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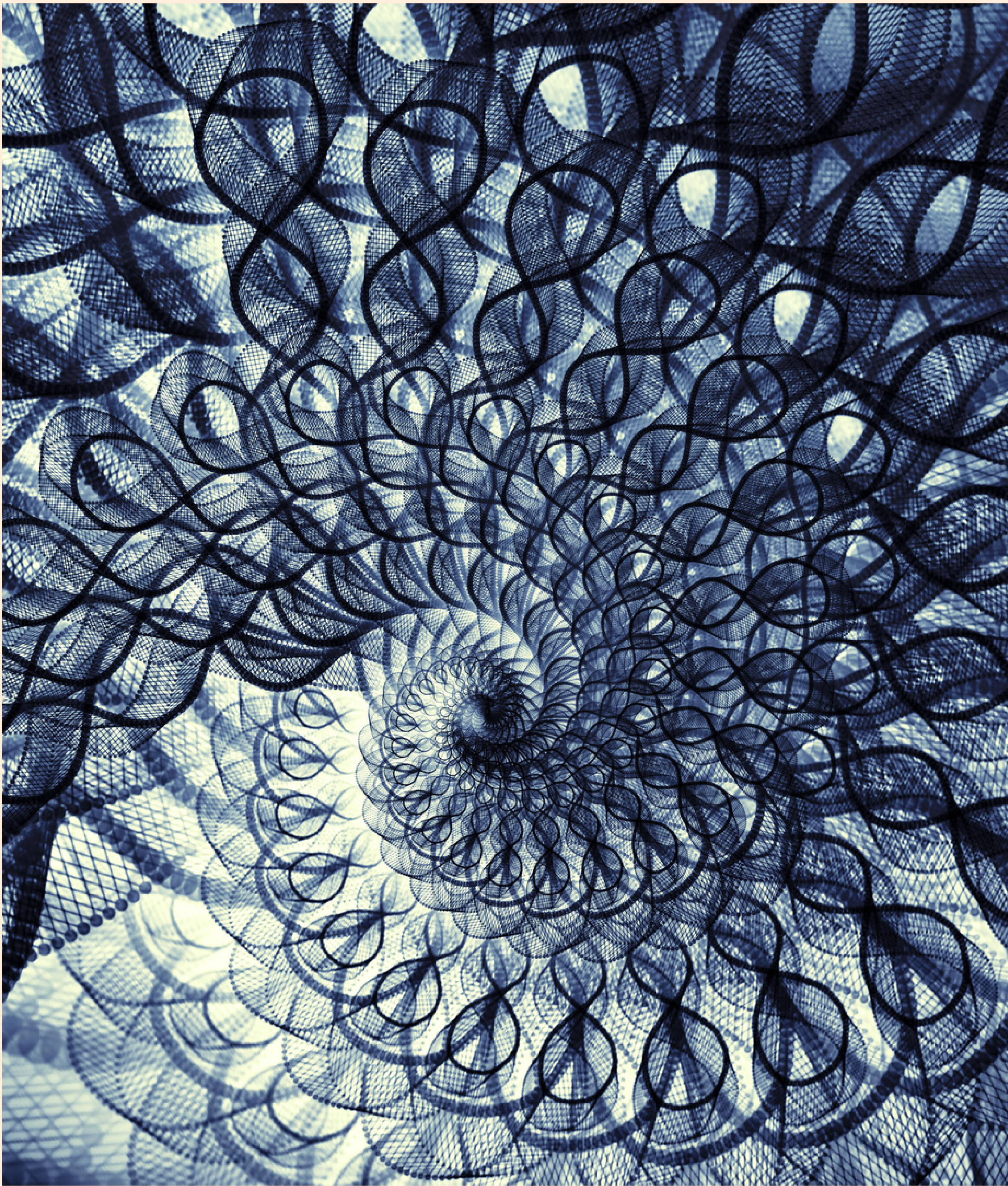

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Contents

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On New Pell Spinor Sequences

Tülay Erişir, Gökhan Mumcu* and Mehmet Ali Güngör

Abstract
Our motivation for this study is to define two new and particular sequences. The most essential feature of these sequences is that they are spinor sequences. In this study, these new spinor sequences obtained using spinor representations of Pell and Pell-Lucas quaternions are expressed. Moreover, some formulas such that Binet formulas, Cassini formulas and generating functions of these spinor sequences, which are called as Pell and Pell-Lucas spinor sequences, are given. Then, some relationships between Pell and Pell-Lucas spinor sequences are obtained. Therefore, an easier and more interesting representations of Pell and Pell-Lucas quaternions, which are a generalization of Pell and Pell-Lucas number sequences, are obtained. We believe that these new spinor sequences will be useful and advantageable in many branches of science, such as geometry, algebra and physics.

Keywords: *Pell, Pell-Lucas, Spinor*

AMS Subject Classification (2020): *53C56, 53Z05*

Corresponding author

1. Introduction and Preliminaries

The number sequences are a subject that is frequently used in mathematics and attracts the attention of readers. The first number sequences that come to mind are the Fibonacci number sequences expressed by Fibonacci (1170-1250), which are frequently encountered in nature [1–3]. The Lucas number sequence, which is obtained by writing the next term as the sum of the previous two terms but with different initial conditions, is another example of a number sequence. In addition, there are many number sequences in the literature, such as the Fibonacci number sequence, whose characteristic equation is different. Moreover, considering different characteristic equations and initial values, different number sequences can be obtained, such as Pell, Pell-Lucas, Modified Pell, Jacobsthal and Jacobsthal-Lucas number sequences etc. [4–6]. Moreover, another studies of this subject are [7, 8, 10, 11, 27]. Horadam discussed Pell numbers and their properties [5]. Patel and Shrivastava obtained some of these with their proofs using Binet forms of some Pell and Pell-Lucas identities [12]. These properties are used to derive generator functions, polynomials, divisibility properties, matrices, determinants of Pell and Pell-Lucas sequences, and many other applications. Koshy mentioned that Pell numbers and Pell-Lucas numbers are special values of Pell and Pell-Lucas polynomials, respectively [13]. Halıcı and Daşdemir studied some relationships between Pell, Pell-Lucas,

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and Modified Pell sequences [14]. Szynal and Wloch studied Pell, Pell-Lucas numbers, quaternions, octonions and recurrence relations [15]. Catarino discussed k-Pell quaternions and octanions and offered some features, including the Binet formula and a generating function [16]. Moreover, Çimen and İpek gave a new quaternion sequence such that Pell and Pell-Lucas quaternion sequence [17].

Spinors can be defined in a simple way as vectors of a space whose transformations are related to spins in physical space. The person who first introduced spinors in a geometric sense was Cartan [18]. Cartan's study [18] is an admirable study in spinor geometry because in this study, spinor representations of the some basic geometric definitions are expressed by Cartan in an easy and understandable way. Another inspiring study on the spinors in geometry was done by Vivarelli [19]. In Vivarelli's study [19], the relationships between quaternions and spinors and spinor representations of 3D rotations were obtained. In the study of Torres del Castillo and Barrales, the spinor representations of the Frenet frame and curvatures of any curve in Euclidean 3-space were given [20]. The spinor representation of the Darboux frame in Euclidean 3-space was obtained [21]. Moreover, in [22], the spinor representation of the Bishop frame in Euclidean 3-space was expressed. On the other hand, the spinor equations for some special curves such as Bertrand, involute-evolute, successor, and Mannheim curves and for Lie groups were obtained [23–27]. Then, for any Minkowski space, hyperbolic spinor equations were given [28–31]. In addition to that, Fibonacci and Lucas spinors were expressed in [32].

Now, the spinors, real quaternions, relationships between them spinors, and Pell, Pell-Lucas quaternions are given.

Assume that any isotropic vector is $\boldsymbol{v} = (v_1,\ v_2,\ v_3) \in \mathbb{C}^3$ where ${v_1}^2 + {v_2}^2 + {v_3}^2 = 0$ and the complex vector space with 3-dimensional is $\mathbb{C}^3$. We can express the set of isotropic vectors in $\mathbb{C}^3$ with the aid of a two-dimensional surface in $\mathbb{C}^2$. Suppose that this two-dimensional surface has coordinates ϖ_1 and ϖ_2. So, we can write $v_1 = {\varpi_1}^2 - {\varpi_2}^2$, $v_2 = \mathbf{i}({\varpi_1}^2 + {\varpi_2}^2)$, $v_3 = -2\varpi_1\varpi_2$ and $\varpi_1 = \pm\sqrt{\frac{v_1 - \mathbf{i}v_2}{2}}$, $\varpi_2 = \pm\sqrt{\frac{-v_1 - \mathbf{i}v_2}{2}}$. Two-dimensional complex vector mentioned above is called as ***spinor*** by Cartan such that

$$\varpi = (\varpi_1,\ \varpi_2) = \begin{bmatrix} \varpi_1 \\ \varpi_2 \end{bmatrix}$$

in spinor space $\mathbb{S}$ [18].

Suppose that any real quaternion is $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ where $q_0, q_1, q_2, q_3 \in \mathbb{R}$. $\{\boldsymbol{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is called the quaternion basis such that

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \quad \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}$$

[33]. We can write $q = S_q + \mathbf{V}_q$ where $q_0 = S_q$ and $\mathbf{V}_q = \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ is called scalar and vector parts of q, respectively [33]. Assume that two any real quaternions $p = S_p + \mathbf{V}_p,\ q = S_q + \mathbf{V}_q$. So, the quaternion product of these quaternions is

$$p \times q = S_pS_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p\mathbf{V}_q + S_q\mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q,$$

where $\langle , \rangle$ is inner product and $\wedge$ is vector product in $\mathbb{R}^3$ [33]. We know that the product of two real quaternions is non-commutative. In addition to that, the quaternion conjugate and the norm of q are given as $q^* = S_q - \boldsymbol{V}_q$ and $N(q) = \sqrt{{q_1}^2 + {q_2}^2 + {q_3}^2 + {q_4}^2}$. Let the norm of q be $N(q) = 1$, then q is defined as unit quaternion [33].

Vivarelli expressed a relationship between spinors and quaternions such that

$$\begin{aligned} f &: \mathbb{H} \to \mathbb{S} \\ q &\to f(q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3) \cong \begin{bmatrix} q_3 + \mathbf{i}q_0 \\ q_1 + \mathbf{i}q_2 \end{bmatrix} \equiv \varpi \end{aligned} \tag{1.1}$$

where $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ is any real quaternion [19]. Then, Vivarelli gave a spinor representation of $q \times p$ such that

$$q \times p \to -\mathbf{i}\hat{\varpi}\rho. \tag{1.2}$$

where the spinor ρ corresponds to the real quaternion p with the aid of the transformation f in the equation (1.1) and the complex, unitary, square matrix $\hat{\varpi}$ can be written as

$$\hat{\varpi} = \begin{bmatrix} q_3 + \mathbf{i}q_0 & q_1 - \mathbf{i}q_2 \\ q_1 + \mathbf{i}q_2 & -q_3 + \mathbf{i}q_0 \end{bmatrix} \tag{1.3}$$

[19]. In addition, the spinor matrix $\varpi_L = -\mathbf{i}\widehat{\varpi}$, namely

$$\varpi_L = \begin{bmatrix} q_0 - \mathbf{i}q_3 & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix} \tag{1.4}$$

was called the left Hamilton spinor matrix or fundamental spinor matrix of q [19, 34].

Now, the some equalities about Pell and Pell-Lucas quaternions given in [17] can be expressed. But before that we would like to touch upon an important issue here. There are many studies in the literature about Pell and Pell Lucas number sequences and Pell and Pell-Lucas quaternion sequences. In these studies, while the initial conditions of Pell number sequences are taken as 0 and 1, there is an information confusion regarding the initial conditions of Pell-Lucas number sequences. That is, in some studies, the initial conditions of Pell-Lucas number sequences are taken as 1, 1, while in some studies, the initial conditions are taken as 2, 2. Additionally, in some studies, the expression "Modified Pell number sequence" was used in studies with initial conditions of 1, 1. Actually, there is no problem so far. The real problem is that if the initial conditions are taken differently, some formulas such as Binet, Cassini and sum formulas turn out to be different. Also, the relationships between Pell and Pell-Lucas are different. For example, if you take the initial condition of Pell-Lucas number sequence as 1, 1, you shouldn't use formulas in another study where the initial condition is 2, 2. Otherwise, an information confusion is created in the literature. In this study, the initial conditions of Pell-Lucas number sequence are taken as $Q_0 = 2$, $Q_1 = 2$ and the formulas are used accordingly. Now, we expressed Pell and Pell-Lucas quaternions.

For $n \geq 2$ the nth Pell quaternion and Pell-Lucas quaternion is defined that

$$QP_n = P_n + \boldsymbol{i}P_{n+1} + \boldsymbol{j}P_{n+2} + \boldsymbol{k}P_{n+3}$$

and

$$QPL_n = Q_n + \boldsymbol{i}Q_{n+1} + \boldsymbol{j}Q_{n+2} + \boldsymbol{k}Q_{n+3}$$

where the nth Pell number and Pell-Lucas number $P_n = 2P_{n-1} + P_{n-2}$ and $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ and $Q_0 = 2$, $Q_1 = 2$ [17]. Moroever, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ coincide with basis vectors given for real quaternions. Therefore, the recurrence relation of Pell and Pell-Lucas quaternions for $n \geq 2$ are

$$QP_n = 2QP_{n-1} + QP_{n-2}$$

with initial conditions $QP_0 = \boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$, $QP_1 = 1 + 2\boldsymbol{i} + 5\boldsymbol{j} + 12\boldsymbol{k}$ and

$$QPL_n = 2QPL_{n-1} + QPL_{n-2}$$

with initial conditions $QPL_0 = 2 + 2\boldsymbol{i} + 6\boldsymbol{j} + 14\boldsymbol{k}$, $QPL_1 = 2 + 6\boldsymbol{i} + 14\boldsymbol{j} + 34\boldsymbol{k}$ [17].

Now, we write the some relationship between Pell and Pell-Lucas quaternions with the aid of [5, 12, 14–16, 35, 36]. Therefore, we can write these relationships that

$$\begin{aligned} &i)\ QP_{n-1} + QP_{n+1} = QPL_n, \\ &ii)\ QPL_n + QPL_{n+1} = 4QP_{n+1}, \\ &iii)\ QPL_{n+1} + QPL_{n-1} = 8QP_n. \end{aligned}$$

Moreover, the Binet formula for Pell and Pell-Lucas quaternions are given that

$$QP_n = \frac{\gamma^n \underline{\gamma} - \mu^n \underline{\mu}}{\gamma - \mu}$$

and

$$QPL_n = \gamma^n \underline{\gamma} + \mu^n \underline{\mu}$$

where the quaternions $\underline{\gamma}$ and $\underline{\mu}$ are $\underline{\gamma} = 1 + \boldsymbol{i}\gamma + \boldsymbol{j}\gamma^2 + \boldsymbol{k}\gamma^3$ and $\underline{\mu} = 1 + \boldsymbol{i}\mu + \boldsymbol{j}\mu^2 + \boldsymbol{k}\mu^3$, $\gamma = 1 + \sqrt{2}$, $\mu = 1 - \sqrt{2}$ are roots of the characteristic equation $x^2 - 2x - 1 = 0$.

On the other hand, we give the generating functions of Pell and Pell-Lucas quaternions such that

$$G_P(t) = \frac{QP_0 + (QP_1 - 2QP_0)\,t}{1 - 2t - t^2}$$

and

$$G_{PL}(t)=\frac{QPL_0+(QPL_1-2QPL_0)\,t}{1-2t-t^2},$$

respectively. In addition to that, Cassini formula for Pell and Pell-Lucas quaternions can be given that

$$QP_{n-1}QP_{n+1}-(QP_n)^2=(-1)^n\left(\frac{\gamma\underline{\mu}\,\underline{\gamma}-\mu\underline{\gamma}\,\underline{\mu}}{\gamma-\mu}\right)$$

and

$$QPL_{n-1}QPL_{n+1}-(QPL_n)^2=(-1)^{n-1}(\gamma-\mu)(\gamma\underline{\mu}\,\underline{\gamma}-\mu\underline{\gamma}\,\underline{\mu}),$$

respectively.

2. Main Theorems and Results

We know that there is a spinor for every real quaternion by means of the transformation f in the equation (1.1). Considering this information, a new transformation between Pell and Pell-Lucas quaternions and spinors can be defined and the spinors corresponding to Pell and Pell-Lucas quaternions can be given. Therefore, these spinors associated with Pell and Pell-Lucas quaternions are called as Pell and Pell-Lucas spinors. Then, some formulas such that Binet, Cassini, sum formulas and generating functions for these quaternions spinors and theorems are given.

Definition 2.1. Let $QP_n=P_n+\boldsymbol{i}P_{n+1}+\boldsymbol{j}P_{n+2}+\boldsymbol{k}P_{n+3}$ be nth Pell quaternion where P_n is nth Pell number and the set of Pell quaternions be $\mathbb{Q}_P$. Therefore, the following linear transformation is defined as

$$\begin{aligned} f_P:\ \ &\mathbb{Q}_P\rightarrow\mathbb{S}\\ &QP_n\mapsto f_P(QP_n)\cong SP_n=\begin{bmatrix}P_{n+3}+\mathbf{i}P_n\\ P_{n+1}+\mathbf{i}P_{n+2}\end{bmatrix}\end{aligned}\tag{2.1}$$

where $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ coincide with basis vectors in $\mathbb{R}^3$ and $\mathbf{i}^2=-1$. So, a new sequence for the spinors related with Pell quaternions is defined and this sequence is called as *"Pell Spinor Sequence"* defined as

$$\{SP_n\}_{n\in\mathbb{N}}^{\infty}=\left\{\begin{bmatrix}5\\1+2\mathbf{i}\end{bmatrix},\begin{bmatrix}12+\mathbf{i}\\2+5\mathbf{i}\end{bmatrix},\begin{bmatrix}29+2\mathbf{i}\\5+12\mathbf{i}\end{bmatrix},\begin{bmatrix}70+5\mathbf{i}\\12+29\mathbf{i}\end{bmatrix},\ldots\right\}$$

where $SP_n=\begin{bmatrix}P_{n+3}+\mathbf{i}P_n\\ P_{n+1}+\mathbf{i}P_{n+2}\end{bmatrix}$ is nth Pell spinor and P_n is nth Pell number.

Similarly, we can give the following definition of Pell-Lucas spinor sequence.

Definition 2.2. Let $QPL_n=Q_n+\boldsymbol{i}Q_{n+1}+\boldsymbol{j}Q_{n+2}+\boldsymbol{k}Q_{n+3}$ be nth Pell-Lucas quaternion where Q_n is nth Pell-Lucas number and the set of Pell-Lucas quaternions be $\mathbb{Q}_{PL}$. Therefore, the following linear transformation is defined as

$$\begin{aligned} f_{PL}:\ \ &\mathbb{Q}_{PL}\rightarrow\mathbb{S}\\ &QPL_n\mapsto f_{PL}(QPL_n)\cong SPL_n=\begin{bmatrix}Q_{n+3}+\mathbf{i}Q_n\\ Q_{n+1}+\mathbf{i}Q_{n+2}\end{bmatrix}.\end{aligned}$$

Therefore, a new sequence for the spinors related with Pell-Lucas quaternions is called as ***"Pell-Lucas Spinor Sequence"*** where

$$\{SPL_n\}_{n\in\mathbb{N}}^{\infty}=\left\{\begin{bmatrix}14+2\mathbf{i}\\2+6\mathbf{i}\end{bmatrix},\begin{bmatrix}34+2\mathbf{i}\\6+14\mathbf{i}\end{bmatrix},\begin{bmatrix}82+6\mathbf{i}\\14+34\mathbf{i}\end{bmatrix},\begin{bmatrix}198+14\mathbf{i}\\34+82\mathbf{i}\end{bmatrix},\ldots\right\}$$

where $SPL_n=\begin{bmatrix}Q_{n+3}+\mathbf{i}Q_n\\ Q_{n+1}+\mathbf{i}Q_{n+2}\end{bmatrix}$ is nth Pell-Lucas spinor and Q_n is nth Pell-Lucas number.

Definition 2.3. The conjugate of Pell quaternion QP_n is QP_n^*, and Pell spinor corresponding to this conjugate is defined as

$$SP_n^*=\begin{bmatrix}-P_{n+3}+\mathbf{i}P_n\\ -P_{n+1}-\mathbf{i}P_{n+2}\end{bmatrix}.$$

Similarly, Pell Lucas spinor corresponding to the conjugate of Pell-Lucas quaternion QPL_n is defined as

$$SPL_n^* = \begin{bmatrix} -Q_{n+3} + \mathbf{i}Q_n \\ -Q_{n+1} - \mathbf{i}Q_{n+2} \end{bmatrix}.$$

Definition 2.4. Pell spinor representation of the norm of Pell quaternion QP_n is

$$\overline{SP_n}^t SP_n.$$

Similarly, Pell-Lucas spinor representation of the norm of Pell-Lucas quaternion QPL_n is

$$\overline{SPL_n}^t SPL_n.$$

Now, the recurrence relations of Pell and Pell-Lucas spinor sequences with the following equations can be obtained.

Theorem 2.1. *The recurrence relation of Pell spinors for $n \geqslant 2$ is*

$$SP_n = 2SP_{n-1} + SP_{n-2}$$

where nth, $(n-1)th$ and $(n+1)th$ Pell spinors are SP_n, SP_{n-1} and SP_{n-2}, respectively. The recurrence relation for Pell-Lucas spinor for $n \geqslant 2$ is

$$SPL_n = 2SPL_{n-1} + SPL_{n-2}$$

where nth, $(n-1)th$ and $(n+1)th$ Pell-Lucas spinors are SPL_n, SPL_{n-1} and SPL_{n-2}, respectively.

