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Binomial Transforms of k-Narayana Sequences and Some Properties

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Abstract

The aim of the study is to obtain new binomial transforms for the $k-$Narayana sequence. The first of these is the binomial transform, which is its normal form, and in the first step, after finding the recurrence relation of this new binomial transform, the generating function and Binet formula were obtained. Finally, Pascal's triangle was calculated. In the rest of the article, $k-$binomial transform was performed for the $k-$Narayana sequence and the recurrence relation, generating function, Binet formula and Pascal's triangle were examined for the new sequence obtained. Then, by performing the falling binomial transform and the rising binomial transform, the features listed above were found again for these sequences.

1. Introduction

Some special sequences of numbers such as Fibonacci, Lucas, Horadam and Narayana have been of great interest to the scientific world in recent years. Generalizations of these number sequences in various ways abound in the literature, in particular you can look at [1]. One of the most popular transforms is the binomial transform and it is sufficiently available in the literature.

Authors [2] presented the $k-$Fibonacci sequence also the same authors for this sequences of numbers [3] introduced different binomial transforms, such as falling and rising binomial transforms. Binomial transforms and properties of $k-$Lucas sequences are presented in [4]. Spivey and Steil [5] gave various binomial transforms. In [6], they obtained some applications for the generalized (s,t) matrix sequences. In [7], authors obtained binomial transforms of Padovan and Perrin numbers from the third order.

The person who discovered the Narayana sequence is Narayana, an Indian mathematician, and is as follows

$$N_m = N_{m-1} + N_{m-3} \text{ with } m \geq 3 \tag{1.1}$$

where

$$N_0 = 0,\ N_1 = 1,\ N_2 = 1,$$

see [8]. The first few terms are $0,1,1,1,2,3,4,6,9,13,19,28,41,60,\cdots$.

The characteric equation of (1.1) is :

$$\Psi^3 - \Psi^2 - 1 = 0,$$

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and roots of the characteristic equation are :

$$\begin{aligned}
\Psi_1 &= \frac{1}{3}\left(\sqrt[3]{\frac{1}{2}(29-3\sqrt{93}}+\sqrt[3]{\frac{1}{2}(3\sqrt{93}+29)}+1\right),\\
\Psi_2 &= \frac{1}{3}-\frac{1}{3}(1-i\sqrt{3})\sqrt[3]{\frac{1}{2}(29-3\sqrt{93}}-\frac{1}{6}(1+i\sqrt{3})\sqrt[3]{\frac{1}{2}(3\sqrt{93}+29)},\\
\Psi_3 &= \frac{1}{3}-\frac{1}{3}(1+i\sqrt{3})\sqrt[3]{\frac{1}{2}(29-3\sqrt{93}}-\frac{1}{6}(1-i\sqrt{3})\sqrt[3]{\frac{1}{2}(3\sqrt{93}+29)}.
\end{aligned}$$

Hence, the Narayana sequence can be obtained by Binet's formula:

$$N_m=\frac{\Psi_1^2}{\Psi_1^3+2}\Psi_1^n+\frac{\Psi_2^2}{\Psi_2^3+2}\Psi_2^n+\frac{\Psi_3^2}{\Psi_3^3+2}\Psi_3^n.$$

Generating function found for Narayana equation is:

$$\frac{1}{1-\Psi-\Psi^3}=\sum_{n=0}^{\infty}N_{m+1}\Psi_1^n \text{ , for } n\geq 1,\ n\in\mathbb{Z}.$$

Narayana sequence which has attracted the attention of more mathematicians in recent years and its generalizations. Some of them are as follows:

Some basic properties of Fibonacci-Narayana numbers are proved in [9]. Bilgici in [10], defined a generalized order k Fibonacci-Narayana sequence and by using this generalization and some matrix properties, established some identities related to Fibonacci-Narayana numbers. Soykan studied on Narayana sequence in [11]. Ramirez and Sirvent in [12], introduced the $k-$Narayana sequence and found the identities between these numbers.

For any nonzero integer number k , $k-$Narayana sequence is defined by the following recurrence relation:

$$N_{k,m}=kN_{m-1}+N_{m-3} \text{ with } m\geq 3 \tag{1.2}$$

where

$$N_{k,0}=0,\ N_{k,1}=1,\ N_{k,2}=k,$$

see [12]. The first few terms are $0,1,k,k^2,k^3+1,k^4+2k,k^5+3k^2,k^6+4k^3+1,k^7+5k^4+3k\cdots$.

The characteric equation of (1.2) is :

$$\lambda^3-k\lambda^2-1=0,$$

and the roots of characteristic equation are :

$$\begin{aligned}
\lambda_1 &= \frac{1}{3}\left(k+k^2\sqrt[3]{\frac{2}{27+2k^3+3\sqrt{81+12k^3}}}+\sqrt[3]{\frac{27+2k^3+3\sqrt{81+12k^3}}{2}}\right),\\
\lambda_2 &= \frac{1}{3}\left(k-\mu k^2\sqrt[3]{\frac{2}{27+2k^3+3\sqrt{81+12k^3}}}+\mu^2\sqrt[3]{\frac{27+2k^3+3\sqrt{81+12k^3}}{2}}\right),\\
\lambda_3 &= \frac{1}{3}\left(k+\mu^2k^2\sqrt[3]{\frac{2}{27+2k^3+3\sqrt{81+12k^3}}}-\mu\sqrt[3]{\frac{27+2k^3+3\sqrt{81+12k^3}}{2}}\right)
\end{aligned}$$

where $\mu=\frac{1+i\sqrt{3}}{2}$ is the primitive cube root of unity.

The generating function of the $k-$Narayana sequence is

$$\frac{1}{1-k\lambda-\lambda^3}.$$

Therefore the $k-$Narayana sequence can be obtained by Binet's formula:

$$N_{k,n}=\frac{\lambda_1^{n+1}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\lambda_2^{n+1}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{\lambda_3^{n+1}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\text{ , } n\geq 0.$$

Other recent research ([13],[14],[15]) has also investigated various binomial transforms for various special sequences. These transforms are valuable because they bring a new approach. For details on the binomial transform, see ([16],[17]).

The focus of this paper is to apply binomial transforms and its generalization (like $k-$binomial transform, rising transform and falling transform) to the $k-$Narayana sequence. In addition to these, the recurrence relation, Binet's formula, generating function, Pascal triangle and matrix representation of related transforms were derived.

2. Binomial transform of $k-$Narayana sequences

The binomial transform of $k-$Narayana sequence $\{N_{k,n}\}_{n\in\mathbb{N}}$ is shown as $\{b_{k,n}\}_{n\in\mathbb{N}}$ where $b_{k,n}$ is dedicated by

$$b_{k,n}=\sum_{i=0}^{n}\binom{n}{i}N_{k,i}.$$

To find the recurrence relation of $\{b_{k,n}\}$, we first need a Lemma.

Lemma 2.1. *Let n is a positive integer greater than 1, then $\{b_{k,n}\}$ contents the next equation*

$$b_{k,n+1}=\sum_{i=0}^{n}\binom{n}{i}(N_{k,i}+N_{k,i+1}).$$

Proof. We have,

$$b_{k,n}=\sum_{i=0}^{n}\binom{n}{i}N_{k,i}.$$

If we bear in mind summation feature of binomial numbers $\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}$, also $\binom{n}{n+1}=0$ for the proof. Then we find

$$\begin{aligned} b_{k,n+1} &= \sum_{i=0}^{n+1}\binom{n+1}{i}N_{k,i} \\ &= N_{k,0}+\sum_{i=1}^{n+1}\binom{n+1}{i}N_{k,i} \\ &= N_{k,0}+\sum_{i=1}^{n+1}\binom{n}{i}N_{k,i}+\sum_{i=1}^{n+1}\binom{n}{i-1}N_{k,i}. \end{aligned}$$

Also thanks to the operations performed on the sums

$$\begin{aligned} N_{k,0}+\sum_{i=1}^{n+1}\binom{n}{i}N_{k,i}+\sum_{i=1}^{n+1}\binom{n}{i-1}N_{k,i} &= N_{k,0}+\sum_{i=1}^{n}\binom{n}{i}N_{k,i}+\sum_{i=1}^{n}\binom{n}{i}N_{k,i+1} \\ &= \sum_{i=0}^{n}\binom{n}{i}N_{k,i}+\sum_{i=0}^{n}\binom{n}{i}N_{k,i+1} \\ &= \sum_{i=0}^{n}\binom{n}{i}(N_{k,i}+N_{k,i+1}). \end{aligned}$$

□

The next theorem presents recurrence relation for $\{b_{k,n}\}$.

Theorem 2.2. *The recurrence relation obtained for $\{b_{k,n}\}$ is as follows:*

$$b_{k,n+3}=(k+3)b_{k,n+2}-(2k+3)b_{k,n+1}+(k+2)b_{k,n} \tag{2.1}$$

where $b_{k,0}=0, b_{k,1}=1$, and $b_{k,2}=k+2$.

Proof. To find the coefficients in (2.1)

$$b_{k,n+3}=A_1b_{k,n+2}+A_2b_{k,n+1}+A_3b_{k,n}.$$

If we take $n=0,1$ and 2, we have the system

$$\begin{aligned} b_{k,3} &= A_1b_{k,2}+A_2b_{k,1}+A_3b_{k,0}=k^2+3k+3 \\ b_{k,4} &= A_1b_{k,3}+A_2b_{k,2}+A_3b_{k,1}=k^3+4k^2+6k+5 \\ b_{k,5} &= A_1b_{k,4}+A_2b_{k,3}+A_3b_{k,2}=k^4+5k^3+10k^2+12k+10. \end{aligned}$$

By Cramer rule for the system, we get

$$A_1=k+3,\ A_2=-2k-3,\ \text{and}\ A_3=k+2.$$

So which is completed the proof . □

The characteristic equation of sequences $b_{k,n}$ in (2.1) is

$$\alpha^3-(k+3)\alpha^2+(2k+3)\alpha-(k+2)=0,$$

whose solutions are

$$\begin{aligned}\alpha_1 &= T+S+\frac{k+3}{3}\\ \alpha_2 &= \frac{T(w-1)}{2}-\frac{S(w+1)}{2}+\frac{k+3}{3}\\ \alpha_3 &= \frac{S(w-1)}{2}-\frac{T(w+1)}{2}+\frac{k+3}{3}\end{aligned}$$

where

$$\begin{aligned}T &= \left(\frac{k^3}{27}+\sqrt{\Delta}\right)^{\frac{1}{3}},\\ S &= \left(\frac{k^3}{27}-\sqrt{\Delta}\right)^{\frac{1}{3}},\\ \Delta &= \frac{k^3}{27}+\frac{1}{4},\ \ w=i\sqrt{3}.\end{aligned}$$

Next we derive the Binet formula for $\left\{b_{k,n}\right\}$.

Theorem 2.3. *The Binet formula for the $k-$Narayana sequence is as follows:*

$$b_{k,n}=\frac{p_1\alpha_1^n}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}+\frac{p_2\alpha_2^n}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}+\frac{p_3\alpha_3^n}{(\alpha_3-\alpha_1)(\alpha_2-\alpha_3)}$$

where

$$\begin{aligned}p_1 &= b_{k,2}-(\alpha_2+\alpha_3)b_{k,1}+\alpha_2\alpha_3b_{k,0}=k+2-(\alpha_2+\alpha_3),\\ p_2 &= b_{k,2}-(\alpha_1+\alpha_3)b_{k,1}+\alpha_1\alpha_3b_{k,0}=k+2-(\alpha_1+\alpha_3),\\ p_3 &= b_{k,2}-(\alpha_1+\alpha_2)b_{k,1}+\alpha_1\alpha_2b_{k,0}=k+2-(\alpha_1+\alpha_2).\end{aligned}$$

Proof. To obtain Binet formula let us write

$$b_{k,n}=B_1\alpha_1^n+B_2\alpha_2^n+B_3\alpha_3^n$$

If we take $n=0,1$ and 2, we have the system

$$\begin{aligned}b_{k,0} &= B_1+B_2+B_3=0\\ b_{k,1} &= B_1\alpha_1+B_2\alpha_2+B_3\alpha_3=1\\ b_{k,2} &= B_1\alpha_1^2+B_2\alpha_2^2+B_3\alpha_3^2=k+2\end{aligned}$$

By Cramer rule for the system, we get

$$\begin{aligned}B_1 &= \frac{b_{k,2}-(\alpha_2+\alpha_3)b_{k,1}+\alpha_2\alpha_3b_{k,0}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}\\ B_2 &= \frac{b_{k,2}-(\alpha_1+\alpha_3)b_{k,1}+\alpha_1\alpha_3b_{k,0}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}\\ B_2 &= \frac{b_{k,2}-(\alpha_1+\alpha_2)b_{k,1}+\alpha_1\alpha_2b_{k,0}}{(\alpha_3-\alpha_1)(\alpha_2-\alpha_3)}\end{aligned}$$

So which is completed the proof . □

Now let's obtain the generating function for the $k-$Narayana binomial transform.

Theorem 2.4. *The generating function of $\left\{b_{k,n}\right\}$ is:*

$$b_k(x)=\frac{(1-kx-3x+2kx^2+3x^2)b_{k,0}+(x-kx^2-3x^2)b_{k,1}+x^2b_{k,2}}{1-(k+3)x-(2k+3)x^2-(k+2)x^3}.$$

Proof. We have, $b_k(x) = b_{k,0} + b_{k,1}x + b_{k,2}x^2 + b_{k,3}x^3 + \cdots + b_{k,n}x^n + \cdots$. After doing simple operations we obtain

$$\begin{aligned} b_k(x) &= b_{k,0} + b_{k,1}x + b_{k,2}x^2 + b_{k,3}x^3 + \cdots \\ -(k+3)xb_k(x) &= -b_{k,0}(k+3)x - b_{k,1}(k+3)x^2 - b_{k,3}(k+3)x^3 + \cdots \\ -(2k+3)x^2b_k(x) &= -b_{k,0}(2k+3)x^2 - b_{k,1}(2k+3)x^3 - b_{k,3}(2k+3)x^4 + \cdots \\ -(k+2)x^3b_k(x) &= -b_{k,0}(k+2)x^3 - b_{k,1}(k+2)x^4 - b_{k,3}(k+2)x^5 + \cdots . \end{aligned}$$

From these equations and (2.1), we get

$$\left[1-(k+3)x-(2k+3)x^2-(k+2)x^3\right] b_k(x) = (1-kx-3x+2kx^2+3x^2)b_{k,0} + (x-kx^2-3x^2)b_{k,1} + x^2b_{k,2}.$$

So the generating function for the binomial transform of the $k-$Narayana sequence is

$$b_k(x) = \frac{(1-kx-3x+2kx^2+3x^2)b_{k,0} + (x-kx^2-3x^2)b_{k,1} + x^2b_{k,2}}{1-(k+3)x-(2k+3)x^2-(k+2)x^3}.$$

□

Let's give a new triangle $\{b_{k,n}\}$ for each k to help with the next rules:

1. The part forming the left corner of the triangle consists of the elements of $k-$ Narayana numbers,
2. When we take any number and think that it is chosen outside the left diagonal, it is considered to be the sum of the number to the left of this number and also the number above its diagonal on the left side.
3. On the right diagonal is $\{b_{k,n}\}$.

The next triangle is an example of the $1-$Narayana sequence:

```
            0
         1     1
      1     2     3
   1     2     4     7
2     3     5     9     16
```

Figure 1: $1-$Narayana sequence

3. The $k-$Binomial transform of the $k-$Narayana sequence

The $k-$binomial transform of the $k-$Narayana sequence $\{N_{k,n}\}_{n\in\mathbb{N}}$ is denoted by $\{w_{k,n}\}_{n\in\mathbb{N}}$ where

$$w_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^n N_{k,i}.$$

Lemma 3.1. *Let n is an integer greater than and equal to 1, and $k-$binomial transform of $k-$Narayana sequence satisfies the following relation*

$$w_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} k^{n+1}(N_{k,i} + N_{k,i+1}).$$

Proof. We know that

$$w_{k,n} = \sum_{i=0}^{n} \binom{n}{i} N_{k,i}.$$

If we take $n+1$ instead of n and consider the binomial properties then we have

$$\begin{aligned}
w_{k,n+1} &= \sum_{i=0}^{n+1}\binom{n+1}{i}N_{k,i} \\
&= k^{n+1}N_{k,0}+\sum_{i=1}^{n+1}\binom{n+1}{i}k^{n+1}N_{k,i} \\
&= k^{n+1}N_{k,0}+\sum_{i=1}^{n+1}\binom{n}{i}\binom{n}{i-1}k^{n+1}N_{k,i} \\
&= k^{n+1}N_{k,0}+\sum_{i=1}^{n+1}\binom{n}{i}N_{k,i}+\sum_{i=1}^{n+1}\binom{n}{i-1}N_{k,i} \\
&= k^{n+1}N_{k,0}+\sum_{i=1}^{n}\binom{n}{i}k^{n+1}N_{k,i}+\sum_{i=1}^{n}\binom{n}{i}k^{n+1}N_{k,i+1}
\end{aligned}$$

so we get

$$w_{k,n}=\sum_{i=0}^{n}\binom{n}{i}k^{n+1}(N_{k,i}+N_{k,i+1}).$$

□

The next theorem will provides the recurrence relation for $\{w_{k,n}\}$.

Theorem 3.2. *The recurrence relation obtained for $\{w_{k,n}\}$ is as follows:*

$$w_{k,n+3}=(k^2+3k)w_{k,n+2}-(2k^3+3k^2)w_{k,n+1}+(k^4+2k^3)w_{k,n}. \tag{3.1}$$

Proof. From the recurrence relation of the corresponding transform, there is a general solution as follows

$$w_{k,n+3}=C_1w_{k,n+2}+C_2w_{k,n+1}+C_3w_{k,n}.$$

If $n=0,1$ and 2, the following system is obtained

$$\begin{aligned}
w_{k,3} &= C_1w_{k,2}+C_2w_{k,1}+C_3w_{k,0}=k^5+3k^4+3k^3 \\
w_{k,4} &= C_1w_{k,3}+C_2w_{k,2}+C_3w_{k,1}=k^7+4k^6+6k^5+5k^4 \\
w_{k,5} &= C_1w_{k,4}+C_2w_{k,3}+C_3w_{k,2}=k^9+5k^8+10k^7+12k^6+10k^5
\end{aligned}$$

By Cramer rule for the system, we get

$$C_1=k^2+3k,\ C_2=-2k^3-3k^2,\ \text{and } C_3=k^4+2k^3.$$

so that the evidence is completed. □

The characteristic equation of sequences $w_{k,n}$ in (3.1) is

$$\beta^3-(k^2+3k)\beta^2+(2k^3+3k^2)\beta-(k^4+2k^3)=0,$$

whose solutions are β_1, β_2, and β_3.
Now we construct the Binet formula for $\{w_{k,n}\}$.

Theorem 3.3. *Whichever term of $\{w_{k,n}\}$ can be computed using the Binet formula. It is indicated by*

$$w_{k,n}=\frac{q_1\beta_1^n}{(\beta_1-\beta_2)(\beta_1-\beta_3)}+\frac{q_2\beta_2^n}{(\beta_2-\beta_1)(\beta_2-\beta_3)}+\frac{q_3\beta_3^n}{(\beta_3-\beta_1)(\beta_2-\beta_3)}$$

where

$$\begin{aligned}
q_1 &= w_{k,2}-(\beta_2+\beta_3)w_{k,1}+\beta_2\beta_3w_{k,0}=k\left[k^2+2k-(\beta_2+\beta_3)\right] \\
q_2 &= w_{k,2}-(\beta_1+\beta_3)w_{k,1}+\beta_1\beta_3w_{k,0}=k\left[k^2+2k-(\beta_1+\beta_3)\right] \\
q_3 &= w_{k,2}-(\beta_1+\beta_2)w_{k,1}+\beta_1\beta_2w_{k,0}=k\left[k^2+2k-(\beta_1+\beta_2)\right]
\end{aligned}$$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 2 & & & \\ & & 2 & & 6 & & 16 & & \\ & 4 & & 12 & & 36 & & 104 & \\ 9 & & 26 & & 76 & & 224 & & 656 \end{array}$$

Figure 2: $2-$Narayana sequence

Proof. To obtain Binet formula let us write

$$w_{k,n} = D_1\alpha_1^n + D_2\alpha_2^n + D_3\alpha_3^n$$

If we take $n = 0, 1$and 2, we have the system

$$\begin{aligned} w_{k,0} &= D_1 + D_2 + D_3 = 0 \\ w_{k,1} &= D_1\beta_1 + D_2\beta_2 + D_3\beta_3 = k \\ w_{k,2} &= D_1\beta_1^2 + D_2\beta_2^2 + D_3\beta_3^2 = k^3 + 2k^2 \end{aligned}$$

By Cramer rule for the system, we get

$$\begin{aligned} D_1 &= \frac{w_{k,2} - (\beta_2+\beta_3)w_{k,1} + \beta_2\beta_3 w_{k,0}}{(\beta_1-\beta_2)(\beta_1-\beta_3)}, \\ D_2 &= \frac{w_{k,2} - (\beta_1+\beta_3)w_{k,1} + \beta_1\beta_3 w_{k,0}}{(\beta_2-\beta_1)(\beta_2-\beta_3)}, \\ D_2 &= \frac{w_{k,2} - (\beta_1+\beta_2)w_{k,1} + \beta_1\beta_2 w_{k,0}}{(\beta_3-\beta_1)(\beta_2-\beta_3)}. \end{aligned}$$

So which is completed the proof . □

Theorem 3.4. *The generating function of* $\{w_{k,n}\}$ *is:*

$$w_k(x) = \frac{(1-k^2x-3kx+2k^3x^2+3k^2x^2)w_{k,0} + (x-k^2x^2-3kx^2)w_{k,1} + x^2w_{k,2}}{1-k^2x-3kx+2k^3x^2+3k^2x^2-k^4x^3-2k^3x^3}.$$

Proof. We have $w_k(x) = w_{k,0} + w_{k,1}x + w_{k,2}x^2 + w_{k,3}x^3 \cdots + w_{k,n}x^n + \cdots$
Then, if multiplication is done $-(k^2+3k)x$, $(2k^3+3k^2)x^2$, and $-(k^4+2k^3)x^3$, we obtain

$$\begin{aligned} w_k(x) &= w_{k,0} + w_{k,1}x + w_{k,2}x^2 + w_{k,3}x^3 + \cdots \\ -(k^2+3k)xw_k(x) &= -w_{k,0}(k^2+3k)x - w_{k,1}(k^2+3k)x^2 - w_{k,3}(k^2+3k)x^3 + \cdots \\ (2k^3+3k^2)x^2w_k(x) &= w_{k,0}(2k^3+3k^2)x^2 + w_{k,1}(2k^3+3k^2)x^3 + w_{k,3}(2k^3+3k^2)x^4 + \cdots \\ -(k^4+2k^3)x^3w_k(x) &= -w_{k,0}(k^4+2k^3)x^3 - w_{k,1}(k^4+2k^3)x^4 - w_{k,3}(k^4+2k^3)x^5 + \cdots \end{aligned}$$

from these equations and (3.1), we get

$$\left[1-k^2x-3kx+2k^3x^2+3k^2x^2-k^4x^3-2k^3x^3\right]w_k(x) = (1-k^2x-3kx+2k^3x^2+3k^2x^2)w_{k,0} + (x-k^2x^2-3kx^2)w_{k,1} + x^2w_{k,2}$$

and so the generating function for the $k-$binomial transform of the $k-$Narayana sequence is

$$w_k(x) = \frac{(1-k^2x-3kx+2k^3x^2+3k^2x^2)w_{k,0} + (x-k^2x^2-3kx^2)w_{k,1} + x^2w_{k,2}}{1-k^2x-3kx+2k^3x^2+3k^2x^2-k^4x^3-2k^3x^3}.$$

□

Now, we present a new triangle of the $k-$binomial transform of the $k-$Narayana sequence for each k. The next triangle is an example of the $2-$Narayana sequence:
Since the proofs in this section are similar to the proof steps in the previous section, the theorems are given without proofs.

4. The rising $k-$binomial transform of the $k-$Narayana sequence

The rising $k-$binomial transform of the $k-$Narayana sequence $\{N_{k,n}\}_{n\in\mathbb{N}}$ is denoted by $\{r_{k,n}\}_{n\in\mathbb{N}}$ where

$$r_{k,n}=\sum_{i=0}^{n}\binom{n}{i}k^{i}N_{k,i}.$$

Theorem 4.1. *The recurrence relation obtained for $\{r_{k,n}\}$ is as follows:*

$$r_{k,n+3}=(k^2+3)r_{k,n+2}-(2k^2+3)r_{k,n+1}+(k^3+k^2+1)r_{k,n}. \tag{4.1}$$

The characteristic equation of $\{b_{k,n}\}$ in (4.1) is

$$\gamma^3-(k^2+3)\gamma^2+(2k^2+3)\gamma-(k^3+k^2+1)=0,$$

whose solutions are γ_1, γ_2, and γ_3.
Next we derive the Binet formula for the rising $k-$binomial transform of the $k-$Narayana sequence.

Theorem 4.2. *Whichever term of $\{r_{k,n}\}$ can be computed using the Binet formula. It is indicated by*

$$r_{k,n}=\frac{u_1\gamma_1^n}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)}+\frac{u_2\gamma_2^n}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)}+\frac{u_3\gamma_3^n}{(\gamma_3-\gamma_1)(\gamma_2-\gamma_3)}$$

where

$$\begin{aligned}u_1&=k^3-\gamma_2k-\gamma_3k+2k\\u_2&=k^3-\gamma_1k-\gamma_3k+2k\\u_3&=k^3-\gamma_1k-\gamma_2k+2k\end{aligned}$$

Theorem 4.3. *The generating function of $\{r_{k,n}\}$ is:*

$$r_k(x)=\frac{(1-kx^2-3+2k^2x^2+3x^2)r_{k,0}+(1-k^2x^2-3x^2)r_{k,1}+x^2r_{k,2}}{1-kx^2-3+2k^2x^2+3x^2-k^3x^3-k^2x^3-x^3}.$$

Now, we present a new triangle of $\{r_{k,n}\}$ for each k. The next triangle is an example of the $2-$Narayana sequence:

				0				
			1		2			
		2		5		12		
	4		10		25		62	
9		22		54		133		328

Figure 3: $2-$Narayana sequence and its rising $2-$binomial transform

5. The falling $k-$Binomial transform of the $k-$Narayana sequence

The falling $k-$binomial transform of the $k-$Narayana sequence $\{N_{k,n}\}_{n\in\mathbb{N}}$ is denoted by $\{f_{k,n}\}_{n\in\mathbb{N}}$ where

$$f_{k,n}=\sum_{i=0}^{n}\binom{n}{i}k^{n-i}N_{k,i}.$$

Theorem 5.1. *The recurrence relation obtained for $\{f_{k,n}\}$ is as follows:*

$$f_{k,n+3}=4kf_{k,n+2}-5k^2f_{k,n+1}+(2k^3+1)f_{k,n}. \tag{5.1}$$

The characteristic equation of sequences $\{b_{k,n}\}$ in (5.1) is

$$\theta^3-4k\theta^2+5k^2\theta-(2k^3+1)=0,$$

whose solutions are θ_1, θ_2, and θ_3.
Next we derive the Binet formula for $\{f_{k,n}\}$.

