$
iv) $\Theta(0) = u_0$ with convergence in Y_β, where u_0 is the (unique) weak solution of (1.1) corresponding to $\lambda = 0$.

Proof. Let $\mathcal{G}$ be the family of the pairs (J, ξ_J) such that:
1) J is an open interval in $\mathbb{R}$, $J \subset (0,\Sigma)$, and $\lambda_0 \in J$.
2) $\xi_J \in C^1\left(J, U_\beta\right)$, $\xi_J(\lambda_0) = w_0$, and $S(\sigma, \xi_J(\sigma)) = 0$ for all $\sigma \in J$.
3) $r_{\sigma,\xi_J(\sigma)} > 1$ for any $\sigma \in J$.
4) $\sigma \to \xi_J(\sigma)$ is nondecreasing on J

5) $\sigma \to r_{\sigma,\xi_J(\sigma)}$ is nonincreasing on J.
By Lemma 5.4, $\mathcal{G} \neq \varnothing$. Notice that, since $\xi_J \in C^1\left(J, U_\beta\right)$ then, by Lemma 4.7,

$$\sigma \to r_{\sigma,\xi_J(\sigma)} \text{ is continuous on } J. \tag{5.5}$$

We claim that:

$$\text{If } (J,\xi_J) \in \mathcal{G}, (J_*,\xi_{J_*}) \in \mathcal{G}, \text{ and } J \cap J_* \neq \varnothing, \text{ then } \xi_J = \xi_{J_*} \text{ in } J \cap J_*. \tag{5.6}$$

Indeed, let $F := \{\sigma \in J \cap J_* : \xi_J(\sigma) = \xi_{J_*}(\sigma)\}$ Then $\lambda_0 \in F$ and so $F \neq \varnothing$. Also, since ξ_J and ξ_{J_*} are continuous in their respective domains, F is closed in $J \cap J_*$. Moreover, if $\sigma \in F$ then, by the uniqueness assertion of Lemma 5.4 i) (used taking there $\lambda = \sigma$), there exists $\varepsilon > 0$ such that $(\sigma - \varepsilon, \sigma + \varepsilon) \subset F$, and so F is open in $J \cap J_*$. Then, since $J \cap J_*$ is a connected set, we conclude that $F = J \cap J_*$, and thus $\xi_J = \xi_{J_*}$ in $J \cap J_*$.
Let $I := \cup_{J:(J,\xi_J)\in\mathcal{G}} J$. Since I is a union of open intervals contained in $(0,\Sigma)$, and all of them contain λ_0, it follows that

$$I \text{ is an open interval, } I \subset (0,\Sigma), \text{ and } \lambda_0 \in I. \tag{5.7}$$

Let $\Theta : I \to U_\beta$ be defined by

$$\Theta(\sigma) := \xi_J(\sigma) \text{ if } \sigma \in J \text{ for some } (J,\xi_J) \in \mathcal{G}. \tag{5.8}$$

By (5.6) Θ is well defined on I and, from 2) and (5.8),

$$\Theta \in C^1\left(I, U_\beta\right), \text{ and } S(\sigma,\Theta(\sigma)) = 0 \text{ for all } \sigma \in I, \tag{5.9}$$

(Later, within the proof of the lemma, we will define also $\Theta(\lambda_*)$, where λ_* is the left endpoint of I, and we will show that Θ is continuous at λ_*. and that $\lambda_* = 0$). For $\sigma \in I$, let $(J,\xi_J) \in \mathcal{G}$ such that $\sigma \in J$. From (5.8) we have $r_{\sigma,\Theta(\sigma)} = r_{\sigma,\xi_J(\sigma)}$ and, by 3), $r_{\sigma,\xi_J(\sigma)} > 1$. Then,

$$r_{\sigma,\Theta(\sigma)} > 1 \text{ for any } \sigma \in J. \tag{5.10}$$

Suppose that $t \in I$, $s \in I$, and $t \leq s \leq \lambda_0$, and let $(J,\xi_J) \in \mathcal{G}$ such that $t \in J$. Then $s \in J$, and so, since by 4) ξ_J is nondecreasing on J, and taking into account the definition (5.8) of Θ we have $\Theta(t) \leq \Theta(s)$. Then

$$\Theta \text{ is nondecreasing on } I. \tag{5.11}$$

For $\sigma \in I$, let $N^{\sigma,\Theta(\sigma)}$ be defined as in Definition 4.4, that is,

$$N^{\sigma,\Theta(\sigma)}(x) = \sigma h_s(x,\Theta(\sigma)(x)) \text{ for any } x \in \Omega.$$

Now, Θ is nondecreasing on I and, by the assumptions on h stated at the introduction, for any $x \in \Omega$, the mapping $s \to h_s(x,s)$ is nondecreasing , then, for any $x \in \Omega$,

$$\sigma \to N^{\sigma,\Theta(\sigma)}(x) \text{ is nondecreasing on } I. \tag{5.12}$$

On the other hand, by the Rayleigh's variational formula for principal eigenvalues we have, for $\sigma \in I$,

$$r_{\sigma,\Theta(\sigma)} = \inf_{z\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[|\nabla z|^2 + \beta a \Theta(\sigma)^{-\beta-1} z^2\right]}{\int_\Omega N^{\sigma,\Theta(\sigma)} z^2}. \tag{5.13}$$

From this expression, and taking into account (5.11), (5.12), and that $N^{\sigma,\Theta(\sigma)}$ is nonnegative for any $\sigma \in I$, it follows that

$$\text{the mapping } \sigma \to r_{\sigma,\Theta(\sigma)} \text{ is nonincreasing on } I. \tag{5.14}$$

Notice that, from (5.7), (5.9), (5.10), (5.11), and (5.14) it follows that

$$(I,\Theta) \in \mathcal{G} \tag{5.15}$$

Notice also that, since $\Theta : I \to U_\beta$ is continuous then, by Lemma 4.7,

$$\sigma \to r_{\sigma,\Theta(\sigma)} \text{ is continuous from } I \text{ into } \mathbb{R} \tag{5.16}$$

In addition, since $\Theta \in C^1\left(I, U_\beta\right)$ and $S(\sigma,\Theta(\sigma)) = 0$ for all $\sigma \in I$, then from (5.2) and the chain rule we have, for any $\sigma \in I$,

$$\begin{cases} -\Delta(\Theta'(\sigma)) = -\beta a(\Theta(\sigma))^{-(1+\beta)}\Theta'(\sigma) + h(.,\Theta(\sigma)) + \sigma\frac{\partial h}{\partial s}(.,\Theta(\sigma))\Theta'(\sigma) \text{ in } \Omega, \\ \Theta'(\sigma) = 0 \text{ on } \partial\Omega.. \end{cases} \tag{5.17}$$

Since for any $\sigma \in I$, $\Theta(\sigma)$ is a weak solution of (1.1) (taking there $\lambda = \sigma$) we have, in weak sense,

$$\begin{cases} -\Delta(\Theta(\sigma)) = a(\Theta(\sigma))^{-\beta} + \sigma h(.,\Theta(\sigma)) \geq a(\Theta(\sigma))^{-\beta} \text{ in } \Omega, \\ \Theta(\sigma) = 0 \text{ on } \partial\Omega \\ \Theta(\sigma) > 0 \text{ in } \Omega. \end{cases} \tag{5.18}$$

On the other hand, the (unique) weak solution u_0 of (1.1) corresponding to $\lambda = 0$ satisfies

$$\begin{cases} -\Delta u_0 = a(u_0)^{-\beta} \text{ in } \Omega, \\ u_0 = 0 \text{ on } \partial\Omega \\ u_0 > 0 \text{ in } \Omega. \end{cases} \tag{5.19}$$

Then, by Lemma 2.1 ii), $\Theta(\sigma) \geq u_0$ for any $\sigma \in I$ and, by Lemma 2.10 iv), there exists a positive constant c_* such that $u_0 \geq c_* \Psi_\beta$ (with Ψ_β given by Definition 2.3). Then, for any $\sigma \in I$,

$$\Theta(\sigma) \geq c_* \Psi_\beta \text{ in } \Omega. \tag{5.20}$$

Let λ_* and λ^* be such that $I = (\lambda_*, \lambda^*)$. By (5.11), $\sigma \to \Theta(\sigma)$ is nondecreasing on I, and clearly $\Theta(\sigma) \geq 0$ in Ω (because $\Theta(\sigma) \in U_\beta$). Then there exists the pointwise limit $\lim_{\sigma \to \lambda_*} \Theta(\sigma)$. Define, for $x \in \Omega$

$$\Theta(\lambda_*)(x) := \lim_{\sigma \to \lambda_*} \Theta(\sigma)(x). \tag{5.21}$$

We are going to show the following three facts:
A) $\Theta(\lambda_*) \in U_\beta$.
B) $\lim_{\sigma \to \lambda_*} \Theta(\sigma) = \Theta(\lambda_*)$ with convergence in Y_β.
C) $\Theta(\lambda_*)$ is a weak solution of (1.1) corresponding to $\lambda = \lambda_*$.
From (5.20),

$$\Theta(\lambda_*) \geq c_* \Psi_\beta \text{ in } \Omega. \tag{5.22}$$

Also, $\Theta(\sigma) \leq \Theta(\lambda_0)$ for any $\sigma \in (\lambda_*, \lambda_0)$ and, since $\Theta(\lambda_0) \in U_\beta$ we have $\Theta(\lambda_0) \leq c_{**} \Psi_\beta$ for some positive constant c_{**}, then

$$\Theta(\lambda_*) \leq c_{**} \Psi_\beta \text{ in } \Omega. \tag{5.23}$$

Now, for any $\sigma \in I$,

$$\begin{cases} -\Delta(\Theta(\sigma)) = a(\Theta(\sigma))^{-\beta} + \sigma h(.,\Theta(\sigma)) \text{ in } \Omega, \\ \Theta(\sigma) = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.24}$$

and so, for $\lambda_* < \sigma < \tau < \lambda_0$ we have, in weak sense,

$$\begin{cases} -\Delta(\Theta(\tau) - \Theta(\sigma)) = a\left(\Theta^{-\beta}(\tau) - \Theta^{-\beta}(\sigma)\right) + \tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma)) \text{ in } \Omega, \\ \Theta(\tau) - \Theta(\sigma) = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, by taking $\Theta(\tau) - \Theta(\sigma)$ as a test function in the above equation we get

$$\begin{aligned} \int_\Omega |\nabla(\Theta(\tau) - \Theta(\sigma))|^2 &= \int_\Omega a\left(\Theta^{-\beta}(\tau) - \Theta^{-\beta}(\sigma)\right)(\Theta(\tau) - \Theta(\sigma)) + \int_\Omega (\Theta(\tau) - \Theta(\sigma))(\tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma))) \\ &\leq \int_\Omega (\Theta(\tau) - \Theta(\sigma))(\tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma))) \end{aligned} \tag{5.25}$$

where, in the last inequality, we have used that $\Theta(\sigma) \leq \Theta(\tau)$. Now, $0 \leq \Theta(\sigma) \leq \Theta(\tau) \leq \Theta(\lambda_0)$ then, by our assumptions on h,

$$0 \leq \tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma)) \leq \lambda_0 h(.,\Theta(\lambda_0)) \in L^\infty(\Omega),$$

and thus, since there exists the (finite) pointwise limit $\lim_{\sigma \to \lambda_*} \Theta(\sigma)$, we have

$$\lim_{\sigma,\tau \to \lambda_*} (\Theta(\tau) - \Theta(\sigma))(\tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma))) = 0 \text{ a.e. in } \Omega.$$

Also,

$$0 \leq (\Theta(\tau) - \Theta(\sigma))(\tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma))) \leq \Theta(\lambda)\lambda h(.,\Theta(\lambda)) \text{ a.e. in } \Omega,$$

and, by our assumptions on h stated at the introduction and by Lemma 2.10, $\Theta(\lambda)\lambda h(.,\Theta(\lambda)) \in L^1(\Omega)$. Then, by the L:ebesgue's dominated convergence theorem,

$$\lim_{\sigma,\tau\to\lambda_*}\int_\Omega (\Theta(\tau)-\Theta(\sigma))(\tau h(.,\Theta(\tau))-\sigma h(.,\Theta(\sigma)))=0$$

Thus, by (5.25), $\lim_{\sigma,\tau\to\lambda_*}\|\Theta(\tau)-\Theta(\sigma)\|_{H_0^1(\Omega)}=0$ and so, by the Cauchy criterion, there exists $\zeta\in H_0^1(\Omega)$ such that $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\zeta$ with convergence in $H_0^1(\Omega)$. Since $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\Theta(\lambda_*)$ with pointwise convergence in Ω, we have $\zeta=\Theta(\lambda_*)$. Then

$$\Theta(\lambda_*)\in H_0^1(\Omega) \text{ and } \lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\Theta(\lambda_*) \text{ with convergence in } H_0^1(\Omega). \tag{5.26}$$

Moreover, from (5.22), (5.23) and (5.26), we have

$$\Theta(\lambda_*)\in U_\beta.$$

Let us show that, in weak sense,

$$\begin{cases} -\Delta(\Theta(\lambda_*))=a(\Theta(\lambda_*))^{-\beta}+\lambda_* h(.,\Theta(\lambda_*)) \text{ in } \Omega,\\ \Theta(\lambda_*)=0 \text{ on } \partial\Omega.\end{cases} \tag{5.27}$$

Indeed, let $\varphi\in H_0^1(\Omega)$. Since $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\Theta(\lambda_*)$ with convergence in $H_0^1(\Omega)$, we have

$$\lim_{\sigma\to\lambda_*^+}\int_\Omega\langle\nabla\Theta(\sigma),\nabla\varphi\rangle=\int_\Omega\langle\nabla\Theta(\lambda_*),\nabla\varphi\rangle.$$

Also,

$$\lim_{\sigma\to\lambda_*^+} a(\Theta(\lambda_*))^{-\beta}\varphi=a(\Theta(\lambda_*))^{-\beta}\varphi \; a.e. \text{ in } \Omega$$

and, since $a\in L^\infty(\Omega)$ and $\Theta(\lambda_*)\geq c'\Psi_\beta$ with c' a positive constant, we have, for $\sigma\in(\lambda_*,\lambda)$,

$$\left|a(\Theta(\lambda_*))^{-\beta}\varphi\right|\leq\left|a(\Theta(\lambda_*))^{-\beta}\varphi\right|\leq c\Psi_\beta^{-\beta}|\varphi|=c\delta_\Omega\Psi_\beta^{-\beta}\left|\frac{\varphi}{\delta_\Omega}\right| \; a.e. \text{ in } \Omega,$$

with c a positive constant independent of σ. By Remark 2.4 we have $\delta_\Omega\Psi_\beta^{-\beta}\in L^2(\Omega)$, and then, by the Hölder's and the Hardy's inequalities,

$$\int_\Omega\delta_\Omega\Psi_\beta^{-\beta}\left|\frac{\varphi}{\delta_\Omega}\right|\leq\left\|\delta_\Omega\Psi_\beta^{-\beta}\right\|_2\left\|\frac{\varphi}{\delta_\Omega}\right\|_2\leq c'\left\|\delta_\Omega\Psi_\beta^{-\beta}\right\|_2\|\varphi\|_{H_0^1(\Omega)}<\infty$$

and so $\delta_\Omega\Psi_\beta^{-\beta}\left|\frac{\varphi}{\delta_\Omega}\right|\in L^1(\Omega)$. Thus, by the Lebesgue's dominated convergence theorem, $\int_\Omega a(\Theta(\lambda_*))^{-\beta}\varphi\in L^1(\Omega)$ and

$$\lim_{\sigma\to\lambda_*^+}\int_\Omega a\Theta^{-\beta}(\sigma)\varphi=\int_\Omega a\Theta^{-\beta}(\lambda_*)\varphi. \tag{5.28}$$

Also, by the assumptions on h stated at the introduction, and since $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\Theta(\lambda_*)$ pointwise in Ω, we have

$$\lim_{\sigma\to\lambda_*^+}\sigma h(.,\Theta(\sigma))\varphi=\lambda_* h(.,\Theta(\lambda_*))\varphi \; a.e. \text{ in } \Omega. \tag{5.29}$$

In addition, for $\lambda_*<\sigma<\lambda_0$, we have $|\sigma h(.,\Theta(\sigma))\varphi|\leq\lambda_0 h(.,\Theta(\lambda_0))|\varphi|$. By Lemma 2.10, $\Theta(\lambda_0)\in C(\overline{\Omega})$ and then, by our assumptions on h, $\lambda_0 h(.,\Theta(\lambda_0))\in C(\overline{\Omega})$. Therefore $\lambda_0 h(.,\Theta(\lambda_0))|\varphi|\in L^1(\Omega)$ and thus, by the Lebesgue's dominated convergence theorem, $\lambda_* h(.,\Theta(\lambda_*))\varphi\in L^1(\Omega)$ and

$$\lim_{\sigma\to\lambda_*^+}\int_\Omega\sigma h(.,\Theta(\sigma))\varphi=\int_\Omega\lambda_* h(.,\Theta(\lambda_*))\varphi. \tag{5.30}$$

By (5.24) we have, for any $\sigma\in I$

$$\int_\Omega\langle\nabla\Theta(\sigma),\nabla\varphi\rangle=\int_\Omega a\Theta^{-\beta}(\sigma)\varphi+\int_\Omega\sigma h(.,\Theta(\sigma))\varphi$$

and then, from (5.28), (5.29), and (5.30), by taking the limit as $\sigma \to \lambda_*^+$ we get

$$\int_\Omega \langle \nabla \Theta(\lambda_*), \nabla \varphi \rangle = \int_\Omega a\Theta^{-\beta}(\lambda_*)\varphi + \int_\Omega \lambda_* h(.,\Theta(\lambda_*))\varphi.$$

Thus $\Theta(\lambda_*)$ is a weak solution of (5.27).
Now we show that $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma) = \Theta(\lambda_*)$ with convergence in Y_β. To do it, it is enough to see that

$$\sup_{\sigma\in(\lambda_*,\lambda)} \left\|\Theta'(\sigma)\right\|_{Y_\beta} < \infty. \tag{5.31}$$

Indeed, if (5.31) holds, then, for $\lambda_* < \sigma < \tau < \lambda$,

$$\|\Theta(\tau) - \Theta(\sigma)\|_{Y_\beta} = \left\|\int_\sigma^\tau \Theta'(s)\,ds\right\|_{Y_\beta} \le \int_\sigma^\tau \left\|\Theta'(s)\right\|_{Y_\beta} ds \le |\tau-\sigma| \sup_{\sigma\in(\lambda_*,\lambda)} \left\|\Theta'(\sigma)\right\|_{Y_\beta},$$

and so by (5.31) and the Cauchy's criterion, there exists $\xi \in Y_\beta$ such that $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma) = \xi$ with convergence in Y_β, and since $\Theta(\sigma)$ converges pointwise to $\Theta(\lambda_*)$ we have $\xi = \Theta(\lambda_*)$ and so $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma) = \Theta(\lambda_*)$ with convergence in Y_β.
To prove (5.31) observe that, for $\sigma \in (\lambda_*, \lambda)$, $\Theta'(\sigma)$ satisfies, in weak sense,

$$\begin{cases} -\Delta\Theta'(\sigma) + \beta a \Theta^{-\beta-1}(\sigma)\Theta'(\sigma) - \sigma\frac{\partial h}{\partial s}(.,\Theta(\sigma))\Theta'(\sigma) = h(.,\Theta(\sigma)) \text{ in } \Omega, \\ \Theta'(\sigma) = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.32}$$

Since $\Theta(\sigma) \le \Theta(\lambda_0)$, $\sigma\frac{\partial h}{\partial s}(.,\Theta(\sigma)) \le \lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))$, and $h(.,\Theta(\sigma)) \le h(.,\Theta(\lambda_0))$, (5.32) gives that, in weak sense,

$$\begin{cases} -\Delta\Theta'(\sigma) + \beta a \Theta^{-\beta-1}(\lambda_0)\Theta'(\sigma) - \lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))\Theta'(\sigma) \le h(.,\Theta(\lambda_0)) \text{ in } \Omega, \\ \Theta'(\sigma) = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.33}$$

