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DYNAMICAL BEHAVIOR OF A DISEASED PREDATOR-PREY MODEL WITH FEAR EFFECT AND PREY HARVESTING

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Abstract. This article consists of a three-species food web model that has been developed by considering the interaction between susceptible prey, infected prey and predator species. It is assumed that susceptible prey species grow logistically in the absence of predators. It is assumed that predators consume susceptible and infected prey . We consider the effect of fear on susceptible prey due to predator species. Again, the harvesting of susceptible and infected prey has been considered. Furthermore, the predator consumes its prey in the form of Holling-type interactions. The positive invariance, positivity, and boundedness of the system are discussed. The conditions of all biologically feasible equilibrium points have been examined. The local stability of the systems around these equilibrium points is investigated. Furthermore, the occurrence of Hopf-bifurcation concerning the harvesting (h) of the system has been investigated. Finally, we demonstrate some numerical simulation results to illustrate our main analytical findings.

1. Introduction

The term ecology (*oecologie*) was coined in 1876 by the German evolutionary biologist Ernst Haeckel [1]. He combined two Greek words, "oikos," meaning "house" or "dwelling place," and "logos," meaning "science" or "study," to form the word. Ecology is the study of plants and animals activities. Plants and animals are the

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scientific study of the relationship of organisms to each other and to their physical environment. Epidemiology is the study of the prevalence and determinants of health-related conditions or events in specific populations and the application of this study to control health problems. Epidemiology began with Adam and Eve, both of whom sought to examine the quality of the "forbidden fruits." Epidemiology is the study of the distribution and determination of health-related conditions or events in specific populations and the application of this study. Mathematical ecology and mathematical epidemiology are major fields of study in their own right. But there are some commonalities between them. A branch of ecology that considers the effects of transmissible diseases is called eco-epidemiology. Eco-epidemiology is a new branch of mathematical biology that considers both ecological and epidemiological problems simultaneously. Eco-epidemiological research deals with diseases that spread in an interactive population in which epidemiologic and demographic features are incorporated into a model [2,3].

Eco-epidemiological systems are used to investigate the dynamic connection between predator and prey in one population or a population of susceptible and infected animals. Mathematical models have become significant instruments in examining the flow and manipulation of prevention. Kermack-Mckendrick's [4] pioneering work on SIRS epidemiological models has drawn a lot of interest from researchers. Many investigators have studied the population ecology of prey, predators, or both. The non-linear relationship between populations of predators and their prey has been and will remain one of the subjects that are most frequently addressed in both mathematical ecology and epidemiology due to its worldwide existence and significance. Although these issues appear straightforward mathematically at first glance, they are challenging and complicated. Ecology and epidemiology are two distinct, essential, and significant areas of research. Lotka [5] and Volterra [6] models, the first advance in current mathematical ecology, can be examined using the system of dynamical equations. Environmental epidemiology is the complete study of epidemiology and ecology. Eco-epidemiology exerts a significant ecological impact. It is referred to as the study of infection spread between interacting organisms. A variety of mathematical and statistical methodologies are available for analysing eco-epidemiological data. Many ecosystems around the world have predator-prey interactions between species, as well as the lion-deer association. In the environment, predator and prey species display oscillations in population increase and decline or abundance. Animal conservationists and mathematicians have long been intrigued by the study of this volatility in seemingly stable patterns. As a result, many others have extensively studied the dynamics of prey-predator interactions over the last three decades. [7], [8], [9]. Population growth models with the spread of diseases frequently exhibit complicated, non-linear mathematical dynamics. The fundamental goal of these models is to investigate points of equilibrium, their analyses of stability, solutions in the type of periodic, bifurcations, system behaviour of chaotic nature, and so on.

Alfred J. Lotka was the first to investigate the relationships between populations of predators and their prey. A biological representation in terms of mathematical modelling of communications among the population density of predators and population density of prey, called "functional response," is the major part of biological modelling in the population density of predators and population density of prey. There are numerous functional responses, namely the types I–III of the Holling response, the Varley-Hassell response, the Beddington-DeAngelis response, and the Crowley-Martin response. Arditi and Ginzburg's [10] relatively popular type of ratio-dependent response. Much more information on predator-prey systems with Crowley-Martin functional responses has become available in recent decades. In the recent era, some renowned authors [11, 12], [13], [14], [15], [16], [17]. studied functional responses to comprehend the importance of the relationship between the prey and predator in the ecosystem. They used some functional responses, such as the Crowley-Martin functional response, to make the model system more realistic and controllable in the ecosystem. Several investigators [18–21] started exploring a non-linear analysis of the predator-prey scenario involving infection in either the prey or predator population or both populations or the two forms of infection in the predator population system with a functional non-linear response that includes the function of type II Holling. The global and local stability investigations explored the prey-predator food web model with the function of type II Holling, which included the bifurcation analysis for the ratio-dependent intraguild predation model. Recently, several investigators have discovered that there is frequently a constant percentage of prey that is shielded from predators by the refuge. The interactions between prey and predators may be stabilised by refugia, according to several studies and mathematical models. In [22], Maynard Smith discovered that the presence of a static proportional size of refuge of any size neutrally altered the static nature of equilibrium, according to the stochastic stability of a Lotka-Volterra unbiased model. A neutrally stable Lotka-Volterra model's dynamic stability was unaffected by the presence of a constant proportionate refuge. Tapan Kumar Kar [23] considered a Holing type II response function integration and predator model with prey refuge. Commercial exploitation of biological resources to meet society's increasing demands has long been a cause of concern for ecologists, bioeconomists, and resource managers of nature. The impact of harvests is extensively used in forestry, wildlife management, and fisheries. This research uncovered a wide range of fascinating dynamics, such as points of equilibrium, analyses of bifurcation, and limit cycles. In eco-epidemiology, we explore predator-prey models that include infection dynamics. We seek to investigate the dynamics of the predator-prey model using this functional response. A form of predator-dependent functional response is a ratio-dependent functional response. The predation rate of the prey is supposed to be the number of prey consumed by a predator per unit of time. When predator-prey interactions involve intensive searching, ratio-dependent predator-prey models are more suitable than other types. Recently, [24], [25], [26] many researchers have investigated the apparent biological and physiological evidence of growth under different conditions. The prey population density is low in a ratio-dependent model, and as the number of prey grows, the reaction to every predator activity becomes more constant (i.e., a type II reaction under Holling). [15]). Recently, several investigators have discovered that there is frequently a constant percentage of prey that is shielded from predators by the refuge. Predator-prey interactions have been included in the Lotka-Volterra model for a very long time. In a similar vein, the seminal work on the interaction of the susceptible, infected, and recovered has been an interesting topic of study. The original predator-prey model was developed, in large part, by Vito Volterra and Alfred James Lotka. Ecology models and epidemiology models are the two basic categories into which mathematical models are often divided. In the ecological framework, the relationship between the population density of some communities is studied. Epidemiology systems are used to investigate the spread of illnesses between wildlife and humans. It is increasingly crucial to do research on the dynamics of illness within ecological systems. On the one hand, several studies of prey-predator dynamics have been conducted in recent decades, taking into account the impact of a range of biological characteristics. Many mathematical models have been created and investigated in the field of epidemiology, taking into consideration various incidence rates and illnesses. Experts were particularly interested in their recommended ecological models since it is well-accepted that species harvesting is necessary for species coexistence. Ecology models and epidemiology models are the two basic categories into which mathematical models are often divided. There are three different forms of harvesting: constant, proportional to density, nonlinear, and others. All of these have been proposed and investigated. There have been several suggestions for harvesting methods based on research, including harvesting continuously and depending on density in proportional harvesting. We research predator-prey models as well as disease dynamics in eco-epidemiology. Using this physiological response, we hope to investigate the dynamics of the predator-prey paradigm. To address this problem, we study the impact of fear in an eco-epidemiological model with infected prey in this paper. To the best of the available information, none of the scholars have explored the three-species food web model of prey-predator relationships that combines species relationships, such as Holling type II function and disease in prey populations, with the influence of fear in prey harvesting. We explore the diseased prey-predator model utilising Holling type II interaction as well as the influence of fear on susceptible prey populations due to predators and prey harvesting, motivated by this fact. The rest of the paper is structured as follows: The mathematical analysis is investigated in Section 2. In Section 3, some preliminary aspects of the model have been studied. Section 4 deals with the point of equilibrium at the boundary and its stability. In Sections 5 and 6, we determine the existence of the interior point of equilibrium ($E^*(s^*, i^*, p^*)$) and investigate its local and global stability. The occurrence of Hopf-bifurcation is shown in Section 7. Numerical simulations are examined for the proposed model in Section 8. The conclusion of the paper and the biological consequences of our mathematical results are found in Section 8, which concludes the paper.

2. Model Formation

The framework demonstrates the relationship between the population density of prey with infection. Which leads to the following structure of non-linear differential equations. The suggested framework was applied to examine the non-linear population density of susceptible, infected prey and predator biological model

$$\left.\begin{aligned} \frac{d\mathcal{S}}{dT} &= \frac{r_1\mathcal{S}}{1+\mathcal{F}\mathcal{P}}(1-\frac{\mathcal{S}+\mathcal{I}}{K})-\lambda\mathcal{I}\mathcal{S}-\frac{\alpha_1\mathcal{S}\mathcal{P}}{a_1+\mathcal{S}}-H_1E_1\mathcal{S},\\ \frac{d\mathcal{I}}{dT} &= \lambda\mathcal{I}\mathcal{S}-d_1\mathcal{I}-\frac{b_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}}-H_2E_2\mathcal{I},\\ \frac{d\mathcal{P}}{dT} &= -d_2\mathcal{P}+\frac{cb_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}}+\frac{c\alpha_1\mathcal{S}\mathcal{P}}{a_1+\mathcal{S}}. \end{aligned}\right\} \quad (1)$$

Here the conditions are $\mathcal{S}(0)\geq 0, \mathcal{I}(0)\geq 0$ and $\mathcal{P}(0)\geq 0$ the table displays specific biological meanings of the parameters.

Table 1. Biological representation of the model

Parameters	Units	Biological representation
$\mathcal{S}$	Number of components per unit area (tons)	Population density of susceptible Prey
$\mathcal{I}$	Number of components per unit area (tons)	Population density of prey with infection
$\mathcal{P}$	Number of components per unit area (tons)	Population density of Predator
r_1	Per day (T^{-1})	Prey population densities growth rate
K	Number of components per unit area (tons)	The carrying ability of nature
λ	Per day (T^{-1})	Infection rate
a_1	Per day (V)	Constant of Half-saturation
α_1	Per day (T^{-1})	Susceptible prey to predator consumption
b_1	Per day (T^{-1})	Capture rate by predator
c	Per day	Conversion rate of prey to predator
d_1	Per day (T^{-1})	density of diseased prey mortality rate
d_2	Per day (T^{-1})	Density of predator population mortality rate
$\mathcal{F}$	Number of components per unit area (tons)	Impact of fear
E_1, E_2	Number of components per unit area (tons)	Harvesting Effect
H_1, H_2	Number of components per unit area (tons)	Prey's catchability coefficient

The condition for the fear effect is

$$\mathcal{F}_1(\beta, p) = \frac{1}{1+\beta p} \quad (2)$$

This describes the level of fear in susceptible prey as a consequence of the predator. Here, β represents the quantity of fear. Given the epidemiological meaning of β, the following condition is strongly acceptable:

$$\mathcal{F}_1(0,p) = \mathcal{F}_1(\beta,0) = 1$$

$$lim_{\beta\to\infty}\mathcal{F}_1(\beta,p) = 0 = lim_{p\to\infty}\mathcal{F}_1(\beta,p)$$

$$\frac{\partial \mathcal{F}_1(\beta,p)}{\partial \beta} < 0,$$
$$\frac{\partial \mathcal{F}_1(\beta,p)}{\partial p} < 0.$$

In this work we incorporate prey and the fear effect β. Then the system change into the non-dimensional .
Here,
$s = \frac{\mathcal{S}}{K}, \quad i = \frac{\mathcal{I}}{K}, \quad p = \frac{\mathcal{P}}{K}.$
Now (1) becomes,

$$\left.\begin{aligned} \frac{ds}{dt} &= \frac{rs}{1+\beta p}(1-s-i) - is - \frac{s\alpha p}{s+a} - h_1 s \\ \frac{di}{dt} &= is - di - \frac{\theta ip}{a+i} - h_2 i \\ \frac{dp}{dt} &= -\delta p + \frac{c\theta ip}{a+i} + \frac{c\alpha sp}{s+a}. \end{aligned}\right\} \tag{3}$$

here the conditions are,

$$r = \frac{r_1}{\lambda K}, \quad \alpha = \frac{\alpha_1}{\lambda K}, \quad h_1 = \frac{H_1 E_1}{\lambda K},$$
$$d = \frac{d_1}{\lambda K}, \quad h_2 = \frac{H_2 E_2}{\lambda K}, \quad \theta = \frac{b_1}{\lambda K},$$
$$a = \frac{a_1}{K}, \quad \delta = \frac{d_2}{\lambda K}, \quad \beta = \frac{\mathcal{F}}{K}.$$

According to the preliminary criteria $\{s(0), i(0), p(0)\} \geq 0$. The operations described over are in $\mathbb{R}^3_+$.

3. Positivity, Existence and Boundedness of solutions

In this section we discusses the positivity and boundedness solution of the system.(3)

3.1. **Positivity of solutions.**

Theorem 1. *In the* $\mathbb{R}^3_+$ *all the (3) systems solutions are non-negative .*

Proof. Since $\{s(0), i(0), p(0)\} \geq 0$, hence the system (3) written as,

$$\begin{aligned} s(t) =& s(0) exp\left(\int_0^1 \left[\frac{r}{1+\beta p}(1-i-s) - i - \frac{p\alpha}{s+a} - h_1\right] ds\right) \geq 0, \\ i(t) =& i(0) exp\left(\int_0^1 \left[-d + s - \frac{\theta p}{a+i} - h_2\right] ds\right) \geq 0, \\ p(t) =& p(0) exp\left(\int_0^1 \left[\frac{c\theta i}{a+i} + \frac{c\alpha s}{s+a} - \delta\right] ds\right) \geq 0. \end{aligned}$$

Existence of the solutions:
For $t < 0$, let $\mathcal{Z} = (s(t) + i(t) + p(t))$, and $\mathcal{E}(\mathcal{Z}) = (\mathcal{O}_1\mathcal{Z}, \mathcal{O}_2\mathcal{Z}, \mathcal{O}_3\mathcal{Z})^T$, where
$\mathcal{O}_1\mathcal{Z} = \frac{rs}{1+\beta p}(1 - s - i) - is - \frac{\alpha sp}{s+a} - h_1 s$,
$\mathcal{O}_2\mathcal{Z} = is - id - \frac{\theta ip}{a+i} - h_2$,
$\mathcal{O}_3\mathcal{Z} = -\delta p + \frac{c\theta ip}{a+i} + \frac{c\alpha sp}{s+a}$.
Then, (3) is then able to be formed as $\frac{d\mathcal{Z}}{dt} = \mathcal{E}(\mathcal{Z})$, where, $\mathcal{O} : \mathcal{C}_+ \to \mathbb{R}^3_+$ with, $\mathcal{Z}_(0) = \mathcal{Z}_0 \in \mathbb{R}^3_+$.
Here, $\mathcal{E}_\rangle \in \mathcal{C}^\infty(\mathbb{R})$ for $i = 1, 2, 3$. As a result, the mathematical operator $\mathcal{O}$ is both locally Lipschitzian and completely continuous on $\mathbb{R}^3_+$. Therefore, the solution of (3) exists and unique. Hence the region $\mathbb{R}^3_+$ is an invariant domain of the system (3) solutions are positive. □

Theorem 2. *If $c < 1$, $Max\ \frac{rs}{1+\beta p}(1 - s) = \frac{r}{8}$, and $\beta = min(h_1, d + h_2, \delta)$ in $\mathbb{R}^3_+$ all the system (3) solutions are bounded.*

Proof. s, i and p denote the model (3) solutions with positive criteria, hence
$\frac{ds}{dt} \le sr(1 - s)$.
We know that $\limsup_{t\to\infty} s \le 1$,. Let, $\mathcal{Z} = s + i + p$.

$$\begin{aligned}
\frac{d\mathcal{Z}}{dt} =& \frac{ds}{dt} + \frac{di}{dt} + \frac{dp}{dt}\\
=& \frac{rs}{1+\beta p}(1 - s - i) - si - h_1 s - \frac{(1-c)s\alpha p}{s+a}\\
& + si - id - \frac{(1-c)\theta ip}{a+i} - p\delta - h_2 i\\
& \le \frac{rs}{1+\beta p} - p\delta - id - h_1 s - h_2 i \ \ (\text{where}, c < 1)\\
& \le \frac{r}{8} - p\delta - id - h_1 s - h_2 i \ \left(\text{since}, (Max(\frac{rs}{1+\beta p}(1 - s) = \frac{r}{8})\right)\\
& \le \frac{r}{8} - \beta\mathcal{Z}, \ \text{where } \beta = min(h_1, d + h_2, \delta),
\end{aligned}$$

we have, $\dfrac{d\mathcal{Z}}{dt} + \beta z \le \dfrac{r}{8}$.
Using the differential inequality theorem, we obtain

$$0 < \mathcal{Z} \le \frac{r}{4\beta}(1 - \exp^{-\beta t}) + \mathcal{Z}(s_0, i_0, p_0)\exp^{-\beta t}.$$

For $t \to \infty$, we have $0 < \mathcal{Z} < \frac{r}{4\beta}$ in the $\mathbb{R}^3_+$ all the systems (3) solutions are uniformly bounded, for $\epsilon > 0$ are in the region,

$$\Omega = \left\{(s, i, p) \in \mathbb{R}^3_+; s + i + p \le \frac{r}{4\beta} + \epsilon\right\}.$$

□

4. The Existence of Point of Equilibrium

This section examines the potential points of equilibrium (3). The system (3) has three points of equilibrium and one endemic point of equilibrium:

$$\begin{aligned}&\frac{rs}{1+\beta p}(1-s-i)-si-\frac{\alpha sp}{s+a}-h_1 s=0,\\&is-di-\frac{\theta ip}{a+i}-h_2 i=0,\\&-\delta p+\frac{c\theta ip}{a+i}+\frac{c\alpha sp}{s+a}=0.\end{aligned}$$

- $E_0(0,0,0)$ is the point of equilibrium, which is trivial,
- $E_1(\frac{r-h_1}{r},0,0)$ is the free of infection and free of predator point of equilibrium that exists for $r>h_1$.
- The absence of predator point of equilibrium is $E_2(\hat{s},\hat{i},0)$, where, $\hat{s}=d+h_2, \hat{i}=\frac{r(1-d-h_2)-h_1}{r+1}$, it exists for $r(1-h_2-d)>h_1$
- Endemic or positive or interior equilibrium is $E^*(s^*,i^*,p^*)$, where $i^*=\frac{a(a\delta+(\delta-c\alpha)s^*)}{(c\alpha s^*+(c\theta-\delta)(s^*+a))}$, $p^*=\frac{ac(s^*-d)(s^*+a)}{(c\alpha s^*+(c\theta-\delta)(s^*+a))}$ and s^* is the unique positive root of the quadratic equation
  $\mathcal{A}S^2+\mathcal{B}S+\mathcal{C}=0$,
  where,
  $$\begin{aligned}\mathcal{A}=&r(\alpha c+\theta c-\delta),\\\mathcal{B}=&(\theta c-\delta)(ar-r)+\alpha c((1+\beta p)-r)+a(\delta(1+\beta p)+(\delta-c\alpha)r),\\\mathcal{C}=&-a(r(1+\beta p))(c\theta-\delta)+(c\alpha(1+\beta p)(d)-a\delta((1+\beta p)+r))).\end{aligned}$$
  Endemic equilibrium exists for $\delta>\alpha c$.

5. Local Stability Analysis

In order to investigate the local stability property of the system(3), we first find the Jacobian matrix of the system in the form $J(E)=\begin{pmatrix}n_{11}&n_{12}&n_{13}\\n_{21}&n_{22}&n_{23}\\n_{31}&n_{32}&n_{33}\end{pmatrix}$.

Here,

$$\begin{aligned}n_{11}=&\frac{r}{1+\beta p}(1-2s)-i\left(\frac{r}{1+\beta p}+1\right)-\frac{\alpha ap}{(s+a)^2}-h_1, n_{12}=-s(\frac{r}{1+\beta p}+1),\\n_{13}=&\frac{prs}{(1+\beta p)^2}(1-s-i)-\frac{\alpha s}{s+a}, n_{21}=i, n_{22}=s-d-h_2-\frac{a\theta p}{(a+i)^2},\\n_{23}=&-\frac{\theta i}{(a+i)}, n_{31}=\frac{ac\alpha p}{(s+a)^2}, n_{32}=\frac{ac\theta p}{(a+i)^2},\end{aligned}$$

$$n_{33} =- \delta + \frac{c\theta i}{a+i} + \frac{\alpha cs}{s+a}.$$

Theorem 3. *$E_0(0,0,0)$ is the point of equilibrium, which is trivial, is stable if $r < h_1$, otherwise unstable.*

Proof. The characteristic equation of the point of equilibrium E_0 is,
$(\lambda_{01} - (r - h_1))(\lambda_{02} - (-d - h_2))(\lambda_{03} + \delta) = 0,$
$\lambda_{01} = r - h_1, \lambda_{02} = -d - h_2, \lambda_{03} = -\delta,$
here,$\lambda_{02} < 0, \lambda_{03} < 0$. $E_0(0,0,0)$ is the point of equilibrium, which is trivial,is stable if $r < h_1$ otherwise it is unstable. □

Theorem 4. *$E_1(\frac{r-h_1}{r}, 0, 0)$, the free of infection and free of the predator point of equilibrium, is stable if $c\alpha < \delta$ and $h_1 > r(1 - d - h_2)$, otherwise unstable.*

Proof. The characteristic equation of the point of equilibrium E_1 is,

$$(\lambda_{11} - ((h_1 - r)))(\lambda_{12} - (1 - d - h_2 - \frac{h_1}{r}))(\lambda_{13} - (\frac{-\alpha(r-h_1)}{ra + (r-h_1)} - \delta)) = 0,$$

$$\lambda_{11} = h_1 - r, \lambda_{12} = 1 - d - h_2 - \frac{h_1}{r}, \lambda_{13} = \frac{-c\alpha(r-h_1)}{ra + (r-h_1)} - \delta,$$

here, $E_1(\frac{r-h_1}{r}, 0, 0)$ being free of infection and free of the predator point of equilibrium, is stable if $c\alpha < \delta$ and $h_1 > r(1 - d - h_2)$, otherwise unstable. □

Theorem 5. *The equilibrium $E_2(\hat{s}, \hat{i}, 0)$ which absence of predator is asymptotically stable if $\delta > c(\theta + \alpha)$.*

Proof. The matrix in the form of Jacobian at E_2 is $J(E_3) = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$,

where,

$$\begin{aligned} q_{11} =& r(1-2\hat{s}) + i(r+1), \quad q_{12} = (-1-r)\hat{s}, \quad q_{13} = -\frac{\alpha\hat{s}}{s+a}, \\ q_{21} =& \hat{i}, \quad q_{22} = 0, \quad q_{23} = -\frac{\theta\hat{i}}{a+\hat{i}}, \\ q_{31} =& 0, \quad q_{32} = 0, \quad q_{33} = \frac{c\alpha\hat{s}}{a+\hat{s}} - \delta + \frac{c\theta\hat{i}}{a+\hat{i}}. \end{aligned}$$

Here, the characteristic equation of the above matrix in the form of Jacobian is, $\lambda^3 + \mathcal{L}\lambda^2 + \mathcal{M}\lambda + \mathcal{N} = 0$. Here,

$$\begin{aligned} \mathcal{L} &= -q_{11} - q_{33}, \\ \mathcal{M} &= -q_{21}q_{12} + q_{33}q_{11}, \\ \mathcal{N} &= q_{12}q_{21}q_{33}. \end{aligned}$$

If and only if $\mathcal{L}, \mathcal{N}$ and $\mathcal{LM} - \mathcal{N}$ are positive, then the negative real parts are the roots of the above characteristic equation. According to the Routh-Hurwitz criterion. now, $\mathcal{LM} - \mathcal{N} = -q_{11}(-q_{12}q_{21} + q_{33}(q_{33} + q_{11}))$. Now, the sufficient conditions for q_{33} to be negative is $\delta > c(\alpha + \theta)$. The E_2 is locally asymptotically stable provided the above condition in theorem satisfied. □

Theorem 6. *The endemic or positive point of equilibrium E^* is asymptotically stable.*

Proof. The matrix in the form of Jacobian at E^* is $J(E^*) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$,

where,

$$\begin{aligned} r_{11} &= -\frac{s^*(h_1 - r + ar + (1+r)i^* + 2rs^*)}{s^* + a}, \quad r_{12} = -s^*(\frac{r}{1+\beta p^*} + 1), \\ r_{13} &= \frac{p^* r s^*}{(1+\beta p^*)^2}(1 - s^* - i^*) - \frac{\alpha s^*}{s^* + a}, \\ r_{21} &= i^*, \quad r_{22} = \frac{a\theta p^* i^*}{(a + i^*)^2}, \quad r_{23} = \frac{\theta i^*}{(a + i^*)}, \\ r_{31} &= \frac{ac\alpha p^*}{(s^* + a)^2}, \quad r_{32} = \frac{ac\theta p^*}{(a + i^*)^2}, \quad r_{33} = 0. \end{aligned}$$

Here, the characteristic equation of the Matrix in the form of Jacobian E^* is

$$\lambda^3 + \mathcal{F}\lambda^2 + \mathcal{G}\lambda + \mathcal{H} = 0, \tag{4}$$

here,

$$\mathcal{F} = -r_{11} - r_{33}, \mathcal{G} = -r_{21}r_{12} + r_{22}r_{11} - r_{13}r_{31} + r_{23}r_{32},$$
$$\mathcal{H} = r_{13}(-r_{22}r_{31} + r_{21}r_{32}) + r_{23}(r_{12}r_{31} - r_{11}r_{32}).$$

If $\mathcal{F} > 0, \mathcal{H} > 0, \mathcal{FG} - \mathcal{H} > 0$. The negative real parts are the roots of the above characteristic equation if and only if $\mathcal{F}, \mathcal{H}$ and $\mathcal{FG} - \mathcal{H}$ are non-negative, according to the Routh-Hurwitz criterion. The E^* is locally asymptotically stable. □

6. Hopf-Bifurcation Analysis

In this part, we use the harvesting (h_1) effect to analyse the model's bifurcation. Using the bifurcating parameter h_1, the following theorem shows the presence of Hope-bifurcation.

Theorem 7. *The model (3) confronts Hopf-bifurcation if the bifurcation parameter h_1 surpasses a critical point. The following Hopf-bifurcation conditions arise at $h_1 = h_1^*$:*
1.$\mathcal{A}_1(h_1^*)A(h_1^*) - \mathcal{A}_3(h_1^*) = 0$.
2.$\frac{d}{df}(Re(\lambda(h_1)))|_{h_1 = h_1{}^*} \neq 0$ *Here lambda is the zero of the parametric solution correlated with the equilibrium's interior point.*

Proof. For $h_1 = h_1^*$, let the equation of characteristic (4) is in the form

$$(\lambda^2(h_1^*) + \mathcal{A}_2(h_1^*))(\lambda(h_1^*) + \mathcal{A}_1(h_1^*)) = 0. \tag{5}$$

This indicates that the roots of the preceding equation are $\pm i\sqrt{\mathcal{A}_2(h_1^*)}$ and $-\mathcal{A}_1(h_1^*)$. To achieve the Hopf-bifurcation at $h_1 = h_1^*$ the following transversality criterion must be fulfilled.

$$\tfrac{d}{dh_1^*}(Re(\lambda(h_1^*)))| \neq 0.$$

For h_1, the above equation (5) has general roots

$$\lambda_1 = r(h_1) + is(h_1),$$
$$\lambda_2 = r(h_1) - is(h_1),$$
$$\lambda_3 = -\mathcal{A}_1(h_1).$$

Weather check the criteria $\frac{d}{dh_1^*}(Re(\lambda(h_1^*)))| \neq 0$.
Let $\lambda_1 = r(h_1) + is(h_1)$ in the (5), we get

$$\mathcal{C}(h_1) + i\mathcal{D}(h_1) = 0.$$

Where,

$$\mathcal{C}(h_1) = r^3(h_1) + r^2(h_1)\mathcal{A}_1(h_1) - 3r(h_1)s^2(h_1) - s^2(h_1)\mathcal{A}_1(h_1) + \mathcal{A}_2(h_1)r(h_1) + \mathcal{A}_1(h_1)\mathcal{A}_2(h_1),$$
$$\mathcal{D}(h_1) = \mathcal{A}_2(h_1)s(h_1) + 2r(h_1)s(h_1)\mathcal{A}_1(h_1) + 3r^2(h_1)s(h_1) + s^3(h_1).$$

In order to fulfill the (5) we must have $\mathcal{C}(h_1) = 0$ and $\mathcal{D}(h_1) = 0$, then calculating $\mathcal{C}$ and $\mathcal{D}$ with respect to h_1. We have

$$\frac{d\mathcal{C}}{dh_1} = \varsigma_1(h_1)r^{'}(h_1) - \varsigma_2(h_1)s^{'}(h_1) + \varsigma_3(h_1) = 0, \tag{6}$$

$$\frac{d\mathcal{D}}{dh_1} = \varsigma_2(h_1)r^{'}(h_1) + \varsigma_1(h_1)s^{'}(h_1) + \varsigma_4(h_1) = 0, \tag{7}$$

where,

$$\varsigma_1 = 3r^2(h_1) + 2r(h_1)\mathcal{A}_1(h_1) - 3s^2(h_1) + \mathcal{A}_2(h_1),$$
$$\varsigma_2 = 6r(h_1)s(h_1) + 2s(h_1)a_1(h_1),$$
$$\varsigma_3 = r^2(h_1)\mathcal{A}_1^{'}(h_1) + s^2(h_1)\mathcal{A}_1^{'}(h_1) + \mathcal{A}_2^{'}(h_1)r(h_1),$$
$$\varsigma_4 = \mathcal{A}_2^{'}(h_1)s(h_1) + 2r(h_1)s(h_1)\mathcal{A}_1^{'}(h_1).$$

On multiplying (6) by $\varsigma_1(h_1)$ and (7) by $\varsigma_2(h_1)$ respectively

$$r(h_1)^{'} = -\frac{\varsigma_1(h_1)\varsigma_3(h_1) + \varsigma_2(h_1)\varsigma_4(h_1)}{\varsigma_1{}^2(h_1) + \varsigma_2{}^2(h_1)}. \tag{8}$$

Substituting $r(h_1) = 0$ and $s(h_1) = \sqrt{\mathcal{A}_2(h_1)}$ at $h_1 = h_1^*$ on $\varsigma_1(h_1), \varsigma_2(h_1), \varsigma_3(h_1)$, and $\varsigma_4(h_1)$, we obtain

$$\begin{aligned}
\varsigma_1(h_1^*) =& -2\mathcal{A}_2(h_2^*),\\
\varsigma_2(h_1^*) =& 2\mathcal{A}_1(h_1^*)\sqrt{\mathcal{A}_2(h_1^*)}\\
\varsigma_3(h_1^*) =& \mathcal{A}_3^{'}(h_1^*) - \mathcal{A}_2(h_1^*)\mathcal{A}_1^{'}(h_1^*),\\
\varsigma_4(h_1^*) =& \mathcal{A}_2^{'}(h_1^*)\sqrt{\mathcal{A}_2 h_1^*}.
\end{aligned}$$

The equation (8), implies

$$r^{'}(h_1^*) = \frac{\mathcal{A}_3^{'}(h_1^*) - (\mathcal{A}_1(h_1^*\mathcal{A}_2(h_1^*)))}{2(\mathcal{A}_2(h_1^*) + \mathcal{A}_1^2(h_1^*))}, \tag{9}$$

if $\mathcal{A}_3^{'}(h_1^*) - (\mathcal{A}_1(h_1^*)\mathcal{A}_2(h_1^*))^{'} \neq 0$ which implies that $\frac{d}{dh_1^*}(Re(\lambda(h_1^*)))| \neq 0$, and $\lambda_3(h_1^*) = -\mathcal{A}_1(h_1^*) \neq 0$.
Therefore the condition $\mathcal{A}_3^{'}(h_1^*) - (\mathcal{A}_1(h_1^*)\mathcal{A}_2(h_1^*))^{'} \neq 0$ It has been guaranteed that the transversality criterion is satisfied, hence the model (3) has attained the Hopf-bifurcation at $h_1 = h_1^*$. □

7. Numerical Simulations

In this section, several numerical experiments on the system (3) are carried out to verify the mathematical findings. The rate of fear β , predation rate α and harvesting h_1 are the essential parameters in this study, and they will be used as control parameters. For the specified fixed parameter values given in Table 2, the numerical simulation is carried out using the MATLAB and MATHEMATICA software packages.

Table 2. Parameter values

Parameters	**Numeric value**
r	0.5
a	0.3
c	0.6
d	0.25
θ	0.4
δ	0.2
β	Variable
α	Variable

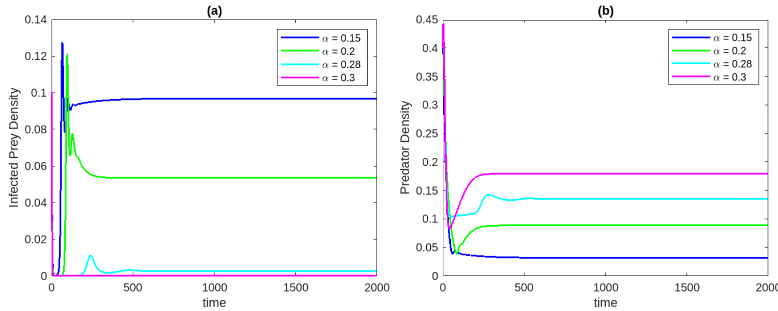

FIGURE 1. *The population of infected prey, and predators for* $\alpha = 0.15, 0.2, 0.28, 0.3$.

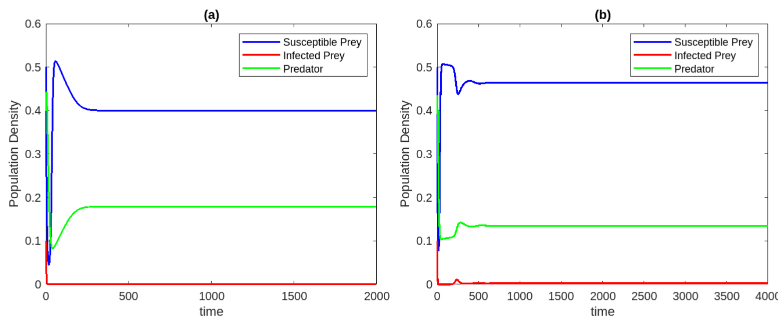

FIGURE 2. *Solutions of time series (3) around the point of equilibrium* E_2 *and the point of equilibrium* E_4.

7.1. **Effect of varying the predation rate** α**.** Let $\beta = 0.3, h_1 = 0.2$ For the parameters specified in Table 2. E_2 is predator free equilibrium and the endemic point of equilibrium E^* exists for $0.1 < \alpha < 035$, respectively, for the given parametric values. The stability of for $\alpha = 0.3$ and $\alpha = 0.28$ is shown in Figure(2). Figure (1) shows that as the predator population grows, so does the predation rate α and the number of infected prey.

7.2. **Effect of varying the harvesting rate** h_1**.** Let $\alpha = 0.3, \beta = 0.15$ For the parameters specified in Table 2.E_2 is predator free equilibrium and the endemic point of equilibrium E^* exists for$0.0140625 < h_1 < 0.307377$, respectively, for the given parametric values. From Figure (3) shows that increasing the rate of harvesting in susceptible prey leads to a decrease in population of susceptible prey and population of predator while increasing the population of infected prey.

7.3. **Bifurcation of harvesting rate** h_1**. Case-I**:(Changing only the parameter value h_1 and $h_2 = 0$)

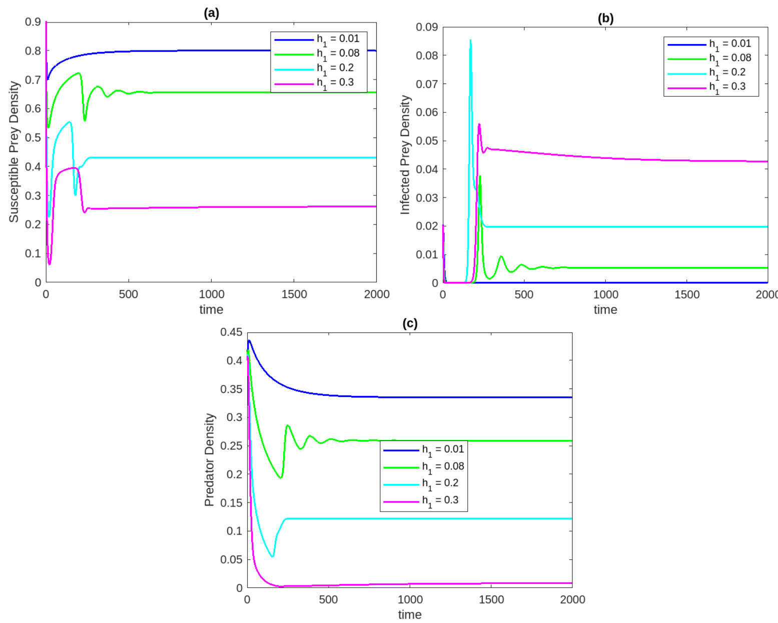

FIGURE 3. *For $\alpha = 0.25$, the population concentrations of susceptible prey, infected prey, and predators are as follows for the parametric values shown in the table. Where* $h_1 = 0.01, 0.08, 0.2, 0.3$

If $h_1 = 0.08$ then the model (3) is locally asymptotically stable about the positive equilibrium point $E^*(0.052861, 0.917829, 0.204774)$ and other parameter values are same, which is shown in Figure (4). Now, we increasing the value of bifurcation parameter $h_1 = 0.133$, then the model (3) lost its stability, arise limit cycle at $E^*(0.04899, 0.920924, 0.220149)$ which shown in figure(5) .

Case-II:(Changing the parameters values both h_1 and h_2)
Now, we choose $h_1 = 0.08$ and $h_2 = 0.15$ then the model (3) will behaves the locally asymptotically stable corresponding to the interior equilibrium point $E^*(0.150488, 0.839649, 0.496640)$, which is shown in Figure (6). We fix $h_2 = 0.15$ and increase the value $h_1 = 0.35$ then the model (3) lost its stability, arise limit cycle and undergoes the Hopf-bifurcation around the positive equilibrium point $E^*(0.151952, 0.838477, 0.465983)$, it is projected in Figure 7. Then the dynamical changes of the model (3) for $h_1 \in (0.01, 0.5)$, $h_2 = 0$ and $h_1 \in (0.2, 0.5)$, $h_2 = 0.15$, respectively displayed in Figure (8) and Figure (9).

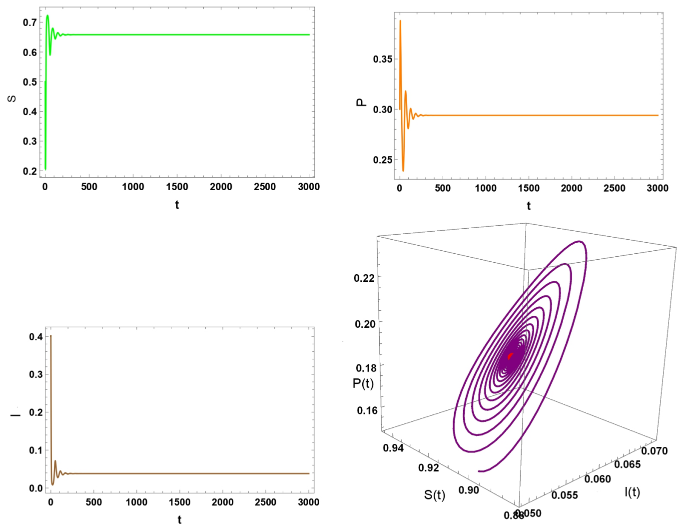

FIGURE 4. *The time analysis and phase portrait for the model(3) when* $h_1 = 0.08$ *and* $h_2 = 0$.

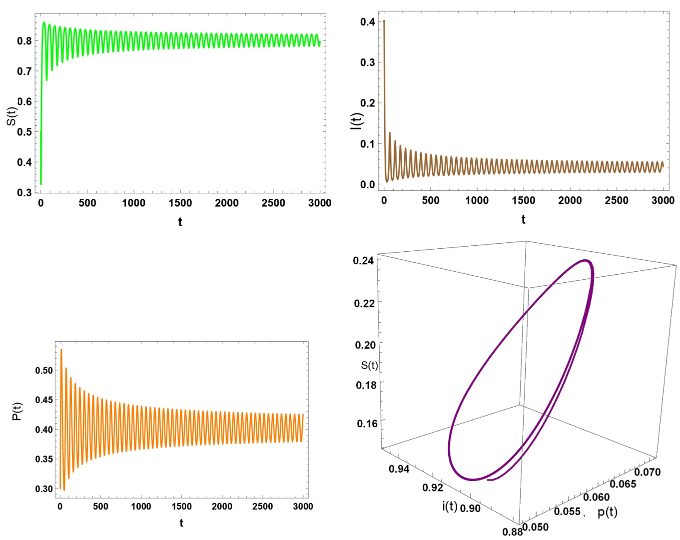

FIGURE 5. *The time analysis and phase portrait for the model (3) when* $h_1 = 0.35$ *and* $h_2 = 0$.

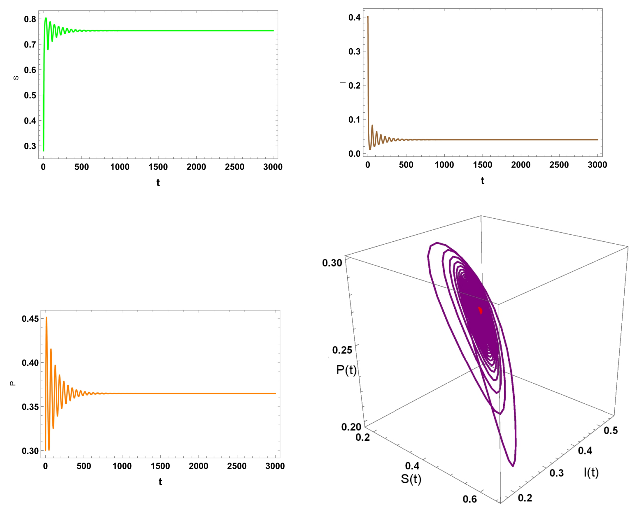

FIGURE 6. *The time analysis and phase portrait for the model (3) when* $h_1 = 0.08$ *and* $h_2 = 0.15$

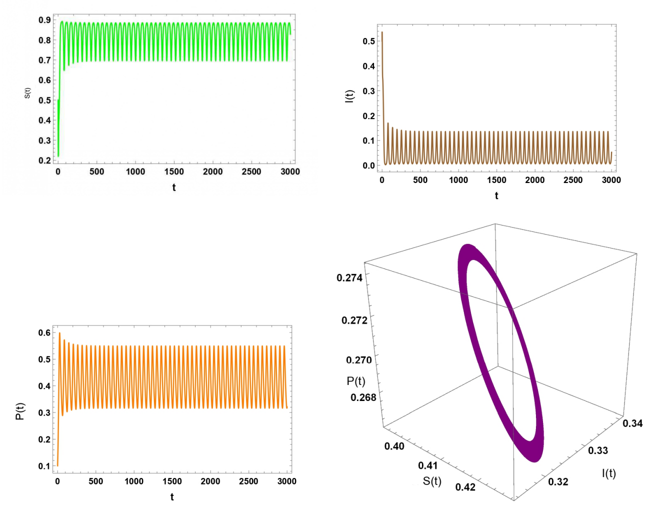

FIGURE 7. *The time analysis and phase portrait for the model (3) when* $h_1 = 0.35$ *and* $h_2 = 0.25$

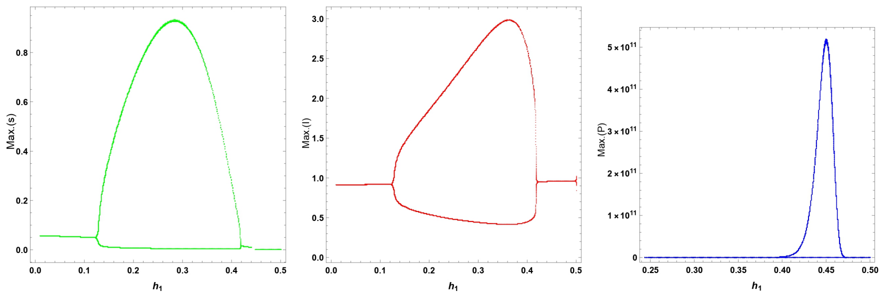

FIGURE 8. *The dynamical changes of the model(3) with* $h_1 \in (0.01, 0.5)$ *and* $h_2 = 0$

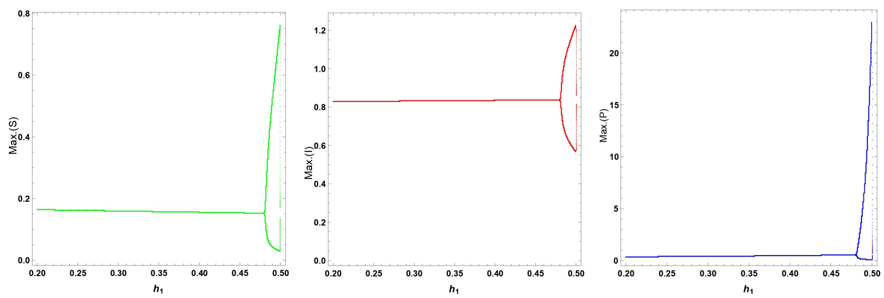

FIGURE 9. *The dynamical changes of the model (3) with* $h_1 \in (0.2, 0.5)$ *and* $h_2 = 0.15$.

8. Conclusion

We researched an eco-epidemiological system that included infection in the population density of prey and fear in the susceptible prey population density as a result of predator attacks on susceptible and diseased prey and harvesting in both prey populations. An eco-epidemic model deals with ecosystems of interacting populations among which a disease spreads. Different control measures and techniques are used to control the disease; harvesting is one of them. It is observed that harvesting plays a very crucial role in preventing the spread of infectious diseases. The positivity ensures that the population cannot be negative, while the boundedness of the solution could be understood as a natural limitation for growth due to limited resources. In addition, each biologically possible point of equilibrium can be represented (3). Furthermore, we investigated the suggested model's local stability (3) and observed the occurrence of Hopf-bifurcation, and we determined that modifying the cost of fear β and modifying the cost of harvesting h_1 has an instantaneous effect on the model's stability (3). As a result, Hopf-bifurcation constrained the developed analytical arguments around the E^* simulation findings. In the proposed models, we deduce that the existence of fear has a higher impact on stability shifts via the Hopf bifurcation. Finally, for the non-delayed models, we examine the time series of the impact of fear and the effect of harvesting in phase portraits and bifurcation diagrams. However, the future direction of the research seems more attractive. Moving forward, we plan to conduct an in-depth analysis of the model and delve into the effect of delay on the dynamics of the model. These future studies will yield exciting results related to the effect of delay.

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

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NONLINEAR SEMILINEAR INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS WITH IMPULSIVE EFFECTS

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Abstract. In this paper, we investigate the existence of a piecewise asymptotically almost automorphic mild solution to some classes of integro-differential equations with impulsive effects in Banach space. The working tools are based on the Mönch's fixed point theorem, the concept of measures of noncompactness theorem and resolvent operator. In order to illustrate our main results, we study the piecewise asymptotically almost automorphic solution of the impulsive differential equations.

1. Introduction

The exploration of impulsive integro-differential equations has witnessed rapid expansion in recent years, finding diverse applications in mathematical models spanning domains such as chemical technology, population dynamics, electrical engineering, medicine, physics, ecology, economics, biology, and beyond. The pioneering work of Milman and Myshkis [36] dates back to 1960 when they first introduced the concept of impulsive differential equations. To delve deeper into the outcomes and practical uses of impulsive integro-differential equations, comprehensive insights can be gleaned from the monographs authored by Bainov and Simeonov [7]. In the books authored by Benchohra *et al.* [9, 10], numerous results concerning differential equations are derived using a range of tools, including the utilization

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of measures of noncompactness and fixed point theory, from which we drew motivation. In the papers [11–15], the authors investigated several types of integro-differential equations under different conditions with qualitative and quantitative results. In [6, 33, 52], the authors considered some fractional integro-differential equations with state-dependent delay. See [2–4, 26–28, 48, 49], for some recent results on impulsive equations.

The notion of almost automorphy stands as a significant extension of Bohr's classical concept of almost periodicity, initially introduced by Bochner in [16] in connection with certain aspects of differential geometry. Since its inception, the realm of almost automorphic functions has witnessed substantial advancement and application across diverse fields such as ordinary differential equations, partial differential equations, functional differential equations, integro-differential equations, fractional differential equations, and even stochastic differential equations. A notable array of references, including [5, 17–19, 24, 25, 32, 35, 37, 38, 41, 42, 45, 50, 51, 54], serve to illustrate these developments. Subsequently, this conceptual framework has undergone compelling, natural, and potent generalizations. To exemplify, N'Guérékata [40] introduced the notion of asymptotically almost automorphic functions, which has been fruitfully applied within the realm of differential equations. For a deeper exploration of outcomes in this domain, one can turn to [1, 34, 44, 47, 53] and their associated references. For a comprehensive understanding of the contemporary theory and applications surrounding asymptotically almost automorphic functions, N'Guérekata's monographs [43] offer valuable insights.

In [29], Goldstein and N'Guérékata studied the following semilinear differential equation in a Banach space $\mathbb{X}$,

$$x'(t) = Ax(t) + F(t, x(t)), \quad t \in \mathbb{R},$$

where A generates an exponentially stable C_0-semigroup and $F(t,x) : \mathbb{R}\times \mathbb{X} \to \mathbb{X}$ is a function of the form $F(t,x) = P(t)Q(x)$. Under appropriate conditions on P and Q, and using the Schauder fixed point theorem, they proved the existence of an almost automorphic mild solution to the above equation.

José and Claudio [46] investigated the existence and uniqueness of an asymptotically almost automorphic mild solution to the following abstract fractional integro-differential neutral equation with unbounded delay:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} D\left(t, u_t\right) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AD\left(s, u_s\right) \mathrm{d}s + g\left(t, u_t\right), \quad t \geq 0, \\
&u_0 = \varphi \in B,
\end{aligned}$$

where $1 < \alpha < 2, D(t,\varphi) = \varphi(0) + f(t,\varphi), A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear densely defined operator of sectorial type on a Banach space $\mathbb{X}$, the history $u_t : (-\infty, 0] \to \mathbb{X}$, defined by $u_t(\theta) = u(t+\theta)$, belongs to an abstract phase space B defined axiomatically, and f, g are functions subject to some additional conditions.

Motivated by the above-mentioned discussions, we are interested in investigating the existence of piecewise asymptotically almost automorphic mild solution for the following integro-differential equations with impulsive differential system

$$\begin{cases} y'(t) = Ay(t) + \displaystyle\int_0^t R(t-s)y(s)ds + f\left(t, y(t), My(t)\right), t \neq t_j, \\ My(t) = \displaystyle\int_0^t H(t,s,y(s))ds, \ t \in \mathbb{R}^+, \\ \Delta y(t_j) = y(t_j^+) - y(t_j^-) = J_j(y(t_j)), \quad j = 1, 2, 3, \ldots, \end{cases} \tag{1}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0} \in E$ and $(E, |\cdot|)$ is a Banach space. Here $R(t)$ is a closed linear operator on E, with domain $D(A) \subset D(R(t))$ which is independent of t. Furthermore, the fixed times t_j satisfy $0 = t_0 < t_1 < t_2 < \ldots < t_j < \ldots$, t_j^+ and t_j^- denote the right and left limits of y at t_j, $\Delta y(t_j) = y(t_j^+) - y(t_j^-)$ represents the jump in the state y at time t_j, where J_j determines the size of the jump. The functions $f : \mathbb{R}^+ \times E \times E \to E$, and $H : D \times E \to E$, $D = \{(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+ : s \leq t\}$, are appropriate functions satisfying certain assumptions that will be specified later.

We note that the results we have obtained and the problem addressed in this paper are regarded as an extension and a natural continuation from the previously cited works, such as [29, 46].

Let us describe the content of this paper. In Section 2, we recall some facts from resolvent operators and measure of noncompactness. In addition, notations about almost automorphic functions and asymptotically almost automorphic functions are also introduced in this section. In Section 3, we study the existence of a piecewise asymptotically almost automorphic mild solutions for system (1) with their proofs, the results are based on Mönch's fixed point theorem under some appropriate assumptions. In last section, we provide an example to illustrate our obtained results.

2. Preliminaries and Basic Results

In this section, we present some mathematical tools needed to demonstrate the main results. Let E and $\widetilde{E}$ be two Banach spaces. For any Banach space E, the norm of E is defined by $|\cdot|$. The space of all bounded linear operators from E to $\widetilde{E}$ is denoted by $L(E, \widetilde{E})$ and $L(E, E)$ is written as $L(E)$. We denote by $\mathfrak{C}(\mathbb{R}^+, E)$ the Banach space of all continuous E-valued function on $\mathbb{R}^+$. We use $\|f\|_{L^p}$ to denote the $L^p(\mathbb{R}^+, E)$ norm of f whenever $f \in L^p(\mathbb{R}^+, E)$ for some p with $1 \leq p < \infty$. We consider the following spaces:

► $\mathfrak{C}^b(\mathbb{R}^+, E)$: the Banach space of all continuous and bounded functions y mapping $\mathbb{R}^+$ into E equipped with the norm

$$\|y\|_{\mathfrak{C}^b} = \sup\{|y(t)| : t \in \mathbb{R}^+\}.$$

► $P\mathfrak{C}(\mathbb{R}^+, E)$: the space formed by all piecewise continuous functions $f : \mathbb{R}^+ \to E$ such that $f(\cdot)$ is continuous at t for any $t \neq (t_j)_{j\in\mathbb{N}}$, $y(t_j^+)$, $y(t_j^-)$ exist, and $y(t_j^-) = y(t_j)$ for all $j \in \mathbb{N}$.

► $P\mathfrak{C}(\mathbb{R}^+ \times \widetilde{E} \times \widetilde{E}, E)$: the space formed by all piecewise continuous functions $f : \mathbb{R}^+ \times \widetilde{E} \times \widetilde{E} \to \widetilde{E}$ such that for any $(y, \nu) \in \widetilde{E} \times \widetilde{E}$, $f(\cdot, y, \nu) \in P\mathfrak{C}(\mathbb{R}^+, E)$, and for any $t \in \mathbb{R}^+$, $f(t, \cdot, \cdot)$ is continuous at $(y, \nu) \in \widetilde{E} \times \widetilde{E}$.

► $P\mathfrak{C}_0(\mathbb{R}^+, E)$: the space formed by all piecewise continuous functions $\Upsilon : \mathbb{R}^+ \to E$ such that $\lim\limits_{t\to\infty} \Upsilon(t) = 0$.

► $P\mathfrak{C}_0(\mathbb{R}^+ \times \widetilde{E} \times \widetilde{E}, E)$: the space of all piecewise continuous functions $\Upsilon : \mathbb{R}^+ \times \widetilde{E} \times \widetilde{E} \to E$ satisfying $\lim\limits_{t\to\infty} \Upsilon(t, y, \nu) = 0$ in t and uniformly for all $(y, \nu) \in K$, where K is any bounded subset of $\widetilde{E} \times \widetilde{E}$.

► $P\mathfrak{C}^b(\mathbb{R}^+, E)$ the subspace of $P\mathfrak{C}(\mathbb{R}^+, E)$ consisting of all bounded functions.

It is well-known that $P\mathfrak{C}^b(\mathbb{R}^+, E)$ is a Banach space with the norm

$$\|y\|_{P\mathfrak{C}^b} = \sup\{|y(t)|, t \in \mathbb{R}^+\}.$$

First, let's recall some basic defnitions and results on the strong continuous evolution family which will be used later.

We consider the following Cauchy problem

$$\begin{cases} y'(t) = Ay(t) + \displaystyle\int_0^t R(t-s)y(s)ds, \quad t \geq 0, \\ y(t) = y_0. \end{cases} \tag{2}$$

Definition 1. *([23, 30]) A resolvent for Equation (2) is a bounded linear operator valued function $S(t) \in L(E)$ for $t \geq 0$, satisfying the following properties:*

- **(a):** *For any $t \in \mathbb{R}^+$, $S(0) = I$ and $\|S(t)\|_{B(E)} \leq \eta e^{-\lambda(t-s)}$ for some constants η and λ.*
- **(b):** *For each $y \in E$, $S(t)y$ is strongly continuous for $t \geq 0$.*
- **(c):** *For $y \in E$, $S(\cdot)y \in \mathfrak{C}^1([0, +\infty), E) \cap \mathfrak{C}([0, +\infty), \widetilde{E})$ and*

$$\begin{aligned} S'(t)y &= AS(t)y + \int_0^t R(t-s)S(s)\widetilde{E}ds \\ &= S(t)Ay + \int_0^t S(t-s)R(s)\widetilde{E}ds. \end{aligned}$$

We introduce the following assumptions:

- **(T_1):** A is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on E.

(T_2): For all $t \geq 0$, $B(t)$ is closed linear operator from $D(A)$ to E and $R(t) \in L(\widetilde{E}, E)$. For any $y \in E$, the map $t \to R(t)y$ is bounded, differentiable and its derivative $R'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 1. *([23,30]) Assume that (T_1) and (T_2) hold. Then there exists a unique resolvent operator for the Cauchy problem (2).*

Definition 2 ([16,41,42]). *Let $u\colon \mathbb{N} \to E$ be a bounded sequence. u is called almost automorphic sequence, if for each real sequence $\{j_i'\}$, there exists a subsequence $\{j_i'\} \subset \{j_i\}$ such that*

$$\widehat{u}(j) = \lim_{n\to\infty} u(j + j_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} u(j + j_n) = \widehat{u}(j),$$

for all $j \in \mathbb{N}$. Represent this class of all sequences as $AA(\mathbb{N}, E)$.

Definition 3. *[1] A bounded piecewise continuous function $f \in P\mathfrak{C}(\mathbb{R}^+, E)$ is said to be almost automorphic if*

(A_1): sequence of impulsive moments $\{t_j\}$ is a almost automorphic sequence,
(A_2): for every sequence of real numbers $\{\sigma_n\}$, there exists a subsequence $\{\sigma_{n_j}\}$ such that
$\mathbb{G}(t) = \lim_{n\to\infty} f(t + \sigma_{n_j})$
is well defined for each $t \in \mathbb{R}$ and
$\lim_{n\to\infty} \mathbb{G}(t - \sigma_{n_j}) = f(t)$
for each $t \in \mathbb{R}$.

Denote by $AA_{P\mathfrak{C}}(\mathbb{R}, E)$ the set of all such functions.

Lemma 1. *[41] $AA_{P\mathfrak{C}}(\mathbb{R}, E)$ is a Banach space with the norm*

$$\|f\|_{P\mathfrak{C}} = \sup_{t\in\mathbb{R}} |f(t)|.$$

Definition 4. *[1,41] A bounded piecewise continuous function $f \in P\mathfrak{C}(\mathbb{R}^+ \times \widetilde{E}, E)$ is called almost automorphic if*

(A_1): sequence of impulsive moments $\{t_j\}$ is a almost automorphic sequence
(A_2): for every sequence of real numbers $\{\sigma_n\}$, there exists a subsequence $\{\sigma_{n_j}\}$ such that

$$\lim_{n\to\infty} f(t + \sigma_{n_j}, y) = g(t, y)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - \sigma_{n_j}, y) = f(t, y)$$

for each $t \in \mathbb{R}$ *and each* $y \in E$.

Denote by $AA_{PC}(\mathbb{R} \times E, E)$ *the set of all such functions.*

The following definition, which is the Bi-almost automorphicity, is a crucial ingredient in our approach.

Definition 5 ([43]). *A bounded piecewise continuous function* $f \in P\mathfrak{C}(\mathbb{R}^+ \times \mathbb{R}^+ \times \widetilde{E}, E))$ *is Bi-almost automorphic if*

(A_1): *sequence of impulsive moments* $\{t_j\}$ *is an almost automorphic sequence*

(A_2): *for every sequence of real numbers* $\{\sigma_n\}$, *there exists a subsequence* $\{\sigma_{n_j}\}$ *such that*

$$\lim_{n\to\infty} f(t+\sigma_{n_j}, s+\sigma_{n_j}, y) = \mathcal{G}(t,s,y)$$

is well defined for each $t \in \mathbb{R}$ *and*

$$\lim_{n\to\infty} \mathcal{G}(t-\sigma_{n_j}, s-\sigma_{n_j}, y) = f(t,s,y)$$

for each $t \in \mathbb{R}$ *and each* $y \in E$.

Definition 6. *[41] A piecewise continuous function* $f \in P\mathfrak{C}(\mathbb{R}^+, E)$ *is said to be asymptotically almost automorphic if it can be decomposed as*

$f(t) = \mathbb{G}(t) + \Upsilon(t)$,

where

$\mathbb{G}(t,y) \in AA_{P\mathfrak{C}}(\mathbb{R}^+, E), \quad \Upsilon(t,y) \in P\mathfrak{C}_0(\mathbb{R}^+, E)$.

The space of these kinds of functions is denoted by $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

Definition 7. *[41] A piecewise continuous function* $f \in P\mathfrak{C}(\mathbb{R}^+ \times \widetilde{E}, E))$ *is said to be asymptotically almost automorphic if it can be decomposed as*

$$f(t,y) = \mathbb{G}(t,y) + \Upsilon(t,y),$$

where

$$\mathbb{G}(t,y) \in AA_{P\mathfrak{C}}(\mathbb{R}^+ \times \widetilde{E}, E), \quad \Upsilon(t,y) \in P\mathfrak{C}_0(\mathbb{R}^+ \times \widetilde{E}, E).$$

This class of functions is denoted by $AAA_{P\mathfrak{C}}(\mathbb{R}^+ \times \widetilde{E}, E)$.

We state a lemma inspired by the paper of J. Cao et al. [19] about the composition result.

Lemma 2. *[20] Let* $y, \nu \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$, $K = \overline{\{\nu(t) : t \in \mathbb{R}^+\}} \times \overline{\{y(t) : t \in \mathbb{R}^+\}}$ *and*

$$f \in AAA_{P\mathfrak{C}}(\mathbb{R}^+ \times E \times E, E) \cap C_K(\mathbb{R}^+ \times E \times E, E),$$

then $f(\cdot, y(\cdot), \nu(\cdot)) \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

The proof of the above lemma is similar to the proof of Lemma 2.5 of [19]. Now, we introduce the Kuratowski measure of noncompactness χ defined by

$$\chi(\Theta) = \inf\{\ \Delta > 0 : \Theta \text{ has a finite cover by sets of diameter} \leq \Delta\},$$

for a bounded set Θ in any space E. Some basic properties of $\chi(\cdot)$ are given in the following lemma. For more details, please see [8].

Lemma 3. *([8]) Let E be a Banach space and $\Theta_1, \Theta_2 \subset E$ be bounded, and the following properties are satisfied:*

- (j_1) *Θ is pre-compact if and only if $\chi(\Theta) = 0$,*
- (j_2) *$\chi(\Theta) = \chi(\overline{\Theta}) = \chi(Conv\Theta)$, where $\overline{\Theta}$ and $conv\Theta$ are the closure and the convex hull of Θ, respectively,*
- (j_3) *$\chi(\Theta_1) \leq \chi(\Theta_2)$ when $\Theta_1 \subset \Theta_2$,*
- (j_4) *$\chi(\Theta_1 + \Theta_2) \leq \chi(\Theta_1) + \chi(\Theta_2)$,*
- (j_5) *$\chi(k\Theta) = |k|\chi(\Theta)$ for any $k \in \mathbb{R}$,*
- (j_6) *$\chi(\Theta_2 + \Theta_1) \leq \chi(\Theta_2) + \chi(\Theta_2)$ where $\Theta_2 + \Theta_1 = \{y + \nu : y \in \Theta, \nu \in \Theta_2\}$,*
- (j_7) *$\chi(\Theta_2 \cup \Theta_1) \leq \max(\chi(\Theta_2), \chi(\Theta_2))$,*
- (j_8) *if $\Gamma : E \to E$ is a Lipschitz continuous map with constant k, then $\chi(\Gamma(\Theta)) \leq k\chi(\Theta)$ for all bounded subset Θ of E.*

Lemma 4. *([21]) Let E be a Banach space, $\Theta \subset E$ be bounded. Then there exists a countable set $\Theta_0 \subset \Theta$, such that*

$$\chi(\Theta) \leq 2\chi(\Theta_0).$$

Lemma 5. *([31]) Let V be a Banach space, and let $\Theta = \{y_n\} \subset \mathfrak{C}([c, d], E)$ be a bounded and countable set for constants $-\infty < c < d < +\infty$. Then $\Psi(v(t))$ is Lebesgue integral on $[c, d]$, and*

$$\chi\Big(\Big\{\int_c^d y_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\int_c^d \chi(\Theta(t))dt.$$

Now, we recall a useful compactness criterion.

