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E-ISSN 2717-6185

FUNDAMENTALS OF CONTEMPORARY MATHEMATICAL SCIENCES

BIANNUALLY SCIENTIFIC JOURNAL

VOLUME 5 - ISSUE 2

2024

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FUNDAMENTALS OF CONTEMPORARY MATHEMATICAL SCIENCES

e-ISSN: 2717-6185

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FCMS
FUNDAMENTALS OF CONTEMPORARY MATHEMATICAL SCIENCES

e-ISSN: 2717-6185

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Volume: 5 Issue: 2 - July 2024

Research Articles

Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1312895
(**2024**) **5**(2) 60 – 71
Research Article

The Alpha Distance Formulae

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Received: 11 June 2023 **Accepted:** 27 July 2024

Abstract: In this study, we define the concept of tangent in two and three-dimensional alpha spaces concerning the alpha circles and the alpha spheres. Then using this concept, we derive the alpha distance formulae between points, a point and a line, between two lines and a point and a plane of the alpha spaces. Finally, we give simple area and volume formulas in the three dimensional space in terms of the alpha distances.

Keywords: Alpha metric, distance formula, metric geometry.

1. Introduction

Metrics with their special properties have been very important keys for many application areas during the recent years. There are many metrics used in the mathematics (see [8]) to measure the distance (similarity or dissimilarity) between points (or vectors). These measurements are important for determining how closely related two pieces of data are in statistical analysis. The alpha metric (α-metric) is a generalization of two famous metrics known as the taxicab and the Chinese checker metric which are used in such applications. They are very suitable for new studies since it includes infinitely many metrics in which the alpha can be considered as a weight that can reflect relative importance of different criteria or dimensions. On the other hand, the derived conclusions are rather wide (for example see [6, 7, 9]).

On the road to the alpha metric, first Menger introduced the taxicab geometry using the taxicab metric [14], and Krause took the first steps to develop it [13]. The taxicab metric is the special case of the l_p-metric for $p = 1$. In [13], Krause asked how to develop a distance function from a point to another which measures the length of ways mimicking the movements of the Chinese checkers in the Cartesian coordinate plane. Then, Chen answered this question defining the Chinese checker metric [2]. After a while, the α-metric for $\alpha \in \left[0, \frac{\pi}{4}\right]$, which includes the taxicab and Chinese checker metrics as special cases for $\alpha = 0$ and $\alpha = \frac{\pi}{4}$, defined by Tian [15].

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2020 *AMS Mathematics Subject Classification*: 51K05, 51K99, 51N20

Later, Gelişgen and Kaya gave n-dimensional α-metric [11, 12]. Finally, Çolakoğlu expanded the interval $\alpha \in \left[0, \frac{\pi}{4}\right]$ to $\alpha \in \left[0, \frac{\pi}{2}\right)$ for the α-metric [3].

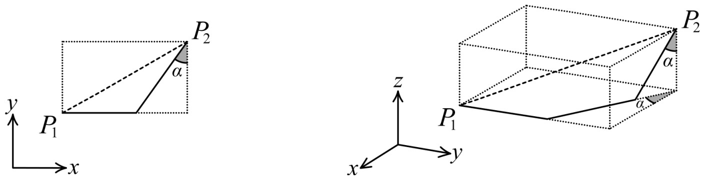

Figure 1: The distances between two points

Geometrically, the α-distance between two points in the plane, is the sum of Euclidean lengths of line segments joining the points, one of which is parallel to a coordinate axis and the other one is parallel to a line making angle α with the other coordinate axis (see Figure 1). So far many studies have been done on this topic (see [1, 4–7, 9, 10]). In this study, we determine the α-distance formulae between two basic elements such as points, lines and planes, whose Euclidean analogs are well-known already, and give simple area and volume formulas in the three dimensional alpha space.

2. Preliminaries

For the positive real number $\lambda(\alpha) = (\sec\alpha - \tan\alpha)$, where $\alpha \in \left[0, \frac{\pi}{2}\right)$, the α-distance between points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the plane is

$$d_\alpha(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + \lambda(\alpha)\min\{|x_1 - x_2|, |y_1 - y_2|\}. \quad (1)$$

Clearly, the unit α-circle has the following equation:

$$\max\{|x|, |y|\} + \lambda(\alpha)\min\{|x|, |y|\} = 1. \quad (2)$$

One can see that in the plane; if $\alpha \in \left(0, \frac{\pi}{2}\right)$, the unit α-circle is an octagon (see Figure 2) having corners $C_1 = (1, 0)$, $C_2 = \left(\frac{1}{\tau}, \frac{1}{\tau}\right)$, $C_3 = (0, 1)$, $C_4 = \left(\frac{-1}{\tau}, \frac{1}{\tau}\right)$, C_1', C_2', C_3', C_4', where C_i' are the symmetric points of C_i about the origin and $\tau = 1 + \lambda(\alpha)$. In addition, if $\alpha = \frac{\pi}{4}$ then the unit α-circle is a regular octagon with the same vertices, and if $\alpha = 0$, the unit α-circle is a square having corners C_1, C_1', C_3, C_3'.

Similarly, the α-distance between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in the three dimensional space is

$$d_\alpha(P_1, P_2) = \Delta_{P_1P_2} + \lambda(\alpha)\delta_{P_1P_2}, \quad (3)$$

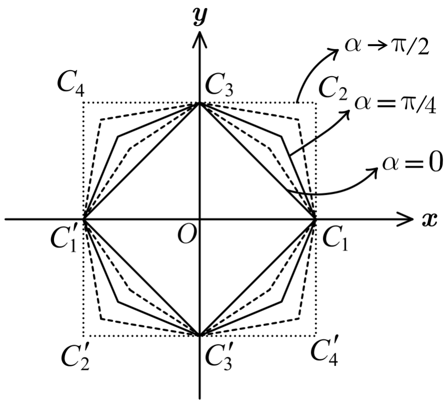

Figure 2: The unit α-circles for $\alpha = 0$, $\alpha = \frac{\pi}{4}$ and $\alpha \to \frac{\pi}{2}$

where

$$\Delta_{P_1P_2} = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}, \tag{4}$$

$$\delta_{P_1P_2} = \min\{|x_1 - x_2| + |y_1 - y_2|, |x_1 - x_2| + |z_1 - z_2|, |y_1 - y_2| + |z_1 - z_2|\}, \tag{5}$$

and the unit α-sphere has the following equation:

$$\max\{|x|, |y|, |z|\} + \lambda(\alpha)\min\{|x| + |y|, |x| + |z|, |y| + |z|\} = 1. \tag{6}$$

One can also see that in three dimensional space; if $\alpha \in \left(0, \frac{\pi}{2}\right)$ then the unit α-sphere is deltoidal icositetrahedron (see Figure 3) having corners $S_1 = (1,0,0)$, $S_2 = (0,1,0)$, $S_3 = (0,0,1)$, $S_4 = \left(\frac{1}{\tau}, \frac{1}{\tau}, 0\right)$, $S_5 = \left(\frac{-1}{\tau}, \frac{1}{\tau}, 0\right)$, $S_6 = \left(\frac{1}{\tau}, 0, \frac{1}{\tau}\right)$, $S_7 = \left(\frac{-1}{\tau}, 0, \frac{1}{\tau}\right)$, $S_8 = \left(0, \frac{1}{\tau}, \frac{1}{\tau}\right)$, $S_9 = \left(0, \frac{-1}{\tau}, \frac{1}{\tau}\right)$, $S_{10} = \left(\frac{1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau}\right)$, $S_{11} = \left(\frac{-1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau}\right)$, $S_{12} = \left(\frac{-1}{\tau}, \frac{-1}{\tau}, \frac{1}{\tau}\right)$, $S_{13} = \left(\frac{1}{\tau}, \frac{-1}{\tau}, \frac{1}{\tau}\right)$, S_1', S_2', S_3', S_4', S_5', S_6', S_7', S_8', S_9', S_{10}', S_{11}', S_{12}', S_{13}', where S_i' are the symmetric points of S_i about the origin and $\tau = 1 + \lambda(\alpha)$, if $\alpha \in \left(0, \frac{\pi}{2}\right)$ then the unit α-sphere is a regular octahedron.

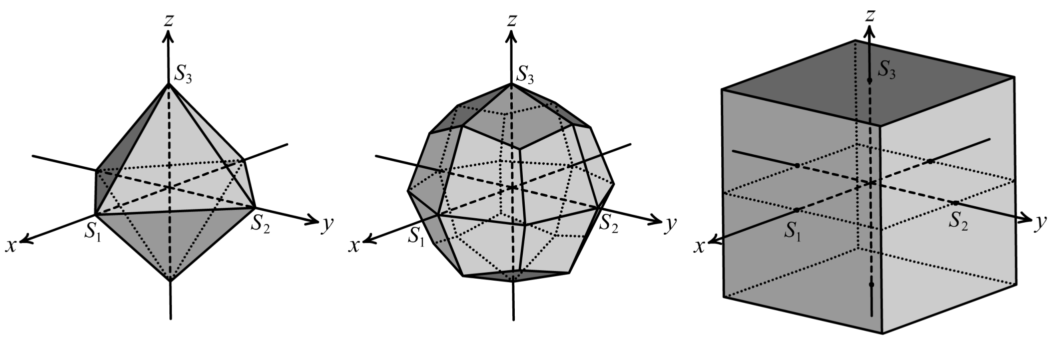

Figure 3: The unit α-spheres for $\alpha = 0$, $\alpha = \frac{\pi}{4}$ and $\alpha \to \frac{\pi}{2}$

3. Main Results

We use the tangent notion to determine α-distance formulae. As a natural analog of the tangent notion in the Euclidean geometry, a line whose α-distance from the center of a given α-circle is the radius of the α-circle, is called a tangent to the α-circle, and a line or a plane whose α-distance from the center of a given α-sphere is the radius of the α-sphere, is called a tangent line or tangent plane to the α-sphere. For instance, in Figure 4, while the lines l_1 and l_2 are tangent to the α-circle with center P_1; the lines l_3, l_4, and the planes Ω_1, Ω_2 are tangent to the α-sphere with center P_2.

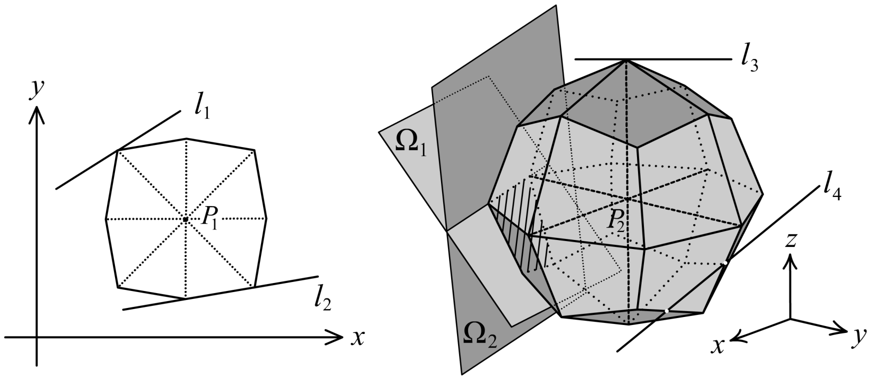

Figure 4: Tangent lines and planes to a α-circle and a α-sphere

Before we start determining the α-distance formulae, let us define three following vector sets that we will use in the proofs as V_1, V_3, V_2, respectively:

$$\{(1,0),(0,1),(1,1),(-1,1)\},$$
$$\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(-1,1,0),(1,0,1),(-1,0,1),(0,1,1),(0,-1,1)\},$$
$$V_3 \cup \{(1,1,1),(-1,1,1),(1,-1,1),(1,1,-1)\}.$$

The formula of the α-distance between a point and a line is given by the following proposition:

Proposition 3.1 *The α-distance between a point $P=(x_0,y_0)$ and a line $l: Ax+By+C=0$ in $\mathbb{R}^2$ is*

$$d_\alpha(P,l) = \frac{|Ax_0+By_0+C|}{\max\left\{|A|,|B|,\frac{|A\mp B|}{1+\lambda(\alpha)}\right\}}. \tag{7}$$

Proof It is clear that

$$d_\alpha(P,l) = \min\{d_\alpha(P,X) : X \in l\},$$

which is equal to the radius of the α-circle with center P, that is tangent to the line l. So, a corner of the α-circle is on l and one of the lines $l_{\mathbf{v}}: \beta(t_{\mathbf{v}}) = (x_0,y_0) + \mathbf{v}t_{\mathbf{v}}$ passing the point P

having the direction vector $\mathbf{v} \in V_1$. Therefore, at least one of the points $Q_{\mathbf{v}} = l \cap l_{\mathbf{v}}$ exists, and l is tangent to the α-circle at one of them. So, we have

$$d_\alpha(P,l) = \min\{d_\alpha(P,Q_{\mathbf{v}}) : \mathbf{v} \in V_1)\}.$$

One can find that

$$Q_{(1,0)} = \left(x_0 + t_{(1,0)}, y_0\right), Q_{(0,1)} = \left(x_0, y_0 + t_{(0,1)}\right),$$

$$Q_{(1,1)} = \left(x_0 + t_{(1,1)}, y_0 + t_{(1,1)}\right), Q_{(-1,1)} = \left(x_0 + t_{(-1,1)}, y_0 + t_{(-1,1)}\right),$$

where $t_{\mathbf{v}} = \frac{-Ax_0 - By_0 - C}{\langle (A,B), \mathbf{v}\rangle}$.

If the line l is not parallel to the lines $l_{\mathbf{v}}$, where $\mathbf{v} \in V_1$, then all of the points $Q_{\mathbf{v}}$ can be obtained and one gets

$$d_\alpha(P,Q_{(1,0)}) = \left|t_{(1,0)}\right| = \frac{|Ax_0 + By_0 + C|}{|A|},$$

$$d_\alpha(P,Q_{(0,1)}) = \left|t_{(0,1)}\right| = \frac{|Ax_0 + By_0 + C|}{|B|},$$

$$d_\alpha(P,Q_{(1,1)}) = \left|t_{(1,1)}\right|(1+\lambda(\alpha)) = \frac{|Ax_0 + By_0 + C|}{|A+B|/(1+\lambda(\alpha))},$$

$$d_\alpha(P,Q_{(-1,1)}) = \left|t_{(-1,1)}\right|(1+\lambda(\alpha)) = \frac{|Ax_0 + By_0 + C|}{|-A+B|/(1+\lambda(\alpha))}.$$

Then one has

$$d_\alpha(P,l) = \min\left\{\frac{|Ax_0+By_0+C|}{|A|}, \frac{|Ax_0+By_0+C|}{|B|}, \frac{|Ax_0+By_0+C|}{|A+B|/(1+\lambda(\alpha))}, \frac{|Ax_0+By_0+C|}{|A-B|/(1+\lambda(\alpha))}\right\},$$

and

$$d_\alpha(P,l) = \frac{|Ax_0 + By_0 + C|}{\max\left\{|A|, |B|, \frac{|A \mp B|}{1+\lambda(\alpha)}\right\}}.$$

Other conditions do not change the result. □

The α-distance between two parallel lines in the plane can be determined by the following formula:

Corollary 3.2 *The α-distance between $l_1 : Ax + By + C_1 = 0$ and $l_2 : Ax + By + C_2 = 0$ in $\mathbb{R}^2$ is*

$$d_\alpha(l_1,l_2) = \frac{|C_1 - C_2|}{\max\left\{|A|, |B|, \frac{|A \mp B|}{1+\lambda(\alpha)}\right\}}. \tag{8}$$

The three dimensional case is similar. One can consider an α-sphere instead of an α-circle. The α-distance from a point to a plane or a line is equal to the radius of the widening α-sphere when the plane or the line becomes tangent to the α-sphere. The following proposition states a formula for the α-distance between a point and a plane in three-dimensional space:

Proposition 3.3 *The α-distance between the point $P=(x_0,y_0,z_0)$ and the plane $\Omega : Ax+By+Cz+D=0$ in $\mathbb{R}^3$ is*

$$d_\alpha(P,\Omega)=\frac{|Ax_0+By_0+Cz_0+D|}{\max\left\{|K|,\frac{|K\mp L|}{1+\lambda(\alpha)},\frac{|K+L\mp M|}{1+2\lambda(\alpha)}\right\}}, \tag{9}$$

where $K,L,M\in\{A,B,C\}$ and $K\neq L\neq M\neq K$.

Proof It is obvious that

$$d_\alpha(P,\Omega)=\min\{d_\alpha(P,X):X\in\Omega\},$$

which is equal to the radius of the α-sphere with center P, that is tangent to the plane Ω. So, at least one vertex of the α-sphere is on Ω and one of the lines $l_\mathbf{v}:\beta(t_\mathbf{v})=(x_0,y_0,z_0)+\mathbf{v}t_\mathbf{v}$ passing the point P having direction vector $\mathbf{v}\in V_2$. Therefore, at least one of the points $Q_\mathbf{v}=\Omega\cap l_\mathbf{v}$ exists, and Ω is tangent to the α-sphere at one of them. So, we have

$$d_\alpha(P,\Omega)=\min\{d_\alpha(P,Q_v):v\in V_2\}.$$

One can find that

$$Q_{(1,0,0)}=\left(x_0+t_{(1,0,0)},y_0,z_0\right),\ Q_{(0,1,0)}=\left(x_0,y_0+t_{(0,1,0)},z_0\right),\ Q_{(0,0,1)}=\left(x_0,y_0,z_0+t_{(0,0,1)}\right),$$
$$Q_{(1,1,0)}=\left(x_0+t_{(1,1,0)},y_0+t_{(1,1,0)},z_0\right),\ Q_{(-1,1,0)}=\left(x_0-t_{(-1,1,0)},y_0+t_{(-1,1,0)},z_0\right),$$
$$Q_{(1,0,1)}=\left(x_0+t_{(1,0,1)},y_0,z_0+t_{(1,0,1)}\right),\ Q_{(-1,0,1)}=\left(x_0-t_{(-1,0,1)},y_0,z_0+t_{(-1,0,1)}\right),$$
$$Q_{(0,1,1)}=\left(x_0,y_0+t_{(0,1,1)},z_0+t_{(0,1,1)}\right),\ Q_{(0,-1,1)}=\left(x_0,y_0-t_{(0,-1,1)},z_0+t_{(0,-1,1)}\right),$$
$$Q_{(1,1,1)}=\left(x_0+t_{(1,1,1)},y_0+t_{(1,1,1)},z_0+t_{(1,1,1)}\right),$$
$$Q_{(-1,1,1)}=\left(x_0-t_{(-1,1,1)},y_0+t_{(-1,1,1)},z_0+t_{(-1,1,1)}\right),$$
$$Q_{(1,-1,1)}=\left(x_0+t_{(1,-1,1)},y_0-t_{(1,-1,1)},z_0+t_{(1,-1,1)}\right),$$
$$Q_{(1,1,-1)}=\left(x_0+t_{(1,1,-1)},y_0+t_{(1,1,-1)},z_0-t_{(1,1,-1)}\right),$$

where $t_\mathbf{v}=\frac{-Ax_0-By_0-Cz_0-D}{\langle(A,B,C),\mathbf{v}\rangle}$.

Thus, if the plane Ω is not parallel to the lines $l_\mathbf{v}$ where $\mathbf{v}\in V_2$, then all of the points $Q_\mathbf{v}$

exist and we obtain

$$
\begin{aligned}
d_\alpha(P,Q_{(1,0,0)}) &= \left|t_{(1,0,0)}\right| = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A|}, \\
d_\alpha(P,Q_{(1,0,0)}) &= \left|t_{(0,1,0)}\right| = \tfrac{|Ax_0+By_0+Cz_0+D|}{|B|}, \\
d_\alpha(P,Q_{(1,0,0)}) &= \left|t_{(0,0,1)}\right| = \tfrac{|Ax_0+By_0+Cz_0+D|}{|C|}, \\
d_\alpha(P,Q_{(1,1,0)}) &= \left|t_{(1,1,0)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A+B|/(1+\lambda(\alpha))},
\end{aligned}
$$

$$
\begin{aligned}
d_\alpha(P,Q_{(-1,1,0)}) &= \left|t_{(-1,1,0)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|-A+B|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(1,0,1)}) &= \left|t_{(1,0,1)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A+C|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(-1,0,1)}) &= \left|t_{(-1,0,1)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|-A+C|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(0,1,1)}) &= \left|t_{(0,1,1)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|B+C|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(0,-1,1)}) &= \left|t_{(0,-1,1)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|-B+C|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(1,1,1)}) &= \left|t_{(1,1,1)}\right|(1+2\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A+B+C|/(1+2\lambda(\alpha))},
\end{aligned}
$$

$$
\begin{aligned}
d_\alpha(P,Q_{(-1,1,1)}) &= \left|t_{(-1,1,1)}\right|(1+2\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|-A+B+C|/(1+2\lambda(\alpha))}, \\
d_\alpha(P,Q_{(1,-1,1)}) &= \left|t_{(1,-1,1)}\right|(1+2\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A-B+C|/(1+2\lambda(\alpha))}, \\
d_\alpha(P,Q_{(1,1,-1)}) &= \left|t_{(1,1,-1)}\right|(1+2\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A+B-C|/(1+2\lambda(\alpha))}.
\end{aligned}
$$

Therefore, we get

$$
d_\alpha(P,\Omega) = \min\left\{\tfrac{|Ax_0+By_0+Cz_0+D|}{|K|}, \tfrac{|Ax_0+By_0+Cz_0+D|}{|K\mp L|/(1+\lambda(\alpha))}, \tfrac{|Ax_0+By_0+Cz_0+D|}{|K+L\mp M|/(1+2\lambda(\alpha))}\right\}
$$

and so

$$
d_\alpha(P,\Omega) = \frac{|Ax_0+By_0+Cz_0+D|}{\max\left\{|K|, \frac{|K\mp L|}{1+\lambda(\alpha)}, \frac{|K+L\mp M|}{1+2\lambda(\alpha)}\right\}},
$$

where $K,L,M \in \{A,B,C\}$ and $K \neq L \neq M \neq K$. Other conditions do not change the result. □

The α-distance between two parallel planes in three-dimensional space can be given as follows:

Corollary 3.4 *The α-distance between $\Omega_1 : Ax+By+Cz+D_1=0$ and $\Omega_2 : Ax+By+Cz+D_2=0$ in $\mathbb{R}^3$ is*

$$
d_\alpha(\Omega_1,\Omega_2) = \frac{|D_1-D_2|}{\max\left\{|K|, \frac{|K\mp L|}{1+\lambda(\alpha)}, \frac{|K+L\mp M|}{1+2\lambda(\alpha)}\right\}}, \tag{10}
$$

where $K, L, M \in \{A, B, C\}$ *and* $K \neq L \neq M \neq K$.

The α-distance between a point and a line in three-dimensional space can be computed by the formula given in the following proposition:

Proposition 3.5 *The* α*-distance between the point* $P = (x_0, y_0, z_0)$ *and the line* l *passing through the point* $P_1 = (x_1, y_1, z_1)$, *with the direction vector* $\mathbf{u} = (u_1, u_2, u_3)$ *in* $\mathbb{R}^3$ *is*

$$d_\alpha(P, l) = \min_{\mathbf{v} \in V_3} \left\{ \max \left\{ \left| \rho_i - \frac{u_i \langle \rho, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle} \right| \right\} + \lambda(\alpha) \min \left\{ \left| \rho_j - \frac{u_j \langle \rho, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle} \right| + \left| \rho_k - \frac{u_k \langle \rho, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle} \right| \right\} \right\}, \tag{11}$$

where $\rho = (\rho_1, \rho_2, \rho_3) = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$, *and* $i, j, k \in \{1, 2, 3\}$ *for* $i \neq j \neq k \neq i$.

Proof We get that

$$d_\alpha(P, l) = \min \{ d_\alpha(P, X) : X \in l \},$$

which is equal to the radius of the α-sphere with center P, that is tangent to the line l. One can see that if the line l tangent to this α-sphere, at least one point on an edge of the sphere is on both the line l and one of the planes $\Omega_{\mathbf{v}}$ passing P having the normal vector $\mathbf{v} \in V_3$. Therefore, at least one of the points $R_{\mathbf{v}} = l \cap \Omega_{\mathbf{v}}$ exists, and l is tangent to the α-sphere at one of them. So, we have

$$d_\alpha(P, l) = \min \{ d_\alpha(P, R_{\mathbf{v}}) : \mathbf{v} \in V_3 \}.$$

Considering $l : \beta(t) = (x_1 + tu_1, y_1 + tu_2, z_1 + tu_3)$ and $\Omega_{\mathbf{v}}$, one can find that

$$R_{\mathbf{v}} = (x_1 + u_1 t_{\mathbf{v}}, y_1 + u_2 t_{\mathbf{v}}, z_1 + u_3 t_{\mathbf{v}}),$$

where $t_v = \frac{\langle \rho, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle}$, $\rho = (\rho_1, \rho_2, \rho_3) = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$ and $\mathbf{v} \in V_3$. If the line l is not parallel to the planes $\Omega_{\mathbf{v}}$, where $\mathbf{v} \in V_3$, then all of the points $R_{\mathbf{v}}$ exist and we obtain

$$
\begin{aligned}
d_\alpha(P, R_{(1,0,0)}) &= \max\left\{\left|\rho_2 - \tfrac{u_2}{u_1}\rho_1\right| \left|\rho_3 - \tfrac{u_3}{u_1}\rho_1\right|\right\} + \lambda(\alpha)\min\left\{\left|\rho_2 - \tfrac{u_2}{u_1}\rho_1\right| \left|\rho_3 - \tfrac{u_3}{u_1}\rho_1\right|\right\}, \\
d_\alpha(P, R_{(0,1,0)}) &= \max\left\{\left|\rho_1 - \tfrac{u_1}{u_2}\rho_2\right| \left|\rho_3 - \tfrac{u_3}{u_2}\rho_2\right|\right\} + \lambda(\alpha)\min\left\{\left|\rho_1 - \tfrac{u_1}{u_2}\rho_2\right| \left|\rho_3 - \tfrac{u_3}{u_2}\rho_2\right|\right\}, \\
d_\alpha(P, R_{(0,0,1)}) &= \max\left\{\left|\rho_1 - \tfrac{u_1}{u_3}\rho_3\right| \left|\rho_2 - \tfrac{u_2}{u_3}\rho_3\right|\right\} + \lambda(\alpha)\min\left\{\left|\rho_1 - \tfrac{u_1}{u_3}\rho_3\right| \left|\rho_2 - \tfrac{u_2}{u_3}\rho_3\right|\right\}, \\
d_\alpha(P, R_{(1,1,0)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1 + k_2, k_1 + k_3, k_2 + k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(\rho_1+\rho_2)}{u_1+u_2}\right| \\
d_\alpha(P, R_{(-1,1,0)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1 + k_2, k_1 + k_3, k_2 + k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(-\rho_1+\rho_2)}{-u_1+u_2}\right| \\
d_\alpha(P, R_{(1,0,1)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1 + k_2, k_1 + k_3, k_2 + k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(\rho_1+\rho_3)}{u_1+u_3}\right| \\
d_\alpha(P, R_{(-1,0,1)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1 + k_2, k_1 + k_3, k_2 + k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(-\rho_1+\rho_3)}{-u_1+u_3}\right| \\
d_\alpha(P, R_{(0,1,1)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1 + k_2, k_1 + k_3, k_2 + k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(\rho_2+\rho_3)}{u_2+u_3}\right| \\
d_\alpha(P, R_{(0,-1,1)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1 + k_2, k_1 + k_3, k_2 + k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(-\rho_2+\rho_3)}{-u_2+u_3}\right|
\end{aligned}
$$

Therefore, we get

$$
d_\alpha(P, l) = \min_{\mathbf{v} \in V_3}\left\{\max\left\{\left|\rho_i - \frac{u_i\langle\rho, \mathbf{v}\rangle}{\langle\mathbf{u}, \mathbf{v}\rangle}\right|\right\} + \lambda(\alpha)\min\left\{\left|\rho_j - \frac{u_j\langle\rho, \mathbf{v}\rangle}{\langle\mathbf{u}, \mathbf{v}\rangle}\right| + \left|\rho_k - \frac{u_k\langle\rho, \mathbf{v}\rangle}{\langle\mathbf{u}, \mathbf{v}\rangle}\right|\right\}\right\},
$$

where $\rho = (\rho_1, \rho_2, \rho_3) = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$, and $i, j, k \in \{1, 2, 3\}$ for $i \neq j \neq k \neq i$. Other conditions do not change the result. □

The α-distance between two skew lines in three dimensional space can be determined by the following proposition:

Proposition 3.6 *Let*

$$
\begin{aligned}
l_1 &: \beta_1(t) = (x_1, y_1, z_1) + t(u_1, u_2, u_3), \\
l_2 &: \beta_2(t) = (x_2, y_2, z_2) + t(v_1, v_2, v_3)
\end{aligned}
$$

be two skew lines. Then the α*-distance between* l_1 *and* l_2 *is*

$$
d_\alpha(l_1, l_2) = \frac{\left|(x_1 - x_2)\mu_{(2,3)} + (y_1 - y_2)\mu_{(3,1)} + (z_1 - z_2)\mu_{(1,2)}\right|}{\max\left\{\left|\mu_{(2,3)}/\lambda_1\right|, \left|\mu_{(3,1)}/\lambda_2\right|, \left|\mu_{(1,2)}/\lambda_3\right|\right\}} \tag{12}
$$

with $\mu_{(m,n)} = u_m v_n - u_n v_m$.

Proof Since the lines l_1 and l_2 are skew, there is only one plane Ω through l_2, parallel to l_1. Then we have

$$
d_\alpha(l_1, \Omega) = d_\alpha(P_1, \Omega)
$$

for any point P_1 on l_1. Thus we get

$$d_\alpha(l_1, l_2) = d_\alpha(P_1, \Omega)$$

since there is an α-sphere whose center at l_1 and radius $d_\alpha(P_1, \Omega)$, that is tangent to l_2. So, since

$$\langle P_2X, (u_1, u_2, u_3) \times (v_1, v_2, v_3)\rangle = 0$$

for $X = (x, y, z)$ and $P_2 = (x_2, y_2, z_2)$ on Ω, we get

$$(x - x_2)\mu_{(2,3)} + (y - y_2)\mu_{(3,1)} + (z - z_2)\mu_{(1,2)} = 0,$$

where $\mu_{(m,n)} = u_m v_n - u_n v_m$, for the equation of the plane Ω. Therefore, by Proposition 3.3, one gets

$$d_\alpha(l_1, l_2) = d_\alpha(P_1, \Omega) = \frac{\left|(x_1 - x_2)\mu_{(2,3)} + (y_1 - y_2)\mu_{(3,1)} + (z_1 - z_2)\mu_{(1,2)}\right|}{\max\left\{|K|, \frac{|K \mp L|}{1+\lambda(\alpha)}, \frac{|K+L\mp M|}{1+2\lambda(\alpha)}\right\}}$$

with $K, L, M \in \{\mu_{(1,2)}, \mu_{(3,1)}, \mu_{(2,3)}\}$ and $K \neq L \neq M \neq K$. □

Clearly, the distance formulae derived here give also the taxicab and Chinese checker distance formulae when $\alpha = 0$ and $\alpha = \frac{\pi}{4}$, respectively (see [4, 10]).

4. Area and Volume in Terms of the Alpha Distance

Here, we give an alpha version of the area and volume formulas in terms of the alpha distance using the following equation which relates the Euclidean distance to the alpha distance between two points in the three dimensional space.

Proposition 4.1 *For any two points P_1 and P_2 in $\mathbb{R}^3$, if $\mathbf{u} = (u_1, u_2, u_3)$ is a direction vector of the line through P_1 and P_2, then*

$$d_E(P_1, P_2) = \rho(\mathbf{u}) d_\alpha(P_1, P_2), \tag{13}$$

where $\rho(\mathbf{u}) = (u_1^2 + u_2^2 + u_3^2)^{1/2} / (\max\{|u_1|, |u_2|, |u_3|\} + \lambda(\alpha) \min\{|u_1 + u_2|, |u_1 + u_3|, |u_2 + u_3|\})$.

Proof Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. Then $\mathbf{u} = k(x_1 - x_2, y_1 - y_2, z_1 - z_2)$ for some $k \in \mathbb{R}^*$. Since

$$\frac{d_E(P_1, P_2)}{d_\alpha(P_1, P_2)} = \frac{\|\mathbf{u}\|_E}{\|\mathbf{u}\|_\alpha},$$

we have

$$d_E(P_1, P_2) = \rho(\mathbf{u}) d_\alpha(P_1, P_2),$$

where

$$\rho(\mathbf{u}) = \frac{\|\mathbf{u}\|_E}{\|\mathbf{u}\|_\alpha} = \frac{(u_1^2 + u_2^2 + u_3^2)^{1/2}}{\max\{|u_1|, |u_2|, |u_3|\} + \lambda(\alpha)\min\{|u_1 + u_2|, |u_1 + u_3|, |u_2 + u_3|\}}.$$

□

The following corollaries gives alpha versions of the standard area and volume formulas in terms of alpha distances. The proofs are straightforward.

Corollary 4.2 *Let PQR be a triangle with the area $\mathcal{A}$ in the three dimensional alpha space, and let $a = d_\alpha(Q, R)$ and $h = d_\alpha(P, H)$, where H is the Euclidean orthogonal projection of the point P on the line QR. If $\mathbf{u}$ and $\mathbf{v}$ are direction vectors of the lines QR and PH, respectively, then*

$$\mathcal{A} = ah/2\rho(\mathbf{u})\rho(\mathbf{v}).$$

Corollary 4.3 *Let $PQRS$ be a tetrahedron having the base QRS in the plane $Ax+By+Cz+D = 0$, and let $h = d_\alpha(P, H)$, where H is the Euclidean orthogonal projection of the point P to the base. If the area of the triangle QRS is $\mathcal{A}$, then the volume of the tetrahedron is*

$$\mathcal{V} = \mathcal{A}h/3\rho(A, B, C).$$

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1353956
(**2024**) **5**(2) 72 – 82
Research Article

On Some Functions Related to e^*-θ-open Sets

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Received: 01 September 2023 **Accepted:** 13 July 2024

Abstract: In this study, we defined the concept of quasi e^*-θ-closed sets by means of e^*-θ-open sets. Depending on this concept, we introduced approximately $e^*\theta$-open functions and investigated some of its basic properties. Also, we defined and studied contra pre $e^*\theta$-open functions, which are stronger than the approximately $e^*\theta$-open functions. Moreover, we characterized the class of $e^*\theta$-$T_{\frac{1}{2}}$ spaces.

Keywords: $e^*\theta$-open functions, quasi e^*-θ-closed sets, approximately $e^*\theta$-open functions, contra pre $e^*\theta$-open functions, $e^*\theta$-$T_{\frac{1}{2}}$ spaces.

1. Introduction

In 2015, Farhan and Yang [10] introduced a new class of open sets called e^*-θ-open. In the following years, some concepts of open functions in relation to e^*-θ-open sets [10] have been investigated. The notion of $e^*\theta$-open functions is introduced by Ayhan [3] as follows: A function $f : X \to Y$ is said to be $e^*\theta$-open if the image of each open set U of X is e^*-θ-open in Y. In 2018, Ayhan and Özkoç [6] defined a new type of open functions called $e^*\theta$-semiopen functions. Again within the same year, Ayhan and Özkoç [5] defined and studied pre $e^*\theta$-open functions. In 2022, Ayhan [4] introduced and investigated weakly $e^*\theta$-open functions and also obtained some characterizations of its.

Rajesh and Salleh [14] gave the definition of quasi-b-θ-closed sets via b-θ-open sets [13] in their work titled "Some more results on b-θ-open sets". Caldas and Jafari [7] introduced and studied $g\beta\theta$-closed sets through β-θ-openness [12], in 2015.

In this paper, we introduce quasi e^*-θ-closed sets [1] defined with the help of e^*-θ-open sets. Moreover, we define and study approximately $e^*\theta$-open functions and contra pre $e^*\theta$-open functions such that these are weaker than $e^*\theta$-open functions.

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2020 *AMS Mathematics Subject Classification*: 54C10, 54D10

2. Preliminaries

Throughout this paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. A point $x \in X$ is called to be δ-cluster point [16] of A if $int(cl(U)) \cap A \neq \emptyset$ for every open neighborhood U of x. The set of all δ-cluster points of A is called the δ-closure [16] of A and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then A is called δ-closed [16] and the complement of a δ-closed set is called δ-open [16]. The set $\{x|(\exists U \in O(X,x))(int(cl(U)) \subseteq A)\}$ is called the δ-interior of A and is denoted by $int_\delta(A)$.

A subset A is called e^*-open [9] $A \subseteq cl(int(cl_\delta(A)))$. The complement of an e^*-open set is called e^*-closed [9]. The intersection of all e^*-closed sets of X containing A is called the e^*-closure [9] of A and is denoted by $e^*\text{-}cl(A)$. The union of all e^*-open sets of X containing in A is called the e^*-interior [9] of A and is denoted by $e^*\text{-}int(A)$. A subset A is said to be e^*-regular [10] set if it is e^*-open and e^*-closed.

