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Some Data Dependence Results From Using $\mathscr{C}$-Class Functions in Partial Metric Spaces

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Abstract

This research paper examines the data dependence of fixed point sets for pseudo-contractive multifunctions in partial metric spaces using the notion of $\mathscr{C}$-class functions. By building upon previous findings from the literature, this work sheds more light on some new perspectives as well as generalizations on this issue. To illustrate how the $\mathscr{C}$-class function can be applied to study the data dependence of fixed point sets for a certain pseudo-contractive multifunction, an illustrative example is given.

1. Introduction

The fixed point theory is a powerful tool with numerous applications in various fields such as biology, chemistry, economics, engineering, game theory, computer science, and mathematical modeling [1, 2]. Recent developments in fixed point theory have focused on extending classical results to more general abstract spaces like b-metric spaces, partial metric spaces, and fuzzy metric spaces [3]. One notable advancement is the introduction of $\mathscr{C}$-class functions [4], which have been used to prove fixed point theorems in different abstract spaces, particularly in the context of partial metric spaces. $\mathscr{C}$-class functions provide a unified framework for studying various types of contractions and have applications in solving differential equations, integral equations, and variational inequalities [5, 6].
The study of partial metric spaces began in 1994, when Matthews introduced this generalization of traditional metric spaces in [7]. Since then, partial metric spaces have found applications across many fields because of their ability to represent asymmetric distance relationships [8]. During the same period, there was significant progress in multivalued mapping research, which greatly contributed to the development of generalized fixed point theory [9–11]. Multifunctions emerged as a natural approach to addressing problems that involved non-uniqueness or set-valued constraints. Initially, the focus was on establishing fixed point results for multifunctions defined on traditional metric and topological spaces.
However, as people became more interested in using these methods in real-life situations modeled by partial metrics, it became necessary to come up with new ways to think about data dependence for multifunctions in this new setting [12]. Data dependence properties play a crucial role in examining how perturbations in the domain affect or propagate to the range sets. In the case of single-valued mappings on metric spaces, classic results have established strong connections between input and output distances (see [13–17]).
However, for multifunctions whose domain and range reside in different partial metric spaces, new approaches were required. Researchers created the partial Hausdorff metric [18] to measure the distance between nonempty subsets using the basic partial metrics. This allowed generalizing key notions like continuity, contraction properties, and more.
In the beginning, researchers came up with the weak contraction and fixed point theorems for multifunctions that behave in certain ways when contracted with respect to the induced partial Hausdorff metric. The mapping had fixed points if the partial Hausdorff distance between images of any two points satisfied a Lipschitz-type condition based on their domain distance.
Today, data dependence results for multifunctions defined in partial and more exotic spaces remain an active area of research. Future directions include investigating new contraction conditions, establishing fixed point theorems for alternative structures, and discovering additional applications inspired by practical problems. Overall, the field has grown significantly since its inception, broadening the scope of generalized fixed point theory.

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The second part of this research paper provides a summary of partial metric spaces, $\mathscr{C}$-class functions, and existing results. The primary emphasis is on the significant contributions made by [12].
This section establishes a solid foundation for our main result, which is outlined in Section 3. Through $\mathscr{C}$-class functions, we aim to enhance our understanding of data dependence in partial metric spaces. The final section deals with the implications of our main result, thereby giving us a better understanding of data dependence in this context.

2. Preliminaries

To fully understand the complexities of data dependence in partial metric spaces, one needs a strong foundation. We propose the concept of partial metric spaces according to Matthews's 1994 study on partial metrics [7]. These metric spaces intriguingly extend to non-zero self-distances.
As a result, let us proceed to review the essential characteristics and definitions of partial metric spaces.

Definition 2.1. *[7] The function $p: X\times X\to\mathbb{R}^+$ defines a partial metric on a nonempty set X, where $\mathbb{R}^+$ includes all nonnegative real numbers. If the following four conditions are satisfied for every $x,y,z\in X$, we call the pair (X,p) a partial metric space:*

$\mathfrak{P}_1$: $p(x,x)=p(y,y)=p(x,y)\Leftrightarrow x=y$.
$\mathfrak{P}_2$: $p(x,x)\leqslant p(x,y)$.
$\mathfrak{P}_3$: $p(x,y)=p(y,x)$
$\mathfrak{P}_4$: $p(x,y)+p(z,z)\leqslant p(x,z)+p(z,y)$.

The partial metric space represented by the pair (X,p).
We define the concept of closed p-balls, $\overline{\mathbb{B}_{p,r}}(x)$, and the open p-balls, $\mathbb{B}_{p,r}(x)$, to simplify our analysis. These sets are defined as

$$\overline{\mathbb{B}_{p,r}}(x)=\{y\in X|p(x,y)\leqslant p(x,x)+r\},\qquad \mathbb{B}_{p,r}(x)=\{y\in X|p(x,y)<p(x,x)+r\}.$$

We denote the full space X as $\overline{\mathbb{B}_{p,+\infty}}(x)$ to keep things simple. By using this notation, we can express important ideas and claims about p-distance thresholds throughout the whole domain X in a more concise and accurate way.
The metric that is related to p, which is a partial metric on X, can be expressed as a new function $p^s: X\times X\to\mathbb{R}^+$. This formula may be used to get the metric p^s:

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y).$$

This metric p^s satisfies all the properties of a metric space: nonnegativity, symmetry, and triangle inequality. Therefore, while p is only a partial metric, the associated metric p^s transforms the partial metric space (X,p) into an actual metric space (X,p^s). We have laid the necessary groundwork to rigorously examine the concept of data dependence within partial metric spaces, a topic we will now explore through theoretical analysis. Let (X,p) be a partial metric space. The following properties hold:

- If $p(x,x)=\lim_{n\to+\infty}p(x,x_n)$, then $\{x_n\}$ is said to converge to a point $x\in X$.
- If a sequence $\{x_n\}$ has a finite limit as n and m approach infinity, it is termed a Cauchy sequence.
- If each Cauchy sequences $\{x_n\}$ in X converge to a point $x\in X$ such that $p(x,x)=\lim_{n,m\to+\infty}p(x_n,x_m)$, then the partial metric space (X,p) is complete.

Consider the collection $C^p(X)$, which represents all nonempty closed subsets of the partial metric space (X,p). In this framework, we introduce the following definitions for $x\in X$ and $A,B\in C^p(X)$:

$$\begin{aligned}H_p(A,B)&=\max\{\delta_p(A,B),\delta_p(B,A)\},\\&=\max\{\sup\{p(a,B)\mid a\in A\},\sup\{p(b,A)\mid b\in B\}\},\end{aligned}$$

such that

$$p(x,A)=\inf\{p(x,a)\mid a\in A\}.$$

Following the established conventions

$$p(x,\emptyset)=+\infty,\quad \delta_p(\emptyset,B)=0. \tag{2.1}$$

Lemma 2.2. *[5, 19] In a partial metric space (X,p) with $A\subset X$, the equivalence relation $a\in\overline{A}\Leftrightarrow p(a,A)=p(a,a)$ holds. Additionally, $p(a,a)=0$ and $a\in\overline{A}\Leftrightarrow p(a,A)=0$, in which $\overline{A}$ represents the closure of A relative to the partial metric p.*

Lemma 2.3. *[20] Consider $x\in X$ and $A\in C^p(X)$ in a partial metric space (X,p). If $\mu>0$ and $p(x,A)<\mu$, we can find that there is an element a in A such that $p(x,a)<\mu$.*

Furthermore, we introduce the intervals J and J' on the nonnegative real numbers, which include the value 0. These intervals can take the form of $[0,a[$, $[0,a]$, or $[0,+\infty[$, where a represents a nonnegative real number.
The following notations are used for a multivalued mapping $T: X\to 2^X$, where 2^X represents any nonempty subsets of X.

- $Fix(T)=\{x\in X|x\in T(x)\}$.
- $M_T(x,y)=\max\left\{p(x,y),p(x,T(x)),p(y,T(y)),\dfrac{p(x,T(y))+p(y,T(x))}{2}\right\}$.

Definition 2.4. *[21] On the interval J, a (c)-comparison function or a Bianchini-Grandolfi gauge function is defined as a non-decreasing function $\varphi : J \to J$ that satisfies the condition:*

$$\mathfrak{s}(t) := \sum_{n=0}^{\infty} \varphi^n(t) \text{ is convergent, for all } t \in J,$$

where φ^n represents the n-th iteration of the function φ and $\varphi^0(t) = t$, i.e.,

$$\varphi^0(t) = t, \varphi^1(t) = \varphi(t), \varphi^2(t) = \varphi(\varphi(t)), \ldots, \varphi^n(t) = \varphi(\varphi^{n-1}(t)).$$

By utilizing Bianchini-Grandolfi gauge functions, we gain a more nuanced understanding of data dependence within partial metric spaces. This allows for a thorough examination of the interconnections between space elements. A theorem has been proven building on prior work, particularly the impactful results of [12]. The theorem elucidates the importance of these relationships identified through application of gauge functions, furthering the theoretical foundations of data dependence within this structure.

Theorem 2.5. *Consider a partial metric space (X, p), with $\bar{x} \in X$, $\lambda \in [0,1]$, and $r > 0$, satisfying the condition that the subspace $\overline{\mathbb{B}_{p,r}}(\bar{x})$ is complete. Let T and F be multivalued mappings from $\overline{\mathbb{B}_{p,r}}(\bar{x})$ to $C^p(X)$. Additionally, let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and upper semicontinuous function, serving as a (c)-comparison function on the interval J. Under the assumption that there exists $\alpha \in J$ satisfying the following two conditions:*

(a) $p(z, F(z)) < \alpha$ *where* $\mathfrak{s}(\alpha) \leqslant (1-\lambda)r, \quad \forall z \in \overline{\mathbb{B}_{p,\lambda r}}(\bar{x})$.
(b) $\delta_p(F(x) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(y)) \leqslant \varphi(M_F(x,y)), \quad \forall x, y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any $K \subseteq \overline{\mathbb{B}_{p,r}}(\bar{x})$, we have

$$\delta_p(Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, Fix(F)) \leqslant \mathfrak{s}(\mathfrak{M}'),$$

where $\mathfrak{M}' := \sup_{x \in \overline{\mathbb{B}_{p,r}}(\bar{x})} \delta_p(T(x) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, F(x))$.

This theorem extends several results within the framework of partial metric spaces. Specifically, it expands upon the findings of:

- Azé et al., who presented Proposition 2.1 in their work [13].
- Lim, who presented Lemma 1 in their work [14].
- Geoffroy et al., who presented Proposition 4.5 in their work [22].
- Mansour et al., who presented Theorem 14 in their work [23].

Theorem 2.5 builds upon and generalizes prior work in the area of partial metric spaces.
Ansari's work in [4] proposed $\mathscr{C}$-class functions which have gone a long way in advancing our understanding and analysis of many mathematical phenomena. According to [6], the idea is useful for generalizing important results in fixed point theory. It is more comprehensive than the gauge function by Bianchini-Grandolfi.
Through the use of $\mathscr{C}$-class functions given by knowledge and structure, we can get deeper insights into, and navigate through the complexities of the issue at hand. Indeed, Ansari's contributions are invaluable as they continue shaping and inspiring further research on this subject area thereby leaving a lasting impact on the field of study.

Definition 2.6 ($\mathscr{C}$-class functions)**.** *[4, 5] Assume that there is a continuous mapping $\mathfrak{F} : J \times J' \to \mathbb{R}$. If $\mathfrak{F}$ satisfies these requirements, we will classify it as a $\mathscr{C}$-class function.*

($\mathfrak{F}_1$) *For any $(s,t) \in J \times J'$, we have $s \geqslant \mathfrak{F}(s,t)$.*
($\mathfrak{F}_2$) *If $\mathfrak{F}(s,t) = s$, then the product $st = 0$.*

In addition, note that $\mathfrak{F}(0,0) = 0$ and that $\mathscr{C}$ is the set of all functions of the $\mathscr{C}$-class on $J \times J'$.

In the work [5], the authors introduced the following collections of $\mathscr{C}$-class functions:

Definition 2.7. *[5] The set of functions of the $\mathscr{C}$-class that satisfy these criteria is called $\mathscr{C}_I$:*

- *$\mathfrak{F}(s,t)$ is non-decreasing for both s and t when $(s,t) \in J \times J'$.*
- *For any fixed $t \in J'$, the series*

  $$\widetilde{\mathfrak{w}}(s,t) := \sum_{n=0}^{\infty} \mathfrak{F}^n(s,t)$$

  converges for all $s \in J$. The function $\mathfrak{F}$ is defined as follows, and $\mathfrak{F}^n$ represents the n-th iteration of this function:

  $$F^0(s,t) = s, \mathfrak{F}^1(s,t) = \mathfrak{F}(s,t), and \mathfrak{F}^{n+1}(s,t) = \mathfrak{F}(\mathfrak{F}^n(s,t),t).$$

Definition 2.8. *[5] $\mathscr{C}_{II}$ comprises a set of $\mathscr{C}$-class functions that adhere to the following specifications:*

- *$\mathfrak{F}(s,t)$ exhibits non-decreasing behavior in s and non-increasing behavior in t.*
- *For any given $t \in J'$, the series*

  $$\widetilde{\mathfrak{w}}(s,t) := \sum_{n=0}^{\infty} \mathfrak{F}^n(s,t)$$

  converges for every $s \in J$, where the n-th iteration of the function $\mathfrak{F}$ with the following recurrence relation is represented as $\mathfrak{F}^n$:

  $$\mathfrak{F}^0(s,t) = s, \mathfrak{F}^1(s,t) = \mathfrak{F}(s,t), and \mathfrak{F}^{n+1}(s,t) = \mathfrak{F}(\mathfrak{F}^n(s,t), \mathfrak{F}^n(s,t))$$

Here are some examples of functions belonging to both $\mathscr{C}_I$ and $\mathscr{C}_{II}$, as presented in Ansari et al. [5]. These examples illustrate the definitions provided in Definition 2.7 and Definition 2.8.

Example 2.9.
- *Given the functions* $\mathfrak{F}(s,t) = s - t$ *and* $\widetilde{\mathfrak{w}}(s,t) = 2s - t$, *it can be concluded that* $\mathfrak{F} \in \mathscr{C}_{II}$.
- *For* $\mathfrak{F}(s,t) = \lambda s$ *with* $\lambda \in [0,1)$, $\widetilde{\mathfrak{w}}(s,t) = \dfrac{s}{1-\lambda}$ *is derived. Hence,* $\mathfrak{F}$ *belongs to* $\mathscr{C}_I \cap \mathscr{C}_{II}$.
- *Since* $\widetilde{\mathfrak{w}}(s,t) = \mathfrak{s}(s)$ *and* φ *is a (c)-comparison function on J,* $\mathfrak{F}$ *belongs to* $\mathscr{C}_I \cap \mathscr{C}_{II}$.
- *Given* $\mathfrak{F}(s,t) = \dfrac{s^2}{2\sqrt{s^2+a^2}}$ *with* $a \geqslant 0$, *the corresponding transformation is* $\widetilde{\mathfrak{w}}(s,t) = s + \sqrt{s^2+a^2} - a$ *for* $s,t \geqslant 0$. *Hence,* $\mathfrak{F}$ *lies in* $\mathscr{C}_I \cap \mathscr{C}_{II}$.
- *When* $\mathfrak{F}(s,t) = st^k$ *with* $k > 1$, *the corresponding transformation is* $\widetilde{\mathfrak{w}}(s,t) = \dfrac{s}{1-t^k}$. *Here,* $\mathfrak{F}$ *is categorized under* $\mathscr{C}_I$.

Remark 2.10. *If* $\mathfrak{F}$ *is a* $\mathscr{C}$*-class function in either* $\mathscr{C}_I$ *or* $\mathscr{C}_{II}$, *then the following functional equations are satisfied by the functions* $\widetilde{\mathfrak{w}}$ *and* $\mathfrak{F}$:
- *For* $\mathfrak{F} \in \mathscr{C}_I$:
$$\widetilde{\mathfrak{w}}(\mathfrak{F}(s,t),t) = \widetilde{\mathfrak{w}}(s,t) - s.$$
- *For* $\mathfrak{F} \in \mathscr{C}_{II}$:
$$\widetilde{\mathfrak{w}}(\mathfrak{F}(s,t),\mathfrak{F}(s,t)) = \widetilde{\mathfrak{w}}(s,t) - s.$$

A class of functions Ξ that were mentioned in [5] are recalled in the following. These functions, represented as $\tau : X^2 \times (2^X)^2 \to J'$, satisfy a crucial condition. To be more precise, for any $x,y \in X$ and $A,C \in 2^X$, $\tau(x,y,A,C) = 0$ implies that $x = y$ or $p(x,y) = 0$ is true. Additionally, we establish the nondecreasing property of $\tau \in \Xi$ within the (X,p) space, as indicated by the following inequality:

$$p(x,y) \leqslant p(a,b) \Rightarrow \tau(x,y,A_x,C_y) \leqslant \tau(a,b,A_a,C_b) \quad \forall A_x,A_a,C_y,C_b \in 2^X.$$

Example 2.11.
- $\tau(x,y,A,C) = \dfrac{p(x,y)}{1+\exp(-p(x,A)+p(y,C))}$,
- $\tau(x,y,A,C) = \log(1+p^s(x,y))$,
- $\tau(x,y,A,C) = p(x,y)^n$, *where n is a positive real number.*

3. Main results

The major finding of our research can be succinctly expressed as follows:

Theorem 3.1. *Consider a partial metric space* (X,p), *where* $\bar{x} \in X$, $\lambda \in [0,1]$, *and* $r > 0$ *such that the subspace* $\overline{\mathbb{B}_{p,r}}(\bar{x})$ *is complete. Let* $T,F : \overline{\mathbb{B}_{p,r}}(\bar{x}) \to C^p(X)$ *be multivalued mappings. Assuming that* $\tau \in \Xi$, $\alpha \in J$, *and* $\mathfrak{F} \in \mathscr{C}$, *which is upper semicontinuous with respect to the first variable, satisfy either of the following conditions:*
- $\mathfrak{F} \in \mathscr{C}_I$ *and* τ *is nondecreasing,*
- $\mathfrak{F} \in \mathscr{C}_{II}$ *and* $\tau(x,y,F(x),F(y)) \geqslant \alpha$ *for* $x,y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

We will establish our assumptions based on the satisfaction of the following two conditions:

(a) $p(z,F(z)) < \alpha$ *where* $\widetilde{\mathfrak{w}}(\alpha,\cdot) \leqslant r(1-\lambda)$, $\quad \forall z \in \overline{\mathbb{B}_{p,\lambda r}}(\bar{x})$.
(b) $\delta_p(F(x) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(y)) \leqslant \mathfrak{F}(M_F(x,y), \tau(x,y,F(x),F(y)))$, $\quad \forall x,y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any given $K \subseteq \overline{\mathbb{B}_{p,r}}(\bar{x})$, *and for every* $y \in Fix(T)$ *and* $w \in F(y)$, *satisfying* $p(y,w) < \alpha$, *we can establish the following inequality:*

$$\delta_p(Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, Fix(F)) \leqslant \widetilde{\mathfrak{w}}(\mathfrak{M}, \tau(y,w,F(y),F(w))) \tag{3.1}$$

where $\mathfrak{M} := \sup_{x \in \overline{\mathbb{B}_{p,r}}(\bar{x})} \delta_p(T(x) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, F(x))$.

Proof. If the quantity $\mathfrak{M} \in \{0,+\infty\}$, there is nothing to prove; therefore, we may assume that $0 < \mathfrak{M} < +\infty$. Moreover, if $Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K = \emptyset$, then according to the convention (2.1), we are finished.
So we assume that $Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K \neq \emptyset$ and we take $x_0 \in Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K$, i.e., $x_0 \in T(x_0) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K$.
Fix $\varepsilon \geqslant \varepsilon' > 1$ such that $\delta_p(T(x_0) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, F(x_0)) \leqslant \mathfrak{M} < \varepsilon\mathfrak{M}$. Thus, using (a), we have

$$p(x_0,F(x_0)) < \min\{\alpha, \varepsilon\mathfrak{M}\}.$$

According to Lemma 2.3, there exists $x_1 \in F(x_0)$ such that

$$p(x_0,x_1) < \min\{\alpha, \varepsilon\mathfrak{M}\}.$$

Moreover, $x_1 \in \overline{\mathbb{B}_{p,r}}(\bar{x})$, indeed,

$$\begin{aligned} p(x_1,\bar{x}) &\leqslant p(x_1,x_0) + p(x_0,\bar{x}) - p(x_0,x_0) \\ &\leqslant \alpha + \lambda r + p(\bar{x},\bar{x}) \\ &\leqslant \widetilde{\mathfrak{w}}(\alpha,\cdot) + \lambda r + p(\bar{x},\bar{x}) \\ &\leqslant (1-\lambda)r + \lambda r + p(\bar{x},\bar{x}) \\ &\leqslant p(\bar{x},\bar{x}) + r. \end{aligned}$$

If $x_1 = x_0$, then $x_0 \in T(x_0) \cap F(x_0)$, and subsequently for any $x_0 \in Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\overline{x}) \cap K$, we have

$$p(x_0, Fix(F)) \leqslant p(x_0, x_0) \leqslant \min\left\{\mathfrak{M}, \widetilde{\mathfrak{w}}\left(\mathfrak{M}, \tau(x_0, x_0, F(x_0), F(x_0))\right)\right\}.$$

This demonstrates that such x_0 satisfy inequality (3.1), since the distance between x_0 and the fixed point set $Fix(F)$ is positive, thereby fulfilling the requirement defined by inequality (3.1).
Moreover, for any $w \in F(x_1)$, we have

$$\begin{aligned} M_F(x_0,x_1) &= \max\left\{p(x_0,x_1), p(x_0,F(x_0)), p(x_1,F(x_1)), \frac{p(x_0,F(x_1)) + p(x_1,F(x_0))}{2}\right\} \\ &\leqslant \max\left\{p(x_0,x_1), p(x_1,w), \frac{p(x_0,w) + p(x_1,x_1)}{2}\right\} \\ &\leqslant \max\left\{p(x_0,x_1), p(x_1,w), \frac{p(x_0,x_1) + p(x_1,w)}{2}\right\} \\ &= \max\left\{p(x_0,x_1), p(x_1,w)\right\}. \end{aligned}$$

Assuming that $\max\{p(x_0,x_1), p(x_1,w)\} = p(x_1,w)$. Then, from condition (b), and the definition of $\mathscr{F}$, we derive a contradiction. Thus, we infer that $M_F(x_0,x_1) \leqslant p(x_0,x_1)$.
Given (b), we can deduce that

$$\begin{aligned} p(x_1,F(x_1)) &\leqslant \delta_p(F(x_0) \cap \overline{\mathbb{B}_{p,r}}(\overline{x}), F(x_1)) \\ &\leqslant \mathfrak{F}\left(M_F(x_0,x_1), \tau(x_0,x_1,F(x_0),F(x_1))\right) \\ &\leqslant \mathfrak{F}\left(p(x_0,x_1), \tau(x_0,x_1,F(x_0),F(x_1))\right) \\ &< \min\{\mathfrak{F}(\alpha, \tau(x_0,x_1,F(x_0),F(x_1)), \mathfrak{F}(\varepsilon'\mathfrak{M}, \tau(x_0,x_1,F(x_0),F(x_1))\} \\ &= \min\{\mathfrak{F}(\alpha,\tau_0), \mathfrak{F}(\varepsilon'\mathfrak{M},\tau_0)\} \end{aligned}$$

where $\tau_0 = \tau(x_0,x_1,F(x_0),F(x_1))$, and subsequently, $\tau_k = \tau(x_k,x_{k+1},F(x_k),F(x_{k+1}))$.
This indicates the existence of $x_2 \in F(x_1) \cap \overline{\mathbb{B}_{p,r}}(\overline{x})$ such that

$$p(x_1,x_2) < \min\{\mathfrak{F}(\alpha,\tau_0), \mathfrak{F}(\varepsilon'\mathfrak{M},\tau_0)\} \in J.$$

Given that $n \in \mathbb{N}$ and a finite sequence $x_0, \ldots, x_n$ has been formed, let's assume that it satisfies:

$$\begin{cases} x_n \in F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\overline{x}), \\ M_F(x_{n-1},x_n) \leqslant p(x_{n-1},x_n) \in J \\ p(x_{n-1},x_n) < \min\{\mathfrak{F}^{n-1}(\alpha,\tau_0), \mathfrak{F}^{n-1}(\varepsilon'\mathfrak{M},\tau_0)\}. \end{cases}$$

If either $x_n = x_{n-1}$ or $x_{n-1} \in F(x_{n-1})$ for any $n \in \mathbb{N}^*$, then our task is complete. Hence, let us assume that for every $n \in \mathbb{N}^*$, it holds that $x_{n-1} \notin F(x_{n-1})$ and $x_{n-1} \neq x_n$, thereby implying that $p(x_{n-1},x_n) > 0$.
Now, let us consider the case of any $n \in \mathbb{N}^*$, and we can proceed as follows:

$$\begin{aligned} M_F(x_{n-1},x_n) &= \max\left\{p(x_{n-1},x_n), p(x_{n-1},F(x_{n-1})), p(x_n,F(x_n)), \frac{p(x_{n-1},F(x_n)) + p(x_n,F(x_{n-1}))}{2}\right\} \\ &= \max\left\{p(x_{n-1},x_n), p(x_n,F(x_n)), \frac{p(x_{n-1},F(x_n)) + p(x_n,x_n)}{2}\right\} \\ &\leqslant \max\left\{p(x_{n-1},x_n), p(x_n,F(x_n)), \frac{p(x_{n-1},x_n) + p(x_n,F(x_n))}{2}\right\} \\ &= \max\left\{p(x_{n-1},x_n), p(x_n,F(x_n))\right\}. \end{aligned}$$