Lemma 6. *[22][Corduneanu]*
A set $C \subset P\mathfrak{C}^b(\mathbb{R}^+, E)$ is relatively compact if the following conditions hold

- ***(i):*** *C is bounded in $P\mathfrak{C}^b(\mathbb{R}^+, E)$,*
- ***(ii):*** *C is a locally equicontinuous family of function, i.e., for any constant $d > 0$, the functions in C are equicontinuous in $[0, d]$,*
- ***(iii):*** *the set $C(t) := \{y(t) : y \in C\}$ is relatively compact on any compact interval of $\mathbb{R}^+$,*
- ***(iv):*** *the functions from C are equiconvergent, i.e For each $\varepsilon > 0$,there exists $d(\varepsilon) > 0$ such that $|y(t) - y(+\infty)| < \varepsilon$ for all $t \geq d(\varepsilon)$ and for all $y \in C$.*

Finally, we will make use of Mönch's fixed point theorem

Theorem 2. *(Mönch fixed point) [39]. Suppose that Ω is a closed convex subset of X; $0 \in \Omega$. If the map $N : \Omega \to X$ is continuous and of Mönch type, namely, Q satisfies the following property*

$$\Theta \subset \Omega, \Theta \ \text{ is countable}, \quad \Theta \subset \overline{Conv}\,(N(\Theta) \cup \{0\}) \Longrightarrow \overline{\Theta} \ \text{ is compact},$$

then, N has a fixed point in Ω.

3. The Main Results

Before starting our main results, we recall the definition of the mild solution of (1).

Definition 8. *A function $y \in P\mathfrak{C}^b(\mathbb{R}^+, E)$ is called a mild solution to the problem (1) if y satisfies the integral equation*

$$y(t) = S(t)y_0 + \sum_{0<t_j<t} S(t-t_j)J_j(y(t_j)) +_0^t S(t-s)f\left(s, y(s), My(s)\right) ds, \ t \in \mathbb{R}^+. \quad (3)$$

The following assumptions are needed to establish our results.

($\mathbb{H}_1$)**:** The resolvent operator given by Theorem 1 satisfies the following condition:

$$\|S(t-s)\|_{L(E)} \leq \eta e^{-\lambda(t-s)} \text{ where } \eta > 0 \text{ and } \lambda > 0.$$

($\mathbb{H}_2$)**:** The function $f : \mathbb{R}^+ \times E \times E \to E$ satisfies:

(i)**:** For a.e. $t \in \mathbb{R}^+$, the function $f(t, \cdot, \cdot) : \mathbb{R}^+ \times E \times E \to E$ is continuous, and for each $(y, \nu) \in E \times E$, the function $f(\cdot, y, \nu) : \mathbb{R}^+ \times E \times E$ is strongly measurable.

(i)**:** The function $f(t, y, \nu)$ asymptotically almost automorphic i.e., $f(t, y, \nu) = \mathbb{G}(t, y, \nu) + \Upsilon(t, y, \nu)$ with

$$\mathbb{G}(t, y, \nu) \in AA_{P\mathfrak{C}}(\mathbb{R} \times E \times E, E), \quad \Upsilon(t, y, \nu) \in \mathfrak{C}_0(\mathbb{R}^+ \times E \times E, E),$$

and $\mathbb{G}(t, y, \nu)$ is uniformly continuous on any bounded subset $K \subset E \times E$ uniformly for $t \in \mathbb{R}$.

(ii)**:** There exists a function $\hbar \in L^{\frac{1}{p_1}}(\mathbb{R}^+, \mathbb{R}^+)$, for a constant $p_1 \in (0, 1)$ such that:

$$|f(t, y, \nu)| \leq \hbar(t)(|y| + |\nu|) \text{ for a.e } t \in \mathbb{R}^+ \text{ and each } y, \nu \in E.$$

(iii)**:** There exists a function $\rho \in L^{\frac{1}{p_2}}(\mathbb{R}^+, \mathbb{R}^+)$, for a constant $p_2 \in (0, 1)$ such that:

$$\chi(f(t, \Omega_1, \Omega_2)) \leq \rho(t)\left(\chi(\Omega_1) + \chi(\Omega_2)\right), t \in \mathbb{R}^+,$$

for any bounded countable subsets $\Omega_1, \Omega_2 \subset P\mathfrak{C}^b(\mathbb{R}^+, E)$.

($\mathbb{H}_3$): The function $H : D \times E \to E$ have the decomposition $H = H^a + H_0^\rho$ in which H^a is Bi-almost automorphic functions which satisfies Bi-almost automorphic in (t,s) uniformly on bounded subsets of E and is ϱ-bounded. Moreover,

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \varrho(t,s)\mathrm{d}s = \varrho^* < +\infty.$$

Also, the Bi-almost automorphic functions H^a is $(\phi, \widehat{\phi})$-Lipschitz (see [20]), with

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \phi(t,s)\mathrm{d}s = \phi^* < +\infty;$$

and for every compact interval $[a,b] \subset \mathbb{R}$, the following limit holds

$$\lim_{t\to+\infty} \int_{a}^{b} \phi(t,s)\mathrm{d}s = 0,$$

we also assume that there exists a function $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ such that $|H(t,s,0)| \le \pi(t,s)$,

$$\lim_{t\to+\infty} \int_{-\infty}^{0} \pi(t,s)\mathrm{d}s = 0$$

and $H_0^\theta \in \mathfrak{C}_0^\theta(D \times E, E)$, with

$$\int_{0}^{d} \theta(t,s)\mathrm{d}s = 0, \text{ for a.e } d > 0.$$

and

$$\sup_{t\in\mathbb{R}^+} \int_{0}^{t} \theta(t,s)\mathrm{d}s = q < +\infty$$

(i): There exists a positive function $\upsilon(t,s) \in L^1(D, \mathbb{R}^+)$ such that:

$$|H(t,s,y)| \le \upsilon(t,s)(1+|y|), \text{ for a.e } t \in \mathbb{R}^+ \text{ and each } y \in E.$$

(ii): There exists a positive function $\vartheta(t,s) \in L^1(D, \mathbb{R}^+)$ such that for any bounded countable $\Omega \subset P\mathfrak{C}^b(\mathbb{R}^+, E)$

$$\chi(H(t,s,\Omega) \le \vartheta(t,s)\chi(\Omega), t \in \mathbb{R}^+.$$

($\mathbb{H}_4$): The impulse functions $J_j : E \to E$ for $j = 1,2,3\ldots,$ is a sequence of almost asymptotically automorphic function and satisfies:

(i): There exist positive constant numbers σ_j and ς_j such that

$$|J_j(y)| \le \sigma_j\, |y| + \varsigma_j, \text{ for a.e } t \in \mathbb{R}^+ \text{ and each } y \in E.$$

(ii): There exist $\theta_j > 0\,;\, j = 1,2,\ldots.$ such that for any bounded countable $\Omega \subset P\mathfrak{C}^b(\mathbb{R}^+, E)$

$$\chi(J_j(\Omega)) \le \theta_j \chi(\Omega).$$

In the proofs of our results, we need the following auxiliary result.

Lemma 7. *[20] Let $f = \mathbb{G}+\Upsilon \in AAA_{P\mathfrak{C}}(\mathbb{R}^+\times E\times E, E)$ with $\mathbb{G} \in AA_{P\mathfrak{C}}(\mathbb{R}, E), \Upsilon \in P\mathfrak{C}_0(\mathbb{R}^+, E)$. Then*

$$E_1(t) := \int_0^t S(t-s)f(s)ds \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E).$$

Lemma 8. *[20] Suppose the functions $H : \mathbb{R}\times\mathbb{R}\times E \to E$ satisfies condition $(\mathbb{H}_3)$. Then, the integral operators E_2 such that*

$$E_2 y(t) = \int_0^t H(t, s, y(s))\mathrm{d}s,\ t \in \mathbb{R}^+,$$

maps $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$ into $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

Theorem 3. *Assume that the hypotheses $(\mathbb{H}_1) - (\mathbb{H}_4)$ are satisfied. Then the problem (1) has an asymptotically almost automorphic mild solution if*

$$\eta \max\left(\frac{\varsigma_j}{1-e^{-\lambda\varpi}} + \frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}, \frac{\theta_j}{1-e^{-\lambda\varpi}} + 4(1+2\omega^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) \le 1. \tag{4}$$

Proof. Let $\overline{\mho}_\kappa = \left\{y \in P\mathfrak{C}^b(\mathbb{R}^+, E) \cap AAA(\mathbb{R}^+, E) : \|y\| \le \kappa\right\}$. Define an operator Q on $\overline{\mho}_\kappa$ by

$$(Qy)(t) = S(t)y_0 + \sum_{0<t_j<t} S(t-t_j)J_j(y(t_j)) +_0^t S(t-s)f\left(s, y(s), My(s)\right)ds,\ t \in \mathbb{R}^+. \tag{5}$$

We next show that Q has a fixed point in $\overline{\mho_\kappa}$. We divide the proof into several steps.

Step 1. For every $y \in \overline{\mho}_\kappa,\ Qy \in P\mathfrak{C}^b(\mathbb{R}^+, E)$.

For $t \in \mathbb{R}^+$, from the hypotheses $(\mathbb{H}_1)$-$(\mathbb{H}_4)$, we get

$$\begin{aligned}
|(Qy)(t)| \le{}& \|S(t)\|_{L(E)} |y_0| + \sum_{0<t_j<t} \|S(t-t_j)\|_{L(E)} |J_j(y(t_j))| \\
&+ \int_0^t \|S(t-s)\|_{L(E)} \hbar(s) \left(|y(s)| +_0^s \upsilon(s,\tau)(1+|y(\tau)|)d\tau\right) ds \\
&\le \eta |y_0| + \eta \sum_{0<t_j<t} \sigma_j |y(t_j)| + \varsigma_j \\
&+ \eta \int_0^t e^{-\lambda(t-s)}\hbar(s) \left(|y(s)| +_0^s \upsilon(s,\tau)(1+|y(\tau)|)d\tau\right) ds \\
&\le \eta|y_0| + \eta \sum_{0<t_j<t} e^{-\lambda(t-t_j)}(\sigma_j |y(t_j)| + \varsigma_j) \\
&+ \eta \int_0^t e^{-\lambda(t-s)}\hbar(s) \left(\sup_{s\in\mathbb{R}^+} |y(s)| + \upsilon^*(1 + \sup_{s\in\mathbb{R}^+} |y(s)|)\right) ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \eta|y_0| + \eta \sum_{0<t_j<t} e^{-\lambda(t-t_j)}(\sigma_j\,|y(t_j)| + \varsigma_j) \\
&+ \eta \int_0^t e^{-\lambda(t-s)}\hbar(s)\left((1+\upsilon^*)(1+\sup_{s\in\mathbb{R}^+}|y(s)|)\right)ds \\
&\leq \eta|y_0| + \eta(\sigma_j\|y\|_{P\mathfrak{C}^b} + \varsigma_j)\sum_{0<t_j<t} e^{-\lambda(t-t_j)} \\
&+ \eta(1+\upsilon^*)\int_0^t e^{-\lambda(t-s)}\hbar(s)ds\|y\|_{P\mathfrak{C}^b} \\
&\leq \eta|y_0| + \frac{\eta(\sigma_j\|y\|_{P\mathfrak{C}^b} + \varsigma_j)}{1-e^{-\lambda\varpi}} \\
&+ \eta(1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\left(\int_0^t e^{-\frac{\lambda}{1-p_1}(t-s)}ds\right)^{1-p_1}(1+\|y\|_{P\mathfrak{C}^b}) \\
&\leq \eta|y_0| + \frac{\eta(\sigma_j\|y\|_{P\mathfrak{C}^b} + \varsigma_j)}{1-e^{-\lambda\varpi}} \\
&+ \eta(1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\left(1-e^{-\frac{\lambda t}{1-p_1}}\right)(1+\|y\|_{P\mathfrak{C}^b}) \\
&\leq \eta|y_0| + \frac{\eta\varsigma_j}{1-e^{-\lambda\varpi}} + \eta\left(\frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\right)(1+\|y\|_{P\mathfrak{C}^b}),
\end{aligned}
$$

which implies that $Qy \in P\mathfrak{C}^b(\mathbb{R}^+, E)$.

Step 2. For every $y \in \overline{\mho}_\kappa$, $Qy \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.
Claim 1. Proving that $(Py)(t)$ belongs to $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$,
where

$$(Py)(t) = S(t)y_0 +_0^t S(t-s)f\left(s, y(s), My(s)\right)ds,\ t \in \mathbb{R}^+.$$

Let

$$E(t) = S(t)y_0,$$

then

$$|E(t)| = |S(t)y_0| \leq |S(t)y_0| \leq \eta e^{-\lambda t}|y_0|.$$

Since $\lambda > 0$, we get $\lim_{t\to+\infty} |(E(t)| = 0$. That is

$$E \in P\mathfrak{C}_0(\mathbb{R}^+, E). \tag{6}$$

Applying Lemma 8 and Lemma 2, we infer that $My(t)$ and $f(\cdot, y(\cdot), My(t)(\cdot))$ belong to $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$. By Lemma 7 and 6, we obtain that P is $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$-valued.

Claim 2. Proving that $\sum_{0<t_j<t} S(t-t_j)J_j(y(t_j))$ belongs to $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

From the assumption $(\mathbb{H}_4)$, $J_j(y(t_j)) \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$. By definition, it can be expressed as

$$J_j(y(t_j)) = J_{j1}(y(t_j)) + J_{j2}(y(t_j))$$

such that $J_{j1}(y(t_j)) \in AAA(\mathbb{R}^+, E)$,$J_{j2}(y(t_j)) \in P\mathfrak{C}_0(\mathbb{R}^+, E)$ Then:

$$\begin{aligned}\sum_{0<t_j<t} S(t-t_j)J_j(y(t_j)) &= \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j)) + \sum_{0<t_j<t} S(t-t_j)J_{j2}(y(t_j)) \\ &= \wp^a(t) + \wp^0(t).\end{aligned}$$

Since $J_{j1} \in AA(\mathbb{R}^+, E)$, for every real sequence $\{t_j\}$, there exists a subsequence $\{t_{j_n}\}$ such that

$$\lim_{n\to\infty} J_{j1}(y(t_j + t_{j_n})) = \mathbb{J}_{j1}(y(t_j))$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \mathbb{J}_{j1}(y(t_j - t_{j_n})) = J_{j1}(y(t_j)),$$

Now, we have

$$\wp^a(t+t_{j_n}) = \sum_{0<t_j<t+t_{j_n}} S(t+t_{j_n}-t_j)J_{j1}(y(t_j)) = \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j+t_{j_n})),$$

then

$$\lim_{n\to\infty} \wp^a(t+t_{j_n}) = \lim_{n\to\infty} \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j+t_{j_n})) = \sum_{0<t_j<t} S(t-t_j)\mathbb{J}_{j1}(y(t_j)) = \overline{\wp}^a(t),$$

Similarly

$$\overline{\wp}^a(t-t_{j_n}) = \sum_{0<t_j<t-t_{j_n}} S(t-t_{j_n}-t_j)\mathbb{J}_{j1}(y(t_j))) = \sum_{0<t_j<t} S(t-t_j)\mathbb{J}_{j1}(y(t_j-t_{j_n}))),$$

then

$$\lim_{n\to\infty} \overline{\wp}^a(t-t_{j_n}) = \lim_{n\to\infty} \sum_{0<t_j<t} S(t-t_j)\mathbb{J}_{j1}(y(t_j-t_{j_n}))) = \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j)),$$

then,

$$\wp^a(t) = \sum_{0<t_j<t} S(t-t_j)J_{j1}(y(t_j))$$

belongs to $AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.
Next, we show that $\wp^0(t) \in \mathfrak{C}_0(\mathbb{R}^+, E)$. Since $J_{j2} \in PC_0(\mathbb{R}^+, E)$, one can choose a $T > 0$ such that

$$|J_{j2}| \leq \varepsilon.$$

This enables us to conclude that for all $t > T$,

$$\begin{aligned}\wp^0(t) = \left|\sum_{0<t_j<t} S(t-t_j)J_{j2}(y(t_j))\right| &\leq \sum_{0<t_j<t} \|S(t-t_j)\|_{L(E)} |J_{j2}(y(t_j))| \\ &\leq \eta \sum_{0<t_j<t} e^{-\lambda(t-t_j)} |J_{j2}(y(t_j))|\end{aligned}$$

$$\begin{aligned} &\leq \eta\,|J_{j2}| \sum_{0<t_j<t} e^{-\lambda(t-t_j)} \\ &\leq \frac{\eta\,|J_{j2}|}{1-e^{-\lambda\varpi}} \\ &\leq \varepsilon \end{aligned}$$

So, $\wp^0(t) \in P\mathfrak{C}_0(\mathbb{R}^+, E)$. Finally by (6), we prove our claim that $Qy \in AAA_{P\mathfrak{C}}(\mathbb{R}^+, E)$.

Step 3. We prove that $Q\left(\overline{\mho}_\kappa\right) \subset \overline{\mho}_\kappa$.
If this condition fails, then for every positive constant $\kappa > 0$ and $t \geq 0$, there exists a function $\widehat{y} \in \overline{\mho}_\kappa$ but $Q(\widehat{y}) \notin \overline{\mho}_\kappa$, i.e $|(Q\widehat{y})(t)| > \kappa$. Thus, by the Hölder inequality, the conditions $(\mathbb{H}_1)-(\mathbb{H}_4)$, based on the above estimations, we can easily demonstrate that

$$|(Q\widehat{y})(t)| \leq \eta|y_0| + \frac{\eta\varsigma_j}{1-e^{-\lambda\varpi}} + \eta\left(\frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\right)(1+\kappa).$$

Thus,

$$\kappa < \eta|y_0| + \frac{\eta\varsigma_j}{1-e^{-\lambda\varpi}} + \eta\left(\frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\right)(1+\kappa).$$

Dividing on both sides by κ and taking the lower limit as $\kappa \to +\infty$, we can obtain that

$$1 < \frac{\eta\varsigma_j}{1-e^{-\lambda\varpi}} + \eta\left(\frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\right).$$

This contradicts (4). Hence, for some positive number κ, we must have $Q(\overline{\mho}_\kappa) \subset \overline{\mho}_\kappa$.
Step 4. We show that Q is continuous $\overline{\mho}_\kappa$.
To demonstrate the continuity of Q, we assume that there exists a sequence y_n, y in $\overline{\mho}_\kappa$ and $y_n \to y$ as $n \to +\infty$.
Case 1. If $t \in [0, d]$, $d > 0$, and $y_n \in \overline{\mho}_\kappa$, we have

$$\begin{aligned} &|(Qy_n)(t) - (Qy)(t)| \\ &\leq_{0<t_j<t} S(t-t_j)\,|J_j(y(t_j)) - J_j(y_n(t_j))| \\ &+\eta \int_0^t \left| f\left(s, y_n(s), \int_0^s H(s,\tau,y_n(\tau))d\tau\right) - f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right)\right| ds. \end{aligned}$$

By the Lebesgue dominated convergence theorem accompanying with $(\mathbb{H}_2)(i)$, we get

$$\|Qy_n - Qy\| \to 0 \qquad \text{as} \qquad n \to +\infty.$$

Case 2. If $t \in (d, +\infty)$, $d > 0$, by $(\mathbb{H}_2)(i)$, we can see that

$$|J_j(y_n(t_j)) - J_j(y(t_j))| \leq \frac{(1-e^{-\lambda\varpi})\varepsilon_j}{2} \quad \text{for } t \geq d. \tag{7}$$

and

$$\left|f\left(t, y_n(t), \int_0^t H(t,s,y_n(s))ds\right) - f\left(t, y(t), \int_0^t H(t,s,y(s))ds\right)\right| \leq \frac{\lambda\varepsilon}{2\eta} \quad \text{for } t \geq d. \quad (8)$$

Hence, according to the dominated convergence theorem and (8), we obtain that for every $t \geq 0$,

$$\begin{aligned} &|(Qy_n)(t) - (Qy)(t)| \\ &\leq \sum_{0<t_j<t} S(t-t_j)\,|J_j(y_n(t_j)) - J_j(y(t_j))| \\ &+_0^t \|S(t-s)\|_{L(E)} \left|f\left(s, y_n(s), \int_0^s H(s,\tau,y(\tau))d\tau\right) - f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right)\right| ds \\ &\leq \frac{1-e^{-\lambda\varpi}}{2} \sum_{0<t_j<t} \varepsilon_j e^{-\lambda(t-t_j)} + \frac{\lambda\varepsilon}{2\eta}\int_0^t e^{-\lambda(t-s)}ds \\ &\leq \frac{\varepsilon}{2} + \frac{\eta}{\lambda}\frac{\lambda\varepsilon}{2\eta}(1-e^{-\lambda t}) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \quad (9)$$

Then the inequality (9) reduces to

$$\|Q(y_n) - Q(y)\|_{P\mathfrak{C}^b} \to 0 \quad \text{as } n \to \infty.$$

This implies that Q is continuous in $\overline{\mho}_\kappa$.

Next, we demonstrate that the operator Q is equi-continuous on every compact interval $[0,d]$ of $[0,+\infty)$, for $d > 0$ and is equi-convergent in $y \in \overline{\mho}_\kappa$.

Step 5. $Q(\overline{\mho}_\kappa)$ is equicontinuous.

Let $0 < d < +\infty$ be an arbitrary constant. Generally, let $0 \leq \tau_1 \leq \tau_2 \leq d$, for any $y \in \overline{\mho}_\kappa$, we know that

$$\begin{aligned} &|(Qy)(\tau_2) - (Qy)(\tau_1)| \\ &= \Bigg| S(\tau_2)y_0 + \sum_{0<t_j<\tau_2} S(\tau_2 - t_j)J_j(y(t_j)) \\ &+ {}_0^{\tau_2} S(s-\tau_2) f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right) ds \\ &- S(\tau_1)y_0 + \sum_{0<t_j<\tau_1} S(\tau_1 - t_j)J_j(y(t_j)) \\ &+ \int_0^{\tau_1} S(t-s) f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right) ds \Bigg| \\ &\leq |S(\tau_2)y_0 - S(\tau_1)y_0| \\ &+ \left|\sum_{0<t_j<\tau_2} S(\tau_1 - t_j)J_j(y(t_j)) - \sum_{0<t_j<\tau_1} S(\tau_2 - t_j)J_j(y(t_j))\right| \end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^{\tau_1}(S(\tau_2,s)-S(\tau_1,s))f\left(s,y(s),\int_0^s H(s,\tau,y(\tau))d\tau\right)ds\right.\\
&\left.+\int_{\tau_1}^{\tau_2} S(\tau_2,\tau)f\left(s,y(s),\int_0^s H(s,\tau,y(\tau))d\tau\right)ds\right|\\
&\le |S(\tau_2)y_0-S(\tau_1)y_0|\\
&+\sum_{0<t_j<\tau_1}\|S(\tau_1-t_j)-S(\tau_2-t_j)\|_{L(E)}\,|J_j(y(t_j))|\\
&+\sum_{\tau_1<t_j<\tau_2}\|S(\tau_1-t_j)\|_{L(E)}\,|J_j(y(t_j))|\\
&+_0^{\tau_1}\|S(\tau_2,\tau)-S(\tau_1,\tau)\|_{B(V)}\,\hbar(s)\left(|y(s)|+_0^s\upsilon(s,\tau)(1+y(\tau)|)d\tau\right)ds\\
&+\eta_{\tau_1}^{\tau_2}e^{-\lambda(t-s)}\hbar(s)\left(|y(s)|+_0^s\upsilon(s,\tau)(1+y(\tau)|)d\tau\right)ds.
\end{aligned}
$$

It follows from the Hölder's inequality that

$$
\begin{aligned}
|(Qy)(\tau_2)-(Qy)(\tau_1)| \le{}& \|S(\tau_2)-S(\tau_1)\|_{L(E)}\,|y_0|\\
&+(\sigma_j\varrho+\varsigma_j)\sum_{0<t_j<\tau_1}\|\mathrm{I}-S(\tau_2-\tau_1)\|_{L(E)}\\
&+\eta(\sigma_j\varrho+\varsigma_j)\sum_{\tau_1<t_j<\tau_2}e^{-\lambda(t-t_j)}\\
&+(1+\upsilon^*)\varrho_0^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|_{B(V)}\,\hbar(s)ds\\
&+\eta\|\hbar\|_{L^{\frac{1}{p_1}}}(1+\upsilon^*)\varrho\|\hbar\|_{L^{\frac{1}{p_1}}}\left(\int_0^t e^{-\frac{\lambda}{1-p_1}(t-s)}ds\right)^{1-p_1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|(Qy)(\tau_2)-(Qy)(\tau_1)| \le{}& \|S(\tau_2)-S(\tau_1\|_{L(E)}\,|v_0|\\
&+(\sigma_j\varrho+\varsigma_j)\sum_{0<t_j<\tau_1}\|S(\tau_1-t_j)-S(\tau_2-t_j)\|_{L(E)}\\
&+\frac{\eta(\sigma_j\varrho+\varsigma_j)(\tau_2-\tau_1)}{\varpi}\\
&+(1+\upsilon^*)\varrho_0^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|_{B(V)}\,\hbar(s)ds\\
&+\frac{\eta\|\hbar\|_{L^{\frac{1}{p_1}}}(1+\upsilon^*)\varrho(1-p_1)^{1-p_1}}{\lambda^{1-p_1}}\left(e^{-\frac{\lambda}{1-p_1}(t-\tau_2)}-e^{-\frac{\lambda}{1-p}(t-\tau_1)}\right)^{1-p_1}.
\end{aligned}
$$

The right-hand side tends to zero as $\tau_2\to\tau_1$. This proves the equicontinuity of $Q(\overline{\mho}_\kappa)$.

Step 6. $\overline{\mho}_\kappa(t)=\left\{(Qy)(t): y\in\overline{\mho}_\kappa\right\}$ is a relatively compact subset of E in each $t\in\mathbb{R}^+$.

Let H be a subset of $\overline{\mho}_\kappa$ such that $H\in\overline{conv}(Q(M)\cup\{0\})$. In addition, by Lemma 4, we know that there is a countable set $\{y\}_{n=0}^{n=+\infty}\subset\Theta$ such that $\chi(Q(\Theta))\le$

$2\chi(Q(\{y\}_{n=0}^{n=+\infty}))$ for any bounded set Θ. Thus for $\{y_n\}_{n=0}^{+\infty} \subset H$, for the appropriate choice of H, for every $t \in [0, d]$, by utilizing Lemma 5 and the properties of the measure χ, we obtain

$$
\begin{aligned}
& \chi(Q(H(t))) \\
\leq\ & 2\chi(Q(\{y_n(t)\}_{n=0}^{\infty})) \\
\leq\ & 2\chi\left(\left\{S(t)y_0 + \sum_{0<t_j<t} S(t-t_j)J_j(y_n(t_j))\right.\right. \\
+\ & \left.\left. {}_0^t S(t-s) f\left(s, y_n(s), {}_0^s H(s,\tau,y_n(\tau))d\tau\right) ds\right\}_{n=0}^{\infty}\right) \\
\leq\ & \sum_{0<t_j<t} S(t-t_j)\chi(J_j(y_n(t_j))) \\
+\ & 2\chi\left(\left\{{}_0^t S(t-s) f\left(s, y_n(s), {}_0^s H(s,\tau,y_n(\tau)d\tau\right) ds\right\}_{n=0}^{\infty}\right) \\
\leq\ & \sum_{0<t_j<t} S(t-t_j)\theta_j\chi((y_n(t_j))) \\
+\ & 2\chi\left({}_0^t S(t-s) f\left(s, \{y_n(s)\}_{n=0}^{\infty}, {}_0^s H(s,\tau,\{y_n(\tau)\}_{n=0}^{\infty})d\tau\right) ds\right) \\
\leq\ & \sum_{0<t_j<t} S(t-t_j)\theta_j \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+\ & 4\eta {}_0^t e^{-\lambda(t-s)}\rho(t)\left(\sup_{s\in[0,t]}\chi(\{y_n(s)\}_{n=0}^{\infty}) + 2 {}_0^s \vartheta(s,\tau) \sup_{\tau\in[0,s]}\chi(\{y_n(\tau)\}_{n=0}^{\infty}))d\tau\right) ds \\
\leq\ & \sum_{0<t_j<t} S(t-t_j)\theta_j \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+\ & 4\eta {}_0^t e^{-\lambda(t-s)}\rho(t)\left(\sup_{s\in[0,t]}\chi(\{y_n(s)\}_{n=0}^{\infty}) + 2\sup_{\tau\in[0,s]}\chi(\{y_n(s)\}_{n=0}^{\infty}) {}_0^s \vartheta(s,\tau)d\tau\right) ds \\
\leq\ & \eta \sup_{\tau\in[0,s]}\theta_j \sum_{0<t_j<t} e^{-\lambda(t-t_j)} \sup_{\tau\in[0,s]}\chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+\ & 4\eta(1+2\vartheta^*) {}_0^t e^{-\lambda(t-s)}\rho(s) \sup_{s\in[0,t]}\chi(\{y_n(s)\}_{n=0}^{\infty}) ds \\
\leq\ & \sum_{0<t_j<t} S(t-t_j)\theta_j \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+\ & 4\eta(1+2\vartheta^*) {}_0^t e^{-\lambda(t-s)}\rho(s) ds \chi(\{y_n\}_{n=0}^{\infty}) \\
\leq\ & \frac{\eta\theta_j}{1-e^{-\lambda\varpi}} \sup_{\tau\in[0,s]}\chi(\{y_n(s)\}_{n=0}^{\infty}) \\
+\ & 4\eta(1+\vartheta^*)\|\rho\|_{L^{\frac{1}{p_2}}}\left(\int_0^t e^{-\frac{\lambda}{1-p_2}(t-s)} ds\right)^{1-p_2} \chi(\{y_n\}_{n=0}^{\infty})
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{\eta\theta_j}{1-e^{-\lambda\varpi}} \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
&+ 4\eta(1+2\vartheta^*)\|\rho\|_{L^{\frac{1}{p_2}}} \left(1-e^{-\frac{\lambda t}{1-p_2}}\right) \chi(\{y_n\}_{n=0}^{\infty}) \\
&\leq \frac{\eta\theta_j}{1-e^{-\lambda\varpi}} \sup_{\tau\in[0,s]} \chi(\{y_n(s)\}_{n=0}^{\infty}) \\
&+ 4\eta(1+2\vartheta^*)\|\rho\|_{L^{\frac{1}{p_2}}} \chi(\{y_n\}_{n=0}^{\infty}),
\end{aligned}$$

which ensures that

$$\chi((Q(H)(t)) \leq \left(\frac{\eta\theta_j}{1-e^{-\lambda\varpi}} + 4\eta(1+2\omega^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) \chi(H(t)).$$

Then

$$\chi(H) \leq \chi(Q(\Theta)(t)) \leq \left(\frac{\eta\theta_j}{1-e^{-\lambda\varpi}} + 4\eta(1+2\omega^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) \chi(H).$$

That is to say

$$\left(\frac{\eta\theta_j}{1-e^{-\lambda\varpi}} + 4\eta(1+2\omega^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right) \chi(H(t)) \leq 0.$$

From (10), we observe that $\chi(H) = 0$.

Step 7. $Q(\overline{\mho}_\kappa)$ is equiconvergent.

Let $y \in \overline{\mho}_\kappa$. For $t \in \mathbb{R}^+$, we have

$$\begin{aligned}
|(Qy)(t)| &\leq |S(t)y_0| + \sum_{0<t_j<t} \|S(t-t_j)\|_{L(E)} |J_j(y(t_j))| \\
&+ {}_0^t S(t-s) f\left(s, y(s), \int_0^s H(s,\tau,y(\tau))d\tau\right) ds \\
&\leq \eta\,|y_0|\, e^{-\lambda t} + \eta \sum_{0<t_j<t} e^{-\lambda(t-t_j})(\sigma_j\,|y(t_j)| + \varsigma_j) \\
&+ \eta \int_0^t e^{-\lambda(t-s)} \hbar(s)\,(|y(s)| +_0^s \upsilon(s,\tau)|y(\tau)|d\tau)\, ds \\
&\leq \eta|y_0|e^{-\lambda t} + \frac{\eta(\sigma_j(1+\|y\|_{P\mathfrak{C}^b}) + \varsigma_j)}{1-e^{-\lambda\varpi}} \\
&+ \eta(1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \left(\int_0^t e^{-\frac{\lambda}{1-p_1}(t-s)} ds\right)^{1-p_1} ds(1+\|y\|_{P\mathfrak{C}^b}) \\
&\leq \eta|y_0|e^{-\lambda t} + \frac{\eta(\sigma_j(1+\|y\|_{P\mathfrak{C}^b}) + \varsigma_j)}{1-e^{-\lambda\varpi}} \\
&+ \eta(1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}} \left(1-e^{-\frac{\lambda t}{1-p_1}}\right)(1+\|y\|_{P\mathfrak{C}^b})
\end{aligned}$$

$$\leq \quad \eta|y_0|e^{-\lambda t} + \frac{\eta\varsigma_j + \kappa\sigma_j}{1 - e^{-\lambda\varpi}} + (1 + \upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}\left(1 - e^{-\frac{\lambda t}{1-p_1}}\right)(1 + \kappa).$$

Then, we get

$$|(Qy)(t)| \to \frac{\eta\varsigma_j + \kappa\sigma_j}{1 - e^{-\lambda\varpi}} + (1 + \upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}(1 + \kappa) \quad as \quad t \to +\infty.$$

Hence, as $t \to +\infty$, we have $|(Qy)(t) - (Qy)(+\infty)| \to 0$.
Thus, from the above results $\overline{\mho_\kappa}$ is a relatively compact set. By Lemma 2, we know that Q has a fixed point in $\overline{\mho_\kappa}$. The proof is complete.

4. Example

To end this work, we apply our abstract results to the study of an integro-differential equation with impulsive effects. Consider the system

$$\begin{cases} \partial\partial t\psi(t,\xi) = \partial^2\partial\xi^2\psi(t,\xi) + \displaystyle\int_0^t f(t-s)\partial^2\partial s^2\psi(t,\xi)ds \\ +2^{-t}\sin\left(\dfrac{1}{2+\cos t+\cos\sqrt{2}t}\right)\left(e^{-\psi(t,\xi)} + \displaystyle\int_0^t a(t)e^{-(t-s)}(1+\psi(t,s))ds\right) \\ +2^{-t}\left(\psi(t,\xi) + \displaystyle\int_0^t a(t)e^{-(t-s)}(1+\psi(t,s))ds\right), t\in\mathbb{R}^+, t\neq t_j, j=1,2,3\ldots,\xi\in[0,1], \\ \Delta\psi(t_j,\xi) = (1-e^{-\lambda\varpi})\ln(1+2^{-j-2})\psi(t_j,\xi) + 2^{-j-2}(1-e^{-\lambda\varpi})\sin(\psi(t_j,\xi)), j=1,2,3\ldots, \\ \psi(t,0)=\psi(t,1)=0, \quad \psi(0,\xi)=\psi_0(\xi), \quad t\in\mathbb{R}^+, \quad \xi\in[0,1], \end{cases} \tag{10}$$

where $t_j = \sin\left(\dfrac{1}{2+\cos j+\cos\sqrt{2}j}\right)$ and the function $a \in AA_{P\mathfrak{C}}(\mathbb{R})$ such that $|a| \leq \dfrac{3-4(\ln 2)^2}{8(\ln 2)^2}$. Here $f : \mathbb{R} \to \mathbb{R}$ is bounded uniformly continuous and continuously differentiable. Set $E = L^2(0,1)$ and let A be be the Laplace operator

$$(A\psi)(\xi) = \partial^2\partial s^2\psi(\xi),$$

then $A : D(A) = H^2(0,1) \cap H_0^1(0,1) \to L^2(0,1)$. Note that, the operator A has eigenvalues $\{-n^2\pi^2\}_1^{+\infty}$ and generates a C_0-semigroup $(S(t))_{t\geq 0}$ on E such that

$$\|S(t)\|_{L(E)} \leq \eta e^{-\lambda t},$$

with $\eta = 1$, $\lambda = \pi^2$ for all $t \geq 0$.
We define the operator $B(t) : B : E \to E$ as follows:

$$B(t)\psi = f(t)A\psi \quad \text{for } t \geq 0 \quad \text{and } \psi \in D(A).$$

Furthermore we set

$$\psi(t)(\xi) = \psi(t,\xi) \ \text{ for } t \in \mathbb{R}^+ \text{ and } \xi \in [0,1].$$

$$\psi(0) = \psi(0,\xi) \ \text{ for } t \in \mathbb{R}^+ \text{ and } \xi \in [0,1].$$

Then the system (10) takes the following abstract form

$$\begin{cases} \psi'(t) = A\psi(t) + \displaystyle\int_0^t B(t-s)\psi(s)ds + f\left(t, \psi(t), \int_0^t H(t,s,\psi(s))ds\right), \ t \geq 0, \\ \psi(0) = \psi_0, \end{cases} \tag{11}$$

where the nonlinear function $f : \mathbb{R}^+ \times E \times E \to E$ given by

$$\begin{aligned} f\left(t, \psi(t), \int_0^t H(t,s,\psi(s))ds\right) &= 2^{-t} \sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right) \\ &\times \left(e^{-\psi} + \int_0^t a(t)e^{-(t-s)}(1+\psi(t,s))ds\right) \\ &+ 2^{-t}\left(\psi(t) + \int_0^t a(t)e^{-(t-s)}(1+\psi(t,s))ds\right). \end{aligned}$$

Let

$$\begin{aligned} \mathbb{G}\left(t, \psi(t), \varphi(t)\right) &= 2^{-t} \sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right)\left(\sin\psi(t) + \varphi(t)\right), \\ \Upsilon(t, \psi(t), \varphi(t)) &= 2^{-t}(\psi(t) + \varphi(t)), \end{aligned}$$

and

$$H(t,s,\psi(s)) = a(t)e^{-(t-s)}(1+\psi(t,s)).$$

Then it is easy to verify that $\mathbb{G}, \Upsilon : \mathbb{R} \times E \times E \to E$ are continuous and $\mathbb{G}(t, \psi(t), \varphi(t)) \in AA(\mathbb{R} \times E \times E \to E)$ and

$$|\Upsilon(t, \psi(t), \varphi(t))| \leq 2^{-t}(|\psi| + |\varphi|),$$

which implies $\Upsilon(t, \psi(t), \varphi(t)) \in C_0(\mathbb{R}^+ \times E \times E \to E)$ and

$$f(t, \psi(t), \varphi(t)) = \mathbb{G}(t, \psi(t), \varphi(t)) + \Upsilon(t, \omega(t), \vartheta(t)) \in AAA_{P\mathfrak{C}}(\mathbb{R}^+ \times E \times E, E).$$

Observe that

$$|f(t, \psi(t), \varphi(t))| \leq 2^{-t}(|\psi(t)| + |\varphi(t)|).$$

Moreover, for a bounded subset Ω_1, Ω_2 of E, and from properties of measure of noncompactness χ, we have

$$\chi(f(t, \Omega_1, \Omega_2)) \leq 2^{-t}\left(\chi(\Omega_1) + \chi(\Omega_2)\right).$$

Moreover, let $p_1 = p_2 = \dfrac{1}{2}$, then, the assumptions $(\mathbb{H}_2)$ hold with

$$\hbar(t) = \rho(t) = 2^{-t}.$$

Similarly, H clearly satisfies the following:

$$|H(t,s,\psi_2) - H(t,s,\psi_1)| \leq |a(t)|\, e^{-(t-s)}\, |\psi_2 - \psi_1|.$$

Now, by the property of measure of noncompactness for bounded subset Ω of E, we have

$$\chi(H(t,s,\Omega)) \leq |a(t)|\, e^{-(t-s)}\chi(\Omega).$$

In addition

$$\|a\|_{P\mathfrak{C}} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-(t-s)}\mathrm{d}s = \|a\|_{P\mathfrak{C}} < +\infty.$$

and for every compact interval $[c,d] \subset \mathbb{R}$, we have

$$\lim_{t\in+\infty} \int_{a}^{b} a(t)e^{-(t-s)}\mathrm{d}s = \lim_{t\in+\infty} \|a\|_{P\mathfrak{C}}(e^{-(t-d)} - e^{-(t-c)}) = 0,$$

and

$$H(t,s,0) = a(t)e^{-(t-s)}.$$

Then the assumptions $(\mathbb{H}_1)$ hold with

$$\varrho(t,s) = \phi(t,s) = \theta(t,s) = \pi(t,s) = a(t)e^{-(t-s)} \text{ and } \widehat{\phi}(t,s) = b(t)e^{-(t-s)},$$

b the limit functions given in Definition 3 with $f = a,\ \mathbb{G} = b$.
Moreover,

$$|J_j(\psi)| \leq (1 - e^{-\lambda\varpi})\ln(1 + 2^{-j})|\psi(t)| + 2^{-j-2}(1 - e^{-\lambda\varpi}).$$

Now, by the property of measure of noncompactness for bounded subset Ω of E, we have

$$\chi(J_j(\Omega)) \leq 2^{-j-2}(1 - e^{-\lambda\varpi})\chi(\Omega).$$

Furthermore, from Theorem 3, we obtain

$$\begin{aligned}
\Delta &= \eta \max\left(\frac{\varsigma_j}{1-e^{-\lambda\varpi}} + \frac{\sigma_j}{1-e^{-\lambda\varpi}} + (1+\upsilon^*)\|\hbar\|_{L^{\frac{1}{p_1}}}, \frac{\theta_j}{1-e^{-\lambda\varpi}} + 4(1+2\vartheta^*)\|\rho\|_{L^{\frac{1}{p_2}}}\right)\\
&\leq \max\left(\frac{1}{2} + \frac{4(1+|a|)}{(\ln 2)^2}, \frac{1}{4} + \frac{16(1+2|a|)}{(\ln 2)^2}\right)\\
&\leq \frac{1}{4} + \frac{4}{(\ln 2)^2}\max\left(1+|a|, 4(1+2|a|)\right)\\
&\leq \frac{1}{4} + \frac{16(1+2|a|)}{(\ln 2)^2}\\
&\leq 1.
\end{aligned}$$

So, all the conditions of Theorem 3 are satisfied. Hence by the conclusion of Theorems 3, it follows that the problem (1) has at least one an asymptotically almost automorphic mild solution $\psi \in \overline{\mho_\kappa}$.

5. Conclusions

In the present research, we have investigated existence for the piecewise asymptotically almost automorphic mild solutions of impulsive integro-differential equations with instantaneous impulses in Banach space. To achieve the desired results for the given problems, the fixed-point approach was used, namely Mönch's fixed point theorem, combined with resolvent operators from the Grimmer perspective and the concept of measures of non-compactnes. An example is provided to demonstrate how the major results can be applied. Our results in the given configuration are novel and substantially contribute to the literature on this field of study. We feel that there are multiple potential study avenues such as coupled systems, problems with infinite delays, problems with inclusions and many more due to the limited number of publications on integro-differential equations and inclusions, particularly with impulses. We hope that this article will serve as a starting point for such an undertaking.

Author Contribution Statements The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

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NEW APPLICATIONS IN THIRD-ORDER STRONG DIFFERENTIAL SUBORDINATION THEORY

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Abstract. The research conducted in this investigation focuses on extending known results from the second-order differential subordination theory for the special case of third-order strong differential subordination. This paper intends to facilitate the development of new results in this theory by showing how specific lemmas used as tools in classical second-order differential subordination theory are adapted for the context of third-order strong differential subordination. Two theorems proved in this study extend two familiar lemmas due to D.J. Hallenbeck and S. Ruscheweyh, and G.M. Goluzin, respectively. A numerical example illustrates applications of the new results but the theorems are hoped to become helpful tools in generating new outcome for this very recently initiated line of research concerning third-order strong differential subordination.

1. Introduction and Preliminaries

For the special case of third-order differential subordinations, J.A. Antonino and S.S. Miller [1] extended differential subordination theory first proposed by S.S. Miller and P.T. Mocanu [2,3], setting a new direction for further research into this topic. Applications of the outcomes discussed in [1] rapidly followed, and this topic of research is currently progressing successfully. By applying fundamental results regarding the third-order differential subordination, a direction of study deals with defining appropriate classes of admissible functions. Specific devellopments of third-order differential subordination continue to be obtained nowadays in view of this approach. For example, p-valent functions connected to a generalized fractional

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differintegral operator are analyzed in [4]. The same approach delivers interesting conclusions for special functions in [5] and [6] as well as for a generalized operator in [7].

Recent studies have started the development of an alternative approach in third-order differential subordination theory concerning another essential concept, that of the the best dominant. New ways of identifying the best dominant of a third-order differential subordination are provided in [8,9], along with techniques for finding the dominants for any third-order differential subordination.

The study presented in this paper intends to show how the classical results concerning third-order differential subordination are extended for the particular context of strong differential subordination theory in general and for the third-order strong differential subordination in particular. The first results in this directions are proposed in the very recent paper [10]. In their work, the authors extend the definitions specific to second-order strong differential subordination adapting them for the third-order strong differential subordination and develop some new results using the approach consisting in choosing appropriate classes of admissible functions. In this research, we propose other extensions form the classical theory of differential subordination to strong differential subordination and we obtain particular third-order strong differential subordination results.

Certain basic aspects concerning strong differential subordination theory were first presented in a published study from 2009 [11], following certain ideas set by J.A. Antonino and S. Romaguera through their work from 1994, [12], where the notion of strong differential subordination was first mentioned in the context of the special case of Briot-Bouquet differential subordination. The paper [11] defined the fundamental concepts of dominant of the solutions of the strong differential subordination and of solution of a strong differential subordination, as well as the three problems that form the basis of the theory and the fundamental tool in the analysis of strong differential subordination that is the class of admissible functions. The theory was further improved by the introduction of certain classes of analytic functions particularly applied in strong differential subordination studies in 2012 [13]. Latest results applying the results presented in [13] include strong differential results involving different operators [14,15], multiplier transformation and Ruscheweyh derivative applications in strong differential subordination theory [16], first order strong differential subordinations [17], and q-calculus aspects included in strong differential subordination studies alongside particular operators [18].

Those classes [13], used also in the present investigation, are:

Analytic functions in $U \times \overline{U}$ represented by $H(U \times \overline{U})$;

$$H\zeta[a,n] = \left\{ f \in H(U \times \overline{U}) : f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots \right\},$$

considering $a_k(\zeta)$ holomorphic in $\overline{U}$, $k \geq n$, $a \in \mathbb{C}$, $n \in \mathbb{N}$, the class derived from the classical:

$$H[a,n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots ,\ z \in U\};$$

$$H\zeta_U(U) = \{f \in H_\zeta[a,n] : f(\cdot,\zeta) \text{ univalent in } U \text{ for all } \zeta \in \overline{U}\};$$

$$A\zeta_n = \{f \in H(U \times \overline{U}) : f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \cdots, \quad z \in U,\ \zeta \in \overline{U}\},$$

with $A\zeta_1 = A\zeta$ and $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n+1$, $n \in \mathbb{N}$, the class derived from the classical:

$$A_n = \{f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + \cdots, \quad z \in U\}, \quad \text{with } A_1 = A;$$

$$S^*\zeta = \{f \in A\zeta : \mathrm{Re}\frac{zf'_z(z,\zeta)}{f(z,\zeta)} > 0, \quad z \in U,\ \zeta \in \overline{U}\},$$

the class of starlike functions in $U \times \overline{U}$ derived from the classical class of starlike functions:

$$S^* = \{f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0\};$$

$$K\zeta = \{f \in A\zeta : \mathrm{Re}\left(\frac{zf''_{z^2}(z,\zeta)}{f'_z(z,\zeta)} + 1\right) > 0, \quad z \in U,\ \zeta \in \overline{U}\},$$

the class of convex functions in $U \times \overline{U}$, derived from the classical class of convex functions:

$$K = \{f \in A : \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0,\ f(0) = 0,\ f'(0) \neq 0, \quad z \in U\}.$$

The notions of strong differential subordination necessary for this research are listed as follows.

Definition 1. *[13] Let $h(z,\zeta)$ and $f(z,\zeta)$ be analytic functions in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $h(z,\zeta)$, or $h(z,\zeta)$ is said to be strongly superordinate to $f(z,\zeta)$ if there exists a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$ such that $f(z,\zeta) = h(w(z),\zeta)$, for all $\zeta \in \overline{U}$, $z \in U$. In such a case, we write*

$$f(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Remark 1. *[13] a) If $f(z,\zeta)$ is analytic in $U \times \overline{U}$ and univalent in U for $\zeta \in \overline{U}$, then Definition 1 is equivalent to:*

$$f(0,\zeta) = h(0,\zeta), \text{ for all } \zeta \in \overline{U} \text{ and } f(U \times \overline{U}) \subset h(U \times \overline{U}).$$

b) If $f(z,\zeta) = f(z)$, $h(z,\zeta) = h(z)$, then the strong superordination becomes the usual superordination.

Definition 2. *[13] We denote by Q_ζ the set of functions $q(\cdot,\zeta)$ that are analytic and injective, as function of z, on $\overline{U} \backslash E(q(z,\zeta))$ where*

$$E(q(z,\zeta)) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z,\zeta = \infty\}$$

and are such that $q'_z(z,\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q(z,\zeta))$, $\zeta \in \overline{U}$.

The subclass of Q_ζ for which $q(0,\zeta) = a$ is denoted by $Q_\zeta(a)$.

Definition 3. *[13] Let Ω_ζ be a set in $\mathbb{C}$, $q(\cdot,\zeta)\in\Omega_\zeta$ and n a positive integer. The class of admissible functions $\phi_n[\Omega_\zeta, q(\cdot,\zeta)]$ consists of those functions $\psi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(r,s,t;\xi,\zeta)\notin\Omega_\zeta$$

whenever

$$r=q(z,\zeta),\ s=nq'_z(z,\zeta),\ Re\left(\frac{t}{s}+1\right)\geq nRe\left[\frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}+1\right],$$

$z\in U$, $\zeta\in\partial U\backslash E(q(\cdot,\zeta))$ and $n\geq 1$. When $n=1$, we write $\phi_1[\Omega_\zeta,q(\cdot,\zeta)]$ as $\phi[\Omega_\zeta,q(\cdot,\zeta)]$.

In the special case when $h(\cdot,\zeta)$ is an analytic mapping of $U\times\overline{U}$ onto $\Omega_\zeta\neq\mathbb{C}$ we denote the class $\phi_n[h(U\times\overline{U}),q(z,\zeta)]$ by $\phi_n[h(z,\zeta),q(z,\zeta)]$.

The class of admissible functions has been extended in [10] for the case of third-order strong differential subordination as it shows the next definition and will be used as such in the present investigation.

Definition 4. *[10] Let Ω_ζ be a set in $\mathbb{C}$, $q(\cdot,\zeta)\in\Omega_\zeta$ and $n\geq 2$. The class of admissible functions $\phi_n[\Omega_\zeta,q(\cdot,\zeta)]$ consists of those functions $\psi:\mathbb{C}^4\times U\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(r,s,t,u;\xi,\zeta)\notin\Omega_\zeta \tag{1}$$

whenever

$$r=q(z,\zeta),\ s=nq'_z(z,\zeta),\ Re\left(\frac{t}{s}+1\right)\geq nRe\left[\frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}+1\right],$$

$$Re\frac{u}{s}\geq n^2Re\frac{z^2q'''_{z^3}(z,\zeta)}{q'_z(z,\zeta)},$$

$z\in U$, $\zeta\in\partial U\backslash E(q(\cdot,\zeta))$ and $n\geq 2$.

An important known result that will be applied for the proofs of the new results is the following lemma used in third-order differential subordination theory and given here having a particular form required by the theory of strong differential subordination:

Lemma 1. *([1], [19]) Let $q(z,\zeta)\in Q_\zeta(a)$ and let $p(z,\zeta)=a+a_n(\zeta)z^n+a_{n+1}(\zeta)z^{n+1}+...\in H\left(U\times\overline{U}\right)$, with $p(z,\zeta)\neq\zeta$, and $n\geq 2$. If $p(\cdot,\zeta)$ is not subordinate to $q(\cdot,\zeta)$, then there exist points $z_0\in U$, $z_0=r_0e^{i\theta_0}$ and $\xi_0\in\partial U\backslash E(q(\cdot,\zeta))$ for which $p\left(U\times\overline{U_{r_0}}\right)\subset q\left(U\times\overline{U}\right)$ and $p(z_0,\zeta)=q(\xi_0,\zeta)$, and an $m\geq n$, such that the following conditions are satisfied:*

(i) $z_0p'_z(z,\zeta)=q(\xi_0,\zeta)$;

(ii) $Re\frac{\xi_0q''_{z^2}(\xi_0,\zeta)}{q'_z(\xi_0,\zeta)}\geq 0$ *and* $\left|\frac{z_0p'_z(z_0,\zeta)}{q'_z(\xi_0,\zeta)}\right|\leq m$;

(iii) $z_0 p'_z(z_0,\zeta) = m\xi_0 q'_z(\xi_0,\zeta)$;

(iv) $Re\left(\frac{z_0 p''_{z^2}(z_0,\zeta)}{p'_z(z_0,\zeta)}+1\right) \geq mRe\left(\frac{\xi_0 q''_{z^2}(\xi_0,\zeta)}{q'_z(\xi_0,\zeta)}+1\right)$;

(v) $Re\frac{z_0^2 p'''_{z^3}(z_0,\zeta)}{p'_z(z_0,\zeta)} \geq m^2 Re\frac{\xi_0^2 q'''_{z^3}(\xi_0,\zeta)}{q'_z(\xi_0,\zeta)}$.

The first part of this lemma was used in [10] for developing a theorem. In the next section, the form of this lemma is adapted for strong differential subordination theory and will be applied for proving the original results contained in the Main results section of this paper.

The main concern of the present investigation is to present applications in third-order strong differential subordination studies of the known results due to D.J. Hallenbeck and S. Ruscheweyh [20] and G.M. Goluzin [21], respectively. The following two lemmas are used in the next section for developing two new theorems.

Lemma 2. *([20]) Let $h \in K$, with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$, $Re\gamma \geq 0$. If $p \in H[a,n]$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t)\, t^{\frac{\gamma}{n}-1} dt.$$

Function $q \in K$ and is the best (a,n)-dominant.

Lemma 3. *([21]) Let $h \in K$. If the following differential subordination is satisfied:*

$$z p'(z) \prec h(z),$$

then

$$p(z) \prec q(z) = \int_0^z \frac{h(t)}{t} dt,$$

and q is the best dominant.

Remark 2. *In 1970, T.J. Suffridge [22] proved that Goluzin's result remains true even if $h \in S^*$.*

Lemma 2 and Lemma 3 have facilitated major developments in second-order differential subordination theory, hence, the new theorems presented in the next section based on those popular results should help for the development of the newly initiated line of research concerning third-order strong differential subordinations.

2. Main Results

The first original outcome of the study extends the results obtained by Hallenbeck and Ruscheweyh [20] shown in Lemma 2. The theorem proved here also provides techniques of finding the best dominant of a third-order strong differential subordination.

Theorem 1. *Take $h(z,\zeta) \in K\zeta$, satisfying $h(0,\zeta) = a \in \mathbb{C}$ for all $\zeta \in \overline{U}$. Consider the functions $p(z,\zeta) \in H[a,n]$, $n \geq 2$, $p(z,\zeta) \neq a$ and $q(z,\zeta) \in H[a,n]$, $q(z,\zeta) \in Q_\zeta(a)$ satisfying:*

(i) $Re\frac{\zeta q''_{z^2}(\xi,\zeta)}{q'_z(\xi,\zeta)} \geq 0$ and $\left|\frac{zp'_z(z,\zeta)}{q'_z(\xi,\zeta)}\right| \leq n$, where $z \in U$, $\xi \in \partial U \backslash E(q(z,\zeta))$, $n \geq 2$;

(ii) $q(z,\zeta) + zq'_z(z,\zeta) + z^2 q''_{z^2}(z,\zeta) = \frac{\gamma}{z^\gamma}\int_0^z h(t,\zeta)\, t^{\gamma-1}dt$, $\gamma \in \mathbb{C}$, $Re\gamma > 0$, $z \in U$, $\zeta \in \overline{U}$.

If $p(z,\zeta) \in Q_\zeta(a)$ and

$$p(z,\zeta) + zp'_z(z,\zeta) + z^2 p''_{z^2}(z,\zeta) + \frac{2zp'_z(z,\zeta) + 3z^2 p''_{z^2}(z,\zeta) + z^3 p'''_{z^3}(z,\zeta)}{\gamma} \in H\left(U \times \overline{U}\right),$$

then

$$p(z,\zeta) + zp'_z(z,\zeta) + z^2 p''_{z^2}(z,\zeta) + \frac{2zp'_z(z,\zeta) + 3z^2 p''_{z^2}(z,\zeta) + z^3 p'''_{z^3}(z,\zeta)}{\gamma} \prec\prec$$

$$q(z,\zeta) + zq'_z(z,\zeta) + z^2 q''_{z^2}(z,\zeta) + \frac{2zq'_z(z,\zeta) + 3z^2 q''_{z^2}(z,\zeta) + z^3 q'''_{z^3}(z,\zeta)}{\gamma} \tag{2}$$

implies

$$p(z,\zeta) \prec\prec q(z,\zeta),$$

where $z \in U$, $\zeta \in \overline{U}$ and $q(z,\zeta)$ is said to be the best dominant.

Proof. The functions $p(z,\zeta)$, $q(z,\zeta)$ and $h(z,\zeta)$ may be assumed to be satifying the conditions of Lemma 1 and the condition $q'_z(z,\zeta) \neq 0$ for $\xi \in \partial U \backslash E(q(z,\zeta))$. Otherwise, the functions can be replaced by $p_\rho(z,\zeta) = p(\rho z,\zeta)$, $q_\rho(z,\zeta) = q(\rho z,\zeta)$ and $h_\rho(z,\zeta) = h(\rho z,\zeta)$, respectively, with $0 < \rho < 1$ and those functions have the necessary properties on $U \times \overline{U}$.

Hence, Lemma 1 will be applied for the proof of this result, also considering the definition given for the class of admissible functions.

Define now the function $\psi : \mathbb{C}^4 \times U \times \overline{U} \to \mathbb{C}$ as

$$\psi(r,s,t,u,z,\zeta) = r + s + t + \frac{2s + 3t + u}{\gamma}, \quad r,s,t,u \in \mathbb{C},\ \mathrm{Re}\gamma > 0. \tag{3}$$

Taking $r = p(z,\zeta)$, $s = zp'_z(z,\zeta)$, $t = z^2 p''_{z^2}(z,\zeta)$, $u = z^3 p'''_{z^3}(z,\zeta)$, the function in (3) becomes:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right) = \tag{4}$$
$$p(z,\zeta) + zp'_z(z,\zeta) + z^2p''_{z^2}(z,\zeta) + \frac{2zp'_z(z,\zeta) + 3z^2p''_{z^2}(z,\zeta) + z^3p'''_{z^3}(z,\zeta)}{\gamma}.$$

Using (4), strong differential subordination (2) becomes:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right) \prec\prec \tag{5}$$
$$q(z,\zeta) + zq'_z(z,\zeta) + z^2q''_{z^2}(z,\zeta) + \frac{2zq'_z(z,\zeta) + 3z^2q''_{z^2}(z,\zeta) + z^3q'''_{z^3}(z,\zeta)}{\gamma},$$

$\mathrm{Re}\gamma > 0$.

Using relation (ii), we can write:

$$z^\gamma\left[q(z,\zeta) + zq'_z(z,\zeta) + z^2q''_{z^2}(z,\zeta)\right] = \gamma\int_0^1 h(t,\zeta)\cdot t^{\gamma-1}dt. \tag{6}$$

By differentiating (6) with respect to z, making simple calculations yield:

$$q(z,\zeta) + zq'_z(z,\zeta) + z^2q''_{z^2}(z,\zeta) + \frac{2zq'_z(z,\zeta) + 3z^2q''_{z^2}(z,\zeta) + z^3q'''_{z^3}(z,\zeta)}{\gamma} = h(z,\zeta). \tag{7}$$

By applying (7), the strong differential subordination (5) can be written as:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right) \prec\prec h(z,\zeta),$$

which can be interpreted in view of Remark 1, part a), as:

$$\left\{\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right)\right\} \subset h\left(U\times\overline{U}\right).$$

Considering $z = z_0 \in U$, we write:

$$\left\{\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right)\right\} \subset h\left(U\times\overline{U}\right).$$

Assume now that $p(z,\zeta) \not\prec\prec q(z,\zeta)$. In this situation, Lemma 1 shows that there exist $z_0 = r_0e^{i\theta_0} \in U$ and $\xi_0 \in \partial U\backslash E(q(z,\zeta))$ such that

$$p(z_0,\zeta) = q(\xi_0,\zeta),\ z_0p'_z(z_0,\zeta) = m\xi_0q'_z(\xi_0,\zeta),$$
$$t = z_0^2p''_{z^2}(z_0,\zeta),\ \ u = z_0^3p'''_{z^3}(z_0,\zeta), \tag{8}$$

satisfy the conditions of Lemma 1.

By replacing $r = q(\xi_0,\zeta)$, $s = m\xi_0q'_z(\xi_0,\zeta)$, t and u in the admissibility condition (1), we obtain:

$$\psi\left(q(\xi_0,\zeta), m\xi_0q'_z(\xi_0,\zeta), t, u\right) \notin h\left(U\times\overline{U}\right).$$

Using the equalities given by (8), we have:

$$\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right) \notin h\left(U\times\overline{U}\right),$$

but this contradicts (2). Hence, we must have that

$$p(z,\zeta) \prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Since $q(z,\zeta) \in H\zeta_U(U)$ and is a solution for the equation (7), it follows that $q(z,\zeta)$ is the best dominant for the strong differential subordination (2). ☐

Remark 3. *This theorem shows that finding the best dominant for a third-order strong differential subordination requires only the existence of a univalent solution for the differential equation associated with the strong differential subordination.*

The next theorem extends the result proved by G.M. Goluzin in 1935 [21] for second-order differential subordinations to fit the theory of third-order strong differential subordination.

Theorem 2. *Let $h(z,\zeta) \in K\zeta$, with $h(0,\zeta) = a \in \mathbb{C}$ for all $\zeta \in \overline{U}$. Consider the functions $p(z,\zeta) \in H[a,n]$, $n \geq 2$, $p(z,\zeta) \neq a$ and $q(z,\zeta) \in Q_\zeta(a)$, $q(z,\zeta) \in H\zeta_U(U)$ satisfying:*

(i) $Re\frac{\xi q''_{z^2}(\xi,\zeta)}{q'_z(\xi,\zeta)} \geq 0$ and $\left|\frac{zp'_z(z,\zeta)}{q'_z(z,\zeta)}\right| \leq n$, where $z \in U$, $\xi \in \partial U \backslash E(q(z,\zeta))$, $n \geq 2$;

(ii) $zq(z,\zeta) \cdot q'_z(z,\zeta) + z^2 q''_{z^2}(z,\zeta) = \int_0^z \frac{h(t,\zeta)}{t}dt$, $z \in U$, $\zeta \in \overline{U}$.

If

$$zp(z,\zeta) \cdot p'_z(z,\zeta) + (zp'_z(z,\zeta))^2 + z^2 p''_{z^2}(z,\zeta)[p(z,\zeta)+2] + z^3 p'''_{z^3}(z,\zeta) \prec\prec$$
$$zq(z,\zeta) \cdot q'_z(z,\zeta) + (zq'_z(z,\zeta))^2 + z^2 q''_{z^2}(z,\zeta)[q(z,\zeta)+2] + z^3 q'''_{z^3}(z,\zeta),$$

implies

$$p(z,\zeta) \prec\prec q(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

with $q(z,\zeta)$ designated as the best dominant of the third-order strong differential subordination (2).

