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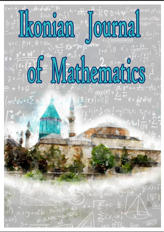

Solvability of two-dimensional system of difference equations with constant coefficients

Ömer Aktaş[1], **Merve Kara**[1], **Yasin Yazlik**[2]

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Difference equations systems,
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Abstract — In the present paper, the solutions of the following system of difference equations

$$u_n = \alpha_1 v_{n-2} + \frac{\delta_1 v_{n-2} u_{n-4}}{\beta_1 u_{n-4} + \gamma_1 v_{n-6}},\ v_n = \alpha_2 u_{n-2} + \frac{\delta_2 u_{n-2} v_{n-4}}{\beta_2 v_{n-4} + \gamma_2 u_{n-6}},\ n \in \mathbb{N}_0,$$

where the initial values u_{-l}, v_{-l}, for $l = \overline{1,6}$ and the parameters α_p, β_p, γ_p, δ_p, for $p \in \{1,2\}$ are non-zero real numbers, are investigated. In addition, the solutions of the aforementioned system of difference equations are presented by utilizing the Fibonacci sequence when the parameters are equal to 1. Finally, the periodic solutions according to some special cases of the parameters are obtained.

Subject Classification (2020): 39A10, 39A20, 39A23.

1. Introduction and Preliminaries

Difference equations are one of the important topics of applied mathematics. Therefore, some mathematicians have studied in this field [1–20]. Some difference equations occur as the recurrence relation of a number sequence. For example, Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is identified by

$$F_{n+1} = F_n + F_{n-1},\ n \in \mathbb{N}, \tag{1.1}$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$ in [21]. Binet's formula for equation (1.1) is

$$F_n = \frac{A^n - B^n}{A - B},\ n \in \mathbb{N}_0, \tag{1.2}$$

where $A = \frac{1+\sqrt{5}}{2}$, $B = \frac{1-\sqrt{5}}{2}$. Equation (1.2) is a solution of equation (1.1) and the general term Fibonacci sequence. In addition, there are some types of nonlinear difference equations for which their general solutions can be found. One of them is Riccati difference equation, which is in the following form:

$$z_{n+1} = \frac{\epsilon z_n + \theta}{\zeta z_n + \eta}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

[1] aktas.omer10@gmail.com; [1] mervekara@kmu.edu.tr (Corresponding Author); [2] yyazlik@nevsehir.edu.tr
[1] Department of Mathematics, Kamil Ozdag Science Faculty, Karamanoglu Mehmetbey University, Karaman, Turkey
[2] Department of Mathematics, Faculty of Science and Art, Nevsehir Hacı Bektaş Veli University, Nevsehir, Turkey
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for $\zeta \neq 0$, $\epsilon\eta - \zeta\theta \neq 0$, where the parameters $\epsilon, \theta, \zeta, \eta$ and the initial condition z_0 are real numbers. The general solution of equation (1.3) can be written as follows

$$z_n = \frac{z_0\left(\theta\zeta - \epsilon\eta\right)s_{n-1} + \left(\epsilon z_0 + \theta\right)s_n}{\left(\zeta z_0 - \epsilon\right)s_n + s_{n+1}}, \quad n \in \mathbb{N}, \tag{1.4}$$

where the sequence $(s_n)_{n\in\mathbb{N}_0}$ is satisfying

$$s_{n+1} - \left(\epsilon + \eta\right)s_n - \left(\theta\zeta - \epsilon\eta\right)s_{n-1} = 0, \quad n \in \mathbb{N},$$

where $s_0 = 0$, $s_1 = 1$, in [22].

The following higher-order difference equation,

$$x_n = \alpha x_{n-k} + \frac{\delta x_{n-k} x_{n-(k+l)}}{\beta x_{n-(k+l)} + \gamma x_{n-l}}, \; n \in \mathbb{N}_0, \tag{1.5}$$

where k and l are fixed natural numbers, the initial conditions x_{-j}, $j = \overline{1, k+l}$ and the parameters α, β, γ, δ are real numbers, was solved by the authors in [23]. In addition, the case $k = 2$, $l = 4$ in equation (1.5), it was obtained the exact solutions and investigated equilibria, local stability and global attractivity in [24]. Similarly, the authors of [25] studied the behavior of the solutions of the difference equation which was obtained by taking $k = 1$, $l = 3$ in equation (1.5).

There are some difference equations that are similar in shape to the difference equation in (1.5). But, they are not particular cases of equation (1.5). For example, in [26], the authors explored the qualitative behavior of the solutions of the following difference equations:

$$y_{n+1} = Ay_{n-1} + \frac{\pm B y_{n-1} y_{n-3}}{C y_{n-3} \pm D y_{n-5}}, \; n \in \mathbb{N}_0, \tag{1.6}$$

where the initial conditions y_{-k}, for $k = \overline{0,5}$, are arbitrary positive real numbers and the parameters A, B, C and D are positive real numbers.

Similarly, the authors studied the behaviour of the rational difference equation

$$y_{n+1} = \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}}, \; n \in \mathbb{N}_0, \tag{1.7}$$

where the initial conditions y_{-k}, for $k = \overline{0,4}$, are positive real numbers and the parameters α, β, A and B are real numbers, in [27].

In addition, in [28], Almatrafi and Alzubaidi studied the local and global stability, periodicity and solutions of the following rational difference equations

$$u_{n+1} = au_{n-1} \pm \frac{bu_{n-1}u_{n-4}}{cu_{n-4} - du_{n-6}}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

where the parameters a,b,c and d are positive real numbers and the initial values u_{-k}, for $k = \overline{0,6}$, are non-zero real numbers.

Moreover, the authors of [29] studied the behavior of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_{n-1} + dx_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.9}$$

where the initial conditions x_{-k}, for $k = \overline{0,2}$ are arbitrary positive real numbers and the parameters a,b,c and d are positive constants. In [30], Elsayed and Al-Rakhami investigated some of the qualitative behavior of the rational difference equation

$$\Psi_{n+1} = \alpha\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3} + \delta\Psi_{n-6}}, \quad n \in \mathbb{N}_0, \tag{1.10}$$

where the parameters α, β, γ and δ are arbitrary positive real numbers.

Further, in [31] Elsayed studied the qualitative behavior of the solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.11}$$

where a,b,c and d, are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers. There are some difference equations as equations in (1.6)-(1.11) in literature (see [32–35]).
In [36], the authors generalized the equation (1.5) to the following two-dimensional system

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, y_n = \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, n \in \mathbb{N}_0, \tag{1.12}$$

where k and l are positive integers, the initial conditions x_{-i}, y_{-i}, $i = \overline{1,k+l}$ and the parameters a, b, c, d, α, β, γ, δ are real numbers. They showed that system (1.12) can be solved in closed form.

A natural question is if equation (1.6) generalizes to a two-dimensional system of difference equations. Here, we give a positive answer. We expand equation (1.6) to the following two-dimensional system of difference equations

$$u_n = \alpha_1 v_{n-2} + \frac{\delta_1 v_{n-2}u_{n-4}}{\beta_1 u_{n-4} + \gamma_1 v_{n-6}}, v_n = \alpha_2 u_{n-2} + \frac{\delta_2 u_{n-2}v_{n-4}}{\beta_2 v_{n-4} + \gamma_2 u_{n-6}}, n \in \mathbb{N}_0, \tag{1.13}$$

where the initial values u_{-l}, v_{-l}, for $l = \overline{1,6}$, are positive real numbers and the parameters α_p, β_p, γ_p and δ_p, for $p \in \{1,2\}$, are positive real numbers.

Our aim to show that system (1.13) is solvable in explicit form. Also, we investigate the periodicity of the solutions depending on special cases of the parameters. Additionally, we gain the solutions for the case $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 1$ by using Fibonacci sequence.

We give the following very well-known definition which used in this paper.

Definition 1.1. [37] (Periodicity) A sequence $(x_n)_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there exists $n_0 \geq -k$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. If $n_0 = -k$ then the sequence $(x_n)_{n=-k}^{\infty}$ is said to be periodic with period p.

2. Explicit Solutions of System (1.13)

The system (1.13) can be written in the following form

$$\frac{u_n}{v_{n-2}} = \frac{\left(\alpha_1\beta_1+\delta_1\right)\frac{u_{n-4}}{v_{n-6}}+\alpha_1\gamma_1}{\beta_1\frac{u_{n-4}}{v_{n-6}}+\gamma_1}, \quad \frac{v_n}{u_{n-2}} = \frac{\left(\alpha_2\beta_2+\delta_2\right)\frac{v_{n-4}}{u_{n-6}}+\alpha_2\gamma_2}{\beta_2\frac{v_{n-4}}{u_{n-6}}+\gamma_2}, \qquad n\in\mathbb{N}_0.$$

By employing the change of variables

$$x_n = \frac{u_n}{v_{n-2}}, \quad y_n = \frac{v_n}{u_{n-2}}, \qquad n\geq -4, \tag{2.1}$$

system (1.13) is transformed into the following system

$$x_n = \frac{\left(\alpha_1\beta_1+\delta_1\right)x_{n-4}+\alpha_1\gamma_1}{\beta_1 x_{n-4}+\gamma_1}, \quad y_n = \frac{\left(\alpha_2\beta_2+\delta_2\right)y_{n-4}+\alpha_2\gamma_2}{\beta_2 y_{n-4}+\gamma_2}, \ n\in\mathbb{N}_0. \tag{2.2}$$

We consider the following equation

$$z_n = \frac{\left(\alpha\beta+\delta\right)z_{n-4}+\alpha\gamma}{\beta z_{n-4}+\gamma}, \qquad n\in\mathbb{N}_0, \tag{2.3}$$

instead of equations in (2.2). If we apply decomposition of indices $n\to 4(m+1)+i$, $i=\overline{-4,-1},\ m\geq -1$, in equation (2.3), then it can be written the following equation

$$z_{m+1}^{(i)} = \frac{\left(\alpha\beta+\delta\right)z_m^{(i)}+\alpha\gamma}{\beta z_m^{(i)}+\gamma}, \tag{2.4}$$

where $z_m^{(i)} = z_{4m+i}, \quad i=\overline{-4,-1}, \quad m\in\mathbb{N}_0,$
From equation (1.4), the general solutions of the equations in (2.4) as follows

$$z_m^{(i)} = \frac{-\delta\gamma z_0^{(i)} s_{m-1} + \left(\left(\alpha\beta+\delta\right)z_0^{(i)}+\alpha\gamma\right)s_m}{\left(\beta z_0^{(i)}-\alpha\beta-\delta\right)s_m + s_{m+1}}, \qquad m\in\mathbb{N}, \tag{2.5}$$

for $i=\overline{-4,-1}$, where sequence of $(s_m)_{m\in\mathbb{N}_0}$ is satisfying

$$s_{m+1}-\left(\alpha\beta+\delta+\gamma\right)s_m+\delta\gamma s_{m-1}=0, \qquad m\in\mathbb{N}. \tag{2.6}$$

From equation (2.5), the solutions of equations in (2.2) are expressed as

$$x_{4m+i} = \frac{-\delta_1\gamma_1 x_i s_{m-1}+\left(\left(\alpha_1\beta_1+\delta_1\right)x_i+\alpha_1\gamma_1\right)s_m}{\left(\beta_1 x_i-\alpha_1\beta_1-\delta_1\right)s_m+s_{m+1}}, \qquad m\in\mathbb{N}_0, \tag{2.7}$$

$$y_{4m+i} = \frac{-\delta_2\gamma_2 y_i s_{m-1}+\left(\left(\alpha_2\beta_2+\delta_2\right)y_i+\alpha_2\gamma_2\right)s_m}{\left(\beta_2 y_i-\alpha_2\beta_2-\delta_2\right)s_m+s_{m+1}}, \qquad m\in\mathbb{N}_0, \tag{2.8}$$

for $i=\overline{-4.-1}$.
From (2.1) , we have

$$u_n = x_n v_{n-2} = x_n y_{n-2} u_{n-4},\ v_n = y_n u_{n-2} = y_n x_{n-2} v_{n-4}, \quad n\geq -2. \tag{2.9}$$

From system (2.9), we obtain

$$u_{4m+j} = x_{4m+j}y_{4m+j-2}u_{4(m-1)+j}, \quad m \in \mathbb{N}_0,$$
$$v_{4m+j} = y_{4m+j}x_{4m+j-2}v_{4(m-1)+j}, \quad m \in \mathbb{N}_0, \tag{2.10}$$

for $j = \overline{-2,1}$.

From system (2.10), we get

$$u_{4m+j} = u_{j-4}\prod_{p=0}^{m} x_{4p+j}y_{4p+j-2}, \quad m \in \mathbb{N}_0,$$
$$v_{4m+j} = v_{j-4}\prod_{p=0}^{m} y_{4p+j}x_{4p+j-2}, \quad m \in \mathbb{N}_0, \tag{2.11}$$

for $j = \overline{-2,1}$.

By putting formulas (2.7) and (2.8) back into system (2.11), we gain

$$u_{4m-2} = u_{-6}\prod_{p=0}^{m}\left(\frac{-\delta_1\gamma_1u_{-2}s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right)u_{-2} + \alpha_1\gamma_1v_{-4}\right)s_p}{\left(\beta_1u_{-2} - \left(\alpha_1\beta_1+\delta_1\right)v_{-4}\right)s_p + v_{-4}s_{p+1}}\right.$$
$$\left.\times\left(\frac{-\delta_2\gamma_2v_{-4}s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right)v_{-4} + \alpha_2\gamma_2u_{-6}\right)s_p}{\left(\beta_2v_{-4} - \left(\alpha_2\beta_2+\delta_2\right)u_{-6}\right)s_p + u_{-6}s_{p+1}}\right)\right), \tag{2.12}$$

$$v_{4m-2} = v_{-6}\prod_{p=0}^{m}\left(\frac{-\delta_2\gamma_2v_{-2}s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right)v_{-2} + \alpha_2\gamma_2u_{-4}\right)s_p}{\left(\beta_2v_{-2} - \left(\alpha_2\beta_2+\delta_2\right)u_{-4}\right)s_p + u_{-4}s_{p+1}}\right.$$
$$\left.\times\left(\frac{-\delta_1\gamma_1u_{-4}s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right)u_{-4} + \alpha_1\gamma_1v_{-6}\right)s_p}{\left(\beta_1u_{-4} - \left(\alpha_1\beta_1+\delta_1\right)v_{-6}\right)s_p + v_{-6}s_{p+1}}\right)\right), \tag{2.13}$$

$$u_{4m-1} = u_{-5}\prod_{p=0}^{m}\left(\frac{-\delta_1\gamma_1u_{-1}s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right)u_{-1} + \alpha_1\gamma_1v_{-3}\right)s_p}{\left(\beta_1u_{-1} - \left(\alpha_1\beta_1+\delta_1\right)v_{-3}\right)s_p + v_{-3}s_{p+1}}\right.$$
$$\left.\times\left(\frac{-\delta_2\gamma_2v_{-3}s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right)v_{-3} + \alpha_2\gamma_2u_{-5}\right)s_p}{\left(\beta_2v_{-3} - \left(\alpha_2\beta_2+\delta_2\right)u_{-5}\right)s_p + u_{-5}s_{p+1}}\right)\right), \tag{2.14}$$

$$v_{4m-1} = v_{-5}\prod_{p=0}^{m}\left(\frac{-\delta_2\gamma_2v_{-1}s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right)v_{-1} + \alpha_2\gamma_2u_{-3}\right)s_p}{\left(\beta_2v_{-1} - \left(\alpha_2\beta_2+\delta_2\right)u_{-3}\right)s_p + u_{-3}s_{p+1}}\right.$$
$$\left.\times\left(\frac{-\delta_1\gamma_1u_{-3}s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right)u_{-3} + \alpha_1\gamma_1v_{-5}\right)s_p}{\left(\beta_1u_{-3} - \left(\alpha_1\beta_1+\delta_1\right)v_{-5}\right)s_p + v_{-5}s_{p+1}}\right)\right), \tag{2.15}$$

$$u_{4m} = u_{-4}\prod_{p=0}^{m}\left(\frac{-\delta_1\gamma_1 u_{-4}s_p + \left(\left(\alpha_1\beta_1+\delta_1\right)u_{-4}+\alpha_1\gamma_1 v_{-6}\right)s_{p+1}}{\left(\beta_1 u_{-4} - \left(\alpha_1\beta_1+\delta_1\right)v_{-6}\right)s_{p+1} + v_{-6}s_{p+2}}\right)$$
$$\times\left(\frac{-\delta_2\gamma_2 v_{-2}s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right)v_{-2}+\alpha_2\gamma_2 u_{-4}\right)s_p}{\left(\beta_2 v_{-2} - \left(\alpha_2\beta_2+\delta_2\right)u_{-4}\right)s_p + u_{-4}s_{p+1}}\right), \quad (2.16)$$
$$v_{4m} = v_{-4}\prod_{p=0}^{m}\left(\frac{-\delta_2\gamma_2 v_{-4}s_p + \left(\left(\alpha_2\beta_2+\delta_2\right)v_{-4}+\alpha_2\gamma_2 u_{-6}\right)s_{p+1}}{\left(\beta_2 v_{-4} - \left(\alpha_2\beta_2+\delta_2\right)u_{-6}\right)s_{p+1} + u_{-6}s_{p+2}}\right)$$
$$\times\left(\frac{-\delta_1\gamma_1 u_{-2}s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right)u_{-2}+\alpha_1\gamma_1 v_{-4}\right)s_p}{\left(\beta_1 u_{-2} - \left(\alpha_1\beta_1+\delta_1\right)v_{-4}\right)s_p + v_{-4}s_{p+1}}\right), \quad (2.17)$$

$$u_{4m+1} = u_{-3}\prod_{p=0}^{m}\left(\frac{-\delta_1\gamma_1 u_{-3}s_p + \left(\left(\alpha_1\beta_1+\delta_1\right)u_{-3}+\alpha_1\gamma_1 v_{-5}\right)s_{p+1}}{\left(\beta_1 u_{-3} - \left(\alpha_1\beta_1+\delta_1\right)v_{-5}\right)s_{p+1} + v_{-5}s_{p+2}}\right)$$
$$\times\left(\frac{-\delta_2\gamma_2 v_{-1}s_{p-1} + \left(\left(\alpha_2\beta_2+\delta_2\right)v_{-1}+\alpha_2\gamma_2 u_{-3}\right)s_p}{\left(\beta_2 v_{-1} - \left(\alpha_2\beta_2+\delta_2\right)u_{-3}\right)s_p + u_{-3}s_{p+1}}\right), \quad (2.18)$$
$$v_{4m+1} = v_{-3}\prod_{p=0}^{m}\left(\frac{-\delta_2\gamma_2 v_{-3}s_p + \left(\left(\alpha_2\beta_2+\delta_2\right)v_{-3}+\alpha_2\gamma_2 u_{-5}\right)s_{p+1}}{\left(\beta_2 v_{-3} - \left(\alpha_2\beta_2+\delta_2\right)u_{-5}\right)s_{p+1} + u_{-5}s_{p+2}}\right)$$
$$\times\left(\frac{-\delta_1\gamma_1 u_{-1}s_{p-1} + \left(\left(\alpha_1\beta_1+\delta_1\right)u_{-1}+\alpha_1\gamma_1 v_{-3}\right)s_p}{\left(\beta_1 u_{-1} - \left(\alpha_1\beta_1+\delta_1\right)v_{-3}\right)s_p + v_{-3}s_{p+1}}\right), \quad (2.19)$$

for $m \in \mathbb{N}_0$.

3. Periodicity

We obtain the periodicity of the solutions of the system (1.13) depending on the parameters are equal either 1 or −1 in this section.

Theorem 3.1. Suppose that α_p, β_p, γ_p, δ_p, for $p \in \{1,2\}$ and the initial values u_{-l}, v_{-l}, for $l = \overline{1,6}$ are non-zero real numbers. Then, the following statements hold.

- **a)** If $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_1 = -1$, $\gamma_2 = -1$, $\delta_1 = -1$, $\delta_1 = -1$, the solutions of the system (1.13) are periodic with period 12.
- **b)** If $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = -1$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\delta_1 = 1$, $\delta_1 = 1$, the solutions of the system (1.13) are periodic with period 12.
- **c)** If $\alpha_1 = -1$, $\alpha_2 = -1$, $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\delta_1 = 1$, $\delta_1 = 1$, the solutions of the system (1.13) are periodic with period 12.
- **d)** If $\alpha_1 = -1$, $\alpha_2 = -1$, $\beta_1 = -1$, $\beta_2 = -1$, $\gamma_1 = -1$, $\gamma_2 = -1$, $\delta_1 = -1$, $\delta_1 = -1$, the solutions of the system (1.13) are periodic with period 12.

Proof.

- **a)** If $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_1 = -1$, $\gamma_2 = -1$, $\delta_1 = -1$, $\delta_1 = -1$, system (1.13) turns into the following system

$$u_n = v_{n-2} - \frac{v_{n-2}u_{n-4}}{u_{n-4} - v_{n-6}},\ v_n = u_{n-2} - \frac{u_{n-2}v_{n-4}}{v_{n-4} - u_{n-6}},\ n \in \mathbb{N}_0. \tag{3.1}$$

From (2.7) and (2.8), we have

$$x_{4m+i} = \frac{-x_i s_{m-1} - s_m}{x_i s_m + s_{m+1}}, \tag{3.2}$$

$$y_{4m+i} = \frac{-y_i s_{m-1} - s_m}{y_i s_m + s_{m+1}}, \tag{3.3}$$

where $m \in \mathbb{N}_0$ and $i = \overline{-4, -1}$.