In the event that $\max\{p(x_{n-1},x_n), p(x_n,F(x_n))\} = p(x_n,F(x_n))$, it leads to a contradiction based on condition (b) and the definitions of δ and $\mathfrak{F}$. Consequently, we can conclude that $M_F(x_{n-1},x_n) \leqslant p(x_{n-1},x_n) \in J$, reinforcing the validity of this inequality.
As $x_n \in F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\overline{x})$, we have

$$\begin{aligned} p(x_n,F(x_n)) &\leqslant \delta_p(F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\overline{x}), F(x_n)) \\ &\leqslant \mathfrak{F}\left(M_F(x_{n-1},x_n), \tau_{n-1}\right) \\ &\leqslant M_F(x_{n-1},x_n). \end{aligned}$$

If we make the assumption that $M_F(x_{n-1},x_n) \leqslant p(x_n,F(x_n))$ or $\tau_{n-1} = 0$ for a certain value of $n \in \mathbb{N}^*$, we can deduce that

$$\mathfrak{F}\left(M_F(x_{n-1},x_n), \tau_{n-1}\right) = M_F(x_{n-1},x_n)$$

which implies that $M_F(x_{n-1},x_n)\tau_{n-1} = 0$. Consequently, we arrive at the contradiction that $x_{n-1} = x_n$ or $p(x_{n-1},x_n) = 0$. So we assume that $p(x_n,F(x_n)) < M_F(x_{n-1},x_n)$ and $\tau_{n-1} \neq 0$ for all $n \in \mathbb{N}^*$ and then there exists $x_{n+1} \in F(x_n)$ such that

$$p(x_n,x_{n+1}) < M_F(x_{n-1},x_n) \leqslant p(x_{n-1},x_n).$$

Furthermore, if τ is nondecreasing and $F \in \mathscr{C}_I$, then

$$\begin{aligned}
p(x_n, x_{n+1}) &\leqslant \delta_p(F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(x_n)) \\
&\leqslant \mathfrak{F}(M_F(x_{n-1}, x_n), \tau_{n-1}) \\
&\leqslant \mathfrak{F}(p(x_{n-1}, x_n), \tau_0) \\
&\leqslant \mathfrak{F}\left(\mathfrak{F}^{n-1}(\alpha, \tau_{n-1}), \tau_0\right) \\
&\leqslant \mathfrak{F}\left(\mathfrak{F}^{n-1}(\alpha, \tau_0), \tau_0\right) \\
&\leqslant \mathfrak{F}^n(\alpha, \tau_0)
\end{aligned}$$

else if $F \in \mathscr{C}_{II}$ and $\tau_{n-1} \geqslant \alpha$

$$\begin{aligned}
p(x_n, x_{n+1}) &\leqslant \delta_p(F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(x_n)) \\
&\leqslant \mathfrak{F}(M_F(x_{n-1}, x_n), \tau_{n-1}) \\
&\leqslant \mathfrak{F}(p(x_{n-1}, x_n), \alpha) \\
&\leqslant \mathfrak{F}(\mathfrak{F}^{n-1}(\alpha, \tau_0), \mathfrak{F}^{n-1}(\alpha, \tau_0)) \\
&\leqslant \mathfrak{F}^n(\alpha, \tau_0).
\end{aligned}$$

So, using induction, we can find $x_{n+1} \in F(x_n)$ with

$$p(x_n, x_{n+1}) < \min\{\mathfrak{F}^n(\alpha, \tau_0), \mathfrak{F}^n(\varepsilon'\mathfrak{M}, \tau_0)\},$$

and

$$\begin{aligned}
p(x_{n+1}, \bar{x}) &\leqslant p(\bar{x}, x_0) + \sum_{j=0}^{n} p(x_{j+1}, x_j) - \sum_{j=0}^{n} p(x_j, x_j) \\
&< p(\bar{x}, x_0) + \sum_{j=0}^{\infty} \mathfrak{F}^j(\alpha, \tau_0) \\
&\leqslant p(\bar{x}, \bar{x}) + \lambda r + \widetilde{\mathfrak{w}}(\alpha, \tau_0) \\
&\leqslant p(\bar{x}, \bar{x}) + r.
\end{aligned}$$

Therefore, it follows that $x_{n+1} \in \overline{\mathbb{B}_{p,r}}(\bar{x})$, and consequently, the sequence $\{x_n\}$ is a Cauchy sequence within $\overline{\mathbb{B}_{p,r}}(\bar{x})$. This observation is further reinforced by the fact that for any integers n and m satisfying $n > m$, we have the following:

$$\begin{aligned}
p(x_n, x_m) &\leqslant \sum_{k=m}^{n-1} p(x_{k+1}, x_k) - \sum_{k=m+1}^{n-1} p(x_k, x_k) \\
&< \sum_{k=m}^{n-1} \mathfrak{F}^k(\alpha, \tau_0) \\
&\leqslant \widetilde{\mathfrak{w}}(\alpha, \tau_0).
\end{aligned}$$

Consequently, we have

$$p^s(x_n, x_m) \leqslant 2p(x_n, x_m) < 2\widetilde{\mathfrak{w}}(\alpha, \tau_0).$$

Consequently, we can deduce that the sequence $\{x_n\}$ is actually a Cauchy sequence within the metric space (X, p^s). This conclusion is supported by the fact that $\widetilde{\mathfrak{w}}(\alpha, \tau_0)$ converges for every $\tau_0 \in J'$. Furthermore, since $(X \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), p)$ is a complete metric space, it follows that $(X \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), p^s)$ is also complete. As a result, the sequence $\{x_n\}$ converges to a point x^* with respect to p^s and satisfies the condition:

$$p(x^*, x^*) = \lim_{n \to +\infty} p(x_n, x^*) = \lim_{n,m \to +\infty} p(x_n, x_m) = 0.$$

Now we assert that $x^* \in F(x^*)$. With the application of the partial metric's property $(\mathfrak{P}_4)$, one can obtain

$$\begin{aligned}
p(x^*, F(x^*)) &\leqslant p(x^*, x_n) + p(x_n, F(x^*)) - p(x_n, x_n) \\
&\leqslant p(x^*, x_n) + \delta_p(F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(x^*)) \\
&\leqslant p(x^*, x_n) + \mathfrak{F}(M_F(x_{n-1}, x^*), \tau(x_{n-1}, x^*, F(x_{n-1}), F(x^*))) \\
&\leqslant p(x^*, x_n) + \mathfrak{F}(p(x_{n-1}, x^*), \tau(x_{n-1}, x^*, F(x_{n-1}), F(x^*))).
\end{aligned}$$

Exploiting the upper semicontinuity property of the function $\mathfrak{F}$ concerning the first variable and employing the limit superior as n approaches infinity, we arrive at $p(x^*, F(x^*)) = 0 = p(x^*, x^*)$. This leads to the conclusion that $x^* \in F(x^*)$ as per Lemma 2.2.

Through meticulous computations, we obtain the following results:

$$\begin{aligned}
p(x_0,x^*) &\leqslant \sum_{j=0}^{\infty} p(x_{j+1},x_j) - \sum_{j=1}^{\infty} p(x_j,x_j) \\
&\leqslant \sum_{j=0}^{\infty} \min\{\mathfrak{F}^j(\alpha,\tau_0),\mathfrak{F}^j(\varepsilon'\mathfrak{M},\tau_0)\} \\
&\leqslant \widetilde{\mathfrak{w}}(\varepsilon'\mathfrak{M},\tau_0).
\end{aligned}$$

Given that $p(x_0,Fix(F)) \leqslant p(x_0,x^*)$, we can deduce that

$$p(x_0,Fix(F)) \leqslant \widetilde{\mathfrak{w}}(\varepsilon\mathfrak{M},\tau_0).$$

This inequality holds for any $y := x_0 \in Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K$. Consequently, we obtain

$$\delta_p(Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, Fix(F)) \leqslant \widetilde{\mathfrak{w}}(\varepsilon\mathfrak{M},\tau_0).$$

By allowing ε to approach 1, we successfully complete the proof. □

You can see how to apply Theorem 3.1 in the following example.

Example 3.2. *Let $X = \mathbb{R}^+ = [0,+\infty[$ be equipped with the partial metric defined as follows:*

$$p(x,y) = \begin{cases} 0, & \text{if } x = y \in \left[0,\frac{125}{216}\right]; \\ \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Now, let $\mathfrak{F}(s,t)$ be defined as follows:

$$\mathfrak{F}(s,t) = \begin{cases} st^2, & \text{if} (s\times t) \in \left[0,\frac{5}{6}\right] \times \left[0,\frac{\sqrt{11}}{4}\right]; \\ 6s-4t, & \text{otherwise.} \end{cases}$$

The function $\mathfrak{F}$ belongs to the set $\mathscr{C}$ over the interval $J \times J' = \left[0,\frac{5}{6}\right] \times \left[0,\frac{\sqrt{11}}{4}\right]$. Specifically, $\mathfrak{F}$ is an element of $\mathscr{C}_I$ and $\widetilde{\mathfrak{w}}(s,t) = \frac{s}{1-t^2}$. Define $F : [0,1] \to C^p(X)$ as follows:

$$F(x) = \begin{cases} \{x^3\}, & \text{if } x \in \left[0,\frac{5}{6}\right]; \\ [1,+\infty[, & \text{if } x \in \left]\frac{5}{6},1\right]. \end{cases}$$

Utilizing the parameters specified as follows, we proceed to apply Theorem 3.1:

$$\bar{x} = \frac{1}{6}, \quad r = 1, \quad \lambda = \frac{1}{5}, \quad \alpha = \frac{1}{4} \in J, \quad \overline{\mathbb{B}_{p,r}}(\bar{x}) = [0,1]$$

and

$$\tau(x,y,F(x),F(y)) = p(x,y),$$

which is a nondecreasing.

First, observe that for every z in the closed interval, $\overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) = \left[0,\frac{1}{5}\right]$, we can express the function as follows:

$$p(z,F(z)) = p(z,\{z^3\}) = \max\{z,z^3\} = z \leqslant \frac{1}{5} < \frac{1}{4} = \alpha.$$

Additionally, it holds that $\widetilde{\mathfrak{w}}(\alpha,t) = \frac{\alpha}{1-t^2} \leqslant \frac{1}{4}\cdot\frac{16}{5} = \frac{4}{5} = r(1-\lambda)$. Therefore, condition (a) of Theorem 3.1 is satisfied. To establish the validity of condition (b) in Theorem 3.1, it is enough to examine the following scenarios:

1. If $x = y \in \left[0,\frac{5}{6}\right]$ then

$$\begin{aligned}
\delta_p(F(x)\cap[0,1],F(y)) = \delta_p(\{x^3\},\{x^3\}) = 0 &\leqslant \begin{cases} p(x,x)\cdot p(x,x)^2, & x \in \left[0,\frac{\sqrt{11}}{4}\right]; \\ 6p(x,x)-4p(x,x), & x \in \left]\frac{\sqrt{11}}{4},\frac{5}{6}\right]. \end{cases} \\
&\leqslant \mathfrak{F}(p(x,y),p(x,y))) \\
&\leqslant \mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y))).
\end{aligned}$$

2. If $x,y\in\left[0,\frac{5}{6}\right]$ and $x\neq y$ then

$$\begin{aligned}\delta_p(F(x)\cap[0,1],F(y))=\delta_p(\{x^3\},\{y^3\})&\leqslant\begin{cases}\max\{x,y\}\cdot(\max\{x,y\})^2, & x,y\in\left[0,\frac{\sqrt{11}}{4}\right];\\ 6\max\{x,y\}-4\max\{x,y\}, & x,y\in\left]\frac{\sqrt{11}}{4},\frac{5}{6}\right].\end{cases}\\ &\leqslant\mathfrak{F}(p(x,y),p(x,y)))\\ &\leqslant\mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y))).\end{aligned}$$

3. If $x,y\in\left]\frac{5}{6},1\right]$, then

$$\begin{aligned}\delta_p(F(x)\cap[0,1],F(y))&=\delta_p(\{1\},[1,+\infty[)=1\\ &\leqslant 6\max\{x,y\}-4\max\{x,y\}\\ &\leqslant\mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y))).\end{aligned}$$

4. If $x\in\left[0,\frac{5}{6}\right]$ and $y\in\left]\frac{5}{6},1\right]$ then

$$\delta_p(F(x)\cap[0,1],F(y))=\delta_p(\{x^3\},[1,+\infty[)=1\leqslant 2p(x,y)=\mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y)))$$

and

$$\delta_p(F(y)\cap[0,1],F(x))=\delta_p(\{1\},\{x^3\})=1\leqslant 2p(x,y)=\mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y))).$$

Hence, the condition (b) of Theorem 3.1 is satisfied, and $x^*\in Fix(F)=\{0,1\}\subset\overline{\mathbb{B}_{p,r}}(\overline{x})$ are the required points.
Hence, for arbitrary multivalued mapping $T:[0,1]\to C^p(X)$, all conditions of Theorem 3.1 are satisfied and then, for any $K\subseteq[0,1]$, and for every $y\in Fix(T)$ and $w\in F(y)$, satisfying $p(y,w)<\alpha$, we have

$$\begin{aligned}\delta_p(Fix(T)\cap[0,\tfrac{1}{5}]\cap K,Fix(F))&\leqslant\sup_{x\in[0,1]}\delta_p(T(x)\cap[0,\tfrac{1}{5}]\cap K,F(x))\cdot\frac{1}{1-p(y,w)^2}\\ &\leqslant\widetilde{\mathfrak{w}}\left(\sup_{x\in[0,1]}\delta_p(T(x)\cap[0,\tfrac{1}{5}]\cap K,F(x)),\tau(y,w,F(y),F(w))\right).\end{aligned}$$

Consider, as an illustrative example, the mapping $T:[0,1]\to C^p(X)$ defined by

$$T(x)=\begin{cases}\left\{\frac{1}{2}x+\frac{1}{12}\right\}, & x\in\left[0,\frac{143}{225}\right];\\ [2,+\infty[, & x\in\left]\frac{143}{225},1\right].\end{cases}$$

Furthermore, we observe that $Fix(T)=\{\frac{1}{6}\}$. Let us consider $y=\frac{1}{6}\in Fix(T)$ and any $w\in F(\frac{1}{6})$, specifically $w=\frac{1}{216}$. In this case, we can calculate:

$$\begin{aligned}p(y,w)&=p(\tfrac{1}{6},\tfrac{1}{216})\\ &=\max\left\{\frac{1}{6},\frac{1}{216}\right\}\\ &=\frac{1}{6}<\frac{1}{4}=\alpha.\end{aligned}$$

Let $K\subseteq[0,1]$, then we have

Case 1. $Fix(T)\cap[0,\frac{1}{5}]\cap K=\emptyset$, then

$$\delta_p(Fix(T)\cap[0,\tfrac{1}{5}]\cap K,Fix(F))=0\leqslant\widetilde{\mathfrak{w}}\left(\sup_{x\in[0,1]}\delta_p(T(x)\cap[0,\tfrac{1}{5}]\cap K,F(x)),\tau(y,w,F(y),F(w))\right).$$

Case 2. $Fix(T)\cap[0,\frac{1}{5}]\cap K\neq\emptyset$, i.e., $\frac{1}{6}\in K$, then

$$\delta_p(Fix(T)\cap[0,\tfrac{1}{5}]\cap K,Fix(F))=\delta_p(\{\tfrac{1}{6}\},\{0,1\})=\min\left\{p(\tfrac{1}{6},0),p(\tfrac{1}{6},1)\right\}=\frac{1}{6}.$$

Let $x\in[0,1]$, we consider the following cases:

- If $x > \frac{7}{30}$, we have $T(x) > \frac{1}{5}$ and then

$$\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x)) = 0.$$

- If $x \leqslant \frac{7}{30}$ and $T(x)\cap K = \emptyset$ then

$$\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x)) = 0.$$

- If $x \leqslant \frac{7}{30}$ and $T(x)\cap K \neq \emptyset$ then

$$\begin{aligned}\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x)) &= \delta_p(\{\frac{1}{2}x+\frac{1}{12}\},\{x^3\})\\ &= p(\frac{1}{2}x+\frac{1}{12},x^3)\\ &= \max\left\{\frac{1}{2}x+\frac{1}{12},x^3\right\}\\ &= \frac{1}{2}x+\frac{1}{12}.\end{aligned}$$

Then

$$\sup_{x\in[0,1]} \delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x)) = \sup_{x\in[0,1]}\left(\frac{1}{2}x+\frac{1}{12}\right) = \frac{7}{12}.$$

Hence,

$$\begin{aligned}\widetilde{\mathfrak{w}}\left(\sup_{x\in[0,1]} \delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x)), \tau(y,w,F(y),F(w))\right) &= \sup_{x\in[0,1]} \delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x))\cdot\frac{1}{1-p(y,w)^2}\\ &= \sup_{x\in[0,1]} \delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x))\cdot\frac{36}{35}\\ &= \frac{36}{35}\cdot\frac{7}{12} = \frac{3}{5} \geqslant \frac{1}{6}\\ &\geqslant \delta_p(Fix(T)\cap\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K, Fix(F)).\end{aligned}$$

4. Some Consequences

This section deals with the ramifications of the theorem 3.1. Through these corollaries, we wish to clarify other insights and implications that have resulted from the study. The corollaries that we have derived from the main theorem are not universal but rather depend on the specific choices of the parameters $\mathfrak{F}$, τ, λ, and r. Different values of these parameters may lead to different outcomes or even invalidate some of the corollaries. Therefore, we need to be careful when applying the corollaries to concrete situations and always check the assumptions and conditions that are required for their validity.
Consider, for instance, $\mathfrak{F}(s,t) = \varphi(s)$ belonging to both $\mathscr{C}_I$ and $\mathscr{C}_{II}$, where $\varphi(s)$ is an arbitrary Bianchini-Grandolfi gauge on J and $\widetilde{\mathfrak{w}}(s,t) = \mathfrak{s}(s)$. Thus, under these conditions, Theorem 2.5 directly follows from Theorem 3.1.

Corollary 4.1. *Consider a partial metric space (X,p), where $\bar{x}\in X$, $\lambda\in[0,1]$, and $r>0$ such that the subspace $\overline{\mathbb{B}_{p,r}}(\bar{x})$ is complete. Let $T,F:\overline{\mathbb{B}_{p,r}}(\bar{x})\to C^p(X)$ be multivalued mappings. Assuming that φ be an increasing and upper semicontinuous function, serving as Bianchini-Grandolfi gauge on J. Under the assumption that there exists $\alpha\in J$ satisfying the following two conditions:*

(a) $p(z,F(z)) < \alpha$ *where* $\mathfrak{s}(\alpha)\leqslant r(1-\lambda)$, $\quad\forall z\in\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})$.
(b) $\delta_p(F(x)\cap\overline{\mathbb{B}_{p,r}}(\bar{x}),F(y)) \leqslant \varphi(M_F(x,y))$, $\quad\forall x,y\in\overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any given $K\subseteq\overline{\mathbb{B}_{p,r}}(\bar{x})$, we can establish the following inequality:

$$\delta_p(Fix(T)\cap\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K, Fix(F)) \leqslant \mathfrak{s}(\mathfrak{M})$$

where $\mathfrak{M} := \sup_{x\in\overline{\mathbb{B}_{p,r}}(\bar{x})} \delta_p(T(x)\cap\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K, F(x))$.

Proof. Given a $\mathscr{C}$-class function $\mathfrak{F}(s,t)=\varphi(s)$ that is independent of the second variable t, it is possible to select any $\tau\in\Xi$ such that τ is either nondecreasing or greater than α. Subsequently, Theorem 3.1 can be applied. □

Let $\mathfrak{F}(s,t) = ks-t$ and $k\in]0,1]$ be an element of $\mathscr{C}_{II}$ for $J\times J' = \mathbb{R}^2_+$. Then, with $\widetilde{\mathfrak{w}}(s,t) = \frac{1}{2-k}(2s-t)$, we can state this corollary:

Corollary 4.2. *Suppose (X,p) is a partial metric space and $\bar{x}$ is in X. Let λ, k, and r be no-negative numbers with $0\leqslant\lambda\leqslant 1$, $k\leqslant 1$, and $r>0$. Then the subspace $\overline{\mathbb{B}_{p,r}}(\bar{x})$ is complete. Suppose that T and F are multivalued mappings from $\overline{\mathbb{B}_{p,r}}(\bar{x})$ to $C^p(X)$. Let τ be an element of Ξ and α be a positive real number such that $\tau(x,y,F(x),F(y))\geqslant\alpha$ for all x and y in $\overline{\mathbb{B}_{p,r}}(\bar{x})$. We will establish our assumptions based on the satisfaction of the following two conditions:*

(a) $p(z,F(z))<\alpha$ *where* $2\alpha\leqslant r(1-\lambda)(2-k)$, $\quad\forall z\in\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})$.
(b) $\delta_p(F(x)\cap\overline{\mathbb{B}_{p,r}}(\bar{x}),F(y))\leqslant kM_F(x,y)-\tau(x,y,F(x),F(y))$, $\quad\forall x,y\in\overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any given $K\subseteq\overline{\mathbb{B}_{p,r}}(\bar{x})$, *and for every* $y\in Fix(T)$ *and* $w\in F(y)$, *satisfying* $p(y,w)<\alpha$, *we can establish the following inequality:*

$$\delta_p(Fix(T)\cap\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K,Fix(F))\leqslant\frac{1}{2-k}\left(2\mathfrak{M}-\tau(y,w,F(y),F(w))\right)$$

where $\mathfrak{M}:=\sup_{x\in\overline{\mathbb{B}p,r}(\bar{x})}\delta_p(T(x)\cap\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K,F(x))$.

When λ is zero, we get a simpler version of theorem 3.1:

Corollary 4.3. *Let* (X,p) *be a partial metric space and let* $T,F:\overline{\mathbb{B}_{p,r}}(\bar{x})\to C^p(X)$ *be multivalued mappings such that* $\bar{x}$ *is a fixed point of* T *and* $\overline{\mathbb{B}_{p,r}}(\bar{x})$ *is a complete subspace for some positive* r. *Assuming that* $\tau\in\Xi$, $\alpha\in J$, *and* $\mathfrak{F}\in\mathscr{C}$, *which is upper semicontinuous with respect to the first variable, satisfy either of the following conditions:*

- $\mathfrak{F}\in\mathscr{C}_I$ *and* τ *is nondecreasing,*
- $\mathfrak{F}\in\mathscr{C}_{II}$ *and* $\tau(x,y,F(x),F(y))\geqslant\alpha$ *for* $x,y\in\overline{\mathbb{B}_{p,r}}(\bar{x})$.

Our assumptions require two conditions:

(a) $p(\bar{x},F(\bar{x}))<\alpha$ *where* $\widetilde{\mathfrak{w}}(\alpha,\cdot)\leqslant r$.
(b) $\delta_p(F(x)\cap\overline{\mathbb{B}_{p,r}}(\bar{x}),F(y))\leqslant\mathfrak{F}\left(M_F(x,y),\tau(x,y,F(x),F(y))\right)$, $\quad\forall x,y\in\overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any given $K\subseteq\overline{\mathbb{B}_{p,r}}(\bar{x})$, *and for every* $w\in F(\bar{x})$, *satisfying* $p(\bar{x},w)<\alpha$, *we can establish the following inequality:*

$$p(\bar{x},Fix(F))\leqslant\widetilde{\mathfrak{w}}\left(\mathfrak{M},\tau(\bar{x},w,F(\bar{x}),F(w))\right)$$

where $\mathfrak{M}:=\sup_{x\in\overline{\mathbb{B}p,r}(\bar{x})}p(\bar{x},F(x))$.

Let λ be a nonzero number, and let r go to infinity. Then $\overline{\mathbb{B}_{p,+\infty}}(x)$ is equal to X, and we have the following corollary:

Corollary 4.4. *Consider the complete partial metric space* (X,p). *Let* $T,F:X\to C^p(X)$ *be multivalued mappings. Assuming that* $\tau\in\Xi$, $\alpha\in J$, *and* $\mathfrak{F}\in\mathscr{C}$, *which is upper semicontinuous with respect to the first variable, satisfy either of the following conditions:*

- $\mathfrak{F}\in\mathscr{C}_I$ *and* τ *is nondecreasing,*
- $\mathfrak{F}\in\mathscr{C}_{II}$ *and* $\tau(x,y,F(x),F(y))\geqslant\alpha$ *for* $x,y\in X$.

Our assumptions require two conditions:

(a) $p(z,F(z))<\alpha$ *where* $\widetilde{\mathfrak{w}}(\alpha,\cdot)<+\infty$, $\quad\forall z\in X$.
(b) $\delta_p(F(x),F(y))\leqslant\mathfrak{F}\left(M_F(x,y),\tau(x,y,F(x),F(y))\right)$, $\quad\forall x,y\in X$.