Also,

$$\begin{cases} -\Delta\Theta'(\lambda_0) + \beta a \Theta^{-\beta-1}(\lambda_0)\Theta'(\lambda_0) - \lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))\Theta'(\lambda_0) = h(.,\Theta(\lambda_0)) \text{ in } \Omega, \\ \Theta'(\lambda_0) = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.34}$$

and so, since $N^{\lambda_0,\Theta(\lambda_0)} = \lambda_0 h(.,\Theta(\lambda_0))$ and $\rho_{N^{\lambda_0,\Theta(\lambda_0)},\Theta(\lambda_0)} > 1$, from (5.33), (5.34) and the maximum principle of ([37] Lemma 4.4 ii)) it follows that $\Theta'(\sigma) \le \Theta'(\lambda_0)$ $a.e$ in Ω. We have also $\Theta'(\sigma) \ge 0$ $a.e$ in Ω (because $\sigma \to \Theta(\sigma)$ is nondecreasing). Therefore

$$\left\|\Psi_\beta^{-1}\Theta'(\sigma)\right\|_\infty \le c \text{ for any } \sigma \in (\lambda_*, \lambda_0) \tag{5.35}$$

where $c := \left\|\Psi_\beta^{-1}\Theta'(\lambda_0)\right\|_\infty$ (note that c is finite because $\Theta'(\lambda_0) \in Y_\beta$). In particular, (5.35) gives that for some constant $c' > 0$,

$$\left\|\Theta'(\sigma)\right\|_\infty \le c' \text{ for any } \sigma \in (\lambda_*, \lambda_0). \tag{5.36}$$

From (5.32) we have also

$$\begin{aligned} \int_\Omega \left|\nabla\Theta'(\sigma)\right|^2 &= -\int_\Omega \beta a \Theta^{-\beta-1}(\sigma)\left(\Theta'(\sigma)\right)^2 + \int_\Omega \sigma\frac{\partial h}{\partial s}(.,\Theta(\sigma))\left(\Theta'(\sigma)\right)^2 + \int_\Omega h(.,\Theta(\sigma))\Theta'(\sigma) \\ &\le \int_\Omega \lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))\left(\Theta'(\sigma)\right)^2 + \int_\Omega h(.,\Theta(\lambda_0))\Theta'(\sigma) \\ &\le M\int_\Omega \left(\Theta'(\sigma)\right)^2 + M\int_\Omega \Theta'(\sigma) \end{aligned}$$

where $M; = \left\|\lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))\right\|_\infty + \|h(.,\Theta(\lambda_0))\|_\infty$. Then, taking into account (5.36), we conclude that for some constant $c'' > 0$,

$$\left\|\Theta'(\sigma)\right\|_{H_0^1(\Omega)} \le c'' \text{ for any } \sigma \in (\lambda_*, \lambda_0).$$

which jointly with (5.35) gives (5.31).
Now we show that $\lambda_* = 0$. We proceed by the way of contradiction. Suppose $\lambda^* > 0$. Then, since $r_{\lambda_0,\Theta(\lambda_0)} > 1$ and since, by 5'), $\sigma \to r_{\sigma,\Theta(\sigma)}$ is nonincreasing on I, we have $r_{\lambda_*,\Theta(\lambda_*)} > 1$. Thus, by Lemma 5.4 there exists $\varepsilon > 0$ such that Θ has an extension (still denoted by Θ) to $I_\varepsilon := (\lambda_* - \varepsilon,\ \lambda^*)$ such that $(I_\varepsilon, \Theta) \in \mathcal{G}$, which contradicts the definition of λ_*.
Then $\lambda_* = 0$, and so, by (5.27), $\Theta(\lambda_*) = u_0$ (where u_0 is the unique solution of (1.1) for $\lambda = 0$).

□

Proof of Theorem 1.6. We proceed by the way of contradiction. Let Σ be as given by Remark 1.5. Let $\lambda_0 \in (0,\Sigma)$ and let u_{λ_0} and v_{λ_0} be the minimal weak solution and the maximal weak solution, respectively, of problem (1.1) corresponding to $\lambda = \lambda_0$. Then $u_{\lambda_0} \neq v_{\lambda_0}$. Suppose, by contradiction, that there exists a weak solution w of (1.1, corresponding to $\lambda = \lambda_0$ such that $u_\lambda \neq w \neq v_\lambda$. By Lemma 5.5, applied with $w_0 = w$, there exists a function $\Theta \in C^1\left((0,\lambda_0+\varepsilon),U_\beta\right) \cap C\left([0,\lambda_0+\varepsilon),U_\beta\right)$ such that $\Theta(\lambda_0) = w$, $\Theta(0) = u_0$ (where u_0 is the unique weak solution of (1.1) corresponding to $\lambda = 0$), and such that $r_{\sigma,\Theta(\sigma)} > 1$ for any $\sigma \in (0,\lambda_0)$, and with Θ satisfying, in weak sense and for any $\sigma \in (0,\lambda_0)$,

$$\begin{cases} -\Delta(\Theta(\sigma)) = a\Theta^{-\beta}(\sigma) + \sigma h(.,\Theta(\sigma)) \text{ in } \Omega, \\ \Theta(\sigma) = 0 \text{ on } \partial\Omega, \\ \Theta(\sigma) > 0 \text{ in } \partial\Omega. \end{cases}$$

Again by Lemma 5.5, but applied now with $w_0 = u_{\lambda_0}$, we have that, for some $\varepsilon' > 0$, there exists a function $\Phi \in C^1\left((0,\lambda_0+\varepsilon'),U_\beta\right) \cap C\left([0,\lambda_0+\varepsilon'),U_\beta\right)$ such that $\Phi(\lambda_0) = u_{\lambda_0}$,
$\Phi(0) = u_0$ where, as above, u_0 is the weak solution of (1.1) for $\lambda = 0$, and such that $r_{\sigma,\Phi(\sigma)} > 1$ for any $\sigma \in (0,\lambda_0)$, and satisfying, in weak sense and for any $\sigma \in (0,\lambda)$,

$$\begin{cases} -\Delta(\Phi(\sigma)) = a\Phi^{-\beta}(\sigma) + \sigma h(.,\Phi(\sigma)) \text{ in } \Omega, \\ \Phi(\sigma) = 0 \text{ on } \partial\Omega, \\ \Phi(\sigma) > 0 \text{ in } \partial\Omega. \end{cases}$$

Observe that, since $w \neq u_{\lambda_0}$, then

$$\Theta(\sigma) \neq \Phi(\sigma) \text{ for any } \sigma \in (0,\lambda_0). \tag{5.37}$$

Indeed, let

$$\lambda_{**} := \sup\{\eta \in [0,\lambda_0] : \Theta(\eta) = \Phi(\eta)\}. \tag{5.38}$$

We claim that $\lambda_{**} = 0$. In fact, since $\Theta(\lambda_0) = w \neq u_{\lambda_0} = \Phi(\lambda_0)$, and since Θ and Φ are continuous at λ_0 we have, necessarily, $\lambda_{**} < \lambda_0$. If $\lambda_{**} > 0$ then $\lambda_{**} \in (0,\lambda_0)$ and so $r_{\lambda_{**},\Theta(\lambda_{**})} > 1$. Thus Lemma 5.4 can be applied taking there $\lambda = \lambda_{**}$ and $u = \Theta(\lambda_{**})$ to obtain a number $\varepsilon > 0$ and an open neighborhood V of $\Theta(\lambda_{**})$ in Y_β such that for any $\sigma \in (\lambda_{**}-\varepsilon,\lambda_{**}+\varepsilon)$ there exists a unique $\xi(\sigma) \in V$ such that $S(\sigma,\xi(\sigma)) = 0$. By diminishing ε if necessary, we can assume that $(\lambda_{**}-\varepsilon,\lambda_{**}+\varepsilon) \subset (0,\lambda_0)$. From the continuity of Θ and Φ at λ_{**} and from (5.38), we have that $\Theta(\lambda_{**}) = \Phi(\lambda_{**}) \in V$ and so, δ positive and small enough, we have that if $\lambda_{**} < \sigma < \lambda_{**}+\delta$ then $\Theta(\sigma) \neq \Phi(\sigma)$ and also $S(\sigma,\Theta(\sigma)) = S(\sigma,\Phi(\sigma)) = 0$, which contradicts the uniqueness assertion of Lemma 5.4. Thus $\lambda_{**} = 0$ and so (5.37) holds.
Now, for $\sigma \in (0,\lambda_0)$,

$$\begin{cases} -\Delta(\Theta(\sigma)-\Phi(\sigma)) = a\left((\Theta(\sigma))^{-\beta} - (\Phi(\sigma))^{-\beta}\right) + \sigma(h(.,\Theta(\sigma)) - h(.,\Phi(\sigma))) \text{ in } \Omega, \\ \Theta(\sigma)-\Phi(\sigma) = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.39}$$

and, by the mean value theorem, $\sigma(h(.,\Theta(\sigma)) - h(.,\Phi(\sigma))) = \sigma\frac{\partial h}{\partial s}(.,\eta_\sigma)(\Theta(\sigma)-\Phi(\sigma))$ for some function η_σ such that, for $x \in \Omega$, $\eta_\sigma(x)$ belongs to the open segment with endpoints $\Phi(\sigma)(x)$ and $\Theta(\sigma)(x)$.
Since $0 \leq \Theta(\sigma) \leq \Theta(\lambda_0)$ and $0 \leq \Phi(\sigma) \leq \Phi(\lambda_0)$, and since $\Theta(\lambda_0)$ and $\Phi(\lambda_0)$ belong to $L^\infty(\Omega)$ (because they belong to Y_β) then there exists a positive constant M_1 such that $0 \leq \eta_\sigma \leq M_1$ for any $\sigma \in (0,\lambda_0)$. Then, from our assumptions on h, it follows that there exists a constant M such that $\left|\frac{\partial h}{\partial s}(.,\eta_\sigma)\right| \leq M$ for any $\sigma \in (0,\lambda_0)$. Then, for such σ,
Now we take the test function $\varphi = \Theta(\sigma)-\Phi(\sigma)$ in (5.39) to obtain

$$\begin{aligned} \|\Theta(\sigma)-\Phi(\sigma)\|^2_{H^1_0(\Omega)} &= \int_\Omega \left((\Theta(\sigma))^{-\beta} - (\Phi(\sigma))^{-\beta}\right)(\Theta(\sigma)-\Phi(\sigma)) + \int_\Omega \sigma(h(.,\Theta(\sigma)) - h(.,\Phi(\sigma)))(\Theta(\sigma)-\Phi(\sigma)) \\ &\leq \int_\Omega \sigma|h(.,\Theta(\sigma)) - h(.,\Phi(\sigma))||\Theta(\sigma)-\Phi(\sigma)| \\ &\leq \sigma M\int_\Omega (\Theta(\sigma)-\Phi(\sigma))^2 \\ &\leq \sigma M c_P^2\|\Theta(\sigma)-\Phi(\sigma)\|^2_{H^1_0(\Omega)} \end{aligned}$$

where c_P is the constant of the Poincaré's inequality in Ω, and where in the first inequality we used that $s \to as^{-\beta}$ is nonincreasing and, in the second one, the Poincaré's inequality was used. Then, since $\Theta(\sigma) \neq \Phi(\sigma)$ for any $\sigma \in (0,\lambda_0)$ we conclude that $1 \leq \sigma M c_P^2$ which, by taking $\lim_{\sigma\to 0^+}$, gives a contradiction that completes the proof of the theorem. □

Remark 5.6. *An inspection of the proof given for Theorem 1.6 shows that, if w is a weak solution of (1.1) then, in order to construct the function Θ (and to prove its properties), the assumption $w \neq v_\lambda$ was used only to guarantee that $\rho_{N\lambda,w,w} > 1$. From this fact one gets that if for some $\lambda \in (0,\Sigma)$, $r_{\lambda,v_\lambda} > 1$ then, proceeding as in the proof of Theorem 1.6, a contradiction is reached. Therefore necessarily $r_{\lambda,v_\lambda} \leq 1$ for any $\lambda \in (0,\Sigma)$.*

6. Conclusion

For a C^2 and bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, we considered the problem

$$-\Delta u = au^{-\beta} + \lambda h(.,u) \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, u > 0 \text{ in } \Omega, \tag{6.1}$$

where λ is a nonnegative parameter and the solutions are understood in weak sense. Under the assumptions H1)-H6) stated at the introduction our main result can be readed as follows. If for some $\lambda \geq 0$ the above problem has at least two weak solutions, then it has exactly two weak solutions (which belong to $H_0^1(\Omega) \cap C^1(\Omega) \cap C(\overline{\Omega})$), namely, a minimal solution u_λ and a maximal solution v_λ, such that $u_\lambda \neq v_\lambda$ and $u_\lambda \leq v_\lambda$ in Ω. This fact, combined with known previous results leads to the following statement: There exists $\Sigma > 0$ such that:
For $\lambda = 0$ and $\lambda = \Sigma$ there exists exactly one weak solution,
For $0 < \lambda < \Sigma$ there exists exactly two weak solutions in $H_0^1(\Omega)$,
For $\lambda > \Sigma$ no weak solutions exist.
Let us stress that although there are many results concerning existence and multiplicity for solutions of singular elliptic problems, exact multiplicity results are far less abundant in the literature .
Our result complements known multiplicity results for these kind of singular problems. As an example, it applies, for instance, when $n \geq 2$, $a \in C(\overline{\Omega})$ is strictly positive in $\overline{\Omega}$ and $h(x,s) = \sum_{j=1}^{m} b_j(x) s^{p_j}$ with $b_j \in C(\overline{\Omega})$, such that $b_j > 0$ in $\overline{\Omega}$, and $1 < p_1 < p_2 < .... < p_m < \frac{n+2}{n-2}$ (with the convention that $\frac{n+2}{n-2} = \infty$ if $n = 2$).
Some possible future directions of research include:
i) Study problem (6.1) in cases where the coefficient a of the singular term of the equation is singular at $\partial(\Omega)$ in order to obtain, again in some of these situations, exact multiplicity results.
ii) For $\beta > 0$ arbitrary search for exact multiplicity results for solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of problem (6.1).
iii) Investigate the situation when, under suitable assumptions, the Laplacian is replaced by the $q-$Laplacian in (6.1) for some $1 < q < \infty$.
Other interesting questions remain. For instance:
By ([36], Theorem 1.2) $\lambda \to u_\lambda$ is nondecreasing on $[0,\Sigma]$, and by ([35], Theorem 1.2), $\lambda = 0$ is a bifurcation point from ∞ for problem (6.1). Then, since for $\lambda \in (0,\lambda)$ u_λ and v_λ are the unique solutions of (6.1), one could suspect that $\lim_{\lambda \to 0^+} \|v_\lambda\|_{C(\overline{\Omega})} = \infty$, and that the map $\lambda \to v_\lambda$ is non increasing on $(0,\Sigma]$. It would interestig to prove these fact (if true). It would be also interesting to investigate the regularity properties (if any) of the mapping $\lambda \to v_\lambda$.

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Solvability of Quadratic Integral Equations of Urysohn Type Involving Hadamard Variable-Order Operator

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Abstract

This study investigates the existence of solutions to integral equations in the form of quadratic Urysohn type with Hadamard fractional variable order integral operator. Due to the lack of semigroup properties in variable-order fractional integrals, it becomes challenging to get the existence and uniqueness of corresponding integral equations, hence the problem is examined by employing the concepts of piecewise constant functions and generalized intervals to address this issue. In this context, the problem is reformulated as integral equations with constant orders to obtain the main results. Both the Schauder and Banach fixed point theorems are employed to prove the uniqueness results. In addition, an illustration is included in order to verify those results in the final step.

1. Introduction

As a subject of mathematical analysis, fractional calculus arose as a result of investigating the matter regarding whether it is feasible to employ the complex or real number powers for integral and differential operators. (see [1, 2]). Over the past three decades, the theory has led to numerous significant results in both pure and applied mathematics alongside other branches of sciences, for example: chemistry, ,signal and image processing, physics,control theory, biology, biophysics, economics etc. (see [3, 4, 5, 6, 7, 8, 9]).
The arbitrary order of integral and differentiation whose order depends on a function of certain variables, which corresponds to a more complicated category, are known as variable-order operators. Following its introduction in 1993 by Samko and Ross [10], the concept of fractional variable-order (FVO) differential and integral operators, along with its basic features, have naturally garnered significant interest from numerous researchers. The investigation of fractional variable models is still in its early stages since addressing the variable fractional order is certainly tough to study in some cases, whose features like the semi-group property are separated from the associated characteristics of systems with conventional fractional orders. However, for recent developments on the theory of fractional variational calculus and numerical methods dealing with fractional problems of variable order see [11, 12, 13, 14, 15, 16, 17, 18, 19] and the references therein.
There have been very recent publications dealing with fractional equations of variable order coupled with auxiliary conditions from a qualitative perspective. For example, in [20], the authors examined the existence of solutions to a boundary value problem for a class of fractional equations of Riemann–Liouville(R-L) variable order with finite delay by employing the Darbo type fixed-point as well as Kuratowski measure of noncompactness. It has been considered in [21] for the first time a Caputo FVO initial value problem(IVP) under impulsive conditions, and the uniqueness and existence of solutions have been examined. Two fixed point theories were employed to show the main results. In [22], monotone iterative technique together with upper and lower solutions have been applied to IVP for linear homogeneous and non-homogeneous diffusion equations involving the conformable operator of variable order to show the existence and uniqueness properties. In [23], existence and uniqueness of a boundary value problem with variable order operators of Hadamard type have been examined with the aid of Schauder,

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and Banach fixed point theorems and stability criteria have been obtained regarding Ulam–Hyers–Rassias(UHR). Suitable criteria ensuring the existence and uniqueness of a class of FVO Riemann-Liouville equations including fractional boundary conditions was discussed in [24] and suitable conditions providing the stability in the UHR sense were also established. For additional papers, one might consult the latest publications[25, 26, 27, 28, 29, 30] and the associated references therein.
The study of integral equations is an essential component of nonlinear analysis and investigated by many scholars in view of their wide range of scientific applications [31, 32]. There have also been a number of studies that examine the existence of solutions of functional integral and integro-differential equations of fractional conventional order [33]. However, only a limited number of papers have discussed the existence of solutions to such problems involving FVO operators [34, 35, 36, 37, 38].
In this work, we study the quadratic integral equation of Urysohn type with fractional variable order (**QIEUFVO**)

$$u(t) = \quad q(t) + (\Phi u)(t)\int_1^t \frac{1}{\Gamma(\omega(t))}(\log\frac{t}{\sigma})^{\omega(t)-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}, \tag{1.1}$$

where $t\in\Upsilon := [1,K]$, $q\in C(\Upsilon,\mathbb{R})$, $\xi:\Upsilon^2\times\mathbb{R}\to\mathbb{R}$ is given function, $1<\omega(t)\leq 2$ and $\Phi: C(\Upsilon,\mathbb{R})\to C(\Upsilon,\mathbb{R})$ is an appropriate operator.
In order to find the existence and uniqueness of (1.1), we employ the notions of generalized interval, partition and piece-wise constant functions, hence converting the equation to fractional integral equations of constant order.

2. Preliminaries

This section provides several concepts and results that will be required throughout the subsequent sections.

Definition 2.1. *([40], [23]) Let $\omega(t):\Upsilon\to(1,2]$, then the left Hadamard fractional integral of variable order (HFIVO) for function ψ is*

$$({}^{H}I_{1^+}^{\omega(t)}\psi)(t) = \frac{1}{\Gamma(\omega(t))}\int_1^t (\log\frac{t}{\sigma})^{\omega(t)-1}\psi(\sigma)\frac{d\sigma}{\sigma},\ t>1, \tag{2.1}$$

As expected, when $\omega(t$ is constant, then HFIVO is corresponds to the standard Hadamard fractional integral operator.