Proof. As seen in the proof of the first theorem, the functions $p(z,\zeta)$, $q(z,\zeta)$ and $h(z,\zeta)$ may be assumed to be satifying the conditions of Lemma 1 on $U \times \overline{U}$ and the condition $q'_z(z,\zeta) \neq 0$ for $\xi \in \partial U \backslash E(q(z,\zeta))$.

By differentiating (ii) with respect to z, we have

$$q(z,\zeta) \cdot q'_z(z,\zeta) + z^2 (q'_z(z,\zeta))^2 + z^2 q(z,\zeta) q''_{z^2}(z,\zeta) + 2z^2 q''_{z^2}(z,\zeta) + z^3 q'''_{z^3}(z,\zeta) = h(z,\zeta). \tag{9}$$

By applying (9), third-order strong differential subordination (2) becomes:

$$zp(z,\zeta) \cdot p'_z(z,\zeta) + [zp'_z(z,\zeta)]^2 + z^2 p''_{z^2}(z,\zeta)[p(z,\zeta)+2] + z^3 p'''_{z^3}(z,\zeta) \prec\prec h(z,\zeta). \tag{10}$$

For finalizing the proof of this theorem, define the function $\psi : \mathbb{C}^4 \times U \times \overline{U} \to \mathbb{C}$ as

$$\psi(r,s,t,u,z,\zeta) = r \cdot s + s^2 + t(r+2) + u, \quad r,s,t,u \in \mathbb{C}. \tag{11}$$

Taking $r = p(z,\zeta)$, $s = zp'_z(z,\zeta)$, $t = z^2 p''_{z^2}(z,\zeta)$, $u = z^3 p'''_{z^3}(z,\zeta)$, relation (11) becomes:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2 p''_{z^2}(z,\zeta), z^3 p'''_{z^3}(z,\zeta)\right) = \tag{12}$$
$$p(z,\zeta) \cdot zp'_z(z,\zeta) + [zp'_z(z,\zeta)]^2 + z^2 p''_{z^2}(z,\zeta)[p(z,\zeta)+2] + z^3 p'''_{z^3}(z,\zeta).$$

Using (12), the third-order strong differential subordination (10) becomes:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right) \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}. \quad (13)$$

Since $h(z,\zeta)\in K\zeta$ we have that $h(z,\zeta)\in H\zeta_U(U)$ and applying part a of Remark 1 we can write an equivalent form of (13):

$$\left\{\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_{z^2}(z,\zeta), z^3p'''_{z^3}(z,\zeta)\right)\right\}\subset h\left(U\times\overline{U}\right). \quad (14)$$

Considering $z=z_0\in U$, from (14) we have:

$$\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right)\in h\left(U\times\overline{U}\right). \quad (15)$$

Assume now that $p(z,\zeta)\not\prec\prec q(z,\zeta)$. Then, according to Lemma 1 there exist $z_0\in U$ and $\xi_0\in\partial U\backslash E(q(z,\zeta))$ such that:

$$\begin{gathered} p(z_0,\zeta)=q(\xi_0,\zeta),\ z_0p'_z(z_0,\zeta)=m\xi_0q'_z(\xi_0,\zeta),\\ t=z_0^2p''_{z^2}(z_0,\zeta),\ \ u=z_0^3p'''_{z^3}(z_0,\zeta), \end{gathered} \quad (16)$$

satisfy the conditions of Lemma 1.

By replacing $r=q(\xi_0,\zeta)$, $s=m\xi_0q'_z(\xi_0,\zeta)$, $t=z_0^2p''_{z^2}(z_0,\zeta)$, $u=z_0^3p'''_{z^3}(z_0,\zeta)$ in the admissibility condition from Definition 3, we have:

$$\psi\left(q(\xi_0,\zeta), m\xi_0q'_z(\xi_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right)\notin h\left(U\times\overline{U}\right).$$

Using the equalities seen in (16), relation (2) is written as:

$$\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_{z^2}(z_0,\zeta), z_0^3p'''_{z^3}(z_0,\zeta)\right)\notin h\left(U\times\overline{U}\right),$$

which contradicts (15). Hence, we must have that

$$p(z,\zeta)\prec\prec q(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}.$$

Since $q(z,\zeta)\in H\zeta_U(U)$ and is a solution for the differential equation (9), it follows that $q(z,\zeta)$ is the best dominant for the third-order strong differential subordination (2). □

Example 1. *Using the outcome of Theorem 1, we can write:*

Let $h(z,\xi)=1+2z\xi$, $z\in U$, $\xi\in\overline{U}$, $h(z,\xi)\in K\xi$, $h(0,\xi)=1\in\mathbb{C}$, $p(z,\xi)=1+z^3\xi$, $q(z,\xi)=1+z\xi$, $z\in U$, $\xi\in\overline{U}$, $\gamma=1$ *satisfying:*

(i) $Re\frac{\xi q''_{z^2}(z,\xi)}{q'_z(z,\xi)}=Re\frac{0}{2\xi}=0\geq 0$ *and* $\left|\frac{z\cdot 3z^2\xi}{\xi}\right|=3\left|z^3\right|\leq 3$, $z\in U$, $\xi\in\partial U\backslash E(q(z,\xi))$;

(ii) $(1+z\xi)+z\xi=\frac{1}{z}\int_0^z(1+2t\xi)\,dt$, $\gamma=1$.

If $\left(1+z^3\xi\right)+z\left(3z^2\xi\right)+z^2\cdot 6z\xi+2z\left(3z^2\xi\right)+3z^2\cdot 6z\xi+z^3\cdot 6\xi=$ $1+z^3\xi+3z^2\xi+6z^3\xi+6z^3\xi+18z^3\xi+6z^3\xi=1+40z^4\xi$, *is analytic in* $U\times\overline{U}$, *then*

$$1+40z^4\xi\prec\prec 1+z\xi+z\xi+2z\xi=1+4z\xi,$$

implies

$$1+z^3\xi\prec\prec 1+z\xi, \quad z\in U,\ \xi\in\overline{U},$$

and $q(z)=1+z\xi$ *is designated as the best dominant.*

3. Conclusion

The new results established in this investigation are contained in Section 2 of the paper, after the necessary notions and previously established results necessary for the investigation are presented. The line of research followed by this study concerns the development of the newly initiated theory of third-order strong differential subordination. Having seen the new recent results obtained by researchers concerning classical third-order differential subordination theory, and considering the nice developments involving the theory of strong differential subordination, this study extends previously known lemmas established in [20, 21], popular in researches in geometric function theory, providing new tools for improving the knowledge related to third-order strong differential subordination theory, recently initiated by the publication [10]. The new results obtained here are given in Theorem 1 and Theorem 2. A numerical example is provided hoping to inspire certain applications for particular functions to be used as best dominants of third-order strong differential subordinations, which could result in obtaining interesting consequences with significant geometrical interpretations. Nevertheless, the main idea of the study doesn't focus on numerical examples but on providing new means of investigation in the field.

Since the initial lemmas that have motivated this study presented as Lemma 2 and 3 concerning second-order differential subordination theory have facilitated major developments of that topic, it is expected that the new results proved during this investigation to have the same effect on motivating future research in third-order strong differential subordination theory.

Author Contribution Statements Both authors jointly worked on the results and they read and approved the final manuscript.

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ON A CLASS OF FOURTH-ORDER NEUTRAL DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENTS

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Abstract. In this paper, we investigate a fourth-order neutral differential equation characterized by piecewise constant arguments. Our study focuses on establishing both the existence and uniqueness of solutions to this equation, incorporating a prescribed initial condition. In addition, we investigate the stability analysis of the above-mentioned equation and show that the zero solution of this equation cannot be asymptotically stable and indicate under what conditions it is unstable. Through rigorous mathematical analysis and theoretical exploration, this research contributes to the deeper understanding of fourth-order neutral differential equations with piecewise constant arguments, offering insights into their solution behavior and stability properties.

1. Introduction

In this work, we investigate the fourth-order neutral differential equation with piecewise constant arguments (NDEPCA)

$$\frac{d^4}{dt^4}\Big(x(t)+px(t-1)\Big)=qx([t-1]),\quad t\geq 0, \tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\quad -1\leq t\leq 0, \tag{2}$$

where p and q are real constants, $[.]$ denotes the greatest integer function and $\varphi\in C([-1,0],\mathbb{R})$ is an initial function.

Our aim is to show the existence and uniqueness of solutions for the initial value problem (1)–(2) and to demonstrate that its zero solution cannot be asymptotically stable. Obtaining the solutions of neutral differential equations with piecewise constant arguments using difference equations offers numerous advantages. In this study, we show that equation (1) exhibits the same asymptotic properties as the

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corresponding fifth-order difference equation. The paper is structured as follows: In the first section, previous studies are presented to provide the necessary motivation for this paper. Additionally, important definitions and theorems related to neutral differential equations and difference equations are provided. In the second section, the existence and uniqueness of solutions for the initial value problem (1)–(2) are demonstrated, and it is shown that the zero solution of a fourth-order equation of type (1) cannot be asymptotically stable. Section 3 consists of a numerical example.

Differential equations with piecewise constant arguments (DEPCA) were pioneered by Busenberg, Cooke, Shah, and Wiener in their seminal works [12, 36]. These equations bridge the realms of difference and differential equations, incorporating both discrete and continuous dynamics at integer points. This connection is particularly evident in epidemic models, where the interplay between discrete events and continuous processes naturally emerges. Following this research, numerous significant problems spanning the vibration of spring-mass systems, biomedicine, electronic processes, epidemic diseases, isolated mechanisms and some significant properties of the solutions have been investigated through the utilization of DEPCA [1]- [3], [5]-[11], [13]-[20], [22, 23, 26, 29, 30, 35, 37].

However there are only a few articles that issued on neutral differential equations with piecewise constant arguments (NDEPCA) (see [4, 24, 27, 28], [31]–[34],[38]). Some stability and oscillation results for NDEPCA have been discussed in [34], where the oscillatory behavior and stability of the trivial solution of first- and second-order NDEPCA were analyzed:

$$\frac{d}{dt}\Big(y(t)+py(t-1)\Big)=-qy([t-1]),$$

and

$$\frac{d^2}{dt^2}\Big(y(t)+py(t-1)\Big)=-qy([t-1]). \tag{3}$$

It was proved that the trivial solution of Eq. (3) is not asymptotically stable. Later, in [33], Papaschinopoulos obtained a unique solution for the third-order NDEPCA

$$\frac{d^3}{dt^3}\Big(y(t)+py(t-1)\Big)=-qy([t-1]), \tag{4}$$

and demonstrated that the zero solution of Eq. (4) is not asymptotically stable.

The gap in the literature, along with these earlier studies, motivates us to explore the asymptotic behavior of Eq. (1).

Now, let us give definition:

Definition 1. *A function $x:[-1,\infty)\to\mathbb{R}$ is a solution of the initial value problem (1)-(2) if the following conditions are satisfied:*

(i) x *and* $\dfrac{dx}{dt}\in C([-1,\infty),\mathbb{R})$,

(ii) $\dfrac{d^2}{dt^2}\Big(x(t)+px(t-1)\Big)=\beta(t)$ *and* $\dfrac{d^3}{dt^3}\Big(x(t)+px(t-1)\Big)=\alpha(t)$ *exist on* $[0,\infty)$ *and* $\beta(t)$ *and* $\alpha(t)$ *are continuous on* $[0,\infty)$,

(iii) $\dfrac{d^4}{dt^4}\Big(x(t)+px(t-1)\Big)$ *exist on* $[0,\infty)$ *with the possible exception at the point* $[t]\in[0,\infty)$ *where one-sided derivatives exists;*

(iv) x *satisfies Eq. (1) on each interval* $[n,n+1)$ *with* $n=0,1,2,...$ *and initial condition (2) on the interval* $[-1,0]$.

Before giving the main theorems, consider the $k-th$ order difference equation

$$x_{n+k}+p_1x_{n+k-1}+p_2x_{n+k-2}+...+p_kx_n=0, \tag{5}$$

where $p_i, i=1,2,...,k$ are real numbers. Also, we can write the corresponding characteristic equation of (5) as follows:

$$p(\lambda)=\lambda^k+p_1\lambda^{k-1}+...+p_k. \tag{6}$$

Now, we should remember the following well-known some theorems for difference equations:

Theorem 1. *([21], p246.) The zero solution of Eq. (5) is asymptotically stable if and only if* $|\lambda|<1$ *for all roots* λ *of Eq. (6).*

Theorem 2. *(Schur-Cohn Criterion or Jury Conditions, [25]) The roots of the Eq. (6) lie inside the unit disk if and only if the following hold:*
(i) $p(1)>0$,
(ii) $(-1)^kp(-1)>0$,
(iii) consider the matrix $A_1^{\pm},A_2^{\pm},...$ *for* $i=1,2,...k$,

$$A_i^{\pm}=\begin{pmatrix}1 & p_{k-1} & p_{k-2} & ... & p_{k-i+1}\\ 0 & 1 & p_{k-1} & ... & p_{k-i+2}\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & ... & 1\end{pmatrix}\pm\begin{pmatrix}p_{i-1} & p_{i-2} & ... & p_1 & p_0\\ p_{i-2} & p_{i-3} & ... & p_0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ p_0 & 0 & ... & 0 & 0\end{pmatrix}$$

and determinants $|A_1^{\pm}|>0,|A_3^{\pm}|>0,...,|A_{k-1}^{\pm}|>0$ *(for* k *is even) or* $|A_2^{\pm}|>0,|A_4^{\pm}|>0,...,|A_{k-1}^{\pm}|>0$ *(for* k *is odd).*

Theorem 3. *([21], Theorem 5.12) The zero solution of Eq. (5) is unstable if*

$$|p_1|-\sum_{i=2}^{k}|p_i|>1.$$

2. Main Results

Theorem 4. *The initial value problem (1)-(2) has a unique solution* $x(t)$ *with* $x(-1)=c_{-1}$ *and* $x(0)=c_0$.

Proof. Let us consider,

$$\frac{d}{dt}\Big(x(t)+px(t-1)\Big)\Big|_{t=0}=\gamma_0,$$
$$\frac{d^2}{dt^2}\Big(x(t)+px(t-1)\Big)\Big|_{t=0}=\beta_0,$$
$$\frac{d^3}{dt^3}\Big(x(t)+px(t-1)\Big)\Big|_{t=0}=\alpha_0,$$

and $x(-1)=\varphi(-1)=c_{-1}, x(0)=\varphi(0)=c_0$. We apply the method of steps to show the existence and uniqueness of the solution of (1)-(2). Let $x_0(t)\equiv x(t)$ on the interval $0\le t<1$,

$$\frac{d^4}{dt^4}\Big(x(t)+px(t-1)\Big)=qx(-1)=q\varphi(-1)=qc_{-1}.$$

Integrating this equation from 0 to t, we obtain

$$\frac{d^3}{dt^3}\Big(x(t)+px(t-1)\Big)=\alpha_0+qc_{-1}t,$$

and again, integrating from 0 to t, we get

$$\frac{d^2}{dt^2}\Big(x(t)+px(t-1)\Big)=\beta_0+\alpha_0t+qc_{-1}\frac{t^2}{2},$$

and also integrating this equation from 0 to t, we obtain

$$\frac{d}{dt}\Big(x(t)+px(t-1)\Big)=\gamma_0+\beta_0t+\alpha_0\frac{t^2}{2}+qc_{-1}\frac{t^3}{6},$$

and finally, if we integrate this equation from 0 to t, we obtain

$$x(t)+px(t-1)=x(0)+px(-1)+\gamma_0t+\beta_0\frac{t^2}{2}+\alpha_0\frac{t^3}{6}+qc_{-1}\frac{t^4}{24}.$$

On the interval $0\le t<1$, we can rewrite this equation as follows:

$$x_0(t)\equiv x(t)=-p\varphi(t-1)+c_0+pc_{-1}+\gamma_0t+\beta_0\frac{t^2}{2}+\alpha_0\frac{t^3}{6}+qc_{-1}\frac{t^4}{24},$$

where $x(0)=c_0, x(-1)=c_{-1}$. Let $x_1(t)\equiv x(t)$ be a solution of (1)-(2) for $t\in[1,2)$. Let us consider,

$$\frac{d}{dt}\Big(x(t)+px(t-1)\Big)\Big|_{t=1}=\gamma_1,$$
$$\frac{d^2}{dt^2}\Big(x(t)+px(t-1)\Big)\Big|_{t=1}=\beta_1,$$
$$\frac{d^3}{dt^3}\Big(x(t)+px(t-1)\Big)\Big|_{t=1}=\alpha_1,$$

with the path followed in the previous step, we obtain

$$x_1(t)\equiv x(t)=-px_0(t-1)+c_1+pc_0+\gamma_1(t-1)+\beta_1\frac{(t-1)^2}{2}+\alpha_1\frac{(t-1)^3}{6}+qc_0\frac{(t-1)^4}{24}. \tag{7}$$

By the continuity of $x(t)$ at $t = 1$, one can write clearly

$$c_1 = (1-p)c_0 + (p + \frac{q}{24})c_{-1} + \gamma_0 + \frac{\beta_0}{2} + \frac{\alpha_0}{6}, \tag{8}$$

and

$$\begin{cases} \alpha_1 = \alpha_0 + qc_{-1}, \\ \beta_1 = \beta_0 + \alpha_0 + \frac{q}{2}c_{-1}, \\ \gamma_1 = \gamma_0 + \beta_0 + \frac{\alpha_0}{2} + \frac{q}{6}c_{-1}. \end{cases} \tag{9}$$

If we put (9) and (8) into Eq. (7), we obtain for $1 \leq t < 2$,

$$x_1(t) \equiv x(t) = -p\Big[-p\varphi(t-2) + c_0 + pc_{-1} + \gamma_0(t-1) + \beta_0\frac{(t-1)^2}{2} + \alpha_0\frac{(t-1)^3}{6} + qc_{-1}\frac{(t-1)^4}{24}\Big] + c_0 + (p+\frac{q}{24})c_{-1} + \gamma_0 + \frac{\beta_0}{2} + \frac{\alpha_0}{6}.$$

We will do it for the general case. Now, let us assume that, respectively, $x_n(t) \equiv x(t)$ be a solution of (1)-(2) on the interval $n \leq t < n+1$ and $x_{n+1}(t) \equiv x(t)$ be a solution of (1)-(2) on the interval $n+1 \leq t < n+2$, let us consider

$$\frac{d}{dt}\Big(x(t) + px(t-1)\Big)\Big|_{t=n} = \gamma_n \text{ and } \frac{d}{dt}\Big(x(t) + px(t-1)\Big)\Big|_{t=n+1} = \gamma_{n+1}, \tag{10}$$

$$\frac{d^2}{dt^2}\Big(x(t) + px(t-1)\Big)\Big|_{t=n} = \beta_n \text{ and } \frac{d^2}{dt^2}\Big(x(t) + px(t-1)\Big)\Big|_{t=n+1} = \beta_{n+1}, \tag{11}$$

$$\frac{d^3}{dt^3}\Big(x(t) + px(t-1)\Big)\Big|_{t=n} = \alpha_n \text{ and } \frac{d^3}{dt^3}\Big(x(t) + px(t-1)\Big)\Big|_{t=n+1} = \alpha_{n+1}, \tag{12}$$

in the same way, $x_n(t) = x(t)$ can be written as

$$x_n(t) \equiv x(t) = -px_{n-1}(t-1) + c_n + pc_{n-1} + \gamma_n(t-n) + \beta_n\frac{(t-n)^2}{2} + \alpha_n\frac{(t-n)^3}{6} + qc_{n-1}\frac{(t-n)^4}{24}, \tag{13}$$

for $t \in [n, n+1)$, where $c_n = x(n)$ and $c_{n-1} = x(n-1)$. Finally, on the interval $n+1 \leq t < n+2$, we derive

$$x_{n+1}(t) \equiv x(t) = -px_{n-1}(t-1) + c_{n+1} + pc_n + \gamma_{n+1}(t-n-1) + \beta_{n+1}\frac{(t-n-1)^2}{2} + \alpha_{n+1}\frac{(t-n-1)^3}{6} + \frac{q}{24}c_n(t-n-1)^4. \tag{14}$$

Because of the continuity of $x(t)$ at $t = n+1$, it must be the case that

$$\lim_{t\to n+1} x_n(t) = \lim_{t \to n+1} x_{n+1}(t) \text{ for } n = 0, 1, 2, ...$$

Therefore from (13) and (14), we get

$$c_{n+1} + (p-1)c_n + (-p - \frac{q}{24})c_{n-1} = \gamma_n + \frac{\beta_n}{2} + \frac{\alpha_n}{6}, \quad n = 0, 1, 2, ... \tag{15}$$

By the continuity at $t = n+1$ and for $n = 0, 1, 2, ...$, from (10), (11) and (12), we can write following equations:

$$\begin{cases} \gamma_{n+1} = \gamma_n + \beta_n + \frac{\alpha_n}{2} + \frac{q}{6} c_{n-1}, \\ \beta_{n+1} = \beta_n + \alpha_n + \frac{q}{2} c_{n-1}, \\ \alpha_{n+1} = \alpha_n + q c_{n-1}. \end{cases}$$

From these equations, we can write α_n, β_n, and γ_n as follows:

$$\begin{cases} \alpha_n = \alpha_{n+1} - q c_{n-1}, \\ \beta_n = \beta_{n+1} - \alpha_{n+1} + \frac{q}{2} c_{n-1}, \\ \gamma_n = \gamma_{n+1} - \beta_{n+1} + \frac{1}{2}\alpha_{n+1} - \frac{q}{6} c_{n-1}. \end{cases} \tag{16}$$

Therefore, from (16) and (15), we obtain

$$c_{n+1} + (p-1)c_n + (-p + \frac{q}{24})c_{n-1} = \gamma_{n+1} - \frac{\beta_{n+1}}{2} + \frac{\alpha_{n+1}}{6}, \quad n = 0, 1, 2, ... \tag{17}$$

If we replace n with $n+1$ in Eq. (15), we get

$$c_{n+2} + (p-1)c_{n+1} + (-p - \frac{q}{24})c_n = \gamma_{n+1} + \frac{\beta_{n+1}}{2} + \frac{\alpha_{n+1}}{6} \tag{18}$$

From Eq. (17), Eq. (18) and by using (16), we can write

$$c_{n+2} + (p-4)c_{n+1} + (6 - 4p - \frac{q}{24})c_n + (-4p + 6p - \frac{11q}{24})c_{n-1}$$
$$+ (1 - 4p - \frac{11q}{24})c_{n-2} + (p - \frac{q}{24})c_{n-3} = 0, \quad n = 2, 3, ...$$

Therefore, we obtain the fifth-order difference equation for $n = -1, 0, 1, ...$

$$\begin{aligned} c_{n+5} + (p-4)c_{n+4} + (6 - 4p - \frac{q}{24})c_{n+3} + (-4 + 6p - \frac{11q}{24})c_{n+2} \\ + (1 - 4p - \frac{11q}{24})c_{n+1} + (p - \frac{q}{24})c_n = 0, \end{aligned} \tag{19}$$

with the initial conditions

$$\begin{aligned} c_{-1} = \varphi(-1), \quad c_0 = \varphi(0), \quad c_1 = (1-p)c_0 + (p + \frac{q}{24})c_{-1} + \gamma_0 + \frac{\beta_0}{2} + \frac{\alpha_0}{6}, \\ c_2 = (p^2 - p + 1 + \frac{q}{24})c_0 + (-p^2 + p + \frac{q(15-p)}{24})c_{-1} + (2-p)\gamma_0 + (\frac{4-p}{2})\beta_0 \\ + (\frac{8-p}{6})\alpha_0 \\ c_3 = (1-p)c_2 + (p + \frac{q}{24})c_1 + \frac{7q}{12}c_0 + \frac{25q}{12}c_{-1} + \gamma_0 + \frac{5\beta_0}{2} + \frac{19\alpha_0}{6}, \end{aligned} \tag{20}$$

This initial value problem has a unique solution. Then the solution $x(t)$ of (1)-(2) defined by (13) is unique on the interval $n \leq t < n+1$. Thus, the proof is completed. □

Now, the solution methodology for (1)-(2) can be succinctly described by referring to Lemma 3 in [34], which offers a comprehensive approach.

$$x(t) + px(t-1) = v(t), \quad t \geq 0,$$

with the initial function

$$x(t) = \varphi(t), \quad -1 \leq t \leq 0,$$

is the continuous function given by

$$x(t) = (-p)^{n+1}\varphi(\theta - 1) + \sum_{k=0}^{n}(-p)^{n-k}v(k+\theta), \quad t \geq 0,$$

where $v(k+\theta)$ can be obtain from (13) as follows:

$$v(t) = c_n + pc_{n-1} + \gamma_n(t-n) + \frac{\beta_n}{2}(t-n)^2 + \frac{\gamma_n}{6}(t-n)^3 + \frac{q}{24}c_{n-1}(t-n)^4,$$

we get the solution of (1)-(2) as in the form

$$x(t) = (-p)^{n+1}\Big[\varphi(\theta-1) + \sum_{k=0}^{n}(-p)^{-k-1}[c_k + (p+\frac{q}{24}\theta^4)c_{k-1} + \gamma_k\theta + \frac{1}{2}\beta_k\theta^2 + \frac{1}{6}\alpha_k\theta^3]\Big], \tag{21}$$

where $\varphi \in C([-1,0],\mathbb{R})$, $t = n + \theta$ with $0 \leq \theta \leq 1$ and $n = -1, 0, 1, ...$
Now, we investigate the stability nature behaviour of solutions of the general fifth order linear difference equation with constant coefficients of the form

$$c_{n+5} + a_4c_{n+4} + a_3c_{n+3} + a_2c_{n+2} + a_1c_{n+1} + a_0c_n = 0, \quad n = -1, 0, 1, ... \tag{22}$$

where $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$. The characteristic equation of Eq. (22) is

$$p(\lambda) = \lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \tag{23}$$

The following lemma gives necessary and sufficient conditions for the asymptotic stability of the zero solution of Eq. (22).

Lemma 1. *The zero solution of Eq.* (22) *is asymptotically stable if and only if the following conditions hold:*

(I) $1 + a_3 + a_1 > |a_4 + a_2 + a_0|$

(II) $1 - a_0^2 > |a_1 - a_4a_0|,$

(III) $a_0^4 + a_0^3a_2 + a_0^3a_4 - a_0^2a_1a_3 - a_0^2a_1 - a_0^2a_3^2 - a_0^2a_3 - a_0^2a_4^2 - 2a_0^2 + a_0a_1^2a_4 + a_0a_1a_2 + a_0a_1a_3a_4 + 2a_0a_1a_4 + 2a_0a_2a_3 - a_0a_2a_4^2 - a_0a_2 + a_0a_3a_4 - a_0a_4^3 - a_0a_4 - a_1^3 - a_1^2a_3 - a_1^2 + a_1a_2a_4 + a_1a_4^2 + a_1 - a_2^2 - a_2a_4 + a_3 + 1 > 0,$

(IV) $a_0^4 - a_0^3a_2 - a_0^3a_4 + a_0^2a_1a_3 + a_0^2a - 1 + 2a_0^2a_2a_4 - a - 0^2a_3^2 + a_0^2a_3 - a_0^2a_4^2 - 2a_0^2 - a_0a_1^2a_4 - 3a_0a_1a_2 + a_0a_1a_3a_4 + 2a_0a_1a_4 + 2a_0a_2a_3 - a_0a_2a_4^2 + a_0a_2 - 3a_0a_3a_4 + a_0a_4^3 + a_0a_4 + a_1^3 - a_1^2a_3 - a_1^2 + a_1a_2a_4 + 2a_1a_3 - a_1a_4^2 - a_1 - a_2^2 + a_2a_4 - a_3 + 1 > 0.$

Proof. By Theorem 1, the zero solution of Eq. (22) is asymptotically stable if and only if each root of λ of Eq. (23) satisfies $|\lambda| < 1$. Using the condition (i) and (ii) in Theorem 2, we can easily obtain

$$1 + a_4 + a_3 + a_2 + a_1 + a_0 > 0 \quad \text{and} \quad 1 - a_4 + a_3 - a_2 + a_1 - a_0 > 0, \tag{24}$$

it can be easily seen that the conditions (24) are equivalent to condition (I). Using the condition (iii) in Theorem 2, we can write

$A_2^+ = \begin{pmatrix} 1+a_1 & a_4+a_0 \\ a_0 & 1 \end{pmatrix}, A_2^- = \begin{pmatrix} 1-a_1 & a_4-a_0 \\ -a_0 & 1 \end{pmatrix}$ and

$$A_4^+ = \begin{pmatrix} 1+a_3 & a_4+a_2 & a_3+a_1 & a_2+a_0 \\ a_2 & 1+a_1 & a_4+a_0 & a_3 \\ a_1 & a_0 & 1 & a_4 \\ a_0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_4^- = \begin{pmatrix} 1-a_3 & a_4-a_2 & a_3-a_1 & a_2-a_0 \\ -a_2 & 1-a_1 & a_4-a_0 & a_3 \\ -a_1 & -a_0 & 1 & a_4 \\ -a_0 & 0 & 0 & 1 \end{pmatrix}.$$

We can say that the determinants of $A_2^{\pm}$ and $A_4^{\pm}$ must be positive. If numerical calculations are performed, the conditions $(II), (III)$, and (IV) are obtained. ☐

Theorem 5. *The zero solution of Eq.* (1) *is not asymptotically stable.*

Proof. Applying Lemma 1 to Eq. (19), we obtain that the zero solution of Eq. (19) is asymptotically stable if and only if

(a) $p < 1$ and $q < 0$,

(b) $2 - 4p - (p-4)(p - \frac{q}{24}) - (p - \frac{q}{24})^2 - \frac{11q}{24} > 0$ and $4p + (p-4)(p - \frac{q}{24}) + (p - \frac{q}{24})^2 + \frac{11q}{24} > 0$,

(c) $-\frac{q}{864}(-3456p^3 - 24p^2(7q + 432) + p(13q^2 - 3504q + 3456) - 121q^2 + 3672q + 10368) > 0$ and $\frac{q^2}{96}(-24p^2 + p(q - 144) - 9(q + 24)) > 0$.

However, these conditions are inconsistent. If we solve these inequalities, we can approximately obtain $p > 87.332$ and $-12(\sqrt{p^2 - 90p + 233} - 3p + 15) < q < 12(\sqrt{p^2 - 90p + 233} + 3p - 15)$. This, however, contradicts the condition that $p < 1$. As a result, the zero solution of Eq. (19) is not asymptotically stable. It is clear that from the Eq. (21), the zero solution of the Eq. (19) is not asymptotically stable then the zero solution of (1) is not asymptotically stable. ☐

Theorem 6. *The zero solution of* (1) *is unstable if the condition*

$$|p-4| - |p - \frac{q}{24}| - |1 - 4p - \frac{11q}{24}| - |-4 + 6p - \frac{11q}{24}| - |6 - 4p - \frac{q}{24}| > 1 \tag{25}$$

is hold.

Proof. We will apply Theorem 3 to prove this result. In difference equation (19), it's clear that $p_1 = p-4, p_2 = 6-4p-\frac{q}{24}, p_3 = -4+6p-\frac{11q}{24}, p_4 = 1-4p-\frac{11q}{24}$ and $p_5 = p-\frac{q}{24}$. So, under the condition of (25), the inequality (3) is satisfied and the solution c_n of the Eq. (19) is unstable. When the solution of Eq. (19) is unstable it is observed that solution $x(t)$ of (1) is unstable. □

3. Example

Example 1. *Let us consider fourth-order neutral differential equation with piece-wise argument*

$$\frac{d^4}{dt^4}\Big(x(t)-x(t-1)\Big) = -x([t-1]), \quad t \geq 0, \tag{26}$$

and initial function

$$\varphi(t) = t, \quad -1 \leq t \leq 0. \tag{27}$$

This initial value problem is a special case of (1)-(2) with $p=-1, q=-1$ and $\varphi(t)=t$. We can obtain corresponding difference equation of Eq. (26) from (19) *as follows:*

$$c_{n+5} - 5c_{n+4} + \frac{241}{24}c_{n+3} - \frac{229}{24}c_{n+2} + \frac{131}{24}c_{n+1} - \frac{23}{24}c_n = 0, \quad n=-1,0,1,... \tag{28}$$

and also, if $\alpha_0 = \beta_0 = \gamma_0 = 0$ is taken in the equations (20), we can write the initial conditions: $c_{-1} = -1, c_0 = 0, c_1 = \frac{25}{24}, c_2 = \frac{8}{3}, c_3 = \frac{3647}{576}$. Thus, the difference equation (28) *has a unique solution c_n. It can be clearly seen that the solution c_n of Eq. (28) is not asymptotically stable. Finally, if the c_n solution is substituted into equation* (21) *for $n=-1,0,1,...$ and the equations* (16) *are used, the $x(t)$ solution of equation* (26) *is found. This solution is not asymptotically stable (See Figure 1).*

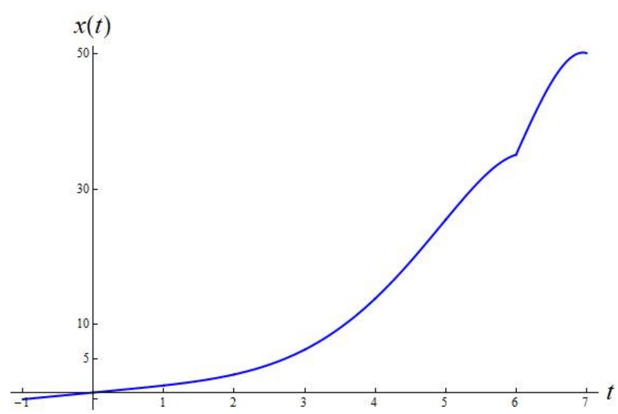

FIGURE 1. Solution $x(t)$ of initial value problem (26)-(27).

4. Conclusion

In this study, we have investigated a fourth-order neutral differential equation with piecewise constant arguments. Our analysis has focused on demonstrating the existence and uniqueness of solutions for the equation, along with a specified initial condition. Through rigorous mathematical analysis, we have established the conditions necessary for stability in the considered equation. Our findings contribute to the understanding of differential equations with piecewise constant arguments and provide valuable insights into their behavior and stability properties. This work not only enhances theoretical understanding but also offers practical implications for various applications where such equations arise. In this study, we have demonstrated that the zero solution of a fourth-order neutral differential equation with piecewise constant arguments of type (1) is not asymptotically stable. Further research could explore extensions of these results to more complex systems or investigate additional properties of similar equations. Also, these analyses can be made more generalized. Moreover, the oscillation state of the solutions of the equations (1) and (4) can be investigated. This is an open problem.

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EXISTENCE AND UNIQUENESS OF A WEAK SOLUTION FOR SINGULAR WEIGHTED ROBIN PROBLEM INVOLVING $p(.)$-BIHARMONIC OPERATOR

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Abstract. The aim of this paper is to find the existence of solutions for the following class of singular fourth order equation involving the weighted $p(.)$-biharmonic operator:

$$\begin{cases} \Delta\left(a(x)\left|\Delta u\right|^{p(x)-2}\Delta u\right) = \lambda b(x)\left|u\right|^{q(x)-2}u + V(x)\left|u\right|^{-\gamma(x)}, & x\in\Omega, \\ a(x)\left|\Delta u\right|^{p(x)-2}\frac{\partial u}{\partial \upsilon} + \beta(x)\left|u\right|^{p(x)-2}u = 0, & x\in\partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\geq 2)$. Using variational methods, we prove the existence at least one nontrivial weak solution of such a Robin problem in weighted variable exponent second order Sobolev spaces $W_a^{2,p(.)}(\Omega)$ under some appropriate conditions. Finally, we deduce some uniqueness results.

1. Introduction

In this paper, the weighted singular Robin problem

$$\begin{cases} \Delta\left(a(x)\left|\Delta u\right|^{p(x)-2}\Delta u\right) = \lambda b(x)\left|u\right|^{q(x)-2}u + V(x)\left|u\right|^{-\gamma(x)}, & x\in\Omega, \\ a(x)\left|\Delta u\right|^{p(x)-2}\frac{\partial u}{\partial \upsilon} + \beta(x)\left|u\right|^{p(x)-2}u = 0, & x\in\partial\Omega, \end{cases} \tag{1}$$

is investigated with respect to some suitable assumptions, where a and b are weight functions and nonnegative, $\frac{\partial u}{\partial \upsilon}$ is the outer unit normal derivative of u on $\partial\Omega$, p,q are continuous functions on $\overline{\Omega}$, i.e. $p,q\in C\left(\overline{\Omega}\right)$ with $1<p^-=\inf_{x\in\Omega}p(x)\leq p(x)\leq p^+=\sup_{x\in\Omega}p(x)<\frac{N}{2}$, $\beta\in L^\infty(\partial\Omega)$ such that $\beta^-=\inf_{x\in\partial\Omega}\beta(x)>0$, and $\Omega\subset\mathbb{R}^N$ $(N>2)$ is a bounded smooth domain, λ is a positive parameter,

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$\gamma : \Omega \to (0,1)$ is a continuous function, $1-\gamma^- < p^-$, $q^+ < p^-$, $V \in L_a^{\frac{p^*(.)}{p^*(.)+\gamma(.)-1}}(\Omega)$, $V > 0$ and $p^*(x) = \frac{Np(x)}{N-2p(x)}$.

In 2018, Chung [12] consider the $p(x)$-Laplacian Robin eigenvalue problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda V(x) |u|^{q(x)-2} u, & x \in \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \upsilon} + \beta(x) |u|^{p(x)-2} u = 0, & x \in \partial\Omega, \end{cases}$$

and prove the existence of a continuous family of eigenvalues in a neighborhood of the origin using variational methods under some suitable conditions on the functions q and V.

In 2024, Chung and Ho [14] use a concentration-compactness principle to solve the lack of compactness of the critical Sobolev imbedding, and obtain the existence of solutions to the following problem involving critical growth

$$\begin{cases} \Delta^2_{p(x)} u - M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \Delta_{p(x)} u = \lambda f(x,u) + |u|^{q(x)-2} u, & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases}$$

In 2011, Ayoujil and Amrouss [8] investigate the following problem:

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2} \Delta u\right) = \lambda |u|^{q(x)-2} u, & x \in \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and obtained that the energy functional associated to the problem (2) has a non-trivial minimum for any positive λ for $\max_{x\in\Omega} q(x) < \min_{x\in\Omega} p(x)$ (see Theorem 3.1 in [8]). When $p(x) = q(x)$, the problem (2) is considered by Ayoujil and Amrouss [7].

In 2015, Ge, Zhou and Wu [20] discuss the following problem:

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2} \Delta u\right) = \lambda V(x) |u|^{q(x)-2} u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where V is an indefinite weight and λ is a positive real number. They obtained several situations concerning the growth rates, and they showed, using the mountain pass lemma and Ekeland's principle, the existence of a continuous family of eigenvalues.

In 2019, Kefi and Saoudi [25] search the existence of solutions for the following inhomogeneous singular equation involving the $p(x)$-biharmonic operator:

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2} \Delta u\right) = g(x) u^{-\gamma(x)} \mp \lambda f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

They study the problem (4), which contains a singular term and indefinite many more general terms than the equation (3), and prove the existence of a weak solution for problem (4).

In 2022, using variational techniques combined with the theory of the generalized Lebesgue-Sobolev spaces Alsaedi, Ali and Ghanmi [1] studied weak solutions for the following class of singular fourth order elliptic equations:

$$\begin{cases} \Delta\left(|x|^{p(x)}|\Delta u|^{p(x)-2}\Delta u\right)=a(x)u^{-\gamma(x)}+\lambda f(x,u), & \text{in } \Omega,\\ u=\Delta u=0, & \text{on } \partial\Omega,\end{cases} \tag{5}$$

and prove the existence at least one nontrivial weak solution in $W_0^{2,p(.)}(\Omega)$.

In 2022, Mbarki [32] discuss the existence of solutions for a class of singular $p(x)$-biharmonic Laplacian problem with Navier boundary conditions:

$$\begin{cases} \Delta\left(|x|^{p(x)}|\Delta u|^{p(x)-2}\Delta u\right)=\lambda V(x)|u|^{q(x)-2}u+a(x)u^{-\gamma(x)}, & \text{in } \Omega,\\ u=\Delta u=0, & \text{on } \partial\Omega.\end{cases} \tag{6}$$

In 2022, Kulak, Aydın and Unal [28] consider the existence of weak solutions of weighted Robin problem involving $p(.)$-biharmonic operator:

$$\begin{cases} \Delta\left(a(x)|\Delta u|^{p(x)-2}\Delta u\right)=\lambda b(x)|u|^{q(x)-2}u, & \text{in } \Omega,\\ a(x)|\Delta u|^{p(x)-2}\frac{\partial u}{\partial \upsilon}+\beta(x)|u|^{p(x)-2}u=0, & \text{on } \partial\Omega.\end{cases} \tag{7}$$

under some conditions in $W_{a,b}^{2,p(.)}(\Omega)$. We refer for instance to see ([2], [13], [22], [24], [26]).

Inspired by the articles mentioned above, we show the existence and uniqueness of nontrivial solutions of problem (1) using compact embedding theorems in $W_a^{2,p(.)}(\Omega)$ and variational methods. Therefore, we will obtain more general results than the problems (4), (5), (6).

2. Abstract setting

Let Ω be a bounded open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Put

$$C_+\left(\overline{\Omega}\right)=\left\{h\in C\left(\overline{\Omega}\right):\inf_{x\in\overline{\Omega}}h(x)>1\right\},$$

For any $p\in C_+\left(\overline{\Omega}\right)$, we set

$$p^-=\inf_{x\in\Omega}p(x) \text{ and } p^+=\sup_{x\in\Omega}p(x)$$

such that $1<p^-\le p^+<\infty$ and

$$L^{p(\cdot)}(\Omega)=\left\{u\,\middle|\,u:\Omega\to\mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx<\infty\right\}$$

with the (Luxemburg) norm

$$\|u\|_{p(.)}=\inf\left\{\lambda>0:\varrho_{p(.)}\left(\frac{u}{\lambda}\right)\le 1\right\},$$

where

$$\varrho_{p(.)}(u)=\int_{\Omega}|u(x)|^{p(x)}\,dx.$$

Moreover, the space $\left(L^{p(.)}(\Omega),\|.\|_{p(.)}\right)$ is a reflexive Banach space [27]. The weighted Lebesgue space $L_a^{p(.)}(\Omega)$ is defined by

$$L_a^{p(.)}(\Omega)=\left\{u\,\middle|\,u:\Omega\longrightarrow\mathbb{R}\text{ measurable and }\int_{\Omega}|u(x)|^{p(x)}\,a(x)dx<\infty\right\}$$

such that $\|u\|_{p(.),a}=\left\|ua^{\frac{1}{p(.)}}\right\|_{p(.)}<\infty$ for $u\in L_a^{p(.)}(\Omega)$, where a is a weight function from Ω to $(0,\infty)$. Moreover, $u\in L_a^{p(.)}(\Omega)$ if and only if $|u|^{p(.)}a\in L^1(\Omega)$ [34].

We can define the space $L_a^{p(.)}(\partial\Omega)$ similarly by

$$L_a^{p(.)}(\partial\Omega)=\left\{u\,\middle|\,u:\partial\Omega\longrightarrow\mathbb{R}\text{ measurable and }\int_{\partial\Omega}|u(x)|^{p(x)}\,a(x)d\sigma<+\infty\right\}$$

with the norm

$$\|u\|_{p(.),a,\partial\Omega}=\inf\left\{\tau>0:\int_{\partial\Omega}\left|\frac{u(x)}{\tau}\right|^{p(x)}a(x)d\sigma\leq 1\right\}$$

for $u\in L_a^{p(.)}(\partial\Omega)$, where $d\sigma$ is the measure on the boundary of Ω. Then $\left(L_a^{p(.)}(\partial\Omega),\|.\|_{p(.),a,\partial\Omega}\right)$ is a a reflexive Banach space. If $a\in L^{\infty}(\Omega)$, then $L_a^{p(.)}=L^{p(.)}$ [15].

Proposition 1. *(see [3], [5], [6], [19], [21], [30], [31]) For all $u,v\in L_a^{p(.)}(\Omega)$, we have*

(i) $\|u\|_{p(.),a}<1$ *(resp.*$=1,>1$*) if and only if* $\varrho_{p(.),a}(u)<1$ *(resp.*$=1,>1$*),*

(ii) $\|u\|_{p(.),a}^{p^-}\leq\varrho_{p(.),a}(u)\leq\|u\|_{p(.),a}^{p^+}$ *with* $\|u\|_{p(.),a}>1$,

(iii) $\|u\|_{p(.),a}^{p^+}\leq\varrho_{p(.),a}(u)\leq\|u\|_{p(.),a}^{p^-}$ *with* $\|u\|_{p(.),a}<1$

(iv) $\min\left\{\|u\|_{p(.),a}^{p^-},\|u\|_{p(.),a}^{p^+}\right\}\leq\varrho_{p(.),a}(u)\leq\max\left\{\|u\|_{p(.),a}^{p^-},\|u\|_{p(.),a}^{p^+}\right\}$,

(v) $\min\left\{\varrho_{p(.),a}(u)^{\frac{1}{p^-}},\varrho_{p(.),a}(u)^{\frac{1}{p^+}}\right\}\leq\|u\|_{p(.),a}\leq\max\left\{\varrho_{p(.),a}(u)^{\frac{1}{p^-}},\varrho_{p(.),a}(u)^{\frac{1}{p^+}}\right\}$,

(vi) $\varrho_{p(.),a}(u-v)\to 0$ *if and only if* $\|u-v\|_{p(.),a}\to 0$.

Proposition 2. *(see [17])Let p and q be two measurable functions such that $p\in L^{\infty}(\Omega)$ and $1\leq p(x)q(x)\leq\infty$ for a.e. $x\in\Omega$. Let $u\in L^{q(.)}(\Omega)$, $u\neq 0$. Then*

$$\min\left\{\|u\|_{p(.)q(.)}^{p^+},\|u\|_{p(.)q(.)}^{p^-}\right\}\leq\left\||u|^{p(.)}\right\|_{q(.)}\leq\max\left\{\|u\|_{p(.)q(.)}^{p^+},\|u\|_{p(.)q(.)}^{p^-}\right\}.$$

Let $a^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\Omega)$ and $k \in \mathbb{Z}^+$. Hence we define the weighted variable exponent Sobolev space $W^{k,p(.)}_a(\Omega)$ is defined by

$$W^{k,p(.)}_a(\Omega) = \left\{ u \in L^{p(.)}_a(\Omega) : D^\alpha u \in L^{p(.)}_a(\Omega), 0 \leq |\alpha| \leq k \right\},$$

where $\alpha \in \mathbb{N}_0^N$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_N$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}_{x_1} ... \partial^{\alpha_N}_{x_N}}$. Then $W^{k,p(.)}_a(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{W^{k,p(.)}_a} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{p(.),a}.$$

Alternatively, the space $W^{k,p(.)}_a(\Omega)$ could also be introduced as

$$W^{k,p(.)}_a(\Omega) = \left\{ u \in W^{k-1,p(.)}_a(\Omega) : D_i u = \frac{\partial u}{\partial x_i} \in W^{k-1,p(.)}_a(\Omega), \forall i = 1, 2, ...N \right\}.$$

To find out solutions of the problem (1), we need some essential theories on the space $W^{2,p(.)}_a(\Omega)$. The space $X = W^{2,p(.)}_a(\Omega)$ consists of all measurable functions $u \in L^{p(.)}_a(\Omega)$ such that $D^\alpha u \in L^{p(.)}_a(\Omega)$ for $0 \leq |\alpha| \leq 2$. Hence for any $u \in X$,

$$\|u\|_X = \|u\|_{p(.),a} + \|\nabla u\|_{p(.),a} + \sum_{|\alpha|=2} \|D^\alpha u\|_{p(.),a}$$

Let

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)}, & \text{if } p(x) < \frac{N}{2}, \\ +\infty, & \text{if } p(x) \geq \frac{N}{2}, \end{cases}$$

for every $x \in \overline{\Omega}$. For $p, q \in C_+(\overline{\Omega})$ in which $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding $W^{2,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ (non-weighted). It is obvious that $p(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

Remark 1. *There is a continuous embedding $X \hookrightarrow L^{p^*(.)}_a(\Omega)$ under some conditions.*

Proof. Firstly, we show by induction on k that $W^{k,p(.)}_a(\Omega) \hookrightarrow L^{p^*(.)}_a(\Omega)$. Let $k = 1$. If $0 < a_1 \leq a(x) < a_2 < \infty$ for a.e. $x \in \Omega$, then it is well known that the embedding $W^{1,p(.)}_a(\Omega) \cong W^{1,p(.)}(\Omega) \hookrightarrow L^{p^*(.)}(\Omega)$ for non-weighted case. Moreover, the embedding $W^{1,p(.)}_a(\Omega) \hookrightarrow L^{p^*(.)}_a(\Omega)$ is also valid for weighted case (see [18], [25], [27]). Suppose that the embedding $W^{k-1,p(.)}_a(\Omega) \hookrightarrow L^{r(.)}_a(\Omega)$ is satisfied for $r(x) = Np(x)/(N - ((k-1)p(x)))$ when $p(x) < \frac{N}{k-1}$. Since $u \in W^{k,p(.)}_a(\Omega)$, then u and $D_j u$ $(1 \leq j \leq N)$ belong to $W^{k-1,p(.)}_a(\Omega)$, where $p(x) < \frac{N}{k}$. So it is easy to see that $u \in W^{1,r(.)}_a(\Omega)$ and

$$\|u\|_{W^{1,r(.)}_a} \leq C_1 \|u\|_{W^{k,p(.)}_a}.$$

Due to $kp(x) < N$, we get $r(x) < N$ and $W_a^{1,r(.)}(\Omega) \hookrightarrow L_a^{p^*(.)}(\Omega)$, where $p^*(x) = Nr(x)/N - r(x) = Np(x)/N - kp(x)$ and

$$\|u\|_{p^*,a} \leq C_2 \|u\|_{W_a^{1,r(.)}} \leq C_3 \|u\|_{W_a^{k,p(.)}},$$

i.e. the embedding $W_a^{k,p(.)}(\Omega) \hookrightarrow L_a^{p^*(.)}(\Omega)$ is continuous. So $X \hookrightarrow L_a^{p^*(.)}(\Omega)$. □

For $A \subset \overline{\Omega}$, denote by $p^-(A) = \inf_{x\in A} p(x)$ and $p^+(A) = \sup_{x\in A} p(x)$. Define

$$p^\partial(x) = (p(x))^\partial = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N, \end{cases}$$

and

$$p^\partial_{r(x)}(x) = \frac{r(x)-1}{r(x)} p^\partial(x)$$

for any $x \in \partial\Omega$ and $r \in C(\partial\Omega, \mathbb{R})$ with $r^- = \inf_{x\in\partial\Omega} r(x) > 1$.

Theorem 1. *(see [15])Assume that the set $\partial\Omega$ possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$. If $q \in C(\partial\Omega)$ and the inequality $1 \leq q(x) < p^\partial_{r(x)}(x)$ is valid for all $x \in \partial\Omega$, then there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L_a^{q(.)}(\partial\Omega)$ for $a \in L^{r(.)}(\partial\Omega)$, $r \in C(\partial\Omega)$ with $r(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ for all $x \in \partial\Omega$. In particular, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega)$, where $1 \leq q(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.*

It is easy to see that $p^\partial_{r(x)}(x) < p^\partial(x)$ and $p(x) < p^\partial(x)$. So we have the following Corollary under conditions in Theorem 1.

Corollary 1. *(see [15])*

(i) There is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\partial\Omega)$, where $1 \leq p(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.

(ii) There is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L_a^{p(.)}(\partial\Omega)$, where $1 \leq p(x) < p^\partial_{r(x)}(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.

Theorem 2. *([5])Let $a^{-\alpha(.)} \in L^1(\Omega)$ with $\alpha(x) \in \left(\frac{N}{p(x)}, \infty\right) \cap \left[\frac{1}{p(x)-1}, \infty\right)$. Then we have the compact embedding $W_a^{1,p(.)}(\Omega) \hookrightarrow W^{1,p_*(.)}(\Omega)$, where $p_*(x) = \frac{\alpha(x)p(x)}{\alpha(x)+1}$.*

Corollary 2. *If the inequality $p(x) < p^\partial_{*,r(x)}(x) < p^\partial_*(x)$ is valid for all $x \in \partial\Omega$, then there exists a compact embedding between $W_a^{1,p(.)}(\Omega)$ and $L_a^{p(.)}(\partial\Omega)$.*

Corollary 3. $X \hookrightarrow W_a^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L_a^{p(.)}(\partial\Omega)$.

Theorem 3. *(see [19])Assume that the set $\partial\Omega$ possesses the cone property and $p \in C(\overline{\Omega})$. Suppose that $b \in L^{r(.)}(\Omega)$, $b(x) > 0$ for $x \in \Omega$, $r \in C(\overline{\Omega})$ and $r^- > 1$. If $q \in C(\overline{\Omega})$ and*

$$1 \leq q(x) < \frac{r(x)-1}{r(x)} p^\blacklozenge(x)$$

for all $x \in \overline{\Omega}$, *then there is a compact embedding* $W^{1,p(.)}(\Omega) \hookrightarrow L_b^{q(.)}(\Omega)$, *where*

$$p^{\blacklozenge}(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

Corollary 4. *If the inequality* $1 \leq q(x) < \frac{r(x)-1}{r(x)} (p_*)^{\blacklozenge}(x)$ *is true for all* $x \in \overline{\Omega}$, *then there exists a compact embedding between* $W_a^{1,p(.)}(\Omega)$ *and* $L_b^{q(.)}(\Omega)$. *So* $X \hookrightarrow\hookrightarrow L_b^{q(.)}(\Omega)$.

If we use the method in Theorem 2.1 in [16] and [4], then we obtain the following theorem. In addition, this theorem plays an important role for the existence of weak solutions of the problem (1).

Theorem 4. *(see Theorem 3 in [28])Let* $u \in X$. *Then the norms* $\|u\|_{\partial}$ *and* $\|u\|_X$ *are equivalent on* X, *where*

$$\|u\|_{\partial} = \|\Delta u\|_{p(.),a} + \|u\|_{p(.),a,\partial\Omega}.$$

Let $\beta \in L^{\infty}(\partial\Omega)$ such that $\beta^- = \inf_{x\in\partial\Omega} \beta(x) > 0$. Then, the norm $\|u\|_{\beta(x)}$ is defined by

$$\|u\|_{\beta(x)} = \inf \left\{ \tau > 0 : \int_{\Omega} a(x) \left| \frac{\Delta u(x)}{\tau} \right|^{p(x)} dx + \int_{\partial\Omega} \beta(x) \left| \frac{u(x)}{\tau} \right|^{p(x)} d\sigma \leq 1 \right\}$$

for any $u \in X$. Moreover, $\|.\|_{\beta(x)}$ and $\|.\|_X$ are equivalent on X by Theorem 4.

Proposition 3. *(see [6], [21], [30], [31]) Let* $I_{\beta(x)}(u) = \int_{\Omega} a(x) |\Delta u(x)|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)} d\sigma$ *with* $\beta^- > 0$. *For any* $u, u_k \in X$ $(k = 1, 2, ...)$, *we have*

- *(i)* $\|u\|_{\beta(x)}^{p^-} \leq I_{\beta(x)}(u) \leq \|u\|_{\beta(x)}^{p^+}$ *with* $\|u\|_{\beta(x)} \geq 1$,
- *(ii)* $\|u\|_{\beta(x)}^{p^+} \leq I_{\beta(x)}(u) \leq \|u\|_{\beta(x)}^{p^-}$ *with* $\|u\|_{\beta(x)} \leq 1$,
- *(iii)* $\min\left\{\|u\|_{\beta(x)}^{p^-}, \|u\|_{\beta(x)}^{p^+}\right\} \leq I_{\beta(x)}(u) \leq \max\left\{\|u\|_{\beta(x)}^{p^-}, \|u\|_{\beta(x)}^{p^+}\right\}$,
- *(iv)* $\|u - u_k\|_{\beta(x)} \to 0$ *if and only if* $I_{\beta(x)}(u - u_k) \to 0$ *as* $k \to \infty$,
- *(v)* $\|u_k\|_{\beta(x)} \to \infty$ *if and only if* $I_{\beta(x)}(u_k) \to \infty$ *as* $k \to \infty$.

Definition 1. *We say that* $u \in X$ *is a weak solution of (1) if*

$$\int_{\Omega} a(x) |\Delta u|^{p(x)-2} \Delta u \Delta v dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)-2} uv d\sigma$$
$$-\lambda \int_{\Omega} b(x) |u|^{q(x)-2} uv dx - \int_{\Omega} V(x) |u|^{-\gamma(x)} v dx = 0$$

for all $v \in X$. *We point out that if* $\lambda \in \mathbb{R}$ *is an eigenvalue of the problem (1), then the corresponding* $u \in X - \{0\}$ *is a weak solution of (1).*

To obtain a weak solution to (1), let us introduce the functional $E_\lambda : X \to \mathbb{R}$ defined by

$$E_\lambda(u) = \phi(u) - \lambda \int_\Omega \frac{b(x)}{q(x)} |u|^{q(x)} dx - \Phi_\lambda(u),$$

for any $\lambda > 0$, where

$$\phi(u) = \int_\Omega \frac{a(x)}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)} d\sigma$$

and

$$\Phi_\lambda(u) = \int_\Omega \frac{V(x)}{1-\gamma(x)} |u|^{1-\gamma(x)} dx.$$

Due to the singular term $V(x)|u|^{-\gamma(x)}$, E_λ is not of class C^1 functional in X, and classical variational methods (e.g Mountain-Pass Lemma of Ambrosetti-Robinowitz) are not applicable. It is easy to see that

$$\begin{aligned} < \quad E'_\lambda(u), u >= & \int_\Omega a(x) |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)} d\sigma \\ & -\lambda \int_\Omega b(x) |u|^{q(x)} dx - \int_\Omega V(x) |u|^{-\gamma(x)} dx \end{aligned}$$

for all $u \in X$.

3. Main Results

In this section, we will show that the problem (1) has at least one nontrivial weak solution. Throughout this paper, assume that $1 < p^- \le p^+ < \frac{N}{2}$, $\beta \in L^\infty(\partial\Omega)$, $V \in L_a^{\frac{p^*(.)}{p^*(.)+\gamma(.)-1}}(\Omega)$, $V > 0$ and $a, b > 0$.

Theorem 5 (Vitali's Theorem). *(see p. 60 in [29])Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions with finite integrals over a measurable set $\Omega \subset \mathbb{R}^N$. Suppose that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for almost all $x \in \Omega$ and let f be an almost everywhere finite function. Suppose that the following condition (P) is satisfied:

(P) (Equi-absolutely-continuous) For every $\varepsilon > 0$ there exists a $\delta > 0$ with the property: if $B \subset \Omega$, $\mu(B) < \delta$, then

$$\int_\Omega |f_n(x)| dx < \varepsilon$$

for all $n \in \mathbb{N}$. Hence, the function f has a finite integral over Ω and

$$\lim_{n\to\infty} \int\limits_{\Omega} |f_n(x)|\, dx = \int\limits_{\Omega} |f(x)|\, dx.$$

Theorem 6 (Absolute Continuity of the Lebesgue Integral)**.** *(see Theorem 12.34 in [23]) Let $f \in L^1(\Omega)$. For every $\varepsilon > 0$ there exists a $\delta > 0$ depending only on ε and f such that for all $A \subset \mathbb{R}^N$ satisfying $\mu(A) < \delta$, we have*

$$\int\limits_{A} |f(x)|\, dx < \varepsilon.$$

Lemma 1. *Let $V \in L_a^{\frac{p^*(.)}{p^*(.)+\gamma(.)-1}}(\Omega)$ and $0 < r < a(x)$ for a.e $x \in \Omega$ and some $r > 0$. Then E_λ is weakly lower semi-continuous.*

Proof. The proof consists of three steps.

Step 1: The functional $\phi : X \to \mathbb{R}$ is convex. Indeed, since the function $t \to t^\theta$ is convex on $[0, \infty)$ for any $\theta > 1$, so for each $x \in \Omega$ (or $x \in \partial\Omega$)

$$\left|\frac{\xi + \mu}{2}\right|^{p(x)} \le \left(\frac{|\xi| + |\mu|}{2}\right)^{p(x)} \le \frac{1}{2}|\xi|^{p(x)} + \frac{1}{2}|\mu|^{p(x)}$$

for all $\xi, \mu \in \mathbb{R}^N$. Hence, we have

$$\left|\frac{\Delta u + \Delta v}{2}\right|^{p(x)} \le \left(\frac{|\Delta u| + |\Delta v|}{2}\right)^{p(x)} \le \frac{1}{2}|\Delta u|^{p(x)} + \frac{1}{2}|\Delta v|^{p(x)} \tag{8}$$

and

$$\left|\frac{u + v}{2}\right|^{p(x)} \le \left(\frac{|u| + |v|}{2}\right)^{p(x)} \le \frac{1}{2}|u|^{p(x)} + \frac{1}{2}|v|^{p(x)}. \tag{9}$$

Multiplying (8) and (9) by $\frac{a(x)}{p(x)}$, $\frac{\beta(x)}{p(x)}$ and integrating over Ω and $\partial\Omega$ respectively, we obtain

$$\phi(\frac{u+v}{2}) \le \frac{1}{2}\phi(u) + \frac{1}{2}\phi(v)$$

for any $u, v \in X$. So ϕ is convex.

Step 2: ϕ is weakly lower semi continuous on X. From Step 1 and Corollary 3.8 in [10] it is enough to show that ϕ is strongly lower semi continuous on X. Let $\varepsilon > 0$, $u, v \in X$ such that

$$\|u - v\|_X < \frac{\varepsilon}{\left\| a^{\frac{p(x)-1}{p(x)}} |\Delta u|^{p(x)-1} \right\|_{\frac{p(x)}{p(x)-1}}} < \frac{\varepsilon}{C_6 + C_7}. \tag{10}$$

Since the functional ϕ is convex, variable Hölder inequality and Proposition 2, we obtain

$$\phi(v) \quad \ge \quad \phi(u) + \left\langle \phi'(u), v - u \right\rangle$$

$$
\begin{aligned}
&\geq \phi(u) - \int_{\Omega} a(x)\,|\Delta u|^{p(x)-1}\,|\Delta (v-u)|\,dx - \int_{\partial\Omega} \beta(x)\,|u(x)|^{p(x)-1}\,|u-v|\,d\sigma \\
&\geq \phi(u) - C_4 \left\| a^{\frac{p(.)-1}{p(.)}}\,|\Delta u|^{p(.)-1} \right\|_{\frac{p(.)}{p(.)-1}} \left\| a^{\frac{1}{p(.)}}\,|\Delta (v-u)| \right\|_{p(.)} \\
&\quad - C_5 \left\| \beta^{\frac{p(.)-1}{p(.)}}\,|u|^{p(.)-1} \right\|_{\frac{p(.)}{p(.)-1},\partial\Omega} \left\| \beta^{\frac{1}{p(.)}}\,|u-v| \right\|_{p(.),\partial\Omega} \\
&\geq \phi(u) - C_4 \max\left\{ \left\| a^{\frac{1}{p(.)}}\,|\Delta u| \right\|_{p(.)}^{p^+-1}, \left\| a^{\frac{1}{p(.)}}\,|\Delta u| \right\|_{p(.)}^{p^--1} \right\} \||\Delta (v-u)|\|_{p(.),a} \\
&\quad - C_5 \max\left\{ \left\| \beta^{\frac{1}{p(.)}}\,|u| \right\|_{p(.),\partial\Omega}^{p^+-1}, \left\| \beta^{\frac{1}{p(.)}}\,|u| \right\|_{p(.),\partial\Omega}^{p^--1} \right\} \||u-v|\|_{p(.),\beta,\partial\Omega} \\
&= \phi(u) - C_4 \max\left\{ \|\Delta u\|_{p(.),a}^{p^+-1}, \|\Delta u\|_{p(.),a}^{p^--1} \right\} \||\Delta (v-u)|\|_{p(.),a} \\
&\quad - C_5 \max\left\{ \|u\|_{p(.),\beta,\partial\Omega}^{p^+-1}, \|u\|_{p(.),\beta,\partial\Omega}^{p^--1} \right\} \||u-v|\|_{p(.),\beta,\partial\Omega} \\
&\geq \phi(u) - C_6 \|u-v\|_X - C_7 \|u-v\|_X \geq \phi(u) - \varepsilon,
\end{aligned}
$$

for some positive constants C_4, C_5, C_6 and C_7. It follows that ϕ is strongly lower semi continuous and convex, so we deduce that the functional I is weakly lower semi continuous.

