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Dear Scientists,

We have prepared the second issue of the seventh year of our journal with your contributions and efforts.

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We thank all our colleagues for their contributions.

We look forward to your support from our esteemed researchers and authors in the next stages of our publication life.

We wish you a scientific life full of success..

Kind regards!

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Editor in-Chief

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PLANE KINEMATICS IN LORENTZIAN HOMOTHETIC MULTIPLICATIVE CALCULUS

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Abstract. In this study, Lorentzian plane homothetic multiplicative calculus kinematics is discussed. Lorentzian plane homothetic multiplicative calculus movement, the pole points of a point X relative to the moving and fixed plane are discussed. In this motion, the velocities and accelerations of a point X are obtained. In this motion, the relations between the velocities and accelerations of a point X are obtained. In addition, new theorems and results are given.

1. Introduction

Using different arithmetic operations based on classical analysis alternative analysis have also been described. In 1887, the Volterra type of analysis was determined by [1]. Since this new approach is based on multiplication, this analysis is called multiplicative analysis (also called multiplicative analysis). In recent years, studies have been carried out by revealing some areas for the application of this analysis [2, 3, 4].

After the definition of Volterra analysis, some new studies were conducted by Michael Grossman and Robert Katz between 1967 and 1970. As a result of the studies, new analysis called geometric analysis, bigeometric analysis and anageometric analysis were defined. Some basic definitions and concepts regarding this new analysis, also called non-Newtonian analysis, are given [5].There are also studies in which non-Newtonian analysis is applied. Among these analysis, Dick Stanley's geometric analysis was referred to as multiplicative analysis [6]. Later, in 2008, studies were conducted in which the basic concepts of multiplicative analysis were defined and some of its applications were discussed [7]. The aim of this article is to examine one-parameter lorentzian homothetic multiplicative analysis plane kinematics using matrices. Selahattin Aslan, Murat Bekar and Yusuf Yaylı defined multiplicative quaternions and achieved some results using quaternions [8]. Semra Kaya Nurkan, Ibrahim Gürgil and Murat Kemal Karacan are given in geometric calculus, vectors and their properties, matrix, determinant, vector product and Gram–Schmidt

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in geometric space [9]. Hasan ES gave plane kinematics in multiplicative calculus [10]. The aim of this article is to examine one-parameter Lorentzian homothetic multiplicative calculus plane kinematics using matrices.

2. BASIC CONCEPTS

In [28], the set of the multiplicative calculus real numbers $\mathbb{R}(G)$ are determined as

$$\mathbb{R}(G) = \left\{\exp(m) = e^m : m \in \mathbb{R}\right\}.$$

Then $(\mathbb{R}(G), \oplus, \otimes)$ is a field with multiplicative calculus(geometric) zero 1 and multiplicative calculus (geometric) identy e.

The relations between the basic multiplicative operations and ordinary arithmetic operations can be given for all $m, n \in \mathbb{R}(G)$ as

$$\begin{aligned}
m \oplus n &= mn, \\
m \ominus n &= \frac{m}{n}, \\
m \otimes n &= m^{\ln n} = n^{\ln m}, \\
m \oslash n &= x^{\frac{1}{\ln n}},\ n \neq 1, \\
\sqrt{m}^{G} &= e^{(\ln m)^{\frac{1}{2}}}, \\
m^{-1_G} &= e^{\frac{1}{\log m}}, \\
\sqrt{m^{2_G}}^{G} &= |m|^{G}, \\
m^{2_G} &= m \otimes m = m^{\ln m}, \\
m \otimes e &= m \\
m \oplus 1 &= m, \\
|m|^{G} &= \begin{cases} m & , \quad m > 1, \\ 1 & , \quad m = 1, \\ m^{-1} & , \quad m < 1, \end{cases}
\end{aligned}$$

Additionally, for each $e^m, e^n \in \mathbb{R}(G)$, the multiplicative addition and multiplicative multiplication operations can be given as follows

$$\begin{aligned}
e^m \oplus e^n &= e^{m+n} \\
e^m \otimes e^n &= e^{mn}
\end{aligned}$$

and thus we can write

$$\begin{aligned}
e^m \otimes e^n = e^{mn}&, e^m \oplus e^n = e^{m+n}, \\
e^m \ominus e^n = e^{m-n}&, e^m \oslash e^n = e^{\frac{m}{n}}, \\
\sqrt{e^m}^{G} &= e^{\sqrt{m}}.
\end{aligned}$$

Positive geometric real numbers and negative geometric real numbers are defined as

$$\mathbb{R}^{+}(G) = \{m \in \mathbb{R}(G) : m > 1\}$$

and

$$\mathbb{R}^{-}(G) = \{m \in \mathbb{R}(G) : 0 < m < 1\},$$

respectively [8, 9, 10, 28].

The sentence $\mathbb{R}^2(G)$ is defined as follows

$$\mathbb{R}^2(G) = \{s^\circ = (e^{s_1}, e^{s_2}) : e^{s_1}, e^{s_2} \in \mathbb{R}(G)\} \subset \mathbb{R}^2$$

$$\begin{aligned} s^\circ \oplus z^\circ &= (e^{s_1}, e^{s_2}) \oplus (e^{z_1}, e^{z_2}) \\ &= (e^{s_1} \oplus e^{z_1}, e^{s_2} \oplus e^{z_2}) \\ &= \left(e^{s_1+z_1}, e^{s_2+z_2}\right) \end{aligned}$$

and the multiplicative scalar multiplication as

$$\begin{aligned} e^c \otimes s^\circ &= e^c \otimes (e^{s_1}, e^{s_2}) \\ &= (e^c \otimes e^{s_1}, e^c \otimes e^{s_2}) \\ &= (e^{cs_1}, e^{cs_2}), \end{aligned}$$

where $e^c \in \mathbb{R}(G),\ s^\circ,\ z^\circ \in \mathbb{R}^2(G)$.

Definition 2.1. The relationship between the multiplicative derivative and the classical derivative is determine as

$$h^{*(n)}(x) = e^{(\ln h(x))^{(n)}}.$$

[11, 12, 13, 17, 20, 25].

Definition 2.2. The multiplicative distance defined by [13, 25]. This allows to define the multiplicative distance $d^G(m,n)$ between $m, n \in \mathbb{R}^+(G)$ as

$$d^G(m,n) = \left|\frac{m}{n}\right|^G$$

[11, 12, 13, 25].

Definition 2.3. The relationship between trigonometry and multiplicative trigonometry is determine as $\sin_g \omega = e^{\sin\omega}, \cos_g \omega = e^{\cos\omega}, \tan_g \omega = e^{\tan\omega} = \frac{\sin_g \omega}{\cos_g \omega}$ [5, 6, 11, 12, 13, 14, 15, 30].

Definition 2.4. An $2x2$ multiplicative matrix is defined by

$$K = \begin{bmatrix} e^{k_{11}} & e^{k_{12}} \\ e^{k_{21}} & e^{k_{22}} \end{bmatrix}$$

where $e^{k_{11}}, e^{k_{12}}, e^{k_{21}}, e^{k_{22}} \in \mathbb{R}(G)$. Let K and M be two multiplicative matrices and $K \otimes M = N$ be the multiplication of these matrices, where

$$N = \begin{bmatrix} e^{k_{11}m_{11}+k_{12}m_{21}} & e^{k_{11}m_{12}+k_{12}m_{22}} \\ e^{k_{21}m_{11}+k_{22}m_{21}} & e^{k_{21}m_{12}+k_{22}m_{22}} \end{bmatrix}.$$

Definition 2.5. $2x2$ type identity matrix in multiplicative calculus is

$$I = \begin{bmatrix} e & 1 \\ 1 & e \end{bmatrix}.$$

If matrix D is a $2x2$ type matrix and $D^T \otimes D = D \otimes D^T = I$, then D is called a multiplicative orthogonal matrix.

3. PLANE KINEMATICS IN LORENTZIAN HOMOTHETIC MULTIPLICATIVE CALCULUS

Definition 3.1. The dot product in $\mathbb{R}^2(G)$ is determined as in the equation 3.1

$$\langle m,n\rangle_L^G = e^{m_1 n_1 - m_2 n_2}, \tag{3.1}$$

where $\langle m,n\rangle_L^G$ is the dot product in the multiplicative Lorentz sense and $m = (m_1, m_2)$, $n = (n_1, n_2) \in \mathbb{R}^2(G)$.

Definition 3.2. The norm of a multiplicative vector $m = (m_1, m_2)$ is

$$\|m\|_L^G = \sqrt{\langle m,m\rangle_L^G}^G = e^{\sqrt{m_1^2 - m_2^2}}. \tag{3.2}$$

Definition 3.3. The multiplicative unit circle $S^1(G)$ in $\mathbb{R}^2(G)$ can be defined as

$$\begin{aligned} S^1(G) &= \left\{ m = (m_1, m_2) \in \mathbb{R}^2(G) : \langle m,m\rangle_L^G = e \right\} \\ &= (\cosh_g \omega, \sinh_g \omega) = \left(e^{\cosh\omega}, e^{\sinh\omega}\right). \end{aligned} \tag{3.3}$$

Definition 3.4. Let $m = (e^{m_1}, e^{m_2})$ and $n = (e^{n_1}, e^{n_2})$ be unit vectors in $\mathbb{R}^2(G)$. Then the equation

$$\begin{bmatrix} \cosh_g \omega & \sinh_g \omega \\ \sinh_g \omega & \cosh_g \omega \end{bmatrix} \otimes \begin{bmatrix} e^{m_1} \\ e^{m_2} \end{bmatrix} = \begin{bmatrix} e^{n_1} \\ e^{n_2} \end{bmatrix} \tag{3.4}$$

represents a rotation in $\mathbb{R}^2(G)$ of the multiplicative vector m by a multiplicative angle $\omega \in \mathbb{R}$ in positive direction around the origin $O = (1,1)$ of the Cartesian coordinate system of $\mathbb{R}^2(G)$. We will call this rotation as multiplicative planar rotation. After this rotation multiplicative vector m turns to the multiplicative vector n as given [8]. Where $A(\omega) = \begin{bmatrix} \cosh_g \omega & \sinh_g \omega \\ \sinh_g \omega & \cosh_g \omega \end{bmatrix}$ is a rotation matrix in multiplicative plane.

Definition 3.5. The Lorentzian homothetic multiplicative plane equation of motion in $\mathbb{R}^2(G)$ is determine as,

$$\begin{aligned} y_1 &= x \otimes (h \otimes \cosh_g \omega) \oplus y \otimes (h \otimes \sinh_g \omega) \oplus c_1 \\ y_2 &= x \otimes (h \otimes \sinh_g \omega) \oplus y \otimes (h \otimes \cosh_g \omega) \oplus c_2 \end{aligned} \tag{3.5}$$

If ω, c_1, and c_2 are given by the functions of time parameter t, then this motion is called as a one-parameter Lorentzian homothetic multiplicative motion.

Definition 3.6. The equation of a one-parameter Lorentzian homothetic multiplicative motion in $\mathbb{R}^2(G)$ is defined by

$$Y(t) = B(t) \otimes X(t) \oplus C(t) \tag{3.6}$$

$$Y = \begin{bmatrix} e^{y_1} \\ e^{y_2} \end{bmatrix}, X = \begin{bmatrix} e^{x_1} \\ e^{x_2} \end{bmatrix}, C = \begin{bmatrix} e^{c_1} \\ e^{c_2} \end{bmatrix},$$

where Y and X are the position vectors of the same point R, respectively, for the multiplicative fixed and multiplicative moving systems, and C is the multiplicative translation vector.

If we take the multiplicative derivative of the 3.6 equation with respect to the parameter t. In that case the equation of

$$Y^\star = B^* \otimes X \oplus B \otimes X^* \oplus C^* \tag{3.7}$$

is obtained. Here, $V_a = Y^\star$ is called the absolute speed of the motion, $V_r = B \otimes X^*$ is determine the relative speed of the motion, and $V_f = B^* \otimes X \oplus C^*$ is defined the sliding speed of the motion.

We represent movements in the E_G^2 plane as $\frac{L_{MC}}{L'_{MC}}$; One of which is Lorentzian homothetic multiplicative fixed plane L'_{MC} and the other one is a Lorentzian homothetic multiplicative moving plane L_{MC} that moves relative to the fixed plane.

If the matrices B and C are functions of a parameter t, this motion is called a one-parameter Lorentzian homothetic multiplicative motion and is denoted by $B_1 = \frac{L_{MC}}{L'_{MC}}$. By taking the derivatives with respect to t in 3.7, we get

$$Y^{\star\star} = B^{**} \otimes X \oplus e^2 \otimes (B^* \otimes X^*) \oplus B \otimes X^{**} \oplus C^{**}, \tag{3.8}$$

$$b_a = b_r \oplus b_c \oplus b_f \tag{3.9}$$

where the velocities

$$b_a = Y^{**}, b_f = B^{**} \otimes X \oplus C^{**}, b_r = B \otimes X^{**} \text{ and } b_c = e^2 \otimes (B^* \otimes X^*) \tag{3.10}$$

are called multiplicative absolute acceleration, multiplicative sliding acceleration, multiplicative relative acceleration and multiplicative Coriolis accelerations, respectively.

Definition 3.7. Let X be a point in the plane L_{MC}. The speed of this point X while drawing its orbit in the plane L_{MC} is called relative speed. And this speed is defined by V_r.

Definition 3.8. The relationship between the speeds of motion B_1 is defined as

$$V_a = V_f \oplus V_r \tag{3.11}$$

If X is a fixed point in plane L_{MC} of motion B_1, V_r is zero in the multiplicative sense. Therefore $V_a = V_f$.

The expression $V_a = V_f \oplus V_r$ is called the law of velocities in the motion B_1.

Theorem 3.9. *In lorentzian homothetic multiplicative motion, the absolute velocity vector is equal to the sum of the sliding velocity vector and the relative velocity vectors. So it is*

$$V_a = V_f \oplus V_r.$$

4. POLES OF ROTATING AND ORBIT IN LORENTZIAN HOMOTHETİC MULTIPLICATIVE CALCULUS

Definition 4.1. In the sense of multiplicative calculus, the points where $V_f = 1$ are both L_{MC} and L'_{MC} fixed points. These points are called pole points of the movement.

Theorem 4.2. *In a motion B_1 whose angular velocity is not zero (in the sense of multiplicative calculus), there is a single point that remains constant in both L_{MC} and L'_{MC} at each time t.*

Proof. $V_r = 1$ because point X is fixed at L_{MC}. and since the same point X is fixed at L'_{MC}, $V_f = 1$. For such points the equation $V_f = 1$ gives

$$B^* \otimes X \oplus C^* = 1, \tag{4.1}$$

and

$$X = e^{-1} \otimes (B^*)^{m-inv} \otimes C^* \tag{4.2}$$

where $(B^*)^{m-inv}$ is the multiplacative inverse of B^*. Since

$$B = e^h \otimes \begin{bmatrix} e^{\cosh\omega} & e^{\sinh\omega} \\ e^{\sinh\omega} & e^{\cosh\omega} \end{bmatrix} = \begin{bmatrix} e^{h\cosh\omega} & e^{h\sinh\omega} \\ e^{h\sinh\omega} & e^{h\cosh\omega} \end{bmatrix}, \quad C = \begin{bmatrix} e^{c_1} \\ e^{c_2} \end{bmatrix},$$

$$B^* = \begin{bmatrix} e^{h'\cosh\omega + h\omega'\sinh\omega} & e^{h'\sinh\omega + h\omega'\cosh\omega} \\ e^{h'\sinh\omega + h\omega'\cosh\omega} & e^{h'\cosh\omega + h\omega'\sinh\omega} \end{bmatrix}, \quad C^* = \begin{bmatrix} e^{c_1'} \\ e^{c_2'} \end{bmatrix}$$

we get $\det^G(B^*) = e^{(h')^2 - (h\omega')^2}$Thus B^* is regular and

$$(B^*)^{m-inv} = e^{\frac{1}{(h')^2-(h\omega')^2}} \otimes \begin{bmatrix} e^{h'\cosh\omega + h\omega'\sinh\omega} & e^{-h'\sinh\omega - h\omega'\cosh\omega} \\ e^{-h'\sinh\omega - h\omega'\cosh\omega} & e^{h'\cosh\omega + h\omega'\sinh\omega} \end{bmatrix}.$$

Therefore, the equation of $V_f = 1$ has only one X solution. This point X is the pole point of L_{MC}. Accordingly, from 4.2 equation the result

$$X = P = e^{\frac{1}{(h')^2-(h\omega')^2}} \otimes \begin{bmatrix} e^{-c_1'(h'\cosh\omega + h\omega'\sinh\omega) + c_2'(h'\sinh\omega + h\omega'\cosh\omega)} \\ e^{c_1'(h'\sinh\omega + h\omega'\cosh\omega) - c_2'(h'\cosh\omega + h\omega'\sinh\omega)} \end{bmatrix} \tag{4.3}$$

$$= e^{\frac{1}{(h')^2-(h\omega')^2}} \otimes \begin{bmatrix} e^{(-c_1'h' + c_2'\omega'h)\cosh\omega + (-c_1'h\omega' + c_2'h')\sinh\omega} \\ e^{(c_1'h\omega' - c_2'h')\cosh v + (c_1'h' - c_2'h\omega')\sinh\omega} \end{bmatrix} \tag{4.4}$$

is reached.

The pole point in the multiplicative fixed plane is

$$P' = B \otimes P \oplus C \tag{4.5}$$

setting these values in their planes and calculating we have

$$P' = e^{\frac{1}{(h')^2-(h\omega')^2}} \otimes \begin{bmatrix} e^{-c_1'h'h + h^2c_2'\omega'} \\ e^{h^2c_1'\omega' - h'hc_2'} \end{bmatrix} \oplus \begin{bmatrix} e^{c_1} \\ e^{c_2} \end{bmatrix} \tag{4.6}$$

$$= \begin{bmatrix} e^{\frac{-c_1'h'h + h^2c_2'\omega'}{(h')^2-(h\omega')^2} + c_1} \\ e^{\frac{h^2c_1'\omega' - h'hc_2'}{(h')^2-(h\omega')^2} + c_2} \end{bmatrix} \tag{4.7}$$

or as a vector

$$P' = \left(e^{\frac{-c_1'h'h + h^2c_2'\omega'}{(h')^2-(h\omega')^2} + c_1}, e^{\frac{h^2c_1'\omega' - h'hc_2'}{(h')^2-(h\omega')^2} + c_2} \right). \tag{4.8}$$

□

Here we assume that multiplicative $\omega^*(t) \neq 1$ for all t. That is, multiplicative angular velocity is not 1. In this case there exists a unique pole point in each of the moving and fixed planes of each moment t.

Corollary 4.1. If $\omega(t) = t$, then equation 4.3 will be obtained as

$$X = P = e^{\frac{1}{(h')^2-h^2}} \otimes \begin{bmatrix} e^{(-c_1'h' + c_2'h)\cosh\omega + (c_1'h - c_2'h')\sinh\omega} \\ e^{(c_1'h - c_2'h')\cosh\omega + (a'h' - b'h)\sinh\omega} \end{bmatrix}.$$

Corollary 4.2. For $\omega(t) = t$ and $h(t) = 1$, then equation 4.3 will be obtained as

$$X = P = \begin{bmatrix} e^{c_1'\sinh\omega - c_2'\cosh\omega} \\ e^{-c_1'\cosh\omega + c_2'\cosh\omega} \end{bmatrix}.$$

Corollary 4.3. Let $\omega(t) = t$, then equation 4.8 will be obtained as

$$P' = \left(e^{\frac{-c_1' h' h + h^2 c_2'}{(h')^2 - h^2} + c_1}, e^{\frac{h^2 c_1' - h' h c_2'}{(h')^2 - h^2} + c_2} \right).$$

Corollary 4.4. For $\omega(t) = t$ and $h(t) = 1$, then equation 4.8 will be obtained as

$$P' = \left(e^{-c_2' + c_1}, e^{-c_1' + c_2} \right).$$

Definition 4.3. The point $P = (p_1, p_2)$ is called multiplicative instantaneous rotation center or the pole at moment t of the one parameter motion $B_1 = L_{MC} / L'_{MC}$

Theorem 4.4. *The length of vector V_f is*

$$\|V_f\|_L^G = \exp\left(\sqrt{\left(\frac{h'}{h}\right)^2 - (\theta')^2} \, \|P'Y\|_L \right).$$

Proof. The pole point in multiplicative moving plane $Y = B \otimes X \oplus C$ implies that

$$X = (B)^{m-inv} \otimes \left(Y \oplus (e^{-1}) \otimes C) \right), \tag{4.9}$$

$$V_f = B^* \otimes X \oplus C^* \text{ and } B^* \otimes X \oplus C^* = 1$$

that leads to$X = P = e^{-1} \otimes (B^*)^{m-inv} \otimes C^*$. Now let us find pole points in multiplicative fixed plane. Then we have from equation

$$Y = B \otimes X \oplus C.$$

$Y' = P' = B \otimes \left(e^{-1} \otimes (B^*)^{m-imv} \otimes C^*) \oplus C \right)$, Hence, we get

$$C^* = B^* \otimes (B)^{m-inv} \otimes \left(C \oplus (e^{-1} \otimes P') \right)$$

we substitute this values in the equation $V_f = B^* \otimes X \oplus C^*$ we have $V_f = B^* \otimes (B)^{m-inv} \otimes P'Y$. Now let us calculate the value of $B^* \otimes (B)^{m-inv} \otimes P'Y$, where $P'Y = (e^{y_1 - p_1}, e^{y_2 - p_2})$, then

$$V_f = \begin{bmatrix} e^{\frac{h'}{h}(y_1 - p_1) - \omega'(y_2 - p_2)} \\ e^{\omega'(y_1 - p_1) + \frac{h'}{h}(y_2 - p_2)} \end{bmatrix}$$

or as a vector

$$V_f = \left(e^{\frac{h'}{h}(y_1 - p_1) + \omega'(y_2 - p_2)}, e^{\omega'(y_1 - p_1) + \frac{h'}{h}(y_2 - p_2)} \right). \tag{4.10}$$

then,

$$\|V_f\|_L^G = \exp\left(\sqrt{\left(\frac{h'}{h}\right)^2 - (\theta')^2} \, \|P'Y\|_L \right).$$

□

Corollary 4.5. If the scalar matrix h is constant, then the length of the sliding velocity vector is

$$\|V_f\|_L^G = \exp\left(|x| \, \|P'Y\|_L \right). \tag{4.11}$$

Corollary 4.6. The speed that occurs when drawing the curve (P) at point L_{MC} at X is called V_r. At the same time, V_a is the speed that occurs when drawing the $(P)'$ curve of this point in the plane L'_{MC}. These velocities are equal to each other at time t.

Proof. Since $V_f = 1$, it is concluded from expression $V_a = V_f \oplus V_r$ that $V_a = V_r$. □

Definition 4.5. The vector V_a is called multiplicative absolute acceleration vector with respect to the plane L'_{MC} of the point X and is denoted by b_a. Since $V_a = Y^*$ then $b_a = V_a^* = Y^{**}$.

Definition 4.6. Let $X \in L_{MC}$ be a fixed point in motion $B_1 = L_{MC}/L'_{MC}$. Multiplicative acceleration vector of X with respect to L'_{MC} is called multiplicative sliding acceleration vector. This multiplicative sliding acceleration vector is denoted by b_f.

Since acceleration of the multiplicative sliding acceleration X is a fixed point of L_{MC}, then $b_f = V_f^* = B^{**} \otimes C^{**}$.

5. Accelerations And Union Of Accelerations In Lorentzian Homothetic Multiplicative Calculus

Definition 5.1. We know that point X is multiplicative relative velocity vector V_r to L_{MC}. The vector b_r obtained by taking the derivative of V_r is called multiplicative relative acceleration vector of X in plane L_{MC}. This multiplicative relative acceleration vector is represented by b_r. Considering point X as a moving point in plane L_{MC}, matrix B is taken as constant

Theorem 5.2. *There is the following relationship between $b_a,\ b_r,\ b_c$ and b_f*

$$b_a = b_r \oplus b_c \oplus b_f.$$

Here $b_c = (e^2 \otimes (B^* \otimes X^*)$ is called multiplicative Corilois acceleration.

Corollary 5.1. Let X be a point in the plane L_{MC} . If point X is fixed at L_{MC}, then $b_a = b_f$.

Proof. Note that

$$V_a = B^* \otimes X \oplus B \otimes X^* \oplus C^*,$$

Differentiating the both sides we have

$$V_a^* = B^{**} \otimes X \oplus e^2 \otimes (B^* \otimes X^*) \oplus B \otimes X^{**} \oplus C^{**},$$

since the point X is constant its derivative is 1. Hence

$$\begin{aligned} b_a &= V_a^* \\ &= B^{**} \otimes X \oplus C^{**} \\ &= b_f. \end{aligned}$$

□

Theorem 5.3. *The result of the multiplicative inner product of vectors b_c and V_r is*

$$\langle b_c, V_r\rangle_L^G = \exp\left(2hh'({x'_1}^2 - {x'_1}^2\right). \tag{5.1}$$

Proof.

$$\begin{aligned} V_r &= B \otimes X^*, \\ b_c &= e^2 \otimes (B^* \otimes X^*), \end{aligned}$$

So it is obvious that

$$\langle b_c, V_r\rangle_L^G = \exp\left(2hh'({x'_1}^2 - {x'_1}^2\right).$$

□

Corollary 5.2. If the h value is taken as constant in 5.1 equation, then the Coriolis acceleration b_c is perpendicular to the relative velocity vector V_r at each instant moment t.

6. The Acceleration Poles Of The Motions

The solution of the equation $V_f^* = B^{**} \otimes X \oplus C^{**}$ gives us multiplicative acceleration pole of multiplicative motion.$V_f^* = B^{**} \otimes X \oplus C^{**}$ implies $X = e^{-1} \otimes (B^{**})^{m-inv} \otimes C^{**}$. Now calculating the matrices $e^{-1} \otimes (B^{**})^{m-inv}$ and C^{**}, and setting these in $X = P_1 = e^{-1} \otimes (B^{**})^{m-inv} \otimes C^{**}$, we obtain

$$X = P_1 = \begin{bmatrix} e^{\frac{1}{T}\left(c_1''(-r\cosh\omega + z\sinh\omega) - c_2''(r\sinh\omega + z\cosh\omega)\right)} \\ e^{\frac{1}{T}\left(c_1''(r\sinh\omega + z\cos\omega) + c_2''(-r\cosh\omega + z\sinh\omega)\right)} \end{bmatrix}, \tag{6.1}$$

where $(B^{**})^{m-inv}$ is the multiplicative inverse of B^{**}. Here P_1 is called multiplicative pole curve in multiplicative moving plane. If multiplicative pole curve in multiplicative fixed plane is denoted by P_1' we get

$$P_1' = B \otimes P_1 \oplus C \tag{6.2}$$

Hence

$$P_1' = \begin{bmatrix} e^{\frac{1}{T}\left(-hrc_1'' - hzc_2''\right) + c_1} \\ e^{\frac{1}{T}\left(hzc_1'' - hrc_2''\right) + c_2} \end{bmatrix} \tag{6.3}$$

where $r = h'' + h(\omega')^2, z = 2h'\omega' + h\omega'', T = r^2 - z^2$

Corollary 6.1. If $\omega(t) = t$, then equation 6.1 will be obtained as

(6.4)

$$X = P_1 = \begin{bmatrix} e^{\frac{1}{(h''+h)^2 - 4(h')^2}\left(c_1''(-(h''+h)\cosh\omega + 2h'\sinh\omega) - c_2''((h''+h)\sinh\omega + 2h'\cos\omega)\right)} \\ e^{\frac{1}{(h''+h)^2 - 4(h')^2}\left(c_1''((h''+h)\sinh\omega + 2h'\cosh\omega) + c_2''(-(h''+h)\cosh\omega + 2h'\sinh\omega)\right)} \end{bmatrix}$$

Corollary 6.2. If $\omega(t) = t$ and $h(t) = 1$, then equation 6.1 will be obtained as

$$X = P_1 = \begin{bmatrix} e^{-c_1''\cosh\omega + c_2''\sinh\omega} \\ e^{c_1''\sinh\omega - c_2''\cosh\omega} \end{bmatrix} \tag{6.5}$$

Corollary 6.3. If $\omega(t) = t$, then equation 6.3 will be obtained as

(6.6)

$$P_1' = \left(e^{\frac{1}{(h''+h)^2 - 4(h')^2}\left(h(h''+h)c_1'' - 2hh'c_2''\right) + c_1}, e^{\frac{1}{(h''+h)^2 - 4(h')^2}\left(-2hh'c_1'' - h(h''+h)c_2''\right) + c_2} \right).$$

Corollary 6.4. If $\omega(t) = t$ and $h(t) = 1$, then equation 6.3,will be obtained as

$$P_1' = \left(e^{-c_1'' + c_1}, e^{-c_2'' + c_2} \right). \tag{6.7}$$

7. CONCLUSIONS

In multiplicative Lorentz multiplicative homothetic motions, velocities in plane motion, the relationship between velocities, pole points, and pole curves are given. Additionally, multiplicative Lorentz accelerations and multiplicative Lorentz acceleration combinations have been found.

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NECESSARY CONDITION FOR IA-AUTOMORPHISMS IN LEIBNIZ ALGEBRAS

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Abstract. Let F denote the free Leibniz algebra, which is generated by the set $X = \{x_1, ..., x_n\}$ over the field K with characteristic 0. Let R be an ideal of F. This investigation begins by obtaining a specific matrix representation for the IA-automorphisms of the Leibniz algebra F/R'. Following this, we establish a necessary condition for an IA-endomorphism of F/R' to qualify as an IA-automorphism. This method is explicitly based on Dieudonné determinant.

1. Introduction

Consider the Leibniz algebra F, the free algebra of finite rank n over a field K. Let R be an ideal of F, and denote by R' the commutator subalgebra of R. The Leibniz algebra F/R' of rank n is defined in the usual way.

In their work [2], Bahturin and Nabiev established an explicit matrix representation for automorphisms of a Lie algebra L/R' that are congruent modulo R/R', where L is a free Lie algebra of rank n and R is an ideal of L. Shpilrain, in [9], provided a necessary condition for the invertibility of a matrix over the integral group ring of a free group, utilizing a non-commutative determinant. Initially given for free Lie algebras in [3], this condition was based on a non-commutative determinant.

Furthermore, in [14], the author and Ekici gave a criterion grounded in the Dieudonné determinant with some applications. Recently, [11] addressed the computation of valuations of Dieudonné determinants of matrices over discrete valuation skew fields, exploring two applications stemming from this problem.

Leibniz algebras, serving as potential non-(anti)commutative extensions of Lie algebras, were thoroughly examined in terms of homological algebra by Loday and Pirashvili in [7]. Numerous findings in Leibniz algebras highlight their close relationship with Lie algebras, prompting attempts to extend specific combinatorial results from varieties of Lie algebras to their Leibniz counterparts.

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In [8], Mikhalev and Umirbaev derived significant results regarding subalgebras of free Leibniz algebras. The author investigated the automorphisms of free Leibniz algebras with rank two in the work documented in [12]. Additionally, Papistas and Drensky, in their work [5] in 2005, examined automorphisms within the domain of a free left nilpotent Leibniz algebra with finite rank. Meanwhile, free metabelian Leibniz algebras were characterized in the reference [6].

On another note, explicit matrix forms for IA-automorphisms of free metabelian Leibniz algebras were established for rank 3 in [15] and for rank n in [16]. A recent study by the author in [13] contributed a necessary and sufficient condition for a set of n elements in F/R' to function as a generating set.

This study initially derives a matrix representation for the IA-automorphisms of the Leibniz algebra F/R', employing similar techniques as presented in [**?**]. Subsequently, we provide a necessary condition for the invertibility of a matrix belonging to $UL(F/R')$. This condition establishes a means for identifying non-automorphisms within the Leibniz algebra F/R'. Notably, our approach is explicitly grounded in a non-commutative determinant: the Dieudonné determinant. Furthermore, we present several applications of this methodology.

2. Preliminaries

Loday and Pirashvili described free Leibniz algebras in [7]. Consider the Leibniz algebra F generated freely by a set $\{x_1, ..., x_n\}$ over a field K of characteristic 0. Let $Ann(F)$ represent the ideal of F generated by elements $\{[x, x] : x \in F\}$. The algebra $F_{Lie} = F/Ann(F)$ is identified as a Lie algebra. The notation $Aut(F)$ refers to the automorphism group of F, while $IAut(F)$ designates the IA-automorphisms of F. These automorphisms induce the identity mapping on the quotient algebra F/F', where F' is the commutator ideal of F. Let R be a subalgebra of F, and designate R' as the derived subalgebra of R. The paper [7] introduces the universal enveloping algebra for the Leibniz algebra F. Denote by $UL(F)$, the universal enveloping algebra of F, i.e., the free associative algebra with the generating set $\{r_1, ..., r_n, l_1, ..., l_n\}$, where $l_i = l_{x_i}$ and $r_i = r_{x_i}$ the universal operators of left and right multiplication on x_i. These elements satisfy the following relations

$$(r_{x_i} + l_{x_i})l_{x_j} = 0$$

Denoted by Δ, the kernel of the homomorphism $\varepsilon : UL(F) \to K$ defined by $\varepsilon(r_{x_i}) = 0$, $\varepsilon(l_{x_i}) = 0$ for $i = 1, 2, ..., n$, that is augmentation ideal of $UL(F)$. That is also an $UL(F)$-module generated by r_{x_i}, l_{x_i}, where $i = 1, 2, ..., n$. We represent the mth associative power of Δ as Δ^m. Denoted by Δ_R, the ideal of $UL(F)$ is defined as the kernel of the natural homomorphism $\sigma_R : UL(F) \to UL(F/R)$.

Let $\widehat{a}$ represent the image of $a \in F/R$ under the natural homomorphism $F/R \to (F/R)_{Lie}$. Utilizing this homomorphism, we establish the mapping

$$\widehat{\ }: UL(F/R) \to U((F/R)_{Lie})$$

where $U((F/R)_{Lie})$ denotes the universal enveloping algebra of $(F/R)_{Lie}$. Throughout the subsequent discussion, we define the Lie algebra $(F/R)_{Lie}$ alongside its corresponding subalgebra in $U((F/R)_{Lie})$. This results in $\widehat{r_x} = \widehat{x}$ and $\widehat{l_x} = -\widehat{x}$. It is evident that the kernel of the homomorphism $\widehat{\ }$ is generated by $r_x + l_x$, $x \in F/R$. This kernel is denoted as $\Delta_{Ann(F/R)}$. According to the reference [7], the mapping

$$\delta : U((F/R)_{Lie}) \to UL(F/R)$$

is defined as $\delta(\widehat{x}) = r_x$. Notably, due to the equality $\widehat{\delta(\widehat{x})} = \widehat{r_x} = \widehat{x}$, we establish the identification of the algebra $U((F/R)_{Lie})$ with its corresponding subalgebra in $UL(F/R)$.