Theorem 5.2. *Whichever term of $\{f_{k,n}\}$ can be computed using the Binet formula. It is indicated by*

$$f_{k,n}=\frac{t_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)}+\frac{t_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)}+\frac{t_3\theta_3^n}{(\theta_3-\theta_1)(\theta_2-\theta_3)}$$

where

$$\begin{aligned}t_1 &= 3k-\theta_2-\theta_3\\ t_2 &= 3k-\theta_1-\theta_3\\ t_3 &= 3k-\theta_1-\theta_2\end{aligned}$$

Theorem 5.3. *The generating function of $\{f_{k,n}\}$ is:*

$$f_k(x)=\frac{(1-4kx+5k^2x^2)f_{k,0}+(x-4kx^2)f_{k,1}+x^2f_{k,2}}{1-4kx+5k^2x^2-2k^3x^3-x^3}.$$

Now, we present a new triangle of $\{f_{k,n}\}$ for each k. For example following triangle is for $2-$Narayana sequence and its falling $2-$binomial transform

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 2 & & 4 & & 6 & & \\ & 4 & & 8 & & 16 & & 28 & \\ 9 & & 17 & & 33 & & 65 & & 121 \end{array}$$

Figure 4: $2-$Narayana sequence and its falling $2-$binomial transform

Declarations

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Investigation of Power-Law Fluid on a Decelerated Rotating Disk

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Abstract

This study explores the behaviour of power-law fluids over decelerating rotating disks. The disk's angular velocity decreases inversely with time, and the unsteady governing equations modeling this flow yield similarity transformations that depend on the nondimensional parameter $\hat{\alpha} = \frac{\alpha}{\Omega_0}$. These transformations, introduced here for the first time in the literature, allow for a comprehensive analysis of the fluid dynamics for shear-thinning fluids within the range $0.5 < n \leq 1$.

We examine the no-slip boundary condition alongside the dimensionless unsteadiness parameter, which quantifies the initial deceleration or acceleration of the disk. We present velocity profiles and the viscosity function for various values of $\hat{\alpha}$. The boundary layer problem, formulated through dimensionless momentum and continuity equations derived via similarity transformations, is solved using the `bvp4c` function in MATLAB. This numerical method, employing the 4th-order Runge-Kutta algorithm, provides approximate solutions for the U, V, and W velocity profiles and the μ viscosity function, considering different deceleration parameters and the power-law index n.

Our findings contribute novel insights into the fluid dynamics of power-law fluids in decelerating rotational systems, offering potential applications in industrial and engineering processes where such conditions are prevalent.

1. Introduction

The study of boundary layer analysis on rotating disks has been a topic of significant interest in fluid dynamics research. This area was first explored by Theodore von Karman [1], who established the foundational equations for steady boundary layer flow. The von Kármán boundary layer flow is part of a broader family of flow types characterized by the differential rotation rate between a solid disk and an incompressible fluid rotating above it as a rigid body. This family of flows is known as BEK system flows [2]. The groundbreaking experimental analysis using the china-clay technique conducted by Gregory and Stuart [3] revealed a notable similarity between the von Kármán rotating-disk flow and flows over swept wings. This work demonstrated that despite the apparent differences between these two flow configurations, they exhibit comparable flow patterns and boundary layer characteristics. This discovery prompted the investigation of rotating disk flow as a prototypical case for both empirical and theoretical research.

The von Kármán rotating-disk flow has indeed become a fundamental model for analyzing the transition from laminar to turbulent flow in three-dimensional boundary layers. This model has been widely applied across various scenarios involving rotating disks due to its simplicity and the rich insights it provides into flow behaviour.

Malik's pioneering numerical study [4], together with Lingwood's investigations [5, 6, 7] and more recent contributions like Appelquist's theoretical work [8], have greatly advanced our understanding of steady flows over smooth rotating disks. These theoretical studies have explored various aspects of the flow, including the stability of laminar boundary layers and the mechanisms leading to turbulence, thereby advancing our knowledge of the transition processes in such flows.

Studies on rotating flows with rough disks have also significantly advanced the understanding of boundary layer dynamics. Notably, research by Harris et al. [9] investigated the effects of surface roughness on the flow characteristics over rotating disks, revealing how roughness influences the transition to turbulence and changes the base flow profiles. Cooper and Carpenter

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[10] further explored these effects, focusing on theoretical aspects of how rough surfaces modify boundary layer behaviour. Additionally, Cooper et al. [11] examined the impact of various types of surface roughness on rotating disk flows, contributing to a broader understanding of how surface imperfections affect flow transition and turbulence. Further insights are provided by studies such as those by Alveroglu and Christian [12, 13], which offer valuable perspectives on the complex interactions between surface roughness and rotating flow dynamics.

Although the studies mentioned above primarily focus on Newtonian fluids, significant progress has been made in extending the von Kármán boundary layer flow analysis to non-Newtonian fluids. Mitschka [14] was pivotal in this extension, utilizing a boundary layer approximation to explore how non-Newtonian behaviours influence the flow characteristics around rotating disks. This work marked a significant shift from classical Newtonian models, offering new insights into the behaviour of fluids with complex viscosity profiles. After that, both Mitschka and Ulbrecht [15] as well as Andersson et al. [16] and Hussain et al. [17] contributed valuable numerical solutions for the basic flow in the context of shear-thickening and shear-thinning fluids. Mitschka and Ulbrecht focused on the numerical analysis of shear-thinning and shear-thickening fluids, which exhibit an increase in viscosity with increasing shear rate, while Andersson et al. examined highly shear-thickening fluids. Their work has provided a deeper understanding of how these non-Newtonian properties affect the flow dynamics around rotating disks, revealing differences in boundary layer behaviour and flow transition characteristics compared to Newtonian fluids. These studies have expanded the applicability of von Kármán's work, making it relevant for a broader range of industrial and scientific applications where non-Newtonian fluids are present. However, Denier and Hewit [18] investigated the problem for both shear-thinning and shear-thickening fluids and showed that there are some fundamental issues regarding the application of power-law models in the boundary layer context. More recently, instability analysis examined in the boundary layer of rotating disks for shear-thinning fluids [19, 20, 21]. Using a sixth-order linear stability equation system, they found that increasing shear thinning stabilizes type I and type II modes in the flow. The results are consistent with established asymptotic estimates and provide new insights into the critical Reynolds number and growth rate. In [22], power-law fluids was conducted by Abdulameer et al., investigating the effect of shear-thinning fluids on convective type I and type II instability modes was analyzed using the Chebyshev polynomial method. Further, Alqarni et al. [23] have demonstrated, the local linear convective instability of boundary-layer flows over rough rotating disks using the Carreau model, a different type of non-Newtonian flow, by determining steady-flow profiles. It has been also shown by Lingwood and Henrik Alfredsson [24] that the rotating disk boundary layer itself exhibits a large number of complex instability behaviours that are not yet fully understood.

In addition to these studies, the decelerated rotating disk case investigated viscous flow and emphasised the relationship between fluid stresses and velocities [25, 26]. Further, Turkyılmazoglu et al. [27] focused on the heat transfer characteristics in nanofluid MHD flow across a decelerated rotating disk with uniform suction, and also the impact of uniform suction and magnetohydrodynamics on several nanofluids, such as silver, alumina, and copper, were studied Rahman et al. [28] over a decelerated rotating disk. Additionally, Fang and Tao [29] studied the laminar unsteady flow across a stretchy decelerated rotating disk. These works contributed to the growing body of knowledge on how velocity changes affect fluid behaviour on decelerating rotating disks.

In particular, the similarity transformations of non-Newtonian fluids and the related velocity profiles and viscosity functions are discussed, focusing on the shear-thinning states and the dimensionless unsteadiness parameter of these fluids. This way, the similarity transformations for power-law fluids are obtained on a decelerating rotating disk and solved numerically using MATLAB's bvp4c function. This function's numerical approach relies on a finite difference code that implements the three-stage Lobatto IIIa formula, equivalent to an implicit Runge-Kutta formula with a continuous interpolant [30].

This research examines a non-Newtonian fluid with non-constant viscosity contained in a container and rotates non-uniformly with an angular velocity that varies with time (t) in an inertial frame. The equation of motion for a fluid element about a reference frame may be found using the conservation of momentum concept. The mathematical description of rotating fluids may be done using a variety of reference frames. Adopting a reference frame whose axes are fixed in a fluid-filled container is physically and mathematically natural for many geophysical problems, such as atmospheric dynamics. This is often referred to as a rotating frame, mantle frame, or body frame so that the bounding surface of the container is constant and there are only minor deviations from rigid body rotation [31].

2. Formulation

We consider the steady incompressible power-law fluid reviews that geometry is an infinite rotating plane, stating at $z^* = 0$ and also we use the symbol * to indicate the dimensional parameter. However, since we use the assumption that the disk is decelerating, we consider the inertial frame. The plane rotates with a angular velocity $\Omega_D^* = \Omega_0(1-\alpha t^*)^{-1}$, decelerating around $z^* = 0$. The continuity and Navier-Stokes equations can be considered as

$$\nabla \cdot \boldsymbol{u}^* = 0 \tag{2.1}$$

$$\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \boldsymbol{u}^* \cdot \nabla \boldsymbol{u}^* + 2\boldsymbol{\Omega}_D^* \times \boldsymbol{u}^* + \boldsymbol{r} \times \left(\frac{\partial \boldsymbol{\Omega}_D}{\partial t^*}\right) = -\frac{1}{\rho^*}\nabla p^* + \frac{1}{\rho^*}\boldsymbol{\nabla} \cdot \boldsymbol{\tau}^*. \tag{2.2}$$

Here U^*, V^*, W^* are the steady velocity components in cylindrical polar coordinates, $\boldsymbol{r} = (r^*, \theta, z^*)$ is the position vector, t^* is the time and $\boldsymbol{\Omega}_D^* = (0,0,\Omega_D^*)$ is the vector form of the angular velocity of the disk that describes the disk just rotates about the z^* axis with angular velocity Ω_D^*. Moreover, ρ^* is the fluid density and p^* is the pressure. The stress tensor $\boldsymbol{\tau}^*$ for generalised Newtonian models, is defined by

$$\boldsymbol{\tau}^* = \mu^* \dot{\gamma}^* \quad \text{with} \quad \mu^* = \mu^*(\dot{\gamma}^*)$$

where $\dot{\boldsymbol{\gamma}}^* = \nabla \boldsymbol{u}^* + (\nabla \boldsymbol{u}^*)^T$ is the rate-of-strain tensor and $\mu^*(\dot{\gamma}^*)$ is the non-Newtonian viscosity. The magnitude of the symmetric rate-of-strain tensor is given by

$$\dot{\gamma}^* = \sqrt{\frac{\dot{\boldsymbol{\gamma}}^* : \dot{\boldsymbol{\gamma}}^*}{2}}.$$

The Navier Stokes Equations written for incompressible fluids together with the inertial frame for the reference whose axes rotate in a container filled with fluid are as follows,

$$\frac{1}{r^*}\frac{\partial (r^* U_0^*)}{\partial r^*} + \frac{1}{r^*}\frac{\partial V_0^*}{\partial \theta} + \frac{\partial W_0^*}{\partial z^*} = 0 \tag{2.3}$$

$$\frac{\partial U_0^*}{\partial t^*} + U_0^*\frac{\partial U_0^*}{\partial r^*} + \frac{V_0^*}{r^*}\frac{\partial U_0^*}{\partial \theta} + W_0^*\frac{\partial U_0^*}{\partial z^*} - \frac{(V_0^* + r^*\Omega^*)^2}{r^*} = \frac{1}{\rho^*}\frac{\partial}{\partial z^*}\left(\mu^*\frac{\partial U_0^*}{\partial z^*}\right), \tag{2.4}$$

$$\frac{\partial V_0^*}{\partial t^*} + U_0^*\frac{\partial V_0^*}{\partial r^*} + \frac{V_0^*}{r^*}\frac{\partial V_0^*}{\partial \theta} + W_0^*\frac{\partial V_0^*}{\partial z^*} + \frac{U_0^*\widetilde{V}_0^*}{r^*} + 2\Omega^* U_0^* + r^*\frac{\partial \Omega^*}{\partial t^*} = \frac{1}{\rho^*}\frac{\partial}{\partial z^*}\left(\mu^*\frac{\partial V_0^*}{\partial z^*}\right), \tag{2.5}$$

$$\begin{aligned}&\frac{\partial W_0^*}{\partial t^*} + U_0^*\frac{\partial W_0^*}{\partial r^*} + \frac{V_0^*}{r^*}\frac{\partial W_0^*}{\partial \theta} + W_0^*\frac{\partial W_0^*}{\partial z^*}\\ &= -\frac{1}{\rho^*}\frac{\partial P_1^*}{\partial z^*} + \frac{1}{\rho^* r^*}\frac{\partial}{\partial r^*}\left(\mu^* r^*\frac{\partial U_0^*}{\partial z^*}\right) + \frac{1}{\rho^* r^*}\frac{\partial}{\partial \theta}\left(\mu^*\frac{\partial V_0^*}{\partial z^*}\right) + \frac{2}{\rho^*}\frac{\partial}{\partial z^*}\left(\mu^*\frac{\partial W_0^*}{\partial z^*}\right).\end{aligned} \tag{2.6}$$

Here U_0^*, ,V_0^*, W_0^* are the leading order velocity components and P_1^* is the leading order pressure term. The rate-of-strain tensor $\dot{\gamma}^*$ can be written to be

$$\begin{aligned}\dot{\gamma}^* = \sqrt{\frac{II}{2}} \quad = \quad &\left\{2\left[\left(\frac{\partial U^*}{\partial r^*}\right)^2 + \left(\frac{1}{r^*}\frac{\partial V^*}{\partial \theta} + \frac{U^*}{r^*}\right)^2 + \left(\frac{\partial W^*}{\partial z^*}\right)^2\right]\right.\\ &\left. + \left[r^*\frac{\partial}{\partial r^*}\left(\frac{V^*}{r^*}\right) + \frac{1}{r^*}\frac{\partial U^*}{\partial \theta}\right]^2 + \left(\frac{\partial U^*}{\partial z^*} + \frac{\partial W^*}{\partial r^*}\right)^2 + \left(\frac{\partial V^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial W^*}{\partial \theta}\right)^2\right\}^{1/2}\end{aligned} \tag{2.7}$$

where

$$II = \sum_i \sum_j \dot{\gamma}_{ij}^{*2} = \dot{\gamma}_{r^*r^*}^{*2} + \dot{\gamma}_{\theta\theta}^{*2} + \dot{\gamma}_{z^*z^*}^{*2} + 2\left(\dot{\gamma}_{r^*\theta}^{*2} + \dot{\gamma}_{r^*z^*}^{*2} + \dot{\gamma}_{\theta z^*}^{*2}\right).$$

To determine the unsteady mean flow relative to the disk, we offer the generalization of the standard Newtonian similarity solution. So, we consider the following transformations,

$$U_0^* = U(\eta) r^* \Omega_D^*,\ V_0^* = V(\eta) r^* \Omega_D^*,\ W_0^* = W(\eta)\chi^*,\ P_1^* = \rho^* {\chi^*}^2, \tag{2.8}$$

where

$$\chi^* = \left[\frac{\nu^*}{{r^*}^{1-n}(\Omega_0(1-\alpha t^*)^{-1})^{1-2n}}\right]^{1/(n+1)}.$$

Here U, V, W are the dimensionless radial, azimuthal and axial base flow velocities, respectively. Additionally, P is the pressure and $\nu^* = \frac{m^*}{\rho^*}$ is the kinematic viscosity. The dimensionless similarity coordinate that can be also named as boundary layer thickness is

$$\eta = \frac{{r^*}^{\frac{1-n}{n+1}} z^*}{{L^*}^{2/(n+1)}},\ \text{where } L^* = \sqrt{\frac{\nu^*}{(\Omega_0(1-\alpha t^*)^{-1})^{2-n}}}.$$

These similarity variables account for the time-dependent variation of boundary layer thickness and represent the first instance of such an introduction for power-law fluids in the literature.

Consequently, the similarity equations for the governing boundary layer equations of a decelerated power-law fluid, represented by equations (2.1) and (2.2), are derived for the first time as follows:

$$2U + \frac{1-n}{1+n}\eta U' + W' = 0$$

$$\hat{\alpha}\left(U-\frac{n-2}{n+1}\eta U'\right)+U^2+\left(\eta\frac{1-n}{n+1}U+W\right)U'-(V+1)^2-(\mu U')'=0,$$
$$\hat{\alpha}\left(V-\frac{n-2}{n+1}\eta V'+1\right)+V'\left(\eta\frac{1-n}{n+1}U+W\right)+2U(V+1)-(\mu V')'=0, \tag{2.9}$$
$$\hat{\alpha}\left(\frac{2n-1}{n+1}W+\frac{2-n}{n+1}\eta W'\right)+\frac{1-n}{n+1}\left[U\left(\eta W'-V\right)+2\mu U'\right]+2\mu' U+P'+WW'-(\mu W')'=0.$$

where the primes denote differentiation with respect to η and the viscosity function is

$$\mu=\left[U'^2+V'^2\right]^{\frac{n-1}{2}}.$$

Owing to (2.9) the non-dimensional boundary conditions are

$$U(0)=V(0)=W(0)=0, U(\eta\to\infty)\to 0 \text{ and } V(\eta\to\infty)\to -1. \tag{2.10}$$

To obtain the flow profiles U, V and W, the dimensionless mean flow equations (2.9) valid for the power-law are expressed as a system of first order ordinary differential equations. This system of equations is written as five coupled first order equations in terms of the new five dependent variables $\psi_n(n=1,2,...,5)$, which are defined as follows:

$$\psi_1=U,\ \psi_2=U',\ \psi_3=V,\ \psi_4=V',\ \ \psi_5=W. \tag{2.11}$$

Using the first order equations defined by (2.11), the transformed system of first order ordinary differential equations with no-slip boundary conditions for power-law fluids is obtained as follows:

$$\begin{aligned}
\psi_1' &= \psi_2\\
\psi_2' &= \frac{(\psi_2^2+n\psi_4^2)f-\psi_2\psi_4(n-1)g}{n\mu(\psi_2^2+\psi_4^2)}\\
\psi_3' &= \psi_4\\
\psi_4' &= \frac{(n\psi_2^2+\psi_4^2)g-\psi_2\psi_4(n-1)f}{n\mu(\psi_2^2+\psi_4^2)}\\
\psi_5' &= -2\psi_1-\frac{1-n}{1+n}\eta\psi_2.
\end{aligned} \tag{2.12}$$

The nondimensional boundary conditions are

$$\begin{aligned}
&\psi_1(0)=\psi_3(0)=\psi(0)=0\\
&\psi_1(\infty)\to 0,\ \psi_3(\infty)\to -1.
\end{aligned} \tag{2.13}$$

Also in (2.12), f and g are given below

$$f=\hat{\alpha}(u-(n-2)/(n+1)\eta\psi_2)+\psi_1^2-(\psi_3+1)^2+(\psi_5+\hat{\psi}\psi_1)\psi_2,$$
$$g=\hat{\alpha}(\psi_3-(n-2)/(n+1)\eta\psi_4+1)+2\psi_1(\psi_3+1)+(\psi_5+\hat{\psi}\psi_1)\psi_4,$$

with $\hat{\psi}=(1-n)/(1+n)\eta$. These equations simplify to the Newtonian case when $n=1$, aligning with established literature. Although these equations are applicable to both shear-thickening ($n>1$) and shear-thinning ($n<1$) fluids, Denier and Hewitt [18] demonstrated that bounded solutions of (2.9) subject to (2.10) exist only for shear-thinning fluids with $n>0.5$. Consequently, Griffits et al. [19, 22] investigated power-law indexed flows within the range $0.5<n\leq 1$ for a steady rotating flow. In line with their work, this study also focuses on shear-thinning fluids with $0.5<n\leq 1$ in the context of a decelerated rotating disk.

3. Results and Discussion

The obtained similarity equations (2.9) were solved approximately in MATLAB using the bvp4c function in accordance with the boundary conditions given in (2.10) and the following velocity profiles were obtained. Hussain et al. [17] expanded the range of $\hat{\alpha}$ to -100, building upon earlier work by Watson and Wang [25], who established that a disk can only have a momentum layer if it is decelerating, i.e., when $\hat{\alpha}<0$. Rahman et al. [28] later provided numerical solutions for eight different values of $\hat{\alpha}$ in the interval $0\leq-\hat{\alpha}\leq 20$. Here, by choosing $0.5<n\leq 1$, we observe that the flow is non-Newtonian and we observe the changes due to the unsteadiness parameter $0\leq-\hat{\alpha}\leq 8$. The unsteadiness parameter was truncated at $\hat{\alpha}=-8$ because of the discrepancies in the results obtained due to the deceleration of the rotating disk.

Before performing the calculations, the numerical scheme needed to be validated using a comparison methodology. Table 2 presents a comparison with the results of [25, 29] the classical Newtonian fluid case. The compared data are in good agreement, which not only validates the code but also demonstrates the accuracy and reliability of the numerical scheme.

$n = 0.6$

$\hat{\alpha}$	$U'(0)$	$V'(0)$	$W(\infty)$
0	0.5	-0.6770	-1.3046
-0.5	0.5416	-0.4858	-0.8241
-2	0.5011	-0.0785	-0.3312
-5	0.3515	-0.2152	-0.0948
-8	0.2783	-0.3293	-0.0430

$n = 0.7$

$U'(0)$	$V'(0)$	$W(\infty)$
0.5015	-0.6530	-1.1967
0.5638	-0.4676	-0.8391
0.6433	-0.0050	-0.4438
0.6637	0.6416	-0.1846
0.6191	1.0941	-0.0989

$n = 0.8$

$\hat{\alpha}$	$U'(0)$	$V'(0)$	$W(\infty)$
0	0.5039	-0.6362	-1.0773
-0.5	0.5826	-0.4523	-0.8436
-2	0.7633	0.0610	-0.5660
-5	1.0053	1.0296	-0.3399
-8	1.1075	1.9083	-0.2331

$n = 0.9$

$U'(0)$	$V'(0)$	$W(\infty)$
0.5069	-0.6243	-0.9688
0.5992	-0.4394	-0.8469
0.8583	0.1141	-0.6923
1.3165	1.2624	-0.5516
1.6824	2.4214	-0.4715

$n = 1$

$\hat{\alpha}$	$U'(0)$	$V'(0)$	$W(\infty)$
0	0.5102	-0.6159	-0.8845
-0.5	0.6143	-0.4284	-0.8510
-2	0.9315	0.1550	-0.8193
-5	1.5628	1.3609	-0.8012
-8	2.1873	2.5887	-0.7941

Table 1: Table of boundary values for U, V and W

$n = 1$

$\hat{\alpha} = -0.5$	[25]	[29]	Present
$U'(0)$	0.614283	0.6143	0.6143
$V'(0)$	-0.428406	-0.4284	-0.4284
$W(\infty)$	0.4255		-0.8510

$\hat{\alpha} = -2$	[25]	[29]	Present
$U'(0)$	0.931507	0.9315	0.9315
$V'(0)$	0.154981	0.1550	0.1550
$W(\infty)$	0.4096		-0.8193

$\hat{\alpha} = -5$	[25]	[29]	Present
$U'(0)$	1.562797	1.5627	1.5628
$V'(0)$	1.360850	1.3609	1.3609
$W(\infty)$	0.4006		-0.8012

Table 2: Comparison of $U'(0)$, $V'(0)$ and $W(\infty)$ obtained for various $\hat{\alpha}$ variables with the results of [25] and [29] the classical Newtonian fluid case.

The power-law shear-thinning flow profiles calculated for the decelerating disk are illustrated in Figures 1, 2, and 3. They reveal that all those profiles decay to their corresponding far field values as decelarating parameter decreases. Figure 1 reveals a notable decrease in the boundary layer thickness as the instability parameter $-\hat{\alpha}$ increases. This decrease becomes more pronounced with increasing $-\hat{\alpha}$, ultimately leading to a significant reduction in the boundary layer thickness. Additionally, the maximum value of the radial jet also decreases as $-\hat{\alpha}$ grows in magnitude. Also, as deceleration increases, the location of the maximum value of the radial jet shifts closer to the disk surface. Despite these changes in boundary layer thickness and radial jet strength, the inflectional shape of the flow profile remains preserved, indicating stability in the overall flow structure.

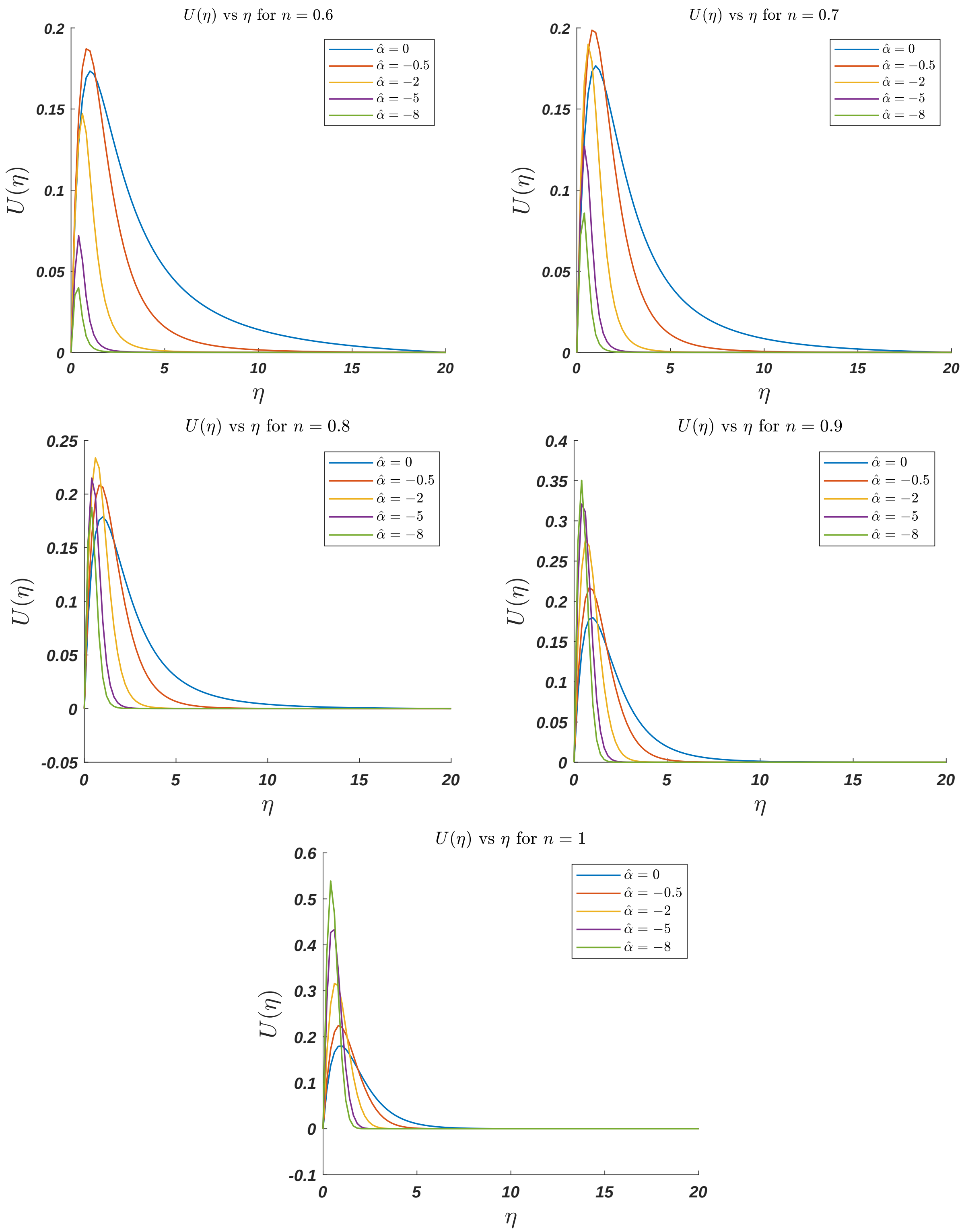

Figure 1: Radial velocity profiles for power-law flow for $0.5 < n \leq 1$ versus η. The boundary layer thickness η axis has been truncated at 20.