Proof. Firstly, we show the recurrence relation for Pell spinors. Therefore, if we calculate $2SP_{n-1} + SP_{n-2}$, then we obtain

$$2SP_{n-1} + SP_{n-2} = 2\begin{bmatrix} P_{n+2} + \mathbf{i}P_{n-1} \\ P_n + \mathbf{i}P_{n+1} \end{bmatrix} + \begin{bmatrix} P_{n+1} + \mathbf{i}P_{n-2} \\ P_{n-1} + \mathbf{i}P_n \end{bmatrix}$$

$$= \begin{bmatrix} 2P_{n+2} + P_{n+1} + \mathbf{i}(2P_{n-1} + P_{n-2}) \\ 2P_n + P_{n-1} + \mathbf{i}(2P_{n+1} + P_n) \end{bmatrix}.$$

Since the recurrence relation for Pell number sequence is $P_n = 2P_{n-1} + P_{n-2}$, we have

$$2SP_{n-1} + SP_{n-2} = \begin{bmatrix} P_{n+3} + \mathbf{i}P_n \\ P_{n+1} + \mathbf{i}P_{n+2} \end{bmatrix} = SPn.$$

Similarly, we can easily obtain for Pell-Lucas spinor sequence such that

$$2SPL_{n-1} + SPL_{n-2} = 2\begin{bmatrix} Q_{n+2} + \mathbf{i}Q_{n-1} \\ Q_n + \mathbf{i}Q_{n+1} \end{bmatrix} + \begin{bmatrix} Q_{n+1} + \mathbf{i}Q_{n-2} \\ Q_{n-1} + \mathbf{i}Q_n \end{bmatrix}$$

$$= \begin{bmatrix} 2Q_{n+2} + Q_{n+1} + \mathbf{i}(2Q_{n-1} + Q_{n-2}) \\ 2Q_n + Q_{n-1} + \mathbf{i}(2Q_{n+1} + Q_n) \end{bmatrix}$$

$$= \begin{bmatrix} Q_{n+3} + \mathbf{i}Q_n \\ Q_{n+1} + \mathbf{i}Q_{n+2} \end{bmatrix} = SPLn$$

where the recurrence relation $Q_n = 2Q_{n-1} + Q_{n-2}$ of Pell Lucas number sequence is used $(n \geqslant 2)$. □

Now, the some relations between Pell and Pell-Lucas spinors can be given.

Theorem 2.2. *Let nth Pell and Pell-Lucas spinors be SP_n and SPL_n, respectively. In this case, for $n \geqslant 2$ there are the following relations between these spinors;*

$$\begin{aligned} i)\ & SP_{n-1} + SP_{n+1} = SPL_n, \\ ii)\ & SPL_n + SPL_{n+1} = 4SP_{n+1}, \\ iii)\ & SPL_{n+1} + SPL_{n-1} = 8SP_n, \\ iv)\ & 2SP_n + 2SP_{n-1} = SPL_n. \end{aligned}$$

Proof. i) Let $(n-1)th$ and $(n+1)th$ Pell spinors be SP_{n-1} and SP_{n+1}, respectively. Then, we can write the equation

$$SP_{n-1} + SP_{n+1} = \begin{bmatrix} P_{n+2} + \mathbf{i}P_{n-1} \\ P_n + \mathbf{i}P_{n+1} \end{bmatrix} + \begin{bmatrix} P_{n+4} + \mathbf{i}P_{n+1} \\ P_{n+2} + \mathbf{i}P_{n+3} \end{bmatrix}$$

$$= \begin{bmatrix} P_{n+2} + P_{n+4} + \mathbf{i}(P_{n-1} + P_{n+1}) \\ P_n + P_{n+2} + \mathbf{i}(P_{n+1} + P_{n+3}) \end{bmatrix}.$$

On the other hand, we know that the relationship between Pell and Pell-Lucas numbers is $Q_n = P_{n-1} + P_{n+1}$ from [35]. If we use this relationship we can write

$$SP_{n-1} + SP_{n+1} = \begin{bmatrix} Q_{n+3} + \mathbf{i}Q_n \\ Q_{n+1} + \mathbf{i}Q_{n+2} \end{bmatrix} = SPL_n.$$

This completes the proof.

ii) Assume that nth and $(n+1)th$ Pell-Lucas spinors are SPL_n and SPL_{n+1}. Therefore, we have

$$SPL_n + SPL_{n+1} = \begin{bmatrix} Q_{n+3} + \mathbf{i}Q_n \\ Q_{n+1} + \mathbf{i}Q_{n+2} \end{bmatrix} + \begin{bmatrix} Q_{n+4} + \mathbf{i}Q_{n+1} \\ Q_{n+2} + \mathbf{i}Q_{n+3} \end{bmatrix}$$

$$= \begin{bmatrix} Q_{n+3} + Q_{n+4} + \mathbf{i}(Q_n + Q_{n+1}) \\ Q_{n+1} + Q_{n+2} + \mathbf{i}(Q_{n+2} + Q_{n+3}) \end{bmatrix}.$$

In addition to that, we know that there is the relationship $4P_{n+1} = Q_n + Q_{n+1}$ between Pell and Pell-Lucas numbers from [37]. So, we get

$$SPL_n + SPL_{n+1} = \begin{bmatrix} 4P_{n+4} + \mathbf{i}4P_{n+1} \\ 4P_{n+2} + \mathbf{i}4P_{n+3} \end{bmatrix} = 4SP_{n+1}$$

iii) Suppose that $(n-1)th$ and $(n+1)th$ Pell-Lucas spinors are SPL_{n-1} and SPL_{n+1}, respectively. Then, we get

$$SPL_{n+1} + SPL_{n-1} = \begin{bmatrix} Q_{n+4} + \mathbf{i}Q_{n+1} \\ Q_{n+2} + \mathbf{i}Q_{n+3} \end{bmatrix} + \begin{bmatrix} Q_{n+2} + \mathbf{i}Q_{n-1} \\ Q_n + \mathbf{i}Q_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} Q_{n+4} + Q_{n+2} + \mathbf{i}(Q_{n+1} + Q_{n-1}) \\ Q_{n+2} + Q_n + \mathbf{i}(Q_{n+3} + Q_{n+1}) \end{bmatrix} = \begin{bmatrix} 8P_{n+3} + \mathbf{i}8P_n \\ 8P_{n+1} + \mathbf{i}8P_{n+2} \end{bmatrix} = 8P_n$$

where $8P_n = Q_{n+1} + Q_{n-1}$.

iv) This proof is clear that $SPL_n = SP_{n-1} + SP_{n+1}$ from option i). Moreover, we know that $SP_{n+1} = 2SP_n + SP_{n-1}$. Consequently,

$$SPL_n = SP_{n-1} + 2SP_n + SP_{n-1} = 2SP_n + 2SP_{n-1}.$$

This completes the proof. □

Theorem 2.3. *Assume that nth Pell and Pell-Lucas spinors are SP_n and SPL_n, respectively. Therefore, the Binet Formulas for these spinors are the following equations. The Binet formula for Pell spinors is*

$$SP_n = \frac{1}{\gamma - \mu}\left(\gamma^n S_\gamma - \mu^n S_\mu \right),$$

the Binet formula for Pell-Lucas spinors is

$$SPL_n = \gamma^n S_\gamma + \mu^n S_\mu$$

where $\gamma = 1 + \sqrt{2}$, $\mu = 1 - \sqrt{2}$ are the roots of characteristic equation $x^2 - 2x - 1 = 0$ and $S_\gamma = \begin{bmatrix} \gamma^3 + \boldsymbol{i} \\ \gamma + \boldsymbol{i}\gamma^2 \end{bmatrix}$ and $S_\mu = \begin{bmatrix} \mu^3 + \boldsymbol{i} \\ \mu + \boldsymbol{i}\mu^2 \end{bmatrix}$.

Proof. First, we prove it for Pell spinors. We know that the Binet formula for Pell number sequence is

$$P_n = \frac{\gamma^n - \mu^n}{\gamma - \mu}$$

where $\gamma = 1+\sqrt{2}$, $\mu = 1-\sqrt{2}$. Therefore, if we write the last equation in the nth Pell spinor we obtain

$$SP_n = \begin{bmatrix} P_{n+3} + \mathbf{i}P_n \\ P_{n+1} + \mathbf{i}P_{n+2} \end{bmatrix} = \frac{1}{\gamma-\mu}\begin{bmatrix} \gamma^{n+3} - \mu^{n+3} + \mathbf{i}(\gamma^n - \mu^n) \\ \gamma^{n+1} - \mu^{n+1} + \mathbf{i}(\gamma^{n+2} - \mu^{n+2}) \end{bmatrix}$$

$$SP_n = \frac{1}{\gamma-\mu}\left(\begin{bmatrix} \gamma^{n+3} + \mathbf{i}\gamma^n \\ \gamma^{n+1} + \mathbf{i}\gamma^{n+2} \end{bmatrix} - \begin{bmatrix} \mu^{n+3} + \mathbf{i}\mu^n \\ \mu^{n+1} + \mathbf{i}\mu^{n+2} \end{bmatrix} \right)$$

$$SP_n = \frac{1}{\gamma-\mu}\left(\gamma^n \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix} - \mu^n \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix} \right)$$

or

$$SP_n = \frac{1}{\gamma-\mu}\left(\gamma^n S_\gamma - \mu^n S_\mu \right)$$

where $S_\gamma = \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix}$ and $S_\mu = \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix}$.

Now, we give the Binet formula for Pell-Lucas spinors. We know that the Binet formula for Pell-Lucas number sequence is $Q_n = \gamma^n + \mu^n$. In this case, we can obtain

$$SPL_n = \begin{bmatrix} Q_{n+3} + \mathbf{i}Q_n \\ Q_{n+1} + \mathbf{i}Q_{n+2} \end{bmatrix} = \begin{bmatrix} \gamma^{n+3} + \mu^{n+3} + \mathbf{i}(\gamma^n + \mu^n) \\ \gamma^{n+1} + \mu^{n+1} + \mathbf{i}(\gamma^{n+2} + \mu^{n+2}) \end{bmatrix}$$

$$SPL_n = \begin{bmatrix} \gamma^{n+3} + \mathbf{i}\gamma^n \\ \gamma^{n+1} + \mathbf{i}\gamma^{n+2} \end{bmatrix} + \begin{bmatrix} \mu^{n+3} + \mathbf{i}\mu^n \\ \mu^{n+1} + \mathbf{i}\mu^{n+2} \end{bmatrix}$$

$$SPL_n = \gamma^n \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix} + \mu^n \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix}$$

or

$$SPL_n = \gamma^n S_\gamma + \mu^n S_\mu$$

where $S_\gamma = \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix}$ and $S_\mu = \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix}$. □

Theorem 2.4. *Let nth Pell and Pell-Lucas spinors be SP_n and SPL_n, respectively. The sum formulas for Pell spinors are the following options;*

$$\text{i)} \quad \sum_{t=0}^{n} SP_t = \frac{1}{4}\Big[SPL_{n+1} - SPL_0\Big],$$

$$\text{ii)} \quad \sum_{t=0}^{n} SP_{2t} = \frac{1}{2}\Big[SP_{2n+1} - SP_{-1}\Big],$$

$$\text{iii)} \quad \sum_{t=0}^{n} SP_{2t-1} = \frac{1}{2}\Big[SP_{2n} - SP_{-2}\Big].$$

Proof. i) We know that for Pell spinors the Binet formula is $SP_n = \frac{1}{\gamma-\mu}(\gamma^n S_\gamma - \mu^n S_\mu)$. Therefore, we can write

$$\begin{aligned} \sum_{t=0}^{n} SP_t &= \sum_{t=0}^{n} \frac{1}{\gamma-\mu}(\gamma^t S_\gamma - \mu^t S_\mu) \\ &= \frac{1}{\gamma-\mu}\Big(\sum_{t=0}^{n} \gamma^t S_\gamma - \sum_{t=0}^{n} \mu^t S_\mu\Big). \end{aligned} \tag{2.2}$$

On the other hand, we know that $\sum_{t=0}^{n}\gamma^{t}=\frac{1-\gamma^{n+1}}{1-\gamma}$ and $\sum_{t=0}^{n}\mu^{t}=\frac{1-\mu^{n+1}}{1-\mu}$. If we use these information in the last equation then, we get

$$\sum_{t=0}^{n}SP_t=\frac{1}{4}\Bigg((\gamma^{n+1}S_\gamma+\mu^{n+1}S_\mu)-(S_\gamma+S_\mu)\Bigg)$$

where $\gamma-\mu=2\sqrt{2}$. Moreover, for Pell-Lucas spinors the Binet formula is $SPL_n=\gamma^nS_\gamma+\mu^nS_\mu$. So, we can obtain that

$$\sum_{t=0}^{n}SP_t=\frac{1}{4}(SPL_{n+1}-SPL_0)$$

and this completes the proof.

ii) Similarly, if we use the Binet formula for Pell spinors then we easily get

$$\begin{aligned}\sum_{t=0}^{n}SP_{2t}&=\sum_{t=0}^{n}\frac{1}{\gamma-\mu}(\gamma^{2t}S_\gamma-\mu^{2t}S_\mu)\\&=\frac{1}{\gamma-\mu}\Bigg(\sum_{t=0}^{n}\gamma^{2t}S_\gamma-\sum_{t=0}^{n}\mu^{2t}S_\mu\Bigg).\end{aligned}$$

Moreover, we know that $\sum_{t=0}^{n}\gamma^{2t}=\frac{1-\gamma^{2n+2}}{1-\gamma^2}$ and $\sum_{t=0}^{n}\mu^{2t}=\frac{1-\mu^{2n+2}}{1-\mu^2}$. Therefore, we have

$$\begin{aligned}\sum_{t=0}^{n}SP_{2t}&=\frac{1}{2(\gamma-\mu)}\Bigg(\frac{1-\mu^{2n+2}}{\mu}S_\mu-\frac{1-\gamma^{2n+2}}{\gamma}S_\gamma\Bigg)\\&=\frac{1}{2(\gamma-\mu)}\Bigg(\mu S_\gamma-\gamma S_\mu+\gamma^{2n+1}S_\gamma-\mu^{2n+1}S_\mu\Bigg)\end{aligned}$$

where $\gamma\mu=-1$. Then, we obtain

$$\sum_{t=0}^{n}SP_{2t}=\frac{1}{2}(SP_{2n+1}+SP_0-\frac{1}{2}SPL_0).$$

In addition to that, if we use $SPL_0=2SP_0+SP_{-1}$ from Theorem (2.2) , we easily get

$$\sum_{t=0}^{n}SP_{2t}=\frac{1}{2}(SP_{2n+1}-SP_{-1}).$$

iii) We use the Binet formula for Pell spinors. So, we can write

$$\begin{aligned}\sum_{t=0}^{n}SP_{2t-1}&=\sum_{t=0}^{n}\frac{1}{\gamma-\mu}(\gamma^{2t-1}S_\gamma-\mu^{2t-1}S_\mu)\\&=\frac{1}{\gamma-\mu}\Bigg(\sum_{t=0}^{n}\gamma^{2t-1}S_\gamma-\sum_{t=0}^{n}\mu^{2t-1}S_\mu\Bigg).\end{aligned}$$

Similar to the other options i) and ii) we can easily obtain that

$$\begin{aligned}\sum_{t=0}^{n}SP_{2t-1}&=\frac{1}{2(\gamma-\mu)}\Bigg(\gamma^2(1-\mu^{2n+2})S_\mu-\mu^2(1-\gamma^{2n+2})S_\gamma\Bigg)\\&=\frac{1}{2(\gamma-\mu)}\Bigg(\gamma^{2n}S_\gamma-\mu^{2n}S_\mu+2\sqrt{2}(S_\gamma+S_\mu)-3(S_\gamma-S_\mu)\Bigg).\end{aligned}$$

If we use Binet formulas for Pell and Pell-Lucas spinors then, we get

$$\sum_{t=0}^{n}SP_{2t-1}=\frac{1}{2}(SP_{2n}-3SP_0+SPL_0)$$

and consequently

$$\sum_{t=0}^{n} SP_{2t-1} = \frac{1}{2}(SP_{2n} - SP_{-2})$$

where we know that $SPL_0 = 2SP_0 + 2SP_{-1}$ and $SP_0 = 2SP_{-1} + SP_{-2}$. This proof is completed. □

Now, considering [18, 34] we express the following definition.

Definition 2.5. Suppose that SP_n and SPL_n are nth Pell and Pell-Lucas spinors. The fundamental Pell and Pell-Lucas spinor matrices are

$$(SP_n)_L = \begin{bmatrix} P_n - \mathbf{i}P_{n+3} & -P_{n+2} - \mathbf{i}P_{n+1} \\ P_{n+2} - \mathbf{i}P_{n+1} & P_n + \mathbf{i}P_{n+3} \end{bmatrix}$$

and

$$(SPL_n)_L = \begin{bmatrix} Q_n - \mathbf{i}Q_{n+3} & -Q_{n+2} - \mathbf{i}Q_{n+1} \\ Q_{n+2} - \mathbf{i}Q_{n+1} & Q_n + \mathbf{i}Q_{n+3} \end{bmatrix}.$$

The fundamental Pell and Pell-Lucas spinor matrices are also called as left Hamilton Pell and Pell Lucas spinor matrices, respectively.

Now, we express the Cassini Formula for Pell and Pell-Lucas spinors.

Theorem 2.5. *The similar formula replacing Cassini formula for Pell spinors is*

$$(SP_{n-1})_L SP_{n+1} - (SP_n)_L SP_n = (-1)^n \frac{1}{\gamma - \mu}(\gamma (S_\mu)_L S_\gamma - \mu (S_\gamma)_L S_\mu)$$

and for Pell-Lucas spinors the similar formula is

$$(SPL_{n-1})_L SPL_{n+1} - (SPL_n)_L SPL_n (-1)^{n-1}(\gamma - \mu)(\gamma (S_\mu)_L S_\gamma - \mu (S_\gamma)_L S_\mu)$$

where $S_\mu = \begin{bmatrix} \mu^3 + \boldsymbol{i} \\ \mu + \boldsymbol{i}\mu^2 \end{bmatrix}$, $(S_\mu)_L = \begin{bmatrix} 1 - \boldsymbol{i}\mu^3 & -\mu^2 - \boldsymbol{i}\mu \\ \mu^2 - \boldsymbol{i}\mu & 1 + \boldsymbol{i}\mu^3 \end{bmatrix}$, $S_\gamma = \begin{bmatrix} \gamma^3 + \boldsymbol{i} \\ \gamma + \boldsymbol{i}\gamma^2 \end{bmatrix}$, $(S_\gamma)_L = \begin{bmatrix} 1 - \boldsymbol{i}\gamma^3 & -\gamma^2 - \boldsymbol{i}\gamma \\ \gamma^2 - \boldsymbol{i}\gamma & 1 + \boldsymbol{i}\gamma^3 \end{bmatrix}$.

Proof. Pell spinor product corresponding to the product of Pell quaternions $QP_{n-1}QP_{n+1} - (QP_n)^2$ is $(SP_{n-1})_L SP_{n+1} - (SP_n)_L SP_n$. In this case, if we use the Binet formula in Theorem (2.3) for Pell spinors $SP_n = \frac{1}{\gamma-\mu}(\gamma^n S_\gamma - \mu^n S_\mu)$, then we get

$$(SP_n)_L = \frac{1}{\gamma - \mu}(\gamma^n L_{S_\gamma} - \mu^n L_{S_\mu}).$$

Therefore, we obtain

$$\begin{aligned}
(SP_{n-1})_L SP_{n+1} - (SP_n)_L SP_n =& \frac{1}{\gamma - \mu}(\gamma^{n-1}(S_\gamma)_L - \mu^{n-1}(S_\mu)_L)\frac{1}{\gamma - \mu}(\gamma^{n+1} S_\gamma - \mu^{n+1} S_\mu) \\
& - \frac{1}{\gamma - \mu}(\gamma^{n}(S_\gamma)_L - \mu^{n}(S_\mu)_L)\frac{1}{\gamma - \mu}(\gamma^{n} S_\gamma - \mu^{n} S_\mu) \\
=& \frac{1}{(\gamma - \mu)^2}\left((-\gamma^{n-1}\mu^{n+1} + \gamma^n\mu^n)(S_\gamma)_L S_\mu + (-\gamma^{n+1}\mu^{n-1} + \gamma^n\mu^n)(S_\mu)_L S_\gamma\right) \\
=& (-1)^{n-1}\frac{1}{\gamma - \mu}(\mu (S_\gamma)_L S_\mu - \gamma (S_\mu)_L S_\gamma) \\
=& (-1)^{n}\frac{1}{\gamma - \mu}(\gamma (S_\mu)_L S_\gamma - \mu (S_\gamma)_L S_\mu)
\end{aligned}$$

where $S_\mu = \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix}$, $(S_\mu)_L = \begin{bmatrix} 1 - \mathbf{i}\mu^3 & -\mu^2 - \mathbf{i}\mu \\ \mu^2 - \mathbf{i}\mu & 1 + \mathbf{i}\mu^3 \end{bmatrix}$, $S_\gamma = \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix}$, $(S_\gamma)_L = \begin{bmatrix} 1 - \mathbf{i}\gamma^3 & -\gamma^2 - \mathbf{i}\gamma \\ \gamma^2 - \mathbf{i}\gamma & 1 + \mathbf{i}\gamma^3 \end{bmatrix}$.