3. Automorphisms of F/R'

Consider the abelian Leibniz algebra R/R' that is freely generated by a set $\{a_1, a_2, \dots, a_n\}$ as a free K-module. Let F/R be a Leibniz algebra over K, functioning as a free K-module. The wreath product of Leibniz algebras R/R' and F/R is defined in a standard manner, akin to the case of Lie algebras [10]. Denoted as $W = (R/R')wr(F/R)$, it takes the form $W = F/R \oplus I_{R/R'}$, where it is the semidirect sum of F/R and the free F/R-module $I_{R/R'}$ with the free generating set $\{a_1, a_2, \dots, a_n\}$. Furthermore, R/R' is not only a module on F/R but also a $UL(F/R)$-module, where the module action is given by

$$\begin{aligned} u * r_v &= [u, v] \\ u * l_v &= [v, u] \end{aligned}$$

for $u \in R/R'$, $v \in F/R$ and $r_v, l_v \in UL(F/R)$. Let $\overline{x}$ represent $x + R' \in F/R'$, and $\overline{\overline{x}}$ denote $x + R \in F/R$.

The proof of the following theorem is identical to the one presented in the case of Lie algebras, as detailed in [10].

Theorem 3.1. *The mapping* $\overline{x_j} \longrightarrow \overline{\overline{x_j}} + a_j$, $j \in \{1, 2, ..., n\}$ *extends to a monomorphism* $\mu : F/R' \longrightarrow (R/R')wr(F/R)$.

Let $AutW$ represent the automorphism group of W. Consider a subgroup of $AutW$ denoted as $\overline{AutW}$. The elements of $\overline{AutW}$ are characterized by their invariance of $I_{R/R'}$ and F/R. In other words, if $\alpha \in \overline{AutW}$, then the automorphism $\alpha : W \to W$ satisfies $\alpha(I_{R/R'}) \subset I_{R/R'}$ and $\alpha(F/R) \subset F/R$.

The subsequent theorem analogies the embedding concept in Lie algebras, initially established by Bahturin and Nabiyev in [**?**]. The same arguments are employed to prove this theorem in the case of Leibniz algebras.

Theorem 3.2. *An embedding denoted by* $\vartheta : \overline{Aut(F/R')} \to \overline{Aut((R/R')wr(F/R))}$ *exists, such that if* $\alpha \in \overline{Aut(F/R')}$ *preserves* R/R'*, and* $\widetilde{\alpha} = \vartheta(\alpha)$*, then* $\widetilde{\alpha}\mu = \mu\alpha$*, where* μ *represents the embedding defined in Theorem 1.*

The proof of the theorem at hand mirrors the demonstration employed by Bahturin and Nabiyev in establishing their result for Lie Algebras [**?**]. The author and Tas Adiyaman have already given similar proofs in [15, 16] to obtain the explicit matrix form of IA-automorphisms of the free metabelian Leibniz algebras, and the theorem is a generalization of the corresponding result in [16].

Theorem 3.3. *Let* F/R' *be a Leibniz algebra of finite rank. Let* G *be the group of invertible matrices of the form* $E + AQ$*, where* E *is the identity matrix,* $A = [a_{kj}]_{n\times m}$ *is a fix matrix,* $Q = [q_{ji}]_{m\times n}$ *is an arbitrary matrix both with coefficients in* $UL(F/R), 1 \le i, k \le n,\ 1 \le j \le m$*. Then* $IAut(F/R') \cong G$*.*

4. The Dieudonné Determinant

Every invertible square matrix belonging to $U((F/R)_{\mathrm{Lie}})$ can be expressed as a multiplication of elementary and diagonal matrices, as detailed in [3]. In this context, elementary matrices differ from the identity matrix by, at most, a single element outside the diagonal. Consider an algebra

$$(UL(F)/\Delta_R)/(\Delta^m/\Delta_R), m \geq 2.$$

Denote by H_m the image of this algebra under the homomorphism $\widehat{}$ and take the multiplicative group H_m^* of all invertible elements of H_m. Since

$$(a+u+\widehat{\Delta_R})(a^{-1}-a^{-2}u+\ldots+(-1)^{m-1}a^{-m}u^{m-1}+\widehat{\Delta_R}) = 1+\widehat{\Delta_R} \text{ modulo } \widehat{\Delta^m/\Delta_R},$$

we have

$$(a+u+\widehat{\Delta_R})^{-1} = a^{-1} - a^{-2}u + ... + (-1)^{m-1}a^{-m}u^{m-1} + \widehat{\Delta_R} \text{ modulo } \widehat{\Delta^m/\Delta_R}.$$

Therefore, the invertible elements in H_m can be expressed as

$$a + u + \widehat{\Delta_R} + \widehat{\Delta^m/\Delta_R}$$

with $u \in \widehat{\Delta}$ and $0 \neq a \in K$. Next, consider the commutator subgroup $[H_m^*, H_m^*]$ within the group H_m^*. This subgroup is generated, modulo $\widehat{\Delta^m/\Delta_R}$, by elements characterized by the following expression

$$(1-u+\widehat{\Delta_R})(1-w+\widehat{\Delta_R})(1-u+\widehat{\Delta_R})^{-1}(1-w+\widehat{\Delta_R})^{-1}$$

Here, u and w belong to the set Δ. Let S_m be the subsemigroup of $\widehat{UL(F)/\Delta_R}$ generated by all such elements. For a matrix A belonging to the general linear group $GL_n(H_m)$ over H_m, its Dieudonné determinant is defined by exploiting the property that every invertible matrix over H_m can be diagonalized. For any arbitrary permutation $\sigma \in S_n$, we link it with the permutation matrix $P(\sigma) = (\delta_{i,\sigma(j)})$, where δ represents the Kronecker symbol.

For every invertible matrix A over a skew field, a decomposition A can be expressed as $A = TDP(\sigma)V$ known as the Bruhat Normal Form, where

$$T = \begin{bmatrix} 1 & * & * \\ & ... & * \\ 0 & & 1 \end{bmatrix}, \; D = diag(a_1, ..., a_n), \; V = \begin{bmatrix} 1 & . & 0 \\ * & .. & . \\ * & * & 1 \end{bmatrix},$$

σ is a permutation, $P(\sigma)$ is the permutation matrix corresponding to σ. The matrices D and σ are unique with these properties (refer to [4]). The Dieudonné determinant of A is given by

$$D_m(A) = \pi(sgn(\sigma)a_1...a_n),$$

where π is the canonical mapping $H_m^* \to H_m^*/\,[H_m^*, H_m^*]$.

Theorem 4.1. *Consider R as an ideal and F/R' as a finitely generated Leibniz algebra. Let $M \in GL_n(UL(F)/\Delta_R)$ and $det_m(M)$ represent any preimage of $D_m(\widehat{M})$ in $UL(F)/\Delta_R$, where $\Delta_R \subset \Delta^m$ for $m \geq 2$. Then, for any arbitrary m,*

$$det_m(M) = (a + r_u)r_{g_m} \quad modulo \; (\Delta^m/\Delta_R \; + \Delta_{Ann(F/R)})$$

where $a \in K \setminus \{0\}, u \in \widehat{\Delta_R}$, $g_m \in S_m$.

Proof. Let $M \in GL_n(UL(F)/\Delta_R)$. Since M is invertible over $U(F)/\Delta_R$, then $\widehat{M}$, the image of M under the homomorphism $\widehat{\ }: UL(F/R) \to U((F/R)_{Lie})$, is an invertible matrix over $\widehat{UL(F)/}\Delta_R$ and it can be written as $\widehat{M} = E.D$, where E is the product of elementary matrices and

$$D = diag(a_1 + \widehat{\Delta_R}, a_2 + \widehat{\Delta_R}, ..., a_n + \widehat{\Delta_R})$$

where $0 \neq a_i \in K$ by [14]. Given that the sole invertible elements within $U(F_{Lie})$ are the elements belonging to the field K, the invertible elements within $\widehat{UL(F)/}\Delta_R$ can be expressed as

$$a_1 + \widehat{\Delta_R}, a_2 + \widehat{\Delta_R}, \ldots, a_n + \widehat{\Delta_R}$$

where the elements $a_1, \ldots, a_n$ are constrained to lie within the field K. Consider the algebra $H_m = (\widehat{UL(F)/}\Delta_R)/(\widehat{\Delta^m/\Delta_R})$. The image of $\widehat{M}$ over H_m remains invertible. Consequently, the Dieudonné determinant of $\widehat{M}$ can be expressed as follows

$$D_m(\widehat{M}) = a_1.a_2...a_n + \widehat{\Delta_R} + (\widehat{\Delta^m/\Delta_R}).$$

This representation implies that the Dieudonné determinant of $\widehat{M}$ can be further written as $a + u + w$, where $a = a_1 \cdot a_2 \ldots a_n \in K$, $u \in \widehat{\Delta_R}$, and $w \in \widehat{\Delta^m/\Delta_R}$. Consider the algebra $H_m = (\widehat{UL(F)/}\Delta_R)/(\widehat{\Delta^m/\Delta_R})$. The image of $\widehat{M}$ over H_m is also invertible. Therefore, the Dieudonné determinant of $\widehat{M}$ takes the form

$$D_m(\widehat{M}) = a_1.a_2...a_n + \widehat{\Delta_R} + (\widehat{\Delta^m/\Delta_R}).$$

This implies that the Dieudonné determinant of $\widehat{M}$ can be expressed as

$$a + u + w$$

where $a = a_1.a_2 \ldots a_n \in K$, $u \in \widehat{\Delta_R}$, $w \in \widehat{\Delta^m/\Delta_R}$. An arbitrary preimage $det_m(\widehat{M})$ of $D_m(\widehat{M})$ in $\widehat{UL(F)/}\Delta_R$ is equal to

$$(a + u)g_m \ \ \text{modulo}(\widehat{\Delta^m/\Delta_R}),$$

where, $a = a_1.a_2...a_n$, $u \in \widehat{\Delta_R}$, $g_m \in S_m$. Through the homomorphism $\delta : U((F/R)_{Lie}) \to UL(F/R)$ defined as $\delta(x) = r_x$, for $x \in (F/R)_{Lie}$, it is clear that any preimage $det_m(M)$ of $det_m(\widehat{M})$ in $UL(F)/\Delta_R$ can be expressed as

$$(a + r_u)r_{g_m} \text{ modulo } (\Delta^m/\Delta_R \ + \Delta_{Ann(F/R)}),$$

where $\Delta_{Ann(F/R)}$ is an ideal of $UL(F/R)$ generated by the element $r_v + l_v$ for $v \in F/R$. □

Now we have

Theorem 4.2. *Let ψ be an element of $IAut(F/R')$. Consider $\widetilde{\psi}$ as the restricted automorphism of ψ to $I_{R/R'}$, as defined in Theorem 3.2. Denote by M the matrix corresponding to $\widetilde{\psi}$, and let $det_m(M)$ represent an arbitrary preimage of $D_m(\widehat{M})$ in $UL(F)/\Delta_R$. It holds*

$$det_m(M) = (1 + r_u)r_{g_m} \quad \text{mod } (\Delta^m/\Delta_R + \Delta_{Ann(F/R)})$$

where $u \in \widehat{\Delta_R}$ and $g_m \in S_m$ for any $m \geq 2$.

Proof. Given an IA-automorphism ψ of F/R'. By the equality $\mu\psi = \widetilde{\psi}\mu$ from Theorem 3.2 and the definition of the $\overline{AutW}$, there exists an automorphism $\widetilde{\psi}$ restricted to $I_{R/R'}$ with an invertible corresponding matrix M over $UL(F)/\Delta_R$. Through the homomorphism

$$\widehat{\ }: UL(F/R) \to U((F/R)_{Lie}),$$

$\widehat{M}$ is also invertible over $\widehat{UL(F)/}\Delta_R$, expressed as

$$\widehat{M} = E.D,$$

where E is the product of elementary matrices and $D = diag(1+\widehat{\Delta_R}, 1+\widehat{\Delta_R}, ..., 1+\widehat{\Delta_R})$. Consequently, this implies

$$D_m(\widehat{M}) = (1+u)g_m \; modulo \; \widehat{\Delta^m/\Delta_R}$$

where, $u \in \widehat{\Delta_R}$ and $g_m \in S_m$. Thus, according to Theorem 4.1, the arbitrary preimage of $det_m(M)$ in $UL(F)/\Delta_R$ is given by

$$(1+r_u)r_{g_m} \;\; \text{modulo} \; (\Delta^m/\Delta_R \; + \Delta_{Ann(F/R)}).$$

□

Remark 4.3. Theorem 4.2 establishes a necessary condition for an IA-endomorphism of F/R' to qualify as an $IA-$automorphism. This condition provides a means to identify the non-invertibility of a square matrix M over $UL(F)/\Delta_R$. The process involves computing $detm(M)$, initiating from $m = 1$, and proceeding until the condition outlined in the Theorem 4.2 is contradicted.

Example 4.4. Let $R = \gamma_m(F)$, m-th term of the lower central series of F, for $m \geq 4$ and ψ be the endomorphism of $F/\gamma_m(F)'$ defined as

$$\begin{aligned}\psi \quad : \quad & \overline{x_1} \to \overline{x_1} + [\overline{x_1}, \overline{x_2}] + [[\overline{x_1}, [\overline{x_2}, \overline{x_3}]], \overline{x_4}] \\ & \overline{x_i} \to \overline{x_i} + w_i, i \neq 1\end{aligned}$$

where $w_i \in \gamma_m(F)$. Through the verification of Theorem 3.3, it is determined that the restriction of $\widetilde{\psi}$ to $I_{R/R'}$ is associated with the matrix M of the form

$$\begin{bmatrix} 1+r_{\overline{\overline{x_2}}} & r_{\overline{\overline{x_1}}} + r_{\overline{\overline{x_3}}} l_{\overline{\overline{x_1}}} r_{\overline{\overline{x_4}}} & l_{\overline{\overline{x_2}}}.l_{\overline{\overline{x_1}}}.r_{\overline{\overline{x_4}}} & ... & 0 \\ u_{21} & 1+u_{22} & u_{23} & ... & u_{2n} \\ u_{31} & u_{32} & 1+u_{33} & ... & u_{3n} \\ . & . & . & ... & . \\ u_{n1} & u_{n2} & u_{n3} & ... & 1+u_{nn} \end{bmatrix}$$

where $u_{ij} \in \Delta^3$. Let M be invertible in $UL(F)/\Delta_{\gamma_m(F)}$. Then, $\widehat{M}$ is also invertible and which is of the form

$$\begin{bmatrix} 1+\overline{\overline{x_2}} & \overline{\overline{x_1}} - \overline{\overline{x_3 x_1 x_4}} & \overline{\overline{x_2 x_1 x_4}} & ... & 0 \\ \widehat{u_{21}} & 1+\widehat{u_{22}} & \widehat{u_{23}} & ... & \widehat{u_{2n}} \\ \widehat{u_{31}} & \widehat{u_{32}} & 1+\widehat{u_{33}} & ... & \widehat{u_{3n}} \\ . & . & . & ... & . \\ \widehat{u_{n1}} & \widehat{u_{n2}} & \widehat{u_{n3}} & ... & 1+\widehat{u_{nn}} \end{bmatrix}.$$

Since $\Delta_{\gamma_m(F)} \subset \Delta^3$ for $m \geq 4$, consider $H_3 = \widehat{U(F)/\Delta_{\gamma_m(F)}}/\widehat{\Delta^3/\Delta_{\gamma_m(F)}}$. The image of elements of $\widehat{M}$ in H_3 determines

$$\begin{bmatrix} 1+\overline{\overline{x_2}} & \overline{\overline{x_1}} & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}.$$

Then, we obtain

$$D_3(\widehat{M}) = 1 + \overline{\overline{x_2}} \text{ modulo } (\widehat{\Delta_{\gamma_m(F)}} + \widehat{\Delta^3/\Delta_{\gamma_m(F)}}).$$

Therefore,

$$det_3(M) = 1 + r_{\overline{\overline{x_2}}} + \Delta_{\gamma_m(F)} + \Delta^3/\Delta_{\gamma_m(F)} + \Delta_{Ann(F/R)}.$$

By Theorem 4.2,

$$1+r_{\overline{\overline{x_2}}}+\Delta_{\gamma_m(F)}+\Delta^3/\Delta_{\gamma_m(F)}+\Delta_{Ann(F/R)} = 1+\Delta_{\gamma_m(F)}+\Delta^3/\Delta_{\gamma_m(F)}+\Delta_{Ann(F/R)}.$$

Hence, it follows that $r_{\overline{\overline{x_2}}} \in \Delta_{\gamma_m(F)} + \Delta^3/\Delta_{\gamma_m(F)} + \Delta_{Ann(F/R)}$. This is impossible, thus, $\widetilde{\psi}$ cannot be an automorphism.

Example 4.5. Let $R = F'$, and consider the endomorphism ψ on F/R' defined by the following mappings

$$\begin{aligned} \psi \quad : \quad & \overline{x_1} \to \overline{x_1} + [[\overline{x_1}, \overline{x_2}], \overline{x_3}] + w_1, \\ & \overline{x_i} \to \overline{x_i} + w_i, i \neq 1. \end{aligned}$$

where $w_i \in F'', i = 1, \cdots, n$. The associated matrix M is given in the form

$$\begin{bmatrix} 1+r_{\overline{\overline{x_2}}}r_{\overline{\overline{x_3}}}+u_{11} & l_{\overline{\overline{x_1}}}r_{\overline{\overline{x_3}}}+u_{12} & u_{13} & .. & u_{1n} \\ u_{21} & 1+u_{22} & u_{23} & ... & u_{2n} \\ u_{31} & u_{32} & 1+u_{33} & ... & u_{3n} \\ . & . & . & ... & . \\ u_{n1} & u_{n2} & u_{n3} & ... & 1+u_{nn} \end{bmatrix},$$

where $w_{ij} \in \Delta^3$. Let M be invertible in $UL(F)/\Delta_{F''}$. Hence, $\widehat{M}$ is

$$\begin{bmatrix} 1+\overline{\overline{x_2 x_3}}+\widehat{u_{11}} & -\overline{\overline{x_1 x_3}}+\widehat{u_{12}} & \widehat{u_{13}} & ... & \widehat{u_{1n}} \\ \widehat{u_{21}} & 1+\widehat{u_{22}} & \widehat{u_{23}} & ... & \widehat{u_{2n}} \\ \widehat{u_{31}} & \widehat{u_{32}} & 1+\widehat{u_{33}} & ... & \widehat{u_{3n}} \\ . & . & . & ... & . \\ \widehat{u_{n1}} & \widehat{u_{n2}} & \widehat{u_{n3}} & ... & 1+\widehat{u_{nn}} \end{bmatrix}$$

Since, $\Delta_{F''} \subset \Delta^3$, take $H_3 = \widehat{UL(F)/\Delta_{F''}}/\widehat{\Delta^3/\Delta_{F''}}$. $\widehat{M}$ in H_3 is

$$\begin{bmatrix} 1+\overline{\overline{x_2 x_3}} & -\overline{\overline{x_1 x_3}} & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}.$$

Then, we obtain

$$det_3(M) = 1 + r_{\overline{\overline{x_2}}}r_{\overline{\overline{x_3}}} + \Delta_{F''} + \Delta^3/\Delta_{F''} + \Delta_{Ann(F/R)}.$$

By Theorem 4.2

$$1+r_{\overline{\overline{x_2}}}r_{\overline{\overline{x_3}}}+\Delta_{F''}+\Delta^3/\Delta_{F''}+\Delta_{Ann(F/R)}=1+\Delta_{F''}+\Delta^3/\Delta_{F''}+\Delta_{Ann(F/R)}.$$

This yields $r_{\overline{\overline{x_2}}}r_{\overline{\overline{x_3}}}\in\Delta_{F''}+\Delta^3/\Delta_{F''}+\Delta_{Ann(F/R)}$. However, this is impossible. Thus, $\widetilde{\psi}$ and ψ are not automorphisms.

Example 4.6. Given an endomorphism ψ of F/F'' defined as

$$\begin{array}{rcl}\psi & : & \overline{x_1}\to\overline{x_1}+[[\overline{x_2},\overline{x_2}]+[\overline{x_2},\overline{x_1}],\overline{x_1}]\\ & & \overline{x_i}\to\overline{x_i}+[\overline{x_i},\overline{x_i}]+[x_i,x_1], i\neq 1\end{array}$$

Its associated matrix M is

$$\begin{bmatrix} 1+l_{\overline{\overline{x_2}}}r_{\overline{\overline{x_1}}} & (r_{\overline{\overline{x_2}}}+l_{\overline{\overline{x_2}}})r_{\overline{\overline{x_1}}} & 0 & ... & 0\\ 0 & 1+r_{\overline{\overline{x_2}}}+l_{\overline{\overline{x_2}}}+r_{\overline{\overline{x_1}}} & 0 & ... & 0\\ 0 & 0 & 1+r_{\overline{\overline{x_3}}}+l_{\overline{\overline{x_3}}}+r_{\overline{\overline{x_1}}} & ... & 0\\ . & . & . & ... & .\\ 0 & 0 & 0 & ... & 1+r_{\overline{\overline{x_n}}}+l_{\overline{\overline{x_n}}}+r_{\overline{\overline{x_1}}}\end{bmatrix}$$

Then, $\widehat{M}$ is

$$\begin{bmatrix} 1-\overline{\overline{x_2x_1}} & 0 & 0 & ... & 0\\ 0 & 1+\overline{\overline{x_1}} & 0 & ... & 0\\ 0 & 0 & 1+\overline{\overline{x_1}} & ... & 0\\ . & . & . & ... & .\\ 0 & 0 & 0 & ... & 1+\overline{\overline{x_1}}\end{bmatrix}$$

Since, $\Delta_{F''}\subset\Delta^2$, consider $H_2=U\widehat{(F)/\Delta_{F''}}/\widehat{\Delta^2/\Delta_{F''}}$. $\widehat{M}$ in H_2 is

$$\begin{bmatrix} 1 & 0 & 0 & ... & 0\\ 0 & 1+\overline{\overline{x_1}} & 0 & ... & 0\\ 0 & 0 & 1+\overline{\overline{x_1}} & .. & 0\\ . & . & . & ... & .\\ 0 & 0 & 0 & ... & 1+\overline{\overline{x_1}}\end{bmatrix}$$

Then,

$$D_3(\widehat{M})=1+n.\overline{\overline{x_1}}+...+(\overline{\overline{x_1}})^n\text{modulo }(\widehat{\Delta_{F''}}+\widehat{\Delta^2/\Delta_{F''}}).$$

Hence, we obtain

$$det_2(M)=1+nr_{\overline{\overline{x_1}}}+...+(r_{\overline{\overline{x_1}}})^n+\Delta_{F''}+\Delta^2/\Delta_{F''}+\Delta_{Ann(F/R)}.$$

By Theorem 4.2, we can express the equation as follows

$$1+nr_{\overline{\overline{x_1}}}+...+(r_{\overline{\overline{x_1}}})^n+\Delta_{F''}+\Delta^2/\Delta_{F''}+\Delta_{Ann(F/R)}=1+\Delta_{F''}+\Delta^2/\Delta_{F''}+\Delta_{Ann(F/R)}.$$

This yields $nr_{\overline{\overline{x_1}}}+...+(r_{\overline{\overline{x_1}}})^n\in\Delta_{F''}+\Delta^2/\Delta_{F''}+\Delta_{Ann(F/R)}$ and consequently, $nr_{\overline{\overline{x_1}}}\in\Delta_{F''}+\Delta^2/\Delta_{F''}+\Delta_{Ann(F/R)}$. However, this is impossible. Therefore, ψ is not an automorphism.

Conclusion

This study initially derives a matrix representation of the IA-automorphisms on the Leibniz algebra F/R'. Following this, we set forth a prerequisite for an IA-endomorphism of F/R' to qualify as an IA-automorphism. In this criterion, we identify the non-invertibility of a square matrix M over $UL(F)/\Delta_R$. This approach explicitly relies on the Dieudonn'e determinant.

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ON INTERACTIVE SOLUTION FOR TWO POINT FUZZY BOUNDARY VALUE PROBLEM

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Abstract. In this manuscript, the eigenvalues and eigenfunctions of the two-point fuzzy boundary value problem (FBVP) are analyzed under the concept of interactivity between the fuzzy numbers found in the boundary conditions. A fuzzy solution is provided for this problem via sup-J extension, which can be seen as a generalization of Zadeh's extension principle. Finally, an example is presented in order to compare the given features.

1. Introduction

In this paper, the FBVP is considered

$$\widehat{u}'' + \lambda \widehat{u} = 0, \quad t \in [a, b] \tag{1.1}$$

which satisfies the conditions

$$\widehat{a}_1 \widehat{u}(a) -^h \widehat{a}_2 \widehat{u}'(a) = 0 \tag{1.2}$$

$$\widehat{b}_1 \widehat{u}(b) -^h \widehat{b}_2 \widehat{u}'(b) = 0 \tag{1.3}$$

where $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2$ non-negative triangular fuzzy numbers, $\lambda > 0$, at least one of the numbers $\widehat{a}_1$ and $\widehat{a}_2$ and at least one of the numbers $\widehat{b}_1$ and $\widehat{b}_2$ are nonzero and $-^h$ is Hukuhara difference.

Fuzzy differential equation (FDE) is utilized to model problems in science and engineering.In most of the problems there are uncertain structural parameters. Instead, many researchers have modeled these uncertain structural parameters as fuzzy numbers in this area [4, 10]. This occurs a fuzzy boundary value problem with fuzzy boundary conditions.

The studies of two-point FBVP have been made with the Hukuhara derivative [11, 14] and generalized Hukuhara derivative [6, 15, 22, 27–29]. But in some cases

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the fuzzy solutions with Hukuhara derivative suffer from certain disadvantages since the diameter of the solutions is unbounded as time increases [13, 14] and the fuzzy solutions with generalized Hukuhara derivative have some not interval solutions which are associated with the existence of switch points [20]. Moreover, Gasilov et al. argued that the solutions obtained with the method of Khastan and Nieto [15] are difficult to interpret because the solutions of the four different problems may not reflect the nature of the studied phenomenon [9].

Another approach to solving FBVP has been proposed, including the Zadeh's extension principle [1, 17]. For a boundary value problem, the associated crisp problem is solved and in the solution, the fuzzy boundary value is substituted instead of the real constant. Then the arithmetic operations are regarded as operations on fuzzy numbers [16].

Recently, several authors have used the concept of interactivity to study fuzzy differential equations (FDEs) [5, 25]. The relation of interactivity between two fuzzy numbers arises in the presence of a joint possibility distribution J for them. In this case, the solution is obtained in terms of the $sup-J$ extension principle of the solution of an associated classical BVP. Moreover, this proposed approach always produces a proper fuzzy solution, in contrast to other methods presented in the literature [14, 15, 23]. This means that its $\alpha-$ cuts are proper intervals. Moreover, the fuzzy solution obtained by this approach always has a smaller or equal to the solution via Zadeh's extension [12, 17].

This paper analyses FBVP with fuzzy boundary values given by interactive fuzzy values. The fuzzy solution is obtained using the $sup-J$ extension principle [5]. In order to illustrate the utility of this $sup-J$ proposal, the solution of a second order FBVP is presented.

2. Preliminaries

2.1. **Solution for a crisp boundary value problem.** Let the fuzzy problem (1.1-1.3) be considered as a crisp problem.

Then we shall make use of solutions of (1.1) defined by initial conditions instead of boundary conditions in a manner similar to Titchmarsh's method [24].

Lemma 2.1. *([24]) For any $\lambda > 0$ the equation*

$$u'' + \lambda u = 0, \quad t \in [a, b]$$

has a unique solution $u = u(t, \lambda)$ satisfying the initial conditions

$$u(a) = a_2, \qquad u'(a) = a_1 \qquad (\text{ or } u(b) = b_2, \quad u'(b) = b_1).$$

For each $t \in [a, b]$, $u(t, \lambda)$ is an entire function of λ

Two solutions $\Phi_\lambda(t)$ and $\Psi_\lambda(t)$ of the equation (1.1) are defined as follows. Let $\Phi_\lambda(t) = \Phi(t, \lambda)$ be the solution of equation (1.1) on $[a, b]$, which satisfies the initial conditions

$$\begin{pmatrix} u(a) \\ u'(a) \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \tag{2.1}$$

and $\Psi_\lambda(t) = \Psi(t,\lambda)$ be the solution of equation (1.1) on $[a,b]$, which satisfies the initial conditions

$$\begin{pmatrix} u(b) \\ u'(b) \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}. \tag{2.2}$$

Let us consider the following linear and homogeneous differential equation with (2.1) and (2.2) initial conditions, where $a_1,\ a_2, b_1, b_2 \in \mathbb{R}$, given by

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(a) = a_2,\ \Phi'(a) = a_1 \end{cases} \tag{2.3}$$

and

$$\begin{cases} \Psi'' + \lambda\Psi = 0 \\ \Psi(b) = b_2,\ \Psi'(b) = b_1. \end{cases} \tag{2.4}$$

First, let's search for the solution of the problem in (2.3) with the help of the algorithm created by Sanchez et al. [19]. Then a solution is found for the problem (2.4) by doing similar operations.

Firstly, the general solution of (2.3) is given

$$\Phi_\lambda(t) = C_1\Phi_1(t) + C_2\Phi_2(t), \tag{2.5}$$

where Φ_1, Φ_2 are linearly independent solutions of the homogeneous differential equation which is given as in (2.3).

The scalar coefficients C_1 and C_2 are determined from the initial values a_2 and $-a_1$:

$$C_1 = \frac{\Phi_2(a)(a_1) + \Phi_2'(a)a_2}{\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a)} \text{ and } C_2 = -\frac{\Phi_1(a)(a_1) + \Phi_1'(a)a_2}{\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a)}.$$

Thus, from (2.5), the general solution of (2.3) is given by

$$\Phi_\lambda(t) = a_1 m_1(t) + a_2 m_2(t), \tag{2.6}$$

where $m_1(t)$ and $m_2(t)$ are defined for $\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a) \neq 0$ as follows [9]:

$$m_1(t) = \frac{\Phi_2(a)\Phi_1(t) - \Phi_1(a)\Phi_2(t)}{\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a)}, \text{ and } m_2(t) = \frac{\Phi_2'(a)\Phi_1(t) - \Phi_1'(a)\Phi_2(t)}{\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a)}. \tag{2.7}$$

Similarly, the general solution of (2.4) is given by

$$\Psi_\lambda(t) = b_1 m_3(t) + b_2 m_4(t), \tag{2.8}$$

where $m_3(t)$ and $m_4(t)$ are defined for $\Phi_1(b)\Phi_2'(b) - \Phi_2(b)\Phi_1'(b) \neq 0$ as follows

$$m_3(t) = \frac{\Phi_2(b)\Phi_1(t) - \Phi_1(b)\Phi_2(t)}{\Phi_1(b)\Phi_2'(b) - \Phi_2(b)\Phi_1'(b)}, \text{ and } m_4(t) = \frac{\Phi_2'(b)\Phi_1(t) - \Phi_1'(b)\Phi_2(t)}{\Phi_1(b)\Phi_2'(b) - \Phi_2(b)\Phi_1'(b)}. \tag{2.9}$$

Then this solutions $\Phi_\lambda(t)$ and $\Psi_\lambda(t)$ are put in the Wronskians function

$$w(\lambda) = W_\lambda(\Phi, \Psi; t) = \Phi_\lambda(t)\Psi_\lambda'(t) - \Phi_\lambda'(t)\Psi_\lambda(t) \tag{2.10}$$

which are independent of $t \in [a,b]$. For each fixed t these functions and derivatives are entire in λ [24].

Lemma 2.2. *([24] If $\lambda = \lambda_0$ is an eigenvalue, then $\Phi(t,\lambda_0)$ and $\Psi(t,\lambda_0)$ are linearly dependent and eigenfunctions corresponding to this eigenvalue .*

Theorem 2.3. *([24]) The eigenvalues of the problem (1.1-1.3) are the zeros of the function* $w(\lambda)$.

In Section 3, equations (2.6) and (2.8) will be used to define the fuzzy solution of the second order two point boundary values problem with fuzzy boundary values.

Before the approach applied to solve an FBV problem is introduced, it is necessary first to review some concepts of fuzzy sets theory.

2.2. **Basic concepts of fuzzy sets.**

Definition 2.4. ([18]) Let E be a universal set. A fuzzy subset $\widehat{A}$ of E is given by its membership function $\mu_{\widehat{A}} : E \to [0,1]$, where $\mu_{\widehat{A}}(t)$ represents the degree to which $t \in E$ belongs to $\widehat{A}$. We denote the class of the fuzzy subsets of E by the sembol $F(E)$.

Definition 2.5. ([16]) The $\alpha-$ cut of a fuzzy set $\widehat{A} \subseteq E$denoted by $\left[\widehat{A}\right]^{\alpha}$, is defined as $\left[\widehat{A}\right]^{\alpha} = \left\{x \in E : \widehat{A}(t) \geq \alpha\right\}$, $\forall \alpha \in (0,1]$. If E is also topological space, then the $0-$ cut is defined as the closure of the support of $\widehat{A}$,that is, $\left[\widehat{A}\right]^{0} = \overline{\left\{x \in E : \widehat{A}(t) > 0\right\}}$. The $1-$ cut of a fuzzy subset $\widehat{A}$ is also called as core of $\widehat{A}$ and denoted by $\left[\widehat{A}\right]^{1} = core\left(\widehat{A}\right)$

Definition 2.6. ([21]) A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy interval), whose $\alpha-$ cut set is denoted by $[\widehat{u}]^{\alpha}$, i.e., $[\widehat{u}]^{\alpha} = \{x : \widehat{u}(t) \geq 0\}$, if it satisfies two axioms:

i. There exists $r \in \mathbb{R}$ such that $\widehat{u}(r) = 1$,

ii. For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u_{\alpha}^{-} \leq u_{\alpha}^{+} < +\infty$ such that $[\widehat{u}]^{\alpha}$ is equal to the closed interval $[u_{\alpha}^{-}, u_{\alpha}^{+}]$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathbb{R}_F$. $F_K(\mathbb{R})$, the family of fuzzy sets of $\mathbb{R}$ whose $\alpha-$ cuts are nonempty compact convex subsets of $\mathbb{R}$. If $\widehat{u} \in \mathbb{R}_F$ and $\widehat{u}(t) = 0$ whenever $t < 0$, then $\widehat{u}$ is called a non-negative fuzzy real number and $\mathbb{R}_F^{+}$ denotes the set of all non-negative fuzzy real numbers. For all $\widehat{u} \in \mathbb{R}_F^{+}$ and each $\alpha \in (0,1]$, real number u_{α}^{-} is positive.