The effect of non-zero $\hat{\alpha}$ on the azimuthal profile is presented in Figure 2. For each value of the shear-thickening parameter n, the figure demonstrates that the transition of the profile to the boundary value at the far field becomes more rapid as the disk's deceleration increases. This observation is consistent with the decrease in boundary layer thickness noted previously.

Figure 3 shows the axial velocity profile for various values of the shear-thickening parameter n. The figure illustrates that an increased deceleration rate leads to a significant reduction in the magnitude of the axial jet, hence reduce the amount of the fluid entering the boundary layer. Furthermore, the decrease in the magnitude of the axial flow becomes more pronounced as the shear-thickening parameter n decreases.

For small unsteadiness parameter values, the fluid ahead of the disk rotates slower than the disk. This phenomenon may be a consequence of the rapid decelerated rotation of the disk, while the inertia of the neighbouring fluid layer allows the fluid to sustain its more significant angular momentum for a long time. Also, as shown in Figures 1, 2, 3 the results are consistent with the boundary layer analysis for a disk rotating at constant velocity ($\hat{\alpha} = 0$). This involves approximate solutions that are directly identical to the result for constant angular velocity rotation of the disk and whose velocity profiles are equivalent to those obtained for flows with known power-law indexes [19].

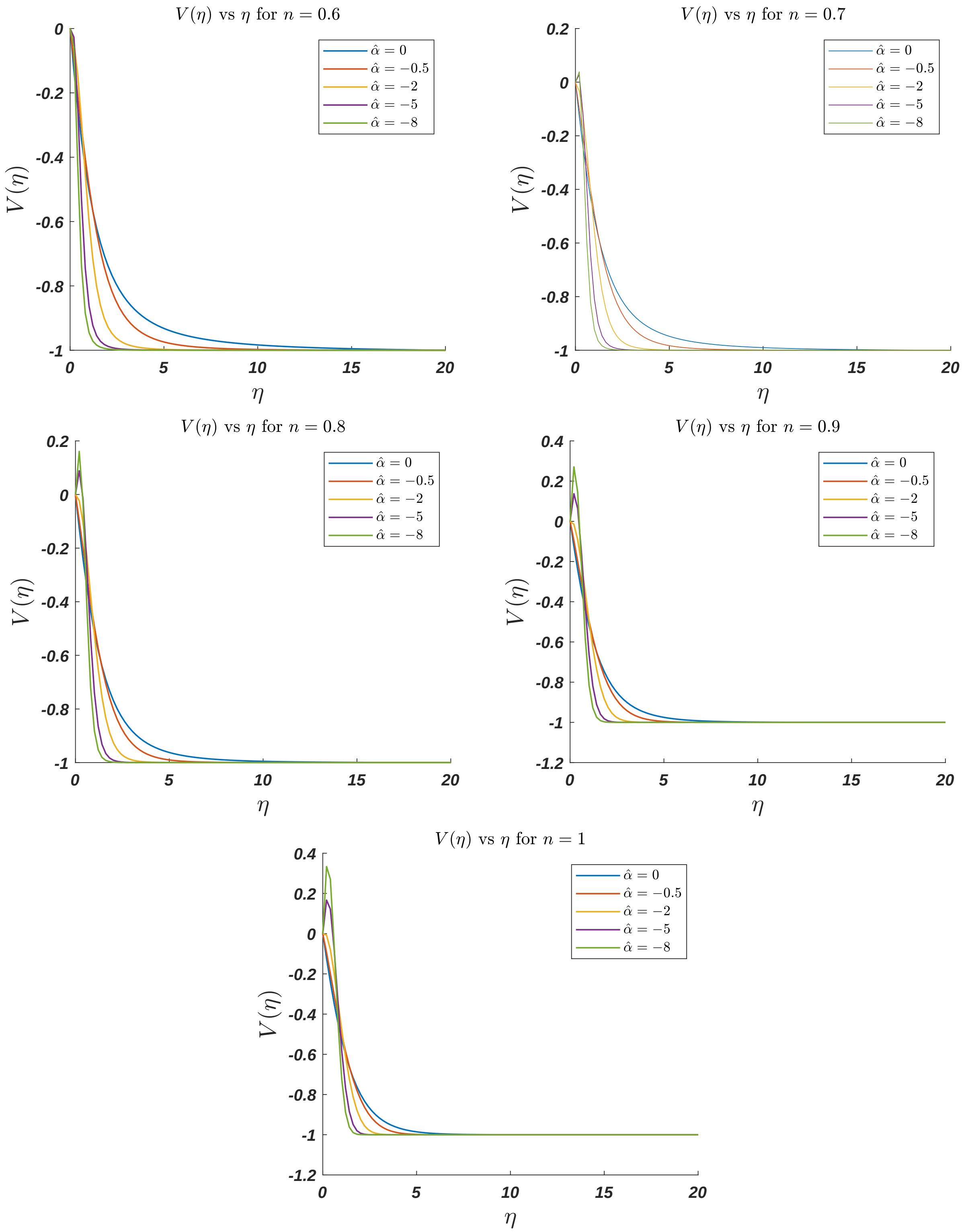

Figure 2: Azimuthal velocity profiles for power-law flow for $0.5 < n \leq 1$ versus η. The boundary layer thickness η axis has been truncated at 20.

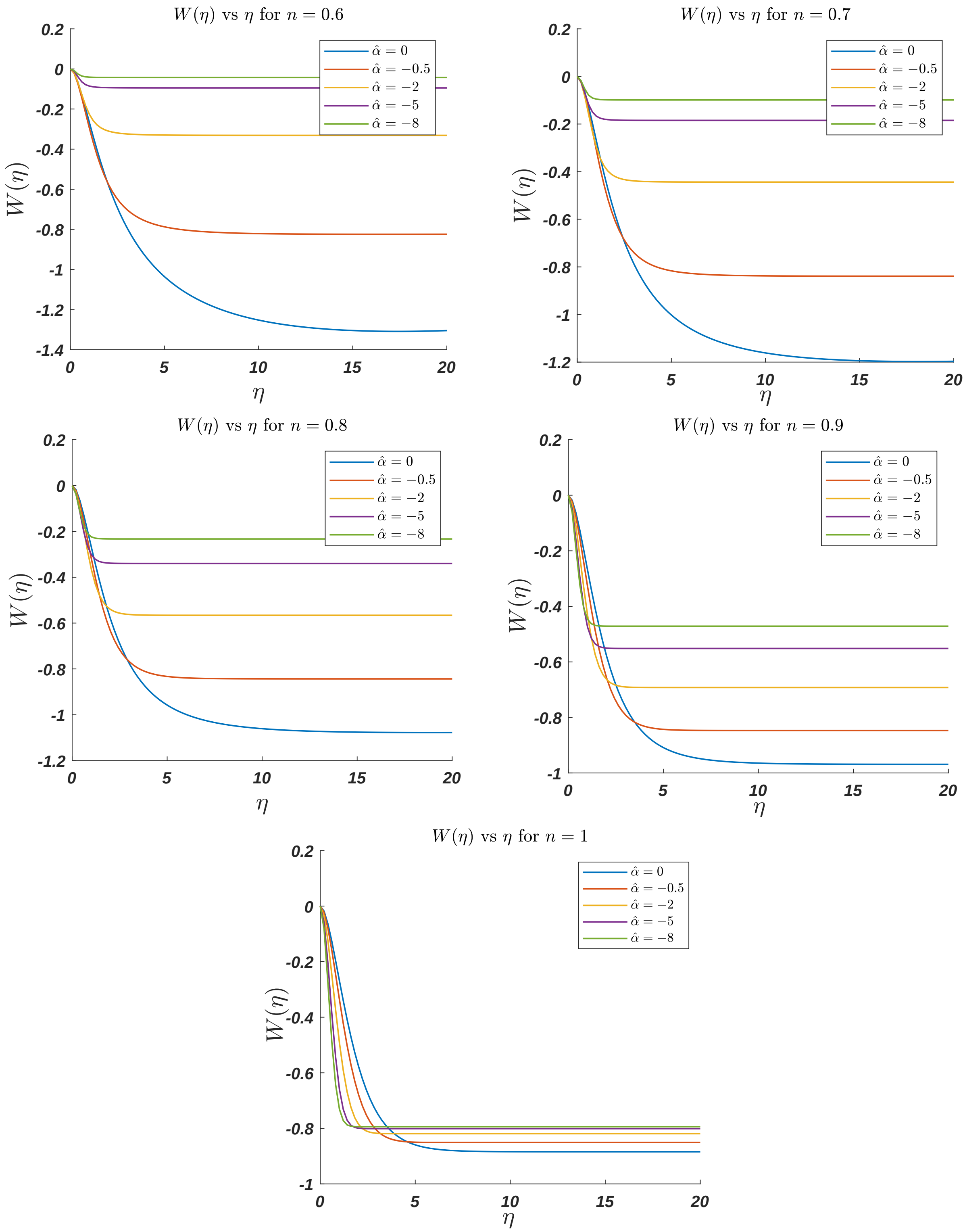

Figure 3: Axial velocity profiles for power-law flow for $0.5 < n \leq 1$ versus η. The boundary layer thickness η axis has been truncated at 20.

Finally, viscosity profiles for the numerical results obtained are given in Figure 4. As the angular velocity of the disk slows down with time, the viscous effects of the flow on the wall surface increase, the numerical data obtained when the power-law index $n = 0.6$ show a decrease by almost half as n increases.

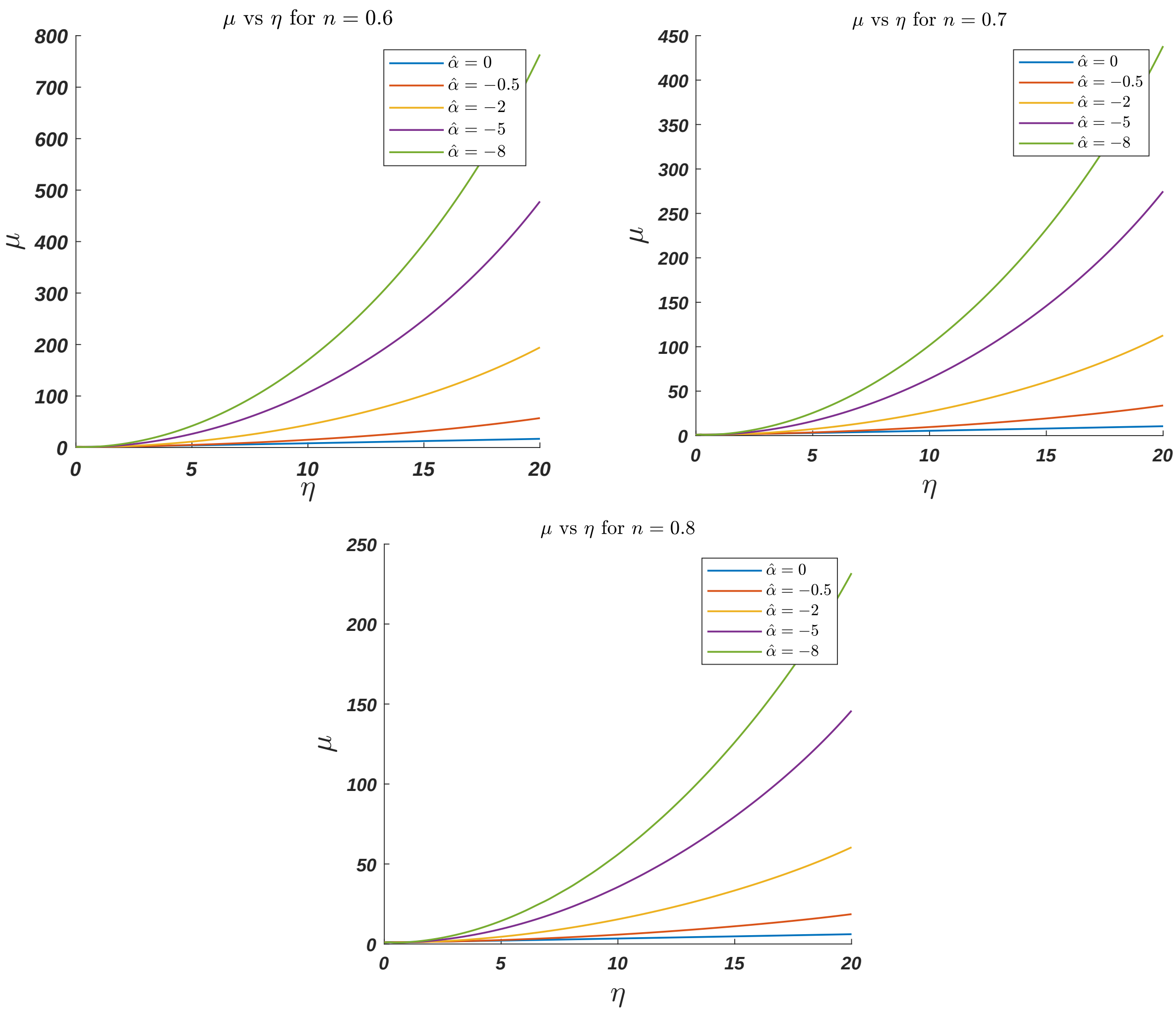

Figure 4: μ versus η for $n = 0.6, 0.7, 0.8$.

4. Conclusion

This study investigates power-law fluids over decelerating rotating disks, with the disk's angular velocity inversely proportional to time. By deriving similarity transformations, we explored the flow characteristics dependent on the nondimensional unsteadiness parameter $\hat{\alpha}$. For $0.5 < n \leq 1$, we analyzed the no-slip condition and the dimensionless unsteadiness parameter, detailing the velocity profiles and viscosity function with respect to deceleration parameters $\hat{\alpha} = 0, -0.5, -2, -5, -8$. The findings revealed that an increased decelerating parameter results in a thinner boundary layer and a reduction in the maximum value of the U profile. It also causes a decrease in the amount of axial flow towards the boundary layer, which is consistent with the observed reduction in boundary layer thickness. Additionally, it was observed that the inflectional profile of mean flow components does not change notably with varying deceleration rates. These findings provide valuable insights into the behaviour of non-Newtonian fluids over decelerating rotating disks, with applications in engineering and industrial processes. The study demonstrates the effectiveness of numerical methods in solving complex fluid dynamics problems, contributing to advancements in the field.

For future work, this study can be extended to include other non-Newtonian fluid models, such as the Bingham and Carreau models, which could offer a more comprehensive understanding of fluid behaviours in different scenarios. Additionally, exploring cases with rough rotating disks could provide insights into how surface texture influences flow dynamics. Finally, it would be valuable to investigate other flow scenarios within the BEK system for non-Newtonian cases. Exploring these different flow configurations could further enhance our understanding of non-Newtonian fluid dynamics and contribute additional insights to the field. Incorporating these elements could enhance the applicability and relevance of our findings. We anticipate that our results will serve as a useful foundation for these extended studies, contributing valuable data and insights to the field.

Declarations

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The Linear Algebra of the Pell-Lucas Matrix

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Abstract

In this paper, we introduce the Pell-Lucas and the symmetric Pell-Lucas matrices. The study delves into the linear algebra aspects of these matrices, analyzing their mathematical properties and relationships. We construct decompositions for both the Pell-Lucas matrix and its inverse matrix. We present the Cholesky factorization of the symmetric Pell-Lucas matrices. Furthermore, we derive some valuable identities and bounds for the eigenvalues of these symmetric matrices through the application of majorization notation.

1. Introduction

Numerous researchers in the disciplines of calculus, applied mathematics, and linear algebra, as well as other branches of mathematics, have been interested in the Fibonacci and Lucas numbers. There are also other relationships that are written and new number sequences, such as Pell and Pell-Lucas number sequences, are derived that are similar to the recurring relationships of the Fibonacci and Lucas number sequences. The Pell numbers P_n and the Pell-Lucas numbers Q_n are defined by

$$P_{n+1} = 2P_n + P_{n-1}, \quad for \quad n \geq 1,$$

where $P_0 = 0$ and $P_1 = 1$, and

$$Q_{n+1} = 2Q_n + Q_{n-1}, \quad for \quad n \geq 1,$$

where $Q_0 = 2$ and $Q_1 = 2$, respectively. In addition, we present several identities associated with the Pell-Lucas numbers and relationship between the Pell numbers and the Pell-Lucas numbers for $k \in \mathbb{N}$.

$$Q_k + Q_{k+1} = 4P_{k+1}, \tag{1.1}$$

$$Q_k + Q_{k+2} = 8P_{k+1}, \tag{1.2}$$

$$Q_1^2 + Q_2^2 + \cdots + Q_k^2 = \frac{Q_k Q_{k+1} - 4}{2}. \tag{1.3}$$

We refer to [1, 2, 3] for further information on the Pell and the Pell-Lucas numbers.

M_n denotes the set of all $n \times n$ matrices. If any matrix $P \in M_n$ may be written as $P = RR^T$ or $P = R^T R$, where $R \in M_n$ is a lower triangular matrix with diagonal entries that are not negative, then this factorization is known as a Cholesky factorization. Moreover, this factorization is unique if R is nonsingular.

A matrix $S \in M_n$ of the form

$$S = \begin{bmatrix} S_{11} & 0 & \cdots & 0 \\ 0 & S_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{nn} \end{bmatrix}$$

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in which $S_{ii} \in M_{n_i}$, $i = 1,2,\ldots,k$, and $\sum_{i=1}^{k} n_i = n$, is called block diagonal. This matrix is frequently described as $S = S_{11} \oplus S_{22} \oplus \cdots \oplus S_{nn}$.

Many issues resulting from linear recurrence relations can be resolved using matrix methods, which are a significant instrument (see, for example, [4]). Before we go on to matrix factorization, we need to first grasp Cholesky factorization of the Pascal matrix (see, for example, [5]). Furthermore, factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices were presented by Lee et al. in [6]. The authors [7] discussed linear algebra of the k-Fibonacci and the symmetric k-Fibonacci matrix. In addition to [7], a factorization of the Pascal matrix are provided in [8]. Zhang [9] also researched the Pascal matrix and its generalization. Irmak and Köme [10] investigated the factorizations of the Lucas and the symmetric Lucas matrix. In [11], factorizations and inverse factorizations of generalized k-Fibonacci matrices were proposed. The authors [12] discussed the decomposition of Jacobsthal matrix and Jacobsthal-Lucas symmetric matrix, along with the inverses of these matrices. Kılıç and Taşcı [13] gave the factorizations and eigenvalues of Pell and symmetric Pell matrices. Furthermore, for the eigenvalues of the symmetric Pell matrix, they provided some relations and boundaries. Motivated by this paper, we define a new matrix as follows. Then, in this paper we consider the factorizations and eigenvalues of Pell-Lucas and symmetric Pell-Lucas matrices.

Definition 1.1. *Let $i, j = 1,2,\ldots,n$. Then, we define the Pell-Lucas matrix such that*

$$A_n = [a_{ij}] = \begin{cases} Q_{i-j+1}\,, & i-j+1 > 0 \\ 0\,, & i-j+1 \leq 0 \end{cases}.$$

Example 1.2. *For $n = 6$ in Definition 1.1, then we have*

$$A_6 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 \\ 14 & 6 & 2 & 0 & 0 & 0 \\ 34 & 14 & 6 & 2 & 0 & 0 \\ 82 & 34 & 14 & 6 & 2 & 0 \\ 198 & 82 & 34 & 14 & 6 & 2 \end{bmatrix},$$

and the first column of A_6 is the vector $(2,6,14,34,82,198)^T$. As a result, the matrix A_n reveals a variety of interesting facts.

2. Factorizations

This section discusses the creation and factorization of our Pell-Lucas matrix of order n using the $(0,1,2)$-matrix, which is defined as a matrix whose elements are all either 0, 1 or 2. Let I_n represents the order n identity matrix. Further, we define the $n \times n$ matrices L_n, $\overline{A_n}$ and X_k by

$$L_0 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\; L_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

and $L_k = L_0 \oplus I_k$, $k = 1,2,\ldots$, $\overline{A_n} = [1] \oplus A_{n-1}$, $X_1 = I_n$, $X_2 = I_{n-3} \oplus L_{-1}$, for $3 \leq k < n$, $X_k = I_{n-k} \oplus L_{k-3}$, and $X_n = L_{n-3}$. Then we reach the following lemma.

Lemma 2.1. *For $k \geq 3$, we have $\overline{A_k} \cdot L_{k-3} = A_k$.*

Proof. For $k = 3$, we have $\overline{A_3} \cdot L_0 = A_3$. Let $k > 3$. By using the familiar Pell-Lucas sequences, and matrix product definition, we get the following conclusion. □

For $i, j = 1,2,\ldots,n$, we define a matrix

$$\Gamma_n = [\gamma_{ij}] = \begin{cases} 2, & i = j \\ 2, & i = j+1 \\ 0, & otherwise \end{cases}. \quad (2.1)$$

Also we can give the inverse of matrix Γ_n as follows:

$$\Gamma_n^{-1} = [\gamma_{ij}] = \begin{cases} (-1)^{i-j}\dfrac{1}{2}, & i \geq j \\ 0, & otherwise \end{cases}. \quad (2.2)$$

We can obtain the following theorem by using Lemma 2.1 and equation (2.1).

Theorem 2.2. *The X_k's and Γ_n can factor the Pell-Lucas matrix A_n in the following way:*

$$A_n = X_1 X_2 \cdots X_n \Gamma_n = \Gamma_n X_1 X_2 \cdots X_n.$$

Now we give the factorization of A_6 in Example 1.2.

Example 2.3. *From Theorem 2.2, for $n = 6$, we have*

$$\begin{aligned} A_6 &= X_1X_2X_3X_4X_5X_6\Gamma_6 \\ &= I_6\,(I_3\oplus L_{-1})\,(I_3\oplus L_0)\,(I_2\oplus L_1)\,(I_1\oplus L_2)\,L_3\Gamma_6 \\ &= \begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&2&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&2&1&0\\0&0&0&1&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&2&1&0&0\\0&0&1&0&1&0\\0&0&0&0&0&1\end{bmatrix} \\ &\cdot\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&2&1&0&0&0\\0&1&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0&0&0\\2&1&0&0&0&0\\1&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}2&0&0&0&0&0\\2&2&0&0&0&0\\0&2&2&0&0&0\\0&0&2&2&0&0\\0&0&0&2&2&0\\0&0&0&0&2&2\end{bmatrix}. \end{aligned}$$

Now, we give another factorization of A_n. For $i, j = 1,2,\ldots,n$, we define a matrix

$$V_n = [v_{ij}] = \begin{cases} Q_i\,, & j=1\,, \\ 1\,, & i=j\,, \\ 0\,, & otherwise \end{cases} \quad , \quad i.e, \quad V_n = \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ Q_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Q_n & 0 & \cdots & 1 \end{bmatrix}.$$

An elementary calculation leads to the next theorem.

Theorem 2.4. *For $n \geq 1$, $A_n = V_n(I_1\oplus V_{n-1})(I_2\oplus V_{n-2})\cdots(I_{n-1}\oplus V_1)$.*

The inverse of the Pell-Lucas matrix A_n is easily found. We know that

$$L_0^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad L_{-1}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad \text{and} \quad L_k^{-1} = L_0^{-1}\oplus I_k.$$

For $k = 1,2,\ldots,n$, we define $Y_k = X_k^{-1}$. Then $Y_1 = X_1^{-1} = I_n$,

$Y_2 = X_2^{-1} = I_{n-3}\oplus L_{-1}^{-1} = I_{n-2}\oplus \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, and $Y_n = L_{n-3}^{-1}$. Also we can derive

$$V_n^{-1} = \begin{bmatrix} Q_1/4 & 0 & 0 & \cdots & 0 \\ -Q_2/2 & 1 & 0 & \cdots & 0 \\ -Q_3/2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Q_n/2 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \text{and} \quad (I_k\oplus V_{n-k})^{-1} = I_k\oplus V_{n-k}^{-1}.$$

Utilizing Theorem 2.2 and Theorem 2.4, we derive the subsequent corollary.

Corollary 2.5. *The inverse of the Pell-Lucas matrix A_n^{-1} can be factored by the Y_k's and Γ_n^{-1} as follows:*

$$\begin{aligned} A_n^{-1} &= \Gamma_n^{-1}Y_nY_{n-1}\ldots Y_2Y_1 = Y_nY_{n-1}\ldots Y_2Y_1\Gamma_n^{-1} \\ &= (I_{n-1}V_1)^{-1}\cdots(I_2\oplus V_{n-2})^{-1}(I_1\oplus V_{n-1})^{-1}V_n^{-1} \end{aligned}$$

By Corollary 2.5, we get

$$A_n^{-1} = [\alpha_{ij}] = \begin{cases} 1/2\,, & i=j \\ -3/2\,, & i-j=1 \\ (-1)^{i-j}, & i-j\geq 2 \\ 0, & otherwise \end{cases}. \tag{2.3}$$

Example 2.6. *By (2.3), the inverse of A_6 in Example 1.2 is*

$$A_6^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ -3/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1 & -3/2 & 1/2 & 0 & 0 & 0 \\ -1 & 1 & -3/2 & 1/2 & 0 & 0 \\ 1 & -1 & 1 & -3/2 & 1/2 & 0 \\ -1 & 1 & -1 & 1 & -3/2 & 1/2 \end{bmatrix}.$$

Definition 2.7. *For $i, j = 1,2,\dots,n$, we define the symmetric Pell-Lucas matrix such that*

$$B_n = [b_{ij}] = [b_{ji}] = \begin{cases} \sum_{k=1}^{i} Q_k^2, & i = j \\ b_{i,j-2} + 2b_{i,j-1} + 4, & i+1 = j \\ b_{i,j-2} + 2b_{i,j-1}, & i+1 < j \end{cases},$$

where $b_{1,0} = 4$.