A point x of X is called an e^*-θ-cluster point of A if $e^*\text{-}cl(U) \cap A \neq \emptyset$ for every e^*-open set U containing x. The set of all e^*-θ-cluster points of A is called the e^*-θ-closure [10] of A and is denoted by $e^*\text{-}cl_\theta(A)$. A subset A is said to be e^*-θ-closed if $A = e^*\text{-}cl_\theta(A)$. The complement of an e^*-θ-closed set is called an e^*-θ-open [10] set. A point x of X said to be an e^*-θ-interior point [10] of a subset A, denoted by $e^*\text{-}int_\theta(A)$, if there exists an e^*-open set U of X containing x such that $e^*\text{-}cl(U) \subseteq A$. Also it is noted in [10] that

$$e^*\text{-regular} \Rightarrow e^*\text{-}\theta\text{-open} \Rightarrow e^*\text{-open}.$$

The family of all open (resp. closed, e^*-θ-open, e^*-θ-closed, e^*-open, e^*-closed, e^*-regular) subsets of X is denoted by $O(X)$ (resp. $C(X)$, $e^*\theta O(X)$, $e^*\theta C(X)$, $e^*O(X)$, $e^*C(X)$, $e^*R(X)$). The family of all open (resp. closed, e^*-θ-open, e^*-θ-closed, e^*-open, e^*-closed, e^*-regular) sets of X containing a point x of X is denoted by $O(X,x)$ (resp. $C(X,x)$, $e^*\theta O(X,x)$, $e^*\theta C(X,x)$, $e^*O(X,x)$, $e^*C(X,x)$, $e^*R(X,x)$).

We shall use the well-known accepted language almost in the whole of the proofs of the theorems in this article.

Lemma 2.1 *[10, 11] Let X be a topological space and $A,B \subseteq X$. Then the following properties are hold:*

(i) $A \subseteq e^*\text{-}cl(A) \subseteq e^*\text{-}cl_\theta(A)$.

(ii) If $A \in e^*\theta O(X)$, *then* $e^*\text{-}cl_\theta(A) = e^*\text{-}cl(A)$.

(iii) If $A \subseteq B$, *then* $e^*\text{-}cl_\theta(A) \subseteq e^*\text{-}cl_\theta(B)$.

(iv) $e^*\text{-}cl_\theta(A) \in e^*\theta C(X)$ *and* $e^*\text{-}cl_\theta(e^*\text{-}cl_\theta(A)) = e^*\text{-}cl_\theta(A)$.

(v) If $A_\alpha \in e^*\theta O(X)$ for each $\alpha \in \Lambda$, then $\bigcup\{A_\alpha | \alpha \in \Lambda\} \in e^*\theta O(X)$.

(vi) $e^*\text{-}cl_\theta(A) = \bigcap\{F | (A \subseteq F)(F \in e^*\theta C(X))\}$.

(vii) $e^*\text{-}cl_\theta(X \backslash A) = X \backslash e^*\text{-}int_\theta(A)$.

(viii) A is e^*-θ-open in X iff for each $x \in A$, there exists $U \in eR(X,x)$ such that $U \subseteq A$.

Definition 2.2 *A function $f : X \to Y$ is called e^*-irresolute [8] if $f^{-1}[A]$ is e^*-θ-open in X for every e^*-θ-open set A of Y.*

3. Quasi e^*-θ-closed Sets

Definition 3.1 *A subset A of a space X is called quasi e^*-θ-closed [2] (briefly, $qe^*\theta$-closed) if $e^*\text{-}cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is e^*-θ-open in X. A subset A of a space X is said to be quasi e^*-θ-open (briefly, $qe^*\theta$-open) if $X \backslash A$ is $qe^*\theta$-closed. The family of all $qe^*\theta$-closed (resp. $qe^*\theta$-open) subsets of X is denoted by $qe^*\theta C(X)$ (resp. $qe^*\theta O(X)$).*

Theorem 3.2 *Every e^*-θ-closed set is $qe^*\theta$-closed.*

Proof Let $A \in e^*\theta C(X)$, $U \in e^*\theta O(X)$ and $A \subseteq U$.

$$\left.\begin{array}{r} A \in e^*\theta C(X) \\ (U \in e^*\theta O(X))(A \subseteq U) \end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(A) = A \subseteq U.$$

□

Remark 3.3 *This implication is not reversible as shown in the following example.*

Example 3.4 *Let $X = \{1,2,3\}$, define a topology $\tau = \{\varnothing, X, \{1\}, \{2\}, \{1,2\}\}$ on X. It is not difficult to see $e^*\theta C(X) = 2^X \backslash \{\{3\}\}$ and the subset $\{1,2\}$ is $qe^*\theta$-closed but it is not e^*-θ-closed (cf. Example 1 in [10]).*

Lemma 3.5 *A subset A of a topological space X is $qe^*\theta$-open if and only if $F \subseteq e^*\text{-}int_\theta(A)$ whenever F is e^*-θ-closed in X and $F \subseteq A$.*

Proof *Necessity.* Let $F \subseteq A$, $F \in e^*\theta C(X)$ and $A \in qe^*\theta O(X)$.

$$\left.\begin{array}{r} A \supseteq F \in e^*\theta C(X) \Rightarrow \backslash A \subseteq \backslash F \in e^*\theta O(X) \\ A \in qe^*\theta O(X) \Rightarrow \backslash A \in qe^*\theta C(X) \end{array}\right\}$$

$$\Rightarrow \backslash e^*\text{-}int_\theta(A) = e^*\text{-}cl_\theta(\backslash A) \subseteq \backslash F$$

$$\Rightarrow F \subseteq e^*\text{-}int_\theta(A).$$

Sufficiency. Let $\backslash F \in e^*\theta O(X)$ and $\backslash A \subseteq \backslash F$.

$$\left.\begin{array}{r}(\backslash F \in e^*\theta O(X))(\backslash A \subseteq \backslash F) \Rightarrow (F \in e^*\theta C(X))(F \subseteq A) \\ \text{Hypothesis}\end{array}\right\}$$

$$\Rightarrow F \subseteq e^*\text{-}int_\theta(A)$$

$$\Rightarrow e^*\text{-}cl_\theta(\backslash A) = \backslash e^*\text{-}int_\theta(A) \subseteq \backslash F$$

Then, $\backslash A \in qe^*\theta C(X)$ and hence $A \in qe^*\theta O(X)$. □

Definition 3.6 *A function $f : X \to Y$ is said to be approximately $e^*\theta$-open (briefly, ap-$e^*\theta$-open) if $e^*\text{-}cl_\theta(B) \subseteq f[A]$ whenever $A \in e^*\theta O(X)$, $B \in qe^*\theta C(Y)$ and $B \subseteq f[A]$.*

Definition 3.7 *A function $f : X \to Y$ is said to be:*

(1) *$e^*\theta$-closed [3] (resp. pre $e^*\theta$-closed [5]), if the image of each closed (resp. e^*-θ-closed) set F of X is e^*-θ-closed in Y.*

(2) *$e^*\theta$-open [3] (resp. pre $e^*\theta$-open [5]), if the image of each open (resp. e^*-θ-open) set U of X is e^*-θ-open in Y.*

Theorem 3.8 *Let $f : X \to Y$ be a function. If $f[A]$ is e^*-θ-closed in Y for every $A \in e^*\theta O(X)$, then f is ap-$e^*\theta$-open.*

Proof Let $B \subseteq f[A]$, where $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Y)$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Y))(B \subseteq f[A]) \\ \text{Hypothesis}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(B) \subseteq e^*\text{-}cl_\theta(f[A]) = f[A]$$

$$\Rightarrow f[A] \in e^*\theta C(Y).$$

□

Theorem 3.9 *Every pre $e^*\theta$-open function is ap-$e^*\theta$-open.*

Proof Let $B \subseteq f[A]$, where $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Y)$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Y))(B \subseteq f[A]) \\ f \text{ is pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow (f[A] \in e^*\theta O(Y))(B \in qe^*\theta C(Y))(B \subseteq f[A])$$

$$\Rightarrow e^*\text{-}cl_\theta(B) \subseteq f[A].$$ □

Remark 3.10 *This implication is not reversible as shown in the following example.*

Example 3.11 *Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f = \{(a, a), (b, b), (c, b)\}$. It isn't difficult to see $e^*\theta O(X) = 2^X \backslash \{\{a, b\}\}$, $qe^*\theta C(X) = 2^X$*

and hence f *is ap-*$e^*\theta$*-open. However* $\{a,c\}$ *is* e^**-*θ*-open in* X*, but* $f[\{a,c\}] = \{a,b\}$ *is not* e^**-*θ*-open in* X*. Therefore,* f *is not pre* $e^*\theta$*-open.*

Theorem 3.12 *Let* $f : X \to Y$ *be a function. If the* e^**-*θ*-open and* e^**-*θ*-closed sets of* Y *coincide, then* f *is ap-*$e^*\theta$*-open if and only if* $f[W] \in e^*\theta C(Y)$ *for every* e^**-*θ*-open subset* W *of* X*.*

Proof *Necessity.* Let A be an arbitrary subset of Y such that $A \subseteq U$, where $U \in e^*\theta O(Y)$ and let $W \in e^*\theta O(X)$.

$$\left.\begin{array}{r}(A \subseteq U)(U \in e^*\theta O(Y)) \\ e^*\theta O(Y) = e^*\theta C(Y)\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(A) \subseteq e^*\text{-}cl_\theta(U) = U$$

Therefore all subset of Y are $qe^*\theta$-closed and hence all are $qe^*\theta$-open.

$$\left.\begin{array}{r}(W \in e^*\theta O(X))(Y \supseteq f[W] \in qe^*\theta O(Y))(f[W] \subseteq f[W]) \\ f \text{ is ap-}e^*\theta\text{-open}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(f[W]) \subseteq f[W]$$

$$\Rightarrow f[W] \in e^*\theta C(Y).$$

Sufficiency. It is obvious from Theorem 3.8. □

Corollary 3.13 *Let* $f : X \to Y$ *be a function. If the* e^**-*θ*-open and* e^**-*θ*-closed sets of* Y *coincide, then* f *is ap-*$e^*\theta$*-open if and only if* f *is pre* $e^*\theta$*-open.*

Definition 3.14 *A function* $f : X \to Y$ *is said to be contra pre* $e^*\theta$*-open (resp. contra pre* $e^*\theta$*-closed) if the image of each* e^**-*θ*-open (resp.* e^**-*θ*-closed) set* U *of* X *is* e^**-*θ*-closed (resp.* e^**-*θ*-open) in* Y*.*

Theorem 3.15 *Every contra pre* $e^*\theta$*-open function is ap-*$e^*\theta$*-open.*

Proof Let $B \subseteq f[A]$, where $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Y)$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Y))(B \subseteq f[A]) \\ f \text{ is contra pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(B) \subseteq e^*\text{-}cl_\theta(f[A]) = f[A].$$

□

Remark 3.16 *This implication is not reversible as shown in the following example.*

Example 3.17 *Consider the same topology in Example 3.11. Define the identity function* $f : (X,\tau) \to (X,\tau)$*. Then,* f *is ap-*$e^*\theta$*-open. However* $\{c\}$ *is* e^**-*θ*-open in* X*, but* $f[\{c\}] = \{c\}$ *is not* e^**-*θ*-closed in* X*. Therefore,* f *is not contra pre* $e^*\theta$*-open.*

Remark 3.18 *The following examples show that contra pre $e^*\theta$-openness and pre $e^*\theta$-openness are independent notions.*

Example 3.19 *Define the same function on the topology in Example 3.11. Since the image of every e^*-θ-open set of X is e^*-θ-closed in X, then f is contra pre $e^*\theta$-open. However, f is not pre $e^*\theta$-open.*

Example 3.20 *Consider the same topology in Example 3.11. Define the identity function $f:(X,\tau)\to(X,\tau)$. Since the image of every e^*-θ-open set of X is e^*-θ-open in X, then f is pre $e^*\theta$-open. However $\{c\}$ is e^*-θ-open in X, but $f[\{c\}]=\{c\}$ is not e^*-θ-closed in X. Therefore, f is not contra pre $e^*\theta$-open.*

Remark 3.21 *From Definitions 3.6, 3.7, 3.14, we have the relation among ap-$e^*\theta$-open functions, contra pre $e^*\theta$-open functions and other well-known functions in topological spaces. The converses of the below implications are not true in general, as shown in the previous examples.*

$$\begin{array}{ccccc}
 & & \textit{pre } e^*\theta\textit{-open function} & & \\
 & \not\nearrow \; \not\swarrow & & \searrow \; \not\nwarrow & \\
e^*\theta\textit{-open function} & & \not\downarrow \not\uparrow & & \textit{ap-}e^*\theta\textit{-open function} \\
 & \not\searrow \; \not\nwarrow & & \nearrow \; \not\swarrow & \\
 & & \textit{contra pre } e^*\theta\textit{-open function} & &
\end{array}$$

Theorem 3.22 *If $f:X\to Y$ is e^*-irresolute and ap-$e^*\theta$-open surjection, then $f^{-1}[B]$ is $qe^*\theta$-open in X whenever B is $qe^*\theta$-open subset of Y.*

Proof Let $B\in qe^*\theta O(Y)$. Suppose that $A\subseteq f^{-1}[B]$, where $A\in e^*\theta C(X)$.

$$\left.\begin{array}{r}
(A\in e^*\theta C(X)\Rightarrow \setminus A\in e^*\theta O(X))(B\in qe^*\theta O(Y)\Rightarrow \setminus B\in qe^*\theta C(Y)) \\
A\subseteq f^{-1}[B]\Rightarrow f^{-1}[\setminus B]\subseteq \setminus A\Rightarrow f[f^{-1}[\setminus B]]\overset{f \text{ is surj.}}{=}\setminus B\subseteq f[\setminus A] \\
f \text{ is ap-}e^*\theta\text{-open}
\end{array}\right\}$$

$$\Rightarrow \setminus e^*\text{-}int_\theta(B)=e^*\text{-}cl_\theta(\setminus B)\subseteq f[\setminus A]$$

$$\left.\begin{array}{r}
\Rightarrow \setminus f^{-1}[e^*\text{-}int_\theta(B)]\subseteq \setminus A\Rightarrow A\subseteq f^{-1}[e^*\text{-}int_\theta(B)] \\
f \text{ is } e^*\text{-irresolute}
\end{array}\right\}\Rightarrow f^{-1}[e^*\text{-}int_\theta(B)]\in e^*\theta O(X)$$

$$\Rightarrow A\subseteq f^{-1}[e^*\text{-}int_\theta(B)]=e^*\text{-}int_\theta(f^{-1}[e^*\text{-}int_\theta(B)])\subseteq e^*\text{-}int_\theta(f^{-1}[B]).$$

This implies that by Lemma 3.5, $f^{-1}[B]$ is $qe^*\theta$-open in X. □

Definition 3.23 *A function $f:X\to Y$ is called quasi $e^*\theta$-irresolute (briefly, $qe^*\theta$-irresolute) if $f^{-1}[A]$ is $qe^*\theta$-closed in X for every $qe^*\theta$-closed set A of Y.*

Theorem 3.24 *Let $f : X \to Y$, $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$. Then:*

(i) $g \circ f$ is ap-$e^*\theta$-open if f is pre $e^*\theta$-open and g is ap-$e^*\theta$-open.

(ii) $g \circ f$ is ap-$e^*\theta$-open if f is ap-$e^*\theta$-open and g is bijective pre $e^*\theta$-closed and $qe^*\theta$-irresolute.

Proof (i): Let $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Z)$, where $B \subseteq (gof)[A]$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Z))(B \subseteq (gof)[A] = g[f[A]]) \\ f \text{ is pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow \left.\begin{array}{r} f[A] \in e^*\theta O(Y) \\ g \text{ is ap-}e^*\theta\text{-open}\end{array}\right\}$$

$$\Rightarrow e^*\text{-}cl_\theta(B) \subseteq g[f[A]] = (gof)[A].$$

This implies that $g \circ f$ is ap-$e^*\theta$-open.

(ii): Let $A \in e^*\theta O(X)$ and $B \in qe^*\theta C(Z)$, where $B \subseteq (gof)[A]$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Z))(B \subseteq (gof)[A] = g[f[A]]) \\ g \text{ is } qe^*\theta\text{-irresolute}\end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{r}(A \in e^*\theta O(X))(g^{-1}[B] \in qe^*\theta C(Y))(g^{-1}[B] \subseteq g^{-1}[g[f[A]]] \overset{\text{g is bijective}}{=} f[A]) \\ f \text{ is ap-}e^*\theta\text{-open}\end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{r} e^*\text{-}cl_\theta(g^{-1}[B]) \subseteq f[A] \\ g \text{ is pre } e^*\theta\text{-closed}\end{array}\right\}$$

$$\Rightarrow e^*\text{-}cl_\theta(B) \subseteq e^*\text{-}cl_\theta(g[g^{-1}[B]]) \subseteq g[e^*\text{-}cl_\theta(g^{-1}[B])] \subseteq g[f[A]] = (gof)[A].$$

This implies that $g \circ f$ is ap-$e^*\theta$-open. □

Theorem 3.25 *Let $f : X \to Y$, $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$. Then:*

(i) $g \circ f$ is contra pre $e^*\theta$-open if f is pre $e^*\theta$-open and g is contra pre $e^*\theta$-open.

(ii) $g \circ f$ is contra pre $e^*\theta$-open if f is contra pre $e^*\theta$-open and g is pre $e^*\theta$-closed.

Proof (i): Let $U \in e^*\theta O(X)$.

$$\left.\begin{array}{r} U \in e^*\theta O(X) \\ f \text{ is pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow \left.\begin{array}{r} f[U] \in e^*\theta O(Y) \\ g \text{ is contra pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow g[f[U]] = (gof)[U] \in e^*\theta C(Z).$$

This implies that $g \circ f$ is contra pre $e^*\theta$-open.

(ii): Let $U \in e^*\theta O(X)$.

$$\left.\begin{array}{r} U \in e^*\theta O(X) \\ f \text{ is contra pre } e^*\theta\text{-open}\end{array}\right\} \Rightarrow \left.\begin{array}{r} f[U] \in e^*\theta C(Y) \\ g \text{ is pre } e^*\theta\text{-closed}\end{array}\right\} \Rightarrow g[f[U]] = (gof)[U] \in e^*\theta C(Z).$$

This implies that $g \circ f$ is contra pre $e^*\theta$-open. □

Theorem 3.26 *Let $f : X \to Y$, $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is contra pre $e^*\theta$-open. Then:*

(i) If f is an e^*-irresolute surjection, then g is contra pre $e^*\theta$-open.

(ii) If g is an e^*-irresolute injection, then f is contra pre $e^*\theta$-open.

Proof (i): Let $U \in e^*\theta O(Y)$.

$$\left.\begin{array}{r} U \in e^*\theta O(Y) \\ f \text{ is } e^*\text{-irresolute} \end{array}\right\} \Rightarrow \left.\begin{array}{r} f^{-1}[U] \in e^*\theta O(X) \\ g \circ f \text{ is contra pre } e^*\theta\text{-open} \end{array}\right\}$$

$$\Rightarrow (gof)[f^{-1}[U]] = g[f[f^{-1}[U]]] \overset{f \text{ is surj.}}{=} g[U] \in e^*\theta C(Z).$$

This implies that g is contra pre $e^*\theta$-open.

(ii): Let $U \in e^*\theta O(X)$.

$$\left.\begin{array}{r} U \in e^*\theta O(X) \\ g \circ f \text{ is contra pre } e^*\theta\text{-open} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (gof)[U] = g[f[U]] \in e^*\theta C(Z) \\ g \text{ is } e^*\text{-irresolute} \end{array}\right\}$$

$$\Rightarrow g^{-1}[g[f[U]]] \overset{g \text{ is inj.}}{=} f[U] \in e^*\theta C(Y).$$

This implies that f is contra pre $e^*\theta$-open. □

Definition 3.27 *Let X and Y be two topological spaces. A function $f : X \to Y$ has an $e^*\theta$-closed graph if its $G(f) = \{(x, f(x)) | x \in X\}$ is e^*-θ-closed in the product space $X \times Y$.*

Definition 3.28 *The product space $X = X_1 \times \ldots \times X_n$ has property $P_{e^*\theta}$ [5] if A_i is an e^*-θ-open set in a topological spaces X_i for $i = 1, 2, \ldots, n$, then $A_1 \times \ldots \times A_n$ is also e^*-θ-open in the product space $X = X_1 \times \ldots \times X_n$.*

Theorem 3.29 *If $f : X \to Y$ is a contra pre $e^*\theta$-open function with $e^*\theta$-closed fibers which has the property $P_{e^*\theta}$, then f has an $e^*\theta$-closed graph.*

Proof Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r} (x,y) \notin G(f) \Rightarrow (x,y) \in X \times Y \setminus G(f) \Rightarrow x \in \setminus f^{-1}[\{y\}] \\ f^{-1}[\{y\}] \text{ is } e^*\theta\text{-closed} \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{r} (\exists E \in e^*\theta O(X,x))(E \subseteq \setminus f^{-1}[\{y\}]) \\ f \text{ is contra pre } e^*\theta\text{-open} \end{array}\right\} \Rightarrow A := \setminus f[E] \in e^*\theta O(Y,y)$$

$$\Rightarrow \left.\begin{array}{r} (x,y) \in E \times A \subseteq X \times Y \setminus G(f) \\ X \times Y \text{ has the property } P_{e^*\theta} \end{array}\right\} \Rightarrow E \times A \in e^*\theta O(X \times Y)$$

$$\Rightarrow X \times Y \backslash G(f) \in e^{*}\theta O(X \times Y)$$

$$\Rightarrow G(f) \in e^{*}\theta C(X \times Y).$$

□

4. Characterizations of $e^{*}\theta$-$T_{\frac{1}{2}}$ Spaces

Definition 4.1 *A topological space X is said to be $e^{*}\theta$-$T_{\frac{1}{2}}$ [1] if every $qe^{*}\theta$-closed set is $e^{*}\theta$-closed.*

Lemma 4.2 *Let X be a topological space and $A \subseteq X$. If $A \in qe^{*}\theta C(X)$, then $F \nsubseteq e^{*}\text{-}cl_{\theta}(A) \backslash A$ where $\varnothing \neq F \in e^{*}\theta C(X)$.*

Proof Let $A \in qe^{*}\theta C(X)$. Suppose that $F \subseteq e^{*}\text{-}cl_{\theta}(A) \backslash A$, where $\varnothing \neq F \in e^{*}\theta C(X)$.

$$\left.\begin{array}{r}(\varnothing \neq F \in e^{*}\theta C(X))(F \subseteq e^{*}\text{-}cl_{\theta}(A) \backslash A) \Rightarrow (\backslash F \in e^{*}\theta O(X))(A \subseteq \backslash F) \\ A \in qe^{*}\theta C(X)\end{array}\right\}$$

$$\left.\begin{array}{r}\Rightarrow e^{*}\text{-}cl_{\theta}(A) \subseteq \backslash F \Rightarrow F \subseteq \backslash e^{*}\text{-}cl_{\theta}(A) \\ F \subseteq e^{*}\text{-}cl_{\theta}(A) \backslash A \Rightarrow F \subseteq e^{*}\text{-}cl_{\theta}(A)\end{array}\right\} \Rightarrow F \subseteq (\backslash e^{*}\text{-}cl_{\theta}(A)) \cap e^{*}\text{-}cl_{\theta}(A) \Rightarrow F = \varnothing.$$

This is a contradiction and hence $e^{*}\text{-}cl_{\theta}(A) \backslash A$ does not contain any non-empty e^{*}-θ-closed set.

□

Theorem 4.3 *For a topological space X, the following statements are equivalent:*

(i) X is $e^{*}\theta$-$T_{\frac{1}{2}}$,

(ii) For each $x \in X$, $\{x\}$ is e^{*}-θ-closed or e^{*}-θ-open.

Proof $(i) \Rightarrow (ii)$: Suppose that for any $x \in X$, $\{x\} \notin e^{*}\theta C(X)$.

$$\left.\begin{array}{r}\{x\} \notin e^{*}\theta C(X) \Rightarrow X \backslash \{x\} \in e^{*}\theta O(X) \\ X \backslash \{x\} \subseteq X \in e^{*}\theta O(X)\end{array}\right\} \Rightarrow e^{*}\text{-}cl_{\theta}(X \backslash \{x\}) \subseteq X$$

$$\left.\begin{array}{r}\Rightarrow X \backslash \{x\} \in qe^{*}\theta C(X) \\ X \text{ is } e^{*}\theta\text{-}T_{\frac{1}{2}}\end{array}\right\} \Rightarrow X \backslash \{x\} \in e^{*}\theta C(X).$$

Thus $X \backslash \{x\} \in e^{*}\theta C(X)$ or equivalently $\{x\} \in e^{*}\theta O(X)$.

$(ii) \Rightarrow (i)$: Let $A \in qe^{*}\theta C(X)$ and $x \in e^{*}\text{-}cl_{\theta}(A)$.

Case I. If $\{x\} \in e^{*}\theta C(X)$:

$$A \in qe^{*}\theta C(X) \overset{\text{Lemma 4.2}}{\Rightarrow} (\{x\} \in e^{*}\theta C(X))(\{x\} \nsubseteq e^{*}\text{-}cl_{\theta}(A) \backslash A) \Rightarrow x \in A.$$

Case II. If $\{x\} \in e^{*}\theta O(X)$:

$$\{x\} \in e^*\text{-}cl_\theta(A) \Rightarrow (\{x\} \in e^*\theta O(X,x))(\{x\} \cap A \neq \varnothing) \Rightarrow x \in A.$$

As can be seen, in both cases $x \in A$. Thus $e^*\text{-}cl_\theta(A) \subseteq A$. Since there is always $A \subseteq e^*\text{-}cl_\theta(A)$, A is $e^*\text{-}\theta$-closed. □

Theorem 4.4 *For a topological space Y, the following statements are equivalent:*

(i) *Y is $e^*\theta\text{-}T_{\frac{1}{2}}$,*

(ii) *For every space X, every map $f: X \to Y$ is ap-$e^*\theta$-open.*

Proof $(i) \Rightarrow (ii)$: Let $B \in qe^*\theta C(Y)$ and let $B \subseteq f[A]$, where $A \in e^*\theta O(X)$.

$$\left.\begin{array}{r}(A \in e^*\theta O(X))(B \in qe^*\theta C(Y))(B \subseteq f[A]) \\ Y \text{ is } e^*\theta\text{-}T_{\frac{1}{2}}\end{array}\right\} \Rightarrow (A \in e^*\theta O(X))(B \in e^*\theta C(Y))(B \subseteq f[A])$$

$$\Rightarrow e^*\text{-}cl_\theta(B) = B \subseteq f[A]$$

Then, f is ap-$e^*\theta$-open.

$(ii) \Rightarrow (i)$: Let $B \in qe^*\theta C(Y)$. Suppose that $B \subseteq f[B]$, where $B \in e^*\theta O(X)$.

$$\left.\begin{array}{r}(B \in qe^*\theta C(Y))(B \in e^*\theta O(X))(B \subseteq f[B]) \\ f \text{ is ap-}e^*\theta\text{-open}\end{array}\right\} \Rightarrow e^*\text{-}cl_\theta(B) \subseteq f[B] = B \Rightarrow B \in e^*\theta C(Y).$$

Then, Y is $e^*\theta\text{-}T_{\frac{1}{2}}$. □

Theorem 4.5 *[2] For a topological space X, the following statements are equivalent:*

(i) *X is $e^*\theta\text{-}T_{\frac{1}{2}}$,*

(ii) *X is $e^*\theta\text{-}T_1$.*

5. Conclusion

Various forms of closed sets have been worked on by many topologist in recent years. This paper is concerned with the concept of quasi $e^*\text{-}\theta$-closed sets and which are defined by utilizing the notion of $e^*\text{-}\theta$-open set. Also, we defined approximately $e^*\theta$-open functions via quasi $e^*\text{-}\theta$-closed sets and $e^*\theta$-open sets. We demonstrated that newly defined these functions are weaker than $e^*\theta$-open functions, pre $e^*\theta$-open functions and contra pre $e^*\theta$-open functions (cf. Remark 3.21). We believe that this study will help researchers to support further studies on continuous functions.

Declaration of Ethical Standards

The author declares that the materials and methods used in her study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1357160
(**2024**) **5**(2) 83 – 93
Research Article

Statistical Convergence in A-Metric Spaces

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Received: 08 September 2023 **Accepted:** 03 April 2024

Abstract: In this paper, we will present the notion of statistical convergence in A-metric spaces, which is an important concept in summability theory. We will define statistical convergence in A-metric spaces and investigate its basic properties. Furthermore, we will explore the relationship between strongly p-Cesàro and statistical convergence in A-metric spaces.

Keywords: Statistical convergence, Cauchy sequence, A-metric spaces.

1. Introduction

Metric spaces are an important topic in mathematics, particularly in analysis and topology. Fréchet [11] was the first to introduce the notion of metric space in 1906. Since then, many researchers have been interested in the generalization of metric spaces and have published various papers on this topic [7, 14, 17–19, 25]. One of the outcomes of these studies was the concept of A-metric space, which was proposed by Abbas et al. [2] in 2015 as a generalization of S-metrics which is a generalization of metric spaces. Some fixed point theorems in this space have been studied.

The concept of statistical convergence was introduced in 1951 by Fast in [9] and Steinhaus in [26]. Afterwards, Shoenberg [24] introduced it in 1959. Since then, the properties of statistical convergence have been studied by different mathematicians and applied in several area (see, [4–6, 8, 10, 12, 13, 15, 22, 23, 27]).

Let's recall the definitions of natural density, statistical convergence and p-strongly Cesàro summability (see the references given above for details).

For a set K of positive integers, the asymptotic (or natural) density is defined as follows;

$$\delta(K) = \lim_k \frac{1}{k} |\{t \leq k : t \in K\}|$$

, where $|\{t \leq k : t \in K\}|$ denotes the number of elements of K not exceeding k.

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2020 *AMS Mathematics Subject Classification*: 40A05, 40A35, 40A99

A sequence (x_t) is said to be statistically convergent to x, if for every $\varepsilon > 0$,

$$\lim_k \frac{1}{k} |\{t \leq k : |x_t - x| \geq \varepsilon\}| = 0.$$

A sequence (x_t) is said to be statistically Cauchy, if for every $\varepsilon > 0$, there exists a positive integer $S = S(\varepsilon)$ such that

$$\lim_k \frac{1}{k} |\{t \leq k : |x_t - x_S| \geq \varepsilon\}| = 0.$$

A sequence (x_t) is said to be p-strongly Cesàro summable to x, if

$$\lim_{k\to\infty} \frac{1}{k} \sum_{t=1}^{k} | x_t - x |^p = 0.$$

Recently, Bilalov and Nazarova [3] investigated statistical convergence in metric spaces. In [16], Kedian et al. defined the concept of statistical convergence in cone metric spaces. Then Nuray [21] examined statistical convergence in 2-metric spaces, and [20] investigated partial metric spaces. Recently, Abazari [1] introduced the definition of statistically convergent sequence and studied its basic properties in g-metric spaces.

In this paper, we examine the concept of a statistically convergent sequence and investigate some of its fundamental properties in A-metric spaces. Then, we discuss statistically compact spaces and characterize the statistical completeness of A-metric spaces. Furthermore, we explore the relationship between strong p-Cesàro convergence and statistical convergence in A-metric spaces.

Now let us give the basic definitions and notations that we will use in our study.

Definition 1.1 *[2] Let X be a nonempty set. A function $A : X^n \to [0, \infty)$ is called an A-metric on X if for any $x_i, a \in X, i = 1, 2, \ldots, n$ the following conditions hold;*

(A1) $A(x_1, x_2, \ldots, x_{n-1}, x_n) \geq 0$,

(A2) $A(x_1, x_2, \ldots, x_{n-1}, x_n) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_n$,

(A3) $A(x_1, x_2, \ldots, x_{n-1}, x_n) \leq \sum_{k=1}^{n} A(\underbrace{x_k, x_k, \ldots, x_k}_{n-1}, a)$.

Also the pair (X, A) is called an A-metric space.

Example 1.2 *[2] Let $X = \mathbb{R}$. A function $A : X^n \to [0, \infty)$ by*

$$A(x_1, x_2, \ldots, x_{n-1}, x_n) = \sum_{k=1}^{n} \sum_{k<j} |x_k - x_j|$$

is an A-metric space on X.

Lemma 1.3 *[2] Let* (X, A) *be an* A*-metric space. Then* $A(x, x, \ldots, x, y) = A(y, y, \ldots, y, x)$ *for all* $x, y \in X$.

Lemma 1.4 *[2] Let* (X, A) *be an* A*-metric space. For all* $x, y \in X$, *we get*

$$A(x, x, \ldots, x, z) \leq (n-1)A(x, x, \ldots, x, y) + A(y, y, \ldots, y, z)$$

and

$$A(x, x, \ldots, x, z) \leq (n-1)A(x, x, \ldots, x, y) + A(z, z, \ldots, z, y).$$

Definition 1.5 *[2] The* A*-metric space* (X, A) *is called bounded if there exists an* $r > 0$ *such that* $A(y, y, \ldots, y, x) \leq r$ *for all* $x, y \in X$. *Otherwise,* X *is unbounded.*

Definition 1.6 *[2] Let* (X, A) *be an* A*-metric space and* (x_t) *be a sequence in this space:*

(1) *The sequence* (x_t) *is said to be convergent to* x, *if for every* $\varepsilon > 0$, *there exists a positive integer* t_0 *such that* $A(x_t, x_t, \ldots, x_t, x) < \varepsilon$ *for every* $t \geq t_0$.

(2) *The sequence,* (x_t) *is said to be a Cauchy sequence if for every* $\varepsilon > 0$, *there exists a positive integer* t_0 *such that* $A(x_t, x_t, \ldots, x_t, x_m) < \varepsilon$ *for all* $t, m \geq t_0$.

2. Main Results

This section introduces the definition of statistical convergence of sequences in A-metric space and studies some basic properties. It also explores the relationship between strongly p-Cesàro and statistical convergence in A-metric spaces.

Definition 2.1 *Let* (X, A) *be an* A*-metric space. We say that a sequence* (x_t) *in* X *is statistically convergent to* $x \in X$, *if for every* $\varepsilon > 0$,

$$\lim_{k \to \infty} \frac{1}{k} |\{t \leq k : A(x_t, x_t, \ldots, x_t, x) \geq \varepsilon\}| = 0$$

or equivalently

$$\lim_{k \to \infty} \frac{1}{k} |\{t \leq k : A(x_t, x_t, \ldots, x_t, x) < \varepsilon\}| = 1$$

and is denoted by $x_t \xrightarrow{AS} x$. *This means that* $A(x_t, x_t, \ldots, x_t, x) < \varepsilon$ *holds for almost all* t. *In this case, we write* $A_{st} - \lim x_t = x$ *in the sense of statistical convergence.*

Definition 2.2 *Let* (X, A) *be an* A*-metric space. We say that a sequence* (x_t) *in* X *statistically Cauchy sequence, if for every* $\varepsilon > 0$, *there exists a positive integer* $l = l(\varepsilon)$

$$\lim_{k \to \infty} \frac{1}{k} |\{t, l \leq k : A(x_t, x_t, \ldots, x_t, x_l) \geq \varepsilon\}| = 0$$

or equivalently

$$\lim_{k\to\infty}\frac{1}{k}|\{t,l\le k : A(x_t,x_t,\ldots,x_t,x_l)<\varepsilon\}|=1.$$

Theorem 2.3 *Any convergent sequence in an $A-$metric space (X,A) is also statistically convergent.*

Proof Let (x_t) be a sequence in A-metric space (X,A) such that converges to $x\in X$. For every $\varepsilon>0$, there exists a positive integer k_0 such that for all $t\ge k_0$,

$$A(x_t,x_t,\ldots,x_t,x)<\varepsilon.$$

The set

$$\mathcal{A}(k)=\{t\le k : A(x_t,x_t,\ldots,x_t,x)<\varepsilon\},$$

then we can write

$$|\mathcal{A}(k)|=|\{t\le k : A(x_t,x_t,\ldots,x_t,x)<\varepsilon\}|\ge k-k_0.$$

Therefore

$$\lim_{k\to\infty}\frac{|\mathcal{A}(k)|}{k}=1.$$

Hence (x_t) is statistically convergent. □

The following example shows that the converse of above theorem is not valid.

Example 2.4 *Let $X=\mathbb{R}$ and A be the metric as follows;*

$$\begin{aligned} A:\mathbb{R}^n &\to \mathbb{R}^+ \\ A(x_1,x_2,\ldots,x_n) = \quad & max\{|x_1-x_2|,|x_1-x_3|,|x_1-x_4|,\ldots,|x_1-x_n| \\ & |x_2-x_3|,|x_2-x_4|,\ldots,|x_2-x_n|, \\ & \cdots \\ & |x_{n-2}-x_{n-1}|,|x_{n-2}-x_n|, \\ & |x_{n-1}-x_n|\}. \end{aligned}$$

Consider the following sequence

$$x_t:=\begin{cases} t, & \text{if } t \text{ is square} \\ 0, & o.w. \end{cases} \quad .$$

It is clear that (x_t) is statistically convergent while it is not convergent normally.