Definition 2.7. ([7]) An arbitrary fuzzy number $\widehat{u}$ in the parametric form is represented by an ordered pair of functions $[u_{\alpha}^{-}, u_{\alpha}^{+}]$, $0 \leq \alpha \leq 1$, which satisfy the following requirements

i. u_{α}^{-} is bounded non-decreasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,

ii. u_{α}^{+} is bounded non- increasing left continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

iii. $u_{\alpha}^{-} \leq u_{\alpha}^{+}$, $0 < \alpha \leq 1$.

Definition 2.8. ([2,10]) A fuzzy number $\widehat{A}$ is said to be triangular if the parametric representation of its $\alpha-$ cut is of the form for all $\alpha \in [0,1]$

$\left[\widehat{A}\right]^{\alpha} = [(m - a_{\alpha}^{-})\,\alpha + a_{\alpha}^{-}, (m - a_{\alpha}^{+})\,\alpha + a_{\alpha}^{+}]$, where $\left[\widehat{A}\right]^{0} = [a_{\alpha}^{-}, a_{\alpha}^{+}]$ and *core* $\left(\widehat{A}\right) = m\dot{A}$ triangular fuzzy number is denoted by the triple $(a_{\alpha}^{-}; m; a_{\alpha}^{+})$.

Zadeh's extension principle is a mathematical method to extend classical functions to deal with fuzzy sets as input arguments [26]. For multiple fuzzy variables as arguments, Zadeh's extension principle is defined as follows.

Definition 2.9. ([1]) Let $f : X_1 \times X_2 \to Z$ a classical function and let $\widehat{A}_i \in F(X_i)$, for $i = 1, 2$. The Zadeh's extension $\widehat{f}$ of f, applied to $(\widehat{A}_1, \widehat{A}_2)$, is the fuzzy set $\widehat{f}$ $(\widehat{A}_1, \widehat{A}_2)$ of Z, whose membership function is defined by

$$\widehat{f}(\widehat{A}_1, \widehat{A}_2)(z) = \begin{cases} \underset{(x_1,x_2)\in f^{-1}(z)}{sup} \quad min\{\widehat{A}_1(x_1), \widehat{A}_2(x_2)\}, & \text{if } f^{-1}(z) \neq \varnothing, \\ 0 & , \text{if } f^{-1}(z) = \varnothing \end{cases}$$

where $f^{-1}(z) = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1, x_2) = z\}$.

We can apply Zadeh's extension principle to define the standard arithmetic for fuzzy numbers [26] . Let $[\widehat{u}]^{\alpha} = [u_{\alpha}^{-}, u_{\alpha}^{+}]$ and $[\widehat{v}]^{\alpha} = [v_{\alpha}^{-}, v_{\alpha}^{+}]$. For all $\alpha \in [0, 1]$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} [\widehat{u} \oplus \widehat{v}]^{\alpha} &= [\widehat{u}]^{\alpha} + [\widehat{v}]^{\alpha} = \{x + y : x \in [\widehat{u}]^{\alpha}, y \in [\widehat{v}]^{\alpha}\}, \\ [\lambda \odot \widehat{u}]^{\alpha} &= \lambda \odot [\widehat{u}]^{\alpha} = \{\lambda x : x \in [\widehat{u}]^{\alpha}\}. \end{aligned}$$

Theorem 2.10. *([2]) Let X and Y be topological spaces, $f : X \to Y$ be a continuous function and $\widehat{A}$ a fuzzy subset of $X\dot{S}o$ for all $\alpha \in [0, 1]$, we have*

$$[\hat{f}(\widehat{A})]^{\alpha} = f\left(\left[\widehat{A}\right]^{\alpha}\right).$$

As a consequence of Theorem 2.10, it is obtained that $\hat{f}(\widehat{A})$ is a fuzzy number whenever the function $f : X \to Y$ be a continuous function and $\widehat{A}$ is a fuzzy number.

The concept of interactivity between fuzzy numbers is based on the notion of joint possibility distributions [5]. More precisely, a fuzzy subset J of $\mathbb{R}^n$ is called a joint possibility distribution of $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ if

$$\widehat{A}_i(x_i) = \underset{x_j\in\mathbb{R}, j\neq i}{sup} J\left(x_1, ..., x_n\right),$$

for all $x_i \in \mathbb{R}$ and for all $i = 1, ..., n$. Moreover, the fuzzy numbers $\widehat{A}_1, ..., \widehat{A}_n$ are said to be non-interactive if their joint possibility distribution is given by and for all $i = 1, ..., n$. Moreover, the fuzzy numbers $\widehat{A}_1, ..., \widehat{A}_n$ are said to be non-interactive if their joint possibility distribution is given by

$$J_{\hat{}}\left(x_1, ..., x_n\right) = min\{\widehat{A}_1(x_1), ..., \widehat{A}_n(x_n)\}, \forall(x_1, ..., x_n) \in \mathbb{R} \tag{2.11}$$

Otherwise, the fuzzy numbers $\widehat{A}_1, ..., \widehat{A}_n$ are said to be interactive . Next, the notion of $sup - J$ extension principle proposed by Carlsson et al. is presented [5].

Definition 2.11. ([5]) Let $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ and $f : \mathbb{R}_n \to \mathbb{R}$. Given a joint possibility distribution J of $\widehat{A}_1, ..., \widehat{A}_n$, the $sup - J$ extension of f at $(\widehat{A}_1, ..., \widehat{A}_n)$ is the fuzzy set $\widehat{f}(J) := f_J(A_1, ..., A_n)$ of $\mathbb{R}$ whose membership function is given by

$$\widehat{f}(J)\,(z) = \sup_{f(x_1,...,x_n)=z} J\,(x_1, ..., x_n)\,, \forall z \in \mathbb{R}$$

for all $z \in \mathbb{R}$, where $f^{-1}(z) = \{(x_1, ..., x_n) : f((x_1, ..., x_n)) = z\}$.

Remark 2.12. If $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ are non-interactive that is, if the corresponding joint possibility distribution J is defined as in (2.11), then the $sup - J$ extension principle corresponds to the Zadeh's extension principle of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $(\widehat{A}_1, ..., \widehat{A}_n) \in \mathbb{R}^n_F$. In this case, the symbol $\widehat{f}(\widehat{A}_1, ..., \widehat{A}_n)$ is used simply instead of $f_J(\widehat{A}_1, ..., \widehat{A}_n)$ to denote the Zadeh's extension of f at $(\widehat{A}_1, ..., \widehat{A}_n)$.

The next corollary is an immediate consequence of Theorem 2.10

Corollary 2.13. ([5]) Let $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. If J is a joint possibility distribution of the fuzzy numbers $\widehat{A}_1, ..., \widehat{A}_n$, then we have

$$\left[f_J(\widehat{A}_1, ..., \widehat{A}_n)\right]_\alpha = f\left([J]_\alpha\right)$$

for all $\alpha \in [0, 1]$.

The usual arithmetic operations of addition, subtraction, multiplication, and division for fuzzy numbers are defined Definition 2.14. Other forms of arithmetic operations between fuzzy numbers can be established using the notion of $sup - J$ extension principle. Next, an arithmetic defined for the class of the linearly correlated (or completely correlated) fuzzy numbers is presented [1].

Definition 2.14. ([3,5]) Two fuzzy numbers A and B are linearly correlated if there exists $q, r \in R$, $q \neq 0$, such that $[B]^\alpha = q[A]^\alpha + r$ for each $\alpha \in [0, 1]$ or, equivalently, if A and B are interactive with respect to J_L given cutwise by

$$[J_L]_\alpha = \{(xq + r : x \in [A]^\alpha\}\,.$$

In this case, we may simply write $B = qA + r$ is written.

If A and B are linearly interactive fuzzy numbers $[B]^\alpha = q[A]^\alpha + r$, with $[A]^\alpha = [a^-_\alpha, a^-_\alpha]$ and $[B]^\alpha = [b^-_\alpha, b^+_\alpha]$, then the addition $(+_L)$ subtraction $(-_L)$ are given bye

$$[B +_L A]^\alpha = (q+1)\,[A]^\alpha + r = \begin{cases} [b^-_\alpha + a^-_\alpha, b^+_\alpha + a^+_\alpha] & \text{if} \quad q > 0, \\ [b^+_\alpha + a^-_\alpha, b^-_\alpha + a^+_\alpha] & \text{if} \quad -1 \leq q < 0, \\ [b^-_\alpha + a^+_\alpha, b^+_\alpha + a^-_\alpha] & \text{if} \quad q < -1, \end{cases} \tag{2.12}$$

$$[B -_L A]^\alpha = (q-1)\,[A]^\alpha + r = \begin{cases} [b^-_\alpha - a^-_\alpha, b^+_\alpha - a^+_\alpha] & \text{if} \quad q \geq 1, \\ [b^+_\alpha - a^+_\alpha, b^-_\alpha - a^-_\alpha] & \text{if} \quad 0 \leq q < 1, \\ [b^-_\alpha - a^+_\alpha, b^+_\alpha - a^-_\alpha] & \text{if} \quad q < 0, \end{cases} \tag{2.13}$$

for all $\alpha \in [0; 1]$ [3].

Definition 2.15. ([23]) Let $\widehat{u} \in E$ and for $\alpha \in [0,1]$, $[\widehat{u}]^\alpha = [u_\alpha^-, u_\alpha^+]$. Then $-^h[\widehat{u}]^\alpha$ is defined as follows:

$$-^h[\widehat{u}]^\alpha = -^h \left[u_\alpha^-, u_\alpha^+\right] = 0 -^h \left[u_\alpha^-, u_\alpha^+\right] = \left[-u_\alpha^-, -u_\alpha^+\right].$$

3. Solution Method of the FBVP

In this section we concern with the fuzzy initial value problems obtained by replacing the initial values a_1, a_2 and b_1, b_2 with fuzzy numbers $\widehat{a}_1$, $\widehat{a}_2$ and $\widehat{b}_1$, $\widehat{b}_2$ in Equations (2.3) and (2.4). More precisely, let us consider the following FIVPs:

$$\left\{ \begin{array}{c} \Phi'' + \lambda\Phi = 0 \\ \Phi(a) = \widehat{a}_2, \quad \Phi'(a) = \widehat{a}_1 \end{array} \right. \tag{3.1}$$

and

$$\left\{ \begin{array}{c} \Psi'' + \lambda\Psi = 0 \\ \Psi(b) = \widehat{b}_2, \quad \Psi'(b) = \widehat{b}_1 \end{array} \right. \tag{3.2}$$

where $\widehat{a}_1 = (a_{10}, a_1, a_{11})$, $\widehat{a}_2 = (a_{20}, a_2, a_{21})$, $\widehat{b}_1 = (b_{10}, b_1, b_{11})$,$\widehat{b}_2 = (b_{20}, b_2, b_{21}) \in \mathbb{R}_F$, λ is crisp number and $\lambda = p^2$, $p > 0$.

We present two different methods such as sup-J and Zadeh extension principle to solve the FIVPs (3.1) and (3.2).

Let $\Phi(.; a_1, a_2)$ and $\Psi(.; b_1, b_2)$ be the deterministic solution of the associated IVPs of equations (3.1) and (3.2), given in (2.6) and (2.8), where a_1, a_2, b_1, b_2 are the initial conditions. Let's first consider solution Φ and then similarly we get solution Ψ. Let U be an open subset of $\mathbb{R}^2$ such that $\left([\widehat{a}_1]^0 \times [\widehat{a}_2]^0\right) \subset U$. For each t, let be the operator $S_t : U \to \mathbb{R}$,given by

$$S_t(\Phi_0) = \Phi(t, \Phi_0)$$

and $J = J_L$ be a joint possibility distribution of $\widehat{a}_1, \widehat{a}_2 \in \mathbb{R}_F$. The fuzzy solution of (3.1) via $sup - J$ extension principle is given by

$$\widehat{\Phi}_J(t) = S_t(\widehat{a}_1, \widehat{a}_2).$$

If S_t is a continuous function, then by Corollary 2.13 , we have ([19]):

$$\begin{aligned} \widehat{\Phi}_J(t) &= [(S_t)_J(\widehat{a}_1, \widehat{a}_2)]^\alpha = (S_t)([J]^\alpha) \\ &= \left\{S_t(z, qz + r) : z \in [\widehat{a}_2]^\alpha = \left[(a_2)_\alpha^-, (a_2)_\alpha^+\right]\right\} \\ &= m_1(t)z + m_2(t)(qz + r) : z \in [\widehat{a}_2]^\alpha \\ &= (m_1(t) + qm_2(t))\left[(a_2)_\alpha^-, (a_2)_\alpha^+\right] + rm_2(t) \end{aligned} \tag{3.3}$$

for all $\alpha \in [0,1]$.

If the initial conditions are non interactive fuzzy numbers, we can use Zadeh's extension principle to obtain a solution given by

$$\begin{aligned} \widehat{\Phi}(t) &= \left[\widehat{S}_t(\widehat{a}_1, \widehat{a}_2)\right]^\alpha = S_t([\widehat{a}_1]^\alpha \times [\widehat{a}_2]^\alpha) \\ &= \left\{S_t(\widehat{a}_1, \widehat{a}_2) : z \in [\widehat{a}_2]^\alpha = \left[(a_2)_\alpha^-, (a_2)_\alpha^+\right]\right\} \end{aligned}$$

for all $t \in [t_0, T]$. If S_t is a continuous function, then by Corollary 2.13, we have:

(3.4)
$$\begin{aligned}\widehat{\Phi}(t) &= \left[\widehat{S}_t\left(\widehat{a}_1, \widehat{a}_2\right)\right] \\ &= \left\{S_t\left(z, qz+r\right) : a_1 \in \left[\widehat{a}_1\right]^{\alpha} = \left[\left(a_1\right)_{\alpha}^{-}, \left(a_1\right)_{\alpha}^{+}\right], a_2 \in \left[\widehat{a}_2\right]^{\alpha} = \left[\left(a_2\right)_{\alpha}^{-}, \left(a_2\right)_{\alpha}^{+}\right]\right\} \\ &= m_1(t)\left[\left(a_2\right)_{\alpha}^{-}, \left(a_2\right)_{\alpha}^{+}\right] + m_2(t)\left[\left(a_1\right)_{\alpha}^{-}, \left(a_1\right)_{\alpha}^{+}\right]\end{aligned}$$

for all $\alpha \in [0, 1]$.

Theorem 3.1. *([8]) Let $\widehat{\Phi}(t)$ and $\widehat{\Phi}_J(t)$ be the Zadeh and linear linear interactive solutions to the FIVP, respectively. Thus, $\widehat{\Phi}_J(t) \subseteq \widehat{\Phi}(t)$ for all $t \in \mathbb{R}$.*

Similarly, we get $\widehat{\Psi}(t)$ and $\widehat{\Psi}_J(t)$ be the Zadeh and linear interactive solutions to the FIVP, respectively.

The above theorem reveals that the linear interactive solution is contained in Zadeh's fuzzy solution. In fact, this result holds for every joint possibility distribution J, such that $J \subseteq J_{\wedge}$ [8].

Since λ is crisp (non-fuzzy) we substitute classical cases of the obtained fuzzy solutions $\widehat{\Phi}_{\lambda}(t) = \widehat{\Phi}(t, \lambda)$ and $\widehat{\Psi}_{\lambda}(t) = \widehat{\Psi}(t, \lambda)$ in (2.10). So we get the Wronskian function as follows

(3.5)
$$w(\lambda) = W_{\lambda}\left(\Phi, \Psi; t\right) = \Phi_{\lambda}(t)\,\Psi_{\lambda}'(t) - \Phi_{\lambda}'(t)\,\Psi_{\lambda}(t).$$

Definition 3.2. ([11]) Let $\left[\widehat{u}(t, \lambda)\right]^{\alpha} = \left[u_{\alpha}^{-}(t, \lambda), u_{\alpha}^{+}(t, \lambda)\right]$ be a solution of the fuzzy differential equation 1.1 where $\alpha \in [0, 1]$. If the fuzzy differential equation 1.1 has the nontrivial solutions such that $u_{\alpha}^{-}(t, \lambda) \neq 0$ and $u_{\alpha}^{+}(t, \lambda) \neq 0$, then the $\lambda = \lambda_0$ is eigenvalue of (1.1)

Theorem 3.3. *([11]) The roots of equations (3.5) coincide with the eigenvalues of the fuzzy boundary value problem (1.1-1.3).*

The next section presents an example of FBVP with interactive and non-interactive boundary values.

4. Example

Consider the two point fuzzy boundary value problem

(4.1)
$$\widehat{u}'' + \lambda \widehat{u} = 0$$

(4.2)
$$\widehat{2}u(0) + \widehat{1}u'(0) = 0$$

(4.3)
$$\widehat{4}u(1) + \widehat{3}u'(1) = 0$$

where $\widehat{1} = (0, 1, 2)$, $\widehat{2} = (1, 2, 3)$, $\widehat{3} = (2, 3, 4)$, $\widehat{4} = (3, 4, 5)$and $\lambda = p^2$, $p > 0$.

From (4.1-4.3) problem and Definition 2.15, we get two FIVPs involving a crisp differential equation (4.1) with fuzzy initial values as follows:

(4.4)
$$\Phi'' + p^2 \Phi = 0, \quad \Phi(0) = \widehat{1}, \quad \Phi'(0) = -^{h}\widehat{2}$$

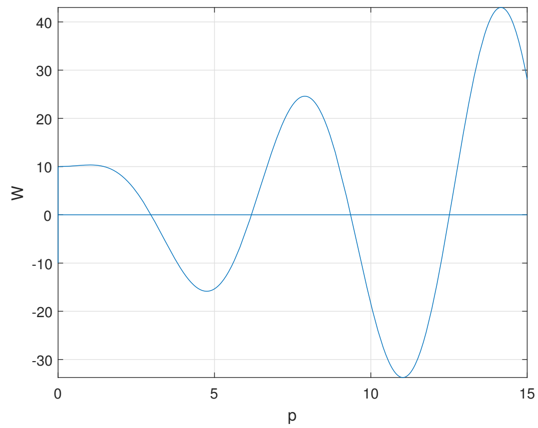

FIGURE 1. The function $W(\lambda) = \left(3p + \frac{8}{p}\right)\sin(p) + (2)\cos(p)$

and

$$\Psi'' + p^2\Psi = 0, \quad \Psi(1) = \widehat{3}, \quad \Psi'(1) = -^h\widehat{4} \tag{4.5}$$

We shall define two solutions $\widehat{\Phi}_\lambda(t)$ and $\widehat{\Psi}_\lambda(t)$ of the equations (4.4) and (4.5). The linearly independent classical solution the homogeneous ODE of (4.4) are given

$$\Phi_1(x) = \cos(pt), \text{ and } \Phi_2(x) = \sin(pt).$$

Thus, we have from 2.7

$$m_1(t) = -\frac{\sin(pt)}{p} \text{ and } m_2(t) = \cos(pt)$$

First the solution obtained from the Zadeh's extension principle (3.4) is provided, which is given by

$$\begin{aligned} \left[\widehat{\Phi}(t,\lambda)\right]^\alpha &= m_1(t)[\alpha, 2-\alpha] + m_2(t)[\alpha+1, 3-\alpha] \\ &= -\frac{1}{p}\sin(pt)[\alpha, 2-\alpha] + \cos(pt)[\alpha+1, 3-\alpha] \end{aligned} \tag{4.6}$$

for all $\alpha \in [0,1]$ and $t \in [0, 3.5]$.

Analogically $\widehat{\Psi}(t,\lambda)$ is obtained as follows

$$\begin{aligned} \left[\widehat{\Psi}(t,\lambda)\right]^\alpha &= m_1(t)[\alpha+2, 4-\alpha] + m_2(t)[\alpha+3, 5-\alpha] \\ &= -\frac{1}{p}\sin(pt-p)[\alpha+2, 4-\alpha] + \cos(pt-p)[\alpha+3, 5-\alpha] \end{aligned} \tag{4.7}$$

for all $\alpha \in [0,1]$ and $t \in [0, 3.5]$.

These $\widehat{\Phi}(t,\lambda)$ and $\widehat{\Psi}(t,\lambda)$ have unique solution [24]. Then putting the classical cases of (4.6) and (4.7) in equation (3.5), Wronskian function is obtained as

$$w(p) = \left(3p + \frac{8}{p}\right)\sin(p) + (2)\cos(p). \tag{4.8}$$

From Theorem 3.3 , the eigenvalues of the fuzzy problem (4.1)-(4.3) are zeros the functions $w(\lambda)$ in (4.8).

If the values satisfying the equation (4.8) compute with Matlab Program, then eigenvalues of fuzzy problem are obtained in Table 1. as follows:

	p_n	λ_n
$n=1$	2.9709	8.8262
$n=2$	6.1827	38.2257
$n=3$	9.3557	87.5291
$n=4$	12.514	156.6
$n\approx$	$n\pi$	$(n\pi)^2$

Table 1. Eigenvalues of the fuzzy problem

The first five eigenvalues are found numerically and then the approximation of the remaining eigenvalues will be used. From Figure 1 It can be seen that the graphs intersect at infinitely many point $p_n \approx n\pi \quad (n = 1,2,3...)$, where the error in this approximation approaches zero as $n \to \infty$. Given this estimate, Matlab program can be used to compute p_n more accurately.

From the equations (4.6) and (4.7)

$$\left[\widehat{\Phi}(t,\lambda_n)\right]^{\alpha} = -\frac{1}{p_n}\sin(p_n t)[\alpha, 2-\alpha] + \cos(p_n t)[\alpha+1, 3-\alpha] \tag{4.9}$$

and

$$\left[\widehat{\Psi}(t,\lambda_n)\right]^{\alpha} = -\frac{1}{p_n}\sin(p_n t - p_n)[\alpha+2, 4-\alpha] + \cos(p_n t - p_n)[\alpha+3, 5-\alpha] \tag{4.10}$$

are eigenfunctions associated with $\lambda_n = (p_n)^2$.

In particular, $p_1 = 2.9709$ is selected in Table 1 and substituted this value respectively in (4.9) and (4.10).

First the solutions obtained from the Zadeh's extension principle 3.4 are provided ,which are given by

$$\begin{aligned} \left[\widehat{\Phi}(t,p_1)\right]^{\alpha} &= m_1(t,p_1)[\alpha, 2-\alpha] + m_2(t,p_1)[\alpha+1, 3-\alpha] \\ &= -\frac{1}{2.9709}\sin(2.9709t)[\alpha, 2-\alpha] + \cos(2.9709t)[\alpha+1, 3-\alpha] \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \left[\widehat{\Psi}(t,p_1)\right]^{\alpha} &= m_1(t,p_1)[\alpha+2, 4-\alpha] + m_2(t,p_1)[\alpha+3, 5-\alpha] \\ &= -\frac{1}{2.9709}\sin(2.9709t - 2.9709)[\alpha+2, 4-\alpha] \\ &\quad + \cos(2.9709t - 2.9709)[\alpha+3, 5-\alpha] \end{aligned} \tag{4.12}$$

for all $\alpha \in [0,1]$ and for all $t \in [0, 3.5]$.

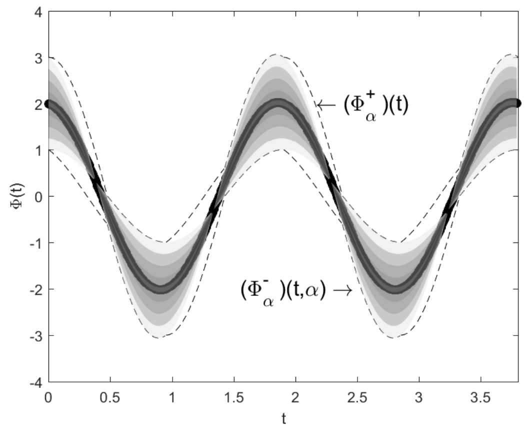

FIGURE 2. The gray-scale lines varying from white to black represent the $\alpha-$cuts of the fuzzy solution (4.11) via $sup-J$ extension principle, where their endpoints for varying from 0 to 1. Black dashed lines represent the 0-cut of the fuzzy solution (4.13) via Zadeh extension principle of $\widehat{\Phi}$

Then, it is assumed that $\widehat{1}$, $\widehat{2}$ and $\widehat{3}$, $\widehat{4}$ are linearly interactive, then there exists (q, r) such that $B = qA + r$ with $(q = 1,\ r = 1$ for $\widehat{1}, \widehat{2}$ and $\widehat{3}, \widehat{4}$ linearly interactive numbers). In this case the solutions $\widehat{\Phi}_j$ and $\widehat{\Psi}_j$ obtained from the $sup - j$ extension principle by means of 3.3 whose $\alpha-$cut is given by

$$\begin{aligned} \left[\widehat{\Phi}_j\left(t, p_1\right)\right]^\alpha &= \left(m_1\left(t, p_1\right) + q m_2\left(t, p_1\right)\right)[\alpha, 2-\alpha] + r m_2\left(t, p_1\right) \\ &= \left(-\frac{1}{2.9709}\sin\left(2.9709t\right) + q\cos\left(2.9709t\right)\right)[\alpha, 2-\alpha] \\ &\quad + r\cos\left(2.9709t\right) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \left[\widehat{\Psi}_j\left(t, p_1\right)\right]^\alpha &= \left(m_1\left(t, p_1\right) + q m_2\left(t, p_1\right)\right)[\alpha, 2-\alpha] + r m_2\left(t, p_1\right) \\ &= -\frac{1}{2.9709}\sin\left(2.9709t - 2.9709\right)[\alpha+2, 4-\alpha] \\ &\quad + \cos\left(2.9709t - 2.9709\right)[\alpha+3, 5-\alpha] \end{aligned} \tag{4.14}$$

for all $\alpha \in [0, 1]$ and for all $t \in [0, 3.5]$. Fig. 2 and Fig. 3 illustrate the fuzzy solutions (4.13) and (4.14) of this FBVP for the cases where the boundary values are interactive as well as non-interactive.

Note that the solution via sup-J extension principle is contained in the solution via Zadeh's extension principle, corroborating the statement provided in [12].

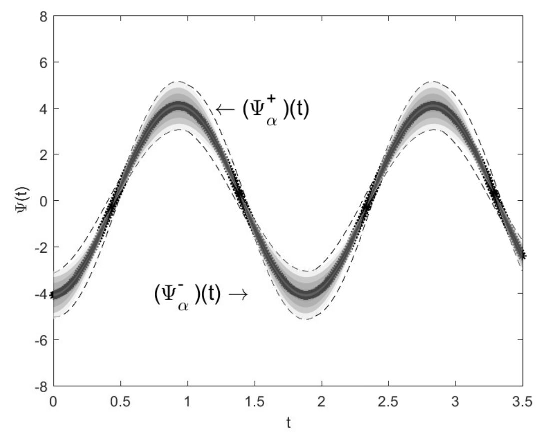

FIGURE 3. The gray-scale lines varying from white to black represent the $\alpha-$cuts of the fuzzy solution (4.12) via $sup - J$ extension principle, where their endpoints for varying from 0 to 1. Black dashed lines represent the 0-cut of the fuzzy solution (4.14) via Zadeh's extension principle of $\widehat{\Psi}$

5. Conclusion

This manuscript studies linear ordinary differential equations with two point boundary values given by interactive fuzzy numbers. The solution is obtained by means of the $sup - J$ and Zadeh's extension principle from the deterministic solutions of the associated BVP. The boundary values are non-interactive fuzzy numbers, then the fuzzy solution is given via Zadeh's extension principle.

We study linear two point FBVP with boundary values given by interactive and non-interactive fuzzy numbers. We show that the interactive fuzzy solution is contained in the non-interactive fuzzy solution by Fig.2 and Fig. 3. So it can be concluded that the fuzzy interactive solution with uncertain boundary conditions (with a membership degree given by their α-cuts) that is closer to the classical deterministic solution.

The approach via H-derivative or gH-derivative for two-point FBVP is equivalent to the study some systems of classical differential equations, which can result in an additional study of switching points. In contrast to this approach, the fuzzy solutions obtained by means of the extension principle are always well defined and do not require the analysis of the existence of switching points. Moreover from Zadeh's extension principle, the sign of the solution is not considered itself and the signs of its first and second derivatives.

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$\mathcal{C}_\alpha-$CURVES AND THEIR $\mathcal{C}_\alpha-$FRAME IN CONFORMABLE DIFFERENTIAL GEOMETRY

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Abstract. The aim of this study is to redesign the space curve and its Frenet framework, which are extremely important in terms of differential geometry, by using conformable derivative arguments. In this context, conformable counterparts of basic geometric concepts such as angle, vector, line, plane and sphere have been obtained. The advantages of the conformable derivative over the classical (Newton) derivative are mentioned. Finally, new concepts produced by conformable derivative are supported with the help of examples and figures.

1. Introduction

Perhaps the most interesting and well representative field of study of differential geometry is the theory of curves. Examination of the local properties of the curves yields different and important results. This theory has very different applications in linear and nonlinear differential equations and physics. Frenet equations are at the forefront of the most widely used and natural structure of the theory of curves. These equations have a very elite status in geometry and have many different uses. These formulas were first used in 1847 and discovered and published by Frenet J.F. Unaware of him, Serret J.A. calculated the same formulas in 1851. For this reason, these formulas are called the Frenet-Serret formulas by giving the names of both today. In this way, many new curve concepts have joined the geometry family with the help of Frenet-Serret vectors. The best examples of this are Bertrand curve pair, Mannheim curve pair and Involute-Evolute curve pair. In addition, Bishop, Darboux and Sabban frames in Euclidean and Minkowski spaces are different approaches to describing the motion of the curve. With the help of these approaches, many studies are carried out for the properties or characterization of curves in 3-dimensional Euclidean and Minkowski spaces according to Frenet, Bishop and Darboux frame [1, 2, 3].

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Classical analysis, a mathematical theory widely used today, was discovered by Leibniz G. and Newton I. in the second half of the 17th century, based on the concepts of derivative and integral, and are also referred to as Newtonian analysis. Over time, alternative analyses to Newton analysis are tried to be produced. Fractional analysis can be considered as the most important of these. Fractional analyses, which is first mentioned in Leibniz's letter to L'Hospital in 1695, aim to expand integer order derivatives to fractional orders. This theory, which is not widely accepted at first, are gained a place in every field today. The most important reason for this is the assumption that fractional analyzes have some advantages over Newtonian analysis. It is a fact that fractional analysis gives more numerical results than Newtonian analysis, especially in the solutions of some special differential equations [4, 5, 6, 7]. In this context, fractional analysis are become extremely popular and as a result, many types of fractional analysis are emerged. In general, fractional derivatives are grouped under two headings: global fractional derivative and local fractional derivative. The most important of the global fractional derivatives are Riemann-Liouville, Caputo, Grünwald-Letnikov, Wely, Riesz [8, 9, 10, 11]. The most important of the local fractional derivatives are proven themselves today as conformable, M-derivative and V-derivative [12, 13, 14, 15]. Global and local fractional derivatives have a big distinction within themselves. The most important difference between them is that global derivatives do not satisfy Leibniz and the chain rule as in the classical derivative, while local fractional derivatives do not have such a disadvantage. In addition, in global fractional derivatives, the derivative of the constant is not zero except for the Caputo derivative, but this is not the case in local derivatives. This situation is made local derivatives more indispensable in some matters.

The theory of curves and surfaces can be defined as the study of the motion of a point in a space with the help of linear algebra and calculus. Moreover, Leibniz and the chain rule are two indispensable elements when making calculations in differential geometry. For this reason, if fractional analysis is to be applied in differential geometry, the most appropriate one is local fractional derivatives. Fractional calculus has been used effectively in the field of differential geometry for the last decade, as it has proven itself in every field. This adventure was first started when Yajima T. and Kamasaki K. examined the Caputo fractional derivative of surfaces [16]. Additionally, Yajima T. et al. succeeded in creating the Frenet frame using fractional calculus [17]. Lazopoulos K.A. and Lazopoulos A.K. are made fractional calculations on manifolds [18]. Evren M.E. explained that local fractional derivatives are more useful and advantageous than global fractional derivatives in differential geometry [19]. Has A. et al obtained some advantages of the conformable derivative in terms of geometry compared to the classical derivative [20]. Gozutok and colleagues created the Frenet frame using conformable derivatives [21]. Following these developments, the use of fractional analysis in differential geometry has increased tremendously and many studies have been carried out on this subject [22, 23, 24, 25, 26, 27, 28, 29, 30].

In this study, the basic geometric properties of the curves were reconstructed using compatible derivative arguments. In the first stage, the main concepts of angle, vector, line, plane and sphere, which are geometric concepts, were redesigned with the help of conformable calculus. In addition, the orthogonal and orthonormal systems, which can be considered the basis of vectors, have been redefined in a similar way. Afterwards, with these of the conformable concepts obtained, the conformable space curve and the conformable Frenet framework at any point of it were created. In the final, examples were given and enriched with figures to make the subject more fluent.

2. Preliminaries

Khalil R. et al. are introduced a new derivative called the conformable fractional derivative of order α of the function f, which is defined as [12]:

$$D_\alpha(f)(s) = \lim_{\varepsilon\to 0} \frac{f(s+\varepsilon s^{1-\alpha}) - f(s)}{\varepsilon}.$$

where $f : [0,\infty) \to \mathbb{R}$ and $0 < \alpha < 1$. The relationship between the conformable derivative and the classical derivative, where $f'(s) = df(s)/ds$, is obtained as follows:

$$D_\alpha f(s) = s^{1-\alpha}\frac{d}{ds}f(s).$$

We say with the next theorem that the conformable derivative satisfies some properties such as linearity, Leibniz's rule and chain rule, as in the conventional derivative.