So we get

$$b_{1j} = b_{j1} = 2Q_j, \quad for \ \ j \geq 1 \tag{2.4}$$
$$b_{2j} = b_{j2} = 8P_{j+1}, \quad for \ \ j \geq 2. \tag{2.5}$$

Example 2.8. *For $n = 6$ in Definition 2.7, then we get*

$$B_6 = \begin{bmatrix} 4 & 12 & 28 & 68 & 164 & 396 \\ 12 & 40 & 96 & 232 & 560 & 1352 \\ 28 & 96 & 236 & 572 & 1380 & 3332 \\ 68 & 232 & 572 & 1392 & 3360 & 8112 \\ 164 & 560 & 1380 & 3360 & 8116 & 19596 \\ 396 & 1352 & 3332 & 8112 & 19596 & 47320 \end{bmatrix}.$$

According to the Definition 2.7, the following lemma is derived.

Lemma 2.9. *For $j \geq 3$, we get $b_{3,j} = P_{j-3}(8P_4 + 4) + P_{j-2}\left(\frac{Q_3Q_4 - 4}{2}\right)$.*

Proof. From (1.3), we know that $b_{3,3} = Q_1^2 + Q_2^2 + Q_3^2 = \frac{Q_3Q_4 - 4}{2}$. On the other hand, since $P_0 = 0$, and $P_1 = 1$, then we have $b_{3,3} = \frac{Q_3Q_4 - 4}{2} = P_0(8P_4 + 4) + P_1\left(\frac{Q_3Q_4 - 4}{2}\right)$. By induction, the proof is completed. □

We know that $b_{3,1} = b_{1,3} = 2Q_3$ and $b_{3,2} = b_{2,3} = 8P_4$ by (2.4) and (2.5). In addition, we get that $b_{4,1} = b_{1,4}$, $b_{4,2} = b_{2,4}$, and $b_{4,3} = b_{3,4}$. By induction, the following lemma is reaced.

Lemma 2.10. *For $j \geq 4$, we have $b_{4,j} = P_{j-4}(8P_4 + 4 + Q_3Q_4) + P_{j-3}\left(\frac{Q_4Q_5 - 4}{2}\right)$.*

By using Lemmas 2.9 and 2.10, we can derive $b_{5,1}, b_{5,2}, b_{5,3}$, and $b_{5,4}$. From these conclusions and Definition 2.7, we reach the following lemma.

Lemma 2.11. *For $j \geq 5$, we get*

$$b_{5,j} = P_{j-5}(8P_4 + 4 + Q_3Q_4 + Q_4Q_5) + P_{j-4}\left(\frac{Q_5Q_6 - 4}{2}\right).$$

Proof. From (1.3), and Definition 2.7, since $b_{5,5} = Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 + Q_5^2 = \frac{Q_5Q_6 - 4}{2}$, by induction, the proof is completed. □

Utilizing Definition 2.7, Lemmas 2.9, 2.10 and 2.11, we arrive at the following lemma through induction on the variable i.

Lemma 2.12. *For $j \geq i \geq 6$, we have*

$$b_{i,j} = P_{j-i}\left(8P_4 + 4 + \sum_{k=4}^{i} Q_{k-1}Q_k\right) + P_{j-i+1}\left(\frac{Q_iQ_{i+1} - 4}{2}\right).$$

We can easily obtain the following corollary by using Pell numbers and Pell-Lucas numbers.

Corollary 2.13. *For the symmetric Pell-Lucas matrix B_n, we get*

$$B_n=[b_{ij}]=\begin{cases}\frac{1}{2}Q_{i+j+1}-\frac{1}{2}Q_{j-i+1+(-1)^{i+1}}+(-1)^{i+1}2P_{j-i}, & i\leq j\\ \frac{1}{2}Q_{i+j+1}-\frac{1}{2}Q_{i-j+1+(-1)^{j+1}}+(-1)^{j+1}2P_{i-j}, & i>j\end{cases}.$$

Lemma 2.14. *Let $i,j\in\mathbb{Z}^+$ and $i\geq 3$. Then we have*

$$\sum_{k=1}^{i-2}(-1)^{i-2-k}b_{k,j}-\frac{3}{2}b_{i-1,j}+\frac{1}{2}b_{i,j}=\begin{cases}Q_{j-i+1}, & i\leq j\\ 0, & i>j\end{cases}. \tag{2.6}$$

Proof. Assume that $i\leq j$. Now, we prove the theorem by the induction method on i. Let $i=3$. From Corollary 2.13, we can derive

$$\begin{aligned} b_{1,j}-\frac{3}{2}b_{2,j}+\frac{1}{2}b_{3,j} &= \left(\frac{1}{2}Q_{j+2}-\frac{1}{2}Q_{j+1}+2P_{j-1}\right)+\left(-\frac{3}{4}Q_{j+3}+\frac{3}{4}Q_{j-2}+3P_{j-2}\right)+\left(\frac{1}{4}Q_{j+4}-\frac{1}{4}Q_{j-1}+P_{j-3}\right)\\ &= Q_{j-2}.\end{aligned}$$

Suppose that the hypothesis is true for i. For $i+1$, by using equations (1.1), (1.2) and Corollary 2.13, we find

$$\begin{aligned}\sum_{k=1}^{i-1}(-1)^{i-1-k}b_{k,j}-\frac{3}{2}b_{i,j}+\frac{1}{2}b_{i+1,j} &= b_{i-1,j}-\sum_{k=1}^{i-2}(-1)^{i-2-k}b_{k,j}-\frac{3}{2}b_{i,j}+\frac{1}{2}b_{i+1,j}\\ &= b_{i-1,j}+\left(-\frac{3}{2}b_{i-1,j}+\frac{1}{2}b_{i,j}-Q_{j-i+1}\right)-\frac{3}{2}b_{i,j}+\frac{1}{2}b_{i+1,j}\\ &= -\frac{1}{2}b_{i-1,j}-b_{i,j}+\frac{1}{2}b_{i+1,j}-Q_{j-i+1}\\ &= \left(-\frac{1}{4}Q_{i+j}+\frac{1}{4}Q_{j-i+2+(-1)^i}-(-1)^iP_{j-i+1}\right)\\ &\quad+\left(-\frac{1}{2}Q_{i+j+1}+\frac{1}{2}Q_{j-i+1+(-1)^{i+1}}+(-1)^i2P_{j-i}\right)\\ &\quad+\left(\frac{1}{4}Q_{i+j+2}-\frac{1}{4}Q_{j-i+(-1)^i}+(-1)^iP_{j-i-1}\right)-Q_{j-i+1}\\ &= \frac{1}{4}\left(-Q_{i+j}-2Q_{i+j+1}+Q_{i+j+2}\right)+\frac{1}{4}\left(Q_{j-i+2+(-1)^i}+2Q_{j-i+1+(-1)^{i+1}}-Q_{j-i+(-1)^i}\right)\\ &\quad+(-1)^i\left(-P_{j-i+1}+2P_{j-i}+P_{j-i-1}\right)-Q_{j-i+1}\\ &= \frac{1}{4}\left(2Q_{j-i+1+(-1)^i}+2Q_{j-i+1+(-1)^{i+1}}\right)-Q_{j-i+1}\\ &= 4P_{j-i+1}-Q_{j-i+1}\\ &= Q_{j-i}+Q_{j-i+1}-Q_{j-i+1}\\ &= Q_{j-i}.\end{aligned}$$

The proof for $i>j$ can be completed in a similar way. □

Theorem 2.15. *For $n\in\mathbb{Z}^+$, we have $Y_nY_{n-1}\ldots Y_2Y_1\Gamma_n^{-1}B_n=A_n^T$ and the Cholesky factorization of B_n is given by $B_n=A_nA_n^T$.*

Proof. By Corollary 2.5, $Y_nY_{n-1}\ldots Y_2Y_1\Gamma_n^{-1}=A_n^{-1}$. So, if we get $A_n^{-1}B_n=A_n^T$, then the theorem holds. Let $A_n^{-1}B_n=[c_{ij}]$. So, from (1.1), (2.3), (2.4), (2.5) and Lemma 2.14, we find the following:

$$
\begin{aligned}
A_n^{-1}B_n = [c_{ij}] &= \begin{cases} \frac{1}{2}b_{1j}, & i=1 \\ -\frac{3}{2}b_{11}+\frac{1}{2}b_{21}, & i=2, j=1 \\ -\frac{3}{2}b_{1j}+\frac{1}{2}b_{2j}, & i=2, j\geq 2 \\ \sum_{k=1}^{i-2}(-1)^{i-2-k}b_{k,j}-\frac{3}{2}b_{i-1,j}+\frac{1}{2}b_{i,j}, & i\geq 3 \end{cases} \\
&= \begin{cases} Q_j, & i=1 \\ -3Q_1+Q_2, & i=2, j=1 \\ -3Q_j+4P_{j+1}, & i=2, j\geq 2 \\ \sum_{k=1}^{i-2}(-1)^{i-2-k}b_{k,j}-\frac{3}{2}b_{i-1,j}+\frac{1}{2}b_{i,j}, & i\geq 3 \end{cases} \\
&= \begin{cases} Q_j, & i=1 \\ 0, & i=2, j=1 \\ Q_{j-1}, & i=2, j\geq 2 \\ Q_{j-i+1}, & i\geq 3, i\leq j \\ 0, & i\geq 3, i>j \end{cases} \\
&= \begin{cases} Q_{j-i+1}, & i\leq j \\ 0, & i>j \end{cases} \\
&= A_n^T.
\end{aligned}
$$

Hence, the Cholesky factorization of B_n is given by $B_n = A_nA_n^T$. □

Now we give the Cholesky factorization of B_6 by using A_6 in Example 1.2.

Example 2.16. *By Theorem 2.15, since the Cholesky factorization of B_6 is $A_6A_6^T$, then we get*

$$
B_6 = \begin{bmatrix} 4 & 12 & 28 & 68 & 164 & 396 \\ 12 & 40 & 96 & 232 & 560 & 1352 \\ 28 & 96 & 236 & 572 & 1380 & 3332 \\ 68 & 232 & 572 & 1392 & 3360 & 8112 \\ 164 & 560 & 1380 & 3360 & 8116 & 19596 \\ 396 & 1352 & 3332 & 8112 & 19596 & 47320 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 0 \\ 14 & 6 & 2 & 0 & 0 & 0 \\ 34 & 14 & 6 & 2 & 0 & 0 \\ 82 & 34 & 14 & 6 & 2 & 0 \\ 198 & 82 & 34 & 14 & 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 & 14 & 34 & 82 & 198 \\ 0 & 2 & 6 & 14 & 34 & 82 \\ 0 & 0 & 2 & 6 & 14 & 34 \\ 0 & 0 & 0 & 2 & 6 & 14 \\ 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.
$$

Moreover, since $B_n^{-1} = \left(A_n^T\right)^{-1} A_n^{-1}$, we obtain

$$B_n^{-1} = [\beta_{ij}] = [\beta_{ji}] = \begin{cases} \dfrac{2(n+1-i)+1}{2}, & i=j<n \\ \dfrac{1}{4}, & i=j=n \\ -\dfrac{4(n-i)+1}{4}, & i+1=j<n \\ -\dfrac{3}{4}, & i+1=j=n \\ (-1)^{j-i}(n+1-j), & i+1<j<n \\ \dfrac{(-1)^{j-i}}{2}, & i+1<j=n \end{cases}. \tag{2.7}$$

Example 2.17. *By (2.7), the inverse of B_6 in Example 2.8 is*

$$B_6^{-1} = \begin{bmatrix} 13/2 & -21/4 & 4 & -3 & 2 & -1/2 \\ -21/4 & 11/2 & -17/4 & 3 & -2 & 1/2 \\ 4 & -17/4 & 9/2 & -13/4 & 2 & -1/2 \\ -3 & 3 & -13/4 & 7/2 & -9/4 & 1/2 \\ 2 & -2 & 2 & -9/4 & 5/2 & -3/4 \\ -1/2 & 1/2 & -1/2 & 1/2 & -3/4 & 1/4 \end{bmatrix}.$$

From Theorem 2.15, we get the following corollary.

Corollary 2.18. *For $n \in \mathbb{Z}^+$, we get*

$$Q_{n+1}Q_n + Q_nQ_{n-1} + \cdots + Q_2Q_1 = \begin{cases} \dfrac{(Q_{n+1})^2}{2} - 2, & if\ n\ is\ even \\ \dfrac{(Q_{n+1})^2}{2} - 6, & if\ n\ is\ odd \end{cases}.$$

3. Eigenvalues of the symmetric Pell-Lucas matrix B_n

In this section, we consider the eigenvalues of the symmetric Pell-Lucas matrix B_n.

Let $W = \{r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n \ : \ r_1 \geq r_2 \geq \cdots \geq r_n\}$. For $r,s \in W$, $r \prec s$ if $\sum_{i=1}^{t} r_i \leq \sum_{i=1}^{t} s_i$, $t = 1,2,\ldots,n-1$, and if $t = n$, then equality holds.It is stated that s majorizes r or that r is majorized by s when $r \prec s$. The condition for majorization can be written as follows: for $r,s \in W$, $r \prec s$ if $\sum_{i=0}^{t} r_{n-i} \geq \sum_{i=0}^{t} s_{n-i}$, $t = 0,1,\ldots,n-2$, and if $t = n-1$, then equality holds.

The following is an exciting simple fact:

$$(\bar{r}, \bar{r}, \ldots, \bar{r}) \prec (r_1, r_2, \ldots, r_n), \quad \text{where} \quad \bar{r} = \frac{\sum_{i=1}^{n} r_i}{n}.$$

We refer to [14] and [15] for more information about majorizations.

An $n \times n$ matrix $D = [d_{ij}]$ is doubly stochastic if $d_{ij} \geq 0$ for $i,j = 1,2,\ldots,n$, $\sum_{i=1}^{n} d_{ij} = 1$, $j = 1,2,\ldots,n$, and $\sum_{j=1}^{n} d_{ij} = 1$, $i = 1,2,\ldots,n$. Hardy et al. [16] show that there must exist a doubly stochastic matrix D such that r=sD. This is the necessary and sufficient condition for $r \prec s$.

It is a well-known fact that the eigenvalues and the main diagonal components of a real symmetric matrix are both real numbers. The concept of majorization provides the precise link between the main diagonal components and the eigenvalues. The diagonal components symmetric matrix majorize the vector of eigenvalues of the matrix.

By Definition 1.1, we have $det(A_n) = 2^n$. Also by Theorem 2.15, since $B_n = A_n A_n^T$, we have $det(B_n) = 2^{2n}$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of B_n. Since $B_n = A_n A_n^T$ and $\sum_{i=1}^{k} Q_i^2 = \frac{Q_{k+1}Q_k}{2} - 2$ by (1.3), the eigenvalues of B_n are all positive and

$$\left(\frac{Q_{n+1}Q_n}{2} - 2, \frac{Q_nQ_{n-1}}{2} - 2, \dots, \frac{Q_2Q_1}{2} - 2\right) \prec (\lambda_1, \lambda_2, \dots, \lambda_n). \tag{3.1}$$

In [17], we arrive at the combinatorial property

$$Q_n = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-m} \binom{n-m}{m} 2^{n-2m}, \ for\ n \neq 0. \tag{3.2}$$

Hence, we obtain the following corollaries.

Corollary 3.1. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of B_n. Then we have*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \begin{cases} \dfrac{\left(\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n-m+1} \binom{n-m+1}{m} 2^{n-2m+1}\right)^2}{4} - 2n - 1, & if\ n\ is\ even \\ \dfrac{\left(\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n-m+1} \binom{n-m+1}{m} 2^{n-2m+1}\right)^2}{4} - 2n - 3, & if\ n\ is\ odd \end{cases}.$$

Proof. From (3.1), and Corollary 2.18, we find

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_n &= \frac{Q_{n+1}Q_n + Q_nQ_{n-1} + \cdots + Q_2Q_1}{2} - 2n \\ &= \begin{cases} \dfrac{(Q_{n+1})^2}{4} - 2n - 1, & if\ n\ is\ even \\ \dfrac{(Q_{n+1})^2}{4} - 2n - 3, & if\ n\ is\ odd \end{cases}. \end{aligned}$$

By (3.2), the proof is completed. □

Corollary 3.2. *If n is an even number, then we have*

$$4n\lambda_n \leq \left(\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n-m+1} \binom{n-m+1}{m} 2^{n-2m+1}\right)^2 - 8n - 4 \leq 4n\lambda_1.$$

If n is an odd number, then we have

$$4n\lambda_n \leq \left(\sum_{m=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n-m+1} \binom{n-m+1}{m} 2^{n-2m+1}\right)^2 - 8n - 12 \leq 4n\lambda_1.$$

Proof. Let $S_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Since

$$\left(\frac{S_n}{n}, \frac{S_n}{n}, \dots, \frac{S_n}{n}\right) \prec (\lambda_1, \lambda_2, \dots, \lambda_n), \tag{3.3}$$

we have $\lambda_n \leq \frac{S_n}{n} \leq \lambda_1$. Then by Corollary 3.1, the proof is completed. □

From (2.7), we have

$$\left(\frac{2n+1}{2}, \frac{2n-1}{2}, \dots, \frac{7}{2}, \frac{5}{2}, \frac{1}{4}\right) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \frac{1}{\lambda_{n-2}}, \dots, \frac{1}{\lambda_3}, \frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right). \tag{3.4}$$

Therefore, there exists a doubly stochastic matrix $H = [h_{ij}]$ such that

$$\left(\frac{2n+1}{2}, \frac{2n-1}{2}, \dots, \frac{7}{2}, \frac{5}{2}, \frac{1}{4}\right) = \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \dots, \frac{1}{\lambda_3}, \frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right) \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{bmatrix}.$$

That is, we find $\frac{1}{\lambda_n} h_{1n} + \frac{1}{\lambda_{n-1}} h_{2n} + \cdots + \frac{1}{\lambda_1} h_{nn} = \frac{1}{4}$ and $h_{1n} + h_{2n} + \cdots + h_{nn} = 1$.

Lemma 3.3. *For all $i=1,2,\ldots,n$, we get $h_{n-(i-1),n} \leq \frac{\lambda_i}{n-1}$.*

Proof. Assume that $h_{n-(i-1),n} > \frac{\lambda_i}{n-1}$. So

$$\begin{aligned} h_{1n}+h_{2n}+\cdots+h_{nn} &> \frac{\lambda_1}{n-1}+\frac{\lambda_2}{n-1}+\cdots+\frac{\lambda_n}{n-1} \\ &= \frac{1}{n-1}\left(\lambda_1+\lambda_2+\cdots+\lambda_n\right). \end{aligned}$$

Since $h_{1n}+h_{2n}+\cdots+h_{nn}=1$ and $\sum_{i=1}^{n}\lambda_i \geq n$, this yields a contradiction, then $h_{n-(i-1),n} \leq \frac{\lambda_i}{n-1}$. □

For $k \in \mathbb{Z}^+$, we define

$$\begin{aligned} T_k &= \sum_{i=1}^{k}\frac{1}{\lambda_i} \\ &= \frac{2k+1}{2}+\frac{2k-1}{2}+\frac{2k-3}{2}+\cdots+\frac{7}{2}+\frac{5}{2}+\frac{1}{4} \\ &= \frac{2k^2+4k-5}{4}. \end{aligned} \tag{3.5}$$

Hence we obtain

$$\left(\frac{T_n}{n},\frac{T_n}{n},\ldots,\frac{T_n}{n},\frac{T_n}{n}\right) \prec \left(\frac{1}{\lambda_n},\frac{1}{\lambda_{n-1}},\ldots,\frac{1}{\lambda_2},\frac{1}{\lambda_1}\right).$$

Theorem 3.4. *Let $2 \leq n \in \mathbb{Z}^+, S_n=\lambda_1+\lambda_2+\cdots+\lambda_n$ and $U_n=\frac{1}{n-1}\left(S_n-\frac{n}{T_n}\right)$. Then we have*

$$\left(\frac{n}{T_n},U_n,U_n,\ldots,U_n\right) \prec \left(\lambda_1,\lambda_2,\ldots,\lambda_n\right).$$

.

Proof. For $i,j=1,2,\ldots,n$, we define an $n\times n$ matrix

$$G_n=[g_{ij}]=\begin{bmatrix} g_{11} & g_{12} & g_{12} & \cdots & g_{12} \\ g_{21} & g_{22} & g_{22} & \cdots & g_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n1} & g_{n2} & g_{n2} & \cdots & g_{n2} \end{bmatrix}, \tag{3.6}$$

where for $i=1,2,\ldots,n$, $g_{i1}=\dfrac{1}{T_n\lambda_i}$ and $g_{i2}=\dfrac{1-g_{i1}}{n-1}$.
From (3.5) and (3.6), for $i=1,2,\ldots,n$, we have

$$\begin{aligned} g_{11}+g_{21}+\cdots+g_{n1} &= \frac{1}{T_n\lambda_1}+\frac{1}{T_n\lambda_2}+\cdots+\frac{1}{T_n\lambda_n}=1, \\ g_{12}+g_{22}+\cdots+g_{n2} &= \frac{1-g_{11}}{n-1}+\frac{1-g_{21}}{n-1}+\cdots+\frac{1-g_{n1}}{n-1}=1, \\ g_{i1}+(n-1)\,g_{i2} &= g_{i1}+(n-1)\,\frac{1-g_{i1}}{n-1}=1, \end{aligned}$$

where $g_{i1}\geq 0$ and $g_{i2}\geq 0$. Then, G_n is a doubly stochastic matrix. Also, we get

$$\begin{aligned} \lambda_1 g_{11}+\lambda_2 g_{21}+\cdots+\lambda_n g_{n1} &= \lambda_1\frac{1}{T_n\lambda_1}+\lambda_2\frac{1}{T_n\lambda_2}+\cdots+\lambda_n\frac{1}{T_n\lambda_n}=\frac{n}{T_n}, \\ \lambda_1 g_{12}+\lambda_2 g_{22}+\cdots+\lambda_n g_{n2} &= \lambda_1\left(\frac{1-g_{11}}{n-1}\right)+\lambda_2\left(\frac{1-g_{21}}{n-1}\right)+\lambda_n\left(\frac{1-g_{n1}}{n-1}\right)=U_n. \end{aligned}$$

Therefore, we have

$$\left(\frac{n}{T_n},U_n,U_n,\ldots,U_n\right)=\left(\lambda_1,\lambda_2,\ldots,\lambda_n\right)G_n,$$

and so, we obtain

$$\left(\frac{n}{T_n},U_n,U_n,\ldots,U_n\right) \prec \left(\lambda_1,\lambda_2,\ldots,\lambda_n\right).$$

□

Lemma 3.5. *For $k = 2, 3, \ldots, n$, we get*

$$\lambda_k \geq \frac{1}{T_k},$$

where $T_k = \frac{2k^2+4k-5}{4}$.

Proof. By using (3.4), for $k \geq 2$, we have

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots + \frac{1}{\lambda_{k-1}} + \frac{1}{\lambda_k} \leq \frac{1}{4} + \frac{5}{2} + \frac{7}{2} + \cdots + \frac{2k-1}{2} + \frac{2k+1}{2} = T_k$$

Therefore, we have

$$\frac{1}{\lambda_k} \leq T_k - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots + \frac{1}{\lambda_{k-1}}\right) \leq T_k,$$

and so, the proof is completed. □

Theorem 3.6. *Let $2 \leq n \in \mathbb{Z}^+, S_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and $U_n = \frac{1}{n-1}\left(S_n - \frac{n}{T_n}\right)$. Then for $k \leq n-2$, we have*

$$\begin{aligned} \lambda_1 &\leq 2^{2n} \prod_{i=2}^{n} T_i, \\ \lambda_{n-k} &\leq (k+1)\,U_n - \sum_{i=0}^{k-1} \frac{1}{T_{n-i}}. \end{aligned}$$

Proof. By Theorem 2.15, we know that $det\,(B_n) = 2^{2n} = \lambda_1 \lambda_2 \cdots \lambda_n$. By Lemma 3.5, we get

$$2^{2n} = \lambda_1 \lambda_2 \cdots \lambda_n \geq \lambda_1 \prod_{i=2}^{n} \frac{1}{T_i},$$

and so, we obtain $\lambda_1 \leq 2^{2n} \prod_{i=2}^{n} T_i$. By Theorem 3.4, for $k \leq n-2$, we have

$$\lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-(k-1)} + \lambda_{n-k} \leq (k+1)\,U_n,$$

and so, by Lemma 3.5, we get

$$\begin{aligned} \lambda_{n-k} &\leq (k+1)\,U_n - \left(\lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-(k-1)}\right) \\ &\leq (k+1)\,U_n - \sum_{i=0}^{k-1} \frac{1}{T_{n-i}}. \end{aligned}$$

Then the proof is completed. □

By applying Theorem 3.4 and Lemma 3.5, we can readily derive the subsequent corollary.

Corollary 3.7. *Let $2 \leq n \in \mathbb{Z}^+$ and $k \leq n-2$. Then we have*

$$\begin{aligned} &\frac{n}{T_n} \leq \lambda_1 \leq 2^{2n} \prod_{i=2}^{n} T_i, \\ &\frac{1}{T_{n-k}} \leq \lambda_{n-k} \leq (k+1)\,U_n - \sum_{i=0}^{k-1} \frac{1}{T_{n-i}}, \\ &\frac{1}{T_n} \leq \lambda_n \leq U_n. \end{aligned}$$

4. Conclusions

In this article, we introduce the Pell-Lucas A_n and the symmetric Pell-Lucas B_n matrices. We consider the linear algebra of these matrices. Firstly, we construct two different factorizations of Pell-Lucas matrices by the new matrix Γ_n. We find the inverse of the Pell-Lucas matrix A_n^{-1}, and present the factorization of A_n^{-1}. Then, we derive the components $[b_{ij}]$ of the Pell-Lucas matrix B_n, and construct the Cholesky factorization of B_n. This factorization is $A_n A_n^T$. We determine the inverse of the symmetric Pell-Lucas matrix B_n^{-1}. We give some interesting relations which include the eigenvalues of Pell-Lucas matrices. Moreover, we obtain the lower and upper boundaries for the eigenvalues of B_n by majorizations.

Declarations

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The Most Powerful Member of the Power-Divergence Family for the Independence Model in Contingency Tables

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Abstract

The family of power-divergence (PD) test statistic contains many well-known test statistics used in the analysis of the contingency tables under the independence model. In this work, we compare the various test statistics for the independence model. The type-I and type-II errors of the test statistics are obtained and compared via simulation study considering the different degree of freedoms and sample sizes. According to the simulation results, we recommend the PD(0.4) test statistic for the small sample size based on its power and type-I error rates. Two applications are given to demonstrate the usefulness of the PD(0.4) test statistic over the chi-square test statistic contingency tables.

1. Introduction

The chi-square test was developed by [1] to evaluate the association or difference between categorical variables. The chi-square test is commonly used in social and medical sciences to test the dependence structures of the levels of the categorical variables. The results of the chi-square test are misinterpreted by the researchers because of the lack of statistical knowledge [2]. Besides, the application of the chi-square test is very problematic for the small sample sizes which is ignored in many researches. It is well-known that the test statistic of the chi-square test follows the χ^2 distribution. However, the asymptotic approximation is only valid for the non-sparse contingency tables and large sample sizes. Working with the less observations than needed reduces the power of the test. Therefore, to obtain the higher power value, one should work with required sample size based on the dimension of the contingency table [3, 4]. The determination of the sample size is done based on four inputs: type-I error, type-II error or power, effect size and degree of freedom (df). The type-I and type-II errors are the pre-determined inputs [5].