From (2.6), we obtain

$$s_{m+1} + s_m + s_{m-1} = 0,$$

where $s_0 = 0$ and $s_1 = 1$.

From this, we get

$$s_{3t+b} = b, \tag{3.4}$$

for $t \in \mathbb{N}_0$ and $b = \overline{-1, 1}$.

From (2.1), we have

$$\begin{aligned} u_{12m+j} =& x_{12m+j}y_{12m+j-2}x_{12m+j-4}y_{12m+j-6} \\ &\times x_{12m+j-8}y_{12m+j-10}u_{12(m-1)+j}, \\ v_{12m+j} =& y_{12m+j}x_{12m+j-2}y_{12m+j-4}x_{12m+j-6} \\ &\times y_{12m+j-8}x_{12m+j-10}v_{12(m-1)+j}, \end{aligned} \tag{3.5}$$

where $m \in \mathbb{N}_0$ and $j = \overline{6, 17}$.

From system (3.5), we obtain

$$u_{12m+j} = u_{j-12}\prod_{p=0}^{m} x_{12p+j}y_{12p+j-2}x_{12p+j-4}y_{12p+j-6} \times x_{12p+j-8}y_{12p+j-10}, \tag{3.6}$$

$$v_{12m+j} = v_{j-12}\prod_{p=0}^{m} y_{12p+j}x_{12p+j-2}y_{12p+j-4}x_{12p+j-6} \times y_{12p+j-8}x_{12p+j-10}, \tag{3.7}$$

where $m \in \mathbb{N}_0$ and $j = \overline{6, 17}$.

By using (3.2), (3.3) and (3.4) into (3.6) and (3.7), we get

$$u_{12m+j} = u_{j-12}, \quad v_{12m+j} = v_{j-12},$$

where $m \in \mathbb{N}_0$ and $j = \overline{6, 17}$.

b) If $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = -1$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\delta_1 = 1$, $\delta_1 = 1$, system (1.13) turns into the system (3.1). Then, it can be proven like (a).

c) If $\alpha_1 = -1$, $\alpha_2 = -1$, $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\delta_1 = 1$, $\delta_1 = 1$, system (1.13) turns into the following system

$$u_n = -v_{n-2} + \frac{v_{n-2}u_{n-4}}{u_{n-4}+v_{n-6}},\ v_n = -u_{n-2} + \frac{u_{n-2}v_{n-4}}{v_{n-4}+u_{n-6}},\ n \in \mathbb{N}_0. \tag{3.8}$$

From (2.7) and (2.8), we obtain

$$x_{4m+i} = \frac{-x_i s_{m-1} - s_m}{x_i s_m + s_{m+1}}, \tag{3.9}$$

$$y_{4m+i} = \frac{-y_i s_{m-1} - s_m}{y_i s_m + s_{m+1}}, \tag{3.10}$$

where $m \in \mathbb{N}_0$ and $i = \overline{-4,-1}$.

We obtain, from (2.6),

$$s_{m+1} - s_m + s_{m-1} = 0,$$

where $s_0 = 0$ and $s_1 = 1$.

From this, we get

$$s_{6t+3r+q} = \begin{cases} 0, & \text{if } 3r+q \in \{0,3\}, \\ 1, & \text{if } 3r+q \in \{1,2\}, \\ -1, & \text{if } 3r+q \in \{4,5\}, \end{cases} \tag{3.11}$$

for $t \in \mathbb{N}_0$, $r \in \{0,1\}$ and $q = \overline{0,2}$.

From (2.1), we have

$$\begin{aligned} u_{12m+j} =& x_{12m+j}y_{12m+j-2}x_{12m+j-4}y_{12m+j-6} \\ &\times x_{12m+j-8}y_{12m+j-10}u_{12(m-1)+j}, \\ v_{12m+j} =& y_{12m+j}x_{12m+j-2}y_{12m+j-4}x_{12m+j-6} \\ &\times y_{12m+j-8}x_{12m+j-10}v_{12(m-1)+j}, \end{aligned} \tag{3.12}$$

where $m \in \mathbb{N}_0$ and $j = \overline{6,17}$.

From system (3.12), we obtain

$$\begin{aligned} u_{12m+j} =& u_{j-12}\prod_{p=0}^{m} x_{12p+j}y_{12p+j-2}x_{12p+j-4}y_{12p+j-6} \\ &\times x_{12p+j-8}y_{12p+j-10}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} v_{12m+j} =& v_{j-12}\prod_{p=0}^{m} y_{12p+j}x_{12p+j-2}y_{12p+j-4}x_{12p+j-6} \\ &\times y_{12p+j-8}x_{12p+j-10}, \end{aligned} \tag{3.14}$$

where $m \in \mathbb{N}_0$ and $j = \overline{6,17}$.

By using (3.9)-(3.11) into (3.13) and (3.14), we get

$$u_{12m+j} = u_{j-12}, \quad v_{12m+j} = v_{j-12},$$

where $m \in \mathbb{N}_0$ and $j = \overline{6,17}$.

d) If $\alpha_1 = -1$, $\alpha_2 = -1$, $\beta_1 = -1$, $\beta_2 = -1$, $\gamma_1 = -1$, $\gamma_2 = -1$, $\delta_1 = -1$, $\delta_1 = -1$, system (1.13) turns into the system (3.8). Then, it can be proven like (c).

4. An Application

We obtain the solutions of the system (1.13) with $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 1$. In this case, we have the following system

$$u_n = v_{n-2} + \frac{v_{n-2}u_{n-4}}{u_{n-4} + v_{n-6}}, \quad v_n = u_{n-2} + \frac{u_{n-2}v_{n-4}}{v_{n-4} + u_{n-6}}, \quad n \in \mathbb{N}_0. \tag{4.1}$$

From (2.6), we obtain

$$s_{m+1} - 3s_m + s_{m-1} = 0, \quad m \in \mathbb{N}, \tag{4.2}$$

where $s_0 = 0, \quad s_1 = 1$.
Binet Formula for (4.2) is

$$s_m = \frac{\left(\frac{3+\sqrt{5}}{2}\right)^m - \left(\frac{3-\sqrt{5}}{2}\right)^m}{\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{3-\sqrt{5}}{2}\right)}, \quad m \in \mathbb{N}_0. \tag{4.3}$$

Note that

$$\left(\frac{1 \mp \sqrt{5}}{2}\right)^2 = \frac{3 \mp \sqrt{5}}{2}. \tag{4.4}$$

Using (4.4) in (4.3), we have

$$s_m = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2m} - \left(\frac{1-\sqrt{5}}{2}\right)^{2m}}{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2} = F_{2m}, \quad m \in \mathbb{N}_0. \tag{4.5}$$

Using (4.5) into (2.12)-(2.19), we get

$$u_{4m-2} = u_{-6} \prod_{p=0}^{m} \frac{\left(u_{-2}F_{2p+1} + v_{-4}F_{2p}\right)\left(v_{-4}F_{2p+1} + u_{-6}F_{2p}\right)}{\left(v_{-4}F_{2p-1} + u_{-2}F_{2p}\right)\left(u_{-6}F_{2p-1} + v_{-4}F_{2p}\right)}, \tag{4.6}$$

$$v_{4m-2} = v_{-6} \prod_{p=0}^{m} \frac{\left(v_{-2}F_{2p+1} + u_{-4}F_{2p}\right)\left(u_{-4}F_{2p+1} + v_{-6}F_{2p}\right)}{\left(u_{-4}F_{2p-1} + v_{-2}F_{2p}\right)\left(v_{-6}F_{2p-1} + u_{-4}F_{2p}\right)}, \tag{4.7}$$

$$u_{4m-1} = u_{-5} \prod_{p=0}^{m} \frac{\left(u_{-1}F_{2p+1} + v_{-3}F_{2p}\right)\left(v_{-3}F_{2p+1} + u_{-5}F_{2p}\right)}{\left(v_{-3}F_{2p-1} + u_{-1}F_{2p}\right)\left(u_{-5}F_{2p-1} + v_{-3}F_{2p}\right)}, \tag{4.8}$$

$$v_{4m-1} = v_{-5} \prod_{p=0}^{m} \frac{\left(v_{-1}F_{2p+1} + u_{-3}F_{2p}\right)\left(u_{-3}F_{2p+1} + v_{-5}F_{2p}\right)}{\left(u_{-3}F_{2p-1} + v_{-1}F_{2p}\right)\left(v_{-5}F_{2p-1} + u_{-3}F_{2p}\right)}, \tag{4.9}$$

$$u_{4m} = u_{-4} \prod_{p=0}^{m} \frac{\left(u_{-4}F_{2p+3} + v_{-6}F_{2p+2}\right)\left(v_{-2}F_{2p+1} + u_{-4}F_{2p}\right)}{\left(v_{-6}F_{2p+1} + u_{-4}F_{2p+2}\right)\left(u_{-4}F_{2p-1} + v_{-2}F_{2p}\right)}, \tag{4.10}$$

$$v_{4m} = v_{-4} \prod_{p=0}^{m} \frac{\left(v_{-4}F_{2p+3} + u_{-6}F_{2p+2}\right)\left(u_{-2}F_{2p+1} + v_{-4}F_{2p}\right)}{\left(u_{-6}F_{2p+1} + v_{-4}F_{2p+2}\right)\left(v_{-4}F_{2p-1} + u_{-2}F_{2p}\right)}, \tag{4.11}$$

$$u_{4m+1} = u_{-3} \prod_{p=0}^{m} \frac{\left(u_{-3}F_{2p+3} + v_{-5}F_{2p+2}\right)\left(v_{-1}F_{2p+1} + u_{-3}F_{2p}\right)}{\left(v_{-5}F_{2p+1} + u_{-3}F_{2p+2}\right)\left(u_{-3}F_{2p-1} + v_{-1}F_{2p}\right)}, \tag{4.12}$$

$$v_{4m+1} = v_{-3} \prod_{p=0}^{m} \frac{\left(v_{-3}F_{2p+3} + u_{-5}F_{2p+2}\right)\left(u_{-1}F_{2p+1} + v_{-3}F_{2p}\right)}{\left(u_{-5}F_{2p+1} + v_{-3}F_{2p+2}\right)\left(v_{-3}F_{2p-1} + u_{-1}F_{2p}\right)}, \tag{4.13}$$

for $m \in \mathbb{N}_0$.

5. Conclusion

In this paper, we have obtained the solutions of two-dimensional system of difference equations in explicit form by using convenient transformation. In addition, we have investigated the periodic solutions of aforementioned system of difference equations when the parameters are equal to 1 or equal to -1. Finally, an application was given to show that the solutions of the mentioned system are related to Fibanacci numbers when all parameters are equal to 1.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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Global Behavior of Solutions of a Two-dimensional System of Difference Equations

Mehmet Gümüş[1], **Raafat Abo-Zeid**[2], **Kemal Türk**[3]

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Discrete dynamical system,
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Abstract — In this paper, we mainly investigate the qualitative and quantitative behavior of the solutions of a discrete system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{ax_{n-1} + by_{n-1}}, \quad n = 0, 1, \ldots,$$

where a, b and the initial values x_{-1}, x_0, y_{-1}, y_0 are non-zero real numbers. For $a \in \mathbb{R}_+ - \{1\}$, we show any admissible solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is either entirely located in a certain quadrant of the plane or there exists a natural number $N > 0$ (we calculate its value) such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is located. Besides, some numerical simulations with graphs are given in the article to emphasize the efficiency of our theoretical results.

Subject Classification (2020): 39A06, 39A20.

1. Introduction

In case of interruption of events developing over time, mathematical models are established with difference equations using discrete variables. In this way, difference equations have an important place in research on real-life problems, especially in fields such as economics, medicine, chemistry and biology. In addition to its importance in practice, difference equations are also used in theoretical research, that is, to obtain solutions of differential equations, delayed differential equations, and fractional differential equations. It is very difficult most of the time to obtain solutions to rational difference equations. Additionally, there is no general technique to obtain or qualitatively investigate solutions. For this reason, the study of non-linear difference equations of order greater than one is truly remarkable and every qualified study in this field is valuable.

Difference equations have a very old history. However, its research has progressed rapidly, especially in the last thirty years. Research in this field can be carried out under three headings: quantitative, qualitative and numerical. Quantitative research is carried out by determining the analytical solutions of the equation, qualitative research is carried out by examining the behavior of the solutions of the equation, and numerical research is carried out by determining the approximate values of the solution of the equation by various

[1]m.gumus@beun.edu.tr (Corresponding Author); [2]abuzead73@yahoo.com; [3]kemalturk@beun.edu.tr
[1,3]Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, Zonguldak, Türkiye
[2]Department of Basic Science, The Higher Institute for Engineering & Technology, Al-Obour, Cairo, Egypt
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methods.
Therefore, this paper can be viewed as both a qualitative and quantitative investigation of a system of difference equations. Now let's give a detailed background of the system we discuss in this article:
In [29], the authors studied the global dynamics of the system

$$x_{n+1} = \frac{\beta_1 x_n}{A_1 + y_n},\ y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n}, \quad n = 0, 1, \ldots,$$

where the parameters $\gamma_2, A_1, \beta_1, \beta_2$ are positive numbers and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers such that $x_0 + y_0 > 0$.
Camouzis et al. [10], studied the global behavior of the system of difference equations

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{x_n},\ y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n}, \quad n = 0, 1, \ldots, \tag{1.1}$$

with nonnegative parameters and positive initial conditions. They studied the boundedness character of the system (1.1) in its special cases.
In [9], Camouzis et al. conjectured that:
Every positive solution of the system

$$x_{n+1} = \frac{y_n}{x_n},\ y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n}, \quad n = 0, 1, \ldots,$$

with nonnegative parameters and positive initial conditions, converges to a finite limit.
Bekker et al. [8] confirmed that conjecture.
In [28], Kudlak et al. studied the existence of unbounded solutions of the system of difference equations

$$x_{n+1} = \frac{x_n}{y_n},\ y_{n+1} = x_n + \gamma_n y_n,\ n = 0, 1, \ldots,$$

where $0 < \gamma_n < 1$ and the initial values are positive real numbers.
There is an increasing interest in the applications of difference and systems of difference equations in various fields. Even if a difference equation appears very plain and simple, its solutions can exhibit very complex behavior. In this paper, we study the global behavior of the admissible solutions of the system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{a x_{n-1} + b y_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where a, b, and the initial values x_{-1}, x_0, y_{-1}, y_0 are nonzero real numbers.
We shall study here, the behavior of the solutions of system (1.2) using their closed form. Other relevant qualitative and quantitative theories of difference equations can be obtained in references ([1]-[7], [12], [15], [16], [22], [26], [30], [32]-[34] and the references therein). For more on discrete systems of difference equations that are solved in closed form in references (see [11], [13], [14], [17]-[21], [23]-[25], [31], [35]-[38]).

2. Linearized Stability and Solution of the System (1.2)

In this section, we investigate the local asymptotic behavior of the equilibrium point of the system (1.2) and derive its solution.
It is clear that the system (1.2) has no equilibrium points when $a = 1$ and it has a unique equilibrium point

$(\frac{b}{1-a}, 1)$ when $a \neq 1$. To study the linearized stability of the unique equilibrium point of the system (1.2), we consider the transformation

$$F\begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{x_{n-1}}{y_{n-1}} \\ x_n \\ \frac{x_{n-1}}{ax_{n-1}+by_{n-1}} \\ y_n \end{pmatrix}.$$

The linearized system associated with the system (1.2) about an equilibrium point $(\bar{x}, \bar{y})$ is

$$Z_{n+1} = J_F(\bar{x}, \bar{y})Z_n, \ n = 0, 1, ...,$$

where

$$Z_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix} \text{ and } J_F(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \frac{1}{\bar{y}} & 0 & -\frac{\bar{x}}{\bar{y}^2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{b\bar{y}^3}{\bar{x}^2} & 0 & -\frac{b\bar{y}^2}{\bar{x}} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For more results on the stability of difference equations, see [27].

Theorem 2.1. Assume that $a \neq 1$. Then the equilibrium point $(\frac{b}{1-a}, 1)$ of the system (1.2) is

1. locally asymptotically stable if $|a| < 1$,
2. unstable (saddle point) if $|a| > 1$.

Proof.

The Jacobian matrix about the equilibrium point $(\frac{b}{1-a}, 1)$ becomes

$$J_F\left(\frac{b}{1-a}, 1\right) = \begin{pmatrix} 0 & 1 & 0 & -\frac{b}{1-a} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{(1-a)^2}{b} & 0 & -(1-a) \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.1}$$

It is enough to see that the eigenvalues of the matrix (2.1) are $0, 0, \sqrt{|a|}, -\sqrt{|a|}$, and the result follows.

Now, returning to the system (1.2), we can write

$$u_{n+1} = au_{n-1} + b, \ n = 0, 1, ..., \tag{2.2}$$

where

$$u_n = \frac{x_n}{y_n}, \text{ with } u_{-1} = \frac{x_{-1}}{y_{-1}}, \text{ and } u_0 = \frac{x_0}{y_0}.$$

Solving (2.2), we obtain the following:

1. If $a \neq 1$, then

$$x_n = \begin{cases} \frac{a^{\frac{n-1}{2}}\alpha_1 + b}{1-a} & , n = 1, 3, ..., \\ \frac{a^{\frac{n}{2}-1}\alpha_2 + b}{1-a} & , n = 2, 4, ..., \end{cases} \tag{2.3}$$

and

$$y_n = \begin{cases} \frac{a^{\frac{n-1}{2}}\alpha_1+b}{a^{\frac{n+1}{2}}\alpha_1+b} & ,n=1,3,..., \\ \frac{a^{\frac{n}{2}}\alpha_2+b}{a^{\frac{n}{2}}\alpha_2+b} & ,n=2,4,..., \end{cases} \tag{2.4}$$

where $\alpha_i = \frac{x_{-2+i}}{y_{-2+i}}(1-a)-b,\ i=1,2.$

2. If $a=1$, then

$$x_n = \begin{cases} \beta_1 + b(\frac{n-1}{2}) & ,n=1,3,..., \\ \beta_2 + b(\frac{n}{2}-1) & ,n=2,4,..., \end{cases} \tag{2.5}$$

and

$$y_n = \begin{cases} \frac{\beta_1+b(\frac{n-1}{2})}{\beta_1+b(\frac{n+1}{2})} & ,n=1,3,..., \\ \frac{\beta_2+b(\frac{n}{2}-1)}{\beta_2+b(\frac{n}{2})} & ,n=2,4,..., \end{cases} \tag{2.6}$$

where $\beta_i = \frac{x_{-2+i}}{y_{-2+i}},\ i=1,2.$

The forbidden set for the system (1.2) depends on the value of a. For the system (1.2) we have the following:

- If $a \neq 1$, then the forbidden set of the system (1.2) is

$$F_1 = \bigcup_{m=0}^{\infty} \{(x,y) \in \mathbb{R}^2 : y = -\frac{a^m}{b}x\}.$$

- If $a = 1$, then the forbidden set of the system (1.2) is

$$F_2 = \bigcup_{m=1}^{\infty} \{(x,y) \in \mathbb{R}^2 : y = -\frac{1}{bm}x\}.$$

From now on, we assume that all solutions are admissible, that is for any solution $\{(x_n,y_n)\}_{n=-1}^{\infty}$ of the system (1.2), the initial points $(x_{-i},y_{-i}) \notin F_1$ if $a \neq 1$ or $(x_{-i},y_{-i}) \notin F_2$ if $a=1$, $i=0,1$.

Theorem 2.2. Assume that $|a|<1$. Then the equilibrium point $(\frac{b}{1-a},1)$ of the system (1.2) is globally asymptotically stable.

Proof.
Using formulas (2.3) and (2.4), we have

$$(x_n,y_n) \to (\frac{b}{1-a},1),\ \text{as}\ n\to\infty. \tag{2.7}$$

That is, the equilibrium point $(\frac{b}{1-a},1)$ of the system (1.2) is a global attractor.
Using Theorem (2.1)(1), the proof follows.

We give the following result without proof as a consequence of the solution form of the system (1.2).

Theorem 2.3. Assume that $a \neq 1$. The following statements are true:

1. If $a>1$, then the solution $\{(x_n,y_n)\}_{n=-1}^{\infty}$ is unbounded, namely:

$$\{(x_{2n+1},y_{2n+1})\}_{n=-1}^{\infty} \to (-\infty\ .\ sgn(\alpha_1),\frac{1}{a}),\ \text{as}\ n\to\infty,$$

and

$$\{(x_{2n+2}, y_{2n+2})\}_{n=-1}^{\infty} \to (-\infty . sgn(\alpha_2), \frac{1}{a}), \text{ as } n \to \infty.$$

2. If $a = 1$, then the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is unbounded, namely:
   $\{(x_n, y_n)\}_{n=-1}^{\infty} \to (\infty . sgn(b), 1)$ as $n \to \infty$.

Theorem 2.4. Assume that $a \neq 1$. Then the set $I = \{(x, y) \in \mathbb{R}^2 : (a-1)x + by = 0\}$ is an invariant set for the system (1.2).