Similarly, for Pell-Lucas Spinors considering $SPL_n = \gamma^n S\gamma + \mu^n S_\mu$ and $(SPL_n)_L = \gamma^n (S\gamma)_L + \mu^n (S_\mu)_L$ we have

$$\begin{aligned}
(SPL_{n-1})_L SPL_{n+1} - (SPL_n)_L SPL_n =& (\gamma^{n-1}(S_\gamma)_L + \mu^{n-1}(S_\mu)_L)(\gamma^{n+1} S_\gamma + \mu^{n+1} S_\mu) \\
& - (\gamma^{n}(S_\gamma)_L + \mu^{n}(S_\mu)_L)(\gamma^{n} S_\gamma + \mu^{n} S_\mu) \\
=& (\gamma^{n-1}\mu^{n+1} - \gamma^n\mu^n)(S_\gamma)_L S_\mu + (\gamma^{n+1}\mu^{n-1} - \gamma^n\mu^n)(S_\mu)_L S_\gamma \\
=& (-1)^{n-1}(\gamma - \mu)(\gamma (S_\mu)_L S_\gamma - \mu (S_\gamma)_L S_\mu)
\end{aligned}$$

and consequently

$$(SPL_{n-1})_L SPL_{n+1} - (SPL_n)_L SPL_n = (-1)^{n-1}(\gamma-\mu)(\gamma(S_\mu)_L S_\gamma - \mu(S_\gamma)_L S_\mu)$$

where $S_\mu = \begin{bmatrix} \mu^3+\mathbf{i} \\ \mu+\mathbf{i}\mu^2 \end{bmatrix}$, $(S_\mu)_L = \begin{bmatrix} 1-\mathbf{i}\mu^3 & -\mu^2-\mathbf{i}\mu \\ \mu^2-\mathbf{i}\mu & 1+\mathbf{i}\mu^3 \end{bmatrix}$, $S_\gamma = \begin{bmatrix} \gamma^3+\mathbf{i} \\ \gamma+\mathbf{i}\gamma^2 \end{bmatrix}$, $(S_\gamma)_L = \begin{bmatrix} 1-\mathbf{i}\gamma^3 & -\gamma^2-\mathbf{i}\gamma \\ \gamma^2-\mathbf{i}\gamma & 1+\mathbf{i}\gamma^3 \end{bmatrix}$. □

Conclusion 2.1. The Cassini formulas for Pell and Pell-Lucas spinors can be obtained that

$$for\ Pell\ spinors\ (SP_{n-1})_L SP_{n+1} - (SP_n)_L SP_n = (-1)^{n-1}\begin{bmatrix} 12+2\mathbf{i} \\ 4+10\mathbf{i} \end{bmatrix},$$

$$for\ Pell-Lucas\ spinors\ (SPL_{n-1})_L SPL_{n+1} - (SPL_n)_L SPL_n = 8(-1)^{n-1}\begin{bmatrix} 12+2\mathbf{i} \\ 4+10\mathbf{i} \end{bmatrix}.$$

Theorem 2.6. *The generator function for Pell spinors is*

$$G_{SP}(t) = \frac{1}{1-2t-t^2}\begin{bmatrix} 5+2t+\boldsymbol{i}t \\ 1+\boldsymbol{i}(2+t) \end{bmatrix}$$

and the generator function for Pell-Lucas spinors is

$$G_{SPL}(t) = \frac{1}{1-2t-t^2}\begin{bmatrix} 14+6t+\boldsymbol{i}(2-2t) \\ 2+2t+\boldsymbol{i}(6+2t) \end{bmatrix}$$

Proof. We take nth Pell spinor is SP_n. Therefore, for nth Pell spinor the generator function is calculated with the aid of the equation $G_{SP}(t) = \sum_{n=0}^{\infty} SP_n t^n$. In this case, using $G_{SP}(t)$, $2tG_{SP}(t)$ and $t^2G_{SP}(t)$ we obtain that

$$\begin{aligned} G_{SP}(t) &= SP_0 + SP_1t + SP_2t^2 + SP_3t^3 + SP_4t^4 + SP_5t^5 + ... \\ -2tG_{SP}(t) &= -2SP_0t - 2SP_1t^2 - 2SP_2t^3 - 2SP_3t^4 - 2SP_4t^5 - 2SP_5t^6 + ... \\ -t^2G_{SP}(t) &= -SP_0t^2 - SP_1t^3 - SP_2t^4 - SP_3t^5 - SP_4t^6 - SP_5t^7 + ... \end{aligned}$$

and

$$G_{SP}(t) = \frac{1}{(1-2t-t^2)}(SP_0 + (SP_1 - 2SP_0)t)$$

where

$$\begin{aligned} SP_0 + (SP_1 - 2SP_0) &= \begin{bmatrix} P_3+\mathbf{i}P_0 \\ P_1+\mathbf{i}P_2 \end{bmatrix} + \left(\begin{bmatrix} P_4+\mathbf{i}P_1 \\ P_2+\mathbf{i}P_3 \end{bmatrix} - \begin{bmatrix} 2P_3+2\mathbf{i}P_0 \\ 2P_1+2\mathbf{i}P_2 \end{bmatrix}\right)t \\ &= \begin{bmatrix} 5 \\ 1+2\mathbf{i} \end{bmatrix} + \left(\begin{bmatrix} 12+\mathbf{i} \\ 2+5\mathbf{i} \end{bmatrix} - \begin{bmatrix} 10 \\ 2+4\mathbf{i} \end{bmatrix}\right)t = \begin{bmatrix} 5+2t+\mathbf{i}t \\ 1+\mathbf{i}(2+t) \end{bmatrix}. \end{aligned}$$

Consequently, we get

$$G_{SP}(t) = \frac{1}{1-2t-t^2}\begin{bmatrix} 5+2t+\mathbf{i}t \\ 1+\mathbf{i}(2+t) \end{bmatrix}.$$

Now, we calculate the generator function for Pell-Lucas spinors. Therefore, if we consider the function $G_{SPL}(t) = \sum_{n=0}^{\infty} SPL_n t^n$, we have

$$G_{SPL}(t) = \frac{1}{1-2t-t^2}(SPL_0 + (SPL_1 - 2SPL_0)t)$$

using $G_{SPL}(t)$, $2tG_{SPL}(t)$ and $t^2G_{SPL}(t)$. Finally, we obtain

$$G_{SPL}(t) = \frac{1}{1-2t-t^2}\begin{bmatrix} 14+6t+\mathbf{i}(2-2t) \\ 2+2t+\mathbf{i}(6+2t) \end{bmatrix}.$$

This completes the proof. □

Theorem 2.7. *Assume that $-nth$ Pell and Pell-Lucas spinors are SP_{-n} and SPL_{-n}. In this case these spinors are calculated as follows; for Pell spinors*

$$SP_{-n} = (-1)^n \begin{bmatrix} P_{n-3} - \boldsymbol{i}P_n \\ P_{n-1} - \boldsymbol{i}P_{n-2} \end{bmatrix}$$

for Pell-Lucas spinors

$$SPL_{-n} = (-1)^n \begin{bmatrix} -Q_{n-3} + \boldsymbol{i}Q_n \\ -Q_{n-1} - \boldsymbol{i}Q_{n-2} \end{bmatrix}.$$

Proof. We know that the Binet formula for nth Pell spinor is $SP_n = \frac{1}{\gamma-\mu}(\gamma^n S_\gamma - \mu^n S_\mu)$ where $S_\mu = \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix}$, $S_\gamma = \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix}$. On the other hand, we can write the equation $\gamma\mu = -1 \Longrightarrow \gamma = (-1)\mu^{-1}$. If we take n powers of both sides then, we get $\gamma^{-n} = (-1)^n\mu^n$. Similarly, we easily see that $\mu^{-n} = (-1)^n\gamma^n$. In this case, considering the Binet formula for $-nth$ Pell spinor $SP_{-n} = \frac{1}{\gamma-\mu}(\gamma^{-n}S_\gamma - \mu^{-n}S_\mu)$ we calculate as

$$SP_{-n} = \frac{1}{\gamma - \mu}((-1)^n\mu^n S_\gamma - (-1)^n\gamma^n S_\mu)$$

and

$$SP_{-n} = \frac{(-1)^n}{\gamma - \mu}(\mu^n S_\gamma - \gamma^n S_\mu).$$

If we make this equation even more detailed, we get

$$SP_{-n} = \frac{(-1)^n}{\gamma - \mu} \begin{bmatrix} \mu^n\gamma^3 - \gamma^n\mu^3 + \mathbf{i}(\mu^n - \gamma^n) \\ \mu^n\gamma - \gamma^n\mu + \mathbf{i}(\mu^n\gamma^2 - \gamma^n\mu^2) \end{bmatrix} \tag{2.3}$$

where $\gamma = 2 - \mu$ and $\mu = 2 - \gamma$. Additionally, if the characteristic equation $x^2 - 2x - 1 = 0$ of Pell number sequence is used, the equations $\gamma^2 = 5 - 2\mu$, $\mu^2 = 5 - 2\gamma$, $\gamma^3 = 12 - 5\mu$ and $\mu^3 = 12 - 5\gamma$ are obtained. Therefore, we obtain the Eq (2.3) as

$$SP_{-n} = \frac{(-1)^n}{\gamma - \mu} \begin{bmatrix} \mu^n(12 - 5\mu) - \gamma^n(12 - 5\gamma) + \mathbf{i}(\mu^n - \gamma^n) \\ \mu^n(2 - \mu) - \gamma^n(2 - \gamma) + \mathbf{i}(\mu^n(5 - 2\gamma) - \gamma^n(5 - 2\gamma)) \end{bmatrix}$$

$$= (-1)^n \begin{bmatrix} -12(\frac{\gamma^n - \mu^n}{\gamma - \mu}) + 5(\frac{\gamma^{n+1} - \mu^{n+1}}{\gamma - \mu}) - \mathbf{i}(\frac{\gamma^n - \mu^n}{\gamma - \mu}) \\ -2(\frac{\gamma^n - \mu^n}{\gamma - \mu}) + (\frac{\gamma^{n+1} - \mu^{n+1}}{\gamma - \mu}) + \mathbf{i}(-5(\frac{\gamma^n - \mu^n}{\gamma - \mu}) + 2(\frac{\gamma^{n+1} - \mu^{n+1}}{\gamma - \mu})) \end{bmatrix}$$

$$= (-1)^n \begin{bmatrix} -12P_n + 5P_{n+1} - \mathbf{i}P_n \\ -2P_n + P_{n+1} - \mathbf{i}(-5P_n + 2P_{n+1}) \end{bmatrix} = (-1)^n \begin{bmatrix} P_{n-3} - \mathbf{i}P_n \\ P_{n-1} - \mathbf{i}P_{n-2} \end{bmatrix}$$

$$= \begin{cases} \text{If } n \text{ is even number,} & \begin{bmatrix} P_{n-3} - \mathbf{i}P_n \\ P_{n-1} - \mathbf{i}P_{n-2} \end{bmatrix} \\ \text{If } n \text{ odd number,} & \begin{bmatrix} P_{n-3} - \mathbf{i}P_n \\ P_{n-1} - \mathbf{i}P_{n-2} \end{bmatrix} \end{cases}.$$

Now, we calculate for Pell-Lucas spinors. Considering Binet formula $SPL_n = \gamma^n S_\gamma + \mu^n S_\mu$ for Pell-Lucas spinor sequence and we can write for $-n$

$$SPL_{-n} = \gamma^{-n}S_\gamma + \mu^{-n}S_\mu.$$

If we use again the equations $\gamma^{-n} = (-1)^n\mu^n$ and $\mu^{-n} = (-1)^n\gamma^n$ then, we have

$$SPL_{-n} = (-1)^n(\mu^n S_\gamma + \gamma^n S_\mu)$$

and

$$SPL_{-n} = (-1)^n \begin{bmatrix} \mu^n\gamma^3 + \gamma^n\mu^3 + \mathbf{i}(\mu^n + \gamma^n) \\ \mu^n\gamma + \gamma^n\mu + \mathbf{i}(\mu^n\gamma^2 + \gamma^n\mu^2) \end{bmatrix}.$$

Finally, we get

$$SPL_{-n} = (-1)^n \begin{bmatrix} -Q_{n-3} + \mathbf{i}Q_n \\ -Q_{n-1} + \mathbf{i}Q_{n-2} \end{bmatrix}.$$

□

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Affiliations

TÜLAY ERIŞIR
ADDRESS: Department of Mathematics, Erzincan Binali Yıldırım University, Erzincan, Turkey
E-MAIL: tulay.erisir@erzincan.edu.tr
ORCID ID:0000-0002-7750-6910

GÖKHAN MUMCU
ADDRESS: Department of Mathematics, Erzincan Binali Yıldırım University, Erzincan, Turkey
E-MAIL: gokhanmumcu@outlook.com
ORCID ID: 0000-0002-5828-1963

MEHMET ALI GÜNGÖR
ADDRESS: Department of Mathematics, Sakarya University, Sakarya, Turkey
E-MAIL: agungor@sakarya.edu.tr
ORCID ID: 0000-0003-1863-3183

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Inextensible Flow of Quaternionic Curves According to Type 2-Quaternionic Frame in the Euclidean Space

Önder Gökmen Yıldız* and Hazer Usta

Abstract
In this paper, we investigate inextensible flows of quaternionic curve according to type 2-quaternionic frame. We give necessary and sufficient conditions for inextensible flow of quaternionic curves. Moreover, we obtain evolution equations of the Frenet frame and curvatures according to type 2-quaternionic frame.

Keywords: *Curvature flows, Quaternionic curve, Real quaternion*

AMS Subject Classification (2020): *53C44, 53A04*

****Corresponding author***

1. Introduction

The quaternions are extensions of the complex numbers. Quaternions were defined as the quotient of two directed lines in a three dimensional space or equivalently as the quotient of two vectors by Sir William Rowan Hamilton [1]. Quaternions can be represented in various ways: as the sum of a real scalar and a real three dimensional vector, as pairs of complex numbers or as four-dimensional vectors with real components. Quaternion multiplication is generally not commutative, so quaternions are not a field.

K. Baharatti and M. Nagaraj studied quaternionic curves in three-dimensional and four-dimensional Euclidean space and obtained their Frenet formulas [2]. In analogy with the Euclidean case, A.C. Coken and A. Tuna defined Frenet formulas for the quaternionic curves in semi-Euclidean space [3]. F. Kahraman Aksoyak introduced a new version of Frenet formulas for quaternionic curves in four-dimensional Euclidean space and called it type 2-quaternionic frame [4]. After that, by using these quaternionic frames, a lot of papers about quaternionic curves have been studied [5–12].

A family of curves parametrized by time can be thought as evolving curves. The time evolution of geometric locus is investigated by using its flow. There have been various studies on flows of curves, but firstly, D.Y. Kwon and F.C. Park introduced inextensible flows of plane curves [13] and D.Y. Kwon et al. investigated inextensible flows of curves and developable surfaces in $\mathbb{R}^3$ [14]. Then in many different spaces, inextensible flows of curves are studied (see, [15–19]). Inextensible flows of curves also studied for quaternionic curves (see, [6, 10, 12]).

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Our aim is to study inextensible flows of quaternionic curve according to type 2-quaternionic frame. We give necessary and sufficient conditions for inextensible flow of quaternionic curves. Moreover, we obtain evolution equations of the Frenet frame and curvatures according to type 2-quaternionic frame.

2. Preliminaries

In this section, a brief summary of the theory of quaternions in the Euclidean space is presented.

The space of quaternions Q is isomorphic to $\mathbb{R}^4$, four-dimensional vector space over the real numbers. There are three operations in Q: addition, scalar multiplication and quaternion multiplication. Addition and scalar multiplication of quaternions are defined to be the same as in $\mathbb{R}^4$.

A real quaternion q is an expression of the form $q = ae_1 + be_2 + ce_3 + de_4$, where a, b, c and d are real numbers, and e_1, e_2, e_3 are quaternionic units which satisfy the non-commutative multiplication rules,

$$\begin{aligned} i) e_i \times e_i &= -e_4, \quad (e_4 = 1,\ 1 \le i \le 3) \\ ii) e_i \times e_j &= e_k = -e_j \times e_i, \quad (1 \le i, j \le 3), \end{aligned}$$

where (ijk) is an even permutation of (123) in the Euclidean space $\mathbb{R}^4$. Further, a real quaternion can be written as $q = S_q + V_q$, where $S_q = d$ is the scalar part and $V_q = ae_1 + be_2 + ce_3$ is the vector part of q. The product of two quaternions can be expanded as

$$p \times q = S_p S_q - < V_p, V_q > + S_p V_q + S_q V_p + V_q \wedge V_q,$$

for every $p, q \in Q$, where $<,>$ and $\wedge$ are inner product and cross product on R^3, respectively. The conjugate of the quaternion q is denoted by $\overline{q}$ and defined as

$$\overline{q} = S_q - V_q = de_4 - ae_1 - be_2 - ce_3,$$

and is called by "Hamiltonian conjugation of q". The h-inner product of two quaternions is defined by

$$h(p, q) = \frac{1}{2}(p \times \overline{q} + q \times \overline{p}),$$

where h is the symmetric, non-degenerate, real-valued and bilinear form. Let p and q be two real quaternions, then $h(p, q) = 0$ if and only if p and q are $h-$orthogonal. The norm of a real quaternion q is defined by

$$\|q\|^2 = h(q, q) = a^2 + b^2 + c^2 + d^2.$$

If $q + \overline{q} = 0$, then q is called a spatial quaternion. The three-dimensional Euclidean space $\mathbb{R}^3$ is identified with the space of spatial quaternion $Q_s = \{\gamma \in Q \,|\, \gamma + \overline{\gamma} = 0\} \subset Q$ in an obvious manner.

Theorem 2.1. *Let*

$$\gamma : [0,1] \subset \mathbb{R} \longrightarrow Q_s, \quad \gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i, \quad (1 \le i \le 3),$$

be a smooth curve with arc-lenght parameter and $\{t, n_1, n_2\}$ be the Frenet trihedron of γ. Then Frenet equations are

$$\begin{aligned} t' &= kn_1 \\ n_1' &= -kt + rn_2 \\ n_2' &= -rn_1, \end{aligned}$$

where t is the unit tangent, n_1 is the unit principal normal, n_2 is the unit binormal vector fields, k is the principal curvature and r is the torsion of the quaternionic curve γ, [2].

Theorem 2.2. *Let*

$$\beta : [0,1] \subset \mathbb{R} \longrightarrow Q, \quad \beta(s) = \sum_{i=1}^{4} \gamma_i(s) e_i, \quad e_4 = 1,$$

be a smooth curve β in Q and $\{T, N_1, N_2, N_3\}$ be the Frenet apparatus of β, then the Frenet equations are

$$\begin{aligned} T' &= KN_1 \\ N_1' &= -KT + kN_2 \\ N_2' &= -kN_1 + (r-K)N_3 \\ N_3' &= -(r-K)N_2, \end{aligned}$$

where $N_1 = t \times T$, $N_2 = n_1 \times T$, $N_3 = n_2 \times T$ and $K = \|T'(s)\|$, [2].

It is obtained the Frenet formulae in [2] and the apparatus for the curve β by making use of the Frenet formulae for a curve γ in E^3. Moreover, there are relationships between curvatures of the curves β and γ. These relations can be explained that the torsion of β is the principal curvature of the curve γ. Also, the bitorsion of β is $(r-K)$, where r is the torsion of γ and K is the principal curvature of β. These relations are only determined for quaternions, [2].

The alternative quaternionic frame for a quaternionic curve in $\mathbb{R}^4$ by using of a similar method in [2] given by Kahraman Aksoyak [4]

Theorem 2.3. *Let*

$$\zeta : [0,1] \subset R \longrightarrow Q, \quad \zeta(s) = \sum_{i=1}^{4} \gamma_i(s) e_i, \quad e_4 = 1,$$

be a smooth curve ζ in Q. The Frenet equations of $\zeta(s)$ for type 2-quaternionic frame are

$$\begin{aligned} T' &= KN_1 \\ N_1' &= -KT + -rN_2 \\ N_2' &= rN_1 + (K-k)N_3 \\ N_3' &= -(K-k)N_2, \end{aligned}$$

where $N_1 = b \times T$, $N_2 = n_1 \times T$, $N_3 = t \times T$ and $K = \|T'\|$, [4].

For further quaternions concepts see [20].

3. Flow of quaternionic curves according to type 2-quaternionic frame

Throughout this section, we investigate flow of quaternionic curve according to type 2-quaternionic frame.

Unless otherwise stated we assume that $\zeta : [0,l] \times [0,w] \to Q$ is a one parameter family of smooth quaternionic curve in Q where l is arclength of initial curve and u is the curve parametrization variable, $0 \leq u \leq l$. Let $\zeta(u,t)$ be a position vector of the semi-real quaternionic curve at time t. The arclength variation of $\zeta(u,t)$ is given by

$$s(u,t) = \int_0^u \left\| \frac{\partial \zeta}{\partial u} \right\| du = \int_0^u v du.$$

The operator $\frac{\partial}{\partial s}$ is given in term of u by $\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$.

Definition 3.1. Let ζ be smooth quaternionic curve. Any flow of ζ can be given by

$$\frac{\partial \zeta}{\partial t} = g_1 T + g_2 N_1 + g_3 N_2 + g_4 N_3, \tag{3.1}$$

where g_1, g_2, g_3 and g_4 are scalar speed functions of ζ.