Then, for any given $K\subseteq X$, *and for every* $y\in Fix(T)$ *and* $w\in F(y)$, *satisfying* $p(y,w)<\alpha$, *we can establish the following inequality:*

$$\delta_p(Fix(T)\cap K,Fix(F))\leqslant\widetilde{\mathfrak{w}}\left(\mathfrak{M},\tau(y,w,F(y),F(w))\right)$$

where $\mathfrak{M}:=\sup_{x\in X}\delta_p(T(x)\cap K,F(x))$.

5. Conclusion

In this work, we have extended the results of [12] about the data dependence of fixed point sets for pseudo-contractive multifunctions in the context of partial metric spaces. By utilizing $\mathscr{C}$-class functions, we established new theorems on the data dependence of fixed point sets and implied corollaries. The practical examples given show our key findings. The present investigation adds to the literature on fixed point theory in partial metric spaces and offers tools that are useful in investigating the data dependence for various classes of multifunctions on their fixed points. This paper therefore opens up the possibility of studying other generalized metric structures and nonlinear operators based on techniques developed here that form an active area of research too.

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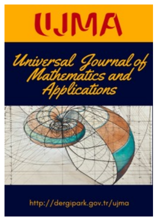

On Linear Combinations of Harmonic Mappings Convex in the Horizontal Direction

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Abstract

The process of creating univalent harmonic mappings which are not analytic is not simple or straightforward. One efficient method for constructing desired univalent harmonic maps is by taking the linear combination of two suitable harmonic maps. In this study, we take into account two harmonic, univalent, and convex in the horizontal direction mappings, which are horizontal shears of $\Psi_m(z) = \frac{1}{2i\sin\gamma_m}\log\left(\frac{1+ze^{i\gamma_m}}{1+ze^{-i\gamma_m}}\right)$, and have dilatations $\omega_1(z) = z$, $\omega_2(z) = \frac{z+b}{1+bz}$, $b \in (-1,1)$. We obtain sufficient conditions for the linear combination of these two harmonic mappings to be univalent and convex in the horizontal direction. In addition, we provide an example to illustrate the result graphically with the help of Maple.

1. Introduction

In the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$, a continuous complex-valued function $f = u + iv$ is harmonic for the real harmonic functions u and v, may be expressed as $f = h + \overline{g}$ in which h and g are analytic in $\mathbb{E}$. Denote by H be the class of harmonic mappings f normalized by $h(0) = g(0) = h'(0) - 1 = 0$, where

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m \quad \text{and} \quad g(z) = \sum_{m=1}^{\infty} b_m z^m.$$

The Jacobian of $f = h + \overline{g}$ is given by $J_f = |h'|^2 - |g'|^2$. In [1], it is proved that $J_f > 0$ in $\mathbb{E}$ if and only if $f \in H$ is locally univalent and sense-preserving. For every z in $\mathbb{E}$, the condition $J_f > 0$ is equal to the dilatation $\omega(z) = g'(z)/h'(z)$ satisfying $|\omega(z)| < 1$ (see [2, 3]).

We denote by $\mathbb{S}_H$ the class of all univalent, harmonic, and sense-preserving mappings $f = h + \overline{g} \in H$. Let $\mathbb{S}_H^0 = \{f \in \mathbb{S}_H : g'(0) = 0\} \subset \mathbb{S}_H$. A domain is said to be convex in the horizontal direction (CHD) (or convex in the vertical direction), if every line parallel to the real axis (or imaginary axis) intersects the domain either with a connected or empty intersection. If $f \in \mathbb{S}_H^0$ maps $\mathbb{E}$ onto a CHD domain, f is said to be a CHD mapping.
A function $f \in \mathbb{S}_H$ is CHD, if

$$h(z) - g(z) = \frac{1}{2i\sin\gamma}\log\left(\frac{1+ze^{i\gamma}}{1+ze^{-i\gamma}}\right) \quad \text{for } \gamma \in \left[\frac{\pi}{2}, \pi\right). \tag{1.1}$$

Let $\mathbb{S}_H(\gamma)$ be the class of all such mappings. Recently, Çakmak et al. [4] studied the convolutions of mappings in the class $\mathbb{S}_H(\gamma)$.
Construction of univalent harmonic mappings is not a very easy and straight forward task. In 1984, Clunie and Sheil-Small introduced a method, known as shear construction or shearing, for constructing a univalent harmonic mapping from a related conformal map. Following method described in the result of Clunie and Sheil-Small [2] creates harmonic mappings that are convex in one direction:

Lemma 1.1. *[2] A harmonic locally univalent function $f = h + \overline{g}$ maps $\mathbb{E}$ univalently onto a domain convex in a direction ϕ if and only if an analytic univalent function $h - e^{2i\phi} g$ maps $\mathbb{E}$ univalently onto a domain convex in the direction of ϕ.*

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Taking the linear combination of two appropriate harmonic maps is another method to create additional examples of non-analytic harmonic mappings. Recently, many researchers have studied this topic such as Dorff and Rolf [5], Long and Dorff [6], and Kumar et al. [7] investigated the linear combination of harmonic univalent mappings which are convex in the vertical direction (CVD). Dorff and Rolf [5] provided the conditions for the linear combination of harmonic mappings which are CVD and have same dilatation to be univalent and CVD. Long and Dorff [6] obtained the conditions (especially conditions of dilatation) for the linear combination of harmonic mappings f_m for $m=1,2$ which satisfy $h_m+g_m=\frac{1}{2i\sin\gamma_m}\log\left(\frac{1+ze^{i\gamma_m}}{1+ze^{-i\gamma_m}}\right)$ $\left(\gamma_m\in\left[\frac{\pi}{2},\pi\right)\right)$ to be univalent and CVD. Wang et al. [8] proved the linear combinations of harmonic right half plane mappings which satisfy $h_m+g_m=\frac{z}{1-z}$ for $m=1,2$ are CHD. Additionally, Demirçay [9], Demirçay and Yaşar [10] examined the conditions for the linear combination of harmonic mappings f_m for $m=1,2$ which satisfy $h_m-g_m=\frac{1}{2i\sin\gamma_m}\log\left(\frac{1+ze^{i\gamma_m}}{1+ze^{-i\gamma_m}}\right)$ to be univalent and CHD.

It is clear from the aforementioned publications that the dilatation functions of the corresponding harmonic functions are significant in determining how their linear combinations behave. In this article, our primary goal is to use two harmonic mappings satisfying (1.1) with particular dilatations $\omega_1(z)=z$, $\omega_2(z)=\frac{z+b}{1+bz}$, $b\in(-1,1)$ to design univalent, sense-preserving, and CHD harmonic mappings. We derive adequate requirements for the univalent and CHD nature of the linear combination of these two harmonic mappings.

2. Preliminary Results

In this section, we state three results obtained by Demirçay [9] and Demirçay and Yaşar [10] and an efficient tool which is known as Cohn's Rule [11].

Theorem 2.1. *[9, 10] Let $f_m=h_m+\overline{g}_m\in\mathbb{S}_H(\gamma_m)$, for $m=1,2$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$ be two harmonic CHD mappings which satisfy (1.1). Then $f_3=\lambda f_1+(1-\lambda)f_2\in S_H$ and CHD for $0\le\lambda\le1$, if f_3 is locally univalent and sense-preserving.*

Lemma 2.2. *[9, 10] Let $f_m=h_m+\overline{g}_m\in\mathbb{S}_H(\gamma_m)$, for $m=1,2$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$ be two harmonic CHD mappings which satisfy (1.1). If $\omega_m=\frac{g_m'}{h_m'}$ are dilatations of f_m, $m=1,2$, respectively, then the dilatation ω of $f_3=\lambda f_1+(1-\lambda)f_2$ $(0\le\lambda\le1)$ is given by*

$$\omega=\frac{I}{II} \tag{2.1}$$

where

$$\begin{aligned} I &= \lambda\omega_1(1-\omega_2)\left(1+2z\cos\gamma_2+z^2\right)\\ &\quad+(1-\lambda)\omega_2(1-\omega_1)\left(1+2z\cos\gamma_1+z^2\right), \end{aligned}$$

and

$$\begin{aligned} II &= \lambda(1-\omega_2)\left(1+2z\cos\gamma_2+z^2\right)\\ &\quad+(1-\lambda)(1-\omega_1)\left(1+2z\cos\gamma_1+z^2\right). \end{aligned}$$

Theorem 2.3. *[9, 10] Let $f_m=h_m+\overline{g}_m\in\mathbb{S}_H(\gamma_m)$, for $m=1,2$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$ be two harmonic CHD mappings which satisfy (1.1). If $\gamma_1=\gamma_2$, then $f_3=tf_1+(1-\lambda)f_2\in S_H$ and CHD for $0\le\lambda\le1$.*

Lemma 2.4. *(Cohn's Rule, see [11]) Suppose a polynomial*

$$r(z)=c_0+c_1z+c_2z^2+\cdots+c_mz^m \tag{2.2}$$

of degree m, and

$$r^*(z)=z^n\overline{r\left(\frac{1}{\overline{z}}\right)}=\overline{c}_m+\overline{c}_{m-1}z+\overline{c}_{m-2}z^2+\cdots+\overline{c}_0z^m.$$

Indicate the number of roots in r inside and on the unit circle, respectively, using the symbols s and t. If $|c_0|<|c_m|$, then

$$r_1=\frac{\overline{c}_mr(z)-c_0r^*(z)}{z}$$

has the number of roots inside and on the unit circle, respectively, $s_1=s-1$ and $t_1=t$.

3. Main Result

Theorem 3.1. *Suppose $f_m=h_m+\overline{g}_m\in\mathbb{S}_H(\gamma_m)$, for $m=1,2$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$ be two harmonic CHD mappings which satisfy (1.1). If $\omega_1(z)=z$, $\omega_2(z)=\frac{z+b}{1+bz}$, $b\in(-1,1)$, then $f_3=\lambda f_1+(1-\lambda)f_2\in S_H$ $(0<\lambda<1)$ and CHD provided $b(\gamma_1-\gamma_2)>0$.*

We require the following lemma in order to demonstrate our primary finding:

Lemma 3.2. *Let $b\in(-1,0)\cup(0,1)$, $\lambda\in(0,1)$, and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$. If $b(\gamma_1-\gamma_2)>0$, then*

$$(i)\quad |1+b(1-2\lambda)|>|b(1-2\lambda)+1+2b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)|; \tag{3.1}$$

$$(ii)\quad |1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)| \tag{3.2}$$

$$>|(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2|. \tag{3.3}$$

Proof of (i). It is obvious that $1+b(1-2\lambda)>0$ holds for $b\in(-1,0)\cup(0,1)$ and $\lambda\in(0,1)$. Then, following inequalities needs to be proved

$$-(1+b(1-2\lambda))<b(1-2\lambda)+1+2b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2), \tag{3.4}$$

$$b(1-2\lambda)+1+2b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)<1+b(1-2\lambda). \tag{3.5}$$

That is,

$$-[1+b(1-2\lambda)]<b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)<0.$$

First, because $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$, then $b(\gamma_1-\gamma_2)>0$ is equivalent to $b(\cos\gamma_1-\cos\gamma_2)<0$. Therefore, for $0<\lambda<1$ we have

$$b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)<0.$$

Now, we contemplate two cases to prove the second inequality.
Case 1: If $b\in(0,1)$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$, then $b(\gamma_1-\gamma_2)>0$ implies $-1<\cos\gamma_1-\cos\gamma_2<0$. Thus,

$$b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)>-b\lambda(1-\lambda)>-[1+b(1-2\lambda)] \tag{3.6}$$

holds for $\lambda\in(0,1)$. (3.6) holds because of $b\left(\lambda^2-3\lambda+1\right)>-1$ for $\lambda\in(0,1)$ and $b\in(0,1)$.
Case 2: If $b\in(-1,0)$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$, then $b(\gamma_1-\gamma_2)>0$ implies $0<\cos\gamma_1-\cos\gamma_2<1$. Thus,

$$b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)>b\lambda(1-\lambda)>-[1+b(1-2\lambda)] \tag{3.7}$$

holds for $\lambda\in(0,1)$. (3.7) holds because of $b\left(\lambda^2+\lambda-1\right)<1$ for $\lambda\in(0,1)$ and $b\in(-1,0)$.
Proof of (ii). If $b(\gamma_1-\gamma_2)>0$, then in view of inequality (i) we know that

$$[1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)]>0$$

for $b\in(-1,0)\cup(0,1)$, $\lambda\in(0,1)$, $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$. So inequality (ii) is equivalent to the inequalities

$$\begin{aligned}&1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)\\ >\;&(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2\\ >\;&-[1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)].\end{aligned} \tag{3.9}$$

Now, let

$$\begin{aligned}f(b,\lambda)\;:=\;&1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)\\ &-[(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2]\\ =\;&(1+b)(1-\cos\gamma_1)\\ &+\lambda[(1+b(1-2\lambda))(\cos\gamma_1-\cos\gamma_2)+2b(\cos\gamma_1-1)].\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial f(b,\lambda)}{\partial b}\;=\;&1-\cos\gamma_1+[3\cos\gamma_1-\cos\gamma_2-2]\lambda\\ &+2[\cos\gamma_2-\cos\gamma_1]\lambda^2,\\ \frac{\partial f(b,\lambda)}{\partial \lambda}\;=\;&4b[\cos\gamma_2-\cos\gamma_1]\lambda+[(3b+1)\cos\gamma_1-(b+1)\cos\gamma_2-2b].\end{aligned}$$

Let $\frac{\partial f(b,\lambda)}{\partial b}=0$ and $\frac{\partial f(b,\lambda)}{\partial \lambda}=0$. Then we have

$$b=b_0=\frac{\cos\gamma_1-\cos\gamma_2}{2-\cos\gamma_1-\cos\gamma_2}$$

and

$$\lambda=\lambda_0=\frac{1}{2},\text{ and }\lambda=\lambda_1=\frac{1-\cos\gamma_1}{\cos\gamma_2-\cos\gamma_1}.$$

Since $\lambda_1\notin(0,1)$, it is obvious that

$$f(b,\lambda)\geq f(b_0,\lambda_0)=1-\frac{\cos\gamma_1}{2}-\frac{\cos\gamma_2}{2}>0$$

which implies that

$$\begin{aligned}&1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)\\ >\;&(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2.\end{aligned}$$

Thus, inequality (3.8) is proved.
Next, let

$$\begin{aligned} I &:= (1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2 \\ &\quad +[1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)] \\ &= (1+b)(1+\cos\gamma_1)+\lambda\left[(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1)\right]. \end{aligned}$$

Let

$$g(b):=(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1).$$

Since

$$g'(b)=-\cos\gamma_1-\cos\gamma_2-2<0,$$

g is decreasing for $b\in(-1,0)\cup(0,1)$ and $\gamma_1,\gamma_2\in\left[\frac{\pi}{2},\pi\right)$.
Also, $g(-1)=2\cos\gamma_2+2>0$ and $g(1)=-2\cos\gamma_1-2<0$.
If $g(b)<0$, then

$$\begin{aligned} I &= (1+b)(1+\cos\gamma_1)+\lambda\left[(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1)\right] \\ &> (1+b)(1+\cos\gamma_1)+\left[(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1)\right] \\ &> (1-b)(1+\cos\gamma_2) \\ &> 0. \end{aligned}$$

If $g(b)>0$, then

$$\begin{aligned} I &= (1+b)(1+\cos\gamma_1)+\lambda\left[(1-b)(\cos\gamma_1+\cos\gamma_2)-2(b+\cos\gamma_1)\right] \\ &> (1+b)(1+\cos\gamma_1) \\ &> 0. \end{aligned}$$

Thus, the proof is complete.
Proof of Theorem 3.1. In view of Theorem 2.1, it suffices to show that $|\omega|<1$ in $\mathbb{E}$. If $b=0$, then $\omega_2(z)=\omega_1(z)=z$. If we substitute these into (2.1), we get $\omega=z$. If $\gamma_1=\gamma_2$, then this case was proved in Theorem 2.3. Thus, we just need to consider the case $b(\gamma_1-\gamma_2)>0$.

Setting $\omega_1(z)=z$ and $\omega_2(z)=\frac{z+b}{1+bz}$ in (2.1), we get

$$\begin{aligned} \omega(z) &= \frac{\lambda z\left(1-\frac{z+b}{1+bz}\right)\left(1+2z\cos\gamma_2+z^2\right)+(1-\lambda)(1-z)\left(1+2z\cos\gamma_1+z^2\right)\frac{z+b}{1+bz}}{\lambda\left(1-\frac{z+b}{1+bz}\right)\left(1+2z\cos\gamma_2+z^2\right)+(1-\lambda)(1-z)\left(1+2z\cos\gamma_1+z^2\right)} \\ &= \frac{\lambda z(1-b)\left(1+2z\cos\gamma_2+z^2\right)+(1-\lambda)\left(1+2z\cos\gamma_1+z^2\right)(z+b)}{\lambda(1-b)\left(1+2z\cos\gamma_2+z^2\right)+(1-\lambda)\left(1+2z\cos\gamma_1+z^2\right)(1+bz)} \\ &= \frac{r(z)}{r^*(z)}, \end{aligned}$$

where

$$\begin{aligned} r(z) &= (1-b\lambda)z^3+[2\lambda(1-b)\cos\gamma_2+2(1-\lambda)\cos\gamma_1+b(1-\lambda)]z^2 \\ &\quad +[1-b\lambda+2b(1-\lambda)\cos\gamma_1]z+b(1-\lambda) \\ &:= c_3z^3+c_2z^2+c_1z+c_0 \end{aligned}$$

and

$$\begin{aligned} r^*(z) &= b(1-\lambda)z^3+[1-b\lambda+2b(1-\lambda)\cos\gamma_1]z^2 \\ &\quad +[2\lambda(1-b)\cos\gamma_2+2(1-\lambda)\cos\gamma_1+b(1-\lambda)]z+(1-b\lambda) \\ &= z^3\overline{p\left(\frac{1}{\overline{z}}\right)}. \end{aligned}$$

Thus if z_0 is a zero of r and $z_0\neq 0$, then $1/\overline{z_0}$ is a zero of r^*, we can rewrite

$$\omega(z)=\frac{(z+\eta)(z+\xi)(z+\zeta)}{(1+\overline{\eta}z)\left(1+\overline{\xi}z\right)\left(1+\overline{\zeta}z\right)}.$$

It is known that, the function $\varphi(z)=\frac{z+\delta}{1+\overline{\delta}z}$ for $|\delta|\leq 1$ maps closed unit disk $\overline{\mathbb{E}}$ onto itself. If we show that $|\eta|\leq 1$, $|\xi|\leq 1$, $|\zeta|\leq 1$ has a modulus that is strictly less than one for at least one of them, then $|\omega|<1$ in $\mathbb{E}$. As $|c_3|=|1-b\lambda|>|c_0|=|b(1-\lambda)|$ grips for all $-1<b<0,\ 0<b<1,$ and $0<\lambda<1$, applying Lemma 2.4 to r, and thus it suffices to prove that all the roots of r_1 lie inside or on the unit circle where

$$r_1(z)=\frac{c_3r(z)-c_0r^*(z)}{z}=(1-b)\widetilde{r_1}(z)$$

and

$$\begin{aligned}\widetilde{r_1}(z) &= [1+b(1-2\lambda)]z^2 \\ &\quad +[2(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+2\lambda(1-b\lambda)\cos\gamma_2]z \\ &\quad +[b(1-2\lambda)+1+2b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)] \\ &:= \widetilde{c_2}z^2+\widetilde{c_1}z+\widetilde{c_0}.\end{aligned}$$

By Lemma 3.2 of (i), we have $|\widetilde{c_2}| > |\widetilde{c_0}|$. Then applying again Lemma 2.4 on $\widetilde{r_1}$, we get

$$r_2(z) = \frac{b_2\widetilde{r_1}(z) - b_0\widetilde{r_1^*}(z)}{z} = -4b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)\widetilde{r_2}(z),$$

and

$$\begin{aligned}\widetilde{r_2}(z) &= [1+b(1-2\lambda)+b\lambda(1-\lambda)(\cos\gamma_1-\cos\gamma_2)]z \\ &\quad +[(1-\lambda)(-b\lambda+1+b)\cos\gamma_1+\lambda(1-b\lambda)\cos\gamma_2] \\ &:= \widetilde{\widetilde{c_1}}z+\widetilde{\widetilde{c_0}}.\end{aligned}$$

By the Lemma 3.2 of (ii), we have $\left|\widetilde{\widetilde{c_1}}\right| > \left|\widetilde{\widetilde{c_0}}\right|$. Hence, the zeros of $\widetilde{r_2}$, r_2, $\widetilde{r_1}$, and r_1 lie in $|z|<1$. Thus, $|\omega|<1$.

Example 3.3. *Let* $\gamma_1 = \frac{5\pi}{6}$, *then*

$$h_1(z) - g_1(z) = -i\log\left(\frac{1+ze^{i\frac{5\pi}{6}}}{1+ze^{-i\frac{5\pi}{6}}}\right).$$

Suppose $\omega_1(z) = z$, *then we get*

$$h_1'(z) - g_1'(z) = \frac{1}{(1-z)\left(1+z^2-\sqrt{3}z\right)}.$$

Using

$$\frac{g_1'(z)}{h_1'(z)} = z,$$

then integration gives

$$h_1(z) = \frac{1+\sqrt{3}}{2}\ln\left(\frac{1-z\sqrt{3}+z^2}{1-2z+z^2}\right)+\tan^{-1}(2z-\sqrt{3})+\frac{\pi}{3},$$

and

$$g_1(z) = \frac{1+\sqrt{3}}{2}\ln\left(\frac{1-z\sqrt{3}+z^2}{1-2z+z^2}\right)-\tan^{-1}(2z-\sqrt{3})-\frac{\pi}{3}.$$

Also, let $\gamma_2 = \frac{\pi}{2}$ *and* $\omega_2(z) = \frac{2z+1}{2+z}$. *Then*

$$h_2(z) - g_2(z) = -\frac{i}{2}\log\left(\frac{1+iz}{1-iz}\right).$$

Thus, we yield

$$h_2(z) = \frac{3}{4}\ln\left(\frac{1+z^2}{1-2z+z^2}\right)+\frac{\tan^{-1}(z)}{2},$$

and

$$g_2(z) = \frac{3}{4}\ln\left(\frac{1+z^2}{1-2z+z^2}\right)-\frac{\tan^{-1}(z)}{2}.$$

Then using Theorem 3.1, we can conclude that $f_3 = \lambda f_1 + (1-\lambda)f_2 \in S_H$ *and CHD. The images of the concentric circles which have radius 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9 in the unit disk* $\mathbb{E}$ *under* f_3 *with* $\lambda = 0, \frac{1}{2}, 1$, *respectively, are shown in Figures 3.1, 3.2, and 3.3.*

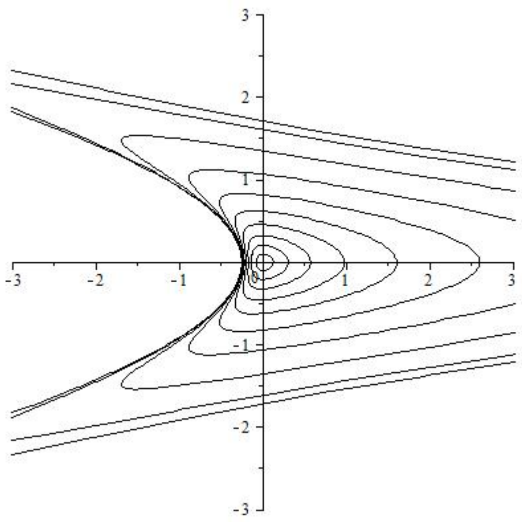

Figure 3.1: The image of $\mathbb{E}$ under f_3 with $\lambda = 0$

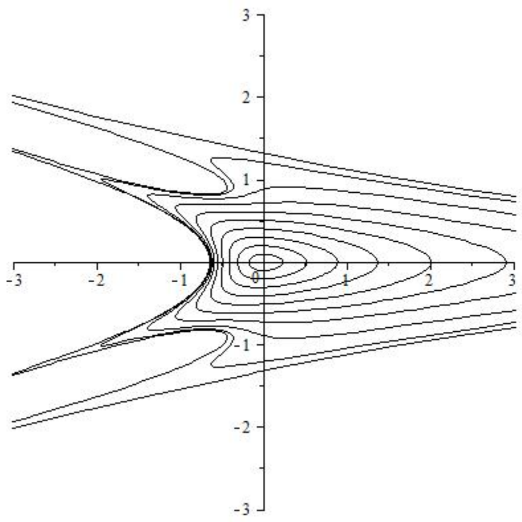

Figure 3.2: The image of $\mathbb{E}$ under f_3 with $\lambda = 1/2$

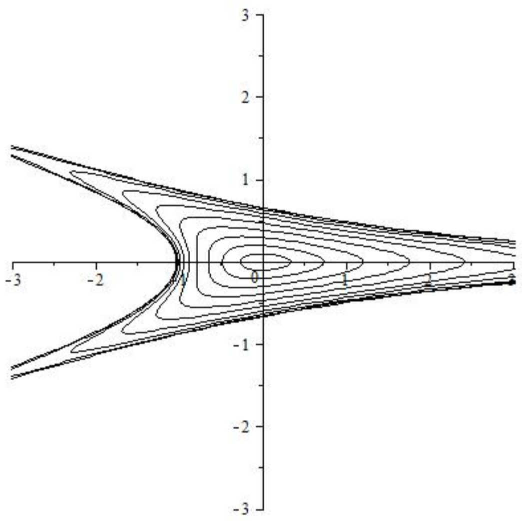

Figure 3.3: The image of $\mathbb{E}$ under f_3 with $\lambda = 1$

4. Conclusion

Fluid flow issues have been studied and resolved using harmonic mapping techniques (see [12]). Specifically, while working with planner fluid dynamical issues, the study of univalent harmonic mappings with unique geometric properties like convexity and convexity in one direction occurs naturally for addressing dynamical planner fluid problems. On the other hand, creating univalent harmonic mappings which are not analytic is not a very easy and straight forward task. To generate new examples of non-analytic desired univalent harmonic mappings, a linear combination of two suitable harmonic mappings can be helpful. In this paper, we considered two harmonic mappings $f_m = h_m + \overline{g}_m$ for $m = 1,2$ which satisfy $h_m - g_m = \frac{1}{2i\sin\gamma_m}\log\left(\frac{1+ze^{i\gamma_m}}{1+ze^{-i\gamma_m}}\right)$ for $\gamma_1, \gamma_2 \in \left[\frac{\pi}{2}, \pi\right)$ and have dilatations $\omega_1(z) = z$ and $\omega_2(z) = \frac{z+b}{1+bz}$ for $b \in (-1,1)$. Our main result is if $b(\gamma_1 - \gamma_2) > 0$ then the linear combination $f_3 = \lambda f_1 + (1-\lambda) f_2$ for $0 < \lambda < 1$ is univalent and CHD. In addition, we provided an example to illustrate the result graphically with the help of Maple.