Proof.
Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of the system (1.2) such that $(x_{-i}, y_{-i}) \in I$, $i = 0, 1$. Then

$$x_1 = \frac{x_{-1}}{y_{-1}} = \frac{b}{1-a} \text{ and } y_1 = \frac{x_{-1}}{ax_{-1} + by_{-1}} = 1.$$

This implies that $(x_1, y_1) \in I$. Similarly, we can show that $(x_2, y_2) \in I$. Assume that $(x_t, y_t) \in I$, $-1 \leq t \leq n_0 - 1$ for a certain $n_0 \in \mathbb{N}$. Then

$$x_{n_0} = \frac{x_{n_0-1}}{y_{n_0-1}} = \frac{b}{1-a} \text{ and } y_{n_0} = \frac{x_{n_0-1}}{ax_{n_0-1} + by_{n_0-1}} = 1.$$

This implies that $(x_{n_0}, y_{n_0}) \in I$ and the proof is completed.

3. Behaviors of Solutions of the System (1.2)

This section is devoted to study the behaviors of the admissible solutions of the system (1.2). During this section, assume that $a \in \mathbb{R}_+ - \{1\}$ and consider the real-valued functions

$$f(x) = a^x \alpha + b, \; g(x) = \frac{a^x \alpha + b}{a^{x+1}\alpha + b}.$$

For $\alpha b < 0$, denote $l_1 = \frac{\ln(-\frac{b}{\alpha})}{\ln a}$.
We shall introduce the following two Lemmas to be used in the subsequent results.

Lemma 3.1. For the function $f(x)$, the following statements are true:

1. When $\alpha b > 0$, then $f(x) > 0$ ($f(x) < 0$) if $\alpha > 0$ ($\alpha < 0$).

2. When $\alpha b < 0$, we have the following:

   (a) If $\alpha > 0$, then we have the following:

      i. If $0 < a < 1$, then $f(x) < 0$ for all $x > 0$ ($x > l_1$) when $-\frac{b}{\alpha} \in]1, \infty[$ ($-\frac{b}{\alpha} \in]0, 1[$).
      ii. If $a > 1$, then $f(x) > 0$ for all $x > 0$ ($x > l_1$) when $-\frac{b}{\alpha} \in]0, 1[$ ($-\frac{b}{\alpha} \in]1, \infty[$).

   (b) If $\alpha < 0$, then we have the following:

      i. If $0 < a < 1$, then $f(x) > 0$ for all $x > 0$ ($x > l_1$) when $-\frac{b}{\alpha} \in]1, \infty[$ ($-\frac{b}{\alpha} \in]0, 1[$).
      ii. If $a > 1$, then $f(x) < 0$ for all $x > 0$ ($x > l_1$) when $-\frac{b}{\alpha} \in]0, 1[$ ($-\frac{b}{\alpha} \in]1, \infty[$).

Proof.

1. The proof is clear and will be omitted.

2. Assume that $\alpha b<0$.

   (a) When $\alpha>0$, we have the following:

   i. If $0<a<1$, then for $-\frac{b}{\alpha}\in]1,\infty[$ we have $f(x)<\alpha+b<0$ for all $x>0$. Otherwise, if $-\frac{b}{\alpha}\in]0,1[$, then
   $f(x)<f(\frac{\ln(\frac{-b}{\alpha})}{\ln a})=a^{\frac{\ln(\frac{-b}{\alpha})}{\ln a}}\alpha+b=0$ for all $x>l_1$.

   ii. If $a>1$, then for $-\frac{b}{\alpha}\in]0,1[$ we have $f(x)>\alpha+b>0$ for all $x>0$. Otherwise, if $-\frac{b}{\alpha}\in]1,\infty[$, then
   $f(x)>f(\frac{\ln(\frac{-b}{\alpha})}{\ln a})=a^{\frac{\ln(\frac{-b}{\alpha})}{\ln a}}\alpha+b=0$ for all $x>l_1$.

   (b) When $\alpha<0$, we have the following:

   i. If $0<a<1$, then for $-\frac{b}{\alpha}\in]1,\infty[$ we have $f(x)>0$ for all $x>0$. Otherwise, if $-\frac{b}{\alpha}\in]0,1[$, then
   $f(x)>f(\frac{\ln(\frac{-b}{\alpha})}{\ln a})=a^{\frac{\ln(\frac{-b}{\alpha})}{\ln a}}\alpha+b=0$.

   ii. If $a>1$, then for $-\frac{b}{\alpha}\in]0,1[$ we have $f(x)<0$ for all $x>0$. Otherwise, if $-\frac{b}{\alpha}\in]1,\infty[$, then
   $f(x)<f(\frac{\ln(\frac{-b}{\alpha})}{\ln a})=a^{\frac{\ln(\frac{-b}{\alpha})}{\ln a}}\alpha+b=0$ for all $x>l_1$.

Lemma 3.2. For the function $g(x)$, the following statements are true:

1. When $\alpha b>0$, then $g(x)>0$ for all $x>0$.

2. When $\alpha b<0$, we have the following:

   (a) If $0<a<1$, then either $g(x)>0$ for all $x>0$ when $-\frac{b}{\alpha}\in]1,\infty[$ or $g(x)>0$ for all $x>l_1$ when $-\frac{b}{\alpha}\in]0,1[$.

   (b) If $a>1$, then either $g(x)>0$ for all $x>0$ when $-\frac{b}{\alpha}\in]0,1[$ or $g(x)>0$ for all $x>l_1$ when $-\frac{b}{\alpha}\in]1,\infty[$.

Proof.

1. The proof is clear and will be omitted.

2. Assume that $\alpha b<0$.

   (a) When $\alpha>0$ and $b<0$, we get $\alpha a+b<\alpha+b$.
   If $-\frac{b}{\alpha}\in]1,\infty[$, then $g(x)>\frac{\alpha+b}{\alpha a+b}=g(0)$ for all $x>0$.
   Otherwise, there exists $l_1=\frac{\ln(\frac{-b}{\alpha})}{\ln a}>0$, $g(x)>g(l_1)=\frac{a_1^l\alpha+b}{a^{l_1+1}\alpha+b}=0$, for all $x>l_1$.
   Now, when $\alpha<0$ and $b>0$, we get $\alpha+b<\alpha a+b$.
   If $-\frac{b}{\alpha}\in]1,\infty[$, then $g(x)>\frac{\alpha+b}{\alpha a+b}=g(0)$ for all $x>0$.
   Otherwise, there exists $l_1=\frac{\ln(\frac{-b}{\alpha})}{\ln a}>0$, $g(x)>g(l_1)=\frac{a_1^l\alpha+b}{a^{l_1+1}\alpha+b}=0$, for all $x>l_1$.

   (b) The proof is similar to that of (2a) and is omitted.

Consider the sets:

$$D_+=\{(x,y)\in\mathbb{R}^2:\frac{x}{y}>\frac{b}{1-a}\},$$

$$D_-=\{(x,y)\in\mathbb{R}^2:\frac{x}{y}<\frac{b}{1-a},\}$$

where $a\neq 1$, b is a nonzero real number and let $(\lceil.\rceil$ denote the ceiling function.

3.1. Case $0 < a < 1$.

Theorem 3.3. Assume for $i = 1,2$ that either $(x_{-2+i}, y_{-2+i}) \in D_+$ with $b > 0$ or $(x_{-2+i}, y_{-2+i}) \in D_-$ with $b < 0$ (respectively). Then except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located either in the 1^{st} quadrant or the 2^{nd} quadrant (respectively).

Proof.
When $(x_{-2+i}, y_{-2+i}) \in D_+$, $i = 1,2$, we get $\alpha_1 > 0$ and $\alpha_2 > 0$.
Using formulas (2.3) and (2.4), we get

$$sgn(x_{2m+i}) = sgn(\frac{a^m\alpha_i + b}{1-a}) = 1, i = 1,2.$$

Similarly,

$$sgn(y_{2m+i}) = sgn\left(\frac{a^m\alpha_i + b}{a^{m+1}\alpha_i + b}\right) = 1,\ i = 1,2.$$

Then we conclude (using Lemma (3.1) (1) and Lemma (3.2) (1)) that, except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant.
When $(x_{-i}, y_{-i}) \in D_-$ with $b < 0$ for $i = 1,2$, the proof is similar and is omitted.

Theorem 3.4. Assume for $i = 1,2$ that either $(x_{-2+i}, y_{-2+i}) \in D_-$ with $b > 0$ or $(x_{-2+i}, y_{-2+i}) \in D_+$ with $b < 0$ (respectively). Then the following statements are true:

1. If $-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2} \in]1,\infty[$, then except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located either in the 1^{st} quadrant or the 2^{nd} quadrant (respectively).
2. If $-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2} \in]0,1[$, then there exists a positive integer n_0 such that $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ is located either in the 1^{st} quadrant or the 2^{nd} quadrant (respectively).

Proof.
We shall prove only when $(x_{-2+i}, y_{-2+i}) \in D_-$ with $b > 0$, $i = 1,2$. For the other case, the proof is similar and will be omitted.
Assume that $(x_{-2+i}, y_{-2+i}) \in D_-$, $i = 1,2$. Then $\alpha_i < 0$, $i = 1,2$.

1. If $-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2} \in]1,\infty[$, then using Lemma (3.1) (2b) and Lemma (3.2) (2a), we get

   $$sgn(x_{2m+i}) = sgn(\frac{a^m\alpha_i + b}{1-a}) = 1,\ i = 1,2,\ m = 0,1,...,$$

   and

   $$sgn(y_{2m+i}) = sgn\left(\frac{a^m\alpha_i + b}{a^{m+1}\alpha_i + b}\right) = 1,\ i = 1,2\ m = 0,1,....$$

   Therefore, except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant.

2. If $-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2} \in]0,1[$, then using Lemma (3.1) (2b) and Lemma (3.2) (2a), we conclude that there exists a positive integer $\lceil\frac{\ln(-\frac{b}{\alpha_i})}{\ln a}\rceil$ such that

   $$sgn(x_{2m+i}) = sgn(\frac{a^m\alpha_i + b}{1-a}) = 1,\ m \geq \lceil\frac{\ln(-\frac{b}{\alpha_i})}{\ln a}\rceil,\ i = 1,2,$$

and

$$sgn(y_{2m+i}) = sgn\left(\frac{a^m\alpha_i + b}{a^{m+1}\alpha_i + b}\right) = 1,\ m \geq \lceil\frac{\ln(-\frac{b}{\alpha_i})}{\ln a}\rceil,\ i = 1,2.$$

Now, we claim that

$$sgn(x_n) = 1,\ n \geq n_0,$$

where

$$n_0 = max\{2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil + 1, 2\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 2\} - 1.$$

To prove the claim, let $n_0' := max\{2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil + 1, 2\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 2\}$. We have three cases to consider:

- If $\alpha_1 = \alpha_2 := \alpha$, then $n_0' = 2\lceil\frac{\ln(-\frac{b}{\alpha})}{\ln a}\rceil + 2$.
  But
  $$sgn(x_n) = 1, n = 2\lceil\frac{\ln(-\frac{b}{\alpha})}{\ln a}\rceil + 1.$$
  Then
  $$sgn(x_n) = 1,\ n \geq n_0' - 1 = n_0.$$
- If $\alpha_1 < \alpha_2$, then $\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil > \lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil$.
  It follows that
  $$n_0' - 1 = 2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil \geq 2\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 2.$$
  Therefore,
  $$sgn(x_n) = 1,\ n \geq n_0' - 1 = n_0.$$
- If $\alpha_1 > \alpha_2$, then $\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil < \lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil$.
  It follows that
  $$n_0' - 1 = 2\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 1 \geq 2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil + 3.$$
  Therefore,
  $$sgn(x_n) = 1,\ n \geq n_0' - 1 = n_0.$$

The claim is proved.
Therefore, for $n \geq n_0 = max\{2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil + 1, 2\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 2\} - 1$, (x_n, y_n) is located in the 1^{st} quadrant.

Theorem 3.5. Assume that $(x_{-1}, y_{-1}) \in D_+$ and $(x_0, y_0) \in D_-$. Then the following statements are true:

1. If $b > 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 1^{st} quadrant when $-\frac{b}{\alpha_2} \in]1, \infty[$ or there exists a positive integer n_2 such that $\{(x_n, y_n)\}_{n=n_2}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_2} \in]0, 1[$.
2. If $b < 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_1} \in]1, \infty[$ or there exists a positive integer n_1 such that $\{(x_n, y_n)\}_{n=n_1}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_1} \in]0, 1[$.

Proof.

1. Assume that $b > 0$ and let $-\frac{b}{\alpha_2} \in]1,\infty[$. Then $\alpha_1 b > 0$ and $\alpha_2 b < 0$. Using Lemma (3.1) and Lemma (3.2), we get
$$sgn(x_{2m+i}) = sgn(\frac{a^m\alpha_i + b}{1-a}) = 1,\ i = 1,2,\ m = 0,1,...,$$
and
$$sgn(y_{2m+i}) = sgn\left(\frac{a^m\alpha_i + b}{a^{m+1}\alpha_i + b}\right) = 1,\ i = 1,2\ m = 0,1,....$$
Therefore, we have except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant.
Otherwise, if $-\frac{b}{\alpha_2} \in]0,1[$, then there exists a positive integer $m_2 := \lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil$ such that
$$sgn(x_{2m+2}) = sgn(\frac{a^m\alpha_2 + b}{1-a}) = 1,\ m \geq m_2,$$
and
$$sgn(y_{2m+2}) = sgn\left(\frac{a^m\alpha_2 + b}{a^{m+1}\alpha_2 + b}\right) = 1,\ m \geq m_2.$$
Then $\{(x_n, y_n)\}_{n=n_2}^{\infty}$ is located in the 1^{st} quadrant, where $n_2 = 2m_2 + 1$.
Note that:
$$sgn(x_{2m_2+1}) = 1 \text{ and } sgn(y_{2m_2+1}) = 1.$$

2. When $b < 0$, the proof is similar and is omitted.

Note: In Theorem (3.5), we have $n_1 = \lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil$ and $n_2 = \lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 1$.

Theorem 3.6. Assume that $(x_{-1}, y_{-1}) \in D_-$ and $(x_0, y_0) \in D_+$.

1. If $b > 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 1^{st} quadrant when $-\frac{b}{\alpha_1} \in]1,\infty[$ or $\{(x_n, y_n)\}_{n=n_1}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_1} \in]0,1[$.

2. If $b < 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_2} \in]1,\infty[$ or $\{(x_n, y_n)\}_{n=n_2}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_2} \in]0,1[$.

Proof.
The proof is similar to that of Theorem (3.5) and is omitted. Note: In Theorem (3.6), the values of n_1 and n_2 are:
$$n_1 = 2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil \text{ and } n_2 = \lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil + 1.$$
To illustrate Theorem (3.3) and Theorem (3.5), we give the following numerical examples:

Example (1) Assume that $a = 0.8, b = -0.4$ and the initial values are $(x_{-1}, y_{-1}) = (3,-1)$, $(x_0, y_0) = (-7,0.2)$ $((x_{-2+i}, y_{-2+i}) \in D_-,\ i = 1,2)$. Then except (possibly) for the initial values, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 2^{nd} quadrant.
Here $\alpha_1 = -0.2$, $\alpha_2 = -6.6$.

The values of the first 30 terms (including the initial values) of the solution are:

$(3,-1)$, $(-7,0.2)$, $(-3,1.07143)$, $(-35,1.23239)$, $(-2.8,1.06061)$, $(-28.4,1.22837)$, $(-2.64,1.05096)$, $(-23.12,1.22354)$, $(-2.512,1.0425)$, $(-18.896,1.21778)$, $(-2.4096,1.03519)$, $(-15.5168,1.21098)$, $(-2.32768,1.02897)$, $(-12.8134,1.20305)$, $(-2.26214,1.02373)$, $(-10.6508,1.19395)$, $(-2.20972,1.01935)$, $(-8.9206,1.18366)$, $(-2.16777,1.01572)$, $(-7.53648,1.17223)$, $(-2.13422,1.01274)$, $(-6.42919,1.1598)$, $(-2.10737,1.0103)$, $(-5.54335,1.14658)$, $(-2.0859,1.0083)$, $(-4.83468,1.13284)$, $(-2.06872,1.00669)$, $(-4.26774,1.11891)$, $(-2.05498,1.00538)$, $(-3.81419,1.10513)$, $(-2.04398,1.00432)$, $(-3.45136,1.09183)$.

(See figure 1).

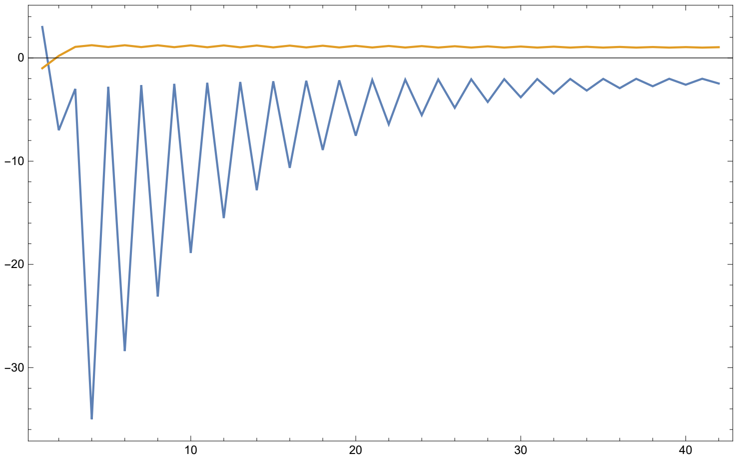

Figure 1. $x_{n+1}=\frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1}=\frac{x_{n-1}}{0.8x_{n-1}-0.4y_{n-1}}$

Example (2) Assume that $a=0.5, b=2$ and the initial values are $(x_{-1},y_{-1})=(3,0.5)$, $(x_0,y_0)=(26,-0.2)$ $((x_{-1},y_{-1})\in D_+,(x_0,y_0)\in D_-)$. Then for $n\geq 13$, (x_n,y_n) is located in the 1^{st} quadrant.
Here $n_2=2\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil+1=13$, where $\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}=\frac{2}{67}\in]0,1]$, $\alpha_2=-67$.
The values of the first 30 terms (including the initial values) of the solution are:

$(3,0.5)$, $(26,-0.2)$, $(6,1.2)$, $(-130,2.06349)$, $(5,1.11111)$, $(-63,2.13559)$, $(4.5,1.05882)$, $(-29.5,2.31373)$, $(4.25,1.0303)$, $(-12.75,2.91429)$, $(4.125,1.01538)$, $(-4.375,23.3333)$, $(4.0625,1.00775)$, $(-0.1875,-0.0983607)$, $(4.03125,1.00389)$, $(1.90625,0.645503)$, $(4.01563,1.00195)$, $(2.95312,0.849438)$, $(4.00781,1.00098)$, $(3.47656,0.92999)$, $(4.00391,1.00049)$, $(3.73828,0.966179)$, $(4.00195,1.00024)$, $(3.86914,0.983371)$, $(4.00098,1.00012)$, $(3.93457,0.991754)$, $(4.00049,1.00006)$, $(3.96729,0.995894)$, $(4.00024,1.00003)$, $(3.98364,0.997951)$.

Clear that $x_n>0$ and $y_n>0$, $n\geq 13$. (See figure 2).

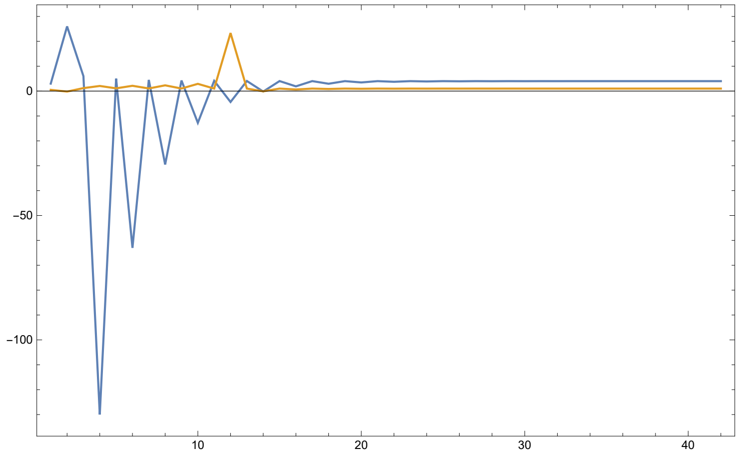

Figure 2. $x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{0.5x_{n-1}+2y_{n-1}}$

3.2. Case $a > 1$.

Theorem 3.7. Assume for $i = 1,2$ that $(x_{-2+i}, y_{-2+i}) \in D_+$. Then we have the following:

1. If $b > 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 1^{st} quadrant when $-\frac{b}{\alpha_i} \in]0,1[$, $i = 1,2$ or there exists a positive integer n_0 such that $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_i} \in]1,\infty[$, $i = 1,2$.
2. If $b < 0$, then except for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant.

Proof.
When $(x_{-2+i}, y_{-2+i}) \in D_+$ for $i = 1,2$, we get $\alpha_i < 0$ for $i = 1,2$.