Theorem 2.5 *Let (X, A) be an A-metric space and (x_t) be a sequence in this space. If $x_t \xrightarrow{AS} x$ and $x_t \xrightarrow{AS} y$, then $x = y$.*

Proof For any $\varepsilon > 0$, we define the following two sets

$$\mathcal{A}(\varepsilon) := \{t \in \mathbb{N} : A(x_t, x_t, \ldots, x_t, x), \geq \frac{\varepsilon}{n}\}$$

$$\mathcal{B}(\varepsilon) := \{t \in \mathbb{N} : A(x_t, x_t, \ldots, x_t, y). \geq \frac{\varepsilon}{n}\}$$

Since $x_t \xrightarrow{AS} x$ and $x_t \xrightarrow{AS} y$, then $\delta_1(\mathcal{A}(\varepsilon)) = 0$ and $\delta_1(\mathcal{B}(\varepsilon)) = 0$. Let $\mathcal{C}(\varepsilon) := \mathcal{A}(\varepsilon) \cup \mathcal{B}(\varepsilon)$, then $\delta_1(\mathcal{C}(\varepsilon)) = 0$. So $\delta_1(\mathcal{C}^c(\varepsilon)) = 1$. Suppose that $t \in \mathcal{C}^c(\varepsilon)$, then by Lemma 1.3 and Lemma 1.4, we can write

$$\begin{aligned} A(x, x, \ldots, x, y) &\leq (n-1)A(x, x, \ldots, x, x_t) + A(x_t, x_t, \ldots, x_t, y) \\ &= (n-1)A(x_t, x_t, \ldots, x_t, x) + A(x_t, x_t, \ldots, x_t, y) \\ &< (n-1)\frac{\varepsilon}{n} + \frac{\varepsilon}{n} \\ &= \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrarty, we have $A(x, x, \ldots, x, y) = 0$. Therefore $x = y$.

□

Theorem 2.6 *Let (x_t) and (y_t) be two sequences in an A-metric space (X, A). If $y_t \xrightarrow{AS} y$ and $A(x_t, x_t, \ldots, x_t, y) \leq A(y_t, y_t, \ldots, y_t, y)$ for each $t \in \mathbb{N}$, then $x_t \xrightarrow{AS} y$.*

Proof Let $y_t \xrightarrow{AS} y$. For each $\varepsilon > 0$, we can write

$$\{t \leq k : A(x_t, x_t, \ldots, x_t, y) < \varepsilon\} \supseteq \{t \leq k : A(y_t, y_t, \ldots, y_t, y) < \varepsilon\}$$

and

$$\delta(\{t \leq k : A(x_t, x_t, \ldots, x_t, y) < \varepsilon\}) \geq \delta(\{t \leq k : A(y_t, y_t, \ldots, y_t, y) < \varepsilon\}) = 1.$$

Therefore $x_t \xrightarrow{AS} y$. □

Definition 2.7 *([4]) A set $\mathcal{A} = \{t_m : m \in \mathbb{N}\}$ is said to be statistically dense in $\mathbb{N}$, if the set $\mathcal{A}(k) = \{t_j \in A : t_j \leq k\}$ has asymptotic density 1, that is,*

$$\delta(\mathcal{A}) = \lim_{k\to\infty} \frac{|\mathcal{A}(k)|}{k} = 1.$$

Definition 2.8 *Let (X, A) be an A-metric space and (x_t) be a sequence in X. A subsequence (x_{t_m}) of sequence (x_t) is said to be statistically dense in X if the index set $\{t_m : m \in \mathbb{N}\}$ is statistically dense subset of $\mathbb{N}$, namely,*

$$\delta(\{t_m : m \in \mathbb{N}\}) = 1.$$

The following theorem shows the equivalence between some properties of A-metric spaces. This result has been previously studied for cone metric spaces. We will give a proof for the A-metric spaces case using some techniques from [16].

Theorem 2.9 *Let (x_t) be a sequence in an A-metric space (X, A). Then the followings are equivalent:*

(i) *(x_t) is statistically convergent in (X, A),*

(ii) *There is a convergent sequence (y_t) in X such that $x_t = y_t$ for almost all $t \in \mathbb{N}$,*

(iii) *There is a statistically dense subsequence (x_{t_m}) of (x_t) such that (x_{t_m}) is convergent,*

(iv) *There is a statistically dense subsequence (x_{t_m}) of (x_t) such that (x_{t_m}) is statistically convergent.*

Proof $(i) \Rightarrow (ii)$ Let $\varepsilon > 0$ and $x_t \xrightarrow{AS} x \in X$. We can write

$$\lim_{k\to\infty} \frac{1}{k} \mid \{t \le k : A(x_t, x_t, \ldots, x_t, x) < \varepsilon\} \mid = 1.$$

We can choose (t_m) as an increasing sequence in $\mathbb{N}$ such that

$$\frac{1}{k} \mid \{s \le k : A(x_s, x_s, \ldots, x_s, x) < \frac{1}{2^m}\} \mid > 1 - \frac{1}{2^m}$$

for every $k > t_m$. We can assume that $t_m < t_{m+1}$ for each $m \in \mathbb{N}$. Now we define (y_s) as follows;

$$y_s := \begin{cases} x_s, & 1 \le s \le t_1 \\ x_s, & t_m < s \le t_{m+1}, A(x_s, x_s, \ldots, x_s, x) < \dfrac{1}{2^m} \\ x, & \text{o.w.} \end{cases}.$$

Choose $m \in \mathbb{N}$ such that $\frac{1}{2^m} < \varepsilon$. Then $A(y_s, y_s, \ldots, y_s, x) < \varepsilon$ for each $s > t_m$, that is, (y_s) is convergent to x. Fix $k \in \mathbb{N}$, for $t_m < k \le t_{m+1}$, we get

$$\{s \le k : y_s \ne x_s\} \subseteq \{1, 2, \ldots, k\} - \{s \le k : A(x_s, x_s, \ldots, x_s, x) < \frac{1}{2^m}\}.$$

Thus

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{k}\mid\{s\le k: y_s\ne x_s\}\mid &\le \lim_{k\to\infty}\frac{1}{k}k-\lim_{k\to\infty}\frac{1}{k}\mid\{s\le k: A(x_s,x_s,\dots,x_s,x)<\frac{1}{2^m}\}\mid\\ &<1-\left(1-\frac{1}{2^m}\right)\\ &=\frac{1}{2^m}\\ &<\varepsilon.\end{aligned}$$

Hence $\lim_{k\to\infty}\frac{1}{k}\mid\{s\le k: y_s\ne x_s\}\mid=0$, that is, $\delta(\{s\in\mathbb{N}: y_s\ne x_s\})=0$. Therefore $x_s=y_s$ for almost all $s\in\mathbb{N}$.

$(ii)\Rightarrow(iii)$ Assume that (y_t) is a convergent sequence in X such that $x_t=y_t$ for almost all $t\in\mathbb{N}$. Let $\mathcal{A}=\{t\in\mathbb{N}: x_t=y_t\}$. Then $\delta(\mathcal{A})=1$. Thus $(y_t)_{t\in\mathcal{A}}$ is both a convergent sequence and a statistically dense subsequence of (x_t).

$(iii)\Rightarrow(iv)$ It is an obvious consequence of Theorem 2.3.

$(iv)\Rightarrow(i)$ Assume that there is a statistically dense subsequence (x_{t_m}) of (x_t) such that $x_{t_m}\xrightarrow{AS}x\in X$. Let $\mathcal{A}=\{t_m: m\in\mathbb{N}\}$. Then $\delta(\mathcal{A})=1$. Since, for each $\varepsilon>0$,

$$\{t\in\mathbb{N}: A(x_t,x_t,\dots,x_t,x)<\varepsilon\}\supseteq\{t_m\in\mathbb{N}: A(x_{t_m},x_{t_m},\dots,x_{t_m},x)<\varepsilon\}$$

, thus

$$\delta\left(\{t\in\mathbb{N}: A(x_t,x_t,\dots,x_t,x)<\varepsilon\}\right)\ge\delta\left(\{t_m\in\mathbb{N}: A(x_{t_m},x_{t_m},\dots,x_{t_m},x)<\varepsilon\}\right)=1,$$

we get $x_t\xrightarrow{AS}x\in X$. □

The following corollary is a direct consequence of Theorem 2.9.

Corollary 2.10 *In A-metric spaces, every statistically convergent sequence has a convergent subsequence.*

Theorem 2.11 *Let (x_t) be any statistically convergent sequence in an A-metric space (X,A), then it is statistically Cauchy.*

Proof Let (x_t) be a statistically convergent sequence in A-metric space (X,A), that is, for $\varepsilon>0$, $\lim_{k\to\infty}\frac{1}{k}\mid\{t\le k: A(x_t,x_t,\dots,x_t,x)<\frac{\varepsilon}{n}\}\mid=1$. Then we can write

$$A(x_t,x_t,\dots,x_t,x)<\frac{\varepsilon}{n}\text{ for a.a.t,}$$

and if $s := s(\varepsilon)$ is chosen so that

$$A(x_s, x_s, \dots, x_s, x) < \frac{\varepsilon}{n}.$$

Then by Lemma 1.3 and Lemma 1.4, we get

$$\begin{aligned} A(x_t, x_t, \dots, x_t, x_s) &\leq (n-1)A(x_t, x_t, \dots, x_t, x) + A(x, x, \dots, x, x_s) \\ &= (n-1)A(x_t, x_t, \dots, x_t, x) + A(x_s, x_s, \dots, x_s, x) \\ &< (n-1)\frac{\varepsilon}{n} + \frac{\varepsilon}{n} \\ &= \varepsilon \text{ for a.a.t.} \end{aligned}$$

So (x_t) is a statistically Cauchy sequence. □

Definition 2.12 *Let (X, A) be an A-metric space. We say that (X, A) is statistically complete if every statistically Cauchy sequence in X is also statistically convergent.*

Lemma 2.13 *Every statistically complete A-metric space is also complete.*

Proof Let (X, A) be a statistically complete A-metric space. If a sequence (x_t) be a Cauchy sequence in X, then it is a statistically Cauchy sequence in this space. Since (X, A) is statistically complete, the sequence (x_t) is statistically convergent. By Corollary 2.10, there is a subsequence (x_{t_m}) of the sequence (x_t) that converges to a point $x \in X$. Since (x_t) is a Cauchy sequence, so, for $\varepsilon > 0$, there exist a positive integer K such that

$$A(x_t, x_t, \dots, x_t, x_s) < \frac{\varepsilon}{n}$$

for each $t, s \geq K$. Also (x_{t_m}) converges to x. So, there exists a positive integer m_0 such that $t_{m_0} \geq K$ and

$$A(x_{t_{m_0}}, x_{t_{m_0}}, \dots, x_{t_{m_0}}, x) < \frac{\varepsilon}{n}.$$

By Lemma 1.3 and Lemma 1.4, we have

$$\begin{aligned} A(x_t, x_t, \dots, x_t, x) &\leq (n-1)A(x_t, x_t, \dots, x_t, x_{t_{m_0}}) + A(x_{t_{m_0}}, x_{t_{m_0}}, \dots, x_{t_{m_0}}, x) \\ &< (n-1)\frac{\varepsilon}{n} + \frac{\varepsilon}{n} \\ &= \varepsilon \end{aligned}$$

for each $t \geq K$, that is, (x_t) converges to x. So, (X, A) is a complete A-metric space. □

In a partial metric space, Nuray [21] examined the relationship between p-strongly Cesàro summability and statistical convergence. Here following the same approach, we explain the relationship between strongly q-Cesàro convergence and statistical convergence in an A-metric space (X, A).

Definition 2.14 *Let (X, A) be an A-metric space, let (x_t) be a sequence in X, and let p be a positive real number. The sequence (x_t) is said to be p-strongly Cesàro summable to $x \in X$ if*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{t=1}^{k} [A(x_t, x_t, \ldots, x_t, x)]^p = 0$$

and is denoted by $x_t \overset{A[C,p]}{\longrightarrow} x$.

Theorem 2.15 *Let (x_t) be a sequence in an A-metric space (X, A) and $p \in \mathbb{R}^{>0}$. Then*

(i) If the sequence (x_t) is strongly p-Cesàro summable to $x \in X$, then it is statistically convergent to x.

(ii) If (X, A) is bounded and the sequence (x_t) is statistically convergent to $x \in X$, then it is strongly p-Cesàro summable to x.

Proof

(i) Let (x_t) be a sequence in an A-metric space (X, A). Assume the sequence (x_t) is strongly p-Cesàro summable to $x \in X$. Then for any $\varepsilon > 0$, we can write

$$\begin{aligned}
\sum_{t=1}^{k} [A(x_t, x_t, \ldots, x_t, x)]^p &= \sum_{\substack{t=1 \\ A(x_t, x_t \ldots, x_t, x) \geq \varepsilon}}^{k} [A(x_t, x_t, \ldots, x_t, x)]^p \\
&\quad + \sum_{\substack{t=1 \\ A(x_t, x_t \ldots, x_t, x) < \varepsilon}}^{k} [A(x_t, x_t, \ldots, x_t, x)]^p \\
&\geq \sum_{\substack{t=1 \\ A(x_t, x_t \ldots, x_t, x) \geq \varepsilon}}^{k} [A(x_t, x_t, \ldots, x_t, x)]^p \\
&\geq \left| \{t \leq k : A(x_t, x_t, \ldots, x_t, x) \geq \varepsilon\} \right| \varepsilon^p.
\end{aligned}$$

This completes the proof.

(ii) Now suppose that (X, A) is bounded and (x_t) is statistically convergent to x. Since (X, A) is bounded, there exists a constant $L > 0$ such that $A(x_t, x_t, \ldots, x_t, x) < L$ for all $x_t, x \in X$.

Let $\varepsilon > 0$ be given and choose K_ε such that

$$\frac{1}{k} \mid \{t \leq k : A(x_t, x_t, \ldots, x_t, x) \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\} \mid < \frac{\varepsilon}{2L^p}$$

for all $k > K_\varepsilon$ and set $\mathcal{A}_k = \{t \leq k : A(x_t, x_t, \ldots, x_t, x) \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\}$. Now for all $k > K_\varepsilon$, we can write

$$\frac{1}{k}\sum_{t=1}^{k}[A(x_t, x_t, \ldots, x_t, x)]^p = \frac{1}{k}\sum_{t\in\mathcal{A}_k}[A(x_t, x_t, \ldots, x_t, x)]^p + \frac{1}{k}\sum_{\substack{t\notin\mathcal{A}_k \\ t\leq k}}[A(x_t, x_t, \ldots, x_t, x)]^p$$

$$< \frac{1}{k}\left(\frac{k\varepsilon}{2L^p}\right)L^p + \frac{1}{k}k\frac{\varepsilon}{2} = \varepsilon.$$

So, (x_t) is strongly p-Cesàro summable to x. □

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1373528
(**2024**) **5**(2) 94 – 122
Research Article

More on the Vague Complex Subhypergroups

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Received: 09 October 2023 **Accepted:** 31 July 2024

Abstract: The concept of a vague complex set provides a more comprehensive perspective than that of a vague set. This article aims to investigate vague complex and anti-vague complex subhypergroups (H_v-subgroups), supported by various examples with the help of vague complex sets and hyperstructures. Moreover, we explore their properties and relationships with vague and anti-vague subhypergroups.

Keywords: Vague complex set, vague complex subhypergroup, anti-vague complex subhypergroup.

1. Introduction

The integration of mathematics with other scientific fields, such as computer science, is highly significant and has been a primary focus of global research conducted by experts in hyperstructure theory over recent decades. Marty [8] introduced algebraic hyperstructures, which are a broader concept than traditional algebraic structures. In traditional algebraic structures, the combination of two elements produces another element, whereas in algebraic hyperstructures, it results in a set. Since this innovation, many extensive works have focused on this area of study (see [2]). Today, hyperstructures have various applications in different branches of mathematics and computer science and are explored in numerous researches around the world.

Vougiouklis [13–15] has made significant contributions to the field of generalized algebraic hyperstructures or in the area of H_v-structures. In traditional hyperstructure axioms, the concept of equality is replaced by the condition of having a non-empty intersection. Recently, many studies have introduced H_v-structures. For example, Davvaz and et al. have conducted extensive research on both hyperstructures and H_v-structures [1, 3]. Moreover, Davvaz [4] introduced the concept of fuzzy subhypergroups (or H_v-subgroups) within the context of hypergroups (or H_v-groups). A concise overview of the theory of fuzzy algebraic hyperstructures can be found in [5].

In fuzzy theory, fuzzy mathematics is a field closely related to the theory of fuzzy sets

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2020 *AMS Mathematics Subject Classification*: 03B52, 03G27, 08A05

and logic. This idea, developed by Zadeh [16], extends traditional set theory. Fuzzy sets include elements that have varying degrees of membership. Unlike conventional set theory, where an element either belongs to the set or does not, fuzzy sets enable a gradual evaluation of membership levels, providing a more detailed perspective. This variability is expressed through a membership function that assigns values within the real interval $[0, 1]$. After Zadeh introduced fuzzy sets, numerous applications in mathematics and related fields emerged, providing researchers with significant motivation to explore various concepts. This has led to the expansion of abstract algebra within the framework of fuzzy settings. The study of fuzzy algebraic structures began with Rosenfeld's description of fuzzy subgroups [9]. The exploration of fuzzy hyperstructures has become a fascinating research area within fuzzy sets, and a considerable amount of research has been focused on examining the connections between fuzzy sets and hyperstructures.

As for vague set theory, a vague set consists of elements, each assigned a membership level within a continuous subinterval of $[0, 1]$. These sets are defined by both an accuracy membership function and an inaccuracy membership function. The concept of vague sets was introduced by Gau and Buehrer [6] as an extension of fuzzy set theory. On the other hand, Ramot [10, 11] introduced complex fuzzy sets, which further extend the idea of fuzzy sets. Then, Singh [12] proposed the concept of complex vague sets based on a lattice structure. Additionally, Husban and Salleh [7] explored complex vague relations, extending the scope of vague relations by expanding the range of accuracy and inaccuracy membership functions from the interval $[0, 1]$ to the unit circle in the complex plane.

The key benefit of vague sets which generalizes the fuzzy sets compared to fuzzy sets is that they distinguish between the positive and negative evidence for an element's membership in the set. With vague sets, we not only have an estimate of the likelihood that an element belongs to the set, but we also have lower and upper bounds on this likelihood. In light of the aforementioned studies, we want to define subhyperstructures using complex vague sets in order to expand the features of subhyperstructures that work with fuzzy complex sets.

The main aim of this work is to investigate the algebraic structures of subhypergroups using vague complex sets. After an Introduction, in Section 2 we present some definitions related to vague sets, hyperstructures and vague, anti-vague subhypergroups. In Section 3, we define vague complex subhypergroups and investigate their algebraic properties supporting different examples. In Section 4, we study anti-vague complex subhypergroups and investigate the relationships between these subhypergroups and vague complex subhypergroups.

2. Preliminaries

This section will provide foundational knowledge about vague complex H_v-subgroups. First, we will present definitions and theorems related to hyperstructures and vague subhyperstructures.

Definition 2.1 *[3] Let S be a non-void set. A mapping $*: S \times S \to P^*(S)$ is called a binary hyperoperation on S, where $P^*(S)$ represents the family of all non-void subsets of S. In this definition, the $(S,*)$ is called a hypergroupoid.*

In the above definition, if K and L are two non-void subsets of S and $e \in S$, then we define:

1. $K * L = \bigcup_{\substack{k \in K \\ l \in L}} k * l,$
2. $e * K = \{e\} * K$ *and* $K * e = K * \{e\}.$

Definition 2.2 *[3] A hypergroupoid $(S,*)$ is called as:*

- *Semihypergroup: If for all $k, l, m \in S$, then $k * (l * m) = (k * l) * m$,*
- *Quasihypergroup: If for all $k \in S$, we have $k * S = S = S * k$ (This condition is known as the reproduction property),*
- *Hypergroup: If it satisfies both the conditions of a semihypergroup and a quasihypergroup,*
- *H_v-group: If it is a quasihypergroup and for all $k, l, m \in S$, we have $k*(l*m) \cap (k*l)*m \neq \emptyset$.*

Definition 2.3 *[3] Let $(S,*)$ be a hypergroup (or H_v-group) and $M \subseteq S$. The pair $(M,*)$ is considered as a subhypergroup (or H_v-subgroup) of $(S,*)$ if for every $m \in M$, the requirement $m * M = M = M * m$ holds.*

Definition 2.4 *[16] A fuzzy set, defined on a universe of discourse A is characterized by a membership function $\lambda_F(x)$ which assigns any element a grade of membership in F. The fuzzy set can be represented as follows:*

$$F = \{(x, \lambda_F(x)) : x \in A\},$$

where $\lambda_F(x)$ belongs to the interval $[0,1]$.

Definition 2.5 *[6] A vague set V in the universe of discourse S is characterized by two membership functions:*

$$\text{An accuracy membership function } t_V : S \to [0,1],$$

$$\text{An inaccurate membership function } f_V : S \to [0,1].$$

Here, $t_V(s)$ *represents the lower bound of grade of membership of* s *derived from the "evidence for* s*" and* $f_V(s)$ *represents the lower bound of negation of membership of* s *derived from the "evidence against* s*".*
Additionally, $0 \le t_V(s) + f_V(s) \le 1$. *The grade of membership of* s *in the vague set* V *is bounded within the subinterval* $[t_V(s), 1-f_V(s)]$ *of* $[0,1]$. *This implies that if the actual grade of membership is represented by* $\lambda_F(s)$, *then* $t_V(s) \le \lambda_F(s) \le 1 - f_V(s)$. *Vague set* V *is denoted by:*

$$V = \{(s, t_V(s), 1 - f_V(s)) : s \in S\}$$

In this representation, $[t_V(s), 1 - f_V(s)]$ *means the "vague value" of* s *in* V.

Definition 2.6 *[6] The complement of a vague set* V *is represented by* V^C *and is characterized as*

$$t_{V^C}(s) = f_V(s)$$

$$1 - f_{V^C}(s) = 1 - t_V(s)$$

Definition 2.7 *[6] A vague set* U *is contained in the other vague set* V, $U \subseteq V$, *if and only if* $t_U \le t_V$ *and* $1 - f_U \le 1 - 1 - f_V$.

Definition 2.8 *[6] Union of two vague sets* K *and* L *is a vague set* M, *written as* $M = K \bigcup L$, *whose accuracy membership and inaccurate membership functions are associated with those of* K *and* L *with*

$$t_M = \max(t_K, t_L)$$

$$1 - f_M = \max(1 - f_K, 1 - f_L) = 1 - \min(f_K, f_L)$$

The union of two vague sets K *and* L *is the smallest vague set containing both* K *and* L.

Definition 2.9 *[6] Intersection of two vague sets* K *and* L *is a vague set* N, *written as* $N = K \bigcap L$, *whose accuracy membership and inaccurate membership functions are associated with those of* K *and* L *with*

$$t_N = \min(t_K, t_L)$$

$$1 - f_N = \min(1 - f_K, 1 - f_L) = 1 - \max(f_K, f_L)$$

The intersection of two vague sets K *and* L *is the largest vague set contained in both* K *and* L.

Definition 2.10 *Suppose that* $(S, *)$ *is a hypergroup (*H_v-*group) and* V *is a vague subset of* S *with accuracy membership function* $t_V(s) \in [0,1]$ *and inaccurate membership function* $f_V(s) \in [0,1]$. *If the following conditions hold,* V *is called a vague subhypergroup (*H_v-*subgroup) of* S *:*

1. $min\{t_V(\gamma), t_V(\delta)\} \leq \; inf\{t_V(\epsilon) : \epsilon \in \gamma * \delta\}$ *for every* $\gamma, \delta \in S$,

2. $min\{1 - f_V(\gamma), 1 - f_V(\delta)\} \leq \; inf\{1 - f_V(\epsilon) : \epsilon \in \gamma * \delta\}$ *for every* $\gamma, \delta \in S$,

3. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $min\{t_V(d), t_V(\gamma)\} \leq t_V(\eta)$,

4. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $min\{1 - f_V(d), 1 - f_V(\gamma)\} \leq 1 - f_V(\eta)$,

5. *For every* $d, \gamma \in S$, *there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $min\{t_V(d), t_V(\gamma)\} \leq t_V(\rho)$,

6. *For every* $d, \gamma \in S$, *there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $min\{1 - f_V(d), 1 - f_V(\gamma)\} \leq 1 - f_V(\rho)$.

Lemma 2.11 *Assume that* $(S, *)$ *is a hypergroup* (H_v*− group) and*

$$V = \{(s, t_V(s), 1 - f_V(s)) : s \in S\}$$

is a vague subhypergroup (H_v*− subgroup) of* S. *Then, for every* $g_1, g_2, \ldots, g_n \in S$

$$min\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \leq inf\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$min\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \leq inf\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Definition 2.12 *Suppose that* $(S, *)$ *is a hypergroup* (H_v*− group) and* V *is a vague subset of* S *with accuracy membership function* $t_V(s) \in [0, 1]$ *and inaccurate membership function* $f_V(s) \in [0, 1]$. *If the following conditions hold,* V *is called an anti-vague subhypergroup* (H_v*− subgroup) of* S*:*

1. $sup\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \; \leq max\{t_V(\gamma), t_V(\delta)\}$ *for every* $\gamma, \delta \in S$,

2. $sup\{1 - f_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{1 - f_V(\gamma), 1 - f_V(\delta)\}$ *for every* $\gamma, \delta \in S$,

3. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $t_V(\eta) \leq \; max\{t_V(d), t_V(\gamma)\}$,

4. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $1 - f_V(\eta) \leq \; max\{1 - f_V(d), 1 - f_V(\gamma)\}$,

5. *For every* $d, \gamma \in S$, *there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $t_V(\rho) \leq \; max\{t_V(d), t_V(\gamma)\}$,

6. For every $d, \gamma \in S$, there exists $\rho \in S$ such that $d \in \rho * \gamma$ and $1 - f_V(\rho) \leq max\{1 - f_V(d), 1 - f_V(\gamma)\}$.

Lemma 2.13 *Assume that $(S, *)$ is a hypergroup (H_v-group) and V an anti-vague subhypergroup (H_v-subgroup) of S. Then, for every $g_1, g_2, \ldots, g_n \in S$*

$$max\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \geq sup\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$max\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \geq sup\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Theorem 2.14 *Assume that $(S, *)$ is a hypergroup (H_v-group) and V a vague subset of S. V is a vague subhypergroup (H_v-subgroup) of S if and only if V^C is an anti-vague subypergroup (H_v-subgroup) of S.*

3. Vague Complex Subhypergroups

In this part, we will present properties of the notion of vague complex subsets to clarify and characterize vague complex subhypergroups (H_v-subgroups). Then, we will explain the properties of vague complex subhypergroups.

Definition 3.1 *Suppose that $V = \{(s, t_V(s), 1 - f_V(s)) : s \in S\}$ is a vague set. Then, the $\pi-$vague set V_π is defined as $V_\pi = \{(s, 2\pi t_V(s), 2\pi(1 - f_V(s)) : s \in S\}$, where it must satisfies $0 \leq 2\pi t_V(s) + 2\pi(1 - f_V(s)) \leq 2\pi$.*

Proposition 3.2 *Suppose that $(S, *)$ is a hypergroup (H_v-group). V is a vague subhypergroup (H_v-subgroup) of S if and only if a $\pi-$vague set V_π is a π-vague subhypergroup (H_v-subgroup) of S.*

Proof The proof is clear. □

Definition 3.3 *[7] A vague complex set V in the universe of discourse S is characterized with two complex membership functions:*
A complex accuracy membership function

$$\widehat{t_V}(s) : S \to \{\alpha : \alpha \in \mathbb{C}, |\alpha| \leq 1\} \text{ and } \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}.$$

A complex inaccurate membership function

$$\widehat{f_V}(s) : S \to \{\alpha : \alpha \in \mathbb{C}, |\alpha| \leq 1\} \text{ and } \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}.$$

Here, $\widehat{t_V}(s)$ *stands for the lower bound of the complex grade of membership* s *derived from "evidence for* s *"and* $\widehat{f_V}(s)$ *stands for the lower bound on the negation of* s *derived from the "evidence against* s *".*

In this definition, $0 \le t_V(s) + f_V(s) \le 1$ *with both* $t_V(s)$ *and* $f_V(s)$ *is real valued, belonging to the interval* $[0,1]$. *Additionally,* $\alpha \in [0,2\pi]$ *and* $w_V^t(s), w_V^f(s) \in [0,1]$. *The complex vague set* V *is symbolized as*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

In this paper, $[t_V(s)e^{i\alpha w_V^t(s)}, 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}]$ *means vague complex rate of* s *in* V.

Definition 3.4 *[7] Let* $V = \{(s, t_V(s)e^{i\alpha w_V^t(s)}, 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$ *be a vague complex subset. The complement of* V *is characterized as*

$$V^C = \{(s, f_V(s)e^{i\alpha w_V^f(s)}, 1 - t_V(s)e^{i\alpha(2\pi - w_V^t(s))}) : s \in S\}.$$

Definition 3.5 *[7] Let* $K = \{(s, t_K(s)e^{i\alpha w_K^t(s)}, 1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}) : s \in S\}$ *and* $L = \{(s, t_L(s)e^{i\alpha w_L^t(s)}, 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}) : s \in S\}$ *be two vague complex subsets of the identical universe* S. *Union of* K *and* L, *represented by* $M = K \bigcup L$, *is determined as:*

$$M = \{(s, \widetilde{t_M}(s), \widetilde{f_M}(s)) : s \in S\},$$

where

$$\widetilde{t_M}(s) = \max(t_K(s), t_L(s))e^{i\alpha \max(w_K^t(s), w_L^t(s))}$$

and

$$\widetilde{f_M}(s) = \max(1 - f_K(s), 1 - f_L(s))e^{i\alpha \max(2\pi - w_K^f(s), 2\pi - w_L^f(s))}.$$

Definition 3.6 *[7] Let* $K = \{(s, t_K(s)e^{i\alpha w_K^t(s)}, 1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}) : s \in S\}$ *and* $L = \{(s, t_L(s)e^{i\alpha w_L^t(s)}, 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}) : s \in S\}$ *be two vague complex subsets of the identical universe* S. *Intersection of* K *and* L, *represented by* $N = K \bigcap L$, *is determined as:*

$$N = \{(s, \widetilde{\widetilde{t_N}}(s), \widetilde{\widetilde{f_N}}(s)) : s \in S\},$$

where

$$\widetilde{\widetilde{t_N}}(s) = \min(t_K(s), t_L(s))e^{i\alpha \min(w_K^t(s), w_L^t(s))}$$

and

$$\widetilde{\widetilde{\widehat{f_N}}}(s) = \min(1 - f_K(s), 1 - f_L(s))e^{i\alpha \min(2\pi - w_K^f(s), 2\pi - w_L^f(s))}.$$

Definition 3.7 *Let* $K = \{(s, t_K(s)e^{i\alpha w_K^t(s)}, 1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}) : s \in S\}$ *and* $L = \{(s, t_L(s)e^{i\alpha w_L^t(s)}, 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}) : s \in S\}$ *be two vague complex sets of a non-void set* S *with the accuracy membership functions* $t_K(s)e^{i\alpha w_K^t(s)}$ *and* $t_L(s)e^{i\alpha w_L^t(s)}$ *and inaccurate membership functions* $1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}$ *and* $1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}$ *respectively. Then*

1. *A vague complex subset* K *is called homogeneous if for every* $m, s \in S$*, we have that*

$$t_K(m) \le t_K(s) \text{ if and only if } w_K^t(m) \le w_K^t(s)$$

   and

   $1 - f_K(m) \le 1 - f_K(s)$ *if and only if* $2\pi - w_K^f(m) \le 2\pi - w_K^f(s)$.

2. *A vague complex subset* K *is called homogeneous by* L *if for every* $m, s \in S$*, we have that*

$$t_K(m) \le t_L(s) \text{ if and only if } w_K^t(m) \le w_L^t(s)$$

   and

$$1 - f_K(m) \le 1 - f_L(s) \text{ if and only if } 2\pi - w_K^f(m) \le 2\pi - w_L^f(s).$$

Notation 3.8 *Let* $K = \{(s, t_K(s)e^{i\alpha w_K^t(s)}, 1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}) : s \in S\}$ *and* $L = \{(s, t_L(s)e^{i\alpha w_L^t(s)}, 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}) : s \in S\}$ *be two vague complex sets of a non-empty set* S *with the accuracy membership functions* $t_K(s)e^{i\alpha w_K^t(s)}$ *and* $t_L(s)e^{i\alpha w_L^t(s)}$ *and inaccurate membership functions* $1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))}$ *and* $1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}$ *respectively.*

- *With* $t_K(s)e^{i\alpha w_K^t(s)} \le t_L(s)e^{i\alpha w_L^t(s)}$*, it refers that* $t_K(s) \le t_L(s)$ *and* $w_K^t(s) \le w_L^t(s)$.
- *With* $1 - f_K(s)e^{i\alpha(2\pi - w_K^f(s))} \le 1 - f_L(s)e^{i\alpha(2\pi - w_L^f(s))}$*, it refers that* $1 - f_K(s) \le 1 - f_L(s)$ *and* $2\pi - w_K^f(s) \le 2\pi - w_L^f(s)$.

In this work, all vague complex sets are considered homogeneous.

Definition 3.9 *Suppose that* $(S, *)$ *is a hypergroup (*H_v*− group) and*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$$

is a (homogeneous) vague complex subset of S*. If the following conditions are satisfied,* V *is called a vague complex subhypergroup (*H_v*− subgroup) of* S*:*

1. $min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\} \leq inf\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\}$ *for every* $\gamma, \delta \in S$,
2. $min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\} \leq \ inf\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\}$ *for every* $\gamma, \delta \in S$,
3. *For every* $d, \gamma \in S$*, there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $min\{\widehat{t_V}(d), \widehat{t_V}(\gamma)\} \leq \widehat{t_V}(\eta)$,
4. *For every* $d, \gamma \in S$*, there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $min\{\widehat{f_V}(d), v_A(\gamma)\} \leq \widehat{f_V}(\eta)$,
5. *For every* $d, \gamma \in S$*, there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $min\{\widehat{t_V}(d), \widehat{t_V}(\gamma)\} \leq \widehat{t_V}(\rho)$,
6. *For every* $d, \gamma \in S$*, there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $min\{\widehat{f_V}(d), v_A(\gamma)\} \leq \widehat{f_V}(\rho)$.

Example 3.10 *Let* $S = \{\gamma, \delta\}$ *and define hypergroup* $(S, *)$ *by next table:*

$*$	γ	δ
γ	γ	S
δ	S	δ

Describe a vague complex subset V *of* S *as:*

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.3e^{i0} \quad and \ 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.6e^{i\pi},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.4e^{i\pi} \quad and \ 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.7e^{i3\pi/2}.$$

In that case, V *is homogeneous vague complex subhypergroup of* S*.*

Example 3.11 *Let* $S = \{\gamma, \delta, \epsilon\}$ *and define the hypergroup* $(S, *)$ *as follows:*

$*$	γ	δ	ϵ
γ	$\{\gamma\}$	$\{\delta\}$	$\{\epsilon\}$
δ	$\{\delta\}$	$\{\gamma\}$	$\{\delta, \epsilon\}$
ϵ	$\{\epsilon\}$	$\{\delta, \epsilon\}$	$\{\gamma\}$

Describe a vague complex subset V *of* S *as:*

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.4e^{i\pi/2} \quad and \ 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.4e^{i\pi/2},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.5e^{i\pi} \quad and \ 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.6e^{i\pi},$$

$$t_V(\epsilon)e^{i\alpha w_V^t(\epsilon)} = 0.5e^{i3\pi/2} \quad and \ 1 - f_V(\epsilon)e^{i\alpha(2\pi - w_V^f(\epsilon))} = 0.7e^{i5\pi/3}.$$

In that case, V is homogeneous vague complex subhypergroup of S.