In our forthcoming research endeavor, we intend to explore the conditions for linear combination and convolution of harmonic mappings involving singular inner functions to be univalent and CHD.

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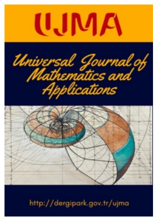

Numerical Approximation Tool Prediction on Potential Broad Application of Subsurface Vertical Flow Constructed Wetland (SSVF CW) Using Chromium and Arsenic Removal Efficiency Study on Pilot Scale

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Abstract

This study investigates the potential broad application of Subsurface Vertical Flow Constructed Wetlands (SSVF CWs) for heavy metal remediation, focusing on Chromium (Cr) and Arsenic (As) removal efficiency. A pilot-scale experimental setup was employed, utilizing a SSVF CW filled with 12 mm gravel and 2 mm coarse sand, planted with Phragmites Australis. The research, conducted over 366 days, aimed to develop a numerical approximation tool to predict the performance and applicability of SSVF CWs in various environmental conditions. The experimental system operated at a hydraulic loading rate of $98-111$ mm/d and a hydraulic retention time of 6 days. Results showed average removal efficiencies of $44.87 \pm 9.52\%$ for Cr and $43.16 \pm 9.43\%$ for As. A mass balance analysis revealed that substrate accumulation was the primary mechanism for heavy metal removal, accounting for 29% of Cr and 26% of As removal. Plant uptake contributed to $3.5-9.9\%$ of Cr and $0.3-$ 8.8% of As removal. Based on these findings, a numerical model was developed to simulate SSVF CW performance under varying environmental and operational parameters. The model incorporated factors such as influent concentrations, hydraulic loading rates, substrate composition, and plant species. Validation against experimental data showed good agreement, with an R^2 value of 0.89 . The numerical tool was then used to predict SSVF CW performance across a range of scenarios, indicating potential broad applications in industrial wastewater treatment, mine drainage remediation, and contaminated groundwater cleanup. This study provides valuable insights into the scalability and versatility of SSVF CWs for heavy metal removal, offering a sustainable and cost-effective solution for water treatment challenges.

1. Introduction

Constructed wetlands offer an economical and environmentally friendly solution for treating various types of wastewaters, including those containing heavy metals [1–3]. Vertical flow constructed wetlands (VFCWs), a specific configuration, are increasingly used for both municipal and industrial wastewater treatment. VFCWs differ from conventional wetlands in their feeding mechanism, filter depth, and operational principles [2, 4] Some studies have shown that VFCWs can effectively remove heavy metals through a combination of physical, chemical, and biological processes. The vertical flow design maximizes contact time between wastewater and filter material, enhancing removal efficiency through microbial action. The layered substrate also supports microorganism growth, improving the treatment of ammonium and organic carbon [2, 4–6]. Wetlands with macrophytes (aquatic plants) have been found to have higher microbial densities

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and metabolic processes compared to unplanted systems, further increasing removal effectiveness. This highlights the important role that vegetation plays in the overall performance of constructed wetlands for wastewater treatment [7].

Constructed wetlands offer an economical and eco-friendly solution for treating various wastewaters, including those containing heavy metals. Vertical flow constructed wetlands (VFCWs), a specific type often used for municipal and industrial wastewater treatment, differ from conventional wetlands in their feeding mechanism, filter depth, and operational principles [4, 8–10]. Some studies have shown VFCWs to be effective in removing heavy metals through a combination of physical, chemical, and biological processes [11, 12]. The vertical flow design maximizes contact between wastewater and filter material, enhancing removal under microbial action. The substrate layers support microorganisms, improving the treatment of ammonium and organic carbon. Wetlands with macrophytes show higher microbial densities and metabolic processes, further increasing removal efficiency. Research has demonstrated high removal rates for total Chromium (Cr), with effluent concentrations consistently below $50\mu g/L$ and an average removal efficiency of 98% [13, 14]. This is attributed to mechanisms such as adsorption to gravel substrate, precipitation reactions, and reduction of Cr(VI) to the less toxic Cr(III) form, enhanced by both wetland design and microbial communities. Dissolved Cr(VI) was also effectively removed, likely through reduction reactions [11, 12]. Arsenic (As) removal was significant but more variable than Cr , with efficiencies between $78-99\%$ and effluent below 100 $\mu g/L$, indicating the wetland was more effective at removing Cr overall [15]. Speciation tests showed preferential removal of As(V) through adsorption to the gravel substrate, while As(III) removal was lower. Improving adsorption capacity could enhance As (III) retention [16, 17].

In present pilot-scale experiment, VFCW demonstrated high potential for removing both Cr and As from synthetic wastewater, comparable to results observed in other studies [15, 18, 19]. The researchers suggest that further optimization of design parameters such as feed rate, hydraulic retention time, bed depth, and plant species selection could further improve performance [2, 20, 21]. The study concludes that VFCWs represent a promising eco-technology for treating heavy metal contaminated effluents. This approach is particularly suitable for small communities and remote locations where land availability and cost considerations are critical factors. The ability of VFCWs to effectively remove heavy metals while offering a sustainable and cost-effective solution underscores their potential for widespread application in wastewater treatment [22–24].

This pilot-scale study investigated the potential of Vertical Flow Constructed Wetlands (VFCWs) to remove chromium (Cr) and arsenic (As) from wastewater, an important mechanism for long-term metal removal. Field implementation should also examine co-treatment of other wastewater pollutants such as organics, nutrients, and additional metals [25–28]. Sequential treatment trains with different wetland configurations may prove beneficial [2, 29]. Despite the need for more research, this study provides valuable proof-of-concept for VFCWs as a sustainable approach to removing toxic heavy metals from wastewater. Constructed wetlands utilize natural processes, offering an energy-efficient and ecologically friendly technology for wastewater treatment [2, 30, 31], especially in small communities and remote locations where land availability and costs are critical factors [2].

2. Materials and Methodologies

2.1. Pilot scale set-up

The experiment was set up on the Aligarh Muslim University campus in India, located between 27°52′N to 27°56′N latitude and 78°3′E to 78°6′E longitude. This research station was part of the "SWINGS" project, an Indo-European collaboration under the FP7 Framework programme. Aligarh, situated 130 km northeast of Delhi in northern India, has a subtropical climate. The average summer temperature is 32.9°C, peaking at 42°C, while the monsoon season averages 26.7°C. Winter temperatures range from 23.3°C to 25°C, with lows around 5°C. The experimental setup consisted of 6 identical beds, each measuring 160 cm × 60 cm × 105 cm. These beds were filled with 40 cm of 12 mm gravel, topped with 50 cm of 2 mm uniformly graded coarse sand. The beds, numbered 1 through 6 , were connected in parallel. For convenience in operation, monitoring, and sampling, the beds were arranged in pairs, as shown in Figure 2.1.

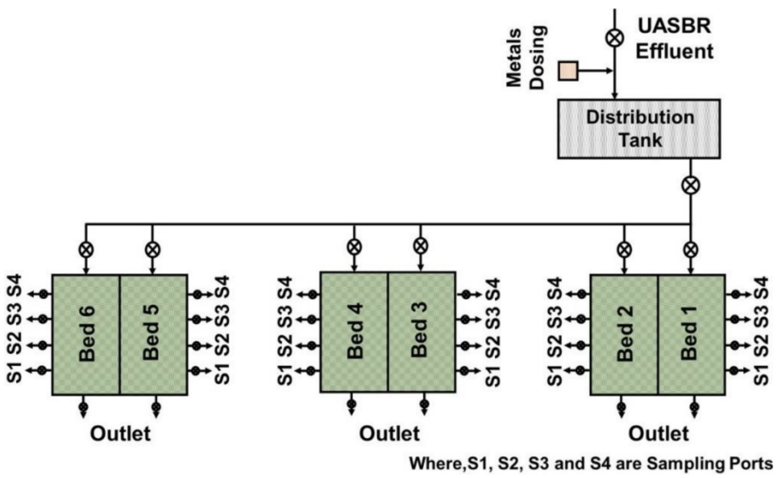

Figure 2.1: Schematic diagram showing the pilot scale used for the present experimental study

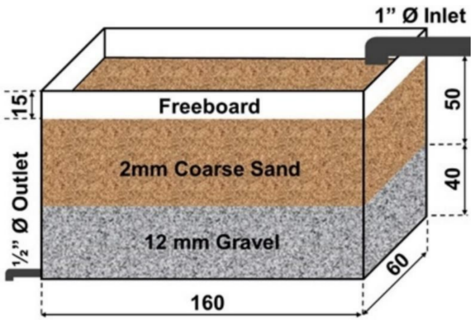

Figure 2.2: Pilot scale CW Beds set-up schematic diagram, where dimensions are in cm

The experiment used constructed wetland (CW) beds, each 160 cm long, 60 cm wide, and 105 cm deep, planted with *Phragmites Australis*. An 800-liter tank distributed water to the beds, receiving effluent from a 50 m3 UASBR via a collection tank. To ensure consistent As and Cr levels, a 5-liter dosing tank was linked to the distribution tank's intake. Each bed featured a 1 -inch inlet, a $1/2$-inch outlet, and four $1/2$-inch sampling ports (S1S4) placed along the bed's length. These ports were set 40 cm apart, with 20 cm between the inlet/outlet and the nearest port. The system operated at a $9.42-10.67$ L/d discharge rate, $0.0984-0.1111$ m/d hydraulic loading rate (HLR), and a 6-day hydraulic retention time (HRT). This configuration allowed researchers to study contaminant removal as water flowed through the beds. To simulate industrial wastewater, the setup used Chromium(VI) Oxide (CrO3) and Sodium Arsenate Dibasic Heptahydrate (Na2HAsO4.7H2O), both ACS reagent grade from Sigma Aldrich.

2.2. Sampling and experimental analysis

The experiment spanned a full year, starting in March 2021 and ending in March 2022. Water samples were collected regularly, every 6th day, from both the influent and effluent of each Vertical Flow Constructed Wetland (VFCW) through grab sampling. To ensure prompt measurements, water temperature (T), conductivity (Cond), and pH were measured using a digital Multi-Parameter Meter (Hach HQ40d).

3. Chromium and Arsenic Mass Balance Calculations

After 366 days of the experiment, a simple mass balance for Cr and As in each bed was calculated using the equations provided by [2] (Eq. (3.1) and Eq. (3.2)). In these equations, the suffixes represent the following: the influent suffix denotes the total mass in the influent, the effluent suffix denotes the total mass in the effluent, the plant suffix represents the total amount absorbed by the plant across all four parts (roots, rhizome, stem, and leaves), the substrate suffix represents the total accumulation in the substrate, and the unaccounted suffix represents the unaccounted amount of Cr and As, which includes any loss or gain from the mass balance calculations. $\mathrm{Cr}_{\text{influent}}$ and $\mathrm{As}_{\text{influent,}}$ as well as $\mathrm{Cr}_{\text{effluent}}$ and $\mathrm{As}_{\text{effluent}}$, were calculated by multiplying the total Cr and As concentrations in the influent and effluent by the water volume, respectively. $\mathrm{Cr}_{\text{plant}}$, $\mathrm{Cr}_{\text{substrate}}$, Asplant, and $\mathrm{As}_{\text{substrate}}$ were determined by multiplying the Cr and As concentrations in the respective components by their weight. $\mathrm{Cr}_{\text{unaccount}}$ and $\mathrm{As}_{\text{unaccount}}$ were calculated using the formulas $\mathrm{Cr}_{\text{influent}} - \left(\mathrm{Cr}_{\text{effluent}} + \mathrm{Cr}_{\text{plant}} + \mathrm{Cr}_{\text{substrate}}\right)$ and $\mathrm{As}_{\text{influent}} - \left(\mathrm{As}_{\text{effluent}} + \mathrm{As}_{\text{plant}} + \mathrm{As}_{\text{substrate}}\right)$, respectively.

$$\mathrm{Cr}_{\text{influent}} = \mathrm{Cr}_{\text{effluent}} + \mathrm{Cr}_{\text{plant}} + \mathrm{Cr}_{\text{substrate}} + \mathrm{Cr}_{\text{unaccount}} \tag{3.1}$$

$$\mathrm{As}_{\text{influent}} = \mathrm{As}_{\text{effluent}} + \mathrm{As}_{\text{plant}} + \mathrm{As}_{\text{substrate}} + \mathrm{As}_{\text{unaccount}} \tag{3.2}$$

3.1. Data analysis

The removal efficiency was calculated using the equation referred to as Eq. (3.3). This efficiency, expressed as a percentage, is determined by subtracting the effluent concentration ($\mathrm{C}_{\text{effluent}}$) from the influent concentration ($\mathrm{C}_{\text{influent}}$), dividing the result by the influent concentration, and then multiplying by 100%. This equation, Eq. (3.3), provides an accurate measure of the removal efficiency. Any values of removal efficiency that are below 0 are treated as 0 in the calculation.

$$\text{Removal efficiency } (\%) = \frac{\mathrm{C}_{\text{influent}} - \mathrm{C}_{\text{effluent}}}{\mathrm{C}_{\text{influent}}} \times 100\% \tag{3.3}$$

To calculate the bioconcentration factor (BCF) and translocation factor (TF) of As in the plants, Eq. (3.4) and Eq. (3.5) were used, respectively. The BCF is determined by dividing the average As concentration in the plant parts by the As concentration in the water. Similarly, the TF is calculated by dividing the average As concentration in the aerial parts (stems and leaves) by the As concentration in the roots. These equations, Eq. (3.4) and Eq. (3.5), provide important insights into the bioconcentration and translocation of As in the plants. It's noteworthy that all the data for these calculations were processed using Microsoft Excel 2021, ensuring precision and reliability in the analysis.

$$\mathrm{BCF} = \frac{\text{Average As concentration in parts}}{\text{As concentration in water}} \tag{3.4}$$

$$\text{TF} = \frac{\text{Average As concentration in aerial parts}}{\text{As concentration in roots}} \quad (3.5)$$

4. Results and Discussion

4.1. Treatment efficiency for traditional pollutants

VFCW bed with added Cr and As and the treatment efficiencies of BOD_5 and COD (mean ±SD) are found to be $86.60 \pm 12.57\%$ and $85.80 \pm 12.62\%$ respectively, on the other hand BOD_5 and COD (mean ±SD) treatment efficiencies in ascending order are found to be ($76.80 \pm 11.45\%$ and $73.10 \pm 11.20\%$) for Bed 1, Bed 4 , and Bed 3 respectively.
Using the ANOVA single-factor test in Microsoft Excel 2021, p-values were found to be less than 0.05 for the experimental study results. These p-values indicate that the findings are statistically significant and satisfactory for the entire experimental investigation.

Month	BOD_5 Removal Efficiency %	COD Removal Efficiency %	Cr Removal Efficiencies %	As Removal Efficiencies %
Mar-21	66.13 ± 6.21	65.02 ± 6.30	23.20 ± 12.42	22.46 ± 12.14
Apr-21	89.00 ± 6.13	88.66 ± 6.27	35.38 ± 3.12	34.39 ± 3.12
May-21	88.22 ± 6.17	87.72 ± 6.24	39.26 ± 5.15	38.19 ± 5.16
Jun-21	90.14 ± 6.31	89.24 ± 6.25	44.85 ± 22.72	44.03 ± 2.97
Jul-21	90.34 ± 6.20	89.34 ± 6.33	45.76 ± 4.10	44.78 ± 4.16
Aug-21	89.90 ± 6.19	88.92 ± 6.28	47.42 ± 3.44	46.40 ± 3.41
Sep-21	90.70 ± 6.12	89.74 ± 6.31	51.63 ± 2.87	50.54 ± 2.86
Oct-21	89.88 ± 6.15	89.08 ± 6.22	50.76 ± 3.07	49.82 ± 3.09
Nov-21	89.26 ± 6.14	88.10 ± 6.26	49.46 ± 0.90	48.33 ± 0.90
Dec-21	82.48 ± 6.22	82.23 ± 6.31	45.60 ± 4.06	44.51 ± 4.06
Jan-22	87.98 ± 6.14	86.98 ± 6.22	48.74 ± 1.95	47.66 ± 1.91
Feb-22	90.28 ± 6.16	89.45 ± 6.23	51.66 ± 3.37	50.68 ± 3.42
Mar-22	91.28 ± 6.17	91.50 ± 6.31	51.66 ± 3.38	50.68 ± 3.43

Table 1: Removal efficiencies (mean ±SD) values for complete experimental study

4.2. Treatment efficiency for Cr and As

The Chromium and Arsenic concentrations in the influent of the Bed were consistently found to be (4.60 ± 1.02mg/l), while the effluent concentrations varied. Throughout the study, the average Cr effluent concentrations in the Bed were (2.51 ± 0.51mg/l), and the average As effluent concentrations were (2.60 ± 0.52mg/l). The Cr removal efficiency for the Bed was recorded at ($44.15 \pm 9.52\%$), and the As removal efficiency was ($43.16 \pm 9.43\%$). The Cr removal efficiency in all VFCWs showed fluctuations, with an initial increase, followed by a decline, and then a subsequent rise towards the end of the study. A similar pattern was observed in As removal efficiency. The treatment efficiencies of Cr and As in each SSVF CW bed exhibited significant fluctuations, initially increasing, then narrowing, decreasing, and eventually rising again. During the early phase of the study (March 2021 to August 2021), the Cr and As removal efficiencies displayed considerable swings. However, starting from the last week of August 2021, the fluctuating pattern became consistent across all six VF CW beds. Notably, the highest Cr removal efficiencies were achieved in September 2021 and March 2022, with values of 55.23% and 57.65%, respectively. Similarly, the highest As removal efficiencies were observed in September 2021 and March 2022, with respective values of 54.13% and 56.75%. Cr and As treatment efficiencies in each SSVF CW bed exhibited significant fluctuations during the first 174 days of the experimental study. The variation narrowed between the 180th and 306th days, with larger peaks observed from the 312th to the 366th day. October also recorded the second and third-highest Cr and As removal efficiencies. The average monthly Cr and As removal efficiencies for all VFCW beds increased steadily from March 2021 to August 2021, with September 2021 showing the highest monthly mean removal rates (as seen in Table 1). Notably, the highest Cr and As removal rates occurred during rainfall events from October 2021 to September 2022. This suggests that the reduction in contaminant concentrations and ambient temperature due to rainfall may have enhanced the Cr and As removal efficiencies in the VF CW beds. Similar outcomes were observed in the treatment of household wastewater in a temperate climate zone with variable conditions in central Europe-specifically in southeast Poland-using an on-site engineered wetland system, which demonstrated the effects of climate conditions on contaminant removal efficiencies [22, 32] when planted with Phragmites australis [32]. During the final six months of the study, the SSVF CW beds displayed nearly identical Cr and As removal patterns, with effluent concentrations being lower than influent concentrations, indicating increased removal efficiencies in all six VFCWs [33, 34].

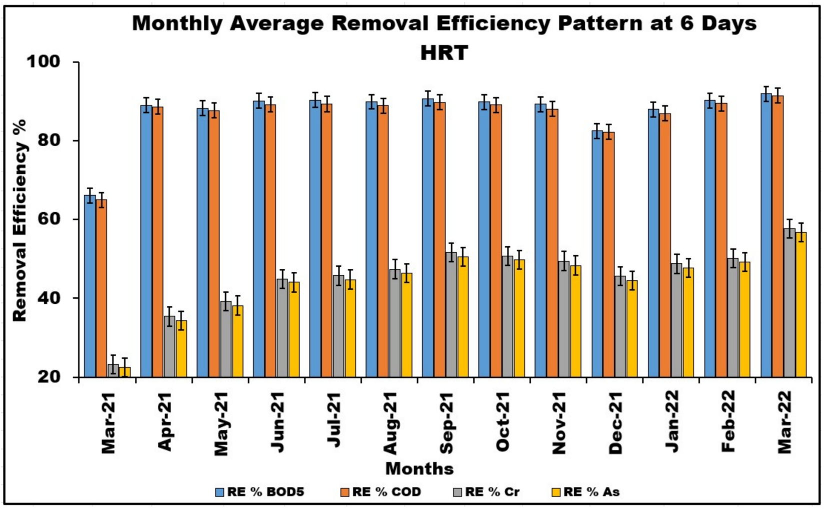

Figure 4.1: Monthly average (mean ±SD) removal efficiency pattern at 6 days HRT

4.3. Adsorption of Cr and As in the media

Chromium and arsenic are among the most prevalent and concerning heavy metal pollutants found in wastewater streams, both of which pose significant risks to the environment and public health [1, 35, 36]. Subsurface vertical flow constructed wetlands (SSVF CWs) have emerged as an effective solution for removing these contaminants through adsorption and accumulation in the substrate [1, 2, 37, 38].

A primary mechanism for removing arsenic and chromium from SSVF CWs is adsorption onto the substrate [39, 40]. The substrate provides additional surface area for pollutants to bind to and become immobilized [2, 37, 38]. Typically composed of materials such as sand, gravel, or organic matter [37–39], the substrate's physicochemical properties-including specific surface area, permeability, and chemical composition-significantly influence adsorption efficiency [2, 37, 38, 41, 42].

To enhance the removal of arsenic and chromium in SSVF CWs through adsorption and accumulation, several operational and design factors must be considered. These include maintaining appropriate environmental conditions, such as pH , redox potential, and nutrient availability, selecting suitable substrate materials, and optimizing hydraulic retention time [38, 43, 44].

It is crucial to consider that the oxidation states and chemical composition of these contaminants can influence the removal processes of arsenic and chromium in SSVF CWs. For instance, the adsorption characteristics and toxicity levels of hexavalent chromium (Cr(VI)) and trivalent chromium (Cr(III)) differ significantly [19, 67]. Similarly, in constructed wetlands, the form of arsenic-whether arsenite (As(III)) or arsenate (As(V))-affects its mobility and removal efficiency [45].

The removal of chromium and arsenic from wastewater through adsorption and accumulation in the substrate within SSVF CWs is a promising and environmentally sustainable approach [2, 46]. Enhancing the design and operation of SSVF CWs can lead to more effective removal, thereby protecting both human health and the environment [2, 37, 38]. In addition to adsorption, the accumulation of chromium and arsenic within the wetland system further contributes to their removal from wastewater [46–49].

5. Numerical Approximation

In this section, we investigate the modeling and control capabilities of the fractional type integral operators considering the data given in Table 1.

In 2020, Kadak [50] constructed a novel family of Bernstein-Kantorovich operators using the fractional mean values of the approximated function. Let $f \in C[0,1]$ and $\alpha > 0$ be fixed parameter. The fractional Bernstein-Kantorovich operator is given by

$$K_n^{\alpha}(f;x) = \alpha \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 (1-s)^{\alpha-1} f\left(\frac{k+s}{n+1}\right) ds, \tag{5.1}$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, x \in [0,1], n \in \mathbb{N}$. Refer for related literature [51–57].
It is known that Bernstein-Kantorovich-type operators are defined over a large class of functions. Also, these operators cover the space $L_p[0,1], 1 \leq p \leq \infty$. According to the definition of fractional Bernstein-Kantorovich operators in Eq. (5.1) the data given in Table-1 have been modeled and analyzed for different values of α in which α denotes the order of Riemann-Lioville fractional integral operators. To accomplish this aim, we will continue in the following steps using fractional mean values of the BOD_5, COD, Cr and As.
Step 1. In Figure 4.1, the months from March 2021 to March 2022 have been mapped on the nodes $x_k = k/n, k = 0, \ldots, n$ and $n = 5$ on the closed interval [0,1]. The function $f(x)$ belonging to $L_p[0,1], 1 \leq p \leq \infty$, is illustrated for BOD_5, COD, Cr and As in Figures 5.1, 5.3, 5.5, and 5.7, respectively.