Theorem 2.1. *Let $f : [0,\infty) \to \mathbb{R}$ and $0 < \alpha < 1$. The following are provided as functions f, g are α-differentiable functions. For all $a, b, p, \lambda \in \mathbb{R}$* [12]*,*

(1) $D_\alpha(af + bg)(s) = aD_\alpha f(s) + bD_\alpha g(s)$,
(2) $D_\alpha(s^p) = ps^{p-\alpha}$,
(3) $D_\alpha(\lambda) = 0$,
(4) $D_\alpha(fg)(s) = f(s)D_\alpha g(s) + g(s)D_\alpha f(s)$,
(5) $D_\alpha(\frac{f}{g})(s) = \frac{g(s)D_\alpha f(s)-f(s)D_\alpha g(s)}{g^2(s)}$,
(6) $D_\alpha(g\circ f)(s) = f^{\alpha-1}(s)D_\alpha f(s)D_\alpha g(f(s))$.

The conformable integral was defined by Khalil R. et al. as the inverse operator of the conformable derivative operator. Accordingly, the conformable integral of the $\alpha-$differentiable function f and for $[t,s]$, is as follows [12]

$$I_t^\alpha f(s) = I_t^s(s^{\alpha-1}f) = \int_t^s \frac{f(s)}{s^{1-\alpha}}ds.$$

In addition, f being a conformable differentiable function is given below for $t > 0$

$$D_\alpha I_\alpha[f(s)] = f(s)$$

The derivative limit of vector-valued functions has also been investigated by means of conformal analysis. We give this in the following theorem.

Theorem 2.2. *Let the function f be a function with n variables and each component is conformable differentiable. Then the conformable derivative of the function f is* [32]

$$D_\alpha f(f_1(s), f_n(s), ...f_n(s)) = f(D_\alpha f_1(s), D_\alpha f_m(s), ..., D_\alpha f_n(s)).$$

3. Some concepts of conformable differential geometry

In this section, the most basic concepts of geometry will be reconstructed with conformable arguments.

Notation: Along the study, expressions that are equal to 1 when $\alpha \to 1$ will be denoted as $\mathbf{1}_\alpha$ and expressions that are equal to 0 when $\alpha \to 1$ will be denoted as $\mathbf{0}_\alpha$. In addition, in order to avoid confusion between classical and conformable concepts, $\mathcal{C}_\alpha$ will be left in charge of conformable concepts.

Remark 3.1 (**A geometric approach to conformable derivative**). The geometric interpretation of the conformable derivative is based on the notion of fractal geometry. In fractal geometry, objects exhibit self-similarity at different scales. The conformable derivative captures this self-similar behavior of a function by considering its local fractional variations. Geometrically, it can be understood as analyzing the "zooming in" behavior of the function at that point, similar to the classical derivative capturing the local linear behavior. Overall, the geometric interpretation of the conformable derivative relates to the self-similarity and scaling properties of functions, enabling us to understand their behavior at different levels of detail and resolution. *More specifically, the conformable derivative can be explained as a measure of how much a straight line and plane bend to form a curve and a surface.* Figure 3 shows how a line is curved with the conformable calculus effect.

There are no Euclidean lines in the $\mathcal{C}_\alpha-$(conformable) space, this only happens when $\alpha \to is$ 1. We present this in Figure 3. This situation leads us to define a new angle in $\mathcal{C}_\alpha-$ space. Because we cannot measure the angle between the classical angle and the lines in $\mathcal{C}_\alpha-$ space. This new angle is called the $\mathcal{C}_\alpha-$ angle, and it measures the angle between the $\mathcal{C}_\alpha-$lines.

Let $\|\mathbf{u}\| = \mathbf{1}_\alpha$ and $\mathbf{v}$ are $\mathcal{C}_\alpha-$unit vector that is, they are vectors of the form $\|\mathbf{u}\| = \mathbf{1}_\alpha$ and $\|\mathbf{v}\| = \mathbf{1}_\alpha$. Then, the *$\alpha-$conformable radian measure* of *$\mathcal{C}_\alpha-$angle* between $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\theta_\alpha = \arccos\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}\right).$$

It is also said that $\mathbf{u}$ and $\mathbf{v}$ are $\mathcal{C}_\alpha-$orthogonal when the following condition is proved,

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{0}_\alpha.$$

When x=1, $\mathcal{C}_\alpha-$ space has a different structure than Euclidean space, so the concept of $\mathcal{C}_\alpha-$ orthogonality will differ from. For example, let's consider the vectors $\mathbf{u} = (s^{1-\alpha}, 1-\alpha, \frac{1}{s^{1-\alpha}})$ and $\mathbf{v} = (\frac{1-\alpha}{s^\alpha}, s^\alpha, 2-2\alpha)$ in $\mathcal{C}_\alpha-$ space. Since $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{0}_\alpha$, vectors $\mathbf{u}$ and $\mathbf{v}$ are $\mathcal{C}_\alpha-$orthogonal. We showed this in Figure 1.

As in Euclidean space, in $\mathcal{C}_\alpha-$space the vector $u \times v$ is $\mathcal{C}_\alpha-$orthogonal to the vectors $\mathbf{u}$ and $\mathbf{v}$. For example, if $\mathbf{u} = (s^{1-\alpha}, 1-\alpha, \frac{1}{s^{1-\alpha}})$ and $\mathbf{v} = (\frac{1-\alpha}{s^\alpha}, s^\alpha, 2-2\alpha)$, $\mathbf{u}\times\mathbf{v} = (2\alpha^2 - 4\alpha - s^{2\alpha-1} + 2, 2\alpha s^{1-\alpha} - 2s^{1-\alpha} - \frac{\alpha}{s} + \frac{1}{s}, -\alpha^2 s^{-\alpha} + 2\alpha s^{-\alpha} - s^{-\alpha} + s)$ is obtained. It is also seen that $\langle \mathbf{u}\times\mathbf{v}, \mathbf{u}\rangle = 0_\alpha$ and $\langle \mathbf{u}\times\mathbf{v}, \mathbf{v}\rangle = 0_\alpha$. We showed this in Figure 2.

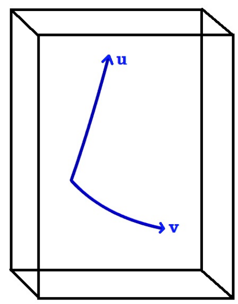

FIGURE 1. $\mathcal{C}_\alpha-$orthogonal vectors.

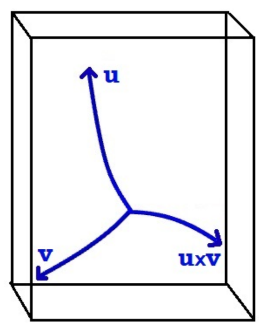

FIGURE 2. $\mathcal{C}_\alpha-$orthogonal system.

Definition 3.2. $\mathcal{C}_\alpha-$*line* l with a direction $\mathbf{v}$ through the point $P_\alpha = ((p_1)_\alpha, (p_2)_\alpha)$ is a subset of $\mathbb{E}^2$ is defined as

$$l = \{X \in \mathbb{E}^2 : X = P_\alpha + \mathbf{v}_\alpha f(t)\}$$

where $f(t) = \int t^{1-\alpha} dt$, $\mathbf{v}_\alpha = ((v_1)_\alpha, (v_2)_\alpha)$ and P_α is the point whose coordinates contain α.

Example 3.3. Let consider the $s \mapsto \mathbf{x}(s) = (s, \int s^{1-\alpha} ds)$, $\mathcal{C}_\alpha-$line passing through the point $P = (0, 0)$ and whose direction is $v = (s^{1-\alpha}, s^{1-\alpha})$.
In Fig. (3) we present the graph of the $\mathcal{C}_\alpha-$line for different α values.

Definition 3.4. $\mathcal{C}_\alpha-$*plane* Γ passing through a point $P_\alpha = ((p_1)_\alpha, (p_2)_\alpha, (p_3)_\alpha)$ and $\mathcal{C}_\alpha-$orthogonal to $\mathbf{v}$ is a subset of $\mathbb{E}^3$ defined by (see Fig. 4)

$$\Gamma = \{X \in \mathbb{E}^3 : \langle X - P_\alpha, \mathbf{v}_\alpha \rangle = 0_\alpha\}$$

where $X = (I_\alpha^a x_1, I_\alpha^a x_2, I_\alpha^a x_3)$, $\mathbf{v}_\alpha = ((v_1)_\alpha, (v_2)_\alpha, (v_3)_\alpha)$ and P_α is the point whose coordinates contain α.

Example 3.5. Let X be a representation point of the $\mathcal{C}_\alpha-$plane that contains the point $P = (0, 0, 0)$ and whose normal is $v = (2^{1-\alpha}, -3^{1-\alpha}, 0)$. If X representative

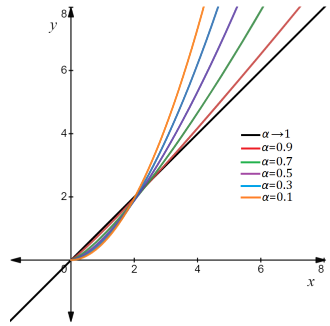

FIGURE 3. $\mathcal{C}_\alpha-$lines.

point is chosen as follows

$$\begin{aligned}\mathbf{x}_1(s) &= \int x^{1-\alpha}dx,\\ \mathbf{x}_2(s) &= \int y^{1-\alpha}dy,\\ \mathbf{x}_3(s) &= 0\end{aligned}$$

we get the $\mathcal{C}_\alpha-$plane. In Fig. (4) we present the graph of the $\mathcal{C}_\alpha-$plane for different α values.

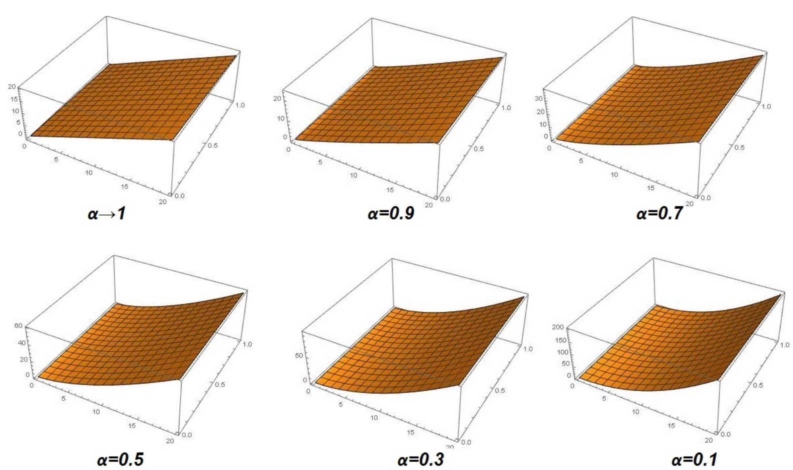

FIGURE 4. $\mathcal{C}_\alpha-$plane for different α values.

Definition 3.6. $\mathcal{C}_\alpha-$*sphere* with radius r_α and centered $C_\alpha = ((x_0)_\alpha, (y_0)_\alpha, (z_0)_\alpha)$ is a subset of $\mathbb{E}^3$ defined by (see Fig. 5)

$$\mathbb{S}^2_\alpha(C_\alpha, r_\alpha) = \{X \in \mathbb{E}^3 : \|X - C_\alpha\| = r_\alpha\}$$

where $X = (I^a_\alpha x_1, I^a_\alpha x_2, I^a_\alpha x_3)$. It should be noted here that the r_α and C_α values are not constant values. That is, the center and radius of the $\mathcal{C}_\alpha$−sphere change for each value of α. We shall denote by $\mathbb{S}^2_\alpha$ the $\mathcal{C}_\alpha$−sphere with radius 1_α and centered at $C_\alpha = (0_\alpha, 0_\alpha, 0_\alpha)$.

Example 3.7. Let $\mathbb{S}^2_\alpha(C_\alpha, r_\alpha)$ be a $\mathcal{C}_\alpha$−sphere in $\mathbb{R}^3$ parameterized by φ. If the coordinate functions of φ is chosen as follows,

$$\begin{aligned} f_1(u,v) &= -\int\int v^{\alpha-1}u^{\alpha-1}\sin u\cos v du dv \\ f_2(u,v) &= \int\int v^{\alpha-1}u^{\alpha-1}\sin u\sin v du dv \\ f_3(u,v) &= \int u^{\alpha-1}\cos u du \end{aligned}$$

we get the $\mathcal{C}_\alpha$−sphere as $\varphi(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$. In Fig. (5), we present the graph of the $\mathcal{C}_\alpha$−sphere for different α values.

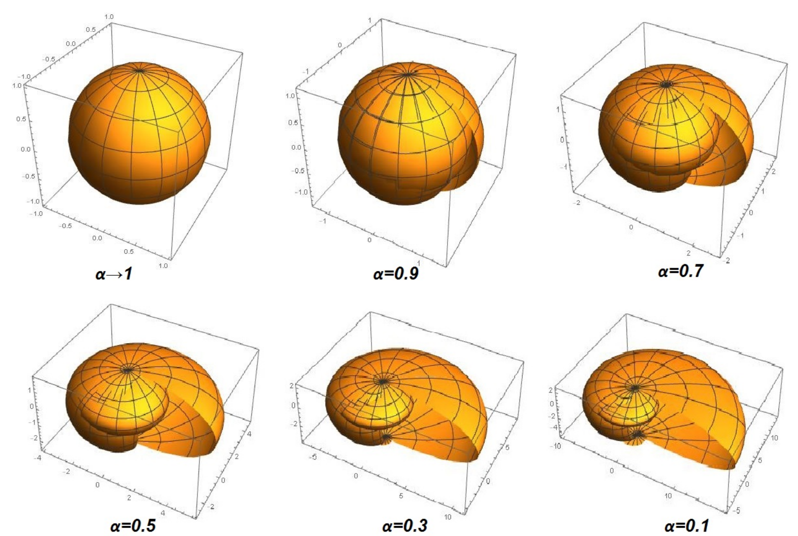

FIGURE 5. $\mathcal{C}_\alpha$−sphere for different α values.

4. $\mathcal{C}_\alpha$−PARAMETRIZED CURVES AND THEIR $\mathcal{C}_\alpha$−FRAME

Given that a 3-dimensional vector valued function to the $\mathcal{C}_\alpha$− space as follow

$$\mathbf{x} : I \subset \mathbb{R} \to \mathbb{E}^3 \tag{4.1}$$

$$s \to \mathbf{x}(s) = (\mathbf{x}_1(s), \mathbf{x}_2(s), \mathbf{x}_3(s)) \tag{4.2}$$

We call $\mathbf{x}$ that satisfies the following equation $\mathcal{C}_\alpha$−naturally parameterized curve.

$$\|D_\alpha \mathbf{x}(s)\| = s^{1-\alpha}$$

where $D_\alpha \mathbf{x}(s) = (D_\alpha \mathbf{x}_1(s), D_\alpha \mathbf{x}_2(s), D_\alpha \mathbf{x}_3(s))$. Also, when $D_\alpha \mathbf{x}(s) \neq 0_\alpha$, $\mathbf{x}$ is called a $\mathcal{C}_\alpha$−regular curve and $D_\alpha \mathbf{x}(s) \times D^2_\alpha \mathbf{x}(s) \neq 0_\alpha$, $\mathbf{x}$ is called a $\mathcal{C}_\alpha$−biregular curve in $\mathcal{C}_\alpha$−space for each $s \in I$.

We named the triple apparatus $\{E_1, E_2, E_3\}$, defined as follows, as the $\mathcal{C}_\alpha-$frame vectors at point $s \in I$ of $\mathcal{C}_\alpha-$naturally parameterized curve $\mathbf{x}$:

$$E_1(s) = D_\alpha \mathbf{x}(s), \quad E_2(s) = \frac{D_\alpha E_1(s)}{\|D_\alpha E_1(s)\|}, \quad E_3(s) = E_1(s) \times E_2(s). \tag{4.3}$$

The E_1, E_2 and E_3 trio are called the $\mathcal{C}_\alpha$ tangent, principle normal and binormal at the point $s \in I$ of $\mathbf{x}$, respectively. Moreover, the vectors of the $\mathcal{C}_\alpha-$frame E_1, E_2 and E_3 are $\mathcal{C}_\alpha-$orthogonal and $\mathcal{C}_\alpha-$orthonormal.

Theorem 4.1. *The conformable derivative change of the $\mathcal{C}_\alpha-$frame at point $s \in I$ of the $\mathcal{C}_\alpha-$naturally parameterized* $\mathbf{x}$ *curve is as follows*

$$\begin{bmatrix} D_\alpha E_1 \\ D_\alpha E_2 \\ D_\alpha E_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}. \tag{4.4}$$

Proof. Considering Eq. (4.3), as follows

$$E_1 = s^{1-\alpha} T, \tag{4.5}$$

$$E_2 = \frac{(1-\alpha)s^{1-2\alpha}}{\sqrt{(1-\alpha)^2 s^{2-4\alpha} + s^{4-4\alpha}\kappa^2}} T + \frac{s^{2-2\alpha}\kappa}{\sqrt{(1-\alpha)^2 s^{2-4\alpha} + s^{4-4\alpha}\kappa^2}} N, \tag{4.6}$$

$$E_3 = \frac{s^{3-3\alpha}\kappa}{\sqrt{(1-\alpha)^2 s^{2-4\alpha} + s^{4-4\alpha}\kappa^2}} B. \tag{4.7}$$

Differentiating of both sides of the above first equation α-th order conformable derivative as for s, we obtain

$$D_\alpha E_1 = (1-\alpha)s^{1-2\alpha} T + s^{2-2\alpha}\kappa N. \tag{4.8}$$

and

$$\|D_\alpha E_1\| = \sqrt{(1-\alpha)^2 s^{2-4\alpha} + s^{4-4\alpha}\kappa^2}. \tag{4.9}$$

Let's consider the $\kappa_\alpha = \|D_\alpha E_1\|$ equation here and use this equation in Eqs. (4.5), (4.6) and (4.7), we get

$$E_1 = s^{1-\alpha} T, \tag{4.10}$$

$$E_2 = \frac{(1-\alpha)s^{1-2\alpha}}{\kappa_\alpha} T + \frac{s^{2-2\alpha}\kappa}{\kappa_\alpha} N, \tag{4.11}$$

$$E_3 = \frac{s^{3-3\alpha}\kappa}{\kappa_\alpha} B. \tag{4.12}$$

Since the triple $\{E_1, E_2, E_3\}$ is $\mathcal{C}_\alpha-$orthogonal basis in $\mathbb{E}^3$,the following equation exist

$$D_\alpha E_1 = a_{11} E_1 + a_{12} E_2 + a_{13} E_3. \tag{4.13}$$

On the other hand, let's consider the definition of κ_α and Eq. (4.3)

$$D_\alpha E_1 = \kappa_\alpha E_2$$

is obtained. Considering this equation in Eq. (4.13), we can be write as

$$a_{12} = \kappa_\alpha.$$

Now, considering Eqs. (4.9), (4.10) and (4.11) by taking advantage of the scalar product of Eq. (4.13) with E_1, we get

$$\langle D_\alpha E_1, E_1 \rangle = a_{11}\langle E_1, E_1 \rangle + a_{12}\langle E_1, E_2 \rangle + a_{13}\langle E_1, E_3 \rangle,$$

$$(1-\alpha)s^{2-3\alpha} = a_{11}s^{2-2\alpha} + \kappa_\alpha \frac{(1-\alpha)s^{2-3\alpha}}{\kappa_\alpha}.$$

So we get the following result as

$$a_{11} = 0.$$

Anologously, considering Eqs. (4.7) and (4.8), by taking advantage of the scalar product of Eq. (4.13) with E_3, we get

$$a_{13} = 0.$$

The other part of the theorem is proved similarly. □

Conclusion 1. There is a relationship between the conformable derivative and the classical (Newton) derivative. When $\alpha \to 1$ is selected in the Conformable derivative, it is possible to return to the classical derivative results. Similarly, since the $\mathcal{C}_\alpha-$frame is obtained by conformable derivative, when $\alpha \to 1$ is selected, the $\mathcal{C}_\alpha-$frame turns into a classical Frenet frame. For this reason, a curve can be examined and compared both in $\mathcal{C}_\alpha-$space and Euclidean space.

Conclusion 2. The conformable derivative has some advantages over the classical derivative in terms of geometric meaning. The most important of these advantages is that conformable derivatives can be defined at points where the classical derivative is not defined. Thus, at points where tangents cannot be created with the classical derivative, alternative tangents can be created with the help of conformable derivative. For example, the derivative of the function $f(x) = 2\sqrt{x}$ is not defined at $x = 0$. Then it is impossible to create a tangent at $x = 0$. However, if the conformable derivative is applied by selecting $\alpha = \frac{1}{2}$ in the function $f(x)$, $D_\alpha f(x) = 0$. In other words, while a classical tangent cannot be mentioned at the point $x = 0$, a conformable tangent can be mentioned. The most important element of the Frenet frame is the tangent vector. Because other Frenet vectors can be obtained depending on the tangent vector. Thus, at points where the Frenet frame cannot be created, the curve can be examined by creating a $\mathcal{C}_\alpha-$frame.

Example 4.2. Let $\mathbf{x} : I \subset \mathbb{R} \to E^3$ be a $\mathcal{C}_\alpha-$naturally parametrized curve in $\mathbb{R}^3$ parameterized by

$$\mathbf{x}(s) = \left(2s^{\frac{1}{2}},\ s^{\frac{3}{2}},\ s^{\frac{5}{2}}\right).$$

The classical derivative of x and the conformable derivative for $\alpha = \frac{1}{2}$ are as follows

$$\mathbf{x}'(s) = \left(s^{\frac{-1}{2}}, \frac{3}{2}s^{\frac{1}{2}}, \frac{5}{2}s^{\frac{3}{2}}\right), \tag{4.14}$$

$$D_{\frac{1}{2}}\mathbf{x}(s) = \left(1, \frac{3}{2}s, \frac{5}{2}s^2\right). \tag{4.15}$$

where $x'(s)$ and $D_{\frac{1}{2}}\mathbf{x}(s)$ are the classical tangent and $\mathcal{C}_\alpha-$tangent of $\mathbf{x}(s)$, respectively. Considering Eqs. (4.14) and (4.14), while $\mathbf{x}'(0)$ is undefined for s=0, $D_{\frac{1}{2}}\mathbf{x}(0))$ is defined. We show this situation in Figure 6.

Moreover, as mentioned in Conclusion 1, while a Frenet frame cannot be obtained at points where there is no derivative of a curve, a $\mathcal{C}_\alpha-$frame can be established at the same point.

Theorem 4.3. *Let $\mathbf{x} = \mathbf{x}(s)$ be $\mathcal{C}_\alpha-$naturally parametrized curve in the Euclidean $3-$space where s measures its $\mathcal{C}_\alpha-$arc length. When $\alpha \to 1$, its curvature and torsion are $\kappa_\alpha \to \kappa$ and $\tau_\alpha \to \tau$, respectively.*

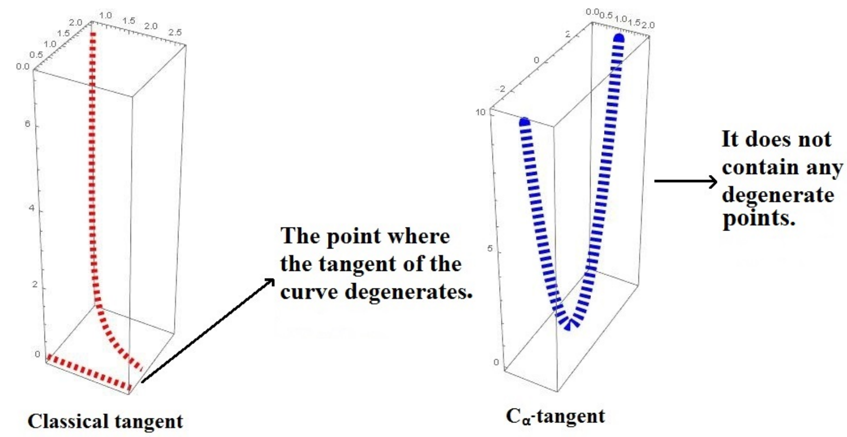

Figure 6. Classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(s)$.

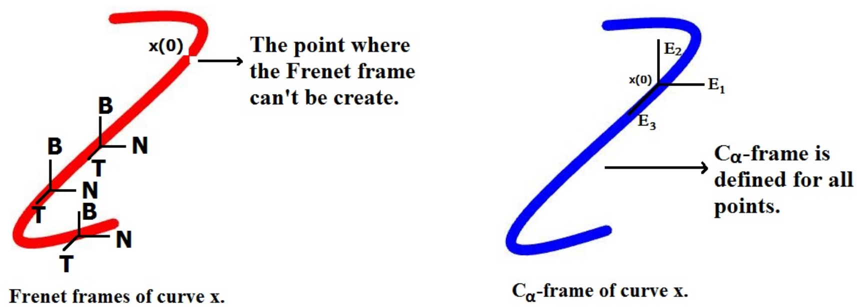

Figure 7. Frenet frame and $\mathcal{C}_\alpha-$frame of the curve $\mathbf{x}(s)$ at $s = 0$.

Proof. Let $\mathbf{x} = \mathbf{x}(s)$ be a regular $\mathcal{C}_\alpha-$curve $\mathbf{x}$. Here let's consider the definition of κ_α and Eq. (4.9), we have

$$\kappa_\alpha = s^{1-\alpha}\sqrt{(1-\alpha)^2 s^{-2\alpha} + s^{2-2\alpha}\kappa^2}. \tag{4.16}$$

Also considering the definition of τ_α and Eqs. (4.11), (4.12) we get

$$\tau_\alpha = \frac{s^{5-5\alpha}\kappa^2}{\kappa_\alpha^2}\tau. \tag{4.17}$$

Here is seen, while $\alpha \to 1$, $\kappa_\alpha \to \kappa$ and $\tau_\alpha \to \tau$. □

Example 4.4. Let $x : I \subset \mathbb{R} \to E^3$ be a $\mathcal{C}_\alpha-$naturally parametrized curve in $\mathbb{R}^3$ parameterized by

$$x(s) = \left(\frac{3}{5}\cos s,\ \frac{3}{5}\sin s,\ \frac{4}{5}s\right).$$

From Eqs. (4.5), (4.6) and (4.7), we get

$E_1 = \left(\frac{-3s^{1-\alpha}}{5}\sin s, \frac{3s^{1-\alpha}}{5}\cos s, \frac{4s^{1-\alpha}}{5}\right),$

$E_2 = \left(\frac{-3(1-\alpha)s^{1-2\alpha}\sin s - 3s^{2-2\alpha}\cos s}{5\kappa_\alpha}, \frac{3(1-\alpha)s^{1-2\alpha}\cos s - 3s^{2-2\alpha}\sin s}{5\kappa_\alpha}, \frac{4(1-\alpha)s^{1-2\alpha}}{5\kappa_\alpha}\right),$

$E_3 = \left(\frac{4s^{3-3\alpha}}{5\kappa_\alpha}\sin s, \frac{-4s^{3-3\alpha}}{5\kappa_\alpha}\cos s, \frac{3s^{3-3\alpha}}{5\kappa_\alpha}\right).$

In addition, the $\mathcal{C}_\alpha-$curvature and torsion of the $\mathcal{C}_\alpha-$curve $\mathbf{x}$ is calculated as in Eqs. (4.16) and (4.17) as follows

$$\kappa_\alpha = \frac{s^{1-\alpha}}{5}\sqrt{25(1-\alpha)^2 s^{-2\alpha} + 9s^{2-2\alpha}},$$

$$\tau_\alpha = \frac{36s^{3-3\alpha}}{125(1-\alpha)^2 s^{-2\alpha} + 45s^{2-2\alpha}}.$$

For different values of α the graphs of the curvature κ_α and torsion τ_α with fractional-order as in following Fig. 8 and Fig. 9.

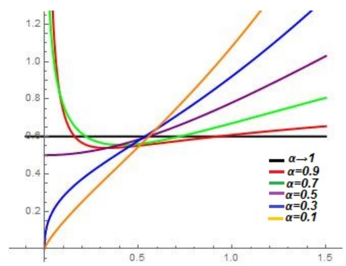

FIGURE 8. $\mathcal{C}_\alpha-$curvatures, κ_α.

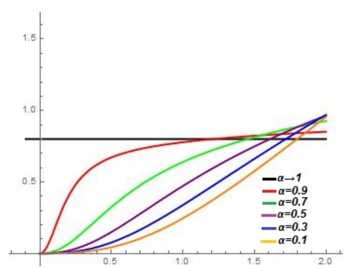

FIGURE 9. $\mathcal{C}_\alpha-$torsion, τ_α.

Example 4.5. Let $\mathbf{x} : I \subset \mathbb{R} \to E^3$ be a $\mathcal{C}_\alpha-$naturally parametrized curve in $\mathbb{E}^3$ parameterized by

$\mathbf{x}(s) = \left(-\frac{225}{16}\int s^{\alpha-1}(\sin 25s + \sin 9s)ds,\ \frac{225}{16}\int s^{\alpha-1}(\cos 25s - \cos 9s)ds,\ -\frac{225}{8}\int s^{\alpha-1}\sin 17s ds\right).$

In Fig. (10) we present the graph of the $\mathcal{C}_\alpha-$naturally parametrized curve for different α values

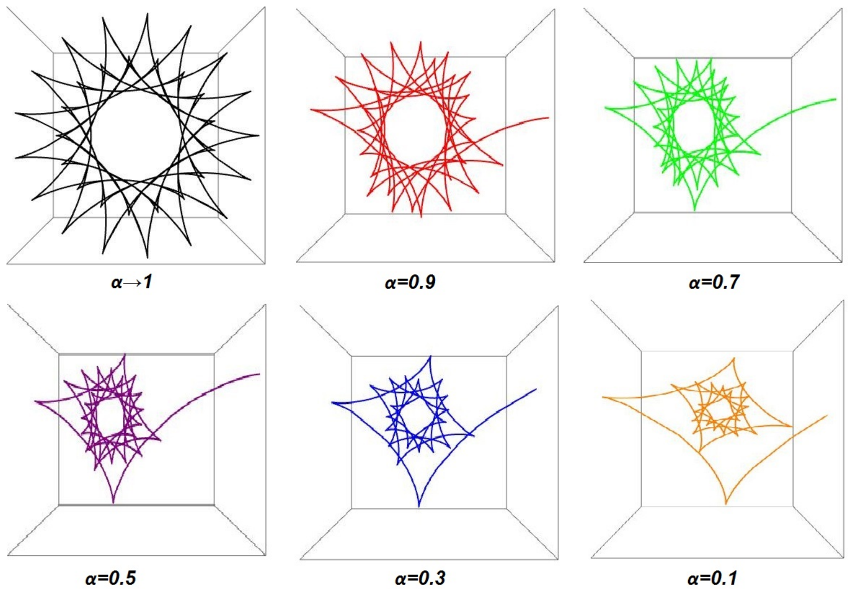

FIGURE 10. $\mathcal{C}_\alpha-$naturally parameterized curve for different α values.

5. Conclusion

In this study, we want to bring a new perspective to some problems that cannot be solved in Euclidean space, with the help of conformable derivatives. Alternatives have been created for some concepts that cannot be defined in Euclidean space with the help of conformable derivatives and have now become examinabla. In addition, one of the most attractive features of this situation is that new concepts can be compared with their classical forms since the return to Euclidean space is achieved at $\alpha \to 1$.

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Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

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The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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SOME COLORING RESULTS ON SPECIAL SEMIGROUPS OBTAINED FROM PARTICULAR KNOTS

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Abstract. For a coloring set $B \subseteq \mathbb{Z}_n$, by considering the Fox n-coloring of any knot K and using the knot semigroup K_S, we show that the set B is actually the same with the set C in the alternating sum semigroup $AS(\mathbb{Z}_n, C)$. Then, by adapting some results on Fox n-colorings to $AS(\mathbb{Z}_n, B)$, we obtain some new results over this semigroup. In addition, we present the existence of different homomorphisms (or different isomorphisms in some cases) between the semigroups K_S and $AS(\mathbb{Z}_n, B)$, and then obtained the number of homomorphisms is in fact a knot invariant. Moreover, for different knots K^1 and K^2, we establish one can obtain a homomorphism or an isomorphism from the different knot semigroups K^1_S and K^2_S to the same alternating sum semigroup $AS(\mathbb{Z}_n, B)$.

1. Introduction

It is known that the knots are equalivance classes of topological inclusions from $\mathbb{S}^1$ to $\mathbb{S}^3$ under ambient isotopes which these isotopes give the smooth deformations between two knots. We may refer the classical book [8] for the details in knot theory. In here, we will mainly give our interest to Torus knots and Pretzel links during the construction of our theories.

As indicated in [7], the fundamental quandle of a knot was defined in a similar manner to the fundamental group of a knot, which made quandles are important tools in knot theory. The number of homomorphisms from the fundamental quandle to a fixed finite quandle has an interpretation as colorings of knot diagrams by quandle elements, and has been widely used as a knot invariant. Furthermore involutary quandles are defined on a single binary operation ([10]). In detail, they are the algebraic way to represent the Reidemeister movements ([1]) and so they are important to obtain new knot invariants and also important to investigate knots. On the other hand, Fox n-colorings are actually the best known involutary quandles. These colorings will be briefly indicated in coming next subsection, and also one part of the main result will be constructed the base on this subject (see Theorem

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2.1 in below). In fact the other whole main theorems (which are about Pretzel links and Torus knots) given in this paper can be thought as consequences of Theorem 2.1.

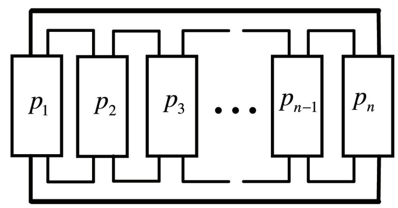
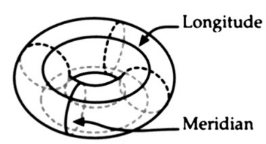


Figure 1.

Let $P(p_1, p_2, \cdots, p_n)$ be an n-Pretzel link $\mathbb{S}^3$ in where $p_i \in \mathbb{Z}$ represents the number of half twists (or, we can call it as *regions*) as depicted in Figure 1. If $n = 3$, then it is called a classical pretzel link $P(p, q, r)$. If n is odd, then an n-Pretzel link $P(p_1, p_2, ..., p_n)$ is a knot if and only if none of two p_i's are even. If n is even, then $P(p_1, p_2, ..., p_n)$ is a knot if and only if only one of the p_i's is even. Generally the number of even p_i's is the number of components unless p_i's are all odd. On the other hand, Torus knots are identified by the number of times the strand wraps around the torus meridionally and longitudinally. We speak of a Torus knot $T_{p,q}$, where p and q are relatively prime; when p and q are not relatively prime, we obtain a link of two or more components ([15]).