Theorem 3.12 *Suppose that $(S,*)$ is a hypergroup (H_v-group), V is a (homogeneous) vague complex subset of S. V is a vague complex subhypergroup (H_v-subgroup) of S if and only if t_V and $1-f_V$ are vague subhypergroups (or H_v-subgroups) of S and w_V^t and $2\pi - w_V^f$ are $\pi-$vague subhypergroups (or H_v-subgroups) of S.*

Proof $(\Longrightarrow)$: Let V be a vague complex subhypergroup (H_v-subgroup) of S. We want to demostrate requirements of description of vague subhypergroups are satisfied for t_V, $1-f_V$ and w_V^t, $2\pi - w_V^f$. For every $\gamma, \delta \in S$, we have

$$min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\} \leq inf\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\},$$

$$min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\} \leq inf\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\}.$$

Notation 3.8 means that

$$inf\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{t_V(\gamma), t_V(\delta)\},$$

$$inf\{1 - f_V(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{1 - f_V(\gamma), 1 - f_V(\delta)\}$$

and

$$inf\{w_V^t(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{w_V^t(\gamma), w_V^t(\delta)\},$$

$$inf\{2\pi - w_V^f(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Let $\gamma, d \in S$. There exist $\eta, \rho \in S$ such that $d \in \gamma * \eta$ and $d \in \rho * \gamma$. Then,

$$min\{\widehat{t_V}(\gamma)\widehat{t_V}(d)\} \leq \widehat{t_V}(\eta)$$

$$min\{\widehat{f_V}(\gamma), \widehat{f_V}(d)\} \leq \widehat{f_V}(\eta)$$

and

$$min\{\widehat{t_V}(\gamma), \widehat{t_V}(d)\} \leq \widehat{t_V}(\rho)$$

$$min\{\widehat{f_V}(\gamma), \widehat{f_V}(d)\} \leq \widehat{f_V}(\rho).$$

Notation 3.8 means that the conditions $3-6$ of definition of vague subhypergroup are satisfied for both t_V , $1-f_V$ and w_V^t , $2\pi - w_V^f$.

$(\Longleftarrow)$: Let t_V and $1-f_V$ be vague subhypergroups (or H_v-subgroups) of S and w_V^t , $2\pi - w_V^f$ be $\pi-$vague subhypergroups (or H_v-subgroups) of S. We want to show that the conditions of vague complex subhypergroups are provided. For every $\gamma, \delta \in S$, we have

$$inf\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{t_V(\gamma), t_V(\delta)\},$$

$$inf\{1 - f_V(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{1 - f_V(\gamma), 1 - f_V(\delta)\}$$

and

$$inf\{w_V^t(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{w_V^t(\gamma) w_V^t(\delta)\},$$

$$inf\{2\pi - w_V^f(\epsilon) : \epsilon \in \gamma * \delta\} \geq min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Notation 3.8 means that

$$min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\} \leq inf\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\},$$

$$min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\} \leq inf\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\}.$$

Suppose that $\gamma, d \in S$. There exist $\eta, \rho \in S$ such that $d \in \gamma * \eta$ and $d \in \rho * \gamma$

$$min\{t_V(\gamma), t_V(d)\} \leq t_V(\eta) \text{ and } min\{w_V^t(\gamma), w_V^t(d)\} \leq w_V^t(\eta),$$

$$min\{1 - f_V(\gamma), 1 - f_V(d)\} \leq 1 - f_V(\eta) \text{ and } min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(d)\} \leq 2\pi - w_V^f(\eta)$$

and

$$min\{t_V(\gamma), t_V(d)\} \leq t_V(\rho) \text{ and } min\{w_V^t(\gamma), w_V^t(d)\} \leq w_V^t(\rho),$$

$$min\{1 - f_V(\gamma), 1 - f_V(d)\} \leq 1 - f_V(\rho) \text{ and } min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(d)\} \leq 2\pi - w_V^f(\rho).$$

Notation 3.8 means that the conditions $3 - 6$ of definition of vague complex subhypergroup are satisfied for V. □

Lemma 3.13 *Suppose that $(S, *)$ is a hypergroup (H_v-group) and V is a (homogeneous) vague complex subhypergroup (H_v-subgroup) of S. For every $g_1, g_2, \ldots, g_n \in S$*

$$min\{\widehat{t_V}(g_1), \widehat{t_V}(g_2), \ldots, \widehat{t_V}(g_n)\} \leq inf\{\widehat{t_V}(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$min\{\widehat{f_V}(g_1), \widehat{f_V}(g_2), \ldots, \widehat{f_V}(g_n)\} \leq inf\{\widehat{f_V}(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Proof Let $g_1, g_2, \ldots, g_n \in S$ and

$$\widehat{t_V}(s) = t_V(s) e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s) e^{i\alpha(2\pi - w_V^f(s))}.$$

To demonstrate the lemma, it suffices to indicate that

$$min\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \leq inf\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$min\{w_V^t(g_1), w_V^t(g_2), \ldots, w_V^t(g_n)\} \leq inf\{w_V^t(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$min\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \leq inf\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$min\{2\pi - w_V^f(g_1), 2\pi - w_V^f(g_2), \ldots 2\pi - w_V^f(g_n)\} \leq inf\{2\pi - w_V^f(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Because V is homogeneous, it suffices to indicate that

$$min\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \leq inf\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$min\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \leq inf\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Theorem 3.12 claims that t_V and $1 - f_V$ are vague subhypergroups (or H_v-subgroups) of S. Using Lemma 2.11 , the proof is completed. □

Definition 3.14 *Assume that*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$$

is a (homogeneous) vague complex subset of a non-empty set S. Level subset $V_{(r,t)}$ of S is described as $V_{(r,t)} = \{s \in S : \widehat{t}_V(s) \geq r$ and $\widehat{f}_V(s) \geq t\}$, where $r = me^{i\varphi}, t = ne^{i\psi}$ such that $m, n \in [0,1]$ and $\varphi, \psi \in [0, 2\pi]$.

Remark 3.15 *$\widehat{t}_V(x) \geq r$ means that $x \in \widehat{t}_r$. Similarly, $\widehat{f}_V(x) \geq t$ means that $x \in \widehat{f}_t$.*

Remark 3.16 *Let $V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$ be a (homogeneous) vague complex subset of a non-empty set S. Followings are true:*

i. Provided that $r_1 \leq r_2$, then $\widehat{t}_{r_2} \subseteq \widehat{t}_{r_1}$,

ii. $\widehat{t}_{0e^{0i}} = S$,

iii. Provided that $t_1 \leq t_2$, then $\widehat{f}_{t_2} \subseteq \widehat{f}_{t_1}$,

iv. $\widehat{f}_{0e^{0i}} = S$.

Theorem 3.17 *Suppose that $(S, *)$ is a hypergroup (H_v-group) and V is a (homogeneous) vague complex subset of S. V is a vague complex subhypergroup (H_v-subgroup) of S if and only if for all $r = ke^{i\psi}$, $t = me^{i\varphi}$ such that $k, m \in [0,1]$ and $\psi, \varphi \in [0, 2\pi]$, $V_{(r,t)} \neq \varnothing$ is a subhypergroup (H_v-subgroup) of S.*

Proof Let V be a vague complex subhypergroup (H_v-subgroup) of S and $\gamma,\delta \in V_{(r,t)} \neq \varnothing$. For every $\epsilon \in \gamma * \delta$, we have

$$\widehat{t_V}(\epsilon) \geq min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\} \geq r,$$

$$\widehat{f_V}(\epsilon) \geq min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\} \geq t.$$

Hence, $\epsilon \in \gamma * \delta \subseteq V_{(r,t)}$ and for all $\epsilon \in V_{(r,t)}$, we have $\epsilon * V_{(r,t)} \subseteq V_{(r,t)}$. Moreover, let $\gamma \in V_{(r,t)}$, using definition of vague complex subhypergroup, there exists $\eta \in S$ such that $\gamma \in \epsilon * \eta$ and

$$\widehat{t_V}(\eta) \geq min\{\widehat{t_V}(\epsilon), \widehat{t_V}(\gamma)\} \geq r,$$

$$\widehat{f_V}(\eta) \geq min\{\widehat{f_V}(\epsilon), \widehat{f_V}(\gamma)\} \geq t.$$

Therefore, this means $\eta \in V_{(r,t)}$. We can get $V_{(r,t)} * \epsilon \subseteq V_{(r,t)}$ using definition of vague complex subhypergroup.

For the converse, let $r = ke^{i\psi}, t = me^{i\varphi}$ such that $k,m \in [0,1]$ and $\psi,\varphi \in [0,2\pi]$ and $V_{(r,t)} \neq \varnothing$ be a subhypergroup (H_v-subgroup) of S. Suppose that

$$r_0 = k_0 e^{i\psi_0} = min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\},$$

$$t_0 = m_0 e^{i\varphi_0} = min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\}.$$

Then,

$$k_0 = \ min\{t_V(\gamma) t_V(\delta)\},$$

$$\psi_0 = min\{w_V^t(\gamma), w_V^t(\delta)\}$$

and

$$m_0 = \ min\{1 - f_V(\gamma), 1 - f_V(\delta)\},$$

$$\varphi_0 = \ min\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Because $\gamma,\delta \in V_{(r_0,t_0)}$ and $V_{(r_0,t_0)}$ is a a subhypergroup (H_v-subgroup) of S, we have $\gamma * \delta \subseteq V_{(r_0,t_0)}$. Thus, for all $\epsilon \in \gamma * \delta$

$$\widehat{t_V}(\epsilon) \geq r_0 = min\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\},$$

$$\widehat{f_V}(\epsilon) \geq t_0 = min\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\}$$

and hence, conditions $1,2$ of the definition of vague complex subhypergroup are satisfied. Similarly, we will prove conditions $3-6$ are obtained. For every $\epsilon,\gamma \in S$, setting by

$$r_1 = k_1 e^{i\psi_1} = min\{\widehat{t_V}(\gamma), \widehat{t_V}(\epsilon)\},$$

$$t_1 = m_1 e^{i\varphi_1} = min\{\widehat{f_V}(\gamma), \widehat{f_V}(\epsilon)\}.$$

Then, $\gamma,\epsilon \in V_{(r_1,t_1)}$. With $V_{(r_1,t_1)}$ is a subhypergroup (H_v-subgroup) of S refers $\epsilon * V_{(r_1,t_1)} = V_{(r_1,t_1)}$. The latter means there exists $\delta \in V_{(r_1,t_1)}$ such that $\gamma \in \epsilon * \delta$. Hence,

$$\widehat{t}_V(\delta) \geq r_1 = min\{\widehat{t}_V(\gamma),\widehat{t}_V(\epsilon)\},$$

$$\widehat{f}_V(\delta) \geq t_1 = min\{\widehat{f}_V(\gamma),\widehat{f}_V(\epsilon)\}.$$

□

Corollary 3.18 *Assume that $(S,*)$ is a hypergroup (H_v-group) and V is a (homogeneous) vague complex subhypergroup (H_v-subgroup) of S with*

$$V = \{(s,\widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

If $0e^{0i} \leq r_1 = s_1e^{i\theta_1} < r_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, then $\widehat{t}_{r_1} = \widehat{t}_{r_2}$ if and only if there is no $\sigma \in S$ such that $r_1 \leq \widehat{t}_V(\sigma) < r_2$ and similarly if $0e^{0i} \leq m_1 = n_1e^{i\varphi_1} < m_2 = n_2e^{i\varphi_2} \leq 1e^{2\pi i}$, then $\widehat{f}_{m_1} = \widehat{f}_{m_2}$ if and only if there is no $\sigma \in S$ such that $m_1 \leq \widehat{f}_V(\sigma) < m_2$.

Proof Suppose that $0e^{0i} \leq r_1 = s_1e^{i\theta_1} < r_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$ such that $\widehat{t}_{r_1} = \widehat{t}_{r_2}$. Assume that there exists $\sigma \in S$ such that $r_1 \leq \widehat{t}_V(\sigma) < r_2$. We have $\sigma \in \widehat{t}_{r_1} = \widehat{t}_{r_2}$. This means that $\widehat{t}_V(\sigma) \geq r_2$ and it is contradiction. Similarly, let $0e^{0i} \leq m_1 = n_1e^{i\varphi_1} < m_2 = n_2e^{i\varphi_2} \leq 1e^{2\pi i}$ such that $\widehat{f}_{m_1} = \widehat{f}_{m_2}$. Assume that there exists $\sigma \in S$ such that $m_1 \leq \widehat{f}_V(\sigma) < m_2$. We have $\sigma \in \widehat{f}_{m_1} = \widehat{f}_{m_2}$. This means that $\widehat{f}_V(\sigma) \geq m_2$ and it is a contradiction.

Because of $0e^{0i} \leq r_1 = s_1e^{i\theta_1} < r_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, it follows by Remark 3.16 that $\widehat{t}_{r_2} \subseteq \widehat{t}_{r_1}$. In order to show that $\widehat{t}_{r_1} \subseteq \widehat{t}_{r_2}$, let $\sigma \in \widehat{t}_{r_1}$. Then, $\widehat{t}_V(\sigma) \geq r_1$. Because there is no $\sigma \in S$ such that $r_1 \leq \widehat{t}_V(\sigma) < r_2$, it follows that $\widehat{t}_V(\sigma) \geq r_2$. Thus, $\sigma \in \widehat{t}_{r_2}$ and $\widehat{t}_{r_1} \subseteq \widehat{t}_{r_2}$. Similarly, since $0e^{0i} \leq m_1 = n_1e^{i\varphi_1} < m_2 = n_2e^{i\varphi_2} \leq 1e^{2\pi i}$, it follows by previous Remark 3.16 that $\widehat{f}_{m_2} \subseteq \widehat{f}_{m_1}$. In order to show that $\widehat{f}_{m_1} \subseteq \widehat{f}_{m_2}$, let $\sigma \in \widehat{f}_{m_1}$. Then, $\widehat{f}_V(\sigma) \geq m_1$. Because there is no $\sigma \in S$ such that $m_1 \leq \widehat{f}_V(\sigma) < m_2$, it follows that $\widehat{f}_A(\sigma) \geq m_2$. Thus, $\sigma \in \widehat{f}_{m_2}$ and $\widehat{f}_{m_1} \subseteq \widehat{f}_{m_2}$. □

Corollary 3.19 *Assume that $(S,*)$ is a hypergroup (H_v-group) and V is a (homogeneous) vague complex subhypergroup (or H_v-subgroup) of S with*

$$V = \{(s,\widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

If the range of $\widehat{t}_V$ is finite set $\{r_1, r_2, \ldots, r_n\}$ and $\widehat{f}_V$ is finite set $\{m_1, m_2, \ldots, m_n\}$, then the sets $\{\widehat{t}_{r_i} : i = 1, 2, \ldots, n\}$ and $\{\widehat{f}_{m_i} : i = 1, 2, \ldots, n\}$ comprises whole the level subhypergroups (or

H_v-subgroups) of S. Besides of this, if $r_1 \geq r_2 \geq \ldots \geq r_n$ then whole the level subhypergroups of S create the chain $\widehat{t}_{r_1} \subseteq \widehat{t}_{r_2} \subseteq \ldots \subseteq \widehat{t}_{r_n}$. Similarly, if $m_1 \geq m_2 \geq \ldots \geq m_n$ then whole the level subhypergroups of S create the chain $\widehat{f}_{m_1} \subseteq \widehat{f}_{m_2} \subseteq \ldots \subseteq \widehat{f}_{m_n}$.

Proof Suppose that $\widehat{t}_q, \widehat{f}_q \neq \varnothing$ are level subhypergroups (or H_v-subgroups) of S such that $\widehat{t}_q \neq \widehat{t}_{r_i}$ and $\widehat{f}_q \neq \widehat{f}_{m_i}$ for all $1 \leq i \leq n$. Let r_z and m_z be closest complex numbers to q. There are two cases : $q < r_z$ and $q < m_z$, $q > r_z$ and $q > m_z$. We think the first case, the second case is like that of the first case. Because the ranges of $\widehat{t}_V$ and $\widehat{f}_V$ are finite sets $\{r_1, r_2, \ldots, r_n\}$ and $\{m_1, m_2, \ldots, m_n\}$ respectively, it follows that there is no $\epsilon \in S$ such that $q \leq \widehat{t}_V(\epsilon) < r_z$ and $q \leq \widehat{f}_V(\epsilon) < m_z$. Using Corollary 3.18, we have a contradiction. □

Proposition 3.20 *Suppose that $(S, *)$ is the biset hypergroup like $\gamma * \delta = \{\gamma, \delta\}$ for all $\gamma, \delta \in S$ and V is any homogeneous vague complex subset of S. Then V is a vague complex subhypergroup of S.*

Proof Assume that $r = me^{i\psi}$ and $t = ne^{i\varphi}$ such that $m, n \in [0,1]$ and $\psi, \varphi \in [0, 2\pi]$. By Theorem 3.17, it suffices to indicate that $V_{(r,t)} \neq \varnothing$ is a subhypergroup of S. We have $V_{(r,t)} \subseteq \upsilon * V_{(r,t)}$ as for all $\tau \in V_{(r,t)}$, $\tau \in \upsilon * \tau = \{\upsilon, \tau\}$. Besides of this, $\upsilon * V_{(r,t)} = V_{(r,t)} * \upsilon = \{\tau * \upsilon : \tau \in V_{(r,t)}\} = \{\tau, \upsilon\} \subseteq V_{(r,t)}$ for every $\upsilon \in V_{(r,t)}$. Therefore, $V_{(r,t)}$ is a subhypergroup of S. □

Proposition 3.21 *Suppose that $(S, *)$ is the total hypergroup like $\gamma * \delta = S$ for every $\gamma, \delta \in S$ and V is any homogeneous vague complex subset of S. Then V is a vague complex subhypergroup of S if and only if $\widehat{t}_V$ and $\widehat{f}_V$ are stable complex functions.*

Proof It is clear that if $\widehat{t}_V$ and $\widehat{f}_V$ are constant complex functions, then V is a vague complex subhypergroup of S. Let V be a vague complex subhypergroup of S and $\widehat{t}_V$ be not a constant complex function. We may find $\gamma, \delta \in S$, $r = se^{i\phi}$ such that $s \in [0,1]$ and $\phi \in [0, 2\pi]$ such that $\widehat{t}_V \ (\gamma) < \widehat{t}_V(\delta) = r$. It is clear that $\gamma \notin \widehat{t}_r$ and $\delta \in \widehat{t}_r$. Because $V_{(r,t)} \neq \varnothing$ is a subhypergroup of S, it follows that $S = \delta * \delta \subseteq \widehat{t}_r$. Similarly, let V be a vague complex subhypergroup of S and $\widehat{f}_V$ be not a constant complex function. We may find $\gamma, \delta \in S$, $u = ke^{i\varphi}$ such that $k \in [0,1]$ and $\varphi \in [0, 2\pi]$ such that $\widehat{f}_V \ (\gamma) < \widehat{f}_V(\delta) = u$. It is clear that $\gamma \notin \widehat{f}_u$ and $\delta \in \widehat{f}_u$. Because $V_{(r,t)} \neq \varnothing$ is a subhypergroup of S, it follows that $S = \delta * \delta \subseteq \widehat{f}_u$. These are contradictions. Hence, $\widehat{t}_V$ and $\widehat{f}_V$ are constant complex functions. □

Proposition 3.22 *Suppose that $(S, *)$ is a hypergroup (H_v-group) and V is a (homogeneous) vague complex subset of S. V is a vague complex subhypergroup (H_v-subgroup) of S if and only* *if for all* $r = ke^{i\psi}$ *and* $t = me^{i\varphi}$ *such that* $k, m \in [0,1]$ *and* $\psi, \varphi \in [0, 2\pi]$, *the following assertions are provided:*

i. $V_{(r,t)} * V_{(r,t)} \subseteq V_{(r,t)}$,

ii. $\upsilon * (S - V_{(r,t)}) - (S - V_{(r,t)}) \subseteq \upsilon * V_{(r,t)}$, *for every* $\upsilon \in V_{(r,t)}$,

iii. $(S - V_{(r,t)}) * \upsilon - (S - V_{(r,t)}) \subseteq V_{(r,t)} * \upsilon$, *for every* $\upsilon \in V_{(r,t)}$.

Proof Let V be a vague complex subhypergroup (H_v−subgroup) of S. Then, $V_{(r,t)}$ is a subhypergroup (H_v−subgroup) of S as $\upsilon * V_{(r,t)} = V_{(r,t)}$ for every $\upsilon \in V_{(r,t)}$. Hence, $V_{(r,t)} * V_{(r,t)} \subseteq V_{(r,t)}$. We need to indicate that $\upsilon * (S - V_{(r,t)}) - (S - V_{(r,t)}) \subseteq \upsilon * V_{(r,t)}$. Suppose that $\eta \in \upsilon * (S - V_{(r,t)}) - (S - V_{(r,t)})$. We have that η is not an element in $(S - V_{(r,t)})$. This implies that $\eta \in V_{(r,t)} = \upsilon * V_{(r,t)}$. Similarly, condition iii may be proved.

Conversely, let the conditions i, ii be satisfied. Using Theorem 3.17, it suffices to indicate $V_{(r,t)}$ is a subhypergroup (H_v−subgroup) of S as $\upsilon * V_{(r,t)} = V_{(r,t)} * \upsilon = V_{(r,t)}$ for every $\upsilon \in V_{(r,t)}$. Suppose that there exists $\gamma \in V_{(r,t)}$ such that γ is not an element in $\upsilon * V_{(r,t)}$. Using reproduction property of $(S, *)$, there exists $\delta \in S$ such that $\gamma \in \upsilon * \delta$. There are two cases for δ:

Case 1: $\delta \in V_{(r,t)}$. We have that $\gamma \in \upsilon * \delta \subseteq \upsilon * V_{(r,t)}$. This is a contradiction.

Case 2: $\delta \notin V_{(r,t)}$. We have that $\delta \in (S - V_{(r,t)})$. $\gamma \in \upsilon * \delta$ means $\gamma \in \upsilon * (S - V_{(r,t)})$. Because $\gamma \in V_{(r,t)}$, it follows that γ is not in $(S - V_{(r,t)})$. Hence, using by assumption we have $\gamma \in \upsilon * (S - V_{(r,t)}) - (S - V_{(r,t)}) \subseteq \upsilon * V_{(r,t)}$. This is a contradiction. Similarly, using condition iii, we can prove that $V_{(r,t)} * \upsilon = V_{(r,t)}$. □

Proposition 3.23 *Assume that* $(S, *)$ *is a hypergroup* (H_v−*group*) *and* V *is a (homogeneous) vague complex subhypergroup* (H_v−*subgroup*) *of* S *with*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w^t_V(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w^f_V(s))}) : s \in S\}.$$

Describe $\bar{\widehat{t}}$ *and* $\bar{\widehat{f}}$ *as follows:*

$$\bar{\widehat{t}} = \{s \in S : \widehat{t}_V(s) = 1e^{2\pi i}\} \text{ and } \bar{\widehat{f}} = \{s \in S : \widehat{f}_V(s) = 1e^{2\pi i}\}.$$

Then, $\bar{\widehat{t}}$ *and* $\bar{\widehat{f}}$ *are empty or subhypergroups* (H_v−*subgroups*) *of* S.

Proof We need to demonstrate that $\tau * \bar{\widehat{t}} = \bar{\widehat{t}} = \bar{\widehat{t}} * \tau$ for every $\tau \in \bar{\widehat{t}}$. Let $d \in \bar{\widehat{t}}$ and $\eta \in \tau * d$. Then, $\widehat{t}_V(\eta) \geq min\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} = 1e^{2\pi i}$ implies that $\widehat{t}_V(\eta) = 1e^{2\pi i}$. Hence, $\eta \in \tau * d \subseteq \bar{\widehat{t}}$. For every $\tau, d \in \overline{\overline{\widehat{t}}}$, there exists $\rho \in S$ such that $d \in \tau * \rho$ and $\widehat{t}_V(\rho) \geq min\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} = 1e^{2\pi i}$. This refers that $\widehat{t}_V(\rho) = 1e^{2\pi i}$ and $\rho \in \overline{\overline{\widehat{t}}}$.

Similarly, we want to indicate $\upsilon * \overline{\overline{\widehat{f}}} = \overline{\overline{\widehat{f}}} = \overline{\overline{\widehat{f}}} * \upsilon$ for every $\upsilon \in \overline{\overline{\widehat{f}}}$. Let $d \in \overline{\overline{\widehat{f}}}$ and $\eta \in \upsilon * d$. Then, $\widehat{f}_V(\eta) \geq min\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} = 1e^{2\pi i}$ implies that $\widehat{f}_V(\eta) = 1e^{2\pi i}$. Hence, $\eta \in \upsilon * d \subseteq \overline{\overline{\widehat{f}}}$. For every $\upsilon, d \in \overline{\overline{\widehat{f}}}$, there exists $\rho \in S$ such that $d \in \upsilon * \rho$ and $\widehat{f}_V(\rho) \geq min\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} = 1e^{2\pi i}$. This refers $\widehat{f}_V(\rho) = 0e^{0i}$ and $\rho \in \overline{\overline{\widehat{f}}}$. □

Proposition 3.24 *Suppose that $(S, *)$ is a hypergroup (H_v−group) and V is a (homogeneous) vague complex subhypergroup (H_v−subgroup) of S with*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

Describe $supp(\widehat{t})$ and $supp(\widehat{f})$ as follows:

$$supp(\widehat{t}) = \{s \in S : \widehat{t}_V(s) > 0e^{0i}\} \text{ and } supp(\widehat{f}) = \{s \in S : \widehat{f}_V(s) > 0e^{0i}\}.$$

Then, $supp(\widehat{t})$ and $supp(\widehat{f})$ are empty or subhypergroups (H_v−subgroups) of S.

Proof We need to demonstrate $\tau *\ supp(\widehat{t}) = supp(\widehat{t}) = supp(\widehat{t}) * \tau$ for every $\tau \in supp(\widehat{t})$. Let $d \in supp(\widehat{t})$ and $\eta \in \tau * d$. Then, $\widehat{t}_V(\eta) \geq min\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} > 0e^{0i}$ implies that $\widehat{t}_V(\eta) > 0e^{0i}$. Hence, $\eta \in \tau * d \subseteq supp(\widehat{t})$. For every $\tau, d \in supp(\widehat{t})$, there exists $\rho \in S$ such that $d \in \tau * \rho$ and $\widehat{t}_V(\rho) \geq min\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} > 0e^{0i}$. This refers that $\widehat{t}_V(\rho) > 0e^{0i}$ and $\rho \in supp(\widehat{t})$.

Similarly, we want to indicate that $\upsilon *\ supp(\widehat{f}) = supp(\widehat{f}) = supp(\widehat{f}) * \upsilon$ for every $\upsilon \in supp(\widehat{f})$. Let $d \in supp(\widehat{f})$ and $\eta \in \upsilon * d$. Then, $\widehat{f}_V(\eta) \geq min\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} > 0e^{0i}$ implies that $\widehat{f}_V(\eta) > 0e^{0i}$. Hence, $\eta \in \upsilon * d \subseteq supp(\widehat{f})$. For every $\upsilon, d \in supp(\widehat{f})$, there exists $\rho \in S$ such that $d \in \upsilon * \rho$ and $\widehat{f}_V(\rho) \geq min\{\widehat{f}_V(\upsilon), \widehat{f}_V(d) > 0e^{0i}$. This refers $\widehat{f}_V(\rho) > 0e^{0i}$ and $\rho \in supp(\widehat{f})$. □

Remark 3.25 *We defined the complement of the vague complex set in Definition 3.4. Now, we present some examples, where V and V^C are vague complex subhypergroups (This situation generally is not valid).*

Example 3.26 *Assume that $S = \{\gamma, \delta\}$ and describe the hypergroup $(S, *)$ by the next table:*

$*$	γ	δ
γ	γ	S
δ	S	δ

Describe a vague complex subset V of S as follows:

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.3e^{i0} \ \textit{and} \ 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.6e^{i\pi},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.4e^{i\pi} \ \textit{and} \ 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.7e^{i3\pi/2}.$$

We have

$$t_{V^C}(\gamma)e^{i\alpha w_{V^C}^t(\gamma)} = 0.4e^{i\pi} \ \textit{and} \ 1 - f_{V^C}(\gamma)e^{i\alpha(2\pi - w_{V^C}^f(\gamma))} = 0.7e^{i2\pi},$$

$$t_{V^C}(\delta)e^{i\alpha w_{V^C}^t(\delta)} = 0.3e^{i\pi/2} \ \textit{and} \ 1 - f_{V^C}(\delta)e^{i\alpha(2\pi - w_{V^C}^f(\delta))} = 0.6e^{i\pi}.$$

In that case, V and V^C are homogeneous vague complex subhypergroups of S.

Example 3.27 *Suppose that $(S,*)$ is any hypergroup (H_v− group) with the vague complex subset V of S, which is described:*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\},$$

where $t_V(s)$, $f_V(s) \in [0,1]$ such that $0 \leq t_V(s) + f_V(s) \leq 1$ and $w_V^t(s), w_V^f(s) \in [0,1]$ and $\alpha \in [0, 2\pi]$ are fixed real numbers. Therefore, V and V^C are homogeneous vague complex subhypergroups (H_v− subgroups) of S.

Remark 3.28 *Assume that $(S,*)$ is a hypergroup (H_v− group), V is a (homogeneous) vague complex subhypergroup (H_v− subgroup) of S. V^C is not necessarily a vague complex subhypergroup (or H_v− subgroup) of S.*

We will explain this previous situation with the next example.

Example 3.29 *Suppose that $S = \{\gamma, \delta, \epsilon\}$, as a H_v− group $(S,*)$ is defined as:*

$*$	γ	δ	ϵ
γ	γ	$\{\delta, \epsilon\}$	ϵ
δ	$\{\delta, \epsilon\}$	ϵ	γ
ϵ	ϵ	γ	δ

Describe vague complex subset V of S

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.3e^{i\pi} \ \textit{and} \ 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.5e^{i3\pi/2},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = t_V(\epsilon)e^{i\alpha w_V^t(\epsilon)} = 0.2e^{i\pi/2} \ \textit{and} \ 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 1 - f_V(\epsilon)e^{i\alpha(2\pi - w_V^f(\epsilon))} = 0.4e^{i\pi}.$$

We have,

$$\widehat{t_r} = \left\{ \begin{array}{ll} S, & if\ r \leq 0.2e^{i\pi/2} \\ \{\gamma\}, & if\ \ 0.2e^{i\pi/2} < r \leq 0.3e^{i\pi} \\ \varnothing, & otherwise \end{array} \right\} \ and\ \ \widehat{f_t} = \left\{ \begin{array}{ll} S, & if\ t \leq 0.4e^{i\pi} \\ \{\gamma\}, & if\ \ 0.4e^{i\pi} < t \leq 0.5e^{i3\pi/2} \\ \varnothing, & otherwise \end{array} \right\}$$

either empty sets or subhypergroups of S, which refers V is homogeneous vague complex subhypergroup of S. Because

$$0.5e^{i\pi/2} = \widehat{t}_{V^C}(\gamma) = \widehat{t}_{V^C}(\delta * \epsilon) < min\{\widehat{t}_{V^C}(\delta), \widehat{t}_{V^C}(\epsilon)\} = 0.6e^{i\pi},$$

$$0.7e^{i\pi} = \ \widehat{f}_{V^C}(\gamma) = \widehat{f}_{V^C}(\delta * \epsilon) < min\{\widehat{f}_{V^C}(\delta), \widehat{f}_{V^C}(\epsilon)\} = 0.8e^{i3\pi/2}.$$

This is a contradiction. Hence, V^C is not vague complex H_v-subgroup of S.

4. Anti-Vague Complex Subhypergroups

In this part, we will examine anti-vague complex subhypergroups (or H_v-subgroup) along with their properties. Additionally, we will elucidate the transitions between vague complex subhypergroups (or H_v-subgroup) and anti-vague complex subhypergroups (or H_v-subgroup).

Definition 4.1 *Suppose that $(S, *)$ is a hypergroup (H_v-group) and*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$$

is a (homogeneous) vague complex subset of S. If the following assertions are provided, V is called an anti-vague complex subhypergroup (H_v-subgroup) of S:

1. $sup\{\widehat{t}_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{\widehat{t}_V(\gamma), \widehat{t}_V(\delta)\}$ *for every* $\gamma, \delta \in S$,
2. $sup\{\widehat{f}_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq \ max\{\widehat{f}_V(\gamma), \widehat{f}_V(\delta)\}$ *for every* $\gamma, \delta \in S$,
3. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $\widehat{t}_V(\eta) \leq max\{\widehat{t}_V(d), \widehat{t}_V(\gamma)\}$,
4. *For every* $d, \gamma \in S$, *there exists* $\eta \in S$ *such that* $d \in \gamma * \eta$ *and* $\widehat{f}_V(\eta) \leq max\{\widehat{f}_V(d), v_A(\gamma)\}$,
5. *For every* $d, \gamma \in S$, *there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $\widehat{t}_V(\rho) \ \leq max\{\widehat{t}_V(d), \widehat{t}_V(\gamma)\}$,
6. *For every* $d, \gamma \in S$, *there exists* $\rho \in S$ *such that* $d \in \rho * \gamma$ *and* $\widehat{f}_V(\rho) \leq max\{\widehat{f}_V(d), v_A(\gamma)\}$.

Now, we will give some examples of anti-vague complex H_v-subgroups.

Example 4.2 *Assume that $S = \{\gamma, \delta\}$ and characterize the hypergroup as $(S, *)$:*

$*$	γ	δ
γ	γ	S
δ	S	δ

Describe an vague complex subset V of S as follows:

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.3e^{i0} \text{ and } 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.6e^{i\pi},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.4e^{i\pi} \text{ and } 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.7e^{i3\pi/2}.$$

We have

$$t_{V^C}(\gamma)e^{i\alpha w_{V^C}^t(\gamma)} = 0.4e^{i\pi} \text{ and } 1 - f_{V^C}(\gamma)e^{i\alpha(2\pi - w_{V^C}^f(\gamma))} = 0.7e^{i2\pi},$$

$$t_{V^C}(\delta)e^{i\alpha w_{V^C}^t(\delta)} = 0.3e^{i\pi/2} \text{ and } 1 - f_{V^C}(\delta)e^{i\alpha(2\pi - w_{V^C}^f(\delta))} = 0.6e^{i\pi}.$$

Then, V and V^C are homogeneous anti-vague complex subhypergroups of S.

Example 4.3 *Assume that $S = \{\gamma, \delta, \epsilon\}$ and define the hypergroup as $(S, *)$:*

$*$	γ	δ	ϵ
γ	$\{\gamma\}$	$\{\delta\}$	$\{\epsilon\}$
δ	$\{\delta\}$	$\{\delta, \epsilon\}$	S
ϵ	$\{\epsilon\}$	S	$\{\delta, \epsilon\}$

Describe a vague complex subset V of S as:

$$t_V(\gamma)e^{i\alpha w_V^t(\gamma)} = 0.6e^{i2\pi} \text{ and } 1 - f_V(\gamma)e^{i\alpha(2\pi - w_V^f(\gamma))} = 0.8e^{i3\pi/2},$$

$$t_V(\delta)e^{i\alpha w_V^t(\delta)} = 0.4e^{i3\pi/2} \text{ and } 1 - f_V(\delta)e^{i\alpha(2\pi - w_V^f(\delta))} = 0.7e^{i\pi},$$

$$t_V(\epsilon)e^{i\alpha w_V^t(\epsilon)} = 0.3e^{i\pi/2} \text{ and } 1 - f_V(\epsilon)e^{i\alpha(2\pi - w_V^f(\epsilon))} = 0.7e^{i\pi/2}.$$

In that case, V is homogeneous anti-vague complex subhypergroup of S.

Example 4.4 *Suppose that $(S, *)$ is any hypergroup (H_v– group) with the vague complex subset V of S, which is defined as:*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\},$$

where $t_V(s), f_V(s) \in [0,1]$ *such that* $0 \le t_V(s) + f_V(s) \le 1$ *and* $w_V^t(s), w_V^f(s) \in [0,1]$ *and* $\alpha \in [0, 2\pi]$ *are fixed real numbers. Therefore,* V *is a homogeneous anti-vague complex subhypergroup* (H_v-*subgroup) of* S.

Proposition 4.5 *Let* $(S, *)$ *be a hypergroup (or* H_v-*group). Then* V *be an anti-vague subhypergroup (*H_v-*subgroup) of* S *if and only if a* $\pi-$*vague set* V_π *is a* $\pi-$*anti-vague subhypergroup (*H_v-*subgroup) of* S.

Proof Proof is straightforward. □

Theorem 4.6 *Suppose that* $(S, *)$ *is a hypergroup (*H_v-*group) and* V *is a (homogeneous) vague complex subset of* S*. Then* V *is an anti-vague complex subhypergroup (*H_v-*subgroup) of* S *if and only if* t_V *and* $1 - f_V$ *are anti-vague subhypergroups (*H_v-*subgroups) of* S *and* w_V^t *and* $2\pi - w_V^f$ *are* $\pi-$*anti-vague subhypergroup (*H_v-*subgroups) of* S.