1. If $b > 0$, then $\alpha_i b < 0$, $i = 1,2$. Using Lemma (3.1) and Lemma (3.2), we conclude that except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 1^{st} quadrant when $\max\{-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2}\} < 1$. When $\min\{-\frac{b}{\alpha_1}, -\frac{b}{\alpha_2}\} > 1$, there exist two positive integers m_1 and m_2 such that the subsequences $\{(x_{2m+1}, y_{2m+1})\}_{n=m_1}^{\infty}$ and $\{(x_{2m+2}, y_{2m+2})\}_{n=m_2}^{\infty}$ are located in the 1^{st} quadrant, where $m_1 = \lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil$ and $m_2 = \lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil$. Therefore, we conclude that $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ is located in the 1^{st} quadrant, where $n_0 = \max\{2\lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil + 1, 2\lceil \frac{\ln(-\frac{b}{\alpha_2})}{\ln a} \rceil + 2\} - 1$.
2. When $b < 0$, the proof is a direct consequence of applying Lemma (3.1) (1) and Lemma (3.2) (1).

Theorem 3.8. Assume for $i = 1,2$ that $(x_{-2+i}, y_{-2+i}) \in D_-$. Then we have the following:

1. If $b > 0$, then except (possibly) for the initial conditions, the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is located in the 2^{nd} quadrant.
2. If $b < 0$, then either the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ (except (possibly) for the initial conditions) is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_i} \in]0,1[$, $i = 1,2$ or $\{(x_n, y_n)\}_{n=n_0}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_i} \in]1,\infty[$, $i = 1,2$.

Proof.
The proof is similar to that of Theorem (3.7) and is omitted. To illustrate Theorem (3.8), we give the follow-

ing numerical example:

Example (3) Assume that $a=1.2, b=-3$ and the initial values are $(x_{-1},y_{-1})=(-2.7,-0.3)$, $(x_0,y_0)=(12.1,1.1)$ $((x_{-2+i},y_{-2+i})\in D_-,\ i=1,2)$. Then for $n\geq 17$, (x_n,y_n) is located in the 2^{nd} quadrant.
Here $n_0=\max\{2\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil+1, 2\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil+2\}-1=17$, where $\lceil\frac{\ln(-\frac{b}{\alpha_1})}{\ln a}\rceil=6$ and $\lceil\frac{\ln(-\frac{b}{\alpha_2})}{\ln a}\rceil=8$.
The values of the first 30 terms (including the initial values) of the solution are:

$$(-2.7,-0.3),\ (12.1,1.1),\ (9,1.15385),\ (11,1.07843),\ (7.8,1.22642),\ (10.2,1.1039),$$
$$(6.36,1.37306),\ (9.24,1.14243),\ (4.632,1.81051),\ (8.088,1.20616),\ (2.5584,36.5068),$$
$$(6.7056,1.3287),\ (0.07008,-0.0240337),\ (5.04672,1.65138),\ (-2.9159,0.448664),$$
$$(3.05606,4.5799),\ (-6.49908,0.601828),\ (0.667277,-0.303409),\ (-10.7989,0.676679),$$
$$(-2.19927,0.390002),\ (-15.9587,0.720469),\ (-5.63912,0.577368),\ (-22.1504,0.748818),$$
$$(-9.76695,0.6635),\ (-29.5805,0.768393),\ (-14.7203,0.712352),\ (-38.4966,0.782516),$$
$$(-20.6644,0.743396),\ (-49.1959,0.793034),\ (-27.7973,0.76457),\ (-62.0351,0.801051),$$
$$(-36.3567,0.779718).$$

Clear that $x_n<0$ and $y_n>0$, $n\geq 17$. (See figure 3).

Theorem 3.9. Assume that $(x_{-1},y_{-1})\in D_-$ and $(x_0,y_0)\in D_+$. Then we have the following:

1. If $b>0$, then except for the initial conditions we have, the subsequence $\{(x_{2m+1},y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant and either the subsequence $\{(x_{2m+2},y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_2}\in]0,1[$, or the subsequence $\{(x_{2m+2},y_{2m+2})\}_{m=m_2}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_2}\in]1,\infty[$.

2. If $b<0$, then except for the initial conditions we have, the subsequence $\{(x_{2m+2},y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant and either the subsequence $\{(x_{2m+1},y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_1}\in]0,1[$, or the subsequence $\{(x_{2m+1},y_{2m+1})\}_{m=m_1}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_1}\in]1,\infty[$.

Proof.
Assume that $(x_{-1},y_{-1})\in D_-$ and $(x_0,y_0)\in D_+$. Then $\alpha_1>0$ and $\alpha_2<0$.

1. When $b>0$, then $\alpha_1 b>0$ and $\alpha_2 b<0$. Using Lemmas (3.1) (1) and (3.2) (1), we conclude that except (possibly) for the initial conditions we have, the subsequence $\{(x_{2m+1},y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant.
If $-\frac{b}{\alpha_2}\in]0,1[$, then using Lemmas (3.1) (2b) and (3.2) (2b), we conclude that except (possibly) for the initial conditions we have, the subsequence $\{(x_{2m+2},y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant.
Otherwise, if $-\frac{b}{\alpha_2}\in]1,\infty[$, then the subsequence $\{(x_{2m+2},y_{2m+2})\}_{m=m_2}^{\infty}$ is located in the 1^{st} quadrant.

2. The proof is similar to (1) and is omitted.

Theorem 3.10. Assume that $(x_{-1},y_{-1})\in D_+$ and $(x_0,y_0)\in D_-$.

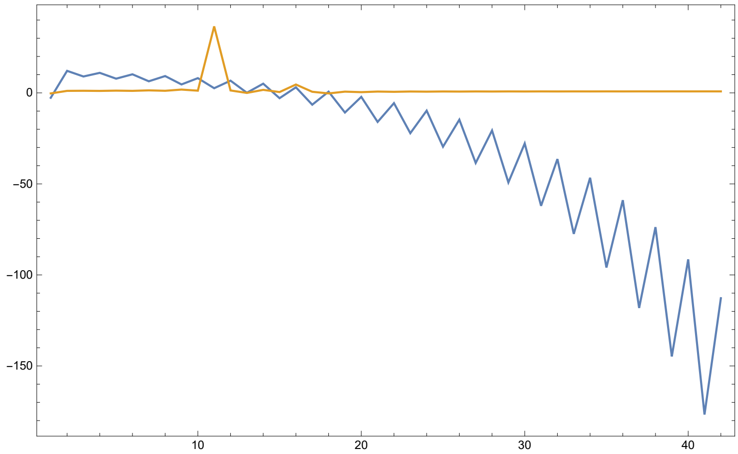

Figure 3. $x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{1.2x_{n-1}-3y_{n-1}}$

1. If $b > 0$, then except (possibly) for the initial conditions we have, the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant and either the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_1} \in]0,1[$, or the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=m_1}^{\infty}$ is located in the 1^{st} quadrant when $-\frac{b}{\alpha_1} \in]1,\infty[$.

2. If $b < 0$, then except (possibly) for the initial conditions we have, the subsequence $\{(x_{2m+1}, y_{2m+1})\}_{m=-1}^{\infty}$ is located in the 1^{st} quadrant and either the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_2} \in]0,1[$, or the subsequence $\{(x_{2m+2}, y_{2m+2})\}_{m=m_2}^{\infty}$ is located in the 2^{nd} quadrant when $-\frac{b}{\alpha_2} \in]1,\infty[$.

Proof.
The proof is similar to that of Theorem (3.9) and is omitted. To illustrate Theorem (3.10), we give the following numerical example:

Example (4) Assume that $a = 1.5, b = 1$ and the initial values are $(x_{-1}, y_{-1}) = (3.9, -2)$, $(x_0, y_0) = (-1.5, 0.5)$ $((x_{-1}, y_{-1}) \in D_+, (x_0, y_0) \in D_-)$. Then the solution $\{(x_n, y_n\}_{n=-1}^{\infty}$ has the property that:
Except (possibly) for the initial values, $\{(x_{2m+2}, y_{2m+2}\}_{m=-1}^{\infty}$ is located in the 2^{nd} quadrant and $\{(x_{2m+1}, y_{2m+1}\}_{m=10}^{\infty}$ is located in the 1^{st} quadrant.
Here $m_1 = \lceil \frac{\ln(-\frac{b}{\alpha_1})}{\ln a} \rceil = 10$, where $\frac{x_{-1}}{y_{-1}} = -1.95$ and $\frac{x_0}{y_0} = -3$, $-\frac{b}{\alpha_1} = 40 \in]1,\infty[$
The values of the first 30 terms (including the initial values) of the solution are:

(3.9, −2), (−1.5, 0.5), (−1.95, 1.01299), (−3., 0.857143), (−1.925, 1.01987), (−3.5, 0.823529), (−1.8875, 1.03072), (−4.25, 0.790698), (−1.83125, 1.0483), (−5.375, 0.761062), (−1.74687, 1.07811), (−7.0625, 0.736156), (−1.62031, 1.13271), (−9.59375, 0.716453), (−1.43047, 1.24855), (−13.3906, 0.701596), (−1.1457, 1.59446), (−19.0859, 0.690796), (−0.718555, 9.23212), (−27.6289, 0.683151), (−0.077832, −0.0881199), (−40.4434, 0.67784), (0.883252, 0.379913), (−59.665, 0.6742), (2.32488, 0.5181), (−88.4976, 0.671727), (4.48732, 0.580433), (−131.746, 0.670057), (7.73098, 0.613742), (−196.62, 0.668935), (12.5965, 0.633157), (−293.929, 0.668182).

(See figure 4).

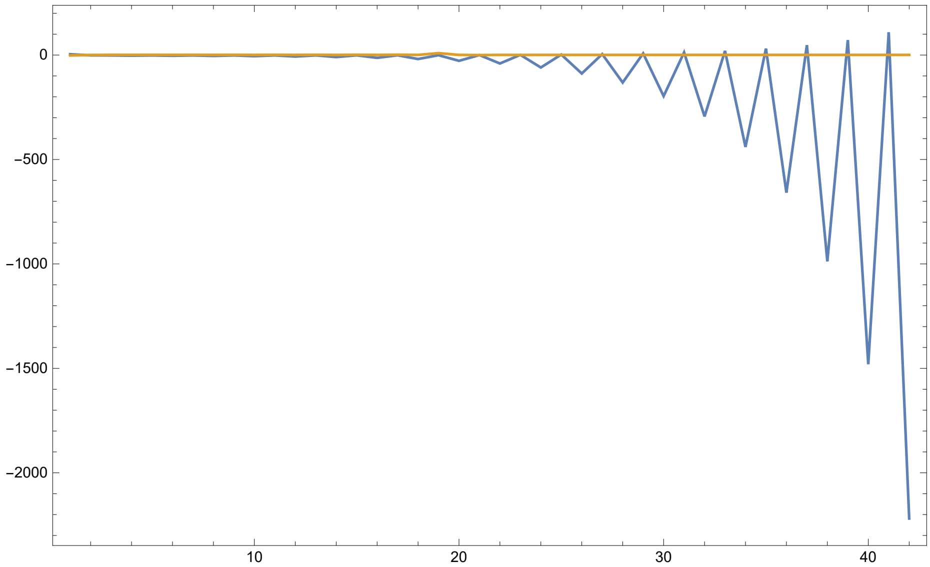

Figure 4. $x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{1.5x_{n-1}+y_{n-1}}$

Discussions and Conclusions

In this paper, we studied the admissible solutions of the non-linear discrete system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{ax_{n-1} + by_{n-1}}, \quad n = 0, 1, \ldots,$$

where a, b and the initial values x_{-1}, x_0, y_{-1}, y_0 are non-zero real numbers. We discussed the linearized and global stability to the steady state $(\frac{b}{1-a}, 1)$ when $a \neq 1$ as well as introducing the forbidden sets. For $a \in \mathbb{R}_+ - \{1\}$, we showed any admissible solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is either entirely located in a certain quadrant of the plane or there exists a natural number $N > 0$ (we calculated its value) such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is located. We conjecture that the same results can be obtained for the discrete system

$$x_{n+1} = \frac{x_{n-k}}{y_{n-k}}, \quad y_{n+1} = \frac{x_{n-k}}{ax_{n-k} + by_{n-k}}, \quad n = 0, 1, \ldots,$$

where a, b are non-zero real numbers and the initial points (x_{-i}, y_{-i}), where $i = 0, 1, ..., k$ are non-zero real numbers.

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Author Contributions

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Conflicts of Interest

The authors declare no conflict of interest.

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A Qualitative Investigation of a System of Third-Order Difference Equations with Multiplicative Reciprocal Terms

Durhasan Turgut Tollu[1], İbrahim Yalçınkaya[2]

Keywords
Boundedness, Equilibrium point, System of difference equations

Abstract — In this paper, we study the system of third-order difference equations

$$x_{n+1} = a + \frac{a_1}{y_n} + \frac{a_2}{y_{n-1}} + \frac{a_3}{y_{n-2}}, \quad y_{n+1} = b + \frac{b_1}{x_n} + \frac{b_2}{x_{n-1}} + \frac{b_3}{x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where the parameters a, a_i, b, b_i $(i = 1,2,3)$ and the initial values x_{-j}, y_{-j} $(j = 0,1,2)$ are positive real numbers. We first prove a general convergence theorem. By applying this convergence theorem to the system, we show that positive equilibrium is a global attractor. We also study the local asymptotic stability of the equilibrium and show that it is globally asymptotically stable. Finally, we study the invariant set of solutions.

Subject Classification (2020): 39A10, 39A20, 39A30.

1. Introduction

Difference equations have been studied with great interest for the last thirty years. Determining the qualitative behavior of solutions, which is very important in applications, forms the basis of these studies. Difference equations have become a significant topic in mathematics and other disciplines because they can be discrete analogs of differential equations or mathematical models of phenomena. For some examples of discrete analogs of differential equations, see [1]. For some mathematical models, see [8]. In our opinion, this fact is the basis of the intense interest mentioned above. But whatever the reason, some classes of difference equations are being studied for the development of the theory of difference equations, even though they are not any mathematical models. The main idea, of course, is to discover new classes of difference equations and to develop new techniques and methods for determining the qualitative behavior of solutions of difference equations.

Since many mathematical models are nonlinear, nonlinear difference equations are studied quite frequently. Rational difference equations, as a subclass of nonlinear difference equations, are also frequently encountered in the literature. Below, we list some old and new studies that we encounter in the literature on the rational difference equations that we think are related to our research.

[1]dttollu@erbakan.edu.tr (Corresponding Author); [2]iyalcinkaya@erbakan.edu.tr
[1]Department of Mathematics and Computer Sciences, Faculty of Science, Necmettin Erbakan University, Konya, Türkiye
[2]Department of Mathematics and Computer Sciences, Faculty of Science, Necmettin Erbakan University, Konya, Türkiye
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In [6], DeVault et al. conducted a boundedness study on positive solutions of the second-order difference equation

$$x_{n+1} = \frac{A}{x_n^p} + \frac{B}{x_{n-1}^q}, \quad n \in \mathbb{N}_0,$$

where p, q, A, B, and the initial values are positive real numbers.

In [7], DeVault et al. showed that every positive solution of the third-order equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where $A \in (0,\infty)$, converges to a two-periodic solution of the equation.

In [28], Philos et al. studied the attractivity of the unique positive equilibrium of the higher-order equation

$$x_{n+1} = a + \sum_{k=1}^{m} \frac{b_k}{x_{n-k}}, \quad n \in \mathbb{N}_0,$$

where a and b_k $(k = 1,2,\ldots,m)$ are nonnegative real parameters with $B = \sum\limits_{k=1}^{m} b_k > 0$.

In [9], El-Metwally et al. established a global convergence result and applied it to the higher-order equation

$$x_{n+1} = \sum_{i=0}^{m} \frac{A_i}{x_{n-2i}}, \quad n \in \mathbb{N}_0,$$

where A_i $(i = 1,2,\ldots,m)$ are nonnegative and the initial values are positive. They showed that every positive solution of the equation converges to a two-periodic solution.

In [10], El-Metwally et al. established a global convergence result and applied it to the higher-order equation

$$x_{n+1} = \sum_{i=0}^{k-1} \frac{A_i}{x_{n-i}}, \quad n \in \mathbb{N}_0,$$

where A_i $(i = 0,1,\ldots,k-1)$ are nonnegative with $A = \sum\limits_{i=1}^{k-1} A_i > 0$, and the initial values are positive. They showed that every positive solution of the equation converges to a $p-$periodic solution.

The study of two-dimensional systems, which are generally symmetric, of difference equations is a process initiated by Papaschinopoulos and Schinas in the late nineties. See, e.g. [22–26, 29]. Their work encouraged other authors, especially in the area of mathematics, to work on such systems. In the 2000s, studies on nonlinear rational difference equations and their systems gathered speed, and a rich literature emerged. Although this speed is not at the initial level, new studies are being published, especially on difference equation systems.

Fuzzy difference equations, which are a type of difference equation that is by definition particularly related to symmetric systems, also began to be studied during this process. For example, in [27], Papaschinopoulos and Papadopoulos considered the fuzzy difference equation

$$x_{n+1} = A + \frac{B}{x_n}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where A, B, x_0 are fuzzy numbers. Due to the nature of fuzzy difference equations, to study the solutions of

Eq.(1.1), they were interested in the system of classical difference equations

$$y_{n+1}=\alpha+\frac{\beta}{z_n},\quad z_{n+1}=\gamma+\frac{\delta}{y_n},\quad n\in\mathbb{N}_0,$$

which is a special case of the system stated in the abstract of this paper.

In [13], in line with [27], Hatir et al. investigated the behavior of the positive solutions of the fuzzy difference equation

$$x_{n+1}=A+\frac{B}{x_{n-1}},\quad n\in\mathbb{N}_0, \tag{1.2}$$

where the parameters A, B, and the initial values x_{-1}, x_0 are fuzzy numbers. Naturally, to study the positive solutions of Eq.(1.1), they discussed the positive solutions of the system of classical difference equations

$$y_{n+1}=\alpha+\frac{\beta}{z_{n-1}},\quad z_{n+1}=\gamma+\frac{\delta}{y_{n-1}},\quad n\in\mathbb{N}_0,$$

which is another special case of the system in the abstract. For similar studies on fuzzy difference equations, see references [34, 35]. Apart from these, many systems of difference equations have been studied. For some examples, see [2, 3, 5, 11, 12, 14–18, 21, 30–33, 36, 37].

In this work, we define the system of difference equations

$$x_{n+1}=a+\frac{a_1}{y_n}+\frac{a_2}{y_{n-1}}+\frac{a_3}{y_{n-2}},\quad y_{n+1}=b+\frac{b_1}{x_n}+\frac{b_2}{x_{n-1}}+\frac{b_3}{x_{n-2}},\quad n\in\mathbb{N}_0, \tag{1.3}$$

where the parameters a, a_i, b, b_i $(i=1,2,3)$ and the initial values x_{-j}, y_{-j} $(j=0,1,2)$ are positive real numbers. We investigate the qualitative behavior of positive solutions of system (1.3). More specifically, we establish a global convergence result and apply it to (1.3) to study the global stability of the positive equilibrium.

For the methods followed in our study, the references [4, 19, 20] can be consulted.

2. Main Results

In this section, the main results of the paper are given and proven. This section is divided into two subsections.

2.1. A result of convergence

The following theorem states a general convergence result and enables us to prove that the unique positive equilibrium of (1.3) is the global attractor.

Theorem 2.1. Let $[\alpha,\beta]$ and $[\gamma,\delta]$ be intervals of positive real numbers and assume that $h_1:[\gamma,\delta]^{k+1}\to[\alpha,\beta]$ and $h_2:[\alpha,\beta]^{k+1}\to[\gamma,\delta]$ are continuous functions satisfying the following properties:

(a) Both $h_1(y_1,y_2,\ldots,y_{k+1})$ and $h_2(x_1,x_2,\ldots,x_{k+1})$ are decreasing in all of the arguments.

(b) If $(m_1,M_1,m_2,M_2)\in[\alpha,\beta]^2\times[\gamma,\delta]^2$ is a solution of the system

$$\begin{aligned} m_1 &= h_1(M_2,M_2,\ldots,M_2),\quad M_1=h_1(m_2,m_2,\ldots,m_2),\\ m_2 &= h_2(M_1,M_1,\ldots,M_1),\quad M_2=h_2(m_1,m_1,\ldots,m_1), \end{aligned} \tag{2.1}$$

then $m_1 = M_1$ and $m_2 = M_2$. Then the system

$$\left.\begin{array}{l} x_{n+1} = h_1(y_n, y_{n-1}, \ldots, y_{n-k}) \\ y_{n+1} = h_2(x_n, x_{n-1}, \ldots, x_{n-k}) \end{array}\right\}, \quad n \in \mathbb{N}_0, \tag{2.2}$$

has a unique positive equilibrium $(\overline{x}, \overline{y}) \in [\alpha, \beta] \times [\gamma, \delta]$ and its every positive solution converges to this equilibrium.