Step 3: E_λ is weakly lower semi-continuous. Let $\{u_n\}$ be a sequence which is weakly converges to u in X. Then, from Step 2, we have

$$\phi(u) \leq \liminf_{n\to\infty} \phi(u_n). \tag{11}$$

By Corollary 4 we have the compact embedding $X \hookrightarrow\hookrightarrow L_b^{q(.)}(\Omega)$. Hence, the sequence $\{u_n\}$ converges strongly to u in $L_b^{q(.)}(\Omega)$ and

$$\lim_{n\to\infty} \int_{\Omega} \frac{b(x)}{q(x)}\,|u_n|^{q(x)}\,dx = \liminf_{n\to\infty} \int_{\Omega} \frac{b(x)}{q(x)}\,|u_n|^{q(x)}\,dx = \int_{\Omega} \frac{b(x)}{q(x)}\,|u|^{q(x)}\,dx. \tag{12}$$

On the other hand, by Vitali's Theorem, we can claim that

$$\lim_{n\to\infty} \int_{\Omega} V(x)\,|u_n|^{1-\gamma(x)}\,dx = \int_{\Omega} V(x)\,|u|^{1-\gamma(x)}\,dx. \tag{13}$$

Indeed, we only need to prove that

$$\left\{ \int_{\Omega} V(x)\,|u_n|^{1-\gamma(x)}\,dx, n \in \mathbb{N} \right\} \tag{14}$$

is equi-absolutely-continuous. It is known that every weakly convergent sequence is bounded. So $(u_n)_{n\in\mathbb{N}}$ is bounded in X. In addition, using the continuous embedding $X \hookrightarrow L_a^{p^*(.)}(\Omega)$ by Remark 1, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L_a^{p^*(.)}(\Omega)$, and there exists a $C_8 > 0$ such that $\|u_n\|_{p^*(.),a} < C_8$ for all $n \in \mathbb{N}$. Now, let $\varepsilon > 0$, then, using Proposition 1 and the absolutely-continuity of $\int_{\Omega} |V(x)|^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} a(x)dx$, there exist two positive constants ς and ξ such that

$$\|V\|^{\varsigma}_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \leq \int_{\Omega} |V(x)|^{\frac{p^*(x)}{p^*(x)+\gamma(x)-1}} a(x)dx < \varepsilon^{\xi} \tag{15}$$

for every $\Omega_2 \subset \Omega$. Consequently, by the Hölder inequality, Proposition 2 and (15) we have

$$\begin{aligned}
\int_{\Omega} |V(x)| |u_n|^{1-\gamma(x)} dx &\leq \int_{\Omega} \left(|V(x)| a(x)^{\frac{p^*(x)+\gamma(x)-1}{p^*(x)}}\right) \left(|u_n|^{1-\gamma(x)} a(x)^{-\frac{p^*(x)+\gamma(x)-1}{p^*(x)}}\right) dx \\
&\leq C_9 \left\| |V(x)| a(x)^{\frac{p^*(x)+\gamma(x)-1}{p^*(x)}} \right\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1}} \left\| |u_n|^{1-\gamma(x)} a(x)^{\frac{1-p^*(x)-\gamma(x)}{p^*(x)}} \right\|_{\frac{p^*(.)}{1-\gamma(.)}} \\
&= C_9 \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \cdot \left\| |u_n|^{1-\gamma(x)} a(x)^{\frac{1-\gamma(x)}{p^*(x)}} a(x)^{-1} \right\|_{\frac{p^*(.)}{1-\gamma(.)}} \\
&\leq C_{10} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \left\| \left(|u_n| a(x)^{\frac{1}{p^*(x)}}\right)^{1-\gamma(x)} \right\|_{\frac{p^*(.)}{1-\gamma(.)}} \\
&\leq C_{10} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \max\left\{ \left\| |u_n| a(x)^{\frac{1}{p^*(x)}} \right\|^{1-\gamma^+}_{p^*(.)}, \left\| |u_n| a(x)^{\frac{1}{p^*(x)}} \right\|^{1-\gamma^-}_{p^*(.)} \right\} \\
&= C_{10} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \max\left\{ \|u_n\|^{1-\gamma^+}_{p^*(.),a}, \|u_n\|^{1-\gamma^-}_{p^*(.),a} \right\} \\
&\leq C_{10} \|V\|_{\frac{p^*(.)}{p^*(.)+\gamma(.)-1},a} \|u_n\|^{d}_{p^*(.),a} < C_{10}\varepsilon^{\xi} \|u_n\|^{d}_{p^*(.),a}
\end{aligned}$$

for $d > 0$. So the claim (13) is obtained because of the boundedness of the sequence $(u_n)_{n\in\mathbb{N}}$ in $L_a^{p^*(.)}(\Omega)$. So we have

$$E_{\lambda}(u) \leq \liminf_{n\to\infty} E_{\lambda}(u_n)$$

by (11), (12) and (13). □

Lemma 2. *E_{λ} is bounded from below and coercive.*

Proof. It is clear that

$$\begin{aligned}
E_{\lambda}(u) &= \int_{\Omega} \frac{a(x)}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)} d\sigma - \lambda \int_{\Omega} \frac{b(x)}{q(x)} |u|^{q(x)} dx \\
&\quad - \int_{\Omega} \frac{V(x)}{1-\gamma(x)} |u|^{1-\gamma(x)} dx \\
&\geq \frac{1}{p^+} I_{\beta(x)} - \frac{\lambda}{q^-} \int_{\Omega} b(x) |u|^{q(x)} dx - \frac{1}{1-\gamma^+} \int_{\Omega} V(x) |u|^{1-\gamma(x)} dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{p^{+}} I_{\beta(x)}(u)-\frac{\lambda}{q^{-}} \max \left\{\|u\|_{q(.), b}^{q^{-}},\|u\|_{q(.), b}^{q^{+}}\right\}-\frac{1}{1-\gamma^{+}} \int_{\Omega}|V(x)||u|^{1-\gamma(x)} d x \\
&\geq \frac{1}{p^{+}} I_{\beta(x)}(u)-\frac{\lambda}{q^{-}}\|u\|_{q(.), b}^{q^{-}}-\frac{1}{1-\gamma^{+}}\|V\|_{\frac{p^{*}(.)}{p^{*}(.)+\gamma(.)-1}, a} \max \left\{\|u\|_{\beta(x)}^{1-\gamma^{+}},\|u\|_{\beta(x)}^{1-\gamma^{-}}\right\} \\
&\geq \frac{1}{p^{+}}\|u\|_{\beta(x)}^{p^{-}}-\frac{\lambda C_{11}}{q^{-}}\|u\|_{\beta(x)}^{q^{-}}-\frac{1}{1-\gamma^{+}}\|V\|_{\frac{p^{*}(.)}{p^{*}(.)+\gamma(.)-1}, a}\|u\|_{\beta(x)}^{1-\gamma^{-}} .
\end{aligned}
$$

Since $1-\gamma^{-}<p^{-}$ and $q^{+}<p^{-}$, we infer that $E_{\lambda}(u) \rightarrow \infty$ as $u \rightarrow \infty$. So E_{λ} is is bounded from below and coercive. □

Lemma 3. *There exists a function $\varphi \in X$ such that $\varphi \neq 0$ and $E_{\lambda}(\varphi)<0$.*

Proof. Let $\varphi \in C_{0}^{\infty}(\Omega)$ such that $\Omega' \subset \operatorname{supp} \varphi \subset \Omega_{1} \subset \Omega$ and $0 \leq \varphi \leq 1$ in Ω_{1}. Then we have

$$
\begin{aligned}
E_{\lambda}(t \varphi) &= \int_{\Omega} \frac{a(x) t^{p(x)}}{p(x)}|\Delta \varphi|^{p(x)} d x+\int_{\partial \Omega} \frac{\beta(x) t^{p(x)}}{p(x)}|\varphi|^{p(x)} d \sigma-\lambda \int_{\Omega} \frac{b(x) t^{q(x)}}{q(x)}|\varphi|^{q(x)} d x \\
&\quad -\int_{\Omega} \frac{V(x) t^{1-\gamma(x)}}{1-\gamma(x)}|\varphi|^{1-\gamma(x)} d x \\
&\leq \frac{t^{p^{-}}}{p^{-}} I_{\beta(x)}(\varphi)-\frac{\lambda}{q^{+}} \int_{\Omega} t^{q(x)}|\varphi|^{q(x)} b(x) d x-\int_{\Omega} \frac{V(x) t^{1-\gamma(x)}}{1-\gamma^{-}}|\varphi|^{1-\gamma(x)} d x \\
&\leq \frac{t^{p^{-}}}{p^{-}} I_{\beta(x)}(\varphi)-\frac{t^{1-\gamma^{-}}}{1-\gamma^{-}} \int_{\Omega} V(x)|\varphi|^{1-\gamma(x)} d x
\end{aligned}
$$

for any $t \in(0,1)$. Since $1-\gamma^{-}<p^{-}$, we obtain $E_{\lambda}(t \varphi)<0$ for any $t<\delta^{\frac{1}{p^{-}-\left(1-\gamma^{-}\right)}}$ with $0<\delta<\min \left\{1, \frac{\frac{p^{-}}{1-\gamma^{-}}}{I_{\beta(x)}(\varphi)} \int_{\Omega} V(x)|\varphi|^{1-\gamma(x)} d x\right\}$. Finally, we point out that $I_{\beta(x)}(\varphi)>0$. In fact, if $I_{\beta(x)}(\varphi)=0$, then $\|\varphi\|_{\beta(x)}=0$ and consequently $\varphi=0$ in Ω, which is a contradiction. □

Theorem 7. *The problem (1) has at least one nontrivial weak solution.*

Proof. From Lemma 2 we can define

$$
m_{\lambda}=\inf _{u \in X} E_{\lambda}(u) .
$$

Let $\left(u_{n}\right)_{n \in \mathbb{N}}$ be a minimizing sequence, that is $E_{\lambda}\left(u_{n}\right) \rightarrow m_{\lambda}$ as $n \rightarrow \infty$. Assume that $\left(u_{n}\right)_{n \in \mathbb{N}}$ is not bounded. So $\left\|u_{n}\right\|_{X} \rightarrow \infty$ as $n \rightarrow \infty$. Since E_{λ} is coercive, we have

$$
E_{\lambda}\left(u_{n}\right) \rightarrow+\infty \text { as }\left\|u_{n}\right\|_{X} \rightarrow \infty .
$$

This contradicts the fact that $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence, so $(u_n)_{n\in\mathbb{N}}$ is bounded in X. Since X is a reflexive Banach space, then there exists a subsequence still denoted by u_n and $u_\lambda \in X$ such that $u_n \rightharpoonup u_\lambda$ weakly in X. From Lemma 1

$$E_\lambda (u_\lambda) \leq \liminf_{n\to\infty} E_\lambda (u_n) = m_\lambda.$$

On the other hand, from the definition of m_λ, we have $m_\lambda \leq E_\lambda (u_\lambda)$. Therefore, u_λ is a global minimum for E_λ, which is a weak solution for the problem (1). Finally, Lemma 3 it follows that $u_\lambda \neq 0$. The proof of the Theorem is completed. □

4. Uniqueness of the Solution

We begin considering the following problem

$$\begin{cases} \Delta \left(a(x) |\Delta u_n|^{p(x)-2} \Delta u_n\right) = \frac{V(x)}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}, & x \in \Omega, \\ a(x) |\Delta u_n|^{p(x)-2} \frac{\partial u_n}{\partial v} + \beta(x) |u_n|^{p(x)-2} u_n = 0, & x \in \partial\Omega, \end{cases} \tag{16}$$

where $u_n = \min\{u, n\}$. By Theorem 7, the problem (16) has a solution $u_n \in X \cap L^\infty(\Omega)$ and $u_n > 0$ for each $n \in \mathbb{N}$ (see Lemma 4.1 in [11] and Lemma 3.1 in [9]). Now we recall the algebraic inequality from Lemma A.0.5 in [33].

Lemma 4. *Let $x, y \in \mathbb{R}^N$ and $\langle ., . \rangle$ the standard scalar product in $\mathbb{R}^N$. Then*

$$\left\langle |x|^{p-2} x - |y|^{p-2} y, x - y \right\rangle \geq c |x-y|^p$$

for $p \geq 2$.

Theorem 8. *The problem (16) has a unique solution in $X \cap L^\infty(\Omega)$.*

Proof. Let $n \in \mathbb{N}$ and $u_n, v_n \in X \cap L^\infty(\Omega)$ solves the problem (16). Then we can write

$$\int_\Omega a(x) |\Delta u_n|^{p(x)-2} \Delta u_n \Delta\varphi dx + \int_{\partial\Omega} \beta(x) |u_n|^{p(x)-2} u_n \varphi d\sigma = \int_\Omega \frac{V(x)\varphi}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} dx \tag{17}$$

and

$$\int_\Omega a(x) |\Delta v_n|^{p(x)-2} \Delta v_n \Delta\varphi dx + \int_{\partial\Omega} \beta(x) |v_n|^{p(x)-2} v_n \varphi d\sigma = \int_\Omega \frac{V(x)\varphi}{\left(v_n + \frac{1}{n}\right)^{\gamma(x)}} dx \tag{18}$$

for all $\varphi \in X$. By choosing $(u_n - v_n)^+ = \max\{u_n - v_n, 0\}$ as a test function for the weak solution, and subtracting (18) from (17) we obtain

$$\int_\Omega V(x) \left\{ \frac{1}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} - \frac{1}{\left(v_n + \frac{1}{n}\right)^{\gamma(x)}} \right\} (u_n - v_n)^+ dx = \tag{19}$$

$$\int_\Omega a(x) \left\{ |\Delta u_n|^{p(x)-2} \Delta u_n - |\Delta v_n|^{p(x)-2} \Delta v_n \right\} \Delta (u_n - v_n)^+ dx$$

$$+\int_{\partial\Omega}\beta(x)\left\{|u_n|^{p(x)-2}u_n-|v_n|^{p(x)-2}v_n\right\}(u_n-v_n)^+\,d\sigma$$

$$\geq C_{12}\int_{\Omega}a(x)\left|\Delta(u_n-v_n)^+\right|^{p(x)-2}dx+C_{13}\int_{\partial\Omega}\beta(x)\left|(u_n-v_n)^+\right|^{p(x)-2}d\sigma\geq 0$$

by Lemma 4. On the other hand, we have

$$\int_{\Omega}V(x)\left\{\frac{1}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}-\frac{1}{\left(v_n+\frac{1}{n}\right)^{\gamma(x)}}\right\}(u_n-v_n)^+\,dx$$

$$=\int_{\Omega}V(x)\left\{\frac{\left(v_n+\frac{1}{n}\right)^{\gamma(x)}-\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}\left(v_n+\frac{1}{n}\right)^{\gamma(x)}}\right\}(u_n-v_n)^+\,dx\leq 0. \tag{20}$$

Hence, we infer that $(u_n-v_n)^+=0$ a.e. in Ω and $u_n\leq v_n$ from (19) and (20). By symmetry, this also implies $u_n=v_n$. □

5. Conclusion

In this paper we obtain the existence of solutions for the class of singular fourth order equation (1) involving the weighted $p(.)$-biharmonic operator. Moreover, we find a unique solution for (16) in $X\cap L^{\infty}(\Omega)$. The existence of multiple weak solutions to the problem (1) can also be investigated in other studies in the future.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

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TZITZEICA CURVES WITH Q-FRAME IN THREE-DIMENSIONAL MINKOWSKI SPACE

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Abstract. In this work, both timelike and spacelike Tzitzeica, spherical, and spherical Tzitzeica curves are analyzed in 3-dimensional Minkowski space by using q-frame. Tzitzeica and spherical curves are characterized using spacelike and timelike q-frames within the context of Minkowski three-space, and the theorems concerning spherical Tzitzeica curves are established.

1. Introduction

At the start of the 20th century, a Romanian Mathematician, named Gheorgha Tzitzeica, defined a space curve called the Tzitzeica curve, where the constant value is the ratio of the torsion to the square of the distance from the curve's origin to the osculating plane at any arbitrary point on the curve [20], [21].

After this Tzitzeica curve was defined, many researchers have studied this subject. Karacan and Bükcü worked on two different hyperbolic cylindrical Tzitzeica curve in 2008 and gave the condition for cylindrical curve being a Tzitzeica curve dealing with third order ordinary differential equation in three-dimensional Minkowski space in 2009 [11], [12]. In 2010, Agnew et al. presented a thorough definition of Tzitzeica curves and surfaces, predating Bobe et al.'s work in 2012. The latter researchers established the connections between Tzitzeica curves and surfaces in Minkowski spaces and their counterparts originating from Euclidean space [4], [7], [9], [18], facilitated by the introduction of three novel centro-affine invariant functions [1], [5]. In [3], both Tzitzeica curves and rectifying curves were

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discussed and Tzitzeica conditions were given for both spacelike and timelike helices and pseudospherical curves in $\mathbb{E}_1^3$. Furthermore, calculations were performed using a different frame, Bishop frame, for fixed-width space curves in Euclidean 3-space [10] and spacelike curves in Minkowski 3-space [6].

In this study, a new frame called q-frame, found in [8], [13]- [15], [23], is used to examine the Tzitzeica, spherical, and spherical Tzitzeica curves in 3-dimensional Minkowski space. The conditions being Tzitzeica curve and spherical Tzitzeica curve are analyzed for the both spacelike and timelike curves.

2. Preliminaries

Consider a real vector space denoted as V. A bilinear form on this vector space can be defined as a function, denoted as $\langle , \rangle : V \times V \to \mathbb{R}$. In three-dimensional Minkowski space $\mathbb{E}_1^3$, this function of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is expressed

$$\langle u, v\rangle = u_1 v_1 + u_2 v_2 - u_3 v_3.$$

A scalar product space is called Lorentz space when $v = 1$ and $dimV \geq 2$ [17].

Definition 1. *A tangent vector $u \in V$ is*

- *spacelike if $\langle u, u\rangle > 0$ or $u = 0$,*
- *timelike if $\langle u, u\rangle < 0$,*
- *null if $\langle u, u\rangle = 0$ and $u \neq 0$ [16], [17].*

The norm of the vector u is given by $\|u\| = |\langle u, u\rangle|^{1/2}$.

Definition 2. *Let Γ be the set of all timelike vectors in a Lorentz vector space V. For $u \in \Gamma$,*

$$C(u) = \{v \in \Gamma | \langle u, v\rangle > 0\}$$

is the timecone of V containing u [17].

Proposition 1. *Let u and v be timelike vectors in a Lorentz vector space. Then, it holds that:*

- *$|\langle u, v\rangle| \geq |u||v|$, with equality if and only if u and v are collinear.*
- *If u and v belong to the same timecone in $C(u)$, there exists a unique non-negative number $\theta \geq 0$, known as the hyperbolic angle between u and v, such that:*

$$\langle u, v\rangle = -|u||v| \cosh \theta.$$

The cross product of u and v in three-dimensional Minkowski space $\mathbb{E}_1^3$ is defined as

$$u \times v = (u_3 v_2 - u_2 v_3, u_1 v_3 - u_3 v_1, u_1 v_2 - u_2 v_1)$$

[2].

Definition 3. *The Lorentzian unit circle in the Lorentz plane* $\mathbb{R}_1^2$ *is given by the set*

$$S_1^1 = \{u \in \mathbb{R}_1^2 | \langle u, u \rangle = 1\}.$$

The tangent vectors of this Lorentzian circle are always timelike type. Besides, the hyperbolic unit circle shown in Figure 1 in the Lorentz plane is given by the set

$$H_0^1 = \{u \in \mathbb{R}_1^2 | \langle u, u \rangle = -1\}.$$

The tangent vectors of this hyperbolic unit circle are always spacelike [22].

Similarly, the Lorentzian unit sphere and hyperbolic sphere shown in Figure 2 are given

$$S_1^2 = \{v \in \mathbb{R}_1^3 | \langle v, v \rangle = 1\},$$

$$H_0^2 = \{v \in \mathbb{R}_1^3 | \langle v, v \rangle = -1\},$$

respectively [22].

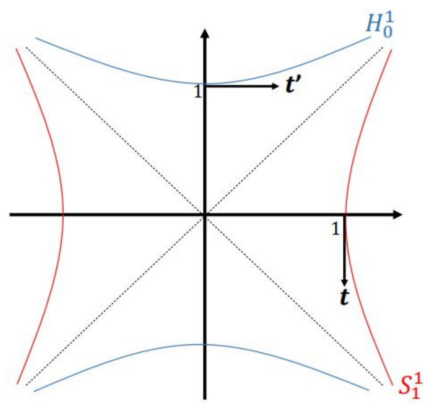

FIGURE 1. Lorentzian and hyperbolic unit circles

Definition 4. *The distance of a point* P *in space to a plane is called the length of the vector* $\overrightarrow{PS}$ *such that* S *is the foot of the perpendicular projection of* P *on the plane is* S*.*

$$l = d(P,S) = || \overrightarrow{PS} || = \frac{| < \overrightarrow{AP}, \overrightarrow{n} > |}{|| \overrightarrow{n} ||}$$

where A *be any point on the plane and* $\overrightarrow{n}$ *be the normal vector of the plane [24].*

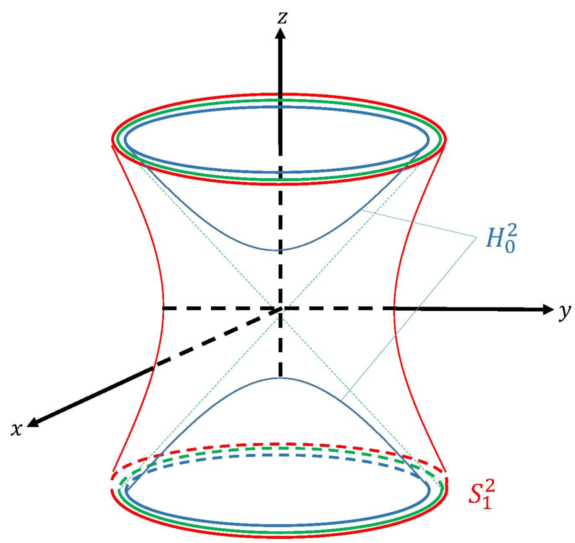

FIGURE 2. Lorentzian and hyperbolic unit spheres

3. Tzitzeica Curves in 3-Dimensional Minkowski Space

In this chapter, Tzitzeica and spherical curves are defined by using q-frame in Minkowski three-space. For these Tzitzeica and spherical curves, some results are given and they are characterized with respect to their curvatures. After defining spherical curve, the condition being Tzitzeica spherical curve is examined.

3.1. **Spacelike Tzitzeica Curves with q-frame in Minkowski 3-space.** In this part of our work, we deal with a spacelike curve that occurs when the projection vector $\mathbf{k}$ is timelike. For that spacelike curve, we examine both Tzitzeica and spherical curves, and then we work on Tzitzeica spherical curve. Lastly, investigations are shown on the Lorentz sphere.

Theorem 1. *The derivative formula of q-frame vectors for spacelike curve when* $\mathbf{t}$ *spacelike,* $\mathbf{k} = (0,0,1)$ *timelike,* $\mathbf{n_q}$ *spacelike, and* $\mathbf{b_q}$ *timelike is given*

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}'_\mathbf{q} \\ \mathbf{b}'_\mathbf{q} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n_q} \\ \mathbf{b_q} \end{bmatrix}.$$

The q-curvatures are written

$$k_1 = \kappa \cosh\theta,\ k_2 = \kappa \sinh\theta \ \text{ve}\ k_3 = -d\theta - \tau$$

[19].

Definition 5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a spacelike curve with arc-length parameter when $k_1 > 0$ and $k_3 \neq 0$. The curve α is called the Tzitzeica curve if the α satisfies the condition*

$$\frac{k_3}{d^2_{qos}} = a$$

with the distance d_{qos} of the curve from the origin of the q-osculator plane at the arbitrary point $\alpha(s)$. Here, $a \neq 0$ is a constant.
Using definition 4 and $\mathbf{b_q} = \mathbf{t} \times \mathbf{n_q}$, one can write

$$\begin{aligned} d(O, qos) &= d_{qos} \\ &= \left| \frac{\langle \alpha(s), \mathbf{t} \times \mathbf{n_q} \rangle}{\| \mathbf{t} \times \mathbf{n_q} \|} \right| \\ &= \left| \frac{\langle \alpha(s), \mathbf{b_q} \rangle}{\| \mathbf{b_q} \|} \right| . \end{aligned}$$

Since the timelike binormal vector b is a unit vector, from which the distance of the q-osculator plane to the origin is found in the form of

$$d_{qos} = \left| \langle \alpha(s), \mathbf{b_q} \rangle \right| . \tag{1}$$

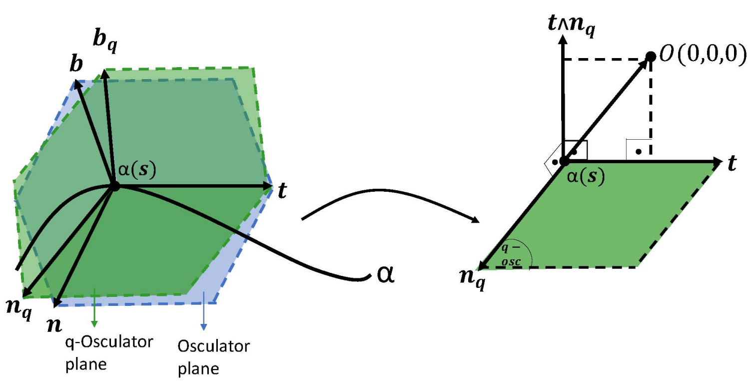

FIGURE 3. The distance d_{qos} of the q-osculator plane to the origin

Theorem 2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a unit spacelike curve in $\mathbb{R}^3_1$. The curve α is called Tzitzeica curve when the following equality satisfies*

$$k_3' \langle \mathbf{b_q}, \alpha \rangle + 2k_2 k_3 \langle \mathbf{t}, \alpha \rangle + 2k_3^2 \langle \mathbf{n_q}, \alpha \rangle = 0.$$

Proof. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a unit spacelike curve in $\mathbb{R}^3_1$. Assume that the curve α is Tzitzeica curve. Using definition 5 and equation (1), one can get

$$\frac{k_3}{\langle \mathbf{b_q}, \alpha \rangle^2} = a \neq 0. \tag{2}$$

Derivative of the last equation gives us

$$\frac{k_3' \langle \mathbf{b_q}, \alpha \rangle^2 - k_3(2 \langle \mathbf{b_q}, \alpha \rangle \langle \mathbf{b_q'}, \alpha \rangle)}{\langle \mathbf{b_q}, \alpha \rangle^4} = 0.$$

When the necessary simplifications are made, we can conclude

$$\frac{2k_3^2 \langle \mathbf{n_q}, \alpha \rangle + 2k_2 k_3 \langle \mathbf{t}, \alpha \rangle + k_3' \langle \mathbf{b_q}, \alpha \rangle}{\langle \mathbf{b_q}, \alpha \rangle^3} = 0.$$

This gives us a proof of theorem. □

The spacelike spherical curve is written

$$\alpha(s) = s_1 \mathbf{t}(s) + s_2 \mathbf{n_q}(s) + s_3 \mathbf{b_q}(s)$$

with respect to q-frame vectors. One can write

$$\left\| \overrightarrow{O\alpha} \right\| = r$$

for the sphere with radius r. Using $\alpha(s) \in S_1^2$, the properties of symmetry of scalar product and the curve being unit speed, we obtain

$$s_1 = \langle \mathbf{t}, \alpha \rangle = 0.$$

The first and second partial derivatives of this equation are as follows

$$k_1 \langle \mathbf{n_q}, \alpha \rangle - k_2 \langle \mathbf{b_q}, \alpha \rangle = 0 \tag{3}$$

and

$$\langle \mathbf{n_q}, \alpha \rangle (k_1' + k_2 k_3) - \langle \mathbf{b_q}, \alpha \rangle (k_2' + k_1 k_3) = 0 \tag{4}$$

respectively. When the equation (3) is multiplied by $-k_2' + k_1 k_3$ and the equation (4) is multiplied by k_2, and added together, the equation

$$\langle \mathbf{n}_q, \alpha \rangle = -\frac{1}{k_1 - \dfrac{k_2(k_1' + k_2 k_3)}{k_2' + k_1 k_3}}$$

is obtained. Similarly, multiplying the equation (3) by $-k_1' - k_2 k_3$ and the equation (4) by k_1, and adding together, we can get

$$\langle \mathbf{b_q}, \alpha \rangle = \frac{1}{k_2 - k_1 \dfrac{(k_2' + k_1 k_3)}{k_1' + k_2 k_3}}.$$

Taking derivative of $\langle \mathbf{n_q}, \alpha \rangle = s_2$ gives us

$$\langle \mathbf{b_q}, \alpha \rangle = \frac{s_2'}{k_3} = s_3.$$

Since $\langle \mathbf{n}_q, \alpha \rangle = s_2$, one can get

$$s_2 = \frac{k_2' + k_1 k_3}{k_1 k_2' + k_1^2 k_3 - k_2 k_1' - k_2^2 k_3}.$$

Now, to find s_3, it is enough to take the square of the above equation, and divide by k_3. In these settings, we are able to find

$$\begin{aligned} s_3 &= \frac{1}{(k_1^2(\frac{k_2}{k_1})' + (k_1^2 - k_2^2)k_3)^2}[-k_1'(k_3(k_1^2 + k_2^2) + k_1k_2' \\ &\quad + \frac{k_1k_2k_3' + k_2k_2''}{k_3}) + \frac{k_2'k_2(2k_1k_3^2 + k_1'' + k_2k_3')}{k_3} \\ &\quad - k_2((k_1')^2 - 2(k_2')^2 - k_1''k_1 + k_2''k_2)]. \end{aligned}$$

Corollary 1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}_1^3$ be a unit speed spacelike curve in $\mathbb{R}_1^3$. α is a spherical Tzitzeica curve if and only if*

$$k_3' \langle \mathbf{b_q}, \alpha \rangle + 2k_3^2 \langle \mathbf{n_q}, \alpha \rangle = 0.$$

Theorem 3. *Let $M \subset \mathbb{R}_1^3$ be a spacelike curve with coordinate neighborhood (I, α). The geometric location of the centers of the spheres, which are the three common points of M and infinity, is*

$$a(s) = \alpha(s) + s_2(s)\mathbf{n_q}(s) + \lambda \mathbf{b_q}(s),$$

so that the q-vectors at the point $\alpha(s)$ corresponding to the point $s \in I$ are $\{\mathbf{t}(s), \mathbf{n_q}(s), \mathbf{b_q}(s)\}$ where $\lambda \in \mathbb{R}$ and $s_2 : I \longrightarrow \mathbb{R}$ is the same as the coefficient of $\mathbf{n_q}$ in the equation of the spherical curve.

Proof. Let $f : I \longrightarrow \mathbb{R},\ f(s) = \langle a - \alpha(s), a - \alpha(s) \rangle - r^2$. Since there are three common points with the spheres

$$S_1^2 = \{x \mid x \in \mathbb{R}_1^3, \langle x - a, x - a \rangle = r^2\}$$

of the point $a(s)$ of M, there must be

$$f(s) = f'(s) = f''(s) = 0.$$

Since $f(s) = 0$, the equality $\langle a - \alpha(s), a - \alpha(s) \rangle = r^2$ must be satisfied. Using this equality, we can get

$$\langle \mathbf{t}(s), a - \alpha(s) \rangle = 0.$$

With the help of derivation of the last equality and $f'(s) = 0$,

$$k_1(s) \langle \mathbf{n_q}(s), a - \alpha(s) \rangle - k_2(s) \langle \mathbf{b_q}(s), a - \alpha(s) \rangle + 1 = 0$$

is found. On the other hand, since $\langle a - \alpha(s), \mathbf{t}(s) \rangle = s_1(s)$, we have $s_1(s) = 0$. Similarly, we can get

$$\langle a - \alpha(s), \mathbf{n_q}(s) \rangle = s_2(s)$$

and

$$\langle a - \alpha(s), \mathbf{b_q}(s) \rangle = -s_3(s).$$

In this setting, we are able to obtain

$$s_1^2(s) + s_2^2(s) - s_3^2(s) = r^2.$$

Using the equalities of s_1 and s_2, it is easily found as

$$a = \alpha(s) + s_2(s)\mathbf{n_q}(s) + \lambda\mathbf{b_q}(s).$$

□

Theorem 4. *Let $M \subset \mathbb{R}^3_1$ be a spacelike curve with coordinate neighborhood (I, α). For any $s \in I$, when $k_2 = 0$ at the point $\alpha(s)$, the radius of osculating sphere is constant if and only if the center of osculating sphere are the same such that $s_3 \neq 0$, $k_3 \neq 0$.*

Proof. The center of osculating sphere is written

$$a(s) = \alpha(s) + s_2(s)\mathbf{n_q}(s) + s_3(s)\mathbf{b_q}(s)$$

such that $\alpha(s) \in M$ and the radius

$$\begin{aligned} r &= \|\overrightarrow{\alpha a}\| \\ &= \|a - \alpha(s)\| \\ &= \sqrt{\langle s_2(s)\mathbf{n_q}(s) + s_3(s)\mathbf{b_q}(s), s_2(s)\mathbf{n_q}(s) + s_3(s)\mathbf{b_q}(s)\rangle} \\ &= \sqrt{s_2^2(s) - s_3^2(s)}. \end{aligned}$$

Taking derivative of $r^2 = s_2^2(s) - s_3^2(s)$, and using $s_3(s) = \dfrac{s_2'(s)}{k_3(s)}$, we can found

$$k_3(s)s_2(s) - s_3'(s) = 0. \tag{5}$$

From the equation $a(s) = \alpha(s) + s_2(s)\mathbf{n_q}(s) + s_3(s)\mathbf{b_q}(s)$, we have

$$D_{\dot{\alpha}} a(s) = (1 - s_2(s)k_1(s) - s_3(s)k_2(s))\mathbf{t}(s) + (-s_2(s)k_3(s) + s_3'(s))\mathbf{b_q}(s).$$

After using the equality of s_2, s_3 and $k_2 = 0$, one can get

$$D_{\dot{\alpha}} a(s) = (-s_2(s)k_3(s) + s_3'(s))\mathbf{b_q}(s).$$

In the light of equation (5), for any $s \in I$, that $a(s)$ is a constant is obtained. On the other hand, let $a(s)$ be a constant for any $s \in I$. Since $r = \|\overrightarrow{\alpha a}\|$, we have

$$\langle a(s) - \alpha(s), a(s) - \alpha(s)\rangle = r^2(s).$$

Taking derivative of this equation gives

$$\left\langle D_{\dot{\alpha}} a(s), a(s) - \alpha(s)\right\rangle = r(s)\left.\frac{dr}{ds}\right|_s.$$

We then have

$$r(s)\left.\frac{dr}{ds}\right|_s = 0.$$

Either $r(s) = 0$ or $\left.\dfrac{dr}{ds}\right|_s = 0$ is provided. Being $r(s) = 0$ contradicts both $s_2 = s_3 = 0$. Therefore, $\left.\dfrac{dr}{ds}\right|_s = 0$. We then conclude the proof by finding $r(s) = 0$ for any $s \in I$. □

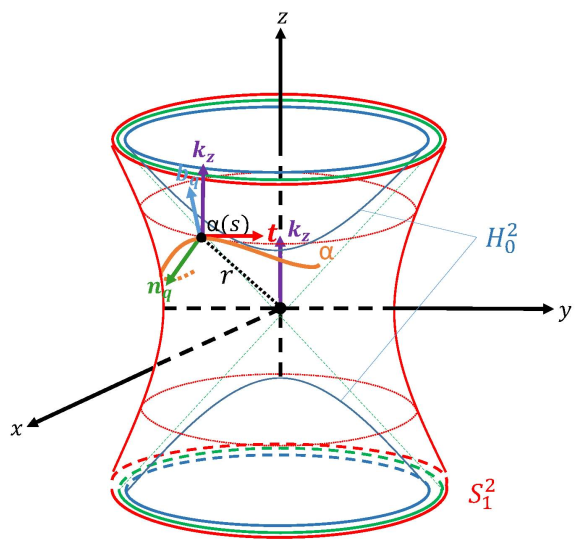

FIGURE 4. Spacelike spherical curve with q-frame in Minkowski 3-space

3.2. **Timelike Tzitzeica Curves with q-frame in Minkowski 3-space.** In this part, we work on the similar theorems in previous section for timelike curve when the projection vector k is spacelike. Since the proofs are also made in similar ways as in the case of the spacelike curve, we omit them in this case.

Theorem 5. *The derivative formula of q-frame vectors for timelike curve when* $\mathbf{t}$ *timelike,* $\mathbf{k} = (0, 1, 0)$ *spacelike,* $\mathbf{n_q}$ *spacelike and* $\mathbf{b_q}$ *spacelike is given*

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}'_{\mathbf{q}} \\ \mathbf{b}'_{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n_q} \\ \mathbf{b_q} \end{bmatrix}.$$

The q-curvatures are written

$$k_1 = \kappa \cos\theta, k_2 = -\kappa \sin\theta, k_3 = d\theta + \tau$$

[19].

Definition 6. *Let* $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ *be a timelike curve with arc-length parameter when* $k_1 > 0$ *and* $k_3 \neq 0$*. The curve* α *is called the Tzitzeica curve if the* α *satisfies the condition*

$$\frac{k_3}{d^2_{qos}} = a$$

with the distance d_{qos} of the curve from the origin of the q-osculator plane at the arbitrary point $\alpha(s)$. Here, $a \neq 0$ is a constant.
Using definition 6 and $\mathbf{b_q}= \mathbf{t} \times \mathbf{n_q}$, one can write

$$\begin{aligned} d(O, qos) &= d_{qos} \\ &= \left| \frac{\langle \alpha(s), \mathbf{t} \times \mathbf{n_q} \rangle}{\|\mathbf{t} \times \mathbf{n_q}\|} \right| \\ &= \left| \frac{\langle \alpha(s), \mathbf{b_q} \rangle}{\|\mathbf{b_q}\|} \right|. \end{aligned}$$

Since the spacelike binormal vector b is a unit vector, from which the distance of the q-osculator plane to the origin is found in the form of

$$d_{qos} = |\langle \alpha(s), \mathbf{b_q} \rangle| = |\langle \mathbf{b_q}, \alpha(s) \rangle|\,.$$

Corollary 2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a unit speed timelike curve in $\mathbb{R}^3_1$. α is a spherical Tzitzeica curve if and only if*

$$k_3' \langle \mathbf{b_q}, \alpha \rangle - 2k_2 k_3 \langle \mathbf{t}, \alpha \rangle + 2k_3^2 \langle \mathbf{n_q}, \alpha \rangle = 0.$$

The timelike spherical curve is given

$$\alpha(s) = s_1 \mathbf{t}(s) + s_2 \mathbf{n_q}(s) + s_3 \mathbf{b_q}(s)$$

with respect to q-frame vectors. In the light of recent calculations given above section, one can find

$$\begin{aligned} s_1 &= \langle \mathbf{t}, \alpha \rangle = 0, \\ s_2 &= \frac{k_2' + k_1 k_3}{-k_1 k_2' - k_1^2 k_3 + k_2 k_1' - k_2^2 k_3}, \\ s_3 = \frac{s_2'}{k_3} &= \frac{1}{(k_2^2 (\frac{k_1}{k_2})' - (k_1^2 + k_2^2) k_3)^2} [\frac{k_2' k_2 (2k_1 k_3^2 - k_1'' + k_2 k_3')}{k_3} \\ &\quad + k_1' (k_3 (k_1^2 - k_2^2) + \frac{k_1 k_2' + k_1 k_2 k_3{}' + k_2 k_2''}{k_3}) \\ &\quad + k_2 (2(k_2')^2 + (k_1')^2 - k_1'' k_1 - k_2'' k_2)]. \end{aligned}$$

Corollary 3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a unit speed timelike curve in $\mathbb{R}^3_1$. α is a spherical Tzitzeica curve if and only if*

$$k_3' \langle \mathbf{b_q}, \alpha \rangle + 2k_3^2 \langle \mathbf{n_q}, \alpha \rangle = 0.$$

Theorem 6. *Let $M \subset \mathbb{R}^3_1$ be a timelike curve with coordinate neighborhood (I, α). The geometric location of the centers of the spheres, which are the three common points of M and infinity, is*

$$a(s) = \alpha(s) + s_2(s) \mathbf{n_q}(s) + \lambda \mathbf{b_q}(s),$$

so that the q-vectors at the point $\alpha(s)$ *corresponding to the point* $s \in I$ *are* $\{\mathbf{t}(s), \mathbf{n_q}(s), \mathbf{b_q}(s)\}$ *where* $\lambda \in \mathbb{R}$ *and* $s_2 : I \longrightarrow \mathbb{R}$ *is the same as the coefficient of* $\mathbf{n_q}$ *in the equation of the spherical curve.*

Theorem 7. *Let* $M \subset \mathbb{R}^3_1$ *be a timelike curve with coordinate neighborhood* (I, α)*. For any* $s \in I$*, when* $k_2 = 0$ *at the point* $\alpha(s)$*, the radius of osculating sphere is constant if and only if the centers of osculating spheres are the same such that* $s_3 \neq 0$*,* $k_3 \neq 0$*.*

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ALMOST INNER DERIVATIONS OF LEIBNIZ ALGEBRAS

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Abstract. This work is presented the study on almost inner derivations of Leibniz algebras. In this note, we demonstrate the natural extensions of some general properties on derivations given for Lie algebras to Leibniz algebras with finite dimension, and also we investigate which statements a mapping have to hold to be an almost inner derivation.

1. Introduction

Leibniz algebras which were first initiated by Loday [10] are as a generalization of Lie algebras. Loday and Pirashvili in [11] investigated such algebras by using homological algebras. In literature, many papers have consisted of the results which show the similarities and the differences between Lie and Leibniz algebras. In the paper [9] M. Ladra and et al. studied the derivations of Leibniz algebras and they extended several common properties of derivations and automorphisms given for Lie algebras to Leibniz algebras with finite dimensions over $\mathbb{C}$. The paper [16] of C. Zargeh is proved that if Leibniz algebra $\mathcal{L}$ has a derivation $\delta : \mathcal{L} \to \mathcal{L}$ satisfying $\mathcal{L}^m \subset \delta(\mathcal{L})$ for some $m > 1$ where $\mathcal{L}^m$ is the m-th terms of lower central series of $\mathcal{L}$, then $\mathcal{L}$ is solvable. The derivations of Leibniz algebras are studied in many papers including [4, 5]. There exist still several open natural questions. One of those questions is on the almost inner derivations which were not considered for Leibniz algebras.
The principal goal of this note is to demonstrate the important consequences on almost inner derivations of Leibniz algebras which are analogs to the consequences in Lie algebras. Our fundamental starting point is presented by the papers [3, 7, 14, 15] which studied on almost inner derivations of Lie algebras.
This paper is planned as follows. Several definitions and notations are introduced in Section 2. Section 3 is presented to the notion of almost inner derivation. First,

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we examine some special types of derivations, this concerns the almost inner ones which form a generalization of the inner derivations. Then we derive a procedure to figure out the set of all almost inner derivations, and we also give an example for this method. In Section 4, we investigate which statements a general mapping have to hold to be an almost inner derivation by using the structure constants. In the concluding Section 5, we focus on fixed basis vectors for an arbitrary derivation. In particular, we prove that if any basis vector for all almost inner derivations is fixed, then the set of all almost inner derivations is equal to the set of all inner derivations.

2. Preliminaries

This section introduces the concepts of Lie algebra and Leibniz algebra which will be used in later sections. The material in this section is based on [1,2,8,10,13]. Given a field K with characteristic zero. Recall that an algebra $\mathcal{L}$ over K is Lie algebra if the algebra satisfies the following properties

$$\begin{array}{ll} \text{(i)} & pp = 0, \ \text{(anti-commutativity)} \\ \text{(ii)} & (pq)r + (qr)p + (rp)q = 0 \ \text{(Jacobi identity)} \end{array}$$

for all $p, q, r \in \mathcal{L}$. Let $\mathcal{L}$ be a Lie algebra and $\mathcal{I}$ be a subspace of $\mathcal{L}$. If $xy \in \mathcal{I}$ for all $x \in \mathcal{I}$ and $y \in \mathcal{L}$, $\mathcal{I}$ is said to be a Lie ideal of $\mathcal{L}$. The set of all linear maps on $\mathcal{L}$, $gl(\mathcal{L})$, becomes a Lie algebra with Lie product given by $[h_1, h_2] = h_1h_2 - h_2h_1$ for every $h_1, h_2 \in gl(\mathcal{L})$.
An algebra $\mathcal{L}$ over K with an operation $[,] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is said to be a left Leibniz algebra if $\mathcal{L}$ holds Leibniz identity

$$[[p, q], r] = [p, [q, r]] - [q, [p, r]]$$

for every p, q, r in $\mathcal{L}$. Similarly, we say a right Leibniz algebra if $\mathcal{L}$ holds Leibniz identity

$$[p, [q, r]] = [[p, q], r] - [[p, r], q].$$

We use left Leibniz algebra the rest of this paper. We give the left normed convention for Leibniz brackets, that is,

$$[p_1, p_2, p_3, \ldots, p_s] = [[\ldots[[p_1, p_2], p_3], \ldots], p_s]$$

for all $p_1, p_2, \ldots, p_s \in \mathcal{L}$.

It is clear that Leibniz algebra is obvious a generalization of Lie algebra. Given a subspace $\mathcal{I}$ of a Leibniz algebra $\mathcal{L}$, $\mathcal{I}$ is a subalgebra if $[p, q] \in \mathcal{I}$ for every $p, q \in \mathcal{I}$. If $[p, q] \in \mathcal{I}$ and $[q, p] \in \mathcal{I}$ for every $p \in \mathcal{L}$ and $q \in \mathcal{I}$, then we say $\mathcal{I}$ an ideal of $\mathcal{L}$ and we denote by $\mathcal{I} \trianglelefteq \mathcal{L}$. The left centre of $\mathcal{L}$ is denoted by $C^l(\mathcal{L}) = \{p \in \mathcal{L} | [p, q] = 0 \text{ for every } q \in \mathcal{L}\}$ and the right centre of $\mathcal{L}$ is represented by $C^r(\mathcal{L}) = \{p \in \mathcal{L} | [q, p] = 0 \text{ for every } q \in \mathcal{L}\}$. The centre of $\mathcal{L}$ is represented by $C(\mathcal{L}) = C^l(\mathcal{L}) \cap C^r(\mathcal{L})$. Given two Leibniz algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ over K, a linear mapping $\theta : \mathcal{L}_1 \to \mathcal{L}_2$ is said to be a homomorphism if it satisfies that $\theta([p,q]) = [\theta(p), \theta(q)]$ for every $p, q \in \mathcal{L}_1$. The series of ideals

$$\mathcal{L} = \mathcal{L}^1 \supseteq \mathcal{L}^2 \supseteq \ldots \supseteq \mathcal{L}^k \supseteq \mathcal{L}^{k+1} \supseteq \ldots$$

where for positive integer m, $\mathcal{L}^{m+1} = [\mathcal{L}, \mathcal{L}^m]$ is called the lower central series of $\mathcal{L}$. We say nilpotent of class c if a Leibniz algebra holds that $\mathcal{L}^{c+1} = 0$ but $\mathcal{L}^c \neq 0$. Hence, if $\mathcal{L}$ is nilpotent of class c, we have $\mathcal{L}^c \subseteq C^r(\mathcal{L})$. We also have $\mathcal{L}^c \subseteq C^l(\mathcal{L})$. Therefore, $\mathcal{L}^c \subseteq C^l(\mathcal{L}) \cap C^r(\mathcal{L}) = C(\mathcal{L})$ and $C(\mathcal{L}) \neq 0$.

3. Derivations of Leibniz Algebras

Definition 1. *Given a Leibniz algebra $\mathcal{L}$ over a field K. A derivation of $\mathcal{L}$ is a K-linear mapping $\delta : \mathcal{L} \to \mathcal{L}$ given by $\delta([p,q]) = [\delta(p), q] + [p, \delta(q)]$ for every $p, q \in \mathcal{L}$.*

By $der\mathcal{L}$, we represent the set of all derivations in $\mathcal{L}$. This set with the following multiplication

$$[,] : der\mathcal{L} \times der\mathcal{L} \to der\mathcal{L}$$

by $[\delta_1, \delta_2] = t(\delta_1)\delta_2 - \delta_2 t(\delta_1)$ where t is a linear operator with $t^2 = t$ is an algebra, it is called derivation algebra. Indeed, for any $\delta_1, \delta_2 \in der\mathcal{L}$ and $p, q \in \mathcal{L}$ we obtain

$$\begin{aligned}
[\delta_1, \delta_2]([p,q]) &= (t(\delta_1)\delta_2 - \delta_2 t(\delta_1))([p,q]) \\
&= f(\delta_1)([\delta_2(p), q] + [p, \delta_2(q)]) - \delta_2([t(\delta_1)(p), q] + [p, t(\delta_1)(q)]) \\
&= [t(\delta_1)\delta_2(p), q] - [\delta_2 t(\delta_1)(p), q] + [p, t(\delta_1)\delta_2(q)] - [p, \delta_2 t(\delta_1)(q)] \\
&= [[\delta_1, \delta_2](p), q] + [p, [\delta_1, \delta_2](q)].
\end{aligned}$$

It means that $[\delta_1, \delta_2]$ is a derivation of $\mathcal{L}$. In addition, $der\mathcal{L}$ is a Leibniz algebra. Clearly, $der\mathcal{L}$ is a Lie algebra if t is the identity map.

For any element a in $\mathcal{L}$, the left multiplication operator $\mathcal{L}_a : \mathcal{L} \to \mathcal{L}$ given by $\mathcal{L}_a(p) = [a, p]$ for $p \in \mathcal{L}$. Given a left multiplication $\mathcal{L}_a$, by Leibniz identity we obtain

$$\begin{aligned}
\mathcal{L}_a([p,q]) &= [a, [p,q]] \\
&= [[a,p], q] + [p, [a,q]] \\
&= [\mathcal{L}_a(p), q] + [p, \mathcal{L}_a(q)]
\end{aligned}$$

for all $p, q \in \mathcal{L}$. This shows that $\mathcal{L}_a$ is a derivation of $\mathcal{L}$ and it is said to be inner derivation. The set of all such derivations is represented by $id(L)$.

Lemma 1. *Given a Leibniz algebra $\mathcal{L}$ over K. Then $id(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$ with Lie product. Also $id(\mathcal{L})$ is a Lie ideal of $der\mathcal{L}$.*

Proof. Let $\mathcal{L}_a$ and $\mathcal{L}_b$ two inner derivations of $\mathcal{L}$. For all $p, q \in \mathcal{L}$ we obtain

$$\begin{aligned}
[\mathcal{L}_a, \mathcal{L}_b]([p,q]) &= (\mathcal{L}_a\mathcal{L}_b - \mathcal{L}_b\mathcal{L}_a)([p,q]) \\
&= \mathcal{L}_a([\mathcal{L}_b(p), q] + [p, \mathcal{L}_b(q)]) - \mathcal{L}_b([\mathcal{L}_a(p), q] + [p, \mathcal{L}_a(q)])
\end{aligned}$$

$$\begin{aligned} &= [\mathcal{L}_{[a,b]}(p), q] + [p, \mathcal{L}_{[a,b]}(q)] \\ &= [[[a,b],p],q] + [p,[[a,b],q]] \\ &= [[a,b],[p,q]] \\ &= \mathcal{L}_{[a,b]}([p,q]). \end{aligned}$$

Hence $id(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$. Moreover, for each element $\mathcal{L}_a \in id(\mathcal{L})$ and $\delta \in der\mathcal{L}$, we obtain

$$\begin{aligned} [\mathcal{L}_a, \delta]([p,q]) &= (\mathcal{L}_a\delta - \delta\mathcal{L}_a)([p,q]) \\ &= \mathcal{L}_a([\delta(p),q] + [p,\delta(q)]) - \delta([a,[p,q]] \\ &= -[\delta(a),[p,q]] \\ &= \mathcal{L}_{-\delta(a)}[p,q], \end{aligned}$$

as required. □

Definition 2. *A derivation $\delta \in der\mathcal{L}$ of a Leibniz algebra $\mathcal{L}$ is called an almost inner derivation if $\delta(p) \in [\mathcal{L}, p]$ for all $p \in \mathcal{L}$.*

By $aid(\mathcal{L})$, we represent the set of all almost inner derivations of $\mathcal{L}$. Since $[\mathcal{L}, p] = \{[q,p] | q \in \mathcal{L}\}$, it is obvious that the set of all inner derivations, $id(\mathcal{L})$, is a subset of $aid(\mathcal{L})$.

Lemma 2. *Given a Leibniz algebra $\mathcal{L}$ over K. Then $aid(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$ with Lie product. Also $aid(\mathcal{L})$ is a Lie ideal of $der\mathcal{L}$.*

Proof. Let $\delta_1, \delta_2 \in aid(\mathcal{L})$ and $p \in \mathcal{L}$. Then there are $q_1, q_2 \in \mathcal{L}$ with $\delta_1(p) = [q_1, p]$ and $\delta_2(p) = [q_2, p]$. By applying the derivation condition and Leibniz identity, we have

$$\begin{aligned} [\delta_1, \delta_2](p) &= (\delta_1\delta_2 - \delta_2\delta_1)(p) \\ &= \delta_1([q_2,p]) - \delta_2([q_1,p]) \\ &= [\delta_1(q_2),p] + [q_2,\delta_1(p)] - [\delta_2(q_1),p] - [q_1,\delta_2(p)] \\ &= [\delta_1(q_2) - \delta_2(q_1) + [q_2,q_1], p] \in [\mathcal{L}, p]. \end{aligned}$$

Hence we obtain $[\delta_1, \delta_2] \in aid(\mathcal{L})$. So $aid(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$. Moreover, given $\delta \in der\mathcal{L}$ and $h \in aid(\mathcal{L})$. Since $h \in aid(\mathcal{L})$, there is an element $q \in \mathcal{L}$ satisfying $h(p) = [q, p]$. Then

$$\begin{aligned} [h, \delta](p) &= (h\delta - \delta h)(p) \\ &= [q, \delta(p)] - \delta([q,p]) \\ &= -[\delta(q), p]. \end{aligned}$$

Therefore we obtain that $aid(\mathcal{L})$ is a Lie ideal of $der\mathcal{L}$. □

Definition 3. *We say an almost inner derivation δ a central almost inner derivation if there is an element $p \in \mathcal{L}$ with $\delta - \mathcal{L}_p$ maps $\mathcal{L}$ to $C(\mathcal{L})$.*

The set of all central almost inner derivations of $\mathcal{L}$ is denoted by $caid(\mathcal{L})$. We have the following inclusions of Lie subalgebras

$$id(\mathcal{L}) \subseteq caid(\mathcal{L}) \subseteq aid(\mathcal{L}) \subseteq der\mathcal{L}.$$

Clearly, $caid(\mathcal{L})$ is a Lie subalgebra of $der\mathcal{L}$. To see that this subalgebra is a Lie ideal of $aid(\mathcal{L})$ we give the next lemma.

Lemma 3. *Given a Leibniz algebra $\mathcal{L}$ over a field K. Then $caid(\mathcal{L})$ is a Lie ideal of $aid(\mathcal{L})$.*

Proof. Let $\delta_1 \in caid(\mathcal{L})$ and $\delta_2 \in aid(\mathcal{L})$. Then there is an element $p \in \mathcal{L}$ satisfying $\delta_1 - \mathcal{L}_p = \delta_3$ maps $\mathcal{L}$ to $C(\mathcal{L})$ and there is an element $q \in \mathcal{L}$ with $\delta_2(p) = [q, p] \in [\mathcal{L}, p]$. To prove that $caid(\mathcal{L})$ is a Lie ideal of $aid(\mathcal{L})$, we need to show that $[\delta_2, \delta_1] \in caid(\mathcal{L})$. Since $aid(\mathcal{L})$ is an ideal of $der\mathcal{L}$ for every derivations of $\mathcal{L}$, it is clear that $[\delta_2, \delta_1] \in aid(\mathcal{L})$. Suppose that $\delta_4 = [\delta_2, \delta_1] - \mathcal{L}_{\delta_2(p)}$. For any element $r \in \mathcal{L}$,

$$\mathcal{L}_{\delta_2(p)}(r) = \mathcal{L}_{[q,p]}(r) = [[q, p], r] = [\delta_2, \mathcal{L}_p](r).$$

Then we get

$$\mathcal{L}_{\delta_2(p)} = [\delta_2, \mathcal{L}_p]. \tag{1}$$

By (1), we have

$$\delta_4 = [\delta_2, \delta_1] - [\delta_2, \mathcal{L}_p] = [\delta_2, \delta_1 - \mathcal{L}_p] = [\delta_2, \delta_3].$$

It follows that δ_3 maps $\mathcal{L}$ to $C(\mathcal{L})$ and δ_2 maps $C(\mathcal{L})$ to $C(\mathcal{L})$. Hence δ_4 maps $\mathcal{L}$ to $C(\mathcal{L})$. Since there is $\delta_2(p) \in \mathcal{L}$ such that $[\delta_2, \delta_1] - \mathcal{L}_{\delta_2(p)} = \delta_4$ maps $\mathcal{L}$ to $C(\mathcal{L})$, $[\delta_2, \delta_1] \in caid(\mathcal{L})$. □

The results obtained for the derivations of Leibniz algebras are given in the next theorem.

Theorem 1. *Given a Leibniz algebra $\mathcal{L}$. Then the following statements satisfy*

(i) *Let $\delta \in aid(\mathcal{L})$. Then $\delta(\mathcal{L}) \subseteq [\mathcal{L}, \mathcal{L}]$, $\delta(C(\mathcal{L})) = 0$ and $\delta(\mathcal{I}) \subseteq \mathcal{I}$ for any ideal of $\mathcal{L}$.*

(ii) *Let $\delta \in caid(\mathcal{L})$. Then there is an element $p \in \mathcal{L}$ such that $\delta|_{[\mathcal{L},\mathcal{L}]} = \mathcal{L}_p|_{[\mathcal{L},\mathcal{L}]}$.*

(iii) *If $\mathcal{L}$ is a Leibniz algebra with the nilpotency class 2, then $caid(\mathcal{L}) = aid(\mathcal{L})$.*

(iv) *If the centre of $\mathcal{L}$ is zero, then $caid(\mathcal{L}) = id(\mathcal{L})$.*

(v) *If $\mathcal{L}$ is a nilpotent Leibniz algebra, then $aid(\mathcal{L})$ is also nilpotent.*

Proof. (i) If $\delta \in aid(\mathcal{L})$, then for every element $p \in \mathcal{L}$ we have

$$\delta(p) \in [\mathcal{L}, p] \subseteq [\mathcal{L}, \mathcal{L}]. \tag{2}$$

Therefore, for each ideal $\mathcal{I}$ of $\mathcal{L}$ and $p \in \mathcal{I}$ we have

$$\delta(p) \in [\mathcal{L}, \mathcal{I}] \subseteq \mathcal{I} \text{ and } \delta(p) \in [\mathcal{I}, \mathcal{L}] \subseteq \mathcal{I}.$$

Thus $\delta(\mathcal{I}) \subseteq \mathcal{I}$. By (2), we obtain that for all $p \in C(\mathcal{L})$, $\delta(p) = 0$, that is, $\delta(C(\mathcal{L})) = 0$.

(ii) If $\delta \in caid(\mathcal{L})$, then there is an element $p \in \mathcal{L}$ such that $\delta_1 = \delta - \mathcal{L}_p$ maps $\mathcal{L}$ to the centre of $\mathcal{L}$. Namely, $\delta_1(\mathcal{L}) \subseteq C(\mathcal{L})$. Since δ_1 is a derivation of $\mathcal{L}$ and for every $a, b \in \mathcal{L}$,

$$\delta_1([a,b]) = [\delta_1(a), b] + [a, \delta_1(b)] = 0.$$

(iii) We know that from the inclusions of Lie subalgebras $caid(\mathcal{L}) \subseteq aid(\mathcal{L})$. Now we must only show that $aid(\mathcal{L}) \subseteq caid(\mathcal{L})$. Suppose that $\delta \in aid(\mathcal{L})$. Then there is an element $p \in \mathcal{L}$ such that $\delta(p) \in [\mathcal{L}, p] \subseteq [\mathcal{L}, \mathcal{L}]$. Moreover, if $\mathcal{L}$ is a nilpotent Leibniz algebra of class m, then $\mathcal{L}^m \subseteq C^l(\mathcal{L})$. By Proposition 4.2 in [6], $\mathcal{L}^m \subseteq C^r(\mathcal{L})$. Hence we obtain $\mathcal{L}^m \subseteq C^l(\mathcal{L}) \cap C^r(\mathcal{L}) = C(\mathcal{L}) \neq 0$. Since $\mathcal{L}$ is a nilpotent of class 2, we can write from (2)

$$\delta(\mathcal{L}) \subseteq [\mathcal{L}, \mathcal{L}] = \mathcal{L}^2 \subseteq C(\mathcal{L}).$$

This means that δ maps $\mathcal{L}$ to the centre of $\mathcal{L}$, that is, $\delta \in caid(\mathcal{L})$ and $aid(\mathcal{L}) \subseteq caid(\mathcal{L})$. Therefore $aid(\mathcal{L}) = caid(\mathcal{L})$.

(iv) We know that from the inclusions of Lie subalgebras $id(\mathcal{L}) \subseteq caid(\mathcal{L})$. Now we need only to show that $caid(\mathcal{L}) \subseteq id(\mathcal{L})$. Suppose that $\delta \in caid(\mathcal{L})$ and $C(\mathcal{L}) = 0$. Then there is an element $p \in \mathcal{L}$ satisfying $\delta - \mathcal{L}_p$ maps $\mathcal{L}$ to the centre of $\mathcal{L}$. Since $C(\mathcal{L}) = 0$, we have $(\delta - \mathcal{L}_p)(q) = 0$ for all $q \in \mathcal{L}$. Namely $\delta - \mathcal{L}_p = 0$. Thus $\delta = \mathcal{L}_p$. This shows that $\delta \in id(\mathcal{L})$ and $caid(\mathcal{L}) \subseteq id(\mathcal{L})$. Therefore, we obtain $caid(\mathcal{L}) = id(\mathcal{L})$.