Let K be an oriented knot (or link) with n crossings. Label those crossings by $1, 2, \cdots, n$ and label the n arcs by $a_1, a_2, \cdots, a_n$. Construct an $n \times n$ matrix M such that each row r corresponds to the crossing labeled by again r and each column s corresponds to the arc labeled by again s. Suppose that at crossing r the over-passing arc is labeled a_i, that the arc a_j ends at crossing r, and that the arc a_k begins at crossing r. Suppose also that i, j and k are mutually distinct. Assume also that crossing r is positive. Then, for a real number t, the entries will be the formed as $M(r, i) = 1 - t$, $M(r, j) = -1$ and $M(r, k) = t$. When crossing r is negative, then $M(r, i) = 1 - t$, $M(r, j) = t$, $M(r, k) = -1$ and other elements of M are zero.

The Alexander matrix A_K is defined as to be the matrix obtained from the matrix M by deleting row n and column n. It is also known that the Alexander polynomial $\Delta_K(t)$ of a knot K is the determinant of it's Alexander matrix (see, for instance, [2, 11]), and the Alexander polynomial at $t = -1$ (and then taking absolute value) defines the determinant of a knot K. We recall that the Alexander polynomial is the first invariant polynomial defined on knots. The invariant property of these polynomials of the knots that belongs to the same equivalence classes are the same. We note that while the Alexander polynomials of a Torus $T_{p,q}$ (cf. [15]) and a

Pretzel link $P(p,q,r)$ (cf. [19]) are

$$\Delta_{T_{p,q}}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} \quad \text{and}$$
$$\Delta_{P(p,q,r)}(t) = \frac{1}{4}\left[(pq+pr+qr)(t-2+t^{-1}) + (t+2+t^{-1})\right] ,$$

respectively, the determinants of them are calculated by

$$\Delta_{T_{p,2}}(-1) = p \quad \text{and} \quad \Delta_{P(p,q,r)}(-1) = pq+pr+qr\,, \tag{1.1}$$

and respectively.

1.1. **Quandles and Fox n-Coloring.** For any set Q, by defining two binary operations $x \triangleright y$ and $x \triangleright^{-1} y$ which satisfy $(x \triangleright y) \triangleright^{-1} y = x$, one can obtain a *quandle* over Q. If only $(x \triangleright y) \triangleright y = x$ holds, then it is named as *involutory quandle*. On the other hand, the other important quandle is the named as *Alexander quandle* which consists of a quandle with a left action given by $a \triangleright b = ta + (1-t)b$. The importance of Alexander quandle comes from the fact that it is another way the computation of Alexander polynomials. On the other hand, if we take $t=-1$ in an Alexander quandle, then we get the *dihedral quandle*. The dihedral quandles are placed into knot colorings (in some sources, authors use the term Fox n-coloring). We may refer, for instance, [4, 5, 6, 7, 9, 10, 14] for more details on quandles, colorings and some other well known types. In this paper, we will apply Fox n-coloring to the knots in terms of dihedral quandles by following the fact that they are knot invariant and very useful for the characterization of a knot.

At this point let us briefly indicate the meaning of Fox n-coloring. For a knot K and a diagram D of K, let A be the set of arcs in D. Now let us matching (not necessarily one to one) the elements of A by the elements of $\mathbb{Z}_n$. Also, for each matching, let us consider the equivalence

$$a \triangleright b \equiv c \equiv -a + 2b \pmod{n} \;\text{ such that }\; n \geq 2 \tag{1.2}$$

such that a and c represent the numerical values in $\mathbb{Z}_n$ for the bottom arcs, respectively, while b represents the numerical value in $\mathbb{Z}_n$ for the upper arc. After all, if whole equivalences satisfy up to $\mathbb{Z}_n$ then we say that the knot K is named as *Fox n-colorable* (or shortly *n-colorable*). The subject Fox n-coloring is actually correspondent to the involutory quandle ([10]). In here, we strongly note that since the matrix obtained by deleting the last row and column of the coefficient matrix of n-coloring equations and the matrix obtained by replacing $t=-1$ in Alexander matrix of K are the same, we get that the positive integer n is the determinant of K itself (in other words $n = \Delta_K(-1)$) or it is a positive integer that divides this determinant (in other words $n \mid \Delta_K(-1)$).

Now let us denote the number of colorings of K in terms of the quandle Q by $Col_Q(K)$. Then we have the following lemma.

Lemma 1.1 ([7])**.** *The quandle Q distinguishes knots K and K' if $Col_Q(K) \neq Col_Q(K')$.*

1.2. **Semigroups K_S and $AS(G,B)$.** Recently, it has been defined a new semigroup under the name of *knot semigroups* and denoted by K_S (cf. [18]). The elements of K_S are the arcs of the knot K and the relations are every crossings on K. In fact for a single crossing as in Figure 2, we have two relations $xy = yz$ and $zy = yx$, where x, y and z are the generators.

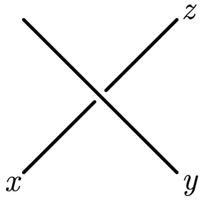


FIGURE 2. Two relations $xy = yz$ and $zy = yx$ obtained from a single crossing

There are some immediate examples that can be given. Firstly, since the *unknot* (or, equivalently, a *circle*), notated by 0_1, contains a unique arc without any crossing, then the knot semigroup of unknot is actually a free semigroup with a single generator which can be expressed as $K_{S(0_1)} = \langle x\,;\,\rangle$. Second example can be given on torus knots $T_{p,q}$, where p and q are relatively prime; when p and q are not relatively prime we obtain a link of two or more components. By taking $q = 2$, we obtain the torus knot semigroups

$$\begin{aligned} K_{ST_{p,2}} = \langle\, a_0, a_1, a_2, ..., a_{p-1} \quad ; \quad & a_0a_1 = a_1a_2,\ a_1a_2 = a_2a_3,\ \cdots,\ a_{p-2}a_{p-1} = a_{p-1}a_0, \\ & a_0a_{p-1} = a_{p-1}a_{p-2},\ a_{p-1}a_{p-2} = a_{p-2}a_{p-3}, \\ & \cdots,\ a_2a_1 = a_1a_0 \,\rangle\,. \end{aligned} \tag{1.3}$$

The diagram for the torus knot semigroup

$$K_{ST_{3,2}} = \langle x, y, z\,;\, xy = yz,\ zy = yx,\ yx = xz,\ zx = xy,\ xz = zy,\ yz = zx\rangle$$

is drawn in Figure 3.

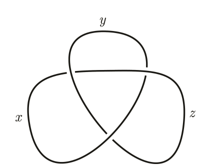


FIGURE 3. The diagram of the knot semigroup $K_{ST_{3,2}}$ and its presentation

In the following, we will give our attention to important terminologies, namely *alternating sum* and *alternating sum semigroups*, for the knot semigroups. The details and some properties on them can be found in [18].

Definition 1.2 ([18])**.** Let G be a group as the form of either $\mathbb{Z}_n$ or $\mathbb{Z}$, and let $B \subseteq G$. For any positive word $b_1b_2b_3 \cdots b_k \in B^+$, the alternating sum of this word is the value of the expression

$$b_1 - b_2 + b_3 \cdots (-1)^{k+1} b_k$$

that is calculated in G. Further, any such two words $u, v \in B^+$ are in relation $\sim$ if and only if the length of u is equal to the length of v and the alternating sum of u is equal to the alternating sum of v.

Moreover since the relation $\sim$ is a congruence on the set B^+, we then get a factor semigroup $B^+/\sim$. Let us denote it shortly by $AS(G, B)$ and call it an alternating sum semigroup.

Another version of the alternating sum semigroup has also been defined in [18] under the name of strong alternating sum semigroups which will not be needed in this paper.

Since one of our main aim is to obtain a homomorphism (or an isomorphism in some special cases) between knot semigroups and alternating sum semigroups, in the following we will give some fundamental facts about it.

Suppose that A^+/κ is a knot semigroup, where A is the set of arcs and κ is the cancellative congruence on the free semigroup A^+ induced by the defining relations of the knot semigroup. Also similarly as above, let $\sim$ be a congruence on B^+, where B is an alphabet of the same size as A. To obtain an isomorphism between A^+/κ and $B^+/\sim$, the following lemma is useful.

Lemma 1.3 ([18])**.** *Let us consider a bijection $\phi : A \to B$ that in fact induces an isomorphism $\phi : A^+ \to B^+$. Consider a congruence κ on A^+ and a congruence $\sim$ on B^+ such that for each $u, v \in A^+$, if $u\kappa v$ then $\phi(u) \sim \phi(v)$. Then ϕ induces not only a mapping but also a homomorphism $\psi : A^+/\kappa \to B^+/\sim$. Additionally let us suppose that there exists a subset, namely set of canonical words, of B^+ such that in each class of $\sim$ there is exactly one canonical word and at least one word of each class of κ is mapped by ϕ to a canonical word. Then ψ is actually an isomorphism.*

By considering Lemma 1.3, it has been proved the following theory in [18].

Proposition 1 ([18])**.** The knot semigroup $K_{ST_{p,2}}$ of the torus knot diagram $T_{p,2}$ (where p is odd) is isomorphic to the alternating sum semigroup $AS(\mathbb{Z}_p, \mathbb{Z}_p)$.

In this paper, it will be detailed this isomorphism defined in Proposition 1 up to decomposition of p. More clearly, we will say that the set B is changed depends on the value of p or the label corresponding an arc on the diagram of the torus. (See Theorem 2.12, Corollary 3 below).

2. Main Results

Under this section, we will present our main theorems to reach the aim of this paper.

2.1. **Connection Fox n-Coloring and Alternating Sum Semigroup.**

In this first result section, by comparing the Fox n-Coloring which is used for coloring of knots and the alternating sum semigroup, we will get the number of homomorphism from first to second, and also solve a conjecture given in [18, Conjecture 24]. In fact our approximation solve a more general case.

Theorem 2.1. *Let C be a set for using n-coloring of the knot K. Then there exists a homomorphism[1] from the knot semigroup K_S of K to the alternating sum semigroup $AS(G, C)$, where the set G is actually $\mathbb{Z}_n$ that is used for n-coloring. In fact the reverse part is also valid.*

Proof. By the meaning of Fox n-coloring, each arc in the knot was matched with an element of $\mathbb{Z}_n$ and the values obtained after each matching had to be satisfied Equation (1.2) which was written for each crossing of the knot. On the other hand, the relations of the knot semigroup K_S are the relations of the form $xy = yz$ and

[1] We should note that when we define such a homomorphism, we assume that the numerical value of each arc in the knot diagram and the values of these arcs in the semigroup $AS(\mathbb{Z}_n, C)$ are equal.

$zy = yx$ that were written for each crossing. It is easy to see that if we carry these relations to any alternating sum semigroup $AS(G, C)$ such that $C \subseteq G$, then they become the form of $x - y = y - z$ and $z - y = y - x$ since the subset C contains the relations that satisfy the equations $x - y = y - z$, $z - y = y - x$ in $AS(G, C)$.

Now let us rewrite the equation given in (1.2) as $c - b \equiv b - a \,(\mathrm{mod}\ n)$, and let us renamed the values a, b and c as z, y and x, respectively. Also take $G = \mathbb{Z}_n$. Then the elements used in Fox n-coloring and the elements of B become same. According to the above replacements and equations, since $x - y = y - z = -(z - y) = -(y - x)$, it will enough to obtain the values that satisfy the equation either $x - y = y - z$ or $z - y = y - x$. □

Example 2.2. For a Torus knot $T_{3,2}$, since the determinant $\Delta_{T_{3,2}}(-1) = 3$ by Equation *(1.1)*, the knot $T_{3,2}$ can be colored in terms of $G = \mathbb{Z}_3$. Further, since the number of colors is 9, it can be defined 9 different homomorphisms from $K_{ST_{3,2}}$ to $AS(\mathbb{Z}_3, C)$. Additionally the total number of the set C using the coloring of $T_{3,2}$ is 4 which are defined as

$$C = \{0\}, \quad C = \{1\}, \quad C = \{2\}, \quad C = \{0, 1, 2\}\,.$$

By considering these sets, the 9 homomorphisms defined from $K_{ST_{3,2}}$ to $AS(\mathbb{Z}_3, C)$ are as presented in Table *2.2*. We strictly note that 6 of among these 9 homomorphisms are actually isomorphisms. As a result of this, one can easily say that the homomorphisms defined from $K_{ST_{3,2}}$ to $AS(\mathbb{Z}_3, C)$ are not unique.

		Homomorphism								
		ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9
Elements	x	0	1	2	0	0	1	1	2	2
	y	0	1	2	1	2	0	2	0	1
	z	0	1	2	2	1	2	0	1	0

The first consequence of Theorem 2.1 is the following.

Corollary 1. *For the value t in the homomorph semigroup $AS(\mathbb{Z}_t, B)$ of K_S, we have either $t = \Delta_K(-1)$ or $t \mid \Delta_K(-1)$.*

Proof. According to Theorem 2.1, the knot K can be colored in terms of the subset B in the semigroup $AS(\mathbb{Z}_t, B)$. However it is well known that to a knot K be colored by modulo n, the value n must satisfy $n = \Delta_K(-1)$ or $n \mid \Delta_K(-1)$. Thus it is seen that $t = n$ or $t \mid n$ which implies that $t = \Delta_K(-1)$ or $t \mid \Delta_K(-1)$. □

Depends on the above corollary, if a knot K can be colored by modulo t then it can be colored by modulo kt as well. In fact the importance of this theory for us is the values of t which satisfies $t \leq \Delta_K(-1)$.

The following lemma is important for the characterization of a knot.

Proposition 2 ([3])**.** If a knot can be colored by modulo $n > 2$, then it cannot be deformed to an unknotted curve.

Now by considering Corollary 1 and Proposition 2 together, one can decide whether a knot can be deformed to an unknot via homomorphisms.

Corollary 2. *If there exists a homomorphism from the knot semigroup K_S of a knot K to any alternating sum semigroup $AS(G,B)$, then K cannot be deformed to an unknot. In here, K_S and $AS(G,B)$ are different than $K_{S(0_1)}$ and $\mathbb{N}$, respectively.*

Proof. Assume that such a homomorphism exists with the certain rule of not every element of K_S mapped to a single element in $AS(G,B)$. Under this rule, since the images of all values are same then the knot is colored by a unique color obviously. On the other hand, by Theorem 2.1, the subset B can be used for Fox n-coloring as well. Then, by Proposition 2, K cannot be unknot, as required.

Note that, since $K_{S(0_1)}$ is actually an unknot (circle), the homomorphism $K_{S(0_1)} \to \mathbb{N}$ obviously cannot imply a deformation as required. □

In [18, Conjecture 24], it has been recently stated that a knot diagram has the knot semigroup isomorphic to $\mathbb{N}$ if and only if it is a diagram of the trivial knot. In the following, by considering a splittable knot, we present a more effective situation.

Lemma 2.3 ([16])**.** *If a link is splittable then it can be colored by modulo $n \geq 2$.*

Therefore we have the following result which has a direct proof by Lemma 2.3 and Theorem 2.1.

Theorem 2.4. *Suppose K is a splittable knot. Then one can define a non-trivial homomorphism from the knot semigroup K_S to the alternating sum semigroup $AS(\mathbb{Z}_n, B)$.*

2.2. **Results on the links $P(u,m,1)$, $P(-u,-u,-u)$ and $T_{p,2}$.** In this section, by obtaining knot semigroups of some special Pretzel and Torus links, we will formulate how one can establish the elements of the alternating sum semigroups $AS(G,B)$ that are homomorph of the knot semigroups of these links. Moreover, depends on these formulas, we will give another formulate concerning about the number of homomorphisms from the knot semigroups of these links to the related semigroups $AS(G,B)$.

Unless stated otherwise throughout this section $n, m, p \in \mathbb{Z}^+$.

First of all, we should note that the diagram of the Pretzel link $P(u_1,u_2,u_3)$ can be drawn as in Figure 4 according to the famous book [12]. Thus, by considering the crossing as indicated in Section 1.2 over the diagram in Figure 4, we obtain the following lemma. In fact the proof of it will be omitted since it is basically based on the idea in Section 1.2.

Lemma 2.5. *The knot semigroup for the Pretzel link $P(u_1,u_2,u_3)$ is defined as $K_{SP(u_1,u_2,u_3)} = \langle A\ ;\ R\rangle$, where $A = \{a_0, a_1, a_2, \cdots, a_{u_1+u_2+u_3-1}\}$ and the relation set R is*

(2.1)

$$\left.\begin{array}{ll}
\textit{From regions } u_1: & a_{u_1+1}a_{u_1} = a_{u_1}a_{u_1-1} = \cdots = a_1a_0\,, \\
& a_0a_1 = a_1a_2 = \cdots a_{u_1-1}a_{u_1} = a_{u_1}a_{u_1+1}\,, \\
& \\
\textit{From regions } u_2: & a_{u_1}a_{u_1+2} = a_{u_1+2}a_{u_1+3} = \cdots = a_0a_{u_1+u_2+1}\,, \\
& a_{u_1+u_2+1}a_0 = a_0a_{u_1+u_2} = \cdots = a_{u_1+3}a_{u_1+2} = a_{u_1+2}a_{u_1}\,, \\
& \\
\textit{From regions } u_3: & a_{u_1+2}a_{u_1+1} = a_{u_1+1}a_{u_1+u_2+2} = \cdots = a_{u_1+u_2+1}a_1\,, \\
& a_1a_{u_1+u_2+1} = a_{u_1+u_2+1}a_{u_1+u_2+u_3-1} = \cdots = a_{u_1+1}a_{u_1+2}\,.
\end{array}\right\}$$

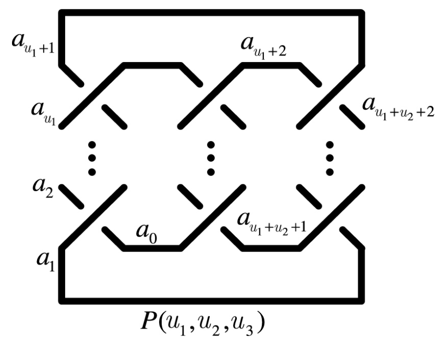


FIGURE 4. The diagram for the Pretzel link $P(u_1, u_2, u_3)$.

Although Lemma 2.5 will not be directly needed in our theories, it will be used as an adaption to the cases $P(u, m, 1)$ and $P(-u, -u, -u)$ in below. By replacing the link $P(u_1, u_2, u_3)$ to the link $P(u, m, 1)$, the first related result is obtained as in the following.

Theorem 2.6. *The knot semigroup $K_{SP(u,m,1)}$ of the Pretzel Link $P(u, m, 1)$ is homomorphic to the semigroup $AS(\mathbb{Z}_t, B)$, where*

(2.2)
$$B = \{\, x_0 + rk \; ; \; r = 0, 1, 2, \cdots , u + 1 \,\} \cup \{\, x_0 + [s(u + 1) - 1]\, k \; ; \; s = 2, 3, 4, \cdots , m \,\}$$

such that $x_0, k \in \mathbb{Z}_t$ are arbitrary elements and

$$\text{either } t = (m + 1)(u + 1) - 1 \text{ or } t \mid (m + 1)(u + 1) - 1\,. \tag{2.3}$$

Remark 2.7. The set B in *(2.2)* is the same set with C in Fox n-coloring (used in Theorem *2.1*), and the number t in *(2.3)* is giving the number n in the Fox n-colorings. These correspondents are also valid for Theorems *2.9* and *2.12*.

Proof. By Lemma 2.5, it is clear that the generating set is given as $A = \{a_0, a_1, a_2, \ldots , a_{u+m}\}$. On the other hand, by considering the diagram in Figure 5 and then replacing the equations in (2.1) to the case $P(u, m, 1)$, the relation set R can be obtained as in Eq. (2.4) below.

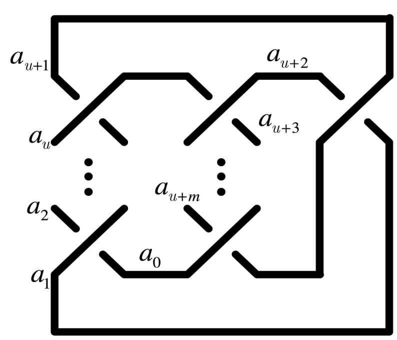


FIGURE 5. Diagram for the Pretzel link $P(n, m, 1)$.

$$
\left.
\begin{array}{lll}
 & \text{From the region } u: & a_{u+1}a_u = a_u a_{u-1} = \cdots = a_1 a_0\,, \\
 & & a_0 a_1 = a_1 a_2 = \cdots = a_{u-1}a_u = a_{u_1}a_{u+1}\,, \\
 & & \\
(2.4) & \text{From the region } m: & a_u a_{u+2} = a_{u+2}a_{u+3} = \cdots = a_{u+m}a_0 = a_0 a_{u+1}\,, \\
 & & a_{u+1}a_0 = a_0 a_{u+m} = \cdots = a_{u+2}a_u\,, \\
 & & \\
 & \text{From the region } 1: & a_{u+2}a_{u+1} = a_{u+1}a_1\,, \\
 & & a_{u+1}a_{u+2} = a_1 a_{u+1}\,.
\end{array}
\right\}
$$

Let us match each a_i with an element $x_i \in \mathbb{Z}_t$. Now, by the diagram in Figure 5, if we translate a general relation $a_i a_j = a_j a_k$ (where i, j and k are the elements of the set $\{a_0, a_1, \cdots, a_{u+1}\}$) to the alternating sum, then we clearly get

$$x_i - x_j = x_j - x_k\,.$$

Thus, if we apply same translation to the first row of "From the region u" in Eq. (2.4), then we get

$$x_{u+1} - x_u = x_u - x_{u-1} = \cdots = x_1 - x_0\,. \tag{2.5}$$

To simplify of the calculation, let us equalize the equation in (2.5) to an arbitrary value $k \in \mathbb{Z}_t$. After that, by assuming the initial value as $x_0 = x_0$, we have

$$x_1 = x_0 + k\,, \quad x_2 = x_0 + 2k\,, \quad \cdots\,, \quad x_u = x_0 + uk\,, \quad x_{u+1} = x_0 + (u+1)k\,. \tag{2.6}$$

Similarly as in (2.5), by applying the alternating sum to the first row of "From the region m" in Eq. (2.4) and by the last term

$$x_0 - x_{u+1} = -(u+1)k \tag{2.7}$$

of Eq. (2.6), we clearly have

$$x_u - x_{u+2} = x_{u+2} - x_{u+3} = \cdots = x_0 - x_{u+1} = -(u+1)k\,.$$

In the last equality, let us think each difference pairs separately as in Eq. (2.7). In that case, we obtain the following systematical equations.

$$
\begin{array}{rcl}
x_u - x_{u+2} = -(u+1)k & \Rightarrow & x_{u+2} = x_0 + 2\left[(u+1) - 1\right]k \\
 & & \text{by the equality in (2.7)} \\
x_{u+2} - x_{u+3} = -(u+1)k & \Rightarrow & x_0 + 2\left[(u+1) - 1\right]k - x_{u+3} = -(u+1)k \\
 & \Rightarrow & x_{u+3} = x_0 + 3\left[(u+1) - 1\right]k \\
 & & \text{by iteratively using of the equality in (2.7)} \\
 & & \\
\vdots & \vdots & \vdots \\
x_{u+m-1} - x_{u+m} = -(u+1)k & \Rightarrow & x_{u+m} = x_{u+m-1} + (u+1)k \Rightarrow \\
(2.8) \qquad & \Rightarrow & x_{u+m} = x_0 + m\left[(u+1) - 1\right]k \\
 & & \text{by iteratively using of the equality in (2.7)} \\
(2.9) \quad x_{u+m} - x_0 = -(u+1)k & \Rightarrow & x_{u+m} = x_0 - (u+1)k\,.
\end{array}
$$

Now, by equalizing the values of the term x_{u+m} in Eqs. (2.8) and (2.9), we obtain

$$(mu + m + u)k = 0 \quad \text{or equivalently} \quad (mu + m + u)k \equiv 0 \pmod{t} \tag{2.10}$$

In here, the congruence $(mu+m+u)k \equiv 0 \pmod{t}$ gives the correctness of equations in (2.3), as required.

At this point we should note that one can also take $t = k$ or $t \mid k$ to be held the congruency in (2.10). So this will also give that since all x_i's are equal to each other, the equations for alternating sum semigroup still hold. However, since such these solutions will imply infinite number of homomorphisms, we only consider the cases $t = \Delta_K(-1)$ or $t \mid \Delta_K(-1)$.

To end up the proof, let us express how can one define a homomorphism as required in theorem. For the semigroups $K_{SP(u,m,1)} = A^+/\kappa$ and $AS(\mathbb{Z}_t, B) = B^+/\sim$, where $A = \{a_0, a_1, a_2, \ldots, a_{u+m}\}$ (which is the set of arcs), B is as in the expression of theorem, κ is the set of relations as given in (2.4) and $\sim$ is the set of relations correspond to the relations in (2.4) which we have already obtained in above. Now, since for each a_i $(0 \leq i \leq u+m)$ we obtain a different corresponds value x_i up to choosing of x_0, k and t, this will imply that we have a finite number of different functions $\phi_j : A \to B$ with the rule $a_i \to x_i$. Thus, by Lemma 1.3, there must exists a unique homomorphism from A^+/κ to $B^+/\sim$ for each of these different functions. In fact the number of such these different homomorphisms is defined in Theorem 2.16 below.

Hence the result. □

Example 2.8. For $P(u, m, 1)$, if one choose $x_0 = 0$, $k = 1$ and $t = (m+1)(u+1)-1$, then the number of elements in sets A and B become equal. Therefore we have a one-to-one matching between each a_i and x_i which implies that we obtain not only a homomorphism from $K_{SP(u,m,1)}$ to $AS(\mathbb{Z}_t, B)$ but also an isomorphism. In here, the set B is defined as

$$\{0,\ 1,\ 2,\ \cdots,\ u,\ u+1,\ 2(u+1)-1,\ 3(u+1)-1,\ \cdots,\ m(u+1)-1\}\ .$$

By applying a quite similar progress as in the case of $P(u, m, 1)$, we can obtain similar results for the Pretzel link $P(-u, -u, -u)$ and the Torus knot $T_{p,2}$. In the following, by omitting the proofs but considering Lemma 1.3, we will indicate the existence of homomorphisms from the knot semigroup $K_{SP(-u,-u,-u)}$ to $AS(\mathbb{Z}_t, B)$ as in the coming result which is another version of Theorem 2.6. We first note that, by [12], the diagram of the Pretzel link $P(-u, -u, -u)$ is drawn as in Figure 6, and so as a consequence of Lemma 2.5 one can easily obtain the generating set $A = \{a_0, a_1, a_2, \ldots, a_{3u-1}\}$ while the set of relations R as defined in Eq. (2.11) below.

(2.11)

$$\left.\begin{array}{rl}
\text{From the first region } -u: & a_{u+1}a_u = a_u a_{u-1} = \cdots = a_1 a_0\ , \\
& a_0 a_1 = a_1 a_2 = \cdots = a_{u-1} a_u = a_u a_{u+1}\ , \\[1ex]
\text{From the second region } -u: & a_{u+2} a_{u+1} = a_{u+1} a_{u+3} = \cdots = a_{2u} a_{2u+1} = a_{2u+1} a_2\ , \\
& a_2 a_{2u+1} = a_{2u+1} a_{2u} = \cdots = a_{u+3} a_{u+1} = a_{u+1} a_{u+2}\ , \\[1ex]
\text{From the third region } -u: & a_u a_{u+2} = a_{u+2} a_{2u+2} = \cdots = a_{3u-1} a_0 = a_0 a_{2u+1}\ , \\
& a_{2u+1} a_0 = a_0 a_{3u-1} = \cdots = a_{2u+2} a_{u+2} = a_{u+2} a_u\ .
\end{array}\right\}$$

Thus the other main result of this paper is the following.

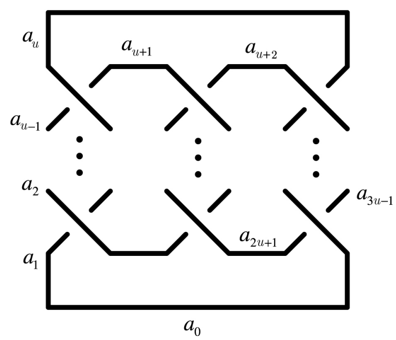


FIGURE 6. Diagram for the Pretzel link $P(-u,-u,-u)$.

Theorem 2.9. *We always have a finite number of homomorphisms from $K_{SP(-u,-u,-u)}$ to the alternating sum semigroup $AS(\mathbb{Z}_t, B)$, where*

$$B = \{\, x_0 + rk \ ; \ r = 0, 1, \cdots, u+1 \,\} \cup \{\, x_0 + (u+2s)k \ ; \ s = 1, 2, \cdots, u-1 \,\}$$

such that x_0, k are arbitrary elements of $\mathbb{Z}_t$ and either $t = 3u$ or $t \mid 3u$.

On the other hand the existence of isomorphism is defined as follows.

Theorem 2.10. *For only $u = 1$, there exists $K_{SP(-u,-u,-u)} \cong AS(\mathbb{Z}_t, B)$.*

Proof. If $u = 1$, then the diagram and knots of the $P(-1,-1,-1)$ are the same with the diagram and knots of Torus knot $T_{3,2}$. So, by Proposition 1, we have $K_{ST_{3,2}} \cong AS(\mathbb{Z}_3, \mathbb{Z}_3)$. On the other hand, if $u \neq 1$, then the number of arcs in the diagram of $P(-u,-u,-u)$ is $3u$ (which gives the cardinality of the generating set A) and so the number of elements in the set B is $(u+2)+(u-1) = 2u+1$. However, for all $u > 1$, since it is always true that $3u > 2u+1$, we obtain the number of arcs in $P(-u,-u,-u)$ is greater than the number of elements of B which implies that it cannot be defined an isomorphism. □

It is known that tricolorability (i.e. Fox n-coloring when $u = 3$) is an invariant under Reidemeister moves (cf. [1]). Since invariant property is an important tool in every branch of mathematics, it is good enough to study tricolorability for our cases. In fact, by the condition $t = 3u$ or $t \mid 3u$ in Theorem 2.9, it is not hard to see that t can be choosed as 3. That means there exists a homomorphism from the semigroup $K_{SP(-u,-u,-u)}$ to $AS(\mathbb{Z}_3, B)$. Therefore we have the following result.

Theorem 2.11. *All Pretzel links $P(-u,-u,-u)$ are tricolorability.*

Now let us give our attention to the Torus knot. In the remaining part of this section, we will adapt the theories on $P(u,m,1)$ and $P(-u,-u,-u)$ to the Torus knot $T_{p,2}$. Recall that the case $p = 3$ in Torus knot gives $P(-1,-1,-1)$ and so there is nothing to do since we have already obtained previously. Therefore in the following result the case $p = 3$ coincides with Theorems 2.9, 2.10 and 2.11.

Theorem 2.12. *Consider the Torus knot $T_{p,2}$ (where p and 2 are relatively prime) as defined in (1.3). Then we have a homomorphism from the Torus knot semigroup $K_{ST_{p,2}}$ to the alternating sum semigroup $AS(\mathbb{Z}_t, B)$, where*

$$B = \{\, x_0 + rk \ ; \ r = 0, 1, 2, \cdots, p-1 \,\} \ ,$$

x_0, $k \in \mathbb{Z}_t$ and either $t \mid \Delta_{T_{p,2}}(-1)$ or $t = \Delta_{T_{p,2}}(-1)$ such that $\Delta_{T_{p,2}}(-1) = p$ by (1.1).

Proof. In the proof, we will actually follow a similar way as in the proof of Theorem 2.6. Now if we translate the relations defined in (1.3) to the relations of $AS(\mathbb{Z}_t, B)$, then we have

$$x_0 - x_1 = x_1 - x_2, x_1 - x_2 = x_2 - x_3, \cdots, x_{p-2} - x_{p-1} = x_{p-1} - x_0\,. \tag{2.12}$$

By rearranging and then equalizing a constant k, we also get

$$x_0 - x_1 = x_1 - x_2 = x_2 - x_3 = \cdots = x_{p-2} - x_{p-1} = x_{p-1} - x_0 = k\,,$$

which can be clearly written as

$$x_1 = x_0 + k, x_2 = x_0 + 2k, \cdots, x_{p-1} = x_0 + (p-1)k\,,$$

In (2.12), as the general term, let us take $x_{p-2} - x_{p-1} = x_{p-1} - x_0 = k$ and then replace the x_i values all the related places. So

$$x_0 + (p-2)k - (x_0 + (p-1)k) = x_0 + (p-1)k - x_0 = -k = (p-1)k$$
$$\Longrightarrow pk \equiv 0 \pmod{t}\,.$$

Therefore, by this last congruence, we must have $t \mid p$ or $t = p$, where $p = \Delta_{T_{p,2}}(-1)$.

The set of arcs (or equivalently the generating set) is defined as $A = \{a_0, a_1, a_2, \cdots, a_{p-1}\}$ while the set of x_i values is given by $B = \{x_0 + rk\ ;\ r = 0, 1, 2, ..., p-1\}$. Hence, by applying Lemma 1.3, we reached that there exists a homomorphism from $K_{ST_{p,2}}$ to $AS(\mathbb{Z}_t, B)$, as required. □

Example 2.13. In Theorem *2.12*, if we choose $x_0 = 0$, $k = 1$ and $t = \Delta_{T_{p,2}}(-1) = p$, then the set B is given by $\{0, 1, 2, \cdots p-2, p-1\}$. Therefore the number of arcs in Torus knot $T_{p,2}$ and the cardinality of B are both p, and so there is a one-to-one correspondence between each arc in A and each element in B. So, by Lemma *1.3*, we obtain an isomorphism $K_{ST_{p,2}} \cong AS(\mathbb{Z}_p, B)$.

Remark 2.14. We strictly note that a similar situation in Example *2.13* (which is an example of Theorem *2.12*) was given as a result in the paper [18, Theorem 3] by considered with only a unique isomorphism. Nevertheless, Example *2.13* actually shows that different choices for arbitrary x_0, k and p will imply different isomorphisms between $K_{ST_{p,2}}$ and $AS(\mathbb{Z}_p, B)$.