When the contingency tables have large number of cells, the frequencies of each cell may be very small or has zero frequencies. So, the contingency tables with large numbers of row and column variables yields the less observations in the cells. In this case, these contingency tables are called as sparse contingency tables [6]. The sparse contingency tables occur when the the values of 0 and 1 in many cells of the contingency table and the total number of cells are higher than the sample size [7, 8]. Besides, the sparseness index (SI) is useful to determine the sparse contingency tables. The SI is defined as

$$SI = \frac{n}{RC},$$

where n is the sample size R is the number of rows and C is the number of columns in the contingency tables.

There are various studies in the literature for the comparison of goodness-of-fit test statistics in small samples. [9] performed a study to find a clear answer about what is the minimum value of the expected frequency and sample size to achieve the reasonable approximation to the χ^2 distribution. [10] implemented a simulation study to compare the χ^2, G^2 and [11] test statistic for 13 contingency tables. [10] found that the χ^2 and Cressie and Read statistics can be used for smaller sample sizes than suggested by [9]. Several rule of thumb were suggested by researchers for χ^2 approximation of the Pearson and likelihood ratio test statistics. [9] suggested that minimum cell expectation should be higher than $5t_5/t$ where t_5 is the number of cells where the expected frequency is smaller than 5 and t is the total number of cells of the corresponding contingency

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table. [12] suggested that the sample size should be higher than 4 or 5 times t. [13] showed that the χ^2 statistic is much more appropriate than G^2 statistic for the small sample size. Recently, [14] performed a comprehensive simulation study to asses the small sample accuracy of the seven members of the power-divergence statistics for testing both independence and homogeneity in contingency tables. The results of the study of [14] showed that G^2 statistic rejects the null hypothesis too often in both sparse and non-sparse contingency tables. They suggested the non-asymptotic variant of χ^2 statistic removes the deficiency of the Pearson χ^2 statistic for sparse contingency tables. [15] investigated the determination of the power divergence parameter under quasi-independence model. More recently, [16] studied the asymptotic properties of T^2 test statistic under the symmetry model and concluded that the approximation of the T^2 test statistic to χ^2 distribution is only valid for very large sample sizes. While the chi-square approach gives healthy results in tables with a degree of freedom greater than 1 and a maximum of 20 % of the expected frequencies below 5, this approach is weak in the sparse contingency table [7].

A general class of the test statistics was proposed by [11] and called as power-divergence (PD) family of statistics. The PD statistics contains Pearson's χ^2, likelihood ratio statistic G^2, Freeman-Tukey's T^2, modified likelihood ratio statistics GM^2 and Neyman's modified χ^2 as its sub-models. Note that these test statistics follow χ^2 distribution [12, 17, 18]. This study compares the members of the PD test statistic using the different values of the parameter λ based on the independence model. The type-I and power values of the test statistics are compared with simulation studies for different dimensions of the contingency tables. The goal of the simulation study is to find the most powerful test statistic for the independence model considering the sample sizes, type-I and type-II errors.

The other sections of the study is designed as follows. Firstly, the independence model is given in Section 2. The PD family of statistics is given in Section 3. The comparison of the test statistics via simulation studies is presented in Section 4. The recommended test statistic and Freeman-Halton (FH) test statistic is compared in Section 5. The power comparison of the most powerful test statistic and χ^2 test statistic based on the real datasets is given in Section 6. The future work and conclusions of the presented study are given in Section 7.

2. Independence model

In the analysis of contingency tables, either "row and column variables are independent of each other" or "the constant levels of one of the variables do not differ between the other variable levels" are tested according to the researcher's purpose. The total chi-square of the calculations for the entire R×C table is divided into row, column and relationship components as follows

$$\chi_T^2 = \chi_R^2 + \chi_C^2 + \chi_{RC}^2. \tag{2.1}$$

In two dimensional tables, the independence hypothesis is expressed with (2.2)

$$H_0 : p_{ij} = p_{i.}p_{.j}, \quad i = 1,2,3,...,R; \; j = 1,2,3,...,C. \tag{2.2}$$

The probability density function for the observed frequencies (n_{ij}) is as follows

$$P\left(n_{ij} \middle| p_{ij}, n\right) = \frac{n!}{\prod\limits_{i,j} n_{ij}!} \prod_{i,j} {p_{ij}}^{n_{ij}}. \tag{2.3}$$

Substituting $p_{ij} = p_{i.}p_{.j}$ in (2.3), we have

$$P\left(n_{ij} \middle| p_{ij}, n\right) = \frac{n!}{\prod\limits_{i,j} n_{ij}!} \prod_{i,j} {p_{ij}}^{n_{ij}} = \frac{n!}{\prod\limits_{i,j} n_{ij}!} \prod_{i} {p_{i.}}^{n_{i.}} \prod_{j} {p_{.j}}^{n_{.j}} \tag{2.4}$$

$$= \frac{n! \prod\limits_{i,j} {p_{i.}}^{n_{i.}}}{\prod\limits_{i} n_{i.}!} \frac{n! \prod\limits_{i,j} {p_{.j}}^{n_{.j}}}{\prod\limits_{j} n_{.j}!} \frac{\prod\limits_{i} n_{i.}! \prod\limits_{j} n_{.j}!}{n! \prod\limits_{i,j} n_{ij}!} \tag{2.5}$$

Equality of $n_{ij} = np_{ij} + e_{ij}$ is written instead of n in (2.5). When the natural logarithm is taken using the Stirling series expansion in (2.5), the three terms on the right side of the equation (2.5) follows approximately the chi-square distribution (see [19].

$$\mathrm{X}_T^2 = \sum_{i=1}^{R} \sum_{j=1}^{C} \frac{\left(n_{ij} - np_{i.}p_{.j}\right)^2}{np_{i.}p_{.j}}. \tag{2.6}$$

The quantity in (2.6) follows the chi-square distribution with (RC-1) degrees of freedom. The first part on the right side of the equation is given by

$$X_R^2 = \sum_i \frac{(n_{i.} - np_{i.})^2}{np_{i.}}, \tag{2.7}$$

which follows the chi-square distribution with R-1 df. The second part is given by

$$X_C^2 = \sum_j \frac{(n_{.j} - np_{.j})^2}{np_{.j}},$$

which follows the chi-square distribution with C-1 df. The third part is the test statistic calculated for the independence hypothesis which is given by

$$X_{RC}^2 = \sum_i \sum_j \frac{(n_{ij} - n_{i.}n_{.j}/n)^2}{n_{i.}n_{.j}/n}.$$

The df can be extracted using the relation given in (2.1). So, the df of the independence model is $(R-1)(C-1)$. The likelihood estimates of expected values e_{ij} under independence model is $e_{ij} = n_{i.}n_{.j}/n$.

3. Power-divergence family

The PD family of statistics, $PD(\lambda)$, is given by

$$PD(\lambda) = \frac{2}{\lambda(\lambda+1)} \sum_{i=1}^{R} \sum_{j=1}^{C} n_{ij} \left[\left(\frac{n_{ij}}{e_{ij}} \right)^{\lambda} - 1 \right], \tag{3.1}$$

where $i = 1,2,3,...,R,\ j = 1,2,3,...,C$ and $\lambda \in \Re$. When the $\lambda = 0$ and $\lambda = -1$, the equation (3.1) is not valid. So, the limiting cases of (3.1) for $\lambda = 0$ and $\lambda = -1$ are given as follows

$$\lim_{\lambda \to 0} \frac{2}{\lambda(\lambda+1)} \sum_{i=1}^{R} \sum_{j=1}^{C} n_{ij} \left[\left(\frac{n_{ij}}{e_{ij}} \right)^{\lambda} - 1 \right] = 2 \sum_{i=1}^{R} \sum_{j=1}^{C} n_{ij} \left[\ln \left(\frac{n_{ij}}{e_{ij}} \right) \right],$$

$$\lim_{\lambda \to -1} \frac{2}{\lambda(\lambda+1)} \sum_{i=1}^{R} \sum_{j=1}^{C} n_{ij} \left[\left(\frac{n_{ij}}{e_{ij}} \right)^{\lambda} - 1 \right] = 2 \sum_{i=1}^{R} \sum_{j=1}^{C} e_{ij} \left[\ln \left(\frac{e_{ij}}{n_{ij}} \right) \right],$$

respectively. As given in Section 1, the PD family of statistics contains various known test statistics as its sub-models.

- PD(1) reduces the Pearson's χ^2 test statistics
- PD(0) reduces the likelihood ratio G^2 test statistics
- PD(-1/2) reduces the Freeman Tukey's T^2 test statistics
- PD(2/3) reduces the Cressie Read test statistic C^2

4. Simulation studies

Simulation studies are performed to evaluate the performance of the test statistics for the independence model. The multinomial distribution is used to generate contingency tables. The below probability matrices are used to obtain type-I errors of the test statistics. The probability matrices are generated by assuming that null hypothesis, H_0 is true.

3x3 contingency table			3x4 contingency table				3x5 contingency table				
0.10	0.06	0.04	0.03	0.03	0.02	0.02	0.03	0.03	0.09	0.06	0.09
0.15	0.09	0.06	0.06	0.06	0.04	0.04	0.02	0.02	0.06	0.04	0.06
0.25	0.15	0.10	0.21	0.21	0.14	0.14	0.05	0.05	0.15	0.10	0.15

Table 1: Probability matrices used to detect type-I errors for $R = 3$ and $C = 3,4,5$

4x4 contingency table				4x5 contingency table				
0.02	0.03	0.01	0.04	0.02	0.02	0.06	0.04	0.06
0.04	0.06	0.02	0.08	0.03	0.03	0.09	0.06	0.09
0.06	0.09	0.03	0.12	0.03	0.03	0.09	0.06	0.09
0.08	0.12	0.04	0.16	0.02	0.02	0.06	0.04	0.06

Table 2: Probability matrices used to detect type-I errors for $R = 4$ and $C = 4,5$

5x5 contingency table				
0.01	0.01	0.03	0.02	0.03
0.01	0.01	0.03	0.02	0.03
0.03	0.03	0.09	0.06	0.09
0.02	0.02	0.06	0.04	0.06
0.03	0.03	0.09	0.06	0.09

Table 3: Probability matrix used to detect type-I errors for $R = 5$ and $C = 5$

Also, the below matrices are used to obtain power of the test statistics.

3x3 contingency table			3x4 contingency table				3x5 contingency table				
0.03	0.11	0.06	0.01	0.04	0.01	0.04	0.09	0.01	0.04	0.12	0.04
0.2	0.03	0.07	0.09	0.03	0.07	0.01	0.07	0.04	0.03	0.01	0.05
0.15	0.22	0.13	0.15	0.3	0.05	0.2	0.1	0.12	0.15	0.05	0.08

Table 4: Probability matrices used to detect powers for $R = 3$ and $C = 3,4,5$

4x4 contingency table				4x5 contingency table				
0.05	0.01	0.03	0.01	0.1	0.05	0.01	0.02	0.02
0.01	0.02	0.1	0.07	0.1	0.1	0.05	0.03	0.02
0.1	0.02	0.08	0.1	0.1	0.1	0.05	0.03	0.02
0.15	0.04	0.1	0.11	0.1	0.05	0.01	0.01	0.03

Table 5: Probability matrices used to detect powers for $R = 4$ and $C = 4,5$

5x5 contingency table				
0.015	0.015	0.04	0.01	0.02
0.02	0.02	0.015	0.03	0.015
0.02	0.04	0.07	0.09	0.08
0.03	0.04	0.03	0.07	0.03
0.04	0.06	0.06	0.09	0.05

Table 6: Probability matrix used to detect powers for $R = 5$ and $C = 5$

These probability matrices are generated by assuming that the alternative hypothesis, H_1 is true. The row and column marginal probabilities are degenerated to create the departure from the independence model. The significance level α is determined as 0.05. The interpretation of the simulation results are done based on the 0.06 value. The test statistics having the type-I error above the 0.06 value are considered as inappropriate. The simulation replication is determined as $N = 10,000$.

Table 7 shows the effect sizes of the contingency tables used for the power calculation. Note that the effect sizes of the contingency tables used for the type-I error is zero. As reported in Table 7, the small and moderate effect sizes are used to compare the power values of the test statistics.

Effect size	Dimension					
	3x3	3x4	3x5	4x4	4x5	5x5
w	0.4341	0.3328	0.4691	0.3328	0.2564	0.2642

Table 7: The effect sizes of the contingency tables used for the power calculation.

4.1. Type-I error

Figure 1 displays the simulation results for the 3x3 contingency table. We also consider the sparseness index to analyze the behaviours of the test statistics for the small sample sizes. When the indicator SI is below 5 value, we call this contingency table as *sparse table*. So, the contingency table is called as sparse table if the number of observations is below 45 for the $R = 3$ and $C = 3$. This value is plotted in the figures vertically. According to the findings in Figure 1, we evaluate the convergence of the test statistics to χ^2 distribution. From Figure 1, we observe that T^2, G^2, $PD(0.1)$, $PD(0.2)$ and $PD(0.3)$ are above the 0.06 value which is evidence that these test statistics do not converge to χ^2 distribution. When the sample size is above 150, all test statistics work well, except T^2.

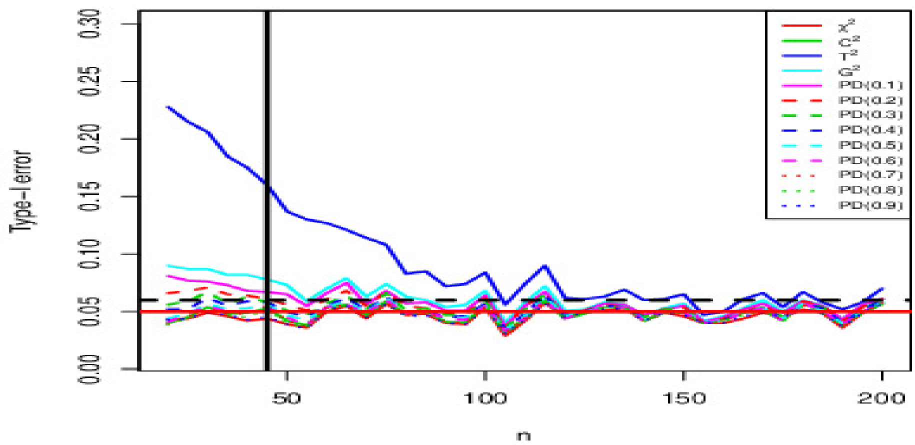


Figure 1: Type-I errors of the test statistics for R=3 and C=3.

Figure 2 displays the simulation results for the 3x4 contingency table. From these results, the T^2, G^2, $PD(0.1)$, $PD(0.2)$ and $PD(0.3)$ test statistics do not converge the χ^2 distribution for both sparse and non-sparse contingency tables. The vertical line represents the sample size for the sparseness index which is 60. Additionally, the convergence of the G^2 statistic to χ^2 distribution needs high sample sizes for $R = 3$ and $C = 4$ contingency tables. The C^2 performs better than the G^2 statistic. All test statistics converge to the χ^2 distribution when the sample size is higher than 150, except T^2 statistic.

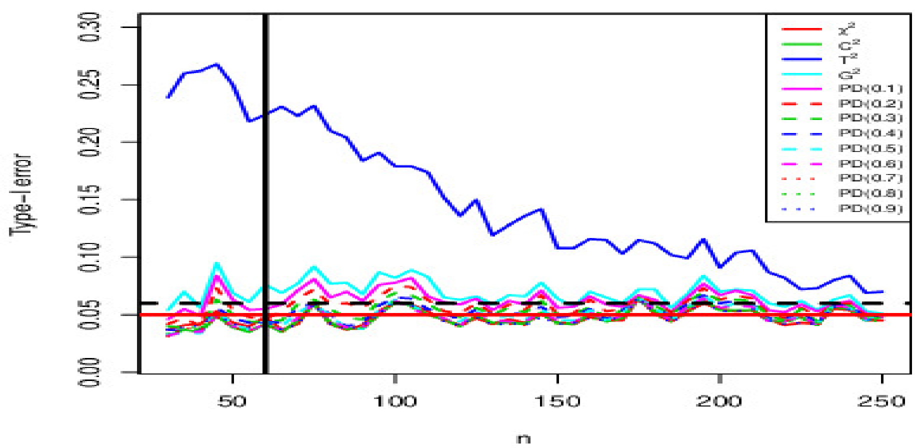


Figure 2: Type-I errors for R=3 and C=4.

Figure 3 displays the simulation results for the 3x5 contingency table. Again, the same test statistics fail to converge the χ^2 distribution. Here, the vertical line is 75 for the sample size of sparseness index. The G^2 needs higher sample sizes to converge to χ^2 distribution for $R = 3$ and $C = 5$.

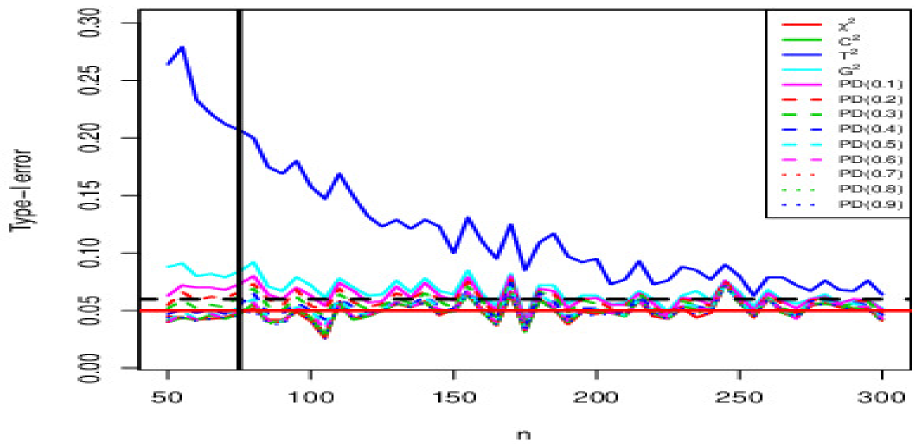


Figure 3: Type-I errors for R=3 and C=5.

The interpretation of the results of the 4x4, 4x5 and 5x5 contingency tables are similar to the previous simulation results. The results of these contingency tables are plotted in Figure 4. The vertical lines of the figures are 80, 120 and 125, respectively. From these figures, we conclude the convergence of the G^2 to χ^2 distribution is not valid for the small sample sizes.

The below findings are observed based on the simulation results of the test statistics for type-I errors.

- The convergence of the G^2 statistic to χ^2 distribution is very problematic for small sample sizes (see [20])
- The C^2 statistic performs better than G^2 statistic.
- The convergence of the T^2 statistic to χ^2 distribution is only valid for the large sample sizes and it cannot be used for any small sample size.
- The dimension of the contingency table effects the convergence of the statistics.
- More sample size is needed for the high dimensional contingency tables.

So, end of the simulation study for the type-I errors of the test statistics, we eliminate the T^2, G^2, $PD(0.1)$, $PD(0.2)$ and $PD(0.3)$ statistics since they do not converge well to χ^2 distribution. In the second step, we compare the power results of the test statistics converges well to χ^2 distribution.

Additionally, we compare the p-values of the test statistics for R=3, C=3 and $n = 50$. Let F_X be the distribution of the test statistic X under the null hypothesis. If F_T is continious, the p-value is distributed as $U(0,1)$ [21]. The distribution of the p-values for test statistics are evaluated via Kolmogorov-Smirnov (KS) test. The histograms of the p-values of the test statistics with the p-values of KS test are displayed in Figure 5. From these figures, it is clear that the distribution of the p-values of the T^2, G^2, $PD(0.1)$, $PD(0.2)$ and $PD(0.3)$ test statistics do not follow the $U(0,1)$ distribution since their p-values are lower than 0.05. It is evidence that these test statistics do not perform well for small sample sizes.

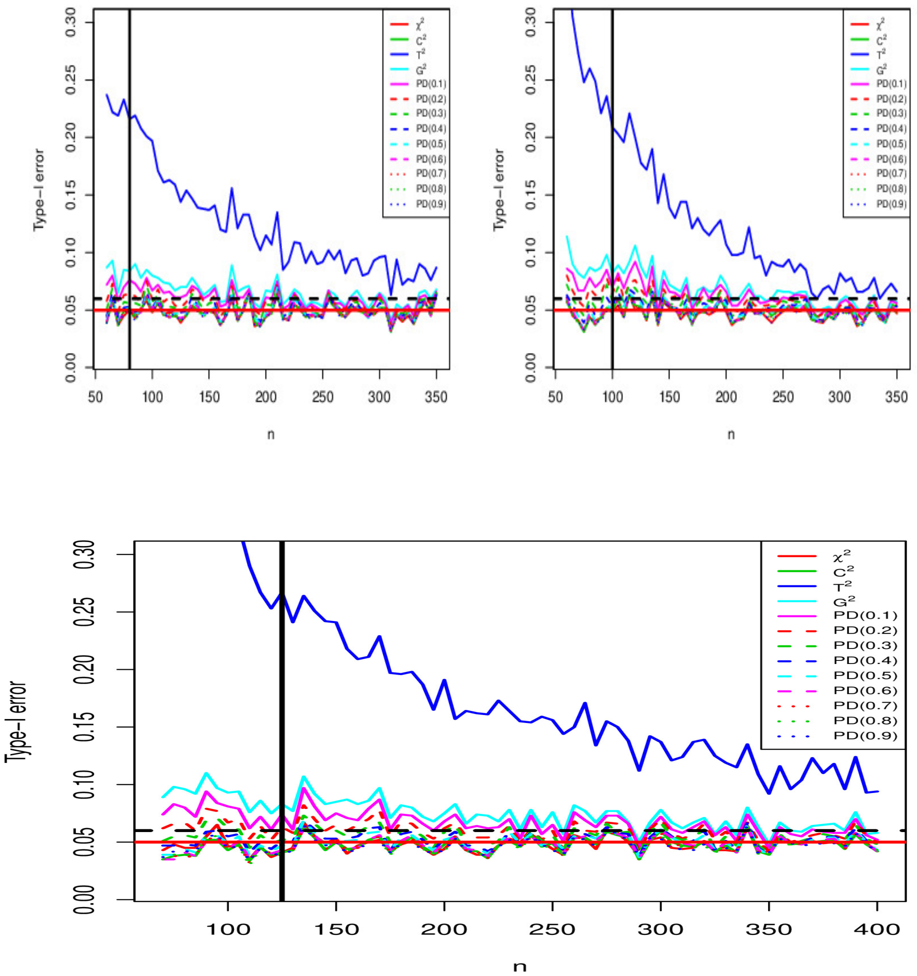


Figure 4: Type-I errors for (top-left) R=4 and C=4, (top-right) R=4 and C=5 and (bottom) R=5 and C=5

The below findings are observed based on the simulation results of the test statistics for type-I errors.

- The convergence of the G^2 statistic to χ^2 distribution is very problematic for small sample sizes (see [20])
- The C^2 statistic performs better than G^2 statistic.
- The convergence of the T^2 statistic to χ^2 distribution is only valid for the large sample sizes and it cannot be used for any small sample size.
- The dimension of the contingency table effects the convergence of the statistics.
- More sample size is needed for the high dimensional contingency tables.

So, end of the simulation study for the type-I errors of the test statistics, we eliminate the T^2, G^2, $PD(0.1)$, $PD(0.2)$ and $PD(0.3)$ statistics since they do not converge well to χ^2 distribution. In the second step, we compare the power results of the test statistics converges well to χ^2 distribution.

Additionally, we compare the p-values of the test statistics for R=3, C=3 and $n = 50$. Let F_X be the distribution of the test statistic X under the null hypothesis. If F_T is continious, the p-value is distributed as $U(0,1)$ [21]. The distribution of the p-values for test statistics are evaluated via Kolmogorov-Smirnov (KS) test. The histograms of the p-values of the test statistics with the p-values of KS test are displayed in Figure 5. From these figures, it is clear that the distribution of the p-values of the T^2, G^2, $PD(0.1)$, $PD(0.2)$ and $PD(0.3)$ test statistics do not follow the $U(0,1)$ distribution since their p-values are lower than 0.05. It is evidence that these test statistics do not perform well for small sample sizes.

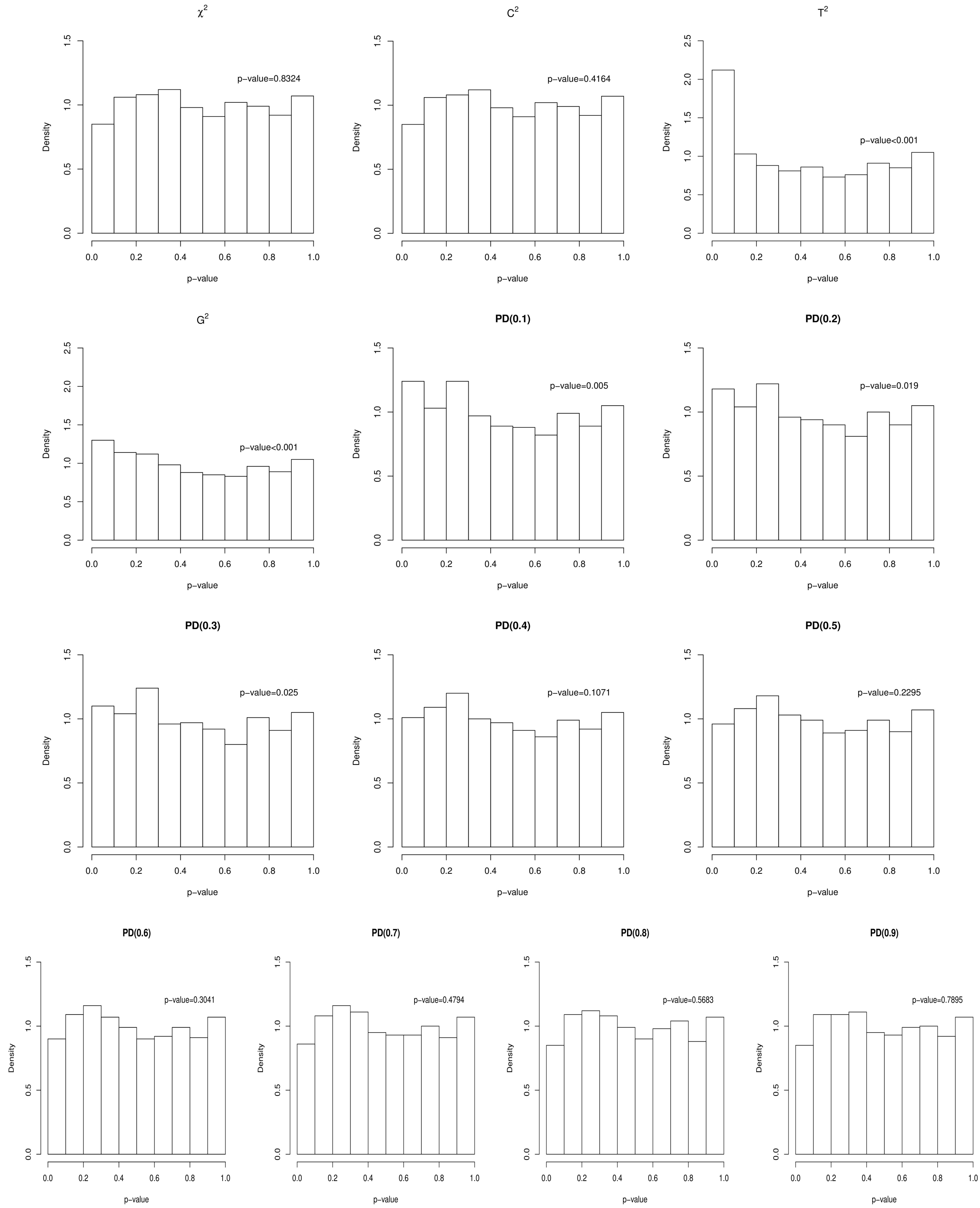


Figure 5: The distribution of p-values for the test statistics under R=3 and C=3 and $n = 50$

4.2. Power of test

Now, we examine the power results of each test statistics which are the members of the power-divergence family. The contingency tables are generated using the probability matrices given in Section 4.1. According to the results of the type-I errors of the test statistics, T^2, G^2, $PD(0.1)$, $PD(0.2)$ and $PD(0.3)$ do not converge the χ^2 distribution. Although the power of test results are reported for all test statistics, T^2, G^2, $PD(0.1)$, $PD(0.2)$ and $PD(0.3)$ are not considered in evaluation of the power results.