Remark 2.2. *([40], [41]) In the general case, the semi-group property is not satisfied for the Integral operator of variable order, i.e.*

$${}^{H}I_{1^+}^{\omega(t)}\,{}^{H}I_{1^+}^{v(t)}\psi(t) \neq {}^{H}I_{1^+}^{\omega(t)+v(t)}\psi(t).$$

Lemma 2.3. *([41] Let $\omega:\Upsilon\to(1,2]$ be a continuous function, then for $\psi\in C_\sigma(\Upsilon,\mathbb{R})$ where*

$$C_\sigma(\Upsilon,\mathbb{R}) = \{\psi(t)\in C(\Upsilon,\mathbb{R}),\ (logt)^\sigma\psi(s)\in C(\Upsilon,\mathbb{R}), 0\leq\sigma\leq 1\},$$

the integral ${}^{H}I_{1^+}^{\omega(t)}\psi(t)$ exists for any $t\in\Upsilon$.

Lemma 2.4. *([41]) If $\omega\in C(\Upsilon,(1,2])$, then ${}^{H}I_{1^+}^{\omega(t)}\psi(t)\in C(\Upsilon,\mathbb{R})$ for any $\psi\in C(\Upsilon,\mathbb{R})$.*

Theorem 2.5. *(Schauder Fixed Point Theorem) ([42]) Let Λ be a convex subset of Banach Space Π and $\mathcal{F}:\Lambda\longrightarrow\Lambda$ be completely continuous map, then $\mathcal{F}$ has at least one fixed point in Λ.*

3. Existence and uniqueness results

We first state the underlying presumption:

(H1) Let

$$\mathcal{P} = \{\Upsilon_1 := [0,K_1], \Upsilon_2 := (K_1,K_2], \Upsilon_3 := (K_2,K_3]\ldots\Upsilon_n := (K_{n-1},K]\}$$

be a partition of the interval Υ and let $\omega(s):\Upsilon\to(1,2]$ be a piece-wise continuous function with respect to $\mathcal{P}$, i.e.,

$$\omega(t) = \sum_{\vartheta=1}^{n}\omega_\vartheta I_\vartheta(t) = \begin{cases} \omega_1, & if\ t\in\Upsilon_1,\\ \omega_2, & if\ t\in\Upsilon_2,\\ . & \\ . & \\ . & \\ \omega_n, & if\ t\in\Upsilon_n,\end{cases}$$

where $1<\omega_\vartheta\leq 2$ are constants, and

$$I_\vartheta(t)=\begin{cases}1, & for\, t\in\Upsilon_\vartheta,\\ 0, & for\, elsewhere.\end{cases}$$

The notion $\Pi_\vartheta=C(\Upsilon_\vartheta,\,\mathbb{R})$ denotes the Banach space of continuous functions from Υ_ϑ into $\mathbb{R}$ with the norm

$$\|u\|_{\Pi_\vartheta}=\sup_{t\in\Upsilon_\vartheta}|u(t)|,\vartheta\in\{1,2,...,n\}.$$

Then, for any $t\in\Upsilon_\vartheta$, $\vartheta=1,2,...,n$, the (HFIVO) for function $\xi(t,\sigma,u(\sigma))\in C(\Upsilon^2\times\mathbb{R},\mathbb{R})$, defined by (2.1), might then be stated as

$$^{H}I_{1^+}^{\omega(t)}\xi(t,\sigma,u(t)) \quad =\sum_{i=1}^{i=\vartheta-1}\int_{K_{i-1}}^{K_i}\frac{1}{\Gamma(\omega_i)}(\log\frac{t}{\sigma})^{\omega_i-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}+\int_{K_{\vartheta-1}}^{t}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}. \quad (3.1)$$

According to (3.1), for any $t\in\Upsilon_\vartheta$, $\vartheta\in\{1,2,...,n\}$, 1.1 can be stated in the following format:

$$u(t) \quad =q(t)+(\Phi u)(t)\Big(\sum_{i=1}^{i=\vartheta-1}\int_{K_{i-1}}^{K_i}\frac{1}{\Gamma(\omega_i)}(\log\frac{t}{\sigma})^{\omega_i-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}+\int_{K_{\vartheta-1}}^{t}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big). \quad (3.2)$$

Let $u\in C(\Upsilon_\vartheta,\mathcal{R})$ be a solution of (3.2), such that $u(t)\equiv 0$ on $t\in[1,K_{\vartheta-1}]$. Then (3.2) is reduced to

$$u(t)=q(t)+(\Phi u)(t)\Big(\int_{K_{\vartheta-1}}^{t}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big),\ t\in\Upsilon_\vartheta. \quad (3.3)$$

We now impose the following assumptions:

(H2) There exists $\varpi_\vartheta>0$ such that

$$|(\Phi u)(t)-(\Phi\tilde{u})(t)|\le\varpi_\vartheta|u(t)-\tilde{u}(t)|$$

for each $u,\tilde{u}\in\Pi_\vartheta$ and $t\in\Upsilon_\vartheta$.

(H3) There are non-negative constants η and γ such that

$$|(\Phi u)(t)|\le\eta+\gamma|u(t)|$$

for each $u\in\Pi_\vartheta$ and $t\in\Upsilon_\vartheta$.

(H4) Let $\xi:\Upsilon_\vartheta^2\times\mathbb{R}\to\mathbb{R}$ be a continuous function and non-decreasing with respect to its three variables, separately, and there exists a constants $0\le\sigma\le 1$, $D_\vartheta>0$ such that

$$(logt)^\sigma|\xi(t,\sigma,u)-\xi(t,\sigma,\tilde{u})|\le D_\vartheta|u-\tilde{u}|$$

for all $(t,\sigma)\in\Upsilon_\vartheta^2$ and $u,\tilde{u}\in\mathbb{R}$.

(H5) There exists a continuous non-decreasing function $g\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $\bar{h}\in C(\Upsilon,\mathbb{R}_+)$ and a constant $0\le\sigma\le 1$ such that for each $(t,\sigma)\in\Upsilon_m^2$ and $u\in\mathbb{R}$ we have

$$(logt)^\sigma|\xi(t,\sigma,u)|\le\bar{h}(\sigma)g(|u|),$$

Theorem 3.1. *Let $\vartheta\in\{1,2,...,n\}$, suppose that hypotheses $(H1)-(H5)$ hold, and there exists a constant r_ϑ, such that*

$$\frac{r_\vartheta}{q^\star+\frac{(logK_\vartheta)^{1-\sigma}-(logK_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)}(\log\frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta-1}(\eta+\gamma r_\vartheta)g(r_\vartheta)\bar{h}^\star}>1, \quad (3.4)$$

where $\bar{h}^\star=sup\{\bar{h}(\sigma):\sigma\in\Upsilon_\vartheta\}$ and $q^\star=sup\{q(t):t\in\Upsilon_\vartheta\}$.
Then, (3.3) has at least solution in Π_ϑ.

Proof. Let the operator

$$S:\Pi_\vartheta\to\Pi_\vartheta$$

given by

$$(Su)(t)=q(t)+(\Phi u)(t)\Big(\int_{K_{\vartheta-1}}^{t}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big).$$

Let the set

$$B_{r_\vartheta}=\{u\in\Pi_\vartheta:\|u\|_{\Pi_\vartheta}\le r_\vartheta\}.$$

Clearly B_{r_ϑ} is nonempty, convex, closed and bounded.

Step 1: Claim: $S(B_{r_\vartheta}) \subseteq (B_{r_\vartheta})$.
For $u \in B_{r_\vartheta}$, we have

$$\begin{aligned}
|(Su)(t)| &\leq |q(t)| + |(\Phi u)(t)| \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} |\xi(t,\sigma,u(\sigma))| \frac{d\sigma}{\sigma} \\
&\leq |q(t)| + (\eta + \gamma |u(t)|) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} (log\sigma)^{-\sigma} \bar{h}(\sigma) g(|u(\sigma)|) \frac{d\sigma}{\sigma} \\
&\leq |q(t)| + (\eta + \gamma |u(t)|)(\log \frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta - 1} \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (log\sigma)^{-\sigma} \bar{h}(\sigma) g(|u(\sigma)|) \frac{d\sigma}{\sigma} \\
&\leq q^\star + \frac{(logK_\vartheta)^{1-\sigma} - (logK_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)} (\log \frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta - 1} (\eta + \gamma \|u\|_{\Pi_\vartheta}) g(\|u\|_{\Pi_\vartheta}) \bar{h}^\star. \\
&\leq r_\vartheta.
\end{aligned}$$

Step 2: Claim: S is continuous.
Let (u_n) be a sequence such that $u_n \to u$ in Π_ϑ then

$$\begin{aligned}
|(Su_n)(t) - (Su)(t)| = \Big| &(\Phi u_n)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u_n(\sigma)) \frac{d\sigma}{\sigma} \\
&- (\Phi u)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u(\sigma)) \frac{d\sigma}{\sigma} \Big| \\
&\leq \Big| (\Phi u_n)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u_n(\sigma)) \frac{d\sigma}{\sigma} \\
&- (\Phi u_n)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u(\sigma)) \frac{d\sigma}{\sigma} \Big| \\
&+ \Big| (\Phi u_n)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u(\sigma)) \frac{d\sigma}{\sigma} \\
&- (\Phi u)(t) \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} \xi(t,\sigma,u(\sigma)) \frac{d\sigma}{\sigma} \Big| \\
&\leq |(\Phi u_n)(t)| \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} |\xi(t,\sigma,u_n(\sigma)) d\sigma - \xi(t,\sigma,u(\sigma))| \frac{d\sigma}{\sigma} \\
&+ |(\Phi u_n)(t) - (\Phi u)(t)| \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} |\xi(t,\sigma,u(\sigma))| \frac{d\sigma}{\sigma} \\
&\leq (\eta + \gamma |u_n(t)|) D_\vartheta \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} (log\sigma)^{-\sigma} |u_n(\sigma) - u(\sigma)| \frac{d\sigma}{\sigma} \\
&+ \varpi_\vartheta |u_n(\sigma) - u(\sigma)| \int_{K_{\vartheta-1}}^{t} \frac{1}{\Gamma(\omega_\vartheta)} (\log \frac{t}{\sigma})^{\omega_\vartheta - 1} (log\sigma)^{-\sigma} \bar{h}(\sigma) g(|u(\sigma)|) \frac{d\sigma}{\sigma} \\
&\leq \Big[\Big(D_\vartheta (\eta + \gamma r_\vartheta) + \bar{h}^\star \varpi_\vartheta g(r_\vartheta) \Big) \frac{(\log K_\vartheta)^{1-\sigma} - (\log K_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)} (\log \frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta - 1} \Big] \|u_n - u\|_{\Pi_\vartheta}
\end{aligned}$$

i.e., we obtain

$$\|(Su_n) - (Su)\|_{\Pi_\vartheta} \to 0 \ \ as \ \ n \to \infty.$$

As a consequence, the operator S is a continuous on Π_n.

Step 3: Claim: S is compact
Step 1 leads to the outcome $\|S(u)\|_{\Pi_\vartheta} \leq r_\vartheta$ for each $u \in B_{r_\vartheta}$, yielding the boundedness of $S(B_{r_\vartheta})$. We shall now demonstrate the equicontinuity of $S(B_{r_\vartheta})$.

For $t_1,t_2\in\Upsilon_\vartheta,\ t_1<t_2$ and $u\in B_{r_\vartheta}$, estimate

$$
\begin{aligned}
|(Su)(t_2)-(Su)(t_1)| &\le |q(t_2)-q(t_1)|+\Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\\
&\quad-(\Phi u)(t_1)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_n-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big|\\
&\le |q(t_2)-q(t_1)|+\Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\\
&\quad-\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big|\\
&\quad+\Big|\Big((\Phi u)(t_2)-(\Phi u)(t_1)\Big)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big|\\
&\le |q(t_2)-q(t_1)|+\Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_1}\Big(\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma))\\
&\quad-\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\Big)\frac{d\sigma}{\sigma}\Big|\\
&\quad+\Big|(\Phi u)(t_2)\int_{t_1}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big|\\
&\quad+\Big|\Big((\Phi u)(t_2)-(\Phi u)(t_1)\Big)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big|\\
&\le |q(t_2)-q(t_1)|+\Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\Big(\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))\Big)\frac{d\sigma}{\sigma}\Big|\\
&\quad+\Big|(\Phi u)(t_2)\int_{K_{\vartheta-1}}^{t_1}\frac{\xi(t_1,\sigma,u(\sigma))}{\Gamma(\omega_\vartheta)}((\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}-(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1})\frac{d\sigma}{\sigma}\Big|\\
&\quad+\Big|(\Phi u)(t_2)\int_{t_1}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\xi(t_2,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big|\\
&\quad+\Big|\Big((\Phi u)(t_2)-(\Phi u)(t_1)\Big)\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\xi(t_1,\sigma,u(\sigma))\frac{d\sigma}{\sigma}\Big|\\
&\le |q(t_2)-q(t_1)|+|(\Phi u)(t_2)|\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\Big|\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))\Big|\frac{d\sigma}{\sigma}\\
&\quad+|(\Phi u)(t_2)||\xi(t_1,t_1,r_\vartheta)|\Big|\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}-(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\frac{d\sigma}{\sigma}\Big|\\
&\quad+|(\Phi u)(t_2)||\xi(t_2,s_2,r_\vartheta)|\Big|\int_{t_1}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\frac{d\sigma}{\sigma}\Big|\\
&\quad+|(\Phi u)(t_2)-(\Phi u)(t_1)||\xi(t_1,t_1,r_\vartheta)|\Big|\int_{K_{\vartheta-1}}^{t_1}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_1}{\sigma})^{\omega_\vartheta-1}\frac{d\sigma}{\sigma}\Big|\\
&\le |q(t_2)-q(t_1)|+|(\Phi u)(t_2)|\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t_2}{\sigma})^{\omega_\vartheta-1}\Big|\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))\Big|\frac{d\sigma}{\sigma}\\
&\quad+|(\Phi u)(t_2)|\frac{|\xi(t_1,t_1,r_\vartheta)|}{\Gamma(\omega_\vartheta+1)}\Big|(\log\frac{t_2}{t_1})^{\omega_\vartheta}-(\log\frac{t_2}{K_{\vartheta-1}})^{\omega_\vartheta}+(\log\frac{t_1}{K_{\vartheta-1}})^{\omega_\vartheta}\Big|\\
&\quad+|(\Phi u)(t_2)|\frac{|\xi(t_2,t_2,r_\vartheta)|}{\Gamma(\omega_\vartheta+1)}(\log\frac{t_2}{t_1})^{\omega_\vartheta}+|(\Phi u)(t_2)-(\Phi u)(t_1)|\frac{|\xi(t_1,t_1,r_\vartheta)|}{\Gamma(\omega_\vartheta+1)}(\log\frac{t_1}{K_{\vartheta-1}})^{\omega_\vartheta}
\end{aligned}
$$

Owing to $(H4)$, we know that the function $\xi(t,\sigma,u)$ is uniformly continuous on $\Upsilon_\vartheta^2\times B_{r_\vartheta}$, then we have

$$\lim_{t_2\to t_1}|\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))|=0$$

uniformly in $\sigma\in\Upsilon_\vartheta$ and $u\in B_{r_\vartheta}$. Hence, we have

$$\Big|\int_{K_{\vartheta-1}}^{t_2}\frac{1}{\Gamma(\omega_\vartheta)}\frac{\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))}{(t_2-\sigma)^{1-\omega_\vartheta}}d\sigma\Big|\le\sup_{\sigma\in\Upsilon_\vartheta,\ u\in B_{r_\vartheta}}\frac{(t_2-K_{\vartheta-1})^{1-\omega_\vartheta}}{\Gamma(\omega_\vartheta+1)}|\xi(t_2,\sigma,u(\sigma))-\xi(t_1,\sigma,u(\sigma))|\to 0 \quad (3.5)$$

as $t_2\to t_1$.

So, $\|(Su)(t_2)-(Su)(t_1)\|_{\Pi_\vartheta}\to 0$ as $|t_2-t_1|\to 0$. It confirms $S(B_{r_\vartheta})$ is equicontinuous.
Thereby, considering Theorem 2.5, (3.3) has at least one solution $\widetilde{u}_\vartheta\in B_{r_\vartheta}$.

□

The subsequent result regarding uniqueness relates to the Banach Contradiction Principle.

Theorem 3.2. *Assume that given conditions in Theorem 3.1 hold, and moreover*

$$\Big(D_\vartheta(\eta+\gamma r_\vartheta)+\bar{h}^\star\varpi_\vartheta g(r_\vartheta)\Big)\frac{(\log K_\vartheta)^{1-\sigma}-(\log K_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)}(\log\frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta-1}\leq 1. \tag{3.6}$$

is satisfied.
Then, (3.3) has a unique solution in Π_ϑ.

Proof. Let $u,\tilde{u}\in B_{r_\vartheta}$, and $t\in\Upsilon_\vartheta$, we have

$$\begin{aligned}
&|(Su)(t)-(S\tilde{u})(t)|\\
=\;&\Big|(\Phi u)(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}-(\Phi\tilde{u})(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,\tilde{u}(\sigma))\frac{d\sigma}{\sigma}\Big|\\
\leq\;&\Big|(\Phi u)(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,u(\sigma))\frac{d\sigma}{\sigma}-(\Phi u)(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,\tilde{u}(\sigma))\frac{d\sigma}{\sigma}\Big|\\
+\;&\Big|(\Phi u)(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,\tilde{u}(\sigma))\frac{d\sigma}{\sigma}-(\Phi\tilde{u})(t)\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}\xi(t,\sigma,\tilde{u}(\sigma))\frac{d\sigma}{\sigma}\Big|\\
\leq\;&|(\Phi u)(t)|\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}|\xi(t,\sigma,u(\sigma))d\sigma-\xi(t,\sigma,\tilde{u}(\sigma))|\frac{d\sigma}{\sigma}\\
+\;&|(\Phi u)(t)-(\Phi\tilde{u})(t)|\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}|\xi(t,\sigma,\tilde{u}(\sigma))|\frac{d\sigma}{\sigma}\\
\leq\;&(\eta+\gamma|u(t)|)D_\vartheta\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}(log\sigma)^{-\sigma}|z(\sigma)-\tilde{u}(\sigma)|\frac{d\sigma}{\sigma}\\
+\;&\varpi_\vartheta|u(\sigma)-\tilde{u}(\sigma)|\int_{K_{\vartheta-1}}^t\frac{1}{\Gamma(\omega_\vartheta)}(\log\frac{t}{\sigma})^{\omega_\vartheta-1}(log\sigma)^{-\sigma}\bar{h}(\sigma)g(|\tilde{u}(\sigma)|)\frac{d\sigma}{\sigma}\\
\leq\;&\Big[\Big(D_\vartheta(\eta+\gamma r_\vartheta)+\bar{h}^\star\varpi_\vartheta g(r_\vartheta)\Big)\frac{(\log K_\vartheta)^{1-\sigma}-(\log K_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)}(\log\frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta-1}\Big]\|u-\tilde{u}\|_{\Pi_n}
\end{aligned}$$

Therefore,

$$\|(Su)(t)-(S\tilde{u})(t)\|_{\Pi_\vartheta}\leq\Big[\Big(D_\vartheta(\eta+\gamma r_\vartheta)+\bar{h}^\star\varpi_\vartheta g(r_\vartheta)\Big)\frac{(\log K_\vartheta)^{1-\sigma}-(\log K_{\vartheta-1})^{1-\sigma}}{(1-\sigma)\Gamma(\omega_\vartheta)}(\log\frac{K_\vartheta}{K_{\vartheta-1}})^{\omega_\vartheta-1}\Big]\|u-\tilde{u}\|_{\Pi_\vartheta}$$

Thus, according to equation (3.6), the operator S is a contraction mapping. Therefore, S has a unique fixed point referring to the uniqueness of solution of (3.3). □

The subsequent result discusses uniqueness property of (1.1).