Step 2. To calculate the relevant fractional mean values for a fixed $\alpha > 0$, we define linear functional $f_i(x)$ on $[x_{i-1}, x_i]$ for $i = 1,\dots,5$ (see Figures 5.2, 5.4, 5.6 and 5.8 for BOD_5, COD, Cr and As, respectively). Then, each sample value in Figure 4.1 will be replaced by the corresponding fractional mean values. *i.e.*

$$f\left(\frac{j}{n}\right) \approxeq \alpha \int_0^1 (1-s)^{\alpha-1} f_j\left(\frac{j+1}{n+1}\right) ds, \tag{5.2}$$

where $f \in L_p[0,1]$, $f_j \in C[x_{j-1}, x_j]$, $j = 0,\dots,5$ and $\alpha > 0$, and $f_0(x) = 0$.

Step 3. In this step, using Eq. (5.2), we get

$$\begin{aligned} K_n^{\alpha}(f;x) &= \alpha \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 (1-s)^{\alpha-1} f_j\left(\frac{j+1}{n+1}\right) ds, \\ &\approxeq \alpha \Bigg\{ b_{n,1}(x) \int_0^1 (1-s)^{\alpha-1} f_1\left(\frac{1+1}{n+1}\right) ds + b_{n,2}(x) \int_0^1 (1-s)^{\alpha-1} f_2\left(\frac{2+1}{n+1}\right) ds \\ &\quad + \cdots + b_{n,5}(x) \int_0^1 (1-s)^{\alpha-1} f_5\left(\frac{5+1}{n+1}\right) ds \Bigg\} \qquad (f_0(x) = 0), \end{aligned}$$

where $f \in L_p[0,1]$, $f_j \in C[x_{j-1}, x_j]$, $j = 0,\dots,5$ and $\alpha > 0$.

Step 4. In the final step, we estimate the trend of the data given in Table-1 for different values of α at the point $x_i \in [0,1]$ for $i = 0,1,\dots,5$(see Figures 5.9, 5.10, 5.11, and 5.12) (for BOD_5, COD, Cr and As, respectively). In Figures 5.9, 5.10, 5.11, and 5.12, using the above steps the approximate values of data are given depending on the different values of $\alpha = 0.1, 0.2, 0.6, 1$. The Figures show that above mentioned data can be obtain approximately by utilizing the operator given in Eq. (5.1). As can be seen the trend values obtained with the help of the operator for the different values of α shows the consistency of the operators. In particular, for increasing values of α, we have good trends.

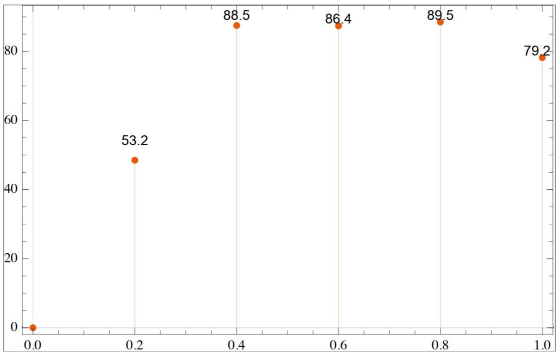

Figure 5.1: The graphs of the function (for BOD_5) $f(x)$ defined on $[0,1]$ with the points $x_k = k/n, k = 0,1,\dots,n, n = 5$.

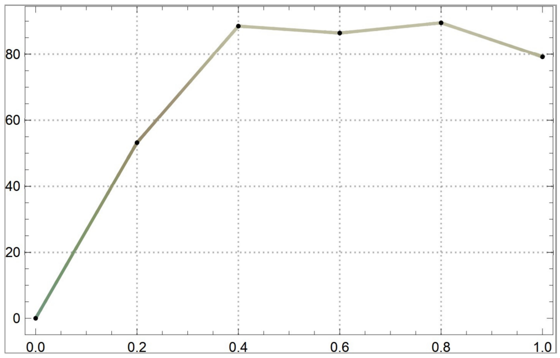

Figure 5.2: The graphs of the functions (for BOD_5) $f_i(x)$ defined on the closed intervals $[x_{i-1}, x_i]$ for $i = 1,\dots,5$

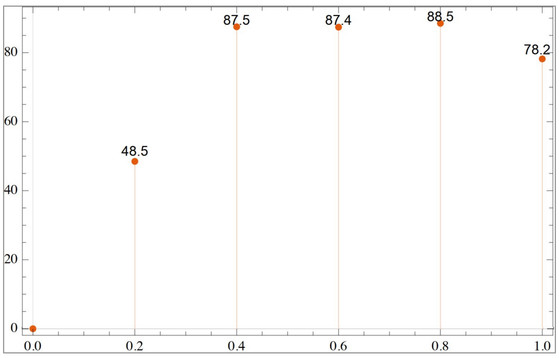

Figure 5.3: The graphs of the function (for **COD**) $f(x)$ defined on $[0,1]$ with the points $x_k = k/n, k = 0,1,\dots,n, n = 5$.

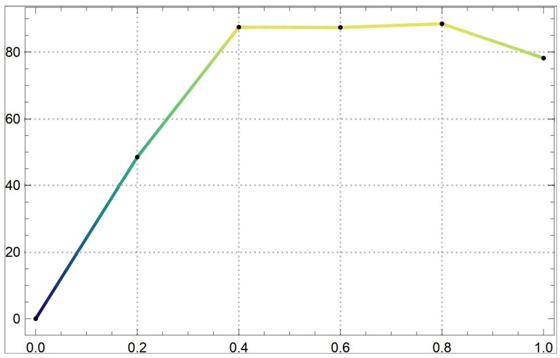

Figure 5.4: The graphs of the functions (for **COD**) $f_i(x)$ defined on the closed intervals $[x_{i-1}, x_i]$ for $i = 1,\dots,5$

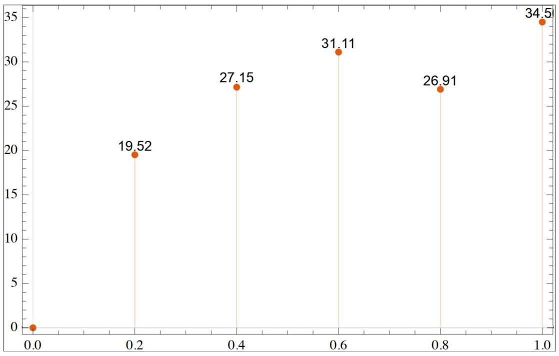

Figure 5.5: The graphs of the function (for **Cr**) $f(x)$ defined on $[0,1]$ with the points $x_k = k/n, k = 0,1,\ldots,n, n = 5$.

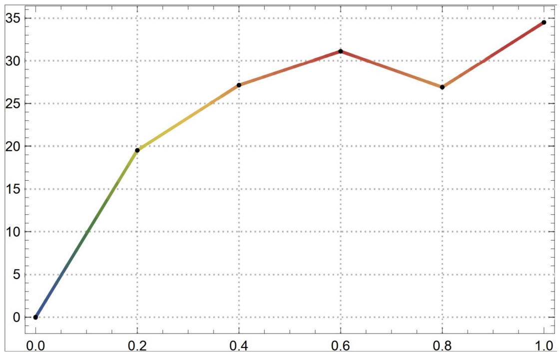

Figure 5.6: The graphs of the functions (**Cr**) $f_i(x)$ defined on the closed intervals $[x_{i-1}, x_i]$ for $i = 1,\ldots,5$

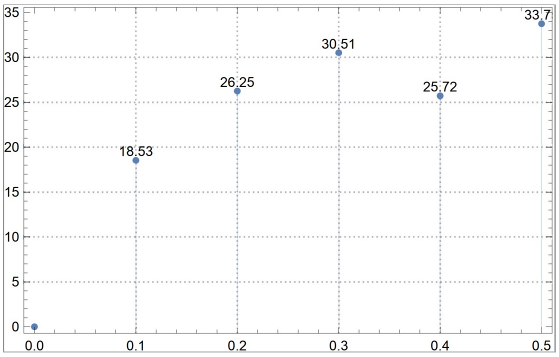

Figure 5.7: The graphs of the function (**As**) $f(x)$ defined on $[0,.5]$ with the points $x_k = k/n, k = 0,1,\ldots,n, n = 5$.

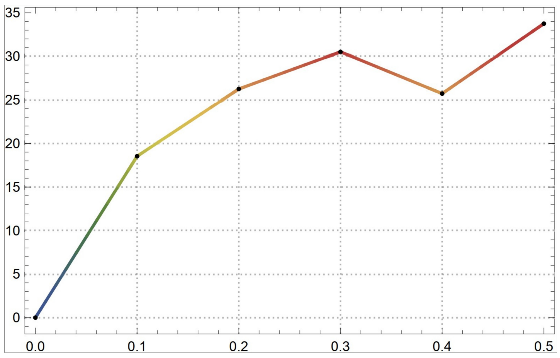

Figure 5.8: The graphs of the functions (**As**) $f_i(x)$ defined on the closed intervals $[x_{i-1}, x_i]$ for $i = 1,\ldots,5$

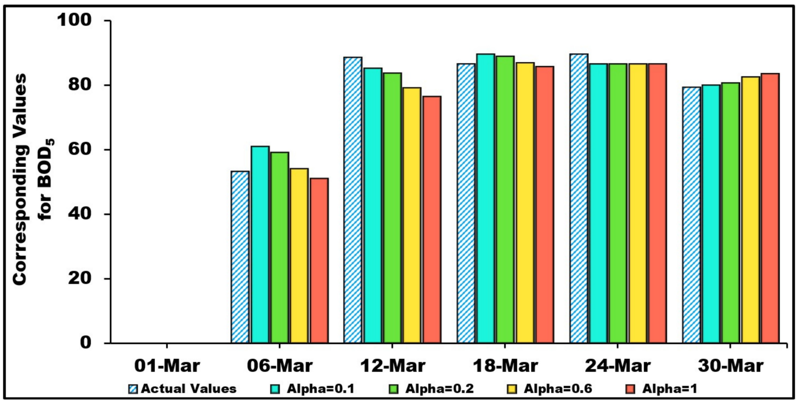

Figure 5.9: The trends of BOD_5 using the fractional type Bernstein Kantorovich operators.

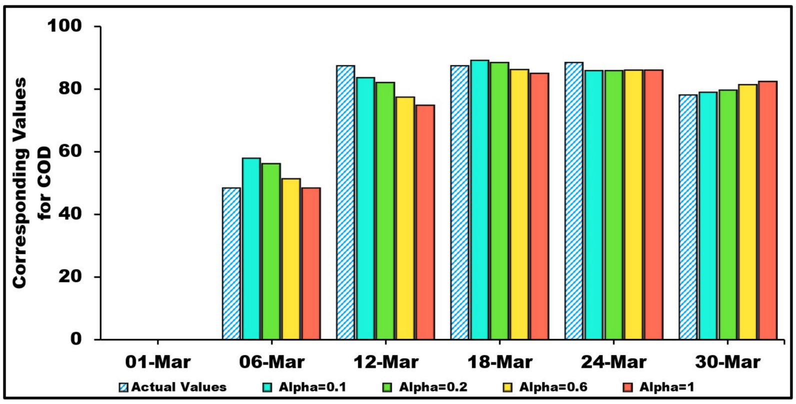

Figure 5.10: The trends of **COD** using the fractional type Bernstein Kantorovich operators.

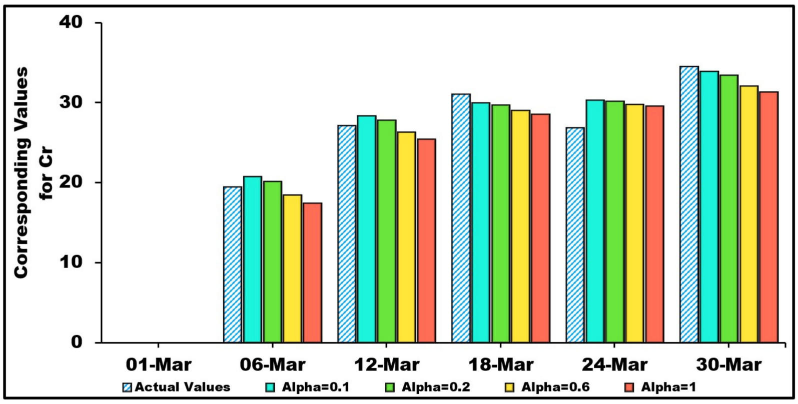

Figure 5.11: The trends of **Cr** using the fractional type Bernstein Kantorovich operators.

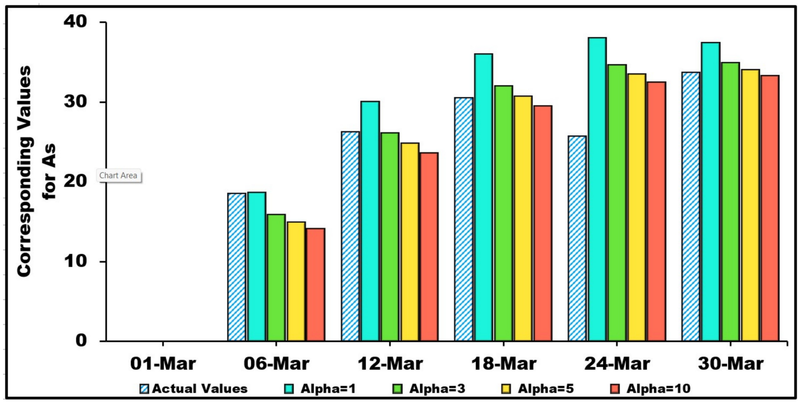

Figure 5.12: The trends of **As** using the fractional type Bernstein Kantorovich operators.

6. Conclusions

When employing vertical flow constructed wetlands (VFCWs) for the elimination of conventional pollutants (BOD_5, COD) from wastewater, irrespective of the presence of chromium (Cr) and arsenic (As), no significant differences were observed in the removal efficiencies. At the point of discharge, with hydraulic loading rates (HLR) and hydraulic retention times (HRT) ranging from 9.42 to 10.67 L/d, 0.0984 to 0.1111 m/d, and 6 days respectively, the removal efficacy exhibited a marginally greater effectiveness for Cr in comparison to As within each constructed wetland bed, demonstrating mean removal efficiencies of ($Cr - As$) at (44.15% - 43.16%) for the respective beds. The effectiveness of Cr and As removal is found to be closely correlated with variables such as the ambient temperature of the influent, prevailing climatic conditions, species of macrophytes present, and the phenological stage of the vegetation. Furthermore, the results findings are parallel to the previous studies on removal mechanisms, related literature can be found in [58–72]. The highest observed removal efficiency of chromium (Cr) and arsenic (As) occurred during the Monsoon and Autumn seasons, periods distinguished by optimal ambient temperatures and the proliferation of vegetation. In contrast, the minimal efficiency was recorded in January, a month that corresponds with the lowest temperatures and macrophytes in a state of senescence. Within each constructed wetland (CW) bed, both the substrate and macrophytes are capable of accumulating Cr and As. The primary mechanism contributing to the removal of Cr and As was the accumulation on the surface of the media, which accounted for 29% to 26% of the influent concentration, whereas the accumulation by plants for Cr ranged from 3.5% to 9.9% of the influent concentration and for As ranged from 0.3% to 8.8% of the influent concentration. The overarching conclusions derived from this experimental investigation indicate that subsurface flow constructed wetlands (SSVF CW) utilizing coarse sand and gravel as substrate, in conjunction with Phragmites australis, exhibit significant potential in effectively mitigating Cr and As contaminants from wastewater.

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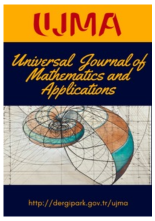

The Essential Gronwall Inequality Demands the $(\rho,\varphi)-$Fractional Operator with Applications in Economic Studies

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Abstract

Gronwall's inequalities are important in the study of differential equations and integral inequalities. Gronwall inequalities are a valuable mathematical technique with several applications. They are especially useful in differential equation analysis, stability research, and dynamic systems modeling in domains spanning from science and math to biology and economics. In this paper, we present new generalizations of Gronwall inequalities of integral versions. The proposed results involve $(\rho,\varphi)-$Riemann-Liouville fractional integral with respect to another function. Some applications on differential equations involving $(\rho,\varphi)-$Riemann-Liouville fractional integrals and derivatives are established.

1. Introduction

In recent years, fractional calculus has been applied to real and complex domains, like physics and engineering, see for instance [1–3], also in chemistry and biology, see for example [4]. It has also been used in relaxation-oscillation phenomena and diffusion vibrations, see the research papers [5–8]. Fractional calculus has some investigations in mechanical systems, we invite the reader to consult the articles [9–12]. Such applications have motivated researchers to work on new fractional calculus theories(see [13–15]. The generalized Riemann-Liouville fractional operators are a class of integral operators that extend the classical Riemann-Liouville fractional integral and derivative operators [16–20]. They are defined for functions that are not necessarily differentiable, but satisfy certain integrability conditions [21–23]. The generalized Riemann-Liouville fractional operators have many properties and applications in mathematics and physics, including fractional differential equations, fractional calculus, and signal processing. They also have connections to other areas of mathematics, such as complex analysis and number theory [24–30].

In the same way, in the present paper, we shall discuss some fractional integral variants of the well-known Gronwall integral inequality and some applications on differential equations that involve "new fractional order derivatives". The Gronwall inequality is a fundamental result in the theory of differential equations. It provides a bound on the growth of a function that satisfies a certain type of differential inequality [31–35]. The Gronwall inequality is often used to study the existence, uniqueness, and stability of solutions to differential equations. It is named after the Swedish mathematician T.H. Gronwall, who first proved it in 1919. Before presenting our results, we need to present to the reader some motivated papers for the present paper. We recall the paper [36] where a generalized Gronwall-type inequality involving Riemann-Liouville derivatives has been considered. Then, in [18], the Gronwall inequality has been proved and some applications for differential equations of the hybrid type, that involve Hadamard derivatives, have been established. Other types of inequalities have also been considered in [37–40].

The fractional Gronwall inequality is a generalization of the classical Gronwall inequality, which is a fundamental result in the theory of ordinary differential equations. It provides an estimate on the growth of a function in terms of an integral involving the function and

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its derivatives. The fractional Gronwall inequality has important applications in the study of fractional differential equations, which are differential equations involving fractional derivatives. It can be used to establish existence, uniqueness, and stability results for solutions of such equations. The main aim of this work is to establish generalizations for Gronwall inequality by applying fractional integrals with respect to another function. Also, our aim is to establish sufficient guaranteeing conditions on the boundedness of solutions for some classes of differential equations involving "$(\rho,\varphi)-$Riemann-Liouville derivatives".

The structure of this research paper is given as follows: In Section 2, we give some used preliminaries. In Section 3, our integral results are proved. In Section 4, we continue with the main results; two classes of differential equations, in the sense of "$(\rho,\varphi)-$Riemann-Liouville", are studied. At the end, a conclusion follows.

2. Preliminaries

We introduce some used preliminaries [6,9,12]. In particular, the generalized fractional derivatives in the sense of $(\rho,\varphi)-$Riemann-Liouville involving (or with respect to) the function φ are introduced, for the first time, in this section.
We begin by noticing that the classical form of the inequality of Gronwall says that if a positive function u over $I:=[t_0,T);T\leq\infty$, that satisfies the inequality

$$u(z)\leq f(z)+\int_{t_0}^{z}L(t)\,u(t)\,dt,\ \ z\in[t_0,T),$$

where f is a continuous function on $[t_0,T)$, and $L(t)\geq 0$ over the same interval,
then, one has the following result:

Lemma 2.1. *We have*

$$u(z)\leq f(z)+\int_{t_0}^{z}f(t)\,L(t)\exp\left(\int_t^z L(\tau)\,d\tau\right)dt,\ \ t_0\leq z<T.$$

The following Lemma is also needed in the present work.

Lemma 2.2. *(Jensen) If we take $n\in\mathbb{N}^*$, and also the nonnegative numbers $r_1,...,r_n$, then, for $m>1$,*

$$\left(\sum_{i=1}^{n}r_i\right)^m\leq n^{m-1}\sum_{i=1}^{n}r_i^m.$$

We are also concerned with the following auxiliary result:

Lemma 2.3. *Let $T\leq\infty, I=[t_0,T)\subset\mathbb{R},\ \ f,g,q\in C(I,\mathbb{R}_+)$. We suppose also that for $u\in C(I,\mathbb{R}_+)$, the inequality holds*

$$u(x)\leq f(x)+\int_{t_0}^{x}g(t)\,u(t)\,dt+\int_{t_0}^{x}q(t)\,u^{\gamma}(t)\,dt,\ \ x\in I,$$

with $0\leq\gamma<1$.
Hence, for any $x\in I$, the inequality

$$u(x)\leq\left[F^{1-\gamma}(x)+(1-\gamma)\int_{t_0}^{x}p(t)\exp\left((\gamma-1)\int_{t_0}^{t}g(\tau)\,d\tau\right)dt\right]^{\frac{1}{1-\gamma}}\times\exp\left(\int_{t_0}^{x}g(t)\,dt\right),$$

is valid, such that $F(x)=\max\limits_{t_0\leq t\leq x}f(t)$.

Under the same interval I, the following estimate of u holds.

Theorem 2.4. *Let consider the nonnegative continuous functions $u,f,g,k_i,i\in\{1,...,n\}$ and suppose there are some positive real numbers $r_1,r_2,...,r_n$. If u satisfies the estimate:*

$$u^r(x)\leq g(x)+f(x)\int_0^x\sum_{i=1}^{n}k_i(t)\,u^{r_i}(t)\,dt,\ x\in I,$$

then, we have

$$u(x)\leq\left[f(x)+g(x)\int_0^x\sum_{i=1}^{n}\%k_i(t)\left(\frac{r_i}{r}f(t)+\frac{r-r_i}{r}\right)\times\exp\left(\int_{\sigma}^{x}g(\tau)\sum_{\%i=1}^{n}\frac{r_i}{r}k_i(\tau)\,d\tau\right)dt\right]^{\frac{1}{r}},$$

for $r\geq\max\{r_i,i=1,...,n\}$.

Now, we recall the following $(\rho,\varphi)-$Riemann-Liouville fractional integrals of a function f on $[a,b]$ with respect to φ, see the paper of M Bezziou et al. [6]:

$${}_{\rho}I^{\alpha}_{a+,\varphi}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,f(t)dt,$$

and

$${}_{\rho}I^{\alpha}_{b-,\varphi}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\varphi^{\rho}(t)-\varphi^{\rho}(x)\right)^{\alpha-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,f(t)dt,$$

where $\alpha,\rho>0$.
Let us now pass to introduce, for the first time, new generalized fractional derivatives in the sense of $(\rho,\varphi)-$Riemann-Liouville with respect to the function φ. We define the proposed derivatives as follows:

Definition 2.5. *Consider $\alpha>0$ and take f as an integrable function over $[a,b]$. If $\varphi\in C^1([a,b],\mathbb{R})$ is an increasing function, such that $\varphi'(x)\varphi^{\rho-1}(x)\neq 0,\forall x\in[a,b],\rho>0$,*
then the left-sided (respectively), the right-sided $(\rho,\varphi)-$ Riemann-Liouville fractional derivative of order α is, respectively, defined by:

$$\rho D^{\alpha}_{a+,\varphi}f(x)=\left(\frac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\frac{d}{dx}\%\right)^n\rho I^{n-\alpha}_{a+,\varphi}f(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\frac{d}{dx}\right)^n\int_a^x(\varphi^\rho(x)-\varphi^\rho(t))^{n-\alpha-1}\varphi\prime(t)\varphi^{\rho-1}(t)f(t)dt,$$

and

$$\rho D^{\alpha}_{b-,\varphi}f(x)=\left(-\frac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\frac{d}{dx}\%\right)^n\rho I^{n-\alpha}_{b-,\varphi}f(x)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\frac{d}{dx}\right)^n\int_x^b(\varphi^\rho(t)-\varphi^\rho(x))^{n-\alpha-1}\varphi\prime(t)\varphi^{\rho-1}(t)f(t)dt,$$

where $n=[\alpha]+1$.

Remark 2.6. *Several existing operators can be derived from Definition 2.5 as follows:*

(i) Letting $\rho=1$ and $\varphi(x)=x$, thus we can obtain the definition of Riemann-Liouville derivative [22, 34].

(ii) Letting $\rho=1$ and $\varphi(x)=\ln(x)$, hence, we can get the definition of Hadamard derivative [22, 34].

(iii) Letting $\rho=1$ and $\varphi(x)=\dfrac{x^{\nu+1}}{\upsilon+1}$, where $\upsilon\neq-1$ is a real number, hence, we can obtain the definition of the Katugampola derivative proposed in [10, 21].