In Q, the inextensible condition of the length of the curve can be expressed by [13]

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0. \tag{3.2}$$

Definition 3.2. A quaternionic curve evolution $\zeta(u,t)$ and its flow $\frac{\partial \zeta}{\partial t}$ in Q are said to be inextensible if

$$\frac{\partial}{\partial t}\left\|\frac{\partial \zeta}{\partial u}\right\| = 0.$$

Lemma 3.1. *The evolution equation for the speed v according to type 2-quaternionic frame is given by*

$$\frac{\partial v}{\partial t} = \frac{\partial g_1}{\partial u} - v\kappa g_2. \tag{3.3}$$

Proof. As $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are commutative and $v^2 = h\left(\frac{\partial \zeta}{\partial u}, \frac{\partial \zeta}{\partial u}\right)$, we have

$$2v\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}h\left(\frac{\partial \zeta}{\partial u}, \frac{\partial \zeta}{\partial u}\right) = 2h\left(\frac{\partial \zeta}{\partial u}, \frac{\partial}{\partial u}\left(\frac{\partial \zeta}{\partial t}\right)\right).$$

By using the equations of type 2-quaternionic frame, we obtain

$$\frac{\partial v}{\partial t} = \frac{\partial g_1}{\partial u} - v\kappa g_2.$$

□

Theorem 3.1. *The flow of quaternionic curve is inextensible according to type 2-quaternionic frame if and only if*

$$\frac{\partial g_1}{\partial s} = \kappa g_2. \tag{3.4}$$

Proof. Let the flow of quaternionic curve be inextensible. From equation (3.2) and (3.3), we have

$$\frac{\partial}{\partial t}s(u,t) = \int_0^u \frac{\partial v}{\partial t}du = \int_0^u \left(\frac{\partial g_1}{\partial u} - v\kappa g_2\right)du = 0.$$

This clearly forces

$$\frac{\partial g_1}{\partial s} = \kappa g_2.$$

□

Lemma 3.2. *Let the flow of $\zeta(u,t)$ be inextensible. Derivatives of the elements of type 2-quaternionic frame with respect to evolution parameter can be given as follows;*

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right)N_1 + \left(-g_2 r + \frac{\partial g_3}{\partial s} - g_4(\kappa - k)\right)N_2 \\
&\quad + \left(g_3(\kappa - k) + \frac{\partial g_4}{\partial s}\right)N_3, \\
\frac{\partial N_1}{\partial t} &= -\left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right)T + \psi_1 N_2 + \psi_2 N_3, \\
\frac{\partial N_2}{\partial t} &= \left(g_2 r - \frac{\partial g_3}{\partial s} + g_4(\kappa - k)\right)T - \psi_1 N_1 + \psi_3 N_3, \\
\frac{\partial N_3}{\partial t} &= -\left(g_3(\kappa - k) + \frac{\partial g_4}{\partial s}\right)T - \psi_2 N_1 - \psi_3 N_2,
\end{aligned}$$

where $\psi_1 = h\left(\frac{\partial N_1}{\partial t}, N_2\right), \psi_2 = h\left(\frac{\partial N_1}{\partial t}, N_3\right), \psi_3 = h\left(\frac{\partial N_2}{\partial t}, N_3\right).$

Proof. Let $\frac{\partial \zeta}{\partial t}$ be inextensible. Then, considering that $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ are commutative, we get

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \zeta}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial \zeta}{\partial t}\right) = \frac{\partial}{\partial s}\left(g_1 T + g_2 N_1 + g_3 N_2 + g_4 N_3\right),$$

substituting (3.4) in the last equation, we have

$$\begin{aligned}\frac{\partial T}{\partial t} &= \left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right) N_1 + \left(-g_2 r + \frac{\partial g_3}{\partial s} - g_4\left(\kappa - k\right)\right) N_2 \\ &+ \left(g_3\left(\kappa - k\right) + \frac{\partial g_4}{\partial s}\right) N_3.\end{aligned}$$

Now, if we consider orthogonality of $\{T, N_1, N_2, N_3\}$, then we get

$$\begin{aligned}0 &= \frac{\partial}{\partial t} h\left(T, N_1\right) = h\left(\frac{\partial T}{\partial t}, N_1\right) + h\left(T, \frac{\partial N_1}{\partial t}\right) \\ &= \left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right) + h\left(T, \frac{\partial N_1}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t} h\left(T, N_2\right) = h\left(\frac{\partial T}{\partial t}, N_2\right) + h\left(T, \frac{\partial N_2}{\partial t}\right) \\ &= \left(-g_2 r + \frac{\partial g_3}{\partial s} - g_4\left(\kappa - k\right)\right) + h\left(T, \frac{\partial N_2}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t} h\left(T, N_3\right) = h\left(\frac{\partial T}{\partial t}, N_3\right) + h\left(T, \frac{\partial N_3}{\partial t}\right) \\ &= \left(g_3\left(\kappa - k\right) + \frac{\partial g_4}{\partial s}\right) + h\left(T, \frac{\partial N_3}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t} h\left(N_1, N_2\right) = h\left(\frac{\partial N_1}{\partial t}, N_2\right) + h\left(N_1, \frac{\partial N_2}{\partial t}\right) \\ &= \psi_1 + h\left(N_1, \frac{\partial N_2}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t} h\left(N_1, N_3\right) = h\left(\frac{\partial N_1}{\partial t}, N_3\right) + h\left(N_1, \frac{\partial N_3}{\partial t}\right) \\ &= \psi_2 + h\left(N_1, \frac{\partial N_3}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t} h\left(N_2, N_3\right) = h\left(\frac{\partial N_2}{\partial t}, N_3\right) + h\left(N_2, \frac{\partial N_3}{\partial t}\right) \\ &= \psi_3 + h\left(N_2, \frac{\partial N_3}{\partial t}\right),\end{aligned}$$

which brings about that

$$\begin{aligned}\frac{\partial N_1}{\partial t} &= -\left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right) T + \psi_1 N_2 + \psi_2 N_3, \\ \frac{\partial N_2}{\partial t} &= \left(g_2 r - \frac{\partial g_3}{\partial s} + g_4\left(\kappa - k\right)\right) T - \psi_1 N_1 + \psi_3 N_3, \\ \frac{\partial N_3}{\partial t} &= -\left(g_3\left(\kappa - k\right) + \frac{\partial g_4}{\partial s}\right) T - \psi_2 N_1 - \psi_3 N_2,\end{aligned}$$

where $\psi_1 = h\left(\frac{\partial N_1}{\partial t}, N_2\right), \psi_2 = h\left(\frac{\partial N_1}{\partial t}, N_3\right), \psi_3 = h\left(\frac{\partial N_2}{\partial t}, N_3\right).$ □

Theorem 3.2. *Let the flow of $\zeta(u, t)$ be inextensible. Then the evolution equation of κ is*

$$\frac{\partial \kappa}{\partial t} = \frac{\partial g_1}{\partial s}\kappa + g_1\frac{\partial \kappa}{\partial s} + \frac{\partial^2 g_2}{\partial s^2} + 2\frac{\partial g_3}{\partial s} r + g_3\frac{\partial r}{\partial s} - g_2 r^2 - g_4 r\left(\kappa - k\right).$$

Proof. Since $\frac{\partial}{\partial s}\left(\frac{\partial T}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial s}\right)$, we have

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial T}{\partial t}\right)&=\left(-g_1\kappa^2-\frac{\partial g_2}{\partial s}\kappa-g_3\kappa r\right)T\\&+\left(\frac{\partial g_1}{\partial s}\kappa+g_1\frac{\partial \kappa}{\partial s}+\frac{\partial^2 g_2}{\partial s^2}+2\frac{\partial g_3}{\partial s}r+g_3\frac{\partial r}{\partial s}-g_2r^2-g_4r\left(\kappa-k\right)\right)N_1\\&+\left(-g_1\kappa r-2\frac{\partial g_2}{\partial s}r-g_3r^2+g_2\frac{\partial r}{\partial s}+\frac{\partial^2 g_3}{\partial s^2}\right.\\&\left.-2\frac{\partial g_4}{\partial s}\left(\kappa-k\right)-g_4\frac{\partial\left(\kappa-k\right)}{\partial s}-g_3\left(\kappa-k\right)^2\right)N_2\\&+\left(-g_2r\left(\kappa-k\right)+2\frac{\partial g_3}{\partial s}\left(\kappa-k\right)-g_4\left(\kappa-k\right)^2\right.\\&\left.+g_3\frac{\partial\left(\kappa-k\right)}{\partial s}+\frac{\partial^2 g_4}{\partial s^2}\right)N_3\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial s}\right)&=\frac{\partial}{\partial t}\left(\kappa N_1\right)=\frac{\partial \kappa}{\partial t}N_1+\kappa\frac{\partial N_1}{\partial t}\\&=\left(-g_1\kappa^2-\frac{\partial g_2}{\partial s}\kappa-g_3\kappa r\right)T+\frac{\partial \kappa}{\partial t}N_1+\psi_1\kappa N_2\\&+\psi_2\kappa N_3.\end{aligned}$$

From equality of the component of N_1 in above equations, we obtain

$$\frac{\partial \kappa}{\partial t}=\frac{\partial g_1}{\partial s}\kappa+g_1\frac{\partial \kappa}{\partial s}+\frac{\partial^2 g_2}{\partial s^2}+2\frac{\partial g_3}{\partial s}r+g_3\frac{\partial r}{\partial s}-g_2r^2-g_4r\left(\kappa-k\right).$$

□

Corollary 3.1. *In theorem (3.2), from rest of the equality, we get*

$$\begin{aligned}\kappa\psi_1&=-g_1\kappa r-2\frac{\partial g_2}{\partial s}r-g_3r^2+g_2\frac{\partial r}{\partial s}+\frac{\partial^2 g_3}{\partial s^2}-2\frac{\partial g_4}{\partial s}\left(\kappa-k\right)-g_4\frac{\partial\left(\kappa-k\right)}{\partial s}-g_3\left(\kappa-k\right)^2,\\\kappa\psi_2&=-g_2r\left(\kappa-k\right)+2\frac{\partial g_3}{\partial s}\left(\kappa-k\right)-g_4\left(\kappa-k\right)^2+g_3\frac{\partial\left(\kappa-k\right)}{\partial s}+\frac{\partial^2 g_4}{\partial s^2}.\end{aligned}$$

Theorem 3.3. *Let the flow of $\zeta\left(u,t\right)$ be inextensible. Then the evolution equation of r is*

$$\frac{\partial r}{\partial t}=g_2\kappa r-\frac{\partial g_3}{\partial s}\kappa+g_4\kappa\left(\kappa-k\right)-\frac{\partial \psi_1}{\partial s}+\psi_2\left(\kappa-k\right).$$

Proof. Noticing that $\frac{\partial}{\partial s}\left(\frac{\partial N_1}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial N_1}{\partial s}\right)$, it is seen that

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial N_1}{\partial t}\right)&=\left(-\frac{\partial g_1}{\partial s}\kappa-g_1\frac{\partial \kappa}{\partial s}+\frac{\partial^2 g_2}{\partial s^2}+\frac{\partial g_3}{\partial s}r+g_3\frac{\partial r}{\partial s}\right)T\\&+\left(-g_1\kappa^2+\frac{\partial g_2}{\partial s}\kappa+g_3\kappa r+\psi_1 k\right)N\\&+\left(\frac{\partial \psi_1}{\partial s}-\psi_2\left(\kappa-k\right)\right)N_2\\&+\left(\psi_1\left(\kappa-k\right)+\frac{\partial \psi_2}{\partial s}\right)N_3\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial N_1}{\partial s}\right) &= \frac{\partial}{\partial t}\left(-\kappa T - rN_2\right)\\
&= \left(-\frac{\partial \kappa}{\partial t} - g_2 r^2 + \frac{\partial g_3}{\partial s} r - g_4 r\left(\kappa - k\right)\right) T\\
&+ \left(-g_1\kappa^2 - \frac{\partial g_2}{\partial s}\kappa - g_3\kappa r + \psi_1 r\right) N_1\\
&+ \left(g_2 r\kappa - \frac{\partial g_3}{\partial s}\kappa + g_4\kappa\left(\kappa - k\right) - \frac{\partial r}{\partial t}\right) N_2\\
&+ \left(-g_3\kappa\left(\kappa - k\right) - \frac{\partial g_4}{\partial s}\kappa - \psi_3 r\right) N_3.\end{aligned}$$

From above equations, we get

$$\frac{\partial r}{\partial t} = g_2\kappa r - \frac{\partial g_3}{\partial s}\kappa + g_4\kappa\left(\kappa - k\right) - \frac{\partial \psi_1}{\partial s} + \psi_2\left(\kappa - k\right).$$

□

Corollary 3.2. *In theorem (3.3), from rest of the equality, we obtain*

$$\psi_1\left(\kappa - k\right) = -\frac{\partial \psi_2}{\partial s} - g_3\kappa\left(\kappa - k\right) - \frac{\partial g_4}{\partial s}\kappa - \psi_3 r.$$

Theorem 3.4. *Let the flow of $\zeta\left(u, t\right)$ be inextensible. Then the evolution equation of $\left(\kappa - k\right)$ is*

$$\frac{\partial\left(\kappa - k\right)}{\partial t} = -\psi_2 r + \frac{\partial \psi_3}{\partial s}.$$

Proof. Noticing that $\frac{\partial}{\partial s}\left(\frac{\partial N_2}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial N_2}{\partial s}\right)$, it is seen that

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial N_2}{\partial t}\right) &= \left(\frac{\partial g_2}{\partial s} r + g_2\frac{\partial r}{\partial s} - \frac{\partial^2 g_3}{\partial s^2} + \frac{\partial g_4}{\partial s}\left(\kappa - k\right) + g_4\frac{\partial\left(\kappa - k\right)}{\partial s} + \psi_1\kappa\right) T\\
&+ \left(g_2\kappa r - \frac{\partial g_3}{\partial s}\kappa + g_4\kappa\left(\kappa - k\right) - \frac{\partial \psi_1}{\partial s}\right) N_1\\
&+ \left(\psi_1 r - \psi_3\left(\kappa - k\right)\right) N_2\\
&+ \left(\frac{\partial \psi_3}{\partial s}\right) N_3\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial N_2}{\partial s}\right) &= \frac{\partial}{\partial t}\left(rN_1 + \left(\kappa - k\right) N_3\right)\\
&= \left(-g_1 r\kappa - \frac{\partial g_2}{\partial s} r - g_3 r^2 - g_3\left(\kappa - k\right)^2 - \frac{\partial g_4}{\partial s}\left(\kappa - k\right)\right) T\\
&+ \left(\frac{\partial r}{\partial t} - \psi_2\left(\kappa - k\right)\right) N_1\\
&+ \left(\psi_1 r - \psi_3\left(\kappa - k\right)\right) N_2\\
&+ \left(\psi_2 k + \frac{\partial\left(\kappa - k\right)}{\partial t}\right) N_3.\end{aligned}$$

From above equations, we obtain

$$\frac{\partial\left(\kappa - k\right)}{\partial t} = -\psi_2 r + \frac{\partial \psi_3}{\partial s}.$$

□

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Affiliations

ÖNDER GÖKMEN YILDIZ
ADDRESS: Bilecik Şeyh Edebali University, Dept. of Mathematics, Bilecik-Türkiye
E-MAIL: ogokmen.yildiz@bilecik.edu.tr
ORCID ID:0000-0002-2760-1223

HAZER USTA
ADDRESS: Bilecik Şeyh Edebali University, Dept. of Mathematics, Bilecik-Türkiye
E-MAIL: usta2611@gmail.com
ORCID ID: 0009-0001-7164-2137

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Taylor's Power Law and Packing Circles

Oğuzhan Kaya* and Zeki Topçu

Abstract
The famous Taylor Power Law is in general observed in ecology and relates the variance of the population of a certain species in a unit area while Circle Packing is an arrangement of circles in a given area. We show that the circle packing problem in $\mathbb{R}^2$ satisfies the Taylor power law formula for $b = 2$.

Keywords: *Density, Distance, Packing circles, Population, Taylor's Law*
AMS Subject Classification (2020): *60C05; 62-07; 65K05*
****Corresponding author***

1. Introduction

In [1], the author presented a linear relationship between the expectation and the variance of a population size in a complex system. Since then, this relation stated explicitly as

$$\text{variance} = a(\text{mean})^b \text{ with } \ a, b > 0.$$

is called Taylor's Power Law (abbreviated as TPL) and has been observed in various ecological and biological systems, including populations of animals, plants, and microorganisms. The exponent b in TPL for the majority of these analyzed systems ranges from 1 to 2, with a clustering around $b = 2$. Different models have been investigated thus far, but no clear cause for this occurrence has yet been found. Our approach here may be a reference to that phenomenon. Note that when $b = 1$ the population is distributed homogeneously across space. In order to predict how populations will behave over time or in determining the spatial distribution of populations TPL is helpful.

In this study, which aims to address the spatial distribution of individuals within a population, we associate TPL with another important concept *the circle packing problem*(abbreviated as CPP), which is about optimizing the maximum radius of n ($n \geq 1$) identical circles placed in a closed region in $\mathbb{R}^d$ ($d \geq 2$) such that none of the circles in the region overlap. There are several variations of CPP, including the problem where circles must be placed within a specific shape or the sizes of the circles are not all equal. The reader can find various packing representations of circles in [2] when $d = 2$; for example, if the closed region is a square in $\mathbb{R}^2$, in the case where $n = 1$ there exists a unique circle in the packing and the radius of the circle is 0.5. In the case where $n = 7$, the best packing is given in Figure 1 below. The best packing means that the region contains the largest number of non-overlapping identical

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circles. By [2], the radius r of each circle in the packing is approximately 0.1744576302 and the greatest distance between two centers is approximately 0.535898384. In [2], the CPP is solved up to $n = 10000$ circles inside the different shapes in $\mathbb{R}^2$.

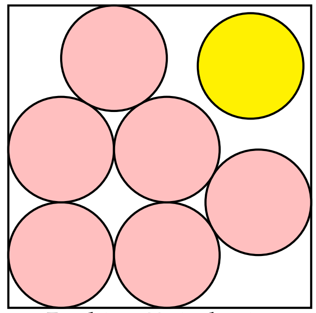


Figure 1. Packing 7 circles in a square

In dimension 3, the circle packing problem becomes the sphere packing problem which begins with a conjecture of Kepler and solved in [3]. So far, the problem has been solved up to the case where $d = 24$.

Here we will see the circle packing problem as the distribution of points in a closed region. More precisely, suppose we are trying to place n distinct points in a closed region in $\mathbb{R}^d$ such that the minimum distance between any two points is as large as possible. Assuming each of these points to be the center of a circle, the distribution of points in this closed region coincides with the problem of finding the radius of circles in the circle packing problem. we show that the distance between the centers of two randomly chosen circles in a packing obeys TPL.

Here, the TPL formula, which has been applied to explain the demographic structure of a living species (insects, microorganisms, humans) is actually thought to be related to the CPP. Our result mainly based on [4] in which the author established TPL as an important tool for understanding population dynamics and spatial patterns in different fields. In the next section, we study the probability distribution of the distance between two randomly chosen points on a line, on a circle and also on a square in $\mathbb{R}^2$. We assume that the distribution of distances between randomly chosen points is independent and uniformly distributed in the fixed region. In the rest of the work, we present the relationship between the expectation and the variance of the distance between the centers of the circles placed in a square with respect to the optimization of the packing and, we show that CPP satisfies TPL.

2. Distance between points in a fixed region and TPL

Let ℓ be a line in $\mathbb{R}^2$ of length $L > 0$. The choice of a randomly chosen point on ℓ is given by a random variable X_1 with the probability density function

$$f_{X_1}(x) = \begin{cases} \frac{1}{L} & \text{if } x \in [0, L] \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Now let us choose a second point on ℓ. It gives the random variable X_2. Obviously, the distance $Y = |X_1 - X_2|$ between the points will also be a random variable. The probability density function of Y is known to be

$$f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{L^2}$$

Proposition 2.1. *With preceding notation, the random variable Y obeys TPL.*

Proof. Consider

$$\varphi(x_1, x_2) = |x_1 - x_2| = \begin{cases} x_1 - x_2, & \text{if } x_1 \geq x_2 \\ x_2 - x_1, & \text{if } x_2 \geq x_1 \end{cases} \tag{2.2}$$

The expected value of the distance between two randomly chosen points is

$$E(Y) = E(\varphi(x_1, x_2)) = \int_0^L \int_0^L \varphi(x_1, x_2) f_{X_1X_2}(x_1, x_2) dx_1 dx_2$$

$$= \frac{1}{L^2}\int_0^L\int_0^L |x_1 - x_2| dx_2 dx_1 = \frac{L}{3}$$

so the variance is

$$Var(Y) = \frac{1}{L^2}\int_0^L\int_0^L |x_1 - x_2|^2 dx_2 dx_1 - \frac{L^2}{9} = \frac{L^2}{18}$$

Hence Y obeys TPL with the values $b = 2$ and $a = \frac{1}{2}$. □

Let $\mathcal{C}$ be a circle with radius $r > 0$ in $\mathbb{R}^2$. By [4, 5], the probability density function of the distance between two randomly selected points on $\mathcal{C}$ is

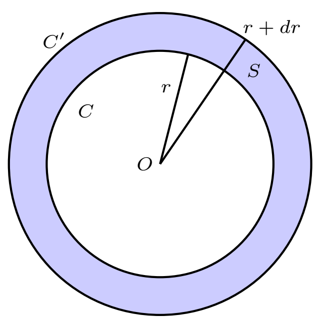
$$f(x) = \frac{4x}{\pi r^2}\left(\arccos\left(\frac{x}{2r}\right) - \frac{x}{2r}\left(1 - \frac{x^2}{4r^2}\right)^{\frac{1}{2}}\right).$$

Proposition 2.2. *The distance between two random points on $\mathcal{C}$ obeys TPL.*

Proof. Let us choose two points $P_1 = (T_1, \Theta_1)$ and $P_2 = (T_2, \Theta_2)$ (in polar coordinates) on $\mathcal{C}$. The randomness of the selection tells us that the probability of one of the points lying in the area dA is proportional to dA:

$$\mathbb{P}\{T_i \in (r_i, r_i + dr_i), \Theta_i \in (\theta_i, \theta_i + d\theta_i)\} = \frac{r_i dr_i d\theta_i}{\pi r^2}, i = 1, 2.$$

Let Y be the distance between P_1 and P_2 which belongs to the interval $(x, x + dx)$. Consider another circle $\mathcal{C}'$ with the same center as $\mathcal{C}$. So, its radius is $r + dr$. Denote by S the annulus between two circles. If two points are in C' we have one of the following cases:
(i) Both points are in C,
(ii) At least one point is in S.