Theorem 3.3. *Assume that hypotheses $(H1)-(H5)$ and inequality (3.6) hold for all $\vartheta\in\{1,2,...,n\}$. Then, the problem (1.1) has a unique solution in $C(\Upsilon,\mathcal{R})$.*

Proof. In light of the proof previously mentioned, we may conclude that (3.3) has a unique solution.Based on the above proofs, we know that (3.3) possesses a unique solution $\widetilde{u}_\vartheta\in\Pi_\vartheta$, $\vartheta\in\{1,2,...,n\}$. This is in accordance with Theorem 3.2.

Let us define the solution function for any $\vartheta\in\{1,2,...,n\}$ as

$$u_\vartheta=\begin{cases}0, & t\in[1,K_{\vartheta-1}],\\ \widetilde{u}_\vartheta, & t\in\Upsilon_\vartheta,\end{cases} \tag{3.7}$$

Thus, $u_\vartheta\in C([1,K_\vartheta],\mathbb{R})$ solves the integral equation (3.2) for $t\in\Upsilon_\vartheta$.
Then, the function

$$u(t)=\begin{cases}u_1(t), \quad t\in\Upsilon_1,\\ u_2(t)=\begin{cases}0, & t\in\Upsilon_1,\\ \widetilde{u}_2, & t\in\Upsilon_2\end{cases}.\\ .\\ .\\ .\\ u_n(t)=\begin{cases}0, & t\in[1,K_{n-1}],\\ \widetilde{u}_n, & t\in\Upsilon_n\end{cases}\end{cases}$$

is a unique solution of (1.1) in $C(\Upsilon,\mathcal{R})$.

□

4. Example

We shall examine the following problem

$$u(t)=\frac{1}{e^{e^{\frac{1}{\sqrt{t+2}}}}}+\frac{|u(t)|}{1+|u(t)|}\int_1^t\frac{1}{\Gamma(\omega(t))}(\log\frac{t}{\sigma})^{\omega(t)-1}\frac{(logt)^{-\frac{1}{10}}}{t+2}\frac{u(\sigma)}{\sigma+10}\frac{d\sigma}{\sigma},\ t\in\Upsilon:=[1,e], \tag{4.1}$$

where

$$\omega(t)=\begin{cases}\frac{3}{2}, & t\in\Upsilon_1:=[1,2],\\ \frac{6}{5}, & t\in\Upsilon_2:=]2,e].\end{cases} \tag{4.2}$$

Let

$$\begin{aligned}K_0 &= 1,\ K_1=2,\ K_2=K=e,\\ q(t) &= \frac{1}{e^{e^{\frac{1}{\sqrt{t+2}}}}},\ t\in\Upsilon\\ (\Phi u)(t) &= \frac{u(t)}{1+u(t)},\ t\in\Upsilon\ \ and\ \ u\in C(\Upsilon,\mathcal{R}_+),\\ \xi(t,\sigma,u) &= \frac{(logt)^{-\frac{1}{10}}}{t+2}.\frac{1}{\sigma+10}u,\ (t,\sigma,u)\in\Upsilon^2\times\mathcal{R}_+,\end{aligned}$$

and $u\in C(\Upsilon,\mathcal{R}_+)$. It is clear that (4.1) can be written as (1.1).
By using (4.2), according to (3.3) we take into consideration the subsequent auxiliary equations:

$$u(t)=\frac{1}{e^{e^{\frac{1}{\sqrt{t+2}}}}}+\frac{|u(t)|}{1+|u(t)|}\int_1^t\frac{1}{\Gamma(\omega_1)}(\log\frac{t}{\sigma})^{\omega_1-1}\frac{(logt)^{-\frac{1}{10}}}{t+2}\frac{u(\sigma)}{\sigma+10}\frac{d\sigma}{\sigma},\ t\in\Upsilon_1, \tag{4.3}$$

and

$$u(t)=\frac{1}{e^{e^{\frac{1}{\sqrt{t+2}}}}}+\frac{|u(t)|}{1+|u(t)|}\int_1^t\frac{1}{\Gamma(\omega_2)}(\log\frac{t}{\sigma})^{\omega_2-1}\frac{(logt)^{-\frac{1}{10}}}{t+2}\frac{u(\sigma)}{\sigma+10}\frac{d\sigma}{\sigma},\ t\in\Upsilon_2. \tag{4.4}$$

Let us show that conditions $(H1)-(H5)$ and inequalities (3.4), (3.6) hold.
For $\vartheta=1$, we have

$$|(\Phi u)(t)-(\Phi\tilde{u})(t)|=|\frac{u(t)}{1+u(t)}-\frac{\tilde{u}(t)}{1+\tilde{u}(t)}|=|\frac{u(t)-\tilde{u}(t)}{(1+u(t))(1+\tilde{u}(t))}|\le\frac{1}{2}|u(t)-\tilde{u}(t)|.$$

It is obvious that (H2) is satisfied with $\varpi_1=\frac{1}{2}$. for each $u,\tilde{u}\in\Pi_1$ and $t\in\Upsilon_1$.

$$|(\Phi u)(t)|=|\frac{u(t)}{1+u(t)}|\le|u(t)|$$

Moreover (H3) holds with $\eta=0$ and $\gamma=1$. for each $u\in\Pi_1$ and $t\in\Upsilon_1$.

$$(logt)^{\frac{1}{10}}|\xi(t,\sigma,u)-\xi(t,\sigma,\tilde{u})|=|\frac{1}{t+2}.\frac{1}{\sigma+10}u-\frac{1}{t+2}.\frac{1}{\sigma+10}.\tilde{u}|=\frac{1}{t+2}.\frac{1}{\sigma+10}|u-\tilde{u}|\le\frac{1}{33}|z-\tilde{u}|$$

Hence, (H4) is satisfied with $\sigma=\frac{1}{10}$ and $D_1=\frac{1}{33}$. for all $(t,\sigma)\in\Upsilon_1^2$ and $u,\tilde{u}\in\Pi_1$.

$$(logt)^{\frac{1}{10}}|\xi(t,\sigma,u)|=\frac{1}{t+2}.\frac{1}{\sigma+10}.|u|\le\bar{h}(\sigma)g(|u|),$$

Then, (H5) holds with $\sigma=\frac{1}{10}$, $g(|u|)=|u|$ and $\bar{h}(\sigma)=\frac{1}{3(\sigma+10)}$ which means that $\bar{h}^\star=\frac{1}{33}$,
and the inequality

$$\frac{r_1}{q^\star+\frac{(logK_1)^{1-\sigma}-(logK_0)^{1-\sigma}}{(1-\sigma)\Gamma(\omega_1)}(\log\frac{K_1}{K_0})^{\omega_1-1}(\eta+\gamma r_1)g(r_1)\bar{h}^\star}>1,$$

is satisfied for each $r_1\in(0.2533,\ 43.8014)$ which means that condition (3.4) holds, and the inequality

$$\Big(D_1(\eta+\gamma r_1)+\bar{h}^\star\varpi_1 g(r_1)\Big)\frac{(\log K_1)^{1-\sigma}-(\log K_0)^{1-\sigma}}{(1-\sigma)\Gamma(\omega_1)}(\log\frac{K_1}{K_0})^{\omega_1-1}\le 1,$$

is satisfied for each $r_1 \in (0, > 29.3685)$ which means that condition (3.6) holds.
Consequently, by Theorem 3.2, (4.3) has a unique solution $\widetilde{u_1}$ in Π_1.

For $\vartheta = 2$, we have

$$|(\Phi u)(t) - (\Phi \tilde{u})(t)| = |\frac{u(t)}{1+u(t)} - \frac{\tilde{u}(t)}{1+\tilde{u}(t)}| = |\frac{u(t)-\tilde{u}(t)}{(1+u(t))(1+\tilde{u}(t))}| \leq \frac{1}{2}|u(t)-\tilde{u}(t)|$$

Then, (H2) is satisfied with $\varpi_2 = \frac{1}{2}$.
for each $u, \tilde{u} \in \Pi_2$ and $t \in \Upsilon_2$.

$$|(\Phi u)(t)| = |\frac{u(t)}{1+u(t)}| \leq |u(t)$$

Then, (H3) holds with $\eta = 0$ and $\gamma = 1$. for each $u \in \Pi_2$ and $t \in \Upsilon_2$.

$$(logt)^{\frac{1}{10}}|\xi(t,\sigma,u) - \xi(t,\sigma,\tilde{u})| = |\frac{1}{t+2}.\frac{1}{\sigma+10}.u - \frac{1}{t+2}.\frac{1}{\sigma+10}.\tilde{u}| = \frac{1}{t+2}.\frac{1}{\sigma+10}|u-\tilde{u}| \leq \frac{1}{48}|u-\tilde{u}|$$

Hence, (H4) is satisfied with $\sigma = \frac{1}{10}$ and $D_2 = \frac{1}{48}$. for all $(t,\sigma) \in \Upsilon_2^2$ and $u, \tilde{u} \in \Pi_2$.

$$(logt)^{\frac{1}{10}}|\xi(t,\sigma,u)| = \frac{1}{t+2}.\frac{1}{\sigma+10}.|u| \leq \bar{h}(\sigma)g(|u|),$$

Then, (H5) holds with $\sigma = \frac{1}{10}$, $g(|u|) = |u|$ and $\bar{h}(\sigma) = \frac{1}{4(\sigma+10)}$, $\bar{h}^\star = \frac{1}{48}$,
and the inequality

$$\frac{r_2}{q^\star + \frac{(logK_2)^{1-\sigma}-(logK_1)^{1-\sigma}}{(1-\sigma)\Gamma(\omega_2)}(\log\frac{K_2}{K_1})^{\omega_2-1}(\eta+\gamma r_2)g(r_2)\bar{h}^\star} > 1,$$

is satisfied for each $r_2 \in (0.2589,\ 178.3125)$ which means that condition (3.4) holds, and the inequality

$$\Big(D_2(\eta+\gamma r_2) + \bar{h}^\star \varpi_2 g(r_2)\Big)\frac{(\log K_2)^{1-\sigma} - (\log K_1)^{1-\sigma}}{(1-\sigma)\Gamma(\omega_2)}(\log\frac{K_2}{K_1})^{\omega_2-1} \leq 1,$$

is satisfied for each $r_2 \in (0,\ 119.0476)$ which implies that condition (3.6) holds.
Consequently, by Theorem 3.2, (4.4) has a unique solution $\widetilde{u_2}$ in Π_2.
Then, according to Theorem 3.3, problem (4.1) has a unique solution

$$u(t) = \begin{cases} \widetilde{u_1}(t), & t \in \Upsilon_1, \\ u_2(t), & t \in \Upsilon_2. \end{cases}$$

where

$$u_2(t) = \begin{cases} 0, & t \in [1, K_1], \\ \widetilde{u_2}, & t \in \Upsilon_2, \end{cases}$$

5. Conclusion

In this work, we deal with integral equations in the form of quadratic Urysohn type involving Hadamard fractional variable order integral operator. A thorough study and an effective mathematical framework for fractional calculus of variable order has been presented recently. The literature contains surveys of the sorts of variable-order derivatives and integrals, along with some physical applications. Regrettably, when applied to variables of order, this attribute does not possess the semi-group property. Because of this, we are unable to simply transform the FVO differential equations into a corresponding integral equation, unlike with constant-order fractional equations. Based on our understanding of this challenge, we have utilized piece-wise constant functions to establish existence and uniqueness results. In the final stage, we apply the results of our method by constructing a numerical example.
As a future study, these findings on the Urysohn integral equation could be applied to other spaces, such as Frechet space combined with different fractional integral operators.

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Mathematical Modeling of Schistosomiasis Transmission Using Reaction-Diffusion Equations

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Abstract

Schistosomiasis, a neglected tropical disease caused by parasitic trematodes of the genus *Schistosoma*, affects millions of people in tropical and subtropical regions lacking access to clean water and proper hygiene. With its impact on health and well-being, the World Health Organization aspires to eliminate schistosomiasis by 2030. This work addresses the challenge of effective control in endemic areas by integrating diffusion in each sub-population using reaction-diffusion equations. The proposed model includes treated individuals who have undergone massive drug administration and a time-dependent function that models the change in human behavior. We present a Partial Differential Equation (PDE) model of schistosomiasis spread that incorporates population movement and human behavior change. Mathematical analysis explores the system's dynamics according to the infection threshold R_0, shedding light on the disease's behavior. Sensitivity analysis is used to identify the key parameters affecting disease spread. Numerical simulations under different scenarios elucidate the impact of human behavior on disease dynamics. This research contributes to a deeper understanding of schistosomiasis transmission and provides insights into control strategies.

1. Introduction

Schistosomiasis, also known as Bilharzia or snail fever, is a neglected tropical disease (NTD) prevalent in tropical and subtropical countries with limited access to safe drinking water and proper hygiene. It is caused by trematode blood flukes of the genus *Schistosoma* and is endemic in 52 countries, affecting over 290 million people in 2018 [1]. The World Health Organization (WHO) has aimed to eliminate schistosomiasis as a public health problem by 2030 [2]. Schistosomiasis is transmitted through contact with fresh water contaminated by the infective larvae of *Schistosoma* parasites [1]. The life cycle of the disease involves the release of parasite eggs into water bodies through human waste, hatching of eggs into miracidia, infecting snails, developing into cercariae and eventually infecting humans through skin penetration, leading to organ damage, abdominal pain, blood in stool or urine, anemia, dysuria and other health complications [3, 4]. Effective control of schistosomiasis remains challenging in endemic regions and the main approach is mass drug administration (MDA) using praziquantel, an anthelmintic drug, to reduce morbidity, mortality and transmission rates [5].
Research into the dynamics of *Schistosoma* infections traces its origins back to 1965 when George Macdonald introduced the inaugural mathematical framework for schistosome epidemiology [6]. This pioneering model, based on differential equations, describes the progression of the average worm burden in the human host, taking into account the complex nature of schistosome. Subsequent to this, researchers have developed the model taking into account the heterogeneity of the intermediate host[7]. Contemporary investigations have leveraged agent-based models (ABMs) and individual-based models (IBMs) to capture the multifaceted diversity in human behaviors and interactions [8, 9]. These innovative models have not only illuminated the pivotal role of water-related activities but have also pinpointed regions of heightened transmission risk [8]. However, human behavior and the movement of individuals between locations, play a significant role in disease spread, as infected individuals can introduce the parasite to new areas, potentially creating new transmission hotpots [10]. Since then, various models have been proposed that include more detailed information such as spatial heterogeneity or seasonality [10, 11]. Zhang et al. [12] studied the spatial distribution of schistosomiasis and the treatment needs in Africa. Manuela Ciddio et al. [10] utilized a multidimensional network model to investigate the spatial spread of schistosomiasis within the Saint-Louis region of

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Senegal. The study emphasizes the crucial role of spatial connectivity in disease propagation and underscores the significance of accounting for various transport pathways to develop effective disease control strategies.
This paper aims to develop a mathematical model for schistosomiasis spread based on reaction-diffusion equations that integrate human behavior change. Reaction-diffusion systems have proven to be effective and appropriate modeling tools for comprehending the spatiotemporal dynamics of diseases. Operating within the domain of spatial continuity, these systems have been pivotal in delving into intricate topics such as nonlinear infection mechanisms and spatial diffusivity. The model we present in this paper takes into account individuals who have undergone mass drug administration (MDA) as detailed in [13]. Moreover, our investigation extends to encompass changes in human behavior and the exploration of diffusion phenomena, contributing to an enhanced understanding of the spatial distribution of the disease.
The remainder of this work is organized as follows. Section 2 presents a Partial Differential Equation (PDE) model that incorporates population mobility and the biological description of the infection parameters. A mathematical analysis of the model to understand the dynamical behavior of the system depending on the value of the threshold of infection R_0 is done in Section 3. Section 4 conducts the sensitivity analysis of R_0 to identify parameters sensitive to the disease spread. Section 5 presents the numerical simulations under different scenarios by taking appropriate parameters to explore the effect of human behavior on disease dynamics. Finally, Section 6 gives a brief discussion and conclusion.

2. Model Formulation

Models of schistosomiasis transmission typically incorporate various aspects of the schistosome life cycle. The populations considered consist of humans (H) and snails (S), with the presence of cercariae (C) and miracidia (M). Cercariae (C) represent larval worms shed into the aquatic environment by infected snails, while miracidia (M) are eggs shed into streams by infected humans engaging in activities like fishing, swimming, or drinking. The human population is divided into sub-populations: susceptible (S_h), exposed (E_h), infected (I_h) and treated (T_h) individuals, while the snail population consists of susceptible (S_s), exposed (E_s) and infected (I_s) snails. The susceptible human reproduces at a constant rate Λ_h and dies naturally at the rate μ_h, The susceptible become infected through contact with fresh water contaminated by cercariae from infected snail at the rate $\beta_{ch}\theta C$. The exposed humans become infectious at a rate γ_h and we assume that a rate σ_h of infected humans receives the MDA, while a fraction λ recovers and returns to the susceptible class. Others may die because of the infections at a rate ρ_h, We assume that the treated humans are not infectious, i.e., they do not produce eggs for miracidia. Shedding of infection within the environment by infected humans is assumed to occur at rate α_m which represents the rate of miracidia produced by infected humans. Susceptible snails reproduce at a constant rate Λ_s and die naturally at the rate μ_s. They become infected upon contact with miracidia from the shedding of infected humans and mammals at the rate $\beta_{ms}M$. The exposed snails become infectious at a rate γ_s and those infected snails shed larva worms (cercariae) in the environment at a rate α_c. The death rates of miracidia and cercariae are μ_m and μ_c, respectively. The model assumes no immigration of infectious individuals. Figure 1 illustrates the transmission diagram of Schistosomiasis.

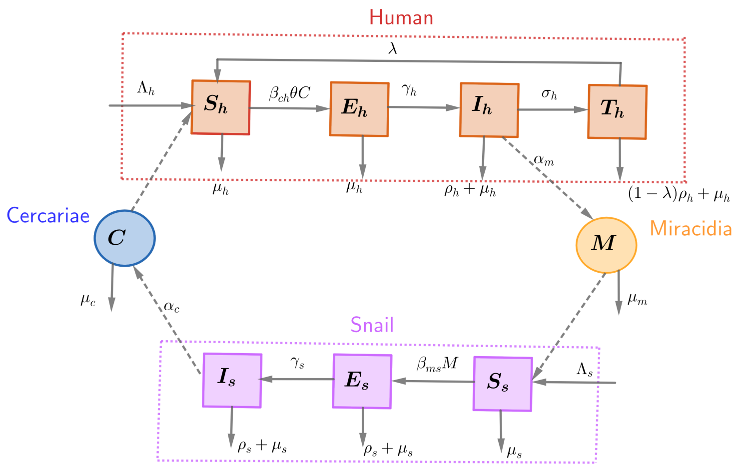


Figure 1: Transmission dynamics of Schistosomiasis. *The disease cycle begins when infected individuals release Schistosoma eggs into freshwater bodies through feces or urine. These eggs hatch, releasing miracidia that infect snails, where they develop into cercariae. The cercariae are then released into the water, actively seeking contact with human skin. Upon skin penetration, they enter the bloodstream and migrate to the liver, maturing into adult worms. The worms then migrate to the veins of the urinary or intestinal systems, where they lay eggs, which starts the whole cycle again. Direct transitions between compartments are represented by the horizontal solid arrows. The mortality rate is represented by the vertical arrows exiting the compartments. The dashed arrow from C to S_h and from M to S_s indicates the contact of susceptible humans with freshwater contaminated by cercariae and the contact of susceptible snails with miracidia, respectively. On the other hand, the dashed arrow from I_h to M and from I_s to C indicates the shedding rate of miracidia and cercariae, respectively.*

We assume that snails, miracidia and cercariae can move within their environment due to factors such as water currents, host movements and other ecological interactions. Diffusion processes allow us to simulate the movement of these populations over time, which affects how they encounter and interact with each other.