We pass to prove the following important two properties

Theorem 2.7. *Consider $0<\alpha<1$, and take f as an integrable function over $[a,b]$. If $\varphi\in C^1([a,b],\mathbb{R})$ is increasing, such that $\varphi'(x)\varphi^{\rho-1}(x)\neq 0,\forall x\in[a,b],\rho>0$, then, the following two properties:*

$$\left(_\rho D^{\alpha}_{a+,\varphi}\ {_\rho I^{\alpha}_{a+,\varphi}}\right)f(x)-f(x)=0 \tag{2.1}$$

and

$$\left(_\rho D^{\alpha}_{b-,\varphi}\ {_\rho I^{\alpha}_{b-,\varphi}}\right)f(x)-f(x)=0 \tag{2.2}$$

hold.

Proof. We begin by proving the left-sided fractional operator (2.1)
Thanks to the Fubini theorem, we can write

$$\begin{aligned}
&\left(_\rho D^{\alpha}_{a+,\varphi}\ {_\rho I^{\alpha}_{a+,\varphi}}\right)f(x)={_\rho}\ D^{\alpha}_{a+,\varphi}\left({_\rho I^{\alpha}_{a+,\varphi}}f(x)\right)=\tfrac{1}{\Gamma(1-\alpha)}\left(\tfrac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\tfrac{d}{dx}\right)\textstyle\int_a^x\dfrac{\varphi\prime(t)\varphi^{\rho-1}(t)}{(\varphi^\rho(x)-\varphi^\rho(t))^\alpha}\left({_\rho I^{\alpha}_{a+,\varphi}}f(t)\right)dt\\
&=\tfrac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\tfrac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\tfrac{d}{dx}\right)\textstyle\int_a^x\left[\tfrac{\varphi\prime(t)\varphi^{\rho-1}(t)}{(\varphi^\rho(x)-\varphi^\rho(t))^\alpha}\times\int_a^t(\varphi^\rho(t)-\varphi^\rho(s))^{\alpha-1}\varphi\prime(s)\varphi^{\rho-1}(s)f(s)ds\right]dt\\
&=\tfrac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\tfrac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\tfrac{d}{dx}\right)\textstyle\int_a^x\left[\varphi\prime(s)\varphi^{\rho-1}(s)f(s)\times\int_s^x\tfrac{(\varphi^\rho(t)-\varphi^\rho(s))^{\alpha-1}}{(\varphi^\rho(x)-\varphi^\rho(t))^\alpha}\varphi\prime(t)\varphi^{\rho-1}(t)dt\right]ds\\
&=\tfrac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\tfrac{\varphi^{1-\rho}(x)}{\varphi\prime(x)}\tfrac{d}{dx}\right)\textstyle\int_a^x\varphi\prime(s)\varphi^{\rho-1}(s)f(s)ds\times\Gamma(1-\alpha)\Gamma(\alpha)\\
&=f(x).
\end{aligned}$$

Notice here that to achieve the proof of (2.1), we can use the transformation:

$$u:=\frac{\varphi^\rho(t)-\varphi^\rho(s)}{\varphi^\rho(x)-\varphi^\rho(s)}.$$

The proof of (2.2) can be achieved by using the same arguments as in the proof of (2.1). □

3. Results

We have first to present the following estimate for the continuous positive function u.

Theorem 3.1. *Consider $\alpha>0$ and γ in $]0,1[$. Then, take f,g and p in $C(I,\mathbb{R}_+)$. If $u\in C(I,\mathbb{R}_+)$ satisfies*

$$u(x)\leq f(x)+\int_{t_0}^x(\varphi^\rho(x)-\varphi^\rho(t))^{\alpha-1}\varphi\prime(t)\varphi^{\rho-1}(t)g(t)u(t)dt+\int_{t_0}^x(\varphi^\rho(x)-\varphi^\rho(t))^{\alpha-1}\varphi\prime(t)\varphi^{\rho-1}(t)p(t)u^\gamma(t)dt,t_0<x, \tag{3.1}$$

then the following two inequalities are valid: (i) If $\alpha-\dfrac{1}{2}>0$, then, we have

$$\begin{aligned}
u(x)\leq&\left[F_1^{1-\gamma}(x)+(1-\gamma)B_1\textstyle\int_{t_0}^x\exp\left((\gamma-1)B_1\int_{t_0}^t g^2(z)dz\right)\times p^2(t)\varphi\prime(t)\varphi^{\rho-1}(t)\exp\left((2\gamma-2)\varphi^\rho(t)\right)dt\right]^{\frac{1}{2(1-\gamma)}}\\
&\times\exp\left(\left(\varphi^\rho(x)+\tfrac{B_1}{2}\%\right)\textstyle\int_{t_0}^x g^2(t)\varphi\prime(t)\varphi^{\rho-1}(t)dt\right),\ \mathrm{t}\in I,
\end{aligned}\tag{3.2}$$

where $F_1(x) = \max_{t_0 \le t \le x} 3e^{-2\varphi^\rho(t)} f^2(t)$, *and* $B_1 = \dfrac{6\Gamma(2\alpha-1)}{\rho 4^\alpha}$.

(ii) *In the case where* α *is in* $\left(0, \frac{1}{2}\right]$. *If* $0 = q - \dfrac{1+\alpha}{\alpha}, 0 = -p+1+\alpha$, *then one has*

$$u(x) \le \left[F_2^{1-\gamma}(x) + (1-\gamma) B_2 \int_{t_0}^x \exp\left((\gamma-1) B_2 \int_{t_0}^t b^q(\tau)\, d\tau\right) \times p^q(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t) \exp\left(q(\gamma-1)\varphi^\rho(t)\right) dt\right]^{\frac{1}{q(1-\gamma)}} \\ \times \exp\left(\left(\varphi^\rho(x) + \tfrac{B_2}{q}\%\right) \int_{t_0}^x g^q(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t)\, dt\right), \tag{3.3}$$

where $F_2(x) = \max_{t_0 \le t \le x} 3^{q-1} e^{-q\varphi^\rho(t)} f^q(t)$, *and* $B_2 := 3^{-1} 3^q \left(\dfrac{\Gamma((p\alpha-p)+1)}{\rho p^{(\alpha p-p)+1}}\right)^{\dfrac{q}{p}}$.

Proof. Taking $x \in I$, we obtain:

$$u(x) - f(x) \le \int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{\alpha-1} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, g(t) u(t) dt + \int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{\alpha-1} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, p(t) u^\gamma(t) dt. \tag{3.4}$$

(i) Using Cauchy-Schwarz inequality to (3.4), we get

$$\begin{aligned} u(x) - f(x) &\le \left(\int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{(2\alpha-2)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, e^{2\varphi^\rho(t)} dt\right)^{1/2} \\ &\times \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, (gu)^2 dt\right)^{1/2} \\ &+ \left(\int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{(2\alpha-2)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, e^{2\varphi^\rho(t)} dt\right)^{1/2} \\ &\times \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, p^2(t) u^{2\gamma}(t) dt\right)^{1/2} \\ &\le \left(\tfrac{2\Gamma(2\alpha-1)}{\rho 4^\alpha} e^{2\varphi^\rho(x)}\right)^{1/2} \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, (gu)^2 dt\right)^{1/2} \\ &+ \left(\tfrac{2\Gamma(2\alpha-1)}{\rho 4^\alpha} e^{2\varphi^\rho(x)}\right)^{1/2} \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, (pu^\gamma)^2(t) dt\right)^{1/2}, \end{aligned} \tag{3.5}$$

where, $\alpha > \dfrac{1}{2}$.

Thanks to Lemma 2.2, with $m = 2, n = 3$, we observe that (3.5) is equivalent to:

$$u^2(x) - 3f^2(x) \le \left(\tfrac{6\Gamma(2\alpha-1)}{\rho 4^\alpha} e^{2\varphi^\rho(x)}\right) \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi^{\rho-1}(t)\, \varphi\prime(t)\, g^2(t) u^2(t) dt\right) \\ + \left(\tfrac{6\Gamma(2\alpha-1)}{\rho 4^\alpha} e^{2\varphi^\rho(x)}\right) \left(\int_{t_0}^x e^{-2\varphi^\rho(t)} \varphi^{\rho-1}(t)\, \varphi\prime(t)\, u^{2\gamma}(t) p^2(t) dt\right). \tag{3.6}$$

We shall now consider $R(x)$ the quantity $\left(u(x) e^{-\varphi^\rho(x)}\right)^2$. Then, (3.6) can be transformed into:

$$R(x) \le F_1(x) + B_1 \left(\int_{t_0}^x \varphi^{\rho-1}(t)\, \varphi'(t)\, g^2(t) R(t) dt\right) + B_1 \left(\int_{t_0}^x e^{2(\gamma-1)\varphi^\rho(t)} \varphi^{\rho-1}(t)\, \varphi\prime(t)\, p^2(t) R^\gamma(t) dt\right).$$

As $F_1(x)$ is nondecreasing, then by Lemma 2.3, we observe that

$$R(x) \le \left[F_1^{1-\gamma}(x) + (1-\gamma) B_1 \left(\int_{t_0}^x e^{2(\gamma-1)\varphi^\rho(t)} \varphi^{\rho-1}(t)\, \varphi\prime(t)\, p^2(t) \ \times \exp\left((1-\gamma) B_1 \int_{t_0}^t \varphi^{\rho-1}(\tau)\, \varphi\prime(\tau)\, g^2(\tau) d\tau\right) dt\right]^{\frac{1}{1-\gamma}} \\ \times \exp\left(B_1 \int_{t_0}^x \varphi^{\rho-1}(t)\, \varphi\prime(t)\, g^2(t) dt\right). \tag{3.7}$$

So, from (3.7), we obtain (3.2).

(ii) Now, for $\alpha \in \left(0, \frac{1}{2}\right]$, $q = \dfrac{\alpha+1}{\alpha}, p = \alpha+1$, and using Holder inequality on the two integrals of (3.1), we get:

$$\begin{aligned} u(x) &\le f(x) + \left(\int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{p(\alpha-1)} \varphi^{\rho-1}(t)\, \varphi\prime(t)\, e^{p\varphi^\rho(t)} dt\right)^{\frac{1}{p}} \\ &\times \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi^{\rho-1}(t)\, \varphi\prime(t)\, g^q(t) u^q(t) dt\right)^{\frac{1}{q}} \\ &+ \left(\int_{t_0}^x (\varphi^\rho(x) - \varphi^\rho(t))^{p(\alpha-1)} \varphi^{\rho-1}(t)\, \varphi\prime(t)\, e^{p\varphi^\rho(t)} dt\right)^{\frac{1}{p}} \\ &\times \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, p^q(t) u^{q\gamma}(t) dt\right)^{\frac{1}{q}} \\ &\le f(x) + \left(\tfrac{\Gamma(p(\alpha-1)+1)}{p^{p(\alpha-1)+1}} e^{p\varphi^\rho(x)}\right)^{\frac{1}{p}} \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi^{\rho-1}(t)\, \varphi\prime(t)\, g^q(t) u^q(t) dt\right)^{\frac{1}{q}} \\ &+ \left(\tfrac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}} e^{p\varphi^\rho(x)}\right)^{\frac{1}{p}} \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, p^q(t) u^{q\gamma}(t) dt\right)^{\frac{1}{q}}. \end{aligned} \tag{3.8}$$

In view of Lemma 2.2, (with $m = q$ and $n = 3$), and thanks to (3.8), we obtain

$$u^q(x) \le 3^{q-1} f^q(x) + 3^{q-1} \left(\tfrac{e^{p\varphi^\rho(x)} \Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}}\right)^{\frac{q}{p}} \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, (gu)^q(t) dt\right) \\ + 3^{q-1} \left(\tfrac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}} e^{p\varphi^\rho(x)}\right)^{\frac{q}{p}} \left(\int_{t_0}^x e^{-q\varphi^\rho(t)} \varphi\prime(t)\, \varphi^{\rho-1}(t)\, p^q(t) u^{q\gamma}(t) dt\right). \tag{3.9}$$

Let us now take $R(x)$ equal to the quantity $\left(u(x)e^{-\varphi^{\rho}(x)}\right)^{q}$. Then (3.9) implies that

$$R(x) \le F_2(x) + B_2\left(\int_{t_0}^{x} \varphi'(t)\varphi^{\rho-1}(t)g^{q}(t)R(t)dt\right) + B_2\left(\int_{t_0}^{x} e^{q(\gamma-1)\varphi^{\rho}(t)}\varphi^{\rho-1}(t)p^{q}(t)\varphi\prime(t)R^{\gamma}(t)dt\right).$$

The function R is nondecreasing on $[t_0,T)$, from Lemma 2.3.
Thus, the reader can see that

$$\begin{aligned}R(x) \le &\Big[F_2^{1-\gamma}(x) + (1-\gamma)B_2\Big(\textstyle\int_{t_0}^{x} e^{q(\gamma-1)\varphi^{\rho}(t)}\varphi^{\rho-1}(t)\varphi\prime(t)p^{q}(t)\\ &\times \exp\big((1-\gamma)B_2\big(\textstyle\int_{t_0}^{t}\varphi^{\rho-1}(\tau)g^{q}(\tau)\varphi\prime(\tau)d\tau\big)dt\Big]^{\frac{1}{1-\gamma}}\\ &\times \exp\big(B_2\textstyle\int_{t_0}^{x}\varphi^{\rho-1}(t)g^{q}(t)\varphi\prime(t)dt\big).\end{aligned}$$

By the relations of $u(x)$ and $R(x)$, we conclude that (3.3) holds. □

Remark 3.2. (1) *When $\rho=1$ and $\varphi(x)=x$ on $[t_0,T)\subset\mathbb{R}$, the inequalities established in Theorem 3.1 can be transformed into the inequalities established in Theorem 4 given in [36].*

(2) *Taking $\rho=1$ and $\varphi(x)=\ln(x)$ on $[t_0,T), t_0\ge 1$, then the inequalities established in Theorem 3.1 become the inequalities established in Theorem 3.1 given in [18].*

The second main result to be presented to the reader is given by.

Theorem 3.3. *Let us take over the interval I the nonnegative and continuous functions u, f and $g_i, i\in\{1,2,..,n\}$.
If*

$$u(x)-f(x) \le \int_{t_0}^{x}\left(\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\varphi\prime(t)\varphi^{\rho-1}(t)\right)\sum_{i=1}^{n} g_i(t)u^{\gamma_i}(t)dt, \tag{3.10}$$

then we have the following cases: (i) If $\alpha>\frac{1}{2}$, then

$$\begin{aligned}u(x) \le &\left[2f^{2}(x) + 2\tfrac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(x)}\textstyle\int_{t_0}^{x}\sum_{i=1}^{n} n\varphi\prime(t)\varphi^{\rho-1}(t)e^{-2\varphi^{\rho}(t)}g_i^{2}(t)\left[\gamma_i\left(2f^{2}(t)-1\right)+1\right]\right.\\ &\left.\times \exp\left(\textstyle\int_{t}^{x} 2\tfrac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(\tau)}\sum_{i=1}^{n} n\varphi\prime(\tau)\varphi^{\rho-1}(\tau)e^{-2\varphi^{\rho}(\tau)}\gamma_i g_i^{2}(\tau)d\tau\right)dt\right]^{\frac{1}{2}}.\end{aligned}$$

(ii) If $\alpha\in\left(0,\frac{1}{2}\right]$, $0=-q+\dfrac{1+\alpha}{\alpha}, p-1-\alpha=0$, then we have

$$\begin{aligned}u(x) \le &\left[2^{q-1}f^{q}(x) + 2^{q-1}\left(\frac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(x)}\right)^{\frac{q}{p}}\right.\\ &\times \textstyle\int_{t_0}^{x}\sum_{i=1}^{n} n^{q-1}\varphi\prime(t)\varphi^{\rho-1}(t)e^{-q\varphi(t)}g_i^{q}(t)\left[\gamma_i\left(2^{q-1}f^{q}(t)-1\right)+1\right]\\ &\left.\times \exp\left(\textstyle\int_{t}^{x} 2^{q-1}\left(\tfrac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(\tau)}\right)^{\frac{q}{p}}\sum_{i=1}^{n} n^{q-1}\varphi\prime(\tau)\varphi^{\rho-1}(\tau)e^{-q\varphi^{\rho}(\tau)}\gamma_i g_i^{q}(\tau)d\tau\right)dt\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. As $x\in[t_0,T)$, we get

$$u(x)-f(x) \le \int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\varphi^{\rho-1}(t)\varphi\prime(t)e^{\varphi^{\rho}(t)}\sum_{i=1}^{n} g_i(t)u^{\gamma_i}(t)e^{-\varphi^{\rho}(t)}dt.$$

(i) By employing Cauchy-Schwarz inequality and Lemma 2.2, we obtain:

$$\begin{aligned}u(x) \le &f(x) + \left(\textstyle\int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{2(\alpha-1)}\varphi^{\rho-1}(t)\varphi\prime(t)e^{2\varphi^{\rho}(t)}dt\right)^{\frac{1}{2}}\\ &\times\left(\textstyle\int_{t_0}^{x}\sum_{i=1}^{n} n\varphi^{\rho-1}(t)\varphi\prime(t)e^{-2\varphi^{\rho}(t)}g_i^{2}(t)u^{2\gamma_i}(t)dt\right)^{\frac{1}{2}}\\ \le &f(x) + \left(\tfrac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(x)}\right)^{\frac{1}{2}}\times\left(\textstyle\int_{t_0}^{x}\sum_{i=1}^{n} n\varphi\prime(t)\varphi^{\rho-1}(t)e^{-2\varphi^{\rho}(t)}g_i^{2}(t)u^{2\gamma_i}(t)dt\right)^{\frac{1}{2}}.\end{aligned}$$

And using Lemma 2.3 for $m=2$, the above inequality becomes

$$u^{2}(x)-2f^{2}(x) \le \left(2\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(x)}\right)\times\left(\int_{t_0}^{x}\sum_{i=1}^{n} n\varphi\prime(t)\varphi^{\rho-1}(t)e^{-2\varphi^{\rho}(t)}g_i^{2}(t)u^{2\gamma_i}(t)dt\right).$$

So,

$$u^{\tilde{p}}(x)-\tilde{f}(x) \le \tilde{g}(x)\left(\int_{0}^{x}\sum_{i=1}^{n}\tilde{h}_i(t)u^{\tilde{p}_i}(t)dt\right),$$

where

$$\tilde{p}=2,\tilde{p}_i=2\gamma_i,\tilde{h}_i(x)=ng_i^2(x)\varphi\prime(x)\varphi^{\rho-1}(x)e^{-2\varphi^{\rho}(x)},\ \tilde{f}(x)=2f^2(x)$$

and

$$\tilde{g}(x)=2\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(x)}.$$

Theorem 2.4 permits us to write

$$u(x)\leq\left[\tilde{f}(t)+\tilde{g}(t)\int_0^x\sum_{i=1}^n\tilde{h}_i(t)\left[\gamma_i\left(\tilde{f}\%(t)-1\right)+1\right]\times\exp\left(\int_t^x\tilde{g}(\tau)\sum_{i=1}^n\gamma_i\tilde{h}_i(\tau)d\tau\right)dt\right]^{\frac{1}{2}}.$$

(ii) Using Holder inequality, 3.10 allows us to write

$$\begin{aligned}u(x)&\leq f(x)+\left(\int_{t_0}^x(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{p(\alpha-1)}\varphi\prime(t)\varphi^{\rho-1}(t)e^{p\varphi^{\rho}(t)}dt\right)^{\frac{1}{p}}\\&\times\left(\int_{t_0}^x\sum_{i=1}^n n^{q-1}\varphi\prime(t)\varphi^{\rho-1}(t)e^{-q\varphi^{\rho}(t)}g_i^q(t)u^{q\gamma_i}(t)dt\right)^{\frac{1}{q}}\\&\leq f(x)+\left(\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{p\varphi^{\rho}(x)}\right)^{\frac{1}{p}}\times\left(\int_{t_0}^x\sum_{i=1}^n n^{q-1}\varphi\prime(t)\varphi^{\rho-1}(t)e^{-q\varphi^{\rho}(t)}g_i^q(t)u^{q\gamma_i}(t)dt\right)^{\frac{1}{q}}.\end{aligned}\tag{3.11}$$

The inequality (3.11) and Lemma 2.3 give us

$$u^q(x)\leq 2^{q-1}f^q(x)+2^{q-1}\left(\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{q\varphi^{\rho}(x)}\right)^{\frac{q}{p}}\times\left(\int_{t_0}^x\sum_{i=1}^n\%n^{q-1}\varphi\prime(t)\varphi^{\rho-1}(t)e^{-q\varphi^{\rho}(t)}g_i^q(t)u^{q\gamma_i}(t)dt\right).$$

We consider

$$\tilde{p}=q,\ \tilde{p}_i=q\gamma_i,\ \tilde{h}_i(x)=n^{q-1}g_i^q(x)\varphi\prime(x)\varphi^{\rho-1}(x)e^{-q\varphi^{\rho}(x)},\tilde{f}(x)=2^{q-1}f^2(x)$$

and

$$\tilde{g}(x)=2^{q-1}\left(\frac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(x)}\right)^{\frac{q}{p}},$$

then, we can write

$$u^{\tilde{p}}(x)\leq\tilde{f}(x)+\tilde{g}(x)\left(\int_0^x\sum_{i=1}^n\tilde{h}_i(t)u^{\tilde{p}_i}(t)dt\right).\tag{3.12}$$

Thus, from Theorem 2.4, we can conclude that

$$u(x)\leq\left[\tilde{f}(x)+\tilde{g}(x)\int_0^x\sum_{i=1}^{\%n}\tilde{h}_i(t)\left[\gamma_i\left(\tilde{f}\%(t)-1\right)+1\right]\times\exp\left(\int_t^x\tilde{g}(\tau)\overset{\%}{n}\sum_{i=1}\gamma_i\tilde{h}_i(\tau)d\tau\right)dt\%\right]^{\frac{1}{q}}.\tag{3.13}$$

□

4. Applications

Differential equations are commonly used in economics to represent the change of economic variables across time. These equations may define connections between variables like production, consumption, and investment. Gronwall's inequality may be used to investigate the behavior and stability of certain differential equation solutions. Assume we have a differential equation that defines the rate of variation of an identified economic variable and you want to assess the solution's long-term pattern or stability. Gronwall's inequality might be implemented to constrain the solution based on initial or boundary circumstances.

In this section, we will use the above "$(\rho,\varphi)-$ theorems" related to Gronwall inequality to investigate bounded solutions for two classes of fractional differential equations that involve $(\rho,\varphi)-$generalized derivatives with initial conditions.
Class 1: Suppose that we have:

$$\begin{cases}{}_{\rho}D^{\alpha}_{t_0,\varphi}u(x)=\psi(x,u(x))+h(x)u(x),\ \ t_0\leq x<T\leq\infty,\\ {}_{\rho}I^{1-\alpha}_{t_0,\varphi}u(x)\Big|_{x=t_0}=u_0,\end{cases}\tag{4.1}$$

where ${}_{\rho}D^{\alpha}_{t_0,\varphi}$, ${}_{\rho}I^{1-\alpha}_{t_0,\varphi}$ are respectively the $(\rho,\varphi)-$ Riemann-Liouville derivative of fractional order and $(\rho,\varphi)-$ Riemann-Liouville integral, $\rho>0,u_0\in\mathbb{R}$, with respect to $\varphi\in C^1([t_0,T),\mathbb{R}_+)$, $\varphi'(x)\varphi^{\rho-1}(x)\neq 0$, $\psi\in C([t_0,T)\times\mathbb{R},\mathbb{R})$ and $h\in C([t_0,T),\mathbb{R}_+)$.
From [37], we know that $u(x)$ satisfies (4.1) if $u(x)$ satisfies the equation:

$$u(x)=\frac{u_0}{\Gamma(\alpha)}(\varphi^{\rho}(x)-\varphi^{\rho}(t_0))^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^x(\varphi^{\rho}(x)-\varphi^{\rho}(t))^{\alpha-1}\times\varphi'(t)\varphi^{\rho-1}(t)[\psi(t,u(t))+h(t)u(t)]dt.\tag{4.2}$$

We consider the following hypothesis:
(H_1) : There exist $g,p\in C([t_0,T),\mathbb{R}^+),0<\gamma<1$, and $|\psi(t,u(x))+h(x)u(x)|\leq g(x)|u(x)|+p(x)|u^{\gamma}(x)|$
is valid.
Under (H_1), we prove the following integral inequalities for the solution of the above differential problem of Class 1.

Theorem 4.1. *Assume that (H_1) is valid. If u is the solution of (4.1), then the following two inequalities are true:*
(i) If $\alpha > \frac{1}{2}$, then, we have:

$$|u(x)| \leq \Big[\tilde{F}_1^{1-\gamma}(x) + (1-\gamma) B_1 \int_{t_0}^{x} \exp\left((\gamma-1) B_1 \int_{t_0}^{t} g^2(\tau)\, d\tau\right) \times p^2(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t) \exp\left(2(\gamma-1)\varphi^{\rho}(t)\right) dt\Big]^{\frac{1}{2(1-\gamma)}} \times \exp\left(\left(\varphi^{\rho}(x) + \frac{B_1}{2}\right) \int_{t_0}^{x} g^2(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t)\, dt\right), \quad x \in I, \tag{4.3}$$

where $\tilde{F}_1(x) = \max_{t_0 \leq t \leq x} 3e^{-2\varphi^{\rho}(t)} \left(\frac{|u_0|}{\Gamma(\alpha)}\right)^2 |\varphi^{\rho}(t) - \varphi^{\rho}(t_0)|^{2(\alpha-1)}$, *and* $B_1 = \frac{6\Gamma(2\alpha-1)}{\rho 4^{\alpha}}$.