Proof.
Let

$$m_1^0 := \alpha, \quad M_1^0 := \beta, \quad m_2^0 := \gamma, \quad M_2^0 := \delta$$

and

$$\begin{aligned} m_1^{i+1} &: = h_1(M_2^i, M_2^i, \ldots, M_2^i), \quad M_1^{i+1} := h_1(m_2^i, m_2^i, \ldots, m_2^i), \\ m_2^{i+1} &: = h_2(M_1^i, M_1^i, \ldots, M_1^i), \quad M_2^{i+1} := h_2(m_1^i, m_1^i, \ldots, m_1^i). \end{aligned}$$

For each $i = 0, 1, \ldots$, we have

$$\begin{aligned} \alpha &\leq h_1(\delta, \delta, \ldots, \delta) \leq h_1(\gamma, \gamma, \ldots, \gamma) \leq \beta, \\ \gamma &\leq h_2(\beta, \beta, \ldots, \beta) \leq h_2(\alpha, \alpha, \ldots, \alpha) \leq \delta \end{aligned}$$

and so,

$$\begin{aligned} m_1^0 &= \alpha \leq h_1(M_2^0, M_2^0, \ldots, M_2^0) = m_1^1 \leq h_1(m_2^0, m_2^0, \ldots, m_2^0) = M_1^1 \leq \beta = M_1^0, \\ m_2^0 &= \gamma \leq h_2(M_1^0, M_1^0, \ldots, M_1^0) = m_2^1 \leq h_2(m_1^0, m_1^0, \ldots, m_1^0) = M_2^1 \leq \delta = M_2^0. \end{aligned}$$

Moreover, we have

$$\begin{aligned} m_1^1 &= h_1(M_2^0, M_2^0, \ldots, M_2^0) \\ &\leq h_1(M_2^1, M_2^1, \ldots, M_2^1) \\ &= m_1^2 \\ &\leq h_1(m_2^1, m_2^1, \ldots, m_2^1) \\ &= M_1^2 \\ &\leq h_1(m_2^0, m_2^0, \ldots, m_2^0) \\ &= M_1^1, \end{aligned}$$

and

$$\begin{aligned} m_2^1 &= h_2(M_1^0, M_1^0, \ldots, M_1^0) \\ &\le h_2(M_1^1, M_1^1, \ldots, M_1^1) \\ &= m_2^2 \\ &\le h_2(m_1^1, m_1^1, \ldots, m_1^1) \\ &= M_2^2 \\ &\le h_2(m_1^0, m_1^0, \ldots, m_1^0) \\ &= M_2^1. \end{aligned}$$

By induction, one can see for $i = 0, 1, \ldots,$ that

$$\begin{aligned} \alpha &= m_1^0 \le m_1^1 \le \cdots \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le \cdots \le M_1^1 \le M_1^0 = \beta, \\ \gamma &= m_2^0 \le m_2^1 \le \cdots \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le \cdots \le M_2^1 \le M_2^0 = \delta. \end{aligned}$$

It follows that the sequences $(m_1^i)_i$ and $(m_2^i)_i$ (resp. $(M_1^i)_i$ and $(M_2^i)_i$) are increasing (resp. decreasing) and also bounded, and therefore they are convergent sequences. Then we can assume that

$$m_1 = \lim_{i\to+\infty} m_1^i, \quad M_1 = \lim_{i\to+\infty} M_1^i, \quad m_2 = \lim_{i\to+\infty} m_2^i, \quad M_2 = \lim_{i\to+\infty} M_2^i.$$

Then,

$$\alpha \le m_1 \le M_1 \le \beta, \quad \gamma \le m_2 \le M_2 \le \delta.$$

By taking limits in the equalities

$$\begin{aligned} m_1^{i+1} &= h_1(M_2^i, M_2^i, \ldots, M_2^i), \quad M_1^{i+1} = h_1(m_2^i, m_2^i, \ldots, m_2^i), \\ m_2^{i+1} &= h_2(M_1^i, M_1^i, \ldots, M_1^i), \quad M_2^{i+1} = h_2(m_1^i, m_1^i, \ldots, m_1^i), \end{aligned}$$

and using that h_1 and h_2 are continuous, we obtain system (2.1). So, from (b), it follows that $m_1 = M_1$ and $m_2 = M_2$. It can be concluded from the hypothesis that

$$m_1^0 = \alpha \le x_n \le \beta = M_1^0, \quad m_2^0 = \gamma \le y_n \le \delta = M_2^0, \quad n = 1, 2, \ldots.$$

Therefore, we obtain

$$\begin{aligned} m_1^1 &= h_1(M_2^0, M_2^0, \ldots, M_2^0) \le h_1(y_n, y_{n-1}, \ldots, y_{n-2}) = x_{n+1} \le h_1(m_2^0, m_2^0, \ldots, m_2^0) = M_1^1, \\ m_2^1 &= h_2(M_1^0, M_1^0, \ldots, M_1^0) \le h_2(x_n, x_{n-1}, \ldots, x_{n-2}) = y_{n+1} \le h_2(m_1^0, m_1^0, \ldots, m_1^0) = M_2^1, \end{aligned}$$

for $n = 2, 3, \ldots,$ and

$$\begin{aligned} m_1^2 &= h_1(M_2^1, M_2^1, \ldots, M_2^1) \le h_1(y_n, y_{n-1}, \ldots, y_{n-2}) = x_{n+1} \le h_1(m_2^1, m_2^1, \ldots, m_2^1) = M_1^2, \\ m_2^2 &= h_2(M_1^1, M_1^1, \ldots, M_1^1) \le h_2(x_n, x_{n-1}, \ldots, x_{n-2}) = y_{n+1} \le h_2(m_1^1, m_1^1, \ldots, m_1^1) = M_2^2. \end{aligned}$$

for $n = 4,5,\ldots$, and

$$\begin{aligned} m_1^3 &= h_1(M_2^2, M_2^2, \ldots, M_2^2) \le h_1(y_n, y_{n-1}, \ldots, y_{n-2}) = x_{n+1} \le h_1(m_2^2, m_2^2, \ldots, m_2^2) = M_1^3, \\ m_2^3 &= h_2(M_1^2, M_1^2, \ldots, M_1^2) \le h_2(x_n, x_{n-1}, \ldots, x_{n-2}) = y_{n+1} \le h_2(m_1^2, m_1^2, \ldots, m_1^2) = M_2^3 \end{aligned}$$

for $n = 6,7,\ldots$. Moreover, by induction, it follows for $i = 0,1,\ldots$, that

$$m_1^i \le x_n \le M_1^i, \quad m_2^i \le y_n \le M_2^i, \quad n \ge 2i+1.$$

It is obvious that $i \to +\infty$ implies $n \to +\infty$. Also, since $m_1 = M_1$ and $m_2 = M_2$, we obtain

$$\lim_{n\to+\infty} x_n = M_1, \quad \lim_{n\to+\infty} y_n = M_2.$$

Moreover, in this case, since system (2.1) reduces to

$$M_1 = h_1(M_2, M_2, \ldots, M_2), \quad M_2 = h_2(M_1, M_1, \ldots, M_1),$$

we obtain

$$M_1 = \overline{x}, \quad M_2 = \overline{y}.$$

Therefore, the proof is completed.

2.2. Dynamics of system (1.3)

We here begin our study on system (1.3). For the sake of simplicity, let $a_1 + a_2 + a_3 = \alpha$ and $b_1 + b_2 + b_3 = \beta$. The equilibrium points of system (1.3) correspond to the solutions of the system

$$\overline{x} = a + \frac{\alpha}{\overline{y}}, \quad \overline{y} = b + \frac{\beta}{\overline{x}}, \tag{2.3}$$

from which it follows that

$$\begin{aligned} \overline{x} &= \frac{\beta - \alpha - ab \pm \sqrt{\Delta}}{2b}, \\ \overline{y} &= \frac{\alpha - \beta - ab \pm \sqrt{\Delta}}{2a}, \end{aligned}$$

where

$$\begin{aligned} \Delta &= (\alpha - \beta - ab)^2 + 4ab\alpha \\ &= (\beta - \alpha - ab)^2 + 4ab\beta \\ &> 0. \end{aligned}$$

Hence, system (1.3) possesses the positive equilibrium point

$$(\overline{x}, \overline{y}) = \left(\frac{\beta - \alpha - ab + \sqrt{\Delta}}{2b}, \frac{\alpha - \beta - ab + \sqrt{\Delta}}{2a} \right).$$

Theorem 2.2. The equilibrium $(\overline{x},\overline{y})$ of system (1.3) is locally asymptotically stable.

Proof.
Let

$$\begin{aligned} f &: = a+\frac{a_1}{y_n}+\frac{a_2}{y_{n-1}}+\frac{a_3}{y_{n-2}}, \\ f_1 &: = x_n, \\ f_2 &: = x_{n-1}, \\ g &: = b+\frac{b_1}{x_n}+\frac{b_2}{x_{n-1}}+\frac{b_3}{x_{n-2}}, \\ g_1 &: = y_n, \\ g_2 &: = y_{n-1}. \end{aligned}$$

Then, we can define a map $T:(0,\infty)^6 \longrightarrow (0,\infty)^6$ and the system corresponding to T as follows:

$$W_{n+1} = T(W_n), \tag{2.4}$$

where $W_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^t$, ($t$ states the transpose operation)

$$T\begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix} = \begin{pmatrix} a+\frac{a_1}{y_n}+\frac{a_2}{y_{n-1}}+\frac{a_3}{y_{n-2}} \\ x_n \\ x_{n-1} \\ b+\frac{b_1}{x_n}+\frac{b_2}{x_{n-1}}+\frac{b_3}{x_{n-2}} \\ y_n \\ y_{n-1} \end{pmatrix}.$$

In this case, the equilibrium of (2.4) is $E = (\overline{x},\overline{x},\overline{x},\overline{y},\overline{y},\overline{y})^t$. For $i = 0,1,2$, we obtain

$$\begin{aligned} \frac{\partial f}{\partial x_{n-i}}|_E &= 0, \quad \frac{\partial f}{\partial y_{n-i}}|_E = -\frac{a_{i+1}}{\overline{y}^2}, \\ \frac{\partial f_1}{\partial x_n}|_E &= 1, \quad \frac{\partial f_1}{\partial x_{n-1}}|_E = \frac{\partial f_1}{\partial x_{n-2}}|_E = 0, \quad \frac{\partial f_1}{\partial y_{n-i}}|_E = 0, \\ \frac{\partial f_2}{\partial x_n}|_E &= 0, \quad \frac{\partial f_2}{\partial x_{n-1}}|_E = 1, \quad \frac{\partial f_2}{\partial x_{n-2}}|_E = 0, \quad \frac{\partial f_2}{\partial y_{n-i}}|_E = 0, \\ \frac{\partial g}{\partial x_{n-i}}|_E &= -\frac{b_{i+1}}{\overline{x}^2}, \quad \frac{\partial g}{\partial y_{n-i}}|_E = 0, \\ \frac{\partial g_1}{\partial x_{n-i}}|_E &= 0, \quad \frac{\partial g_1}{\partial y_n}|_E = 1, \quad \frac{\partial g_1}{\partial y_{n-1}}|_E = \frac{\partial g_1}{\partial y_{n-2}}|_E = 0, \\ \frac{\partial g_1}{\partial x_{n-i}}|_E &= 0, \quad \frac{\partial g_1}{\partial y_n}|_E = 0, \quad \frac{\partial g_1}{\partial y_{n-1}}|_E = 1, \quad \frac{\partial g_1}{\partial y_{n-2}}|_E = 0. \end{aligned}$$

By these partial derivatives, one can obtain the Jacobian of the map T evaluated at E as follows:

$$J_T(E) = \begin{pmatrix} 0 & 0 & 0 & -\frac{a_1}{\overline{y}^2} & -\frac{a_2}{\overline{y}^2} & -\frac{a_3}{\overline{y}^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{b_1}{\overline{x}^2} & -\frac{b_2}{\overline{x}^2} & -\frac{b_3}{\overline{x}^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix $J_F(E)$ has the characteristic polynomial

$$\begin{aligned} P(\lambda) &= \lambda^6 - \frac{a_1 b_1 \lambda^4 + (a_1 b_2 + a_2 b_1)\lambda^3 + (a_1 b_3 + a_2 b_2 + a_3 b_1)\lambda^2 + (a_2 b_3 + a_3 b_2)\lambda + a_3 b_3}{\overline{x}^2 \overline{y}^2} \\ &= \lambda^6 - \frac{\left(a_1 \lambda^2 + a_2 \lambda + a_3\right)\left(b_1 \lambda^2 + b_2 \lambda + b_3\right)}{\overline{x}^2 \overline{y}^2}. \end{aligned}$$

We need to ensure that all roots of P are less than 1 in absolute value. For this, let

$$\Phi(\lambda) = \lambda^6$$

and

$$\Psi(\lambda) = -\frac{\left(a_1 \lambda^2 + a_2 \lambda + a_3\right)\left(b_1 \lambda^2 + b_2 \lambda + b_3\right)}{\overline{x}^2 \overline{y}^2}.$$

It is easily seen that every root of Φ satisfies the condition $|\lambda| < 1$. That is, those are all less than 1 in absolute value. So, if we assume

$$|\Psi(\lambda)| \leq \frac{(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)}{\overline{x}^2 \overline{y}^2} < 1 = |\Phi(\lambda)|, \quad \forall \lambda \in \mathbb{C}, \quad |\lambda| = 1,$$

then every root of P will satisfy the condition $|\lambda| < 1$ according to Rouché's theorem. After some arrangements, we get the inequality

$$\alpha\beta < \overline{x}^2 \overline{y}^2. \tag{2.5}$$

From (2.3), we obtain

$$\overline{xy} = ab + \frac{b\alpha}{\overline{y}} + \frac{a\beta}{\overline{x}} + \frac{\alpha\beta}{\overline{xy}} \quad \Leftrightarrow \quad \overline{x}^2 \overline{y}^2 = ab\overline{xy} + b\alpha\overline{x} + a\beta\overline{y} + \alpha\beta,$$

and therefore

$$\overline{x}^2 \overline{y}^2 - \alpha\beta = ab\overline{xy} + b\alpha\overline{x} + a\beta\overline{y} > 0,$$

which shows that the inequality in (2.5) is always satisfied. This completes the proof.

Theorem 2.3. Every positive solution of (1.3) is bounded.

Proof.
Let $\left\{(x_n, y_n)\right\}_{n=-2}^{\infty}$ be a positive solution of (1.3). Then, we obtain from (1.3) that

$$x_n \geq a > 0, \quad y_n \geq b > 0 \tag{2.6}$$

for all $n \in \mathbb{N}$. That is, x_n and y_n are bounded from below. Also, it follows from system (1.3) and (2.6) that

$$\begin{aligned} x_{n+1} &= a+\frac{a_1}{y_n}+\frac{a_2}{y_{n-1}}+\frac{a_3}{y_{n-2}} \le a+\frac{\alpha}{b} < \infty, \\ y_{n+1} &= b+\frac{b_1}{x_n}+\frac{b_2}{x_{n-1}}+\frac{b_3}{x_{n-2}} \le b+\frac{\beta}{a} < \infty \end{aligned}$$

for all $n \in \mathbb{N}$. That is, x_n and y_n are bounded from above. This completes the proof.

Theorem 2.4. The positive equilibrium $(\overline{x}, \overline{y})$ of system (1.3) is globally asymptotically stable.

Proof.
Theoretically, for the equilibrium $(\overline{x}, \overline{y})$ to be globally asymptotically stable, it must be locally asymptotically stable. See [19]. But we have already proven this in Theorem 2.2. Then, we only need to show that $(\overline{x}, \overline{y})$ is the global attractor of the positive solutions. That is, we will show that

$$\lim_{n\to\infty} x_n = \bar{x} \text{ and } \lim_{n\to\infty} y_n = \overline{y}.$$

To do this, we apply Theorem 2.1 to (1.3). We know from Theorem 2.3 that x_n and y_n are bounded for all $n \ge 1$. Then, it follows that $a \le m_1 := \lim_{n\to\infty} \inf x_n \le \lim_{n\to\infty} \sup x_n := M_1 \le a + \frac{\alpha}{b}$ and $b \le m_2 := \lim_{n\to\infty} \inf y_n \le \lim_{n\to\infty} \sup y_n := M_2 \le b + \frac{\beta}{a}$. It suffices to show that $m_1 = M_1$ and $m_2 = M_2$.
Consider the system

$$M_1 = a+\frac{a_1}{m_2}+\frac{a_2}{m_2}+\frac{a_3}{m_2}, \tag{2.7}$$

$$m_1 = a+\frac{a_1}{M_2}+\frac{a_2}{M_2}+\frac{a_3}{M_2}, \tag{2.8}$$

$$M_2 = b+\frac{b_1}{m_1}+\frac{b_2}{m_1}+\frac{b_3}{m_1}, \tag{2.9}$$

$$m_2 = b+\frac{b_1}{M_1}+\frac{b_2}{M_1}+\frac{b_3}{M_1}. \tag{2.10}$$

Then, from (2.7) and (2.10), it follows that

$$bM_1^2+\left(\beta-\alpha-ab\right)M_1-a\beta = 0, \tag{2.11}$$

$$bm_1^2+\left(\beta-\alpha-ab\right)m_1-a\beta = 0, \tag{2.12}$$

from (2.8) and (2.9), it follows that

$$aM_2^2+\left(\alpha-\beta-ab\right)M_2-b\alpha = 0, \tag{2.13}$$

$$am_2^2+\left(\alpha-\beta-ab\right)m_2-b\alpha = 0. \tag{2.14}$$

Note that (2.11) and (2.12) are equations that have the same solutions. Also, since

$$\left(\beta-\alpha-ab\right)^2+4ab\beta > 0, \quad -\frac{a}{b}\beta < 0,$$

(2.11) and (2.12) have simple real roots such that one is positive and another is negative. Therefore, the positive solutions of them are the same, and so we have $M_1 = m_1$. Similarly, (2.13) and (2.14) are equations that have the same solutions, and since

$$\left(\alpha-\beta-ab\right)^2+4ab\alpha>0, \quad -\frac{b}{a}\alpha<0,$$

(2.13) and (2.14) have simple real roots such that one is positive and another is negative. Therefore, the positive solutions of them are the same, and so we have $M_2 = m_2$. Consequently, by Theorem 2.1, $(\overline{x},\overline{y})$ is a global attractor and thus globally asymptotically stable.The proof is complete.

According to Theorem 2.3, for all $n \in \mathbb{N}$, the inequalities $a \le x_n \le a+\frac{\alpha}{b}$ and $b \le y_n \le b+\frac{\beta}{a}$ exist. That is, the positive solutions of system (1.3) are bounded. However, depending on the subset that initial conditions are found, the solutions in question can be always found in this subset. Such subsets are called invariant sets. In the next theorem, the invariant sets of system (1.3) are examined.

Theorem 2.5. The following statements are true:

(a) $\left[a,\overline{x}\right]\times\left[\overline{y},b+\frac{\beta}{a}\right]$ is an invariant set of system (1.3).

(b) $\left[\overline{x},a+\frac{\alpha}{b}\right]\times\left[b,\overline{y}\right]$ is an invariant set of system (1.3).

Proof.
Let the functions

$$\widehat{h}_1\left(\overline{x}\right)=a+\frac{\alpha}{b+\frac{\beta}{\overline{x}}}-\overline{x}, \quad \widehat{h}_2\left(\overline{y}\right)=b+\frac{\beta}{a+\frac{\alpha}{\overline{y}}}-\overline{y}$$

be defined, taking into account the system in (2.3). In this case, we can see that

$$\begin{aligned}
\widehat{h}_1(a) &= a+\frac{\alpha}{b+\frac{\beta}{a}}-a=\frac{\alpha}{b+\frac{\beta}{a}}>0,\\
\widehat{h}_1\left(a+\frac{\alpha}{b}\right) &= a+\frac{\alpha}{b+\frac{\beta}{a+\frac{\alpha}{b}}}-a-\frac{\alpha}{b}\\
&= \frac{\alpha}{b+\frac{b\beta}{ab+\alpha}}-\frac{\alpha}{b}\\
&= \frac{\alpha}{b}\left(\frac{1}{1+\frac{\beta}{ab+\alpha}}-1\right)\\
&< 0,
\end{aligned}$$

and

$$\begin{aligned}
\widehat{h}_2(b) &= b+\frac{\beta}{a+\frac{\alpha}{b}}-b=\frac{\beta}{a+\frac{\alpha}{b}}>0,\\
\widehat{h}_2\left(b+\frac{\beta}{a}\right) &= b+\frac{\beta}{a+\frac{\alpha}{b+\frac{\beta}{a}}}-b-\frac{\beta}{a}\\
&= \frac{\beta}{a+\frac{a\alpha}{ab+\beta}}-\frac{\beta}{a}\\
&= \frac{\beta}{a}\left(\frac{1}{1+\frac{\alpha}{ab+\beta}}-1\right)\\
&< 0.
\end{aligned}$$

Hence, we obtain

$$\left(\overline{x},\overline{y}\right)\in\left[a,a+\frac{\alpha}{b}\right]\times\left[b,b+\frac{\beta}{a}\right].$$

(a) Assume that $\left(x_{-j},y_{-j}\right)\in\left[a,\overline{x}\right]\times\left[\overline{y},b+\frac{\beta}{a}\right]$ for $j=0,1,2$. Then, from system (1.3), we have

$$\begin{aligned}
a &\leq x_1=a+\frac{a_1}{y_0}+\frac{a_2}{y_{-1}}+\frac{a_3}{y_{-2}}\leq a+\frac{a_1}{\overline{y}}+\frac{a_2}{\overline{y}}+\frac{a_3}{\overline{y}}=\overline{x},\\
b+\frac{\beta}{a} &\geq y_1=b+\frac{b_1}{x_0}+\frac{b_2}{x_{-1}}+\frac{b_3}{x_{-2}}\geq b+\frac{b_1}{\overline{x}}+\frac{b_2}{\overline{x}}+\frac{b_3}{\overline{x}}=\overline{y},\\
a &\leq x_2=a+\frac{a_1}{y_1}+\frac{a_2}{y_0}+\frac{a_3}{y_{-1}}\leq a+\frac{a_1}{\overline{y}}+\frac{a_2}{\overline{y}}+\frac{a_3}{\overline{y}}=\overline{x},\\
b+\frac{\beta}{a} &\geq y_2=b+\frac{b_1}{x_1}+\frac{b_2}{x_0}+\frac{b_3}{x_{-1}}\geq b+\frac{b_1}{\overline{x}}+\frac{b_2}{\overline{x}}+\frac{b_3}{\overline{x}}=\overline{y},\\
&\vdots
\end{aligned}$$

In this case, by induction, one can see that $\left(x_n,y_n\right)\in\left[a,\overline{x}\right]\times\left[\overline{y},b+\frac{\beta}{a}\right]$ for $n\geq-2$.