Proof $(\Longrightarrow)$: Let V be an anti-vague complex subhypergroup (H_v-subgroup) of S. We need to demonstrate that requirements of description of anti-vague subhypergroups are satisfied for t_V, $1 - f_V$ and w_V^t, $2\pi - w_V^f$. For every $\gamma, \delta \in S$, we have

$$sup\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\} \le max\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\},$$

$$sup\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\} \le max\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\}.$$

Notation 3.8 implies that

$$sup\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \le max\{t_V(\gamma), t_V(\delta)\},$$

$$sup\{1 - f_V(\epsilon) : \epsilon \in \gamma * \delta\} \le max\{1 - f_V(\gamma), 1 - f_V(\delta)\},$$

and

$$sup\{w_V^t(\epsilon) : \epsilon \in \gamma * \delta\} \le max\{w_V^t(\gamma), w_V^t(\delta)\},$$

$$sup\{2\pi - w_V^f(\epsilon) : \epsilon \in \gamma * \delta\} \le max\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Assume that $\gamma, d \in S$. There exist $\eta, \rho \in S$ such that $d \in \gamma * \eta$ and $d \in \rho * \gamma$. Then,

$$max\{\widehat{t_V}(\gamma)\widehat{t_V}(d)\} \ge \widehat{t_V}(\eta),$$

$$max\{\widehat{f_V}(\gamma), \widehat{f_V}(d)\} \ge \widehat{f_V}(\eta)$$

and

$$max\{\widehat{t_V}(\gamma), \widehat{t_V}(d)\} \ge \widehat{t_V}(\rho),$$

$$max\{\widehat{f_V}(\gamma), \widehat{f_V}(d)\} \ge \widehat{f_V}(\rho).$$

Notation 3.8 means that the conditions $3-6$ of description of anti-vague subhypergroup are yielded for both t_V , $1-f_V$ and w_V^t , $2\pi - w_V^f$.

$(\Longleftarrow)$: Suppose that t_V and $1-f_V$ are anti-vague subhypergroups (H_v-subgroups) of S and w_V^t , $2\pi - w_V^f$ are $\pi-$anti-vague subhypergroups (H_v-subgroups) of S. We want to demonstrate that requirements of anti-vague complex subhypergroups are provided. For every $\gamma, \delta \in S$, we have

$$sup\{t_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{t_V(\gamma), t_V(\delta)\},$$

$$sup\{1-f_V(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{1-f_V(\gamma), 1-f_V(\delta)\}$$

and

$$sup\{w_V^t(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{w_V^t(\gamma) w_V^t(\delta)\},$$

$$sup\{2\pi - w_V^f(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(\delta)\}.$$

Notation 3.8 means that

$$sup\{\widehat{t_V}(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{\widehat{t_V}(\gamma), \widehat{t_V}(\delta)\},$$

$$sup\{\widehat{f_V}(\epsilon) : \epsilon \in \gamma * \delta\} \leq max\{\widehat{f_V}(\gamma), \widehat{f_V}(\delta)\}.$$

Suppose that $\gamma, d \in S$. There exist $\eta, \rho \in S$ such that $d \in \gamma * \eta$ and $d \in \rho * \gamma$

$$max\{t_V(\gamma), t_V(d)\} \geq t_V(\eta) \text{ and } max\{w_V^t(\gamma), w_V^t(d)\} \geq w_V^t(\eta),$$

$$max\{1-f_V(\gamma), 1-f_V(d)\} \geq 1-f_V(\eta) \text{ and } max\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(d)\} \geq 2\pi - w_V^f(\eta),$$

and

$$max\{t_V(\gamma), t_V(d)\} \geq t_V(\rho) \text{ and } max\{w_V^t(\gamma), w_V^t(d)\} \geq w_V^t(\rho),$$

$$max\{1-f_V(\gamma), 1-f_V(d)\} \geq 1-f_V(\rho) \text{ and } max\{2\pi - w_V^f(\gamma), 2\pi - w_V^f(d)\} \geq 2\pi - w_V^f(\rho).$$

Notation 3.8 means that the conditions $3-6$ of description of anti-vague complex subhypergroup are yielded for V. □

Lemma 4.7 *Suppose that $(S, *)$ is a hypergroup (H_v-group), V is (homogeneous) an anti-vague complex subhypergroup (or H_v-subgroup) of S. For every $g_1, g_2, \ldots, g_n \in S$,*

$$max\{\widehat{t_V}(g_1), \widehat{t_V}(g_2), \ldots, \widehat{t_V}(g_n)\} \geq sup\{\widehat{t_V}(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$max\{\widehat{f_V}(g_1), \widehat{f_V}(g_2), \ldots, \widehat{f_V}(g_n)\} \geq sup\{\widehat{f_V}(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Proof Let $g_1, g_2, \ldots, g_n \in S$ and $\widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}$. To demonstrate this lemma, it suffices to indicate that

$$max\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \geq sup\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$max\{w_V^t(g_1), w_V^t(g_2), \ldots, w_V^t(g_n)\} \geq sup\{w_V^t(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}$$

and

$$max\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \geq sup\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$max\{2\pi - w_V^f(g_1), 2\pi - w_V^f(g_2), \ldots, 2\pi - w_V^f(g_n)\} \geq sup\{2\pi - w_V^f(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Because V is homogeneous, it is adequate to indicate

$$max\{t_V(g_1), t_V(g_2), \ldots, t_V(g_n)\} \geq sup\{t_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\},$$

$$max\{1 - f_V(g_1), 1 - f_V(g_2), \ldots, 1 - f_V(g_n)\} \geq sup\{1 - f_V(x) : x \in g_1 * (g_2 * (\ldots) * g_n)\}.$$

Theorem 4.6 claims that t_V and $1 - f_V$ are anti-vague subhypergroups (H_v-subgroups) of S. Using Lemma 2.13, the proof is completed. □

Definition 4.8 *Let $V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$ be a (homogeneous) vague complex subset of a non-empty set S. Lower level subset $\overline{V}_{(r,t)}$ of S is described as $\overline{V}_{(r,t)} = \{s \in S : \widehat{t}_V(s) \leq r$ and $\widehat{f}_V(s) \leq t\}$, where $r = me^{i\varphi}, t = ne^{i\psi}$ such that $m, n \in [0,1]$ and $\varphi, \psi \in [0, 2\pi]$.*

Remark 4.9 *Let $V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$ be a (homogeneous) vague complex subset of a non-empty set S. Then followings are true:*

i. If $r_1 \leq r_2$, then $\overline{\widehat{t}_{r_1}} \subseteq \overline{\widehat{t}_{r_2}}$,

ii. $\overline{\widehat{t}}_{1e^{2\pi i}} = S$,

iii. If $t_1 \leq t_2$, then $\overline{\widehat{f}}_{t_1} \subseteq \overline{\widehat{f}}_{t_2}$,

iv. $\overline{\widehat{f}}_{1e^{2\pi i}} = S$.

Theorem 4.10 *Assume that $(S, *)$ is a hypergroup (H_v-group), V is a (homogeneous) vague complex subset of S. Then V is an anti-vague complex subhypergroup (H_v-subgroup) of S if and only if for every $r = ke^{i\psi}$, $t = me^{i\varphi}$ such that $k, m \in [0,1]$ and $\psi, \varphi \in [0, 2\pi]$, $\overline{V}_{(r,t)} \neq \varnothing$ is a subhypergroup (H_v-subgroup) of S.*

Proof The proof is similar to that of Theorem 3.17. □

Corollary 4.11 *Assume that* $(S,*)$ *is a hypergroup* (H_v*−group*)*,* V *is an (homogeneous) anti-vague complex subhypergroup* (H_v*−subgroup*) *of* S *with*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

If $0e^{0i} \le r_1 = s_1e^{i\theta_1} < r_2 = s_2e^{i\theta_2} \le 1e^{2\pi i}$*, then* $\overline{\widehat{t}}_{r_1} = \overline{\widehat{t}}_{r_2}$ *if and only if there is no* $\sigma \in S$ *such that* $r_1 < \widehat{t}_V(\sigma) \le r_2$ *and similarly if* $0e^{0i} \le m_1 = n_1e^{i\varphi_1} < m_2 = n_2e^{i\varphi_2} \le 1e^{2\pi i}$*, then* $\overline{\widehat{f}}_{m_1} = \overline{\widehat{f}}_{m_2}$ *if and only if there is no* $\sigma \in S$ *such that* $m_1 < \widehat{f}_V(\sigma) \le m_2$.

Proof Suppose that $0e^{0i} \le r_1 = m_1e^{i\varphi_1} < r_2 = m_2e^{i\varphi_2} \le 1e^{2\pi i}$ such that $\overline{\widehat{t}}_{r_1} = \overline{\widehat{t}}_{r_2}$. Assume that there exists $\sigma \in S$ such that $r_1 < \widehat{t}_V(\sigma) \le r_2$. Then, $\sigma \in \overline{\widehat{t}}_{r_2} = \overline{\widehat{t}}_{r_1}$. This refers $\widehat{t}_V(\sigma) \le r_1$ and it is a contradiction. Similarly, suppose that $0e^{0i} \le m_1 = n_1e^{i\varphi_1} < m_2 = n_2e^{i\varphi_2} \le 1e^{2\pi i}$, such that $\overline{\widehat{f}}_{m_1} = \overline{\widehat{f}}_{m_2}$. Assume that there exists $\sigma \in S$ such that $m_1 < \widehat{f}_V(\sigma) \le m_2$. Then, $\sigma \in \overline{\widehat{f}}_{m_2} = \overline{\widehat{f}}_{m_1}$. This refers that $\widehat{f}_V(\sigma) \le m_1$ and it is a contradiction.

Because of $0e^{0i} \le r_1 = m_1e^{i\varphi_1} < r_2 = m_2e^{i\varphi_2} \le 1e^{2\pi i}$, it follows by Remark 4.9 $\overline{\widehat{t}}_{r_1} \subseteq \overline{\widehat{t}}_{r_2}$. In order to indicate $\overline{\widehat{t}}_{r_2} \subseteq \overline{\widehat{t}}_{r_1}$, suppose that $\sigma \in \overline{\widehat{t}}_{r_2}$. Then, $\widehat{t}_V(\sigma) \le r_2$. Because there is no $\sigma \in S$ such that $r_1 < \widehat{t}_V(\sigma) \le r_2$, it follows $\widehat{t}_V(\sigma) \le r_1$. Hence, $\sigma \in \overline{\widehat{t}}_{r_1}$ and $\overline{\widehat{t}}_{r_2} \subseteq \overline{\widehat{t}}_{r_1}$. Similarly, since $0e^{0i} \le m_1 = n_1e^{i\varphi_1} < m_2 = n_2e^{i\varphi_2} \le 1e^{2\pi i}$, it follows by Remark 4.9, $\overline{\widehat{f}}_{m_1} \subseteq \overline{\widehat{f}}_{m_2}$. In order to show $\overline{\widehat{f}}_{m_2} \subseteq \overline{\widehat{f}}_{m_1}$, suppose that $\sigma \in \overline{\widehat{f}}_{m_2}$. Then, $\widehat{f}_V(\sigma) \le m_2$. Because there is no $\sigma \in S$ such that $m_1 < \widehat{f}_V(\sigma) \le m_2$, it follows that $\widehat{f}_V(\sigma) \le m_1$. Hence, $\sigma \in \overline{\widehat{f}}_{m_1}$ and $\overline{\widehat{f}}_{m_2} \subseteq \overline{\widehat{f}}_{m_1}$. □

Corollary 4.12 *Assume that* $(S,*)$ *is a hypergroup* (H_v*−group*)*,* V *is an (homogeneous) anti-vague complex subhypergroup* (H_v*−subgroup*) *of* S *with*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(x)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}.$$

If the range of $\widehat{t}_V$ *is finite set* $\{r_1, r_2, \ldots, r_n\}$ *and* $\widehat{f}_V$ *is finite set* $\{s_1, s_2, \ldots, s_n\}$*, then the sets* $\{\overline{\widehat{t}}_{r_i} : i = 1, 2, \ldots, n\}$ *and* $\{\overline{\widehat{f}}_{s_i} : i = 1, 2, \ldots, n\}$ *comprises whole the lower level subhypergroups* (H_v*−subgroups*) *of S. Besides of this, if* $r_1 \le r_2 \le \ldots \le r_n$*, then whole the lower level subhypergroups of* S *create the chain* $\overline{\widehat{t}}_{r_1} \subseteq \overline{\widehat{t}}_{r_2} \subseteq \ldots \subseteq \overline{\widehat{t}}_{r_n}$*. Similarly, if* $s_1 \le s_2 \le \ldots \le s_n$*, then whole the lower level subhypergroups of* S *create the chain* $\overline{\widehat{f}}_{s_1} \subseteq \overline{\widehat{f}}_{s_2} \subseteq \ldots \subseteq \overline{\widehat{f}}_{s_n}$.

Proof Assume that $\bar{\bar{t}}_m, \bar{\bar{f}}_m \neq \emptyset$ are lower level subhypergroups (H_v-subgroups) of S such that $\bar{\bar{t}}_m \neq \bar{\bar{t}}_{r_i}$ and $\bar{\bar{f}}_m \neq \bar{\bar{f}}_{s_i}$ for all $1 \leq i \leq n$. Suppose that r_q and s_q are closest complex numbers to m. There are two cases : $m < r_q$ and $m < s_q$, $m > r_q$ and $m > s_q$. We think the first case, the second case is like that of the first case. Because the ranges of $\widehat{t}_V$ and $\widehat{f}_V$ are finite sets $\{r_1, r_2, \ldots, r_n\}$ and $\{s_1, s_2, \ldots, s_n\}$ respectively, it follows that there is no $\epsilon \in S$ such that $m < \widehat{t}_V(\epsilon) \leq r_q$ and $m < \widehat{f}_V(\epsilon) \leq s_q$. Using by Corollary 4.11, we have a contradiction. □

Proposition 4.13 *Assume that $(S, *)$ is the biset hypergroup like $\gamma * \delta = \{\gamma, \delta\}$ for every $\gamma, \delta \in S$ and V is any homogeneous vague complex subset of S. Then V is an anti-vague complex subhypergroup of S.*

Proof The proof is similar to that of Proposition 3.20. □

Proposition 4.14 *Assume that $(S, *)$ is the total hypergroup like $\gamma * \delta = S$ for every $\gamma, \delta \in S$ and V is any homogeneous vague complex subset of S. Then V is an anti-vague complex subhypergroup of S if and only if $\widehat{t}_V$ and $\widehat{f}_V$ are stable complex functions.*

Proof The proof is similar to that of Proposition 3.21. □

Proposition 4.15 *Assume that $(S, *)$ is a hypergroup (H_v-group), V is a (homogeneous) vague complex fuzzy subset of S. Then V is an anti-vague complex subhypergroup (H_v-subgroup) of S if and only if for every $r = ke^{i\psi}$ and $t = me^{i\varphi}$ such that $k, m \in [0, 1]$ and $\psi, \varphi \in [0, 2\pi]$, the following properties are provided:*

i. $\overline{V}_{(r,t)} * \overline{V}_{(r,t)} \subseteq \overline{V}_{(r,t)}$,

ii. $\upsilon * (S - \overline{V}_{(r,t)}) - (S - \overline{V}_{(r,t)}) \subseteq \upsilon * \overline{V}_{(r,t)}$, *for every* $\upsilon \in \overline{V}_{(r,t)}$,

iii. $(S - \overline{V}_{(r,t)}) * \upsilon - (S - \overline{V}_{(r,t)}) \subseteq \overline{V}_{(r,t)} * \upsilon$, *for every* $\upsilon \in \overline{V}_{(r,t)}$.

Proof Suppose that V is an anti-vague complex subhypergroup (H_v-subgroup) of S. Then by Theorem 4.10, $\overline{V}_{(r,t)}$ is a subhypergroup (H_v-subgroup) of S as $\upsilon * \overline{V}_{(r,t)} = \overline{V}_{(r,t)}$ for every $\upsilon \in \overline{V}_{(r,t)}$. Hence, $\overline{V}_{(r,t)} * \overline{V}_{(r,t)} \subseteq \overline{V}_{(r,t)}$. We need to indicate $\upsilon * (S - \overline{V}_{(r,t)}) - (S - \overline{V}_{(r,t)}) \subseteq \upsilon * \overline{V}_{(r,t)}$. Suppose that $\eta \in \upsilon * (S - \overline{V}_{(r,t)}) - (S - \overline{V}_{(r,t)})$. We have η is not an element in $(S - \overline{V}_{(r,t)})$. This refers $\eta \in \overline{V}_{(r,t)} = \upsilon * \overline{V}_{(r,t)}$. Similarly, condition iii may be proved.

Conversely, assume that the conditions i, ii are yielded. Using Theorem 4.10, it is adequate to demonstrate $\overline{V}_{(r,t)}$ is a subhypergroup (H_v-subgroup) of S like $\upsilon * \overline{V}_{(r,t)} = \overline{V}_{(r,t)} * \upsilon = \overline{V}_{(r,t)}$

for every $\upsilon \in \overline{V}_{(r,t)}$. Suppose that there exists $\gamma \in \overline{V}_{(r,t)}$ such that γ is not an element in $\upsilon * \overline{V}_{(r,t)}$. Using reproduction properties of $(S,*)$, there exists $\delta \in S$ such that $\gamma \in \upsilon * \delta$. Two situations are here for δ :

Case 1: $\delta \in \overline{V}_{(r,t)}$. We have $\gamma \in \upsilon * \delta \subseteq \upsilon * \overline{V}_{(r,t)}$. This is a contradiction.

Case 2 : $\delta \notin \overline{V}_{(r,t)}$. We have $\delta \in (S - \overline{V}_{(r,t)})$. $\gamma \in \upsilon * \delta$ means $\gamma \in \upsilon * (S - \overline{V}_{(r,t)})$. Because $\gamma \in \overline{V}_{(r,t)}$, it follows that γ is not in $(S - \overline{V}_{(r,t)})$. Hence, using by assumption $\gamma \in \upsilon * (S - \overline{V}_{(r,t)}) - (S - \overline{V}_{(r,t)}) \subseteq \upsilon * \overline{V}_{(r,t)}$. This is a contradiction. Similarly, using condition *iii*, we can prove $\overline{V}_{(r,t)} * \upsilon = \overline{V}_{(r,t)}$. □

Proposition 4.16 *Assume that* $(S,*)$ *is a hypergroup (*H_v*– group),* V *is an (homogeneous) anti-vague complex subhypergroup (*H_v*– subgroup) of* S *with*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\},$$

Describe $\overline{\overline{\widehat{t}}}$ *and* $\overline{\overline{\widehat{f}}}$ *as follows:*

$$\overline{\overline{\widehat{t}}} = \{s \in S : \widehat{t}_V(s) = 0e^{0i}\} \text{ and } \overline{\overline{\widehat{f}}} = \{s \in S : \widehat{f}_V(s) = 0e^{0i}\},$$

Then, $\overline{\overline{\widehat{t}}}$ *and* $\overline{\overline{\widehat{f}}}$ *are empty or subhypergroups (*H_v*– subgroups) of* S.

Proof We need to demonstrate that $\tau * \overline{\overline{\widehat{t}}} = \overline{\overline{\widehat{t}}} = \overline{\overline{\widehat{t}}} * \tau$ for every $\tau \in \overline{\overline{\widehat{t}}}$. Let $d \in \overline{\overline{\widehat{t}}}$ and $\eta \in \tau * d$. Then, $\widehat{t}_V(\eta) \leq \max\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} = 0e^{0i}$ implies that $\widehat{t}_V(\eta) = 0e^{0i}$. Hence, $\eta \in \tau * d \subseteq \overline{\overline{\widehat{t}}}$. For every $\tau, d \in \overline{\overline{\widehat{t}}}$, there exists $\rho \in S$ such that $d \in \tau * \rho$ and $\widehat{t}_V(\rho) \leq \max\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} = 0e^{0i}$. This refers that $\widehat{t}_V(\rho) = 0e^{0i}$ and $\rho \in \overline{\overline{\widehat{t}}}$.

Similarly, we want to indicate that $\upsilon * \overline{\overline{\widehat{f}}} = \overline{\overline{\widehat{f}}} = \overline{\overline{\widehat{f}}} * \upsilon$ for every $\upsilon \in \overline{\overline{\widehat{f}}}$. Let $d \in \overline{\overline{\widehat{f}}}$ and $\eta \in \upsilon * d$. Then, $\widehat{f}_V(\eta) \leq \max\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} = 0e^{0i}$ implies that $\widehat{f}_V(\eta) = 0e^{0i}$. Hence, $\eta \in \upsilon * d \subseteq \overline{\overline{\widehat{f}}}$. For every $\upsilon, d \in \overline{\overline{\widehat{f}}}$, there exists $\rho \in S$ such that $d \in \upsilon * \rho$ and $\widehat{f}_V(\rho) \leq \max\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} = 0e^{0i}$. This refers that $\widehat{f}_V(\rho) = 0e^{0i}$ and $\rho \in \overline{\overline{\widehat{f}}}$. □

Proposition 4.17 *Assume that* $(S,*)$ *is a hypergroup (*H_v*– group),* V *is an (homogeneous) anti-vague complex subhypergroup (*H_v*– subgroup) of* S *with*

$$V = \{(s, \widehat{t}_V(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f}_V(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\}$$

Describe $\overline{supp}(\widehat{t})$ and $\overline{supp}(\widehat{f})$ as:

$$\overline{supp}(\widehat{t}) = \{s \in S : \widehat{t}_V(s) < 1e^{2\pi i}\} \text{ and } \overline{supp}(\widehat{f}) = \{s \in S : \widehat{f}_V(s) < 1e^{2\pi i}\}$$

Then, $\overline{supp}(\widehat{t})$ and $\overline{supp}(\widehat{f})$ are empty or subhypergroups (H_v-subgroups) of S.

Proof We need to demonstrate that $\tau * \overline{supp}(\widehat{t}) = \overline{supp}(\widehat{t}) = \overline{supp}(\widehat{t}) * \tau$ for every $\tau \in \overline{supp}(\widehat{t})$. Let $d \in \overline{supp}(\widehat{t})$ and $\eta \in \tau * d$. Then, $\widehat{t}_V(\eta) \leq \max\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} < 1e^{2\pi i}$ implies that $\widehat{t}_V(\eta) < 1e^{2\pi i}$. Hence, $\eta \in \tau * d \subseteq \overline{supp}(\widehat{t})$. For every $\tau, d \in \overline{supp}(\widehat{t})$, there exists $\rho \in S$ such that $d \in \tau * \rho$ and $\widehat{t}_V(\rho) \leq \max\{\widehat{t}_V(\tau), \widehat{t}_V(d)\} < 1e^{2\pi i}$. This refers that $\widehat{t}_V(\rho) < 1e^{2\pi i}$ and $\rho \in \overline{supp}(\widehat{t})$.

Similarly, we want to demonstrate that $\upsilon * \overline{supp}(\widehat{f}) = \overline{supp}(\widehat{f}) = \overline{supp}(\widehat{f}) * \upsilon$ for every $\upsilon \in \overline{supp}(\widehat{f})$. Let $d \in \overline{supp}(\widehat{f})$ and $\eta \in \upsilon * d$. Then, $\widehat{f}_V(\eta) \leq \max\{\widehat{f}_V(\upsilon), \widehat{f}_V(d)\} < 1e^{2\pi i}$ implies that $\widehat{f}_V(\eta) < 1e^{2\pi i}$. Hence, $\eta \in \upsilon * d \subseteq \overline{supp}(\widehat{f})$. For every $\upsilon, d \in \overline{supp}(\widehat{f})$, there exists $\rho \in S$ such that $d \in \upsilon * \rho$ and $\widehat{f}_V(\rho) \leq \max\{\widehat{f}_V(\upsilon), \widehat{f}_V(d) < 1e^{2\pi i}$. This refers that $\widehat{f}_V(\rho) < 1e^{2\pi i}$ and $\rho \in \overline{supp}(\widehat{f})$.

□

Theorem 4.18 *Assume that $(S, *)$ is a hypergroup (H_v-group), V is a homogeneous vague complex subset of S. Then V is a vague complex subhypergroup (H_v-subgroup) of S if and only if V^C is an anti-vague complex subhypergroup (H_v-subgroup) of S.*

Proof $(\Longrightarrow)$: Let V be a vague complex subhypergroup (H_v-subgroup) of S. Using Theorem 3.12, we have that $t_V, 1 - f_V$ are vague subhypergroups (H_v-subgroups) of S and $w^t_V, 2\pi - w^f_V$ are $\pi-$vague subhypergroups (H_v-subgroups) of S. Using Theorem 2.14, $t_{V^C}, 1 - f_{V^C}$ are anti-vague subhypergroups (H_v-subgroups) of S and $w^t_{V^C}, 2\pi - w^f_{V^C}$ are $\pi-$anti-vague subhypergroups (or H_v-subgroups) of S. From Theorem 4.6, V^C is an anti-vague complex subhypergroup (H_v-subgroup) of S.

$(\Longleftarrow)$: The proof is similar to the previous part. □

Corollary 4.19 *Assume that $(S, *)$ is a hypergroup (H_v-group), V is a homogeneous vague complex subset of S. Then V is a vague complex and anti-vague complex subhypergroup (H_v-subgroup) of S if and only if V^C is a vague complex and anti-vague complex subhypergroup (H_v-subgroup) of S.*

Proof The proof is clear from Theorem 4.18. □

Example 4.20 *Assume that $(S, *)$ is the biset hypergroup and V is any homogeneous vague* *complex subset of* S. *Using Proposition 3.20 and 4.13,* V *and* V^C *are vague complex and anti-vague complex subhypergroups of* S.

Example 4.21 *Assume that* $(S,*)$ *is any hypergroup* (H_v*−group) with the vague complex subset* V *of* S*, which is described:*

$$V = \{(s, \widehat{t_V}(s) = t_V(s)e^{i\alpha w_V^t(s)}, \widehat{f_V}(s) = 1 - f_V(s)e^{i\alpha(2\pi - w_V^f(s))}) : s \in S\},$$

where $t_V(s)$, $f_V(s) \in [0,1]$ *such that* $0 \le t_V(s) + f_V(s) \le 1$ *and* $w_V^t(s), w_V^f(s) \in [0,1]$ *and* $\alpha \in [0, 2\pi]$ *are fixed real numbers. Therefore,* V *and* V^C *are homogeneous vague complex and anti-vague complex subhypergroups* (H_v*−subgroups) of* S.

5. Conclusion

This study has examined the concepts of vague complex and anti-vague complex subhypergroups (H_v−subgroups) using the framework of vague complex sets and hyperstructures. With this work, subhyperstructures were studied from a broader perspective with the help of vague complex sets. Various examples were provided to illustrate these concepts and their properties. We have also explored the relationships between vague and anti-vague subypergroups, establishing key distinctions and connections.

The findings of this research contribute to a deeper understanding of the structure and behavior of H_v−subgroups under vague and anti-vague conditions. However, several open problems remain for further investigation. One such problem is the study of neutrosophic complex H_v−subgroups, which could provide a more generalized and nuanced framework for analyzing uncertainties and indeterminacies in mathematical structures. Researchers are encouraged to explore these areas to broaden the scope and applicability of hyperstructure theory.

Acknowledgements

The third section of this study were originally presented at the *15th Algebraic Hyperstructures and Its Applications Conference - AHA 2023.* This paper is an advanced version of the presentation.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Sanem Yavuz]: Collected the data, contributed to completing the research and solving the problem, wrote the manuscript (%40).

Author [Serkan Onar]: Contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%35).
Author [Bayram Ali Ersoy]: Thought and designed the research/problem, contributed to research method or evaluation of data (%25).

Conflicts of Interest
The authors declare no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1387316
(**2024**) **5**(2) 123 – 133
Research Article

On the Boundary Functional of a Semi-Markov Process

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Received: 07 November 2023 **Accepted:** 09 March 2024

Abstract: In this paper, we consider the semi-Markov random walk process with negative drift, positive jumps. An integral equation for the Laplace transform of the conditional distribution of the boundary functional is obtained. In this work, we define the residence time of the system by generalized exponential distributions with different parameters via fractional order integral equation. The purpose of this paper is to reduce an integral equation for the Laplace transform of the conditional distribution of a boundary functional of the semi-Markov random walk processes to fractional order differential equation with constant coefficients.

Keywords: Laplace transform, random variable, semi-Markov random walk process, Riemann-Liouville fractional derivative.

1. Introduction

A semi-Markov processes are investigated in different directions. In recent years, a semi-Markov random walk with one or two barriers are being used to solve a number of very interesting problems in the fields of inventory, queues and reliability theories, mathematical biology etc. It is well known that the semi-Markov processes have been introduced by Levy [14], Smith [22] and Takàcs [23] in order to reduce the limitation induced by the exponential distribution of the corresponding time intervals. This is the immediate generalization of Markov chains since the Markov property is the typical consequence of the lack of memory of the exponential distribution. The semi-Markov process is constructed by the so-called Markov renewal process. The Markov renewal process is defined by the transition probabilities matrix, called the renewal kernel, and by an initial distribution. The essential developments of semi-Markov processes theory were proposed by Pyke [21], Feller [7], Cinlar [6], Gikhman and Skorokhod [9], Limnios and Oprisan [15] and Grabski [8]. In the work of Abdel-Rehim, Hassan and El-Sayed [3] a simulation of the short and long memory of ergodic Markov and non-Markov genetic diffusion processes on the long run was

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2020 *AMS Mathematics Subject Classification*: 60K15, 60G50, 34A08

investigated. Using asymptotic methods and factorization methods similar problems were studied by Lebowitz and Percus [13] and Lotov and Orlova [16]. In many cases asymptotic analysis of the factorization representations of dual transforms leads to the complete asymptotic expansions of the distributions under consideration [4]. But in particular case of semi-Markov random walk process we can obtain the explicit form for probability characteristics. The authors of [10, 12, 19, 20] have found the Laplace transform of the distribution of the first moment reaching level zero of the semi-Markov random walk processes. It should be noted that finding the Laplace transform of the semi-Markov random walk processes is a powerful tool in applied mathematics and engineering. It is well known the connection between semigroup theory and the Markov processes. In the semigroup theory of Markov processes, a particular process is usually represented as a semigroup of contraction operators in some concrete Banach space, and the properties of the particular process are deduced from the properties of the associated semigroup of operators. From this point of view, by Atangana in [1] it was shown that the Atangana Baleanu fractional derivative possesses the Markovian and non-Markovian properties. We also refer to [2] for more results on fractional modeling of probabilistic processes. We recall that, in [5] the authors studied a stepwise semi-Markov processes. Then authors used the fractional Riemann-Liouville derivative. Moreover, the obtained solution for the fractional differential equation was in the form of a threefold sum. But in the presented paper, we obtained a mathematical model of a semi-Markov process with negative drift, positive jumps for a certain general class of probability distributions, and in the class of gamma distributions we managed to reduce the study of a mathematical model through a fractional differential equation with a fractional Riemann-Liouville derivative. In conclusion, we were able to find solution for the fractional differential equation.

In this paper, jump processes with a waiting time between jumps that is not necessarily given by an exponential random variable is consider. In the present paper, we study the semi-Markov random walk process with negative drift, positive jumps and delaying barrier.

An integral equation for the Laplace transform of the conditional distribution of the boundary functional is constructed. In particular, constructed integral equation is reduce to the fractional order differential equation in the class of gamma distributions. Finally, solution of the fractional order differential equation is obtained. The paper is organized in five section. In Section 2, we introduce analytic expression of a stochastic process and some notation. Section 3 is devoted to construct an integral equation for the Laplace transform of the conditional distribution of the boundary functional, also it is shown that the obtained integral equation is reduce a fractional differential equation in the class of gamma distributions. The main result is obtained in Section 4. Finally, the conclusion is given in Section 5.

2. Problem Statement and Preliminaries

Let's assume that sequences of independent and identically distributed pairs of random variables $\{\xi_k, \zeta_k\}_{k=1}^{\infty}$,be given on the any probability space (Ω, F, P), where the random variables ξ_k and ζ_k are independent, positive. Now, we can construct the stochastic process $X_1(t)$ as follows

$$X_1(t) = z - t + \sum_{i=0}^{k-1} \zeta_i, \text{ if } \sum_{i=0}^{k-1} \xi_i \le t < \sum_{i=0}^{k} \xi_i,$$

where $\xi_0 = \zeta_0 = 0$. This stochastic model is called "a semi-Markov random walk process with negative drift, positive jumps". Let this process is delayed by a barrier zero:

$$X(t) = X_1(t) - \inf_{0 \le s \le t} \{0, X_1(s)\}.$$

Now, we introduce the following random variable

$$\tau_0 = \inf\{t : X(t) = 0\}.$$

We set $\tau_0 = \infty$ if $X(t) > 0$ for every t. Notice that the random variable τ_0 is the time of the first crossing of the process $X(t)$ into the delaying barrier at zero level. τ_0 is called boundary functional of the semi-Markov random walk process with negative drift, positive jumps.

The aim of this study is to find the Laplace transform of the conditional distribution of the random variable τ_0. Laplace transform of the conditional distribution of the random variable τ_0 by

$$L(\theta|z) = E\left[e^{-\theta\tau_0} | X(0) = z\right], \quad \theta > 0, \ z \ge 0.$$

Let us denote the conditional distribution of random variable of τ_0 and the Laplace transform of the conditional distribution are defined by

$$N(t|z) = P\left[\tau_0 > t | X(0) = z\right],$$

and

$$\tilde{N}(\theta|z) = \int_0^{\infty} e^{-\theta t} N(t|z) dt,$$

respectively.

Thus, it is easy to see that

$$\tilde{N}(\theta|z) = \frac{1 - L(\theta|z)}{\theta}$$

or, equivalently,

$$L(\theta|z) = 1 - \theta\tilde{N}(\theta|z).$$

3. The Construction of an Integral Equation for the $\tilde{N}(\theta|z)$ and Reduction to the Fractional Order Differential Equation

Theorem 3.1 *Function $\tilde{N}(\theta|z)$ satisfy the following integral equation:*

$$\tilde{N}(\theta|z) = \int_0^z e^{-\theta t} P\{\xi_1 > t\}\, dt + \int_z^\infty \tilde{N}(\theta|y) \int_0^z e^{-\theta t} d_t P\{\xi_1 < t\}\, d_y P\{\zeta_1 < y - z + t\}$$

$$+ \int_0^z \tilde{N}(\theta|y) \int_{z-y}^z e^{-\theta t} d_t P\{\xi_1 < t\}\, d_y P\{\zeta_1 < y - z + t\}. \tag{1}$$

Proof Using by the law of total probability, we can get

$$N(t|z) = P\{\tau_0 > t;\ \xi_1 > t\,|X(0) = z\} + P\{\tau_0 > t;\ \xi_1 < t\,|X(0) = z\}$$

$$= P\{z - t > 0; \xi_1 > t\} + \int_0^t \int_0^\infty P\{\xi_1 \in ds;\ z - s > 0; z - s + \zeta_1 \in dy\} P\{\tau_0 > t - s|X(0) = z\}.$$

Then

$$N(t,\ |z) = P\{z - t > 0\}\ P\{\xi_1 > t\}$$

$$+ \int_0^t \int_0^\infty P\{\xi_1 \in ds; z - s > 0; z - s + \zeta_1 \in dy\}\ N(t - s\,|y). \tag{2}$$

By applying the Laplace transform with respect to t both sides of the equation (2) we get:

$$\tilde{N}(\theta|z) = \int_0^\infty e^{-\theta t} P\{z - t > 0; \xi_1 > t\}\, dt$$

$$+ \int_0^\infty e^{-\theta t} \int_0^t \int_0^\infty P\{\xi_1 \in ds; z - s > 0; z - s + \zeta_1 \in dy\}\ N(t - s\,|y),\ \ \theta > 0$$

or

$$\tilde{N}(\theta|z) = \int_0^z e^{-\theta t} P\{\xi_1 > t\}\, dt$$

$$+ \int_0^\infty \tilde{N}(\theta|y) \int_0^\infty e^{-\theta t} P\{\xi_1 \in dt;\ z - t > 0\}\, d_y P\{\zeta_1 < y - z + t\},\ \ \theta > 0.$$

The following equation can be easily obtained from the last equation:

$$\tilde{N}(\theta|z) = \int_0^z e^{-\theta t} P\{\xi_1 > t\}\, dt$$

$$+ \int_0^\infty \tilde{N}(\theta|y) \int_0^z e^{-\theta t} d_t P\{\xi_1 < t\}\, d_y P\{\zeta_1 < y - z + t\}.$$

It is clear that it should be taken $y - z + t > 0$. From this condition, it follows that $t > max(0, z - y)$. Then the last equation can be rewritten as follows

$$\tilde{N}(\theta|z) = \int_0^z e^{-\theta t} P\{\xi_1 > t\}\, dt$$

$$+\int_0^\infty \tilde{N}(\theta|y) \int_{\max(0,z-y)}^z e^{-\theta t} d_t P\{\xi_1 < t\}\, d_y P\{\zeta_1 < y - z + t\}\,. \tag{3}$$

Finally, from integral equation (3), (1) is obtained. □

This completes the proof.