Figure 6 displays the power results of the test statistics for 3x3 contingency table. As seen from these Figure, T^2 has the highest power value among others. However, since it does not converge the χ^2 distribution, its power result is not meaningful. Similarly, the power results of the G^2, $PD(0.1)$, $PD(0.2)$ and $PD(0.3)$ statistics are also not meaningful. After eliminating these test statistics, the most powerful test statistics is $PD(0.4)$ for 3x3 contingency table. Vertical lines of the Figure 6 shows the minimum required sample size to achieve the 0.80 and 0.90 power values. The minimum sample size is 60 for the power 0.80 and minimum sample size is 75 for the power 0.90.

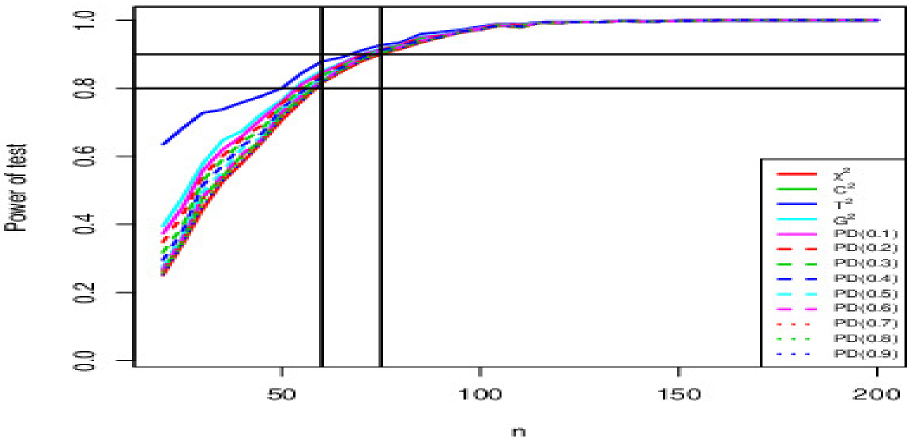


Figure 6: Power results under R=3 and C=3.

Figure 7 displays the power results of the test statistics for 3x4 contingency tables. These results are also in favour of the $PD(0.4)$ test statistics. The minimum sample size for the powers 0.80 and 0.90 are 65 and 80, respectively.

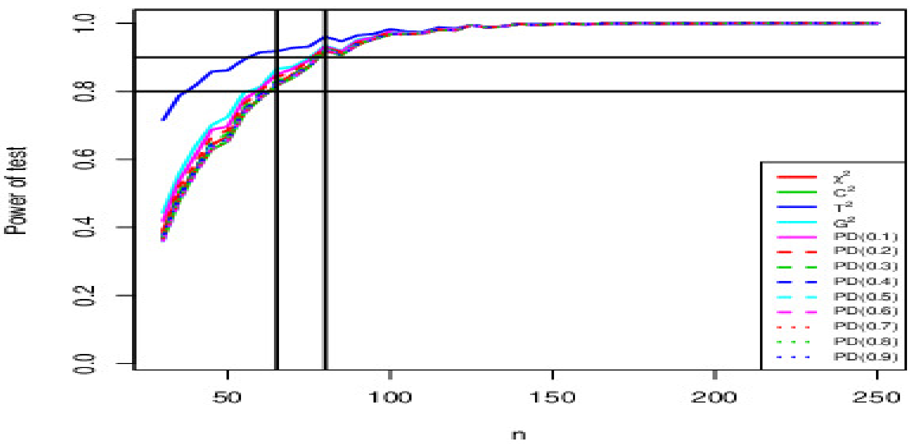


Figure 7: Power results under R=3 and C=4.

Similarly, Figure 8 displays the power results of the test statistics for 3x5 contingency table. The most powerful test statistic is $PD(0.4)$. As in previous results, the $PD(0.4)$ test statistics has the highest power among others. The minimum sample size for the powers 0.80 and 0.90 are 70 and 90, respectively.

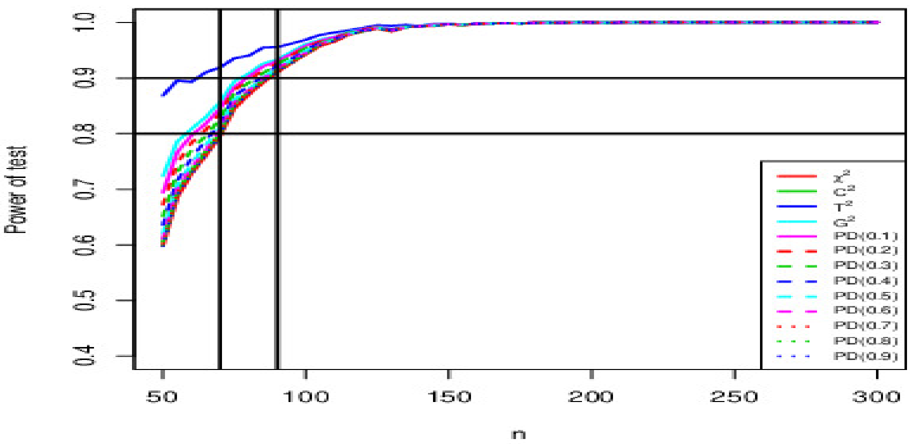


Figure 8: Power results under R=3 and C=5.

Figure 9 displays the power results of the test statistics for 4x4 contingency table. $PD(0.4)$ is again the most powerful test statistic among others. From these results, we conclude that the minimum required sample size is 130 to achieve at least 0.80 power and required sample size is 150 to achieve at least 0.90 power.

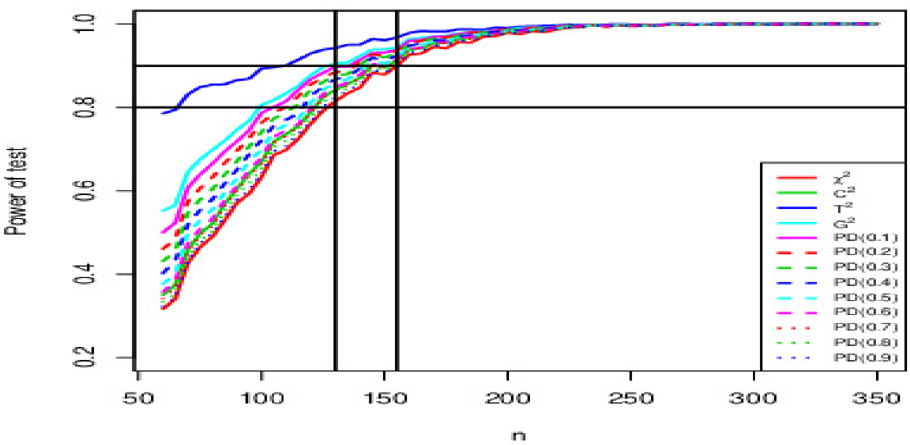


Figure 9: Power results under R=4 and C=4.

Figure 10 displays the power results of the test statistics for 4x5 contingency table. Results show that $PD(0.4)$ has the highest power. According to the vertical lines, the required sample size is 160 for the power 0.80 and 195 for the power 0.90.

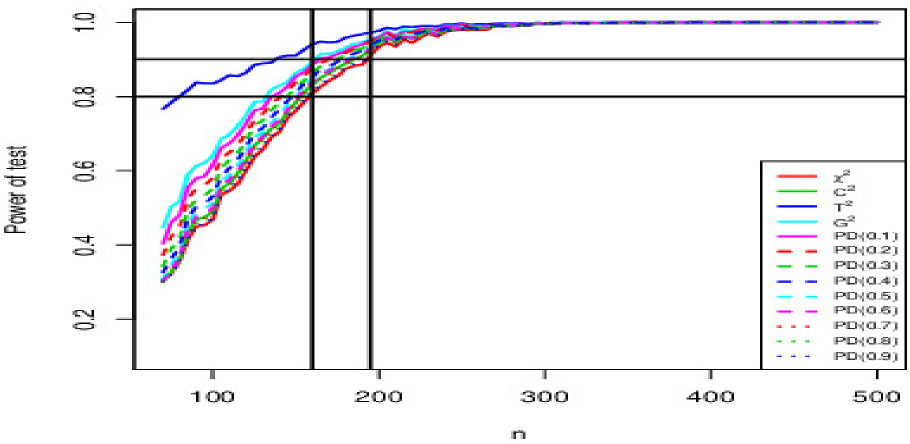


Figure 10: Power results under R=4 and C=5.

In a similar vein, Figure 11 displays the power results of the test statistics for 5x5 contingency table. Again, $PD(0.4)$ has the highest value of the power results. The required sample size is 280 for the power 0.80 and 350 for the power 0.90. As seen from these results, once the dimension of the contingency table increases, the required sample size increases to reach higher power values.

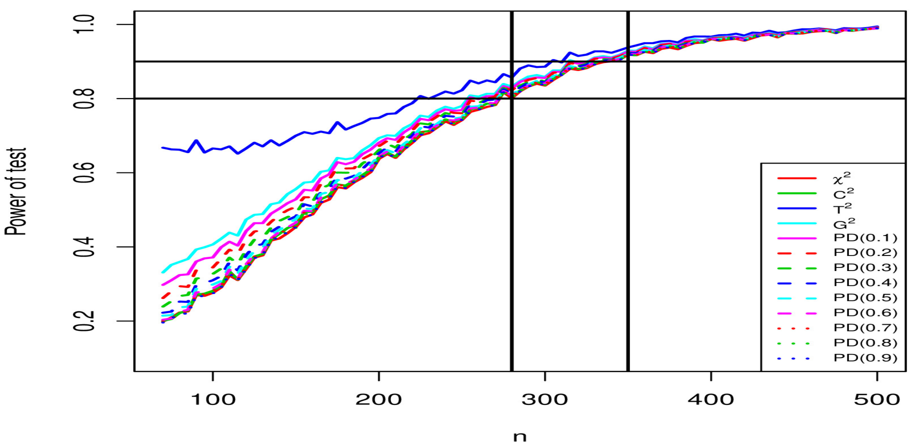


Figure 11: Power results under R=5 and C=5.

Table 8 shows the minimum required sample sizes for the contingency tables to reach the minimum 0.80 and 0.90 power values. As seen these results, the required sample size is an increasing function of the dimension of the contingency table. Therefore, higher dimension needs more sample size. The determined effect sizes for each table dimension are reported in Table 7.

Table dimensions	Power	
	0.8	0.9
3x3	60	75
3x4	65	80
3x5	70	90
4x4	130	150
4x5	160	195
5x5	280	350

Table 8: Minimum required sample sizes for the powers 0.8 and 0.9

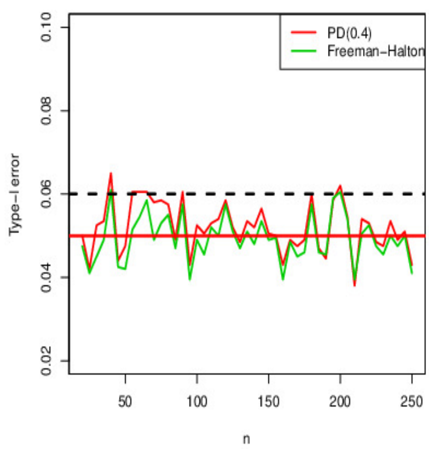
As given in Section 1, the sample size is function of type-I error, power, df and effect size (see, Section 5). The powers are calculated by considering the different values of the effect size, df and sample sizes for the fixed type-I error 0.05. The results are given in Table 9. From these results, it is seen that when the effect size is low, the required sample size should be large to obtain the high power. Also, when the df is high, the sample size should be large to obtain the high power. Under these results, if Table 8 is revisited, the sample sizes given in this table are determined based on the high effect sizes.

Sample size (df=4)	w=0.05	w=0.15	w=0.30	w=0.50	Sample size (df=6)	w=0.05	w=0.15	w=0.30	w=0.50
50	0.056	0.113	0.358	0.820	50	0.055	0.100	0.303	0.758
100	0.063	0.189	0.663	0.989	100	0.060	0.161	0.589	0.980
150	0.069	0.272	0.852	1.000	150	0.065	0.229	0.796	0.999
200	0.076	0.358	0.943	1.000	200	0.071	0.303	0.911	1.000
250	0.083	0.443	0.980	1.000	250	0.076	0.378	0.965	1.000
500	0.121	0.773	1.000	1.000	500	0.106	0.705	1.000	1.000
Sample size (df=8)	w=0.05	w=0.15	w=0.30	w=0.50	Sample size (df=9)	w=0.05	w=0.15	w=0.30	w=0.50
50	0.054	0.092	0.267	0.706	50	0.054	0.089	0.253	0.683
100	0.058	0.143	0.534	0.968	100	0.058	0.137	0.510	0.962
150	0.063	0.202	0.747	0.998	150	0.062	0.192	0.725	0.998
200	0.067	0.267	0.879	1.000	200	0.066	0.253	0.863	1.000
250	0.072	0.334	0.948	1.000	250	0.070	0.317	0.939	1.000
500	0.097	0.650	1.000	1.000	500	0.094	0.626	1.000	1.000

Table 9: The calculated powers for the different values of the effect size, df and sample sizes

5. Comparison of PD(0.4) and Fisher-Freeman-Halton exact test statistics

It is well-known that the Fisher exact test is used for R=2 and C=2 contingency tables when more than 20% of cells have expected frequencies less than 5. However, when the table dimension is larger than 2×2, the FH test is used [22]. In this section, we compare the empirical type-I error rates of the PD(0.4) and the FH test statistics based on the simulation study. The same probability matrices given in Section 4 are used. The type-I errors of the PD(0.4) and FH test statistics are reported graphically in Figures 12, 13 and 14. As seen from these figures, it is observed that the PD(0.4) and FH produce similar results in terms of their type-I error rates. Both test statistics can be used for sparse and non-sparse contingency tables. The obtained type-I errors of the PD(0.4) and FH test statistics are below the desired value, 0.05. Also, the empirical power values of the PD(0.4) and FH test statistics are reported in Figures 15, 16 and 17. PD(0.4) and FH test statistics produce similar results for their power values, as in type-I error rates.



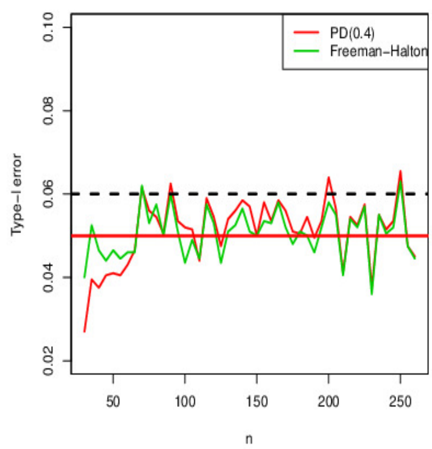


Figure 12: Type-I errors of the PD(0.4) and FH test for R=3 and C=3(left) and R=3 and C=4 (right)

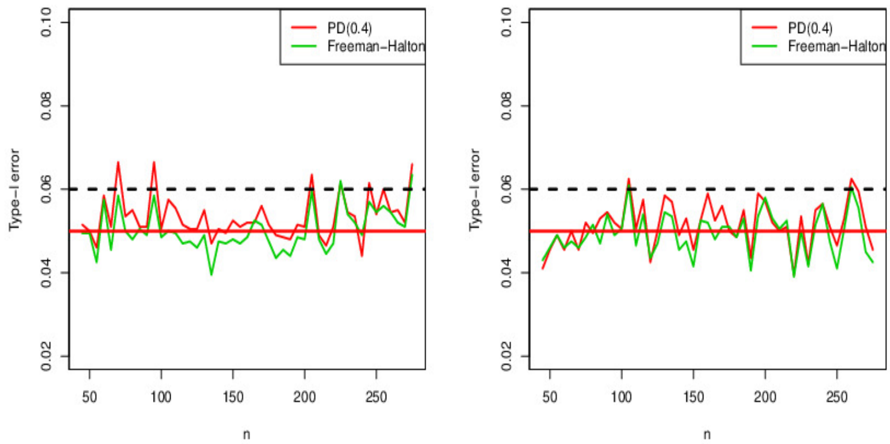


Figure 13: Type-I errors of the PD(0.4) and FH test for R=3 and C=5(left) and R=4 and C=4 (right)

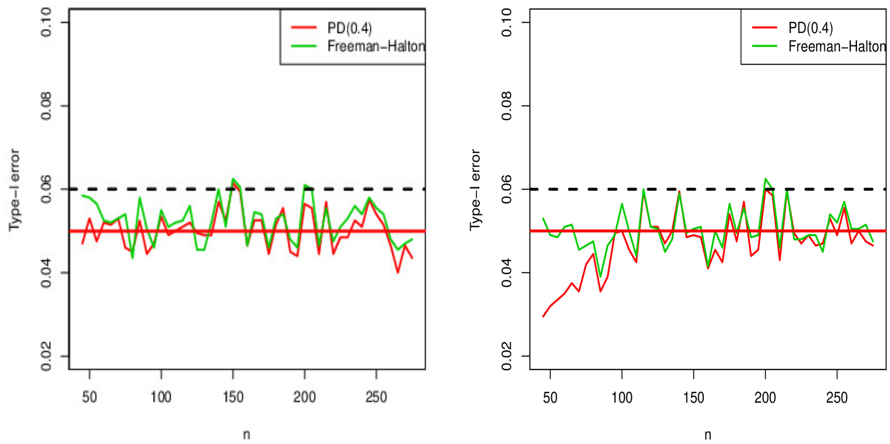


Figure 14: Type-I errors of the PD(0.4) and FH test for R=4 and C=5(left) and R=5 and C=5 (right)

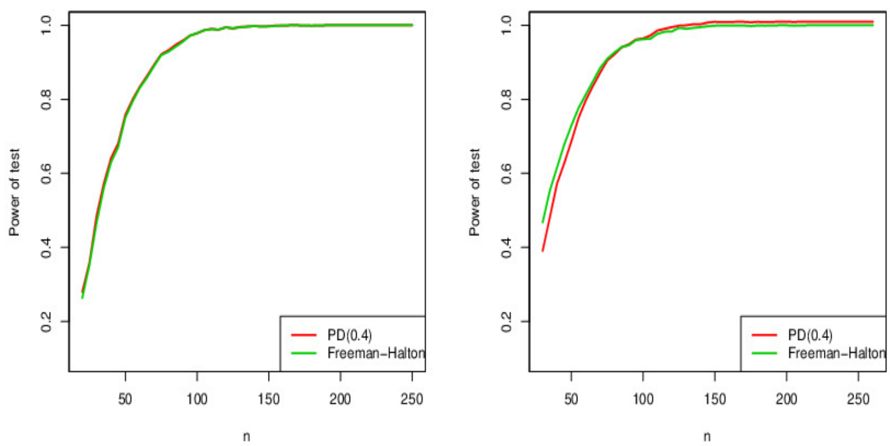


Figure 15: Power values of the PD(0.4) and FH test for R=3 and C=3(left) and R=3 and C=4 (right)

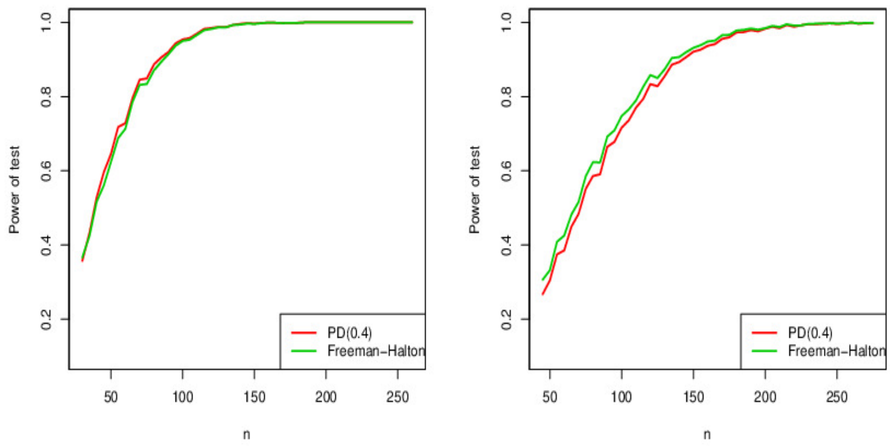


Figure 16: Power values of the PD(0.4) and FH test for R=3 and C=5(left) and R=4 and C=4 (right)

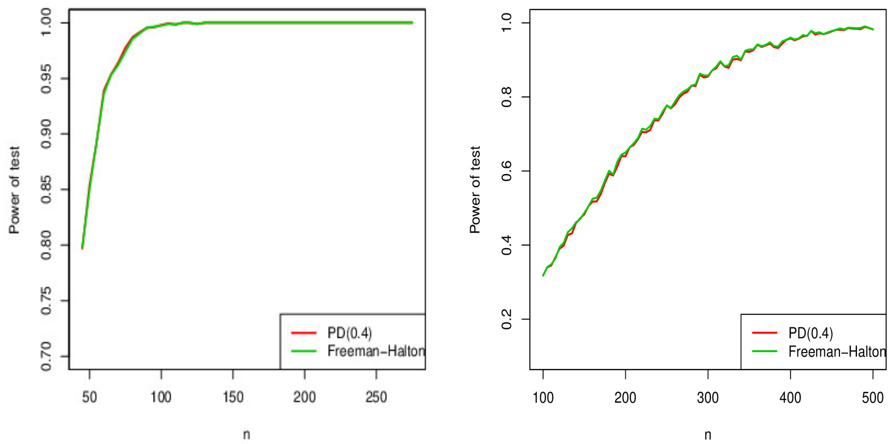


Figure 17: Power values of the PD(0.4) and FH test for R=4 and C=5(left) and R=5 and C=5 (right)

6. Power comparison of the PD(0.4) and χ^2 test statistics via real data application

The sample size determination is an important step of any field work. Before collecting the data, the researcher should know how many observations is needed to reach the desired power value. The sample size is a function of three parameters. These are effect size, type-I error and power to detect H_1 hypothesis.

Let $\mathrm{PD}\,(0.4)_c$ be the calculated value of the $\mathrm{PD}\,(0.4)$ test statistic which is calculated by

$$\mathrm{PD}\,(0.4)_c = \frac{2}{0.4\,(0.4+1)} \sum_{i=1}^{R} \sum_{j=1}^{C} \left[\left(\frac{n_{ij}}{e_{ij}} \right)^{0.4} - 1 \right],$$

where n_{ij} and e_{ij} are the observed and expected frequencies, respectively . When the null hypothesis (H_0) is true, the test statistic is distributed as χ^2 distribution with $(R-1)(C-1)$ df. The null hypothesis is rejected when $\mathrm{PD}\,(0.4)_c > \chi^2_{(R-1)(C-1),\alpha}$ where α is the significance level which is called as type-I error. When the null hypothesis is not true, the distribution of $\mathrm{PD}\,(0.4)_c$ follows the non-central χ^2 distribution with non-centrality parameter λ and df $(R-1)(C-1)$. The non-centrality parameter λ is a function of n and effect size w. We have the following equation to calculate the parameter λ (see [3])

$$\lambda = nw^2. \tag{6.1}$$

The effect size is calculated by $w = \sqrt{\mathrm{PD}\,(0.4)_c/n}$. So, replacing w in (6.1), we have $\lambda = \mathrm{PD}\,(0.4)_c$. So, the power of the $\mathrm{PD}\,(0.4)_c$ test statistic can be obtained by

$$\text{Power} = 1 - \Pr\left(\chi^2_{(R-1)(C-1),\lambda}\left(\text{PD}\,(0.4)_c\right) < \chi^2_{(R-1)(C-1),\alpha}\right). \tag{6.2}$$

The power of the χ^2 test statistic can be easily computed by changing the $\text{PD}\,(0.4)_c$ in (6.2) with the test statistic value of the χ^2. In the remaining part of these section, we analyze two data sets to compare the PD(0.4) with χ^2 test statistics. Note that the calculated power values in the remaining part of this section are empirical powers.

6.1. Pneumonia data

To compare the power value of the PD(0.4) and χ^2 test statistics, we use the data set on the vaccination program for the pneumonia patients. The data can be found in the work of [23]. Also, the data set is given in Table 10. Here, the research question is that *Does the vaccine protect the individuals from the pneumococcal pneumonia disease?*.

Health outcome	Unvaccinated	Vaccinated
Sick with pneumococcal pneumonia	23	5
Sick with non-pneumococcal pneumonia	8	10
No pneumonia	61	77

Table 10: The data set for vaccination program

The data is analyzed using the PD(0.4) and χ^2 test statistics. Obtained results are given in Table 11. The significance level α is selected 0.05 for both test statistics. According to the Table 11, both of the test statistics reject the null hypothesis. However, the power value of the PD(0.4) test statistic is higher than the χ^2 test statistic. So, we recommend the usage of the PD(0.4) test statistic to obtain higher power value than those of the χ^2 test statistic.

Test statistics	Value	df	p-value	Power
χ^2	13.649	2	0.001	0.921
PD(0.4)	14.095	2	< 0.001	0.930

Table 11: Results of the test statistics for the pneumonia data

6.2. Epidemiological data

The second data is on the obesity risk of children based on their race. The data set can be found in [24]. Here, the research question is that *Does the obesity risk differ by the race?*. To answer this question, we analyze the data set given in Table 12 with PD(0.4) and χ^2 test statistics.

Risk	Black	White	Others
At risk	185	140	90
Not at risk	80	17	23

Table 12: Epidemiological data for the children

The obtained results are given in Table 13. Based on the results in Table 13, since the power of PD(0.4) is higher than the χ^2, we recommend the PD(0.4) test statistic to analyze the current data set.