(v) Suppose that $\mathcal{L}$ is a Leibniz algebra with the nilpotency class m ($\mathcal{L}^{m+1} = 0, \mathcal{L}^m \neq 0$). For $\delta \in aid(\mathcal{L})$ and $p \in \mathcal{L}$, from (2) we can define nilpotent operator,

$$\begin{aligned}
\delta^1(p) &\in [\mathcal{L}, p] \subseteq [\mathcal{L}, \mathcal{L}] = \mathcal{L}^2 \\
\delta^2(p) &\in [\mathcal{L}, [\mathcal{L}, p]] \subseteq [\mathcal{L}, [\mathcal{L}, \mathcal{L}]] = [\mathcal{L}, \mathcal{L}^2] = \mathcal{L}^3 \\
&\vdots \\
\delta^m(p) &\in [\mathcal{L}, [\ldots, [\mathcal{L}, p] \ldots]] \subseteq [\mathcal{L}, [\ldots, [\mathcal{L}, \mathcal{L}] \ldots]] = [\mathcal{L}, \mathcal{L}^m] = \mathcal{L}^{m+1}.
\end{aligned}$$

Since $\mathcal{L}$ is nilpotent of class m, then $\mathcal{L}^{m+1} = 0$, so $\delta^m = 0$. Therefore δ is a nilpotent. By Engel theorem [[6], Theorem 4.5], $aid(\mathcal{L})$ is nilpotent. □

Example 1. *Given a Leibniz algebra $\mathcal{L}$ over a field K with the basis $\{e_1, e_2, e_3, e_4, e_5\}$ by the following multiplication*

$$\begin{array}{lll}
[e_1, e_2] = e_2, & [e_2, e_1] = -e_2, & [e_4, e_1] = e_5, \\
[e_1, e_4] = e_4, & [e_2, e_3] = e_4, & [e_5, e_1] = -e_5, \\
[e_1, e_5] = e_5, & [e_3, e_2] = e_5, & [e_i, e_j] = 0
\end{array}$$

for other multiplications. We will compute $id(\mathcal{L})$, $aid(\mathcal{L})$ and $caid(\mathcal{L})$. Since every derivation δ of $\mathcal{L}$ is of the following form $\delta(e_1) = \alpha_1 e_2 + \alpha_2 e_5$, $\delta(e_2) = \beta_1 e_2 + \beta_2 e_5$, $\delta(e_3) = \gamma_1 e_3 + \gamma_2 e_4$, $\delta(e_4) = \sigma_1 e_4$, $\delta(e_5) = \tau_1 e_5$, we obtain

$$\begin{aligned}
der\mathcal{L} = \; &\{\delta | \delta(e_1) \in Span\{e_2, e_5\}, \delta(e_2) \in Span\{e_2, e_5\}, \delta(e_3) \in Span\{e_3, e_4\}, \\
&\delta(e_4) \in Span\{e_4\}, \delta(e_5) \in Span\{e_5\}\}.
\end{aligned}$$

By using the definition of inner derivation of $\mathcal{L}$. We obtain the following results

$$\begin{array}{lllll} \mathcal{L}_{e_1}(e_1)=0, & \mathcal{L}_{e_2}(e_1)=-e_2, & \mathcal{L}_{e_3}(e_1)=0, & \mathcal{L}_{e_4}(e_1)=e_5, & \mathcal{L}_{e_5}(e_1)=-e_5, \\ \mathcal{L}_{e_1}(e_2)=e_2, & \mathcal{L}_{e_2}(e_2)=0, & \mathcal{L}_{e_3}(e_2)=e_5, & \mathcal{L}_{e_4}(e_2)=0, & \mathcal{L}_{e_5}(e_2)=0, \\ \mathcal{L}_{e_1}(e_3)=0, & \mathcal{L}_{e_2}(e_3)=e_4, & \mathcal{L}_{e_3}(e_3)=0, & \mathcal{L}_{e_4}(e_3)=0, & \mathcal{L}_{e_5}(e_3)=0, \\ \mathcal{L}_{e_1}(e_4)=e_4, & \mathcal{L}_{e_2}(e_4)=0, & \mathcal{L}_{e_3}(e_4)=0, & \mathcal{L}_{e_4}(e_4)=0, & \mathcal{L}_{e_5}(e_4)=0, \\ \mathcal{L}_{e_1}(e_5)=e_5, & \mathcal{L}_{e_2}(e_5)=0, & \mathcal{L}_{e_3}(e_5)=0, & \mathcal{L}_{e_4}(e_5)=0, & \mathcal{L}_{e_5}(e_5)=0. \end{array}$$

It is clear to see that $\mathcal{L}_{e_5}=-\mathcal{L}_{e_4}=\mathcal{L}_{-e_4}$. *Hence we have*

$$id(\mathcal{L})=Span\{\mathcal{L}_{e_1},\mathcal{L}_{e_2},\mathcal{L}_{e_3},\mathcal{L}_{e_4}\}.$$

To obtain $aid(\mathcal{L})$ *we must calculate* $[\mathcal{L},e_i]$ *for all* $1\le i\le 5$,

$$[\mathcal{L},e_1]=Span\{e_2,e_5\},[\mathcal{L},e_2]=Span\{e_2,e_5\},$$
$$[\mathcal{L},e_3]=Span\{e_4\}=[\mathcal{L},e_4],[\mathcal{L},e_5]=Span\{e_5\}.$$

Hence we obtain

$$\begin{aligned} aid(\mathcal{L}) \;=\; & \{\delta|\delta(e_1)\in Span\{e_2,e_5\},\delta(e_2)\in Span\{e_2,e_5\},\delta(e_3)\in Span\{e_4\}, \\ & \delta(e_4)\in Span\{e_4\},\delta(e_5)\in Span\{e_5\}\}. \end{aligned}$$

To determine the set of all of the central almost inner derivation we need the centre of $\mathcal{L}$, $C(\mathcal{L})=0$. *Take* $\delta\in aid(\mathcal{L})$ *such that* $\delta(e_1)=0,\delta(e_2)=e_2,\delta(e_3)=0,\delta(e_4)=e_4$ *and* $\delta(e_5)=e_5$. *Now we need to show that there exists an element* p *in* $\mathcal{L}$ *such that* $\delta-\mathcal{L}_p$ *maps* $\mathcal{L}$ *to the centre of* $\mathcal{L}$. *Then we have for* $e_1\in\mathcal{L}$

$$\begin{aligned} (\delta-\mathcal{L}_{e_1})(e_1) &= \delta(e_1)-\mathcal{L}_{e_1}(e_1)=0-[e_1,e_1]=0-0=0, \\ (\delta-\mathcal{L}_{e_1})(e_2) &= \delta(e_2)-\mathcal{L}_{e_1}(e_2)=e_2-[e_1,e_2]=e_2-e_2=0, \\ (\delta-\mathcal{L}_{e_1})(e_3) &= \delta(e_3)-\mathcal{L}_{e_1}(e_3)=0-[e_1,e_3]=0-0=0, \\ (\delta-\mathcal{L}_{e_1})(e_4) &= \delta(e_4)-\mathcal{L}_{e_1}(e_4)=e_4-[e_1,e_4]=e_4-e_4=0, \\ (\delta-\mathcal{L}_{e_1})(e_5) &= \delta(e_5)-\mathcal{L}_{e_1}(e_5)=e_5-[e_1,e_5]=e_5-e_5=0. \end{aligned}$$

It follows that $\delta-\mathcal{L}_{e_1}$ *maps* $\mathcal{L}$ *to the centre of* $\mathcal{L}$. *For* e_2,e_3,e_4 *we have a similar result, that is why,* $aid(\mathcal{L})$ *consists of only inner derivations. Therefore,* $caid(\mathcal{L})=id(\mathcal{L})$. *As a result,* $\delta\in caid(\mathcal{L})$.

Definition 4. *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be two Leibniz algebras over* K. *The direct sum of the Leibniz algebras* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *which is denoted by* $\mathcal{L}_1\oplus\mathcal{L}_2$ *is the vector space direct sum with* $[\mathcal{L}_1,\mathcal{L}_2]=0$ *and* $[\mathcal{L}_2,\mathcal{L}_1]=0$.

Theorem 2. *Let* $\mathcal{G}$ *and* T *be two Leibniz algebras over* K. *Then* $aid(\mathcal{G}\oplus T)=aid(\mathcal{G})\oplus aid(T)$.

Proof. Let $\delta\in aid(\mathcal{G}\oplus T)$ and $p\in\mathcal{G}\oplus T$. Then $p=p_1+p_2$, where $p_1\in\mathcal{G},p_2\in T$. By the definition of almost inner derivation, $\delta(p)\in[\mathcal{G}\oplus T,p]$ and there is an element $q=q_1+q_2\in\mathcal{G}\oplus T$, where $q_1\in\mathcal{G},q_2\in T$ satisfying $\delta(p)=[q,p]\in[\mathcal{G}\oplus T,p]$. Doing some calculations we get

$$\begin{aligned} \delta(p)=[q,p] &= [q_1+q_2,p_1+p_2] \\ &= [q_1,p_1]+[q_1,p_2]+[q_2,p_1]+[q_2,p_2] \end{aligned}$$

$$= \quad [q_1, p_1] + [q_2, p_2].$$

So $\delta(p) \in \mathcal{G} \oplus T$. We say $\delta_1 = \delta|_{\mathcal{G}} \in aid(\mathcal{G})$, similarly $\delta_2 = \delta|_T \in aid(T)$. Hence δ can be written as $\delta = \delta_1 + \delta_2$, which is defined as

$$\begin{aligned} \delta : \mathcal{G} \oplus T &\rightarrow \mathcal{G} \oplus T \\ p_1 + p_2 &\mapsto (\delta_1 + \delta_2)(p_1 + p_2) = \delta_1(p_1) + \delta_2(p_2). \end{aligned}$$

Furthermore, let $p = p_1 + p_2 \in \mathcal{G} \oplus T, \delta_1 \in aid(\mathcal{G}) \oplus 0$ and $\delta_2 \in 0 \oplus aid(T)$. Then

$$\begin{aligned} [\delta_1, \delta_2](p) &= [\delta_1, \delta_2](p_1 + p_2) \\ &= (\delta_1\delta_2 - \delta_2\delta_1)(p_1) + (\delta_1\delta_2 - \delta_2\delta_1)(p_2) \\ &= (\delta_1\delta_2)(p_1) - (\delta_2\delta_1)(p_1) + (\delta_1\delta_2)(p_2) - (\delta_2\delta_1)(p_2) \\ &= 0. \end{aligned}$$

This means that $[aid(\mathcal{G}), aid(T)] = 0$ and $[aid(T), aid(\mathcal{G})] = 0$. Thus, $aid(\mathcal{G} \oplus T) \subseteq aid(\mathcal{G}) \oplus aid(T)$. Conversely, let $\delta_1 + \delta_2 \in aid(\mathcal{G}) \oplus aid(T)$, where $\delta_1 \in aid(\mathcal{G}) \oplus 0$, $\delta_2 \in 0 \oplus aid(T)$ and

$$\begin{aligned} \delta_1 + \delta_2 : \mathcal{G} \oplus T &\rightarrow \mathcal{G} \oplus T \\ p_1 + p_2 &\mapsto (\delta_1 + \delta_2)(p_1 + p_2) = \delta_1(p_1) + \delta_2(p_2). \end{aligned}$$

By the definition of $aid(\mathcal{G} \oplus T)$, there are $q_1 \in \mathcal{G}$ and $q_2 \in T$ such that $\delta_1(p_1) = [q_1, p_1]$ and $\delta_2(p_2) = [q_2, p_2]$. Therefore, by the definition, there exists $p_1 + p_2 \in \mathcal{G} \oplus T$, where $p_1 \in \mathcal{G}$ and $p_2 \in T$. Since $[q_1, p_2] = [p_1, q_2] = 0$, we have

$$\begin{aligned} (\delta_1 + \delta_2)(p_1 + p_2) &= \delta_1(p_1) + \delta_2(p_2) \\ &= [q_1, p_1] + [q_2, p_2] \\ &= [q_1 + q_2, p_1 + p_2]. \end{aligned}$$

Then $[q_1 + q_2, p_1 + p_2] \in \mathcal{G} \oplus T$ and we obtain $q_1 + q_2 \in \mathcal{G} \oplus T$. This shows that $\delta_1 + \delta_2 \in \mathcal{G} \oplus T$, that is, $aid(\mathcal{G}) \oplus aid(T) \subseteq aid(\mathcal{G} \oplus T)$. As a result, we obtain $aid(\mathcal{G}) \oplus aid(T) = aid(\mathcal{G} \oplus T)$, as required. □

Theorem 3. *Let $\mathcal{L}$ be a Leibniz algebra and $id(\mathcal{L})$ be an ideal of $der\mathcal{L}$ in which each element is nilpotent. Then $aid(\mathcal{L})$ is nilpotent.*

Proof. If each element of $id(\mathcal{L})$ is nilpotent, then there is a positive integer m such that $\mathcal{L}_p^m \neq 0$ and $\mathcal{L}_p^{m+1} = 0$. We have $\mathcal{L}_p^{m+1}(q) \in [\mathcal{L}^{m+1}, q]$. By Corollary 4.8 in [6], $\mathcal{L}$ is nilpotent and so by Theorem 1 (v), $aid(\mathcal{L})$ is nilpotent. □

4. Structure Constants

In this section firstly we derive which conditions a general map have to satisfy to be an almost inner derivation. Recall that if $\mathcal{L}$ is a Leibniz algebra over K with basis $P = \{p_1, p_2, \ldots, p_m\}$, then all elements in $\mathcal{L}$ can be determined by the products

$[p_i, p_j]$. Moreover, each product $[p_i, p_j]$ is expressed by a linear combination of the elements of basis as the following

$$[p_i, p_j] = \sum_{l=1}^{m} c_{ij}^{l} p_l, \tag{3}$$

where for $1 \leq i, j, l \leq m$, c_{ij}^{l} are scalars in K. We say that the c_{ij}^{l} are the structure constants of $\mathcal{L}$ with respect to this basis. The structure constants of $\mathcal{L}$ depend on the choice of basis of $\mathcal{L}$, that is, for different bases, we have different structure constants (more details in [13]).
Since a derivation δ of $\mathcal{L}$ is linear, we have

$$\delta(p_i) = \sum_{j=1}^{m} \alpha_{ij} p_j,$$

where $A = [\alpha_{ij}]_{m\times m}$ is the corresponding matrix of derivation δ. Let p_i and p_j be arbitrary two basis vectors in P. Then

$$\delta([p_i, p_j]) = \sum_{l=1}^{m} c_{ij}^{l} \delta(p_l) = \sum_{k=1}^{m} (\sum_{l=1}^{m} \alpha_{lk} c_{ij}^{l}) p_k \tag{4}$$

and

$$\begin{aligned} [\delta(p_i), p_j] + [p_i, \delta(p_j)] &= \sum_{l=1}^{m} \alpha_{il} [p_l, p_j] + \sum_{l=1}^{m} \alpha_{jl} [p_i, p_l] \\ &= \sum_{k=1}^{m} (\sum_{l=1}^{m} (\alpha_{il} c_{lj}^{k} + \alpha_{jl} c_{il}^{k})) p_k. \end{aligned} \tag{5}$$

Hence by (4) and (5), we obtain

$$\sum_{l=1}^{m} \alpha_{lk} c_{ij}^{l} = \sum_{l=1}^{m} (\alpha_{il} c_{lj}^{k} + \alpha_{jl} c_{il}^{k})$$

for every $1 \leq i, j, k \leq m$. As every derivation, an inner derivation can also be represented by a matrix. Let $\mathcal{L}_{p_i}$ be an inner derivation. Then we have

$$\mathcal{L}_{p_i}(p_j) = [p_i, p_j] = \sum_{k=1}^{m} \beta_{jk} p_k.$$

Hence by the equation (3), we obtain $\beta_{jk} = c_{ij}^{k}$ for all $1 \leq i, j, k \leq m$. Given an arbitrary $p = \sum_{i=1}^{m} t_i p_i \in \mathcal{L}$, where $t_i \in K$ and let $B = [\beta_{ji}]_{m\times m}$ be the matrix representation of $\mathcal{L}_p$. By using bilinearity of Leibniz bracket, the entries of B are given by

$$\beta_{jk} = \sum_{i=1}^{m} t_i c_{ij}^{k}.$$

Moreover, there are other conditions imposed by the definition of an almost inner derivation. Indeed, take $\delta \in aid(\mathcal{L})$, there exists a_{ij} with $1 \leq i, j \leq m$, so that

$$\delta(p_i) \quad = \quad \sum_{j=1}^{m} a_{ij}[p_j, p_i] = \sum_{k=1}^{m}\sum_{j=1}^{m} a_{ij}c_{ji}^{k}p_k. \tag{6}$$

These values a_{ij} with $1 \leq i, j \leq m$ are referred to as the parameters of δ with respect to the basis P. By using the bilinearity of a derivation and the equation (6) for any $p = \sum_{i=1}^{m} \beta_i p_i \in \mathcal{L}$ where $\beta_i \in K$ for all $1 \leq i \leq m$, we have

$$\delta(p) = \sum_{i=1}^{m} \beta_i \delta(p_i) = \sum_{k=1}^{m}(\sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i a_{ij} c_{ji}^{k})p_k. \tag{7}$$

Besides, there exist $\gamma_j \in K$ for $1 \leq j \leq m$, so that

$$\delta(p) = [\sum_{j=1}^{m} \gamma_j p_j, p] = \sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i \gamma_j [p_j, p_i] = \sum_{k=1}^{m}(\sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i \gamma_j c_{ji}^{k})p_k. \tag{8}$$

Therefore, we have two ways to write $\delta(p)$. The equations (7) and (8) give a system of linear equations

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i a_{ij} c_{ji}^{k} = \sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i \gamma_j c_{ji}^{k}$$

for all $1 \leq i, j \leq m$. Equivalently,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \beta_i (a_{ij} - \gamma_j) c_{ji}^{k} = 0. \tag{9}$$

The aim is to obtain the conditions on the parameters a_{ij} with $1 \leq i, j \leq m$ such that there exist γ_j for which the system of equations (9) has a solution for all possible values of β_i. An arbitrary almost inner derivation $\delta : \mathcal{L} \to \mathcal{L}$ can be expressed as $p \mapsto A.p$ where $A = [\alpha_{ij}]$ is the matrix representation of δ and . is matrix multiplication. By the equation (6), the entries of A are given by

$$\alpha_{ij} = \sum_{k=1}^{m} a_{ij} c_{ji}^{k}.$$

5. Fixed Basis Vectors

Let $\mathcal{L}$ be an m-dimensional Leibniz algebra over K with the basis $P = \{p_1, p_2, \ldots, p_m\}$. We denote by $C_{\mathcal{L}}(p)$ the centralizer of p which is defined by

$$C_{\mathcal{L}}(p) = \{q \in \mathcal{L} | [p, q] = [q, p] = 0\}.$$

Let $\delta \in aid(\mathcal{L})$, then there is a mapping $\varphi_\delta : \mathcal{L} \to \mathcal{L}$ satisfying $\delta(p) = [\varphi_\delta(p), p] \in [\mathcal{L}, p]$ for all $p \in \mathcal{L}$. This is not unique because for any $q \in C_{\mathcal{L}}(p)$, we can take

$\varphi_\delta(p)+q$ instead of $\varphi_\delta(q)$. Namely,

$$\delta(p) = [\varphi_\delta(p)+q,p] = [\varphi_\delta(p),q]+[q,p] = [\varphi_\delta(p),p].$$

In general, this map need not be linear. Let $p \in \mathcal{L}$, then p can be written as a linear combination of the basis P such that $p = \sum_{j=1}^{m} \alpha_j p_j$, where $\alpha_j \in K$. We represent by $t_i(p) = \alpha_i$ the i-th projection mapping of p with respect to the given basis.

Definition 5. *A basis vector p_i is called a fixed vector for δ with $\alpha \in K$ iff $t_i(\varphi_\delta(p_j)) = \alpha$ where $p_j \notin C_{\mathcal{L}}(p_i)$ for every $1 \le j \le m$.*

Example 2. *Let $\mathcal{L}$ be a 3-dimensional Leibniz algebra with the basis $\{p,q,r\}$ by the following rule $[p,p]=q$ and $[p,q]=r$. Then the centralizers for $p,q,r \in \mathcal{L}$, $C_{\mathcal{L}}(p) = Span\{r\}$, $C_{\mathcal{L}}(q) = Span\{q,r\}$, $C_{\mathcal{L}}(r) = Span\{p,q,r\}$. Let $\delta \in aid(\mathcal{L})$ and φ_δ be a mapping with*

$$\delta(p) = [\varphi_\delta(p),p] = q, \delta(q) = [\varphi_\delta(q),q] = r, \delta(r) = [\varphi_\delta(r),r] = 0.$$

Hence we obtain $\varphi_\delta(p) = p, \varphi_\delta(q) = p$ and $\varphi_\delta(r) \in Span\{p,q,r\}$. In particular, we take a map φ_δ with the following rule

$$\varphi_\delta(p) = p, \varphi_\delta(q) = p, \text{ and } \varphi_\delta(r) = q.$$

Thus, for $p \in \mathcal{L}$ we have

$$\begin{aligned} &p \notin C_{\mathcal{L}}(p), t_1(\varphi_\delta(p)) = t_1(p) = t_1(1.p+0.q+0.r) = 1,\\ &v \notin C_{\mathcal{L}}(p), t_1(\varphi_\delta(q)) = t_1(p) = t_1(1.p+0.q+0.r) = 1,\\ &r \in C_{\mathcal{L}}(p). \end{aligned}$$

p is fixed basis vector for δ with fixed value $\alpha = 1$. For $q \in \mathcal{L}$

$$\begin{aligned} &p \notin C_{\mathcal{L}}(q), t_2(\varphi_\delta(q)) = t_2(p) = t_2(1.p+0.q+0.r) = 0,\\ &q \in C_{\mathcal{L}}(q),\\ &r \in C_{\mathcal{L}}(q). \end{aligned}$$

q is fixed basis vector for δ with fixed value $\beta = 0$. Finally, for $r \in \mathcal{L}$ we obtain $p \in C_{\mathcal{L}}(r)$, $q \in C_{\mathcal{L}}(r)$, $w \in C_{\mathcal{L}}(r)$, this means that r is also fixed basis vector for δ.

Lemma 4. *Let $\mathcal{L}$ be a Leibniz algebra and $\delta \in aid(\mathcal{L})$ which is defined by a mapping $\varphi_\delta : \mathcal{L} \to \mathcal{L}$. If p_i is a fixed basis vector with fixed value α, then $\delta^{'} = \delta + \mathcal{L}_{\alpha p_i} \in aid(\mathcal{L})$ which is determined by a mapping $\varphi_{\delta'} : \mathcal{L} \to \mathcal{L}$ holding*

$$t_i(\varphi_{\delta'}(p_k)) = 2t_i(\varphi_\delta(p_k)), \quad t_j(\varphi_{\delta'}(p_k)) = 0$$

for every $1 \le i,j,k \le m$ and $i \ne j$.

Proof. Firstly, we will show that $\delta^{'} \in aid(\mathcal{L})$. For any $p \in \mathcal{L}$,

$$\begin{aligned} \delta^{'}(p) &= (\delta + \mathcal{L}_{\alpha p_i})(p)\\ &= [\varphi_\delta(p),p] + [\alpha p_i, p]\\ &= [\varphi_\delta(p) + \alpha p_i, p]. \end{aligned}$$

This implies that $[\varphi_\delta(p) + \alpha p_i, p] \in [\mathcal{L}, p]$. So $\delta^{'} \in aid(\mathcal{L})$ and $\delta^{'}$ is defined by the mapping

$$\begin{array}{rcl} {\varphi_{\delta'}}^* : \mathcal{L} & \to & \mathcal{L} \\ p & \mapsto & \varphi_\delta(p) + \alpha p_i. \end{array}$$

Now we define the mapping $\varphi_{\delta'} : \mathcal{L} \to \mathcal{L}$ such that

$$p \mapsto \begin{cases} \varphi_\delta(p) + \alpha p_i, & \text{if } p \notin \{p_1, p_2, \dots, p_m\} \\ \varphi_\delta(p) + t_i(\varphi_\delta(p))p_i, & \text{if } p \in \{p_1, p_2, \dots, p_m\}. \end{cases}$$

We need to prove that $\delta^{'}$ is determined by this map. Indeed for all $p \notin \{p_1, p_2, \dots, p_m\}$ we have $\varphi_{\delta'}(p) = {\varphi_{\delta'}}^*(p)$ and for all $p \in \{p_1, p_2, \dots, p_m\}$ there are two cases:
Case 1. If $p_j \notin C_{\mathcal{L}}(p_i)$, then we have $f_i(\varphi_\delta(p_j)) = \alpha$. Thus, ${\varphi_{\delta'}}^* = \varphi_{\delta'}$.
Case 2. If $p_j \in C_{\mathcal{L}}(p_i)$, then we have

$$\begin{array}{rcl} \delta^{'}(p_j) & = & (\delta + \mathcal{L}_{\alpha p_i})(p_j) \\ & = & \delta(p_j) + [\alpha p_i, p_j] \\ & = & [\varphi_\delta(p_j), p_j] \\ & = & [\varphi_\delta(e_j) + t_i(\varphi_\delta(e_j))p_i, p_j] \\ & = & [\varphi_{\delta'}(p_j), p_j]. \end{array}$$

Hence $\delta^{'}$ is given by $\varphi_{\delta'}$. By the definition of $\varphi_{\delta'}$, it is clear to show that $t_i(\varphi_{\delta'}(p_k)) = 2t_i(\varphi_\delta(p_k)), \quad t_j(\varphi_{\delta'}(p_k)) = 0$ for every $1 \leq i, j, k \leq m$ and $i \neq j$. □

Corollary 1. *Given a Leibniz algebra $\mathcal{L}$ and $\delta \in aid(\mathcal{L})$ which is defined by a mapping φ_δ. If each basis vector is fixed, then $\delta \in id(\mathcal{L})$.*

Proof. Let α_i be the fixed value of p_i. By Lemma 4, we obtain that

$$\delta^{'} = \delta + \mathcal{L}_p$$

where $p = \sum_{i=1}^{m} \alpha_i p_i$ is an almost inner derivation which is given by a mapping $\varphi_{\delta'}$ with $\varphi_{\delta'}(p_i) = 0$ for every $1 \leq i \leq m$. It follows that

$$\delta^{'}(p_i) = [\varphi_{\delta'}(p_i), p_i] = 0$$

for all p_i basis vectors. Hence we obtain $\delta^{'} = 0$ and $\delta = -\mathcal{L}_p$. This shows that $\delta \in id(\mathcal{L})$. □

Corollary 2. *If any basis vector for all almost inner derivation is fixed, then $aid(\mathcal{L}) = id(\mathcal{L})$.*

Proof. We know from the inclusions of Lie subalgebras $id(\mathcal{L}) \subseteq aid(\mathcal{L})$. By Corollary 1, we obtain that $aid(\mathcal{L}) \subseteq id(\mathcal{L})$. Therefore, $aid(\mathcal{L}) = id(\mathcal{L})$. □

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SEMI-SLANT LIGHTLIKE SUBMERSIONS

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Abstract. In this paper, we intend to study semi-slant lightlike submersions from indefinite Kaehler manifolds onto lightlike manifolds. After giving definitions and basic properties, we obtain conditions for a lightlike submersion to be a semi-slant lightlike submersion. We indicate some relevant examples. Finally, we investigate the geometric properties of foliations that appeared with a semi-slant lightlike submersion.

1. Introduction

Studying differentiable maps defined between manifolds are one of the methods used to compare geometric structures. One of these maps is submersion, in which the rank of the transformation is equal to the dimension of the target manifold. Moreover, if this map is isometric, it is called Riemannian submersion.

Riemannian submersions were first defined by O' Neill and Gray independently of each other [15], [7]. This definition was extended to manifolds with different differentiable structures. After some important developments in complex and contact geometry, the Riemannian submersions have become interesting. The differential geometry of manifolds with special structures have been examined by using different kind of Riemannian submersions [1, 6, 8–10, 17, 23, 24, 26].

On the other hand, a major shortcoming of the semi-Riemannian manifold is that there are no suitable types of functions from one manifold to the next to satisfy its geometrical properties.This flaw was fixed by O' Neill in 1983 [16]. As the generalizations of Riemannian submersions, O' Neill introduced the notion of semi-Riemannian submersions. A well known fact is that for a defined Riemannian submersion between two Riemannian manifolds, the fibers are always Riemannian but the fibers of semi-Riemannian manifolds on a semi-Riemann submersions may not be semi-Riemannian manifold.

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The another importance of such maps is their applications in mathematics [24], in theoretical physics (supergravity and superstring theories [13,14],Yang-Mills theory [3,27] and Kaluza-Klein theory [4,11]) and robotic theory [2].

On the other hand, although there have been many publications on the geometry of Riemannian submersions, there have been very few on semi-Riemannian submersions and lightlike submersions. First, Şahin investigated submersions between lightlike manifolds and semi-Riemannian manifolds [21,22]. Here, he obtained O'Neill's tensors defined on Riemannian submersions for lightlike submersions and showed the differences between the two maps for these tensors. Moreover, he studied lightlike harmonic map.

In [5], Duggal investigated harmonic maps between two semi-Riemannian manifolds. He showed that these maps behave differently. Moreover, he obtained that harmonic maps between two semi-Riemannian manifolds must be subject to some restricted classes of semi-Riemannian manifolds. Thus, harmonic maps from a semi-Riemannian manifold into a lightlike manifold were studied only when the target manifold is a Riemannian hypersurface of a lightlike manifold.

In [25], Şahin and Gündüzalp investigated lightlike submersions from a semi-Riemannian manifold onto lightlike manifold. After this definition, different structure in Riemannian submersion theory began to be examined for lightlike submersion as well. Firstly, Sachdeva et all. introduced slant lightlike submersions [19]. Later, Prasad et all. studied slant lightlike submersion for indefinite nearly Kaehler manifold [18]. They established the existence theorems for slant lightlike submersions and investigated geometry of foliations. Kaushal et all. introduced pointwise slant lightlike submersions [12]. Shukla et all. studied screen slant lightlike submersions [20].

Under the motivations and the light of these studies, we defined semi-slant lightlike submersion from indefinite Kaehler manifold onto lightlike manifold. We aim is to present some general properties of this type of submersions and after that to obtain major results on the geometry of them. In Section 2 we review some the standard facts on semi-Riemann submersions and lightlike submersions. After giving the definition of semi-slant lightlike submersions from indefinite Kaehler manifold into lightlike manifold in Section 3 we indicate related examples. In section 4 we study of minimality, integrability and totally geodesic conditions of distributions.

2. Preliminaries

In this section, we introduce lightlike submersions. We define lightlike submersions and O'Neill's tensors for lightlike submersions.

Let (M, g_M) and (B, g_B) be a semi-Riemannian manifold and an r-lightlike manifold, respectively. Therefore, we have a submersion $\psi : M \to B$. Moreover, $\psi^{-1}(q)$ is a submanifold of M, where $dim\psi^{-1} = dimM - dimB$. Then, for $q \in B$, $\psi^{-1}(q)$ is said to be fiber.

Thus, the kernel of ψ_* at the point p is defined by

$$\ker\psi_* = \{X \in T_pM : \psi_*(X) = 0\}.$$

On the other hand, we denote

$$(\ker\psi_*)^\perp = \{Y \in T_pM : g_M(X,Y) = 0, \forall X \in \ker\psi_*\}.$$

Since T_pM is a semi-Riemannian manifold, $(\ker\psi_*)^\perp$ cannot be a supplement to $\ker\psi_*$.

Assume $\triangle = \ker\psi_* \cap (\ker\psi_*)^\perp \neq \{0\}$. Therefore, we have different four cases of submersions:

Case1: Then consider $0 < dim\triangle < \min\{\dim(\ker\psi_*), \dim(\ker\psi_*)^\perp\}$.

Thus $\triangle$ is the radical subspace of T_pM.

On the other hand, $\ker\psi_*$ is a reel lightlike vector space.Then, there is supplementary non degenerate sub-space to $\triangle$. Let $S(\ker\psi_*)$ be a supplementary non degenerate sub space to $\triangle$ in $\ker\psi_*$. Thus we given by

$$\ker\psi_* = \triangle \perp S(\ker\psi_*).$$

By a similar method, we see that

$$(\ker\psi_*)^\perp = \triangle \perp S(\ker\psi_*)^\perp,$$

where $S(\ker\psi_*)^\perp$ is a supplementary sub-space of $\triangle$ in $(\ker\psi_*)^\perp$. However $S(\ker\psi_*)$ is non-degenerate in T_pM, we have

$$T_pM = S(\ker\psi_*) \perp S(\ker\psi_*)^\perp$$

where $S(\ker\psi_*)^\perp$ is the supplementary sub-space of $S(\ker\psi_*)$ in T_pM. On the other hand $S(\ker\psi_*)$ and $S(\ker\psi_*)^\perp$ are non degenerate, we deduce,

$$(S(\ker\psi_*))^\perp = S(\ker\psi_*)^\perp \perp (S(\ker\psi_*)^\perp)^\perp.$$

In that case, for all $\alpha, \beta \in \{1, ..., t\}$ and $i, j \in \{1, ..., r\}$, we get

$$g_M(\xi_i, \xi_j) = g_M(N_i, N_j) = 0, \quad g_M(\xi_i, N_j) = \delta_{ij}$$

$$g_M(W_\alpha, \xi_j) = g_M(W_\alpha, N_j) = 0, \quad g_M(W_\alpha, W_\beta) = \epsilon_\alpha \delta_{\alpha\beta},$$

where $\{\xi_i\}$ is base of $\triangle$, $\{N_i\}$ are null vector fields of $(S(\ker\psi_*)^\perp)^\perp$, $\{W_\alpha\}$ are bases of $S(\ker\psi_*)^\perp$. We can construct the set of vector fields $\{N_i\}$ for $ltr(\ker\psi_*)$, therefore, we arrive

$$tr(\ker\psi_*) = ltr(\ker\psi_*) \perp S(\ker\psi_*)^\perp.$$

We emphasize that $\ker\psi_*$ and $ltr(\ker\psi_*)$ are not orthogonal. Therefore, we show that $\mathcal{H} = tr(\ker\psi_*)$ the horizontal space and $\mathcal{V} = \ker\psi_*$ the vertical space of T_pM as is usual in the theory of Riemannian submersions. Hence we have,

$$T_pM = \mathcal{V}_p \oplus \mathcal{H}_p.$$

We note that $\mathcal{H}$ and $\mathcal{V}$ are not orthogonal.

Now, we can give the definition of a lightlike submersion.

Definition 1. *[25], Let $\psi : (M, g_M) \to (B, g_B)$ be a submersion, where M and B are a semi-Riemannian manifold and an r-lightlike manifold, respectively. Therefore, ψ is said to be an r-lightlike submersion if,*

(i) $\dim \triangle = \dim\{\ker \psi_* \cap (\ker \psi_*)^{\perp}\} = r, 0 < r < \min\{\dim(\ker \psi_*), \dim(\ker \psi_*)^{\perp}\}$.

(ii) $g_M(X, Y) = g_B(\psi_* X, \psi_* Y)$ *for all* $X, Y \in \Gamma(\mathcal{H})$.

Case2: $\dim \triangle = \dim \ker \psi_* < \dim(\ker \psi_*)^{\perp}$.
Therefore, $\mathcal{H} = S(\ker \psi_*)^{\perp} \perp ltr(\ker \psi_*)$ and $\mathcal{V} = \triangle$. Then, ψ is said to be an isotropic submersion.

Case3: $\dim \triangle = \dim(\ker \psi_*)^{\perp} < \dim \ker \psi_*$.
Therefore $\mathcal{H} = ltr(\ker \psi_*)$ and $\mathcal{V} = S(\ker \psi_*) \perp \triangle$. Then, ψ is said to be a co-isotropic submersion.

Case4: $\dim \triangle = \dim(\ker \psi_*)^{\perp} = \dim \ker \psi_*$.
Therefore $\mathcal{H} = ltr(\ker \psi_*)$ and $\mathcal{V} = \triangle$. Then, ψ is said to be a totally lightlike submersion.

Now, we follow the lemma that we will use in the definition of semi-slant lightlike submersion.

Lemma 1. *[19], Let $\psi : (M, J, g_M) \to (B, g_B)$ be a r-lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Let $J\triangle$ be a distribution on M such that $\triangle \cap J\triangle = 0$. Then any distribution complementary to $J\triangle \oplus J(ltr(\ker \psi_*))$ in $S(\ker \psi_*)$ is Riemannian.*

On the other hand, O'Neill was defined tensors $\mathcal{T}$ and $\mathcal{A}$ for Riemannian submersions [15]. Şahin and Gündüzalp are characterized tensors $\mathcal{T}$ and $\mathcal{A}$ for lightlike submersions as follows:

$$\mathcal{T}_E F = h\overset{M}{\nabla}_{vE} vF + v\overset{M}{\nabla}_{vE} hF \tag{1}$$

and

$$\mathcal{A}_E F = h\overset{M}{\nabla}_{hE} hF + v\overset{M}{\nabla}_{hE} vF, \tag{2}$$

for all $E, F \in \Gamma(TM)$, where h and v are the horizontal and vertical projections. Therefore from (1) and (2), we have

$$\overset{M}{\nabla}_U V = \mathcal{T}_U V + v\overset{M}{\nabla}_U V \tag{3}$$

$$\overset{M}{\nabla}_U X = \mathcal{T}_U X + h\overset{M}{\nabla}_U X \tag{4}$$

$$\overset{M}{\nabla}_X U = v\overset{M}{\nabla}_X U + \mathcal{A}_X U \tag{5}$$

$$\overset{M}{\nabla}_X Y = \mathcal{A}_X Y + h\overset{M}{\nabla}_X Y, \tag{6}$$

for all $U, V \in \Gamma(\ker \psi_*)$ and $X, Y \in \Gamma(tr(\ker \psi_*))$, [25].

Now, let's remember the definition of indefinite Kaehler manifold. A 2m-dimensional differentiable manifold $M = (M, J, g_M)$ is said to be indefinite Kaehler manifold if there exist a semi-Riemannian metric g_M and a complex structure J,

$$J^2 = -I, \quad g_M(JE, JF) = g_M(E, F) \tag{7}$$

and

$$(\nabla_E J)F = 0, \tag{8}$$

for any $E, F \in \Gamma(TM)$.

3. Semi-Slant Lightlike Submersions

Firstly, let's define the semi-slant lightlike submersions and give examples.

Definition 2. *Let (M, J, g_M) and (B, g_B) be an indefinite Kaehler manifold and r-lightlike manifold, respectively. Let $\psi : (M, J, g_M) \to (B, g_B)$ be an r-lightlike submersion. Therefore, ψ is called a semi-slant lightlike submersion if there exist on M two non-degenere orthogonal distributions D_1 and D_2 such that*

(i) $J\triangle$ is a distribution in $\ker\psi_*$ such that $\triangle \cap J\triangle = 0$;

(ii)

$$S(\ker\psi_*) = (J\triangle \oplus J(ltr(\ker\psi_*))\perp D_1 \perp D_2;$$

(iii) D_1 is an invariant distribution, under J, that is $JD_1 = D_1$;

(iv) D_2 is slant distribution with angle $\theta(X)$, such that for all $x \in M$ and $X \in (D_2)_x$.

Moreover, the angle θ is saidto be the semi-slant angle of the lightlike submersion. In particular, if $D_1 = 0$, therefore M is a slant lightlike submersion.

Hence we get,

$$\begin{aligned} TM &= \mathcal{V} \oplus \mathcal{H} \\ &= \{\triangle \perp (J\triangle \oplus J(ltrker\psi_*)) \perp D_1 \perp D_2\} \oplus \{\psi(D_2) \perp \mu \perp ltr(ker\psi_*)\}, \end{aligned}$$

where μ is the orthogonal sub-bundle complementary to $\psi(D_2)$ in $S(ker\psi_*)$.

Example 1. *Every slant lightlike submersion from indefinite Kaehler manifold onto r-lightlike manifold is semi-slant lightlike manifold with $D_1 = 0$.*

Example 2. *Let $(\mathbb{R}^{12}_{0,2,10}, g_1, J)$ and $(\mathbb{R}^7_{1,0,6}, g_2)$ be an indefinite Kaehler manifold and lightlike manifold, $g_1 = -(dx_1)^2 - (dx_2)^2 + \sum_{i=3}^{12} (dx_i)^2$ is semi-Riemannian metric and $g_2 = \sum_{j=2}^{7} (dy_j)^2$ is a degenerate metric, where x_i, $i = 1, ...12$ and $y_j, j = 1, ...7$ are the canonical coordinates on $\mathbb{R}^{12}$ and $\mathbb{R}^7$ respectively. If we set $J(x_1, x_2, ..., x_{11}, x_{12}) = (-x_2, x_1, ..., -x_{12}, x_{11})$ then $J^2 = -I$ and J is complex structure on $\mathbb{R}^{12}$. We define the following map*

$$\psi : \mathbb{R}^{12} \to \mathbb{R}^7$$

$$(x_1, ..., x_{12}) \to (x_1 + x_4, x_2, x_3, \frac{x_5 + x_7}{\sqrt{2}}, \frac{x_6 + x_8}{\sqrt{2}}, \sin\alpha x_9 - \cos\alpha x_{11}, x_{12}).$$

On the other hand, kernel of ψ_* *is*

$$\begin{aligned}\ker\psi_* &= Sp\{V_1=-\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_4}, V_2=\frac{\partial}{\partial x_2}, V_3=\frac{\partial}{\partial x_3}, V_4=\frac{\partial}{\partial x_5}-\frac{\partial}{\partial x_7},\\ V_5 &= \frac{\partial}{\partial x_6}-\frac{\partial}{\partial x_8}, V_6=-\cos\alpha\frac{\partial}{\partial x_9}-\sin\alpha\frac{\partial}{\partial x_{11}}, V_7=\frac{\partial}{\partial x_{10}}\}.\end{aligned}$$

Then, we arrive

$$\begin{aligned}(\ker\psi_*)^{\perp} &= Sp\{Z_1=-\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_4}, Z_2=\frac{\partial}{\partial x_5}+\frac{\partial}{\partial x_7}, Z_3=\frac{\partial}{\partial x_6}+\frac{\partial}{\partial x_8}\\ Z_4 &= \sin\alpha\frac{\partial}{\partial x_9}-\cos\alpha\frac{\partial}{\partial x_{11}}, Z_5=\frac{\partial}{\partial x_{12}}\}.\end{aligned}$$

On the other hand, we have $\ker\psi_*\cap(\ker\psi_*)^{\perp}=Sp\{V_1\}$. *Moreover, we get* $ltr(\ker\psi_*)=Sp\{N=\frac{1}{2}(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_4})\}$. *Then the horizontal and vertical spaces are given by*

$$\mathcal{H}=\{N,Z_2,Z_3,Z_4,Z_5\}, \mathcal{V}=Sp\{V_1,V_2,V_3,V_4,V_5,V_6,V_7\},$$

Also by direct computations we obtain, $g_1(N,N)=g_2(\psi_*N,\psi_*N)$, *and* $g_1(Z_i,Z_i)=g_2(\psi_*Z_i,\psi_*Z_i)$ *for all* $i=2,...,5$. *Hence* ψ *is a 1-lightlike submersion. On the other hand, we have* $JV_4=-V_5, JV_5=V_4$. *Thus it follows that* $D_1=Sp\{V_4,V_5\}$ *and* $D_2=Sp\{V_6,V_7\}$ *are a invariant and slant distribution with slant angle* $\theta=\alpha$, *respectively. Moreover* $JV_1=V_2+V_3$, $JN=\frac{1}{2}(-V_2+V_3)$ *such that* $J\triangle$ *and* $J(ltr(\ker\psi_*))$ *are distributions on* $\mathbb{R}^{12}_2$. *Thus* ψ *is a semi slant lightlike submersion.*

Example 3. *Let* $(\mathbb{R}^{12}_{0,2,10},g_1,J)$ *and* $(\mathbb{R}^{6}_{2,0,4},g_2)$ *be an indefinite Kaehler manifold and lightlike manifold,* $g_1=-(dx_1)^2-(dx_2)^2+\sum_{i=3}^{12}(dx_i)^2$ *is semi-Riemannian metric and* $g_2=\sum_{j=3}^{6}(dy_j)^2$ *is a degenerate metric, where* x_i, $i=1,...12$ *and* $y_j, j=1,...6$ *are the canonical coordinates on* $\mathbb{R}^{12}$ *and* $\mathbb{R}^6$ *respectively. If we set* $J(x_1,x_2,...,x_{11},x_{12})=(-x_2,x_1,...,-x_{12},x_{11})$ *then* $J^2=-I$ *and* J *is complex structure on* $\mathbb{R}^{12}$. *We define the following map*

$$\psi:\mathbb{R}^{12}\to\mathbb{R}^7$$

$$(x_1,...,x_{12})\to(x_1+x_5,x_2+x_6,\frac{x_3-x_7}{\sqrt{2}},\frac{x_4-x_8}{\sqrt{2}},\frac{x_9-x_{12}}{\sqrt{2}},x_{11}).$$

On the other hand, kernel of ψ_* *is*

$$\begin{aligned}\ker\psi_* = \;& Sp\{V_1=\tfrac{\partial}{\partial x_1}-\tfrac{\partial}{\partial x_5}, V_2=\tfrac{\partial}{\partial x_2}-\tfrac{\partial}{\partial x_6}, V_3=\tfrac{\partial}{\partial x_3}+\tfrac{\partial}{\partial x_7}, V_4=\tfrac{\partial}{\partial x_4}+\tfrac{\partial}{\partial x_8},\\ & V_5=\tfrac{\partial}{\partial x_9}+\tfrac{\partial}{\partial x_{12}}, V_6=\tfrac{\partial}{\partial x_{10}}\}.\end{aligned}$$

Then, we arrive

$$(\ker\psi_*)^{\perp} = Sp\{Z_1=\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_5}, Z_2=\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_6}, Z_3=\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_7}$$

$$Z_4 = \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_8}, Z_5 = \frac{\partial}{\partial x_9} - \frac{\partial}{\partial x_{12}}, Z_6 = \frac{\partial}{\partial x_{11}}\}.$$

On the other hand, we have $\ker\psi_* \cap (\ker\psi_*)^{\perp} = Sp\{V_1, V_2\}$. *Moreover, we get* $ltr(\ker\psi_*) = Sp\{N_1 = \frac{1}{2}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}), N_2 = \frac{1}{2}(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6})\}$. *Then the horizontal and vertical spaces are given by*

$$\mathcal{H} = \{N_1, N_2, Z_3, Z_4, Z_5, Z_6\}, \mathcal{V} = Sp\{V_1, V_2, V_3, V_4, V_5, V_6\},$$

Also by direct computations we obtain, $g_1(N_j, N_j) = g_2(\psi_* N_j, \psi_* N_j)$, *and* $g_1(Z_i, Z_i) = g_2(\psi_* Z_i, \psi_* Z_i)$ *for all* $i = 3, ..., 6$. *Hence* ψ *is a 2-lightlike submersion. On the other hand, we have* $JV_3 = -V_4, JV_4 = V_3$. *Thus it follows that* $D_1 = Sp\{V_3, V_4\}$ *and* $D_2 = Sp\{V_5, V_6\}$ *are a invariant and slant distribution with slant angle* $\theta = \frac{\alpha}{4}$, *respectively. Moreover* $JV_1 = V_2 + V_3$, $JN = \frac{1}{2}(-V_2 + V_3)$ *such that* $J\triangle$ *and* $J(ltr(\ker\psi_*))$ *are distributions on* $\mathbb{R}^{12}_2$. *Thus* ψ *is a semi slant lightlike submersion.*

Now, let ψ be a r-lightlike submersion. Therefore for $U \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$, we get

$$JU = \phi U + wU, \quad JX = BX + CX, \tag{9}$$

where $wU(CX\ )$ and $\phi U(BX)$ are the transversal component and tangential of $JU(JX)$, respectively.

Denote by P_1, P_2, P_3, P_4, P_5 the projections onto the distributions $\triangle$, $J\triangle$, $J(ltr(\ker\psi_*))$, D_1, D_2, respectively.

Thus, for any $U \in \Gamma(\mathcal{V})$, we can write

$$U = P_1U + P_2U + P_3U + P_4U + P_5U.$$

We applying J to last equation, we get

$$JU = JP_1U + JP_2U + JP_3U + JP_4U + \phi P_5U + wP_5U, \tag{10}$$

where $\phi P_5U(resp.\ wP_5U)$ denotes the tangential (resp. transversal) component of JP_5U. Then, we have

$$JP_1U = \phi P_1U \in \Gamma(J\triangle), \quad wP_1U = 0,$$

$$JP_2U = \phi P_2U \in \Gamma(\triangle), \quad wP_2U = 0,$$

$$JP_3U = wP_3U \in \Gamma(ltr(\ker\psi_*)), \quad \phi P_3U = 0,$$

$$JP_4U = \phi P_4U \in \Gamma(D_1),$$

$$\phi P_5U \in \Gamma(D_2), \quad wP_5U \in \Gamma(\psi(D_2)).$$

Therefore, we can write

$$\phi U = \phi P_1U + \phi P_2U + \phi P_5U.$$

Theorem 1. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold, respectively. Therefore ψ is a semi-slant lightlike submersion if and only if*

i) $Jltr(\ker\psi_*)$ is a distribution on M,

ii) for all $U \in \Gamma(\ker\psi_*)$,

$$\phi^2 P_5 U = \lambda P_5 U, \tag{11}$$

where, $\lambda = -\cos^2\theta$ and θdenotes the semi-slant angle of D_2.

Proof. Firstly, let ψ be a semi-slant lightlike submersion. Therefore $J\triangle$ is a distribution on $S(\ker\psi_*)$. Then, using *Lemma 1*, $J(ltr(\ker\psi_*))$ is a distribution on M.

Further, since ψ is semi-slant lightlike submersion, the slant angle betwen JU and D_2 is constant. Then using (10) and (7), we get

$$\cos\theta_{D_2} = -\frac{g_M(U, (\phi P_5)^2 U)}{\|JU\|\,\|\phi P_5 U\|}.$$

On the other hand, from (7), we obtain

$$\cos\theta_{D_2} = \frac{\|JU\|}{\|\phi P_5 U\|}.$$

By the last two equations, we have

$$\cos\theta_{D_2}^2 = -\frac{g_M(U, (\phi P_5)^2 U)}{\|\phi P_5 U\|^2}.$$

Since the angle θ is constant on D_2, we give

$$\phi^2 P_5 U = \lambda^2 P_5 U,$$

where $\lambda = -\cos^2\theta$.

Conversely, from (i), $J\triangle$ is a distribution on $S(\ker\psi_*)$. Moreover, if *lemma 2* is used, the proof is complete. □

Corollary 1. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore, for all $U, V \in \Gamma(\ker\psi_*)$*

$$g_M(\phi U, \phi V) = \cos^2\theta g_M(U, V), \tag{12}$$

$$g_M(wU, wV) = \sin^2\theta g_M(U, V). \tag{13}$$

4. Minimality, Integrability and Totally Geodesic Foliations

In this section, we investigate minimality,totally geodesic and integrability of distributions.

Theorem 2. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore D_1 is integrable if and only if*

i)$\mathcal{T}_U \phi P_4 V - \mathcal{T}_V \phi P_4 U \notin \Gamma(\psi(D_2))$

ii) $g_M(v\nabla_U \phi P_4 V - v\nabla_V \phi P_4 U, BN) = g_M(\mathcal{T}_V \phi P_4 U - \mathcal{T}_U \phi P_4 V, CN)$

iii) $v\nabla_U \phi P_4 V - v\nabla_V \phi P_4 U \notin \Gamma(\triangle)$,

where $U, V \in \Gamma(D_1), K \in \Gamma(D_2), W \in \Gamma(Jltr(\ker\psi_*)), N \in \Gamma(ltr(\ker\psi_*))$.

Proof. For all $U, V \in \Gamma(D_1)$, since $[U, V] \in \Gamma(\mathcal{V})$ we arrive $g_M([U, V], X) = 0$, where $X \in \Gamma(\mathcal{H})$. Thus, for all $K \in \Gamma(D_2), W \in \Gamma(Jltr(\ker\psi_*))$ and $N \in \Gamma(ltr(\ker\psi_*))$, we get D_1 is integrable if and only if $g_M([U, V], K) = 0, g_M([U, V], N) = 0$ and $g_M([U, V], W) = 0$. Firsly using (7) and (8), we have

$$\begin{aligned} g_M(\nabla_U V, K) &= -g_M(\nabla_U JV - (\nabla_U J)V, JK) \\ &= -g_M(\nabla_U JV, JK). \end{aligned} \tag{14}$$

Then, from (7), (8) and (10) we get

$$\begin{aligned} g_M([U, V], K) &= -g_M(\nabla_U V, J\phi P_5 K) + g_M(\nabla_U JV, wP_5 K) \\ &\quad + g_M(\nabla_V U, J\phi P_5 K) - g_M(\nabla_V JU, wP_5 K). \end{aligned}$$

Also, using (10), (12), (3) and (4) we have

$$\begin{aligned} g_M([U, V], K) &= \cos^2\theta g_M(\nabla_U V, K) + g_M(\mathcal{T}_U \phi P_4 V, wP_5 K) \\ &\quad - \cos^2\theta g_M(\nabla_V U, K) - g_M(\mathcal{T}_V \phi P_4 U, wP_5 K). \end{aligned}$$

After some calculations, we obtain

$$\sin^2\theta g_M([U, V], K) = g_M(\mathcal{T}_U \phi P_4 V - \mathcal{T}_V \phi P_4 U, wP_5 K)$$

which proves (i).

For $N \in \Gamma(ltr(\ker\psi_*))$, from (10), (14), we obtain

$$g_M([U, V], N) = g_M(\nabla_U \phi P_4 V - \nabla_V \phi P_4 U, JN).$$

Thus, using (3) and (9), we get

$$\begin{aligned} g_M([U, V], N) &= g_M(v\nabla_U \phi P_4 V - v\nabla_V \phi P_4 U, BN) \\ &\quad + g_M(\mathcal{T}_U \phi P_4 V - \mathcal{T}_V \phi P_4 U, CN) \end{aligned}$$

which gives (ii).

Finally, $W \in \Gamma(Jltr(\ker\psi_*))$, from (10), (14) and (3), we arrive at

$$g_M([U, V], W) = g_M(v\nabla_U \phi P_4 V - v\nabla_V \phi P_4 U, \phi P_2 W)$$

which proves (iii). □

Theorem 3. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold, where g_M is semi-Riemannian metric of index 2r. Therfore, the invariant distribution D_1 is minimal.*

Proof. The distribuiton D_1 is minimal iff $T_V V + T_{JV} JV = 0$, for all $V \in \Gamma(D_1)$. By virtue of (7), (8) and (3), we obrtain

$$g(T_V V + T_{JV} JV, X) = g(\nabla_V JV, JX) - g(\nabla_{JV} V, JX)$$

which gives our assertion. □

Theorem 4. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore D_2 is integrable if and only if*
i) $g_M(v\nabla_K \phi P_5 L - v\nabla_L \phi P_5 K, \phi P_4 U) = -g_M(\mathcal{T}_K w P_5 L - \mathcal{T}_L w P_5 K, \phi P_4 U)$
ii) $g_B(\psi_*(h\nabla_K w P_5 L) - \psi_*(h\nabla_L w P_5 K), \psi_*(CN)) = -g_M(\mathcal{T}_K w P_5 L - \mathcal{T}_L w P_5 K, BN)$
iii) $g_B(\psi_*(h\nabla_K w P_5 L) - \psi_*(h\nabla_L w P_5 K), \psi_*(w P_3 W)) = g_M(\mathcal{T}_K \phi P_5 L - \mathcal{T}_L \phi P_5 K, w P_3 W)$,
where $K, L \in \Gamma(D_2), U \in \Gamma(D_1), W \in \Gamma(Jltr(\ker \psi_*)), N \in \Gamma(ltr(\ker \psi_*))$.

Proof. For all $K, L \in \Gamma(D_2), U \in \Gamma(D_1)$, using (10), (14), (3) and (4), we get

$$g_M(\nabla_K L, U) = g_M(\mathcal{T}_K \phi P_5 L + v\nabla_K \phi P_5 L + h\nabla_K w P_5 L + \mathcal{T}_K w P_5 L, \phi P_4 U).$$

After some calculations, we have

$$\begin{aligned} g_M([K, L], U) &= g_M(\nabla_K \phi P_5 L - \nabla_L \phi P_5 K, \phi P_4 U) \\ &\quad + g_M(\mathcal{T}_K \phi P_5 L - \mathcal{T}_L \phi P_5 K, \phi P_4 U) \end{aligned}$$

which proves (i).

For $N \in \Gamma(ltr(\ker \psi_*))$, from (10), (14) and (12), we arrive at

$$\begin{aligned} g_M([K, L], N) &= \cos^2\theta g_M(\nabla_K L, N) + g_M(\mathcal{T}_K w P_5 L, BN) + g_M(h\nabla_K w P_5 L, CN) \\ &\quad - \cos^2\theta g_M(\nabla_L K, N) - g_M(\mathcal{T}_L w P_5 K, BN) - g_M(h\nabla_L w P_5 K, CN). \end{aligned}$$

Now, using the character of ψ, we obtain

$$\begin{aligned} \sin^2\theta g_M([K, L], N) &= g_M(\mathcal{T}_K w P_5 L - \mathcal{T}_L w P_5 K, BN) \\ &\quad + g_B(\psi_*(h\nabla_K w P_5 L) - \psi_*(h\nabla_L w P_5 K), \psi_*(CN)) \end{aligned}$$

which proves (ii).

For $W \in \Gamma(Jltr(\ker \psi_*))$

$$\begin{aligned} g_M([K, L], W) &= g_M(\mathcal{T}_K \phi P_5 L - \mathcal{T}_L \phi P_5 K, w P_3 W) \\ &\quad + g_M(h\nabla_K w P_5 L - h\nabla_L w P_5 K, w P_3 W). \end{aligned}$$

Then, using the character of ψ, we have

$$\begin{aligned} g_M([K, L], W) &= g_M(\mathcal{T}_K \phi P_5 L - \mathcal{T}_L \phi P_5 K, w P_3 W) \\ &\quad + g_B(\psi_*(h\nabla_K w P_5 L) - \psi_*(h\nabla_L w P_5 K), \psi_*(w P_3 W)) \end{aligned}$$

which proves (iii). □

Theorem 5. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore $\triangle$ is a totally geodesic foliation on M if and only if*

$$g_M(\mathcal{T}_K JP_3Z + \mathcal{T}_K wP_5Z, JL) = g_M(\nabla_K JP_2Z + \nabla_K JP_4Z + \nabla_K \phi P_5Z, JL),$$

for any $K, L \in \Gamma(\triangle), Z \in S(\ker \psi_*)$.

Proof. For any $K, L \in \Gamma(\triangle), Z \in S(\ker \psi_*)$, using, (10) in (14), we have

$$\begin{aligned} g_M(\nabla_K L, Z) &= -g_M(\nabla_K JP_2Z, JL) - g_M(\nabla_K JP_3Z, JL) - g_M(\nabla_K JP_4Z, JL) \\ &\quad -g_M(\nabla_K \phi P_5Z, JL) - g_M(\nabla_K wP_5Z, JL). \end{aligned}$$

Then by (3) and (4), imply

$$\begin{aligned} g_M(\nabla_K L, Z) &= -g_M(\nabla_K JP_2Z, JL) - g_M(\mathcal{T}_K JP_3Z, JL) - g_M(\nabla_K JP_4Z, JL) \\ &\quad -g_M(\nabla_K \phi P_5Z, JL) - g_M(\mathcal{T}_K wP_5Z, JL) \end{aligned}$$

which gives our assertion. □

Theorem 6. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore D_1 is a totally geodesic foliation on M if and only if*

$$g_M(\mathcal{T}_U \phi P_5Z, JV) = -g_M(\nabla_U \phi P_5Z, JV)$$

and

$$\nabla_U JN \notin \Gamma(D_1),\ \nabla_U JW \notin \Gamma(D_1),$$

for all $U, V \in \Gamma(D_1), Z \in \Gamma(D_2), W \in \Gamma(Jltr(\ker \psi_*)), N \in \Gamma(ltr(\ker \psi_*))$.

Proof. Invariant distribution D_1 defines a totally geodesic foliation iff $g_M(\nabla_U V, Z) = 0$, $g_M(\nabla_U V, Z) = 0$ and $g_M(\nabla_U V, W) = 0$ for any $U, V \in \Gamma(D_1), Z \in \Gamma(D_2)$, $N \in \Gamma(ltr(\ker \psi_*))$, $W \in \Gamma(Jltr(\ker \psi_*))$.

For $U, V \in \Gamma(D_1), Z \in \Gamma(D_2)$, using (7) and (8), we have

$$g_M(\nabla_U V, Z) = -g_M(\nabla_U JZ, JV). \tag{15}$$

By virtue of (10), (3) and (4) in (15) imply that

$$g_M(\nabla_U V, Z) = -g_M(\mathcal{T}_U \phi P_5Z, JV) - g_M(\nabla_U \phi P_5Z, JV).$$

Moreover, for $N \in \Gamma(ltr(\ker \psi_*)), W \in \Gamma(Jltr(\ker \psi_*))$, using (7), (8), (3) and (5), we arrive at

$$\begin{aligned} g_M(\nabla_U V, N) &= -g_M(\nabla_U JN, JV) \\ &= -g_M(v\nabla_U JN, JV) \end{aligned}$$

and $W \in \Gamma(Jltr(\ker \psi_*))$

$$\begin{aligned} g_M(\nabla_U V, W) &= -g_M(\nabla_U JW, JV) \\ &= -g_M(v\nabla_U JW, JV), \end{aligned}$$

which gives our assertion. □

Theorem 7. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore slant distribution D_2 is a totally geodesic foliation on M if and only if*

$$g_M(\mathcal{T}_U JZ, wV) = -g_M(v\nabla_U JZ, \phi V),$$
$$g_M(\mathcal{T}_U JN, wV) = -g_M(v\nabla_U JN, \phi V)$$

and

$$g_M(\mathcal{T}_U JW, \phi V) = -g_B(\psi_*(h\nabla_U JW), \psi_*(wV)),$$

for all $U, V \in \Gamma(D_2), Z \in \Gamma(D_1), W \in \Gamma(Jltr(\ker\psi_*)), N \in \Gamma(ltr(\ker\psi_*))$.

Proof. For all $U, V \in \Gamma(D_2), Z \in \Gamma(D_1)$,using (7) and (8), we give

$$g_M(\nabla_U V, Z) = -g_M(\nabla_U JZ, JV). \tag{16}$$

Now, from (3) and (9), we arrive at

$$g_M(\nabla_U V, Z) = -g_M(\mathcal{T}_U JZ, wV) - g_M(v\nabla_U JZ, \phi V).$$

Moreover, for $W \in \Gamma(Jltr(\ker\psi_*))$ and $N \in \Gamma(ltr(\ker\psi_*))$, using (9), (5) and (4), we have

$$g_M(\nabla_U V, N) = -g_M(\mathcal{T}_U JN, wV) - g_M(v\nabla_U JN, \phi V)$$

and

$$g_M(\nabla_U V, W) = -g_M(\mathcal{T}_U JW, \phi V) - g_M(h\nabla_U JW, wV)$$

which gives our assertion. □

Theorem 8. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore $\mathcal{V}$ is a totally geodesic foliation on M if and only if*

$$g_M(v\nabla_E BN + T_E CN, JF) = -g_M(T_E BN + h\nabla_E CN, JF)$$

and

$$v\nabla_E\phi P_1F + v\nabla_E\phi P_2F + v\nabla_E\phi P_4F + v\nabla_E\phi P_5F + T_E wP_3F + T_E wP_5F \notin \Gamma(D_2),$$

where $E, F \in \Gamma(\mathcal{V}), N \in \Gamma(ltr(\ker\psi_*))$.

Proof. For any $E, F \in \Gamma(\mathcal{H}), N \in \Gamma(ltr(\ker\psi_*))$, using (7), (8) and (9), we have

$$g_M(\nabla_E F, N) = -g_M(\nabla_E BN + \nabla_E CN, F).$$

Then, from (3) and (4), we arrive at

$$g_M(\nabla_E F, N) = -g_M(T_E BN + v\nabla_E BN + T_E CN + h\nabla_E CN, JF).$$

On the other hand, for $K \in \Gamma(D_2)$, using (7), (8) and (10), we get

$$g_M(\nabla_E F, JK) = g_M(J\nabla_E\phi P_1F + J\nabla_E\phi P_2F + J\nabla_E\phi P_3F + J\nabla_E\phi P_4F + J\nabla_E\phi P_5F, JK).$$

By virtue of (3) and (4), we arrive at

$$g_M(\nabla_E F, N) = g_M(w(v\nabla_E \phi P_1 F + v\nabla_E \phi P_2 F + v\nabla_E \phi P_4 F + v\nabla_E \phi P_5 F + T_E w P_3 F + T_E w P_5 F), JK)$$

which completes proof. □

Theorem 9. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore $\mathcal{H}$ is a totally geodesic foliation on M if and only if*

$$g_M(A_X BY + h\nabla_X CY, wP_5 K) = g_M(v\nabla_X BY + A_X CY, \phi P_5 K),$$

$$A_X CY + v\nabla_X BY \notin \Gamma(D_1),$$

and

$$A_X BY + h\nabla_X CY \notin \Gamma(ltr(\ker \psi_*)),$$

where $X, Y \in \Gamma(\mathcal{H}), K \in \Gamma(D_2)$.

Proof. For all $X, Y \in \Gamma(\mathcal{H}), K \in \Gamma(D_2)$, from (7), (8) and (9), we have

$$g_M(\nabla_X Y, K) = -g_M(\nabla_X BY + \nabla_X CY, JK).$$

By virtue of (5) and (6), we get

$$g_M(\nabla_X Y, K) = -g_M(v\nabla_X BY + A_X BY + A_X CY + h\nabla_X CY, \phi P_5 K + wP_5 K).$$

Moreover, for $U \in \Gamma(D_1)$ and for $W \in \Gamma(Jltr(\ker \psi_*))$, by virtue of (5) and (6) we arrive at

$$g_M(\nabla_X Y, K) = -g_M(v\nabla_X BY + A_X CY, K)$$

and

$$g_M(\nabla_X Y, W) = -g_M(A_X BY + h\nabla_X CY, wP_3 W),$$

which gives our assertion. □

Theorem 10. *Let $\psi : (M, J, g_M) \to (B, g_B)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold to an r-lightlike manifold. Therefore M is a locally product manifold of the leaves of $\mathcal{V}$ and $\mathcal{H}$ if and only if*

$$g_M(v\nabla_E BN + T_E CN, JF) = -g_M(T_E BN + h\nabla_E CN, JF),$$

$$v\nabla_E \phi P_1 F + v\nabla_E \phi P_2 F + v\nabla_E \phi P_4 F + v\nabla_E \phi P_5 F + T_E w P_3 F + T_E w P_5 F \notin \Gamma(D_2),$$

and

$$g_M(A_X BY + h\nabla_X CY, wP_5 K) = g_M(v\nabla_X BY + A_X CY, \phi P_5 K),$$

$$A_X CY + v\nabla_X BY \notin \Gamma(D_1),$$

$$A_X BY + h\nabla_X CY \notin \Gamma(ltr(\ker \psi_*)),$$

where $E, F \in \Gamma(\mathcal{V}), N \in \Gamma(ltr(\ker \psi_*)),\ X, Y \in \Gamma(\mathcal{H}), K \in \Gamma(D_2)$.