This leads to the following system of partial differential equations:

$$\begin{cases} \dfrac{\partial S_h(x,t)}{\partial t} &= d_1\Delta S_h + \Lambda_h + \lambda T_h - \beta_{ch}\theta(t)CS_h - \mu_h S_h, \\ \dfrac{\partial E_h(x,t)}{\partial t} &= d_2\Delta E_h + \beta_{ch}\theta(t)CS_h - \gamma_h E_h - \mu_h E_h, \\ \dfrac{\partial I_h(x,t)}{\partial t} &= d_3\Delta I_h + \gamma_h E_h - \sigma_h I_h - \rho_h I_h - \mu_h I_h, \\ \dfrac{\partial T_h(x,t)}{\partial t} &= d_4\Delta T_h + \sigma_h I_h - \lambda T_h - (1-\lambda)\,\rho_h T_h - \mu_h T_h, \\ \dfrac{\partial M(x,t)}{\partial t} &= d_5\Delta M + \alpha_m I_h - \mu_m M, \\ \dfrac{\partial S_s(x,t)}{\partial t} &= d_6\Delta S_s + \Lambda_s - \beta_{ms} M S_s - \mu_s S_s, \\ \dfrac{\partial E_s(x,t)}{\partial t} &= d_7\Delta E_s + \beta_{ms} M S_s - \gamma_s E_s - \rho_s E_s - \mu_s E_s, \\ \dfrac{\partial I_s(x,t)}{\partial t} &= d_8\Delta I_s + \gamma_s E_s - \rho_s I_s - \mu_s I_s, \\ \dfrac{\partial C(x,t)}{\partial t} &= d_9\Delta C + \alpha_c I_s - \mu_c C. \end{cases} \tag{2.1}$$

Where S_h, E_h, I_h and T_h, represent the populations of susceptible, exposed, infected and treated humans at position x and time t, respectively. S_s, E_s, and I_s represent the populations of susceptible, exposed and infected snails at position x and time t, respectively. M and C represent the populations of miracidia and cercariae at position x and time t. We assume that the human, snail, miracidia and cercariae population moves in the region Ω according to Fick's second law [14], with d_i $(i = 1,\dots,9)$, being the diffusion coefficients. Each diffusion coefficient d_i determines how quickly each sub-population spreads through space. The Laplacian operator Δ represents the spatial diffusion between neighboring locations and computes the difference between a compartment's value at a specific location and the average of its neighboring compartments.

By incorporating the model the time-dependent function $\theta(t)$ into the model, we identified human behavioral changes such as avoiding wading, swimming and other forms of contact with contaminated water, as well as adopting improved sanitation and gaining access to clean water. This function is given by

$$\theta(t) = \begin{cases} 1 & \text{No intervention,} \\ (1+\zeta e^{rt})^{-1} & \text{with intervention} \end{cases} \tag{2.2}$$

This type of function is often used to capture the gradual change in behavior from initial resistance to eventual widespread adoption [15]. Here, ζ represents the maximum level of behavior change effectiveness that can be achieved. We have $\zeta \in (0,1)$, where 0 represents no behavior change, 1 represents full behavior change compliance and r determines how quickly behavior change is adopted and becomes effective over time.

The following initial conditions are associated with the system (2.1) :

$$\begin{cases} S_h(x,0) = \phi_1(x),\ E_h(x,0) = \phi_2(x),\ I_h(x,0) = \phi_3(x),\ T_h(x,0) = \phi_4(x),\ M(x,0) = \phi_5(x), \\ S_s(x,0) = \phi_6(x),\ E_s(x,0) = \phi_7(x),\ I_s(x,0) = \phi_8(x),\ C(x,0) = \phi_9(x), \\ x \in \Omega \ \text{ and } \phi_i \in C^2(\Omega) \cap C(\Omega),\ i = 1,\dots,9, \end{cases} \tag{2.3}$$

and homogeneous Neumann boundary conditions are imposed:

$$\frac{\partial S_h}{\partial \eta} = \frac{\partial E_h}{\partial \eta} = \frac{\partial I_h}{\partial \eta} = \frac{\partial T_h}{\partial \eta} = \frac{\partial M}{\partial \eta} = \frac{\partial S_s}{\partial \eta} = \frac{\partial E_s}{\partial \eta} = \frac{\partial I_s}{\partial \eta} = \frac{\partial C}{\partial \eta} = 0, \quad x \in \partial\Omega,\ t > 0, \tag{2.4}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and η is the unit outer normal to $\partial\Omega$. The biological description of all the parameters in the system (2.1) is given in Table 1.

Table 1: Description of the model parameters.

Param.	Biological description	Value	Unit	Source
Λ_h	Recruitment rate of humans	0.62	humans per day	[16]
Λ_s	Recruitment rate of snails	2.5	snails per day	[16]
β_{ch}	Infection rate of cercariae on humans	4×10^{-6}	per day	[10]
β_{ms}	Infection rate of miracidia on snails	5×10^{-5}	per day	[10]
θ	Time-dependent function describing human intervention	-	-	-
ρ_h	Death rate of humans due to infection	0.000274	per day	[13]
ρ_s	Death rate of snails due to infection	0.011	per day	[10]
γ_h	Rate of transmission of humans from exposure to infection	0.0238	per day	[13]
γ_s	Rate of transmission of snails from exposure to infection	0.0286	per day	[13]
σ_h	Transmission rate of humans from infection to treatment	0.03	per day	[17]
α_m	Rate individuals produce miracidia	6.96	miracidia per human per day	[18]
α_c	Rate snails produce cercariae	2.6	cercariae per snail per day	[18]
λ	Treatment efficacy (for *Schistosoma* mansoni)	0.767	-	[19]
μ_h	Natural death rate of humans	0.00004379	per day	[13]
μ_s	Natural death rate of snails	2.7×10^{-3}	per day	[10]
μ_m	Natural death rate of miracidia	3.04	per day	[10]
μ_c	Natural death rate of cercariae	0.91	per day	[10]

3. Mathematical Analysis of the Model

This section is devoted to the theoretical study of the transmission model of the spread of Schistosomiasis described by a system of 9-PDE of the system (2.1). The existence and uniqueness of positive solutions and the existence of equilibria and their stability are established depending on the value of the basic reproduction number.
The system (2.1) can be expressed as:

$$\frac{\partial X(x,t)}{\partial t} = DX(x,t)+f(X(x,t)), \tag{3.1}$$

with $X=(S_h,E_h,I_h,T_h,M,S_s,E_s,I_s,C)$, $D=diag(d_1,d_2,d_3,d_4,d_5,d_6,d_7,d_8,d_9)$, and function f represent the right hand side of the system (2.1) without the diffusive part, i.e.

$$f(X(x,t))=\begin{pmatrix} \Lambda_h+\lambda T_h-\beta_{ch}\theta CS_h-\mu_hS_h, \\ \beta_{ch}\theta CS_h-\gamma_hE_h-\mu_hE_h, \\ \gamma_hE_h-\sigma_hI_h-\rho_hI_h-\mu_hI_h, \\ \sigma_hI_h-\lambda T_h-(1-\lambda)\rho_hT_h-\mu_hT_h, \\ \alpha_mI_h-\mu_mM, \\ \Lambda_s-\beta_{ms}MS_s-\mu_sS_s, \\ \beta_{ms}MS_s-\gamma_sE_s-\rho_sE_s-\mu_sE_s, \\ \gamma_sE_s-\rho_sI_s-\mu_sI_s, \\ \alpha_cI_s-\mu_cC \end{pmatrix}. \tag{3.2}$$

3.1. Existence, Uniqueness and Positivity

We said that $X^-=(S_h^-,E_h^-,I_h^-,T_h^-,M^-,S_s^-,E_s^-,I_s^-,C^-)$ and $X^+=(S_h^+,E_h^+,I_h^+,T_h^+,M^+,S_s^+,E_s^+,I_s^+,C^+)$ and in $C(\overline{\Omega}\times[0,\infty))\cap C^{1,2}(\Omega\times[0,\infty))$ are lower and upper solutions of system (2.1), respectively, if $X^-\leq X^+$ in $\overline{\Omega}\times[0,\infty)$ and the following differential inequalities hold:

$$\begin{cases} \dfrac{\partial X^-(x,t)}{\partial t}\leq DX^-(x,t)+f(X^-(x,t)), \\ \dfrac{\partial X^+(x,t)}{\partial t}\geq DX^+(x,t)+f(X^+(x,t)), \quad \text{for } (x,t)\in\Omega\times(0,\infty) \end{cases} \tag{3.3}$$

and

$$\begin{cases} \dfrac{\partial X^-}{\partial \eta} \leq 0 \leq \dfrac{\partial X^+}{\partial \eta}, \text{ for } (x,t) \in \partial\Omega \times (0,\infty), \\ X^-(x,t) \leq \Phi(x,t) \leq X^+(x,t) \quad \text{ for } (x,t) \in \overline{\Omega} \times (0,\infty). \end{cases} \tag{3.4}$$

Where $\leq$ is the standard order relation in $\mathbb{R}^n$ ($x \leq y \Leftrightarrow x_i \leq y_i$, for $i = 1,...,n$) and $\Phi = (\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6,\phi_7,\phi_8,\phi_9)$.

Theorem 3.1. *Let suppose that the initial functions ϕ_i ($i = 1,2,\dots,9$) are continuous in Ω. Then problem (2.1) has exactly one regular solution $X(x,t) = (S_h(x,t),E_h(x,t),I_h(x,t),T_h(x,t),M(x,t),S_s(x,t),E_s(x,t),I_s(x,t),C(x,t))$. This solution is characterized by positivity and boundedness in the region $\Omega \times [0,\infty)$.*

Proof. The existence and uniqueness of the solution are obtained using the Lemma 1 in [20].

Let $\Gamma := C(\Omega,\mathbb{R})$. We observe that $0_{\mathbb{R}^9} = (0,0,0,0,0,0,0,0,0)$ and $W = (w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9)$ are respectively lower and upper solutions of the system (2.1), where

$$\begin{aligned} w_1 &= \max\left\{\tfrac{\Lambda_h}{\mu_h}, \|\phi_1\|_\Gamma\right\}, \quad w_6 = \max\left\{\tfrac{\Lambda_s}{\mu_s}, \|\phi_6\|_\Gamma\right\}, \\ w_2 &= \max\left\{\tfrac{\Lambda_h}{\mu_h}, \|\phi_2\|_\Gamma\right\}, \quad w_7 = \max\left\{\tfrac{\Lambda_s}{\mu_s}, \|\phi_7\|_\Gamma\right\}, \\ w_3 &= \max\left\{\tfrac{\gamma_h\Lambda_h}{\mu_h^2}, \|\phi_3\|_\Gamma\right\}, \quad w_8 = \max\left\{\tfrac{\gamma_s\Lambda_s}{\mu_s^2}, \|\phi_8\|_\Gamma\right\}, \\ w_4 &= \max\left\{\tfrac{\sigma_h\gamma_h\Lambda_h}{\mu_h^3}, \|\phi_4\|_\Gamma\right\}, \quad w_9 = \max\left\{\tfrac{\alpha_c\Lambda_s}{\mu_c\mu_s}, \|\phi_9\|_\Gamma\right\}, \\ w_5 &= \max\left\{\tfrac{\alpha_m\Lambda_h}{\mu_m\mu_h}, \|\phi_5\|_\Gamma\right\}. \end{aligned} \tag{3.5}$$

By applying the Redinger's Lemma, we conclude that the problem (2.1) has exactly one regular solution $X(x,t)$ such that $0_{\mathbb{R}^9} \leq X(x,t) \leq W$ in $\Omega \times [0,\infty)$.

Hence, $0 \leq S_h(x,t) \leq w_1$, $0 \leq E_h(x,t) \leq w_2$, $0 \leq I_h(x,t) \leq w_3$, $0 \leq T_h(x,t) \leq w_4$, $0 \leq M(x,t) \leq w_5$, $0 \leq S_s(x,t) \leq w_6$, $0 \leq E_s(x,t) \leq w_7$, $0 \leq I_s(x,t) \leq w_8$, $0 \leq C(x,t) \leq w_9$.

Furthermore, if $\phi_i(x) \neq 0$ for $i = 1,...,9$, then from the maximum principle, we have $S_h(x,t) > 0$, $E_h(x,t) > 0$, $I_h(x,t) > 0$, $T_h(x,t) > 0$, $M(x,t) > 0, S_s(x,t) > 0$, $E_s(x,t) > 0$, $I_s(x,t) > 0$, $C(x,t) > 0$ for all $t > 0$, $x \in \Omega$. □

3.2. Equilibria and Basic Reproduction Number

3.2.1. Equilibria

The equilibria of the system (2.1) are found by solving

$$\frac{dX(t)}{dt} = f(X(t)) = 0, \tag{3.6}$$

with $X = (S_h,E_h,I_h,T_h,M,S_s,E_s,I_s,C)$ and f given by (3.2). Hence, the system (2.1) has two equilibrium points, namely the disease-free equilibrium point (DFE) and endemic equilibrium point (EE).

1. The DFE is given by

$$E^0 = (S_h^0,0,0,0,0,S_s^0,0,0,0) = \left(\frac{\Lambda_h}{\mu_h},0,0,0,0,\frac{\Lambda_s}{\mu_s},0,0,0\right),$$

   and it translates to the ideal case where the disease disappears into the human and snail population and always exists.

2. The EE is given by $E^* = (S_h^*, E_h^*, I_h^*, T_h^*, M^*, S_s^*, E_s^*, I_s^*, C^*)$, where

$$\begin{cases} S_h^* &= \dfrac{\mu_c(\rho_s+\mu_s)(\gamma_s+\rho_s+\mu_s)\left(\beta_{ms}\alpha_m I_h^*+\mu_s\mu_m\right)(\lambda+(1-\lambda)\rho_h+\mu_h)\Lambda_h+\lambda\sigma_h I_h^*}{\gamma_s\beta_{ms}\alpha_m\alpha_c\Lambda_s(\lambda+(1-\lambda)\rho_h+\mu_h)I_h^*+\mu_h\mu_c(\rho_s+\mu_s)(\gamma_s+\rho_s+\mu_s)\left(\beta_{ms}\alpha_m I_h^*+\mu_s\mu_m\right)}, \\ E_h^* &= \dfrac{\sigma_h+\rho_h+\mu_h}{\gamma_h} I_h^*, \\ I_h^* &= \dfrac{\mu_h\mu_s\mu_c\mu_m(\gamma_h+\mu_h)(\rho_s+\mu_s)(\sigma_h+\rho_h+\mu_h)(\gamma_s+\rho_s+\mu_s)(\lambda+(1-\lambda)\rho_h+\mu_h)(R_e-1)}{A_1+A_2}, \\ T_h^* &= \dfrac{\sigma_h}{\lambda+(1-\lambda)\rho_h+\mu_h} I_h^*, \\ M^* &= \dfrac{\alpha_m}{\mu_m} I_h^*, \\ S_s^* &= \dfrac{\mu_m\Lambda_s}{\beta_{ms}\alpha_m I_h^*+\mu_s\mu_m}, \\ E_s^* &= \dfrac{\beta_{ms}\alpha_m\Lambda_s}{(\gamma_s+\rho_s+\mu_s)\left(\beta_{ms}\alpha_m I_h^*+\mu_s\mu_m\right)} I_h^*, \\ I_s^* &= \dfrac{\gamma_s\beta_{ms}\alpha_m\Lambda_s}{(\rho_s+\mu_s)(\gamma_s+\rho_s+\mu_s)\left(\beta_{ms}\alpha_m I_h^*+\mu_s\mu_m\right)} I_h^*, \\ C^* &= \dfrac{\gamma_s\beta_{ms}\alpha_m\alpha_c\Lambda_s}{\mu_c(\rho_s+\mu_s)(\gamma_s+\rho_s+\mu_s)\left(\beta_{ms}\alpha_m I_h^*+\mu_s\mu_m\right)} I_h^*. \end{cases} \tag{3.7}$$

With

$$\begin{aligned} R_e &= \frac{\beta_{ch}\theta\beta_{ms}\alpha_c\alpha_m\gamma_h\gamma_s\Lambda_h\Lambda_s}{\mu_h\mu_s\mu_c\mu_m(\gamma_h+\mu_h)(\rho_s+\mu_s)(\sigma_h+\rho_h+\mu_h)(\gamma_s+\rho_s+\mu_s)}, \\ A_1 &= \gamma_s\beta_{ch}\theta\beta_{ms}\alpha_m\alpha_c\Lambda_s(\rho_h\sigma_h((1-\lambda)\rho_h+\mu_h)+(\lambda+(1-\lambda)\rho_h+\mu_h)(\gamma_h(\rho_s+\mu_s)+\mu_h(\sigma_h+\rho_h+\mu_h))), \\ A_2 &= \alpha_m\mu_h\mu_c\beta_{ms}(\gamma_h+\mu_h)(\rho_s+\mu_s)(\sigma_h+\rho_h+\mu_h)(\gamma_s+\rho_s+\mu_s)(\lambda+(1-\lambda)\rho_h+\mu_h). \end{aligned}$$

This equilibrium translates the situation of persistence of the disease into the population and exists if $R_e > 1$.