(b) Assume that $\left(x_{-j},y_{-j}\right)\in\left[\overline{x},a+\frac{\alpha}{b}\right]\times\left[b,\overline{y}\right]$ for $j=0,1,2$. Then, from system (1.3), we have

$$\begin{aligned}
a+\frac{\alpha}{b} &\geq x_1=a+\frac{a_1}{y_0}+\frac{a_2}{y_{-1}}+\frac{a_3}{y_{-2}}\geq a+\frac{a_1}{\overline{y}}+\frac{a_2}{\overline{y}}+\frac{a_3}{\overline{y}}=\overline{x},\\
b &\leq y_1=b+\frac{b_1}{x_0}+\frac{b_2}{x_{-1}}+\frac{b_3}{x_{-2}}\leq b+\frac{b_1}{\overline{x}}+\frac{b_2}{\overline{x}}+\frac{b_3}{\overline{x}}=\overline{y},\\
a+\frac{\alpha}{b} &\geq x_2=a+\frac{a_1}{y_1}+\frac{a_2}{y_0}+\frac{a_3}{y_{-1}}\geq a+\frac{a_1}{\overline{y}}+\frac{a_2}{\overline{y}}+\frac{a_3}{\overline{y}}=\overline{x},\\
b &\leq y_2=b+\frac{b_1}{x_1}+\frac{b_2}{x_0}+\frac{b_3}{x_{-1}}\leq b+\frac{b_1}{\overline{x}}+\frac{b_2}{\overline{x}}+\frac{b_3}{\overline{x}}=\overline{y},\\
&\vdots
\end{aligned}$$

In this case, by induction, one can see that $\left(x_n,y_n\right)\in\left[\overline{x},a+\frac{\alpha}{b}\right]\times\left[b,\overline{y}\right]$ for $n\geq-2$.

3. Numerical Simulation

This section aims to verify the theoretical results obtained in Section 2 using some specific values of the parameters and the initial values $x_{-2}:=5.21$, $x_{-1}:=2.55$, $x_0:=3.75$, $y_{-2}:=2.13$, $y_{-1}:=4.86$, $y_0:=5.50$. The solutions will be represented by drawings of numerical values.

Example 3.1. Let $a:=2.9$, $a_1:=1.2$, $a_2:=1.55$, $a_3:=4.1$, $b:=3.1$, $b_1:=1.1$, $b_2:=1.40$, $b_3:=3.9$. Then the solution of system (1.3) becomes as in Figure 1.

Example 3.2. Let $a:=2.99$, $a_1:=5.2$, $a_2:=2.55$, $a_3:=0.5$, $b:=0.01$, $b_1:=6.1$, $b_2:=15.4$, $b_3:=0.3$. Then the solution of system (1.3) becomes as in Figure 2.

Example 3.3. Let $a:=0.50$, $a_1:=1.21$, $a_2:=6.05$, $a_3:=14.51$, $b:=0.80$, $b_1:=0.17$, $b_2:=12.42$, $b_3:=2.35$. Then the solution of system (1.3) becomes as in Figure 3.

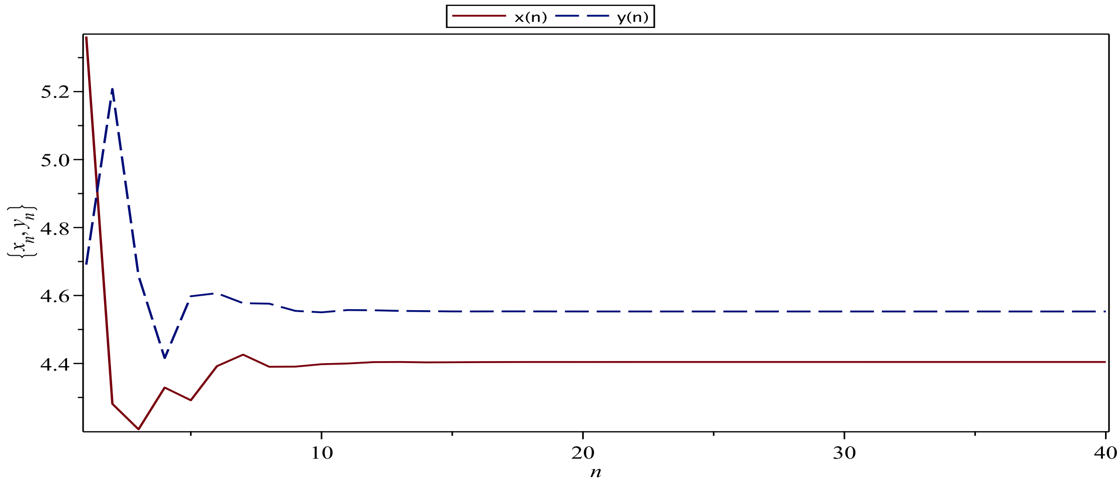

Figure 1. For $a := 2.9$, $a_1 := 1.2$, $a_2 := 1.55$, $a_3 := 4.1$, $b := 3.1$, $b_1 := 1.1$, $b_2 := 1.40$, $b_3 := 3.9$, the solution of system (1.3).

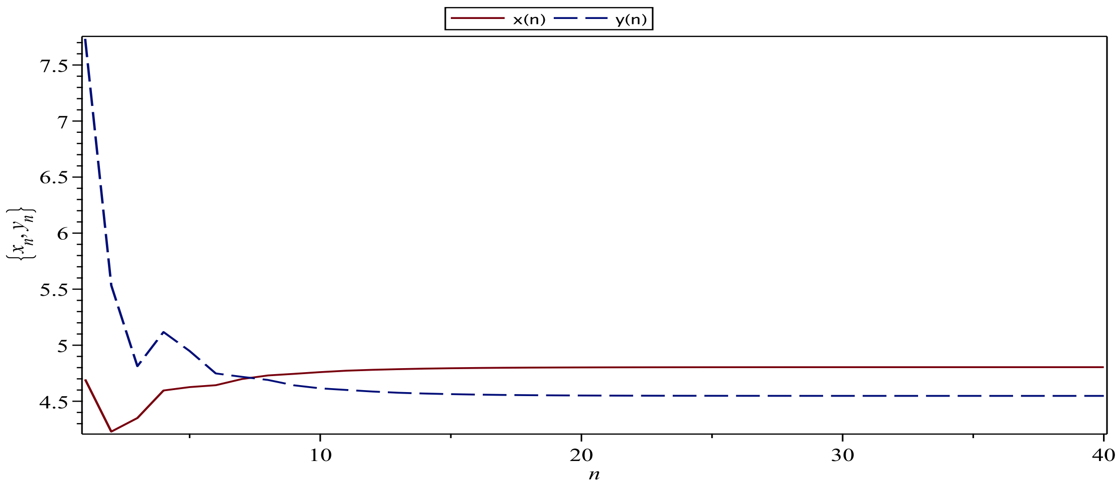

Figure 2. For $a := 2.99$, $a_1 := 5.2$, $a_2 := 2.55$, $a_3 := 0.5$, $b := 0.01$, $b_1 := 6.1$, $b_2 := 15.4$, $b_3 := 0.3$, the solution of system (1.3).

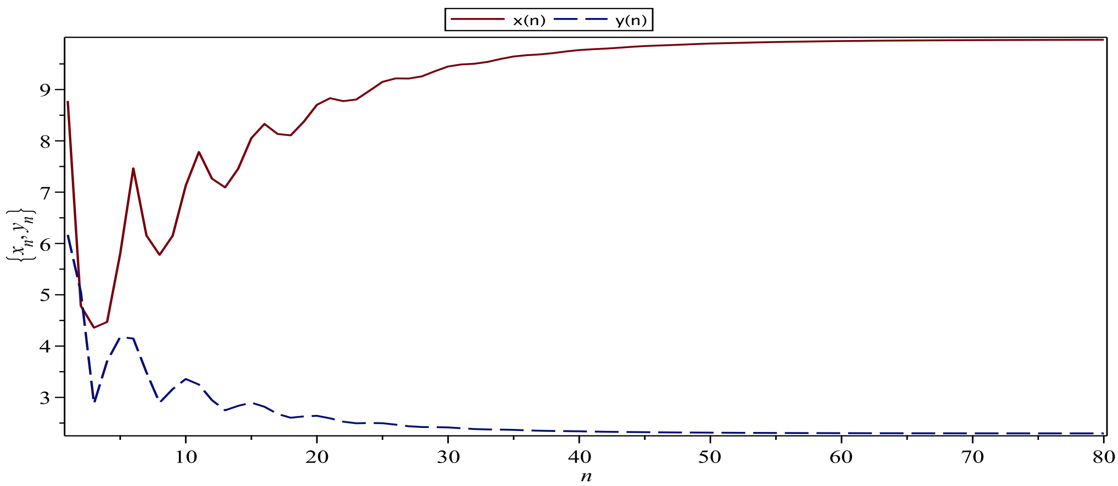

Figure 3. For $a := 2.99$, $a_1 := 5.2$, $a_2 := 2.55$, $a_3 := 0.5$, $b := 0.01$, $b_1 := 6.1$, $b_2 := 15.4$, $b_3 := 0.3$, the solution of system (1.3).

4. Conclusion

In this study, the local and global stability of the positive equilibrium of the system

$$x_{n+1} = a + \frac{a_1}{y_n} + \frac{a_2}{y_{n-1}} + \frac{a_3}{y_{n-2}}, \quad y_{n+1} = b + \frac{b_1}{x_n} + \frac{b_2}{x_{n-1}} + \frac{b_3}{x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where a, a_i, b, b_i $(i = 1,2,3)$ and x_{-j}, y_{-j} $(j = 0,1,2)$ are positive and real, was investigated. It was concluded that for all positive values of all parameters seen in the system, positive solutions converge to the unique positive equilibrium. Also, it was handled invariant sets to better understand the behavior of the solutions. Finally, the theoretical results were confirmed numerically and illustrated with visuals.

Although the system is a third-order system, it can be expanded to a higher order and similar research can be conducted. One option would be to increase the rational terms. In such a case, the system may be

$$x_{n+1} = a + \sum_{s=1}^{k} \frac{a_s}{y_{n-s+1}}, \quad y_{n+1} = b + \sum_{s=1}^{k} \frac{b_s}{x_{n-s+1}}, \quad n \in \mathbb{N}_0,$$

with positive parameters and positive initial values. Note that this system is a generalization of the above and reduces to it for $k = 3$.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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Qualitative Behavior of Solutions of a Two-Dimensional Rational System of Difference Equations

Durhasan Turgut Tollu[1], **Merve Kayhan**[2]

Keywords
Equilibrium point, Periodicity, Rate of convergence, Stability, System of difference equations

Abstract — In this study, the rational system

$$x_{n+1}=\frac{\alpha_1+\beta_1 y_{n-1}}{a_1+b_1 y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2 x_{n-1}}{a_2+b_2 x_n},\quad n\in\mathbb{N}_0,$$

where α_i, β_i, a_i, b_i, $(i=1,2)$, and x_{-j}, y_{-j}, $(j=0,1)$, are positive real numbers, is defined and its qualitative behavior is discussed. The system in question is a two-dimensional extension of an old difference equation in the literature. The results obtained generalize the results in the literature on the equation in question.

Subject Classification (2020): 39A20, 39A23, 39A30.

1. Introduction

Difference equations have occurred in many scientific areas such as biology, physics, engineering, and economics. Particularly, rational difference equations and their systems have great importance in applications. See [4, 11, 23, 24]. As a natural consequence of this, it is very worthy to examine the qualitative analyses of such equations and their systems. Over the past two decades, many studies have been published on the qualitative behavior of difference equations and systems. For example, see [1–3, 5, 6, 8–10, 12–15, 21, 22, 25, 29, 30, 32, 34, 36, 38, 40–42, 44] and therein references. Below, we present a prototype, among others, that caught our attention, along with its two extensions. Gibbons et al. [16] analyzed the boundedness, the oscillatory and periodicity, and the global stability of the nonnegative solutions of the rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{\gamma+x_n},\quad n\in\mathbb{N}_0,\tag{1.1}$$

where the parameters α, β and γ are nonnegative and real. Din et al. [8] investigated the boundedness, the local and global stability, the periodicity, and the rate of convergence of positive solutions of the system of difference equations

$$x_{n+1}=\frac{\alpha_1+\beta_1 x_{n-1}}{a_1+b_1 y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2 y_{n-1}}{a_2+b_2 x_n},\quad n\in\mathbb{N}_0,\tag{1.2}$$

[1]dttollu@erbakan.edu.tr (Corresponding Author); [2]mervekayhan.294@gmail.com
[1]Necmettin Erbakan University, Faculty of Science, Department of Mathematics and Computer Sciences, Konya, Türkiye
[2]TOKİ Kayaşehir Secondary School, İstanbul, Türkiye
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where α_i, β_i, a_i, b_i, $(i=1,2)$, and x_{-j}, y_{-j}, $(j=0,1)$, are positive real numbers. Din [10] investigated the boundedness, the local and global stability behavior, the periodicity, and the rate of convergence of positive solutions of the system of rational difference equations

$$x_{n+1}=\frac{\alpha_1+\beta_1 y_{n-1}}{a_1+b_1 x_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2 x_{n-1}}{a_2+b_2 y_n},\quad n\in\mathbb{N}_0, \tag{1.3}$$

where α_i, β_i, a_i, b_i, $(i=1,2)$, and x_{-j}, y_{-j}, $(j=0,1)$, are positive real numbers.

Studies on the qualitative behavior of the difference equations and systems still continue actively. For recent studies, see, for example [7, 17–20, 26–28, 32, 33, 35, 37, 39, 43] and therein references.

The systems in (1.2) and (1.3) are two-dimensional symmetric extensions of (1.1). Apart from these, there is another two-dimensional symmetric extension of (1.1). In this paper, we define the aforementioned extension of (1.1). That is, we define the rational system

$$x_{n+1}=\frac{\alpha_1+\beta_1 y_{n-1}}{a_1+b_1 y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2 x_{n-1}}{a_2+b_2 x_n}, \tag{1.4}$$

where α_i, β_i, a_i, b_i, $(i=1,2)$ are positive real parameters, and x_{-j}, y_{-j}, $(j=0,1)$ are positive real initial conditions, and discuss qualitative behavior of its solutions. More concretely, we investigate existence of a unique positive equilibrium, local and global stability of the equilibrium, rate of convergence of a solution converging to the equilibrium, existence of unbounded solutions and the periodicity of solutions.

2. Preliminaries

Assume that I, J are some intervals of real numbers and

$$f_1: I^2\times J^2\to I,\quad f_2: I^2\times J^2\to J$$

are continuously differentiable functions. Then, for every set of initial conditions $x_{-1}, x_0\in I$ and $y_{-1}, y_0\in J$, the system of difference equations

$$x_{n+1}=f_1\left(x_n,x_{n-1},y_n,y_{n-1}\right),\quad y_{n+1}=f_2\left(x_n,x_{n-1},y_n,y_{n-1}\right),\quad n\in\mathbb{N}_0, \tag{2.1}$$

has a unique solution denoted by $\left\{(x_n,y_n)\right\}_{n=-1}^{\infty}$. An equilibrium point of system (2.1) is a point $\left(\overline{x},\overline{y}\right)\in I\times J$ that satisfies

$$\overline{x}=f_1\left(\overline{x},\overline{x},\overline{y},\overline{y}\right),\quad \overline{y}=f_2\left(\overline{x},\overline{x},\overline{y},\overline{y}\right).$$

For stability analysis, we use some key results of the multivariable calculus. Hence we transform system (2.1) into the vector system

$$X_{n+1}=F\left(X_n\right),\quad n\in\mathbb{N}_0, \tag{2.2}$$

where $X_n=\left(x_n,y_n,x_{n-1},y_{n-1}\right)^T$, F is a vector map such that $F: I^2\times J^2\to I^2\times J^2$ and

$$F\begin{pmatrix} x_n\\ y_n\\ x_{n-1}\\ y_{n-1}\end{pmatrix}=\begin{pmatrix} f_1\left(x_n,y_n,x_{n-1},y_{n-1}\right)\\ f_2\left(x_n,y_n,x_{n-1},y_{n-1}\right)\\ x_n\\ y_n\end{pmatrix}.$$

It is obvious that if an equilibrium point of system (2.1) is $(\overline{x}, \overline{y})$, then the corresponding equilibrium point of system (2.2) is the point $\overline{X} = (\overline{x}, \overline{y}, \overline{x}, \overline{y})^T$.

By $\|\cdot\|$, we denote any convenient vector norm and the corresponding matrix norm. Also, $X_0 \in I \times J \times I \times J$ is an initial condition of the vector system (2.2) corresponding to the initial conditions $x_{-1}, x_0 \in I$ and $y_{-1}, y_0 \in J$ of system (2.1).

Definition 2.1. [23] Let $\overline{X}$ be an equilibrium of system (2.2). Then,

i) The equilibrium $\overline{X}$ is called stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\left\| X_0 - \overline{X} \right\| < \delta$ implies $\left\| X_n - \overline{X} \right\| < \epsilon$, for all $n \geq 0$. Otherwise, the equilibrium point $\overline{X}$ is called unstable.

ii) The equilibrium $\overline{X}$ is called locally asymptotically stable if it is stable and there exists $\gamma > 0$ such that $\left\| X_0 - \overline{X} \right\| < \gamma$ and $X_n \to \overline{X}$ as $n \to \infty$.

iii) The equilibrium $\overline{X}$ is called a global attractor if $X_n \to \overline{X}$ as $n \to \infty$.

iv) The equilibrium $\overline{X}$ is called globally asymptotically stable if it is both locally asymptotically stable and global attractor.

The linearized system of (2.2) about the equilibrium $\overline{X}$ is of the form

$$Z_{n+1} = J_F Z_n, \quad n \in \mathbb{N}_0, \tag{2.3}$$

where J_F is the Jacobian of the map F at the equilibrium $\overline{X}$. The characteristic polynomial of (2.3) at the equilibrium $\overline{X}$ is

$$P(\lambda) = a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \tag{2.4}$$

with real coefficients and $a_0 > 0$.

Theorem 2.2. [23] Let $\overline{X}$ be any equilibrium of (2.2). If all eigenvalues of J_F at $\overline{X}$ lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{X}$ is local asymptotically stable. If one of the eigenvalues has a modulus greater than one, then the equilibrium point $\overline{X}$ is unstable.

The next results deal with the rate of convergence for a solution converging to an equilibrium of a system of difference equations. See [11, 31] for more details.

Consider the system of difference equations

$$X_{n+1} = (A + B_n) X_n, \quad n \in \mathbb{N}_0, \tag{2.5}$$

where X_n is an $m-$dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : Z^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B_n\| \to 0 \tag{2.6}$$

as $n \to \infty$.

Theorem 2.3 (Perron's First Theorem)**.** Suppose that condition (2.6) holds. If X_n is a solution of (2.5), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{2.7}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 2.4 (Perron's Second Theorem)**.** Suppose that condition (2.6) holds. If X_n is a solution of (2.5), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{1/n} \tag{2.8}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

The following lemma is the second part of Lemma 3.1 in [30].

Lemma 2.5. Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions and a_1, b_1, a_2, b_2 be positive numbers such that $a_1 < b_1$, $a_2 < b_2$. Suppose that

$$f : [a_2, b_2] \times [a_2, b_2] \to [a_1, b_1], \quad g : [a_1, b_1] \times [a_1, b_1] \to [a_2, b_2].$$

In addition, assume that $f(u, v)$ is a decreasing (resp. increasing) function with respect to u (resp. v) for every v (resp. u) and $g(z, w)$ is a decreasing (resp. increasing) function with respect to z (resp. w) for every w (resp. z). Finally suppose that if the real numbers m, M, r, R satisfy the system

$$M = f(r,R), \quad m = f(R,r), \quad R = g(m,M), \quad r = g(M,m)$$

then $m = M$ and $r = R$. Then the system of difference equations

$$x_{n+1} = f(y_n, y_{n-1}), \quad y_{n+1} = g(x_n, x_{n-1}), \quad n \in \mathbb{N}_0, \tag{2.9}$$

has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of the system (2.9) which satisfies

$$x_{n_0} \in [a_1, b_1], \quad x_{n_0+1} \in [a_1, b_1], \quad y_{n_0} \in [a_2, b_2], \quad y_{n_0+1} \in [a_2, b_2], \quad n_0 \in \mathbb{N}$$

tends to the unique positive equilibrium of (2.9).

3. Main results

In this section, we express and prove our main results on the system of difference equations (1.4).