Conclusion 1. *Submersions, lightlike manifolds and semi-Riemannian manifolds have potential for applications in many fields of physics, engineering and mathematics. In particular it is applicable to the theory of liquid crystals (Harmonic morphisms), theory of spacetimes, theory of relativity. Research in this theory has been increasing in recent years After the defination of submersions from semi-Riemannian manifolds onto lightlike manifolds, slant lightlike submersions were studied. In this paper, the idea of examining semi-slant lightlike submersions are emphasized. We defined and studied semi-slant lightlike submersions from an indefinite Kaehler manifold to an r-lightlike manifold. We introduced geometry of foliatons. The works on this subject will be useful tools for the applications of semi-slant lightlike submersion with various manifolds.*

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APPLICATION OF NEUTROSOPHIC POISSON DISTRIBUTION SERIES ON HARMONIC CLASSES OF ANALYTIC FUNCTIONS DEFINED BY $q-$DERIVATIVE OPERATOR AND SIGMOID FUNCTION

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Abstract. There are several authors who have obtained various forms of properties for some subclasses of analytic univalent functions related to different distribution series, such as Binomial, Generalized Discrete Probability, Geometric, Mittag-Leffler, Pascal, and Poisson distribution series. The authors, in this paper, proved the inclusion relation of the harmonic analytic function class $H_q^\alpha(\theta, \gamma(s), \Psi)$ established by applying convolution operators regarding neutrosophic distribution series equipped with the Sigmoid function (activation function). The present results are capable of handling both accurate (determinate) data and inaccurate (indeterminate) data.

1. Introduction

Indicate by $\mathcal{A}$ the family of functions analytic in $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$, of the form $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ which fulfill the normalization $f(0) = f'(0) - 1 = 0$ and also indicate by S the subfamily of $\mathcal{A}$ including univalent functions in $\mathbb{U}$. Further, for the function $g(z) = z + b_2 z^2 + \cdots$, the convolution $f * g$ is expressed as

$$(f * g)(z) = z + \Sigma_{n=2}^{\infty} a_n b_n z^n.$$

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A harmonic function is a type of function that arises in various areas of mathematics, including complex analysis, partial differential equations, and physics. The real-valued function $v(x, y)$ is named harmonic in a domain $B \subset \mathbb{C}$ if it has continuous second order partial derivative in B, which fulfills

$$\Delta v := \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

A harmonic mapping f of the simply connected domain B is a complex-valued function of the form $f = \phi + \overline{\lambda}$, where ϕ, λ are analytic and $\phi(0) = \phi'(0) - 1 = 0$, $\lambda(0) = 0$. We call ϕ and λ analytic and co-analytic part of f, respectively. $J_{f(z)} = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = \left|\phi'(z)\right|^2 - \left|\lambda'(z)\right|^2$ is defined as the Jacobian of f. Also, f is locally univalent iff its Jacobian is never zero, and is sense-preserving provided that the Jacobian is positive. To this end, without loss of generality, indicate by H the family of all harmonic functions of the form $f = \phi + \bar{\lambda}$, where

$$\phi(z) = z + \sum_{v=2}^{\infty} a_v z^v, \quad \lambda(z) = \sum_{v=1}^{\infty} b_v z^v \quad (|b_1| < 1) \tag{1}$$

are analytic in $\mathbb{U}$. We further indicate by S_H the family of functions $f = \phi + \bar{\lambda}$ that are harmonic univalent and sense preserving in $\mathbb{U}$. Consider the subfamily S_H^0 of S_H as $S_H^0 = \left\{f = \phi + \bar{\lambda} \in S_H : \lambda'(0) = b_1 = 0\right\}$. A sense-preserving harmonic mapping $f \in S_H^0$ is in the class S^* if the range $f(\mathbb{C})$ is starlike with respect to the origin. A function $f \in S_H^*$ is named a harmonic starlike mapping in $\mathbb{U}$. On the other hand, a function $f \in \mathbb{U}$ is included in K_H if $f \in S_H^0$ and if $f(\mathbb{U})$ is a convex domain. A function $f \in K_H$ is named convex harmonic in $\mathbb{U}$. Analytically, $f \in S_H^*$ iff $\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right) \geq 0$, and $f \in K_H$ iff $\frac{\partial}{\partial\theta}\left\{\arg\left(\arg\left(\frac{\partial}{\partial\theta} f(re^{i\theta})\right)\right)\right\} \geq 0$, where $z = re^{i\theta} \in \mathbb{U}$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. For further details on the harmonic classes of analytic functions, we may refer to some papers (see [3], [7]– [9], [11]–[13], [17], [18], [22], [24], [26]– [30]) and the relevant literature cited in there.

Indicate by T_H the family of functions in S_H that are expressible as $f = \phi + \bar{\lambda}$, where

$$\phi(z) = z - \sum_{v=2}^{\infty} |a_v| z^v, \quad \lambda(z) = \sum_{v=1}^{\infty} |b_v| z^v \quad (|b_1| < 1). \tag{2}$$

Then, for $0 \leq \nu < 1$, the following geometric representations are possible

$$N_H(\nu) = Re\left\{f \in H : \Re\left[\frac{f'(z)}{z'}\right] \geq \nu, \;\; z = re^{i\theta} \in \mathbb{U}\right\}$$

and

$$R_H(\nu) = Re\left\{f \in H : \Re\left[\frac{f''(z)}{z''}\right] \geq \nu, \;\; z = re^{i\theta} \in \mathbb{U}\right\},$$

where

$$z' = \frac{\partial}{\partial\theta}(z = re^{i\theta}), \;\; z'' = \frac{\partial}{\partial\theta}(z'), \;\; f'(z) = \frac{\partial}{\partial\theta} f(re^{i\theta}), \;\; f'' = \frac{\partial}{\partial\theta}(f'(z)).$$

Define

$$TN_H(\nu) = N_H(\nu) \cap T_H \quad \text{and} \quad TR_H(\nu) = R_H(\nu) \cap T_H.$$

The classes T_H, $N_H(\nu)$, $TN_H(\nu)$, $R_H(\nu)$ and $TR_H(\nu)$ were defined and investigated in [1], [14], [24], [27].

The $q-$derivative, also known as the Jackson $q-$derivative [15], is a concept from the theory of $q-$calculus, which is a generalization of calculus that incorporates a parameter q (often interpreted as a complex number) and extends various concepts from classical calculus.

Next, for $0 < q < 1$, the Jackson's $q-$derivative of a function $f \in S_H$ is expressed as

$$D_q\phi(z) = \begin{cases} \frac{\phi(z)-\phi(qz)}{(1-q)z}, & z \neq 0 \\ \phi'(0), & z = 0 \end{cases} \tag{3}$$

and

$$D_q\lambda(z) = \begin{cases} \frac{\lambda(z)-\lambda(qz)}{(1-q)z}, & z \neq 0 \\ \lambda'(0), & z = 0 \end{cases}. \tag{4}$$

From (3) and (4), we obtain

$$D_q\phi(z) = 1 + \sum_{v=2}^{\infty} [v]_q a_v z^{v-1}$$

and

$$D_q\lambda(z) = \sum_{v=1}^{\infty} [v]_q b_v z^{v-1}.$$

For more details, we can refer to reference [16].

A harmonic function $f = \phi + \bar{\lambda}$ expressed by (1) is said to be $q-$harmonic, locally univalent, and sense preserving in $\mathbb{U}$ if and only if second dilatation w_q fulfills

$$|w_q(z)| = \left| \frac{D_q\phi(z)}{D_q\lambda(z)} \right| < 1 \quad (z \in \mathbb{U}).$$

Let us indicate this class by S_{H_q}. As $q \to 1^-$, S_{H_q} reduces to the class SH (see [2]).

The concept of neutrosophic theory, a new branch of philosophy as a generalization for the fuzzy logic, and also a generalization of the intrinsic fuzzy logic, introduced by Smarandache [25]. This generalization provided a new foundation for handling with the issues of indeterminate data. The usage of neutrosophic crisp sets theory by means of the classical probability distributions, particularly, Poisson, Exponential and uniform distributions provide a new pathway to deal with issues that follow the classical distribution, and also contain data not specified accurately.

A discrete random variable Y is said to have a neutrosophic Poisson distribution if it has a probability mass function

$$P(Y=v) = m_N^v \frac{e^{-m_N}}{v!}, \quad v = 0, 1, 2, \cdots$$

and m_N is the parameter of the distribution. Further,

$$NE(Y) = NV(Y) = m_N$$

where $N = d + I$ is a neutrosophic number [25].

Recently, Alhabib et al. [4] studied a power series of neutrosophic Poisson, which was further exploited in [5] via coefficient inequalities defined by the power series

$$K(m_N, z) = z + \sum_{v=2}^{\infty} \frac{m_N^{v-1}}{(v-1)!} e^{-m_N} z^v \quad (z \in \mathbb{U})$$

and by ratio test, the radius of convergence of the above series was shown to be infinite.

Now for $m_{N1}, m_{N2} > 0$, we establish the operator $\Theta(m_{N1}, m_{N2})$ for $f \in S_H$ as

$$\begin{aligned} Y(f) &= Y(m_{N1}, m_{N2}) f(z) \\ &= K(m_{N1}, z) * \phi(z) + \overline{K(m_{N2}, z) * \lambda(z)} \\ &= \Phi(z) + \overline{\Omega(z)}, \end{aligned}$$

where

$$\Phi(z) = z + \sum_{v=2}^{\infty} \frac{m_{N1}^{v-1}}{(v-1)!} e^{-m_{N1}} a_v z^v, \quad \Omega(z) = b_1 z + \sum_{v=2}^{\infty} \frac{m_{N2}^{v-1}}{(v-1)!} e^{-m_{N2}} b_v z^v \tag{5}$$

for $f = \phi + \overline{\lambda} \in H$.

The present investigation builds on the foundational works of Smarandache and Khalid [25], Oladipo [21], and the recent contributions by Frasin and Lupas [14]. This study explores the innovative application of neutrosophic Poisson distribution series, augmented with an artificial neural network (Sigmoid function), to analyze harmonic data. This approach effectively handles both determinate (accurate) and indeterminate (inaccurate) data, offering a robust method for dealing with uncertainty in mathematical and statistical analyses.

We define and study the class $H_q^{\alpha}(\theta, \gamma(s), \Psi)$ of the function of the form (1) that fulfills the condition

$$\Re \left\{ \frac{\alpha(1+e^{i\theta})}{2\gamma(s)} [z(D_q\phi(z))' + z(D_q\lambda(z))'] + [D_q\phi(z) + D_q\lambda(z)] \right\} > \Psi \tag{6}$$

for $\alpha \geq 0,\ 0 \leq \Psi < 1,\ -\pi < \theta \leq \pi,\ q \in (0,1)$ and $\gamma(s) = \frac{2}{1+e^{-s}},\ s \geq 0$ (real) is modified Sigmoid functions studied in [6], [19], [20].

By suitably specializing the parameters, the class $H_q^{\alpha}(\theta, \gamma(s), \Psi)$ reduces to the various subclasses of harmonic univalent functions:

(i) $H_q^{\alpha}(\theta, \gamma(s), \Psi) = H^{\alpha}(\theta, \gamma(s), \Psi)$ as $q \to 1^-$.

(ii) $H_q^\alpha(\theta,\gamma(s),\Psi) = H^\alpha(0,\gamma(0),\Psi) = H^\alpha(\Psi)$ as $q \to 1^-$ [28].
(iii) $H_q^\alpha(0,0,\gamma(0)) = H^\alpha$ as $q \to 1^-$ [10].
(iv) $H_q^\alpha(\theta,\gamma(0),\Psi) = H_q^\alpha(\theta,\Psi)$.
(v) $H_q^\alpha(0,\gamma(s),\Psi) = H_q^\alpha(\gamma(s),\Psi)$.

The aim of this paper is to present some inclusion properties of the harmonic class $H_q^\alpha(\theta,\gamma(s),\Psi)$ and its related classes.

2. Preliminary Lemmas

Before presenting our main outcomes, we need to state some lemmas that will be used in the sequel.

Lemma 1. *A function f of the form (1) belongs to class $H_q^\alpha(\theta,\gamma(s),\Psi)$ if and only if*

$$\begin{aligned}&\textstyle\sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]|a_v|\\&+\textstyle\sum_{v=1}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]|b_v| \le 2\gamma(s)(1-\Psi).\end{aligned} \tag{7}$$

Proof. Assume $f \in H_q^\alpha(\theta,\gamma(s),\Psi))$. From (6), we note that

$$\Re\left\{1-\textstyle\sum_{v=2}^{\infty}[v]_q\left[\frac{2\gamma(s)+\alpha(v-1)(1+cos\theta)}{2\gamma(s)}\right]|a_v|\,z^{v-1}+\sum_{v=1}^{\infty}[v]_q\left[\frac{2\gamma(s)+\alpha(v-1)(1+cos\theta)}{2\gamma(s)}\right]|b_v|\,z^{v-1}\right\} > \Psi.$$

Choosing z to be real and lettting $z \to 1^-$, we arrive at

$$1-\sum_{v=2}^{\infty}[v]_q\left[\frac{2\gamma(s)+\alpha(v-1)(1+cos\theta)}{2\gamma(s)}\right]|a_v|+\sum_{v=1}^{\infty}[v]_q\left[\frac{2\gamma(s)+\alpha(v-1)(1+cos\theta)}{2\gamma(s)}\right]|b_v| > \Psi,$$

which is equivalent to (7). Conversely, assume that (7) is true, then

$$\begin{aligned}&\left|\frac{\alpha(1+e^{i\theta})}{2\gamma(s)}\left[z(D_q\phi(z))'+z(D_q\lambda(z))'\right]+\left[D_q\phi(z)+D_q\lambda(z)\right]\right|\\&<\sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]|a_v|\\&+\sum_{v=1}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]|b_v|\\&\le 2\gamma(s)(1-\Psi),\end{aligned}$$

which implies that $f \in H_q^\alpha(\theta,\gamma(s),\Psi)$.

When $f \in H_q^\alpha(\theta,\gamma(s),\Psi)$, then

$$|a_v| \le \frac{2\gamma(s)(1-\Psi)}{[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]}, \quad v \ge 2$$

and

$$|b_v| \le \frac{2\gamma(s)(1-\Psi)}{[v]_q\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]}, \quad v \ge 1.$$

As $q \to 1^{-}$, we arrive at

$$|a_v| \leq \frac{2\gamma(s)(1-\Psi)}{v\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]}, \quad v \geq 2$$

and

$$|b_v| \leq \frac{2\gamma(s)(1-\Psi)}{v\left[2\gamma(s)+\alpha(v-1)(1+cos\theta)\right]}, \quad v \geq 1.$$

□

Lemma 2. *A function f of the form (2) belongs to class $TN_H(\nu)$ if and only if*

$$\sum_{v=2}^{\infty} v\,|a_v| + \sum_{v=1}^{\infty} v\,|a_v| \leq 1-\nu.$$

Then

$$|a_v| \leq \frac{1-\nu}{v}, \quad v \geq 2, \quad |b_v| \leq \frac{1-\nu}{v}, \; v \geq 1.$$

Lemma 3. *A function f of the form (2) belongs to class $TR_H(\nu)$ if and only if*

$$\sum_{v=2}^{\infty} v^2\,|a_v| + \sum_{v=1}^{\infty} v^2\,|a_v| \leq 1-\nu.$$

Then

$$|a_v| \leq \frac{1-\nu}{v^2}, \quad v \geq 2, \quad |b_v| \leq \frac{1-\nu}{v^2}, \quad v \geq 1.$$

Lemma 4. *Consider $f \in S_H^*$, where the function f is of the form (1) and $b_1 = 0$, then*

$$|a_v| \leq \frac{(2v+1)(v+1)}{6}, \qquad |b_v| \leq \frac{(2v-1)(v-1)}{6}.$$

Lemma 5. *Consider $f \in K_H$, where the function f is of the form (1) and $b_1 = 0$, then*

$$|a_v| \leq \frac{(v+1)}{2}, \qquad |b_v| \leq \frac{(v-1)}{2}.$$

For easy handling throughout the sequel, we designate the notations:

$$\textstyle\sum_{v=2}^{\infty} \frac{m_{N1}^{v-1}}{(v-1)!} = e^{m_{N1}} - 1, \qquad \sum_{v=2}^{\infty} \frac{m_{N2}^{v-1}}{(v-1)!} = e^{m_{N2}} - 1,$$

$$\textstyle\sum_{v=j}^{\infty} \frac{m_{N1}^{v-1}}{(v-j)!} = m_{N1}^{j-1} e^{m_{N1}}, \quad \sum_{v=j}^{\infty} \frac{m_{N2}^{v-1}}{(v-j)!} = m_{N2}^{j-1} e^{m_{N2}}, \; (j \geq 2).$$

3. Main Results

Theorem 1. *Assume* $m_{N1}, m_{N2} > 0$ *and* $0 \leq \Psi < 1$, $q \in (0,1)$. *If*

$$2\alpha(1+\cos\theta)(m_{N1}^4 + m_{N2}^4) + [21\alpha(1+\cos\theta) + 4\gamma(s)]m_{N1}^3 + 5[4\alpha(1+\cos\theta) + 5\gamma(s)]m_{N1}^2$$
$$+6[5\alpha(1+\cos\theta) + 8\gamma(s)]m_{N1} + 8\gamma(s)[1 - e^{-m_{N1}}] + [15\alpha(1+\cos\theta) + 4\gamma(s)]m_{N2}^3$$
$$+6[4\alpha(1+\cos\theta) + 3\gamma(s)]m_{N2}^2 + 6[\alpha(1+\cos\theta) + 2\gamma(s)]m_{N2} \leq 12\gamma(s)(1-\Psi), \tag{8}$$

then $Y(S_H^*) \subset H_q^\alpha(\theta, \gamma(s), \Psi)$.

Proof. Let $f = \phi + \bar{\lambda} \in S_H^*$ such that ϕ and λ are represented by (1) with $b_1 = 0$. We aim to establish that $Y(f) = \Phi + \Omega \in H_q^\alpha(\theta, \gamma(s), \Psi)$, where Φ and Ω are analytic functions in $\mathbb{U}$ as shown by (5) with $b_1 = 0$. According to Lemma 1, we need to show that

$$\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq 2\gamma(s)(1-\Psi),$$

where

$$\begin{aligned}\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) &= \textstyle\sum_{v=2}^{\infty}[v]_q\left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}a_v\right| \\ &+ \textstyle\sum_{v=2}^{\infty}[v]_q\left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}b_v\right|.\end{aligned}$$

Applying the inequalities from Lemma 1 and letting $q \to 1^-$, we obtain

$$\begin{aligned}&\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) \\ &\leq \tfrac{1}{6}\left[\textstyle\sum_{v=2}^{\infty} v(2v+1)(v+1)\left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right|\right] \\ &+\tfrac{1}{6}\left[\textstyle\sum_{v=2}^{\infty} v(2v-1)(v-1)\left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right|\right] \\ &= \tfrac{1}{6}\left[\textstyle\sum_{v=2}^{\infty}\left\{2\alpha[1+cos\theta)v^4 + [4\gamma(s) + \alpha(1+cos\theta)]v^3 + Q_1\right\}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right] \\ &+\tfrac{1}{6}\left[\textstyle\sum_{v=2}^{\infty}\left\{2\alpha[1+cos\theta)v^4 + [4\gamma(s) - 5\alpha(1+cos\theta)]v^3 + Q_1\right\}\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right]\end{aligned} \tag{9}$$

where

$$Q_1 = 2[2\gamma(s) - \alpha(1+cos\theta)]v^2 + [2\gamma(s) - \alpha(1+cos\theta)]v$$

and

$$Q_2 = 2[2\alpha(1+cos\theta) - 3\gamma(s)]v^2 + [2\gamma(s) - \alpha(1+cos\theta)]v.$$

Setting

$$v = (v-1) + 1, \qquad v^2 = (v-1)(v-2) + 3(v-1) + 1,$$
$$v^3 = (v-1)(v-2)(v-3) + 6(v-1)(v-2) + 7(v-1) + 1,$$

$v^4 = (v-1)(v-2)(v-3)(v-4)+10(v-1)(v-2)(v-3)+25(v-1)(v-2)+15(v-1)+1$

and using these equalities in (9), we can obtain

$$\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta)$$

$$\leq \frac{1}{6}\Bigg[\sum_{v=2}^{\infty}\{2\alpha(1+cos\theta)(v-1)(v-2)(v-3)(v-4)+Q_3+Q_4+Q_5$$

$$+12\gamma(s)\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\Bigg]+\frac{1}{6}\Bigg[\sum_{v=2}^{\infty}\{2\alpha(1+cos\theta)(v-1)(v-2)(v-3)(v-4)$$

$$+Q_6+Q_7+Q_8\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\Bigg],$$

where

$$Q_3 = [21\alpha(1+cos\theta)+4\gamma(s)](v-1)(v-2)(v-3),$$

$$Q_4 = 5[9\alpha(1+cos\theta)+5\gamma(s)](v-1)(v-2),$$

$$Q_5 = 6[5\alpha(1+cos\theta)+8\gamma(s)](v-1),$$

$$Q_6 = [15\alpha(1+cos\theta)+4\gamma(s)](v-1)(v-2)(v-3),$$

$$Q_7 = 6[4\alpha(1+cos\theta)+3\gamma(s)](v-1)(v-2), \quad Q_8 = 6[\alpha(1+cos\theta)+2\gamma(s)](v-1).$$

Thus

$$\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq \frac{1}{6}\left[2\alpha(1+cos\theta)\sum_{v=5}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-5)!}\right.$$

$$\left.+[21\alpha(1+cos\theta)+4\gamma(s)]\sum_{v=4}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-4)!}+12\sum_{v=1}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right]$$

$$+\frac{1}{6}\left[5[9\alpha(1+cos\theta)+5\gamma(s)]\sum_{v=3}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-3)!}+6[5\alpha(1+cos\theta)+8\gamma(s)]\sum_{v=2}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-2)!}\right]$$

$$+\frac{1}{6}\left[2\alpha(1+cos\theta)\sum_{v=5}^{\infty}\frac{m_{N1}^{v-1}e^{-m_{N2}}}{(v-5)!}+[15\alpha(1+cos\theta)+4\gamma(s)]\sum_{v=4}^{\infty}\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-4)!}\right]$$

$$+\frac{1}{6}\left[6[4\alpha(1+cos\theta)+3\gamma(s)]\sum_{v=3}^{\infty}\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-3)!}+6[\alpha(1+cos\theta)+2\gamma(s)]\sum_{v=2}^{\infty}\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-2)!}\right]$$

$$=\frac{1}{6}\left[2\alpha(1+cos\theta)(m_{N1}^4+m_{N2}^4)+[21\alpha(1+cos\theta)+4\gamma(s)]m_{N1}^3+5[4\alpha(1+cos\theta)+5\gamma(s)]m_{N1}^2\right]$$

$$+\frac{1}{6}\left[6[5\alpha(1+cos\theta)+8\gamma(s)]m_{N1}+8\gamma(s)[1-e^{-m_{N1}}]+[15\alpha(1+cos\theta)+4\gamma(s)]m_{N2}^3\right]$$

$$+\frac{1}{6}\left[6[4\alpha(1+cos\theta)+3\gamma(s)]m_{N2}^2+6[\alpha(1+cos\theta)+2\gamma(s)]m_{N2}\right].$$

This expression is bounded by $2\gamma(s)(1-\Psi)$ if condition (8) holds. □

Theorem 2. *Let* $m_{N1}, m_{N2} > 0$ *and* $0 \leq \Psi < 1$. *If*

$$\alpha(1+\cos\theta)[m_{N1}^3 + m_{N2}^3] + 2[3\alpha(1+\cos\theta) + \gamma(s)][m_{N1}^2 + m_{N2}^2]$$
$$+6[\alpha(1+\cos\theta) + \gamma(s)]m_{N1} + 2[\alpha(1+\cos\theta) + 3\gamma(s)]m_{N2} \tag{10}$$
$$+2\gamma(s)[2 - e^{-m_{N1}} - e^{-m_{N2}}] \leq 4\gamma(s)(1-\Psi),$$

then $Y(K_H) \subset H_q^\alpha(\theta, \gamma(s), \Psi)$.

Proof. Let $f = \phi + \bar{\lambda} \in K_H$ such that w and φ are given by (1) with $b_1 = 0$. We need to establish that $Y(f) = \Phi + \Omega \in H_q^\alpha(\theta, \gamma(s), \Psi)$, where Φ and Ω are analytic functions in $\mathbb{U}$ as shown by (5) with $b_1 = 0$. According to Lemma 1, we must show that

$$\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq 2\gamma(s)(1-\Psi),$$

where

$$\begin{aligned}\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) &= \textstyle\sum_{v=2}^\infty [v]_q \left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right] \left|\frac{m_{N1}^{v-1} e^{-m_{N1}}}{(v-1)!} a_v\right| \\ &+ \textstyle\sum_{v=2}^\infty [v]_q \left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right] \left|\frac{m_{N2}^{v-1} e^{-m_{N2}}}{(v-1)!} b_v\right|.\end{aligned}$$

Applying Lemma 5 and the condition $q \to 1^-$, we obtain

$$\begin{aligned}\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) &\leq \frac{1}{2}\left[\sum_{v=2}^\infty v(v+1)\left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right|\right] \\ &+ \frac{1}{2}\left[\sum_{v=2}^\infty v(v-1)\left[2\gamma(s) + \alpha(v-1)(1+cos\theta)\right]\left|\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right|\right] \\ &= \frac{1}{2}\left[\sum_{v=2}^\infty \left[\alpha(1+cos\theta)v^3 + 2\gamma(s)v^2 - \alpha(1+cos\theta)v\right]\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right] \\ &+ \frac{1}{2}\Bigg[\sum_{v=2}^\infty [\alpha(1+cos\theta)v^3 + 2(\gamma(s) - \alpha(1+cos\theta)v^2 + \alpha(1+cos\theta)v] \\ &\times \frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\Bigg].\end{aligned}$$

Next, we have

$$\begin{aligned}&\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) \\ &\leq \frac{1}{2}\left[\sum_{v=2}^\infty \{\alpha(1+cos\theta)(v-1)(v-2)(v-3) + K_1 + Q_2 + 2\gamma(s)\}\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right] \\ &+ \frac{1}{2}\left[\sum_{v=2}^\infty \{\alpha(1+cos\theta)(v-1)(v-2)(v-3) + K_3 + Q_4 + 2\gamma(s)\}\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right],\end{aligned}$$

where

$$K_1 = 2[3\alpha(1+cos\theta)+\gamma(s)](v-1)(v-2), \quad K_2 = 6[\alpha(1+cos\theta)+\gamma(s)](v-1),$$

$$K_3 = 2[2\alpha(1+cos\theta)+\gamma(s)](v-1)(v-2), \quad K_4 = 2[\alpha(1+cos\theta)+3\gamma(s)](v-1).$$

Thus

$$\begin{aligned}\Gamma_q(m_{N1}, m_{N2}, \gamma(s), \theta) &\leq \frac{1}{2}\big[\alpha(1+cos\theta)[m_{N1}^3 + m_{N2}^3] + 2[3\alpha(1+cos\theta)+\gamma(s)] \\ &\times [m_{N1}^2 + m_{N2}^2]\big] + \frac{1}{2}\Big[6[\alpha(1+cos\theta)+\gamma(s)]m_{N1} \\ &+ 2[\alpha(1+cos\theta)+3\gamma(s)]m_{N2} + 2\gamma(s)[2 - e^{-m_{N1}} - e^{-m_{N2}}]\Big].\end{aligned}$$

The last relation is bounded by $2\gamma(s)(1-\Psi)$ provided (10) holds. □

Theorem 3. *Assume* $m_{N1}, m_{N2} > 0$ *and* $0 \leq \Psi < 1$. *If*

$$(1-\nu)[\alpha(1+\cos\theta)(m_{N1}+m_{N2}) + 2\gamma(s)(2 - e^{-m_{N1}} - e^{-m_{N2}})] + b_1 \leq 2\gamma(s)(1-\Psi),$$

then $Y(TN_H(\nu)) \subset H_q^\alpha(\theta, \gamma(s), \Psi)$.

Proof. Let $f \in TN_H(\nu)$. In view of Lemma 1, we need to establish that

$$\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq 2\gamma(s)(1-\Psi),$$

where

$$\begin{aligned}\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) &= \sum_{v=2}^{\infty}[v]_q\,[2\gamma(s) + \alpha(v-1)(1+\cos\theta)]\left|\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}a_v\right| + b_1 \\ &+ \sum_{v=2}^{\infty}[v]_q\,[2\gamma(s) + \alpha(v-1)(1+\cos\theta)]\left|\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}b_v\right|.\end{aligned}$$

Application of Lemma 2 and the condition $q \to 1^-$ yields

$$\begin{aligned}\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) &\leq (1-\nu)\left[\sum_{v=2}^{\infty}[2\gamma(s) + \alpha(v-1)(1+\cos\theta)]\,\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right] \\ &+ (1-\nu)\left[\sum_{v=2}^{\infty}[2\gamma(s) + \alpha(v-1)(1+\cos\theta)]\,\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right] + b_1 \\ &= (1-\nu)[\alpha(1+\cos\theta)(m_{N1}+m_{N2}) + 2\gamma(s)(2 - e^{-m_{N1}} - e^{-m_{N2}})] \\ &+ b_1 \\ &\leq 2\gamma(s)(1-\Psi),\end{aligned}$$

which completes the proof. □

Theorem 4. *Assume* $m_{N1}, m_{N2} > 0$ *and* $0 \leq \Psi < 1$. *If*

$$(1-\nu)\left[\alpha(1+cos\theta)(2-e^{-m_{N1}}-e^{-m_{N2}})+\tfrac{1}{m_{N1}}(1-e^{-m_{N1}}-m_{N1}e^{-m_{N1}})\right]$$

$$+(1-\nu)\left[\tfrac{1}{m_{N2}}(1-e^{-m_{N2}}-m_{N2}e^{-m_{N2}})\right] \leq 2\gamma(s)(1-\Psi),$$

then $Y(TR_H(\nu)) \subset H_q^{\alpha}(\theta, \gamma(s), \Psi)$.

Proof. Assume $f \in TR_H(\nu)$. In view of Lemma 1, we need to establish that

$$\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq 2\gamma(s)(1-\Psi),$$

where

$$\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) = \sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]\left|\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}a_v\right|$$
$$+ b_1 + \sum_{v=2}^{\infty}[v]_q\left[2\gamma(s)+\alpha(v-1)(1+\cos\theta)\right]\left|\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}b_v\right|.$$

From Lemma 3, we have

$$\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) \leq (1-\nu)\left[\sum_{v=2}^{\infty}\left[\alpha(1+\cos\theta)+\frac{2\gamma(s)-\alpha(1+\cos\theta)}{v}\right]\frac{m_{N1}^{v-1}e^{-m_{N1}}}{(v-1)!}\right]$$

$$+ (1-\nu)\left[\sum_{v=2}^{\infty}\left[\alpha(1+\cos\theta)+\frac{2\gamma(s)-\alpha(1+\cos\theta)}{v}\right]\frac{m_{N2}^{v-1}e^{-m_{N2}}}{(v-1)!}\right] + b_1$$
$$= (1-\nu)\left[\alpha(1+\cos\theta)(2-e^{-m_{N1}}-e^{-m_{N2}})+\frac{1}{m_{N1}}(1-e^{-m_{N1}}-m_{N1}e^{-m_{N1}})\right]$$
$$+ (1-\nu)\left[\frac{1}{m_{N2}}(1-e^{-m_{N2}}-m_{N2}e^{-m_{N2}})\right] \leq 2\gamma(s)(1-\Psi).$$

□

Theorem 5. *Let* $m_{N1}, m_{N2} > 0$ *and* $0 \leq \Psi < 1$. *If*

$$e^{-m_{N1}} + e^{-m_{N2}} \leq 1 + \frac{b_1}{2\gamma(s)(1-\Psi)},$$

then $Y(H_q^{\alpha}(\theta, \gamma(s), \Psi)) \subset H_q^{\alpha}(\theta, \gamma(s), \Psi)$.

Proof. From Lemma 1, we established that

$$\begin{aligned}
&\rho_q(m_{N1}, m_{N2}, \gamma(s), \theta) \\
&\leq 2\gamma(s)(1-\Psi)\left[\sum_{v=2}^{\infty} \frac{m_{N1}^{v-1} e^{-m_{N1}}}{(v-1)!} + \sum_{v=2}^{\infty} \frac{m_{N2}^{v-1} e^{-m_{N2}}}{(v-1)!}\right] + b_1 \\
&= 2\gamma(s)(1-\Psi)[2 - e^{-m_{N1}} - e^{-m_{N2}}] + b_1 \\
&= 2\gamma(s)(1-\Psi)[2 - e^{-m_{N1}} - e^{-m_{N2}}] + b_1 \leq 2\gamma(s)(1-\Psi).
\end{aligned}$$

□

4. Conclusion

In this paper, we have established the inclusion relations for the harmonic analytic function class $H_q^{\alpha}(\theta, \gamma(s), \Psi)$ by applying convolution operators associated with the neutrosophic distribution series and incorporating the Sigmoid activation function. Our results extend the existing body of knowledge on analytic univalent functions, which previously encompassed distributions such as Binomial, Generalized Discrete Probability, Geometric, Mittag-Leffler, Pascal, and Poisson.

The innovative approach of utilizing the Sigmoid function within the framework of neutrosophic distribution series has demonstrated the potential to handle both accurate (determinate) and inaccurate (indeterminate) data effectively. This dual capability is particularly significant in applications where data uncertainty and variability are prevalent.

Our findings contribute to the broader understanding of harmonic analytic functions and offer new pathways for future research in the domain of mathematical analysis, particularly in the context of univalent functions and their applications. Further exploration may involve extending these results to other classes of functions and distributions, as well as investigating the practical implications of these theoretical advancements in real-world scenarios.

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ON THE FINITENESS OF SOME p-DIVISIBLE SETS

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Abstract. For any positive integer n, let H_n denote the n^{th} harmonic number. Given a prime number p, it is not known whether the set of integers $J(p) = \{n \in \mathbb{N} : p \mid H_n\}$ is finite. In this paper, we first investigate a variant of this set, namely, we work on the divisibility properties of the differences of harmonic numbers. For any prime p and a positive integer w, we define the set $D(p,w)$ as $\{n \in \mathbb{N} : p \mid H_n - H_w\}$ and work on the structure of this set. We present some finiteness results on $D(p,w)$ and obtain upper bounds for the number of elements in the set. Next, we consider the differences of generalized harmonic numbers and present an upper bound for the corresponding counting function. Moreover, under some plausible conditions, we prove that the difference set of generalized harmonic numbers is finite. Finally, we point out some directions to pursue.

1. Introduction

The n^{th} harmonic number H_n is defined as the sum

$$\sum_{k=1}^{n} \frac{1}{k}$$

for any positive integer n. These numbers have been investigated in different aspects, where one of the paths is to work on their integerness and related properties, such as divisibilities. It is known that these numbers are non-integers except for the case $n = 1$. Moreover, the difference of two harmonic numbers

$$H_n - H_m$$

is also not an integer whenever $n > m \geq 1$ by [23]. However, we focus on the divisibility properties of these differences as they come with intriguing features.

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Let p be a prime number. We use the notation $p \mid \frac{a}{b} \in \mathbb{Q}$ to mean that p divides the numerator of $\frac{a}{b}$ in its lowest terms. In 1991, the set $J_p = J(p) = \{n \in \mathbb{N} : p \mid H_n\}$ was presented in [16]. Some conjectures were also given in the paper and one of the conjectures was that the set is finite for any prime number p. They showed that J_p is finite for the prime numbers $\{2, 3, 5, 7\}$. Later on, the finiteness of the set was obtained for primes p up to 547, except for $\{83, 127, 397\}$, in [10], but the problem is still open.

However, there are some asymptotic results on the set. Let $J_p(x)$ count the number of elements in $J(p)$ that are less than x, for any positive real number x. Then, it is known by [27] that $J_p(x) < 129p^{\frac{2}{3}}x^{0.765}$, hence one has that

$$J_p(x) = o(x).$$

The upper bound was improved later to $3x^{\frac{2}{3}+\frac{1}{25\log p}}$ in [30].

Moreover, it is known that for any prime p, the elements $\{p-1, p(p-1), p^2-1\}$ are always in the set J_p and if the set consists of only those elements, the prime number p is called harmonic (see [16]).

We, in this paper, will work on a variant of this set, namely we will pick a prime number p, a positive integer w and look for positive integers n so that the prime p divides the difference $H_n - H_w$. We will use the following notation for the set.

Definition 1. *For any prime p and a positive integer w, we define*

$$D(p, w) := \{n \in \mathbb{N} : p \mid H_n - H_w\}.$$

Remark 1. *For any prime number p, if J_p is finite, then $D(p, w)$ is also finite. (See [19], Remark.4.12).*

As we mentioned, it is known [23] that the difference $H_n - H_m$ is never an integer whenever $n > m \geq 1$. In addition to this fact, it was shown in [15] that the equality $H_k - H_m = H_\ell - H_n$ is valid only if $k = \ell$ and $m = n$ holds. However, we work around the divisibility properties of $D(p, w)$ as the differences are interesting enough for this purpose. Consequently, we will need the p-adic order ν_p defined on the rational numbers. Let n be any integer and p be a prime number. We have

$$\nu_p(n) = \begin{cases} k & \text{if } p^k \parallel n \\ \infty & \text{if } n = 0 \end{cases}$$

where $p^k \parallel n$ means that $p^k \mid n$ but $p^{k+1} \nmid n$ with $k \in \mathbb{Z}$. If $n = \frac{a}{b}$ is a rational number, we set

$$\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b).$$

We will start by investigating the congruence relations on $D(p, w)$, and then we will give an upper bound for the counting function

$$D_{p,w}(x) = |\{n \in D(p, w) \ : \ n \leq x\}|.$$

To obtain the upper bound, we first need to bound the number of elements in the intervals of length at most p, lying inside the set $D(p, w)$. The idea is based on the argument given in [27]. Eventually, we will obtain our first main result.

Theorem A. *Let p be a prime number, w be a positive integer and $x \geq 1$ be a real number. Then, we have*

$$D_{p,w}(x) < 3x^{\frac{2}{3}+\frac{1}{25\log p}}.$$

Next, we consider an extension of the harmonic numbers, the generalized harmonic numbers. They are defined as

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}$$

for any positive integers n and s. We extend the difference set to these numbers:

Definition 2. *Let p be a prime number and s, w be any positive integers. Then, we define*

$$G(p, s, w) = G_{p,s,w} = \{n \in \mathbb{N} : p \mid H_n^{(s)} - H_w^{(s)}\}.$$

Next, we define the corresponding counting function

$$G_{p,s,w}(x) = |\{n \in G(p, s, w) \colon n \leq x\}|$$

and obtain our second main result.

Theorem B. *Assume that p is a prime number, s, w are any positive integers and $x \geq 1$ is any real number. Then,*

$$G_{p,s,w}(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}+\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}$$

holds. Furthermore, whenever $p > se^{\frac{3}{25}}$ holds, we have

$$G_{p,s,w}(x) = o(x).$$

Moreover, we show that $G(p, s, w)$ is finite in some cases and this will be our third main result.

Theorem C. *Let p be a prime number, s, w be positive integers with $s \geq 2$ and $p - 1 \nmid s$. If the inequality*

$$\nu_p\left(H_k^{(s)}\right) \leq s - 1$$

holds for any $k \in \{1, 2, \ldots p - 1\}$, then $G(p, s, w)$ is finite.
Moreover, if $p^m \leq w < p^{m+1}$ for some integer $m \geq 0$, then we have $G(p, s, w) \subseteq \{1, \ldots, p^{m+1} - 1\}$.

In Section 5, we obtain some difference sets using [26] together with some of our results, including a counter example for the case when the condition in Theorem C fails, and also discuss the computational process.

Then, in the last section, we present some generalizations of the harmonic numbers and point out some directions to work on the divisibility properties of the differences.

A generalization of the harmonic numbers is the Dedekind harmonic numbers [4]. For any number field K, a finite field extension of the rationals, the n^{th} Dedekind harmonic number is defined as

$$h_K(n) = \sum_{\substack{0 \neq I \subseteq \mathcal{O}_K \\ N(I) \leq n}} \frac{1}{N(I)},$$

such that the sum ranges over all non-zero ideals of $\mathcal{O}_K$ with norm less than or equal to n. These numbers also come with plenty of properties and it was shown in the same paper [4] that the difference of these numbers are non-integer after a while.

Moreover, another generalization of the harmonic numbers is the hyperharmonic numbers. These numbers were defined in [13] recursively as

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}$$

for $r \geq 2$, such that $h_n^{(1)} = H_n$.

The integerness of these numbers was an open question proposed in [25]. This property was studied by various authors (see [3,7,8,18]) and recently, it was shown that there are in fact hyperharmonic integers [28]. The set J_p was also extended to the hyperharmonic numbers in [19] and for divisibility properties of the generalized hyperharmonic numbers, which is an simultaneous extension of both generalized harmonic and hyperharmonic numbers, we refer interested readers to [20] and [21].

In [13], it was stated that the n^{th} hyperharmonic number of order r can be written as

$$h_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}).$$

Hence, one may work on this identity to continue the investigation on the differences. In fact, the binomial coefficients leads to a conjecture on the harmonic differences, which arises from central binomial coefficients and the Catalan numbers [24].

Lastly, we direct interested readers to [6] for intriguing results on the differences of hyperharmonic numbers.

2. Properties of $D(p,w)$

In this section, we will investigate the structure of the set. First, let us consider the case where $w<p$ and start with an observation.

We have by [9] that

$$H_{p-1}=1+\frac{1}{2}+\cdots+\frac{1}{p-2}+\frac{1}{p-1}\equiv 0 \pmod p \tag{1}$$

for any prime number $p>2$. Therefore, we may split the sum

$$H_{p-1}=\left(1+\frac{1}{2}+\cdots+\frac{1}{r}\right)+\left(\frac{1}{r+1}+\cdots+\frac{1}{p-2}+\frac{1}{p-1}\right)\equiv 0 \pmod p$$

and write

$$-H_r\equiv\left(\frac{1}{r+1}+\cdots+\frac{1}{p-2}+\frac{1}{p-1}\right) \pmod p$$

for any integer $1\le r\le p-1$.

In particular, we have

$$\frac{1}{k}+\frac{1}{p-k}\equiv 0 \pmod p \tag{2}$$

for any $1\le k\le p-1$, which implies the following result.

Proposition 1. *For any prime p and $1\le r\le p-1$, we have*

$$H_r\equiv H_{p-1-r} \pmod p.$$

Proof. Notice for any prime p and $1\le r\le p-1$ that

$$\begin{aligned}H_{p-1-r}-H_r&=\left(1+\frac{1}{2}+\cdots+\frac{1}{p-1-r}\right)-\left(\frac{1}{r}+\cdots+\frac{1}{2}+1\right)\\&\equiv\left(1+\frac{1}{2}+\cdots+\frac{1}{p-r-1}\right)+\left(\frac{1}{p-r}+\cdots+\frac{1}{p-2}+\frac{1}{p-1}\right)\\&=H_{p-1}\equiv 0 \pmod p\end{aligned}$$

and we are done. □

Corollary 1. *Let p be a prime number, w be a positive integer and a,b be positive integers with $1\le a<b\le p-1$ such that $a+b=p-1$. Then, if $a\in D(p,w)$ then we also have $b\in D(p,w)$.*

This corollary indicates that we have a symmetry about $\frac{p-1}{2}$ for any odd prime p.

We can generalize Corollary 1 for integers greater than $p-1$, but first let us introduce some notations. Given a positive integer n and a prime p, we may write $\hat{n}$ to mean that $\lfloor \frac{n}{p} \rfloor$. The interval

$$[pk, p(k+1)-1]$$

will be denoted by I_k for any $k \in \mathbb{Z}^{\geq 0}$. Moreover, if $a = pk + r \in I_k$ for some k, we will use $\bar{a}$ for the integer $pk + (p-1-r)$. Therefore, a quick observation is as follows:

$$a \in D(p, w) \implies \bar{a} \in D(p, w). \tag{3}$$

Before we give the proof, let us first show an argument that will be quite useful when dealing with modular equivalences. Suppose that p is a prime number and $n = pk + r$ is a positive integer with non-negative integers k and $0 \leq r \leq p-1$. Also, note that for any integers a and b, we have

$$\frac{1}{a} \equiv \frac{1}{pb+a} \pmod{p}.$$

Then, as we have

$$\left(1 + \frac{1}{2} + \cdots + \frac{1}{p-1}\right) \equiv 0,$$

we also have

$$\frac{1}{(pm+1)} + \frac{1}{(pm+2)} + \cdots + \frac{1}{(pm+p-1)}$$

for any integer m (see also [11]). Therefore, for a given n as above, we will write

$$H_n \equiv \frac{1}{p} H_k + H_r \pmod{p}$$

throughout the paper (see [16]). Now, we can proceed with the proof.

Proposition 2. *Let $a = pk + r$ be a positive integer with $r, k \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq p-1$. Then, we have*

$$a \in D(p, w) \implies \bar{a} = pk + (p-1-r) \in D(p, w)$$

for any $w \in \mathbb{Z}^{>0}$.

Proof. Let $a = pk + R$ and $w = pm + r$ for some integers k, m with $0 \leq r, R \leq p-1$. Suppose that $a \in D(p, w)$ so that we write

$$H_a - H_w \equiv \frac{1}{p}\left(H_k - H_m\right) + \left(H_R - H_r\right) \equiv 0 \pmod{p}.$$

Then, setting $\bar{a} = pk + (p-1-R)$ yields that

$$H_{\bar{a}} - H_w \equiv \frac{1}{p}\left(H_k - H_m\right) + \left(H_{p-1-R} - H_r\right) \equiv \frac{1}{p}\left(H_k - H_m\right) + \left(H_R - H_r\right) \equiv 0 \pmod{p}$$

by Proposition 1. □

Remark 2. *For any prime number p and positive integer w, we have*

$$\{w, \bar{w}\} \subseteq D(p, w).$$

Moreover, the equality $D(p, w) = D(p, \bar{w})$ holds.

Furthermore, we actually have

$$H_{p-1} \equiv 0 \pmod{p^2} \tag{4}$$

for primes $p > 3$ by [29]. This congruence points out some more elements in $D(p, w)$ whenever $w < p$.

Proposition 3. *Suppose that $p > 3$ is a prime and $0 < w < p$ is an integer. Then if we let $n = p(p-1) + w$, we have $\{n, \bar{n}\} \in D(p, w)$.*

Proof. Equation (4) states that $\nu_p(H_{p-1}) \geq 2$. As a consequence, we have

$$H_n - H_w \equiv \frac{1}{p} H_{p-1} + H_w - H_w \equiv \frac{1}{p} H_{p-1} \equiv 0 \pmod{p}.$$

□

Proposition 4. *Let p be a prime number and $w < p$ be a positive integer. If $n = p\hat{n} + r \in D(p, w)$ then $\hat{n} \in J_p$.*

Proof. Suppose that we have $n = p\hat{n} + r \in D(p, w)$ for some prime p and an integer $0 < w < p$. Then, $H_n - H_w \equiv \frac{1}{p} H_{\hat{n}} + \left(H_r - H_w\right) \equiv 0 \pmod{p}$ implies $\nu_p\left(H_{\hat{n}}\right) \geq 1$ so that we have $\hat{n} \in J_p$. □

The symmetry

$$n \in D(p, w) \iff \bar{n} \in D(p, w)$$

actually points out that there is a symmetry for the set J_p too, which can be seen by taking w as some element in J_p. Namely, the elements of J_p come in pairs. We omit the case when $n = \bar{n}$, so that $n \equiv \frac{p-1}{2} \pmod{p}$.

We note that we did not consider the case when $0 \in J_p$ throughout our investigation. If we set $H_0 = \frac{0}{1}$ as in [16], then we can see that $\{0, p-1, p(p-1), p^2-1\} \subseteq J_p$ where the pairs are $\{0, p-1\}$ and $\{p(p-1), p^2-1\}$ since

$$\bar{0} = p - 1, \ \ \overline{p(p-1)} = p(p-1) + p - 1 = p^2 - 1.$$

However, we may omit this case. Now, if we remove the restriction $w < p$, we obtain the following result.

Lemma 1. *Let $w \in I_k$ for some non-negative integer k. Then, we have*

$$I_{k+1} \cap D(p, w) = \emptyset.$$

Moreover, if n belongs to $D(p, w)$ then $\hat{n}$ belongs to $D(p, \hat{w})$ for any n and w.

Proof. If $w \in I_k = [pk, p(k+1)-1]$ then we can write $w = pk + r$ for some integer $0 \le r \le p-1$. Now, let us take any $n = p(k+1) + R \in I_{k+1} \cap D(p,w)$ with $0 \le R \le p-1$ and write the difference as

$$H_n - H_w \equiv \frac{1}{p}\big(H_{k+1} - H_k\big) + \big(H_R - H_r\big) = \frac{1}{p}\frac{1}{k+1} + \big(H_R - H_r\big) \pmod p. \quad (5)$$

The p-adic valuation of $H_R - H_r$ is always non-negative as both $R, r < p$. However, we have $\nu_p\left(\frac{1}{p}\frac{1}{k+1}\right) \le -1$ so that we end up with $\nu_p\big(H_n - H_w\big) \le -1$.

For the last part, suppose that $w \in I_k$ and there is $n = p\hat{n} + R \in D(p,w)$. Writing $H_n - H_w$ as in (5), we deduce that

$$H_n - H_w \equiv \frac{1}{p}\big(H_{\hat{n}} - H_{\hat{w}}\big) + \big(H_R - H_r\big) \equiv 0 \pmod p$$

so that $\nu_p\big(H_{\hat{n}} - H_{\hat{w}}\big) \ge 0$ yields that $\hat{n} \in D(p,\hat{w})$. □

Lemma 2. *Let p be an odd prime and w be a positive integer. If $D(p,w)$ is finite, then $D(p, pw+r)$ is also finite for any integer $0 \le r \le p-1$.*

Proof. Suppose that $D(p, pw+r)$ is infinite for some $0 \le r \le p-1$ and write $D(p,w) = \{n_1 < n_2 < \cdots < n_k\}$ for some $k \in \mathbb{Z}^{>0}$. Then, choose some

$$n = pk + R \in D(p, pw+r)$$

with $k > \lfloor n_k/p \rfloor$. As $n \in D(p, pw+r)$ we have

$$H_n - H_{pw+r} \equiv \frac{1}{p}\left(H_k - H_w\right) + \Bigg(H_R - H_r\Bigg) \equiv 0 \pmod p$$

so that

$$\nu_p\left(H_k - H_w\right) \ge 1.$$

Thus, $k \in D(p,w)$ must hold but the fact $k > \lfloor n_k/p \rfloor$ yields a contradiction. □

3. Proof of Theorem A

In this section, we prove our first main result, which is to bound the function

$$D_{p,w}(x) = |\{n \in D(p,w) \;:\; n \le x\}|.$$

We begin by dividing the set into intervals of length at most p, next we bound them and then provide the upper bound for the whole set $D(p,w) \cap [1,x]$.

Before we prove Theorem A, we first prove a weaker version of it, with the use of arguments of [27]. Then, using the tools from [30] we will obtain Theorem A.

For any positive integer d, we let

$$f_d(x) = (x+1)(x+2)\dots(x+d). \quad (6)$$

Consequently we get

$$\frac{f_d'(x)}{f_d(x)} = \frac{1}{x+1} + \frac{1}{x+2} + \cdots + \frac{1}{x+d}.$$

Then, if $n \in D(p,w)$ for some $n > w$ we can say that

$$H_n - H_w = \frac{1}{w+1} + \frac{1}{w+2} + \cdots + \frac{1}{n} = \frac{f_{n-w}'(w)}{f_{n-w}(w)} \equiv 0 \pmod{p}.$$

Now, we will bound the number of elements in the intersection of $D(p,w)$ with intervals of length at most p. To do so, we use the polynomial $f_d'(x)$. However, we will not consider the particular case $d = n-w, x = w$, as the polynomial $f_{n-w}'(w)$ leads to some other direction that we do not investigate in this paper (see Section 6). Moreover, the condition $n > w$ will not be a concern, as we see in the proof of the next lemma.

Lemma 3. *Assume that p is a prime number, w is a positive integer and x, y are real numbers with $1 \le y < p$. Then, we have*

$$|D(p,w) \cap [x, x+y]| < \frac{3}{2} y^{\frac{2}{3}} + 1.$$

Proof. Let us write

$$D(p,w) \cap [x, x+y] = \{n_1 < n_2 < \cdots < n_k\}$$

for some $k \ge 2$ because otherwise there is nothing to show. Therefore, suppose that $k = |D(p,w) \cap [x, x+y]| > 1$. For any $1 \le i < j \le k$ we have

$$H_{n_i} - H_{n_j} = (H_{n_i} - H_w) - (H_{n_j} - H_w) \equiv 0 \pmod{p}. \tag{7}$$

Then, let us set $d_i = n_{i+1} - n_i$ for $i = 1, 2, \ldots, k-1$ and observe for any i that

$$\frac{f_{d_i}'(n_i)}{f_{d_i}(n_i)} = \frac{1}{n_i+1} + \frac{1}{n_i+2} + \cdots + \frac{1}{n_{i+1}} = H_{n_{i+1}} - H_{n_i} \equiv 0 \pmod{p}. \tag{8}$$

by (7) above. Then, the result follows from [27, Lemma 2.2]. □

A partition of J_p was given in [16] as follows. Inductively, we define the sets $J_p^{(1)} = [1, p-1] \cap J_p$ and $J_p^{(k+1)} = \{pn + r \in J_p \ : \ n \in J_p^{(k)},\ 0 \le r \le p-1,\ p \mid H_n\}$ for any positive integer k. It was shown that $J_p^{(k)} = [p^{k-1}, p^k - 1]$. Hence, we can write

$$J_p = \bigcup_{k=1}^{\infty} J_p^{(k)}$$

Fact 5. *By the definition*

$$J_p^{(k+1)} = \{pn + r \in J_p \ : \ n \in J_p^{(k)},\ 0 \le r \le p-1,\ p \mid H_n\},$$

notice that if $J_p^{(k)} = \emptyset$ *for some positive integer* k*, then* $J_p^{(t)} = \emptyset$ *for any* $t \geq k$ *and we get*

$$J_p = \bigcup_{t=1}^{k-1} J_p^{(t)}.$$

Now, we give a partition of $D(p,w)$ for any $w < p$ using the notation above.

Definition 3. *Let* p *be a prime and* $w < p$ *be a positive integer. We define* $D_{p,w}^{(1)} = D(p,w) \cap [1, p-1]$ *and* $D_{p,w}^{(k+1)} = \{pn + r \in D(p,w) : n \in J_p^{(k)},\ \ 0 \leq r \leq p-1\}$.

Next, a similar result can be obtained.

Proposition 6. *The equality*

$$D_{p,w}^{(k)} = D(p,w) \cap [p^{k-1}, p^k - 1]$$

holds for any prime number p *and positive integer* k*.*

Proof. Let us prove by induction on k. For $k = 1$, the result follows. Now, suppose that the equality $D_{p,w}^{(k)} = [p^{k-1}, p^k - 1]$ holds and let

$$pn + r \in D_{p,w}^{(k+1)} = \{pn + r \in D(p,w) \mid n \in J_p^{(k)} \ \ : \ 0 \leq r \leq p-1\}.$$

Then, as $n \in J_p^{(k)}$ we know that $p^{k-1} \leq n \leq p^k - 1$ holds, which implies that

$$pn + r \in [p^k, p^{k+1} - 1]$$

and we are done. Conversely, if $m \in D(p,w) \cap [p^k, p^{k+1} - 1]$ then we can write $m = pn + r$ for some $n \in [p^{k-1}, p^k - 1]$ and $0 \leq r \leq p - 1$. Furthermore, as the integer $m = pn + r \in D(p,w)$, we have

$$H_m - H_w = H_{pn+r} - H_w \equiv \frac{1}{p} H_n + \big(H_r - H_w\big) \equiv 0.$$

That is, as $\nu_p\big(H_r - H_w\big) \geq 0$ holds, we obtain that $n \in J_p$. The proof is now complete. □

Now, we can prove a weaker version of Theorem A.

Lemma 4. *Let* p *be a prime,* $w < p$ *be a positive integer and* $x \geq 1$ *be a real number. Then, we have*

$$D_{p,w}(x) < 129 p^{\frac{2}{3}} x^{0.765}.$$

Proof. First, let us set $N = \frac{3}{2}(p-1)^{2/3} + 1$. With the help of Lemma 3 and [27, Lemma 2.2] we obtain that

$$|D_{p,w}^{(1)}| = |J_p^{(1)}| < N.$$

Next, we have

$$|D_{p,w}^{(k+1)}| = \sum_{n \in J_p^{(k)}} |D(p,w) \cap [pn, pn + p - 1]| < |J_p^{(k)}| N.$$

Moreover, $|J_p^{(k)}| < N^k$ holds by the proof of [27, Theorem 1.1]. Consequently, we get

$$|D_{p,w}^{(k)}| < N^k$$

and the rest is similar to the cited proof. □

Now, we can prove Theorem A.

Proof of Theorem A. Our aim is to improve the upper bound presented in Lemma 4. To improve the upper bound for $D_{p,w}(x)$, we need to modify Definition 3, investigate the different cases, and then follow the procedure presented in [30] for J_p. In the proof of Lemma 3, we had

$$D(p,w) \cap [x, x+y] = \{n_1 < \cdots < n_k\}$$

with some positive integer $w < p$, real numbers x, y with $1 \leq y < p$ and set $d_i = n_{i+1} - n_i$ for $i = 1, 2, \ldots, k-1$. Then, we observed in (8) that

$$\frac{f'_{d_i}(n_i)}{f_{d_i}(n_i)} = H_{n_{i+1}} - H_{n_i} \equiv 0 \pmod{p}.$$

As $f_d(x)$ is a polynomial of degree d and the intersection interval has length at most p, we deduce that there are at most $d-1$ many solutions of

$$f'_{d_i}(n_i) \equiv 0 \pmod{p}.$$

This fact leads that

$$|\{i : n_{i+1} - n_i = d\}| \leq d - 1$$

for any positive integer $d \geq 1$ with $i = 1, 2, \ldots, k$.

At this point, we need to consider the cases where $w \in [p^t, p^{t+1} - 1]$ for some $t \in \mathbb{Z}^{\geq 0}$.

Case 1. $w \in [1, p-1]$.

In this case, we can continue with Definition 3 and set $D_{p,w}^{(1)} = D(p,w) \cap [1, p-1]$ and $D_{p,w}^{(k+1)} = \{pn + r \in D(p,w) : n \in J_p^{(k)},\ 0 \leq r \leq p-1\}$ for any k. Then, together with our argument presented in the proof of Lemma 4, our setup becomes identical with the set up given in [30, Theorem 1.1].

Namely, [30, Lemma 2.4] applies to the difference set so that we have

$$|D(p,w) \cap [x, x+y]| \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} \tag{9}$$

for any prime number p, any positive integer w and any real numbers x, y with $\frac{8}{3} \leq y < p$. Here, we do not have to bound w with p by our observation in the proof of Lemma 3. So, we continue with the improved upper bound.

Let us set

$$N = \left(\frac{9}{8}\right)^{\frac{1}{3}} (p-1)^{\frac{2}{3}}.$$

Then, given a real number x, we can find the positive integer m satisfying

$$p^{m-1} \leq x < p^m.$$

Then, we can write

$$D_{p,w}(x) = D_{p,w}(p^{m-1}-1) + |D(p,w) \cap [p^{m-1}, x]|. \tag{10}$$

For the first summand, we can write by Definition 3 and Proposition 6, together with (9) that

$$\begin{aligned} D_{p,w}(p^{m-1}-1) &= \sum_{i=1}^{m-1} |D(p,w) \cap [p^{i-1}, p^i - 1]| \\ &= \sum_{i=1}^{m-1} |D_{p,w}^{(i)}| \leq \sum_{i=1}^{m-1} N^i = \frac{N}{N-1} N^{m-1}. \end{aligned} \tag{11}$$

Here, we also use the fact that $|D_{p,w}^{(i)}| \leq N^i$ for $i \geq 1$ via Lemma 4. For the second summand, we have

$$|D(p,w) \cap [p^{m-1}, x]| \leq \sum_{\substack{n \in J_p^{(m-1)} \\ pn \leq x}} |D(p,w) \cap [pn, pn + p - 1]|$$

so that

$$\begin{aligned} |D(p,w) \cap [p^{m-1}, x]| \leq N \sum_{\substack{n \in J_p^{(m-1)} \\ pn \leq x}} 1 &= N \left| D(p,w) \cap \left[p^{m-2}, \frac{x}{p}\right] \right| \\ &\leq N^2 \left| D(p,w) \cap \left[p^{m-3}, \frac{x}{p^2}\right] \right| \\ &\leq \dots \\ &= N^{m-1} \left| D(p,w) \cap \left[1, \frac{x}{p^{m-1}}\right] \right|. \end{aligned}$$

Here, if $x < 3p^{m-1}$ then

$$\left| D(p,w) \cap \left[1, \frac{x}{p^{m-1}}\right] \right| \leq 1 \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} \left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}}.$$

Otherwise, if $x \geq 3p^{m-1}$ then by (9) we get

$$\left| D(p,w) \cap \left[1, \frac{x}{p^{m-1}}\right] \right| \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} \left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}}.$$

Thus, we obtain that

$$|D(p,w)\cap[p^{m-1},x]| \le N^{m-1}\left|D(p,w)\cap\left[1,\frac{x}{p^{m-1}}\right]\right| \le N^{m-1}\left(\frac{9}{8}\right)^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}}.$$

Then, combining this result with (11), we write for (10) that

$$D_{p,w}(x) \le \frac{N}{N-1}N^{m-1} + N^{m-1}\left(\frac{9}{8}\right)^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}}.$$

The rest is similar to the proof of [30, Theorem 1.1] and we are done.

Case 2. $w \in [p, p^2-1]$.
In the first case, when we have $w \in I_0 = [1, p-1]$, we had the sets

$$D^{(1)}_{p,w} = D(p,w)\cap[1,p-1]$$

and $D^{(k+1)}_{p,w} = \{pn + r \in D(p,w) : n \in J^{(k)}_p,\ \ 0 \le r \le p-1\}$ for any integer $k \ge 1$. Now, if w belongs to the interval $[p, p^2-1]$, we will need to modify the sets $D^{(k+1)}_{p,w}$ for $k \ge 1$.

We know by Lemma 1 that if $pn + r \in D(p,w)$ then $n \in D(p,\hat{w})$ holds where $0 \le r \le p-1$ and $\hat{w} = \lfloor \frac{w}{p} \rfloor$. However, the positive integer w in the lemma was strictly less than p, and we get $\hat{w} = 0$ so that $D(p,w)$ becomes J_p. That is why we had $J^{(k)}_p$ in Definition 3. However, we need the following definition to have a partition of $D(p,w)$ when $w \in [p, p^2-1]$:

Definition 4. *For any prime number p and a positive integer w, we define*

$$D^{(1)}_{p,w} = D(p,w)\cap[1,p-1] \text{ and } D^{(k+1)}_{p,w} = \{pn + r \in D(p,w) : n \in D^{(k)}_{p,\hat{w}}, r \in [0,p-1]\}$$

where $\hat{w} = \lfloor \frac{w}{p} \rfloor$, $k \in \mathbb{Z}^{>0}$.