3.2.2. Basic reproduction number

The epidemiological concept of the basic reproduction number (R_0) pertains to the average count of fresh infections within a susceptible population caused by a single infectious individual (human or snail). To determine this metric we use the same approach as [21] and compute the next generation matrix.
Let the infective compartment be $X_I = (E_h, I_h, M, E_s, I_s, C)$, considering the following system:

$$\begin{cases} \dfrac{\partial E_h(x,t)}{\partial t} &= d_2\Delta E_h+\beta_{ch}\theta C S_h-\gamma_h E_h-\mu_h E_h, \\ \dfrac{\partial I_h(x,t)}{\partial t} &= d_3\Delta I_h+\gamma_h E_h-\sigma_h I_h-\rho_h I_h-\mu_h I_h, \\ \dfrac{\partial M(x,t)}{\partial t} &= d_5\Delta M+\alpha_m I_h-\mu_m M, \\ \dfrac{\partial E_s(x,t)}{\partial t} &= d_7\Delta E_s+\beta_{ms} M S_s-\gamma_s E_s-\rho_s E_s-\mu_s E_s, \\ \dfrac{\partial I_s(x,t)}{\partial t} &= d_8\Delta I_s+\gamma_s E_s-\rho_s I_s-\mu_s I_s, \\ \dfrac{\partial C(x,t)}{\partial t} &= d_9\Delta C+\alpha_c I_s-\mu_c C. \end{cases} \tag{3.8}$$

Let's consider the two vectors F and V. Where F represents the rate of new infections appearing in a compartment and V represents the rate of infectives leaving the system, defined as follows:

$$F = \begin{pmatrix} \beta_{ch}\theta C S_h \\ 0 \\ 0 \\ \beta_{ms} S_s \\ 0 \\ 0 \end{pmatrix}, \quad \text{and } V = \begin{pmatrix} (\gamma_h+\mu_h)E_h \\ (\sigma_h+\rho_h+\mu_h)I_h-\gamma_h E_h \\ \mu_m M-\alpha_m I_h \\ (\gamma_s+\rho_s+\mu_s)E_s \\ (\rho_s+\mu_s)I_s-\gamma_s E_s \\ \mu_c C-\alpha_c I_s \end{pmatrix}.$$

The Jacobian matrices of F and V at the DFE E^0 are given by:

$$J_F=\begin{pmatrix}0&0&0&0&0&\frac{\beta_{ch}\theta\Lambda_h}{\mu_h}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&\frac{\beta_{ms}\Lambda_s}{\mu_s}&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad J_V=\begin{pmatrix}\gamma_h+\mu_h&0&0&0&0&0\\-\gamma_h&\sigma_h+\rho_h+\mu_h&0&0&0&0\\0&-\alpha_m&\mu_m&0&0&0\\0&0&0&\gamma_s+\rho_s+\mu_s&0&0\\0&0&0&-\gamma_s&\rho_s+\mu_s&0\\0&0&0&0&-\alpha_c&\mu_c\end{pmatrix}.$$

Then, the next generation matrix is given by:

$$J_FJ_V^{-1}=\begin{pmatrix}0&0&0&\frac{\beta_{ch}\theta\Lambda_h\alpha_c\gamma_c}{\mu_h\mu_c(\gamma_s+\rho_s+\mu_s)(\rho_s+\mu_s)}&\frac{\beta_{ch}\theta\mu_h\alpha_c}{\mu_h\mu_c(\rho_s+\mu_s)}&\frac{\beta_{ch}\theta\Lambda_h}{\mu_h\mu_c}\\0&0&0&0&0&0\\0&0&0&0&0&0\\\frac{\beta_{ms}\Lambda_s\alpha_m}{\mu_s(\gamma_h+\mu_h)(\sigma_h+\rho_h+\mu_h)}&\frac{\beta_{ms}\Lambda_s\alpha_m}{\mu_s\mu_m(\sigma_h+\rho_h+\mu_h)}&\frac{\beta_{ms}\Lambda_s}{\mu_s\mu_m}&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}.$$

The reproduction number is the spectral radius of the next generation matrix. Hence, we have

$$R_0:=\rho(J_FJ_V^{-1})=\left(\frac{\beta_{ch}\theta\beta_{ms}\alpha_c\alpha_m\gamma_h\gamma_s\Lambda_h\Lambda_s}{\mu_h\mu_s\mu_c\mu_m(\gamma_h+\mu_h)(\rho_s+\mu_s)(\sigma_h+\rho_h+\mu_h)(\gamma_s+\rho_s+\mu_s)}\right)^{\frac{1}{2}}. \tag{3.9}$$

Using the notations

$$R_{0,hs}=\frac{\beta_{ms}\alpha_m\gamma_s\Lambda_s}{\mu_s\mu_m(\rho_s+\mu_s)(\gamma_s+\rho_s+\mu_s)},\ \text{and}\ R_{0,sh}=\frac{\beta_{ch}\theta\alpha_c\gamma_h\Lambda_h}{\mu_h\mu_c(\gamma_h+\mu_h)(\sigma_h+\rho_h+\mu_h)}, \tag{3.10}$$

the expression of R_0 takes the form:

$$R_0=\sqrt{R_{0,hs}\cdot R_{0,sh}}. \tag{3.11}$$

The quantity $R_{0,hs}$ and $R_{0,sh}$ reflect the transmission from human to snail and from snail to human, respectively. This expression of R_0 as a geometric mean of $R_{0,hs}$ and $R_{0,sh}$, effectively demonstrates how the different population parameters in the life cycle (Human-Snail-Human), such as birth, death and infection rates impact the transmission intensity as shown in Section 4.

Lemma 3.2. *If $R_0>1$, then the endemic equilibrium point E^* of system (2.1) given by (3.7) exists and is unique.*

Proof. It is easy to observe that $R_e=R_0^2$. Hence $R_e>1$ if and only if $R_0>1$. Therefore the necessary and sufficient condition for the existence of the endemic equilibrium E^* is $R_0>1$.

□

The nature of the system (2.1) is determined by the time-dependent intervention function $\theta(t)$. The analysis of the stability of the system is divided into two cases: one where there is no human intervention ($\theta(t)=1$) and the other where there is human intervention ($\theta(t)=(1+\zeta e^{rt})^{-1}$).

3.3. Stability of autonomous dynamical system

3.3.1. Local stability of the equilibrium

To establish the local stability of the equilibrium, A similar methodology as in prior works such as [22, 23] is employed. Consider the eigenvalues of $-\Delta$ on Ω with homogeneous Neumann boundary conditions: $0=\nu_0<\nu_i<\nu_{i+1},\quad i=1,2,\ldots$ and $E(\nu_i)$ the associated eigenspace. Let denote by $\mathbb{B}_i$, the orthogonal basis for $E(\nu_i)$. Consequently, the solution space $\mathbb{B}=\{(S_h,E_h,I_h,T_h,M,S_s,E_s,I_s,C)\}$ of the system (2.1) can be partitioned as follows:

$$\mathbb{B}=\bigoplus_{i=1}^{\infty}\mathbb{B}_i.$$

If we denote by $J(E)$ the Jacobian matrix of the system (2.1) at the equilibrim E, then as prove in [24] the eigenvalues of $J(E)$ are equivalent to the eigenvalue of the matrix

$$M(E)=-\nu_iD+J_f(E).$$

Where $D=diag(d_1,d_2,d_3,d_4,d_5,d_6,d_7,d_8,d_9)$ is a diagonal matrix of the diffusion coefficients and $J_f(E)$ is the Jacobian matrix of the function f given in (3.2) at the equilibrium E.

Theorem 3.3. *If $R_0 < 1$ and $\theta(t) = 1$, then the disease-free equilibrium point E^0 of system (2.1) is locally asymptotically stable (LAS).*

Proof. Let $J(E^0)$ the Jacobian matrix of the system (2.1) at the DFE. The eigenvalue value of $J(E^0)$ are equivalent to that the matrix

$$M(E^0) = -\nu_i D + J_f(E^0) = \begin{pmatrix} -a_1 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -\beta_{ch}\theta\dfrac{\Lambda_h}{\mu_h} \\ 0 & -a_2 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{ch}\theta\dfrac{\Lambda_h}{\mu_h} \\ 0 & \gamma_h & -a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_h & -a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_m & 0 & -a_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_{ms}\dfrac{\Lambda_s}{\mu_s} & -a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_{ms}\dfrac{\Lambda_s}{\mu_s} & 0 & -a_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_s & -a_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_c & -a_9 \end{pmatrix}.$$

Where $a_1 = \mu_h + \nu_i d_1$, $a_2 = (\gamma_h + \mu_h) + \nu_i d_2$, $a_3 = (\sigma_h + \rho_h + \mu_h) + \nu_i d_3$, $a_4 = (\lambda + (1-\lambda)\rho_h + \mu_h) + \nu_i d_4$, $a_5 = \mu_m + \nu_i d_5$, $a_6 = \mu_s + \nu_i d_6$, $a_7 = (\sigma_s + \rho_s + \mu_s) + \nu_i d_7$, $a_8 = (\rho_s + \mu_s) + \nu_i d_8$ and $a_9 = \mu_c + \nu_i d_9$.
The characteristic polynomial of this matrix is given by:

$$P_1(x) = -(a_1 + x)(a_4 + x)(a_6 + x)Q_1(x) \quad \text{with}$$

$Q_1(x) = x^6 + k_5 x^5 + k_4 x^4 + k_3 x^3 + k_2 x^2 + k_1 x + k_0$ and the values of the coefficients are:

$$\begin{aligned}
k_0 &= a_2a_3a_5a_7a_8a_9\left(1 - \frac{\beta_{ch}\theta\beta_{ms}\alpha_c\alpha_m\gamma_h\gamma_s\Lambda_h\Lambda_s}{\mu_h\mu_s a_2a_3a_5a_7a_8a_9}\right), \\
k_1 &= a_2a_3a_5a_7a_8 + a_2a_3a_5a_7a_9 + a_2a_3a_5a_8a_9 + a_2a_3a_7a_8a_9 + a_2a_5a_7a_8a_9 + a_3a_5a_7a_8a_9 > 0, \\
k_2 &= a_2a_3a_5a_7 + a_2a_3a_5a_8 + a_2a_3a_5a_9 + a_2a_3a_7a_8 + a_2a_3a_7a_9 + a_2a_3a_8a_9 + a_2a_5a_7a_8 \\
&+ a_2a_5a_7a_9 + a_2a_5a_8a_9 + a_3a_5a_7a_8 + a_3a_5a_7a_9 + a_3a_5a_8a_9 + a_5a_7a_8a_9 > 0, \\
k_3 &= a_2a_3a_7 + a_2a_3a_8 + a_2a_3a_9 + a_2a_5a_7 + a_2a_5a_8 + a_2a_5a_9 + a_3a_5a_7 + a_3a_5a_8 + a_3a_5a_9 + a_5a_7a_8 \\
&+ a_5a_7a_9 + a_5a_8a_9 + a_2a_7a_8 + a_2a_7a_9 + a_2a_8a_9 + a_3a_7a_8 + a_3a_7a_9 + a_3a_8a_9 + a_7a_8a_9 > 0. \\
k_4 &= a_2a_7 + a_2a_8 + a_2a_9 + a_3a_7 + a_3a_8 + a_3a_9 + a_7a_8 + a_7a_9 + a_8a_9 > 0, \\
k_5 &= a_2 + a_3 + a_5 + a_7 + a_8 + a_9 > 0.
\end{aligned}$$

It is easy to see that P_1 has three negative eigenvalues: $x_1 = -a_1$, $x_2 = -a_4$ and $x_3 = -a_6$. The other eigenvalues are roots of $Q_1(x)$.
Since $k_1, k_2, k_3, k_4, k_5 > 0$, then by using the Routh-Hurwitz criteria [25] and the conditions of Heffernan [26] that the polynomial $Q_1(x)$ has negative real roots if $k_5k_4 > k_3$, $k_4k_2 > k_0$, $k_2k_1 > k_3k_0$. We already have:

$$\begin{aligned}
k_5k_4 - k_3 &= a_2^2a_7 + a_2^2a_8 + a_2^2a_9 + a_2a_3a_7 + a_2a_3a_8 + a_2a_3a_9 + a_2a_5a_7 + a_2a_5a_8 \\
&+ a_2a_5a_9 + a_3a_7a_8 + a_3a_7a_9 + a_3a_8a_9 + a_5a_7a_8 + a_5a_7a_9 + a_5a_8a_9 \\
&> 0.
\end{aligned}$$

If $R_0 < 1$, then we have:

$$\frac{\beta_{ch}\theta\beta_{ms}\alpha_c\alpha_m\gamma_h\gamma_s\Lambda_h\Lambda_s}{\mu_h\mu_s a_2a_3a_5a_7a_8a_9} \le R_0^2 < 1 \Rightarrow 0 < k_0 < a_2a_3a_5a_7a_8a_9. \tag{3.12}$$

Hence,

$$k_4k_2 > a_2a_3a_5a_7a_8a_9 > k_0, \;\; \text{and} \;\; k_2k_1 - k_3k_0 > k_2k_1 - a_2a_3a_5a_7a_8a_9k_3 = 0. \tag{3.13}$$

Thus all the eigenvalues of P_1 have a negative real part, which implies that the disease-free equilibrium E^0 is locally asymptotically stable if $R_0 < 1$. □

Theorem 3.4. *If $R_0 > 1$ and $\theta(t) = 1$, then the endemic equilibrium point E^* of system (2.1) is locally asymptotically stable (LAS).*

Proof. Let $J(E^*)$ the Jacobian matrix of the system (2.1) at the EE. The eigenvalue value of $J(E^*)$ are equivalent to that the matrix

$$M(E^*) = \nu_i D + J_f(E^*) = \begin{pmatrix} -b_1 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -\beta_{ch}\theta S_h^* \\ \beta_{ch}\theta C^* & -b_2 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{ch}\theta S_h^* \\ 0 & \gamma_h & -b_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_h & -b_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_m & 0 & -b_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_{ms}S_s^* & -b_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_{ms}S_s^* & 0 & -b_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_s & -b_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_c & -b_9 \end{pmatrix}.$$

Where $b_1 = \beta_{ch}\theta C^* + \mu_h + \nu_i d_1,\ b_2 = (\gamma_h + \mu_h) + \nu_i d_2,\ b_3 = (\sigma_h + \rho_h + \mu_h) + \nu_i d_3,\ b_4 = (\lambda + (1-\lambda)\rho_h + \mu_h) + \nu_i d_4$, $b_5 = \mu_m + \nu_i d_5$, $b_6 = \beta_{ms}M^* + \mu_s + \nu_i d_6$, $b_7 = -\beta_{ms}M^* + (\sigma_s + \rho_s + \mu_s) + \nu_i d_7$, $b_8 = (\rho_s + \mu_s) + \nu_i d_8$ and $b_9 = \mu_c + \nu_i d_9$. We are employing the approach as [27, 28, 29]. Assuming the linearized equation at the equilibrium point E^* takes the form:

$$U' = M(E^*)U, \tag{3.14}$$

Here, we consider a solution characterized by the expression:

$$U(t) = U_0 e^{tz}, \quad z \in \mathbb{C}^9, \tag{3.15}$$

where $U_0 = (U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9)$. Upon substituting this particular solution form (3.15) into the linearized system (3.14), we obtain the relationship $zU = M(E^*)U$, which can be rephrased as the subsequent system:

$$\begin{cases} zU_1 &= -b_1U_1 + \lambda U_4 - \beta_{ch}\theta S_h^* U_9, \\ zU_2 &= \beta_{ch}\theta C^* U_1 - b_2U_2 + \beta_{ch}\theta S_h^* U_9, \\ zU_3 &= \gamma_h U_2 - b_3 U_3, \\ zU_4 &= \sigma_h U_3 - b_4 U_4, \\ zU_5 &= \alpha_m U_3 - b_5 U_5, \\ zU_6 &= -\beta_{ms} S_s^* U_5 - b_6 U_6, \\ zU_7 &= \beta_{ms} S_s^* U_5 - b_7 U_7, \\ zU_8 &= \gamma_s U_7 - b_8 U_7, \\ zU_9 &= \alpha_c U_8 - b_9 U_9. \end{cases} \tag{3.16}$$

The system (3.16) can be rewritten as

$$(1 + F_i(z))U_i + G_i(U) = (HU)_i, \quad i = 1, \ldots, 9 \tag{3.17}$$

where

$$F_1(z) = \tfrac{1}{b_1}, \quad F_2(z) = \tfrac{1}{b_2}, \quad F_3(z) = \tfrac{1}{b_3}, \quad F_4(z) = \tfrac{1}{b_4}, \quad F_5(z) = \tfrac{1}{b_5},$$
$$F_7(z) = \tfrac{1}{b_6}, \quad F_7(z) = \tfrac{1}{b_7}, \quad F_8(z) = \tfrac{1}{b_8}, \quad F_9(z) = \tfrac{1}{b_9},$$

and

$$G_1(U) = \tfrac{\beta_{ch}\theta S_h^*}{b_1} U_9, \quad G_6(U) = \tfrac{\beta_{ms} S_s^*}{b_6} U_5,$$
$$G_2(U) = G_3(U) = G_4(U) = G_5(U) = G_7(U) = G_8(U) = G_9(U) = 0,$$

and a non-negative matrix H given by

$$H = \begin{pmatrix} 0 & 0 & 0 & \frac{\lambda}{b_1} & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_{ch}\theta C^*}{b_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_{ch}\theta S_h^*}{b_2} \\ 0 & \frac{\gamma_h}{b_3} & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & \frac{\sigma_h}{b_4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_m}{b_5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta_{ms}S_s^*}{b_7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\gamma_s}{b_8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha_c}{b_9} & 0 \end{pmatrix}.$$

The equilibrium state denoted as $E^* = (S_h^*, E_h^*, I_h^*, T_h^*, M^*, S_s^*, E_s^*, I_s^*, C^*)$ is defined as the endemic equilibrium, satisfying the condition $E^* = HE^*$. Since all the components of E^* are positive when $R_0 > 1$. Let U denote a solution of equation (3.17), there exists a minimal positive real c_0 (as established in [29]), such that the following inequality holds:

$$|U| \leq c_0 E^*, \tag{3.18}$$

where $|U| = (|U_1|, |U_2|, |U_3|, |U_4|, |U_5|, |U_6|, |U_7|, |U_8|, |U_9|)$. The objective is to demonstrate $Re(z) < 0$. Let us assume by contradiction that $Re(z) \geq 0$.
Given that $U \neq 0$, we conclude that $Re(z) > 0$, leading to $|1 + F_i(z)| > 1$ for all $i = 1, \ldots, 9$. Hence

$$\frac{c_0}{\Psi(z)} < c_0, \text{ where } \Psi(z) = \min_{i=1,..,8} |1 + F_i(z)| > 1.$$

Hence, by the minimality of c_0, it is follows that:

$$|U| > \frac{c_0}{\Psi(z)} E^*. \tag{3.19}$$

By applying the norm to both sides of the third equation in (3.17) and using the non-negativity of matrix H, we get:

$$|1 + F_3(z)||U_3| = |(HU)_3| \leq H|U_3| \leq c_0 H(E^*)_3 = c_0 I_h^*, \tag{3.20}$$

This implies that $|U_3| \leq \frac{c_0}{\Psi(z)} I_h^*$ and then, contradicts equation (3.19). Hence, $Re(z) < 0$, which means that all eigenvalues of the matrix $M(E^*)$ have a negative real part. Therefore, the endemic equilibrium E^* is locally asymptotically stable if $R_0 > 1$. □

3.3.2. Global stability of the disease-free equilibrium

Theorem 3.5. *If $\theta(t) = 1$, then disease-free equilibrium point E^0 of system (2.1) is globally asymptotically stable (GAS) if $R_0 < 1$ and unstable if $R_0 > 1$.*

Proof. The Lyapunov-LaSalle technique is used to prove the global asymptotic stability of E^0. Let's consider the Lyapunov function defined as follows :

$$L = \int_\Omega [c_1 E_h(x,t) + c_2 I_h(x,t) + c_3 M(x,t) + c_4 E_s(x,t) + c_5 I_s(x,t) + c_6 C(x,t)]\, dx, \tag{3.21}$$

where

$$\begin{aligned}
c_1 &= \alpha_m \alpha_c \gamma_h \gamma_s \beta_s \Lambda_s (\gamma_h + \mu_h), \\
c_2 &= \alpha_m \alpha_c \gamma_s \beta_s \Lambda_s (\gamma_h + \mu_h) \\
c_3 &= \alpha_c \gamma_s \beta_s \Lambda_s (\gamma_h + \mu_h)(\sigma_h + \rho_h + \mu_h), \\
c_4 &= \alpha_c \gamma_s \mu_m \mu_s (\gamma_h + \mu_h)(\sigma_h + \rho_h + \mu_h), \\
c_5 &= \mu_m \mu_s (\rho_s + \mu_s)(\gamma_h + \mu_h)(\gamma_s + \rho_s + \mu_s)(\sigma_h + \rho_h + \mu_h), \\
c_6 &= \mu_m \mu_s \alpha_c (\gamma_h + \mu_h)(\gamma_s + \rho_s + \mu_s)(\sigma_h + \rho_h + \mu_h).
\end{aligned}$$

We have:

$$\begin{aligned}
\frac{dL}{dt} &= \int_\Omega \left[c_1 \frac{\partial E_h(x,t)}{\partial t} + c_2 \frac{\partial I_h(x,t)}{\partial t} + c_3 \frac{\partial M(x,t)}{\partial t} + c_4 \frac{\partial E_s(x,t)}{\partial t} + c_5 \frac{\partial I_s(x,t)}{\partial t} + c_6 \frac{\partial C(x,t)}{\partial t} \right] dx \\
&= \int_\Omega [c_1 (d_2 \Delta E_h + \beta_{ch} \theta C S_h - (\gamma_h + \mu_h E_h) E_h) + c_2 (d_3 \Delta I_h + \gamma_h E_h - (\sigma_h + \rho_h + \mu_h) I_h) \\
&+ \quad c_3 (d_5 \Delta M + \alpha_m I_h - \mu_m M) + c_4 (d_7 \Delta E_s + \beta_{ms} M S_s - (\gamma_s + \rho_s + \mu_s) E_s) \\
&+ \quad c_5 (d_8 \Delta I_s + \gamma_s E_s - (\rho_s + \mu_s) I_s) + c_6 (d_9 \Delta C + \alpha_c I_s - \mu_c C)] dx \\
&= \int_\Omega \left[c_1 \beta_h \left(S_h - \frac{c_6 \mu_c}{c_1 \beta_h} \right) C + (c_2 \gamma_h - c_1 (\gamma + \mu_h)) E_h + (c_3 \alpha_m - c_2 (\sigma_h + \rho_h + \mu_h)) I_h \right. \\
&+ \quad \left. (c_5 \gamma_s - c_4 (\gamma_s + \rho_s + \mu_s)) E_s + (c_6 \alpha_c - c_5 (\rho_s + \mu_s)) I_s + c_4 \beta_s \left(S_s - \frac{c_3 \mu_m}{c_4 \beta_s} \right) M \right] dx \\
&+ \quad \int_\Omega \left[c_1 d_2 \Delta E_h + c_2 d_3 \Delta I_h + c_3 d_5 \Delta M + c_4 d_7 \Delta E_s + c_5 d_8 \Delta I_s + c_6 d_9 \Delta C \right] dx
\end{aligned}$$