3.1. Boundedness and persistence of the system

In this subsection, the boundedness and the persistence of (1.4) are investigated. The following theorem states the result obtained.

Theorem 3.1. If $\beta_1\beta_2 < a_1 a_2$, then every solution of the system of difference equations (1.4) is bounded and persist.

Proof.
From (1.4), we have the following system of difference inequalities

$$x_{n+1} \le \frac{\alpha_1}{a_1} + \frac{\beta_1}{a_1} y_{n-1}, \quad y_{n+1} \le \frac{\alpha_2}{a_2} + \frac{\beta_2}{a_2} x_{n-1}, \quad n \in \mathbb{N}_0. \tag{3.1}$$

We pay regard to the system of nonhomogeneous linear difference equations

$$u_{n+1} = \frac{\alpha_1}{a_1} + \frac{\beta_1}{a_1} v_{n-1}, \quad v_{n+1} = \frac{\alpha_2}{a_2} + \frac{\beta_2}{a_2} u_{n-1}, \quad n \in \mathbb{N}_0, \tag{3.2}$$

with $u_{-1} = x_{-1}$, $u_0 = x_0$, $v_{-1} = y_{-1}$ and $v_0 = y_0$. System (3.2) yields the following independent equatios

$$u_{n+1} = \frac{\alpha_1}{a_1} + \frac{\beta_1}{a_1}\frac{\alpha_2}{a_2} + \frac{\beta_1}{a_1}\frac{\beta_2}{a_2} u_{n-3}, \quad n \geq 2, \tag{3.3}$$

and

$$v_{n+1} = \frac{\alpha_2}{a_2} + \frac{\beta_2}{a_2}\frac{\alpha_1}{a_1} + \frac{\beta_1}{a_1}\frac{\beta_2}{a_2} v_{n-3}, \quad n \geq 2. \tag{3.4}$$

The general solutions of (3.3) and (3.4) are given by

$$\begin{aligned} u_n \;\; = \;\; & \frac{\alpha_1 a_2 + \alpha_2 \beta_1}{a_1 a_2 - \beta_1 \beta_2} + c_1 \left(\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}} \right)^n \\ & + c_2 \left(-\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}} \right)^n + c_3 \left(-i\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}} \right)^n + c_4 \left(i\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}} \right)^n \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} v_n \;\; = \;\; & \frac{\alpha_2 a_1 + \alpha_1 \beta_2}{a_1 a_2 - \beta_1 \beta_2} + c_5 \left(\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}} \right)^n \\ & + c_6 \left(-\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}} \right)^n + c_7 \left(-i\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}} \right)^n + c_8 \left(i\sqrt[4]{\frac{\beta_1}{a_1}\frac{\beta_2}{a_2}} \right)^n, \end{aligned} \tag{3.6}$$

where c_s, $(s = 1, 2, \ldots, 8)$, are arbitrary constants and i is the imaginary unit. From (3.5) and (3.6), it follows that if $\beta_1 \beta_2 < a_1 a_2$, then there exist the limits

$$\lim_{n\to\infty} u_n = \frac{\alpha_1 a_2 + \alpha_2 \beta_1}{a_1 a_2 - \beta_1 \beta_2} \tag{3.7}$$

and

$$\lim_{n\to\infty} v_n = \frac{\alpha_2 a_1 + \alpha_1 \beta_2}{a_1 a_2 - \beta_1 \beta_2}, \tag{3.8}$$

and so the sequences $\{u_n\}$ and $\{v_n\}$ are bounded. Also, since $u_{-1} = x_{-1}$, $u_0 = x_0$, $v_{-1} = y_{-1}$ and $v_0 = y_0$, by comparison method, we find $x_n \leq u_n$ and $y_n \leq v_n$, and so

$$x_n \leq \frac{\alpha_1 a_2 + \alpha_2 \beta_1}{a_1 a_2 - \beta_1 \beta_2} = U_1 \tag{3.9}$$

and

$$y_n \leq \frac{\alpha_2 a_1 + \alpha_1 \beta_2}{a_1 a_2 - \beta_1 \beta_2} = U_2. \tag{3.10}$$

Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ are also bounded. On the other hand, from (1.4), (3.9) and (3.10), it follows that

$$x_{n+1} \geq \frac{\alpha_1}{a_1 + b_1 y_n} \geq \frac{\alpha_1}{a_1 + b_1 \frac{a_1 \alpha_2 + \alpha_1 \beta_2}{a_1 a_2 - \beta_1 \beta_2}} = \frac{\alpha_1 \left(a_1 a_2 - \beta_1 \beta_2 \right)}{a_1 (a_1 a_2 - \beta_1 \beta_2) + b_1 \left(a_1 \alpha_2 + \alpha_1 \beta_2 \right)} = L_1 \tag{3.11}$$

and

$$y_{n+1} \geq \frac{\alpha_2}{a_2 + b_2 x_n} \geq \frac{\alpha_2}{a_2 + b_2 \frac{\alpha_1 a_2 + \beta_1 \alpha_2}{a_1 a_2 - \beta_1 \beta_2}} = \frac{\alpha_2 \left(a_1 a_2 - \beta_1 \beta_2\right)}{a_2 \left(a_1 a_2 - \beta_1 \beta_2\right) + b_2 \left(\alpha_1 a_2 + \beta_1 \alpha_2\right)} = L_2. \tag{3.12}$$

Consequently, from (3.9), (3.10), (3.11) and (3.12), for $n \geq 1$, we have

$$L_1 \leq x_n \leq U_1, \quad L_2 \leq y_n \leq U_2 \tag{3.13}$$

which means that $\{x_n\}$ and $\{y_n\}$ are bounded and persist. The proof is completed.

Theorem 3.2. If $\beta_1 \beta_2 < a_1 a_2$, then the set $[L_1, U_1] \times [L_2, U_2]$ is invariant set of (1.4).

Proof.
Let $\left\{(x_n, y_n)\right\}_{n=-1}^{\infty}$ be an arbitrary positive solution of (1.4). If $\beta_1 \beta_2 < a_1 a_2$, then the bounds L_1, U_1, L_2 and U_2 exist. Also, let $x_{-1}, x_0 \in [L_1, U_1]$ and $y_{-1}, y_0 \in [L_2, U_2]$. Then, from (1.4), we have

$$\begin{aligned}
x_1 &= \frac{\alpha_1 + \beta_1 y_{-1}}{a_1 + b_1 y_0} \leq \frac{\alpha_1 + \beta_1 U_2}{a_1} = U_1, \quad y_1 = \frac{\alpha_2 + \beta_2 x_{-1}}{a_2 + b_2 x_0} \leq \frac{\alpha_2 + \beta_2 U_1}{a_2} = U_2, \\
x_2 &= \frac{\alpha_1 + \beta_1 y_0}{a_1 + b_1 y_1} \leq \frac{\alpha_1 + \beta_1 U_2}{a_1} = U_1, \quad y_2 = \frac{\alpha_2 + \beta_2 x_0}{a_2 + b_2 x_1} \leq \frac{\alpha_2 + \beta_2 U_1}{a_2} = U_2, \\
x_3 &= \frac{\alpha_1 + \beta_1 y_1}{a_1 + b_1 y_2} \leq \frac{\alpha_1 + \beta_1 U_2}{a_1} = U_1, \quad y_3 = \frac{\alpha_2 + \beta_2 x_1}{a_2 + b_2 x_2} \leq \frac{\alpha_2 + \beta_2 U_1}{a_2} = U_2, \\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
x_1 &= \frac{\alpha_1 + \beta_1 y_{-1}}{a_1 + b_1 y_0} \geq \frac{\alpha_1}{a_1 + b_1 U_2} = L_1, \quad y_1 = \frac{\alpha_2 + \beta_2 x_{-1}}{a_2 + b_2 x_0} \geq \frac{\alpha_2}{a_2 + b_2 U_1} = L_2, \\
x_2 &= \frac{\alpha_1 + \beta_1 y_0}{a_1 + b_1 y_1} \geq \frac{\alpha_1}{a_1 + b_1 U_2} = L_1, \quad y_2 = \frac{\alpha_2 + \beta_2 x_0}{a_2 + b_2 x_1} \geq \frac{\alpha_2}{a_2 + b_2 U_1} = L_2, \\
x_3 &= \frac{\alpha_1 + \beta_1 y_1}{a_1 + b_1 y_2} \geq \frac{\alpha_1}{a_1 + b_1 U_2} = L_1, \quad y_3 = \frac{\alpha_2 + \beta_2 x_1}{a_2 + b_2 x_2} \geq \frac{\alpha_2}{a_2 + b_2 U_1} = L_2, \\
&\vdots
\end{aligned}$$

Considering inductively, it can be easily shown that $x_n \in [L_1, U_1]$ and $y_n \in [L_2, U_2]$ for $n \geq -1$. So the proof is completed.

3.2. Stability analysis

In this subsection, the existence of the unique positive equilibrium of (1.4) and local asymptotic stability and global asymptotic stability of the equilibrium are investigated.

Lemma 3.3. System (1.4) possesses a unique positive equilibrium point. If $\beta_1 \beta_2 < a_1 a_2$, then the equilibrium point is in the set $[L_1, U_1] \times [L_2, U_2]$.

Proof.

For the equilibrium points of (1.4) we consider the system

$$\overline{x} = \frac{\alpha_1 + \beta_1 \overline{y}}{a_1 + b_1 \overline{y}}, \quad \overline{y} = \frac{\alpha_2 + \beta_2 \overline{x}}{a_2 + b_2 \overline{x}}. \tag{3.14}$$

From (3.14) we have the independent quadratic equations

$$D_1 \overline{x}^2 + (C_1 - B_1)\overline{x} - A_1 = 0, \quad D_2 \overline{y}^2 + (C_2 - B_2)\overline{y} - A_2 = 0, \tag{3.15}$$

where

$$\begin{aligned}
A_1 &= \alpha_1 a_2 + \beta_1 \alpha_2, \\
B_1 &= \alpha_1 b_2 + \beta_1 \beta_2, \\
C_1 &= a_1 a_2 + b_1 \alpha_2, \\
D_1 &= a_1 b_2 + b_1 \beta_2, \\
A_2 &= a_1 \alpha_2 + \alpha_1 \beta_2, \\
B_2 &= \alpha_2 b_1 + \beta_1 \beta_2, \\
C_2 &= a_1 a_2 + b_2 \alpha_1, \\
D_2 &= a_2 b_1 + b_2 \beta_1.
\end{aligned}$$

Hence, from (3.15), we have

$$\Delta_{\overline{x}} = (C_1 - B_1)^2 + 4A_1 D_1 > 0, \quad \Delta_{\overline{y}} = (C_2 - B_2)^2 + 4A_2 D_2 > 0$$

which implies that they have two real simple roots. Also, since $-A_1/D_1 < 0$ and $-A_2/D_2 < 0$, both equations in (3.15) have one negative and one positive root. Therefore there exists the unique positive equilibrium point of (1.4).

Consider the inequalities

$$\overline{x} \leq \frac{\alpha_1 + \beta_1 \overline{y}}{a_1}, \quad \overline{y} \leq \frac{\alpha_2 + \beta_2 \overline{x}}{a_2},$$

which is obtained from (3.14). Using these two inequalities within each other we get the following inequalities

$$\begin{aligned}
\overline{x} &\leq \frac{\alpha_1}{a_1} + \frac{\beta_1}{a_1}\overline{y} \leq \frac{\alpha_1}{a_1} + \frac{\beta_1}{a_1}\frac{\alpha_2}{a_2} + \frac{\beta_1}{a_1}\frac{\beta_2}{a_2}\overline{x}, \\
\overline{y} &\leq \frac{\alpha_2}{a_2} + \frac{\beta_2}{a_2}\overline{x} \leq \frac{\alpha_2}{a_2} + \frac{\beta_2}{a_2}\frac{\alpha_1}{a_1} + \frac{\beta_2}{a_2}\frac{\beta_1}{a_1}\overline{y}.
\end{aligned}$$

If $\beta_1 \beta_2 < a_1 a_2$, from the last inequalities, it follows that

$$\overline{x} \leq \frac{\alpha_1 a_2 + \alpha_2 \beta_1}{a_1 a_2 - \beta_1 \beta_2} = U_1, \quad \overline{y} \leq \frac{\alpha_2 a_1 + \alpha_1 \beta_2}{a_1 a_2 - \beta_1 \beta_2} = U_2.$$

Moreover, from (3.14) and the inequalities $\overline{x} \le U_1$, $\overline{y} \le U_2$, we obtain the inequalities

$$\overline{x} \ge \frac{\alpha_1}{a_1 + b_1\overline{y}} \ge \frac{\alpha_1}{a_1 + b_1 U_2} = L_1, \quad \overline{y} \ge \frac{\alpha_2}{a_2 + b_2\overline{x}} \ge \frac{\alpha_2}{a_2 + b_2 U_1} = L_2.$$

Thus, for the aforementioned equilibrium point, we have $\left(\overline{x}, \overline{y}\right) \in [L_1, U_1] \times [L_2, U_2]$. So the proof is completed.

Theorem 3.4. If $\beta_1\beta_2 < a_1 a_2$, then the unique positive equilibrium of system (1.4) is locally asymptotically stable.

Proof.

We know from Lemma 3.3 that (1.4) has the unique positive equilibrium $\left(\overline{x}, \overline{y}\right)$. In this case, the vector system corresponding to (1.4) also has the equilibrium point $\overline{X} = \left(\overline{x}, \overline{y}, \overline{x}, \overline{y}\right)^T$. The aforementioned vector system is given by the vector map

$$F\begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1+\beta_1 y_{n-1}}{a_1+b_1 y_n} \\ \frac{\alpha_2+\beta_2 x_{n-1}}{a_2+b_2 x_n} \\ x_n \\ y_n \end{pmatrix}$$

The linearized system of the vector system about $\overline{X} = \left(\overline{x}, \overline{y}, \overline{x}, \overline{y}\right)^T$ is the system

$$Z_{n+1} = J_F(\overline{X}) Z_n, \tag{3.16}$$

where the vector Z_n is

$$Z_n = \begin{pmatrix} z_n \\ z_{n-1} \\ z_{n-2} \\ z_{n-3} \end{pmatrix}$$

and J_F at $\overline{X}$ is

$$J_F(\overline{X}) = \begin{pmatrix} 0 & -\frac{b_1\overline{x}}{a_1+b_1\overline{y}} & 0 & \frac{\beta_1}{a_1+b_1\overline{y}} \\ -\frac{b_2\overline{y}}{a_2+b_2\overline{x}} & 0 & \frac{\beta_2}{a_2+b_2\overline{x}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

The characteristic polynomial of (3.16) at $\overline{X}$ is

$$P(\lambda) = \lambda^4 - \frac{b_1 b_2 \overline{x}\overline{y}}{\left(a_2 + b_2\overline{x}\right)\left(a_1 + b_1\overline{y}\right)}\lambda^2 + \frac{b_1\beta_2\overline{x} + \beta_1 b_2\overline{y}}{\left(a_2 + b_2\overline{x}\right)\left(a_1 + b_1\overline{y}\right)}\lambda - \frac{\beta_1\beta_2}{\left(a_2 + b_2\overline{x}\right)\left(a_1 + b_1\overline{y}\right)}$$

or

$$P(\lambda) = \lambda^4 - \frac{\left(\beta_1 - b_1\overline{x}\lambda\right)\left(\beta_2 - b_2\overline{y}\lambda\right)}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)}. \tag{3.18}$$

Let us consider the polynomial equation $P(\lambda) = 0$. Obviously, since $\beta_1\beta_2 \ne 0$, $\lambda \ne 0$. In this case, it can be seen from (3.18) that there are two cases to consider.

(i) If $\beta_1 < b_1\overline{x}\lambda$ and $\beta_2 < b_2\overline{y}\lambda$, then we have

$$\lambda^4 = \frac{\left(\beta_1 - b_1\overline{x}\lambda\right)\left(\beta_2 - b_2\overline{y}\lambda\right)}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)} < \frac{b_1\overline{x}\lambda b_2\overline{y}\lambda}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)} < \frac{b_1\overline{x}\lambda b_2\overline{y}\lambda}{b_1\overline{y}b_2\overline{x}} = \lambda^2$$

from which it follows that $|\lambda| < 1$.

(ii) If $\beta_1 > b_1\overline{x}\lambda$ and $\beta_2 > b_2\overline{y}\lambda$, then we have

$$\lambda^4 = \frac{\left(\beta_1 - b_1\overline{x}\lambda\right)\left(\beta_2 - b_2\overline{y}\lambda\right)}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)} < \frac{\beta_1\beta_2}{\left(a_1 + b_1\overline{y}\right)\left(a_2 + b_2\overline{x}\right)} < \frac{\beta_1\beta_2}{a_1a_2}.$$

Hence if $\beta_1\beta_2 < a_1a_2$, then we obtain that $|\lambda| < 1$. Therefore the proof is completed.

Theorem 3.5. If $\beta_1\beta_2 < a_1a_2$, then the unique positive equilibrium point of (1.4) is a global attractor.

Proof.

We will use Lemma 2.5 to prove the theorem. Let $\left\{(x_n, y_n)\right\}_{n=-1}^{\infty}$ be any solution of system (1.4). We know that if the inequality $\beta_1\beta_2 < a_1a_2$ is satisfied, then $\left\{(x_n, y_n)\right\}_{n=-1}^{\infty}$ is bounded and persist. Suppose that

$$f(u, v) = \frac{\alpha_1 + \beta_1 v}{a_1 + b_1 u}, \quad g\left(x, y\right) = \frac{\alpha_2 + \beta_2 y}{a_2 + b_2 x}.$$

Then we have

$$f_u(u, v) = -\frac{\left(\alpha_1 + \beta_1 v\right) b_1}{\left(a_1 + u b_1\right)^2} < 0, \quad f_v(u, v) = \frac{\beta_1}{a_1 + u b_1} > 0$$

for $(u, v) \in (L_2, U_2) \times (L_2, U_2)$ and

$$g_x\left(x, y\right) = -\frac{\left(\alpha_2 + \beta_2 y\right) b_2}{\left(a_2 + x b_2\right)^2} < 0, \quad g_y\left(x, y\right) = \frac{\beta_2}{a_2 + x b_2} > 0$$

for $\left(x, y\right) \in (L_1, U_1) \times (L_1, U_1)$. Therefore, the function $f(u, v)$ is decreasing with respect to u for every $v \in (L_2, U_2)$ and it is increasing with respect to v for every $u \in (L_2, U_2)$, and also the function $g(x, y)$ is decreasing with respect to x for every $y \in (L_1, U_1)$ and it is increasing with respect to y for every $x \in (L_1, U_1)$.

Let

$$\limsup_{n\to\infty} x_n = M_1, \quad \liminf_{n\to\infty} x_n = m_1, \quad \limsup_{n\to\infty} y_n = M_2, \quad \liminf_{n\to\infty} y_n = m_2.$$

In this case we can define the system

$$M_1 = \frac{\alpha_1 + \beta_1 M_2}{a_1 + b_1 m_2}, \quad m_1 = \frac{\alpha_1 + \beta_1 m_2}{a_1 + b_1 M_2}, \quad M_2 = \frac{\alpha_2 + \beta_2 M_1}{a_2 + b_2 m_1}, \quad m_2 = \frac{\alpha_2 + \beta_2 m_1}{a_2 + b_2 M_1}. \tag{3.19}$$

From (3.19), we have

$$a_1 M_1 + b_1 M_1 m_2 = \alpha_1 + \beta_1 M_2, \qquad a_1 m_1 + b_1 m_1 M_2 = \alpha_1 + \beta_1 m_2, \tag{3.20}$$

$$a_2 M_2 + b_2 M_2 m_1 = \alpha_2 + \beta_2 M_1, \qquad a_2 m_2 + b_2 m_2 M_1 = \alpha_2 + \beta_2 m_1. \tag{3.21}$$

Furthermore, from (3.20) and (3.21), we have

$$a_1 (M_1 - m_1) + b_1 (M_1 m_2 - m_1 M_2) = \beta_1 (M_2 - m_2) \tag{3.22}$$

and

$$a_2 (M_2 - m_2) + b_2 (m_1 M_2 - M_1 m_2) = \beta_2 (M_1 - m_1), \tag{3.23}$$

respectively. If $M_1 = m_1$, then it is seen from (3.22) that $m_2 = M_2$. On the other hand, if $m_2 = M_2$, then it is seen from (3.23) that $M_1 = m_1$. Therefore, we will just show that $M_2 = m_2$. After some operations, the equalities (3.22) and (3.23) yield the equality

$$\left(\frac{a_1}{b_1} - \frac{\beta_2}{b_2}\right)(M_1 - m_1) + \left(\frac{a_2}{b_2} - \frac{\beta_1}{b_1}\right)(M_2 - m_2) = 0. \tag{3.24}$$

We rewrite (3.24) as

$$M_1 - m_1 = \frac{\frac{a_2}{b_2} - \frac{\beta_1}{b_1}}{\frac{a_1}{b_1} - \frac{\beta_2}{b_2}} (m_2 - M_2). \tag{3.25}$$

If $\beta_1 \beta_2 < a_1 a_2$, then (3.2) becomes

$$M_1 - m_1 = \frac{a_2}{\beta_2} (M_2 - m_2).$$

Using this result in (3.23), we obtain

$$m_1 M_2 - M_1 m_2 = 0.$$

Using the last two results in (3.22), we obtain

$$\left(a_1 a_2 - \beta_1 \beta_2\right)(M_2 - m_2) = 0$$

which implies that $M_2 = m_2$. So the proof is completed. In order to verify the theoretical result we obtained in Theorem 3.5, a special case obtained by giving some values to the parameters and initial conditions of system (1.4) is given in the example below.