The probability that two points are in C is

$$\mathbb{P}\{r + dr\} = \mathbb{P}\{r + dr| \text{ case}(i) \} \times \mathbb{P}\{ \text{case}(i) \} + \mathbb{P}\{r + dr| \text{ case}(ii) \} \times \mathbb{P}\{ \text{case}(ii) \} \tag{2.3}$$

Let us consider each point separately to compute $\mathbb{P}\{r + dr| \text{ case}(i) \}$:

$$\mathbb{P}\{P_1 \text{ is in } C\} = \frac{\text{area(C)}}{\text{area(C')}} = \frac{\pi r^2}{\pi(r + dr)^2} == \frac{1}{1 + 2dr/r + dr^2/r^2} = 1 - \frac{2dr}{r} + o(dr)$$

Since the cases (i) and (ii) are independent we obtain

$$\mathbb{P}\{\text{case}(i)\} = (1 - \frac{2dr}{r} + o(dr))^2 = 1 - \frac{4dr}{r} + o(dr)$$

Hence $\mathbb{P}\{r + dr \mid \text{ case}(ii)\} = \dfrac{2xdx}{\pi r^2}\arccos\dfrac{x}{2r}$ and $\mathbb{P}\{ \text{case}(ii)\} = \frac{4dr}{r} + o(dr)$.

Substitution of these values in (3) gives

$$\mathbb{P}\{r+dr\} = \mathbb{P}\{r\}\left(1-\frac{4dr}{r}\right) + \frac{2xdx}{\pi r^2}\arccos\left(\frac{x}{2r}\right)\left(\frac{4dr}{r}\right) + o(dr)$$

Denote by P. We then get

$$dP = \mathbb{P}\{r+dr\} - \mathbb{P}\{r\} = \left[\frac{-4P}{r} + \frac{8x\ dx}{\pi r^3}\arccos\left(\frac{x}{2r}\right)\right]dr + o(dr)$$

$$r^4 dP + 4r^3 P dr = \frac{8x\ dx\ r}{\pi}\arccos\left(\frac{x}{2r}\right)dr + o(dr)$$

$$\frac{d}{dr}(Pr^4) = \frac{8x\ dx\ r}{\pi}\arccos\left(\frac{x}{2r}\right)$$

The integration of both sides gives

$$Pr^4 = \frac{4x^2dx}{\pi}\int \frac{2r}{x}\arccos\left(\frac{x}{2r}\right)dr + C$$

Therefore,

$$Pr^4 = \frac{4x^2dx}{\pi}\left(\frac{\arccos\left(\frac{x}{2r}\right)r^2}{x} - \frac{\sqrt{4r^2-x^2}}{4}\right)$$

$$P = \frac{4xdx}{\pi r^2}\left(\arccos\left(\frac{x}{2r}\right) - \frac{x}{2r}\left(1-\frac{x^2}{4r^2}\right)^{\frac{1}{2}}\right)$$

Now let us compute $E(Y)$ where Y is the distance between the points.

$$E(Y) = \int_0^{2R} x\frac{4x}{\pi R^2}\left(\arccos\left(\frac{x}{2R}\right) - \frac{x}{2R}\left(1-\frac{x^2}{4R^2}\right)^{\frac{1}{2}}\right)dx$$

First, replace $\frac{x}{2R} = u$ for computing $I_1 = \dfrac{4}{\pi R^2}\int x^2 \arccos\left(\dfrac{x}{2R}\right)dx$, :

$$I_1 = \frac{4}{\pi R^2}8R^3\int u^2\arccos(u)du$$

Integration by parts with $f = arccos(u), g' = u^2$ gives

$$I_1 = \frac{4}{\pi R^2}8R^3\left(\frac{u^3 arccos(u)}{3} + \frac{(1-u^2)^{\frac{3}{2}}}{9} - \frac{\sqrt{1-u^2}}{3}\right)$$

$$I_1 = \frac{4}{\pi R^2}\left(\frac{x^3\arccos(\frac{x}{2R})}{3} + \frac{8R^3(1-(\frac{x}{2R})^2)^{\frac{3}{2}}}{9} - \frac{8R^3\sqrt{1-(\frac{x}{2R})^2)^2}}{3}\right)$$

$$I_1 = -\frac{4\left(\sqrt{4R^2-x^2}\left(Rx^2+8R^3\right) - 3\,|R|\,x^3\arccos\left(\frac{x}{2R}\right)\right)}{9\pi R^2\left|R(\frac{x}{2R})\right|}$$

Substitute $u = 4R^2 - x^2$. We get

$$I_2 = -\int \frac{2x^3\sqrt{1-\frac{x^2}{4R^2}}}{\pi R^3}dx = -\frac{1}{\pi R^4}\int x^3\sqrt{4R^2-x^2}$$

$$= -\frac{1}{\pi R^4}\left(\frac{1}{2}\int u^{\frac{3}{2}} - 4R^2\sqrt{u}\,du\right) = \frac{\left(24x^2+64R^2\right)\left(1-\frac{x^2}{4R^2}\right)^{\frac{3}{2}}}{15\pi R}$$

Since $E(Y) = (I_1 + I_2)|_0^{2R}$ we finally obtain

$$E(Y) = \left(-\frac{\sqrt{4R^2 - x^2}\left(9x^4 + 8R^2x^2 + 64R^4\right) - 60R^2x^3 \arccos\left(\frac{x}{2R}\right)}{45\pi R^4} \right)\Bigg|_0^{2R} = \frac{128}{45\pi} R$$

Hence the mean is $E(Y) \approx 0.9054R$. Let us compute the variance of Y: Let $x_1 = (x, y)$ and $x_2 = (x', y')$. So, the square of the distance between x_1 and x_2 is:

$$d^2(x_1, x_2) = (x - x')^2 + (y - y')^2$$

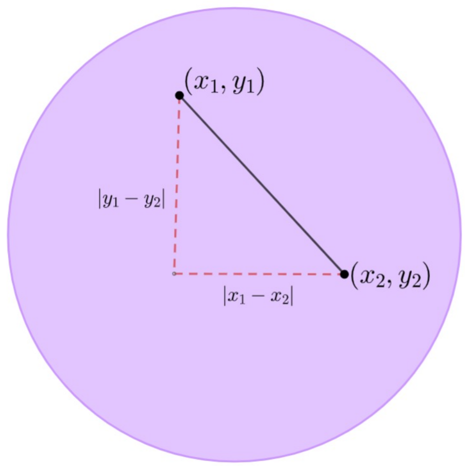


Figure 2. The distance on a circle

Therefore,

$$E(Y^2) = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (x - x')^2 + (y - y')^2 \, dx \, dy \, dx' \, dy'$$

By the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x' = r'\cos\theta'$, $y' = r'\sin\theta'$ we get

$$E(Y^2) = \frac{1}{\pi^2 R^4}\int_0^{2\pi}\int_0^{2\pi}\int_0^{R}\int_0^{R} (r\cos\theta - r'cos\theta')^2 + (r\sin\theta - r'\sin\theta')rr' \, dr \, dr' \, d\theta \, d\theta'$$

$$= \frac{1}{\pi^2 R^4}\int_0^{2\pi}\int_0^{2\pi}\int_0^{R}\int_0^{R} r^3r' + (r')^3 r \, dr \, dr' \, d\theta \, d\theta' = R^2$$

Therefore $Var(Y) = R^2 - \left(\frac{128R}{45\pi}\right)^2 \approx 0.0934R^2$, which concludes the affirmation of the proposition. □

Proposition 2.3. [4] *Let S be a square of size $R > 0$ in $\mathbb{R}^2$. The distance d between two randomly selected points in S obeys TPL.*

Proof. To evaluate the expectation of the distance $d = \sqrt{(x - x')^2 - (y - y')^2}$, without loss of generality, we assume $R = 1$ and calculate the integral

$$I = \int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{(x - x')^2 + (y - y')^2} \, dx' \, dy' \, dy \, dx$$

By symmetry, we write:

$$I = 4\int_0^1\int_0^1\int_0^y\int_0^x \sqrt{(x - x')^2 + (y - y')^2} \, dx' \, dy' \, dy \, dx$$

First substitute $x' \mapsto xx', \quad y' \mapsto yy'$:

$$I = 4\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{x^2(1-x')^2 + y^2(1-y')^2}yxdx^{'}dy^{'}dydx$$

and then substitute $x' \mapsto 1-x', \quad y' \mapsto 1-y'$ we have:

$$I = 4\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{x^2x'^2 + y^2y'^2}\; yx\, dx'\, dy'\, dydx$$

After another substitution $y^2 = u, \quad x^2 = v$:

$$I = \int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{vx'^2 + uy'^2}\;\; dx'\, dy'\, dudv$$

Finally with $vx'^2 = p, \quad uy'^2 = q$:

$$I = \int_0^1\int_0^1\int_0^{y'^2}\int_0^{x'^2} \sqrt{p+q}\;\, dpdq\frac{dydw}{y^2w^2}$$

$$I = \frac{2}{3}\int_0^1\int_0^1\int_0^{y'^2} \left((q+w^2)^{3/2} - q^{3/2}\right) dq\frac{dy'dx'}{y'^2y^2}$$

$$I = \frac{4}{15}\int_0^1\int_0^1 \left((y'^2+y^2)^{5/2} - y'^5 - y^5\right)\frac{dy'dy}{y'^2y^2}$$

By symmetry:

$$I = \frac{8}{15}\int_0^1\int_0^y \left((y'^2+y^2)^{5/2} - y'^5 - y^5\right)\frac{dy'dy}{y'^2y^2}$$

Substitute $y' = ys$:

$$I = \frac{8}{15}\int_0^1\int_0^1 y^2\left((1+s^2)^{5/2} - s^5 - 1\right)\frac{dsdy}{s^2}$$

$$I = \frac{8}{45}\int_0^1 \left((1+s^2)^{5/2} - s^5 - 1\right)\frac{ds}{s^2}$$

$$I = \left.\frac{15s\ln\left(\left|\sqrt{s^2+1}+s\right|\right) - 2s^5 + \sqrt{s^2+1}\left(2s^4+9s^2-8\right)+8}{45s}\right|_0^1$$

$$I = \frac{5\,\mathrm{arsinh}\,(1) + \sqrt{2} + 2}{15}$$

This says that the mean of d is $E(d) = \dfrac{R}{15}(\mathrm{arsinh}\,(1) + \sqrt{2} + 2) \approx 0.5214R$. Hence the variance $Var(d)$ is

$$E(d^2) - E(d)^2 = \frac{R^2}{3} - \left(\frac{R}{15}\left(\mathrm{arsinh}\,(1) + \sqrt{2} + 2\right)\right)^2 \approx 0.0615R^2$$

where

$$E(d^2) = \frac{1}{(R^2)^2}\int_0^R\int_0^R\int_0^R\int_0^R (x-x')^2 + (y-y')^2\, dx\, dx'\, dy\, dy'$$

Therefore TPL is $Var(d) = a(E(d))^b$ which is satisfied for $b = 2$. As expectation is a function of R where variance is a function of R^2. □

Listing 1. Python simulation of mean and variance calculation

```
import math
import matplotlib.pyplot as plt
import pandas as pd
import os
from os.path import exists
import glob
from itertools import combinations

# Auxiliar functions

def Read_file(file_name):
    with open(file_name) as file:
        points = [(float(line.split()[-1]), float(line.split()[-2])) for line in
file]
    return points

def Mean_and_Variance(file_name):
    distances = []
    points = Read_file(file_name)
    for p1, p2 in combinations(points, 2):
        distances.append(math.sqrt((p1[0] - p2[0])**2 + (p1[1] - p2[1])**2))
    mean = sum(distances) / len(distances)
    variance = sum(d**2 for d in distances) / len(distances) - mean**2
    return mean, variance

# Execution of the code

file_list = sorted(glob.glob("/path/to/files/*.txt"), key=os.path.getsize)[1:]
means = []
variances = []
for file_name in file_list:
    mean, variance = Mean_and_Variance(file_name)
    means.append(mean)
    variances.append(variance)

coefA = [v / (e**2) for e, v in zip(means, variances)]
```

3. Main result

In this section, we answer the following question:
(*) *Does the distance between the centers of the randomly chosen circles in a best packing in a square obeys TPL?*
Let us consider a square in $\mathbb{R}^2$ and let n be the number of circles in a best packing. Using the data from [2] (In the page, circles in square is used) , we proceed as follows:

1st operation. Assign the coordinates to n points $P_1, P_2 \dots P_n$ each of which represent the center of a circle in the best packing. For example, for $n = 7$ we list the data as

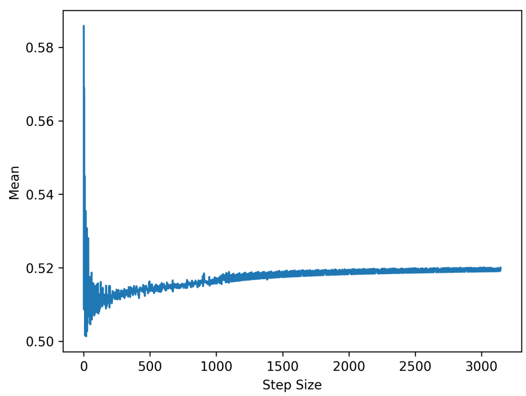
Circle	x-coordinate	y-coordinate
Circle 1	-0.325542369812990561040572795501	-0.325542369812990561040572795501
Circle 2	0.023372890561028316878281613499	-0.325542369812990561040572795500
Circle 3	0.325542369812990561040572795500	-0.151084739625981122081145591001
Circle 4	-0.325542369812990561040572795500	0.023372890561028316878281613499
Circle 5	0.023372890561028316878281613499	0.023372890561028316878281613499
Circle 6	0.300000000000000000000000000000	0.300000000000000000000000000000
Circle 7	-0.151084739625981122081145591001	0.325542369812990561040572795500

2nd operation. Make a list of pairs (P_i, P_j) for all $i \neq j$.
3rd operation. Compute the distance d_{ij} between each pair (P_i, P_j) and transfer the results to the list named as "distances".
4th operation. Compute the mean and the variance using the d_{ij}'s in the list "distances" and transfers the results to the lists named "AllMeans" and "AllVariances" respectively.
5th operation. Store the mean and variance values in the same level in "AllMeans" and "AllVariances" respectively. Then, compute the coefficient a in the formula TPL. Transfer the result to the list named "coefA".
6th operation. Constructing a loop on n. Note that, in [2], the author presents 3146 packings.

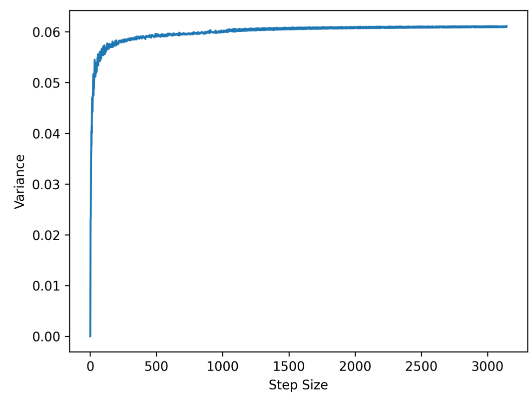
Theorem 3.1. *The distance between two randomly chosen centers in a best circle packing satisfies TPL.*

Proof. We will present a visual proof with results that we get from previous simulations. Figure 3a below resulting from our algorithm represent the change of means with respect to the change of number of circles. On the other hand, the Figure 3b the change of variances with respect to the change of number of circles.

Since the data in [2] contains 3146 packings, the step size is not 10000. The graph in Figure 4 shows the change of variance with respect to the change of the number of circles.



(a) Mean change by step



(b) Variance change by step

Figure 3. Mean and variance change by step

Fixing $b = 2$ in the formula TPL, the graph representing the change of coefficient a with respect to the change of number of circles shows that a converges which concludes the empirical proof of TPL in our specific case.

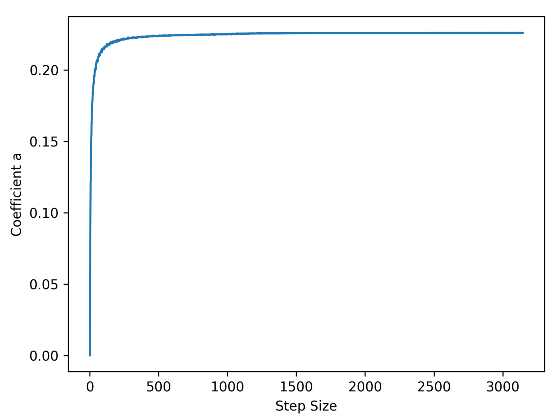


Figure 4. Convergence of *a*

□

Taylor's Power Law can be applied to the population density problems of a city or country ([6]). In this paper, we showed that Circle Packing can be used as another method for population density problems.

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Affiliations

Oğuzhan Kaya
Address: Galatasaray University, Dept. of Mathematics, İstanbul-Turkey
e-mail: okaya@gsu.edu.tr
ORCID ID: 0000-0001-9756-1346

Zeki Topçu
Address: Ecole Normale Superieure, Mathematics Vision and Learning , Paris-France
e-mail: zekitopccu@gmail.com
ORCID ID: 0000-0002-8910-5612

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Lacunary Statistically Convergence via Modulus Function Sequences

Erdal Bayram* and Çiğdem A. Bektaş

Abstract

Modifying the definition of density functions is one method used to generalise statistical convergence. In the present study, we use sequences of modulus functions and order $\alpha \in (0,1]$ to introduce a new density. Based on this density framework, we define strong (f_k)-lacunary summability of order α and (f_k)-lacunary statistical convergence of order α for a sequence of modulus functions (f_k). This concept holds an intermediate position between the usual convergence and the statistical convergence for lacunary sequences. We also establish inclusion theorems and relations between these two concepts in the study.

Keywords: *Lacunary statistical convergence, Lacunary summability, Modulus function, Weighted density*

AMS Subject Classification (2020): *40A05; 40A35; 46A45*

****Corresponding author***

1. Introduction

The concept of statistical convergence was initially proposed by Zygmund [1] in his research. Independently, Steinhaus [2] and Fast [3] also introduced this idea. Subsequently, Schoenberg [4] and numerous other mathematicians further explored and analyzed this concept. Statistical convergence and some derived concepts were introduced and studied in a variety of sequences. Following the demonstration of statistical convergence, the subject has been approached from many angles and various extensions have been produced. In particular, using functions belonging to different classes and sequences belonging to some classes, classes of sequences with statistical convergence have been derived. Meanwhile, it has been established that there is a relationship between statistical convergence and Cesàro summability, and this relationship has been revealed. Since the pioneering studies of Salat [5] and Fridy [6], statistical convergence has become a highly active area of research within summability theory.

The concept of asymptotic (or natural) density is the fundamental tool in statistical convergence, and it is defined for a set $K \subseteq \mathbb{N}^+$ as $\delta(K) = \lim_{n\to\infty} n^{-1} |\{k \leq n : k \in K\}|$ whenever the limit exists. Here, the vertical bars indicate the cardinality of the enclosed set. So, $\delta(A) = 0$ for the finite set A, $\delta(\mathbb{N} \diagdown A) = 1 - \delta(A)$ and $\delta(A) \leq \delta(B)$ whenever $A \subseteq B$.

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Base on the concept of natural density, a sequence of numbers (x_k) is said to be statistical convergent to some number x if for each positive number $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-1} \left|\{k \le n : |x_k - x| \ge \varepsilon\}\right| = 0$$

whenever limit exists. In that case, x is called statistical limit of (x_k) and is written as $S - \lim x_k = x$ or $x_k \to x\,(S)$.

In literature, there exists generalizations of statistical convergence. For instance, a sequence (x_k) is statistically convergent of order $\alpha \in (0, 1]$ to some number x if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-\alpha} \left|\{k \le n : |x_k - x| \ge \varepsilon\}\right| = 0.$$

whenever limit exists (see [7] and [8]).

All statistically convergent sequences and all statistically convergent sequences of order α will be denoted by S and S^{α} respectively.

The notions of lacunary summability and convergence with lacunary sequences were established by Fridy and Orhan ([9] and [10]). A lacunary sequence $\theta = (k_r)_{r\in\mathbb{N}}$ is an increasing sequence of integers such that $k_0 = 0$ and $\lim\limits_{n\to\infty}(k_r - k_{r-1}) = \infty$. For lacunary sequences, we use the notations $h_r = k_r - k_{r-1}$, $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. For the sake of brevity, the set of all lacunary sequences of integers will be denoted by $\mathcal{LS}(\mathbb{Z})$.

A sequence (x_k) is lacunary statistically convergent and, respectively, lacunary statistically convergent of order α to some number L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : |x_k - L| \ge \varepsilon\}\right| = 0$$

and, respectively,

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \left|\{k \in I_r : |x_k - L| \ge \varepsilon\}\right| = 0$$

whenever limit exists. All lacunary statistically convergent sequences and all lacunary statistically convergent sequences of order α are denoted by S_θ and S_θ^{α}, respectively. Lacunary statistically convergent, lacunary boundedness order α and strongly summable sequences of order α have been studied by Connor [11], Çolak [12], Şengül and Et in [13], [14]. Pehlivan and Fisher [15] introduce the concept of lacunary strong convergence with respect to a sequence of modulus functions in a Banach spaces.

The modulus function is the other idea we employ in our research. A function $f : [0, \infty) \to [0, \infty)$ is referred modulus provided that the following conditions hold:

i. $f(v) = 0 \Leftrightarrow v = 0$

ii. $f(v_1 + v_2) \le f(v_1) + f(v_2)$ for every $v_1, v_2 \in [0, \infty)$

iii. f is increasing

iv. f is continuous from the right at 0.

Although the continuity of any modulus function is obvious, a modulus function need not to be bounded. For instance, the modulus $f(v) = \log(v + 1)$ is unbounded, while $g(v) = \frac{v}{v+1}$ is a bounded modulus function. For any modulus f and for every $m \in \mathbb{N}^+$, the inequality $f(mv) \le mf(v)$ and so that $f(m) \le mf(1)$ holds from the condition (ii). The notion of modulus was first established by Nakano [16] and subsequently, Ruckle [17], established a new sequence spaces by a modulus function f and these sequence spaces were then used in many researches (for example see [18], [19], [20], [21]).