The situation depicted in Example 2.13 and Remark 2.14 can be summarized with the following theorem.

Theorem 2.15. *To define an isomorphisms between $K_{ST_{p,2}}$ and $AS(\mathbb{Z}_t, B)$, it must be held $k \neq 0$, $k \nmid p$ and $t = p$.*

Proof. Without loss of the generality, let us investigate the cases as $k = 0$, $k \mid p$ and $t \neq p$, respectively.

- Let $k = 0$. If we write 0 instead of k in the set B in Theorem 2.12, then we have $B = \{x_0\}$. But, in this case, whole elements of $T_{p,2}$ map to a single element in the homomorphism from $K_{ST_{p,2}}$ to $AS(\mathbb{Z}_t, B)$ which clearly breaks down the isomorphism.
- Assume $k \mid p$. Let us reconsider the set B in Theorem 2.12. By the assumption, for any $r_i \neq 0$ $(0 \leq i \leq p-1)$, we get $r_i k \equiv 0 \pmod{p}$. But, since this will imply that $x_0 + 0k = x_0 + r_i k$ (as the meaning of congruence classes), we cannot reach the isomorphism.

- Suppose $t < p$ and $t \mid p$. Remember that the set B in Theorem 2.12 was obtained by considered the equivalence over modulo $t = p$. However, when we take it as $t < p$ and $t \mid p$, clearly the cardinality t of B will be definitely less than p. On the other hand, the number of arcs in the knot diagram (or equivalently, the number of generators in the knot semigroup) is still p. This means that we cannot define an isomorphism between $K_{ST_{p,2}}$ and $AS(\mathbb{Z}_t, B)$ since the cardinality of B is less than p.

As a result of these above facts, we say that to define an isomorphism from $K_{ST_{p,2}}$ to $AS(\mathbb{Z}_t, B)$ (or vice versa), all conditions in theorem must be satisfied. $\square$

Finally, we can bring together Theorems 2.6, 2.9 and 2.12 in a common point as in the following.

Theorem 2.16. *For simplicity, let $\mathcal{N}$ denotes one of $P(u, m, 1)$, $P(-u, -u, -u)$ or $T_{p,2}$. Then the number of homomorphisms from each of the knot semigroups $K_{S\mathcal{N}}$ to the alternating sum semigroup $AS(\mathbb{Z}_t, B)$ is*

$$\sum_{i=1}^{\chi-1} t_i^2$$

such that $t_i \mid \Delta_{\mathcal{N}}(-1)$ and χ is the number of t_i's that divides t.

Proof. In Theorems 2.6, 2.9 and 2.12, we established that if one wants to define a homomorphism from one of the knot semigroups of $P(u, m, 1)$, $P(-u, -u, -u)$ ve $T_{p,2}$ to the alternating sum semigroup $AS(\mathbb{Z}_t, B)$, then the value t must be satisfied $t \mid \Delta_{\mathcal{N}}(-1)$ or $t = \Delta_{\mathcal{N}}(-1)$, and additionally, for each of these theorems, we presented the related B set while $t = \Delta_{\mathcal{N}}(-1)$. Remember that the elements $x_0, k \in \mathbb{Z}_t$ were chosen arbitrarily in these B sets. It easy to verify that each of x_0 and t can be chosen t different ways from $\mathbb{Z}_t$ which imply that the values of x_0 and t can be totally chosen as t^2 different options. On the other hand, since we obtain different B sets up to for each different choices of x_0 and k, we get t^2 different homomorphisms that can be defined on these B sets.

For $t = \Delta_{\mathcal{N}}(-1)$, now let us consider the $t_i \mid t$ values and say χ to the number of such t_i's. In here we must consider 1 does not count in χ since Fox n-colorings start always from $n \geq 2$ (by Lemma 2.3 or more generally Equation (1.2)) and so $t_i \neq 1$. Let B_i denotes a congruence class of the elements in B depends on the value t_i. According to Theorems 2.6, 2.9 and 2.12, one can define a homomorphism from the knot semigroup to the semigroup $AS(\mathbb{Z}_{t_i}, B_i)$ in which x_0, $k \in \mathbb{Z}_{t_i}$. With the same idea as in the above paragraph, ${t_i}^2$ different choices can be applied to x_0 and k in $\mathbb{Z}_{t_i}$, and since each of those gives a new homomorphism, we get total ${t_i}^2$ different homomorphisms for each t_i from the knot semigroup to the semigroup $AS(\mathbb{Z}_{t_i}, B)$. Hence, since this situation can be seen for all $t_i \mid t$, we say that the total number of homomorphisms is $\sum_{i=1}^{\chi-1} t_i^2$, as required. $\square$

Remember that the number of colorings of a knot K in terms of the quandle Q was denoted by $Col_Q(K)$. By considering Lemma 1.1, we can give the following result as a consequence of Theorems 2.1 and 2.16.

Theorem 2.17.

$$Col_Q(\mathcal{N}) = \sum_{i=1}^{\chi-1} t_i^2 \,.$$

One may also present the following particular corollary as a consequence of Theorems 2.1, 2.12 and 2.16.

Corollary 3. *For a prime p, there are total p^2 homomorphisms and $p^2 - p$ isomorphisms from $K_{ST_{p,2}}$ to $AS(\mathbb{Z}_p, B)$.*

3. CONCLUSION

In this study, the homomorphism relations between the nodal semigroups and the alternative total semigroups of some pretzel chains and torus chains are investigated and the number of homomorphisms and isomorphisms in some special cases are given.

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GENERALIZED TOPOLOGICAL OPERATOR ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-OPERATOR) THEORY IN GENERALIZED TOPOLOGICAL SPACES ($\mathscr{T}_{\mathfrak{g}}$-SPACES) PART IV. GENERALIZED DERIVED ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-DERIVED) AND GENERALIZED CODERIVED ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-CODERIVED) OPERATORS

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Abstract. In a recent paper (Cf. [1]), we have introduced the definitions and studied the essential properties of the generalized topological operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived operators*) in a generalized topological space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ ($\mathscr{T}_{\mathfrak{g}}$-*space*). Mainly, we have shown that $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is a pair of both *dual and monotone* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* that is $(\varnothing, \Omega)$, $(\cup, \cap)$-*preserving*, and $(\subseteq, \supseteq)$-*preserving* relative to $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(open, closed) sets. We have also shown that $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is a pair of *weaker* and *stronger* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators*. In this paper, we define by transfinite recursion on the class of successor ordinals the δ^{th}-iterates $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-*coderived operators*) of $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and study their basic properties in a $\mathscr{T}_{\mathfrak{g}}$-space. Moreover, we establish the necessary and sufficient conditions for $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\right) : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ to be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathfrak{T}_{\mathfrak{g}}$. Finally, we diagram various relationships amongst $\text{der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}$, $\text{cod}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and present a nice application to support the overall study.

1. Introduction

Axiomatically, a generalized derived operator ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived operator*) in an ordinary ($\mathfrak{a} = \mathfrak{o}$) or generalized ($\mathfrak{a} = \mathfrak{g}$) topological space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ ($\mathscr{T}_{\mathfrak{a}}$-*space*) is a set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{a}} : \begin{array}{ll} \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ satisfying the following $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived operator axioms*:

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– $\mathrm{Ax}_{\mathrm{DE},1}\left(\mathfrak{g}\text{-Der}_{\mathfrak{a}}\right) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{a}}\left(\varnothing\right) = \varnothing$

– $\mathrm{Ax}_{\mathrm{DE},2}\left(\mathfrak{g}\text{-Der}_{\mathfrak{a}}\right) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-Der}_{\mathfrak{a}}\big(\mathscr{R}_{\mathfrak{a}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{a}}\left(\{\xi\}\right)\big)$

– $\mathrm{Ax}_{\mathrm{DE},3}\left(\mathfrak{g}\text{-Der}_{\mathfrak{a}}\right) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{a}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{a}}\right) \subseteq \mathscr{R}_{\mathfrak{a}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{a}}\right)$

– $\mathrm{Ax}_{\mathrm{DE},4}\left(\mathfrak{g}\text{-Der}_{\mathfrak{a}}\right) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{a}} \cup \mathscr{S}_{\mathfrak{a}}\right) = \bigcup_{\mathscr{U}_{\mathfrak{a}}=\mathscr{R}_{\mathfrak{a}},\mathscr{S}_{\mathfrak{a}}} \mathfrak{g}\text{-Der}_{\mathfrak{a}}\left(\mathscr{U}_{\mathfrak{a}}\right)$

for any $\left(\{\xi\}, \mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{a}}\left(\mathscr{R}_{\mathfrak{a}}\right)$ [1, 2, 3, 4, 5, 6, 7, 8, 9]. A generalized coderived operator ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived operator*) in the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$ is a set-valued map $\begin{array}{rll}\mathfrak{g}\text{-Cod}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ satisfying the following $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived operator axioms*:

– $\mathrm{Ax}_{\mathrm{CD},1}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{a}}\right) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\left(\Omega\right) = \Omega$

– $\mathrm{Ax}_{\mathrm{CD},2}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{a}}\right) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\big(\mathscr{U}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\big(\mathscr{U}_{\mathfrak{a}} \cup \{\zeta\}\big)$

– $\mathrm{Ax}_{\mathrm{CD},3}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{a}}\right) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{a}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\left(\mathscr{U}_{\mathfrak{a}}\right) \supseteq \mathscr{U}_{\mathfrak{a}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\left(\mathscr{U}_{\mathfrak{a}}\right)$

– $\mathrm{Ax}_{\mathrm{CD},4}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{a}}\right) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\left(\mathscr{U}_{\mathfrak{a}} \cap \mathscr{V}_{\mathfrak{a}}\right) = \bigcap_{\mathscr{W}_{\mathfrak{a}}=\mathscr{U}_{\mathfrak{a}},\mathscr{V}_{\mathfrak{a}}} \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\left(\mathscr{W}_{\mathfrak{a}}\right)$

for any $\left(\{\zeta\}, \mathscr{U}_{\mathfrak{a}}, \mathscr{V}_{\mathfrak{a}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ [1, 2, 3, 4, 5, 6, 7, 8, 9]. Alternative axiomatic descriptions for $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-coderived operators in $\mathscr{T}_{\mathfrak{o}}$-spaces can be found in the paper of Lei and Zhang [10].

If $\left(\mathscr{S}_{\mathfrak{a}}, \mathfrak{g}\text{-Ope}_{\mathfrak{a}}\right) \in \mathscr{P}(\Omega) \times \{\mathfrak{g}\text{-Der}_{\mathfrak{a}}, \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\}$ be arbitrarily given, then β factors $\begin{array}{rll}\mathfrak{g}\text{-Ope}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathfrak{g}\text{-Ope}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ yields:

$$\mathbb{Z}_+^0 \ni \beta \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) = \mathfrak{g}\text{-Ope}_{\mathfrak{a}} \circ \cdots \circ \mathfrak{g}\text{-Ope}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right) \overset{\text{def}}{=} \bigcirc_{\alpha \in I_\beta^0} \mathfrak{g}\text{-Ope}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)$$

Thus, $\begin{array}{rll}\left(\mathfrak{g}\text{-Der}_{\mathfrak{a}}^{(\beta)}, \mathfrak{g}\text{-Cod}_{\mathfrak{a}}^{(\beta)}\right): & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \left(\mathfrak{g}\text{-Der}_{\mathfrak{a}}^{(\beta)}, \mathfrak{g}\text{-Cod}_{\mathfrak{a}}^{(\beta)}\right)\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ is the β^{th} *order* of $\begin{array}{rll}\left(\mathfrak{g}\text{-Der}_{\mathfrak{a}}, \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\right): & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \left(\mathfrak{g}\text{-Der}_{\mathfrak{a}}, \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\right)\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ and, for any pair $\left(\mathscr{S}_{\mathfrak{a}}, \mathfrak{g}\text{-Ope}_{\mathfrak{a}}\right) \in \mathscr{P}(\Omega) \times \{\mathfrak{g}\text{-Der}_{\mathfrak{a}}, \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\}$, it holds that:

$$\left[\left(\exists \beta \in \mathbb{Z}_+^0\right)\left(\mathfrak{g}\text{-Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) = \varnothing\right)\right] \veebar \left[\left(\forall \beta \in \mathbb{Z}_+^0\right)\left(\mathfrak{g}\text{-Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) \neq \varnothing\right)\right]$$

If $\mathfrak{g}\text{-Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) = \varnothing$ for some $\beta \in \mathbb{Z}_+^0$, then β is a type of *density measure* of $\mathscr{S}_{\mathfrak{a}}$ to achieve *emptiness* (if this is ever achieved). But if $\mathscr{S}_{\mathfrak{a}}^{(\lambda)} \overset{\text{def}}{=} \bigcap_{\beta \in \mathbb{Z}_+^*} \mathfrak{g}\text{-Ope}_{\mathfrak{a}}^{(\beta)}\left(\mathscr{S}_{\mathfrak{a}}\right) \neq \varnothing$, then λ is a type of *limit order* of $\mathscr{S}_{\mathfrak{a}}$, in which case the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operators $\mathfrak{g}\text{-Ope}_{\mathfrak{a}}^{(1)}$, $\mathfrak{g}\text{-Ope}_{\mathfrak{a}}^{(2)}$, $\ldots : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ can again be applied on $\mathscr{S}_{\mathfrak{a}}^{(\omega)} \in \mathscr{P}(\Omega)$, yielding $\mathfrak{g}\text{-Ope}_{\mathfrak{a}}^{(\lambda+1)}\left(\mathscr{S}_{\mathfrak{a}}\right)$, $\mathfrak{g}\text{-Ope}_{\mathfrak{a}}^{(\lambda+2)}\left(\mathscr{S}_{\mathfrak{a}}\right)$, ... Viewing $\delta = 0, 1, 2, \ldots$ as *successor ordinals* while $\delta = \lambda$ as *limit ordinal*, the foregoing descriptions surprisingly introduce by transfinite recursion on the class of successor ordinals the definitions of the δ^{th}-iterates $\begin{array}{rll}\mathfrak{g}\text{-Der}_{\mathfrak{a}}^{(\delta)}, \ \mathfrak{g}\text{-Cod}_{\mathfrak{a}}^{(\delta)}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{a}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{a}}\right), \ \mathfrak{g}\text{-Cod}_{\mathfrak{a}}^{(\delta)}\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$

($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$*-derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$*-coderived operators*) of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators $\begin{array}{rl} \mathfrak{g}\text{-Der}_{\mathfrak{a}},\ \mathfrak{g}\text{-Cod}_{\mathfrak{a}}: & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\ \mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}), \end{array}$ respectively, in a $\mathscr{T}_{\mathfrak{a}}$-space.

In a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, by virtue of $\mathrm{Ax}_{\mathrm{DE},1}(\mathfrak{g}\text{-Cod}_{\mathfrak{g}})$, …, $\mathrm{Ax}_{\mathrm{DE},4}(\mathfrak{g}\text{-Cod}_{\mathfrak{g}})$ and $\mathrm{Ax}_{\mathrm{CD},1}(\mathfrak{g}\text{-Cod}_{\mathfrak{g}})$, …, $\mathrm{Ax}_{\mathrm{CD},4}(\mathfrak{g}\text{-Cod}_{\mathfrak{g}})$, generalized characterizations of $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ can be realized by specifying either the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator $\begin{array}{rl} \mathfrak{g}\text{-Der}_{\mathfrak{g}}: & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{array}$ or the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator $\begin{array}{rl} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}: & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \end{array}$ respectively [1]. Moreover, if the δ^{th}-iterates $\begin{array}{rl} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)},\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}: & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}),\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \end{array}$ are also themselves $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, then similar roles can be played, thereby realizing other generalized characterizations of $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathfrak{T}_{\mathfrak{g}}$.

Although the literature of $\mathscr{T}_{\mathfrak{a}}$-spaces contains a wealth of information on the study of different types of $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in $\mathscr{T}_{\mathfrak{g}}$-spaces [2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], including the study of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}^{(\delta)}$-coderived operators in $\mathscr{T}_{\mathfrak{o}}$-spaces [23, 24, 25, 26, 27], it does, unfortunately, not contain a study of any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

In investigating the convergence of Fourier series, Cantor [23, 24] has introduced and considered $\begin{array}{rl} \mathrm{der}_{\mathfrak{o}|\mathbb{R}}: & \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R}) \\ & \mathscr{S}_{\mathfrak{o}} \longmapsto \mathrm{der}_{\mathfrak{o}|\mathbb{R}}(\mathscr{S}_{\mathfrak{o}}) \end{array}$ in $\mathbb{R}$. He has also considered its iteration, thereby introducing the notion of ordinal and then the definition of $\begin{array}{rl} \mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}: & \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R}) \\ & \mathscr{S}_{\mathfrak{o}} \longmapsto \mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}}) \end{array}$ in $\mathbb{R}$ for some ordinal δ. Later on, Rutt [25] has introduced a weaker form of $\begin{array}{rl} \mathrm{der}_{\mathfrak{o}}: & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{o}} \longmapsto \mathrm{der}_{\mathfrak{o}}(\mathscr{S}_{\mathfrak{o}}) \end{array}$ and investigated some of its properties as well as the properties of its δ^{th}-order iterate $\begin{array}{rl} \mathrm{der}_{\mathfrak{o}}^{(\delta)}: & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{o}} \longmapsto \mathrm{der}_{\mathfrak{o}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}}) \end{array}$ from a sequential point of view. Adopting a point of view similar to Rutt [25], Tucker [26] has presented a theorem concerning the period of periodic sequences of $\mathfrak{T}_{\mathfrak{o}}$-derived sets with respect to the $\mathfrak{T}_{\mathfrak{o}}^{(\delta)}$-derived operator $\begin{array}{rl} \mathrm{der}_{\mathfrak{o}}^{(\delta)}: & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{o}} \longmapsto \mathrm{der}_{\mathfrak{o}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}}) \end{array}$ and has studied other properties in a $\mathscr{T}_{\mathfrak{o}}$-space. Noticing that, for a large class of real $\mathscr{T}_{\mathfrak{o}|\mathbb{R}}$-spaces of the type $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}} = (\mathbb{R}, \mathscr{T}_{\mathfrak{o}|\mathbb{R}})$, the $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}$-derived operator $\begin{array}{rl} \mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}: & \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R}) \\ & \mathscr{S}_{\mathfrak{o}} \longmapsto \mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}}) \end{array}$ itself realizes an ordinary characterization of $\mathscr{T}_{\mathfrak{o}|\mathbb{R}} : \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R})$ in the $\mathscr{T}_{\mathfrak{o}|\mathbb{R}}$-space $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}$, Higgs [27] has given characterizations of $\mathscr{T}_{\mathfrak{o}|\mathbb{R}}$-spaces for which the δ^{th}-iterate $\begin{array}{rl} \mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}: & \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R}) \\ & \mathscr{S}_{\mathfrak{g}} \longmapsto \mathrm{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}}) \end{array}$ is a $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}$-derived operator. He has also considered the unfortunate extent to which δ^{th}-iteration fails to relate well to several $\mathscr{T}_{\mathfrak{o}|\mathbb{R}}$-concepts and defined the limit δ^{th}-iterate of the $\mathfrak{T}^{(\delta)}_{\mathfrak{o}|\mathbb{R}}$-derived operator in $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}$.

Having introduced the definitions and then investigated the properties of a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces [1], it may be another good research investigation to introduce the definitions and then investigate the properties of the δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces. Such inquiry is what we endeavor to undertake in the present paper.

Hereafter, the paper is structured as thus: In § 2, the preliminary and main concepts are described in §§ 2.1 and §§ 2.2, respectively. The main results are reported in § 3. In § 4, the various relationships amongst the $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators in a $\mathscr{T}_{\mathfrak{a}}$-space are diagrammed in §§ 4.1, and a nice application supporting the overall study is presented in §§ 4.2. Finally, the work is concluded in § 5.

2. Theory

2.1. **Preliminary Concepts.** The standard reference for $\mathscr{T}_{\mathfrak{a}}$-space notations and notions is the Ph.D. Thesis of Khodabocus, M. I. [9], whereas that for $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators notations and preliminary concepts in $\mathscr{T}_{\mathfrak{a}}$-spaces is our recent paper on the subject matter [1] (Cf. [2, 3, 4, 5, 6, 7, 8]).

The notation $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ designates a topological structure called *$\mathscr{T}_{\mathfrak{a}}$-space* on which no separation axioms are assumed unless otherwise mentioned [7, 8, 9]. The relation $(\alpha_1, \alpha_2, \ldots) \,\mathrm{R}\, \mathscr{A}_1 \times \mathscr{A}_2 \times \cdots$ is made a rule to mean $\alpha_1 \,\mathrm{R}\, \mathscr{A}_1$, $\alpha_2 \,\mathrm{R}\, \mathscr{A}_2$, … where $\mathrm{R} = \in, \subset, \supset, \ldots$. Accordingly, $\left(I^0_n, I^*_n\right) = \left([\![0,n]\!], [\![1,n]\!]\right) \subset \mathbb{Z}^0_+ \times \mathbb{Z}^*_+$ and $\left(I^0_\infty, I^*_\infty\right) = \left([\![0,\infty]\!], [\![1,\infty]\!]\right) \sim \mathbb{Z}^0_+ \times \mathbb{Z}^*_+$ are pairs of *finite* and *infinite index sets*, respectively, [8, 9]. For any $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the relations $\Gamma \subset \Omega$, $\mathscr{O}_{\mathfrak{a}} \in \mathscr{T}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}} \in \neg\mathscr{T}_{\mathfrak{a}} \stackrel{\text{def}}{=} \left\{\mathscr{K}_{\mathfrak{a}} : \complement_{\Omega}(\mathscr{K}_{\mathfrak{a}}) \in \mathscr{T}_{\mathfrak{a}}\right\}$ and $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ state that Γ, $\mathscr{O}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}}$ and $\mathscr{S}_{\mathfrak{a}}$ are a *Ω-subset*, *$\mathscr{T}_{\mathfrak{a}}$-open set*, *$\mathscr{T}_{\mathfrak{a}}$-closed set* and *$\mathfrak{T}_{\mathfrak{a}}$-set*, respectively [8, 9]. The $\mathfrak{T}_{\mathfrak{a}}$-operators $\begin{array}{rl} \mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} : & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} \longmapsto \mathrm{int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\ \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ are the *$\mathfrak{T}_{\mathfrak{a}}$-interior* and *$\mathfrak{T}_{\mathfrak{a}}$-closure operators*, respectively [8, 9]. Let the class of all possible pairs of compositions of these $\mathfrak{T}_{\mathfrak{a}}$-operators in $\mathfrak{T}_{\mathfrak{a}}$ be $\mathscr{L}_{\mathfrak{a}}[\Omega] \stackrel{\text{def}}{=} \left\{\mathfrak{op}_{\mathfrak{a},\nu} = \left(\mathrm{op}_{\mathfrak{a},\nu}, \neg\,\mathrm{op}_{\mathfrak{a},\nu}\right) : \nu \in I^0_3\right\}$, where

$$\begin{aligned} \left\langle \mathrm{op}_{\mathfrak{a},\nu} : \nu \in I^0_3 \right\rangle &= \left\langle \mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}} \right\rangle \\ \left\langle \neg\,\mathrm{op}_{\mathfrak{a},\nu} : \nu \in I^0_3 \right\rangle &= \left\langle \mathrm{cl}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}} \right\rangle \end{aligned}$$

Then, $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-set if and only if it holds that

$$(\exists \xi)\left[\left(\xi \in \mathscr{S}_{\mathfrak{a}}\right) \wedge \left(\left(\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}})\right) \vee \left(\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a}})\right)\right)\right] \tag{2.1}$$

for some $(\mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}, \mathrm{op}_{\mathfrak{a}}) \in \mathscr{T}_{\mathfrak{a}} \times \neg\mathscr{T}_{\mathfrak{a}} \times \mathscr{L}_{\mathfrak{a}}[\Omega]$. In this way, the derived class $\mathfrak{g}$-ν-$\mathrm{S}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{\mathrm{E} \in \{\mathrm{O},\mathrm{K}\}} \mathfrak{g}$-$\nu$-$\mathrm{E}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ collects all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets of category $\nu \in I^0_3$ ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$*-sets*), whereas

$$\mathfrak{g}\text{-}\mathrm{S}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{\nu \in I^0_3} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{(\nu,\mathrm{E}) \in I^0_3 \times \{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{\mathrm{E} \in \{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{a}}\right] \tag{2.2}$$

collects all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets irrespective of their categories in $\mathfrak{T}_{\mathfrak{a}}$ [8, 9]. In particular, $\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{(\nu,\mathrm{E})\in\{0\}\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$ collects all $\mathfrak{T}_{\mathfrak{a}}$-sets in $\mathfrak{T}_{\mathfrak{a}}$ [8, 9].

Definition 2.1 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Interior,* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Closure Operators* [2, 3])**.** In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the one-valued maps

$$(2.3)\qquad \begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{a}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{O}_{\mathfrak{a}} \end{aligned}$$

$$(2.4)\qquad \begin{aligned} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{a}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{K}_{\mathfrak{a}} \end{aligned}$$

where $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] : \ \mathscr{O}_{\mathfrak{a}} \subseteq \mathscr{S}_{\mathfrak{a}}\}$ and $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] : \ \mathscr{K}_{\mathfrak{a}} \supseteq \mathscr{S}_{\mathfrak{a}}\}$ are called $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively. Then, $\mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} : \ \nu \in I_3^0\}$ and $\mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \nu \in I_3^0\}$ are the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively.

Definition 2.2 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator* [2, 3])**.** In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the two-valued map

$$(2.5)\qquad \begin{aligned} \mathfrak{g}\text{-}\mathrm{Ic}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto \big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\big) \end{aligned}$$

is called a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-vector operator. Then, $\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Ic}_{\mathfrak{a},\nu} = (\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}) : \nu \in I_3^0\}$ is the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.

Remark 2.3 ($\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator* [1]). For each $\nu \in I_3^0$, $\mathfrak{g}\text{-}\mathrm{Ic}_{\mathfrak{a},\nu} = \mathrm{ic}_{\mathfrak{a}} \stackrel{\mathrm{def}}{=} (\mathrm{int}_{\mathfrak{a}}, \mathrm{cl}_{\mathfrak{a}})$ if based on $\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]\times\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$. Then, $\begin{aligned} \mathbf{ic}_{\mathfrak{a}} : \ \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto \big(\mathrm{int}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\big) \end{aligned}$ is a $\mathfrak{T}_{\mathfrak{a}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.

Definition 2.4 (*Complement* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Operator* [2, 3])**.** Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, the one-valued map

$$(2.6)\qquad \begin{aligned} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \complement_{\mathscr{R}_{\mathfrak{a}}}(\mathscr{S}_{\mathfrak{a}}) \end{aligned}$$

where $\complement_{\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denotes the relative complement of its operand with respect to $\mathscr{R}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]$, is called a natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator on $\mathscr{P}(\Omega)$.

For the sake of clarity, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a}}$ whenever $\mathscr{R}_{\mathfrak{a}} = \Omega$, and $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} = \mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}}$ whenever $\mathscr{R}_{\mathfrak{a}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]$ in which case, the term natural complement $\mathfrak{T}_{\mathfrak{g}}$-operator is employed and it stand for $\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

Definition 2.5 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Derived,* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Coderived Operators* [1])**.** Let $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, denote the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closure operators and, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denote the absolute complement

$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$. Then, the one-valued maps

$$(2.7)\qquad \begin{aligned} \mathfrak{g}\text{-Der}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \big\{\xi \in \mathfrak{T}_{\mathfrak{a}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu}\big(\mathscr{S}_{\mathfrak{a}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{a}}(\{\xi\})\big)\big\} \end{aligned}$$

$$(2.8)\qquad \begin{aligned} \mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \big\{\zeta \in \mathfrak{T}_{\mathfrak{a}} :\ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{a},\nu}\big(\mathscr{S}_{\mathfrak{a}} \cup \{\zeta\}\big)\big\} \end{aligned}$$

on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived operator of category ν and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operator of category ν. The classes $\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\}$ and $\mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\}$ are called, respectively, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived operators and the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators.

Remark 2.6 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Derived,* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Coderived Sets* [1]). In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$, suppose $\big(\mathfrak{g}\text{-Der}_{\mathfrak{a}}(\xi;\mathscr{S}_{\mathfrak{a}}), \mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\zeta;\mathscr{S}_{\mathfrak{a}})\big)$ denotes a pair $(\xi,\zeta) \in \mathfrak{T}_{\mathfrak{a}} \times \mathfrak{T}_{\mathfrak{a}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived points* of $\mathscr{S}_{\mathfrak{a}} \in \mathscr{P}(\Omega)$, then $\big(\mathfrak{g}\text{-Der}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}), \mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\big)$ denotes the pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived sets* of $\mathscr{S}_{\mathfrak{a}}$ in $\mathfrak{T}_{\mathfrak{a}}$, where

$$(2.9)\qquad \begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Der}_{\mathfrak{a}}(\xi;\mathscr{S}_{\mathfrak{a}}) :\ \xi \in \mathfrak{T}_{\mathfrak{a}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\zeta;\mathscr{S}_{\mathfrak{a}}) :\ \xi \in \mathfrak{T}_{\mathfrak{a}}\} \end{cases}$$

denote the pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived sets* of $\mathscr{S}_{\mathfrak{a}}$ in $\mathfrak{T}_{\mathfrak{a}}$.

Definition 2.7 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator* [1])**.** Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, an operator of the type

$$(2.10)\qquad \begin{aligned} \mathfrak{g}\text{-Dc}_{\mathfrak{a},\nu} : \times_{\alpha\in I_2^*}\mathscr{P}(\Omega) &\longrightarrow \bigtimes_{\alpha\in I_2^*}\mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto \big(\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\big) \end{aligned}$$

on $\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ ranging in $\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operator of category ν and, $\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Dc}_{\mathfrak{a},\nu} = \big(\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu}\big) :\ \nu \in I_3^0\}$ is called the class of all such $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.

Remark 2.8 ($\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator* [1]). For any $\nu \in I_3^0$, $\mathfrak{g}\text{-Dc}_{\mathfrak{a},\nu} = \text{dc}_{\mathfrak{a}} \stackrel{\text{def}}{=} \big(\text{der}_{\mathfrak{a}}, \text{cod}_{\mathfrak{a}}\big)$ if based on $\big(\text{cl}_{\mathfrak{g}}, \text{int}_{\mathfrak{g}}\big)$. Then, $\begin{aligned} \mathbf{dc}_{a} : \ \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto \big(\text{der}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \text{cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\big) \end{aligned}$ is a $\mathfrak{T}_{\mathfrak{a}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.

Accordingly,

$$\begin{aligned} \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{a}}] &\stackrel{\text{def}}{=} \big\{\mathfrak{g}\text{-Dc}_{\mathfrak{a},\nu} = \big(\mathfrak{g}\text{-Der}_{\nu}, \mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu}\big) : \nu \in I_3^0\big\} \\ (2.11)\qquad &\subseteq \big\{\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\big\} \times \big\{\mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu} : \nu \in I_3^0\big\} \stackrel{\text{def}}{=} \mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{a}}] \times \mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{a}}] \end{aligned}$$

Then, $\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{a}}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators in the $\mathscr{T}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$; $\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{a}}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}$-derived operators while $\mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{a}}]$ denotes the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators in the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.

2.2. **Main Concepts.** The main concepts underlying the δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces, $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, are now presented.

For any $\left(\mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Ope}_{\mathfrak{g}}\right) \in \mathscr{P}(\Omega) \times \left\{\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right\}$, consider the description:

$$\begin{array}{rcl} 0 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(0)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_0^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ 1 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_1^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ 2 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_2^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \vdots & \\ \beta - 1 & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta-1)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_{\beta-1}^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ \beta & \longleftrightarrow & \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcirc_{\alpha \in I_\beta^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{array} \tag{2.12}$$

where $\bigcirc_{\alpha \in I_0^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}}$; next, $\bigcirc_{\alpha \in I_1^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\bigcirc_{\alpha \in I_2^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$; more generally,

$$\bigcirc_{\alpha \in I_\beta^0} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Ope}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Ope}_{\mathfrak{g}} \circ \cdots \circ \mathfrak{g}\text{-Ope}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$$

β factors $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}$. Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(0)}$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}$, $\ldots$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\beta)}$, $\ldots : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the 0^{th}, 1^{st}, 2^{nd}, $\ldots$, β^{th}, $\ldots$ *order derivative* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operators* of $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(0)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}$, $\ldots$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\beta)}$, $\ldots : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the 0^{th}, 1^{st}, 2^{nd}, $\ldots$, β^{th}, $\ldots$ *order derivative* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators* of $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Then, for any pair $\left(\mathscr{S}_{\mathfrak{g}}, \mathfrak{g}\text{-Ope}_{\mathfrak{g}}\right) \in \mathscr{P}(\Omega) \times \left\{\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right\}$, it holds that:

$$\left[\left(\exists \beta \in I_\infty^0\right)\left(\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}(\mathscr{S}_{\mathfrak{g}}) = \varnothing\right)\right] \veebar \left[\left(\forall \beta \in I_\infty^0\right)\left(\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}(\mathscr{S}_{\mathfrak{g}}) \neq \varnothing\right)\right]$$

Suppose the statement preceding $\veebar$ hold, then the number of iterations of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator $\mathfrak{g}\text{-Ope}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ required to achieve *emptiness* (if this is ever achieved) is a type of *density measure* of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. But if the statement following $\veebar$ holds, then $\mathscr{S}_{\mathfrak{g}}^{(\lambda)} \stackrel{\text{def}}{=} \bigcap_{\beta \in I_\infty^*} \mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}(\mathscr{S}_{\mathfrak{g}}) \neq \varnothing$. Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(1)}$, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(2)}$, $\ldots$, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\beta)}$, $\ldots : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ can again be applied on $\mathscr{S}_{\mathfrak{g}}^{(\omega)} \in \mathscr{P}(\Omega)$, yielding $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\lambda+2)}(\mathscr{S}_{\mathfrak{g}})$, $\ldots$, $\mathfrak{g}\text{-Ope}_{\mathfrak{g}}^{(\lambda+\beta)}(\mathscr{S}_{\mathfrak{g}})$, $\ldots$, with $\mathfrak{g}\text{-Ope}_{\mathfrak{g}} \in \left\{\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right\}$.