According to the Green's formula and the homogeneous Neumann boundary conditions (2.4), we have

$$\int_\Omega \Delta E_h dx = \int_\Omega \Delta I_h dx = \int_\Omega \Delta M dx = \int_\Omega \Delta E_s dx = \int_\Omega \Delta I_s dx = \int_\Omega \Delta C dx = 0,$$

Hence

$$\begin{aligned}
\frac{dL}{dt} &= \int_\Omega \left[c_1\beta_h\left(S_h-\frac{c_6\mu_c}{c_1\beta_h}\right)C+c_4\beta_s\left(S_s-\frac{c_3\mu_m}{c_4\beta_s}\right)M\right]dx\\
&\le \int_\Omega \left[c_1\beta_h\left(\frac{\Lambda_h}{\mu_h}-\frac{c_6\mu_c}{c_1\beta_h}\right)C+c_4\beta_s\left(\frac{\Lambda_s}{\mu_s}-\frac{c_3\mu_m}{c_4\beta_s}\right)M\right]dx\\
&\le \int_\Omega \left[c_1\beta_h\frac{\Lambda_h}{\mu_h}\left(R_0^2-1\right)C\right]dx
\end{aligned}$$

Therefore, $\frac{dL}{dt}\le 0$ whenever $R_0<1$. Furthermore, $\frac{dL}{dt}=0$ if and only if $M=C=0$. These conditions are only satisfied by the DFE E^0. It follows that the largest invariant set $\{(S_h,E_h,I_h,T_h,M,S_s,E_s,I_s,C)\mid \dot{L}=0\}$ when $R_0<1$ is reduced to the singleton E^0. Based on LaSalle's Invariance Principle [30], the DFE E^0 is globally asymptotically stable when $R_0<1$ and unstable if $R_0>1$. $\square$

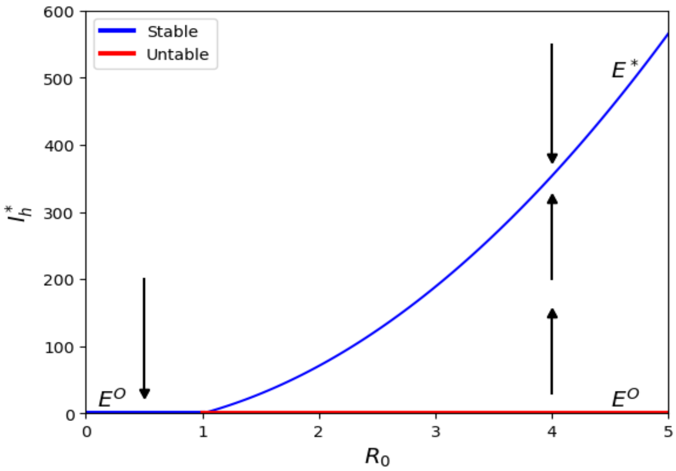


Figure 2: Birfucation plot. *This plot shows the stability of equilibrium points of the system (2.1) for $\theta(t)=1$ as a function of R_0. The horizontal line represents the stable and unstable states of the DFE E^0. The half parabola represents the stable states of the EE E^*. The blue lines denote the stable states and the red lines the unstable states. The black arrows indicate the direction of the vector field.*

3.4. Stability of non-autonomous dynamical system

Theorem 3.6. *If $\theta(t)=(1+\zeta e^{rt})^{-1}$, then the arbitrary equilibrium point $\bar{E}=(\bar{S}_h,\bar{E}_h,\bar{I}_h,\bar{T}_h,\bar{M},\bar{S}_s,\bar{E}_s,\bar{I}_s,\bar{C})$ of the non-autonomous dynamical system (2.1) is uniformly stable.*

Proof. We are employing the approach as in [15]. Let $X(x,t)=(S_h(x,t),E_h(x,t),I_h(x,t),T_h(x,t),M(x,t),S_s(x,t),E_s(x,t),I_s(x,t),C(x,t))$ be a solution of the system (2.1). According to the positive and boundedness of the solution in Theorem 3.1, we have

$$\limsup_{t\to+\infty,x\in\Omega} X(x,t)\le W,$$

with $W=(w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9)$ given in (3.5). Let assume that

$$\|\phi_1\|_{C(\Omega,\mathbb{R})}\le\frac{\Lambda_h}{\mu_h}\text{ and }\|\phi_6\|_{C(\Omega,\mathbb{R})}\le\frac{\Lambda_s}{\mu_s}.$$

Then, for $t>0$, we can derive the norm of the equilibrium point

$$\begin{aligned}
\|E(x,t)\|_\infty &= \|(S_h(x,t),E_h(x,t),I_h(x,t),T_h(x,t),M(x,t),S_s(x,t),E_s(x,t),I_s(x,t),C(x,t))\|_\infty\\
&\le \|W\|_\infty=\max\{w_1,w_6\}=\max\left\{\tfrac{\Lambda_h}{\mu_h},\tfrac{\Lambda_s}{\mu_s}\right\}.
\end{aligned}$$

A time $t=0$, we have:

$$\|E(x,0)\|_\infty = \|(S_h(x,0),E_h(x,0),I_h(x,0),T_h(x,0),M(x,0),S_s(x,0),E_s(x,0),I_s(x,0),C(x,0))\|_\infty$$

$$= \quad \| \left(\frac{\Lambda_h}{\mu_h}, 0,0,0,0, \frac{\Lambda_s}{\mu_s}, 0,0,0 \right) \|_\infty$$
$$= \quad \max \left\{ \frac{\Lambda_h}{\mu_h}, \frac{\Lambda_s}{\mu_s} \right\}.$$

Let consider a class K function $\alpha(.)$ such that

$$\alpha(\|E\|_\infty) = c\|E\|_\infty, \text{ with the constant } c \geq \max\left\{1, \frac{\Lambda_h}{\mu_h}, \frac{\Lambda_s}{\mu_s}\right\}.$$

Therefore,

$$\|E(x,0)\|_\infty < c \Rightarrow \|E(x,t)\|_\infty < c\|E(x,0)\|_\infty = \alpha(\|E(x,0)\|_\infty), \quad \forall t \geq 0.$$

By applying Lemma 4.1 in [31] and the fact that all p-norms in $\mathbb{R}^n$ are equivalent, it result that an arbitrary equilibrium point $\bar{E} = (\bar{S}_h, \bar{E}_h, \bar{I}_h, \bar{T}_h, \bar{M}, \bar{S}_s, \bar{E}_s, \bar{I}_s, \bar{C})$ of the non-autonomous dynamical system (2.1) when $\theta(t) = \theta_0(1+\zeta e^{rt})$ is uniformly stable.

□

4. Sensitivity Analysis

To assess how model parameters influence schistosomiasis spread, we employed global sensitivity analysis. This approach computed partial rank correlation coefficients (PRCC) for model parameters affecting the basic reproduction number R_0 [32], assuming statistical independence for each parameter of interest. This analysis identifies critical parameters significantly impacting the output R_0, guiding accurate measurements.

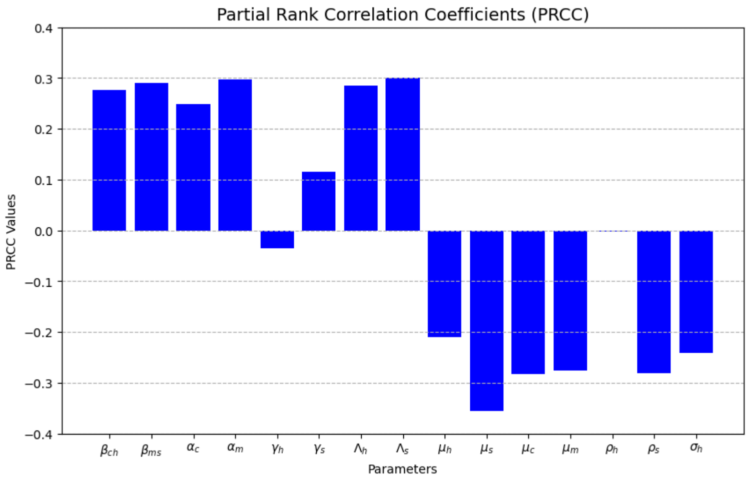


Figure 3: Plot of PRCC R_0. *The PRCC calculation was performed for* R_0 *using Latin Hypercube Sampling (LHS) technique. Parameters in Table 1 were sampled from uniform distributions.*

Figure 3 presents the PRCC values of the model parameters. We observe that parameters such as β_{ch}, β_{ms}, γ_s, α_c, α_m, Λ_h and Λ_s contribute to an increase in the value of R_0, while parameters μ_h μ_s, μ_c, μ_m, ρ_s and σ_h are influential in reducing the burden of schistosomiasis within the population. Notably, the parameter with the highest sensitivity to R_0 is the natural death rate of the snail population μ_s. This suggests that an increase in the snail death rate effectively curtails the spread of schistosomiasis within the population.

Local sensitivity analysis is also used to examine the impact of parameter changes on disease spread using R_0, which determines disease persistence or eradication. The normalized direct sensitivity index of R_0 with respect to a parameter v is given by :

$$S_v^{R_0} = \frac{\partial R_0}{\partial v} \times \frac{v}{R_0}. \tag{4.1}$$

This index quantifies how R_0 changes as v varies. More precisely, if v grows by $x\%$ then R_0 grows by $S_v^{R_0} \times x\%$. A positive index implies a proportional increase (decrease) in R_0 with parameter growth (reduction). Conversely, a negative index signals an opposite relationship. The local sensitivity indexes for R_0related parameters are presented in Table 2.

Table 2: Local sensitivity index for model parameters.

Param.(v)	Λ_h	Λ_s	β_{ch}	β_{ms}	α_c	α_m	γ_h	γ_s
$S_v^{R_0}$	+0.5	+0.5	+0.5	+0.5	+0.5	+0.5	+0.000918	+0.0162
Param.(v)	μ_h	μ_s	μ_c	μ_m	ρ_s	ρ_h	σ_h	
$S_v^{R_0}$	-0.501	-0.631	-0.5	-0.5	-0.531	-0.0574	-0.442	

Table 2 demonstrates that decreasing the recruitment rates of humans and snails results in a substantial decrease in the number of human infections. Specifically, a 1% reduction in either the human or snail recruitment rate would result in a 0.5% decrease in R_0. Conversely, a 1% increase in the treatment rate σ_h would result in a 0.442% reduction in R_0. It is noteworthy that the parameter ρ_h, which characterizes the human death rate attributed to the infection, exerts a relatively low influence on the disease spread threshold. An augmentation of 1% in the parameter ρ_h leads to a mere 0.057% decrease in the threshold. Conversely, the natural death rate of snails, denoted as μ_s, exhibits the most significant local sensitivity index. If μ_s were to increase by 1%, R_0 would decrease notably by 0.631%. Additionally, our investigation highlights that the rates of transmission from exposure to infection, namely γ_h for humans and γ_s for snails, do not wield a significant impact on the reproductive number of the infection. This observation can be rationalized by considering the incubation period required for an exposed human or snail to become infected, which can be quite prolonged.

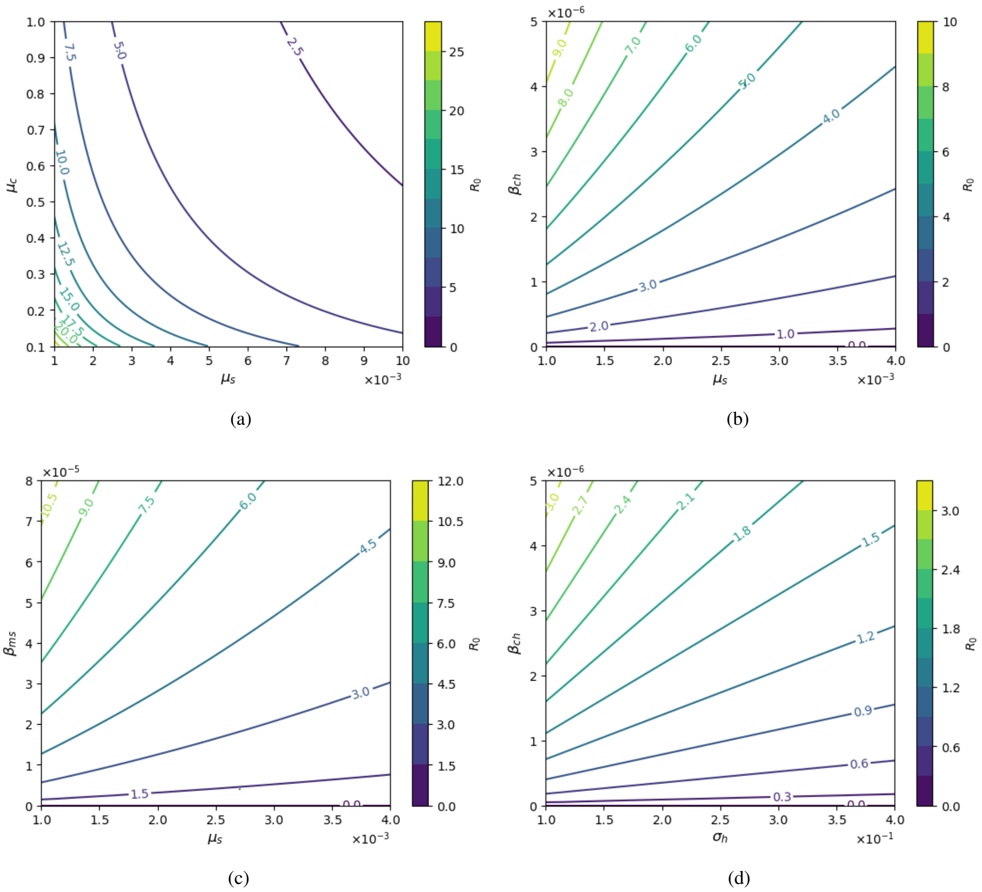


Figure 4: Contour plot of R_0. *(a) Simulated the basic reproduction number R_0 as a function of the and the natural death rate of snail μ_s and death rate of cercariae μ_c. (b) Simulated R_0 as a function of the natural death rate of snail μ_s and the infection rate of cercariae on human β_{ch}. (c) Simulated R_0 as a function of the natural death rate of snail μ_s and the infection rate of miracidia on snail β_{ms}. (d) Simulated R_0 as a function of the treatment rate of infected human σ_h and the infection rate of cercariae on human β_{ch}. The other parameters are taken at their base value in Table 1.*

In Figure 4, we illustrate the influence of parameter changes μ_s, μ_c, β_{ch}, β_{ms} and σ_h on R_0 using a contour plot. When μ_s is increased while μ_c remains constant, an observable decrease in R_0 follows (see Fig 4 (a)). Conversely, a decrease in μ_s while keeping β_{ch} or β_{ms} constant results in an increase in R_0 (see Figure 4 (b)-(c)). Figure 4 (d) demonstrates that even with a high rate of infection, increasing the rate of treatment for infected humans can substantially reduce the value of R_0. Effective control of these parameters can bring R_0 below one, meaning that disease-free equilibrium can be achieved, as proved by Theorem 3.3. This implies that disease-free equilibrium can be achieved by judiciously controlling these parameters.

5. Numerical Simulations

In this section, we conduct numerical simulations to examine disease spread in continuous space and validate theoretical analysis. Numerical computations and plots are performed in MATLAB using the built-in function `pdepe` (For more information, visit: https://www.mathworks.com/help/matlab/ref/pdepe.html). This function employs the finite difference method on a spatial domain $0 \leq x \leq L$ with a grid width set to 10^{-2}. This discretization transforms the system of partial differential equations (PDEs) into a large system of ordinary differential equations (ODEs), which is then solved using the built-in solver `ode15s` with a time step of $\delta t = 10^{-2}$. The 3D plots are generated using the `plotsurface` function of MATLAB, which takes as parameters the time vector, the space vector, and the numerical solution produced by the `pdepe` function.

Spatiotemporal behavior: We consider the model (2.1) with homogeneous Neumann boundary conditions (2.4). For convenience, we set $\Omega = [0,1]$. In our model, human population movement is influenced by factors like migration and commuting behaviors, and it is assumed to occur downstream along the river, reflecting the natural flow of infected individuals and the spread of contamination. Snails, acting as intermediate hosts, can move within the water, primarily influenced by water currents and environmental factors. This movement contributes to the downstream distribution of cercariae. Miracidia, the parasite larvae, and cercariae, the infectious stage, are carried downstream by water flow, facilitating their transmission to susceptible snails and humans in downstream areas. To capture such movement dynamics, we fix the following diffusion coefficients in units of $\text{m}^2\text{day}^{-1}$: $d_1 = 0.1$, $d_2 = 0.05$, $d_3 = 0.02$, $d_4 = 0.1$, $d_5 = 0.0005$, $d_6 = 0.001$, $d_7 = 0.0005$, $d_8 = 0.0003$, $d_9 = 0.0002$.

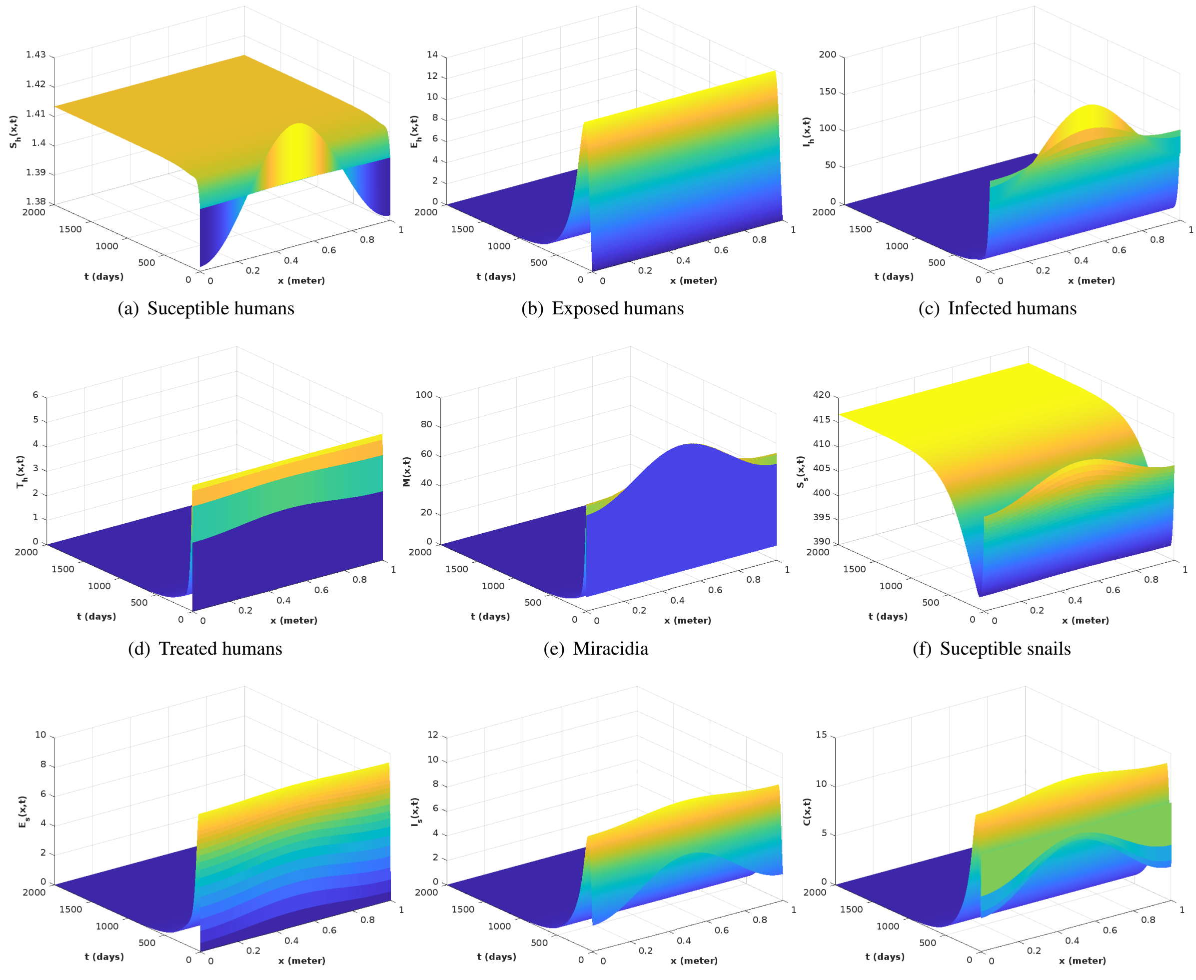
Additionally, we adopt the subsequent initial conditions: $S_h(0) = 0.99\frac{\Lambda_h}{\mu_h} - 200\cos(2\pi x)$, $E_h(0) = 0$, $I_h(0) = 0.01\frac{\Lambda}{\mu_s} - 50\cos(2\pi x)$, $T_h(0) = 0$, $M = 10$, $S_s(0) = 0.99\frac{\Lambda_s}{\mu_s} - 3\cos(2\pi x)$, $E_s(0) = 0$, $I_s(0) = 0.01\frac{\Lambda_s}{\mu_s} - 2\cos(2\pi x)$, $C = 10$.