The space of sequences of unbounded modulus functions $F = (f_k)$ such that $\lim\limits_{u\to 0^+} \sup\limits_{k\in\mathbb{N}} f_k(u) = 0$ will be denoted by $\mathcal{M}^{ub}$.

Changing definition of the density function is one method used to distinguish the statistical convergence. Researchers have explored various generalizations of the concept of asymptotic density. One of these is the density $f-$ given by Aizpuru et al. [22], which is obtained by employing modulus functions.

Definition 1.1. [22] Let f be an unbounded modulus function. The f-density of a set $\mathbb{N}$ is defined by

$$d_f(A) = \lim_{n\to\infty} \frac{f(|A|)}{f(n)}$$

in case this limit exists.

Base on this density, Aizpuru et al. [22] defined f -statistical convergence in normed space as follows.

Definition 1.2. [22] Let f be an unbounded modulus function. The sequence (x_n) in the normed space X is called $f-$statistical convergence to $x \in X$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{f(|\{n \in \mathbb{N} : \|x_n - x\| > \varepsilon\}|)}{f(n)} = 0.$$

Obviously, if the modulus function is the identity function, f-statistical convergence coincides with statistical convergence, and since the f-density of a finite set is zero, topological convergence coincides with f-statistical convergence. Consequently, f-statistical convergence lies between ordinary convergence and statistical convergence. Recently, Bhardwaj and Dhawan [23] proposed f-statistical convergence of order α and strong Cesàro summability of order α with respect to a modulus f, using the f_α-density of a set $A \subseteq \mathbb{N}$. León-Saavedra [24] proved results related to a characterization of the modulus f for cases where f-strong Cesàro convergence coincides with f-statistical convergence and uniform integrability. In addition, İbrahim and Çolak [25] introduced strong lacunary summability of order α via a modulus function.

This paper aims to introduce and study the concept of lacunary statistical convergence and lacunary summability according to a sequence of modulus for number sequences, using f_α-density. This study is motivated by the work of Pehlivan and Fisher [15], Bhardwaj and Dhawan [23], and İbrahim and Çolak [25].

2. Main results

For each $\alpha \in \mathbb{R}$ such that $\alpha > 1$, lacunary statistical convergence is not well defined (see [14], [13]). Therefore, in the rest of article, we consider the case $\alpha \in (0, 1]$.

2.1 Lacunary summability using a sequence of modulus

We proposed a slight generalisation of strongly lacunary summability of order α by using a sequence of modulus functions. Depending on this definition, inclusion relations are given under certain conditions.

Definition 2.1. Suppose $F = (f_k) \subset \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$. The sequence $(z_k) \subset \mathbb{C}$ is strongly $F^\alpha-$lacunary summable (briefly $N_\theta^\alpha(F)-$summable) to some $L \in \mathbb{C}$ provided that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} f_k(|z_k - L|) = 0.$$

holds and this is denoted by $z_k \to L(N_\theta^\alpha(F))$ or $N_\theta^\alpha(F) - \lim_k z_k = L$. The set of all $N_\theta^\alpha(F)$-summable sequences is denoted by $N_\theta^\alpha(F)$, i.e.

$$N_\theta^\alpha(F) = \{(z_k) : \lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} f_k(|z_k - L|) = 0 \text{ for some } L \in \mathbb{C}\}.$$

Remark 2.1. Note that in this definition, the modulus functions f_k are not required to be unbounded. On the other hand, for a sequence of modulus functions $F = (f_k)$,

i. $N_\theta^\alpha(F)$- summability is reduced to $N_\theta-$ summability in the particular case $\alpha = 1$ and $f_k(v) = v$ for all $k \in \mathbb{N}$ (see [26]).

ii. $N_\theta^\alpha(F)$- summability is reduced to $N_\theta^\alpha-$ summability in the particular case $f_k(v) = v$ for all $k \in N$ (see [7]).

iii. $N_\theta^\alpha(F)$- summability is reduced to $N_\theta^\alpha(f)$ $-$summability in the particular case $\alpha = 1$ and $f_k = f$ for all $k \in N$ and for a modulus function f (see [15]).

Theorem 2.1. *Suppose $F = (f_k) \subset \mathcal{M}^{ub}$ and $G = (g_k) \subset \mathcal{M}^{ub}$, $\alpha_1, \alpha_2 \in (0, 1]$ such that $\alpha_1 \leq \alpha_2$ and $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$.*

i. *If* $\sup\limits_{u,k} \frac{f_k(u)}{g_k(u)} < \infty$ *holds then* $N_\theta^{\alpha_1}(G) \subset N_\theta^{\alpha_2}(F)$.

ii. *If* $\inf\limits_{u,k} \frac{f_k(u)}{g_k(u)} > 0$ *holds then* $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}(G)$.

iii. *If* $0 < \inf\limits_{u,k} \frac{f_k(u)}{g_k(u)} \leq \sup\limits_{u,k} \frac{f_k(u)}{g_k(u)} < \infty$ *holds then* $N_\theta^{\alpha_1}(F) = N_\theta^{\alpha_1}(G)$.

Note that infimum and supremum are taken over all $u \in (0, \infty)$ *and* $k \in \mathbb{N}$.

Proof. Choose $z = (z_k) \in N_\theta^{\alpha_1}(G)$. If $p = \sup\limits_{u,k} \frac{f_k(u)}{g_k(u)} < \infty$ holds then $0 < \frac{f_k(u)}{g_k(u)} \leq p$ and hence $f_k(u) \leq p g_k(u)$ holds for all $k \in \mathbb{N}$ and for any $u \in \mathbb{R}^+ \cup \{0\}$. On the other hand, since $0 < \alpha_1 \leq \alpha_2 \leq 1$, we have the following inequalities:

$$\frac{1}{h_r^{\alpha_2}} \sum_{k \in I_r} f_k(|z_k - l|) \leq \frac{1}{h_r^{\alpha_1}} \sum_{k \in I_r} f_k(|z_k - l|) \leq \frac{1}{h_r^{\alpha_1}} \sum_{k \in I_r} p g_k(|z_k - l|)$$

Taking limit as $r \to \infty$, strongly $N_\theta^{\alpha_1}(G)$ $-$summability to $l \in \mathbb{C}$ of (z_k) implies that $z = (z_k) \in N_\theta^{\alpha_1}(F)$.

In the proof of (ii), *if* $q = \inf\limits_{u,k} \frac{f_k(u)}{g_k(u)} > 0$ holds then $g_k(u) \leq \frac{1}{q} f_k(u)$ for every $u \in \mathbb{R}^+ \cup \{0\}$ and for all $k \in \mathbb{N}$. Thus, the rest of the proof is exactly similar to (i). Moreover, (iii) is a consequence of (i) and (ii). □

Remark 2.2. Let us choose $\alpha_1 = \alpha_2 = 1$ and $F = (f_k) \subset \mathcal{M}^{ub}$ and $G = (g_k) \subset \mathcal{M}^{ub}$ such that $f_k(u) = \frac{ku}{u+1}$ and $g_k(u) = 2^k u$ for all $k \in \mathbb{N}$. Considering the sequence in Example 3.1 in [25], we obtain that the inclusion $N_\theta^{\alpha_1}(G) \subset N_\theta^{\alpha_2}(F)$ is strict.

Corollary 2.1. *Suppose that* $F = (f_k) \subset \mathcal{M}^{ub}$ *and* $G = (g_k) \subset \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$, *and* $\alpha_1, \alpha_2 \in (0, 1]$ *such that* $\alpha_1 \leq \alpha_2$. *Then the following assertions hold:*

i. *If* $\sup\limits_{u,k} \frac{f_k(u)}{g_k(u)} < \infty$ *or* $\inf\limits_{u,k} \frac{g_k(u)}{f_k(u)} > 0$ *then* $N_\theta^{\alpha_1}(G) \subset N_\theta^{\alpha_1}(F)$,

ii. *If* $\sup\limits_{u,k} \frac{f_k(u)}{g_k(u)} < \infty$ *or* $\inf\limits_{u,k} \frac{g_k(u)}{f_k(u)} > 0$ *then* $N_\theta(G) \subset N_\theta(F)$,

iii. $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}(F)$.

Note that, supremum is taken over all $u \in (0, \infty)$ *and* $k \in \mathbb{N}$ *in* (i) *and* (ii).

Corollary 2.2. *Suppose that* $F = (f_k) \subset \mathcal{M}^{ub}$ *and* $G = (g_k) \subset \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$, *and* $\alpha_1, \alpha_2 \in (0, 1]$ *such that* $\alpha_1 \leq \alpha_2$. *Then the following assertions hold:*

i. *If* $\sup\limits_{u,k} \frac{f_k(u)}{u} < \infty$ *then* $N_\theta^{\alpha_1} \subset N_\theta^{\alpha_2}(F)$,

ii. *If* $\sup\limits_{u,k} \frac{f_k(u)}{u} < \infty$ *then* $N_\theta^{\alpha_1} \subset N_\theta^{\alpha_1}(F)$,

iii. *If* $\inf\limits_{u,k} \frac{f_k(u)}{u} > 0$ *then* $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}$,

iv. *If* $\inf\limits_{u,k} \frac{f_k(u)}{u} > 0$, *then* $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_1}$

v. *If* $0 < \inf\limits_{u,k} \frac{f_k(u)}{u} \leq \sup\limits_{u,k} \frac{f_k(u)}{u} < \infty$ *then* $N_\theta^{\alpha_1}(F) = N_\theta^{\alpha_1}$.

Corollary 2.3. *Suppose that* $F = (f_k) \subset \mathcal{M}^{ub}$, $\alpha_1, \alpha_2, \gamma \in (0, 1]$ *such that* $\alpha_1 \leq \alpha_2 \leq \gamma$, *and* $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$. *Then the following assertions hold:*

i. *If there exists a modulus function* f *such that* $f_k \leq f$ *for every* $k \in \mathbb{N}$ *then* $N_\theta^{\alpha_1}(f) \subset N_\theta^{\alpha_2}(F)$ *holds,*

ii. *If there exists a modulus function* f *such that* $g \leq f_k$ *for every* $k \in \mathbb{N}$ *then* $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}(g)$ *holds,*

iii. *If there exists a modulus function f and g such that $g \leq f_k \leq f$ for every $k \in \mathbb{N}$ then $N_\theta^{\alpha_1}(f) \subset N_\theta^{\alpha_2}(F) \subset N_\theta^{\gamma}(g)$ hold.*

Proof. The proof of (i) is clear from Theorem 2.1(i) since the inequality $f_k \leq f$ for every $k \in \mathbb{N}$ implies $\sup\limits_{u,k} \frac{f_k(u)}{f(u)} < \infty$. Similarly, the proof of (ii) is follows from Theorem 2.1(ii) since the inequality $g \leq f_k$ for every $k \in \mathbb{N}$ implies $\inf\limits_{u,k} \frac{f_k(u)}{g(u)} > 0$. Hence, (iii) is a consequence of (i) and (ii). □

2.2 Lacunary statistically convergence using a sequence of modulus

In this section, we introduced a new concept of lacunary statistical convergence of order α by using sequences of modulus functions. By some given inclusion theorems, we establish some relations between lacunary summability and lacunary statistical convergence under certain conditions.

We firstly define a density with the help of sequence of modulus functions and order $\alpha \in (0,1]$ as follows:

Definition 2.2. The density of $A \subseteq \mathbb{N}^+$ with respect to a sequence of unbounded modulus functions $F = (f_k) \subset \mathcal{M}^{ub}$ and order $\alpha \in (0,1]$ is defined by the following limit

$$\delta_{F_\alpha}(A) = \lim_{r\to\infty} \frac{f_r(|\{k \leq r : k \in A\}|)}{f_r(r^\alpha)}$$

whenever the limit exists. The abbreviation for this density is referred to as F_α-density.

Remark 2.3. Obviously,

i. If $\alpha = 1$ and $f_k(x) = x$ for all $k \in \mathbb{N}$ then $F_\alpha-$density is reduced to the natural density (see [3]),

ii. If $\alpha \in (0,1]$ and $f_k(x) = x$ for all $k \in \mathbb{N}$ then $F_\alpha-$density is reduced to the $\alpha-$density (see [7]),

iii. If $\alpha = 1$ and $f_k(x) = f(x)$ for all $k \in \mathbb{N}$ and for $f \in \mathcal{M}$ then $F_\alpha-$density is reduced to the $f-$density (see [22]),

iv. If $\alpha \in (0,1]$ and $f_k(x) = f(x)$ for all $k \in \mathbb{N}$ and for $f \in \mathcal{M}$ then $F_\alpha-$density is reduced to the f_α-density (see [23]).

Similar to other density types, we may provide an alternative form of the lacunary statistical convergence in relation to the $F_\alpha-$density in the following manner.

Definition 2.3. Suppose that $F = (f_k) \subset \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$. The sequence $(z_k) \subset \mathbb{C}$ is F^α-lacunary statistically convergent (shortly $S_\theta^\alpha(F)$-convergent) to some $l \in \mathbb{C}$ provided that for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{f_r(h_r^\alpha)} f_r(|\{k \in I_r : |z_k - l| \geq \varepsilon\}|) = 0.$$

holds and this is denoted by $z_k \to l(S_\theta^\alpha(F))$ or $S_\theta^\alpha(F) - \lim_k z_k = l$. The class of all $S_\theta^\alpha(F)$-convergent sequences is denoted by $S_\theta^\alpha(F)$, i.e.

$$S_\theta^\alpha(F) = \{(z_k) : \lim_{r\to\infty} \frac{1}{f_r(h_r^\alpha)} f_r(|\{k \in I_r : |z_k - l| \geq \varepsilon\}|) = 0 \text{ for some } l \in \mathbb{C}\}.$$

Now, we can establish some inclusion theorems between $F-$lacunary summability of order α and $F-$lacunary statistically convergence of order α.

Theorem 2.2. *Suppose that $F = (f_k) \subset \mathcal{M}^{ub}$, $G = (g_k) \subset \mathcal{M}^{ub}$, $\alpha_1, \alpha_2 \in (0,1]$ such that $\alpha_1 \leq \alpha_2$ and $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$. If $\inf\limits_{u,k} \frac{f_k(u)}{g_k(u)} > 0$ and $\lim\limits_{u\to\infty} \frac{g_k(u)}{u} > 0$ for all k, then $N_\theta^{\alpha_1}(F)$-summability implies $S_\theta^{\alpha_2}(G)-$statistically convergence, i.e. $N_\theta^{\alpha_1}(F) \subset S_\theta^{\alpha_2}(G)$.*

Proof. Choose a sequence (z_k) which is $N_\theta^{\alpha_1}(F)$-summable to $l \in \mathbb{C}$. From the assumption, $q = \inf\limits_{u,k} \frac{f_k(u)}{g_k(u)} > 0$ implies that $qg_k(u) \leq f_k(u)$ holds for every $k \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$. Due to (z_k) is $N_\theta^{\alpha_1}(F)-$summable to $l \in \mathbb{C}$, we have □

$$\begin{aligned}\frac{1}{h_r^{\alpha_1}}\sum_{k\in I_r} f_k(|z_k-l|) &\geq q\frac{1}{h_r^{\alpha_1}}\sum_{k\in I_r} g_k(|z_k-l|)\\ &\geq q\frac{1}{h_r^{\alpha_2}}\sum_{k\in I_r} g_k(|z_k-l|)\\ &= q\frac{1}{h_r^{\alpha_2}}\sum_{k\in I_r,|z_k-l|\geq\varepsilon} g_k(|z_k-l|)+q\frac{1}{h_r^{\alpha_2}}\sum_{k\in I_r,|z_k-l|<\varepsilon} g_k(|z_k-l|)\\ &\geq q\frac{1}{h_r^{\alpha_2}}\sum_{k\in I_r,|z_k-l|\geq\varepsilon} g_k(|z_k-l|)\\ &\geq q\frac{1}{h_r^{\alpha_2}}\left|\{k\epsilon I_r:|z_k-l|\geq\varepsilon\}\right|\ g_r(\varepsilon).\end{aligned}$$

where $g_r(\varepsilon)=\inf\limits_{k\in I_r} g_k(\varepsilon)$. Since $|\{k\in I_r:|z_k-l|\geq\varepsilon\}|\in\mathbb{Z}^+$, the following inequality holds:

$$\begin{aligned}\frac{1}{h_r^{\alpha_1}}\sum_{k\in I_r} f_k(|z_k-l|) &\geq \frac{1}{h_r^{\alpha_2}}\inf_{k\in I_r} g_k(|\{k\in I_r:|z_k-l|\geq\varepsilon\}|)\frac{\inf\limits_{k\in I_r} g_k(\varepsilon)}{\inf\limits_{k\in I_r} g_k(1)}q\\ &=\frac{g_r(|\{k\in I_r:|z_k-l|\geq\varepsilon\}|)}{g_r\left(h_r^{\alpha_2}\right)}\frac{g_r\left(h_r^{\alpha_2}\right)}{h_r^{\alpha_2}}\frac{g_r(\varepsilon)}{g_r(1)}q.\end{aligned}$$

As the limit $r\to\infty$, we conclude that $(z_k)\in N_\theta^{\alpha_1}(F)$ implies $(z_k)\in S_\theta^{\alpha_2}(G)$.

Remark 2.4. However, $S_\theta^{\alpha_2}(G)-$statistically convergent a sequence do not need to be $N_\theta^{\alpha_1}(F)-$summable. This observation is evident by referring to Example 3.2 in [25] where we consider $f_k(u)=g_k(u)=u$ for all $k\in\mathbb{N}$.

Corollary 2.4. *Suppose that $F\in\mathcal{M}^{ub}$, $\theta=(k_r)\in\mathcal{LS}(\mathbb{Z})$, and $\alpha_1,\alpha_2\in(0,1]$ such that $\alpha_1\leq\alpha_2$. If $\lim\limits_{u\to\infty}\frac{f_k(u)}{u}>0$ holds for all $k\in\mathbb{N}$ then $N_\theta^{\alpha_1}(F)$-summability implies $S_\theta^{\alpha_2}(F)$-statistical convergence, i.e. $N_\theta^{\alpha_1}(F)\subseteq S_\theta^{\alpha_2}(F)$.*

Proof. Proof is clear by taking $F=G$ in the last Theorem 2.2. □

Corollary 2.5. *Suppose that $F,G\in\mathcal{M}^{ub}$, $\alpha\in(0,1]$ and $\theta=(k_r)\in\mathcal{LS}(\mathbb{Z})$. If $\inf\limits_{u,k}\frac{f_k(u)}{g_k(u)}>0$ and $\lim\limits_{u\to\infty}\frac{g_k(u)}{u}>0$ for all $k\in\mathbb{N}$, then $N_\theta^{\alpha}(F)$-summability implies $S_\theta^{\alpha}(G)$-statistical convergence, i.e. $N_\theta^{\alpha}(F)\subseteq S_\theta^{\alpha}(G)$.*

Proof. It is consequence of Theorem 2.2 by taking $\alpha_2=\alpha$. □

Corollary 2.6. *Suppose that $F\in\mathcal{M}^{ub}$, $\alpha\in(0,1]$ and $\theta=(k_r)\in\mathcal{LS}(\mathbb{Z})$. If $\inf\limits_{u,k}\frac{f_k(u)}{u}>0$ then $N_\theta^{\alpha}(F)$-summability implies S_θ^{α}-statitically convergence, i.e. $N_\theta^{\alpha}(F)\subseteq S_\theta^{\alpha}$ and particularly, $N_\theta^{\alpha}(F)\subseteq S_\theta$ whenever $\alpha=1$.*

Proof. Proof is clear by taking $g_k(u)=u$ for all $k\in\mathbb{N}$ and $\alpha=\alpha_2$ in Corollary 2.4. □

Theorem 2.3. *Suppose that $F=(f_k),G=(g_k)\in\mathcal{M}^{ub}$, $0<\alpha_1\leq\alpha_2\leq1$, and $\theta=(k_r),\psi=(s_r)\in\mathcal{LS}(\mathbb{Z})$ such that $I_r\subset J_r$ for each $r\in\mathbb{N}$. If $\sup\limits_{u,k}\frac{g_k(u)}{u}<\infty$ and $\lim\limits_{r\to\infty}\frac{s_r-s_{r-1}}{(k_r-k_{r-1})^{\alpha_2}}=1$, then each $S_\theta^{\alpha_1}(F)$-convergent bounded sequence is $N_\psi^{\alpha_2}(G)-$summable, i.e. $\ell_\infty\cap S_\theta^{\alpha_1}(F)\subset N_\psi^{\alpha_2}(G)$.*

Proof. Suppose that $I_r=(k_{r-1},k_r]$, $J_r=(s_{r-1},s_r]$, $h_r=k_r-k_{r-1}$, $v_r=s_r-s_{r-1}$ and $0<\alpha_1\leq\alpha_2\leq1$. Choose $(z_k)\in\ell_\infty\cap S_\theta^{\alpha_1}(F)$ such that $z_k\to l(S_\theta^{\alpha_1}(F))$. Firstly, we will show that $S_\theta^{\alpha_1}(F)\subset S_\theta^{\alpha_1}$. Since $(z_k)\in S_\theta^{\alpha_1}(F)$, for every $\varepsilon>0$, we have

$$\lim_{r\to\infty}\frac{1}{f_r(h_r^{\alpha_1})}f_r(|\{k\in I_r:|z_k-l|\geq\varepsilon\}|)=0.$$

Hence, given $p\in\mathbb{N}$ we can find a natural number r_0 such that, □

$$f_r(|\{k\epsilon I_r : |z_k - l| \geq \varepsilon\}|) \leq \frac{1}{p} f_r(h_r^{\alpha_1}) \leq \frac{1}{p} p f_r(\frac{h_r^{\alpha_1}}{p}) = f_r(\frac{h_r^{\alpha_1}}{p})$$

for $r > r_0$. Since f_r are increasing modulus functions, we have,

$$\frac{1}{h_r^{\alpha_1}} |\{k\epsilon I_r : |z_k - l| \geq \varepsilon\}| \leq \frac{1}{p}.$$

It means that the inclusion $S_\theta^{\alpha_1}(F) \subset S_\theta^{\alpha_1}$ holds and hence $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset \ell_\infty \cap S_\theta^{\alpha_1}$. From the assumptions $\lim\limits_{r\to\infty} \frac{v_r}{h_r^{\alpha_2}} = 1$ and $I_r \subset J_r$ for each $r \in \mathbb{N}$, we have $\ell_\infty \cap S_\theta^{\alpha_1} \subset N_\psi^{\alpha_2}$ (see *Theorem 2.14, [14]*) and $N_\theta^{\alpha_2} \subset N_\psi^{\alpha_2}(G)$ holds by Corollary 2.2(ii) since the assumption $\sup\limits_{u,k} \frac{g_k(u)}{u} < \infty$. It follows that $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset N_\psi^{\alpha_2}(G)$ holds.