In view of the above descriptions, 1, 2, $\ldots$, β, $\ldots$ may be viewed as *successor ordinals* while λ as *limit ordinal* and, despite the absence of a *predecessor ordinal*, 0 may, for conveniency, be included in the class of successor ordinals. To define the notion of *ordinal*, the concepts of *everywhere-ordered set*, *similarity* and *order-type* in chronological order have first to be defined. The definition of the first concept (*everywhere-ordered set*) follows.

Definition 2.9 (*Everywhere-Ordered Set*)**.** An "everywhere-ordered set" is an ordered structure of the type

$$\mathfrak{W} \stackrel{\text{def}}{=} (\mathscr{W}, \preccurlyeq) \stackrel{\text{def}}{\longleftrightarrow} \langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots \rangle \tag{2.13}$$

in which $\mathscr{W} \subset \mathfrak{U}$ is an "underlying set" and,

$$\begin{array}{rcll} \preccurlyeq : \ \mathscr{W} \times \mathscr{W} & \longrightarrow & \mathbb{W} \stackrel{\text{def}}{=} \{\alpha \preccurlyeq \beta : \ (\alpha,\beta) \in \mathscr{W} \times \mathscr{W}\} \\ (\alpha,\beta) & \longmapsto & \alpha \preccurlyeq \beta \end{array} \tag{2.14}$$

is a "2-ary rule" satisfying these "everywhere-ordering relation axioms:"

- $\mathrm{Ax}_1(\preccurlyeq) \stackrel{\text{def}}{\longleftrightarrow} (\forall \alpha \in \mathfrak{W})[\alpha \preccurlyeq \alpha \longleftrightarrow \alpha = \alpha]$
- $\mathrm{Ax}_2(\preccurlyeq) \stackrel{\text{def}}{\longleftrightarrow} (\forall (\alpha,\beta) \in \mathfrak{W}^2)[(\alpha \preccurlyeq \beta) \wedge (\beta \preccurlyeq \alpha) \longrightarrow \alpha = \beta]$
- $\mathrm{Ax}_3(\preccurlyeq) \stackrel{\text{def}}{\longleftrightarrow} (\forall (\alpha,\beta,\gamma) \in \mathfrak{W}^3)[(\alpha \preccurlyeq \beta) \wedge (\beta \preccurlyeq \gamma) \longrightarrow \alpha \preccurlyeq \gamma]$
- $\mathrm{Ax}_4(\preccurlyeq) \stackrel{\text{def}}{\longleftrightarrow} (\forall \mathfrak{V} \subseteq \mathfrak{W})[\mathfrak{V} \stackrel{\text{def}}{\longleftrightarrow} \langle \beta_0, \beta_1, \beta_2, \ldots \rangle \longrightarrow \beta_0 \preccurlyeq \beta_1 \preccurlyeq \beta_2 \preccurlyeq \cdots]$

The above definition requires some few explanations. By $\mathrm{Ax}_1(\preccurlyeq)$, $\mathrm{Ax}_2(\preccurlyeq)$ and $\mathrm{Ax}_3(\preccurlyeq)$ are meant that $\begin{array}{rcl} \preccurlyeq : \ \mathscr{W} \times \mathscr{W} & \longrightarrow \mathbb{W} \\ (\alpha,\beta) & \longmapsto \alpha \preccurlyeq \beta \end{array}$ is *reflexive*, *antisymmetric* and *transitive*, respectively; by $\mathrm{Ax}_4(\preccurlyeq)$ is meant that any ordered structure $\mathfrak{V} = (\mathscr{V}, \preccurlyeq)$ derived from $\mathfrak{W} = (\mathscr{W}, \preccurlyeq)$ has a *first* element (*i.e.*, $\beta_0 \in \mathfrak{V} \subseteq \mathfrak{W}$). Moreover, the following statement holds true:

$$(\forall (\alpha,\beta) \in \mathfrak{W} \times \mathfrak{W})[(\alpha \preccurlyeq \beta) \veebar (\beta = \alpha) \veebar (\beta \preccurlyeq \alpha)] \tag{2.15}$$

Thus, given $(\alpha,\beta) \in \mathfrak{W} \times \mathfrak{W}$ then, either α *preceeds* β (*i.e.*, $\alpha \preccurlyeq \beta$), α *is of the same order as* β (*i.e.*, $\beta = \alpha$) or α *succeeds* β (*i.e.*, $\beta \preccurlyeq \alpha$). The remark below is presented in order to avoid any danger of confusing the notations of *underlying* (not ordered) and *everywhere-ordered* sets.

Remark 2.10 (*Everywhere-Ordered Set*). Instead of such *plain sets* notations as $\alpha \in \mathscr{W}$, $(\alpha,\beta) \in \mathscr{W} \times \mathscr{W}$, ... which, in actual fact, are improper, the *ordered sets* notations $\alpha \in \mathfrak{W}$, $(\alpha,\beta) \in \mathfrak{W} \times \mathfrak{W}$, ... are employed solely to stress that α, β, ... are elements of their *ordered set* $\mathfrak{W}$, not of the *underlying set* $\mathscr{W}$ of the ordered set $\mathfrak{W}$. Indeed, in the present context, it does not hold that $\langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots \rangle \neq \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots\}$, though it does hold that $\{\alpha : \ \alpha \in \mathfrak{W}\} = \{\alpha : \ \alpha \in \mathscr{W}\}$.

For each $\mathscr{U} \in \{\mathscr{V}, \mathscr{W}\}$, set $\mathbb{W}_{\mathscr{U}} = \{\alpha \preccurlyeq_{\mathscr{U}} \beta : \ (\alpha,\beta) \in \mathscr{W}_{\mathscr{U}} \times \mathscr{W}_{\mathscr{U}}\}$. Then, the second concept (*similarity*) may be defined as thus.

Definition 2.11 (*Similarity*)**.** The everywhere-ordered sets $\mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}})$ and $\mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}})$, where $\begin{array}{rcl} \preccurlyeq_{\mathscr{V}} : \ \mathscr{V} \times \mathscr{V} & \longrightarrow \mathbb{W}_{\mathscr{V}} \\ (\alpha,\beta) & \longmapsto \alpha \preccurlyeq_{\mathscr{V}} \beta \end{array}$ and $\begin{array}{rcl} \preccurlyeq_{\mathscr{W}} : \ \mathscr{W} \times \mathscr{W} & \longrightarrow \mathbb{W}_{\mathscr{W}} \\ (\alpha,\beta) & \longmapsto \alpha \preccurlyeq_{\mathscr{W}} \beta, \end{array}$ respectively, are said to be "similar," written $\mathfrak{V} \approx \mathfrak{W}$, if and only if there is an "order isomorphism" $\varphi : \mathfrak{V} \cong \mathfrak{W}$ relating the elements α_0, α_1, α_2, ... of $\mathfrak{V}$ to the elements β_0, β_1, β_2, ... of $\mathfrak{W}$ as:

$$\begin{array}{ccc} \mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}}) & \stackrel{\text{def}}{\longleftrightarrow} & \langle \alpha_0, \ \alpha_1, \ \alpha_2, \ \ldots \rangle \\ \approx & & \updownarrow \varphi \\ \mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}}) & \stackrel{\text{def}}{\longleftrightarrow} & \langle \beta_0, \ \beta_1, \ \beta_2, \ \ldots \rangle \end{array} \tag{2.16}$$

From this definition, given $\mathfrak{Y} = (\mathscr{Y}, \preccurlyeq_{\mathscr{Y}}) \longleftrightarrow \langle \gamma_0, \ \gamma_1, \ \gamma_2, \ \ldots, \ \gamma_\nu, \ \ldots \rangle$ with $(\mathfrak{Y}, \mathscr{Y}, \gamma) \in \{(\mathfrak{V}, \mathscr{V}, \alpha), (\mathfrak{W}, \mathscr{W}, \beta)\}$ and $\varphi : \mathfrak{V} \cong \mathfrak{W}$, then $\alpha_0 \preccurlyeq_{\mathscr{V}} \alpha_1 \stackrel{\varphi}{\longrightarrow} \beta_0 \preccurlyeq_{\mathscr{W}} \beta_1$, $\alpha_1 \preccurlyeq_{\mathscr{V}} \alpha_2 \stackrel{\varphi}{\longrightarrow} \beta_1 \preccurlyeq_{\mathscr{W}} \beta_2$, ..., $\alpha_{\nu-1} \preccurlyeq_{\mathscr{V}} \alpha_\nu \stackrel{\varphi}{\longrightarrow} \beta_{\nu-1} \preccurlyeq_{\mathscr{W}} \alpha_\nu$, ... For any

$(\mathfrak{V},\mathfrak{W},\mathfrak{Y}) \in \times_{\mu\in I_3^*}\{\mathfrak{W}_\nu = (\mathscr{W}_\nu, \preccurlyeq_\nu) : \nu \in I_\infty^*\}$, the relations $\mathfrak{V} \approx \mathfrak{V}$, $\mathfrak{V} \approx \mathfrak{W} \longleftrightarrow \mathfrak{W} \approx \mathfrak{V}$ and $(\mathfrak{V} \approx \mathfrak{W}) \wedge (\mathfrak{W} \approx \mathfrak{Y}) \longrightarrow (\mathfrak{V} \approx \mathfrak{Y})$ hold. Therefore, the relation of similarity $\approx : (\mathfrak{V},\mathfrak{W}) \longmapsto \mathfrak{V} \approx \mathfrak{W}$ is *reflexive*, *symmetrical* and *transitive*.

The definition of the third concept (*order-type*) may be stated as thus.

Definition 2.12 (*Order-Type*)**.** An operator of the type

$$\mathrm{OTyp} : \mathfrak{W} \longmapsto \mathrm{OTyp}(\mathfrak{W}) \stackrel{\mathrm{def}}{=} \tau_{\mathscr{W}} \tag{2.17}$$

assigning to any everywhere-ordered set $\mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}})$ a uniquely determined symbol $\tau_{\mathscr{W}}$ is called the "order-type" of $\mathfrak{W}$, provided that if $\mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}})$ be any other everywhere-ordered set together with its uniquely determined order-type $\mathrm{OTyp}(\mathfrak{V}) \stackrel{\mathrm{def}}{=} \tau_{\mathscr{V}}$, the following statement holds:

$$\mathfrak{V} \approx \mathfrak{W} \longleftrightarrow \tau_{\mathscr{V}} = \tau_{\mathscr{W}}. \tag{2.18}$$

Clearly, the manner of proceeding from the relation of similarity to the concept of order-type is exactly the same as that from the relation of equivalence to the concept of cardinal number. For, given any $\mathfrak{V} = (\mathscr{V}, \preccurlyeq_{\mathscr{V}})$ and $\mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}})$, then $\mathfrak{V} \approx \mathfrak{W} \longleftrightarrow \mathrm{OTyp}(\mathfrak{V}) = \mathrm{OTyp}(\mathfrak{W})$ is analogous to $\mathscr{V} \sim \mathscr{W} \longleftrightarrow \mathrm{card}(\mathscr{V}) = \mathrm{card}(\mathscr{W})$.

Remark 2.13. By $\mathfrak{V} \approx \mathfrak{W} \longleftrightarrow \tau_{\mathscr{V}} = \tau_{\mathscr{W}}$ is meant that a uniquely determined symbol actually is assigned not to a single set but to a class of everywhere-ordered sets which are similar to each other.

Granted the definitions of the concepts of *everywhere-ordered set*, *similarity* and *order-type*, the definition of the concept of *ordinal* may be stated as thus.

Definition 2.14 (*Ordinal*)**.** The order-type $\mathrm{OTyp}(\mathfrak{W}) = \tau_{\mathscr{W}}$ of an everywhere-ordered set $\mathfrak{W} = (\mathscr{W}, \preccurlyeq_{\mathscr{W}})$ is called "ordinal," written $\mathrm{ord}(\mathfrak{W}) \stackrel{\mathrm{def}}{=} \delta_{\mathscr{W}}$. Moreover:

– I. $\delta_{\mathscr{W}}$ is called a "predecessor ordinal" if and only if there exists no ordinal $\mathrm{ord}(\mathfrak{W})$ such that $\delta_{\mathscr{W}} = \mathrm{ord}(\mathfrak{W}) + 1$.

– II. $\delta_{\mathscr{W}}$ is called a "successor ordinal" if and only if there exists an ordinal $\mathrm{ord}(\mathfrak{W})$ such that $\delta_{\mathscr{W}} = \mathrm{ord}(\mathfrak{W}) + 1$.

– III. $\delta_{\mathscr{W}}$ is called a "limit ordinal," denoted as $\delta_{\mathscr{W}} \stackrel{\mathrm{def}}{=} \lambda_{\mathscr{W}}$, if and only if it has no immediate predecessor.

Let the symbols 0, δ, and λ (instead of the symbols $0_{\mathscr{W}}$, $\delta_{\mathscr{W}}$, and $\lambda_{\mathscr{W}}$) stand for *predecessor ordinal*, *successor ordinal* and *limit ordinal*, respectively. Then, the definitions of the notions of *ordered derivative* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators* of $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, may well be stated as thus.

Definition 2.15 (δ^{th}*-Iterations:* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$*-Derived,* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$*-Coderived Operators*)**.** Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators of category ν in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. The "δ^{th}-iterate of $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$" is a set-valued map $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ defined by transfinite recursion on the class of successor ordinals as,

– I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(0)}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathscr{S}_{\mathfrak{g}}$

– II. $\mathfrak{g}\text{-Der}^{(1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$

– III. $\mathfrak{g}\text{-Der}^{(\delta+1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$

– IV. $\mathfrak{g}\text{-Der}^{(\lambda)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$

— II. The "δ^{th}-iterate of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$" is a set-valued map $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ defined by transfinite recursion on the class of successor ordinals as,

– I. $\mathfrak{g}\text{-Cod}^{(0)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathscr{S}_{\mathfrak{g}}$

– II. $\mathfrak{g}\text{-Cod}^{(1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$

– III. $\mathfrak{g}\text{-Cod}^{(\delta+1)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} \circ \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$

– IV. $\mathfrak{g}\text{-Cod}^{(\lambda)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$

In the following remark, the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets of category ν and order δ ($\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$*-derived,* $\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$*-coderived sets*) are presented.

Remark 2.16 ($\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$*-Derived,* $\mathfrak{g}$-ν-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$*-Coderived sets*). Suppose $(\mathscr{R}^{(\delta)}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I^*_2} \mathscr{P}(\Omega)$ such that $\mathscr{R}^{(\delta)}_{\mathfrak{g}} = \mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ for some ordinal δ, then $\mathscr{R}^{(\delta)}_{\mathfrak{g}}$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of category* ν *and order* δ. Likewise, given $(\mathscr{U}^{(\delta)}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I^*_2} \mathscr{P}(\Omega)$ such that $\mathscr{U}^{(\delta)}_{\mathfrak{g}} = \mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g},\nu}(\mathscr{V}_{\mathfrak{g}})$ for some ordinal δ, then $\mathscr{U}^{(\delta)}_{\mathfrak{g}}$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived set of* $\mathscr{V}_{\mathfrak{g}}$ *of category* ν *and order* δ. Hence, any $\{\xi\} \in \mathscr{P}(\Omega)$ such that $(\xi \in \mathscr{R}^{(\delta)}_{\mathfrak{g}} \in \mathscr{P}(\Omega)) \wedge (\xi \notin \mathscr{R}^{(\delta+1)}_{\mathfrak{g}} \in \mathscr{P}(\Omega))$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived unit set of* $\mathscr{R}_{\mathfrak{g}}$ *of category* ν *and order* δ, and any $\{\zeta\} \in \mathscr{P}(\Omega)$ such that $(\zeta \in \mathscr{U}^{(\delta)}_{\mathfrak{g}} \in \mathscr{P}(\Omega)) \wedge (\zeta \notin \mathscr{U}^{(\delta+1)}_{\mathfrak{g}} \in \mathscr{P}(\Omega))$ may be called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived unit set of* $\mathscr{U}_{\mathfrak{g}}$ *of category* ν *and order* δ.

Evidently, the use of $\text{der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Der}_{\nu}$, $\text{der} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ introduce the notions of $\mathfrak{T}_{\mathfrak{g}}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ, $\mathfrak{g}$-$\mathfrak{T}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of category* ν *and order* δ, and $\mathfrak{T}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ, respectively; the use of $\text{doc}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\nu}$, $\text{cod} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ introduce the notions of $\mathfrak{T}_{\mathfrak{g}}$*-coderived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ, $\mathfrak{g}$-$\mathfrak{T}$*-coderived set of* $\mathscr{S}_{\mathfrak{g}}$ *of category* ν *and order* δ, and $\mathfrak{T}$*-coderived set of* $\mathscr{S}_{\mathfrak{g}}$ *of order* δ, respectively.

Of the notations $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$ and $\mathfrak{T} = (\Omega, \mathscr{T})$, either the first will be used instead of the second, or both will be used interchangeably.

3. Main Results

In this section, the basic properties of the $\mathfrak{g}$-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}^{(\delta)}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}^{(\delta)}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}^{(\delta)}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, are studied in $\mathscr{T}_{\mathfrak{g}}$-spaces.

In a $\mathscr{T}_{\mathfrak{g}}$-space, every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set is contained in all the preceding $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived sets and, every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set contains all the preceding $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived sets. The theorem follows.

Theorem 3.1. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \leqslant \delta < \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \leqslant \delta < \lambda)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \leqslant \delta < \lambda)$$

Then, to prove Item i., it only suffices to prove that,

$$\big(\forall\delta:\ 1 \leqslant \delta < \lambda\big)\Big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\delta) = 1 \longrightarrow \mathrm{P}(\delta+1) = 1\big)\Big]$$

– Case i. Let $1 = \delta$. Since $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}} :\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad &\longmapsto \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\subseteq \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\
&\longleftrightarrow \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\
&\longleftrightarrow \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 < \delta < \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But, for each $\eta \in \{\delta, \delta+1\}$,

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{((\delta+1)+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 < \delta < \lambda$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}\Bigg(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\subseteq \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\qquad\qquad \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

for all δ such that $1 < \delta < \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

— II. Introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta:\ 1 \leqslant \delta < \lambda)$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)\big[\big(\mathrm{Q}(0)=1\big)\wedge\big(\mathrm{Q}(\delta)=1\ \longrightarrow\ \mathrm{Q}(\delta+1)=1\big)\big]$$

– CASE I. Let $1=\delta$. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it results that

$$\begin{array}{rcl}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \longmapsto & \big\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cup\{\zeta\}\big)\big\}\\ & \supseteq & \big\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\}\\ & \longleftrightarrow & \big\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \zeta\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\}\\ & \longleftrightarrow & \big\{\zeta\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\zeta\}\big)\big\}\\ & \longleftrightarrow & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\end{array}$$

Thus, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}})\ \longleftrightarrow\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus, the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1)=1$. The base case therefore holds.

– CASE II. Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But, for each $\eta\in\{\delta,\delta+1\}$,

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\ \longleftrightarrow\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{((\delta+1)+1)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1)=1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta)=1$ for all δ such that $1\prec\delta\prec\lambda$, it follows that

$$\begin{array}{rcl}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) & \longleftrightarrow & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\ & \longleftrightarrow & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\Big(\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Big)\\ & \supseteq & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\\ & & \qquad\qquad\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\end{array}$$

for all δ such that $1\prec\delta\prec\lambda$, from which $\mathrm{Q}(\lambda)=1$ follows. The proof of the theorem is complete. ☐

The corollary stated below is an immediate consequence of the above theorem.

Corollary 3.2. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*. Then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$

In a $\mathscr{T}_{\mathfrak{g}}$-space, just as $\mathfrak{g}\text{-Der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ (or, $\mathrm{der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$) [1], so is $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ (or, $\mathrm{der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$); likewise, just as $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ [1], so is $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *finer* (or, *larger*, *stronger*) than $\text{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (or, $\text{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$). Accordingly, the proposition follows.

Proposition 1. If $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathbf{dc}_g \in \text{DC}[T_g]$ be a given pair of $\mathfrak{T}_{\mathfrak{g}}$-operators $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{dc}_g \in \text{DC}[T_g]$ be given and $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary. Then:

— I. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \text{P}(\delta) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall \delta : 1 \preccurlyeq \delta \prec \lambda\big)\big[\big(\text{P}(1) = 1\big) \wedge \big(\text{P}(\delta) = 1 \longrightarrow \text{P}(\delta + 1) = 1\big)\big]$$

– CASE I. Let $1 = \delta$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\text{P}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\text{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\subseteq \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\subseteq \quad \text{der}_{\mathfrak{g}} \circ \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \text{der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\text{P}(\delta + 1) = 1$. The inductive case therefore holds.

Since $\text{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\subseteq \quad \text{der}_{\mathfrak{g}} \circ \text{der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \text{der}_{\mathfrak{g}}\Bigg(\bigcap_{\delta \prec \lambda} \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\subseteq \quad \text{der}_{\mathfrak{g}} \circ \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \text{der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\qquad \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\text{P}(\lambda) = 1$ follows.

— II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta:\ 1 \leqslant \delta < \lambda)$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall \delta:\ 1 \leqslant \delta < \lambda\big)\Big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\delta) = 1 \longrightarrow \mathrm{P}(\delta+1) = 1\big)\Big]$$

– CASE I. Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 < \delta < \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\supseteq \mathrm{cod}_{\mathfrak{g}} \circ \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathrm{cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 < \delta < \lambda$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\supseteq \mathrm{cod}_{\mathfrak{g}} \circ \mathrm{cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathrm{cod}_{\mathfrak{g}}\Bigg(\bigcap_{\delta<\lambda} \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\supseteq \mathrm{cod}_{\mathfrak{g}} \circ \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathrm{cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\qquad \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

for all δ such that $1 < \delta < \lambda$, from which $\mathrm{Q}(\lambda) = 1$ follows. The proof of the proposition is complete. □

For any δ such that $1 \leqslant \delta < \lambda$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Accordingly, the following corollary is an immediate consequence of the above proposition.

Corollary 3.3. *If* $\mathfrak{g}\text{-}\mathrm{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

– I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta:\ 1 \leqslant \delta < \lambda)$

– II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta:\ 1 \leqslant \delta < \lambda)$

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the notions of δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operators can be interrelated among themselves and presented *δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operators finness-coarseness diagrams*; similarly, the notions of δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operators can be interrelated among themselves and presented *δ^{th}-order $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operators finness-coarseness diagrams*. A further corollary follows.

Corollary 3.4. *If $\mathfrak{g}\text{-}\mathrm{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathbf{dc}_{g} \in \mathrm{DC}[T_{g}]$ be a given pair of $\mathfrak{T}_{\mathfrak{g}}$-operators $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

– I. *For any $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,*

$$\begin{array}{ccc} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) & \longrightarrow & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ & \nwarrow & \uparrow \\ & & \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda) \end{array} \tag{3.1}$$

– II. *For any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$,*

$$\begin{array}{ccc} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \longleftarrow & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \searrow & \downarrow \\ & & \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda) \end{array} \tag{3.2}$$

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator is *$\varnothing$-grounded* (alternatively, *$\varnothing$-preserving*); the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is *Ω-grounded* (alternatively, *Ω-preserving*). These are embodied in the following theorem.

Theorem 3.5. *Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:*

– I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

– II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

Proof. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove Item I., it only suffices to prove that,

$$(\forall \delta : 1 \preccurlyeq \delta \prec \lambda)\big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\delta) = 1 \longrightarrow \mathrm{P}(\delta + 1) = 1\big)\big]$$

– Case I. Let $1 = \delta$. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}(\varnothing) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\varnothing) = \varnothing$. Thus, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}(\varnothing) = \varnothing$, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing$ and consequently, it follows that

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\varnothing) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\varnothing) = \varnothing$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\varnothing) = \varnothing$, implying $\mathrm{P}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\varnothing) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\varnothing) \\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\bigcap_{\delta \prec \lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\varnothing)\right) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\varnothing) = \varnothing \end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

— II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega \quad (\forall \delta:\ 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove ITEM II., it only suffices to prove that,

$$(\forall \delta:\ 1 \preccurlyeq \delta \prec \lambda)\big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\delta) = 1 \ \longrightarrow\ \mathrm{Q}(\delta+1) = 1\big)\big]$$

– CASE I. Let $1 = \delta$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}(\Omega) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega) = \Omega$. Thus, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)}(\Omega) = \Omega$, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega$ and consequently, it follows that

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\Omega) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega) = \Omega$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\Omega) = \Omega$, implying $\mathrm{Q}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \lambda$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+1)}(\Omega) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\Omega) \\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\bigcap_{\delta \prec \lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\Omega)\right) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega) = \Omega \end{aligned}$$

for all δ such that $1 \prec \delta \prec \lambda$, from which $\mathrm{Q}(\lambda) = 1$ follows. The proof of the theorem is complete. ☐

For any δ such that $1 \preccurlyeq \delta \prec \lambda$, the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator is $\cup$-*additive* (alternatively, $\cup$-*distributive*); the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is $\cap$-*additive* (alternatively, $\cap$-*distributive*). The theorem follows.

Theorem 3.6. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta:\ 1 \preccurlyeq \delta \prec \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall\delta:\ 1\leqslant\delta<\lambda)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni\mathrm{P}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall\delta:1\leqslant\delta<\lambda)$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall\delta:\ 1\leqslant\delta<\lambda\big)\big[\big(\mathrm{P}(1)=1\big)\wedge\big(\mathrm{P}(\delta)=1 \longrightarrow \mathrm{P}(\delta+1)=1\big)\big]$$

– CASE I. Let $1=\delta$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$ holds true, implying $\mathrm{P}(1)=1$. The base case therefore holds.

– CASE II. Let $1<\delta<\lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\\
&= \mathfrak{g}\text{-Der}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Bigg)\\
&= \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\\
&\longleftrightarrow \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}})
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta+1)=1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta)=1$ for all δ such that $1<\delta<\lambda$, it follows that $\mathrm{P}(\lambda)=1$ states that

$$\bigcap_{\delta<\lambda}\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Bigg) \longleftrightarrow \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Bigg(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Bigg)$$

and it is evident that any element in $\bigcap_{\delta<\lambda}\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ is contained in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$. Thus, in order to prove that any element in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ is also in $\bigcap_{\delta<\lambda}\Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$, let it be supposed that $\xi\in\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\Big(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})\Big)$ such that, for some $(\alpha,\beta)<(\lambda,\lambda)$

where $\alpha \leqslant \beta$, say, the statement $\xi \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta=\alpha,\beta} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ holds true. Then, $\xi \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\alpha)}(\mathscr{W}_{\mathfrak{g}})$ and therefore $\xi \in \bigcap_{\delta<\lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$, implying $\mathrm{P}(\lambda) = 1$ holds.

— II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \ \ (\forall \delta : 1 \leqslant \delta < \lambda)$$

Then, to prove Item ii., it only suffices to prove that,

$$(\forall \delta :\ 1 \leqslant \delta < \lambda)\big[(\mathrm{Q}(1) = 1) \wedge (\mathrm{Q}(\delta) = 1 \ \longrightarrow\ \mathrm{Q}(\delta+1) = 1)\big]$$

– Case i. Let $1 = \delta$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 < \delta < \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}})$ and consequently, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \\
&= \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right) \\
&= \quad \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \\
&\longleftrightarrow \quad \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}})
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{W}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 < \delta < \lambda$, it follows that $\mathrm{Q}(\lambda) = 1$ states that

$$\bigcap_{\delta<\lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right) \quad \longleftrightarrow \quad \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$$

and it is evident that any element in $\bigcap_{\delta<\lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ is contained in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$. Thus, in order to prove that any element in $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ is also in $\bigcap_{\delta<\lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$, let it be supposed that $\zeta \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ such that, for some $(\alpha,\beta) < (\lambda,\lambda)$

where $\alpha \preccurlyeq \beta$, say, the statement $\zeta \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \left(\bigcap_{\delta=\alpha,\beta} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ holds true. Then, $\zeta \in \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\alpha)}(\mathscr{W}_{\mathfrak{g}})$ and therefore, it follows that the statement $\zeta \in \bigcap_{\delta \prec \lambda} \left(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \right)$ holds, implying $\mathrm{Q}(\lambda) = 1$ holds. The proof of the theorem is complete. □

The corollary stated below is an immediate consequence of the above theorem.

Corollary 3.7. *If* $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \subseteq \bigcup_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \supseteq \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

For any (δ, η) such that $1 \preccurlyeq \delta \prec \eta \prec \lambda$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Accordingly, the proposition follows.

Proposition 2. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta, \eta) : 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta, \eta) : 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Set $\eta = \delta + \varepsilon$, where $1 \preccurlyeq \varepsilon$, introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\varepsilon) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \varepsilon : 1 \preccurlyeq \varepsilon)$$

Then, to prove Item i., it only suffices to prove that,

$$\big(\forall \varepsilon : 1 \preccurlyeq \varepsilon\big)\Big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\varepsilon) = 1 \longrightarrow \mathrm{P}(\varepsilon + 1) = 1\big)\Big]$$

– Case i. Let $1 = \varepsilon$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \varepsilon$ and assume that the inductive hypothesis $\mathrm{P}(\varepsilon) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But,

$$\begin{aligned} \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}}) \\ &\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\varepsilon+1) = 1$. The inductive case therefore holds.

Since $\mathrm{P}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\left(\bigcap_{\eta \prec \lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\right) \\ &\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\ &\qquad \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, from which $\mathrm{P}(\lambda) = 1$ follows.

— II. Set $\eta = \delta + \varepsilon$, where $1 \preccurlyeq \varepsilon$, introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\varepsilon) \quad \overset{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \varepsilon : \ 1 \preccurlyeq \varepsilon)$$

Then, to prove ITEM II., it only suffices to prove that,

$$(\forall \varepsilon : \ 1 \preccurlyeq \varepsilon)\Big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\varepsilon) = 1 \ \longrightarrow \ \mathrm{Q}(\varepsilon+1) = 1\big)\Big]$$

– CASE I. Let $1 = \varepsilon$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– CASE II. Let $1 \prec \varepsilon$ and assume that the inductive hypothesis $\mathrm{Q}(\varepsilon) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$. But,

$$\begin{aligned} \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\varepsilon)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}}) \\ &\supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+(\varepsilon+1))}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\varepsilon+1) = 1$. The inductive case therefore holds.

Since $\mathrm{Q}(\delta) = 1$ for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\Bigg(\bigcap_{\eta \prec \lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\qquad \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

for all δ such that $1 \prec \delta \prec \eta \prec \lambda$, from which $\mathrm{Q}(\lambda) = 1$ follows. The proof of the proposition is complete. □

The corollary stated below is an immediate consequence of the above proposition.

Corollary 3.8. *If* $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$*,* $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ 1 \preccurlyeq \delta \prec \eta \prec \lambda\big)$

For any (δ, η) such that $1 \preccurlyeq \delta \prec \eta \prec \lambda$, the $(\delta+\eta)^{\mathrm{th}}$-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator is equivalent to the composition of the δ^{th}-order and the η^{th}-order of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator; likewise, the $(\delta+\eta)^{\mathrm{th}}$-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is equivalent to the composition of the δ^{th}-order and the η^{th}-order of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator. The proposition follows.

Proposition 3. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta)\big)$
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta)\big)$

where $(\delta, \eta) \prec (\lambda, \lambda)$.

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0, 1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta,\eta) \overset{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall\,(\delta,\eta) :\ (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\big)$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall\,(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)\big)$$
$$\Big[\big(\mathrm{P}\,(1,1)=1\big)\wedge\big(\mathrm{P}\,(\delta,\eta)=1\ \longrightarrow\ \mathrm{P}\,(\delta+1,\eta+1)=1\big)\Big]$$

– CASE I. Let $(1,1)=(\delta,\eta)$. Then,

$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1+1)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\,(\mathscr{S}_{\mathfrak{g}})$$

implying $\mathrm{P}\,(1,1)=1$. The base case therefore holds.

– CASE II. Let $(1,1)\prec(\delta,\eta)\prec(\lambda,\lambda)$ and assume that the inductive hypothesis $\mathrm{P}\,(\delta,\eta)=1$ holds true. Then, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$ and consequently, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}\,(\mathscr{S}_{\mathfrak{g}})$ and,

$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$$
$$\updownarrow$$
$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta-1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$$
$$\updownarrow$$
$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta+1)}\,(\mathscr{S}_{\mathfrak{g}})$$

Hence, it follows that $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta+1)}\,(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}\,(\delta+1,\eta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{P}\,(\delta,\eta)=1$ holds for all (δ,η) such that $(1,1)\prec(\delta,\eta)\prec(\lambda,\lambda)$. Then,

$$\bigcap_{\delta+\eta\prec\lambda+\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})\quad=\quad\bigcap_{\delta+\eta\prec\lambda+\lambda}\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$$
$$\updownarrow$$
$$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda+\lambda)}\,(\mathscr{S}_{\mathfrak{g}})\quad=\quad\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}\circ\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\lambda)}\,(\mathscr{S}_{\mathfrak{g}})$$

from which $\mathrm{P}\,(\lambda,\lambda)=1$ follows.

— II. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni\mathrm{Q}\,(\delta,\eta)\overset{\text{def}}{\longleftrightarrow}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta+\eta)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\eta)}\,(\mathscr{S}_{\mathfrak{g}})$$
$$\big(\forall\,(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)\big)$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall\,(\delta,\eta):\ (1,1)\preccurlyeq(\delta,\eta)\prec(\lambda,\lambda)\big)$$
$$\Big[\big(\mathrm{Q}\,(1,1)=1\big)\wedge\big(\mathrm{Q}\,(\delta,\eta)=1\ \longrightarrow\ \mathrm{Q}\,(\delta+1,\eta+1)=1\big)\Big]$$

– CASE I. Let $(1,1)=(\delta,\eta)$. Then,

$$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1+1)}\,(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}\circ\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(1)}\,(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(2)}\,(\mathscr{S}_{\mathfrak{g}})$$

implying $\mathrm{Q}\,(1,1)=1$. The base case therefore holds.

– Case ii. Let $(1,1) \prec (\delta,\eta) \prec (\lambda,\lambda)$ and assume that the inductive hypothesis $\mathrm{Q}(\delta,\eta) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ and,

$$\begin{array}{c} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \updownarrow \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta-1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ \updownarrow \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}}) \end{array}$$

Hence, it follows that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{((\delta+1)+(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1,\eta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta,\eta) = 1$ holds for all (δ,η) such that $(1,1) \prec (\delta,\eta) \prec (\lambda,\lambda)$. Then,

$$\begin{array}{ccc} \bigcap_{\delta+\eta \prec \lambda+\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta+\eta \prec \lambda+\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \\ & \updownarrow & \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \end{array}$$

from which $\mathrm{Q}(\lambda,\lambda) = 1$ follows. The proof of the proposition is complete. □

The corollary stated below is an immediate consequence of the above proposition.