Test statistics	Value	df	p-value	Power
χ^2	21.595	2	< 0.001	0.991
PD(0.4)	22.386	2	< 0.001	0.992

Table 13: Results of the test statistics for the epidemiological data

7. Conclusion

We compare the various members of the PD family as well as different values of λ using the extensive simulation study based on the different settings such as dimensions of the contingency tables, type-I error, sample sizes and powers. When the parameter $\lambda = 0.4$, the test statistic reaches the maximum value of the power. Also, we compare the PD(0.4) test statistic with χ^2 test statistics based on the power values. Two applications to the real datasets show that PD(0.4) provides higher powers than the χ^2 test statistic. As a future work, we plan to develop the web-tool to calculate the required sample size and displays the results of the PD(0.4) test statistic.

Declarations

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Conflict of Interest Disclosure: The authors declare no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

Use of AI tools: The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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On a Class of Difference Equations System of Fifth-Order

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Abstract

In the current paper, we investigate the following new class of system of difference equations

$$u_{n+1} = f^{-1}\left(g(v_{n-1})\frac{A_1 f(u_{n-2}) + B_1 g(v_{n-4})}{C_1 f(u_{n-2}) + D_1 g(v_{n-4})}\right),$$
$$v_{n+1} = g^{-1}\left(f(u_{n-1})\frac{A_2 g(v_{n-2}) + B_2 f(u_{n-4})}{C_2 g(v_{n-2}) + D_2 f(u_{n-4})}\right),\ n \in \mathbb{N}_0,$$

where the initial conditions u_{-p}, v_{-p}, for $p = \overline{0,4}$ are real numbers, the parameters A_r, B_r, C_r, D_r, for $r \in \{1,2\}$ are real numbers, $A_r^2 + B_r^2 \neq 0 \neq C_r^2 + D_r^2$, for $r \in \{1,2\}$, f and g are continuous and strictly monotone functions, $f(\mathbb{R}) = \mathbb{R}$, $g(\mathbb{R}) = \mathbb{R}$, $f(0) = 0$, $g(0) = 0$. In addition, we solve aforementioned general two dimensional system of difference equations of fifth-order in explicit form. Moreover, we obtain the solutions of mentioned system according to whether the parameters being zeros or not. Finally, we present an interesting application.

1. Introduction

The notation of $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, stand for the set of natural, non-negative integer, integer and real number, respectively. If $\gamma, \delta \in \mathbb{Z}$, $\gamma \leq \delta$ the notation $\beta = \overline{\gamma,\delta}$ means $\{\beta \in \mathbb{Z} : \gamma \leq \beta \leq \delta\}$.

Difference equations emerge from mathematical models of physical events, numerical solutions of differential equations or generation functions. There has been an intense interest in nonlinear difference equations. Some mathematicians are interested in nonlinear difference equations in these days in [1], [2], [3], [4], [5]. In addition, systems of difference equations are studied by some authors in [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21].

One of the interesting difference equations is

$$w_{n+2} = \Phi w_{n+1} + \Psi w_n,\ n \in \mathbb{N}_0, \tag{1.1}$$

where the initial values w_0, w_1 and the parameters Φ and Ψ are real numbers. Equation (1.1) is solved by De Moivre in [22].

The solution of (1.1) is given by

$$w_n = \frac{(w_1 - \lambda_2 w_0)\lambda_1^n - (w_1 - \lambda_1 w_0)\lambda_2^n}{\lambda_1 - \lambda_2},\ n \in \mathbb{N}_0, \tag{1.2}$$

when $\Psi \neq 0$ and $\Phi^2 + 4\Psi \neq 0$,

$$w_n = ((w_1 - \lambda_1 w_0)n + \lambda_1 w_0)\lambda_1^{n-1},\ n \in \mathbb{N}_0, \tag{1.3}$$

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when $\Psi \neq 0$ and $\Phi^2 + 4\Psi = 0$, where λ_1 and λ_2 are the roots of the polynomial $P(\lambda) = \lambda^2 - \Phi\lambda - \Psi = 0$. Also, the roots of characteristic equation are $\lambda_{1,2} = \frac{\Phi \pm \sqrt{\Phi^2+4\Psi}}{2}$.

Another well-known difference equation, that is Riccati difference equation, is given by

$$w_{n+1} = \frac{\alpha w_n + \beta}{\gamma w_n + \delta}, \; n \in \mathbb{N}_0, \tag{1.4}$$

for $\gamma \neq 0$, $\alpha\delta \neq \beta\gamma$, where the initial condition w_0 and the parameters α, β, γ, δ are real numbers. Equation (1.4) is reduced to equation (1.1) by using the convenient transformation.

There are general forms of the difference equations reduced to equation (1.4) by changing variables in literature. For example, the following difference equation

$$w_{n+1} = \alpha w_{n-k} + \frac{\delta w_{n-k} w_{n-k-l}}{\beta x_{n-k-l} + \gamma x_{n-l}}, \; n \in \mathbb{N}_0, \tag{1.5}$$

where k, l are fixed natural numbers, the parameters α, β, γ, δ and the initial conditions w_{-i}, $i = \overline{1, k+l}$ are real numbers and $\beta^2 + \gamma^2 \neq 0$, is solved in [23].

Some authors solved special cases of equation (1.5) in [24], [25], [26], [27], [28]. A different form of equation (1.5) continued to be studied in the literature [29], [30], [31].

In an earlier paper, Elsayed et al., deal with the following difference equation

$$u_{n+1} = \gamma_0 u_{n-1} + \frac{\gamma_1 u_{n-1} u_{n-4}}{\gamma_2 u_{n-4} + \gamma_3 u_{n-2}}, \; n \in \mathbb{N}_0, \tag{1.6}$$

where the initial values u_{-p}, for $p = \overline{0,4}$ are arbitrary positive real numbers and the coefficients γ_l, for $l = \overline{0,3}$ are real numbers in [32].

Recently, Stević et al., investigate the following difference equations

$$x_{n+1} = \Phi^{-1}\left(\Phi(x_{n-1}) \frac{\alpha\Phi(x_{n-2}) + \beta\Phi(x_{n-4})}{\gamma\Phi(x_{n-2}) + \delta\Phi(x_{n-4})}\right), \; n \in \mathbb{N}_0, \tag{1.7}$$

where the initial values x_{-p}, for $p = \overline{0,4}$ and the parameters α, β, γ and δ are real numbers in [33]. Note that, the different form of equation (1.6) is equation (1.7).

Equations (1.7) can be expand in various ways. For instance, increasing order, adding periodic coefficients, expanding the dimensional, etc.

In this paper, we are interested in the following general two dimensional form of equation (1.7)

$$\begin{aligned} u_{n+1} &= f^{-1}\left(g(v_{n-1}) \frac{A_1 f(u_{n-2}) + B_1 g(v_{n-4})}{C_1 f(u_{n-2}) + D_1 g(v_{n-4})}\right), \\ v_{n+1} &= g^{-1}\left(f(u_{n-1}) \frac{A_2 g(v_{n-2}) + B_2 f(u_{n-4})}{C_2 g(v_{n-2}) + D_2 f(u_{n-4})}\right), \; n \in \mathbb{N}_0, \end{aligned} \tag{1.8}$$

where the initial conditions u_{-p}, v_{-p}, for $p = \overline{0,4}$ are real numbers, the parameters A_r, B_r, C_r, D_r, for $r \in \{1,2\}$, are real numbers, $A_r^2 + B_r^2 \neq 0 \neq C_r^2 + D_r^2$, for $r \in \{1,2\}$, f and g are continuous and strictly monotone functions, $f(\mathbb{R}) = \mathbb{R}$, $g(\mathbb{R}) = \mathbb{R}$, $f(0) = 0$, $g(0) = 0$. We obtain the solutions of system (1.8) in explicit form according to states of parameters by changing the variable. In addition, we present an application, which indicates that some conclusions in [32] are not correct.

2. Explicit-form solutions of system (1.8)

In this section, we investigate the solutions of system (1.8) in explicit-form.

Theorem 2.1. *Assume that A_r, B_r, C_r, $D_r \in \mathbb{R}$, for $r \in \{1,2\}$, $A_1^2 + B_1^2 \neq 0 \neq C_1^2 + D_1^2$, $A_2^2 + B_2^2 \neq 0 \neq C_2^2 + D_2^2$, f and g are continuous and strictly monotone functions, $f(\mathbb{R}) = \mathbb{R}$, $g(\mathbb{R}) = \mathbb{R}$, $f(0) = 0$, $g(0) = 0$. So, the general system (1.8) is solvable in explicit-form.*

Proof. If at least one of the initial values $u_{-p}=0$ or $v_{-p}=0$, for $p=\overline{0,4}$, then the solution of system (1.8) is not defined. Moreover, assume that $u_{n_0}=0$ for some $n_0\in\mathbb{N}_0$. Then from system (1.8) we have $v_{n_0+2}=0$. These facts along with (1.8) imply that v_{n_0+5} is not defined. Similarly, suppose that $v_{n_0}=0$ for some $n_0\in\mathbb{N}_0$. Then from system (1.8) we have $u_{n_0+2}=0$. These facts along with (1.8) imply that u_{n_0+5} is not defined. Hence, for every well-defined solution of system (1.8), we have

$$u_nv_n\neq 0,\ n\geq -4. \tag{2.1}$$

From (2.1) and the conditions of the theorem we have

$$f(u_n)\neq 0,\ g(v_n)\neq 0,\ n\geq -4.$$

Now, we examine the solutions of system (1.8) for two cases:

Case 1: First, suppose that $A_1D_1-B_1C_1\neq 0$, $A_2D_2-B_2C_2\neq 0$ and $C_1\neq 0\neq C_2$. Let

$$x_n=\frac{f(u_n)}{g(v_{n-2})},\ y_n=\frac{g(v_n)}{f(u_{n-2})},\ n\geq -2. \tag{2.2}$$

From (1.8) and monotonicity of f and g, we obtain

$$\begin{aligned} f(u_{n+1}) &= g(v_{n-1})\frac{A_1f(u_{n-2})+B_1g(v_{n-4})}{C_1f(u_{n-2})+D_1g(v_{n-4})},\\ g(v_{n+1}) &= f(u_{n-1})\frac{A_2g(v_{n-2})+B_2f(u_{n-4})}{C_2g(v_{n-2})+D_2f(u_{n-4})},\ n\in\mathbb{N}_0. \end{aligned} \tag{2.3}$$

By using the change of variables (2.2) in (2.3) we get

$$x_{n+1}=\frac{A_1x_{n-2}+B_1}{C_1x_{n-2}+D_1},\ y_{n+1}=\frac{A_2y_{n-2}+B_2}{C_2y_{n-2}+D_2},\ n\in\mathbb{N}_0. \tag{2.4}$$

Let

$$k_m^{(j)}=x_{3m+j},\ l_m^{(j)}=y_{3m+j},\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\}. \tag{2.5}$$

Then from (2.4) and (2.5) we obtain

$$k_{m+1}^{(j)}=\frac{A_1k_m^{(j)}+B_1}{C_1k_m^{(j)}+D_1},\ l_{m+1}^{(j)}=\frac{A_2l_m^{(j)}+B_2}{C_2l_m^{(j)}+D_2}, \tag{2.6}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. The equations in (2.6) are named Riccati type difference equations in literature.

Let

$$k_m^{(j)}=\frac{z_{m+1}^{(j)}}{z_m^{(j)}}+p_j,\ l_m^{(j)}=\frac{t_{m+1}^{(j)}}{t_m^{(j)}}+h_j,\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\}, \tag{2.7}$$

for some $p_j,h_j\in\mathbb{R}$, $j\in\{-2,-1,0\}$.

From (2.6) and (2.7) we have

$$\left(\frac{z_{m+2}^{(j)}}{z_{m+1}^{(j)}}+p_j\right)\left(C_1\frac{z_{m+1}^{(j)}}{z_m^{(j)}}+C_1p_j+D_1\right)-\left(A_1\frac{z_{m+1}^{(j)}}{z_m^{(j)}}+A_1p_j+B_1\right)=0,$$

$$\left(\frac{t_{m+2}^{(j)}}{t_{m+1}^{(j)}}+h_j\right)\left(C_2\frac{t_{m+1}^{(j)}}{t_m^{(j)}}+C_2h_j+D_2\right)-\left(A_2\frac{t_{m+1}^{(j)}}{t_m^{(j)}}+A_2h_j+B_2\right)=0,$$

for some $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$.

Let

$$p_j=-\frac{D_1}{C_1},\ h_j=-\frac{D_2}{C_2},\ j\in\{-2,-1,0\}.$$

Then, we get

$$\begin{aligned}
&C_1^2 z_{m+2}^{(j)}-C_1\left(A_1+D_1\right)z_{m+1}^{(j)}+\left(A_1D_1-B_1C_1\right)z_m^{(j)}=0,\\
&C_2^2 t_{m+2}^{(j)}-C_2\left(A_2+D_2\right)t_{m+1}^{(j)}+\left(A_2D_2-B_2C_2\right)t_m^{(j)}=0,
\end{aligned}\tag{2.8}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$.

Assume that $\Delta_1:=\left(A_1+D_1\right)^2-4\left(A_1D_1-B_1C_1\right)\neq 0$, $\Delta_2:=\left(A_2+D_2\right)^2-4\left(A_2D_2-B_2C_2\right)\neq 0$. Then by employing formula (1.2), we have

$$\begin{aligned}
z_m^{(j)}&=\frac{\left(z_1^{(j)}-\lambda_2 z_0^{(j)}\right)\lambda_1^m-\left(z_1^{(j)}-\lambda_1 z_0^{(j)}\right)\lambda_2^m}{\lambda_1-\lambda_2},\\
t_m^{(j)}&=\frac{\left(t_1^{(j)}-\widehat{\lambda}_2 t_0^{(j)}\right)\widehat{\lambda}_1^m-\left(t_1^{(j)}-\widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_2^m}{\widehat{\lambda}_1-\widehat{\lambda}_2},
\end{aligned}\tag{2.9}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$, where $\lambda_{1,2}=\frac{(A_1+D_1)\pm\sqrt{\Delta_1}}{2C_1}$, $\widehat{\lambda}_{1,2}=\frac{(A_2+D_2)\pm\sqrt{\Delta_2}}{2C_2}$. Equations in (2.9) are the general solutions to equations in (2.8).

By using (2.9) in (2.7), we obtain

$$\begin{aligned}
k_m^{(j)}&=\frac{\left(z_1^{(j)}-\lambda_2 z_0^{(j)}\right)\lambda_1^{m+1}-\left(z_1^{(j)}-\lambda_1 z_0^{(j)}\right)\lambda_2^{m+1}}{\left(z_1^{(j)}-\lambda_2 z_0^{(j)}\right)\lambda_1^{m}-\left(z_1^{(j)}-\lambda_1 z_0^{(j)}\right)\lambda_2^{m}}-\frac{D_1}{C_1}\\
&=\frac{\left(k_0^{(j)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{m+1}-\left(k_0^{(j)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{m+1}}{\left(k_0^{(j)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{m}-\left(k_0^{(j)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{m}}-\frac{D_1}{C_1},\\
l_m^{(j)}&=\frac{\left(t_1^{(j)}-\widehat{\lambda}_2 t_0^{(j)}\right)\widehat{\lambda}_1^{m+1}-\left(t_1^{(j)}-\widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_2^{m+1}}{\left(t_1^{(j)}-\widehat{\lambda}_2 t_0^{(j)}\right)\widehat{\lambda}_1^{m}-\left(t_1^{(j)}-\widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_2^{m}}-\frac{D_2}{C_2}\\
&=\frac{\left(l_0^{(j)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{m+1}-\left(l_0^{(j)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{m+1}}{\left(l_0^{(j)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{m}-\left(l_0^{(j)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{m}}-\frac{D_2}{C_2},
\end{aligned}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$, from the last equalities with (2.5) we have

$$\begin{aligned}
x_{3m+j}&=\frac{\left(x_j+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{m+1}-\left(x_j+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{m+1}}{\left(x_j+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{m}-\left(x_j+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{m}}-\frac{D_1}{C_1},\\
y_{3m+j}&=\frac{\left(y_j+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{m+1}-\left(y_j+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{m+1}}{\left(y_j+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{m}-\left(y_j+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{m}}-\frac{D_2}{C_2},
\end{aligned}\tag{2.10}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$.

From (2.2), we get

$$\begin{aligned} f(u_n) &= x_n g(v_{n-2}) = x_n y_{n-2} f(u_{n-4}) = x_n y_{n-2} x_{n-4} g(v_{n-6}) = x_n y_{n-2} x_{n-4} y_{n-6} f(u_{n-8}) \\ &= x_n y_{n-2} x_{n-4} y_{n-6} x_{n-8} g(v_{n-10}) = x_n y_{n-2} x_{n-4} y_{n-6} x_{n-8} y_{n-10} f(u_{n-12}),\ n \geq 8, \\ g(v_n) &= y_n f(u_{n-2}) = y_n x_{n-2} g(v_{n-4}) = y_n x_{n-2} y_{n-4} f(u_{n-6}) = y_n x_{n-2} y_{n-4} x_{n-6} g(v_{n-8}) \\ &= y_n x_{n-2} y_{n-4} x_{n-6} y_{n-8} f(u_{n-10}) = y_n x_{n-2} y_{n-4} x_{n-6} y_{n-8} x_{n-10} g(v_{n-12}), n \geq 8. \end{aligned} \tag{2.11}$$

From (2.11), we have

$$\begin{aligned} f(u_{12m+i}) &= x_{12m+i} y_{12m+i-2} x_{12m+i-4} y_{12m+i-6} x_{12m+i-8} y_{12m+i-10} f\left(u_{12(m-1)+i}\right), \\ g(v_{12m+i}) &= y_{12m+i} x_{12m+i-2} y_{12m+i-4} x_{12m+i-6} y_{12m+i-8} x_{12m+i-10} g\left(v_{12(m-1)+i}\right), \end{aligned} \tag{2.12}$$

for $m \in \mathbb{N}_0$, $i = \overline{8,19}$. Multiplying the equalities which are obtained from (2.12), from 0 to m, it follows that

$$\begin{aligned} f(u_{12m+3s+p}) = f(u_{3s+p-12}) \prod_{r=0}^{m} \Big(& x_{12r+3s+p} y_{12r+3s+p-2} x_{12r+3s+p-4} y_{12r+3s+p-6} x_{12r+3s+p-8} y_{12r+3s+p-10} \Big), \\ g(v_{12m+3s+p}) = g(v_{3s+p-12}) \prod_{r=0}^{m} \Big(& y_{12r+3s+p} x_{12r+3s+p-2} y_{12r+3s+p-4} x_{12r+3s+p-6} y_{12r+3s+p-8} x_{12r+3s+p-10} \Big), \end{aligned} \tag{2.13}$$

for $m \in \mathbb{N}_0$, $s = \overline{3,6}$, $p = \overline{-1,1}$. From (2.13), we obtain

$$\begin{aligned} f(u_{12m+3s+p}) &= f(u_{3s+p-12}) \prod_{r=0}^{m} \Big(x_{3\left(4r+s+\lfloor \frac{p+2}{3} \rfloor\right)+p-3\lfloor \frac{p+2}{3} \rfloor} y_{3\left(4r+s+\lfloor \frac{p}{3} \rfloor\right)+p-2-3\lfloor \frac{p}{3} \rfloor} \\ &\quad \times x_{3\left(4r+s+\lfloor \frac{p-2}{3} \rfloor\right)+p+2+3\lfloor \frac{p-2}{3} \rfloor} y_{3\left(4r+s-1+\lfloor \frac{p-1}{3} \rfloor\right)+p-3-3\lfloor \frac{p-1}{3} \rfloor} \\ &\quad \times x_{3\left(4r+s-1+\lfloor \frac{p-3}{3} \rfloor\right)+p-5-3\lfloor \frac{p-3}{3} \rfloor} y_{3\left(4r+s-1+\lfloor \frac{p-5}{3} \rfloor\right)+p-7-3\lfloor \frac{p-5}{3} \rfloor} \Big), \\ g(v_{12m+3s+p}) &= g(v_{3s+p-12}) \prod_{r=0}^{m} \Big(y_{3\left(4r+s+\lfloor \frac{p+2}{3} \rfloor\right)+p-3\lfloor \frac{p+2}{3} \rfloor} x_{3\left(4r+s+\lfloor \frac{p}{3} \rfloor\right)+p-2-3\lfloor \frac{p}{3} \rfloor} \\ &\quad \times y_{3\left(4r+s+\lfloor \frac{p-2}{3} \rfloor\right)+p+2+3\lfloor \frac{p-2}{3} \rfloor} x_{3\left(4r+s-1+\lfloor \frac{p-1}{3} \rfloor\right)+p-3-3\lfloor \frac{p-1}{3} \rfloor} \\ &\quad \times y_{3\left(4r+s-1+\lfloor \frac{p-3}{3} \rfloor\right)+p-5-3\lfloor \frac{p-3}{3} \rfloor} x_{3\left(4r+s-1+\lfloor \frac{p-5}{3} \rfloor\right)+p-7-3\lfloor \frac{p-5}{3} \rfloor} \Big), \end{aligned} \tag{2.14}$$

for $m \in \mathbb{N}_0$, $s = \overline{3,6}$, $p = \overline{-1,1}$. By substituting the equations in (2.10) into (2.14) and by using equations in (2.2), we have

$$\begin{aligned} & u_{12m+3s+p} \\ = \ & f^{-1}\Bigg[f(u_{3s+p-12}) \\ & \times \prod_{r=0}^{m} \left(\left(\frac{\left(\frac{f\left(u_{p-3\lfloor \frac{p+2}{3} \rfloor}\right)}{g\left(v_{p-3\lfloor \frac{p+2}{3} \rfloor-2}\right)} + \frac{D_1}{C_1} - \lambda_2 \right) \lambda_1^{4r+s+\lfloor \frac{p+2}{3} \rfloor+1} - \left(\frac{f\left(u_{p-3\lfloor \frac{p+2}{3} \rfloor}\right)}{g\left(v_{p-3\lfloor \frac{p+2}{3} \rfloor-2}\right)} + \frac{D_1}{C_1} - \lambda_1 \right) \lambda_2^{4r+s+\lfloor \frac{p+2}{3} \rfloor+1}}{\left(\frac{f\left(u_{p-3\lfloor \frac{p+2}{3} \rfloor}\right)}{g\left(v_{p-3\lfloor \frac{p+2}{3} \rfloor-2}\right)} + \frac{D_1}{C_1} - \lambda_2 \right) \lambda_1^{4r+s+\lfloor \frac{p+2}{3} \rfloor} - \left(\frac{f\left(u_{p-3\lfloor \frac{p+2}{3} \rfloor}\right)}{g\left(v_{p-3\lfloor \frac{p+2}{3} \rfloor-2}\right)} + \frac{D_1}{C_1} - \lambda_1 \right) \lambda_2^{4r+s+\lfloor \frac{p+2}{3} \rfloor}} - \frac{D_1}{C_1} \right) \right. \\ & \times \left(\frac{\left(\frac{g\left(v_{p-2-3\lfloor \frac{p}{3} \rfloor}\right)}{f\left(u_{p-4-3\lfloor \frac{p}{3} \rfloor}\right)} + \frac{D_2}{C_2} - \widehat{\lambda}_2 \right) \widehat{\lambda}_1^{4r+s+\lfloor \frac{p}{3} \rfloor+1} - \left(\frac{g\left(v_{p-2-3\lfloor \frac{p}{3} \rfloor}\right)}{f\left(u_{p-4-3\lfloor \frac{p}{3} \rfloor}\right)} + \frac{D_2}{C_2} - \widehat{\lambda}_1 \right) \widehat{\lambda}_2^{4r+s+\lfloor \frac{p}{3} \rfloor+1}}{\left(\frac{g\left(v_{p-2-3\lfloor \frac{p}{3} \rfloor}\right)}{f\left(u_{p-4-3\lfloor \frac{p}{3} \rfloor}\right)} + \frac{D_2}{C_2} - \widehat{\lambda}_2 \right) \widehat{\lambda}_1^{4r+s+\lfloor \frac{p}{3} \rfloor} - \left(\frac{g\left(v_{p-2-3\lfloor \frac{p}{3} \rfloor}\right)}{f\left(u_{p-4-3\lfloor \frac{p}{3} \rfloor}\right)} + \frac{D_2}{C_2} - \widehat{\lambda}_1 \right) \widehat{\lambda}_2^{4r+s+\lfloor \frac{p}{3} \rfloor}} - \frac{D_2}{C_2} \right) \end{aligned}$$

$$
\begin{aligned}
&\times\left(\frac{\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}-\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}}{\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor}-\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor}}-\frac{D_1}{C_1}\right)\\
&\times\left(\frac{\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor}-\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor}}{\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}-\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}}-\frac{D_2}{C_2}\right)\\
&\times\left(\frac{\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor}-\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor}}{\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}-\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}}-\frac{D_1}{C_1}\right)\\
&\times\left(\frac{\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor}-\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor}}{\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}-\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}}-\frac{D_2}{C_2}\right)\Bigg)\Bigg],
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
v_{12m+3s+p}=g^{-1}\Bigg[&g\left(v_{3s+p-12}\right)\\
&\times\prod_{r=0}^{m}\Bigg(\left(\frac{\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}-\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}}{\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor}-\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor}}-\frac{D_2}{C_2}\right)\\
&\times\left(\frac{\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor+1}-\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor+1}}{\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor}-\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor}}-\frac{D_1}{C_1}\right)\\
&\times\left(\frac{\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}-\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}}{\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor}-\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor}}-\frac{D_2}{C_2}\right)\\
&\times\left(\frac{\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor}-\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor}}{\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}-\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}}-\frac{D_1}{C_1}\right)
\end{aligned}
\tag{2.16}
$$

$$\times\left(\frac{\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor}-\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor}}{\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_2\right)\widehat{\lambda}_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}-\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}}-\frac{D_2}{C_2}\right)$$

$$\times\left(\frac{\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor}-\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor}}{\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}-\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}}-\frac{D_1}{C_1}\right)\Bigg)\Bigg],$$

for $m\in\mathbb{N}_0$, $s=\overline{3,6}$, $p=\overline{-1,1}$. The formulas in (2.15) and (2.16) are the solutions of system (1.8) if $\Delta_1\neq 0\neq\Delta_2$. Assume that $\Delta_1=(A_1+D_1)^2-4(A_1D_1-B_1C_1)=0$ and $\Delta_2=(A_2+D_2)^2-4(A_2D_2-B_2C_2)=0$. So, by employing formula (1.3), we obtain

$$\begin{aligned} z_m^{(j)} &= \left(\left(z_1^{(j)}-\lambda_1 z_0^{(j)}\right)m+\lambda_1 z_0^{(j)}\right)\lambda_1^{m-1},\\ t_m^{(j)} &= \left(\left(t_1^{(j)}-\widehat{\lambda}_1 t_0^{(j)}\right)m+\widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_1^{m-1}, \end{aligned}\tag{2.17}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$, where

$$\lambda_1=\frac{A_1+D_1}{2C_1}\neq 0,\ \widehat{\lambda}_1=\frac{A_2+D_2}{2C_2}\neq 0.$$