Remark 2.5. For $F, G \in \mathcal{M}^{ub}$, the inclusion $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}(G)$ may be strict. This fact can be seen from Example 3.3 in *[25]* if we take $f_k(u) = g_k(u) = u$ for all $k \in \mathbb{N}$.

Corollary 2.7. *Suppose that $F = (f_k) \in \mathcal{M}^{ub}$, $\theta = (k_r), \psi = (\omega_r) \in \mathcal{LS}(\mathbb{Z})$ such that $I_r \subset J_r$ for every $r \in \mathbb{N}$, and $0 < \alpha_1 \leq \alpha_2 \leq 1$. If $\sup\limits_{u,k} \frac{f_k(u)}{u} < \infty$ for all $k \in \mathbb{N}$ and $\lim\limits_{r\to\infty} \frac{v_r}{h_r^{\alpha_2}} = 1$, then the following assertions hold:*

i. $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset N_\psi^{\alpha_2}(F)$,

ii. $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset N_\psi^{\alpha_1}(F)$,

iii. $\ell_\infty \cap S_\theta^{\alpha_1} \subset N_\psi^{\alpha_1}(F)$.

In case modulus functions are bounded, a result similar to the above can be obtained.

Theorem 2.4. *Suppose that $F = (f_k) \in \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$ and $0 < \alpha_1 \leq \alpha_2 \leq 1$. Then, $S_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_1}(F)$ holds provided that $\sup\limits_{u>0} \sup\limits_{n\in\mathbb{N}} f_n(u) < \infty$.*

Proof. Assume that $\sup\limits_{u>0}\sup\limits_{n\in\mathbb{N}} f_n(u) < \infty$ holds and define $T = \sup\limits_{u>0} T(u)$ where $T(u) = \sup\limits_{n\in\mathbb{N}} f_n(u)$. Choose an arbitrary element $z = (z_k) \in S_\theta^{\alpha_1}(F)$ which is $S_\theta^{\alpha_1}(F)-$convergent to $l \in \mathbb{C}$. As shown in the proof of Theorem 2.3, $S_\theta^{\alpha_1}(F) \subset S_\theta^{\alpha_1}$ is satisfied. Hence $\lim\limits_{r\to\infty} \frac{1}{h_r^{\alpha_1}} |\{k\epsilon I_r : |z_k - l| \geq \varepsilon\}| = 0$ holds. On the other hand, we obtain the following inequality:

$$\begin{aligned} \frac{1}{h_r^{\alpha_1}} \sum_{k\in I_r} f_k(|z_k - l|) &= \frac{1}{h_r^{\alpha_1}} \sum_{k\in I_r, |z_k - l| \geq \varepsilon} f_k(|z_k - l|) + \frac{1}{h_r^{\alpha_1}} \sum_{k\in I_r, |z_k - l| < \varepsilon} f_k(|z_k - l|) \\ &\leq \frac{1}{h_r^{\alpha_1}} T |\{k\epsilon I_r : |z_k - l| \geq \varepsilon\}| + \frac{1}{h_r^{\alpha_1}} h_r T. \end{aligned}$$

Taking the limit $r \to \infty$, it follows that $\lim\limits_{r\to\infty} \frac{1}{h_r^{\alpha_1}} \sum_{k\in I_r} f_k(|z_k - l|) = 0$, i.e. $z = (z_k) \in N_\theta^{\alpha_1}(F)$. □

3. Concluding remarks and future directions

In this study, the categories of strongly lacunary summable sequences and lacunary statistically convergent sequences of numbers were introduced by employing a sequence of modulus functions. Furthermore, inclusion theorems have been established to compare these sets, which depend on parameters such as α, lacunary sequences, and sequences of modulus.

Statistical convergence is frequently employed in applied mathematics. Typically, a sequence is considered to converge statistically to a point when the majority of its elements approximate that point closely. However, achieving this majority often necessitates disregarding many terms in practice. In numerous applications, this approach to statistical convergence can be overly abrupt, resulting in the exclusion of elements from the sequence. Employing modulus functions offers a precise method for maintaining terms without discarding them.

As similar to other types of density functions, in this study, a density function defined by a sequence of unbounded modules and a real number has been used. By using a sequence of module functions instead of a single constant module function, the number of neglected terms will be much lower. Therefore, it can be considered as a method to somewhat improve statistical convergence and summability methods.

This research paper could be a resource for obtaining further advanced results. For example, by selecting different sequences of modulus functions used in various applied fields, application-specific sequence spaces can be obtained.

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Affiliations

Erdal Bayram
Address: Tekirdağ Namık Kemal University, Department of Mathematics, 59030, Tekirdağ-Türkiye
e-mail: ebayram@nku.edu.tr
ORCID ID: 0000-0001-8488-359X

Çiğdem A. Bektaş
Address: Fırat University, Department of Mathematics, 23119, Elazığ-Türkiye
e-mail: cbektas@firat.edu.tr
ORCID ID: 0000-0003-0397-3193

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Characterizations of Some New Classes of Four-Dimensional Matrices on the Double Series Spaces of First Order Cesàro Means

Okan Bodur and Canan Hazar Güleç*

Abstract
The main purpose in this study is to investigate some topological and algebraic properties of the absolutely double series spaces $|C_{1,1}|_k$ defined by combining the first order Cesàro means with the concept of absolute summability for $k \geq 1$. Beside this, we determine the $\alpha-$dual of the space $|C_{1,1}|_1$ and the $\beta(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $k \geq 1$. Finally, we characterize some new four-dimensional matrix classes $\left(|C_{1,1}|_k, \upsilon\right)$, $\left(|C_{1,1}|_1, \upsilon\right)$, $\left(|C_{1,1}|_1, \mathcal{L}_k\right)$, $\left(|C_{1,1}|_k, \mathcal{L}_u\right)$, $\left(\mathcal{L}_u, |C_{1,1}|_k\right)$ and $\left(\mathcal{L}_k, |C_{1,1}|_1\right)$, where $\upsilon \in \{\mathcal{M}_u, \mathcal{C}_{bp}\}$ for $1 \leq k < \infty$. Hence, some important results concerned on Cesàro matrix summation methods have been extended to double sequences.

Keywords: *Double sequences, Dual spaces, Four dimensional Cesàro matrix, Four dimensional matrix transformations, Pringsheim convergence*

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****Corresponding author***

1. Introduction

Recently, studies on the generalization of single sequence spaces to double sequence spaces have increased. Important studies on some double sequence spaces are included in [1–12]. Using Cesàro and weighted means for single series, Hazar, Hazar and Sarıgöl [13–15] have defined new series spaces. Later, Sarıgöl has extended some results to doubly infinite series by two dimensional weighted means [16]. Further, Başar and Sever have introduced the Banach space $\mathcal{L}_k$ of double sequences corresponding to the well-known classical sequence space ℓ_k of single sequences [17].Also, for the special case $k = 1$, the space $\mathcal{L}_k$ is reduced to the space $\mathcal{L}_u$, which was introduced by Zeltser [18].

A double sequence $x = (x_{rs})$ is a double infinite array of elements x_{rs} for all $r, s \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, ...\}$. We denote the set of all complex-valued double sequences by Ω which is a vector space with coordinatewise addition

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and scalar multiplication of double sequences. Any vector subspace of Ω is called as a double sequence space.

A double sequence $x=(x_{rs})$ of complex numbers is called bounded if $\|x\|_\infty=\sup_{m,n\in\mathbb{N}}|x_{mn}|<\infty$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$ which is a Banach space with the norm $\|.\|_\infty$. Consider the double sequence $x=(x_{mn})\in\Omega$. If for every $\epsilon>0$ there exists $n_0=n_0(\epsilon)\in\mathbb{N}$ and $L\in\mathbb{C}$ such that $|x_{mn}-L|<\epsilon$ for all $m,n>n_0$, then we say that the double sequence $x=(x_{mn})$ is convergent in the Pringsheim's sense to the limit point L, where $\mathbb{C}$ denotes the complex field. Then, we write $p-\lim_{m,n\to\infty}x_{mn}=L$ and $L\in\mathbb{C}$ is called the Pringsheim limit of x. The space of all convergent double sequences in the Pringsheim's sense is denoted by $\mathcal{C}_p$. Unlike single sequences, $p-$convergent double sequences need not be bounded. Namely, the set $\mathcal{C}_p-\mathcal{M}_u$ is not empty. So, we consider the set $\mathcal{C}_{bp}$ of double sequences which are both convergent in the Pringsheim's sense and bounded, i.e, $\mathcal{C}_{bp}=\mathcal{C}_p\cap\mathcal{M}_u$. Hardy [19] proved that a sequence in the space $\mathcal{C}_p$ is said to be regularly convergent if it is a single convergent sequence with respect to each index and the space of all such double sequences is denoted by $\mathcal{C}_r$.

Here and after, we assume that υ denotes any of the symbols p, bp or r, and k' denotes the conjugate of k, that is, $\frac{1}{k}+\frac{1}{k'}=1$ for $1<k<\infty$, and $\frac{1}{k'}=0$ for $k=1$.

Let $x=(x_{mn})$ be a double sequence and define the sequence $s=(s_{mn})$ as

$$s_{mn}=\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}$$

for all $m,n\in\mathbb{N}$. For brevity, here and in what follows we use the abbreviation $\sum_{i,j=1}^{m,n}x_{ij}$ for the summation $\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}$. Then, the pair of (x,s) is called as a double series and is denoted by $\sum_{i,j=1}^{\infty}x_{ij}$, or briefly by $\sum_{i,j}x_{ij}$. Let λ be a space of double sequence, converging with respect to some linear convergence rule $\upsilon-\lim:\lambda\to\mathbb{C}$. The sum of a double series $\sum_{i,j}x_{ij}$ according to this rule is defined by $\upsilon-\sum_{i,j}x_{ij}=\upsilon-\lim_{m,n\to\infty}s_{mn}$.

Let us consider double sequence spaces λ and μ, and four dimensional infinite matrix $A=(a_{mnij})$. Then we say that A defines a matrix mapping from λ into μ if for every double sequence $x=(x_{ij})\in\lambda$, $Ax=\{(Ax)_{mn}\}_{i,j\in\mathbb{N}}$, the A- transform of x, is in μ, where

$$(Ax)_{mn}=\upsilon-\sum_{i,j}a_{mnij}x_{ij} \tag{1.1}$$

provided that the double series exists for each $m,n\in\mathbb{N}$. By (λ,μ), we denote the set of such all four dimensional matrices transforming the space λ into the space μ. Thus, $A=(a_{mnij})\in(\lambda,\mu)$ if and only if the double series on the right side of (1.1) converges in the sense of υ for each $m,n\in\mathbb{N}$ and $Ax\in\mu$ for all $x\in\lambda$.

The $\alpha-dual$ λ^α, $\beta(\upsilon)-dual$ $\lambda^{\beta(\upsilon)}$ in regard to the $\upsilon-$convergence for $\upsilon\in\{p,bp,r\}$, and the $\gamma-dual$ λ^γ of a double sequence space λ are respectively described as

$$\lambda^\alpha:=\left\{\varepsilon=(\varepsilon_{ij})\in\Omega:\sum_{i,j}|\varepsilon_{ij}x_{ij}|<\infty\text{ for all }(x_{ij})\in\lambda\right\},$$

$$\lambda^{\beta(\upsilon)}:=\left\{\varepsilon=(\varepsilon_{ij})\in\Omega:\upsilon-\sum_{i,j}\varepsilon_{ij}x_{ij}\text{ exists for all }(x_{ij})\in\lambda\right\},$$

and

$$\lambda^\gamma:=\left\{\varepsilon=(\varepsilon_{ij})\in\Omega:\sup_{m,n\in\mathbb{N}}\left|\sum_{i,j=1}^{m,n}\varepsilon_{ij}x_{ij}\right|<\infty\text{ for all }(x_{ij})\in\lambda\right\}.$$

The $\upsilon-$summability domain $\lambda_A^{(\upsilon)}$ of a four dimensional complex infinite matrix $A=(a_{mnij})$ in a space λ of double sequences is introduced by

$$\lambda_A^{(\upsilon)}=\left\{x=(x_{ij})\in\Omega:Ax=\left(\upsilon-\sum_{i,j}a_{mnij}x_{ij}\right)_{m,n\in\mathbb{N}}\text{ exists and is in }\lambda\right\}.$$

The four dimensional Cesàro matrix $C=(c_{mnij})$ of order one is defined by

$$c_{mnij}=\begin{cases}\frac{1}{mn}, & 1\le i\le m,\ 1\le j\le n\\ 0, & \text{otherwise}\end{cases}$$

for all $m,n,i,j\in\mathbb{N}$.

Let $\sum_{i,j} x_{ij}$ be a doubly infinite series with partial sums (s_{mn}). The Cesàro mean T_{mn} of order one of a double sequence $s=(s_{mn})$ is defined by

$$T_{mn}=\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}s_{ij},\ \ (m,n\in\mathbb{N}).$$

We say that $s=(s_{mn})$ is $(C,1,1)$ summable or double Cesàro summable to some number ℓ if

$$p-\lim T_{mn}=\ell.$$

From the notation of Rhoades [20], a double series $\sum_{i,j} x_{ij}$ is called absolutely double Cesàro summable $|C,1,1|_k$, $k\geq 1$, if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\bar{\Delta}T_{mn}\right|^{k}<\infty,$$

where, for $m,n\geq 2$,

$$\bar{\Delta}T_{m1}=T_{m1}-T_{m-1,1}\ ,$$

$$\bar{\Delta}T_{1n}=T_{1n}-T_{1,n-1},$$

$$\bar{\Delta}T_{mn}=T_{mn}-T_{m-1,n}-T_{m,n-1}+T_{m-1,n-1.}$$

Further, it is easily seen that

$$T_{mn}=\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}s_{ij}=\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}\left(m-i+1\right)\left(n-j+1\right).$$

So, we have for $m,n=1$,

$$\bar{\Delta}T_{11}=x_{11}, \tag{1.2}$$

and, for $m,n\geq 2$,

$$\bar{\Delta}T_{m1}=\frac{1}{m\left(m-1\right)}\sum_{i=2}^{m}x_{i1}\left(i-1\right), \tag{1.3}$$

$$\bar{\Delta}T_{1n}=\frac{1}{n\left(n-1\right)}\sum_{j=2}^{n}x_{1j}\left(j-1\right), \tag{1.4}$$

and

$$\bar{\Delta}T_{mn}=\sum_{i=2}^{m}\sum_{j=2}^{n}\frac{x_{ij}\left(i-1\right)\left(j-1\right)}{\left(m-1\right)\left(n-1\right)mn}. \tag{1.5}$$

Now, referring Sarıgöl [16], we show the double series space $|C_{1,1}|_k$ by the set of all series summable by absolutely double Cesàro summability method of order one $|C,1,1|_k$, that is,

$$|C_{1,1}|_k=\left\{x=(x_{ij})\in\Omega:\sum_{i,j}x_{ij}\text{ is summable }\ |C,1,1|_k\right\}.$$

More recently, Mursaleen and Başar [12] have introduced the spaces $\tilde{\mathcal{M}}_u, \tilde{\mathcal{C}}_p, \tilde{\mathcal{C}}_{bp,} \tilde{\mathcal{C}}_r$ and $\tilde{\mathcal{L}}_u$ of double sequences whose Cesàro transforms of order one are in the spaces $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{bp,} \mathcal{C}_r$ and $\mathcal{L}_u$, respectively. Also, they examine some properties of those sequence spaces, determine certain dual spaces and give some matrix characterizations. In this paper, we investigate some topological and algebraic properties of the absolutely double series spaces $|C_{1,1}|_k$ for $k\geq 1$ taking account of the first order double Cesàro means with the concept of absolute summability. Beside this, we determine the alpha-dual of the space $|C_{1,1}|_1$ and the $\beta\,(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $k\geq 1$. Finally, we characterize some new four-dimensional matrix classes $\left(|C_{1,1}|_k, \upsilon\right)$, $\left(|C_{1,1}|_1, \upsilon\right)$, $\left(|C_{1,1}|_1, \mathcal{L}_k\right)$, $\left(|C_{1,1}|_k, \mathcal{L}_u\right)$, $\left(\mathcal{L}_u, |C_{1,1}|_k\right)$ and $\left(\mathcal{L}_k, |C_{1,1}|_1\right)$, where $\upsilon\in\{\mathcal{M}_u, \mathcal{C}_{bp}\}$ for $1\leq k<\infty$.

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2. Double series spaces of first order Cesàro means

In this section, we give some new results on the absolutely double Cesàro spaces $|C_{1,1}|_k$ for $k \geq 1$. Also, we determine the $\alpha-$ dual of the space $|C_{1,1}|_1$, $\beta\,(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $1 \leq k < \infty$.

Theorem 2.1. *The set $|C_{1,1}|_k$ becomes a linear space with the coordinatewise addition and scalar multiplication, and $|C_{1,1}|_k$ is a Banach space with the norm*

$$\|x\|_{|C_{1,1}|_k} = \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left| \bar{\Delta} T_{mn} \right|^k \right)^{1/k}, \tag{2.1}$$

which is linearly norm isomorphic to the space $\mathcal{L}_k$ for $1 \leq k < \infty$.

Proof. Since the initial assertion is routine verification and so we omit it.

To prove the fact that $|C_{1,1}|_k$ is norm isomorphic to the space $\mathcal{L}_k$, we should show the existence of a linear and norm preserving bijection between the spaces $|C_{1,1}|_k$ and $\mathcal{L}_k$ for $1 \leq k < \infty$. Consider the transformation B defined by

$$B : |C_{1,1}|_k \to \mathcal{L}_k$$

$$x \to y = B\,(x)$$

where $B\,(x) = (y_{mn})$ is defined by

$$B_{mn}\,(x) = y_{mn} = (mn)^{1-1/k}\, \bar{\Delta} T_{mn} \tag{2.2}$$

for $m, n \geq 1$ and $\bar{\Delta} T_{mn}$ is as in $(1.2 - 1.5)$. The linearity of B is clear. Also, $x = \theta$ whenever $B\,(x) = \theta$, which says us that B is injective.

Let $y = (y_{mn}) \in \mathcal{L}_k$ and define the sequence $x = (x_{mn})$ via y by

$$\begin{aligned} x_{mn} = \quad & \frac{1}{(n-1)(m-1)} \Big[m^{1/k} (m-1) \Big(y_{mn} n^{1/k} (n-1) - y_{m,n-1} (n-1)^{1/k} (n-2) \Big) \\ & - (m-1)^{1/k} (m-2) \Big(y_{m-1,n} n^{1/k} (n-1) - y_{m-1,n-1} (n-1)^{1/k} (n-2) \Big) \Big], \end{aligned} \tag{2.3}$$

$$x_{m1} = \frac{1}{m-1} \Big[m^{1/k} (m-1)\, y_{m1} - (m-1)^{1/k} (m-2)\, y_{m-1,1} \Big], \tag{2.4}$$

$$x_{1n} = \frac{1}{n-1} \Big[n^{1/k} (n-1)\, y_{1n} - (n-1)^{1/k} (n-2)\, y_{1,n-1} \Big], \tag{2.5}$$

for $m, n \geq 2$, and

$$x_{11} = y_{11}. \tag{2.6}$$

In that case, it seen that

$$\|x\|_{|C_{1,1}|_k} = \|B\,(x)\|_{\mathcal{L}_k} = \left(\sum_{m,n} |B_{mn}\,(x)|^k \right)^{1/k} = \|y\|_{\mathcal{L}_k} < \infty$$

for $1 \leq k < \infty$. So, this yields that B is surjective and norm preserving. Thus, B is a linear and norm preserving bijection which says the spaces $|C_{1,1}|_k$ and $\mathcal{L}_k$ are norm isomorphic for $1 \leq k < \infty$, as desired.

Now, we may show that $|C_{1,1}|_k$ is a Banach space with norm defined by (2.1). To prove this, we can consider "Let (X, ρ) and (Y, σ) be semi-normed spaces and $F : (X, \rho) \to (Y, \sigma)$ be an isometric isomorphism. Then (X, ρ) is complete if and only if (Y, σ) is complete. In particular, (X, ρ) is a Banach space if and only if (Y, σ) is a Banach space." which can be found section (b) of Corollary 6.3.41 in [21]. Since the transformation B defined from $|C_{1,1}|_k$ into $\mathcal{L}_k$ by (2.2) is an isometric isomorphism and the double sequence space $\mathcal{L}_k$ is a Banach space from Theorem 2.1 in [17], we deduce that the space $|C_{1,1}|_k$ is a Banach space. This is the result that we desired. $\square$

Now we have the following significant lemma, which will be used in the following theorems in order to calculate the $\alpha-$, $\beta\,(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $k \geq 1$.