Consequently, using Lemma 3, we can write that

$$|D^{(1)}_{p,w}| < \frac{3}{2}(p-1)^{2/3} + 1 = N.$$

In fact, we have

$$|D^{(k+1)}_{p,w}| = \sum_{n\in D^{(k)}_{p,\hat{w}}} |D(p,w)\cap[pn, pn+p-1]| < |D^{(k)}_{p,\hat{w}}|N < N^kN = N^{k+1}$$

by the first case, as $\hat{w} \in [1, p-1]$.

As a consequence, we again obtain the same setup in [30] to bound our set. Moreover, as Definition 4 applies to any $w \in [p^t, p^{t+1}-1]$ with $t \in \mathbb{Z}^{\ge 0}$, we can cover all the cases. The proof of Theorem A is now complete. □

Remark 3. *The authors of [30] examined general harmonic numbers in [12], defined as follows. Let $a, b \geq 1$ be two integers. They introduced*

$$H_{a,b}(n) = \sum_{k=0}^{n-1} \frac{1}{ak+b},$$

such that by setting $a = b = 1$, we recover $H_{1,1}(n) = H_n$. Furthermore, for positive integers $w \leq n$, we can express

$$H_n - H_w = \frac{1}{w+1} + \cdots + \frac{1}{n} = H_{1,w+1}(n-w)$$

and encourage readers to consult [12] for further interesting results.

4. Proof of Theorem B

In this section, we work with the differences of generalized harmonic numbers and then prove Theorem B. Let us introduce these numbers. For any positive integers n and s, the n^{th} generalized harmonic number of order s is defined as

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}.$$

First of all, as

$$1 < \sum_{k=1}^{\infty} \frac{1}{k^s} < 2$$

holds, they are non-integer except for the case, when $n = 1$. Also, one can easily show that the difference $H_n^{(s)} - H_m^{(r)}$ is never an integer, except for the trivial case: $n = m$ and $s = r$.

These numbers also satisfy a Wolstenholme [29] type congruence, the generalized version of (4) by [17]:

Fact 7. *For any prime number p and a positive integer s, the congruence*

$$H_{p-1}^{(s)} \equiv 0 \pmod{p}$$

holds whenever $p - 1 \nmid s$.

This fact is quite useful when we deal with the divisibility properties. In particular, we know that most of the time,

$$\nu\left(H_{p-1}^{(s)}\right) = 1$$

holds (see [22]).

Similar to the harmonic numbers, given a positive integer $n = p\hat{n} + r$ with p a prime number and an integer $0 \le r \le p-1$, we have that

$$H_n^{(s)} = H_{p\hat{n}+r}^{(s)} \equiv \frac{1}{p^s} H_{\hat{n}}^{(s)} + H_r^{(s)} \equiv 0 \pmod{p}, \tag{12}$$

whenever $p-1 \nmid s$ by Fact 7.

Moreover, an extension of $J(p)$ can also be defined as

$$J(p,s) = J_{p,s} := \{n \in \mathbb{N} : \ p \mid H_n^{(s)}\}.$$

Fact 8. *If $p\hat{n} + r \in J(p,s)$, then we have $\hat{n} \in J(p,s)$ whenever $p-1 \nmid s$.*

The fact comes from (12) as if $\nu_p\left(\frac{1}{p^s} H_{\hat{n}}^{(s)} + H_r^{(s)}\right) \ge 0$, then

$$\nu_p\left(\frac{1}{p^s} H_{\hat{n}}^{(s)}\right) \ge 0$$

must hold by the Archimedean property of ν_p. In other words, $\nu_p\left(H_{\hat{n}}^{(s)}\right) \ge s$ must hold so that we get $\hat{n} \in J(p,s)$ (see also the proof of Proposition 4).

Also, similar to Fact 5, setting

$$J_{p,s}^{(1)} = J(p,s) \cap [1, p-1] \text{ and } J_{p,s}^{(k+1)} = \{pn + r \in J_{p,s} : n \in J_{p,s}^{(k)}, \ \ 0 \le r \le p-1, \ \ p \mid H_n^{(s)}\}$$

for any $k \ge 1$, we have that $J_{p,s}^{(k)} = [p^{k-1}, p^k - 1]$ (see [5, Lemma 3.1]). Hence, we have the following fact.

Fact 9. *If $J_{p,s}^{(k)} = J(p,s) \cap [p^{k-1}, p^k - 1] = \emptyset$ for some positive integer k, then we have $J(p,s) = \bigcup_{t=1}^{k-1} J_{p,s}^{(t)}$.*

Now, let us define the corresponding difference set for generalized harmonic numbers.

Definition 5. *Let p be a prime number and s, w be any positive integers. Then, we define*

$$G(p,s,w) = G_{p,s,w} = \{n \in \mathbb{N} : p \mid H_n^{(s)} - H_w^{(s)}\}.$$

Note by definition that if $w \in J(p,s)$, then the difference set $G(p,s,w)$ becomes identical with $J(p,s)$.

Let us extend some of our results to the generalized harmonic numbers. For the rest of this section, suppose that $p-1 \nmid s$ holds. Under this condition, we can extend our results from Section 2. For instance, we generalize Lemma 1 and we obtain the following result.

Lemma 5. *Let n, w be positive integers and p be a prime number. Also let $n = p\hat{n}+R$, $w = p\hat{w}+r$ for some non-negative integers $\hat{n}, \hat{w}$ and $0 \leq r, R \leq p-1$. If $n \in G(p, s, w)$, then we have $\hat{n} \in G(p, s, \hat{w})$ for any positive integer s. In particular, if $w < p$, then $G(p, s, \hat{w}) = J(p, s)$.*

Proof. The idea is similar to the proof of Lemma 1. Using (12), we write

$$H_n^{(s)} - H_w^{(s)} = H_{p\hat{n}+R}^{(s)} - H_{p\hat{w}+r}^{(s)} \equiv \frac{1}{p^s}\left(H_{\hat{n}}^{(s)} - H_{\hat{w}}^{(s)}\right) + \left(H_R^{(s)} - H_r^{(s)}\right) \equiv 0 \pmod p$$

where

$$\nu_p\left(H_R^{(s)} - H_r^{(s)}\right) \geq 0$$

as both $r, R \leq p-1$. Thus, we have $\nu_p\left(\frac{1}{p^s}\left(H_{\hat{n}}^{(s)} - H_{\hat{w}}^{(s)}\right)\right) \geq 0$. Therefore, $\hat{n}$ lies in the set $G(p, s, \hat{w})$ with

$$\left(H_{\hat{n}}^{(s)} - H_{\hat{w}}^{(s)}\right) \equiv 0 \pmod{p^s}.$$

Moreover, if $w < p$, then $\hat{w} = \lfloor \frac{w}{p} \rfloor = 0$ and we are done. □

We can also generalize Lemma 2 as follows, which we state without the proof as the process is similar.

Lemma 6. *Let p be an odd prime and w, s be positive integers. If $G(p, s, w)$ is finite, then $G(p, s, pw + r)$ is also finite for any integer $0 \leq r \leq p-1$.*

Now, let us define the counting function for $G(p, s, w)$.

Definition 6. *For any real number $x \geq 1$, a prime number p and a positive integer w, we define*

$$G(p, s, w)(x) = G_{p,s,w}(x) = |G(p, s, w) \cap [1, x]|.$$

We are ready to prove Theorem B.

Proof of Theorem B. To begin with, our first step is to divide the difference set into smaller sets.

Definition 7. *For any prime number p and a positive integer w, we define*

$$G_{p,s,w}^{(1)} = G_{p,s,w} \cap [1, p-1] \text{ and } G_{p,s,w}^{(k+1)} = \{pn + r \in G_{p,s,w} : n \in G_{p,s,\hat{w}}^{(k)}, r \in [0, p-1]\}$$

where $\hat{w} = \lfloor \frac{w}{p} \rfloor$, $k \in \mathbb{Z}^{>0}$.

Recall by Proposition 6 that

$$D_{p,w}^{(k)} = D(p, w) \cap [p^{k-1}, p^k - 1]$$

for any prime number p and positive integer k. By extending this result, we obtain the following proposition which we present without proof.

Proposition 10. *The equality*

$$G_{p,s,w}^{(k)} = G_{p,s,w} \cap [p^{k-1}, p^k - 1]$$

holds for any prime number p and positive integers s, k and w.

Hence, we have

$$G(p, s, w) = \bigcup_{k=1}^{\infty} G_{p,s,w}^{(k)}.$$

Now, in order to count the elements of $G(p, s, w)$, we can consider the intersection of the set with intervals of length at most p. That is, one may first bound the set

$$G(p, s, w) \cap [x, x + y]$$

for some positive real numbers x, y with $1 \leq y < p$. Therefore, we may consider to generalize Lemma 3.

Given two positive integers $n_1, n_2 \in G(p, s, w) \cap [x, x + y]$, with for some prime p, positive integers s, w and real numbers x, y with $1 \leq y < p$, the equivalences

$$H_{n_1}^{(s)} - H_w^{(s)} \equiv 0 \pmod p \text{ and } H_{n_2}^{(s)} - H_w^{(s)} \equiv 0 \pmod p$$

imply that

$$H_{n_2}^{(s)} - H_{n_1}^{(s)} \equiv 0 \pmod p. \tag{13}$$

On the other hand, if we have $n_1, n_2 \in J(p, s) \cap [x, x+y]$ under the same conditions above, we end up with (13). This fact is valid for any finite number of elements inside $G(p, s, w) \cap [x, x + y]$. Consequently, the counting of $G(p, s, w) \cap [x, x + y]$ is essentially equivalent to the counting of $J(p, s) \cap [x, x + y]$, similar to the argument in the proof of Lemma 3. The process was covered broadly in [5, Lemma 3.3, Lemma 3.4] by the author.

Now, as we observed the fact that counting $J(p, s)$ is equivalent to the counting of the difference set, we rely on the proof of bounding $J(p, s)$ given by the author as below.

Theorem ([5, Theorem A])**.** *Suppose that p is a prime number, s is any positive integer and $x \geq 1$ is any real number. Then,*

$$J_{p,s}(x) \leq 3x^{\frac{2}{3} + \frac{1}{25 \log p} + \frac{\log s}{3 \log p} + \frac{\log s}{3 \log x}}$$

holds. Moreover, whenever $p > se^{\frac{3}{25}}$ holds, we have

$$J_{p,s}(x) = o(x).$$

Hence, when our setup becomes identical with the cited theorems proof, we are done. Eventually, we need the following lemma from [5].

Lemma 7 ([5, Lemma 3.5]). *Let p be a prime number and x, y be real numbers with $\frac{8}{3} \leq y < p$. Then, the inequality*

$$|J(p,s) \cap [x, x+y]| \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}$$

holds for any positive integer s.

Now, if we set

$$A = \left(\frac{9}{8}\right)^{\frac{1}{3}} (p-1)^{\frac{2}{3}} s^{\frac{1}{3}},$$

we obtain that

$$|G^{(1)}_{p,s,w}| = |G_{p,s,w} \cap [1, p-1]| \leq A.$$

Moreover, we have

$$|G^{(k+1)}_{p,s,w}| = \sum_{n \in G^{(k)}_{p,s,\hat{w}}} |G(p,s,w) \cap [pn, pn+p-1]| \leq |G^{(k)}_{p,s,\hat{w}}| A$$

so that

$$|G^{(k)}_{p,s,\hat{w}}| \leq A^k$$

holds for any $k \in \mathbb{Z}^{>0}$. Finally, as the upper bounds do not contain w, our setup is now complete. Hence, the upper bound for $J(p,s)$ is also valid for $G(p,s,w)$.

For the last part of the theorem, namely, to obtain the equality

$$G_{p,s,w}(x) = o(x),$$

we only need to work on the inequality

$$\frac{1}{25 \log p} + \frac{\log s}{3 \log p} + \frac{\log s}{3 \log x} < \frac{1}{3}$$

and end up with the condition $p > se^{\frac{3}{25}}$, which can be easily shown. The proof is now complete. □

Now, we prove our last result, Theorem C, which is a direct consequence of [5, Theorem B.(i)].

Theorem C. *Let p be a prime number, s, w be positive integers with $s \geq 2$ and $p - 1 \nmid s$. If the inequality*

$$\nu_p\left(H^{(s)}_k\right) \leq s - 1$$

holds for any $k \in \{1, 2, \ldots p-1\}$, then $G(p,s,w)$ is finite. Moreover, if

$$p^m \leq w < p^{m+1}$$

for some integer $m \geq 0$, then we have $G(p,s,w) \subseteq \{1, \ldots, p^{m+1} - 1\}$.

Proof. Using Fact 9, we first obtain that $J(p,s)$ is finite for any p,s as in the statement, by showing that $J^{(2)}_{p,s}=\emptyset$. Suppose that $\nu_p\left(H^{(s)}_k\right)\le s-1$ holds for any integer $1\le k\le p-1$, for some prime number p, and a positive integer $s\ge 2$ with $p-1\nmid s$. Assume also that $pn+r\in J^{(2)}_{p,s}\ne\emptyset$ for some integers n and $0\le r\le p-1$. Note that we have $n\in[1,p-1]$ as $pn+r\in J^{(2)}_{p,s}=J(p,s)\cap[p,p^2-1]$ via [5, Lemma 3.1]. Now,

$$H^{(s)}_{pn+r}\equiv\frac{1}{p^s}H^{(s)}_n+H^{(s)}_r\equiv 0 \pmod p$$

implies that

$$\nu_p\left(H^{(s)}_n\right)\ge s$$

by the Archimedean property as $\nu_p\left(H^{(s)}_r\right)\ge 0$. On the other hand, the inequality $\nu_p\left(H^{(s)}_n\right)\ge s$ contradicts with our assumption, as $n\in J(p,s)$ with $1\le n\le p-1$. Thus, $J^{(2)}_{p,s}=\emptyset$ and we have

$$J(p,s)=J^{(1)}_{p,s}=J(p,s)\cap[1,p-1].$$

Next, let us take any positive integer w. By Lemmas 5 and 6, if we show that $G(p,s,\hat{w})$ is finite, then we are done. We can bound w as $p^m\le w<p^{m+1}$ for some integer $m\ge 0$. Now, since $J(p,s)$ is finite, the set

$$G\big(p,s,\lfloor\frac{w}{p^m}\rfloor\big)$$

is also finite, since

$$1\le\left\lfloor\frac{w}{p^m}\right\rfloor\le p-1$$

and $J(p,s)=G(p,s,\big\lfloor\lfloor\frac{w}{p^m}\rfloor/p\big\rfloor)$. Also, as $G\big(p,s,\lfloor\frac{w}{p^m}\rfloor\big)$ is finite, $G\big(p,s,\lfloor\frac{w}{p^{m-1}}\rfloor\big)$ is also finite. Continuing the process, we end up with the finiteness of $G(p,s,w)$ and the first part of the theorem is done.

Now, let us obtain the upper bound for the set $G(p,s,w)$. Take any $n\in G(p,s,w)$ so that $p^m\le w\le n$. Again by Lemma 5, we have

$$\left\lfloor\frac{n}{p^m}\right\rfloor\in G\big(p,s,\lfloor\frac{w}{p^m}\rfloor\big)$$

where $1\le\lfloor\frac{w}{p^m}\rfloor\le p-1$. Now, assume that $\lfloor\frac{n}{p^m}\rfloor\ge p$ holds. Then, let us write $\lfloor\frac{n}{p^m}\rfloor=pk+r$ for some k,r with $k\ge 1$ and $0\le r\le p-1$. As we have

$$\left\lfloor\frac{n}{p^m}\right\rfloor=pk+r\in G\big(p,s,\lfloor\frac{w}{p^m}\rfloor\big),$$

we may write

$$H^{(s)}_{pk+r}\equiv\frac{1}{p^s}H^{(s)}_k+\left(H^{(s)}_r-H^{(s)}_{\lfloor\frac{w}{p^m}\rfloor}\right)\equiv 0 \pmod p$$

such that $\nu_p\big(H_k^{(s)}\big) \geq s$, thus $k \in J(p,s)$. However, $J(p,s)$ is bounded above by $p-1$ and for any $k \in J(p,s)$, we have

$$\nu_p\big(H_k^{(s)}\big) \leq s-1.$$

Hence the assumption $\lfloor \frac{n}{p^m} \rfloor \geq p$ fails and

$$\frac{n}{p^m} - 1 < \lfloor \frac{n}{p^m} \rfloor \leq p-1$$

yields that $p^m \leq w \leq n < p^{m+1}$. The proof is now complete. □

5. Computations

In this section, we begin by computing the difference sets $D(p,w)$ for some prime p and positive integers w.

Example 1. $p=5, w=2$. *To compute* $D(5,2)$*, recall that we have* $D_{5,2}^{(1)} = D(5,2) \cap [1,4]$. *Next, as* $\hat{w} = \hat{2} = \lfloor \frac{2}{5} \rfloor = 0$*, we have*

$$D_{5,2}^{(k+1)} = \{5n + r \in D(5,2) : \ n \in D_{5,0}^{(k)}, \ 0 \leq r \leq 4\}$$

for positive integers k.

Also, as

$$D(5,0) = \{n \in \mathbb{N} : \ p \mid H_n - H_0 = H_n\}$$

we have $D(5,0) = J(5) = J_5$. *The prime* 5 *is a harmonic prime so that*

$$J(5) = \{4, 20, 24\}$$

by [16]. Therefore, we have $J_5^{(1)} = \{4\}, J_5^{(2)} = J_5 \cap [5,24] = \{20,24\}$ *and* $J_5^{(3)} = \emptyset$. *Then, by Fact 5, we can write*

$$J_5 = J_5^{(1)} \cup J_5^{(2)}.$$

Moreover, we also have $J_5^{(k)} = \emptyset$ *for any* $k \geq 3$ *by the same fact.*

The equality yields that

$$D_{5,2}^{(k+1)} = \{5n + r \in D(5,2) : \ n \in J_5^{(k)}, \ 0 \leq r \leq 4\} = \emptyset$$

for any $k \geq 3$. *Consequently, we have*

$$D(5,2) = D_{5,2}^{(1)} \cup D_{5,2}^{(2)} \cup D_{5,2}^{(3)}.$$

Now, we can say that 2 *and* $\bar{2} = 5-1-2 = 2$ *is already in the set* $D(5,2)$ *via Remark 2. Then, with the help of [26], we see that there is not any other element in the first level so* $D_{5,2}^{(1)} = \{2\}$. *Next,*

$$D_{5,2}^{(2)} = \{5n + r \in D(5,2) : \ n \in J_5^{(1)}, \ 0 \leq r \leq 4\}$$

and as $J_5^{(1)} = \{4\}$ *we only need to check* $\{5 \cdot 4 + r\}$ *for* $r \in \{0, 1, 2, 3, 4\}$. *By Proposition 3 we already know that* $p(p-1) + w = 22 \in D(5, 2)$. *Eventually, we see that* $D_{5,2}^{(2)} = \{22\}$ *using [26]. Then, for* $D_{5,2}^{(3)}$ *we consider the set*

$$\{5n + r \in D(5, 2) : \ n \in J_5^{(2)} = \{20, 24\}, \ 0 \le r \le 4\}.$$

That is, we check

$$\{100 + r : r \in \{0, 1, 2, 3, 4\}\} \cap D(5, 2) \text{ and } \{120 + r : r \in \{0, 1, 2, 3, 4\}\} \cap D(5, 2).$$

Finally, we obtain $D_{5,2}^{(3)} = \{101, 103, 121, 123\}$ *so that*

$$D(5, 2) = \{2\} \cup \{22\} \cup \{101, 103, 121, 123\} = \{2, 22, 101, 103, 121, 123\}.$$

In the next example, we will see that one do not need to compute each level

$$D_{p,w}^{(k)}$$

to determine $D(p, w)$, as long as $D(p, \hat{w})$ is known.

Example 2. $p = 5, w = 11$. *Now, let us consider the case* $w = 11 > p = 5$. *First, let us write* $\hat{w} = \hat{11} = \lfloor \frac{11}{5} \rfloor = 2$ *as we need* $D(5, 2)$ *to determine* $D(5, 11)$. *So, we have* $D_{5,11}^{(1)} = D(5, 11) \cap [1, 4]$ *and*

$$D_{5,11}^{(k+1)} = \{5n + r \in D(5, 11) : \ n \in D_{5,2}^{(k)}, \ 0 \le r \le 4\}$$

for any $k \ge 1$. *By the first example, we know that* $D_{5,2}^{(k)} = \emptyset$ *for any* $k \ge 4$. *Thus,* $D_{5,11}^{(k)} = \emptyset$ *for any* $k \ge 4$ *and*

$$D(5, 11) = D_{5,11}^{(1)} \cup D_{5,11}^{(2)} \cup D_{5,11}^{(3)}.$$

By following our steps in the first example, we can completely determine $D(5, 11)$. *However, we can use Lemma 1 and quickly get the result:*

if n *belongs to* $D(p, w)$ *then* $\hat{n}$ *belongs to* $D(p, \hat{w})$ *for any* n *and* w.

That is,

$$D(5, 11) = D(5, 11) \cap \{5 \cdot n + r : n \in \{2, 22, 101, 103, 121, 123\}, \ 0 \le r \le 4\}.$$

Thus, using [26] we conclude that

$$D(5, 11) = \{11, 13, 506, 508, 515, 519, 617\}.$$

Example 3. $p = 5, w = 59$. *In this case,* $D(5, 59)$ *can be determined by* $D(5, 11) = \{11, 13, 506, 508, 515, 519, 617\}$ *as* $\hat{w} = \lfloor \frac{w}{p} \rfloor = \lfloor \frac{59}{5} \rfloor = 11$. *Hence, using [26] again, we have that*

$$D(5, 59) = \{55, 59, 65, 69, 2532, 2541, 2543, 2576, 2578, 2596, 2598, 3085, 3089\}.$$

Recall by Fact 5 that if $J_p^{(k)} = \emptyset$ for some $k \in \mathbb{Z}^{>0}$, then $J_p^{(t)} = \emptyset$ for any $t \geq k$ and we have

$$J_p = \bigcup_{t=1}^{k-1} J_p^{(t)}.$$

However, this might not be the case with the difference sets. For instance, if we choose $p = 7$, then we know by [16] that

$$J_7 = \{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}$$

with $J_7 = \bigcup_{t=1}^{6} J_7^{(t)}$. We have $J_7^{(7)} = \emptyset$ and hence, $J_7^{(t)} = \emptyset$ for any $t \geq 7$. Now if we pick $w = 2$, we obtain that

$$D_{7,2}^{(1)} = \{2, 4\} \text{ and } D_{7,2}^{(2)} = \{44, 46\}$$

by Remark 2 and Proposition 3. On the other hand, even if

$$D_{7,2}^{(3)} = D(7, 2) \cap [7^2, 7^3 - 1] = \emptyset$$

holds, we cannot conclude that $D(7, 2) = D_{7,2}^{(1)} \cup D_{7,2}^{(2)}$ as

$$D_{7,2}^{(4)} = \{2094, 2098, 2359, 2365, 2388, 2392\} \neq \emptyset.$$

On the other hand, one may observe that as $J_7^{(7)} = \emptyset$ then $D_{7,2}^{(8)}$ is also empty as

$$D_{7,2}^{(8)} = \{6n + r \in D(7, 2) : \ n \in J_7^{(7)}, \ 0 \leq r \leq 6\}.$$

Hence, the number of non-empty $D_{p,w}^{(k)}$'s cannot exceed the number of non-empty $J_p^{(k)}$'s for $k \in \mathbb{Z}^{>0}$.

To sum up, given a prime number p and a positive integer w, it may be time consuming to determine $D(p, w)$ completely. However, we can find the integer m satisfying $p^m \leq w < p^{m+1}$, namely $m = \lfloor \log_p w \rfloor$. Then, $\lfloor \frac{w}{p^m} \rfloor$ yields the base step to start with. To determine $D(p, \lfloor \frac{w}{p^m} \rfloor)$ we need to determine J_p (see Example 1). This process is done by finding the integer k where $J_p^{(k)} = \emptyset$.

First, we find

$$D\left(p, \left\lfloor \frac{w}{p^m} \right\rfloor\right) \cap [1, p - 1] = D^{(1)}_{p, \lfloor \frac{w}{p^m} \rfloor}.$$

Then, we check

$$D\left(p, \left\lfloor \frac{w}{p^m} \right\rfloor\right) \cap [pn, pn + p - 1]$$

for each $n \in J_p$ so that we completely obtain $D\left(p, \lfloor \frac{w}{p^m} \rfloor\right)$. Next, we determine $D\left(p, \lfloor \frac{w}{p^{m-1}} \rfloor\right)$ by proceeding as we did in the examples above. After m steps, we finally have $D(p, w)$.

TABLE 1. The number of elements in the sets $J(p)$ and $D(p, w)$ for several p, w values.

p	$\|J(p)\|$	$\|D(p,1)\|$	$\|D(p,2)\|$
3	3	3	3
5	3	4	6
7	13	10	20
13	3	10	12
17	3	6	12
23	3	4	8

TABLE 2. The elements in the sets $J(p)$ and $D(p, 1)$ for several p values.

p	$J(p)$	$D(p,1)$
3	$\{2, 7, 22\}$	$\{1, 66, 68\}$
5	$\{4, 20, 24\}$	$\{1, 3, 21, 23\}$
7	$\{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}$	$\{1, 5, 43, 47, 2067, 2069, 2362, 117120, 117148, 719099\}$
13	$\{12, 156, 168\}$	$\{1, 4, 8, 11, 157, 160, 164, 167, 2034, 2190\}$
17	$\{16, 272, 288\}$	$\{1, 15, 273, 287, 4632, 4904\}$
23	$\{22, 506, 528\}$	$\{1, 21, 507, 527\}$

TABLE 3. The elements in the sets $J(p)$ and $D(p, 2)$ for several p values.

p	$J(p)$	$D(p,2)$
3	$\{2, 7, 22\}$	$\{2, 7, 22\}$
5	$\{4, 20, 24\}$	$\{2, 22, 101, 103, 121, 123\}$
7	$\{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}$	$\{2, 4, 44, 46, 2094, 2098, 2359, 2365, 2388, 2392, 14673, 14677, 102726, 102730, 117117, 117123, 117145, 117151, 719096, 719102\}$
13	$\{12, 156, 168\}$	$\{2, 10, 158, 166, 2029, 2032, 2036, 2039, 2185, 2188, 2192, 2195\}$
17	$\{16, 272, 288\}$	$\{2, 7, 9, 14, 274, 279, 281, 286, 4624, 4640, 4896, 4912\}$
23	$\{22, 506, 528\}$	$\{2, 20, 508, 526, 11643, 11655, 12149, 12161\}$

Finally, let us check some examples for $G(p, s, w)$. If we choose $p = 5$ and $s = 2$, we have the following generalized harmonic numbers $H_n^{(s)}$ with the corresponding 5-adic orders:

n	$H_n^{(2)}$	$\nu_5\left(H_n^{(2)}\right)$
1	1	0
2	5/4	1
3	49/36	0
4	205/144	1

Hence via Theorem C, if we take an integer w satisfying $p^m \leq w < p^{m+1}$, we expect to get $G(p, s, w) \subseteq \{1, \ldots, p^{m+1} - 1\}$. We first find $J(5, 2) = \{2, 4\}$ and obtained the following results via [26]:

w	$G(p, s, w)$
$2 \in [1, 4]$	$\{2, 4\}$
$13 \in [5, 24]$	$\{13, 20, 22, 24\}$
$66 \in [25, 124]$	$\{66, 120, 122, 124\}$
$331 \in [125, 624]$	$\{331, 623\}$

On the other hand, we have $H_3^{(2)} = \frac{49}{36}$ and

$$\nu_7\left(H_3^{(2)}\right) = 2 \nleq 1,$$

such that our condition in Theorem C fails. In fact, we have

$$26 \in G(7, 2, 3), \ \ 27 \in G(7, 2, 21), \ \ 182 \in G(7, 2, 43).$$

Lastly, let us close the section with another counter example. One may check that the case $p = 37$ and $s = 3$ yields some elements in $G(p, s, w)$ that are greater than 37. That is because we have $\nu_{37}\left(H_{36}^{(3)}\right) = 3 \nleq 3 - 1 = 2$. For instance, if we pick $w = 10$, we obtain that $1344 \in G(37, 3, 10)$ and $1344 > p - 1 = 37 - 1 = 36$.

6. Conclusion

In this section, we first present some of the generalizations of the harmonic numbers. The first one of those is the Dedekind harmonic numbers. Let K be a number field. Then, the n^{th} Dedekind harmonic number, denoted by $h_K(n)$ is defined as

$$\sum_{\substack{0\neq I\subseteq\mathcal{O}_K\\ N(I)\leq n}} \frac{1}{N(I)},$$

where the sum is ranging over all the non-zero ideals of $\mathcal{O}_K$ with norm less than or equal to n. This idea was inspired by the Dedekind zeta function $\zeta_K(s)$ for K and these are indeed an extension of harmonic numbers as taking $K=\mathbb{Q}$ yields that

$$h_K(n)=H_n$$

as $\zeta_K(s)=\zeta(s)$ in that case.

In [4], it was shown that almost all of these numbers are non-integer. Moreover, the differences of these numbers was also studied. In fact, it was proven under the Riemann hypothesis for $\zeta_K(s)$ that the difference

$$h_K(n)-h_K(m)$$

is not an integer after a while. Namely, there exist constants $\alpha, x_0>0$ such that $h_K(n)-h_K(m)\notin\mathbb{Z}$ for any positive integers $n>m\geq x_0$ whenever

$$n-m\geq\alpha(d_K\log m+\log\Delta_K)\sqrt{m}$$

holds, where d_K is the degree of K and Δ_K denotes the absolute value of the discriminant of K.

Euler introduced the harmonic zeta function given as

$$\zeta_H(s)=\sum_{n=1}^{\infty}\frac{H_n}{n^s},$$

where $\Re(s)>1$. He showed that the identity

$$2\zeta_H(m)=(m+2)\zeta(m+1)-\sum_{k=1}^{m-2}\zeta(m-k)\zeta(k+1)$$

holds for any integers $k\geq 2$, provided that the sum vanishes if $m=2$. In particular, if we let $m=2$, we get

$$2\zeta_H(2)=2\left(\sum_{n=1}^{\infty}\frac{H_n}{n^2}\right)=4\zeta(3)$$

so that

$$\zeta_H(2)=\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\zeta(3)$$

and for $m=3$, we have that

$$\zeta_H(3)=\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{5}{4}\zeta(4)=\frac{\pi^4}{72}.$$

Consequently, one may obtain the special values of the harmonic zeta function via the special values of the Riemann zeta function. One of the applications of the special values of the harmonic zeta function is to approximate the real numbers given in [1, 2]. Moreover, $\zeta_H(s)$ is just one example of a Dirichlet series. It was shown lately that not only this function can be used for the approximation purpose but all Dirichlet series can be used [14].

Now, we point out a direction that has another generalization of the harmonic numbers and the harmonic differences for interested readers. Conway and Guy presented a generalization in their book, *The Book of Numbers* [13] called the hyperharmonic numbers. The hyperharmonic numbers were defined recursively as

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}$$

where $r \geq 2$, and that $h_n^{(1)} = h_n$. These numbers are also endowed with a variety of arithmetic and analytical features. In particular, the integerness of the difference of hyperharmonic numbers was studied in [6] and it was shown that almost all of the differences

$$h_n^{(r)} - h_m^{(s)}$$

are non-integer. However, there are also some cases that the difference is an integer, infinitely many times.

To relate the differences of harmonic numbers with hyperharmonic numbers, one may consider the following identity given by Conway and Guy. They stated that the n^{th} hyperharmonic number of order r can be written as

$$h_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}). \tag{14}$$

The identity (14) points out that in order to work on the p-adic order of harmonic differences, we may consider to work on the p-adic valuations of the binomial coefficient and the corresponding hyperharmonic number.

Now, recall the polynomial at (6)

$$f_d(x) = (x+1)(x+2)\dots(x+d)$$

for some positive integer d. Notice that the polynomial appears in the numerator of the binomial coefficient, as we have

$$\binom{n+r-1}{r-1} = \frac{(n+r-1)(n+r-2)\dots(r)}{n!}$$

so that one direction is to study this polynomial. Moreover, by feeding with the harmonic difference, we may write for (14) that

$$\begin{aligned} h_n^{(r)} &= \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}) \\ &= \frac{(n+r-1)(n+r-2)\dots(r)}{n!}\left(\frac{1}{r} + \frac{1}{r+1} + \dots + \frac{1}{n+r-1}\right) \\ &= \frac{f_n'(r-1)}{n!} \end{aligned}$$

and the focus completely turns on to the hyperharmonic numbers.

Also, if we consider a particular case for the binomial coefficient, some fruitful relations appear, together with a conjecture on the differences [24]. Following the same notation as [24], we let

$$c_n = \binom{2n}{n},$$

to be the n^{th} central binomial coefficient, for any $n \geq 0$. Also, let

$$C_n = \frac{1}{n+1}c_n,$$

be the n^{th} Catalan number with $n \geq 0$.

The main concern of the paper was the p-adic order of the differences

$$c_{ap^{n+1}+b} - c_{ap^n+b} \text{ and } C_{ap^{n+1}+b} - C_{ap^n+b},$$

where a, b are integers with p being a prime number satisfying $(a, p) = 1$ and $n \geq n_k$ for some integer $n_k \geq 0$. Consequently, some identites involving these numbers were presented. For instance, one of the results which were given was as follows.

Fact 11 ([24, Theorem 2.2]). *The equality*

$$\nu_p\left(C_{ap^{n+1}} - C_{ap^n}\right) = n + \nu_p\left(\binom{2a}{a}\right)$$

holds for any integers $n, a \geq 1$ and any prime $p \geq 2$ with $(a, p) = 1$.

The identities yield the function

$$g(k) = 2\binom{2k}{k}(H_{2k} - H_k) \;\; k \geq 1,$$

which is needed to work on the p-adic order of those differences. Finally, a conjecture was proposed, which is still open:

Conjecture ([24, Conjecture 2.9]). *The inequality*

$$\nu_p(g(k)) \leq 2$$

holds for any prime $p \geq 5$ and $k \geq 1$.

In other words, for any prime $p \geq 5$ and $k \geq 1$,

$$\nu_p\left(H_{2k} - H_k\right) \leq 2$$

holds [24, Conjecture 2.10].

So, one may consider to pursue the above case about the differences as an another alternative. Finally, notice that if we let $r = k + 1$ in the function $g(k)$ and set $n = k$, we obtain that

$$g(k) = 2\binom{2k}{k}(H_{2k} - H_k) = 2\binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}) = 2h_n^{(r)} = 2h_n^{(n+1)}$$

by (14).

Thus, we are back to the hyperharmonic numbers. Finally, let us finalize the discussion with an equivalent conjecture to those above:

Conjecture 12. *Let $p \geq 5$ be a prime number. Then,*

$$\nu_p(h_n^{(n+1)}) \leq 2$$

holds for any positive integer n.

Declaration of Competing Interests The author declares that this work does not have any conflict of interest.

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ON SECOND-ORDER q-DIFFERENCE OPERATORS

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Abstract. The minimal and maximal operators defined by second-order q-difference operator are discussed in this paper. Spectrum sets of these defined operators have been determined. In addition, two extensions of the minimal operator is also mentioned.

1. Introduction

Euler [8] initiated the q-analysis in 18th cent., while Jackson [11] gave the definition of q-integral in 1910. Jackson [12] reintroduced q-derivative or $q-$difference operator as

$$D_q u(t) = \frac{u(t) - u(qt)}{(1-q)t}, \quad t \in \mathbb{K} \setminus \{0\}.$$

When the zero is an element of $\mathbb{K}$, the q-derivative, provided that it is independent of the t point, is defined for $|q| < 1$ is follows

$$D_q u(0) = \lim_{n \to +\infty} \frac{u(tq^n) - u(0)}{tq^n}, \quad t \in \mathbb{K} \setminus \{0\}.$$

q-difference operator turns into the classical derivative for $q \to 1$. Also, q-integral denoted by

$$\int_c^d u(t)\, d_q t = \int_0^d u(t)\, d_q t - \int_0^c u(t)\, d_q t, \quad 0 < c < d,$$

is given by Jackson [11] where

$$\int_0^x u(t)\, d_q t := (1-q) \sum_{n=0}^{+\infty} xq^n u(xq^n), \quad x \in \mathbb{K}$$

when two series converge. In addition, it has been proved by Bromwich [7] that the q-integral turns into a classical integral as q approaches zero in parallel with the q-derivative.

In hypergeometric functions, quantum theory, fractal geometry, the variation calculus, orthogonal polynomials and relativity theory, the q-calculus plays an unforeseen role. On addition, research in the q-calculus has been ongoing, such evidenced

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by Phillips and Aral [5, 10]. Also, many problems for second order q-difference operator are studied by many mathematicians such as [1, 2, 9, 15]. However, in our research, we have not encountered any study in terms of second-order q-difference operator, operator theory on a finite interval.

In [16], an operator T which has dense domain is said to be q-hyponormal by Ota if and only if it is ensured that $D(T) \subset D(T^*)$ and $\|T^*x\| \leqslant \sqrt{q}\|Tx\|$ for any $x \in D(T)$ with $q > 0$ and $q \neq 1$. Also, any q-hyponormal operator is closable. It can be defined an operator T as q-cohyponormal if the adjoint operator of T is q-hyponormal.

Annaby and Mansour investigated a q-analogue of Sturm-Liouville problems in $L_q^2(0,a)$, $0 < a < +\infty$ in [4]. However, they need to extend the domains of functions in $L_q^2(0,a)$ to $[0, q^{-1}a]$, because they can write the formal adjoint operator of q-difference operator as q^{-1}-difference operator. This is not necessary, since it is well known that a dense define operator has always the adjoint operator. With the same idea, a minimal operator with a definite set containing the boundary condition $u(a) = 0$ cannot be densely defined [17]. However, the definition set of the minimal operator defined by the second order expressed by the classical derivative is densely defined although it contains the same boundary condition. In some studies in the literature, the density of minimal operator domain is overlooked. For example, the minimal operator defined by the q-Sturm-Liouville expression in [3] is not dense and is not a symmetric operator since its definition set contains the condition $u(a) = 0$. However, when we look at the definition of a symmetric operator, its domain must be dense [13]. The motivation for this study is that there is some discrepancy between the results obtained and those expected according to classical theory. We address this discrepancy in this study.

In this paper we give some basic results for the q-difference operator and give the definitions of the minimal and maximal operators defined by the second order q-difference operator. Then the adjoint operators of the minimal operator is defined and the cohyponormality problem of the maximal operator is considered. In the last section the spectral problem of the minimal and maximal operators is considered. Moreover, the spectrum sets of two different closed extensions of the minimal are given.

Throughout this article, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is considered.

2. The Minimal and the Maximal Operators Definitions

In the literature, $L_q^2(0,1)$ is defined as the set of complex-valued functions defined on $[0,1]$ such that

$$\|v\|^2_{L_q^2(0,1)} = \int_0^1 |v\left(x\right)|^2 d_q x := (1-q)\sum_{k=0}^{\infty} q^k \left|v(q^k)\right|^2 < +\infty.$$

It can be easily seen that $L_q^2(0,1)$ is a linear vector space of classes $[v]$. Besides, u and v are in the same class iff $v\left(q^k\right) = u\left(q^k\right),\ k \in \mathbb{N}_0$. $L_q^2(0,1)$ is a Hilbert space and its inner product [4] is defined as

$$(u,v)_{L_q^2(0,1)} = \int_0^1 u(t)\,\overline{v(t)} d_q t.$$

Lemma 1. *If $D_q^2 u(t)$ is an element in $L_q^2(0,1)$, then the limits $\lim_{n\to+\infty} D_q u(q^n)$ and $\lim_{n\to+\infty} u(q^n)$ exist in $\mathbb{C}$.*

Proof. Suppose $D_q^2 u(t) \in L_q^2(0,1)$, since the constant function $f(t) = 1$ is an element of $L_q^2(0,1)$ then

$$\begin{aligned} \left(D_q^2 u(t), f(t)\right)_{L_q^2(0,1)} &= \int_0^1 D_q^2 u(t)\, d_q \\ &= \sum_{k=0}^{\infty} D_q u(q^n) - D_q u(q^{n+1}) \\ &= \frac{u(1)-u(q)}{1-q} - \lim_{n\to\infty} D_q u(q^n) \end{aligned}$$

is true. This means that the limit $\lim_{n\to+\infty} D_q u(q^n)$ exists. Since the sequence $\{D_q u(q^n)\}$ is bounded, from the definition of $L_q^2(0,1)$ it is obtained that $D_q u(t)$ is in $L_q^2(0,1)$. Similarly, the existence of the limit $\lim_{n\to+\infty} u(q^n)$ is also proved. □

Corollary 2. *If $D_q^2 u(t) \in L_q^2(0,1)$, then $u(t)$ and $D_q u(t)$ are elements in the Hilbert space $L_q^2(0,1)$.*

Corollary 3. *If $D_q^m u(t)$, $m \in \mathbb{N}$ is an element in $L_q^2(0,1)$, then the limits $\lim_{n\to+\infty} D_q^k u(q^n)$ exist in $\mathbb{C}$ and $D_q^k u(t) \in L_q^2(0,1)$ for $0 \le k \le m-1$.*

The operator $L_0 : D_0 \subset L_q^2(0,1) \to L_q^2(0,1)$ is defined as the form $Lu(t) = D_q^2 u(t)$ such that

$$D_0 = \left\{ u(t) \in L_q^2(0,1) : D_q^2 u(t) \in L_q^2(0,1),\ \lim_{n\to+\infty} u(q^n) = \lim_{n\to+\infty} D_q u(q^n) = 0 \right\}.$$

We call that $L_0 : D_0 \subset L_q^2(0,1) \to L_q^2(0,1)$ is the minimal operator introduced by second order q-difference derivative.

Theorem 4. *The operator $L_0 : D_0 \subset L_q^2(0,1) \to L_q^2(0,1)$ has dense domain and is closed in $L_q^2(0,1)$.*

Proof. Firstly, it is obviously seen that D_0 is dense in $L_q^2(0,1)$ because, D_0 contains the set of functions

$$\phi_n(t) := \begin{cases} \dfrac{1}{q^{\frac{n}{2}}\sqrt{1-q}}, & t = q^n \\ 0, & \textit{otherwise} \end{cases}, \quad n \in \mathbb{N}_0$$

which is an orthogonal basis of $L_q^2(0,1)$.

For the closeness of the minimal operator L_0 we suppose that $\{u_n\} \subset D_0$ such that $u_n \xrightarrow[n\to\infty]{} u$ and $L_0 u_n \xrightarrow[n\to\infty]{} g$. Then

$$\|u_n - u\|^2_{L^2_q(0,1)} = (1-q)\sum_{k=0}^{+\infty} q^k \left|u_n\left(q^k\right) - u\left(q^k\right)\right|^2 \xrightarrow[n\to\infty]{} 0.$$

From this result, we have

$$\lim_{n\to\infty} u_n\left(q^k\right) = u\left(q^k\right) \tag{1}$$

for all $k \in \mathbb{N}_0$. Because of this limit, there is an integer $n_0 \in \mathbb{N}_0$ for any $\epsilon > 0$ such that

$$\left|u_n\left(q^k\right) - u\left(q^k\right)\right| < \epsilon$$

where $n_0 \leq n$, $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$. Therefore,

$$0 \leq \left|u(q^k)\right| \leq \left|u_n(q^k) - u(q^k)\right| + \left|u_n(q^k)\right| < \epsilon + \left|u_n(q^k)\right|$$

is hold. From this relation and $\{u_n\} \subset D_0$ it is get that

$$\lim_{k\to+\infty} u\left(q^k\right) = 0.$$

Similarly, we can choose as $\epsilon = (1-q)q^{2k}$ and the following inequality

$$\begin{aligned} \left|D_q u(q^k)\right| = \left|\frac{u(q^k) - u(q^{k+1})}{(1-q)q^k}\right| &\leq \left|\frac{u_n(q^k) - u(q^k)}{(1-q)q^k}\right| + \left|\frac{u_n(q^{k+1}) - u(q^{k+1})}{(1-q)q^k}\right| + \left|D_q u_n(q^k)\right| \\ &< q^k + q^{k+2} + \left|D_q u_n(q^k)\right| \end{aligned}$$

is true. Because of this and $\left\{D_q^2 u_n\right\} \subset D_0$ is a bounded sequence,

$$\lim_{k\to+\infty} D_q u(q^k) = 0$$

is seen, and so $u \in D_0$. On the other hand, from the limit (1) and the uniqueness of the limit, it is gained that

$$\lim_{n\to+\infty} D_q^2 u_n(q^k) = D_q^2 u(q^k) = g\left(q^k\right).$$

The proof is complete with this result. □

Theorem 5. *The adjoint operator* $L_0^* : D\left(L_0^*\right) \subset L_q^2\left(0,1\right) \to L_q^2\left(0,1\right)$ *is*

$$L_0^* u\left(t\right) = \begin{cases} \frac{u(1)}{(1-q)^2} & , q < t \leqslant 1 \\ -\frac{(1+q)u(1) - u(q)}{q^2(1-q)^2} & , q^2 < t \leqslant q \\ \frac{1}{q^2} D^2_{q^{-1}} u\left(t\right) & , 0 < t \leqslant q^2 \end{cases}$$

where $D\left(L_0^*\right) = \{u(t) \in L_q^2\left(0,1\right) : \ D_q^2 u(t) \in L_q^2\left(0,1\right)\}$.

Proof. Suppose $u \in D\left(L_0\right)$ and $D_q^2 v(t) \in L_q^2\left(0,1\right)$,

$$\begin{aligned} &\left(D_q^2 u\left(t\right), v\left(t\right)\right) = \lim_{n\to+\infty} (1-q)\sum_{k=0}^{n} q^k \left(\frac{qu\left(q^k\right) - (1+q)\,u\left(q^{k+1}\right) + u\left(q^{k+2}\right)}{q(1-q)^2 q^{2k}}\right)\overline{v\left(q^k\right)} \\ &= (1-q)u(1)\frac{\overline{v(1)}}{(1-q)^2} + (1-q)q\left(u(q)\left(-\frac{\overline{(1+q)v(1) - v\left(q\right)}}{q^2(1-q)^2}\right)\right) \\ &+ (1-q)\sum_{k=2}^{\infty} q^k u\left(q^k\right)\overline{\left(\frac{1}{q^2} D^2_{q^{-1}} v\left(q^k\right)\right)} + \lim_{n\to+\infty} u\left(q^n\right)\frac{1}{q}\overline{D_q v\left(q^n\right)} - D_q u\left(q^n\right)\overline{v\left(q^n\right)} \end{aligned}$$

$$= (1-q)u(1)\overline{\frac{v(1)}{(1-q)^2}} + (1-q)q\left(u(q)\left(-\overline{\frac{(1+q)v(1)-v(q)}{q^2(1-q)^2}}\right)\right)$$
$$+ (1-q)\sum_{k=2}^{\infty} q^k u\left(q^k\right)\overline{\left(\frac{1}{q^2}D^2_{q^{-1}}v\left(q^k\right)\right)}.$$

Because the inner product definition on $L^2_q(0,1)$ and the equation

$$D^2_{q^{-1}}u(t) = q^2\frac{qu(q^{-2}t)-(1+q)u(q^{-1}t)+u(t)}{(1-q)^2t^2} = \frac{1}{q}D^2_q u\left(q^{-2}t\right), \quad 0<t\le q^2$$

is true,

$$L_0^*u(t) = \begin{cases} \frac{u(1)}{(1-q)^2} & ,q<t\leqslant 1 \\ -\frac{(1+q)u(1)-u(q)}{q^2(1-q)^2} & ,q^2<t\leqslant q \\ \frac{1}{q^2}D^2_{q^{-1}}u(t) & ,0<t\leqslant q^2 \end{cases}$$

is hold and

$$D\left(L_0^*\right) = \left\{u\in L^2_q(0,1) : D^2_q u(t)\in L^2_q(0,1)\right\}$$

is obtained. □

It can be defined $D = \left\{u\in L^2_q(0,1) : D^2_q u(t)\in L^2_q(0,1)\right\}$ and $L : D\subset L^2_q(0,1)\longrightarrow L^2_q(0,1)$, $Lu(t) = D^2_q u(t)$. We call that L is the maximal operator defined by second order q-difference derivative. The maximal operator L is closed on $L^2_q(0,1)$ from Theorem 4. It is true that $L_0\subset L$, $D(L_0^*) = D(L)$ and $D(L^*) = D(L_0)$. In addition, there are two extensions of the minimal operator L_0 different from the operator L defined as following

$$\tilde{L_1}u(t) = D^2_q u(t), \quad D\left(\tilde{L_1}\right) := \left\{u(t)\in L^2_q(0,1) : \lim_{n\to\infty} u(q^n) = 0\right\}$$

and

$$\tilde{L_2}u(t) = D^2_q u(t), \quad D\left(\tilde{L_2}\right) := \left\{u(t)\in L^2_q(0,1) : \lim_{n\to\infty} D_q u(q^n) = 0\right\}.$$

Moreover, $D\left(\tilde{L_k}^*\right) = D\left(\tilde{L_k}\right)$ is easily seen for $k=1,2$.

Corollary 6. *The operator L is a q^4-cohyponormal on $L^2_q(0,1)$.*

Proof. It can be easily seen that $D(L^*) = D_0\subset D = D(L)$ and for any $u\in D(L^*)$

$$\begin{aligned}\|Lu(t)\|^2_{L^2_q(0,1)} &= \int_0^1 \left|D^2_q u(t)\right|^2 d_qt = (1-q)\sum_{k=0}^{+\infty} q^k\left|D^2_q u\left(q^k\right)\right|^2 \\ &= (1-q)\sum_{k=0}^{+\infty} q^k\left|\frac{D_q u\left(q^k\right)-D_q u\left(q^{k+1}\right)}{(1-q)q^k}\right|^2\end{aligned}$$

and

$$\begin{aligned}\|L^*u(t)\|^2_{L^2_q(0,1)} &= (1-q)^{-1}|u(1)|^2 + q^{-3}(1-q)^{-3}|u(q)-(1+q)u(1)|^2 \\ &+ (1-q)\sum_{k=2}^{+\infty} q^k\left|\frac{1}{q^2}D^2_{q^{-1}}u\left(q^k\right)\right|^2 \\ &\geqslant (1-q)\sum_{k=2}^{+\infty} q^k\left|\frac{1}{q^2}D^2_{q^{-1}}u\left(q^k\right)\right|^2\end{aligned}$$

$$
\begin{aligned}
&= (1-q)\sum_{k=2}^{+\infty} q^k \left| \frac{1}{q^3} D_q^2 u\left(q^{k-2}\right) \right|^2 \\
&= \frac{1}{q^4}(1-q)\sum_{k=0}^{+\infty} q^k \left| D_q^2 u\left(q^k\right) \right|^2 \\
&= \frac{1}{q^4} \left\| Lu\left(t\right) \right\|^2_{L_q^2(0,1)} .
\end{aligned}
$$

This means that

$$\left\| Lu\left(t\right) \right\| \leqslant q^2 \left\| L^* u\left(t\right) \right\|, \quad u \in D(L^*)$$

and so the proof is complete. □

Remark 7. *In [17], the maximal operator introduce by first order q-difference derivative is q-cohyponormal operator in* $L_q^2(0,1)$. *Therefore, it is usual to predict that the maximal operator defined by second order q-difference derivative will be the* q^2*-cohyponormal operator. But, the maximal operator defined by the second order q-difference derivative is* q^4*-cohyponormal. This is because a consequence of the equation* $D_q D_{q^{-1}} = \frac{1}{q} D_{q^{-1}} D_q$.

3. Spectrum Sets of the Minimal and Maximal Operators

Now the spectrum problem, which is the important problem of operators, is discussed for the minimal and maximal operators that we defined in the previous section.

Theorem 8. *The continuous and residual spectrum sets of* L_0 *defined by second order* $q-$*difference derivative are*

$$\sigma_r\left(L_0\right) = \sigma_c\left(L_0\right) = \emptyset.$$

Proof. Let $\lambda^2 \in \mathbb{C} \setminus \sigma_p\left(L_0\right)$ and solve the following problem with the boundary value

$$\begin{cases} \left(L_0 - \lambda^2 E\right) u\left(t\right) = f\left(t\right) \\ \lim\limits_{n\to+\infty} u\left(q^n\right) = \lim\limits_{n\to+\infty} D_q u\left(q^n\right) = 0 \end{cases}$$

It can be written

$$\begin{cases} \left(D_q - \lambda E\right)\left(D_q + \lambda E\right) u\left(t\right) = f\left(t\right) \\ \lim\limits_{n\to+\infty} u\left(q^n\right) = \lim\limits_{n\to+\infty} D_q u\left(q^n\right) = 0 \end{cases}$$

Because $\lambda \neq \pm\lambda\left(1-q\right)q^m$, $m \in \mathbb{N}_0$ and Theorem 3.2 proof in [17] the function $g(t)$ exists such that $(D_q + \lambda E)g(t) = f(t)$,

$$
\begin{aligned}
g\left(q^{k+1}\right) \quad &= \left(\prod_{n=0}^{k}\left(1+\lambda\left(1-q\right)q^n\right)\right) g\left(1\right) \\
&- \left(1-q\right)\left(\prod_{n=1}^{k}\left(1+\lambda\left(1-q\right)q^n\right)\right) f\left(1\right) \\
&- \left(1-q\right)\left(\prod_{n=2}^{k}\left(1+\lambda\left(1-q\right)q^n\right)\right) q f\left(q\right)
\end{aligned}
$$

$$-\ldots-(1-q)\left[\left(\prod_{n=k-1}^{k}\left(1+\lambda\left(1-q\right)q^{n}\right)\right)q^{k-1}f\left(q^{k-1}\right)+q^{k}f\left(q^{k}\right)\right]$$

$k\in\mathbb{N}_0$ and $\lim\limits_{n\to+\infty}g\left(q^{n}\right)=0$. From the same reasons there exists a function $u(t)$ the following:

$$\begin{aligned}u\left(q^{k+1}\right) &= \left(\prod_{n=0}^{k}\left(1-\lambda\left(1-q\right)q^{n}\right)\right)u\left(1\right)\\ &-\left(1-q\right)\left(\prod_{n=1}^{k}\left(1-\lambda\left(1-q\right)q^{n}\right)\right)g\left(1\right)\\ &-\left(1-q\right)\left(\prod_{n=2}^{k}\left(1-\lambda\left(1-q\right)q^{n}\right)\right)qg\left(q\right)\\ &-\ldots-(1-q)\left[\left(\prod_{n=k-1}^{k}\left(1-\lambda\left(1-q\right)q^{n}\right)\right)q^{k-1}f\left(q^{k-1}\right)+q^{k}g\left(q^{k}\right)\right]\end{aligned}$$

$k\in\mathbb{N}_0$ and $\lim\limits_{n\to+\infty}u\left(q^{n}\right)=\lim\limits_{n\to+\infty}D_qu\left(q^{n}\right)=0$. Thus, the proof is finished.

□

Theorem 9. *The minimum and maximal operators point spectrum sets are of the following forms*

$$\sigma_p\left(L_0\right)=\left\{\frac{1}{\left(1-q\right)^2q^{2k}}:k\in\mathbb{N}_0\right\},\quad \sigma_p\left(L\right)=\mathbb{C}.$$

Proof. Suppose that $\lambda^2\in\sigma_p\left(L_0\right)$. In this case, a nonzero element $u\left(t\right)$ exists in D_0 and

$$L_0u\left(t\right)=\lambda^2u\left(t\right).$$

Therefore, from [4,6]

$$(D_q-\lambda)(D_q+\lambda)u(t)=0,\quad u(t)\in D_0.$$

Because of this,

$$\frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{\left(1-q\right)q^k}=\lambda u\left(q^k\right)$$

or

$$\frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{\left(1-q\right)q^k}=-\lambda u\left(q^k\right)$$

$k\in\mathbb{N}_0$. From the last equations,

$$u\left(q^{k+1}\right)=c_1\left(\prod_{n=0}^{k}\left(1-\left(1-q\right)\lambda q^{n}\right)\right)+c_2\left(\prod_{n=0}^{k}\left(1+\left(1-q\right)\lambda q^{n}\right)\right),\ k\in\mathbb{N}_0$$

is hold where $c_1\neq0$ or $c_2\neq0$. Because of $u\in D_0$, $u\left(q^k\right)\underset{k\to+\infty}{\rightarrow}0$ and $D_qu\left(q^k\right)\underset{k\to+\infty}{\rightarrow}0$, it is true that

$$c_1\left(\prod_{n=0}^{\infty}\left(1-\left(1-q\right)\lambda q^{n}\right)\right)+c_2\left(\prod_{n=0}^{\infty}\left(1+\left(1-q\right)\lambda q^{n}\right)\right)=0,$$

$$\lambda c_1\left(\prod_{n=0}^{\infty}\left(1-\left(1-q\right)\lambda q^n\right)\right)-\lambda c_2\left(\prod_{n=0}^{\infty}\left(1+\left(1-q\right)\lambda q^n\right)\right)=0.$$

From this, it must be $c_1=0$ or $c_2=0$. In this case,

$$\prod_{n=0}^{\infty}\left(1-\left(1-q\right)\lambda q^n\right)=0$$

or

$$\prod_{n=0}^{\infty}\left(1+\left(1-q\right)\lambda q^n\right)=0$$

iff there is $m\in\mathbb{N}_0$ and

$$1-\lambda\left(1-q\right)q^m=0$$

or

$$1+\lambda\left(1-q\right)q^m=0$$

[14]. Therefore, it is get that $\lambda^2=\frac{1}{(1-q^2)q^{2m}}$, $m\in\mathbb{N}_0$ i.e.

$$\sigma_p\left(L_0\right)=\left\{\frac{1}{\left(1-q\right)^2q^{2k}}:k\in\mathbb{N}_0\right\}$$

is gotten.

Since there are no boundary conditions, the elements defined as

$$u\left(q^{k+1}\right)=c_1\left(\prod_{n=0}^{k}\left(1-\left(1-q\right)\lambda q^n\right)\right)+c_2\left(\prod_{n=0}^{k}\left(1+\left(1-q\right)\lambda q^n\right)\right),\ k\in\mathbb{N}_0$$

is eigenvector of L for any $\lambda^2\in\mathbb{C}$. Thence, $\sigma_p\left(L\right)=\mathbb{C}$ is true.

□

Corollary 10. *The following relation*

$$\sigma\left(L_0\right)=\sigma_p\left(L_0\right)=\left\{\frac{1}{\left(1-q\right)^2q^{2k}}:k\in\mathbb{N}_0\right\}$$

is hold.

Corollary 11. *The spectrum sets of the operators* $\tilde{L}_i : D(\tilde{L})\subset L_q^2\left(0,1\right)\to L_q^2\left(0,1\right)$, $i=1,2$ *are*

$$\sigma_p\left(\tilde{L}_i\right)=\mathbb{C}.$$

Theorem 12. *The spectrum set of* L_0^* *is equal to only the point spectrum and*

$$\sigma_p\left(L_0^*\right)=\left\{\frac{1}{\left(1-q\right)^2q^{2k}}:k\in\mathbb{N}_0\right\}.$$

Proof. Suppose that λ^2 is an eigenvalue of the adjoint operator L_0^*. In this case, there is a nonzero function $u(t)$ in $D(L_0^*)$ such that

$$L_0^*u\left(t\right)=\lambda^2u\left(t\right).$$

From here,

$$\frac{1}{\left(1-q\right)^2}u\left(1\right)=\lambda^2u\left(1\right),$$

$$\frac{u(q)-(1+q)u(1)}{q^2(1-q)^2}u\left(1\right)=\lambda^2 u\left(q\right),$$
$$\left(-\frac{1}{q}D_{q^{-1}}-\lambda\right)\left(-\frac{1}{q}D_{q^{-1}}+\lambda\right)u(t)=0,\quad 0<t\leq q^2.$$

If $u(1)\neq 0$, then $\lambda^2=\frac{1}{(1-q)^2}$ and

$$u(q)=\frac{1}{1-q}u(1),$$
$$u\left(q^k\right)=c_1\prod_{j=2}^{k}\left(1-q^j\right)^{-1}+c_2\prod_{j=2}^{k}\left(1+q^j\right)^{-1},\ k\geqslant 2$$

where $\frac{1+q}{1-q^2}u(q)-\frac{q}{1-q^2}u(1)=\frac{c_1}{1-q^4}+\frac{c_2}{1-q^4}$. In the same idea, if $u(1)=\cdots=u(q^{m-1})=0$, $m\geqslant 1$ and $u(q^m)\neq 0$, then $\lambda^2=(1-q)^2q^{2m}$ and

$$u\left(q^k\right)=c_1\prod_{j=m+1}^{k}\left(1-q^{j-m}\right)^{-1}+c_2\prod_{j=m+1}^{k}\left(1+q^{j-m}\right)^{-1},\ m\in\mathbb{N}.$$

Since there is not any boundary condition, $u(t)$ is an eigenvector of the adjoint operator L_0^*. As a result, the set

$$\sigma_p\left(L_0^*\right)=\left\{\frac{1}{(1-q)^2q^{2k}}:k\in\mathbb{N}_0\right\}$$

is gotten. □

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BIAS CORRECTED MAXIMUM LIKELIHOOD ESTIMATORS FOR THE PARAMETERS OF THE GENERALIZED NORMAL DISTRIBUTION

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Abstract. The generalized normal (GN) distribution was defined as a generalization of the normal, Laplace, and uniform distributions, with extensive application areas modeling different data settings. At the same time, its maximum likelihood estimators (MLEs) are biased in finite samples. Since such biases may affect the accuracy of estimates, we consider constructing unbiased estimators for unknown parameters of GN distribution. This article adopts the bias-corrected approach, following the analytical methodology suggested by Cox and Snell [1]. Additionally, we explore both regular biases and parametric Bootstrap bias correction techniques. A comprehensive Monte Carlo simulation is conducted to compare the performances of these estimators in estimating GN parameters. Finally, a real data example is presented to illustrate the application of methods.

1. Introduction

It is well-known that the most popular distribution is the normal distribution, widely used due to its tractability and extensive application areas. However, it is quite common to encounter non-normality in real-world examples. Distributions like the Laplace distribution can handle non-normality, for instance, in modeling speech signals. Moreover, a more flexible generalized normal (GN) distribution can contain both the normal and Laplace distributions. The GN distribution has defined a generalization of the normal, Laplace, and uniform distributions, providing

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extensive application fields and enabling the modeling of various datasets. Some researchers used the GN distribution in their studies. Some researchers have utilized the GN distribution in their studies. For instance, Briassouli et al. [2] used GN distribution to add watermarks to images; Kokkinakis and Nandi [3] modeled speech signals with the GN distribution; atmospheric noise, Sharifi and Leon-Garcia [4] considered GN distribution for subband coding of audio and video signals; Choi et al. [5] applied in impulsive noise, the direction of arrival, modeling of the independent component analysis; and Wu and Principe [6] proposed to use GN distribution for blind signal separation.

There are different types of GN distribution in literature. This study considers Version 1, a parametric family of symmetrical distributions known as exponential power distribution or generalized error distribution. Subbotin [7] first proposed the exponential power distribution, and later Nadarajah [8] renamed it the GN distribution. Several studies have focused on the parameter estimation of the GN distribution. Notable contributions include those by Varanasi and Aazhang [9], Nadarajah [8], and Roenko et al. [10]. More recently, Eskin [11] and Eskin and Doğru [12] proposed methods for parameter estimation in joint location and scale models of the GN distribution.

When estimating parameters from any probability distribution, the choice of estimation methodology is very important. Among all the classical estimation methods, the most frequently used method is the maximum likelihood estimation (MLE) due to its several attractive properties. For instance, ML estimators are asymptotically unbiased, consistent, and asymptotically normally distributed. However, most of these properties depend on the large sample size condition. Therefore, properties such as unbiasedness may not hold for small or moderate sample sizes.

The primary objective of this article is to develop modified MLEs that are nearly unbiased, with a particular focus on obtaining second-order unbiasedness. To achieve this, we focus on two different approaches.

First, we propose bias-corrected MLEs (BCEs) for the parameters of GN distribution, following the methodology introduced by [1]. This method corrects the bias by subtracting the estimated bias from the original MLEs.

Next, we consider the parametric bootstrap-based bias-correction approach introduced by Efron [13], with further details provided by Efron and Tibshirani [14]. This estimator, referred to as the bootstrap bias-corrected estimator (PBE), applies bias correction numerically, without needing an analytical bias expression.