Note that equations in (2.17) are the solutions to the system (2.8) if $\Delta_1=0=\Delta_2$. From (2.7) and (2.17), we get

$$\begin{aligned} k_m^{(j)} &= \frac{\left(\left(z_1^{(j)}-\lambda_1 z_0^{(j)}\right)(m+1)+\lambda_1 z_0^{(j)}\right)\lambda_1}{\left(z_1^{(j)}-\lambda_1 z_0^{(j)}\right)m+\lambda_1 z_0^{(j)}}-\frac{D_1}{C_1}\\ &= \frac{\left(\left(k_0^{(j)}+\frac{D_1}{C_1}-\lambda_1\right)(m+1)+\lambda_1\right)\lambda_1}{\left(k_0^{(j)}+\frac{D_1}{C_1}-\lambda_1\right)m+\lambda_1}-\frac{D_1}{C_1},\\ l_m^{(j)} &= \frac{\left(\left(t_1^{(j)}-\widehat{\lambda}_1 t_0^{(j)}\right)(m+1)+\widehat{\lambda}_1 t_0^{(j)}\right)\widehat{\lambda}_1}{\left(t_1^{(j)}-\widehat{\lambda}_1 t_0^{(j)}\right)m+\widehat{\lambda}_1 t_0^{(j)}}-\frac{D_2}{C_2}\\ &= \frac{\left(\left(l_0^{(j)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)(m+1)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(l_0^{(j)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)m+\widehat{\lambda}_1}-\frac{D_2}{C_2}, \end{aligned}\tag{2.18}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. By using (2.5) in (2.18), we obtain

$$\begin{aligned} x_{3m+j} &= \frac{\left(\left(x_j+\frac{D_1}{C_1}-\lambda_1\right)(m+1)+\lambda_1\right)\lambda_1}{\left(x_j+\frac{D_1}{C_1}-\lambda_1\right)m+\lambda_1}-\frac{D_1}{C_1},\\ y_{3m+j} &= \frac{\left(\left(y_j+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)(m+1)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(y_j+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)m+\widehat{\lambda}_1}-\frac{D_2}{C_2}, \end{aligned}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. From (2.14), we have

$$
\begin{aligned}
u_{12m+3s+p} = {} & f^{-1}\Bigg[f\left(u_{3s+p-12}\right)\times\prod_{r=0}^{m}\Bigg(\left(\frac{\left(\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p+2}{3}\rfloor+1\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+\lambda_1}-\frac{D_1}{C_1}\right)\\
& \times\left(\frac{\left(\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p}{3}\rfloor+1\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p}{3}\rfloor\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)\\
& \times\left(\frac{\left(\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-2}{3}\rfloor+1\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+\lambda_1}-\frac{D_1}{C_1}\right)\\
& \times\left(\frac{\left(\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-1}{3}\rfloor\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-1}{3}\rfloor-1\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)\\
& \times\left(\frac{\left(\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-3}{3}\rfloor\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-3}{3}\rfloor-1\right)+\lambda_1}-\frac{D_1}{C_1}\right)\\
& \times\left(\frac{\left(\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-5}{3}\rfloor\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-5}{3}\rfloor-1\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)\Bigg)\Bigg],
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
v_{12m+3s+p} = {} & g^{-1}\Bigg[g\left(v_{3s+p-12}\right)\\
& \times\prod_{r=0}^{m}\Bigg(\left(\frac{\left(\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p+2}{3}\rfloor+1\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)\\
& \times\left(\frac{\left(\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p}{3}\rfloor+1\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p}{3}\rfloor\right)+\lambda_1}-\frac{D_1}{C_1}\right)
\end{aligned}
$$

$$\times\left(\frac{\left(\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-2}{3}\rfloor+1\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right) \tag{2.20}$$

$$\times\left(\frac{\left(\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-1}{3}\rfloor\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-1}{3}\rfloor-1\right)+\lambda_1}-\frac{D_1}{C_1}\right)$$

$$\times\left(\frac{\left(\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-3}{3}\rfloor\right)+\widehat{\lambda}_1\right)\widehat{\lambda}_1}{\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{D_2}{C_2}-\widehat{\lambda}_1\right)\left(4r+s+\lfloor\frac{p-3}{3}\rfloor-1\right)+\widehat{\lambda}_1}-\frac{D_2}{C_2}\right)$$

$$\times\left(\frac{\left(\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-5}{3}\rfloor\right)+\lambda_1\right)\lambda_1}{\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{D_1}{C_1}-\lambda_1\right)\left(4r+s+\lfloor\frac{p-5}{3}\rfloor-1\right)+\lambda_1}-\frac{D_1}{C_1}\right)\Bigg)\Bigg],$$

for $m\in\mathbb{N}_0$, $s=\overline{3,6}$, $p=\overline{-1,1}$, if $\Delta_1=0=\Delta_2$.
Now assume that $C_1=0=C_2$, $D_1\neq 0\neq D_2$. In this case, equations in (2.4) turn into

$$x_{n+1}=\frac{A_1}{D_1}x_{n-2}+\frac{B_1}{D_1},\ y_{n+1}=\frac{A_2}{D_2}y_{n-2}+\frac{B_2}{D_2},\ n\in\mathbb{N}_0.$$

Thus,

$$k_{m+1}^{(j)}=\frac{A_1}{D_1}k_m^{(j)}+\frac{B_1}{D_1},\ l_{m+1}^{(j)}=\frac{A_2}{D_2}l_m^{(j)}+\frac{B_2}{D_2},\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\}. \tag{2.21}$$

If $A_1=D_1$ and $A_2=D_2$ then from (2.21), we have

$$k_m^{(j)}=\frac{B_1}{D_1}m+k_0^{(j)},\ l_m^{(j)}=\frac{B_2}{D_2}m+l_0^{(j)},\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\},$$

so

$$x_{3m+j}=\frac{B_1}{D_1}m+x_j,\ y_{3m+j}=\frac{B_2}{D_2}m+y_j,\ m\in\mathbb{N}_0,\ j\in\{-2,-1,0\}. \tag{2.22}$$

From (2.2), (2.14) and (2.22), we get

$$u_{12m+3s+p} = f^{-1}\Bigg[f\left(u_{3s+p-12}\right) \tag{2.23}$$

$$\times\prod_{r=0}^{m}\Bigg(\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}\right)\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p}{3}\rfloor\right)+\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}\right)$$

$$\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}\right)\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p-1}{3}\rfloor-1\right)+\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}\right)$$

$$\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p-3}{3}\rfloor-1\right)+\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}\right)\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p-5}{3}\rfloor-1\right)+\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}\right)\Bigg)\Bigg],$$

$$\begin{aligned} v_{12m+3s+p} &= g^{-1}\Bigg[g\left(v_{3s+p-12}\right)\times\prod_{r=0}^{m}\Bigg(\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p+2}{3}\rfloor\right)+\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}\right)\\ &\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p}{3}\rfloor\right)+\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}\right)\\ &\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p-2}{3}\rfloor\right)+\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}\right)\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p-1}{3}\rfloor-1\right)+\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}\right)\\ &\times\left(\frac{B_2}{D_2}\left(4r+s+\lfloor\frac{p-3}{3}\rfloor-1\right)+\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}\right)\times\left(\frac{B_1}{D_1}\left(4r+s+\lfloor\frac{p-5}{3}\rfloor-1\right)+\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}\right)\Bigg)\Bigg], \end{aligned} \tag{2.24}$$

for $m\in\mathbb{N}_0$, $s=\overline{3,6}$, $p=\overline{-1,1}$. Hence, the formulas in (2.24) and (2.24) are solutions of system (1.8) in this case. Suppose that $A_1\neq D_1$ and $A_2\neq D_2$. By using (2.21), we get

$$\begin{aligned} k_m^{(j)} &= \left(\frac{A_1}{D_1}\right)^m k_0^{(j)}+\frac{B_1}{A_1-D_1}\left(\left(\frac{A_1}{D_1}\right)^m-1\right),\\ l_m^{(j)} &= \left(\frac{A_2}{D_2}\right)^m l_0^{(j)}+\frac{B_2}{A_2-D_2}\left(\left(\frac{A_2}{D_2}\right)^m-1\right), \end{aligned}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. That is,

$$\begin{aligned} x_{3m+j} &= \left(\frac{A_1}{D_1}\right)^m\left(x_j+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1},\\ y_{3m+j} &= \left(\frac{A_2}{D_2}\right)^m\left(y_j+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}, \end{aligned} \tag{2.25}$$

for $m\in\mathbb{N}_0$, $j\in\{-2,-1,0\}$. From (2.2), (2.14) and (2.25), we get

$$\begin{aligned} u_{12m+3s+p} &= f^{-1}\Bigg[f\left(u_{3s+p-12}\right)\times\prod_{r=0}^{m}\Bigg(\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p+2}{3}\rfloor}\left(\frac{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p}{3}\rfloor}\left(\frac{g\left(v_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{f\left(u_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right)\\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p-2}{3}\rfloor}\left(\frac{f\left(u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{g\left(v_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}\left(\frac{g\left(v_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{f\left(u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right)\\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}\left(\frac{f\left(u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{g\left(v_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}\left(\frac{g\left(v_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{f\left(u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right)\Bigg)\Bigg], \end{aligned} \tag{2.26}$$

$$\begin{aligned} v_{12m+3s+p} &= g^{-1}\Bigg[g\left(v_{3s+p-12}\right)\times\prod_{r=0}^{m}\Bigg(\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p+2}{3}\rfloor}\left(\frac{g\left(v_{p-3\lfloor\frac{p+2}{3}\rfloor}\right)}{f\left(u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right)\\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p}{3}\rfloor}\left(\frac{f\left(u_{p-2-3\lfloor\frac{p}{3}\rfloor}\right)}{g\left(v_{p-4-3\lfloor\frac{p}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p-2}{3}\rfloor}\left(\frac{g\left(v_{p+2+3\lfloor\frac{p-2}{3}\rfloor}\right)}{f\left(u_{p+3\lfloor\frac{p-2}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right)\\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}\left(\frac{f\left(u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}\right)}{g\left(v_{p-5-3\lfloor\frac{p-1}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\\ &\times\left(\left(\frac{A_2}{D_2}\right)^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}\left(\frac{g\left(v_{p-5-3\lfloor\frac{p-3}{3}\rfloor}\right)}{f\left(u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}\right)}+\frac{B_2}{A_2-D_2}\right)-\frac{B_2}{A_2-D_2}\right)\\ &\times\left(\left(\frac{A_1}{D_1}\right)^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}\left(\frac{f\left(u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}\right)}{g\left(v_{p-9-3\lfloor\frac{p-5}{3}\rfloor}\right)}+\frac{B_1}{A_1-D_1}\right)-\frac{B_1}{A_1-D_1}\right)\Bigg)\Bigg], \end{aligned} \tag{2.27}$$

for $m\in\mathbb{N}_0$, $s=\overline{3,6}$, $p=\overline{-1,1}$. Then, the solutions of system (1.8) are given by the equations in (2.26) and (2.27) in this case.
Case 2: Assume that $A_1D_1=B_1C_1$, $A_2D_2=B_2C_2$. If $A_1=0$ and $B_1\neq 0$. Then $C_1=0$ and $D_1\neq 0$. If $A_2=0$ and $B_2\neq 0$. Then $C_2=0$ and $D_2\neq 0$. From system (1.8), we have

$$u_{n+1}=f^{-1}\left(\frac{B_1}{D_1}g\left(v_{n-1}\right)\right),\ v_{n+1}=g^{-1}\left(\frac{B_2}{D_2}f\left(u_{n-1}\right)\right),\ n\in\mathbb{N}_0. \tag{2.28}$$

From (2.28) we easily get

$$u_n=f^{-1}\left(\frac{B_1B_2}{D_1D_2}f\left(u_{n-4}\right)\right),\ v_n=g^{-1}\left(\frac{B_1B_2}{D_1D_2}g\left(v_{n-4}\right)\right),\ n\geq 3. \tag{2.29}$$

By using (2.29), we obtain

$$u_{4m+i}=f^{-1}\left(\left(\frac{B_1B_2}{D_1D_2}\right)^{m+1}f\left(u_{i-4}\right)\right),\ v_{4m+i}=g^{-1}\left(\left(\frac{B_1B_2}{D_1D_2}\right)^{m+1}g\left(v_{i-4}\right)\right), \tag{2.30}$$

$m\in\mathbb{N}_0$, $i=\overline{3,6}$.
If $A_1\neq 0$ and $B_1=0$. Then $D_1=0$ from which it follows that $C_1\neq 0$. If $A_2\neq 0$ and $B_2=0$. Then $D_2=0$ from which it follows that $C_2\neq 0$. From system (1.8), we get

$$u_{n+1}=f^{-1}\left(\frac{A_1}{C_1}g\left(v_{n-1}\right)\right),\ v_{n+1}=g^{-1}\left(\frac{A_2}{C_2}f\left(u_{n-1}\right)\right),\ n\in\mathbb{N}_0. \tag{2.31}$$

From (2.31) we easily get

$$u_n=f^{-1}\left(\frac{A_1A_2}{C_1C_2}f\left(u_{n-4}\right)\right),\ v_n=g^{-1}\left(\frac{A_1A_2}{C_1C_2}g\left(v_{n-4}\right)\right),n\geq 1. \tag{2.32}$$

By using (2.32), we obtain

$$u_{4m+i}=f^{-1}\left(\left(\frac{A_1A_2}{C_1C_2}\right)^{m+1}f\left(u_{i-4}\right)\right),\ v_{4m+i}=g^{-1}\left(\left(\frac{A_1A_2}{C_1C_2}\right)^{m+1}g\left(v_{i-4}\right)\right), \tag{2.33}$$

$m \in \mathbb{N}_0$, $i = \overline{3,6}$.
If $D_1 = 0$ so $C_1 \neq 0$. This means $B_1 = 0$, $A_1 \neq 0$. If $D_2 = 0$ so $C_2 \neq 0$. This means $B_2 = 0$, $A_2 \neq 0$. Then we have system (2.31). Moreover, the equalities in (2.33) are solutions of system (2.31).
Assume that $C_1 = 0$ so $D_1 \neq 0$. This means $A_1 = 0$, $B_1 \neq 0$. Suppose that $C_2 = 0$ so $D_2 \neq 0$. This means $A_2 = 0$, $B_2 \neq 0$. So, we obtain system (2.28). In addition, equalities in (2.30) are solutions of system (2.28).
Suppose that $A_1B_1C_1D_1 \neq 0$ and $A_2B_2C_2D_2 \neq 0$. It means $A_1 = \frac{B_1C_1}{D_1}$ and $A_2 = \frac{B_2C_2}{D_2}$. Moreover, we have system (2.28). Similarly, it means $B_1 = \frac{A_1D_1}{C_1}$ and $B_2 = \frac{A_2D_2}{C_2}$. □

3. An application

In this section, we give an application for system (1.8).

Remark 3.1. *If $f = g$, $A_1 = A_2$, $B_1 = B_2$, $C_1 = C_2$, $D_1 = D_2$, $u_{-p} = v_{-p}$, $p = \overline{0,4}$, then, the system (1.8) turns into the following equation*

$$u_{n+1} = f^{-1}\left(f(u_{n-1})\frac{A_1 f(u_{n-2}) + B_1 f(u_{n-4})}{C_1 f(u_{n-2}) + D_1 f(u_{n-4})}\right),\ n \in \mathbb{N}_0. \tag{3.1}$$

Behavior of solutions to equation (1.6) is mentioned in [32]. But somethings are not correct in [32].
Equation (1.6) can be expressed as

$$u_{n+1} = u_{n-1}\frac{\gamma_0\gamma_3 u_{n-2} + (\gamma_0\gamma_2 + \gamma_1)u_{n-4}}{\gamma_2 u_{n-4} + \gamma_3 u_{n-2}},\ n \in \mathbb{N}_0. \tag{3.2}$$

Firstly, the authors of [32] studied to obtain the equilibrium point of equation (1.6). Then, using a great deal calculations, they found $\overline{u} = 0$. If

$$(1-\gamma_0)(\gamma_2+\gamma_3) \neq \gamma_1,$$

an unique equilibrium point of equation (1.6) is $\overline{u} = 0$.
Suppose that an equilibrium point of equation (1.6) is $\overline{u}$. So we get the following equation

$$\overline{u} = \gamma_0\overline{u} + \frac{\gamma_1\overline{u}^2}{(\gamma_2+\gamma_3)\overline{u}}. \tag{3.3}$$

From (3.3), we see that it must be

$$(\gamma_2+\gamma_3) \neq 0 \text{ and } \overline{u} \neq 0.$$

This exterminates the probability $\overline{u} = 0$.
Suppose that $\overline{u} \neq 0$. Moreover, equation (3.3) means

$$\overline{u}\left(1-\gamma_0-\frac{\gamma_1}{\gamma_2+\gamma_3}\right) = 0,$$

so we have

$$1-\gamma_0-\frac{\gamma_1}{\gamma_2+\gamma_3} = 0. \tag{3.4}$$

From equation (3.4), the equilibrium point of the difference equation is $\overline{u} \neq 0$. It implies that the idea in [32] Theorem 3, under the condition, zero equilibrium point of equation (1.6) is local asymptotic stable is not corect, because it is not an equilibrium point at all.
In addition, Theorem 4 in [32] is expressed as:

Theorem 3.2. *If $\gamma_2(1-\gamma_0) \neq \gamma_1$, then the unique equilibrium point of Equation (1.6) is globally asymptotically stable.*

The particular case of equation (3.1) is equation (3.2) with

$$f(x) = x,\ A_1 = \gamma_0\gamma_3,\ B_1 = \gamma_0\gamma_2 + \gamma_1,\ C_1 = \gamma_2,\ D_1 = \gamma_3.$$

Example 3.3. *Keep in mind the equation (1.6) with*

$$\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 1,$$

and then we get the following equation

$$u_{n+1} = u_{n-1}\frac{u_{n-2} + 2u_{n-4}}{u_{n-2} + u_{n-4}},\ n \in \mathbb{N}_0. \tag{3.5}$$

Equation (3.5) is derived from equation (3.1) with $f(x) = x$ and $x \in \mathbb{R}$,

$$A_1 = C_1 = D_1 = 1,\ B_1 = 2. \tag{3.6}$$

By using (3.6) the first equation in (2.8), we get

$$p_1(\lambda) = \lambda^2 - 2\lambda - 1,$$

and its roots are

$$\lambda_1 = 1 + \sqrt{2} \text{ and } \lambda_2 = 1 - \sqrt{2}.$$

Then, we obtain

$$\gamma_2(1-\gamma_0) - \gamma_1 = -1 \neq 0,$$

the restriction $\gamma_2(1-\gamma_0) \neq \gamma_1$ in Theorem 3.2 is valid.
By using the parameters A_1, B_1, C_1, D_1 are as in (3.6) and (2.15)-(2.16), where $f(x) = x$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}
u_{12m+3s+p} &= u_{3s+p-12} \\
&\times \prod_{r=0}^{m}\Bigg(\left(\frac{\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}-\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}}{\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor}-\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor}}-1\right) \\
&\times\left(\frac{\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor+1}-\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor+1}}{\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor}-\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor}}-1\right) \\
&\times\left(\frac{\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}-\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}}{\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor}-\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor}}-1\right) \\
&\times\left(\frac{\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor}-\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor}}{\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}-\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}}-1\right) \\
&\times\left(\frac{\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor}-\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor}}{\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}-\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}}-1\right)
\end{aligned} \tag{3.7}$$

$$\times\left(\frac{\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor}-\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor}}{\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}-\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}}-1\right)\Bigg),$$

for $m\in\mathbb{N}_0,\ s=\overline{3,6},\ p=\overline{-1,1}$.
Note that

$$\begin{aligned}
&\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}-\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor+1}}{\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p+2}{3}\rfloor}-\left(\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p+2}{3}\rfloor}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor+1}-\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor+1}}{\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p}{3}\rfloor}-\left(\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p}{3}\rfloor}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}-\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor+1}}{\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-2}{3}\rfloor}-\left(\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-2}{3}\rfloor}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor}-\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor}}{\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}-\left(\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-1}{3}\rfloor-1}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor}-\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor}}{\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}-\left(\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-3}{3}\rfloor-1}}-1\right)\\
&=\lim_{m\to\infty}\left(\frac{\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor}-\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor}}{\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_2\right)\lambda_1^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}-\left(\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}}+1-\lambda_1\right)\lambda_2^{4r+s+\lfloor\frac{p-5}{3}\rfloor-1}}-1\right)\\
&=\lambda_1-1=\sqrt{2}>1,
\end{aligned}$$

when

$$\begin{aligned}
&\frac{u_{p-3\lfloor\frac{p+2}{3}\rfloor}}{u_{p-3\lfloor\frac{p+2}{3}\rfloor-2}}\neq\lambda_2-1=-\sqrt{2},\neq\frac{u_{p-2-3\lfloor\frac{p}{3}\rfloor}}{u_{p-4-3\lfloor\frac{p}{3}\rfloor}},\\
&\frac{u_{p+2+3\lfloor\frac{p-2}{3}\rfloor}}{u_{p+3\lfloor\frac{p-2}{3}\rfloor}}\neq\lambda_2-1=-\sqrt{2},\neq\frac{u_{p-3-3\lfloor\frac{p-1}{3}\rfloor}}{u_{p-5-3\lfloor\frac{p-1}{3}\rfloor}},p=\overline{-1,1}.\\
&\frac{u_{p-5-3\lfloor\frac{p-3}{3}\rfloor}}{u_{p-7-3\lfloor\frac{p-3}{3}\rfloor}}\neq\lambda_2-1=-\sqrt{2},\neq\frac{u_{p-7-3\lfloor\frac{p-5}{3}\rfloor}}{u_{p-9-3\lfloor\frac{p-5}{3}\rfloor}},
\end{aligned}\tag{3.8}$$

By selecting positive initial conditions providing (3.8) and using equations in (3.7), we obtain

$$\lim_{m\to\infty}u_m=\infty.$$

Now, we give numerical example to support the last equation.

Example 3.4. *Consider the equation (3.5) with the initial values* $u_{-4} = 0.195$, $u_{-3} = 0.1$, $u_{-2} = 2.4$, $u_{-1} = 3$, $u_0 = 7.62$, *the solution is given as in Figure (1).*

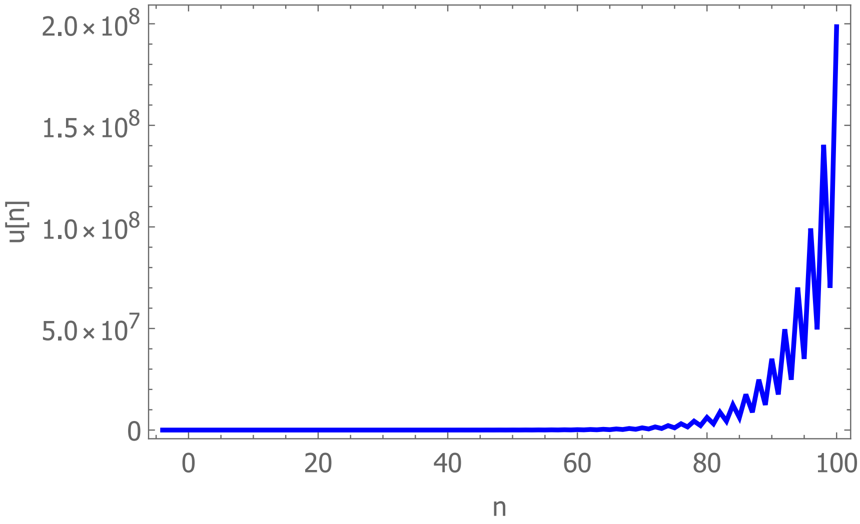

Figure 1: Plots of u_n

Then, the solution is not convergent. It is a counterexample to the claim in Theorem 3.2 (Theorem 4 in [32]).

4. Conclusion

In this study, we have solved the following general two dimensional system of difference equations

$$u_{n+1} = f^{-1}\left(g(v_{n-1})\frac{A_1 f(u_{n-2}) + B_1 g(v_{n-4})}{C_1 f(u_{n-2}) + D_1 g(v_{n-4})}\right), v_{n+1} = g^{-1}\left(f(u_{n-1})\frac{A_2 g(v_{n-2}) + B_2 f(u_{n-4})}{C_2 g(v_{n-2}) + D_2 f(u_{n-4})}\right), \ n \in \mathbb{N}_0,$$

where the parameters A_j, B_j, C_j, D_j, for $j \in \{1,2\}$ are real numbers, the initial values u_{-k}, v_{-k}, for $k = \overline{0,4}$ are real numbers, f and g are continuous and strictly monotone functions, $f(\mathbb{R}) = \mathbb{R}$, $g(\mathbb{R}) = \mathbb{R}$, $f(0) = 0$, $g(0) = 0$. The following particular cases are considered:

1. if $A_1 D_1 \neq B_1 C_1$ and $A_2 D_2 \neq B_2 C_2$
   (a) if $C_1 \neq 0$, $C_2 \neq 0$,
      i. if $(A_1 + D_1)^2 - 4(A_1 D_1 - B_1 C_1) \neq 0$, $(A_2 + D_2)^2 - 4(A_2 D_2 - B_2 C_2) \neq 0$, then the general solutions of system (1.8) is given by formulas in (2.15) and (2.16).
      ii. if $(A_1 + D_1)^2 - 4(A_1 D_1 - B_1 C_1) = 0$, $(A_2 + D_2)^2 - 4(A_2 D_2 - B_2 C_2) = 0$, then the general solutions of system (1.8) is given by formulas in (2.19) and (2.20).
   (b) if $C_1 = 0$, $C_2 = 0$,
      i. if $A_1 = D_1$, $A_2 = D_2$, then the general solutions of system (1.8) is given by formulas in (2.24) and (2.24).
      ii. if $A_1 \neq D_1$, $A_2 \neq D_2$, then the general solutions of system (1.8) is given by formulas in (2.26) and (2.27).
2. if $A_1 D_1 = B_1 C_1$, $A_2 D_2 = B_2 C_2$,
   (a) if $A_1 = 0$, $A_2 = 0$, then the general solutions of system (1.8) is given by formulas in (2.30).
   (b) if $A_1 \neq 0$, $A_2 \neq 0$, then the general solutions of system (1.8) is given by formulas in (2.33).
   (c) if $D_1 = 0$, $D_2 = 0$, then the general solutions of system (1.8) is given by formulas in (2.33).
   (d) if $D_1 \neq 0$, $D_2 \neq 0$, then the general solutions of system (1.8) is given by formulas in (2.30).

(e) if $A_1B_1C_1D_1 \neq 0$, $A_2B_2C_2D_2 \neq 0$.

i. if $A_1 = \frac{B_1C_1}{D_1}$, $A_2 = \frac{B_2C_2}{D_2}$, then the general solutions of system (1.8) is given by formulas in (2.30).

ii. if $B_1 = \frac{A_1D_1}{C_1}$, $B_2 = \frac{A_2D_2}{C_2}$, then the general solutions of system (1.8) is given by formulas in (2.33).

In addition, an application is given.

Declarations

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