Lemma 2.1. *[22] Let $A=(a_{mnij})$ be any four dimensional infinite matrix. At that case, the following statements are satisfied:*
(a) *Let $0<k\leq 1$. Then, $A\in(\mathcal{L}_k,\mathcal{M}_u)$ iff*

$$\xi_1=\sup_{m,n,i,j\in\mathbb{N}}|a_{mnij}|<\infty. \tag{2.7}$$

(b) *Let $1<k<\infty$. Then, $A\in(\mathcal{L}_k,\mathcal{M}_u)$ iff*

$$\xi_2=\sup_{m,n\in\mathbb{N}}\sum_{i,j}|a_{mnij}|^{k'}<\infty. \tag{2.8}$$

(c) *Let $0<k\leq 1$ and $1\leq k_1<\infty$. Then, $A\in(\mathcal{L}_k,\mathcal{L}_{k_1})$ iff*

$$\sup_{i,j\in\mathbb{N}}\sum_{m,n}|a_{mnij}|^{k_1}<\infty.$$

(d) *Let $0<k\leq 1$.Then, $A\in(\mathcal{L}_k,\mathcal{C}_{bp})$ iff the condition (2.7) holds and there exists a $(\lambda_{ij})\in\Omega$ such that*

$$bp-\lim_{m,n\to\infty}a_{mnij}=\lambda_{ij}. \tag{2.9}$$

(e) *Let $1<k<\infty$. Then, $A\in(\mathcal{L}_k,\mathcal{C}_{bp})$ iff (2.8) and (2.9) are satisfied.*

Lemma 2.2. *[23] Let $1<k<\infty$ and $A=(a_{mnrs})$ be a four dimensional infinite matrix of complex numbers. Define $W_k(A)$ and $w_k(A)$ by*

$$W_k(A)=\sum_{r,s=1}^{\infty}\left(\sum_{m,n=1}^{\infty}|a_{mnrs}|\right)^k,$$

$$w_k(A)=\sup_{M\times N}\sum_{r,s=1}^{\infty}\left|\sum_{(m,n)\in M\times N}a_{mnrs}\right|^k,$$

where the supremum is taken through all finite subsets M and N of $\mathbb{N}$. Then, the following statements are equivalent:

$$i)\ W_{k'}(A)<\infty\ ,\qquad ii)\ A\in(\mathcal{L}_k,\mathcal{L}_u)$$
$$iii)\ A^t\in(\mathcal{L}_\infty,\mathcal{L}_{k'})<\infty\ ,\qquad ii) w_{k'}(A)<\infty.$$

To shorten the following theorems and their proofs let us define the sets ψ_p with $p\in\{1,2,3,4\}$ as follows:

$$\psi_1=\left\{b=(b_{mn})\in\Omega:\sup_{i,j\in\mathbb{N}}\sum_{m,n}|g_{mnij}|<\infty\right\}, \tag{2.10}$$

$$\psi_2=\left\{b=(b_{mn})\in\Omega:\sup_{r,s,i,j\in\mathbb{N}}\left|\sum_{m=i}^{r}\sum_{n=j}^{s}b_{mn}f^{(1)}_{mnij}\right|<\infty\right\}, \tag{2.11}$$

$$\psi_3=\left\{b=(b_{mn})\in\Omega:bp-\lim_{r,s\to\infty}\sum_{m=i}^{r}\sum_{n=j}^{s}b_{mn}f^{(k)}_{mnij}\text{ exists}\right\}, \tag{2.12}$$

$$\psi_4=\left\{b=(b_{mn})\in\Omega:\sup_{r,s\in\mathbb{N}}\sum_{i,j}\left|\sum_{m=i}^{r}\sum_{n=j}^{s}b_{mn}f^{(k)}_{mnij}\right|^{k'}<\infty\right\}, \tag{2.13}$$

where the 4-dimensional matrices $G=(g_{mnij})$ and $F^{(k)}=\left(f^{(k)}_{mnij}\right)$ are defined by

$$g_{mnij}=\begin{cases}\dfrac{b_{mn}}{(n-1)(m-1)}(-1)^{m+n-(i+j)}(i-1)(j-1)ij,\ m-1\leq i\leq m\text{ and }n-1\leq j\leq n\\ \dfrac{b_{m1}}{m-1}(-1)^{m-i}(i-1)i,\ m-1\leq i\leq m\text{ and }n=1\\ \dfrac{b_{1n}}{n-1}(-1)^{n-j}(j-1)j,\ n-1\leq j\leq n\text{ and }m=1\\ b_{11},\ n=m=1\end{cases} \tag{2.14}$$

and

$$f_{mnij}^{(k)}=\begin{cases}\dfrac{(-1)^{m+n-(i+j)}}{(n-1)(m-1)}(i-1)(j-1)(ij)^{1/k}\,,\ m-1\le i\le m \text{ and } n-1\le j\le n\\ \dfrac{(-1)^{m-i}}{m-1}(i-1)(i)^{1/k}\,,\ m-1\le i\le m \text{ and } n=1\\ \dfrac{(-1)^{n-j}}{n-1}(j-1)(j)^{1/k}\,,\ n-1\le j\le n \text{ and } m=1\\ 1,\ n=m=1\end{cases}, \tag{2.15}$$

respectively.

Now we give theorems determining the $\alpha-$dual of the space $|C_{1,1}|_1$ and $\beta-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$.

Theorem 2.2. *Let the set ψ_1 and the 4-dimensional matrix $G=(g_{mnij})$ be defined as in (2.10) and (2.14), respectively. Then, $\left(|C_{1,1}|_1\right)^{\alpha}=\psi_1$.*

Proof. Let $b=(b_{mn})\in\Omega$, $x=(x_{mn})\in|C_{1,1}|_1$ and $y=(y_{ij})\in\mathcal{L}_u$. Taking account of relations in $(2.3-2.6)$ for $m,n\ge 1$, we obtain the following equalities: for $m,n\ge 2$

$$b_{mn}x_{mn}=\frac{b_{mn}}{(n-1)(m-1)}\sum_{i=m-1}^{m}\sum_{j=n-1}^{n}(-1)^{m+n-(i+j)}(i-1)(j-1)\,ijy_{ij}=(Gy)_{mn}\,,$$

for $n=1$ and $m\ge 2$

$$b_{m1}x_{m1}=\frac{b_{m1}}{m-1}\sum_{i=m-1}^{m}(-1)^{m-i}(i-1)\,iy_{i1}=(Gy)_{m1}\,,$$

for $m=1$ and $n\ge 2$

$$b_{1n}x_{1n}=\frac{b_{1n}}{n-1}\sum_{j=n-1}^{n}(-1)^{n-j}(j-1)\,jy_{1j}=(Gy)_{1n}$$

and for $n=m=1$

$$b_{11}x_{11}=b_{11}y_{11}=(Gy)_{11}\,,$$

where the four-dimensional matrix $G=(g_{mnij})$ defined by (2.14). In this fact, we see that $bx=(b_{mn}x_{mn})\in\mathcal{L}_u$ whenever $x\in|C_{1,1}|_1$ iff $Gy\in\mathcal{L}_u$ whenever $y\in\mathcal{L}_u$. This leads that $b=(b_{mn})\in\left(|C_{1,1}|_1\right)^{\alpha}$ iff $G\in(\mathcal{L}_u,\mathcal{L}_u)$. Then, we deduce by using (c) of Lemma 2.1 with $k_1=k=1$ that

$$\sup_{i,j\in\mathbb{N}}\sum_{m,n}|g_{mnij}|<\infty.$$

Hence, we have $\left(|C_{1,1}|_1\right)^{\alpha}=\psi_1$, as desired. This step concludes the proof. □

Theorem 2.3. *Let the sets ψ_2, ψ_3, ψ_4 and the 4-dimensional matrix $F^{(k)}=\left(f_{mnij}^{(k)}\right)$ be defined as in $(2.11-2.13)$ and (2.15), respectively. Then, $\left(|C_{1,1}|_1\right)^{\beta(bp)}=\psi_2\cap\psi_3$ for $k=1$ and $\left(|C_{1,1}|_k\right)^{\beta(bp)}=\psi_3\cap\psi_4$ for $1<k<\infty$.*

Proof. Let $b=(b_{mn})\in\Omega$ and $x=(x_{mn})\in|C_{1,1}|_k$ be given. Then, we write from Theorem 2.1 that there exists a double sequence $y=(y_{ij})\in\mathcal{L}_k$. Therefore, by using the equations $(2.3-2.6)$ we obtain that

$$z_{rs}=\sum_{m=1}^{r}\sum_{n=1}^{s}b_{mn}x_{mn}=\sum_{i=1}^{r}\sum_{j=1}^{s}\left(\sum_{m=i}^{r}\sum_{n=j}^{s}b_{mn}f_{mnij}^{(k)}\right)y_{ij}=(Dy)_{rs}$$

for every $r,s\in\mathbb{N}$. Thus, we see that $bx=(b_{mn}x_{mn})\in\mathcal{CS}_{bp}$ whenever $x=(x_{mn})\in|C_{1,1}|_k$ iff $z=(z_{rs})\in\mathcal{C}_{bp}$ whenever $y=(y_{ij})\in\mathcal{L}_k$. This leads to the fact that $b=(b_{mn})\in\left(|C_{1,1}|_k\right)^{\beta(bp)}$ iff $D\in(\mathcal{L}_k,\mathcal{C}_{bp})$, where the four-dimensional matrix $D=(d_{rsij})$ is defined by

$$d_{rsij}=\begin{cases}\sum_{m=i}^{r}\sum_{n=j}^{s}b_{mn}f_{mnij}^{(k)}, 1\le i\le r \text{ and } 1\le j\le s\\ 0,\ \text{otherwise}\end{cases}$$

for every $r,s,i,j\in\mathbb{N}$. Hence, we deduce $\left(|C_{1,1}|_1\right)^{\beta(bp)}=\psi_2\cap\psi_3$ and $\left(|C_{1,1}|_k\right)^{\beta(bp)}=\psi_3\cap\psi_4$ for $1<k<\infty$ from parts (d) and (e) of Lemma 2.1, respectively. □

Theorem 2.4. *Let the sets ψ_2, ψ_4 and the 4-dimensional matrix $F^{(k)} = \left(f^{(k)}_{mnij}\right)$ be defined as in (2.11), (2.13) and (2.15), respectively. Then, $\left(|C_{1,1}|_1\right)^{\gamma} = \psi_2$ and $\left(|C_{1,1}|_k\right)^{\gamma} = \psi_4$ for $1 < k < \infty$.*

Proof. This theorem can be proved by analogy with the proof Theorem 2.3 using Parts (a) and (b) of Lemma 2.1 in place of parts (d) and (e) of Lemma 2.1, respectively. So we leave the details to readers. □

3. Characterizations of some classes of four-dimensional matrices

In the present section, we characterize some matrix mappings from double series spaces $|C_{1,1}|_1$ and $|C_{1,1}|_k$ to the double sequence spaces $\mathcal{M}_u$, $\mathcal{C}_{bp}$, $\mathcal{L}_u$ and $\mathcal{L}_k$ for $1 \le k < \infty$. Although the theorem characterizing matrix mappings from double series spaces $|C_{1,1}|_1$ and $|C_{1,1}|_k$ to the double sequence space $\mathcal{M}_u$ is given with proof, other theorems characterizing other mappings are given without proof since the proof techniques are similar.

Theorem 3.1. *Suppose that $A = (a_{mnij})$ be an arbitrary $4-$dimensional infinite matrix and the 4-dimensional matrix $F^{(k)} = \left(f^{(k)}_{mnij}\right)$ be defined as in (2.15) for $1 \le k < \infty$. In that case, the following statements hold:*

(a) *$A \in \left(|C_{1,1}|_1, \mathcal{M}_u\right)$ if and only if*

$$A_{mn} \in \left(|C_{1,1}|_1\right)^{\beta(bp)} \tag{3.1}$$

and

$$\sup_{m,n,u,v\in\mathbb{N}} \left| \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f^{(1)}_{ijuv} \right| < \infty. \tag{3.2}$$

(b) *Let $1 < k < \infty$. Then, $A \in \left(|C_{1,1}|_k, \mathcal{M}_u\right)$ if and only if*

$$A_{mn} \in \left(|C_{1,1}|_k\right)^{\beta(bp)} \tag{3.3}$$

and

$$\sup_{m,n\in\mathbb{N}} \sum_{u,v} \left| \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f^{(k)}_{ijuv} \right|^{k'} < \infty. \tag{3.4}$$

Proof. The part (a) can be proved by using Lemma 2.1 (a) in a similar way to that used in the proof of the part (b) of Theorem, so, we give the proof only for $1 < k < \infty$ to avoid the repetition of similar statements.

(b) Let $1 < k < \infty$ and $x = (x_{ij}) \in |C_{1,1}|_k$. Then, there exists a double sequence $y = (y_{mn}) \in \mathcal{L}_k$. By using the equalities $(2.3 - 2.6)$, for (s,t)th rectangular partial sum of the series $\sum_{i,j} a_{mnij} x_{ij}$, we have

$$\begin{aligned}
(Ax)^{[s,t]}_{mn} &= \sum_{i,j=1}^{s,t} a_{mnij} x_{ij} \\
&= \sum_{i,j}^{s,t} a_{mnij} \sum_{u,v}^{i,j} f_{ijuv} y_{uv} \\
&= \sum_{u,v=1}^{s,t} \left(\sum_{i=u}^{s}\sum_{j=v}^{t} f_{ijuv} a_{mnij} \right) y_{uv} \\
&= \sum_{u,v=1}^{s,t} h^{mn}_{stuv} y_{uv}
\end{aligned} \tag{3.5}$$

for every $m, n, s, t \in \mathbb{N}$, where the $4-$ dimensional matrix $H_{mn} = (h^{mn}_{stuv})$ is defined by

$$h^{mn}_{stuv} = \begin{cases} \sum_{i=u}^{s}\sum_{j=v}^{t} f_{ijuv} a_{mnij}, & 1 \le u \le s \text{ and } 1 \le v \le t \\ 0, & \text{otherwise} \end{cases}$$

for every $s, t, u, v \in \mathbb{N}$. Then, the equality (3.5) can be written as

$$(Ax)^{[s,t]}_{mn} = (H_{mn} y)_{[s,t]}. \tag{3.6}$$

Therefore, it follows from (3.6) that the *bp*-convergence of $(Ax)_{mn}^{[s,t]}$ and the statement $H_{mn} \in (\mathcal{L}_k, \mathcal{C}_{bp})$ are equivalent for all $x \in |C_{1,1}|_k$ and $m, n \in \mathbb{N}$. Hence, the condition (3.3) is satisfied for each fixed $m, n \in \mathbb{N}$, that is, $A_{mn} \in \left(|C_{1,1}|_k\right)^{\beta(bp)}$ for each fixed $m, n \in \mathbb{N}$ and $1 < k < \infty$.

If we take *bp*-limit in the terms of the matrix $H_{mn} = (h_{stuv}^{mn})$ while $s, t \to \infty$, we obtain that

$$bp - \lim_{s,t\to\infty} h_{stuv}^{mn} = \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}. \tag{3.7}$$

With the relation (3.7), we can define the 4-dimensional matrix $H = (h_{mnuv})$ as

$$h_{mnuv} = \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}$$

for all $m, n, u, v \in \mathbb{N}$. In this situation, we deduce from the equations (3.6) and (3.7) that

$$bp - \lim_{s,t\to\infty} (Ax)_{mn}^{[s,t]} = bp - \lim (Hy)_{mn}. \tag{3.8}$$

Thus, one can write that $A = (a_{mnij}) \in \left(|C_{1,1}|_k, \mathcal{M}_u\right)$ if and only if $H \in (\mathcal{L}_k, \mathcal{M}_u)$, by having in mind the relation (3.8).

Therefore, using Lemma 2.1 (b), we obtain that

$$\sup_{m,n\in\mathbb{N}} \sum_{u,v} \left| \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}^{(k)} \right|^{k'} < \infty,$$

which satisfies the condition (3.4).

So, we conclude that $A = (a_{mnij}) \in \left(|C_{1,1}|_k, \mathcal{M}_u\right)$ if and only if the conditions (3.3) and (3.4) are satisfied. This completes the proof. □

Theorem 3.2. *Suppose that $A = (a_{mnij})$ be an arbitrary $4-$dimensional infinite matrix and the 4-dimensional matrix $F^{(k)} = \left(f_{mnij}^{(k)}\right)$ be defined as in (2.15) for $1 \le k < \infty$. In that case, the following statements hold:*

(a) *$A \in \left(|C_{1,1}|_1, \mathcal{C}_{bp}\right)$ if and only if (3.1), (3.2) hold and there exists $\left(\alpha_{uv}^{(1)}\right) \in \Omega$ such that*

$$bp - \lim_{m,n\to\infty} \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}^{(1)} = \alpha_{uv}^{(1)}.$$

(b) *Let $1 < k < \infty$. Then, $A \in \left(|C_{1,1}|_k, \mathcal{C}_{bp}\right)$ if and only if (3.3), (3.4) hold and there exists $(\alpha_{uv}) \in \Omega$ such that*

$$bp - \lim_{m,n\to\infty} \sum_{i=u}^{\infty}\sum_{j=v}^{\infty} a_{mnij} f_{ijuv}^{(k)} = \alpha_{uv}.$$

Proof. This theorem can be proved by using Lemma 2.1 (d) and (e) in a similar way to that used in the proof of Theorem 3.1. □

Theorem 3.3. *Suppose that $A = (a_{mnij})$ be an arbitrary $4-$dimensional infinite matrix and the 4-dimensional matrix $F^{(k)} = \left(f_{mnij}^{(k)}\right)$ be defined as in (2.15) for $1 \le k < \infty$. In that case, the following statements hold:*

(a) *Let $1 \le k < \infty$. Then, $A \in \left(|C_{1,1}|_1, \mathcal{L}_k\right)$ if and only if (3.1) holds and*

$$\sup_{r,s\in\mathbb{N}} \sum_{m,n} \left| \sum_{i=r}^{\infty}\sum_{j=s}^{\infty} a_{mnij} f_{ijrs}^{(1)} \right|^{k} < \infty.$$

(b) *Let* $1 < k < \infty$. *Then,* $A = (a_{mnij}) \in \left(|C_{1,1}|_k, \mathcal{L}_u\right)$ *if and only if* (3.3) *holds and*

$$\sum_{r,s=1}^{\infty}\left(\sum_{m,n}^{\infty}\left|\sum_{i=r}^{\infty}\sum_{j=s}^{\infty}a_{mnij}f_{ijrs}^{(k)}\right|\right)^{k'} < \infty.$$

Proof. This theorem can be proved by using Lemma 2.1 (c) and Lemma 2.2 in a similar way to that used in the proof of Theorem 3.1. □

Lemma 3.1. *[22] Let* λ *and* μ *be two double sequence spaces in* Ω, $A = (a_{mnij})$ *an arbitrary* 4*-dimensional infinite matrix and* $\Phi = (\phi_{mnuv})$ *be triangle* 4*-dimensional infinite matrix. Then,* $A \in (\lambda, \mu_\Phi)$ *if and only if* $\Phi A \in (\lambda, \mu)$.

Now, we can give the final results of our work by considering the Lemma 2.1, 2.2 and 3.1.

Corollary 3.1. *Let* $A = (a_{mnij})$ *and* $\Phi = (\phi_{mnuv})$ *four dimensional matrices be given by the relation*

$$\phi_{mnuv} = \sum_{i,j=1}^{m,n} b_{mnij}a_{ijuv},$$

where $B = (b_{mnij})$ *is defined as*

$$b_{mnij} = \begin{cases} 1, \ m = n = 1 \\ \dfrac{(i-1)}{m^{1/k}(m-1)}, \ 2 \leq i \leq m \text{ and } n = 1 \\ \dfrac{(j-1)}{n^{1/k}(n-1)}, \ 2 \leq j \leq n, \text{ and } m = 1 \\ \dfrac{(i-1)(j-1)}{(m-1)(n-1)(mn)^{1/k}}, \ 2 \leq i \leq m \text{ and } 2 \leq j \leq n \\ 0, \ \text{otherwise} \end{cases}$$

and, by considering the relation (2.1). *Then, the necessary and sufficient conditions for the classes* $\left(\mathcal{L}_u, |C_{1,1}|_k\right)$ *and* $\left(\mathcal{L}_q, |C_{1,1}|_1\right)$ *can be found as follows:*

(a) $A = (a_{mnij}) \in \left(\mathcal{L}_u, |C_{1,1}|_k\right)$ *if and only if*

$$\sup_{u,v\in\mathbb{N}}\sum_{m,n}|\phi_{mnuv}|^k < \infty$$

holds for $1 \leq k < \infty$.

(b) $A = (a_{mnij}) \in \left(\mathcal{L}_q, |C_{1,1}|_1\right)$ *if and only if*

$$\sum_{u,v=1}^{\infty}\left(\sum_{m,n=1}^{\infty}|\phi_{mnuv}|\right)^{q'} < \infty$$

holds for $1 < q < \infty$ *and* $k = 1$.

4. Conclusion

In this study, we investigate some topological and algebraic properties of the absolutely double series spaces $|C_{1,1}|_k$ defined by combining the first order Cesàro means with the concept of absolute summability for $k \geq 1$. Beside this, we determine the $\alpha-$dual of the space $|C_{1,1}|_1$ and the $\beta(bp)-$ and $\gamma-$duals of the spaces $|C_{1,1}|_k$ for $k \geq 1$. Finally, we characterize some new four-dimensional matrix classes on the absolutely double series spaces $|C_{1,1}|_k$. Hence, some important results concerned on Cesàro matrix summation methods have been extended to double sequences.

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Affiliations

OKAN BODUR
ADDRESS: Pamukkale University, Dept. of Mathematics, 20070, Denizli-Turkey
E-MAIL: okan.bodur@hotmail.com
ORCID ID:0000-0002-6403-4627

CANAN HAZAR GÜLEÇ
ADDRESS: Pamukkale University, Dept. of Mathematics, 20070, Denizli-Turkey
E-MAIL: gchazar@pau.edu.tr
ORCID ID: 0000-0002-8825-5555