(ii) For $\alpha \in \left(0, \frac{1}{2}\right]$, $q = \frac{1+\alpha}{\alpha}$, and $p = 1+\alpha$, we have

$$|u(x)| \leq \Big[\tilde{F}_2^{1-\gamma}(x) + (1-\gamma) B_2 \int_{t_0}^{x} \exp\left((\gamma-1) B_2 \int_{t_0}^{t} b^q(\tau)\, d\tau\right) \times p^q(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t) \exp\left(q(\gamma-1)\varphi^{\rho}(t)\right) dt\Big]^{\frac{1}{q(1-\gamma)}} \times \exp\left(\left(\varphi^{\rho}(x) + \frac{B_2}{q}\right) \int_{t_0}^{x} g^q(t)\, \varphi\prime(t)\, \varphi^{\rho-1}(t)\, dt\right), \tag{4.4}$$

where $\tilde{F}_2(x) = \max_{t_0 \leq t \leq x} 3^{q-1} e^{-q\varphi^{\rho}(t)} \left(\frac{|u_0|}{\Gamma(\alpha)}\right)^q |\varphi^{\rho}(t) - \varphi^{\rho}(t_0)|^{q(\alpha-1)}$, *and* $B_2 = 3^{q-1} \left(\frac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}}\right)^{\frac{q}{p}}$.

Proof. Let $x \in [t_0, T)$, then thanks to (H_1), we have

$$|u(x)| \leq \left|\frac{u_0}{\Gamma(\alpha)} \left(\varphi^{\rho}(x) - \varphi^{\rho}(t_0)\right)^{\alpha-1}\right| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{x} \left(\varphi^{\rho}(x) - \varphi^{\rho}(t)\right)^{\alpha-1} \times \varphi'(t)\, \varphi^{\rho-1}(t) \left(g(t)\, |u(t)| + p(t)\, |u^{\gamma}(t)|\right) dt.$$

Applying Theorem 3.1, we deduce the desired result. □

Let us now consider another class of differential equations.

Example 4.2. *Assume the following data:*

- *let $\varphi(x) = x, u_0 = 0, (\psi + h)(x) = c, c \in \mathbb{R}$. Then the solution u is given by the form, using Mathematica 13.3, as follows (see Figs.4.1 and 4.2)*

$$u(x) = \frac{c}{\Gamma(\alpha)} \frac{x^{\alpha\rho}}{\alpha\rho}, \quad x > 0, \alpha > 0, \rho > 0, c \in \mathbb{R}.$$

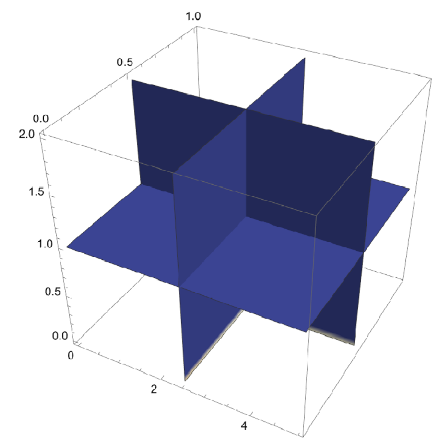

Figure 4.1: The SliceDensityPlot3D of $u(x) = \frac{c}{\Gamma(\alpha)} \frac{x^{\alpha\rho}}{\alpha\rho}$ for $(x, \alpha, \rho), c = 1$.

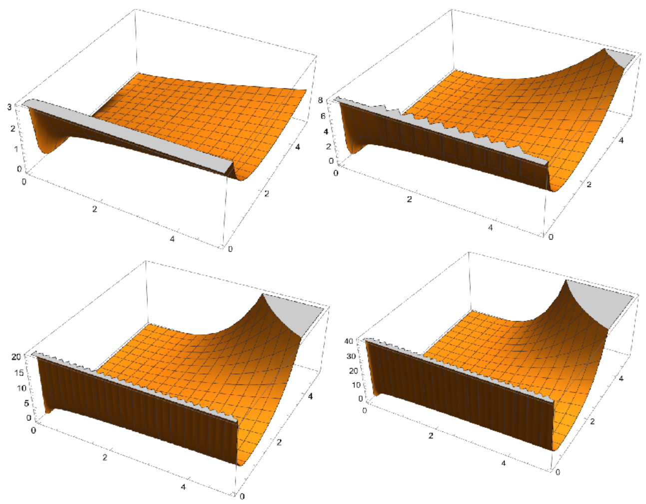

Figure 4.2: The Plot of $u(x) = \frac{c}{\Gamma(\alpha)} \frac{x^{\alpha\rho}}{\alpha\rho}$ for $(x,\rho), c = 1$ and $\alpha = 0.25, 0.5, 0.75, 0.95$.

- *let* $\varphi(x) = exp(x), u_0 = 0, (\psi + h)(x) = c, c \in \mathbb{R}$. *Then the solution u is given by the form (see Fig.4.3 and 4.4),*

$$u(x) = \frac{c}{\Gamma(\alpha)} \frac{\left((e^x)^\rho - 1\right)^\alpha}{\alpha\rho}, \quad x > 0, \alpha > 0, \rho > 0, c \in \mathbb{R}.$$

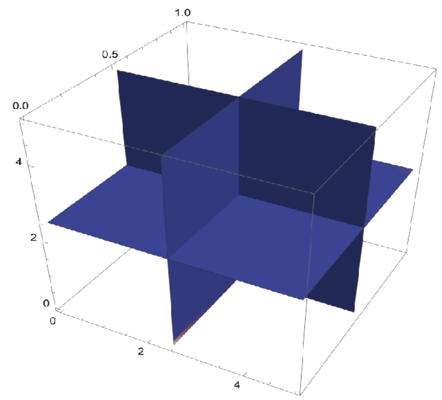

Figure 4.3: The SliceDensityPlot3D of $u(x) = \frac{c}{\Gamma(\alpha)} \frac{\left((e^x)^\rho - 1\right)^\alpha}{\alpha\rho}$ for $(x, \alpha, \rho), c = 1$.

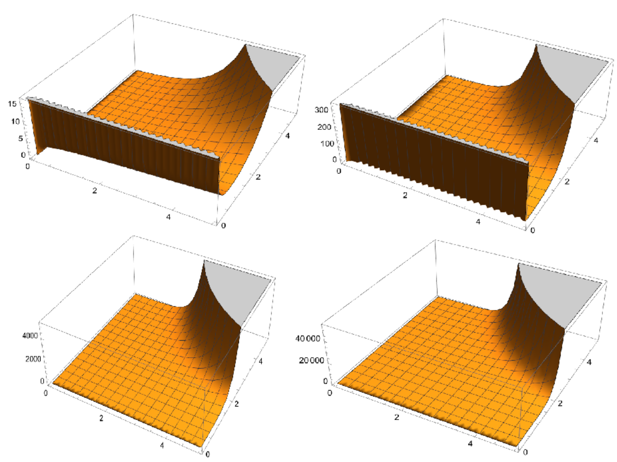

Figure 4.4: The Plot of $u(x) = \frac{c}{\Gamma(\alpha)} \frac{\left((e^x)^\rho - 1\right)^\alpha}{\alpha\rho}$ for $(x,\rho), c = 1$ and $\alpha = 0.25, 0.5, 0.75, 0.95$.

Class 2: We take the following differential problem:

$$\begin{cases} {}_{\rho}D^{\alpha}_{t_0,\varphi}u(x) = \sum\limits_{i=1}^{n} g_i\left(x,u(x)\right), \ t_0 \le x < T \le \infty, \\ {}_{\rho}I^{1-\alpha}_{t_0,\varphi}u(x)\Big|_{x=t_0} = u_0, \end{cases} \tag{4.5}$$

where ${}_{\rho}D^{\alpha}_{t_0,\varphi}$, ${}_{\rho}I^{1-\alpha}_{t_0,\varphi}$ are respectively the $(\rho,\varphi)-$ Riemann-Liouville fractional derivative and $(\rho,\varphi)-$ Riemann-Liouville fractional integral, $\rho > 0, u_0 \in \mathbb{R}$, with respect to $\varphi \in C^1\left([t_0,T),\mathbb{R}_+\right)$, $\varphi'(t)\,\varphi^{\rho-1}(x) \neq 0$ and $g_i \in C\left([t_0,T)\times\mathbb{R},\mathbb{R}\right), i \in \{1,2,...,n\}$.
The fractional integral solution of (4.5) is given by:

$$u(x) = \frac{u_0}{\Gamma(\alpha)}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t_0)\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1} \times\varphi'(t)\,\varphi^{\rho-1}(t)\sum_{i=1}^{n} g_i\left(t,u(t)\right)dt.$$

We consider the following hypothesis.
(H_2) : Assume that there are some $\psi_i \in C([t_0,T),\mathbb{R}^+), i \in \{1,2,...,n\}$ and $0<\gamma_i<1$, such that $\sum\limits_{i=1}^{n}\left|g_i\left(x,u(x)\right)\right| \le \sum\limits_{i=1}^{n}\psi_i(x)\left|u^{\gamma_i}(x)\right|$.

Based on (H_2), we prove the following estimates for the solution of Class 2.

Theorem 4.3. *If (H_2) holds, then the following two inequalities are valid:*

(i) For $\alpha > \frac{1}{2}$, we have

$$|u(x)| \le \Bigg[2Q^2(x) + 2\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(x)}\int_{t_0}^{x}\sum_{i=1}^{n} n\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-2\varphi^{\rho}(t)}g_i^2(t)\left[\gamma_i\left(2Q^2(t)-1\right)+1\right] \times \exp\left(\int_t^x 2\frac{\Gamma(2\alpha-1)}{\rho 4^{\alpha-1}}e^{2\varphi^{\rho}(\tau)}\sum_{i=1}^{n} n\varphi\prime(\tau)\,\varphi^{\rho-1}(\tau)\,e^{-2\varphi^{\rho}(\tau)}\gamma_i g_i^2(\tau)\,d\tau\right)dt\Bigg]^{\frac{1}{2}},$$

where, $Q(x) = \dfrac{|u_0|}{\Gamma(\alpha)}\left|\varphi^{\rho}(x)-\varphi^{\rho}(t_0)\right|^{\alpha-1}$;

(ii) For $\alpha \in \left(0,\frac{1}{2}\right]$, $q = \dfrac{1+\alpha}{\alpha}$, and $p = 1+\alpha$, we have

$$|u(x)| \le \Bigg[2^{q-1}Q^q(x) + 2^{q-1}\left(\frac{\Gamma\left(p(\alpha-1)+1\right)}{\rho p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(x)}\right)^{\frac{q}{p}} \times \int_{t_0}^{x}\sum_{i=1}^{n} n^{q-1}\varphi\prime(t)\,\varphi^{\rho-1}(t)\,e^{-q\varphi(t)}g_i^q(t)\left[\gamma_i\left(2^{q-1}Q^q(t)-1\right)+1\right] \times \exp\left(\int_t^x 2^{q-1}\left(\frac{\Gamma(p(\alpha-1)+1)}{\rho p^{p(\alpha-1)+1}}e^{p\varphi^{\rho}(\tau)}\right)^{\frac{q}{p}}\sum_{i=1}^{n} n^{q-1}\varphi\prime(\tau)\,\varphi^{\rho-1}(\tau)\,e^{-q\varphi^{\rho}(\tau)}\gamma_i g_i^q(\tau)\,d\tau\right)dt\Bigg]^{\frac{1}{q}}.$$

Proof. Let us take $x \in [t_0,T)$. So, we have

$$|u(x)| \le \frac{|u_0|}{\Gamma(\alpha)}\left|\varphi^{\rho}(x)-\varphi^{\rho}(t_0)\right|^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\times\varphi'(t)\,\varphi^{\rho-1}(t)\sum_{i=1}^{n}\left|g_i\left(t,u(t)\right)\right|dt.$$

Using (H_2), we get

$$|u(x)| \le \frac{|u_0|}{\Gamma(\alpha)}\left|\varphi^{\rho}(x)-\varphi^{\rho}(t_0)\right|^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{x}\left(\varphi^{\rho}(x)-\varphi^{\rho}(t)\right)^{\alpha-1}\times\varphi'(t)\,\varphi^{\rho-1}(t)\sum_{i=1}^{n}\psi_i(t)\left|u^{\gamma_i}(t)\right|dt.$$

Thanks to Theorem 3.3, the proof is achieved. □

Example 4.4. *Assume the following data: let $\varphi(x) = x, u_0 = 0, (\psi+h)(x) = c, c \in \mathbb{R}$ and $g_(x) = x, g_2(x) = x^2$. Then the solution u is given by the form, using Mathematica 13.3, as follows (see Fig.4.5 and 4.6)*

$$u(x) = \frac{c}{\Gamma(\alpha)}\left(\frac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-2/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{2\Gamma\left(\frac{2}{\rho}\right)}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^2}\right), \quad x>0, \alpha>0, \rho>0, c\in\mathbb{R}.$$

And for $g_(x) = x, g_2(x) = x^2$ and $g_3(x) = x^3$ the becomes (see Figs. 4.7 and 4.8)

$$u(x) = \frac{c}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-3/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{3\Gamma\left(\frac{3}{\rho}\right)}{\Gamma\left(\alpha+\frac{3}{\rho}+1\right)}+\frac{2\Gamma\left(\frac{2}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{2/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^2}$$

The solution $u(x)$ could symbolize an economic variable in a dynamic economic framework, and the inequality may be utilized for establishing conditions in which the variable stays limited or comes together to a stable equilibrium over time. The parameters α and ρ are complex and self-replicating patterns that may be discovered at various sizes. Some economists have proposed that some patterns found in economic and financial data display fractal-fractional like features. Financial time series data, for instance stock prices or currency rates, may, for example, show self-similar patterns at multiple time scales. This indicates that short-term and long-term movements exhibit comparable patterns or tendencies. The examination of these shapes is known as fractional finance.

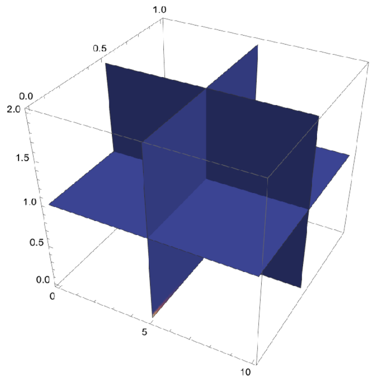

Figure 4.5: The SliceDensityPlot3D of $u(x) = \frac{c}{\Gamma(\alpha)}\left(\frac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-2/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{2\Gamma\left(\frac{2}{\rho}\right)}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^{2}}\right)$ for $(x,\alpha,\rho), c = 1$.

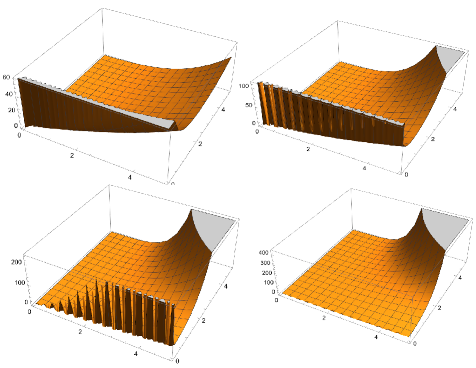

Figure 4.6: The Plot of $u(x) = \frac{c}{\Gamma(\alpha)}\left(\frac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-2/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{2\Gamma\left(\frac{2}{\rho}\right)}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^{2}}\right)$ for $(x,\rho), c = 1$ and $\alpha = 0.25, 0.5, 0.75, 0.95$.

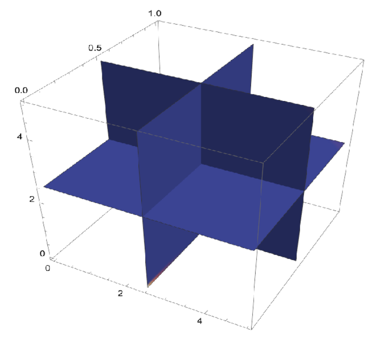

Figure 4.7: The SliceDensityPlot3D of $u(x) = \frac{c}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-3/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{3\Gamma\left(\frac{3}{\rho}\right)}{\Gamma\left(\alpha+\frac{3}{\rho}+1\right)}+\frac{2\Gamma\left(\frac{2}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{2/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^{2}}$ for $(x,\alpha,\rho), c = 1$.

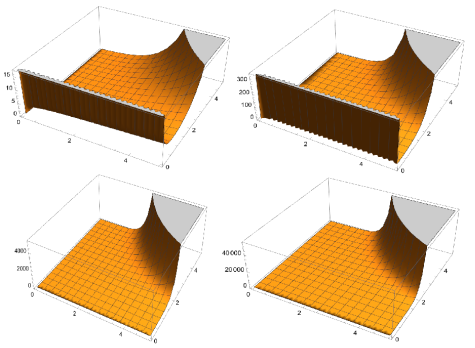

Figure 4.8: The Plot of $u(x)=\dfrac{c}{\Gamma(\alpha)}\dfrac{\Gamma(\alpha)\left(x^{-\rho}\right)^{-3/\rho}\left(x^{\rho}\right)^{\alpha}\left(\frac{3\Gamma\left(\frac{3}{\rho}\right)}{\Gamma\left(\alpha+\frac{3}{\rho}+1\right)}+\frac{2\Gamma\left(\frac{2}{\rho}\right)\left(x^{-\rho}\right)^{1/\rho}}{\Gamma\left(\alpha+\frac{2}{\rho}+1\right)}+\frac{\Gamma\left(\frac{1}{\rho}\right)\left(x^{-\rho}\right)^{2/\rho}}{\Gamma\left(\alpha+\frac{1}{\rho}+1\right)}\right)}{\rho^{2}}$ for $(x,\rho), c=1$ and $\alpha=0.25,0.5,0.75,0.95$.

5. Conclusion

The use of Gronwall's inequality in economics is part of a larger subject that is referred to as mathematical economics, which use mathematical strategies and instruments to understand economic events. It is crucial to note that the exact application of Gronwall's inequality in economics would be dependent on the specifics of the economic model under consideration. We have used one of our recent papers on $(\rho,\varphi)-$Riemann Liouville integrals to prove new results on Gronwall integral inequalities. Then, we have introduced, for the first time, the so-called $(\rho,\varphi)-$Riemann Liouville derivatives with respect to another function. We have presented some of their properties (Theorem 2.7). At the end, we have discussed two classes of differential equations that involve such derivatives. The boundedness of the solutions of these two classes has been established. We invite the interested reader to work on this "new" derivative approach since it has been shown in the study of the above two classes that the introduced derivatives are important to study differential equations. It has also been proved that they generalize several existing derivatives, see Remark 2.6.

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State Feedback Control of Multiagent Singular Linear Systems Representing Brain Neural Networks

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Abstract

A multi-agent singular system is an extension of a traditional multi-agent system. The behavior of neural networks within the brain is crucial for cognitive functions, making it essential to understand the learning processes and the development of potential disorders. This study utilizes the analysis of singular linear systems representing brain neural networks to delve into the complexities of the human brain. In this context, the digraph approach is a powerful method for unraveling the intricate neural interconnections. Directed graphs, or digraphs, provide an intuitive visual representation of the causal and influential relationships among different neural units, facilitating a detailed analysis of network dynamics. This work explores the use of digraphs in analyzing singular linear multi-agent systems that model brain neural networks, emphasizing their significance and potential in enhancing our understanding of cognition and brain function.

1. Introduction

The brain's architecture constitutes a complex recurrent neuronal network that can depicted through a digraph representation (see Figure 1.1). In this depiction, nodes symbolize distinct brain regions, while edges signify the intensity of connections formed between these regions during specific task execution.

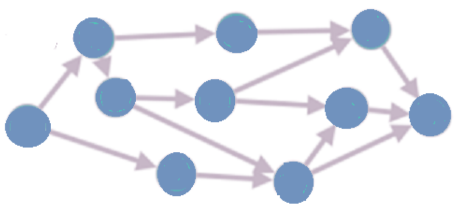

Figure 1.1: Recurrent Neuronal Network.

The term "neuronal network" refers to a specific framework for understanding brain function, wherein neurons serve as the fundamental computational entities and computational processes are interpreted through the lens of network interactions.
It has been shown ([1]) that cognitive control and the ability to control brain dynamics powerfully suggest improving cognitive functions and reversing the possible disorder in learning processes. The human brain can travel between diverse cognitive states. Its most significant function is in linking multiple sources of information in large-scale networks that are required to solve complex cognitive problems and strengthen memory.
Kriegeskorte in [2], states that neuronal network models fix a starting point of a new time for computational neuroscience, in which participants bear a part in real-world labors that require broad knowledge and elaborate calculations.
Neural networks that control their functions have been managed using dynamic linear control systems. In this study, neural networks are considered multi-agent systems, meaning they are systems of linear dynamic systems interconnected through a predetermined topology.

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Digraphs offer an intuitive visual representation of the causal and influential relationships between different neural units, allowing a detailed analysis of the network dynamics, [3].
Various fields of engineering employ multi-agent systems to solve synchronization problems and address consensus problems of the systems (see, for example, [4,5]). On the other hand, it should be said that neural networks are also being studied as non-linear dynamic systems (see, for example, [6]).
In many instances, it is challenging and costly to convert a description of the brain into a multi-agent linear dynamic system. This process involves a combination of differential equations and purely algebraic constraints, naturally transforming them into state-space equations such as:

$$\left.\begin{aligned} \dot{x}^1(t) &= A_1x^1(t)+B_1u^1(t) \\ &\vdots \\ \dot{x}^k(t) &= A_kx^k(t)+B_ku^k(t) \end{aligned}\right\}, \tag{1.1}$$

where $A_i \in M_n(\mathbb{R})$, $B_i \in M_{n\times m}(\mathbb{R})$, $x^i(t) \in \mathbb{R}^n$, $u^i(t) \in \mathbb{R}^m$, $1 \leq i \leq k$. García-Planas in [7] showed that a description of the system using equations of the type

$$\left.\begin{aligned} E_1\dot{x}^1(t) &= A_1x^1(t)+B_1u^1(t) \\ &\vdots \\ E_k\dot{x}^k(t) &= A_kx^k(t)+B_ku^k(t) \end{aligned}\right\}, \tag{1.2}$$

where $E_i \in M_n(\mathbb{R})$ are allowed to be singular matrices, may be much more suitable.
A block diagram is plotted in Figure 1.2.

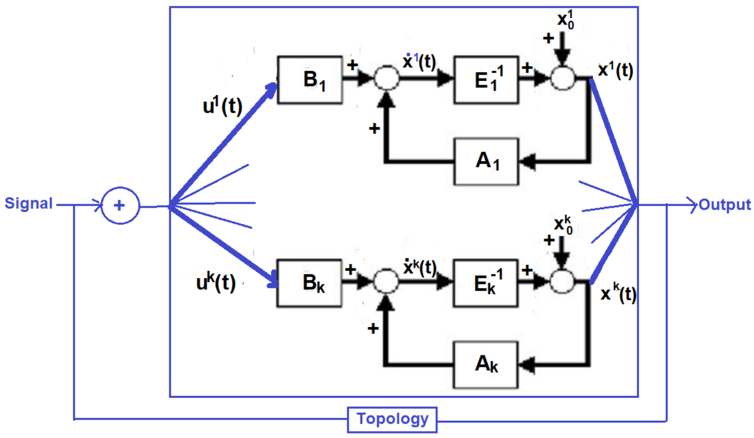

Figure 1.2: Diagram of a multi-agent singular system.

Systems and control theory can provide insights into the theoretical control of the human brain. Research on brain interfaces and neuromodulation indicates that changes in regional activity, measured by evoked potentials or other methods, can lead to alterations in brain function dynamics, [8].
Although fully understanding the relationship between mathematical control measures and the limited knowledge of cognitive control in neuroscience is challenging, even slight advances in this field can encourage further research and efforts to address learning difficulties such as dyscalculia and other issues like the phenomenon of forgetting, [8].
Structural controllability theory can be a powerful method for managing structured linear systems. In [7], Garcia-Planas demonstrated that structural controllability is a mathematical tool applicable to multi-agent singular systems, where each agent follows a predetermined structure.

2. Preliminaries

To investigate the proposed control problems, addressing the complexity inherent in the brain's structure is essential. This complexity necessitates dividing the global model into several local submodels, each representing distinct brain regions with intricate and interconnected network structures. By breaking down the global model into these local submodels, we can better understand and manage each region's specific dynamics and interactions. This approach enables us to conceptualize the brain as a collection of neuronal subnetworks, each contributing to the overall function and working together towards common cognitive and physiological objectives. By focusing on these localized models, we can develop more targeted and effective control strategies, ultimately enhancing our ability to understand and influence brain function as a cohesive multi-network system.
Let's consider a group of k agents as described in (1.2).
In our specific setup, the agents communicate according to the topology defined by the graph $\mathscr{G}$ with

i) Set of Vertices: $V = \{1,\dots,k\}$,
ii) Set of Edges: $\mathscr{E} = \{(i,j) \mid i,j \in V\} \subset V \times V$.