We divide the simulations into different cases corresponding to the stability of each one of the equilibrium points of the model (2.1) as follows:

Case 1: We consider the values $\beta_{ch} = 2\times 10^{-6}$, $\beta_{ms} = 3\times 10^{-5}$, $\alpha_c = 1.5$, $\alpha_m = 2.96$, $\mu_s = 6\times 10^{-3}$, $\mu_c = 1.01$, $\mu_m = 5$ and $\theta(t) = 1$, the other parameters are given in a Table 1. The corresponding threshold is $R_0 = 0.5839 < 1$ and from Theorem 3.5 the DFE is GAS. As depicted in Figure 5, the numbers of infected individuals $I_h(t,x)$ and infected snails $I_s(t,x)$ converge to zero.

Case 2: We consider all the value the parameter values given in a Table 1 with $\theta(t) = 1$. The corresponding threshold is $R_0 = 4.989 > 1$ and it follows from Theorem 3.4 that the EE is LAS. As shown in Figure 6, the numbers of infected individuals $I_h(t,x)$ and infected snails $I_s(t,x)$ converge to the endemic points I_h^* and I_s^*, respectively.

Case 3: We consider the same parameters as presented in Table 1, but with $\theta(t) = (1+\zeta e^{rt})^{-1}$, where $\zeta = 0.02$ and $r = 0.005$. As demonstrated in Theorem 3.6, the equilibrium of the non-autonomous dynamical system displays uniform stability. Illustrated in Figure 7, the intervention function's effect, $\theta(t)$, leads to a gradual reduction of the reproduction number below 1 over time. Consequently, both the numbers of infected individuals, $I_h(t,x)$ and infected snails, $I_s(t,x)$, decrease and converge to zero over time, while the populations of susceptible humans and snails increase. We find that the spatio-temporal evolution of exposed and infected humans are similar, indicating that human interventions have the same effect on exposed individuals as they do on infected individuals.

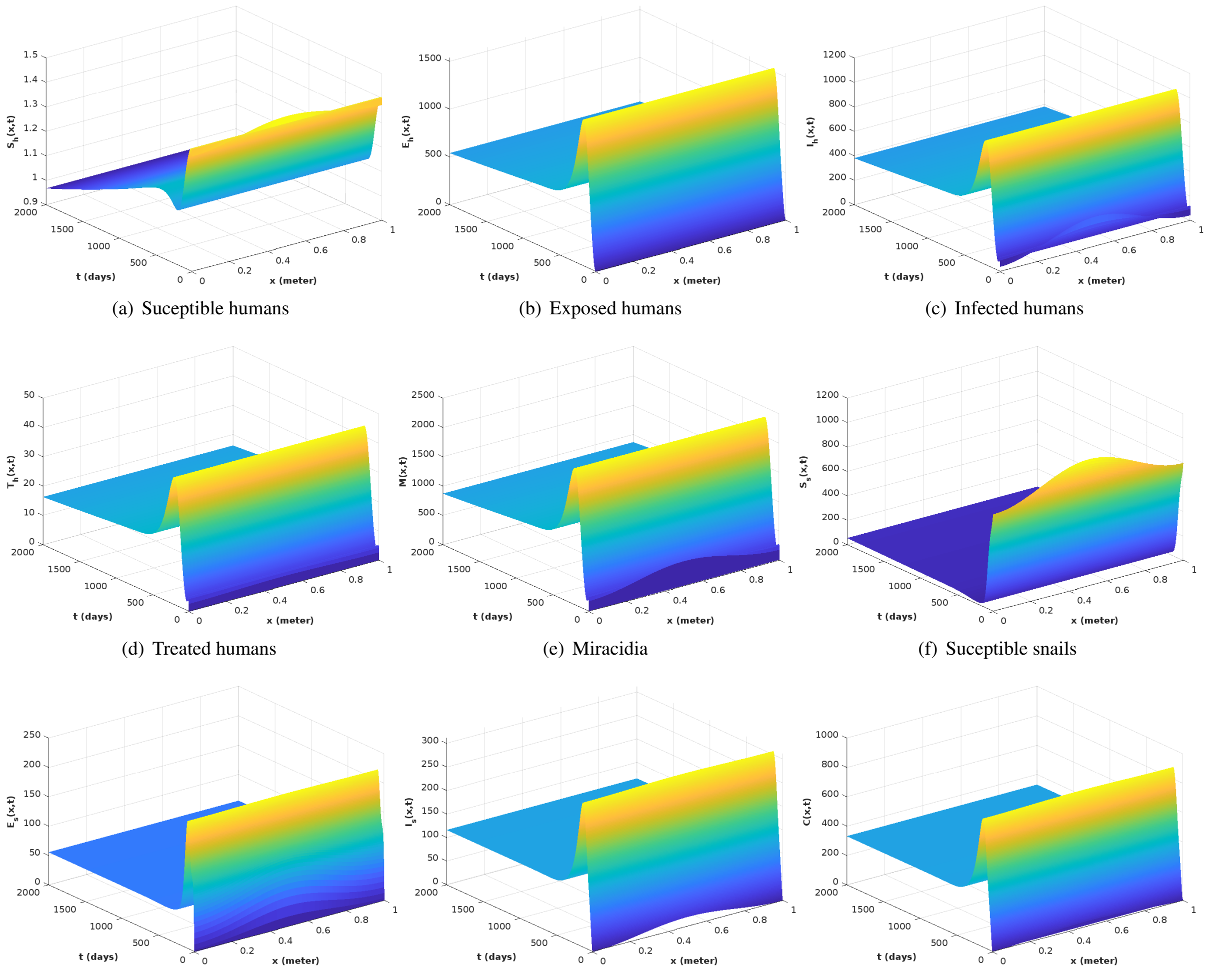


(a) Suceptible humans (b) Exposed humans (c) Infected humans

(d) Treated humans (e) Miracidia (f) Suceptible snails

(g) Exposed snails (h) Infected Snails (i) Cercariae

Figure 5: Spatiotemporal evolution of schistosomiasis transmission when $R_0 < 1$ and $\theta(t) = 1$. The disease-free equilibrium E^0 is globally asymptotically stable.

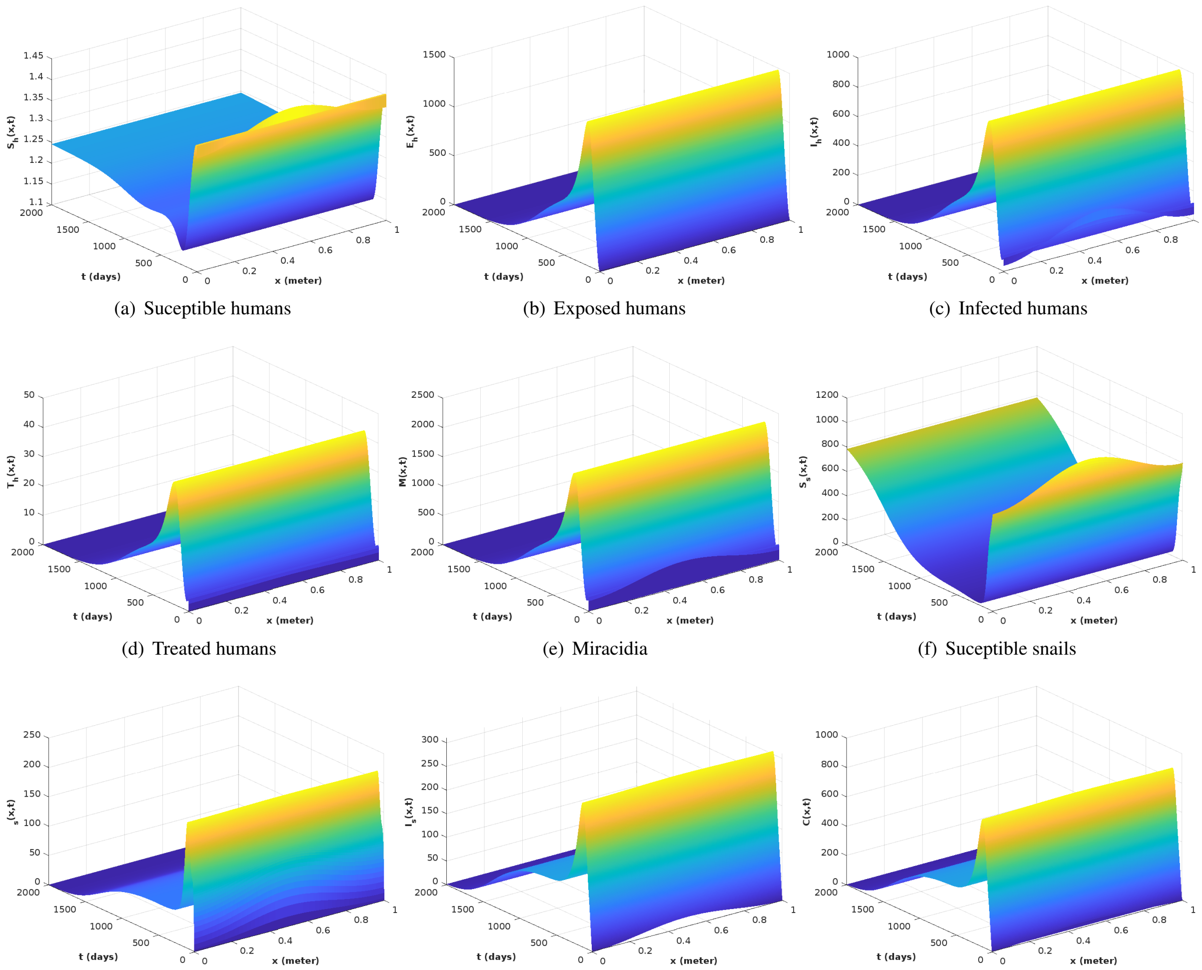


(a) Suceptible humans (b) Exposed humans (c) Infected humans

(d) Treated humans (e) Miracidia (f) Suceptible snails

(g) Exposed snails (h) Infected Snails (i) Cercariae

Figure 6: Spatiotemporal evolution of schistosomiasis transmission when $R_0 > 1$ and $\theta(t) = 1$. The endemic equilibrium E^* is globally asymptotically stable.



(a) Suceptible humans (b) Exposed humans (c) Infected humans

(d) Treated humans (e) Miracidia (f) Suceptible snails

(g) Exposed snails (h) Infected Snails (i) Cercariae

Figure 7: Spatiotemporal evolution of schistosomiasis transmission when $R_0 > 1$ and human intervention $\theta(t) = (1+\zeta e^{rt})^{-1}$. The arbitrary equilibrium of the non-autonomous system is uniformly stable

The simulations above illustrate that in a homogeneous system, while the early phase may exhibit variation depending on the spatial location x, the eventual state of the infectious disease appears to be independent of its dispersal rate.

Control Strategies: Here, an examination of the temporal evolution of the disease progression under different control measures, namely Mass Drug Administration (MDA) and human interventions is conducted. Figure 8 illustrates the progression of disease prevalence while varying treatments for infected humans. For all cases, the baseline parameter values listed in Table 1 are used and only manipulate the parameters σ_h (infected treatment rate) and β_{ch} (reinfection rate). The graphs in Figure 8 are generated using the ODE version of the equation (2.1), with each curve representing the proportion (in percentage) of infected individuals over the total population.

Figure 8 demonstrates the impact of human interventions on the spread of *Schistosoma*. The curves colored in red, yellow, green and blue represent infection prevalence with no treatment ($\sigma_h \approx 0$), low treatment ($\sigma_h = 0.03$), moderate treatment ($\sigma_h = 0.12$) and high treatment ($\sigma_h = 0.25$), respectively. Notably, the most severe outbreaks manifest during the early phase across the four scenarios mentioned. These findings suggest that MDA is an effective control strategy not only in the initial stages of transmission but also throughout the transmission process (see Fig 8 (a)-(c)). Applying appropriate treatment to infected individuals can substantially diminish disease prevalence. Nevertheless, as depicted in Fig 8 (a)-(c), relying solely on MDA becomes insufficient when the reinfection rate becomes high. Therefore, it becomes imperative to encourage individuals to adopt additional control measures such as avoiding contact with contaminated water through wading, swimming and other activities, along with implementing improved sanitation and securing access to clean water. Fig 8 (d) underscores that combining MDA with human interventions ($\zeta = 0.02$ and $r = 0.005$) can lead to a significant reduction in prevalence.

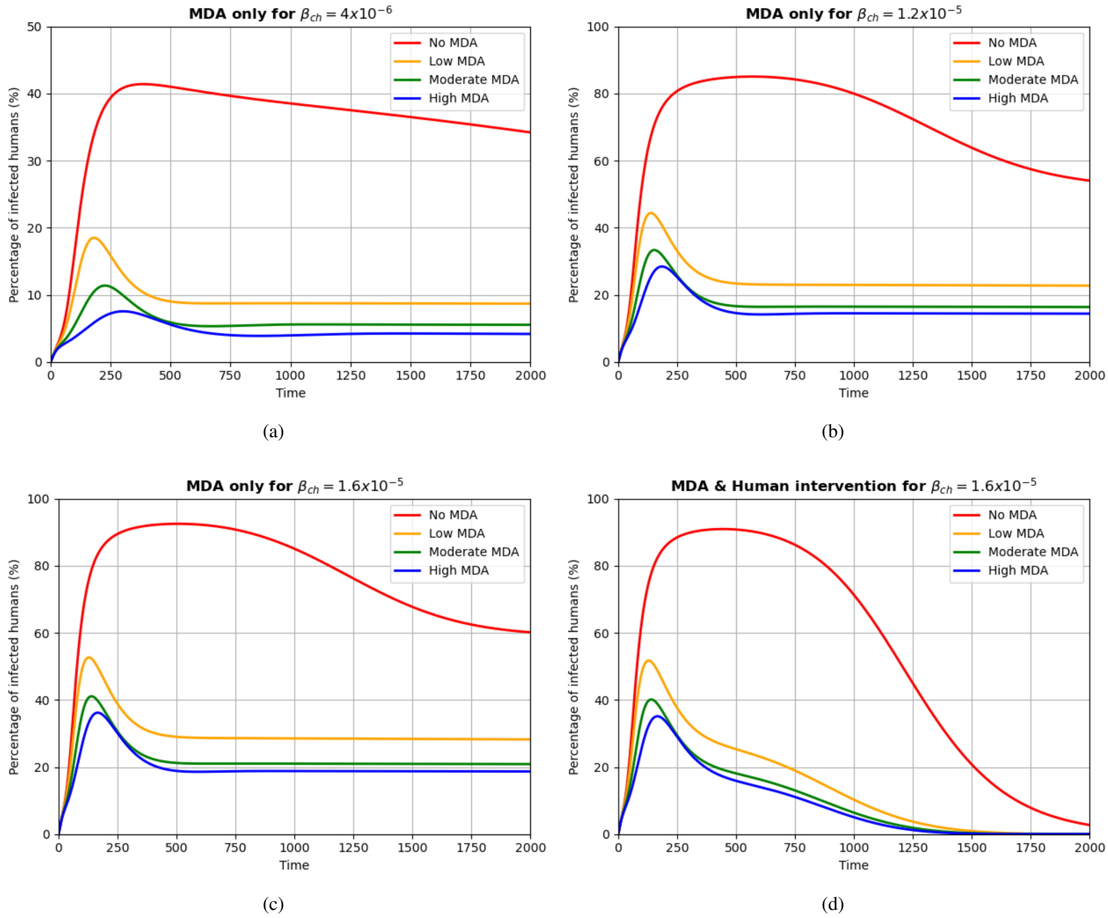


Figure 8: Influence of varying MDA rate on infection prevalence.

6. Conclusion

In this paper, an enhanced mathematical model has been developed utilizing diffusion equations to depict the dynamics of schistosomiasis, thereby expanding upon a previously established framework. By integrating both the influence of treated individuals and the temporal function of human interventions, the extended model has been thoroughly analyzed both temporally and spatially, delving into critical aspects such as the existence, uniqueness, and positivity of solutions, as well as the existence and stability of endemic and disease-free equilibria, contingent upon the threshold value of the basic reproduction number, R_0. It has been demonstrated that when $R_0 < 1$, the global asymptotic stability of the disease-free equilibrium has been conclusively established. Conversely, for $R_0 > 1$, the endemic equilibrium has been firmly established as locally stable within the autonomous system. Furthermore, the results have been extended to non-autonomous systems, showcasing the uniform stability of any arbitrary equilibrium, irrespective of the value of R_0. Additionally, a comprehensive sensitivity analysis of R_0 has been conducted, employing PRCC and the local sensitivity index to unravel the intricate dynamics influenced by individual parameters. It has been determined that a 1% increase in the treatment rate σ_h would result in a 0.442% reduction in R_0. The theoretical findings have been rigorously validated through numerical simulations, which corroborate the conclusions drawn from the qualitative analysis, notably emphasizing the profound impact of various control measures. These findings underscore the efficacy of Mass Drug Administration (MDA) as a control strategy not only during the initial stages of transmission but also throughout the transmission process. However, it has been elucidated through numerical simulations that relying solely on MDA becomes inadequate when the reinfection rate escalates. Consequently, it becomes imperative to advocate for the adoption of additional control measures by individuals, such as avoiding contact with contaminated water through activities like wading and swimming, in addition to implementing improved sanitation and securing access to clean water. Furthermore, the combined implementation of mass drug administration (MDA) and targeted human interventions has been identified as a potent approach, substantially diminishing the prevalence of infection and aligning with the targets set by the World Health Organization. This holistic strategy not only addresses the immediate challenges posed by schistosomiasis but also lays the groundwork for sustainable long-term management of the disease. In our future research, we plan to explore the optimization of intervention strategies by considering socioeconomic factors, geographical variations, and the evolution of drug resistance.

Furthermore, incorporating predictive modeling techniques could facilitate the development of proactive intervention strategies, thereby enhancing the overall effectiveness of schistosomiasis control efforts.

Declarations

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