Corollary 3.9. *If* $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

– i. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)} \circ \mathrm{der}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta,\eta) : (1,1) \preccurlyeq (\delta,\eta)\big)$

– ii. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)} \circ \mathrm{cod}_{\mathfrak{g}}^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall (\delta,\eta) : (1,1) \preccurlyeq (\delta,\eta)\big)$

where $(\delta,\eta) \prec (\lambda,\lambda)$.

For any (δ,η) such that $(1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)$, the $\delta\eta^{\text{th}}$-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator is equivalent to the η^{th}-order of the δ^{th}-order of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator; likewise, the $\delta\eta^{\text{th}}$-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator is equivalent to the η^{th}-order of the δ^{th}-order of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator. Accordingly, the following proposition presents itself.

Proposition 4. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

– i. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \ \big(\forall (\delta,\eta) : (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\big)$

– ii. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \ \big(\forall (\delta,\eta) : (1,1) \preccurlyeq (\delta,\eta) \prec (\lambda,\lambda)\big)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\eta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \eta:\ 1 \preccurlyeq \eta \prec \lambda)$$

Then, to prove Item i., it only suffices to prove that,

$$\left(\forall \eta:\ 1 \preccurlyeq \eta \prec \lambda\right)\left[\left(\mathrm{P}(1) = 1\right) \wedge \left(\mathrm{P}(\eta) = 1 \ \longrightarrow\ \mathrm{P}(\eta+1) = 1\right)\right]$$

– Case i. Let $1 = \eta$. Then,

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\times 1)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\right)^{(1)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$$

implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 \prec \eta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\eta) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$. But, the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\eta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ holds true and on the other hand, the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$ also holds true. Hence, it follows that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\eta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{P}(\delta,\eta) = 1$ holds for all (δ,η) such that $(1,1) \prec (\delta,\eta) \prec (\lambda,\lambda)$. Then,

$$\begin{array}{ccc}
\displaystyle\bigcap_{\delta\prec\lambda}\left(\bigcap_{\eta\prec\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}})\right) & = & \displaystyle\bigcap_{\delta\prec\lambda}\left(\bigcap_{\eta\prec\lambda} \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\right) \\
 & \updownarrow & \\
\displaystyle\bigcap_{\delta\prec\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \displaystyle\bigcap_{\delta\prec\lambda} \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\right)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
 & \updownarrow & \\
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda\times\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}\right)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

from which $\mathrm{P}(\lambda,\lambda) = 1$ follows.

— II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\eta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \eta:\ 1 \preccurlyeq \eta \prec \lambda)$$

Then, to prove Item ii., it only suffices to prove that,

$$\left(\forall \eta:\ 1 \preccurlyeq \eta \prec \lambda\right)\left[\left(\mathrm{Q}(1) = 1\right) \wedge \left(\mathrm{Q}(\eta) = 1 \ \longrightarrow\ \mathrm{Q}(\eta+1) = 1\right)\right]$$

– Case i. Let $1 = \eta$. Then,

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\times 1)}(\mathscr{S}_{\mathfrak{g}}) = \left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\right)^{(1)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$$

implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 < \eta < \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\eta) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it results that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \big(\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\big)$. But, the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\eta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}})$ holds true and on the other hand, the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \big(\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\big) \longleftrightarrow \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$ also holds true. Hence, it follows that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta(\eta+1))}(\mathscr{S}_{\mathfrak{g}}) = \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta+1)}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\eta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta,\eta) = 1$ holds for all (δ,η) such that $(1,1) < (\delta,\eta) < (\lambda,\lambda)$. Then,

$$\begin{array}{ccc}
\bigcap_{\delta<\lambda}\Big(\bigcap_{\eta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}})\Big) & = & \bigcap_{\delta<\lambda}\Big(\bigcap_{\eta<\lambda} \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}})\Big) \\
 & \updownarrow & \\
\bigcap_{\delta<\lambda} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \bigcap_{\delta<\lambda} \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
 & \updownarrow & \\
\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda\times\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}\big)^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

from which $\mathrm{Q}(\lambda,\lambda) = 1$ follows. The proof of the proposition is complete. □

An immediate consequence of the above proposition is the following corollary.

Corollary 3.10. *If* $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

– i. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \big(\mathrm{der}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall(\delta,\eta) :\ (1,1) \leq (\delta,\eta) < (\lambda,\lambda)\big)$

– ii. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta\eta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \big(\mathrm{cod}_{\mathfrak{g}}^{(\delta)}\big)^{(\eta)}(\mathscr{S}_{\mathfrak{g}}) \quad \big(\forall(\delta,\eta) :\ (1,1) \leq (\delta,\eta) < (\lambda,\lambda)\big)$

For any δ such that $1 \leq \delta < \lambda$, the union of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set includes the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set operator composition with itself; the intersection of a $\mathfrak{T}_{\mathfrak{g}}$-set and its $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set is included in the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived set operator composition with itself. These are embodied in the following theorem.

Theorem 3.11. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Then:*

– i. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta :\ 1 \leq \delta < \lambda)$

– ii. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall\delta :\ 1 \leq \delta < \lambda)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space

$\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{P}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta:\ 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove Item i., it only suffices to prove that,

$$(\forall \delta:\ 1 \preccurlyeq \delta \prec \lambda)\big[\big(\mathrm{P}(1) = 1\big) \wedge \big(\mathrm{P}(\delta) = 1 \longrightarrow \mathrm{P}(\delta+1) = 1\big)\big]$$

– Case i. Let $1 = \delta$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{P}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$ and, on the other hand, the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ also holds true. Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{P}(\delta+1) = 1$. The inductive case therefore holds.

Suppose $\mathrm{P}(\delta) = 1$ holds for all δ such that $1 \prec \delta \prec \lambda$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)}\Bigg(\bigcap_{\delta+\delta \prec \lambda+\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg) \\
&\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}}) \\
&\subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

from which $\mathrm{P}(\lambda) = 1$ follows.

— II. Introduce $\mathbb{B} = \{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B} \ni \mathrm{Q}(\delta) \overset{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad (\forall \delta:\ 1 \preccurlyeq \delta \prec \lambda)$$

Then, to prove Item ii., it only suffices to prove that,

$$(\forall \delta:\ 1 \preccurlyeq \delta \prec \lambda)\big[\big(\mathrm{Q}(1) = 1\big) \wedge \big(\mathrm{Q}(\delta) = 1 \longrightarrow \mathrm{Q}(\delta+1) = 1\big)\big]$$

– Case i. Let $1 = \delta$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true, implying $\mathrm{Q}(1) = 1$. The base case therefore holds.

– Case ii. Let $1 \prec \delta \prec \lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta) = 1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently, it follows that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. But, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$ and, on the other hand, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathscr{S}_{\mathfrak{g}} \cap$

$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, implying $\mathrm{Q}(\delta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta)=1$ holds for all δ such that $1\prec\delta\prec\lambda$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+1)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+1)}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}})\\
&\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda+\lambda)}(\mathscr{S}_{\mathfrak{g}})\\
&\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\Bigg(\bigcap_{\delta+\delta\prec\lambda+\lambda}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}})\Bigg)\\
&\subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(2)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+\delta)}(\mathscr{S}_{\mathfrak{g}})\\
&\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})\\
&\supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\\
&\supseteq \mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

from which $\mathrm{Q}(\lambda)=1$ follows. The proof of the theorem is complete. ☐

The following corollary is an immediate consequence of the above theorem.

Corollary 3.12. *If* $\mathfrak{g}\text{-Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g\in\mathrm{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, *and let* $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, *then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$

For any δ such $1\preccurlyeq\delta\prec\lambda$, the image of a $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator is equivalent to the image of the relative complement of any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived unit set in the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator; the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator is equivalent to the image of the union of the $\mathfrak{T}_{\mathfrak{g}}$-set and any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived unit set under the δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator. The theorem follows.

Theorem 3.13. *Let* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, and let* $(\{\xi\},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. *Then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$

Proof. Let $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, and let $(\{\xi\},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

— I. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni\mathrm{P}(\delta)\quad\overset{\text{def}}{\longleftrightarrow}\quad\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\quad(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda)$$

Then, to prove ITEM I., it only suffices to prove that,

$$\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)\big[\big(\mathrm{P}(1)=1\big)\wedge\big(\mathrm{P}(\delta)=1\ \longrightarrow\ \mathrm{P}(\delta+1)=1\big)\big]$$

– CASE I. Let $1=\delta$. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$ holds true, implying $\mathrm{P}(1)=1$. The base case therefore holds.

– CASE II. Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{P}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$ and consequently, it follows that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})$ and, on the other hand, the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\longleftrightarrow\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$ also holds true. Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$, implying $\mathrm{P}(\delta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{P}(\delta)=1$ holds for all δ such that $1\prec\delta\prec\lambda$. Then,

$$\begin{array}{ccc}\displaystyle\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) & = & \displaystyle\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\\ & \updownarrow & \\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\lambda)}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\end{array}$$

from which $\mathrm{P}(\lambda)=1$ follows.

— II. Introduce $\mathbb{B}=\{0,1\}$ as Boolean domain and introduce the Boolean-valued propositional formula

$$\mathbb{B}\ni\mathrm{Q}(\delta)\ \stackrel{\text{def}}{\longleftrightarrow}\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\quad\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)$$

Then, to prove ITEM II., it only suffices to prove that,

$$\big(\forall\delta:\ 1\preccurlyeq\delta\prec\lambda\big)\big[\big(\mathrm{Q}(1)=1\big)\wedge\big(\mathrm{Q}(\delta)=1\ \longrightarrow\ \mathrm{Q}(\delta+1)=1\big)\big]$$

– CASE I. Let $1=\delta$. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$ holds true, implying $\mathrm{Q}(1)=1$. The base case therefore holds.

– CASE II. Let $1\prec\delta\prec\lambda$ and assume that the inductive hypothesis $\mathrm{Q}(\delta)=1$ holds true. Then, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$ and consequently, it follows that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$. But on the one hand, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\big)\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\big)$ and, on the other hand, the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$ also holds true. Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta+1)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)$, implying $\mathrm{Q}(\delta+1)=1$. The inductive case therefore holds.

Suppose $\mathrm{Q}(\delta)=1$ holds for all δ such that $1\prec\delta\prec\lambda$. Then,

$$\begin{array}{ccc}\displaystyle\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) & = & \displaystyle\bigcap_{\delta\prec\lambda}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\\ & \updownarrow & \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{S}_{\mathfrak{g}}) & = & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\lambda)}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\end{array}$$

from which $\mathrm{Q}(\lambda)=1$ follows. The proof of the theorem is complete. □

The corollary stated below is an immediate consequence of the above theorem.

Corollary 3.14. *If* $\mathfrak{g}\text{-Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_g \in \text{DC}[T_g]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, and let* $(\{\xi\}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ *be arbitrary in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

Our research objective concerning the definitions and the essential properties of the concepts of δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces is now complete. Of the notions of the δ^{th}-iterates of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, we conclude the present section with two corollaries and two axiomatic definitions derived from these two corollaries.

The first corollary stated below gives the necessary and sufficient condition for a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator.

Corollary 3.15. *A necessary and sufficient condition for the* δ^{th}*-iterate* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ *of* $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for every* $(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$*, it satisfies:*

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- II. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- III. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- IV. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

The second corollary stated below gives the necessary and sufficient condition for a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator.

Corollary 3.16. *A necessary and sufficient condition for the* δ^{th}*-iterate* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ *of* $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for every* $(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$*, it satisfies:*

- I. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \text{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- III. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$
- IV. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}}) \quad (\forall \delta : 1 \preccurlyeq \delta \prec \lambda)$

Hence, in a strong $\mathscr{T}_{\mathfrak{g}}$-space, for the δ^{th}-iterate of a set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator conditions* (Items I.–IV. of Cor. 3.15), and similarly, for the δ^{th}-iterate of a set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator conditions* (Items v.–viii. of Cor. 3.16).

Evidently, Items i., ii., iii. and iv. of Cor. 3.15 state that the δ^{th}-iterate of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\varnothing$*-grounded* (alternatively, $\varnothing$*-preserving*), ξ*-invariant* (alternatively, ξ*-unaffected*), $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$*-intensive* and $\cup$*-additive* (alternatively, $\cup$*-distributive*), respectively. On the other hand, Items i., ii., iii. and iv. of Cor. 3.16 state that the δ^{th}-iterate of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω*-grounded* (alternatively, Ω*-preserving*), ζ*-invariant* (alternatively, ζ*-unaffected*), $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$*-extensive* and $\cap$*-additive* (alternatively, $\cap$*-distributive*), respectively.

Viewing the δ^{th}-order derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator conditions (Items i.–iv. of Cor. 3.15 above) as δ^{th}*-order* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator axioms*, the axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator, then, can be defined as a δ^{th}-order set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of δ^{th}*-order* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived operator axioms.* The axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in strong $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

Definition 3.17 (*Axiomatic Definition:* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Derived Operator*)**.** The δ^{th}-iterate $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ of $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator of δ^{th} order" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ for some ordinal δ such that $1 \leqslant \delta < \lambda$ if and only if, for any $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies each "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axiom" in $\text{AX}\big[\mathfrak{g}\text{-DE}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \stackrel{\text{def}}{=} \big\{\text{Ax}_{\text{DE},\nu}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) : \nu \in I_4^*\big\}$, where the mapping $\text{Ax}_{\text{DE},\nu} : \mathfrak{g}\text{-DE}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \stackrel{\text{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

- $\text{Ax}_{\text{DE},1}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\varnothing) = \varnothing$
- $\text{Ax}_{\text{DE},2}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$
- $\text{Ax}_{\text{DE},3}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$
- $\text{Ax}_{\text{DE},4}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}\big) \stackrel{\text{def}}{\longleftrightarrow} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}})$

Similarly, viewing the δ^{th}-order derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator conditions (Items i.–iv. of Cor. 3.16 above) as δ^{th}*-order* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axioms*, the axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator, then, can be defined as a δ^{th}-order set-valued map $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axioms.* The axiomatic definition of the concept of a δ^{th}-order $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator in $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

Definition 3.18 (*Axiomatic Definition:* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Coderived Operator*)**.** The δ^{th}-iterate $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longmapsto \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}})$ of $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator of δ^{th} order" on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ for some ordinal δ such that $1 \leqslant \delta < \lambda$ if and only if, for any $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies each "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axiom" in

$\mathrm{AX}\big[\mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \overset{\mathrm{def}}{=} \big\{\mathrm{Ax}_{\mathrm{CD},\nu}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) : \nu \in I_4^*\big\}$, where the mapping $\mathrm{Ax}_{\mathrm{CD},\nu}$: $\mathfrak{g}\text{-}\mathrm{CD}^{(\delta)}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \overset{\mathrm{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

– $\mathrm{Ax}_{\mathrm{CD},1}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\Omega) = \Omega$

– $\mathrm{Ax}_{\mathrm{CD},2}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}\big)$

– $\mathrm{Ax}_{\mathrm{CD},3}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$

– $\mathrm{Ax}_{\mathrm{CD},4}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}\big) \overset{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{U}_{\mathfrak{g}})$

On the essential properties of the δ^{th}-order derivative $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators defined by transfinite recursion on the class of successor ordinals in $\mathscr{T}_{\mathfrak{g}}$-spaces, the discussion of the present section terminates here.

4. Discussion

4.1. Categorical and Ordinal Classifications. In the present section, based on the notions of *coarseness* (or, *smallness*, *weakness*), or alternatively, *finness* (or, *largeness*, *strongness*), the various relationships amongst the $\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$-derived and $\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$-coderived operators

$$\begin{cases} \mathrm{der}_{\mathfrak{a}}^{(\delta)},\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\delta)} \\ \mathrm{cod}_{\mathfrak{a}}^{(\delta)},\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\delta)} \end{cases} : \begin{array}{ccc} \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} & \longmapsto & \begin{cases} \mathrm{der}_{\mathfrak{a}}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}),\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}) \\ \mathrm{cod}_{\mathfrak{a}}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}),\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}) \end{cases} \end{array} \tag{4.1}$$

are established in $\mathscr{T}_{\mathfrak{a}}$-spaces ($\mathfrak{a} \in \{\mathfrak{o},\mathfrak{g}\}$) with respect to their category $\nu \in I_3^0$ and their ordinal $\delta \in [o] = \{\delta : 1 \preccurlyeq \delta \prec \lambda\}$, taking into account the required properties of the corresponding $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators established in $\mathscr{T}_{\mathfrak{a}}$-spaces ($\mathfrak{a} \in \{\mathfrak{o},\mathfrak{g}\}$) in a recent paper [1].

For illustrative purposes, the discussion will be furnished by $\big(\mathfrak{T}_{\mathfrak{a}}^{(\delta)}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}\big)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*derived operators* and $\big(\mathfrak{T}_{\mathfrak{a}}^{(\delta)}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}\big)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*coderived operators diagrams.* For clarity, the notations $\mathfrak{T} = (\Omega, \mathscr{T})$, der, $\mathfrak{g}$-Der, cod, $\mathfrak{g}$-Cod, …, $\mathrm{der}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}^{(\delta)}$, $\mathrm{cod}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}^{(\delta)}$, … will be considered instead of $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$, $\mathrm{der}_{\mathfrak{o}}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{o}}$, $\mathrm{cod}_{\mathfrak{o}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{o}}$, …, $\mathrm{der}_{\mathfrak{o}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{o}}^{(\delta)}$, $\mathrm{cod}_{\mathfrak{o}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{o}}^{(\delta)}$, …, respectively, or both will be considered interchangeably.

In a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}}) \supseteq (\Omega, \mathscr{T}_{\mathfrak{o}}) = \mathfrak{T}_{\mathfrak{o}}$, the so-called $\big(\mathfrak{T}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}\big)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*coderived sets diagram* [*See* [1]: Diag. (4.1), §§ 4.1, p. 213.]

$$\begin{array}{ccccccc}
\mathrm{cod}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathrm{cod}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathrm{cod}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathrm{cod}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_0(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_1(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_3(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Cod}_2(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\
\mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

as well as the so-called $\left(\mathfrak{T}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}\right)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*derived sets diagram* [*See* [1]: DIAG. (4.2), §§ 4.1, p. 214.]

$$\begin{array}{ccccccc}
\operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathfrak{g}\text{-}\mathrm{Der}_{0}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Der}_{1}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Der}_{3}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Der}_{2}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} \\
\operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{array}$$

Let it be granted some pair $(\nu,\mu) \in I_3^0 \times I_3^0$ of categories and some pair of ordinals $(\delta,\eta) \in [o] \times [o]$. Suppose the relations $\begin{cases} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\eta)} \precsim \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\mu}^{(\delta)} \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\eta)} \succsim \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\mu}^{(\delta)} \end{cases}$ stand for $\begin{cases} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}) \end{cases}$ or equivalently, $\begin{cases} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\mu}^{(\delta)} \succsim \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\eta)} \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)} \precsim \mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)} \end{cases}$ stand for $\begin{cases} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{a}}) \\ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\mu}^{(\delta)}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\eta)}(\mathscr{S}_{\mathfrak{a}}), \end{cases}$ respectively, in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$. Then, $\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\nu}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\nu}^{(\eta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}$,$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\eta)}$,$\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\mu}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\eta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In view of the above descriptions, for any pair $(\delta,\eta) \in [o] \times [o]$, the following $\left(\mathfrak{T}_{\mathfrak{a}}^{(\delta)}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}\right)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*coderived operators diagram*, which is to be read horizontally, from left to right and vertically, from top to bottom, presents itself:

$$\begin{array}{ccccccc}
\mathrm{cod}^{(\eta)} & \precsim & \mathrm{cod}^{(\eta)} & \precsim & \mathrm{cod}^{(\eta)} & \succsim & \mathrm{cod}^{(\eta)} \\
\rotatebox{90}{$\precsim$} & & \rotatebox{90}{$\precsim$} & & \rotatebox{90}{$\precsim$} & & \rotatebox{90}{$\precsim$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_{0}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{1}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{3}^{(\eta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Cod}_{2}^{(\eta)} \\
\rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_{0}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{1}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{3}^{(\delta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Cod}_{2}^{(\delta)} \\
\rotatebox{90}{$\precsim$} & & \rotatebox{90}{$\precsim$} & & \rotatebox{90}{$\precsim$} & & \rotatebox{90}{$\precsim$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}^{(\delta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}^{(\delta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}^{(\delta)} \\
\rotatebox{90}{$\precsim$} & & \rotatebox{90}{$\precsim$} & & \rotatebox{90}{$\precsim$} & & \rotatebox{90}{$\precsim$} \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}^{(\eta)} & \precsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}^{(\eta)} & \succsim & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}^{(\eta)} \\
\rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} & & \rotatebox{-90}{$\precsim$} \\
\mathrm{cod}_{\mathfrak{g}}^{(\eta)} & \precsim & \mathrm{cod}_{\mathfrak{g}}^{(\eta)} & \precsim & \mathrm{cod}_{\mathfrak{g}}^{(\eta)} & \succsim & \mathrm{cod}_{\mathfrak{g}}^{(\eta)}
\end{array} \tag{4.2}$$

On the other hand, for any pair $(\delta,\eta)\in[o]\times[o]$, the following $\left(\mathfrak{T}_{\mathfrak{a}}^{(\delta)},\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}\right)_{\mathfrak{a}=\mathfrak{o},\mathfrak{g}}$-*derived operators diagram*, which is to be read horizontally, from left to right and vertically, from top to bottom, also presents itself:

$$
\begin{array}{ccccccc}
\mathrm{der}^{(\eta)} & \gtrsim & \mathrm{der}^{(\eta)} & \gtrsim & \mathrm{der}^{(\eta)} & \lesssim & \mathrm{der}^{(\eta)} \\
\wr\vee & & \wr\vee & & \wr\vee & & \wr\vee \\
\mathfrak{g}\text{-}\mathrm{Der}_{0}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{1}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{3}^{(\eta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Der}_{2}^{(\eta)} \\
\wr\wedge & & \wr\wedge & & \wr\wedge & & \wr\wedge \\
\mathfrak{g}\text{-}\mathrm{Der}_{0}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{1}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{3}^{(\delta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Der}_{2}^{(\delta)} \\
\wr\vee & & \wr\vee & & \wr\vee & & \wr\vee \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}^{(\delta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}^{(\delta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}^{(\delta)} \\
\wr\vee & & \wr\vee & & \wr\vee & & \wr\vee \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}^{(\eta)} & \gtrsim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}^{(\eta)} & \lesssim & \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}^{(\eta)} \\
\wr\wedge & & \wr\wedge & & \wr\wedge & & \wr\wedge \\
\mathrm{der}_{\mathfrak{g}}^{(\eta)} & \gtrsim & \mathrm{der}_{\mathfrak{g}}^{(\eta)} & \gtrsim & \mathrm{der}_{\mathfrak{g}}^{(\eta)} & \lesssim & \mathrm{der}_{\mathfrak{g}}^{(\eta)}.
\end{array} \tag{4.3}
$$

The relationships amongst the $\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$-derived operators $\mathrm{der}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ and the $\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{a}}^{(\delta)}$-coderived operators $\mathrm{cod}_{\mathfrak{a}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, are, therefore, established in $\mathscr{T}_{\mathfrak{a}}$-spaces $(\mathfrak{a}\in\{\mathfrak{o},\mathfrak{g}\})$ with respect to their category $\nu\in I_3^0$ and their ordinal $\delta\in[o]$.

4.2. **A Nice Application.** It is the intent of the present section to present a nice application, highlighting some essential properties of the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operators $\mathrm{der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ and $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators $\mathrm{cod}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\delta)}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space with respect to their category $\nu\in I_3^0$ and their ordinal $\delta\in[o]$.

In considering the same $\mathscr{T}_{\mathfrak{g}}$-space upon which a nice application was presented in a recent paper [1, §§ 4.2], namely the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ based on the 7-point set $\Omega=\left\{\xi_\nu:\ \nu\in I_7^*\right\}$, and the latter topologized by the choice:

$$
\begin{aligned}
\mathscr{T}_{\mathfrak{g}}(\Omega) &= \left\{\varnothing,\ \{\xi_1\},\ \{\xi_1,\xi_3,\xi_5\},\ \{\xi_1,\xi_3,\xi_4,\xi_5,\xi_7\}\right\} \\
&= \left\{\mathscr{O}_{\mathfrak{g},1},\ \mathscr{O}_{\mathfrak{g},2},\ \mathscr{O}_{\mathfrak{g},3},\ \mathscr{O}_{\mathfrak{g},4}\right\} \\
\neg\mathscr{T}_{\mathfrak{g}}(\Omega) &= \left\{\Omega,\ \{\xi_2,\xi_3,\xi_4,\xi_5,\xi_6,\xi_7\},\ \{\xi_2,\xi_4,\xi_6,\xi_7\},\ \{\xi_2,\xi_6\}\right\} \\
&= \left\{\mathscr{K}_{\mathfrak{g},1},\ \mathscr{K}_{\mathfrak{g},2},\ \mathscr{K}_{\mathfrak{g},3},\ \mathscr{K}_{\mathfrak{g},4}\right\}
\end{aligned}
$$

with $\mathscr{R}_{\mathfrak{g}}=\left\{\xi_1,\xi_2,\xi_4\right\}$, $\mathscr{S}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}\cup\left\{\xi_7\right\}$, $\mathscr{U}_{\mathfrak{g}}=\left\{\xi_3,\xi_5,\xi_6,\xi_7\right\}$, and $\mathscr{V}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}}\smallsetminus\left\{\xi_3\right\}$, it was shown through calculations [*See* [1]: SYS. OF EQS (4.11), §§ 4.2, p. 216.] that the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ on the

$\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, result in:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},4} & \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},1} & \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},4} & \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},1} & \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

implying

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},0}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},1}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},3}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},2}(\mathscr{W}_{\mathfrak{g}}) \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},0}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},1}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},3}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},2}(\mathscr{Y}_{\mathfrak{g}}) \end{cases}$$

for each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}$ [*See* [1]: Sys. of Eqs (4.13), §§ 4.2, p. 216.]. It was also shown through calculations [*See* [1]: Sys. of Eqs (4.12), §§ 4.2, p. 216.] that the $\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ result in:

$$\begin{cases} \mathrm{der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},2} & \forall \mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathrm{cod}_{\mathfrak{g}}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},2} & \forall \mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

implying

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ [*See* [1]: Sys. of Eqs (4.14), §§ 4.2, p. 216.].

Consider again the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}} = \{\xi_1,\xi_2,\xi_4\}$, $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \{\xi_7\}$, $\mathscr{U}_{\mathfrak{g}} = \{\xi_3,\xi_5,\xi_6,\xi_7\}$, and $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}} \setminus \{\xi_3\}$. Then, for any $\delta \in [o]$, the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operation of $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operation of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, produce the following results:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},4} & \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},1} & \forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},4} & \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},1} & \forall (\nu,\mathscr{Y}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \end{cases} \tag{4.4}$$

Likewise, for any $\delta \in [o]$, the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operation of $\mathrm{der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operation of $\mathrm{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, also produce the following results:

$$\begin{cases} \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},2} & \forall \mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},2} & \forall \mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \end{cases} \tag{4.5}$$

By virtue of Sys. of Eqs (4.4), it follows that

$$\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\lambda)}(\mathscr{W}_{\mathfrak{g}}) = \bigcap_{\delta<\lambda} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) = \varnothing \;\forall (\nu,\mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \tag{4.6}$$

Again, by virtue of SYS. OF EQS (4.4), it also follows that

$$(4.7)\quad \begin{cases} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\lambda)}(\mathscr{W}_{\mathfrak{g}}) & = \bigcap_{\delta<\lambda} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \neq \varnothing \quad \forall\,(\nu,\mathscr{W}_{\mathfrak{g}}) \in \{0,2\}\times\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\lambda)}(\mathscr{Y}_{\mathfrak{g}}) & = \bigcap_{\delta<\lambda} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \neq \varnothing \quad \forall\,(\nu,\mathscr{Y}_{\mathfrak{g}}) \in I_3^0\times\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \\ \mathrm{der}_{\mathfrak{g}}^{(\lambda)}(\mathscr{W}_{\mathfrak{g}}) & = \bigcap_{\delta<\lambda} \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \neq \varnothing \quad \forall\,(\nu,\mathscr{W}_{\mathfrak{g}}) \in I_3^0\times\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathrm{cod}_{\mathfrak{g}}^{(\lambda)}(\mathscr{Y}_{\mathfrak{g}}) & = \bigcap_{\delta<\lambda} \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \neq \varnothing \quad \forall\,(\nu,\mathscr{Y}_{\mathfrak{g}}) \in I_3^0\times\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

Hence, for any $\delta \in [o]$, it results that the following results hold true for each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\}$:

$$(4.8)\quad \begin{cases} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \end{cases}$$

For any $\delta \in [o]$, the $(\precsim,\succsim)$-relations $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},0}^{(\delta)} \succsim \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},1}^{(\delta)} \succsim \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},3}^{(\delta)} \precsim \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},2}^{(\delta)}$ and $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}^{(\delta)} \precsim \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}^{(\delta)} \precsim \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}^{(\delta)} \succsim \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}^{(\delta)}$ are thus verified. Clearly, for any $\delta \in [o]$, the following results also hold true:

$$(4.9)\quad \begin{cases} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad \forall\,(\nu,\mathscr{W}_{\mathfrak{g}}) \in I_3^0\times\{\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\} \\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\delta)}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{W}_{\mathfrak{g}}) \quad \forall\,(\nu,\mathscr{Y}_{\mathfrak{g}}) \in I_3^0\times\{\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}\} \end{cases}$$

Thus, the $(\precsim,\succsim)$-relations $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)} \precsim \mathrm{der}_{\mathfrak{g}}^{(\delta)}$ and $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\delta)} \succsim \mathrm{cod}_{\mathfrak{g}}^{(\delta)}$, for all $\nu \in I_3^0$, are also verified.

The presentation of this nice application, highlighting some essential properties of the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operators $\mathrm{der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators $\mathrm{cod}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space with respect to their category $\nu \in I_3^0$ and their ordinal $\delta \in [o]$ are, therefore, accomplished and ends here.

If the presentation be explored a step further, other interesting properties can be deduced from the study of other essential properties of $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operators and $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

5. Conclusion

In a recent paper [1], we introduced the definitions and studied the essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces. Mainly, we showed that $(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}) : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ is a pair of both *dual and monotone* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* that is $(\varnothing,\Omega)$, $(\cup,\cap)$-*preserving*, and $(\subseteq,\supseteq)$-*preserving* relative to $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(open, closed) sets [*See* [1]: CORS 3.15 & 3.16, §§ 2.2, p. 198.]. We also showed that $\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ is a pair of *weaker* and *stronger* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* [*See* [1]: THM. 3.1, §§ 2.2, p. 187.]. In the present paper, we have introduced by transfinite recursion on the class of successor ordinals the definitions and investigated the essential of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces [*See* § 3: THMS 3.1–3.13; CORS 3.2–3.16; PROPS 1–4; DEFS 3.17 & 3.18].

The following three statements sum up the outstanding facts resulting from the investigation of the essential of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces:

– I. For any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, $\langle \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \rangle_{\delta \in [o]}$ is a monotone decreasing sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived sets while $\langle \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \rangle_{\delta \in [o]}$ is a monotone increasing sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ [*See* § 3: Thm. 3.1 & Cor. 3.2].

– II. The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operator $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *weaker* than the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operator $\text{der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ while the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operator $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *stronger* than the $\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operator $\text{cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ [*See* § 3: Prop. 1; Cors 3.3 & 3.4].

– III. For any $(\{\xi\}, \mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{a}}]$, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived operator $\mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\varnothing$*-grounded* (alternatively, $\varnothing$*-preserving*), ξ*-invariant* (alternatively, ξ*-unaffected*), $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$*-intensive* and $\cup$*-additive* (alternatively, $\cup$*-distributive*) [*See* § 3: Cor. 3.15: Items I.–IV.] while the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operator $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω*-grounded* (alternatively, Ω*-preserving*), ξ*-invariant* (alternatively, ξ*-unaffected*), $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$*-extensive* and $\cap$*-additive* (alternatively, $\cap$*-distributive*) [*See* § 3: Cor. 3.16: Items I.–IV.] in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

Hence, it follows that the study of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators $\begin{array}{rl} \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}, \ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)} : & \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}), \ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)}(\mathscr{S}_{\mathfrak{g}}) \end{array}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ has resulted in several advantages. Indeed, it has resulted in axiomatic definitions of these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ [*See* § 3: Defs 3.17 & 3.18]. The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\delta)}$-coderived structures $\mathfrak{D}_{\mathfrak{g}}^{(\delta)} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-Der}_{\mathfrak{g}}^{(\delta)})$ and $\mathfrak{C}_{\mathfrak{g}}^{(\delta)} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}^{(\delta)})$, then, are both themselves $\mathscr{T}_{\mathfrak{g}}$-spaces which may well be called $\mathscr{T}_{\mathfrak{g},\text{der}}^{(\delta)}$, $\mathscr{T}_{\mathfrak{g},\text{cod}}^{(\delta)}$*-spaces*, respectively. Accordingly, if Cantor [23, 24] had also considered the $\mathfrak{T}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}$-derived operator $\begin{array}{rl} \text{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)} : & \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R}) \\ & \mathscr{S}_{\mathfrak{o}} \longmapsto \text{der}_{\mathfrak{o}|\mathbb{R}}^{(\delta)}(\mathscr{S}_{\mathfrak{o}}) \end{array}$ in his investigations of the convergence of Fourier series in $\mathbb{R}$, then the study of convergence in any of the $\mathscr{T}_{\mathfrak{g},\text{der}}^{(\delta)}$, $\mathscr{T}_{\mathfrak{g},\text{cod}}^{(\delta)}$-spaces $\mathfrak{D}_{\mathfrak{g}}^{(\delta)}$, $\mathfrak{C}_{\mathfrak{g}}^{(\delta)}$, respectively, might be made another subject of inquiry. The discovery of properties in this direction would definitely bring some benefits to the field of Mathematical Analysis, and the discussion of this paper ends here.

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