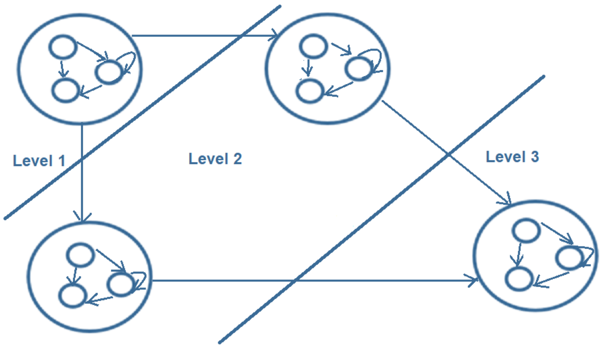

Figure 2.1: Multiagent graph.

The Figure (2.1) shows the graph that defines the topology on the participating agents in the system.
It is well known that each graph has an associated matrix called Laplacian; this matrix is defined as

$$\mathscr{L}=(l_{ij})=\begin{cases}|\mathscr{N}_i| & \text{if } i=j,\\ -1 & \text{if } j\in\mathscr{N}_i,\\ 0 & \text{otherwise.}\end{cases}$$

Writing:

$$\mathscr{X}(t)=\begin{pmatrix}x^1(t)\\ \vdots\\ x^k(t)\end{pmatrix},\quad \dot{\mathscr{X}}(t)=\begin{pmatrix}\dot{x}^1(t)\\ \vdots\\ \dot{x}^k(t)\end{pmatrix},\quad \mathscr{U}(t)=\begin{pmatrix}u^1(t)\\ \vdots\\ u^k(t)\end{pmatrix},$$

$$\mathscr{E}=\begin{pmatrix}E_1 & & \\ & \ddots & \\ & & E_k\end{pmatrix},\ \mathscr{A}=\begin{pmatrix}A_1 & & \\ & \ddots & \\ & & A_k\end{pmatrix},\ \mathscr{B}=\begin{pmatrix}B_1 & & \\ & \ddots & \\ & & B_k\end{pmatrix}.$$

With these notations, it is possible to describe the multisystem as a system:

$$\mathscr{E}\dot{\mathscr{X}}(t)=\mathscr{A}\mathscr{X}(t)+\mathscr{B}\mathscr{U}(t).$$

The description of the local interrelation between systems defined by the considered topology is given by the control:

$$u^i(t)=F_i\sum_{j\in\mathscr{N}_i}(x^i(t)-x^j(t)),\ 1\leq i\leq k. \tag{2.1}$$

That in a matrix description is

$$\mathscr{F}\mathscr{U}(t)=\mathscr{F}(\mathscr{L}\otimes I_n)\mathscr{X}(t),$$

where $\mathscr{F}=\begin{pmatrix}F_1 & & \\ & \ddots & \\ & & F_k\end{pmatrix}$.
Then, the multisystem with interrelation control is described as:

$$\mathscr{E}\dot{\mathscr{X}}(t)=\mathscr{A}\mathscr{X}(t)+\mathscr{B}\mathscr{F}\mathscr{U}(t)=(\mathscr{A}+\mathscr{B}\mathscr{F}(\mathscr{L}\otimes I_n))\mathscr{X}(t). \tag{2.2}$$

3. Controllability Character of a Singular Linear System

Controllability is a crucial property of dynamical systems, which is why there is an extensive body of literature addressing this concept ([7, 9, 10], among others).
Typical features of singular systems $E\dot{x}(t)=Ax(t)+Bu(t)$, which are unknown in the realm of state-space systems $\dot{x}(t)=Ax(t)+Bu(t)$ are possible impulsive responses to nonimpulsive excitations as well as provision for the consistency of initial conditions, complicating the analysis and design of control strategies. Understanding these features is essential for determining the controllability of such systems.
The controllability character can be defined using the following rank conditions which are generalizations of Hautus [9], and Kalman [10], conditions.

Proposition 3.1. *The dynamical Singular system $E\dot{x}(t) = Ax(t) + Bu(t)$ is controllable if and only if:*

$$\begin{array}{l}\operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = n, \\ \operatorname{rank}\begin{pmatrix} sE-A & B \end{pmatrix} = n,\ \forall s \in \mathbb{C}.\end{array}$$

Proposition 3.2 ([11]). *The dynamical system $E\dot{x}(t) = Ax(t) + Bu(t)$ is controllable if and only if the rank of the following matrix in $M_{n^2\times((n-1)n+nm)}(\mathbb{C})$ is n^2*

$$\operatorname{rank}\begin{pmatrix} E & 0 & 0 & \dots & 0 & B & 0 & 0 & \dots & 0 & 0 \\ A & E & 0 & \dots & 0 & 0 & B & 0 & \dots & 0 & 0 \\ 0 & A & E & \dots & 0 & 0 & 0 & B & \dots & 0 & 0 \\ & & \dots & \dots & & & & \dots & & & \\ 0 & 0 & 0 & \dots & E & 0 & 0 & 0 & \dots & B & 0 \\ 0 & 0 & 0 & \dots & A & 0 & 0 & 0 & \dots & 0 & B \end{pmatrix} = n^2.$$

We give evidence of the work applying it to simple example of a graph represented in Figure 3.1.

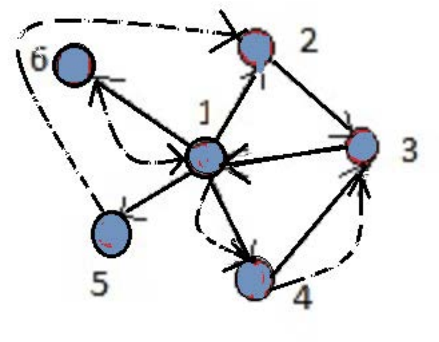

Figure 3.1: Example of a graph.

The system is

$$E = \begin{pmatrix} 0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0 \end{pmatrix},\quad A = \begin{pmatrix} 0&0&1&0&0&0\\1&0&0&0&0&0\\0&1&0&1&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0 \end{pmatrix},\quad B = \begin{pmatrix} 1&0&0\\2&0&0\\1&0&1\\0&1&0\\3&0&1\\1&0&0 \end{pmatrix}.$$

We have that

$$\begin{array}{l}\operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = 6, \\ \operatorname{rank}\begin{pmatrix} sE-A & B \end{pmatrix} = 6,\ \forall s \in \mathbb{C}.\end{array}$$

Then, the singular system is controllable. So, there exist matrices F and G in such away that$(E+BF)$ is invertible and the standard system $\dot{x}(t) = (E+BF)^{-1}(A+BG)x(t) + Bu(t)$ is controllable. For example, we can consider $F = \begin{pmatrix} 0&0&1&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0 \end{pmatrix}$ and $G = \begin{pmatrix} 0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{pmatrix}$.

The fact that

$$\begin{aligned} \operatorname{rank} B &= 3, \\ \operatorname{rank}\begin{pmatrix} B & (E+BF)^{-1}(A+BG)B \end{pmatrix} &= 5, \\ \operatorname{rank}\begin{pmatrix} B & (E+BF)^{-1}(A+BG)B & ((E+BF)^{-1}(A+BG))^2B \end{pmatrix} &= 6, \end{aligned}$$

ensures that taking suitable controls u_1, u_2, u_3 it is possible to reach a desired state from a fixed initial state in a finite time.
For example, taking $u_1 = (1, 0, -2.5)$, $u_2 = (-0.5, 0, 3)$ and $u_3 = (-1.5, 0, 3.5)$, it is possible to reach the node 5 from the node 6.
Writing $\mathbf{A} = (E+BF)^{-1}(A+BG)$, we have:

$$\mathbf{A}^3x_6 + \mathbf{A}^2Bu_1 + \mathbf{A}Bu_2 + Bu_3 = x_5.$$

Determining which B matrices ensure system controllability is challenging, especially when these matrices require the minimum number of inputs. Liu et al. [12] suggest “the maximum coincidence algorithm” based on the network representation of the matrix to select the control

nodes to ensure that systems are controllable. Yuan et al. in [13] exhibit a general framework based on the maximum multiplicity theory to investigate the exact controllability of multiplex interrelated networks, focussing the study on the controllability amount defined by the minimum set of drivers that are needed to control steering the whole system toward any desired state but the authors do not construct the possible drivers. García-Planas in [14] builds the matrices (drivers) based on the matrix A eigenvalues and its geometric multiplicity.
Given a linear dynamical system as $E\dot{x}(t) = Ax(t) + Bu(t)$ that for plainness, from now on, we can write as the triple of matrices (E,A,B). It is well known that the system has many possible control matrices B that can assure the controllability condition; for that, it suffices to consider invertible matrices $B \in Gl(n;\mathbb{R})$.
The objective is to identify the set of all possible matrices B with the minimum number of columns, corresponding to the minimum number $n_B(E,A)$ of independent controllers necessary to manage the entire network.
Controllability with the minimum number of inputs is referred to as *exact controllability*.

Definition 3.3. *Let (E,A) be a pair of matrices defining the homogeneous singular system $E\dot{x}(t) = Ax(t)$. The exact controllability $n_B(E,A)$ is the minimum of the rank of all possible matrices B making the system $E\dot{x}(t) = Ax(t) + Bu(t)$ controllable.*

$$n_B(E,A) = \min\{\operatorname{rank} B, \forall B \in M_{n\times i}\, 1 \leq i \leq n, (E,A,B) \text{ controllable}\}.$$

If confusion is not possible we will write simply n_B.
To want to know n_B makes it attractive to be able to use systems with the same properties, but due to the simplicity of their expression, the computation is immediate. In this sense, we consider the following equivalence relation:

Definition 3.4. *We say that two systems (E,A,B) and $(\bar{E},\bar{A},\bar{B})$ are equivalent if and only if, there exist invertible matrices P and Q such that $(\bar{E},\bar{A},\bar{B}) = (QEP,QAP,QB)$.*

This equivalence relation corresponds with strict equivalence of the pencil $\begin{pmatrix} sE-A & B \end{pmatrix}$. So, the collection of invariants of the pencil are the invariants for the system.
Besides, it is easy to prove that n_B is invariant under this equivalence relation,

Proposition 3.5. *The exact controllability n_B is invariant under equivalence relation considered, that is to say: for any couple of invertible matrices (Q,P),*

$$n_B(E,A) = n_{QB}(QEP,QAP).$$

Proof.

$$\operatorname{rank}\begin{pmatrix} QEP & QB \end{pmatrix} = \operatorname{rank} Q \begin{pmatrix} E & B \end{pmatrix} \begin{pmatrix} P & \\ & I \end{pmatrix} = \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix},$$

$$\operatorname{rank}\begin{pmatrix} sQEP - QAP & QB \end{pmatrix} = \operatorname{rank} Q \begin{pmatrix} sE-A & B \end{pmatrix} \begin{pmatrix} P & \\ & I \end{pmatrix} = \operatorname{rank}\begin{pmatrix} sE-A & B \end{pmatrix}.$$

□

As a consequence, if necessary we can consider (E,A,B) in a simpler form. In particular, the triple of matrices (E,A,B) can be reduced to a weaker form called "Quasi-Weierstraß form" (see [15]) in the following manner:
Let $P = \begin{pmatrix} V & W \end{pmatrix}$ and $Q = \begin{pmatrix} EV & AW \end{pmatrix}^{-1}$. Matrices $V \in M_{n\times r}(C)$ and $W \in M_{n\times(n-r)}(C)$ are in such a way that $\begin{pmatrix} V & W \end{pmatrix}$ and $\begin{pmatrix} EV & AW \end{pmatrix}$ are invertible.

$$(QEP,QAP,QB) = \left(\begin{pmatrix} I_r & \\ & N \end{pmatrix}, \begin{pmatrix} A_r & \\ & I_{n-r} \end{pmatrix}, \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix} \right) = (\tilde{E},\tilde{A},\tilde{B}),$$

where A_r is some matrix and N is nilpotent.
The vector spaces $\operatorname{Im} V$ and $\operatorname{Im} W$ are spanned by the generalized eigenvector at the finite and infinite eigenvalues respectively, and they are derived by the following recursive subspace iteration with a limited number of steps called Wong sequences [16].

$$\begin{array}{ll} V_0 = C^n, & V_{i+1} = \{v \in C^n \mid Av \in E(V_i)\}, \\ W_0 = \{0\}, & W_{i+1} = \{v \in C^n \mid Ev \in A(W_i)\}. \end{array}$$

verifying

$$\begin{array}{l} V_0 \supseteq V_1 \supseteq \ldots \supseteq V_\ell = V_{\ell+1} = \ldots V_{\ell+q} = V^* \supseteq \operatorname{Ker} A, \\ W_0 \subseteq W_1 \subseteq \ldots \subseteq w_m = W_{m+1} = \ldots W_{m+q} = W^*. \end{array}$$

It is easy to prove that $\ell = m$ and satisfy $AV^* \subseteq EV^*$ and $EW^* \subseteq AW^*$.
Matrices V and W are defined in such away that $V^* = \operatorname{Im} V$ and $W^* = \operatorname{Im} W$.

Remark 3.6. *Not every matrix B having n_B columns is valid to make the system controllable. For example if $E = I$, $A = \operatorname{diag}(1,2,3)$ and $B = (1,0,0)^t$, the system (A,B) is not controllable,* (rank $\begin{pmatrix} B & AB & A^2B \end{pmatrix} = 1 < 3$, *or equivalently* rank $\begin{pmatrix} A-\lambda I & B \end{pmatrix} = 2$ *for* $\lambda = 2,3$.

For standard systems we have the following result.

Proposition 3.7 ([17]). *Let $\mu(\lambda_i) = \dim \operatorname{Ker}(A - \lambda_i I)$ be the geometric multiplicity of the eigenvalue λ_i. Then,*

$$n_B = \max_i \{\mu(\lambda_i)\}$$

In [14] a manner to obtain a set of minimal number of controls is presented.
We now extend the theorem to the case of singular systems.

Theorem 3.8. *Let (E,A) the fixed homogeneous part of a singular system. The exact controllability n_B is computed in the following manner.*

$$n_B = \max\{n_E, \mu(\lambda_i)\}$$

where $n_E = \operatorname{rank}(E,B)$, $\mu(\lambda_i) = \dim \operatorname{Ker}(\lambda_i E - A)$ and λ_i (for each i) is the eigenvalue of pencil $sE - A$.

Proof. Proposition 3.5 permit us to consider the system in its canonical reduced form

$$\operatorname{rank}(E,B) = \operatorname{rank}\left(\begin{pmatrix} I & \\ & N \end{pmatrix}\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}\right) = n_1 + \operatorname{rank}(N, \bar{B}_2),$$

$$N = \operatorname{diag}(N_1, \dots, N_{n_E}) \text{ with } N_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}.$$

Taking

$$\bar{B}_2 = \begin{pmatrix} 0 & & \\ \vdots & & \\ 1 & & \\ & \ddots & \\ & & 0 \\ & & \vdots \\ & & 1 \\ & & 0 \\ & & \vdots \\ & & 0 \end{pmatrix} = \begin{pmatrix} w_1^\infty & \dots & w_{n_E}^\infty \end{pmatrix},$$

we have that $\operatorname{rank}(N, \bar{B}_2) = n_2$.

$$\operatorname{rank}\begin{pmatrix} \lambda_i E - A & B \end{pmatrix} = \operatorname{rank}\left(\lambda_i \begin{pmatrix} I & \\ & N \end{pmatrix} - \begin{pmatrix} J & \\ & I \end{pmatrix}\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}\right) = n_2 + \operatorname{rank}\begin{pmatrix} \lambda_i I - J & \bar{B}_1 \end{pmatrix}.$$

$$J = \operatorname{diag}(J_1(\lambda_1), \dots, J_r(\lambda_r)), J_i(\lambda_i) = \operatorname{diag}(J_{i_1}(\lambda_i), \dots, J_{i_{r_i}}(\lambda_i)),$$

and

$$J_{i_j}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{pmatrix}.$$

Taking

$$\bar{B}_{1_i} = \begin{pmatrix} 0 & & \\ \vdots & & \\ 1 & & \\ & \ddots & \\ & & 0 \\ & & \vdots \\ & & 1 \\ & & 0 \\ & & \vdots \\ & & 0 \end{pmatrix} = \begin{pmatrix} w_1^{\lambda_i} & \dots & w_{\mu_i(\lambda_i)}^{\lambda_i} \end{pmatrix},$$

we have that rank $(\lambda_i I - J, \bar{B}_{i_i}) = \mu(\lambda_i)$.
Consider now the following collection of vectors

$$\begin{array}{l} w_1^{\lambda_1}, \dots, w_\ell^{\lambda_1}, \\ \vdots \\ w_1^{\lambda_r}, \dots, w_\ell^{\lambda_r}, \\ w_1^{\infty}, \dots, w_\ell^{\infty}, \end{array}$$

where $\ell = \max(\mu(\lambda_1), \dots, \mu(\lambda_r), n_E)$ and we complete each series of vectors with the zero vectors in case its length is less than ℓ.
Finally we construct the family

$$w_1 = w_1^{\lambda_1} + \dots + w_1^{\lambda_r} + w_1^{\infty}, \dots, w_\ell = w_\ell^{\lambda_1} + \dots + w_\ell^{\lambda_r} + w_\ell^{\infty}.$$

Clearly,

$$\begin{array}{l} \text{rank}\begin{pmatrix} E & B \end{pmatrix} = n, \\ \text{rank}\begin{pmatrix} \lambda E - A & B \end{pmatrix} = n, \text{ for all } \lambda \in \mathbb{C}. \end{array}$$

Now, it suffices to remark that if we consider $B = (b_{ij}) \in M_{n\times m}(\mathbb{C})$ with $m < \ell$, we have

i) if $\ell = n_E$ then rank $\begin{pmatrix} E & B \end{pmatrix} < n$,
ii) if $\ell = \mu(\lambda_i)$ then rank $\begin{pmatrix} \lambda_i E - A & B \end{pmatrix} < n$.

□

4. Controllability of Multiagent Singular Neural Networks

We are concerned about bringing the output of the system (1.1) to a reference value and keeping it there, we can ensure that it is possible when the system is controllable. If topology relating systems is not considered, unquestionably, the system (1.1) is controllable if and only if each subsystem is controllable, and, in this case, there is feedback in which we obtain the requested solution.
We can be interested with the control (2.1) and ask for the stability of the system (2.2)
If having considered this control the resulting system (2.2) has not the desired eigenvalues, we can try to consider different feedback F_i so that, with the new control)with feedback $= K_i$),

$$u^i(t) = K_i \sum_{j\in\mathcal{N}_i} (x^i(t) - x^j(t)), \ 1 \le i \le k, \tag{4.1}$$

the system has appointed, eigenvalues to take a requested output of the system.
In some cases could be attentive in a solution such that

$$\lim_{t\to\infty} \|x^i - x^j\| = 0, \ 1 \le i, j \le k.$$

Namely, finding solutions for each subsystem reaching all, the same point.

Proposition 4.1. *Considering the control $u^i(t) = K_i \sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t)),\ 1 \le i \le k$ the closed-loop system can be detailed as*

$$\mathcal{E}\dot{\mathcal{X}}(t) = (\mathcal{A} + \mathcal{B}\mathcal{K}(\mathcal{L}\otimes I_n))\mathcal{X}(t).$$

where $\mathcal{K}$ is the diagonal matrix $\begin{pmatrix} K_1 & & \\ & \ddots & \\ & & K_k \end{pmatrix}$.
Computing the matrix $\mathcal{A} + \mathcal{B}\mathcal{K}(\mathcal{L}\otimes I_n)$ we obtain

$$\begin{pmatrix} A_1 + l_{11}B_1K_1 & l_{12}B_1K_1 & \dots & l_{1k}B_1K_1 \\ l_{21}B_2K_2 & A_2 + l_{22}B_2K_2 & \dots & l_{2k}B_2K_2 \\ \vdots & \vdots & \ddots & \vdots \\ l_{k1}B_kK_k & l_{k2}B_kK_k & \dots & A_k + l_{kk}B_kK_k \end{pmatrix}.$$

In the special case where all systems in the multi-system have the same dynamics, this is $E_i = E$, $A_i = A$, $B_i = B$ and $K_i = K$ Proposition 4.1 can be rewritten as follows:

Proposition 4.2. *Considering the control $u^i(t) = K\sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t)),\ 1 \le i \le k$ the closed-loop system for a multiagent with identical linear dynamical mode, is detailed as*

$$(I_k \otimes E)\dot{\mathcal{X}} = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L}\otimes I_n))\mathcal{X}.$$

It is also interesting to study the case that we can also consider external controls that allow us to obtain the desired eigenvalues.
It is of interest to recognize the minimum set of driver nodes needed to achieve full control of networks having arbitrary structures and link-weight distributions.
In our particular setup, the objective is to find the collection of all possible matrices E, having the minimum number of columns corresponding to the minimum number $n_D((\mathscr{A}+\mathscr{B}\mathscr{K}(\mathscr{L}\otimes I_n)))$ of independent drivers that are necessary to control the whole network.
Given the protocol as (4.1) with K the feedback gain matrix, and defining

$$\mathscr{U}_{\text{ext}}(t)=\begin{pmatrix}u^1_{\text{ext}}(t)\\ \vdots\\ u^k_{\text{ext}}(t)\end{pmatrix},\ \mathscr{D}=\begin{pmatrix}D_1 & & \\ & \ddots & \\ & & D_k\end{pmatrix}.$$

Proposition 4.3. *With these notations the system can be described as*

$$\mathscr{E}\dot{\mathscr{X}}(t)=(\mathscr{A}+\mathscr{B}\mathscr{K}(\mathscr{L}\otimes I_n))\mathscr{X}(t)+\mathscr{D}\mathscr{U}_{ext}(t).$$

And the minimum number of controls $\mathscr{D}$ necessary to ensure the controllability of the system is

$$n_{\mathscr{D}}=\max_i(n_{\mathscr{E}},\mu(\lambda_i))$$

where $n_{\mathscr{E}}=nk-\text{rank}\mathbb{E}$ and $\mu(\lambda_i)=\dim\text{Ker}\,(s\mathscr{E}-(\mathscr{A}+\mathbb{B}\mathbb{K}(\mathscr{L}\otimes I_n)),\mathscr{D})$.

Example 4.4. *Consider the case where* $E_i=E=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $A_i=A=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $B_i=B=\begin{pmatrix}1\\0\end{pmatrix}$ *and* $K_i=K=\begin{pmatrix}1&1\end{pmatrix}$ *corresponding to the Figure 4.1.*

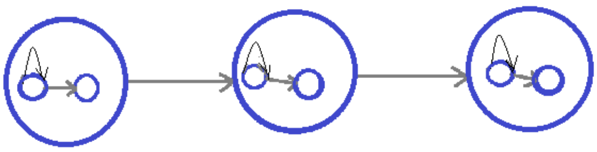

Figure 4.1: Neural network

The Laplacian matrix is

$$\mathscr{L}=\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&0\end{pmatrix}.$$

The matrix $\mathscr{A}+\mathscr{B}\mathscr{K}(\mathscr{L}\otimes I_n)$ *is*

$$\begin{pmatrix}1&0&-1&0&0&0\\1&0&0&0&0&0\\0&0&0&1&-1&0\\0&0&1&0&0&0\\0&0&0&0&1&-1\\0&0&0&0&1&0\end{pmatrix}.$$

In this case $n_{\mathscr{D}}=3$.
Taking

$$\mathscr{D}=\begin{pmatrix}D_1&&\\&D_2&\\&&D_3\end{pmatrix}=\begin{pmatrix}1&0&0\\2&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1\end{pmatrix},$$

we have

$$\text{rank}\begin{pmatrix}E&0&0&0&0&B&0&0&0&0&0\\A&E&0&0&0&0&B&0&0&0&0\\0&A&E&0&0&0&0&B&0&0&0\\0&0&A&E&0&0&0&0&B&0&0\\0&0&0&A&E&0&0&0&0&B&0\\0&0&0&0&A&0&0&0&0&0&B\end{pmatrix}=36.$$

Then, the multisystem is controllable.

5. Discussion and Conclusion

The concept of "control" implies taking action and represents the human endeavor to intervene in the environment to ensure survival and continually enhance the quality of life. Many control problems can be addressed through a mathematical model that describes the physical system in question with equations representing the system's state.
Although control is a fundamental issue in numerous network systems, more studies are still needed to quantitatively explore the control of directed networks, which are the most common configuration in real-world systems.
The primary issue is the network's size. Liu et al. [12] have developed tools to investigate the controllability of networks with arbitrary sizes and topologies using the controllability matrix, focusing on a few driver nodes within the network.
In [8], Gu et al. define different types of controllability (global, regional, average, modal, and boundary) from various perspectives to apply to neural systems. These different viewpoints can help analyze the various roles in controlling the dynamic trajectories of brain network functions.
This paper considers the brain network as a linear, discrete-time, time-invariant multisystem, which allows for considering a more significant number of nodes. In 2018, Abiodun et al. ([18]) conducted a survey on the state of the art in the applications of artificial neural networks.

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