Example 3.6. If $\alpha_1 = 1$, $\beta_1 = 13.1$, $a_1 = 7$, $b_1 = 3$, $\alpha_2 = 12$, $\beta_2 = 3.5$, $a_2 = 8.2$, $b_2 = 1$, then (1.4) becomes

$$x_{n+1} = \frac{1 + 12.1 y_{n-1}}{7 + 3 y_n}, \quad y_{n+1} = \frac{12 + 3.5 x_{n-1}}{6 + x_n}. \tag{3.26}$$

The unique positive equilibrium of (3.26) is $(2.364109242, 1.919175757)$. Plot of the corresponding solution to $x_{-1} = 5.4$, $x_0 = 9.5$, $y_{-1} = 7$ and $y_0 = 1.7$ is given by Figure 1 and Figure 2.

According to the item iv) of Definition 2.1, we give the next result from Theorem 3.4 and Theorem 3.5.

Theorem 3.7. If $\beta_1 \beta_2 < a_1 a_2$, then the unique positive equilibrium point of (1.4) is globally asymptotically stable.

3.3. Rate of convergence of solutions

In this subsection, the rate of convergence of a solution converging to the unique positive equilibrium of (1.4) is studied.

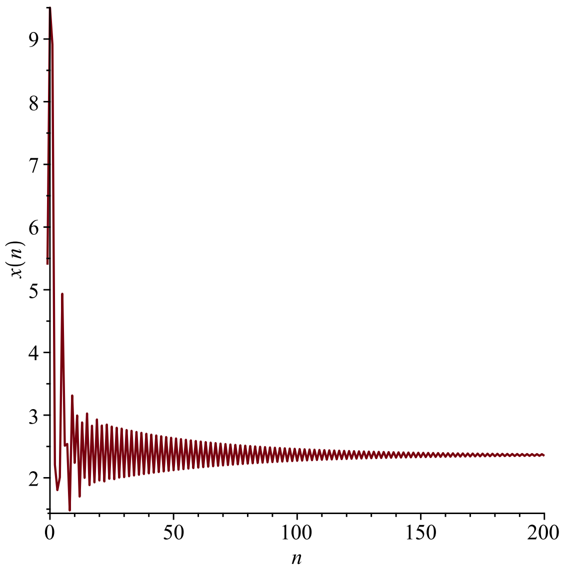

Figure 1. Plot of (x_n) converging to $\overline{x}$

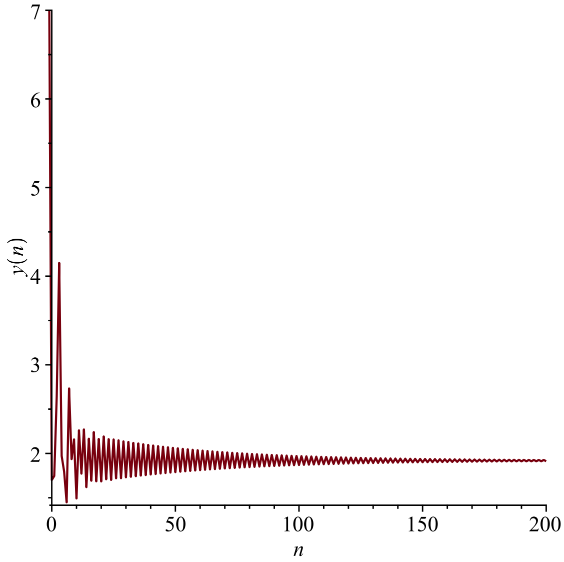

Figure 2. Plot of (y_n) converging to $\overline{y}$

Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be any solution of (1.4) such that

$$\lim_{n\to\infty} x_n = \overline{x} \quad and \quad \lim_{n\to\infty} y_n = \overline{y}, \tag{3.27}$$

where $\overline{x} \in [L_1, U_1]$ and $\overline{y} \in [L_2, U_2]$. From (1.4), we have

$$\begin{aligned} x_{n+1} - \overline{x} &= \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 y_n} - \frac{\alpha_1 + \beta_1 \overline{y}}{a_1 + b_1 \overline{y}} \\ &= \frac{-b_1\left(\alpha_1 + \beta_1 \overline{y}\right)}{(a_1 + b_1 y_n)\left(a_1 + b_1 \overline{y}\right)}\left(y_n - \overline{y}\right) + \frac{\beta_1\left(a_1 + b_1 \overline{y}\right)}{(a_1 + b_1 y_n)\left(a_1 + b_1 \overline{y}\right)}\left(y_{n-1} - \overline{y}\right) \end{aligned}$$

or after some operations and by using (3.14)

$$x_{n+1} - \overline{x} = \frac{-b_1 \overline{x}}{\left(a_1 + b_1 y_n\right)}\left(y_n - \overline{y}\right) + \frac{\beta_1}{\left(a_1 + b_1 y_n\right)}\left(y_{n-1} - \overline{y}\right). \tag{3.28}$$

Similarly, from (1.4), we have

$$\begin{aligned} y_{n+1} - \overline{y} &= \frac{\alpha_{2+} \beta_2 x_{n-1}}{a_2 + b_2 x_n} - \frac{\alpha_2 + \beta_2 \overline{x}}{a_2 + b_2 \overline{x}} \\ &= \frac{-b_2(\alpha_2 + \beta_2 \overline{x})}{(a_2 + b_2 x_n)\left(a_2 + b_2 \overline{x}\right)}\left(x_n - \overline{x}\right) + \frac{\beta_2\left(a_2 + b_2 \overline{x}\right)}{(a_2 + b_2 x_n)\left(a_2 + b_2 \overline{x}\right)}\left(x_{n-1} - \overline{x}\right) \end{aligned}$$

and so, by (3.14),

$$y_{n+1} - \overline{y} = \frac{-b_2 \overline{y}}{(a_2 + b_2 x_n)}\left(x_n - \overline{x}\right) + \frac{\beta_2}{(a_2 + b_2 x_n)}\left(x_{n-1} - \overline{x}\right). \tag{3.29}$$

If the error terms $e_n^1 = x_n - \bar{x}$, $e_n^2 = y_n - \bar{y}$, then we can write the system of the error terms as follows

$$\begin{aligned} e_{n+1}^1 &= a_n e_n^2 + b_n e_{n-1}^2, \\ e_{n+1}^2 &= c_n e_n^1 + d_n e_{n-1}^1, \end{aligned}$$

where

$$a_n = \frac{-b_1\overline{x}}{a_1 + b_1 y_n}, \quad b_n = \frac{\beta_1}{a_1 + b_1 y_n}, \quad c_n = \frac{-b_2\overline{y}}{a_2 + b_2 x_n}, \quad d_n = \frac{\beta_2}{a_2 + b_2 x_n}. \tag{3.30}$$

From (3.30), we obtain the limits

$$\lim_{n\to\infty} a_n = \frac{-b_1\overline{x}}{a_1 + b_1\overline{y}}, \tag{3.31}$$

$$\lim_{n\to\infty} b_n = \frac{\beta_1}{a_1 + b_1\overline{y}}, \tag{3.32}$$

$$\lim_{n\to\infty} c_n = \frac{-b_2\overline{y}}{a_2 + b_2\overline{x}}, \tag{3.33}$$

$$\lim_{n\to\infty} d_n = \frac{\beta_2}{a_2 + b_2\overline{x}}. \tag{3.34}$$

Consequently, from (3.31)-(3.34), we have the following system

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \\ e^1_n \\ e^2_n \end{pmatrix} = \begin{pmatrix} 0 & \frac{-b_1\overline{x}}{a_1 + b_1\overline{y}} & 0 & \frac{\beta_1}{a_1 + b_1\overline{y}} \\ \frac{-b_2\overline{y}}{a_2 + b_2\overline{x}} & 0 & \frac{\beta_2}{a_2 + b_2\overline{x}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^2_n \\ e^1_{n-1} \\ e^2_{n-1} \end{pmatrix}, \tag{3.35}$$

which resembles the linearized system of (1.4) about the equilibrium $\overline{X}$. In this case, one can obtain from Theorem 2.3 and Theorem 2.4 the following results.

Theorem 3.8. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be any positive solution of (1.4) satisfying (3.27). Then, the error vector $\left(e^1_n, e^2_n, e^1_{n-1}, e^2_{n-1}\right)^T$ of the solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of (1.4) satisfies the asymptotic relations

$$\lim_{n\to\infty} (||e_n||)^{\frac{1}{n}} = \left|\lambda_{1,2,3,4} J_F\left(\overline{x}, \overline{y}\right)\right|$$

and

$$\lim_{n\to\infty} \frac{||e_{n+1}||}{||e_n||} = \left|\lambda_{1,2,3,4} J_F\left(\overline{x}, \overline{y}\right)\right|,$$

where the values $\lambda_{1,2,3,4}$ are the eigenvalues of the Jacobian $J_F(\overline{x}, \overline{y})$.

3.4. Existence of unbounded solutions

In this subsection, the existence of unbounded solutions of (1.4) is proven.

Theorem 3.9. If $\beta_1\beta_2 > a_1 a_2$, then every positive solution of (1.4) is unbounded.

Proof.
From (1.4) we have the system of difference inequalities

$$x_{n+1} = \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 y_n} \geq \frac{\alpha_1 + \beta_1 y_{n-1}}{a_1 + b_1 U_2}, \tag{3.36}$$

and

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 x_n} \geq \frac{\alpha_2 + \beta_2 x_{n-1}}{a_2 + b_2 U_1}, \tag{3.37}$$

where U_1 and U_2 are given by (3.9) and (3.10), respectively. Now we can consider the system of nonhomogeneous linear equations

$$w_{n+1} = c_2 + d_2 z_{n-1}, \quad z_{n+1} = c_1 + d_1 w_{n-1}, \quad n \in \mathbb{N}_0, \tag{3.38}$$

where

$$c_1 = \frac{\alpha_1}{a_1 + b_1 U_2}, \quad d_1 = \frac{\beta_1}{a_1 + b_1 U_2}, \quad c_2 = \frac{\alpha_2}{a_2 + b_2 U_1}, \quad d_2 = \frac{\beta_2}{a_2 + b_2 U_1}$$

and $w_{-1} = x_{-1}$, $w_0 = x_0$, $z_{-1} = y_{-1}$, $z_0 = y_0$. The general solution of (3.38) is given by the formulas

$$w_n = \frac{c_2 + c_1 d_2}{1 - d_1 d_2} + k_1 \left(\sqrt[4]{d_1 d_2}\right)^n + k_2 \left(-\sqrt{d_1 d_2}\right)^n + k_3 \left(-i\sqrt[4]{d_1 d_2}\right)^n + k_4 \left(i\sqrt[4]{d_1 d_2}\right)^n \tag{3.39}$$

and

$$z_n = \frac{c_1 + c_2 d_1}{1 - d_1 d_2} + k_5 \left(\sqrt[4]{d_1 d_2}\right)^n + k_6 \left(-\sqrt{d_1 d_2}\right)^n + k_7 \left(-i\sqrt[4]{d_1 d_2}\right)^n + k_8 \left(i\sqrt[4]{d_1 d_2}\right)^n, \tag{3.40}$$

where k_s, $(s = 1, 2, ..., 8)$ are arbitrary constants and i is the imaginary unit. It is easy to see from (3.39) and (3.40) that if $d_1 d_2 > 1$, that is,

$$\beta_1 \beta_2 > (a_1 + b_1 U_2)(a_2 + b_2 U_1) > a_1 a_2$$

then the sequences (w_n) and (z_n) are unbounded. Therefore, since $w_{-1} = x_{-1}$, $w_0 = x_0$, $z_{-1} = y_{-1}$ and $z_0 = y_0$, by comparison method, we have the inequalities $x_n \geq w_n$, $y_n \geq z_n$. Hence the sequences $\{x_n\}$ and $\{y_n\}$ are unbounded. The proof is completed.

Example 3.10. If $\alpha_1 = 1$, $\beta_1 = 12.1$, $a_1 = 3.6$, $b_1 = 3$, $\alpha_2 = 12$, $\beta_2 = 3.5$, $a_2 = 6$, $b_2 = 1$, then (1.4) becomes

$$x_{n+1} = \frac{1 + 12.1 y_{n-1}}{3.6 + 3 y_n}, \quad y_{n+1} = \frac{12 + 3.5 x_{n-1}}{6 + x_n}. \tag{3.41}$$

The unique positive equilibrium of (3.41) is $(2.808100791, 2.478213327)$ and unstable. Plot of the corresponding solution to $x_{-1} = 5.4$, $x_0 = 9.5$, $y_{-1} = 7$ and $y_0 = 1.7$ is given by Figure 3 and Figure 4.

3.5. Period two solutions

In this subsection, the existence of two-periodic solutions of (1.4) is investigated. The next result states the existence of such solutions.

Theorem 3.11. If $a_1 a_2 = \beta_1 \beta_2$, then the system of difference equations (1.4) has two-periodic solutions.

Proof.
Let a two-periodic solution of (1.4) be

$$..., (p_1, q_1), (p_2, q_2), (p_1, q_1), (p_2, q_2), ..., \tag{3.42}$$

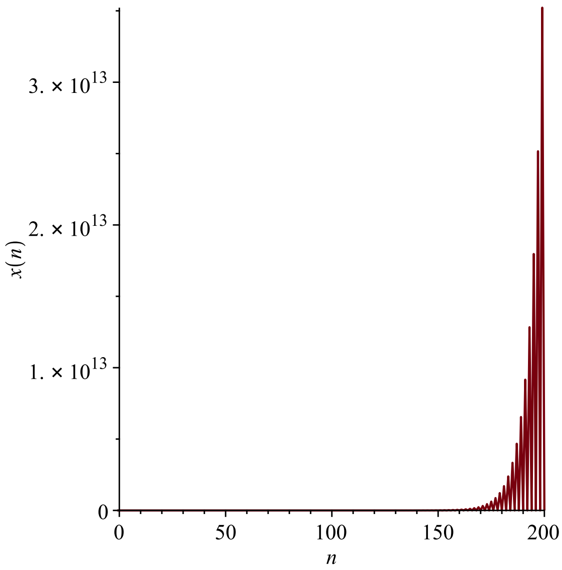

Figure 3. Plot of unbounded (x_n)

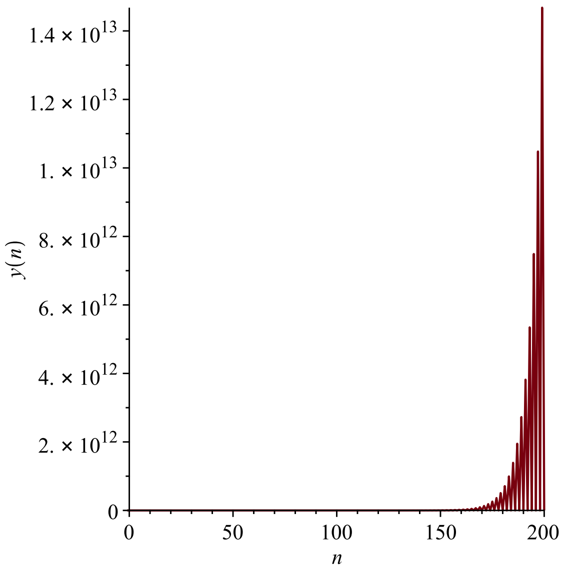

Figure 4. Plot of unbounded (y_n)

where p_1, p_2, q_1, q_2 are positive real numbers such that $p_1 \neq p_2$ ve $q_1 \neq q_2$. Then, from (1.4) and (3.42), we have the system

$$p_1 = \frac{\alpha_1 + \beta_1 q_1}{a_1 + b_1 q_2}, \quad p_2 = \frac{\alpha_1 + \beta_1 q_2}{a_1 + b_1 q_1}, \quad q_1 = \frac{\alpha_2 + \beta_2 p_1}{a_2 + b_2 p_2}, \quad q_2 = \frac{\alpha_2 + \beta_2 p_2}{a_2 + b_2 p_1},$$

from which it follows that

$$a_1 p_1 + b_1 p_1 q_2 = \alpha_1 + \beta_1 q_1, \quad a_1 p_2 + b_1 p_2 q_1 = \alpha_1 + \beta_1 q_2 \tag{3.43}$$

and

$$a_2 q_1 + b_2 q_1 p_2 = \alpha_2 + \beta_2 p_1, \quad a_2 q_2 + b_2 q_2 p_1 = \alpha_2 + \beta_2 p_2. \tag{3.44}$$

After some basic operations, from (3.43) and (3.44), we get the equalities

$$a_1 \left(p_1 - p_2\right) + b_1 \left(p_1 q_2 - p_2 q_1\right) = \beta_1 \left(q_1 - q_2\right)$$

and

$$a_2 \left(q_1 - q_2\right) + b_2 \left(q_1 p_2 - q_2 p_1\right) = \beta_2 \left(p_1 - p_2\right).$$

The last equalities yield

$$(a_1 b_2 - b_1 \beta_2)(p_1 - p_2) + (a_2 b_1 - b_2 \beta_1)(q_1 - q_2) = 0. \tag{3.45}$$

It is obvious from (3.45) and the assumptions $p_1 \neq p_2$ and $q_1 \neq q_2$ that if

$$a_1 b_2 - b_1 \beta_2 = 0 \quad \text{and} \quad a_2 b_1 - b_2 \beta_1 = 0, \tag{3.46}$$

then system (1.4) has two-periodic solutions. Note that (3.46) is equivalent to the desired equality $a_1 a_2 = \beta_1 \beta_2$. So the proof is completed.

Example 3.12. If $\alpha_1 = 3$, $\beta_1 = 6$, $a_1 = 12$, $b_1 = 9$, $\alpha_2 = 2$, $\beta_2 = 4$, $a_2 = 2$, $b_2 = 3$, then system (1.4) becomes

$$x_{n+1} = \frac{3+6y_{n-1}}{12+9y_n}, \quad y_{n+1} = \frac{2+4x_{n-1}}{2+3x_n}. \tag{3.47}$$

The unique positive equilibrium point of (3.47) is (0.4413911092, 1.132782218) and it is unstable. Also, the solution converges a two-periodic solution of the system. Plot of the corresponding solution with $x_{-1} = 3$, $x_0 = 2$, $y_{-1} = 1.3$ and $y_0 = 7$ is given by Figure 5 and Figure 6.

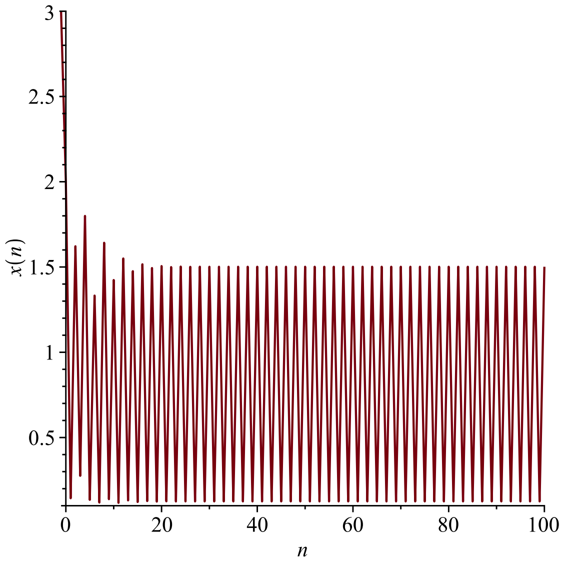

Figure 5. (x_n) converging to a two-periodic solution

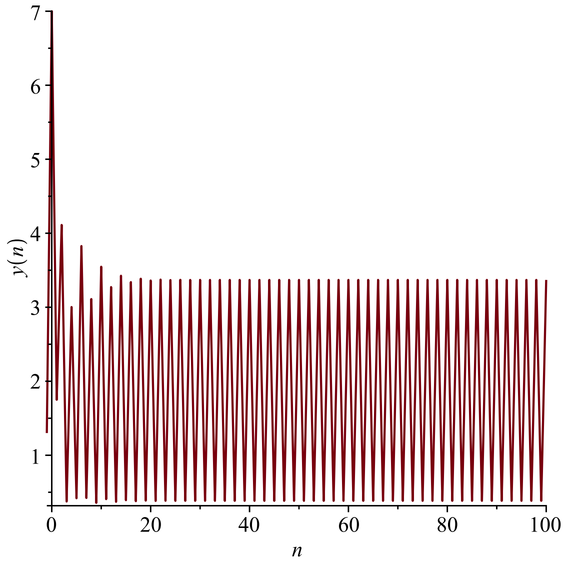

Figure 6. (y_n) converging to a two-periodic solution

4. Conclusion

In this study, the qualitative behavior of the positive solutions of (1.4) was investigated. The results obtained are summarized below.

1. If $\beta_1\beta_2 < a_1a_2$, then the solutions of the system are bounded and persist. In addition, the unique positive equilibrium of the system is globally asymptotically stable.
2. If $\beta_1\beta_2 = a_1a_2$, then the system has two-periodic solutions.
3. If $\beta_1\beta_2 > a_1a_2$, then the system has unbounded solutions.

Availability of data and materials Not applicable.

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