Suppose that the distribution of the random variable ξ_1 has the density function $p_{\xi_1}(s)$, $s > 0$ and the distribution of the random variable ζ_1 has the density function $p_{\zeta_1}(s)$, $s > 0$. Then equation (1) has the form

$$\tilde{N}(\theta|z) = \frac{1}{\theta}\left[1 - e^{-\theta z}\right] + \frac{1}{\theta} e^{-\theta z} P\{\xi_1 < z\} - \frac{1}{\theta}\int_0^z e^{-\theta t} p_{\xi_1}(t) dt$$

$$+\int_z^\infty \tilde{N}(\theta|y) \int_0^z e^{-\theta t} p_{\xi_1}(t) p_{\zeta_1}(y - z + t)\, dtdy$$

$$+\int_0^z \tilde{N}(\theta|y) \int_{z-y}^z e^{-\theta t} p_{\xi_1}(t) p_{\zeta_1}(y - z + t)\, dtdy\,. \tag{4}$$

Let's assume that random variable ξ_1 has the gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, while random variable ζ_1 has Erlang distribution of first order with the parameters μ:

$$p_{\xi_1}(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x > 0 \\ 0, & x \le 0, \end{cases} \qquad \rho_{\zeta_1}(x) = \begin{cases} \mu e^{-\mu x}, & x \ge 0 \\ 0, & x < 0. \end{cases}$$

In the class of these distributions the integral equation (4) can be written as follows

$$\tilde{N}(\theta|z) = \frac{1}{\theta}\left[1 - e^{-\theta z}\right] + \frac{\beta^\alpha e^{-\theta z}}{\theta\Gamma(\alpha)} \int_0^z e^{-\beta y} y^{\alpha-1} dy - \frac{\beta^\alpha}{\theta\Gamma(\alpha)} \int_0^z e^{-(\theta+\beta)t} t^{\alpha-1} dt$$

$$+\frac{\mu\beta^\alpha e^{\mu z}}{\Gamma(\alpha)} \int_z^\infty e^{-\mu y} \tilde{N}(\theta|y) \int_0^z e^{-(\mu+\theta+\beta)t} t^{\alpha-1}\, dtdy$$

$$+\frac{\mu\beta^\alpha e^{\mu z}}{\Gamma(\alpha)} \int_0^z e^{-\mu y} \tilde{N}(\theta|y) \int_{z-y}^z e^{-(\mu+\theta+\beta)t} t^{\alpha-1}\, dtdy\,. \tag{5}$$

Multiplying both sides of equation (5) by $e^{-\mu z}$ and differentiating both sides with respect to z, we obtain

$$e^{-\mu z}\tilde{N}'(\theta|z) - \mu e^{-\mu z}\tilde{N}(\theta|z) = \frac{1}{\theta}\left[(\mu+\theta)e^{-(\mu+\theta)z} - \mu e^{-\mu z}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)e^{-(\mu+\theta)z}}{\theta\Gamma(\alpha)}\int_0^z e^{-\beta y}y^{\alpha-1}dy+\frac{\mu\beta^{\alpha}e^{-\mu z}}{\theta\Gamma(\alpha)}\int_0^z e^{-(\theta+\beta)t}t^{\alpha-1}dt$$

$$+\frac{\mu\beta^{\alpha}e^{-(\mu+\theta+\beta)z}}{\Gamma(\alpha)}z^{\alpha-1}\int_0^{\infty}e^{-\mu y}\tilde{N}\left(\theta\,|\,y\right)dy$$

$$-\frac{\mu\beta^{\alpha}e^{-(\mu+\theta+\beta)z}}{\Gamma(\alpha)}\int_0^z e^{(\theta+\beta)y}\tilde{N}\left(\theta\,|\,y\right)(z-y)^{\alpha-1}dy. \quad (6)$$

It is easy to see that, $\int\limits_0^z e^{-\beta y}y^{\alpha-1}dy=e^{-\beta z}z^{\alpha}\sum\limits_{n=0}^{\infty}\frac{\beta^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}$.

Multiplying both sides of last equation (6) by $e^{-(\mu+\theta+\beta)z}$, we obtain

$$e^{(\theta+\beta)z}\tilde{N}'\left(\theta\,|\,z\right)-\mu e^{(\theta+\beta)z}\tilde{N}\left(\theta\,|\,z\right)=\frac{1}{\theta}\left[(\mu+\theta)e^{\beta z}-\mu e^{(\theta+\beta)z}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta\Gamma(\alpha)}z^{\alpha}\sum_{n=0}^{\infty}\frac{\beta^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}+\frac{\mu\beta^{\alpha}}{\theta\Gamma(\alpha)}z^{\alpha}\sum_{n=0}^{\infty}\frac{(\theta+\beta)^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}$$

$$+\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}z^{\alpha-1}\int\limits_0^{\infty}e^{-\mu y}\tilde{N}\left(\theta\,|\,y\right)dy$$

$$-\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}\int\limits_0^z e^{(\theta+\beta)y}\tilde{N}\left(\theta\,|\,y\right)(z-y)^{\alpha-1}dy. \quad (7)$$

We denote

$$Q(\theta\,|z)=e^{(\theta+\beta)z}\tilde{N}\left(\theta\,|\,z\right). \quad (8)$$

Then equation (7) can be rewritten as follows

$$Q'\left(\theta\,|\,z\right)-(\mu+\theta+\beta)Q\left(\theta\,|\,z\right)=\frac{1}{\theta}\left[(\mu+\theta)e^{\beta z}-\mu e^{(\theta+\beta)z}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta\Gamma(\alpha)}z^{\alpha}\sum_{n=0}^{\infty}\frac{\beta^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}+\frac{\mu\beta^{\alpha}}{\theta\Gamma(\alpha)}z^{\alpha}\sum_{n=0}^{\infty}\frac{(\theta+\beta)^n z^n}{\alpha(\alpha+1)\ldots(\alpha+n)}$$

$$+\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}z^{\alpha-1}\int\limits_0^{\infty}e^{-\mu y}\tilde{N}\left(\theta\,|\,y\right)dy-\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}\int_0^z Q\left(\theta\,|\,y\right)(z-y)^{\alpha-1}dy. \quad (9)$$

It is known that the Riemann-Liouville integral can be expressed by (see, [17, 18])

$$D_z^{-\alpha}(Q\left(\theta\,|z\right))=\frac{1}{\Gamma(\alpha)}\int_0^z Q(\theta\,|y)(z-y)^{\alpha-1}dy,\ \ 0<\alpha\le 1.$$

Taking into account the last equality in (9), we obtain

$$Q'\left(\theta\,|\,z\right)-(\mu+\theta+\beta)Q\left(\theta\,|\,z\right)=\frac{1}{\theta}\left[(\mu+\theta)e^{\beta z}-\mu e^{(\theta+\beta)z}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\beta^{n}z^{n+\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)}+\frac{\mu\beta^{\alpha}}{\theta\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\theta+\beta)^{n}z^{n+\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)}$$

$$+\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}z^{\alpha-1}\int_{0}^{\infty}e^{-\mu y}\tilde{N}(\theta|y)\,dy-\mu\beta^{\alpha}D_{z}^{-\alpha}Q(\theta|z). \tag{10}$$

By applying Riemann-Liouville fractional derivative of order α to both sides equation (10), we obtain

$$D_{z}^{\alpha+1}Q(\theta|z)-(\mu+\theta+\beta)D_{z}^{\alpha}Q(\theta|z)+\mu\beta^{\alpha}Q(\theta|z)$$

$$=\frac{1}{\theta}\left[(\mu+\theta)D_{z}^{\alpha}e^{\beta z}-\mu D_{z}^{\alpha}e^{(\theta+\beta)z}\right]-\frac{\beta^{\alpha}(\mu+\theta)}{\theta\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\beta^{n}D_{z}^{\alpha}z^{n+\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)}$$

$$+\frac{\mu\beta^{\alpha}}{\Gamma(\alpha)}D_{z}^{\alpha}z^{\alpha-1}\int_{0}^{\infty}e^{-\mu y}\tilde{N}(\theta|y)\,dy+\frac{\mu\beta^{\alpha}}{\theta\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\theta+\beta)^{n}D_{z}^{\alpha}z^{n+\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)}. \tag{11}$$

It is well known that the Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{\mathrm{z}^{\mathrm{k}}}{(\alpha\mathrm{k}+\beta)},\quad \alpha,\beta>0,\quad \alpha,\beta\in R.$$

Obviously, $E_{1,1}(\beta z)=e^{\beta z}$. The Riemann-Liouville fractional derivative of the power and the exponential functions are given, respectively, by

$$D_{z}^{\alpha}z^{\alpha-1}=0,\quad D_{z}^{\alpha}z^{n+\alpha}=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\,z^{n},\quad D_{z}^{\alpha}e^{\beta z}=z^{-\alpha}E_{1,1-\alpha}(\beta z).$$

Therefore, the (11) can be rewritten as

$$D_{z}^{\alpha+1}Q(\theta|z)-(\mu+\theta+\beta)D_{z}^{\alpha}Q(\theta|z)+\mu\beta^{\alpha}Q(\theta|z)$$

$$=\frac{1}{\theta}\left[(\mu+\theta)z^{-\alpha}E_{1,1-\alpha}(\beta z)-\mu z^{-\alpha}E_{1,1-\alpha}((\theta+\beta)z)\right]-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}E_{1,1}(\beta z)$$

$$+\frac{\mu\beta^{\alpha}}{\theta}E_{1,1}((\theta+\beta)z). \tag{12}$$

4. Solution of Fractional Differential Equation (12)

Theorem 4.1 *Let* $s>\theta+\beta$ *and* $\left|s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}\right|>|\mu\beta^{\alpha}|$. *Then, a solution of the fractional order differential equation* (12) *has the form*

$$Q(\theta|z)=\frac{(\mu+\theta)}{\theta}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)n+\ell+1}E_{1,(\alpha+1)n+\ell+2}(\beta z)$$

$$-\frac{\mu}{\theta}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}\left((\theta+\beta)z\right)$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}\left(\beta z\right)$$

$$-\frac{\beta^{\alpha}\mu}{\theta}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\left(-\mu\beta^{\alpha}\right)^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}\left((\theta+\beta)z\right)$$

$$+\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-\mu\beta^{\alpha})^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}z^{\ell+n(\alpha+1)+\alpha}}{\Gamma(\ell+(n+1)(\alpha+1))}D_{z}^{\alpha}Q\left(\theta\,|\,0\right). \tag{13}$$

Proof Applying to the equation (12) Laplace transform by z and taking into account $D_{z}^{\alpha-1}(Q\left(\theta|0\right))=0$, we can write

$$L\left[Q\left(\theta\,|\,z\right)\right]=\frac{(\mu+\theta)}{\theta}\frac{s^{\alpha}}{(s-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}$$

$$-\frac{\mu}{\theta}\frac{s^{\alpha}}{(s-\theta-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}\frac{1}{(s-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}$$

$$-\frac{\beta^{\alpha}\mu}{\theta}\frac{1}{(s-\theta-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}$$

$$+\frac{D_{z}^{\alpha}Q\left(\theta|\,0\right)}{s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}}, \tag{14}$$

where $s>\theta+\beta$.

Now, applying to the equation (14) inverse Laplace transform by s, we obtain

$$Q\left(\theta\,|\,z\right)=\frac{(\mu+\theta)}{\theta}L^{-1}\left[\frac{s^{\alpha}}{(s-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}\right]$$

$$-\frac{\mu}{\theta}L^{-1}\left[\frac{s^{\alpha}}{(s-\theta-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}\right]$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}L^{-1}\left[\frac{1}{(s-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}\right]$$

$$-\frac{\beta^{\alpha}\mu}{\theta}L^{-1}\left[\frac{1}{(s-\theta-\beta)\left[s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}\right]}\right]$$

$$+D_{z}^{\alpha}Q\left(\theta\,|\,0\right)L^{-1}\left[\frac{1}{s^{\alpha+1}-(\mu+\theta+\beta)s^{\alpha}+\mu\beta^{\alpha}}\right]. \tag{15}$$

It is known that ([11], Lemma 5), for $s > \theta + \beta$ and $\left|s^{\alpha+1} - (\mu + \theta + \beta)s^{\alpha}\right| > |\mu\beta^{\alpha}|$, we obtain

$$L^{-1}\left[\frac{s^{\alpha}}{(s-\beta)\left[s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}\right]}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}\left(\beta z\right),$$

$$L^{-1}\left[\frac{s^{\alpha}}{(s-\theta-\beta)\left[s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}\right]}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}\left((\theta+\beta)z\right),$$

$$L^{-1}\left[\frac{1}{(s-\beta)\left[s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}\right]}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}\left(\beta z\right),$$

$$L^{-1}\left[\frac{1}{(s-\theta-\beta)\left[s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}\right]}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}\left((\theta+\beta)z\right)$$

and

$$L^{-1}\left[\frac{1}{s^{\alpha+1} - (\mu+\theta+\beta)s^{\alpha} + \mu\beta^{\alpha}}\right] = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-\mu\beta^{\alpha})^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}z^{\ell+n(\alpha+1)+\alpha}}{\Gamma(\ell+(n+1)(\alpha+1))}.$$

This concludes the proof of theorem. □

Taking into account (13) in (8), expression of the function $\tilde{N}(\theta|z)$ can be given as follows

$$\tilde{N}\left(\theta\,|\,z\right) = \frac{(\mu+\theta)}{\theta}e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}\left(\beta z\right)$$

$$-\frac{\mu}{\theta}e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)n+\ell+1}E_{1,\,(\alpha+1)n+\ell+2}\left((\theta+\beta)z\right)$$

$$-\frac{\beta^{\alpha}(\mu+\theta)}{\theta}e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}\left(\beta z\right)$$

$$-\frac{\beta^{\alpha}\mu}{\theta}e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}(-\mu\beta^{\alpha})^{n}\,(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}\,z^{(\alpha+1)(n+1)+\ell}E_{1,\,(\alpha+1)(n+1)+\ell+1}\left((\theta+\beta)z\right)$$

$$+e^{-(\theta+\beta)z}\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-\mu\beta^{\alpha})^{n}(\mu+\theta+\beta)^{\ell}C_{n+\ell}^{\ell}z^{\ell+n(\alpha+1)+\alpha}}{\Gamma(\ell+(n+1)(\alpha+1))}D_{z}^{\alpha}Q\left(\theta\,|\,0\right).$$

5. Conclusion

The main purpose of this study is to investigate the semi-Markov random walk process with negative drift, positive jumps. In general case, we construct an integral equation for the Laplace transform of the conditional distribution of the random variable. In particular, the fractional order differential equation is obtained from constructed integral equation in the class of gamma distributions. The fractional derivatives are described in the Riemann-Liouville sense. In conclusion, we find solution of the fractional order differential equation.

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1393349
(**2024**) **5**(2) 134 – 142
Research Article

On Some Positive Linear Operators Preserving the $\mathbb{B}^{-1}$-Convexity of Functions

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Received: 21 November 2023 **Accepted:** 19 February 2024

Abstract: In the study, we first give an inequality that non-negative differentiable functions must satisfy to be $\mathbb{B}^{-1}$-convex. Then, using the inequality, we show that the $\mathbb{B}^{-1}$-convexity property of functions is preserved by Bernstein-Stancu operators, Szász-Mirakjan operators and Baskakov operators. In addition, we compare the concepts $\mathbb{B}^{-1}$-convexity and $\mathbb{B}$-concavity of functions.

Keywords: $\mathbb{B}^{-1}$-convexity, Bernstein-Stancu operators, Szász-Mirakjan operators, Baskakov operators, shape preserving approximation.

1. Introduction and Preliminaries

Approximation to continuous functions with positive and linear operators is a useful method in the field of approximation theory of mathematics. According to necessity, it may also be required to preserve the convexity characteristics of functions in this type of approach. Therefore, it is significant to determine which kinds of convexities of functions are preserved by positive linear operators. There are many studies in this topic which is called shape preserving approximation: In [6] and references therein, one can find many results in detail for various operators and convexities.

Similarly, the main purpose of working is to determine whether the following positive and linear operators preserve the $\mathbb{B}^{-1}$-convexity property of the functions:

The Bernstein-Stancu operators which is generalization of the Bernstein operators, $B_n^{\alpha,\beta}$ on $C[0,1]$ are defined as follows [9]:

$$B_n^{\alpha,\beta}(h;t) = \sum_{j=0}^{n} \rho_{n,j}(t) h\left(\frac{j+\alpha}{n+\beta}\right), \qquad h \in C[0,1],$$

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2020 *AMS Mathematics Subject Classification*: 26A51, 41A36

where $\rho_{n,j}(t) = \binom{n}{j} t^j (1-t)^{n-j}$, $n \in \mathbb{N}$ and $0 \le \alpha \le \beta$.

In [10], for a function h on $C[0,\infty)$, the Szász-Mirakjan Operators were defined by

$$S_n(h;t) = \sum_{j=0}^{\infty} s_{n,j}(t) h\left(\frac{j}{n}\right)$$

, where $s_{n,j}(t) = \frac{e^{-nt}(nt)^j}{j!}$ and $n \in \mathbb{N}$.

In [2], the Baskakov operators V_n were introduced as

$$V_n(h;t) = \sum_{j=0}^{\infty} v_{n,j}(t) h\left(\frac{j}{n}\right)$$

on $C[0,\infty)$, where $v_{n,j}(t) = \binom{n+j-1}{j} \frac{t^j}{(1+t)^{n+j}}$ and $n \in \mathbb{N}$.

We used the following notations throughout the study, for $t = (t_1, \ldots, t_k)$, $s = (s_1, \ldots, s_k) \in \mathbb{R}^k$,

$$t \vee s := (\max\{t_1, s_1\}, \max\{t_2, s_2\} \ldots, \max\{t_k, s_k\}),$$

$$t \wedge s := (\min\{t_1, s_1\}, \min\{t_2, s_2\} \ldots, \min\{t_k, s_k\}).$$

$$\mathbb{R}^k_+ := \{t = (t_1, t_2, \ldots, t_k) \in \mathbb{R}^k \mid t_i \ge 0 \text{ for each } i \in \{1, 2, \ldots, k\}\},$$

$$\mathbb{R}^k_{++} := \{t = (t_1, t_2, \ldots, t_k) \in \mathbb{R}^k \mid t_i > 0 \text{ for each } i \in \{1, 2, \ldots, k\}\}.$$

Let us now recall the main definitions and theorems necessary for the following sections.

Definition 1.1 *[3, 7] A set $V \subset \mathbb{R}^k_+$ is called $\mathbb{B}$-convex if $\lambda t \vee s \in V$ for all $t, s \in V$, $\lambda \in [0,1]$.*

Remark 1.2 *[13] $\mathbb{B}$-convex subsets of $\mathbb{R}_+$ are intervals which are open, closed or half-open.*

Definition 1.3 *[7] A function $h : V \subset \mathbb{R}^k_+ \to \mathbb{R}_+$ is called $\mathbb{B}$-convex function if the set V is $\mathbb{B}$-convex and the inequality*

$$h(\lambda t \vee s) \le \lambda h(t) \vee h(s)$$

holds for all $t, s \in V$ and $\lambda \in [0,1]$. If V is $\mathbb{B}$-convex and the inequality

$$h(\lambda t \vee s) \ge \lambda h(t) \vee h(s)$$

holds for all $t, s \in V$, $\lambda \in [0,1]$, then h is called $\mathbb{B}$-concave function.

For $\mathbb{B}$-concave functions, the following expression was obtained in [11]:

Theorem 1.4 *[11] A function $h:[0,1]\to\mathbb{R}_+$ is $\mathbb{B}$-concave if and only if the following conditions hold:*

(i) h is increasing on $[0,1]$,

(ii) The inequality $h(\lambda t)\geq \lambda h(t)$ holds for all $\lambda,t\in[0,1]$. Also, let h be a differentiable function. Then h is $\mathbb{B}$-concave if and only if h is increasing and the inequality $th'(t)-h(t)\leq 0$ holds for all $t\in[0,1]$.

Theorem 1.5 *[12] For a $\mathbb{B}$-concave function $h:[0,1]\to\mathbb{R}_+$, the continuity at $t=0$ is sufficient for continuity on $[0,1]$ of h.*

The $\mathbb{B}$-concavity and $\mathbb{B}$-convexity preserving properties of functions by one and two dimensional Bernstein operators were studied in recent years. Some results in the studies as follows:

- Bernstein operators with one variable do not preserve the $\mathbb{B}$-convexity of functions, but preserve the $\mathbb{B}$-concavity of functions [11],
- Two dimensional Bernstein operators do not preserve both of the convexities [12],
- In [8], author presented a sufficient condition for Bernstein operators on bidemensional simplex to be $\mathbb{B}$-concave.

$\mathbb{B}^{-1}$-convex sets were introduced and studied in [1, 4]. One of the results in the studies was obtained for the any subset of $\mathbb{R}_{++}$. See the following theorem:

Theorem 1.6 *Let $W\subset\mathbb{R}_{++}$. W is $\mathbb{B}^{-1}$-convex set if and only if $\lambda t\wedge s\in W$ for all $t,s\in W$ and $\lambda\in[1,\infty)$.*

Remark 1.7 *[13] $\mathbb{B}^{-1}$-convex subsets of $\mathbb{R}_{++}$ are intervals which are open, closed or half-open.*

Then, in [7], $\mathbb{B}^{-1}$-convex functions were defined as the following theorem:

Theorem 1.8 *If a set $W\subset\mathbb{R}_{++}$ is $\mathbb{B}^{-1}$-convex and $h:W\to\mathbb{R}_{++}$, then h is $\mathbb{B}^{-1}$-convex function if and only if the inequality*

$$h(\lambda t\wedge s)\leq \lambda h(t)\wedge h(s) \tag{1}$$

holds for all $t,s\in W$ and $\lambda\in[1,\infty)$.

Corollary 1.9 *[5] If a set $W\subset\mathbb{R}_{++}$ is $\mathbb{B}^{-1}$-convex, then $h:W\to\mathbb{R}_{++}$ is $\mathbb{B}^{-1}$-convex if and only if the following conditions hold:*

(i) h is increasing on W,

(ii) The inequality $h(\lambda t)\leq\lambda h(t)$ holds for all $\lambda t\in W$, where $t\in W$ and $\lambda\in[1,\infty)$.

2. Preservation of $\mathbb{B}^{-1}$-Convex Functions

In this section, an alternative way is given to determine whether the differentiable functions defined on a subset of $\mathbb{R}_{++}$ are $\mathbb{B}^{-1}$-convex. Then using this, results are given regarding the positive linear operators mentioned above preserve the $\mathbb{B}^{-1}$-convexity property of functions.

Lemma 2.1 *Let $W \subset \mathbb{R}_{++}$ be a $\mathbb{B}^{-1}$-convex set and $h : W \to \mathbb{R}_{++}$ be a differentiable function. Then, h is $\mathbb{B}^{-1}$-convex function if and only if h is increasing on W and the inequality $th'(t) - h(t) \le 0$ holds for all $t \in W$.*

Proof Assume h is $\mathbb{B}^{-1}$-convex function. Let $t \in W$ and (λ_n) be a sequence in $(1,\infty)$ with $\lim_{n\to\infty} \lambda_n = 1$ and $\lambda_n t \in W$ for each $n \in \mathbb{N}$. Due to the $\mathbb{B}^{-1}$-convexity of h, we have $h(\lambda_n t) \le \lambda_n h(t)$ for all $n \in \mathbb{N}$. Then, we get

$$\frac{h(\lambda_n t) - h(t)}{(\lambda_n - 1)t} \le \frac{h(t)}{t} \Rightarrow h'(t) = \lim_{n\to\infty} \frac{h(\lambda_n t) - h(t)}{(\lambda_n - 1)t} \le \frac{h(t)}{t}.$$

Therefore, the inequality $th'(t) - h(t) \le 0$ holds. Conversely, showing the second condition in Corollary 1.9 is sufficient for the $\mathbb{B}^{-1}$-convexity of h. Due to the inequality $th'(t) - h(t) \le 0$, we have $\left(\frac{h(t)}{t}\right)' \le 0$, that is, $\left(\frac{h(t)}{t}\right)$ is decreasing on W. Finally, we get the inequality

$$\frac{h(\lambda t)}{\lambda t} \le \frac{h(t)}{t} \Rightarrow h(\lambda t) \le \lambda h(t)$$

for all $\lambda t \in W$, where $t \in W$ and $\lambda \in [1,\infty)$. □

Theorem 2.2 *Let $g : [0,1] \to \mathbb{R}_{++}$ be a function such that the restriction of g to $(0,1]$ is $\mathbb{B}^{-1}$-convex. Then, $B_n^{\alpha,\beta}(g)$ is also $\mathbb{B}^{-1}$-convex on $(0,1]$ for all $n \in \mathbb{N}$ and $0 \le \alpha \le \beta$.*

Proof Let $t \in (0,1]$ and $n \in \mathbb{N}$. Because of the equation (2), $B_n^{\alpha,\beta}(g;t)$ is increasing:

$$B'^{\alpha,\beta}_n(g;t) = n\sum_{j=0}^{n-1} \rho_{n-1,j}(t)\left[g\left(\frac{j+1+\alpha}{n+\beta}\right) - g\left(\frac{j+\alpha}{n+\beta}\right)\right]. \tag{2}$$

Also, considering the equation (2) and the following inequality

$$\begin{aligned}\frac{B_n^{\alpha,\beta}(g;t)}{t} &= t^{-1}\sum_{j=0}^{n} \rho_{n,j}(t)g\left(\frac{j+\alpha}{n+\beta}\right)\\ &\ge t^{-1}\sum_{j=1}^{n} \rho_{n,j}(t)g\left(\frac{j+\alpha}{n+\beta}\right)\\ &= \frac{n}{j+1}\sum_{j=0}^{n-1} \rho_{n-1,j}(t)g\left(\frac{j+1+\alpha}{n+\beta}\right),\end{aligned} \tag{3}$$

we obtain that:

$$B'^{\alpha,\beta}_n(g;t) - \frac{B^{\alpha,\beta}_n(g;t)}{t} \le n\sum_{j=0}^{n-1}\rho_{n-1,j}(t)\left[\frac{j}{j+1}g\left(\frac{j+1+\alpha}{n+\beta}\right) - g\left(\frac{j+\alpha}{n+\beta}\right)\right].$$

For $j=0$,

$$0g\left(\frac{1}{n}\right) - g(0) \le 0$$

and considering conditions in Corollary 1.9, we get the following inequality for $0<j\le n-1$:

$$\frac{j+1}{j}g\left(\frac{j+\alpha}{n+\beta}\right) \ge g\left(\frac{j+1}{j}\frac{j+\alpha}{n+\beta}\right) \ge g\left(\frac{j+\alpha+1}{n+\beta}\right)$$

and therefore,

$$\left[\frac{j}{j+1}g\left(\frac{j+1+\alpha}{n+\beta}\right) - g\left(\frac{j+\alpha}{n+\beta}\right)\right] \le 0. \tag{4}$$

Finally, since $\rho_{n-1,j}(t)\ge 0$ for all $t\in(0,1]$ and the ineqaulity (4) holds for each j $(0\le j\le n-1)$, we obtain that

$$tB'^{\alpha,\beta}_n(g;t) - B^{\alpha,\beta}_n(g;t) \le 0.$$

□

Theorem 2.3 *Let* $g:[0,\infty)\to\mathbb{R}_{++}$ *be a function such that the restriction of* g *to* $(0,\infty)$ *is* $\mathbb{B}^{-1}$*-convex, then* $S_n(g)$ *is also* $\mathbb{B}^{-1}$*-convex on* $(0,\infty)$ *for all* $n\in\mathbb{N}$*.*

Proof Let $t\in(0,\infty)$ and $n\in\mathbb{N}$. As a result of the following statements,

$$S'_n(g;t) = n\sum_{j=0}^{\infty}s_{n,j}(t)\left[g\left(\frac{j+1}{n}\right) - g\left(\frac{j}{n}\right)\right] \tag{5}$$

and

$$\begin{aligned}\frac{S_n(g;t)}{t} &= t^{-1}\sum_{j=0}^{\infty}s_{n,j}(t)g\left(\frac{j}{n}\right)\\ &\ge t^{-1}\sum_{j=1}^{\infty}s_{n,j}(t)g\left(\frac{j}{n}\right)\\ &= \frac{n}{j+1}\sum_{j=0}^{\infty}s_{n,j}(t)g\left(\frac{j+1}{n}\right),\end{aligned} \tag{6}$$

we see that $S_n(g;t)$ is increasing from (5) and we get the following inequality by using (5) and (6):

$$S_n'(g;t)-\frac{S_n(g;t)}{t}\leq n\sum_{j=0}^{\infty}s_{n,j}(t)\left[\frac{j}{j+1}g\left(\frac{j+1}{n}\right)-g\left(\frac{j}{n}\right)\right].$$

Due to $\mathbb{B}^{-1}$-convexity of g, we get the following inequalities: in case $j=0$,

$$0g'\left(\frac{1}{n}\right)-g(0)\leq 0$$

and for each $j\in\mathbb{N}$

$$\frac{j+1}{j}g\left(\frac{j}{n}\right)\geq g\left(\frac{j+1}{j}\frac{j}{n}\right)=g\left(\frac{j+1}{n}\right).$$

Thus, from the above expressions, we obtain the following inequality for each $j\in\mathbb{N}_0$:

$$\left[\frac{j}{j+1}g\left(\frac{j+1}{n}\right)-g\left(\frac{j}{n}\right)\right]\leq 0. \tag{7}$$

Moreover, considering $s_{n,j}(t)\geq 0$ and (7), we obtain that

$$tS_n'(g;t)-S_n(g;t)\leq 0.$$

□

Theorem 2.4 *Let $g:[0,\infty)\to\mathbb{R}_{++}$ be a function which is $\mathbb{B}^{-1}$-convex on $(0,\infty)$. Then $V_n(g)$ is also $\mathbb{B}^{-1}$-convex on $(0,\infty)$ for all $n\in\mathbb{N}$.*

Proof The proof can be easily seen by using the following inequality and similar operations in Theorem 2.3.

$$\begin{aligned}V_n'(g;t)-\frac{V_n(g;t)}{t}&=\frac{(n+j)}{1+t}\sum_{j=0}^{\infty}v_{n,j}(t)\left[g\left(\frac{j+1}{n}\right)-g\left(\frac{j}{n}\right)\right]\\&-t^{-1}\sum_{j=0}^{\infty}v_{n,j}(t)g\left(\frac{j}{n}\right)\\&\leq\frac{(n+j)}{1+t}\sum_{j=0}^{\infty}v_{n,j}(t)\left[g\left(\frac{j+1}{n}\right)-g\left(\frac{j}{n}\right)\right]\\&-t^{-1}\sum_{j=1}^{\infty}v_{n,j}(t)g\left(\frac{j}{n}\right)\\&=\frac{(n+j)}{1+t}\sum_{j=0}^{\infty}v_{n,j}(t)\left[g\left(\frac{j+1}{n}\right)-g\left(\frac{j}{n}\right)\right]\end{aligned}$$

$$- \frac{(n+j)}{1+t} \sum_{j=0}^{\infty} v_{n,j}(t) \frac{1}{j+1} g\left(\frac{j+1}{n}\right)$$

$$= \frac{(n+j)}{1+t} \sum_{j=0}^{\infty} v_{n,j}(t) \left[\frac{j}{j+1} g\left(\frac{j+1}{n}\right) - g\left(\frac{j}{n}\right) \right].$$

□

3. Comparison of $\mathbb{B}^{-1}$-Convexity and $\mathbb{B}$-Concavity of Functions Defined on a Subset of $\mathbb{R}_{++}$

In this section, it is shown that there is no difference between $\mathbb{B}^{-1}$-convex functions and $\mathbb{B}$-concave functions on subsets of $\mathbb{R}_{++}$.

It is clear that, the statement in Teorem 1.4 can be expanded into any $\mathbb{B}$-convex set in $\mathbb{R}_{+}$ as follows:

Lemma 3.1 *For a $\mathbb{B}$-convex set $V \subset \mathbb{R}_+$, the function $h : V \to \mathbb{R}_+$ is $\mathbb{B}$-concave if and only if*

(i) h is increasing on V,

(ii) The inequality $h(\lambda t) \geq \lambda h(t)$ holds for all $\lambda t \in V$, where $t \in V$ and $\lambda \in [0,1]$.

If h is a differentiable function on V, then h is $\mathbb{B}$-concave function if and olny if h is increasing on V and the inequality $th'(t) - h(t) \leq 0$ holds for all $t \in V$.

Proof The technique of the proof is the same as the proof of Theorem 1.4. □

Remark 3.2 *According to Remarks 1.2 and 1.7, $\mathbb{B}^{-1}$-convex sets in $\mathbb{R}_{++}$ are also $\mathbb{B}$-convex sets. Then, considering Lemma 2.1 and Lemma 3.1 , $\mathbb{B}^{-1}$-convexity and $\mathbb{B}$-concavity have the same inequality for differentiable functions defined on a $\mathbb{B}^{-1}$-convex set $W \subset \mathbb{R}_{++}$. Therefore, $\mathbb{B}^{-1}$-convexity of a differentiable function is equivalent to $\mathbb{B}$-concavity. Moreover, the following corollary shows that this is remain true even if a function is not differentiable.*

Corollary 3.3 *Let $W \subset \mathbb{R}_{++}$ be a $\mathbb{B}^{-1}$-convex set. Then, for a function $h : W \to \mathbb{R}_{++}$, $\mathbb{B}^{-1}$-convexity of h is equivalent to $\mathbb{B}$-concavity of h.*

Proof The property of increasing is common for both convexities. Let h be a $\mathbb{B}^{-1}$-convex function. Then we have $h(\lambda t) \leq \lambda h(t)$ for all $\lambda t \in W$, where $t \in W$ and $\lambda \in [1, \infty]$. Given $\mu \in (0,1]$ and $t \in W$ with $\mu t \in W$, we obtain

$$h(t) = h\left(\mu \frac{1}{\mu} t\right) \leq \frac{1}{\mu} h(\mu t).$$

Conversely, let h be a $\mathbb{B}$-concave function. Then we have $h(\mu t) \geq \mu h(t)$ for all $\mu t \in W$, where $t \in W$ and $\mu \in [0,1]$. Given $\lambda \in [1,\infty)$ and $t \in W$ with $\lambda t \in W$, we obtain

$$h(t) = h\left(\lambda \frac{1}{\lambda} t\right) \geq \frac{1}{\lambda} h(\lambda t).$$

□

Based on the Corallary 3.3, we conclude the following corollaries:

Corollary 3.4 *If a function $h:(0,1) \to \mathbb{R}_{++}$ is $\mathbb{B}^{-1}$-convex, then h is continuous on $(0,1)$.*

Proof The proof can be seen from Theorem 1.5. □

Corollary 3.5 *According to algebraic operations in Theorems 2.2, 2.3 and 2.4, it is clear that the theorems are also valid for $\mathbb{B}$-concavity property of functions.*

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Mustafa Uzun]: Collected the data, contributed to research method or evaluation of data, contributed to completing the research and solving the problem, wrote the manuscript (%50).
Author [Tuncay Tunç]: Thought and designed the research/problem, contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1398394
(**2024**) **5**(2) 143 – 158
Research Article

A New Numerical Simulation for Modified Camassa-Holm and Degasperis-Procesi Equations via Trigonometric Quintic B-spline

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Received: 01 December 2023 **Accepted:** 05 July 2024

Abstract: In this study, the soliton solutions of the modified Camassa-Holm (mCH) and Degasperis-Procesi (mDP) equations, known as modified b-equations with significant physical properties, have been obtained. The movement and positions of soliton waves formed by solving the mCH and mDP equations are calculated. Ordinary differential equation systems have been derived using trigonometric quintic B-spline bases for the derivatives in the position and time directions to obtain numerical solutions. An algebraic equation system is then created by applying Crank-Nicolson type approximations for time and position-dependent terms. The stability analysis of this system has been examined using the von Neumann Fourier series method. L_2, L_∞, and absolute error norms are used to measure the convergence of the numerical results to the real solution. The calculated numerical results have been compared with the exact solution and some studies in the literature.

Keywords: Modified Camassa-Holm, modified Degasperis-Procesi equation, soliton waves, trigonometric quintic B-spline bases, collocation method.

1. Introduction

Nonlinear partial differential equations have important applications in science, such as engineering and physics. Often, it is not easy to solve a nonlinear partial differential equation analytically. Therefore, mathematicians and engineers need to obtain numerical solutions to such equations. The structures of Camassa-Holm (CH) [4] and Degasperis-Procesi (DP) [5] equations, which have many physical properties, are as follows:

$$U_t - U_{xxt} + 3UU_x - 2U_xU_{xx} - UU_{xxx} = 0, \tag{1}$$

$$U_t - U_{xxt} + 4UU_x - 3U_xU_{xx} - UU_{xxx} = 0. \tag{2}$$

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2020 *AMS Mathematics Subject Classification*: 65L60, 65M06, 76M10, 65L80

CH (1) and DP (2) equations are not only bi-Hamiltonian but are also linked to the isospectral problem [12]. The (1) and (2) equations are formally integrable according to the scattering / inverse scattering approaches, as well as peakon solitary wave solutions. Although both equations are similar, they are quite different in terms of the isospectral problem. The CH (1) equation accepts second-order isospectral solution while the DP (2) equation accepts third-order isospectral solution [12, 25]. The CH (1) and DP (2) equations are both integrable equations that model shallow water dynamics [18]. In this study, the numerical solutions of the equation called b-equation are obtained by modifying the convection UU_x term in the equations (1) and (2) as U^2U_x. The b-equation family is given by Wazwaz [18] in the form

$$U_t - U_{xxt} + (b+1)U^2U_x - bU_xU_{xx} - UU_{xxx} = 0. \tag{3}$$

The equation (3) gives mCH for $b = 2$ and mDP for $b = 3$ respectively. Wazwaz obtained analytical solutions of the mCH and mDP equations for $b = 2, 3$ in [18, 19], respectively, using the sine-cosine and extended tanh methods as follows:

$$U_t - U_{xxt} + 3U^2U_x - 2U_xU_{xx} - UU_{xxx} = 0,$$

$$U(x,t) = -2\sec h^2(\frac{x}{2} - t)$$

and

$$U_t - U_{xxt} + 4U^2U_x - 3U_xU_{xx} - UU_{xxx} = 0,$$

$$U(x,t) = -\frac{15}{8}\sec h^2(\frac{x}{2} - \frac{5t}{4}).$$

Obtaining both analytical and numerical solutions to such nonlinear partial differential equations is a scientifically important task. Zhou [25] investigated how the solution's derivative blows up in finite time for the Degasperis-Procesi equation. Wazwaz [18] established new solitary wave solutions for mCH and mDP using the extended tanh method. Wazwaz [19] used the tanh method and the sine-cosine method to get solitary wave solutions to the mCH and mDP equations. Abbasbandy [1] applied the homotopy analysis method to obtain the soliton wave solutions for the mCH and mDP equations. Ganji et al. [6] studied Adomian's decomposition method to solve the mCH and mDP equations. Manafian et al. [11] constructed solitary wave solutions of the mCH and mDP equations via the generalized (G'/G)-expansion and generalized tanh-coth methods. Liu and Ouyang [8] found bell-shaped solitary wave and peakon coexist for the same wave speed for the mCH and mDP equations. Lundmark and Szmigielski [10] presented an inverse scattering approach for computing n-peakon solutions of the DP equation. Yousif et al. [20] obtained solitary wave solutions of the mCH and mDP equations by the variational homotopy perturbation method. Behera and Mehra [3] applied the wavelet-optimized finite difference method to solve the mCH and mDP equations. Wang and Tang [17] obtained four new exact solutions for the mCH and mDP equations using some particular phase orbits. Wasim et al. [16] solved mCH and mDP equations numerically by the collocation finite difference scheme based on Quartic B-spline. Yusufoğlu [21] investigated the mCH and mDP equations' analytic treatment using the Exp-function method. Zada and Nawaz [22] introduced an optimal homotopy asymptotic method for finding the approximate solutions of the mCH and mDP equations. Zhang et al. [23] applied the homotopy perturbation method directly to obtain solitary wave solutions of mCH and mDP equations. Zhang et al. [24] investigated the mCH and mDP equations via the auxiliary equation method. They obtained smooth solitary wave solutions, peakons, singular solutions, periodic wave solutions, and Jacobi elliptic solutions for these equations. Ali et al. [2] proposed and analyzed a novel spectral scheme to get the numerical solutions of the two-dimensional time-fractional diffusion equation.

This study aims to obtain numerical solutions of the mCH (4) and mDP (5) equations with initial and boundary conditions as follows:

$$
\begin{gathered}
U_t - U_{xxt} + 3U^2U_x - 2U_xU_{xx} - UU_{xxx} = 0, \ \ x \in (a,b),\, t \geq 0, \\
U(x,0) = -2\sec h^2(\frac{x}{2}),\, U(a,t) = f_1(t),\, U(b,t) = f_2(t), \\
U'(a,t) = f_3(t),\, U'(b,t) = f_4(t),\, U''(a,t) = f_5(t),\, U''(b,t) = f_6(t)
\end{gathered}
\tag{4}
$$

and

$$
\begin{gathered}
U_t - U_{xxt} + 4U^2U_x - 3U_xU_{xx} - UU_{xxx} = 0, \ \ x \in (a,b),\, t \geq 0, \\
U(x,0) = -\frac{15}{8}\sec h^2(\frac{x}{2}),\, U(a,t) = g_1(t),\, U(b,t) = g_2(t), \\
U'(a,t) = g_3(t),\, U'(b,t) = g_4(t),\, U''(a,t) = g_5(t),\, U''(b,t) = g_6(t).
\end{gathered}
\tag{5}
$$

This article is planned as follows: in Section 2, to obtain numerical solutions of the b-equation which contains very strong nonlinear terms with higher order derivatives, a powerful computationally hybrid technique utilizing finite element and finite difference methods together has been presented. Also, in this section, the stability of the scheme is examined by the von Neumann method so that the approximate solutions obtained from the numerical scheme resulting in the algebraic equation system remain close to the analytical solutions of the b-equation with acceptable accuracy, and it is shown that the scheme is unconditionally stable. In Section 3, numerical schemes obtained by applying the presented method to both mCH and mDP equations subject to initial and boundary conditions are given. In addition, to show the accuracy and reliability of the numerical schemes, some numerical results calculated using the same parameters are compared with themselves and also those obtained by other researchers. In Section 4 which is the last section, a brief conclusion is given and a suggestion for future work is also made.

2. Trigonometric Quintic B-spline Collocation Method

Since it is impossible to implement a numerical method on all $x \in R$ and semi-infinite $t \in R^{+}$, the solution region of the considered problems below for numerical simulations is taken as $a \leq x \leq b$ and $0 \leq t \leq T$. The principal idea of a finite element formulation using for obtaining an approximate solution of a physical problem is to result in algebraic equation systems rather than solving differential equations [9, 13]. For this purpose, let x_m be a uniform finite fragmentation of the solution region $[a,b]$, and $a = x_0 < x_1 < \ldots < x_N = b$, where $m = 0,1,\ldots,N$. Taking $h = x_{m+1} - x_m$, $T_m(x)$, $m = -2(1)N+2$, quintic trigonometric B-spline functions on the range $[a,b]$ in terms of nodes x_m as

$$T_m(x) = \frac{1}{\theta}\begin{cases} \tau^5(x_{m-3}) \ , & [x_{m-3}, x_{m-2}] \\ \tau^4(x_{m-3})\phi(x_{m-1}) + \tau^3(x_{m-3})\phi(x_m)\tau(x_{m-2}) + \\ \tau^2(x_{m-3})\phi(x_{m+1})\tau^2(x_{m-2}) + & [x_{m-2}, x_{m-1}] \\ \tau(x_{m-3})\phi(x_{m+2})\tau^3(x_{m-2}) + \phi(x_{m+3})\tau^4(x_{m-2}), \\ \tau^3(x_{m-3})\phi^2(x_m) + \tau^2(x_{m-3})\phi(x_{m+1})\tau(x_{m-2})\phi(x_m) \\ +\tau^2(x_{m-3})\phi^2(x_{m+1})\tau(x_{m-1}) + \tau(x_{m-3})\phi(x_{m+2})\tau^2(x_{m-2})\phi(x_m) + \\ \tau(x_{m-3})\phi(x_{m+2})\tau(x_{m-2})\phi(x_{m+1})\tau(x_{m-1}) + \\ \tau(x_{m-3})\phi^2(x_{m+2})\tau^2(x_{m-1}) + & [x_{m-1}, x_m] \\ \phi(x_{m+3})\tau^3(x_{m-2})\phi(x_m) + \phi(x_{m+3})\tau^2(x_{m-2})\phi(x_{m+1})\tau(x_{m-1}) + \\ \phi(x_{m+3})\tau(x_{m-2})\phi(x_{m+2})\tau^2(x_{m-1}) + \phi^2(x_{m+3})\tau^3(x_{m-1}), \\ \tau^2(x_{m-3})\phi^3(x_{m+1}) + \tau(x_{m-3})\phi(x_{m+2})\tau(x_{m-2})\phi^2(x_{m+1}) + \\ \tau(x_{m-3})\phi^2(x_{m+2})\tau(x_{m-1})\phi(x_{m+1}) + \tau(x_{m-3})\phi^3(x_{m+2})\tau(x_m) + \\ \phi(x_{m+3})\tau^2(x_{m-2})\phi^2(x_{m+1}) + \\ \phi(x_{m+3})\tau(x_{m-2})\phi(x_{m+2})\tau(x_{m-1})\phi(x_{m+1}) + & [x_m, x_{m+1}] \\ \phi(x_{m+3})\tau(x_{m-2})\phi^2(x_{m+2})\tau(x_m) + \phi^2(x_{m+3})\tau^2(x_{m-1})\phi(x_{m+1}) + \\ \phi^2(x_{m+3})\tau(x_{m-1})\phi(x_{m+2})\tau(x_m) + \phi^3(x_{m+3})\tau^2(x_m), \\ \tau(x_{m-3})\phi^4(x_{m+2}) + \phi(x_{m+3})\tau(x_{m-2})\phi^3(x_{m+2}) + \\ \phi^2(x_{m+3})\tau(x_{m-1})\phi^2(x_{m+2}) + & [x_{m+1}, x_{m+2}] \\ \phi^3(x_{m+3})\tau(x_m)\phi(x_{m+2}) + \phi^4(x_{m+3})\tau(x_{m+1}), \\ \phi^5(x_{m+3}), & [x_{m+2}, x_{m+3}] \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

where $\theta = \sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})\sin(2h)\sin(\frac{5h}{2})$, $\tau(x_m) = \sin(\frac{x-x_m}{2})$ and $\phi(x_m) = \sin(\frac{x_m-x}{2})$. It is clear that the set $\{T_{-2}(x), T_{-1}(x), \ldots, T_{N+1}(x), T_{N+2}(x)\}$ forms a base on the interval $[a,b]$ [7, 15]. A typical element $[x_m, x_{m+1}]$ transforms into the interval $[0,1]$ by using $h\xi = x - x_m$. Hence each $[x_m, x_{m+1}]$ element is covered by six trigonometric quıntıc B-splines such as $T_{m-2}(x)$, $T_{m-1}(x)$, $T_m(x)$, $T_{m+1}(x)$, $T_{m+2}(x)$, $T_{m+3}(x)$. Thus, the approximate solution via trigonometric quintic B-spline functions on the element $[x_m, x_{m+1}]$ can be written as

$$U(x,t) \approx U_N(x,t) = \sum_{i=m-2}^{m+3} T_i(x)\delta_i(t).$$

Using (6), the nodal values of $U_N(x,t)$ and its third order derivatives at the nodes x_m are obtained as

$$\begin{aligned}
U_N(x_m,t) &= U_m = a_1\delta_{m-2} + a_2\delta_{m-1} + a_3\delta_m + a_2\delta_{m+1} + a_1\delta_{m+2},\\
U_m' &= -a_4\delta_{m-2} - a_5\delta_{m-1} + a_5\delta_{m+1} + a_4\delta_{m+2},\\
U_m'' &= a_6\delta_{m-2} + a_7\delta_{m-1} + a_8\delta_m + a_7\delta_{m+1} + a_6\delta_{m+2},\\
U_m''' &= -a_9\delta_{m-2} + a_{10}\delta_{m-1} - a_{10}\delta_{m+1} + a_9\delta_{m+2},
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
a_1 &= \sin^5(\frac{h}{2})/\theta,\ a_2 = \sin^4(\frac{h}{2})\sin(h)(8\cos(h)+5)/\theta,\ a_3 = 2\sin^5(\frac{h}{2})(6\cos(2h)+16\cos(h)+11)/\theta,\\
a_4 &= 5\sin^3(\frac{h}{2})\sin(h)/4\theta,\ a_5 = 5\sin^4(\frac{h}{2})\cos^2(\frac{h}{2})(4\cos(h)+1)/\theta,\ a_6 = 5\sin^3(\frac{h}{2})(5\cos(h)+3)/8\theta,\\
a_7 &= 5\sin^3(\frac{h}{2})\cos(\frac{h}{2})(4\cos(2h)+\cos(h)+3)/4\theta,\ a_8=-5\sin^3(\frac{h}{2})(\cos(3h)+6\cos(2h)+10\cos(h)+7)/4\theta,\\
a_9 &= 5\sin^2(\frac{h}{2})\cos(\frac{h}{2})(25\cos(h)-1)/16\theta,\ a_{10} = -5\sin^2(h)(2\cos(2h)-27\cos(h)+1)/32\theta.
\end{aligned}$$

If the expressions given in (7) are used in Equation (3), an ordinary differential equation system is obtained as follows:

$$\begin{aligned}
&a_1\overset{\circ}{\delta}_{m-2} + a_2\overset{\circ}{\delta}_{m-1} + a_3\overset{\circ}{\delta}_m + a_2\overset{\circ}{\delta}_{m+1} + a_1\overset{\circ}{\delta}_{m+2}\\
&-(a_6\overset{\circ}{\delta}_{m-2} + a_7\overset{\circ}{\delta}_{m-1} + a_8\overset{\circ}{\delta}_m + a_7\overset{\circ}{\delta}_{m+1} + a_6\overset{\circ}{\delta}_{m+2})\\
&+z_m^2(b+1)\left(-a_4\delta_{m-2} - a_5\delta_{m-1} + a_5\delta_{m+1} + a_4\delta_{m+2}\right)\\
&-bt_m\left(a_6\delta_{m-2} + a_7\delta_{m-1} + a_8\delta_m + a_7\delta_{m+1} + a_6\delta_{m+2}\right)\\
&-z_m\left(-a_9\delta_{m-2} + a_{10}\delta_{m-1} - a_{10}\delta_{m+1} + a_9\delta_{m+2}\right) = 0,
\end{aligned} \tag{8}$$

where the symbol "$\circ$"is the derivative concerning time and

$$\begin{aligned}
z_m &= \delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2},\\
d_m &= \frac{5}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}).
\end{aligned}$$

Instead of the parameters δ_m and $\overset{\circ}{\delta}_m$, $\frac{\delta_m^{n+1}+\delta_m^n}{2}$ Crank-Nicolson and $\frac{\delta_m^{n+1}-\delta_m^n}{\Delta t}$ forward finite difference approaches are written in Equation (8) respectively, a recurrence relation is obtained between time steps n and $(n+1)$ as

$$\begin{aligned}&\kappa_1\delta_{m-2}^{n+1}+\kappa_2\delta_{m-1}^{n+1}+\kappa_3\delta_m^{n+1}+\kappa_4\delta_{m+1}^{n+1}+\kappa_5\delta_{m+2}^{n+1}\\&=\kappa_6\delta_{m-2}^{n}+\kappa_7\delta_{m-1}^{n}+\kappa_8\delta_m^{n}+\kappa_9\delta_{m+1}^{n}+\kappa_{10}\delta_{m+2}^{n},\end{aligned}\tag{9}$$

where

$$\begin{aligned}\kappa_1&=a_1-a_6-\frac{a_4(b+1)z_m^2\Delta t}{2}-\frac{ba_6d_m\Delta t}{2}+\frac{a_9z_m\Delta t}{2},\\
\kappa_2&=a_2-a_7-\frac{a_5(b+1)z_m^2\Delta t}{2}-\frac{ba_7d_m\Delta t}{2}-\frac{a_{10}z_m\Delta t}{2},\\
\kappa_3&=a_3-a_8-\frac{ba_8d_m\Delta t}{2},\ \kappa_4=a_2-a_7+\frac{a_5(b+1)z_m^2\Delta t}{2}-\frac{ba_7d_m\Delta t}{2}+\frac{a_{10}z_m\Delta t}{2},\\
\kappa_5&=a_1-a_6+\frac{a_4(b+1)z_m^2\Delta t}{2}-\frac{ba_6d_m\Delta t}{2}-\frac{a_9z_m\Delta t}{2},\\
\kappa_6&=a_1-a_6+\frac{a_4(b+1)z_m^2\Delta t}{2}+\frac{ba_6d_m\Delta t}{2}-\frac{a_9z_m\Delta t}{2},\\
\kappa_7&=a_2-a_7+\frac{a_5(b+1)z_m^2\Delta t}{2}+\frac{ba_7d_m\Delta t}{2}+\frac{a_{10}z_m\Delta t}{2},\ \kappa_8=a_3-a_8+\frac{ba_8d_m\Delta t}{2},\\
\kappa_9&=a_2-a_7-\frac{a_5(b+1)z_m^2\Delta t}{2}+\frac{ba_7d_m\Delta t}{2}-\frac{a_{10}z_m\Delta t}{2},\\
\kappa_{10}&=a_1-a_6-\frac{a_4(b+1)z_m^2\Delta t}{2}+\frac{ba_6d_m\Delta t}{2}+\frac{a_9z_m\Delta t}{2}.\end{aligned}$$

The algebraic equation system (9) contains $(N+1)$ equations and $(N+5)$ time-dependent parameters $\delta_m(t)$, $m=0(1)N$. To have a unique solution for this system, the parameters δ_{-2}, δ_{-1}, δ_{N+1} and δ_{N+2} must be eliminated with the help of boundary conditions. If the approaches U_m and U'_m are used to transform the system (9) into an $(N+1)\times(N+1)$ pentadiagonal system, the following relations are obtained for the parameters δ_{-2}, δ_{-1}, δ_{N+1}, and δ_{N+2}

$$\begin{aligned}\delta_{-2}&=m_1\delta_0+m_2\delta_1+m_3\delta_2+\lambda_1,\ \delta_{-1}=m_4\delta_0+m_5\delta_1+m_6\delta_2+\lambda_2,\\
\delta_{N+1}&=m_6\delta_{N-2}+m_5\delta_{N-1}+m_4\delta_N+\lambda_3,\ \delta_{N+2}=m_3\delta_{N-2}+m_2\delta_{N-1}+m_1\delta_N+\lambda_4,\end{aligned}$$

where

$$\begin{aligned}
m_1 &= \frac{a_3a_5}{a_2a_4 - a_1a_5},\ m_2 = \frac{2a_2a_5}{a_2a_4 - a_1a_5},\ m_3 = \frac{a_1a_5 + a_2a_4}{a_2a_4 - a_1a_5},\\
m_4 &= \frac{a_3a_4}{a_1a_5 - a_2a_4},\ m_5 = \frac{a_2a_4 + a_1a_5}{a_1a_5 - a_2a_4},\ m_6 = \frac{2a_1a_4}{a_1a_5 - a_2a_4},\\
\lambda_1 &= \frac{a_2U'(a,t) + a_5U(a,t)}{a_1a_5 - a_2a_4},\ \lambda_2 = \frac{a_4U(a,t) + a_1U'(a,t)}{a_2a_4 - a_1a_5},\\
\lambda_3 &= \frac{a_1U'(b,t) - a_4U(b,t)}{a_1a_5 - a_2a_4},\ \lambda_4 = \frac{a_2U'(b,t) - a_5U(b,t)}{a_2a_4 - a_1a_5}.
\end{aligned}$$

To start the solution of the system (9), it is necessary to find the initial vector δ_m^0. For this purpose, similar to the above, after finding relations for δ_{-2}, δ_{-1}, δ_{N+1}, and δ_{N+2} with the help of $U_m^{'}$ and $U_m^{''}$, the initial vector is generated from the solution of the below matrix system.

$$\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & & & & \\
\alpha_4 & \alpha_5 & \alpha_6 & a_1 & & & \\
a_1 & a_2 & a_3 & a_2 & a_1 & & \\
 & & & \ddots & & & \\
 & & a_1 & a_2 & a_3 & a_2 & a_1\\
 & & & a_1 & \alpha_6 & \alpha_5 & \alpha_4\\
 & & & & \alpha_3 & \alpha_2 & \alpha_1
\end{bmatrix}
\begin{bmatrix}\delta_0^0\\ \delta_1^0\\ \delta_2^0\\ \vdots\\ \delta_{N-2}^0\\ \delta_{N-1}^0\\ \delta_N^0\end{bmatrix}
=
\begin{bmatrix}U_0\\ U_1\\ U_2\\ \vdots\\ U_{N-2}\\ U_{N-1}\\ U_N\end{bmatrix}
+
\begin{bmatrix}\gamma_1\\ \gamma_2\\ 0\\ \vdots\\ 0\\ \gamma_3\\ \gamma_4\end{bmatrix},$$

where

$$\alpha_1 = a_1\beta_4 + a_2\beta_1 + a_3,\ \alpha_2 = a_1\beta_5 + a_2\beta_2 + a_2,\ \alpha_3 = a_1\beta_6 + a_2\beta_3 + a_1,$$

$$\alpha_4 = a_1\beta_1 + a_2,\ \alpha_5 = a_1\beta_2 + a_3,\ \alpha_6 = a_1\beta_3 + a_2,\ \beta_1 = \frac{a_4a_8}{a_5a_6 - a_4a_7},$$

$$\beta_2 = \frac{a_4a_7 + a_5a_6}{a_5a_6 - a_4a_7},\ \beta_3 = \frac{2a_4a_6}{a_5a_6 - a_4a_7},\ \beta_4 = \frac{a_5a_8}{a_4a_7 - a_5a_6},\ \beta_5 = \frac{2a_5a_7}{a_4a_7 - a_5a_6},\ \beta_6 = \frac{a_5a_6 + a_4a_7}{a_4a_7 - a_5a_6},$$

$$\gamma_1 = \frac{(a_1a_5 - a_2a_4)U^{''}(a,0) + (a_1a_7 - a_2a_6)U^{'}(a,0)}{a_5a_6 - a_4a_7},\ \gamma_2 = \frac{a_1a_4U^{''}(a,0) + a_1a_6U^{'}(a,0)}{a_4a_7 - a_5a_6},$$

$$\gamma_3 = \frac{a_1a_6U^{'}(b,0) - a_1a_4U^{''}(b,0)}{a_5a_6 - a_4a_7},\ \gamma_4 = \frac{(a_2a_6 - a_1a_7)U^{'}(b,0) + (a_1a_5 - a_2a_4)U^{''}(b,0)}{a_5a_6 - a_4a_7}.$$

2.1. Stability Analysis

To investigate the stability analysis of the system (9), the Fourier series method of von Neumann [14] is used. In this method, $\delta_m^n = \xi^n e^{i\beta mh}$ is taken, where $i = \sqrt{-1}$, β is the mode number, ξ is the amplification factor, and h is the space step. Since this method is valid for linear schemes, the z_m and d_m constants are taken to be zero. If $\delta_m^n = \xi^n e^{i\beta mh}$ is written in the system (9) and if necessary operations are performed, the following expressions are obtained:

$$\frac{\xi(t^{n+1})}{\xi(t^n)}=\frac{\kappa_6 e^{-2i\beta h}+\kappa_7 e^{-i\beta h}+\kappa_8+\kappa_9 e^{i\beta h}+\kappa_{10}e^{2i\beta h}}{\kappa_1 e^{-2i\beta h}+\kappa_2 e^{-i\beta h}+\kappa_3+\kappa_4 e^{i\beta h}+\kappa_5 e^{2i\beta h}}$$

or

$$\begin{aligned}\frac{\xi(t^{n+1})}{\xi(t^n)}&=\frac{[(\kappa_6+\kappa_{10})\cos(2\beta h)+(\kappa_7+\kappa_9)\cos(\beta h)+\kappa_8]+i\,[(\kappa_{10}-\kappa_6)\sin(2\beta h)+(\kappa_9-\kappa_7)\sin(\beta h)]}{[(\kappa_1+\kappa_5)\cos(2\beta h)+(\kappa_2+\kappa_4)\cos(\beta h)+\kappa_3]+i\,[(\kappa_5-\kappa_1)\sin(2\beta h)+(\kappa_4-\kappa_2)\sin(\beta h)]}\\&=\frac{P+iQ}{R+iS},\end{aligned}$$

where

$$\begin{aligned}P&=[(\kappa_6+\kappa_{10})\cos(2\beta h)+(\kappa_7+\kappa_9)\cos(\beta h)+\kappa_8],\\Q&=[(\kappa_{10}-\kappa_6)\sin(2\beta h)+(\kappa_9-\kappa_7)\sin(\beta h)],\\R&=[(\kappa_1+\kappa_5)\cos(2\beta h)+(\kappa_2+\kappa_4)\cos(\beta h)+\kappa_3],\\S&=[(\kappa_5-\kappa_1)\sin(2\beta h)+(\kappa_4-\kappa_2)\sin(\beta h)].\end{aligned}$$

For the stability of the method, the condition $\left|\frac{\xi(t^{n+1})}{\xi(t^n)}\right|\le 1$ must be provided. Namely, the inequality $\left|P^2\right|+\left|Q^2\right|\le\left|R^2\right|+\left|S^2\right|$ must be ensured. Thus, the following expression is obtained:

$$\left|P^2\right|+\left|Q^2\right|-\left|R^2\right|-\left|S^2\right|\le 0.$$

Since $\left|P^2\right|+\left|Q^2\right|-\left|R^2\right|-\left|S^2\right|\le 0$, the method is unconditionally stable. Besides, it should still be taken into account that the solutions are not distorted when choosing h and Δt.

3. Numerical Applications

In this section, two test problems have been considered for the numerical simulations. To confirm the accuracy and efficiency of the proposed method, we have calculated L_2, L_∞ error norms, and absolute error (AE) that measure the difference between exact (u) and numerical (u_N) solutions as follows:

$$L_2=\sqrt{h\sum_{j=0}^{N}\left|U(x_j,t)-U_N(x_j,t)\right|^2},\ L_\infty=\max_{0\le j\le N}\left|U(x_j,t)-U_N(x_j,t)\right|,\ \text{AE}=\left|U(x_j,t)-U_N(x_j,t)\right|.$$

Table 1: The error norms L_2 and L_∞ for $h = 0.1$ and $\Delta t = 0.01,\ 0.001$ over $-40 \le x \le 40$ and the maximum amplitude and positions of the soliton waves for mCH

Δt	t	L_2	L_∞	x	Present	Exact
0.01	0	0	0	0.00	−2.0000000000	−2.0000000000
	2	0.257512E-3	0.137435E-3	4.00	−1.9999984012	−2.0000000000
	4	0.507308E-3	0.268631E-3	8.00	−1.9999981491	−2.0000000000
	6	0.755920E-3	0.399642E-3	12.00	−1.9999980667	−2.0000000000
	8	1.004526E-3	0.530661E-3	16.00	−1.9999979597	−2.0000000000
	10	1.253169E-3	0.661694E-3	20.00	−1.9999978238	−2.0000000000
0.001	2	0.143124E-5	0.568181E-6	4.00	−1.9999997739	−2.0000000000
	4	0.204605E-5	0.845279E-6	8.00	−1.9999997094	−2.0000000000
	6	0.269423E-5	0.133650E-5	12.00	−1.9999997070	−2.0000000000
	8	0.356980E-5	0.189265E-5	16.00	−1.9999997069	−2.0000000000
	10	0.456604E-5	0.247223E-5	20.00	−1.9999997069	−2.0000000000

3.1. Soliton Solutions of mCH Equation

As a first application, consider the modified Camassa-Holm (4) equation with initial and boundary conditions as follows:

$$U_t - u_{xxt} + 3U^2U_x - 2U_xU_{xx} - UU_{xxx} = 0, \ \ x \in (a,b),\ t \ge 0,$$

$$U(x,0) = -2\sec h^2(\frac{x}{2}),\ U(a,t) = f_1(t),\ U(b,t) = f_2(t),$$

$$U'(a,t) = f_3(t),\ U'(b,t) = f_4(t),\ U''(a,t) = f_5(t),\ U''(b,t) = f_6(t).$$

The numerical results obtained by solving the mCH (4) equation are presented in Table 1. As shown from the Table 2, as Δt gets smaller, the error norms L_2 and L_∞ also decrease significantly. The wave's amplitude at $t = 0$ is calculated as -2 at $x = 0$ and -1.9999997069 at $x = 20$ at $t = 10$, and this change is 2.931×10^{-7}. It is seen that the numerical results given in different locations and times in the Table 2 are very close to the exact solution. In the Table 3, some nodal values have been compared to the analytic solution. In addition to comparing the absolute errors with Ref. [16], the error norms L_2 and L_∞ are also given. The numerical results given by the current method in the Table 3 converge to a better exact solution than those given in Ref. [16]. In Figure 1, the physical behavior of the numerical and analytic solution (*upper*) and the absolute error graphs (*bottom*) at different times are given. It can be seen from Figure 1 that the waves are moving to the right, keeping their speed and height almost perfect.

Table 2: Some nodal values of $U(x,t)$ for $h=0.1$ and $\Delta t=0.001$ over $-15\le x\le 15$ of mCH

t	x	Numeric	Analytic	t	x	Numeric	Analytic
0.05	−12	−0.0000444755	−0.0000444756	0.15	−12	−0.0000364134	−0.0000364136
	−10	−0.0003286090	−0.0003286094		−10	−0.0002690455	−0.0002690467
	−9	−0.0008931260	−0.0008931270		−9	−0.0007312574	−0.0007312601
	−8	−0.0024268374	−0.0024268396		−8	−0.0019871407	−0.0019871468
	−6	−0.0178627158	−0.0178627244		−6	−0.0146366096	−0.0146366342
	6	−0.0217959864	−0.0217959772		6	−0.0265895200	−0.0265894909
	8	−0.0029637527	−0.0029637503		8	−0.0036193465	−0.0036193389
	9	−0.0010908149	−0.0010908138		9	−0.0013322461	−0.0013322426
	10	−0.0004013577	−0.0004013572		10	−0.0004902094	−0.0004902079
	12	−0.0000543226	−0.0000543225		12	−0.0000663497	−0.0000663495
0.10	−12	−0.0000402431	−0.0000402432	0.20	−12	−0.0000329482	−0.0000329484
	−10	−0.0002973396	−0.0002973404		−10	−0.0002434435	−0.0002434450
	−9	−0.0008081500	−0.0008081519		−9	−0.0006616795	−0.0006616830
	−8	−0.0021960179	−0.0021960221		−8	−0.0017981220	−0.0017981298
	−6	−0.0161697309	−0.0161697477		−6	−0.0132483638	−0.0132483957
	6	−0.0240744628	−0.0240744439		6	−0.0293653701	−0.0293653302
	8	−0.0032752003	−0.0032751954		8	−0.0039996179	−0.0039996072
	9	−0.0012055034	−0.0012055012		9	−0.0014723091	−0.0014723042
	10	−0.0004435646	−0.0004435636		10	−0.0005417587	−0.0005417565
	12	−0.0000600357	−0.0000600356		12	−0.0000733277	−0.0000733274

Table 3: Some nodal values of $U(x,t)$ and comparison of absolute error with Ref. [16] for $h=0.1$ and $\Delta t=0.001$ over $-15\le x\le 15$ of mCH

t	x	Numeric	Analytic	Present (AE)	[16] (AE)
0.05	6	−0.0217959864	−0.0217959772	0.918E−8	3.349E−04
	8	−0.0029637527	−0.0029637503	0.236E−8	4.359E−05
	9	−0.0010908149	−0.0010908138	0.108E−8	1.596E−05
	10	−0.0004013577	−0.0004013572	0.475E−9	5.860E−06
	12	−0.0000543226	−0.0000543225	0.852E−10	7.900E−07
0.10	6	−0.0240744628	−0.0240744439	0.189E−7	8.847E−04
	8	−0.0032752003	−0.0032751954	0.491E−8	1.159E−04
	9	−0.0012055034	−0.0012055012	0.225E−8	4.248E−05
	10	−0.0004435646	−0.0004435636	0.993E−9	1.560E−05
	12	−0.0000600357	−0.0000600356	0.178E−9	2.100E−06
0.15	8	−0.0036193465	−0.0036193389	0.766E−8	2.238E−04
	9	−0.0013322461	−0.0013322426	0.352E−8	8.208E−05
	10	−0.0004902094	−0.0004902079	0.155E−8	3.014E−05
0.20	8	−0.0039996179	−0.0039996072	0.106E−7	3.765E−04
	9	−0.0014723091	−0.0014723042	0.490E−8	1.381E−04
	10	−0.0005417587	−0.0005417565	0.217E−8	5.073E−05

	$t=0.05$	$t=0.1$	$t=0.15$	$t=0.20$
$L_2\times 10^6$	0.045859	0.091547	0.136939	0.181932
$L_\infty\times 10^6$	0.202146	0.401670	0.590314	0.768287

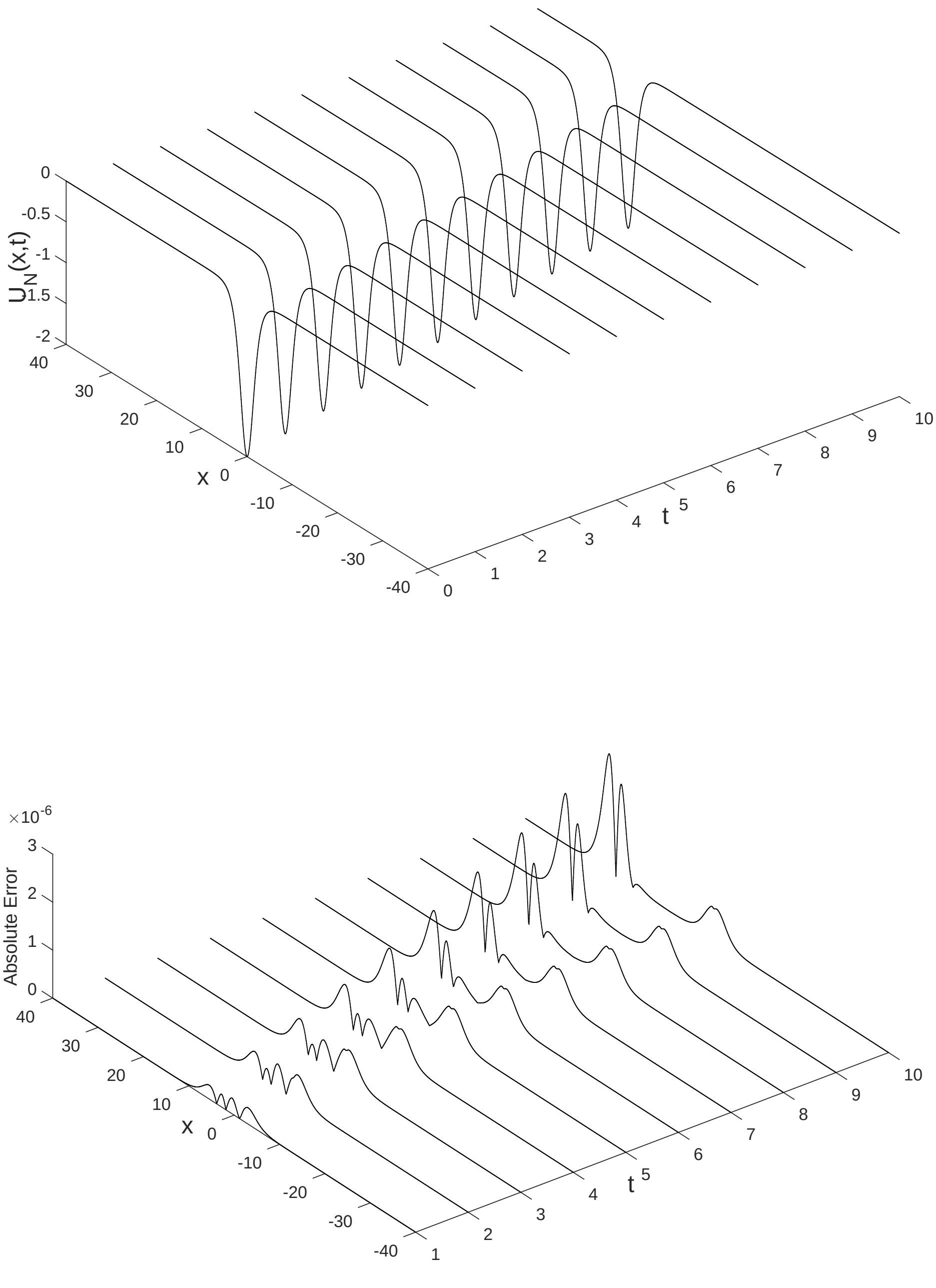


Figure 1: Soliton wave progression (*upper*) and absolute error (*bottom*) of the mCH equation for $h = 0.1$, $\Delta t = 0.001$

Table 4: The error norms L_2 and L_∞ for $h = 0.1$ and $\Delta t = 0.01,\ 0.001$ over $-40 \le x \le 40$ and the maximum amplitude and positions of the waves for mDP

Δt	t	L_2	L_∞	x	Present	Exact
0.01	0	0	0	0.00	−1.8750000000	−1.8750000000
	2	0.502693E-3	0.269075E-3	5.00	−1.8749969077	−1.8750000000
	4	0.987224E-3	0.524039E-3	10.00	−1.8749966431	−1.8750000000
	6	1.470814E-3	0.778917E-3	15.00	−1.8749963319	−1.8750000000
	8	1.954544E-3	1.033844E-3	20.00	−1.8749959041	−1.8750000000
	10	2.438395E-3	1.288820E-3	25.00	−1.8749953600	−1.8750000000
0.001	2	0.284646E-5	0.159639E-5	5.00	−1.8749997931	−1.8750000000
	4	0.372017E-5	0.199237E-5	10.00	−1.8749997716	−1.8750000000
	6	0.426197E-5	0.230974E-5	15.00	−1.8749997713	−1.8750000000
	8	0.481290E-5	0.262623E-5	20.00	−1.8749997709	−1.8750000000
	10	0.537967E-5	0.297078E-5	25.00	−1.8749997512	−1.8750000000

3.2. Soliton Solutions of mDP Equation

As a second and last application, consider the modified Degasperis-Procesi (5) equation with initial and boundary conditions as follows:

$$U_t - U_{xxt} + 4U^2U_x - 3U_xU_{xx} - uu_{xxx} = 0,\ \ x \in (a,b),\ t \ge 0,$$

$$U(x,0) = -\frac{15}{8}\sec h^2(\frac{x}{2}),\ U(a,t) = g_1(t),\ U(b,t) = g_2(t),$$

$$U'(a,t) = g_3(t),\ U'(b,t) = g_4(t),\ U''(a,t) = g_5(t),\ U''(b,t) = g_6(t).$$

The numerical results obtained by solving the mDP (5) equation are given in the Table 4. As shown from the table, the error norms L_2 and L_∞ decrease significantly as Δt gets smaller. Also, the wave's amplitude at $t = 0$ is -1.875 at $x = 0$ and -1.8749997512 at $x = 25$ at $t = 10$, and this change is 2.488×10^{-7}. It is seen that the numerical results given in different locations and times in the Table 5 are quite close to the analytic solution. In the Table 6, some nodal values are compared with the exact solution as well as Ref. [16] a comparison of with absolute errors and L_2, L_∞ error norms are also given. The numerical results obtained with the proposed method converge better than those given in Ref. [16]. In Figure 2, the physical behavior of the numerical and exact solution (*upper*) and the absolute error graphs (*bottom*) at different times are given. It can be seen from Figure 2 that the waves move to the right, keeping their speed and height almost admirably.

4. Conclusions

In this study, soliton wave solutions of the modified Camassa-Holm (mCH) and Degasperis-Process (mDP) equations were obtained by the trigonometric quintic B-spline collocation finite element

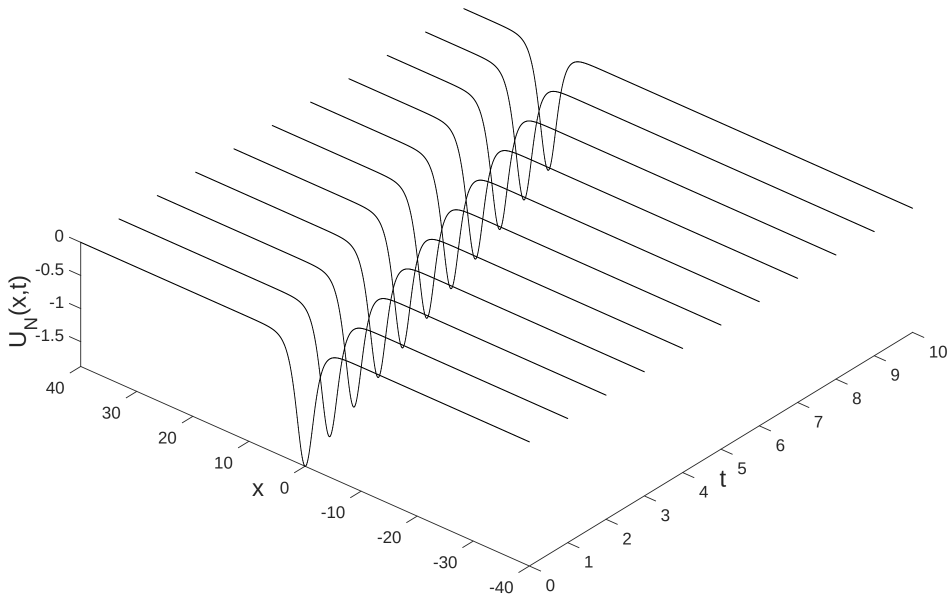
Table 5: Some nodal values of $U(x,t)$ for $h = 0.1$ and $\Delta t = 0.001$ over $-15 \leq x \leq 15$ of mDP

t	x	Numeric	Analytic	t	x	Numeric	Analytic
0.05	−12	−0.0000406663	−0.0000406664	0.15	−12	−0.0000316709	−0.0000316711
	−10	−0.0003004651	−0.0003004657		−10	−0.0002340057	−0.0002340070
	−9	−0.0008166367	−0.0008166379		−9	−0.0006360257	−0.0006360289
	−8	−0.0022190189	−0.0022190214		−8	−0.0017283949	−0.0017284019
	−6	−0.0163346314	−0.0163346415		−6	−0.0127336964	−0.0127337247
	6	−0.0209481240	−0.0209481133		6	−0.0268551982	−0.0268551645
	8	−0.0028488037	−0.0028488009		8	−0.0036571525	−0.0036571434
	9	−0.0010485202	−0.0010485189		9	−0.0013462223	−0.0013462180
	10	−0.0003857973	−0.0003857968		10	−0.0004953602	−0.0004953584
	12	−0.0000522167	−0.0000522166		12	−0.0000670475	−0.0000670471
0.10	−12	−0.0000358879	−0.0000358880	0.20	−12	−0.0000279494	−0.0000279497
	−10	−0.0002651615	−0.0002651625		−10	−0.0002065102	−0.0002065120
	−9	−0.0007206966	−0.0007206988		−9	−0.0005613007	−0.0005613047
	−8	−0.0019584109	−0.0019584157		−8	−0.0015253831	−0.0015253920
	−6	−0.0144226482	−0.0144226677		−6	−0.0112419288	−0.0112419653
	6	−0.0237196488	−0.0237196269		6	−0.0304018338	−0.0304017877
	8	−0.0032277936	−0.0032277878		8	−0.0041435608	−0.0041435480
	9	−0.0011880860	−0.0011880834		9	−0.0015253979	−0.0015253920
	10	−0.0004371602	−0.0004371590		10	−0.0005613073	−0.0005613047
	12	−0.0000591692	−0.0000591690		12	−0.0000759746	−0.0000759742

Table 6: Some nodal values of $U(x,t)$ and comparison of absolute error with Ref. [16] for $h = 0.1$ and $\Delta t = 0.001$ over $-15 \leq x \leq 15$ of mDP

t	x	Numeric	Analytic	Present (AE)	[16] (AE)
0.05	6	−0.0209481240	−0.0209481133	0.106E−7	4.490E−04
	8	−0.0028488037	−0.0028488009	0.277E−8	6.312E−05
	9	−0.0010485202	−0.0010485189	0.127E−8	2.332E−05
	10	−0.0003857973	−0.0003857968	0.560E−9	8.590E−06
	12	−0.0000522167	−0.0000522166	0.101E−9	1.160E−06
0.10	6	−0.0237196488	−0.0237196269	0.219E−7	9.037E−04
	8	−0.0032277936	−0.0032277878	0.581E−8	1.276E−04
	9	−0.0011880860	−0.0011880834	0.267E−8	4.720E−05
	10	−0.0004371602	−0.0004371590	0.118E−8	1.740E−05
	12	−0.0000591692	−0.0000591690	0.212E−9	2.350E−06
0.15	8	−0.0036571525	−0.0036571434	0.912E−8	1.932E−04
	9	−0.0013462223	−0.0013462180	0.421E−8	1.461E−05
	10	−0.0004953602	−0.0004953584	0.186E−8	2.635E−05
0.20	8	−0.0041435608	−0.0041435480	0.127E−7	2.585E−04
	9	−0.0015253979	−0.0015253920	0.590E−8	9.568E−05
	10	−0.0005613073	−0.0005613047	0.285E−8	3.529E−05

	$t = 0.05$	$t = 0.1$	$t = 0.15$	$t = 0.20$
$L_2 \times 10^6$	0.094519	0.188815	0.282694	0.375984
$L_\infty \times 10^6$	0.054802	0.109182	0.162013	0.212648



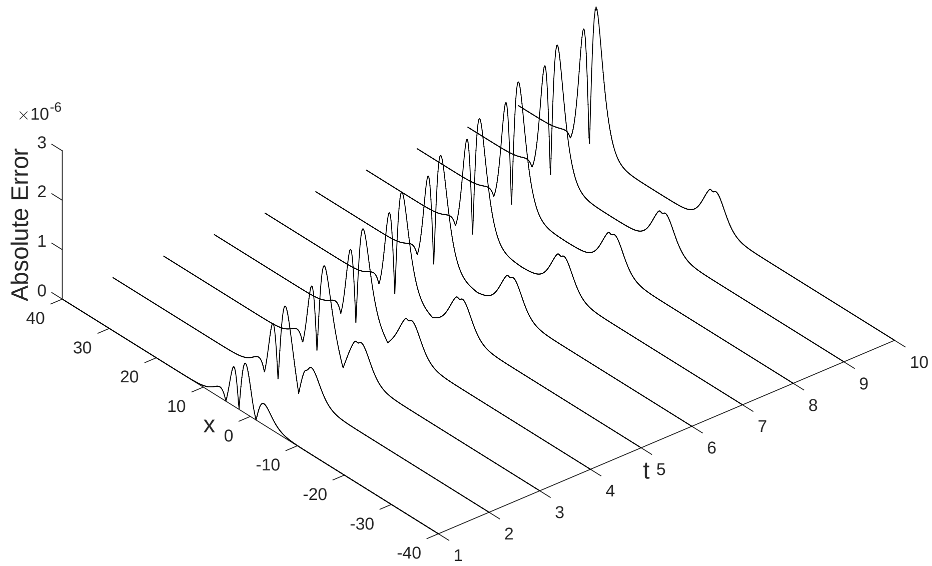


Figure 2: Soliton wave progression (*upper*) and absolute error (*bottom*) of the mDP equation for $h = 0.1,\ \Delta t = 0.001$

method. The presented method's efficiency and power were demonstrated by comparing the values of the numerical and the exact solution at various times and positions, in addition to the calculation of the error norms L_2 and L_∞. The numerical results calculated were compared with the [16] study using the quartic B-spline collocation method. The results indicate that the absolute error found in the proposed method is significantly smaller than those found in the [16]. As can be seen from the Tables 1 and 4, the soliton wave resulting from the solution of the mDP equation moves faster than the soliton wave formed by the solution of mCH. It has been observed that the applied method preserves the physical structure of the solution very well, and it is also speedy and effective since tiny h and Δt are not used.

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1412066
(**2024**) **5**(2) 159 – 172
Research Article

$\phi-$Multiplicative Calculus

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Received: 29 December 2023 **Accepted:** 21 May 2024

Abstract: In this paper, we present a novel mathematical framework termed "ϕ-multiplicative calculus", which serves as a Golden Fibonacci calculus to fundamental concepts in multiplicative calculus. This innovative calculus introduces a parameter ϕ (Golden ratio), offering a nuanced extension of traditional calculus. Our work encompasses the establishment of ϕ-multiplicative calculus and the demonstration of essential theorems concerning derivatives, integrals, and their operation properties within this mathematical framework. The paper contributes to the academic discourse by providing a comprehensive exploration of the proposed ϕ-multiplicative calculus, presenting a robust foundation for further investigations in this specialized mathematical domain.

Keywords: Fibonacci numbers, multiplicative derivative, Golden calculus, integral.

1. Introduction

The most practical mathematical theory, differential and integral calculus, was developed separately by Isaac Newton and Gottfried Wilhelm Leibnitz in the second part of the 17th century [4]. Then, Leonard Euler redirected calculus by making the idea of function essential, and so created analysis [5].

During the time from 1967 to 1970, Michael Grossman and Robert Katz defined a new type of derivative and integral, reversing the roles of subtraction and addition and establishing a new calculus known as multiplicative calculus. It is also known as an alternate or non-Newtonian calculus at times. Unfortunately, multiplicative calculus isn't as well known as Newton's and Leibnitz's calculus, despite the fact that it completely answers all of the requirements demanded of a calculus theory. Multiplicative calculus has a more limited range of applications than Newton and Leibniz calculus. It indeed covers only positive functions. Therefore, one might question the rationale behind developing a new tool with a restrictive scope when a broader, well-developed tool is already in existence. Bashirov and et al. respond to this question similarly to why

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2020 *AMS Mathematics Subject Classification*: 11B39, 26A33

mathematicians use polar coordinates when there is already a rectangular coordinate system that well describes the points on a plane. Bashirov and et al. believe that multiplicative calculus can be particularly useful as a mathematical tool for economics and finance due to the interpretation given to the multiplicative derivative [1, 10, 21]. Subsequently, the multiplicative finite difference method was developed for computing the numerical solutions of high-order multiplicative limit value problems [19]. Furthermore, applications such as modeling multiplicative differential equations, analyzing multiplicative gradients in noisy images, exploring the impact of multiplicative analysis on biomedical image analysis, and investigating double multiplicative integrals have been seen in the literature [2, 3, 6, 9, 16].

Özvatan develops the Golden Fibonacci calculus in her master's thesis under the supervisor of Pashayev, and numerous applications of this calculus are achieved. The calculus is built around the Golden derivative as a finite difference operator with Golden and Silver ratio bases, allowing us to introduce Golden polynomials and Taylor expansions in terms of these polynomials. The Golden binomial and its expansion in terms of Fibonomial coefficients is derived. They demonstrated that Golden binomials correspond to Carlitz' characteristic polynomials. The Golden-heat and Golden-wave equations are introduced and solved using Golden Fibonacci exponential functions and associated whole functions. They build the higher order Golden Fibonacci calculus by presenting higher order Golden Fibonacci derivatives that are connected to powers of golden ratio. This calculus has higher order Fibonacci numbers, higher Golden periodic functions, and higher Fibonomials. They present the generating function for a new form of polynomial, the Bernoulli-Fibonacci polynomials, and investigate their characteristics using the Golden Fibonacci exponential function [17].

As mentioned above, valuable studies have been conducted on the ϕ-multiplicative calculus. As is known, the classical derivative method relies on the definition of a limit for the derivation of a function. Therefore, the classical differentiation method can introduce errors. However, since the Golden Fibonacci calculation is not dependent on the limit definition and performs algebraic operations, there is no margin for error. Similarly, in the golden product calculation, no error is incurred. ϕ-multiplicative calculus exhibits a narrower domain of application in comparison to Golden Fibonacci calculus. However, although the use of the natural logarithm in the ϕ-multiplicative calculation is expected to increase the time, it completes solutions more quickly in examples involving exponential functions, often demonstrating a more effective solution to specific problems. Thus, it provides an alternative approach to problem-solving. Against this backdrop, the current study aims to spotlight ϕ-multiplicative calculus within the realm of analysis and elucidate its applications.

Now, we examine the necessary information about Golden Fibonacci calculus.

The Fibonacci numbers satisfy the recursion relation

$$F_{n+2} = F_{n+1} + F_n$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. First few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13.

The Binet formule of the Fibonacci sequence is

$$F_n = \frac{\phi^n - \hat{\phi}^n}{\phi - \hat{\phi}},$$

where

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1,618033 \quad \text{and} \quad \hat{\phi} = \frac{1-\sqrt{5}}{2} = -\frac{1}{\phi}$$

are roots of quadratic equation of the Fibonacci sequence

$$x^2 - x - 1 = 0.$$

The number ϕ is known as the Golden ratio. The Golden ratio has been applied in an extensive variety, from natural occurrences to architecture and music. Some authors have satisfied and studied many generalizations of Fibonacci numbers. More information can be found in [12].

Let's define a Fibonomial Calculus for the sequence $\{F_n\}_{n\geq 0}$ in order to explain what it is:

- $F-$factorial: $F_n! = F_n F_{n-1} F_{n-2} \dots F_2 F_1$, $F_0! = 1$.
- Fibonomial coefficients: $\binom{n}{k}_F = \frac{F_n!}{F_{n-k}! F_k!}$, $\binom{n}{0}_F = 1$.

  So, the following identity is valid:

$$\binom{n}{k}_F = \binom{n}{n-k}_F.$$

- The binomial theorem for the $F-$analog given by

$$(x +_F y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}.$$

- The $F-$exponential function e_F^x defined by

$$e_F^x = \sum_{n=0}^{\infty} \frac{x^n}{F_n!}.$$

Further insights can be found in the works of [13, 14, 17].

Let $f:\mathbb{R}-\{0\}\to\mathbb{R}$. The Golden derivative operator D_F of $f(x)$ is given as

$$D_F[f(x)]=\frac{f(\phi x)-f(\hat{\phi}x)}{(\phi-\hat{\phi})x}=\frac{f(\phi x)-f(\hat{\phi}x)}{x\sqrt{5}}. \tag{1}$$

The Golden derivative operator is a linear operator because the following conditions apply for any pair of functions f and g and scalar α;

$$D_F[f(x)+g(x)]=D_F[f(x)]+D_F[g(x)],$$

$$D_F[\alpha f(x)]=\alpha D_F[f(x)].$$

The Golden derivative operator D_F on x^n yields

$$D_F[x^n]=F_n x^{n-1}.$$

This, $F-$exponential function under Golden derivative are given by

$$D_F[e_F^{ax}]=ae_F^{ax} \quad (a \quad \text{any constant}).$$

The $F-$analogues of the sine and cosine functions can be described in terms of the exponential function by analogy with their well-known Euler formulas:

$$\sin_F x=\frac{e_F^{ix}-e_F^{-ix}}{2i} \quad \text{and} \quad \cos_F x=\frac{e_F^{ix}+e_F^{-ix}}{2},$$

and the Golden derivatives of these equations are as follows:

$$D_F[\sin_F \lambda x]=\lambda\cos_F \lambda x \quad \text{and} \quad D_F[\cos_F \lambda x]=-\lambda\sin_F \lambda x.$$

The Golden Leibnitz rule using the Golden derivative operator D_F is derived

$$D_F[f(x)g(x)]=D_F[f(x)]g(\phi x)+f(\hat{\phi}x)D_F[g(x)].$$

The Golden derivative of the quotient of $f(x)$ and $g(x)$ may now be computed

$$D_F\left[\frac{f(x)}{g(x)}\right]=\frac{D_F[f(x)]g(\phi x)-f(\phi x)D_F[g(x)]}{g(\phi x)g(\hat{\phi}x)}.$$

Consult prior studies for further details [13, 17, 18].

The function $H(x)$ is called the Golden antiderivative of any function $h(x)$ if $D_F[H(x)]=h(x)$. It is indicated by

$$H(x)+C=\int h(x)d_F x,$$

where C is the constant term. Let a be a real number. The Jackson integral of $h(x)$ is defined to be the series

$$\int_b^a h(x)d_F x = \int_0^a h(x)d_F x - \int_0^b h(x)d_F x,$$

where

$$H(x) = \int_0^y h(x)d_F x = y\sqrt{5}\sum_{k=0}^{\infty} \frac{\hat{\phi}^k}{\phi^{k+1}} f\left(\frac{\hat{\phi}^k}{\phi^{k+1}} y\right).$$

If $H(x)$ is an antiderivative of $h(x)$ and $H(x)$ is continuous at $x = 0$, we have

$$\int_b^a D_F[h(x)]d_F x = h(a) - h(b),$$

where $0 \leq b < a \leq \infty$. Refer to the works of [11, 18, 20].

Now, we will look at the information about multiplicative derivatives that are required.

A real function f is said to be differentiable at a point $x \in \mathbb{R}$ if and only if f is defined on some open interval I containing x and

$$D[f(x)] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

exists. In that case, $f(x)$ is referred to as the derivative of f at x.

Here, first we write $f(x+h)/f(x)$ instead of $f(x+h) - f(x)$ in the (2) equation. Then, if $1/h$ is substituted for h, the multiplicative derivative is obtained as follows:

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, the multiplicative derivative $D^*[f(x)]$ of f at $x \in A$ is defined

$$\begin{aligned} D^*[f(x)] &= \lim_{h \to 0} \left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} \\ &= \lim_{h \to 0} e^{\ln\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}} \\ &= e^{\lim_{h \to 0} \frac{\ln(f(x+h)) - \ln(f(x))}{h}} \\ &= e^{D[\ln(f(x))]} = e^{\frac{D[f(x)]}{f(x)}}. \end{aligned}$$

The multiplicative integral of f is represented by the symbol

$$* \int f(x)^{dx}.$$

If f is a positive function and $\ln f$ on $[a,b]$ is integrable, then f on $[a,b]$ has a multiplicative integral, which is defined by

$$* \int_b^a f(x)^{dx} = e^{\int_b^a \ln f(x) dx}, \quad 0 < b < a.$$

Reference previous works for more insights [7, 8, 15, 22].

2. ϕ-Multiplicative Calculus

2.1. Golden Multiplicative Derivative

Subtraction and division are the operations in the difference quotient in (1). The multiplicative derivative of a function g, on the other hand, is based on the ratio (3). This is as follows similar to the difference quotient (1) with subtraction of $f(\phi x) - f(\hat{\phi} x)$ replaced with division by $f(\phi x)/f(\hat{\phi} x)$ and division by $x\sqrt{5}$ replaced with taking an $1/(x\sqrt{5})$ power:

$$\left(\frac{f(\phi x)}{f(\hat{\phi} x)}\right)^{\frac{1}{x\sqrt{5}}}. \tag{3}$$

Definition 2.1 *Assume that the function f is Golden differentiable and positive ($f(\phi x) > 0$ for all x). The Golden multiplicative derivative of f is defined as follows:*

$$D_F^*[f(x)] = \left(\frac{f(\phi x)}{f(\hat{\phi} x)}\right)^{\frac{1}{x\sqrt{5}}}. \tag{4}$$

Theorem 2.2 *If a positive function f is D_F- differentiable at x, then it is also D_F^*- differentiable at x, and*

$$D_F^*[f(x)] = e_F^{D_F[\ln(f(x))]} \tag{5}$$

Proof We may compute the derivative in equation (4) using what we know about the Golden derivative of f:

$$\begin{aligned} D_F^*[f(x)] = \left(\frac{f(\phi x)}{f(\hat{\phi} x)}\right)^{\frac{1}{x\sqrt{5}}} &= e_F^{\ln\left(\frac{f(\phi x)}{f(\hat{\phi} x)}\right)^{\frac{1}{x\sqrt{5}}}} \\ &= e_F^{\frac{\ln(f(\phi x)) - \ln(f(\hat{\phi} x))}{x\sqrt{5}}} \\ &= e_F^{D_F[\ln(f(x))]}, \end{aligned}$$

where $\ln f(x) = (\ln o f)(x)$. □

Higher-order Golden multiplicative derivatives are simply the Golden multiplicative derivative of a Golden multiplicative derivative. You would use the same Golden multiplicative derivative rules that you learned for finding the first Golden multiplicative derivative of a function.

Corollary 2.3 *Let f positive function. The nth Golden multiplicative derivative of f is given by*

$$D_F^{*(n)}[f(x)] = e_F^{D_F^{(n)}[\ln(f(x))]}, \quad n = 0, 1, 2, \ldots.$$

Proof Utilizing equality (5), we get second Golden multiplicative derivative as

$$\begin{aligned} D_F^{**}[f(x)] &= e_F^{D_F[\ln(e_F^{D_F[\ln(f(x))]})]} \\ &= e_F^{D_F[D_F[\ln(f(x))]]} \\ &= e_F^{D_F^2[\ln(f(x))]}. \end{aligned}$$

To identify further high-level derivations, we continue with the same transaction:

$$\begin{aligned} D_F^{*(3)}[f(x)] &= e_F^{D_F[\ln(e_F^{D_F^2[\ln(f(x))]})]} \\ &= e_F^{D_F[D_F^2[\ln(f(x))]]} \\ &= e_F^{D_F^3[\ln(f(x))]}. \end{aligned}$$

By induction method, if the nth Golden multiplicative derivative of f exists at x, then it is acquired by

$$D_F^{*(n)}[f(x)] = e_F^{D_F^{(n)}[\ln(f(x))]}.$$

□

2.2. The Operation Properties of the Golden Multiplicative Derivative

Here are a few guidelines that are supported by Definition 2.1. Assume that, λ, μ is a positive constant and that f and g are D_F^*−differentiable. The following list may thus be displayed with ease:

- $D_F^*[\lambda f(x)] = D_F^*[f(x)]$ (Constant rule),
- $D_F^*[f(x)g(x)] = D_F^*[f(x)]D_F^*[g(x)]$ (Product rule),
- $D_F^*[f(x)/g(x)] = D_F^*[f(x)]/D_F^*[g(x)]$ (Quotient rule),

- $D_F^*[fog(x)] = D_{F,g}^*[f(g(x))]^{D_F[g(x)]}$ (Chain rule),

- $D_F^*[f(x)^{g(x)}] = (D_F^*[f(x)])^{g(\phi x)} \left(f(\hat{\phi}x)\right)^{D_F[g(x)]}$ (Power rule).

The proofs of the rules are shown as follows:

- The proof of the constant rule is

$$\begin{aligned} D_F^*[\lambda f(x)] &= \left(\frac{\lambda f(\phi x)}{\lambda f(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= \left(\frac{f(\phi x)}{f(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= D_F^*[f(x)]. \end{aligned}$$

- The proof of the product rule is

$$\begin{aligned} D_F^*[f(x)g(x)] &= \left(\frac{f(\phi x)g(\phi x)}{f(\hat{\phi}x)g(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= \left(\frac{f(\phi x)}{f(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \left(\frac{g(\phi x)}{g(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= D_F^*[f(x)]D_F^*[g(x)]. \end{aligned}$$

- The proof of the quotient rule is

$$\begin{aligned} D_F^*[f(x)/g(x)] &= \left(\frac{f(\phi x)/g(\phi x)}{f(\hat{\phi}x)/g(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= \left(\frac{f(\phi x)}{f(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} / \left(\frac{g(\phi x)}{g(\hat{\phi}x)}\right)^{\frac{1}{x\sqrt{5}}} \\ &= D_F^*[f(x)]/D_F^*[g(x)] \end{aligned}$$

- The proof of the chain rule is

$$\begin{aligned} D_F^*[fog(x)] &= \left(\frac{f(g(\phi x))}{f(g(\hat{\phi}x))}\right)^{\frac{1}{x\sqrt{5}}} \\ &= \left(\frac{f(g(\phi x))}{f(g(\hat{\phi}x))}\right)^{\frac{1}{g(\phi x)-g(\hat{\phi}x)} \frac{g(\phi x)-g(\hat{\phi}x)}{x\sqrt{5}}} \\ &= D_{F,g}^*[f(g(x))]^{D_F[g(x)]}. \end{aligned}$$

- The proof of the power rule is

$$D_F^*[f(x)^{g(x)}] = \left(\frac{f(\phi x)^{g(\phi x)}}{f(\hat{\phi}x)^{g(\hat{\phi}x)}}\right)^{\frac{1}{x\sqrt{5}}}$$

$$= \left(\frac{f(\phi x)^{g(\phi x)}}{f(\hat{\phi}x)^{g(\phi x)}}\frac{f(\hat{\phi}x)^{g(\phi x)}}{f(\hat{\phi}x)^{g(\hat{\phi}x)}}\right)^{\frac{1}{x\sqrt{5}}}$$

$$= \left(\frac{f(\phi x)}{f(\hat{\phi}x)}\right)^{\frac{g(\phi x)}{x\sqrt{5}}} \left(f(\hat{\phi}x)\right)^{\frac{g(\phi x)-g(\hat{\phi}x)}{x\sqrt{5}}}$$

$$= (D_F^*[f(x)])^{g(\phi x)} \left(f(\hat{\phi}x)\right)^{D_F[g(x)]}.$$

Theorem 2.4 *Let $f(x)$ and $g(x)$ be two functions, for $n \in \mathbb{Z}^+$, then*

$$D_F^{*(n)}[f(x)^{g(x)}] = e_F^{\sum_{k=0}^{n}\binom{n}{k}_F D_F^k[\ln f(\phi^{n-k}x)]D_F^{n-k}[g(x)]}.$$

Proof We can prove it by induction method. It is obviously true for $n = 1$. Let's assume it is true for $n = m$. Let's show that it is true for $n = m + 1$.

$$D_F^{*(m+1)}[f(x)^{g(x)}] = D_F^*[D_F^{*(m)}[f(x)^{g(x)}]]$$

$$= D_F^*[e_F^{\sum_{k=0}^{m}\binom{m}{k}_F D_F^k[\ln f(\phi^{m-k}x)]D_F^{m-k}[g(x)]}]$$

$$= e_F^{D_F[\sum_{k=0}^{m}\binom{m}{k}_F D_F^k[\ln f(\phi^{m-k}x)]D_F^{m-k}[g(x)]]}$$

$$= e_F^{\sum_{k=0}^{m+1}\binom{m+1}{k}_F D_F^k[\ln f(\phi^{m-k+1}x)]D_F^{m-k+1}[g(x)]}.$$

□

Example 2.5 *Find the golden multiplicative derivative of the exponential function $f(x) = \alpha^x$ with $\alpha > 0$.*

$$D_F^*[f(x)] = D_F^*[\alpha^x] = e_F^{D_F[\ln \alpha^x]} = e_F^{\ln \alpha D_F[x]} = \alpha.$$

In the Table 1, we compare the classical derivative, multiplicative derivative, Golden derivative, and Golden multiplicative derivative of of an arbitrary function.

2.3. Golden Multiplicative Antiderivative (Integral)

This part aims to explore the fundamental principles of the Golden multiplicative integral, shedding light on its diverse applications in various mathematical contexts.

Table 1: Some Golden multiplicative derivative

$f(x)$	$D[f(x)]$	$D^*[f(x)]$	$D_F[f(x)]$	$D_F^*[f(x)]$
t	0	1	0	1
te^{nx}	nte^{nx}	e^n	$t\sum_{m=0}^{\infty}\frac{F_m(nx)^m}{m!}$	e_F^n
ta^x	$ta^x\ln a$	a	$t\frac{a^{\phi x}-a^{\hat{\phi}x}}{x\sqrt{5}}$	a
tx^n	tnx^{n-1}	$e^{\frac{n}{x}}$	tF_nx^{n-1}	$e_F^{nD_F[\ln x^n]}$
$te^{h(x)}$	$tD[h(x)]e^{h(x)}$	$e^{D[h(x)]}$	$t\sum_{m=0}^{\infty}\frac{F_m(h(x))^m}{m!}$	$e_F^{D_F[h(x)]}$
$\frac{1}{h(x)}$	$\frac{-D[h(x)]}{h^2(x)}$	$e^{\frac{-D[h(x)]}{h^3(x)}}$	$\frac{-D_F[h(x)]}{h(\phi x)h(\hat{\phi}x)}$	$e_F^{D_F[-\ln(h(x))]}$
$t(h(x))^n$	$ntD[h(x)](h(x))^{n-1}$	$e^{\frac{nD[h(x)]}{h(x)}}$	$t\frac{(h(\phi x))^n-(h(\hat{\phi}x))^n}{x\sqrt{5}}$	$e_F^{nD_F[\ln(h(x))]}$
$e^{\sin_F bx}$	$b\cos bxe^{\sin_F bx}$	$e^{b\cos_F bx}$	$\frac{e^{\sin_F b\phi x}-e^{\sin_F b\hat{\phi}x}}{x\sqrt{5}}$	$e_F^{D_F[\sin_F bx]}$
$e^{\cos_F bx}$	$-b\sin_F bxe^{\cos_F bx}$	$e^{-b\sin_F bx}$	$\frac{e^{\cos_F b\phi x}-e^{\cos_F b\hat{\phi}x}}{x\sqrt{5}}$	$e_F^{D_F[\cos_F bx]}$

Definition 2.6 *Let h be a positive, bounded function on the range $0 < a < b$: The Golden multiplicative integral, the $F-$analogue of the multiplicative integral, may be defined by*

$$*\int h(x)^{d_Fx} = e_F^{\int \ln h(x)d_Fx}$$

and the definition of the definite golden multiplicative integral is

$$*\int_b^a h(x)^{d_Fx} = e_F^{\int_0^a \ln h(x)d_F(x)-\int_0^b \ln h(x)d_F(x)}.$$

2.4. The Operation Properties of Golden Multiplicative Integral

If f and g are $F-$integrable on $[a,b]$, we can then simply demonstrate the following rules of $F-$integral:

- $*\int_b^a (f(x)^k)^{d_Fx} = *\int_b^a (f(x)^{d_Fx})^k$ (Constant rule),
- $*\int_b^a (f(x)g(x))^{d_Fx} = *\int_b^a f(x)^{d_Fx} * \int_b^a g(x)^{d_Fx}$ (Product rule),
- $*\int_b^a (f(x)/g(x))^{d_Fx} = *\int_b^a f(x)^{d_Fx} / *\int_b^a g(x)^{d_Fx}$ (Quotient rule),
- $*\int_b^a f(x)^{d_Fx} = *\int_b^c f(x)^{d_Fx} * \int_c^a f(x)^{d_Fx}, \quad b \le c \le a.$

The proofs of the rules are shown as follows:

- The proof of the constant rule is

$$\begin{aligned} * \int_b^a (f(x)^k)^{d_F x} &= e_F^{\int_b^a \ln f(x)^k d_F x} \\ &= e_F^{k \int_b^a \ln f(x) d_F x} \\ &= \left(e_F^{\int_b^a \ln f(x) d_F x} \right)^k \\ &= * \int_b^a (f(x)^{d_F x})^k. \end{aligned}$$

- The proof of the product rule is

$$\begin{aligned} * \int_b^a (f(x)g(x))^{d_F x} &= e_F^{\int_b^a \ln(f(x)g(x)) d_F x} \\ &= e_F^{\int_b^a \ln f(x) d_F x + \int_b^a \ln g(x) d_F x} \\ &= e_F^{\int_b^a \ln f(x) d_F x} e_F^{\int_b^a \ln g(x) d_F x} \\ &= * \int_b^a f(x)^{d_F x} * \int_b^a g(x)^{d_F x}. \end{aligned}$$

- The proof of the quotient rule is

$$\begin{aligned} * \int_b^a (f(x)/g(x))^{d_F x} &= e_F^{\int_b^a \ln(f(x)/g(x)) d_F x} \\ &= e_F^{\int_b^a \ln f(x) d_F x - \int_b^a \ln g(x) d_F x} \\ &= \frac{e_F^{\int_b^a \ln f(x) d_F x}}{e_F^{\int_b^a \ln g(x) d_F x}} \\ &= * \int_b^a f(x)^{d_F x} / * \int_b^a g(x)^{d_F x}. \end{aligned}$$

-

$$\begin{aligned} * \int_b^a f(x)^{d_F x} &= e_F^{\int_0^a \ln f(x) d_F x - \int_0^b \ln f(x) d_F x} \\ &= e_F^{\int_0^c \ln f(x) d_F x - \int_0^b \ln f(x) d_F x + \int_0^a \ln f(x) d_F x - \int_0^c \ln f(x) d_F x} \\ &= e_F^{\int_b^c \ln f(x) d_F x + \int_c^a \ln f(x) d_F x} \\ &= * \int_b^c f(x)^{d_F x} * \int_c^a f(x)^{d_F x}. \end{aligned}$$

Example 2.7 *Let* $f(x) = e_F^{\cos_F(\lambda x)}$*, where* λ *is a constant. Then Golden multiplicative integral* *of $f(x)$ is obtained by*

$$
\begin{aligned}
{}^{*}\!\int \left(e_F^{\cos_F(\lambda x)}\right)^{d_F x} &= e_F^{\int \ln e_F^{\cos_F(\lambda x)} d_F x} \\
&= e_F^{\int \cos_F(\lambda x) d_F x} \\
&= e_F^{\int \cos_F(\lambda x) d_F x} \\
&= e_F^{\frac{1}{\lambda}\sin_F(\lambda x)}.
\end{aligned}
$$

Example 2.8 *Let $f(x) = e_F^{\lambda x}$, where $\lambda \in \mathbb{Z}^{+}$. Then Golden multiplicative integral of $f(x)$ is obtained by*

$$
\begin{aligned}
{}^{*}\!\int \left(e_F^{\lambda x}\right)^{d_F x} &= e_F^{\int \ln e_F^{\lambda x} d_F x} \\
&= e_F^{\int \lambda x d_F x} \\
&= e_F^{c} e_F^{\frac{\lambda}{F_2} x^2}.
\end{aligned}
$$

In the Table 2, we compare the classical integral, multiplicative integral, Golden integral, and Golden multiplicative integral of of an arbitrary function.

Table 2: Some Golden Multiplicative Integral

$f(x)$	$\int f(x)dx$	$*\int f(x)^{dx}$	$\int f(x)d_F x$	$*\int f(x)^{d_F x}$
1	x	e^c	x	e_F^c
t	tx	$e^c t^x$	tx	$e_F^c t^x$
e^{nx}	$\frac{e^{nx}}{n}$	$e^c e^{\frac{nx^2}{2}}$	$\frac{e_F^{nx}}{n}$	$e_F^c e_F^{\frac{nx^2}{F_2}}$
$e^{\cos x}$	–	$e^c e^{\sin x}$	–	$e_F^c e_F^{\sin x}$
$e^{\sin x}$	–	$e^c e^{-\cos x}$	–	$e_F^c e_F^{-\cos x}$

3. Conclusion

In conclusion, our paper introduces and establishes the ϕ-multiplicative calculus, a novel mathematical framework that extends fundamental concepts in multiplicative calculus by incorporating the Golden ratio (ϕ) as a key parameter. We have successfully demonstrated the essential theorems pertaining to derivatives, integrals, and operational properties within the ϕ-multiplicative calculus. This work contributes significantly to the academic discourse by providing a comprehensive exploration of this specialized mathematical domain, thereby laying a solid foundation for future research and investigations in the field.

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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