The bias-correction technique has been extensively applied to various distributions in the literature. For instance, Cordeiro et al. [15] applied it to the Beta distribution, while Saha and Paul [16] utilized it for the negative binomial distribution. It was also used by Lemonte et al. [17] for the Birnbaum-Saunders distribution and by Giles and Feng [18] for the Gamma distribution. Other applications include the Kumaraswamy distribution by Lemonte [19], the Topp-Leone distribution by Giles [20], the Lomax distribution by Giles et al. [21], and the Nakagami distribution by Schwartz et al. [22]. Zhang and Liu [23] applied the technique to the skew-normal distribution, while Schwartz and Giles [24] focused on the zero-inflated Poisson distribution. Wang and Wang [25] extended it to the weighted Lindley distribution, and Reath et al. [26] used it for the log-logistic distribution. Further examples include the generalized half-normal distribution by Mazucheli and Dey [27], the unit-Gamma distribution by Mazucheli et al. [28], the inverse Weibull distribution by Mazucheli et al. [29], the Johnson S_B distribution by Menezes and Mazucheli [30], and the unit-Weibull distribution by Menezes et al. [31].

The article is designed in this manner. Section 2 defines the GN distribution and outlines its key distributional properties. Section 3 introduces the bias-corrected approach for deriving MLEs that are bias-free to the second error, along with the MLE and PBE methods. Section 4 conducts a Monte Carlo simulation to compare these methods, supplemented by real data for practical illustration. Finally, Section 5 concludes the article.

2. Generalized Normal Distribution

Let X be a GN-distributed random variable with location parameter μ, scale parameter σ, and the shape parameter s as considered in [8]. The probability density function (pdf) of GN distribution is defined as:

$$f(x) = \frac{s}{2\sigma\Gamma(1/s)} exp\left\{-\left|\frac{x-\mu}{\sigma}\right|^{s}\right\}, x\in\mathbb{R}, \mu\in\mathbb{R},\ \sigma\in\mathbb{R}^{+},\ \ s\in\mathbb{R}^{+}. \tag{1}$$

It is known that the pdf given in 1 has two special cases in terms of s values. When s equals 1, the distribution reduces to the Laplace distribution. Further, when s equals 2, the distribution is normal. Figure 1 shows the different pdf graphs of the GN distribution. We can see from Figure 1 that the distribution is leptokurtic and heavy-tailed for small values of shape parameter s. The pdf is the bell-shaped curve for certain values of s, and the pdfs have the peaky shape of maximum. It can also be observed from the figure that the GN distribution is symmetric around the location parameter, and varying the shape and scale parameters allows for different types of pdfs ([9]). This flexibility gives the GN distribution a wide range of tail behavior, from thinner to thicker tails compared to the normal distribution.

The cumulative distribution function (cdf) of the GN distribution, as given in [8], can be written as:

$$F(x) = \frac{\mathit{\Gamma}\left(1/s, \left((\mu-x)/\sigma\right)^{s}\right)}{2\mathit{\Gamma}(1/s)},\ \ for\ x\leq\mu, \tag{2}$$

$$F(x) = 1-\frac{\mathit{\Gamma}\left(1/s, \left((\mu-x)/\sigma\right)^{s}\right)}{2\mathit{\Gamma}(1/s)},\ \ for\ x>\mu, \tag{3}$$

where $\mathit{\Gamma}(\alpha, x) = \int_x^\infty t^{\alpha-1}\exp(-t)\,dt$ shows the incomplete gamma function. The n^{th} moment of the GN distribution about the origin for each positive n integer was

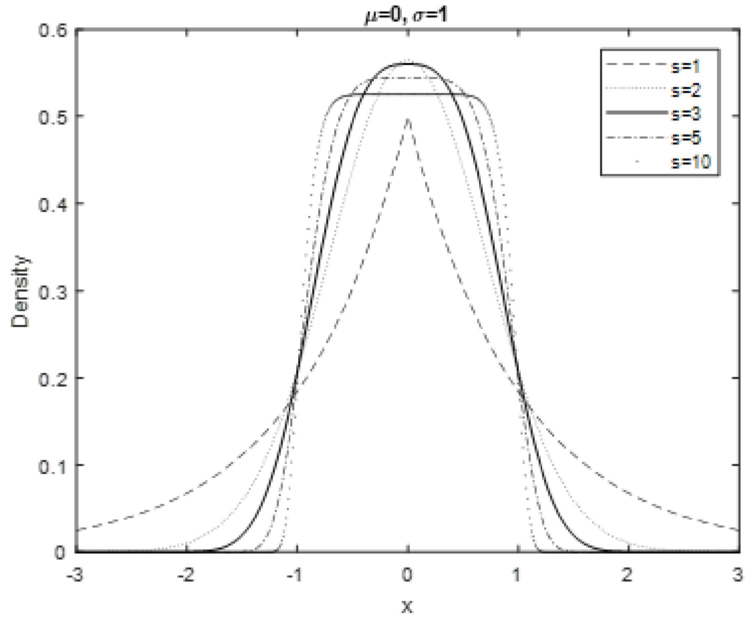

FIGURE 1. Some pdf examples of the GN distribution for $\mu = 0, \sigma = 1$ and different parameter values of s

introduced by [8] to obtain some distributional measures of the GN distribution. This moment is:

$$E\left(X^n\right) = \frac{\mu^n \sum_{k=0}^{n} \binom{n}{k} \left(\sigma/\mu\right)^k \left\{1 + (-1)^k\right\} \Gamma\left((k+1)/s\right)}{2\Gamma\left(1/s\right)}. \tag{4}$$

Further, the expectation, variance, kurtosis, and skewness of the GN distribution can be obtained with the help of the n^{th} moment of the GN distribution given in 5:

$$E\left(X\right) = \mu, \quad Var\left(X\right) = \frac{\sigma^2 \Gamma\left(3/s\right)}{\Gamma\left(1/s\right)},$$

$$Skewness = 0, Kurtosis = \frac{\Gamma\left(1/s\right)\Gamma\left(5/s\right)}{\Gamma^2\left(\frac{3}{s}\right)}. \tag{5}$$

It can be observed that the center of the distribution is μ and the skewness is zero. The variance and kurtosis are related to the parameter s, their values change concerning s.

3. Parameter Estimation

This section presents the ML estimation, along with Cox-Snell bias-corrected, and bootstrap-based bias-corrected inferences for the location and scale parameters of the GN distribution. To simplify computations, the shape parameter s will be estimated using the ML method across all estimation techniques.

3.1. **ML Estimation.** Let $x = x_1, x_2, \ldots, x_n$ be a random sample of size n from a GN distribution with parameter vector $\theta = (\mu, \sigma, s)$. The log-likelihood function for this sample can be written as:

$$l(\theta|x) = nlog\left(\frac{s}{2\sigma\Gamma(1/s)}\right) - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\sigma}\right|^{s}. \tag{6}$$

The maximum likelihood estimates for the parameters μ, σ, and s, $\widehat{\mu}$, $\widehat{\sigma}$, and $\hat{s}$ respectively, can be obtained by the maximization of 6, or equivalently solving the following nonlinear equations:

$$\frac{\partial l}{\partial \mu} = \frac{s}{\sigma^s}\left\{\sum_{x_i \geq \mu}(x_i-\mu)^{s-1} - \sum_{x_i<\mu}(\mu - x_i)^{s-1}\right\}, \tag{7}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{s}{\sigma^{s+1}}\sum_{i=1}^{n}|x_i-\mu|^{s}, \tag{8}$$

$$\frac{\partial l}{\partial s} = \frac{n}{s}\left\{\frac{1}{s}\Psi\left(\frac{1}{s}\right)+1\right\} - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\sigma}\right|\log\left|\frac{x_i-\mu}{\sigma}\right|. \tag{9}$$

We note that solving these nonlinear equations requires the use of numerical methods due to their complexity. Some numerical optimization algorithms can be employed to find the related ML estimates.

3.2. **Cox-Snell Bias-Corrected ML Estimation.** Let $l(\theta)$ be the log-likelihood function based on a sample of n observations and p-dimensional parameter vector θ. $l(\theta)$ is assumed to be regular, meaning that all derivatives up to and including the third order exist and are continuous. Here, we first estimate the parameter s using the ML estimation method. Then, for a given estimate $\hat{s}$, the joint cumulants of derivates of $l(\theta^*)$ are defined as follows, where $\theta^* = (\mu, \sigma)$:

$$\kappa_{ij} = E\left(\frac{\partial^2 l}{\partial\theta_i^*\partial\theta_j^*}\right), \qquad i,j = 1,2,\ldots,p, \tag{10}$$

$$\kappa_{ijl} = E\left(\frac{\partial^3 l}{\partial\theta_i^*\partial\theta_j^*\partial\theta_l^*}\right), \qquad i,j,l = 1,2,\ldots,p, \tag{11}$$

$$\kappa_{ij,l} = E\left(\frac{\partial^3 l}{\partial\theta_i^*\partial\theta_j^*\partial\theta_l^*}\right)E\left[\left(\frac{\partial^2 l}{\partial\theta_i^*\partial\theta_j^*}\right)\left(\frac{\partial l}{\partial\theta_l^*}\right)\right], \qquad i,j,l = 1,2,\ldots,p. \tag{12}$$

The derivates of the second-order cumulants are denoted as follows:

$$\kappa_{ij}^{(l)} = \frac{\partial \kappa_{ij}}{\partial \theta_l^*}, \qquad i, j, l = 1, 2, \ldots, p. \tag{13}$$

All of the expressions in 10 to 13 are assumed to be of the order $O_{(n)}$. [1] showed that when the sample data are independent (but not necessarily identically distributed), the bias of the r^{th} element of the ML of θ^*, denoted as $\widehat{\theta}^*$, can be expressed as:

$$Bias\left(\widehat{\theta}_r^*\right) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} \kappa^{ri}\kappa^{jl}\left[0.5k_{ijl} + k_{ij,l}\right] + \mathcal{O}\left(n^{-2}\right), \tag{14}$$

where $r = 1, \ldots, p$, κ_{ijl} is the $(i,j)^{th}$ element of the inverse of the information matrix denoted as $K = \{-\kappa_{ij}\}$. Corderio and Klein [32] noted that even when all equations in 10 to 13 are of the order $O_{(n)}$, Eq. 14 still holds if the data are non-independent. Eq. 14 can be rewritten in the following form:

$$Bias\left(\widehat{\theta}_r^*\right) = \sum_{r=1}^{p} \kappa^{ri} \sum_{j=1}^{p}\sum_{l=1}^{p}\left[\kappa_{ij}^{(l)} - \frac{1}{2}\kappa_{ijl}\right] + \mathcal{O}\left(n^{-2}\right), \qquad r = 1, 2, \ldots, p. \tag{15}$$

Now, let $a_{ij}^{(l)} = \kappa_{ij}^{(l)} - (\kappa_{ijl}/2)$, for $i, j, l = 1, 2, \ldots, p$ and define the following matrices:

$$A^{(l)} = \left\{a_{ij}^{(l)}\right\}, \qquad i, j, l = 1, 2, \ldots, p, \tag{16}$$

$$A = \left[A^{(1)}|A^{(2)}|\cdots|A^{(p)}\right]. \tag{17}$$

They also showed that the $\mathcal{O}\left(n^{-1}\right)$ bias of the MLE of θ^* in Eq.15 can be re-expressed as:

$$Bias\left(\widehat{\theta}_r^*\right) = K^{-1}Avec\left(K^{-1}\right) + \mathcal{O}\left(n^{-2}\right). \tag{18}$$

Then, the BCE for θ^* can be obtained as:

$$\widehat{\theta}_{r(BCE)}^* = \widehat{\theta}_r^* - \hat{K}^{-1}\hat{A}vec\left(\hat{K}^{-1}\right), \tag{19}$$

where $\hat{K} = K|_{\widehat{\theta}^*}$ and $\hat{A} = A|_{\widehat{\theta}^*}$, and it can be shown that the bias of $\widetilde{\theta}^*$ will be $\mathcal{O}\left(n^{-2}\right)$.

3.3. **Some Inferential Aspects.** To proceed, we require the derivates of the log-likelihood function up to the third order. The derivates can be obtained as:

$$\begin{aligned}
\frac{\partial^2 l}{\partial \mu^2} =& -\frac{s(s-1)}{\sigma^2}\left|\frac{x-\mu}{\sigma}\right|^{s-2} \\
\frac{\partial^3 l}{\partial \mu^3} =& \frac{s(s-1)(s-2)}{\sigma^3}\left|\frac{x-\mu}{\sigma}\right|^{s-3} \\
\frac{\partial^2 l}{\partial \sigma^2} =& \frac{1}{\sigma^2}\left\{1-s(s+1)\left|\frac{x-\mu}{\sigma}\right|^{s}\right\} \\
\frac{\partial^3 l}{\partial \sigma^3} =& -\frac{1}{\sigma^3}\left\{2-s(s+1)(s+2)\left|\frac{x-\mu}{\sigma}\right|^{s}\right\} \\
\frac{\partial^2 l}{\partial \mu \partial \sigma} =& \frac{s}{\sigma^2} sign(\mu-x)\left|\frac{x-\mu}{\sigma}\right|^{s-1} \\
\frac{\partial^3 l}{\partial \mu \partial \sigma^2} =& -\frac{(s+1)s^2}{\sigma^3} sign(\mu-x)\left|\frac{x-\mu}{\sigma}\right|^{s-1} \\
\frac{\partial^3 l}{\partial \mu^2 \partial \sigma} =& -\frac{s^2(s-1)}{\sigma^3} sign(\mu-x)\left|\frac{x-\mu}{\sigma}\right|^{s-2}.
\end{aligned}$$

The joint cumulants of the derivates of the log-likelihood function are found as follows:

$$\begin{aligned}
\kappa_{11} =& E\left[\partial^2 l/\partial \mu^2\right] = -\frac{s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^2\Gamma\left(\frac{1}{s}\right)} \\
\kappa_{12} =& \kappa_{21} = E\left[\partial^2 l/\partial \mu \partial \sigma\right] = 0 \\
\kappa_{22} =& E\left[\partial^2 l/\partial \sigma^2\right] = -\frac{s}{\sigma^2} \\
\kappa_{111} =& E\left[\partial^3 l/\partial \mu^3\right] = \frac{s(s-1)(s-2)}{\sigma^3\Gamma\left(\frac{1}{s}\right)} \\
\kappa_{112} =& \kappa_{121} = \kappa_{211} = E\left[\partial^3 l/\partial \mu^2 \partial \sigma\right] = \frac{s^2(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)} \\
\kappa_{222} =& E\left[\partial^3 l/\partial \sigma^3\right] = \frac{s(s+3)}{\sigma^3} \\
\kappa_{122} =& \kappa_{212} = \kappa_{221} = E\left[\partial^3 l/\partial \sigma^2 \partial \mu\right] = \frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}.
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\kappa_{11}^{(1)} =& \partial \kappa_{11}/\partial \mu = 0 \\
\kappa_{12}^{(1)} =& \partial \kappa_{12}/\partial \mu = 0 \\
\kappa_{22}^{(1)} =& \partial \kappa_{22}/\partial \mu = 0
\end{aligned}$$

$$\kappa_{11}^{(2)} = \partial\kappa_{11}/\partial\sigma = \frac{2s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}$$
$$\kappa_{12}^{(2)} = \partial\kappa_{12}/\partial\sigma = 0$$
$$\kappa_{22}^{(2)} = \partial\kappa_{22}/\partial\sigma = \frac{2s}{\sigma^3}.$$

So, we obtain the elements of $A^{(1)}$:

$$a_{11}^{(1)} = \kappa_{11}^{(1)} - 0.5\kappa_{111} = -0.5\left[\frac{s(s-1)(s-2)\Gamma\left(\frac{s-2}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right]$$
$$a_{12}^{(1)} = \kappa_{12}^{(1)} - 0.5\kappa_{121} = -0.5\left[\frac{s^2(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right]$$
$$a_{22}^{(1)} = \kappa_{22}^{(1)} - 0.5\kappa_{122} = -0.5\left[\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right].$$

The elements of $A^{(2)}$ are:

$$a_{11}^{(2)} = \kappa_{11}^{(2)} - 0.5\kappa_{112} = \frac{s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\left[2-\frac{s}{2}\right]$$
$$a_{12}^{(2)} = \kappa_{12}^{(2)} - 0.5\kappa_{122} = 0 - 0.5\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}$$
$$a_{22}^{(2)} = \kappa_{22}^{(2)} - 0.5\kappa_{222} = \frac{s^2}{\sigma^3}\left[2-\frac{(s+3)}{2}\right].$$

Finally, the information matrix yields as:

$$K = \{-\kappa_{ij}\} = n\begin{bmatrix} -\frac{s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^2\Gamma\left(\frac{1}{s}\right)} & 0 \\ 0 & -\frac{s}{\sigma^2}\end{bmatrix}.$$

Let define $A = \left[A^{(1)}|A^{(2)}\right]$. Then, we have:

$$A = n\begin{bmatrix} -0.5\left[\frac{s(s-1)(s-2)\Gamma\left(\frac{s-2}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & -0.5\left[\frac{s^2(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & \frac{s(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\left[2-\frac{s}{2}\right] & -0.5\left[\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] \\ -0.5\left[\frac{s^2(s-1)\Gamma\left(\frac{s-1}{s}\right)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & -0.5\left[\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & -0.5\left[\frac{s^2(s+1)}{\sigma^3\Gamma\left(\frac{1}{s}\right)}\right] & \frac{s^2}{\sigma^3}\left[2-\frac{(s+3)}{2}\right]\end{bmatrix}$$

Using Corderio and Klein's [32] modification of Coxx and Snell's [1]) method, we can write the bias of $\widehat{\theta}_r^*$ in the following way:

$$Bias\left(\widehat{\theta}_r^*\right) = Bias\begin{pmatrix}\widehat{\mu} \\ \widehat{\sigma}\end{pmatrix} = K^{-1}Avec\left(K^{-1}\right),$$

where $\hat{K} = K|_{\mu=\widehat{\mu};\sigma=\widehat{\sigma}}$ and $\hat{A} = A|_{\mu=\widehat{\mu};\sigma=\widehat{\sigma}}$.

3.4. **Bootstrap-Based Bias Corrected ML Estimation.** An alternative approach that we consider to derive bias-corrected MLEs for the unknown parameters of GN distribution is a bootstrap resampling method by [13]. The bootstrap method uses the MLEs of the data to generate random samples from GN distribution to estimate the bias, and then subtract the bias from the MLE. For a parameter vector θ^*, the estimated bias of $\widehat{\theta}$ is given by:

$$Bias\left(\widehat{\theta}^*_r\right) = \frac{1}{B}\sum_{j=1}^{B}\widehat{\theta}^*_{r(j)} - \widehat{\theta}^*_r, \tag{20}$$

where $\widehat{\theta}^*_{r(j)}$ is the MLE of θ^* obtained from the *j-th* Bootstrap sample. Hence, the bootstrap bias-corrected estimator is defined as:

$$\widehat{\theta}^*_{r(PBE)} = 2\widehat{\theta}^*_r - \frac{1}{B}\sum_{j=1}^{B}\widehat{\theta}^*_{r(j)}. \tag{21}$$

4. Applications

This section includes a simulation study and a real data example to demonstrate the proposed estimators' performance for the GN distribution's location and scale parameters. Some computational aspects are provided below:

Computational details:

1) We used MATLAB R2017b software for all numerical calculations.

2) The *Nelder-Mead* algorithm for the *fminsearch* function in MATLAB was employed to obtain the ML estimates.

3) For generating the random sample from the GN distribution in the simulation study, we followed the random number-generating algorithm outlined below:

Random number generating algorithm from GN distribution:

Step 1. Sample $X \sim Gamma\,(1/s, 1)$.

Step 2. Generate a random sample from the independent random variable Z:

$$Z \sim \frac{1}{2}\left[Z = -1\right] + \frac{1}{2}\left[Z = 1\right].$$

Step 3. Generate a random sample from the $GN\,(\mu, \sigma, s)$ with the help of the following equation:

$$Y = \mu + \sigma Z|X|^{1/s} \sim GN\,(\mu, \sigma, s)\,.$$

4.1. **Simulation study.** This section presents the results of a Monte Carlo simulation study that compares the performances of the MLE, BCE, and PBE. This evaluation is based on biases and mean squared error (MSE) values, calculated using the following formulas:

$$\widehat{bias}\left(\widehat{\theta}\right) = \overline{\theta} - \theta, \quad \widehat{MSE}\left(\widehat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta\right)^2,$$

where θ is the true parameter value, $\widehat{\theta}_j$ is the estimate of θ for the ith simulated dataset and $\overline{\theta} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\theta}_j$. The simulation studies are carried out $N = 10000$ times with sample sizes n=10, 20, 30, 40, and 50. The true parameters are μ= 1 and 2, s=2, 3 and 5, and σ=0.5, 1, 1.5, 2, 2.5, and 3. Note that the ML estimation method is employed for estimating the shape parameter s in all scenarios. The simulation focuses on comparing the location and scale parameters of the GN distribution.

To assess and compare the performances of the estimators, we employed the integrated bias squared (IBSQ) and average root mean square error (ARMSE) criteria, as introduced by Cribari-Neto and Vasconcellos [33]. These criteria are defined as follows:

$$IBSQ_{(n)} = \sqrt{\frac{1}{36}\sum_{h=1}^{36}(r_{h,n})^2}, \quad ARMSE_{(n)} = \frac{1}{36}\sum_{h=1}^{36}RMSE_{h,n},$$

where $r_{h,n}$ and $RMSE_{h,n}$ present the estimated bias and estimated root mean squared error for the h-th scenario, $h = 1, \ldots, 36$. These criteria provide comprehensive measures for the overall performance of the estimators across a range of scenarios.

Simulation results:

The accuracy of the parameter estimates is evaluated by reporting bias and MSE values in Tables 1 - 6. The Monte Carlo experiments involve 1000 bootstrap replications. The results, show that, for the location parameter μ, PBE yields the smallest absolute biases, indicating that the bootstrap correction effectively reduces bias. BCE achieves the smallest MSE values, suggesting that the Cox-Snell correction improves the estimator's precision. MLE and PBE demonstrate similar performances according to MSE criteria for μ, and all estimates show consistency as MSE values decrease with increasing sample sizes. Regarding the scale parameter σ, BCE consistently provides the best results in terms of absolute bias in most simulation cases, with PBE following closely. Moreover, BCE outperforms other estimation methods in terms of MSE values for σ. Similar to μ, all estimates exhibit consistency for σ as indicated by decreasing MSE values with increasing sample sizes.

The results are presented in Tables 7 and 8, which include IBSQ and ARMSE values for different sample sizes. Upon examination of these tables, it is noted that for the parameter μ, IBSQ values display similarity among the estimators. However, ARMSE values highlight the superior performance of BCE. Additionally, concerning the parameter σ, both IBSQ and ARMSE values indicate that BCE and

PBE outperform MLE. Consequently, while BCE and PBE methods demonstrate similar performance, BCE is computationally more straightforward than PBE.

4.2. **Real Data Application.** This section investigates the analysis of a real dataset to illustrate the efficacy of the proposed BCE in comparison to MLE for the parameters of the GN distribution. The dataset relates to hurricanes in the Atlantic, USA, sourced from the Atlantic track files maintained by the US National Hurricane Center. Covering major storm events from 1851 to 2000, this dataset characterizes the weeks of the hurricane season. The hurricane season, as reported by the National Hurricane Center, commences on June 1, and the hurricane dates are transformed into "weeks of the season". These weeks are aggregated in 2-week intervals, starting from June 1-14 (weeks 1 and 2) and concluding in early January (weeks 31 and 32). The dataset covers a total of 755 weeks of the season, and it can be accessed through the link: `https://seattlecentral.edu/qelp/sets/070/070.html#About`.

In this part, we compare the performances of the estimation methods discussed in this paper by applying them to the hurricane dataset. To evaluate the goodness of fit of each estimation method for the given dataset, we utilize the Kolmogorov-Smirnov (KS) test statistics. The steps for calculating the KS statistics are summarized as follows:

- Sort the dataset in descending order.
- Calculate the maximum absolute difference:

$$D = \max_{i=1,2\ldots,n} \{|F_n(x) - F(x)|\} \tag{22}$$

  where, $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \le x}$ is the empirical cdf and $F(x)$ the theoretical cdf of the distribution being tested.
- The smallest value of D indicates the best fit between the theoretical distribution and the observed data.

The estimation results for MLE, BCE, and PBE are summarized in Table 9. Notably, the shape parameter (s) is estimated as 1.1084 using the ML estimation method. Alternatively, we can estimate the shape parameter s using the profile likelihood estimation method. To illustrate this, we present the profile log-likelihood graph for different values of s in Figure 2. From this figure, we observed that the estimated s value is about 1.0, which aligns closely with the ML estimate of s.

Table 9 provides the KS values, estimates, and bootstrap standard errors in brackets. The results indicate that MLE and BCE yield similar results for the parameter μ. Conversely, PBE demonstrates superior performance for estimating the parameter σ. This suggests that, in the context of this real dataset, PBE outperforms both MLE and BCE in estimating the scale parameter of the GN distribution. Notably, the KS values for all estimators are closely aligned. In terms of the overall fit of the estimators to the dataset, the ML estimator exhibits the smallest KS test statistic. Furthermore, Figure 3 displays the histogram of the

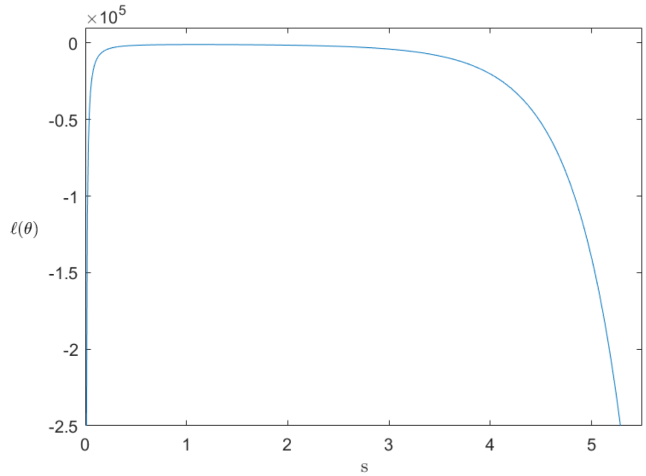

FIGURE 2. The profile log-likelihood graph for different values of s

hurricane dataset alongside the estimated pdfs obtained from MLE, BCE, and PBE. It is evident from the figure that all estimators provide similar results, effectively capturing the characteristics of the entire dataset.

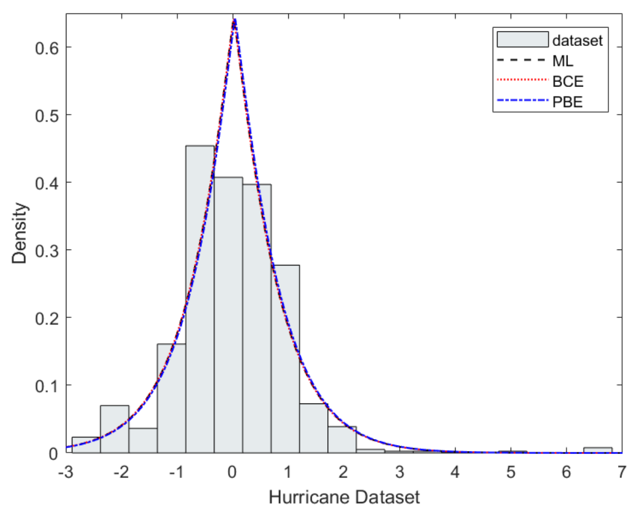

FIGURE 3. The histogram of the hurricane dataset along with the estimated pdfs obtained from MLE, BCE and PBE

TABLE 1. Estimated bias (MSE) for μ and σ, $\mu = 2, s = 2$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0737 (0.0057)	-0.0045 (0.0252)	-0.0009 (0.0247)	-0.0491 (0.0025)	-0.1529 (0.0475)	0.0484 (0.0298)
	20	-0.0384 (0.0015)	-0.0014 (0.0125)	-0.0004 (0.0131)	-0.0256 (0.0007)	-0.1812 (0.0451)	0.0270 (0.0151)
	30	-0.0259 (0.0007)	-0.0006 (0.0083)	-0.0001 (0.0083)	-0.0173 (0.0003)	-0.1895 (0.0441)	0.0172 (0.0092)
	40	-0.0195 (0.0004)	-0.0004 (0.0062)	-0.0001 (0.0063)	-0.0130 (0.0002)	-0.1939 (0.0440)	0.0130 (0.0068)
	50	-0.0157 (0.0003)	-0.0002 (0.0050)	-0.0000 (0.0051)	-0.0105 (0.0001)	-0.1965 (0.0437)	0.0104 (0.0052)
1.0	10	-0.1044 (0.0115)	-0.0017 (0.0494)	-0.0002 (0.0496)	-0.0696 (0.0051)	-0.0741 (0.0533)	0.0683 (0.0599)
	20	-0.0541 (0.0030)	-0.0019 (0.0253)	-0.0001 (0.0248)	-0.0361 (0.0013)	-0.0380 (0.0261)	0.0365 (0.0290)
	30	-0.0367 (0.0014)	-0.0004 (0.0169)	-0.0002 (0.0168)	-0.0244 (0.0006)	-0.0250 (0.0174)	0.0249 (0.0188)
	40	-0.0277 (0.0008)	-0.0003 (0.0126)	-0.0001 (0.0126)	-0.0184 (0.0003)	-0.0194 (0.0129)	0.0186 (0.0137)
	50	-0.0222 (0.0005)	-0.0002 (0.0100)	-0.0000 (0.0100)	-0.0148 (0.0002)	-0.0138 (0.0102)	0.0148 (0.0108)
1.5	10	-0.1271 (0.0171)	-0.0013 (0.0737)	-0.0001 (0.0737)	-0.0847 (0.0076)	-0.3692 (0.2080)	0.0830 (0.0901)
	20	-0.0665 (0.0045)	-0.0002 (0.0376)	-0.0002 (0.0375)	-0.0443 (0.0020)	-0.3219 (0.1395)	0.0449 (0.0410)
	30	-0.0449 (0.0020)	-0.0003 (0.0250)	-0.0003 (0.0252)	-0.0299 (0.0009)	-0.3058 (0.1183)	0.0302 (0.0275)
	40	-0.0339 (0.0012)	-0.0003 (0.0186)	-0.0001 (0.0187)	-0.0226 (0.0005)	-0.2989 (0.1082)	0.0227 (0.0202)
	50	-0.0272 (0.0007)	-0.0005 (0.0153)	-0.0001 (0.0153)	-0.0181 (0.0003)	-0.2935 (0.1011)	0.0183 (0.0162)
2.0	10	-0.1472 (0.0229)	-0.0007 (0.0974)	-0.0007 (0.0975)	-0.0981 (0.0102)	-0.6927 (0.5760)	0.1061 (0.1361)
	20	-0.0768 (0.0061)	-0.0009 (0.0505)	-0.0004 (0.0506)	-0.0512 (0.0027)	-0.6424 (0.4613)	0.0529 (0.0593)
	30	-0.0519 (0.0027)	-0.0004 (0.0329)	-0.0002 (0.0329)	-0.0346 (0.0012)	-0.6234 (0.4220)	0.0353 (0.0376)
	40	-0.0391 (0.0015)	-0.0009 (0.0252)	-0.0002 (0.0252)	-0.0260 (0.0007)	-0.6120 (0.3996)	0.0268 (0.0270)
	50	-0.0314 (0.0010)	-0.0005 (0.0197)	-0.0001 (0.0197)	-0.0209 (0.0004)	-0.6084 (0.3897)	0.0210 (0.0217)
2.5	10	-0.1650 (0.0288)	-0.0024 (0.1233)	-0.0012 (0.1235)	-0.1099 (0.0128)	-1.0373 (1.1964)	0.1185 (0.1659)
	20	-0.0857 (0.0075)	-0.0008 (0.0617)	-0.0006 (0.0618)	-0.0571 (0.0033)	-0.9810 (1.0244)	0.0588 (0.0721)
	30	-0.0578 (0.0034)	-0.0004 (0.0415)	-0.0001 (0.0416)	-0.0385 (0.0015)	-0.9596 (0.9616)	0.0389 (0.0464)
	40	-0.0438 (0.0019)	-0.0002 (0.0316)	-0.0001 (0.0316)	-0.0292 (0.0009)	-0.9443 (0.9229)	0.0288 (0.0334)
	50	-0.0351 (0.0012)	-0.0010 (0.0249)	-0.0000 (0.0249)	-0.0234 (0.0006)	-0.9425 (0.9130)	0.0239 (0.0261)
3.0	10	-0.1807 (0.0345)	-0.0023 (0.1480)	-0.0003 (0.1482)	-0.1204 (0.0153)	-1.3970 (2.0950)	0.1310 (0.1983)
	20	-0.0938 (0.0090)	-0.0005 (0.0741)	-0.0002 (0.0741)	-0.0625 (0.0040)	-1.3361 (1.8589)	0.0655 (0.0883)
	30	-0.0634 (0.0041)	-0.0003 (0.0498)	-0.0001 (0.0499)	-0.0422 (0.0018)	-1.3118 (1.7704)	0.0433 (0.0553)
	40	-0.0480 (0.0023)	-0.0002 (0.0379)	-0.0001 (0.0379)	-0.0320 (0.0010)	-1.3015 (1.7307)	0.0321 (0.0411)
	50	-0.0384 (0.0015)	-0.0011 (0.0298)	-0.0000 (0.0299)	-0.0256 (0.0007)	-1.2941 (1.7045)	0.0258 (0.0323)

TABLE 2. Estimated bias (MSE) for μ and σ, $\mu = 2, s = 3$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0686 (0.0049)	-0.0014 (0.0172)	-0.0004 (0.0174)	-0.0537 (0.0030)	-0.1464 (0.0391)	0.0565 (0.0258)
	20	-0.0360 (0.0013)	-0.0006 (0.0086)	-0.0001 (0.0086)	-0.0282 (0.0008)	-0.1759 (0.0393)	0.0284 (0.0106)
	30	-0.0244 (0.0006)	-0.0013 (0.0055)	-0.0001 (0.0056)	-0.0191 (0.0004)	-0.1870 (0.0405)	0.0188 (0.0066)
	40	-0.0184 (0.0003)	-0.0002 (0.0041)	-0.0000 (0.0042)	-0.0144 (0.0002)	-0.1920 (0.0410)	0.0143 (0.0049)
	50	-0.0148 (0.0001)	-0.0001 (0.0033)	-0.0000 (0.0033)	-0.0116 (0.0001)	-0.1963 (0.0418)	0.0115 (0.0036)
1.0	10	-0.0970 (0.0098)	-0.0018 (0.0343)	-0.0007 (0.0347)	-0.0760 (0.0060)	-0.0855 (0.0413)	0.0802 (0.0523)
	20	-0.0510 (0.0026)	-0.0013 (0.0172)	-0.0002 (0.0172)	-0.0399 (0.0016)	-0.0434 (0.0190)	0.0403 (0.0217)
	30	-0.0344 (0.0012)	-0.0007 (0.0111)	-0.0001 (0.0111)	-0.0270 (0.0007)	-0.0284 (0.0119)	0.0272 (0.0133)
	40	-0.0260 (0.0007)	-0.0015 (0.0082)	-0.0000 (0.0082)	-0.0204 (0.0005)	-0.0196 (0.0088)	0.0204 (0.0095)
	50	-0.0210 (0.0004)	-0.0005 (0.0066)	-0.0000 (0.0066)	-0.0164 (0.0003)	-0.0172 (0.0071)	0.0161 (0.0072)
1.5	10	-0.1188 (0.0147)	-0.0011 (0.0515)	-0.0007 (0.0521)	-0.0930 (0.0090)	-0.3821 (0.1982)	0.0975 (0.0769)
	20	-0.0624 (0.0040)	-0.0003 (0.0258)	-0.0003 (0.0259)	-0.0489 (0.0024)	-0.3289 (0.1336)	0.0489 (0.0308)
	30	-0.0422 (0.0018)	-0.0008 (0.0166)	-0.0001 (0.0167)	-0.0330 (0.0011)	-0.3105 (0.1128)	0.0330 (0.0197)
	40	-0.0319 (0.0010)	-0.0003 (0.0123)	-0.0000 (0.0124)	-0.0250 (0.0006)	-0.3004 (0.1028)	0.0248 (0.0141)
	50	-0.0257 (0.0007)	-0.0013 (0.0100)	-0.0000 (0.0100)	-0.0201 (0.0004)	-0.2950 (0.0970)	0.0199 (0.0109)
2.0	10	-0.1372 (0.0196)	-0.0011 (0.0687)	-0.0006 (0.0965)	-0.1074 (0.0120)	-0.7133 (0.5781)	0.1124 (0.1030)
	20	-0.0721 (0.0053)	-0.0021 (0.0344)	-0.0001 (0.0345)	-0.0564 (0.0032)	-0.6453 (0.4501)	0.0568 (0.0421)
	30	-0.0487 (0.0024)	-0.0014 (0.0222)	-0.0003 (0.0222)	-0.0381 (0.0015)	-0.6243 (0.4122)	0.0379 (0.0258)
	40	-0.0368 (0.0014)	-0.0027 (0.0164)	-0.0001 (0.0165)	-0.0288 (0.0008)	-0.6140 (0.3940)	0.0286 (0.0191)
	50	-0.0296 (0.0009)	-0.0004 (0.0133)	-0.0000 (0.0133)	-0.0232 (0.0005)	-0.6112 (0.3870)	0.0231 (0.0145)
2.5	10	-0.1534 (0.0245)	-0.0017 (0.0858)	-0.0005 (0.0869)	-0.1201 (0.0150)	-1.0563 (1.2034)	0.1281 (0.1312)
	20	-0.0806 (0.0066)	-0.0009 (0.0430)	-0.0002 (0.0431)	-0.0631 (0.0041)	-0.9868 (1.0161)	0.0624 (0.0528)
	30	-0.0545 (0.0030)	-0.0016 (0.0277)	-0.0001 (0.0278)	-0.0426 (0.0018)	-0.9615 (0.9527)	0.0428 (0.0326)
	40	-0.0412 (0.0017)	-0.0012 (0.0206)	-0.0002 (0.0206)	-0.0322 (0.0010)	-0.9532 (0.9298)	0.0320 (0.0237)
	50	-0.0331 (0.0011)	-0.0008 (0.0166)	-0.0001 (0.0166)	-0.0259 (0.0007)	-0.9435 (0.9068)	0.0257 (0.0182)
3.0	10	-0.1681 (0.0294)	-0.0018 (0.1030)	-0.0007 (0.1042)	-0.1316 (0.0180)	-1.4215 (2.1238)	0.1394 (0.1573)
	20	-0.0883 (0.0079)	-0.0022 (0.0516)	-0.0004 (0.0517)	-0.0691 (0.0049)	-1.3427 (1.8539)	0.0701 (0.0644)
	30	-0.0597 (0.0036)	-0.0010 (0.0333)	-0.0001 (0.0334)	-0.0467 (0.0022)	-1.3154 (1.7641)	0.0465 (0.0394)
	40	-0.0451 (0.0021)	-0.0004 (0.0247)	-0.0001 (0.0247)	-0.0353 (0.0013)	-1.3040 (1.7254)	0.0349 (0.0283)
	50	-0.0363 (0.0013)	-0.0010 (0.0199)	-0.0001 (0.0199)	-0.0284 (0.0008)	-1.2964 (1.7014)	0.0281 (0.0221)

TABLE 3. Estimated bias (MSE) for μ and σ, $\mu = 2, s = 5$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0648 (0.0043)	-0.0005 (0.0121)	-0.0008 (0.0127)	-0.0570 (0.0034)	-0.1352 (0.0303)	0.0590 (0.0204)
	20	-0.0344 (0.0012)	-0.0004 (0.0055)	-0.0003 (0.0056)	-0.0303 (0.0009)	-0.1737 (0.0357)	0.0296 (0.0075)
	30	-0.0233 (0.0006)	-0.0006 (0.0035)	-0.0001 (0.0036)	-0.0206 (0.0004)	-0.1864 (0.0383)	0.0197 (0.0044)
	40	-0.0176 (0.0003)	-0.0007 (0.0026)	-0.0001 (0.0026)	-0.0155 (0.0002)	-0.1906 (0.0390)	0.0148 (0.0031)
	50	-0.0142 (0.0002)	-0.0008 (0.0021)	-0.0000 (0.0021)	-0.0125 (0.0001)	-0.1940 (0.0397)	0.0120 (0.0024)
1.0	10	-0.0916 (0.0087)	-0.0012 (0.0242)	-0.0001 (0.0254)	-0.0807 (0.0067)	-0.1020 (0.0348)	0.0851 (0.0414)
	20	-0.0487 (0.0024)	-0.0005 (0.0110)	-0.0001 (0.0112)	-0.0428 (0.0019)	-0.0473 (0.0133)	0.0420 (0.0146)
	30	-0.0330 (0.0011)	-0.0012 (0.0071)	-0.0000 (0.0072)	-0.0291 (0.0009)	-0.0307 (0.0080)	0.0281 (0.0087)
	40	-0.0250 (0.0006)	-0.0007 (0.0052)	-0.0000 (0.0052)	-0.0220 (0.0005)	-0.0242 (0.0058)	0.0213 (0.0064)
	50	-0.0201 (0.0004)	-0.0008 (0.0042)	-0.0000 (0.0042)	-0.0177 (0.0003)	-0.0184 (0.0045)	0.0171 (0.0047)
1.5	10	-0.1122 (0.0130)	-0.0015 (0.0363)	-0.0003 (0.0380)	-0.0988 (0.0101)	-0.4002 (0.1970)	0.1028 (0.0599)
	20	-0.0596 (0.0036)	-0.0017 (0.0165)	-0.0001 (0.0168)	-0.0525 (0.0028)	-0.3329 (0.1273)	0.0500 (0.0220)
	30	-0.0404 (0.0016)	-0.0013 (0.0106)	-0.0001 (0.0107)	-0.0356 (0.0013)	-0.3132 (0.1089)	0.0341 (0.0132)
	40	-0.0306 (0.0009)	-0.0014 (0.0078)	-0.0000 (0.0078)	-0.0269 (0.0007)	-0.3025 (0.0994)	0.0258 (0.0092)
	50	-0.0246 (0.0006)	-0.0012 (0.0062)	-0.0000 (0.0063)	-0.0216 (0.0005)	-0.2968 (0.0943)	0.0209 (0.0071)
2.0	10	-0.1296 (0.0173)	-0.0002 (0.0484)	-0.0004 (0.0507)	-0.1141 (0.0134)	-0.7257 (0.5749)	0.1194 (0.0807)
	20	-0.0688 (0.0048)	-0.0002 (0.0220)	-0.0009 (0.0224)	-0.0606 (0.0037)	-0.6536 (0.4497)	0.0590 (0.0404)
	30	-0.0467 (0.0022)	-0.0008 (0.0142)	-0.0002 (0.0143)	-0.0411 (0.0017)	-0.6315 (0.4129)	0.0392 (0.0254)
	40	-0.0353 (0.0013)	-0.0001 (0.0103)	-0.0001 (0.0104)	-0.0311 (0.0010)	-0.6183 (0.3928)	0.0301 (0.0188)
	50	-0.0284 (0.0008)	-0.0005 (0.0083)	-0.0000 (0.0084)	-0.0250 (0.0006)	-0.6121 (0.3831)	0.0240 (0.0094)
2.5	10	-0.1449 (0.0216)	-0.0014 (0.0605)	-0.0015 (0.0634)	-0.1275 (0.0168)	-1.0798 (1.2271)	0.1329 (0.1014)
	20	-0.0770 (0.0060)	-0.0012 (0.0275)	-0.0007 (0.0280)	-0.0678 (0.0046)	-0.9969 (1.0215)	0.0661 (0.0375)
	30	-0.0522 (0.0027)	-0.0013 (0.0177)	-0.0004 (0.0179)	-0.0460 (0.0021)	-0.9704 (0.9592)	0.0443 (0.0213)
	40	-0.0395 (0.0016)	-0.0006 (0.0129)	-0.0002 (0.0130)	-0.0347 (0.0012)	-0.9567 (0.9282)	0.0336 (0.0153)
	50	-0.0318 (0.0010)	-0.0007 (0.0104)	-0.0001 (0.0105)	-0.0279 (0.0008)	-0.9479 (0.9089)	0.0271 (0.0119)
3.0	10	-0.1587 (0.0260)	-0.0021 (0.0726)	-0.0018 (0.0761)	-0.1397 (0.0201)	-1.4400 (2.1469)	0.1445 (0.1207)
	20	-0.0843 (0.0072)	-0.0012 (0.0330)	-0.0004 (0.0336)	-0.0742 (0.0056)	-1.3491 (1.8544)	0.0727 (0.0443)
	30	-0.0572 (0.0033)	-0.0003 (0.0213)	-0.0001 (0.0215)	-0.0503 (0.0026)	-1.3234 (1.7728)	0.0483 (0.0258)
	40	-0.0432 (0.0019)	-0.0007 (0.0155)	-0.0000 (0.0156)	-0.0380 (0.0015)	-1.3087 (1.7289)	0.0364 (0.0186)
	50	-0.0348 (0.0012)	-0.0003 (0.0125)	-0.0000 (0.0126)	-0.0306 (0.0009)	-1.3010 (1.7055)	0.0295 (0.0142)

TABLE 4. Estimated bias (MSE) for μ and σ, $\mu = 1, s = 2$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0735 (0.0057)	-0.0006 (0.0248)	-0.0004 (0.0248)	-0.0490 (0.0025)	-0.1545 (0.0486)	0.0532 (0.0342)
	20	-0.0383 (0.0015)	-0.0003 (0.0124)	-0.0001 (0.0124)	-0.0256 (0.0007)	-0.1799 (0.0446)	0.0264 (0.0146)
	30	-0.0259 (0.0007)	-0.0003 (0.0083)	-0.0000 (0.0083)	-0.0173 (0.0003)	-0.1885 (0.0437)	0.0177 (0.0092)
	40	-0.0196 (0.0004)	-0.0007 (0.0063)	-0.0000 (0.0063)	-0.0131 (0.0002)	-0.1949 (0.0442)	0.0132 (0.0068)
	50	-0.0157 (0.0002)	-0.0004 (0.0051)	-0.0000 (0.0051)	-0.0105 (0.0001)	-0.1956 (0.0433)	0.0104 (0.0053)
1.0	10	-0.1039 (0.0114)	-0.0008 (0.0499)	-0.0001 (0.0500)	-0.0692 (0.0051)	-0.0761 (0.0550)	0.0761 (0.0686)
	20	-0.0543 (0.0030)	-0.0005 (0.0251)	-0.0004 (0.0251)	-0.0362 (0.0013)	-0.0382 (0.0266)	0.0377 (0.0300)
	30	-0.0366 (0.0014)	-0.0022 (0.0165)	-0.0002 (0.0166)	-0.0244 (0.0006)	-0.0253 (0.0169)	0.0248 (0.0183)
	40	-0.0276 (0.0008)	-0.0003 (0.0126)	-0.0001 (0.0126)	-0.0184 (0.0003)	-0.0190 (0.0127)	0.0185 (0.0134)
	50	-0.0222 (0.0005)	-0.0012 (0.0101)	-0.0000 (0.0101)	-0.0148 (0.0002)	-0.0156 (0.0101)	0.0151 (0.0107)
1.5	10	-0.1278 (0.0173)	-0.0008 (0.0740)	-0.0006 (0.0741)	-0.0851 (0.0077)	-0.3693 (0.2119)	0.0932 (0.1054)
	20	-0.0664 (0.0045)	-0.0007 (0.0370)	-0.0001 (0.0371)	-0.0442 (0.0020)	-0.3242 (0.1420)	0.0459 (0.0439)
	30	-0.0448 (0.0020)	-0.0010 (0.0249)	-0.0004 (0.0249)	-0.0299 (0.0009)	-0.3042 (0.1147)	0.0305 (0.0278)
	40	-0.0339 (0.0012)	-0.0026 (0.0189)	-0.0002 (0.0189)	-0.0226 (0.0005)	-0.2968 (0.1068)	0.0230 (0.0205)
	50	-0.0272 (0.0007)	-0.0008 (0.0149)	-0.0001 (0.0149)	-0.0181 (0.0003)	-0.2924 (0.1002)	0.0182 (0.0157)
2.0	10	-0.1475 (0.0230)	-0.0053 (0.1987)	-0.0016 (0.0988)	-0.0983 (0.0102)	-0.6940 (0.5794)	0.1063 (0.1365)
	20	-0.0766 (0.0060)	-0.0006 (0.0494)	-0.0001 (0.0494)	-0.0511 (0.0027)	-0.6407 (0.4599)	0.0541 (0.0590)
	30	-0.0517 (0.0027)	-0.0004 (0.0332)	-0.0000 (0.0332)	-0.0345 (0.0012)	-0.6192 (0.4155)	0.0354 (0.0362)
	40	-0.0392 (0.0016)	-0.0031 (0.0253)	-0.0001 (0.0253)	-0.0261 (0.0007)	-0.6089 (0.3959)	0.0263 (0.0276)
	50	-0.0314 (0.0010)	-0.0010 (0.0199)	-0.0000 (0.0199)	-0.0209 (0.0004)	-0.6073 (0.3888)	0.0210 (0.0215)
2.5	10	-0.1650 (0.0288)	-0.0047 (0.1233)	-0.0018 (0.1235)	-0.1099 (0.0128)	-1.0382 (1.2006)	0.1227 (0.1737)
	20	-0.0857 (0.0075)	-0.0012 (0.0617)	-0.0003 (0.0618)	-0.0571 (0.0033)	-0.9781 (1.0186)	0.0585 (0.0727)
	30	-0.0578 (0.0034)	-0.0006 (0.0415)	-0.0001 (0.0416)	-0.0385 (0.0015)	-0.9587 (0.9615)	0.0394 (0.0474)
	40	-0.0438 (0.0019)	-0.0015 (0.0316)	-0.0000 (0.0316)	-0.0292 (0.0009)	-0.9508 (0.9343)	0.0295 (0.0332)
	50	-0.0351 (0.0012)	-0.0019 (0.0249)	-0.0000 (0.0249)	-0.0234 (0.0006)	-0.9436 (0.9157)	0.0235 (0.0273)
3.0	10	-0.1807 (0.0345)	-0.0006 (0.1480)	-0.0021 (0.1481)	-0.1204 (0.0153)	-1.3986 (2.0994)	0.1306 (0.1997)
	20	-0.0938 (0.0090)	-0.0016 (0.0741)	-0.0010 (0.0741)	-0.0625 (0.0040)	-1.3327 (1.8492)	0.0652 (0.0871)
	30	-0.0634 (0.0041)	-0.0002 (0.0498)	-0.0003 (0.0499)	-0.0422 (0.0018)	-1.3151 (1.7787)	0.0428 (0.0554)
	40	-0.0480 (0.0023)	-0.0007 (0.0379)	-0.0004 (0.0379)	-0.0320 (0.0010)	-1.2994 (1.7252)	0.0324 (0.0402)
	50	-0.0384 (0.0015)	-0.0021 (0.0298)	-0.0001 (0.0299)	-0.0256 (0.0007)	-1.2946 (1.7057)	0.0259 (0.0318)

TABLE 5. Estimated bias (MSE) for μ and σ, $\mu = 1, s = 3$

		Estimator for μ			Estimator for σ		
σ	n	BCE	MLE	PBE	BCE	MLE	PBE
0.5	10	-0.0686 (0.0049)	-0.0015 (0.0172)	-0.0002 (0.0174)	-0.0537 (0.0030)	-0.1461 (0.0388)	0.0566 (0.0262)
	20	-0.0360 (0.0013)	-0.0005 (0.0086)	-0.0001 (0.0086)	-0.0282 (0.0008)	-0.1769 (0.0394)	0.0281 (0.0102)
	30	-0.0244 (0.0006)	-0.0007 (0.0057)	-0.0000 (0.0057)	-0.0191 (0.0004)	-0.1868 (0.0404)	0.0190 (0.0064)
	40	-0.0184 (0.0003)	-0.0005 (0.0042)	-0.0000 (0.0042)	-0.0144 (0.0002)	-0.1919 (0.0410)	0.0144 (0.0046)
	50	-0.0148 (0.0002)	-0.0002 (0.0033)	-0.0000 (0.0033)	-0.0116 (0.0001)	-0.1952 (0.0414)	0.0115 (0.0036)
1.0	10	-0.0972 (0.0099)	-0.0006 (0.0341)	-0.0006 (0.0345)	-0.0761 (0.0060)	-0.0889 (0.0426)	0.0801 (0.0515)
	20	-0.0509 (0.0026)	-0.0012 (0.0163)	-0.0002 (0.0163)	-0.0398 (0.0016)	-0.0436 (0.0193)	0.0399 (0.0216)
	30	-0.0345 (0.0012)	-0.0003 (0.0115)	-0.0001 (0.0115)	-0.0270 (0.0007)	-0.0281 (0.0125)	0.0269 (0.0136)
	40	-0.0261 (0.0007)	-0.0011 (0.0083)	-0.0000 (0.0083)	-0.0204 (0.0004)	-0.0214 (0.0088)	0.0200 (0.0093)
	50	-0.0210 (0.0004)	-0.0004 (0.0067)	-0.0000 (0.0067)	-0.0164 (0.0003)	-0.0175 (0.0071)	0.0163 (0.0074)
1.5	10	-0.1188 (0.0147)	-0.0003 (0.0532)	-0.0006 (0.0540)	-0.0930 (0.0090)	-0.3824 (0.1981)	0.0981 (0.0779)
	20	-0.0623 (0.0040)	-0.0006 (0.0258)	-0.0006 (0.0259)	-0.0488 (0.0024)	-0.3256 (0.1314)	0.0489 (0.0315)
	30	-0.0421 (0.0018)	-0.0008 (0.0169)	-0.0002 (0.0169)	-0.0330 (0.0011)	-0.3114 (0.1136)	0.0328 (0.0193)
	40	-0.0319 (0.0010)	-0.0012 (0.0124)	-0.0001 (0.0124)	-0.0249 (0.0006)	-0.3001 (0.1025)	0.0248 (0.0139)
	50	-0.0256 (0.0007)	-0.0025 (0.0104)	-0.0001 (0.0104)	-0.0201 (0.0004)	-0.2961 (0.0978)	0.0201 (0.0111)
2.0	10	-0.1372 (0.0196)	-0.0019 (0.0687)	-0.0022 (0.0695)	-0.1074 (0.0120)	-0.7104 (0.5738)	0.1132 (0.1035)
	20	-0.0721 (0.0053)	-0.0003 (0.0344)	-0.0001 (0.0345)	-0.0564 (0.0032)	-0.6450 (0.4503)	0.0561 (0.0427)
	30	-0.0487 (0.0024)	-0.0005 (0.0222)	-0.0003 (0.0222)	-0.0381 (0.0015)	-0.6264 (0.4155)	0.0380 (0.0264)
	40	-0.0368 (0.0014)	-0.0005 (0.0164)	-0.0000 (0.0165)	-0.0288 (0.0008)	-0.6146 (0.3948)	0.0289 (0.0186)
	50	-0.0296 (0.0009)	-0.0022 (0.0133)	-0.0000 (0.0133)	-0.0232 (0.0005)	-0.6121 (0.3882)	0.0230 (0.0145)
2.5	10	-0.1539 (0.0247)	-0.0019 (0.0872)	-0.0001 (0.0886)	-0.1205 (0.0151)	-1.0565 (1.2028)	0.1271 (0.1291)
	20	-0.0808 (0.0067)	-0.0005 (0.0420)	-0.0008 (0.0422)	-0.0633 (0.0041)	-0.9880 (1.0188)	0.0628 (0.0528)
	30	-0.0546 (0.0030)	-0.0010 (0.0280)	-0.0005 (0.0281)	-0.0427 (0.0018)	-0.9616 (0.9534)	0.0424 (0.0331)
	40	-0.0412 (0.0017)	-0.0009 (0.0208)	-0.0002 (0.0208)	-0.0323 (0.0011)	-0.9509 (0.9252)	0.0318 (0.0239)
	50	-0.0331 (0.0011)	-0.0009 (0.0167)	-0.0001 (0.0167)	-0.0259 (0.0007)	-0.9457 (0.9114)	0.0259 (0.0184)
3.0	10	-0.1680 (0.0294)	-0.0036 (0.1059)	-0.0012 (0.1074)	-0.1315 (0.0180)	-1.4258 (2.1359)	0.1389 (0.1572)
	20	-0.0881 (0.0079)	-0.0008 (0.0508)	-0.0013 (0.0509)	-0.0690 (0.0048)	-1.3434 (1.8572)	0.0698 (0.0640)
	30	-0.0597 (0.0036)	-0.0007 (0.0332)	-0.0003 (0.0333)	-0.0467 (0.0022)	-1.3168 (1.7671)	0.0472 (0.0398)
	40	-0.0451 (0.0021)	-0.0017 (0.0244)	-0.0002 (0.0245)	-0.0353 (0.0013)	-1.3047 (1.7281)	0.0350 (0.0278)
	50	-0.0363 (0.0013)	-0.0012 (0.0199)	-0.0001 (0.0200)	-0.0284 (0.0008)	-1.2955 (1.6980)	0.0281 (0.0220)

Table 7. Integrated bias squared norm for BCE, MLE, and PBE

	Estimator for μ			Estimator for σ		
n	BCE	MLE	PBE	BCE	MLE	PBE
10	0.1304	0.0020	0.0009	0.0999	0.7983	0.1057
20	0.0679	0.0012	0.0004	0.0525	0.7456	0.0526
30	0.0459	0.0009	0.0002	0.0356	0.7287	0.0352
40	0.0347	0.0011	0.0001	0.0269	0.7204	0.0265
50	0.0279	0.0010	0.0001	0.0216	0.7158	0.0213

Table 8. Average root-mean-squared error for BCE, MLE, and PBE

	Estimator for μ			Estimator for σ		
n	BCE	MLE	PBE	BCE	MLE	PBE
10	0.1227	0.2339	0.2336	0.0951	0.6491	0.2809
20	0.0636	0.1604	0.1609	0.0494	0.5831	0.1797
30	0.0431	0.1299	0.1302	0.0334	0.5602	0.1412
40	0.0324	0.1120	0.1122	0.0251	0.5485	0.1198
50	0.0257	0.1002	0.1004	0.0200	0.5416	0.1052

Table 9. Estimation results for MLE, BCE, and PBE (Bootstrap standard errors) for hurricane dataset

Estimators	μ	σ	KS-value
MLE	0.0168 (0.0604)	0.8078 (0.1186)	0.0855
BCE	0.0182 (0.0604)	0.8084 (0.1188)	0.0892
PBE	0.0403 (0.1288)	0.8076 (0.1181)	0.1010

5. Conclusion

In this paper, we introduced two bias correction methods, BCE and PBE, for estimating the parameters of the GN distribution. Our simulation study results showed that the BCE for the scale parameter consistently outperforms both MLE and PBE across all sample sizes. Notably, the BCE demonstrates significantly improved accuracy and reliability.

On the other hand, for the location parameter, the PBE exhibits minimal biases, while the BCE shows small MSE values in all cases. This numerical analysis highlights the effectiveness of the proposed bias correction methodology, particularly in improving the precision of parameter estimates for both the location and scale parameters.

To further illustrate the practical applicability of the MLE, BCE, and PBE, we provided a real-data example involving the analysis of hurricane data from the Atlantic, USA. The findings from the real-data example indicated that the PBE outperforms bootstrap standard errors for the scale parameter, highlighting its superiority in estimating the scale parameter's variability. Additionally, the bootstrap standard errors for both the MLE and BCE are comparable for the location parameter.

In conclusion, our study suggests that the BCE offers a practical and useful alternative to the MLE, particularly in cases with small to moderate sample sizes. By incorporating Efron's bootstrap procedure, our proposed BCE method contributes to more accurate and reliable parameter estimation within the GN distribution framework.

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GE-FILTERS, ORDERING FILTERS AND LEFT MAPPINGS IN GE-ALGEBRAS

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Abstract. The notions of ordering filter and left mapping in a GE-algebra are introduced, and their properties are investigated. Relations between ordering filters and GE-filters are established. Conditions for an ordering filter to be a GE-filter, and vice versa, are provided. The conditions under which a left mapping becomes injective or an identity are explored. The conditions under which the GE-kernel of a self-mapping will be a GE-filter are provided. It is confirmed that the sets of all left mappings form a semigroup, and that the sets of all idempotent left mappings form a subsemigroup. The conditions under which the sets of all left mappings can be closed with respect to a binary operation are investigated.

1. Introduction

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [8] proved that Hilbert algebras form a variety which is locally finite. Later, several authors introduced many concepts to explore the concept of Hilbert algebras (see [5–7, 9, 10, 14–16]). Bandaru et al. introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1]). Also, Bandaru et al. introduced several concepts in GE-algebras and investigated its related properties (see [2–4, 12, 13, 17, 18]). Left mappings is very useful concept and many researchers have used it in various mathematical fields. For example, Kondo introduced the

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notion of left mapping on BCK-algebras and investigated some properties of it (see [11]). He showed that in a positive implicative BCK-algebra, if a left mapping is surjective, then it is also an injective one.

In this paper, we introduce the notion of ordering filter in a GE-algebra and provide the conditions for an ordering filter to be a GE-filter. Also, we explore the necessary condition for a GE-filter to be an ordering filter. We introduce the concept of left mapping on GE-algebras and investigate related properties. We define the GE-kernel of a left mapping of a GE-algebra and provide the conditions under which GE-kernel to be a GE-filter. We prove that the set $L(X)$ of all left mappings of a GE-algebra X is closed under the function composition $\circ$ and also a semigroup. We define the operation "$\circledast$" on $L(X)$ by $(f \circledast g)(x) = f(x) * g(x)$ for all $x \in X$ and $f, g \in L(X)$ and observe that the set $L(X)$ is not closed under $\circledast$. Finally, we investigate the conditions under which $L(X)$ be closed with respect to $\circledast$.

This study particularly focuses on ordering filters and left mappings within these algebras, offering a comprehensive exploration of their properties and interrelations. Ordering filters in GE-algebras serve as critical tools for understanding the hierarchical structure and organization within these algebraic systems. Ordering filters help identify and analyze hierarchical relationships and dependencies among elements in a GE-algebra, offering a clearer picture of the overall structure. Establishing relations between ordering filters and GE-filters not only bridges the concepts but also enhances the understanding of how different filters interact and coexist within the algebraic framework. The comprehensive study of ordering filters and left mappings in GE-algebras offers valuable contributions to the understanding of these algebraic structures. By exploring their properties, interrelations, and conditions for specific behaviors, this research paves the way for further advancements in the field of algebra and its applications in logic, computation, and beyond. The motivation lies in the quest for deeper knowledge, the development of new mathematical tools, and the potential for practical applications arising from a robust understanding of GE-algebras.

2. Preliminaries

Definition 1 ([1]). *By a GE-algebra we mean a non-empty set Y with a constant 1 and a binary operation $*$ satisfying the following axioms:*

(GE1) $\gamma_1 * \gamma_1 = 1$,

(GE2) $1 * \gamma_1 = \gamma_1$,

(GE3) $\gamma_1 * (\varpi_2 * \sigma_3) = \gamma_1 * (\varpi_2 * (\gamma_1 * \sigma_3))$

for all $\gamma_1, \varpi_2, \sigma_3 \in Y$.

In a GE-algebra Y, a binary relation "$\leq$" is defined by

$$(\forall \wp_3, \wp_4 \in Y)\,(\wp_3 \leq \wp_4 \;\Leftrightarrow\; \wp_3 * \wp_4 = 1)\,. \tag{1}$$

Definition 2 ([1,2,4]). *A GE-algebra Y is said to be*

- *transitive if it satisfies:*

$$(\forall \wp_3, \wp_4, \wp_5 \in Y)\left(\wp_3 * \wp_4 \leq (\wp_5 * \wp_3) * (\wp_5 * \wp_4)\right). \tag{2}$$

- *commutative if it satisfies:*

$$(\forall \wp_3, \wp_4 \in Y)\left((\wp_3 * \wp_4) * \wp_4 = (\wp_4 * \wp_3) * \wp_3\right). \tag{3}$$

- *left exchangeable if it satisfies:*

$$(\forall \wp_3, \wp_4, \wp_5 \in Y)\left(\wp_3 * (\wp_4 * \wp_5) = \wp_4 * (\wp_3 * \wp_5)\right). \tag{4}$$

- *belligerent if it satisfies:*

$$(\forall \wp_3, \wp_4, \wp_5 \in Y)\left(\wp_3 * (\wp_4 * \wp_5) = (\wp_3 * \wp_4) * (\wp_3 * \wp_5)\right). \tag{5}$$

- *antisymmetric if the binary relation "$\leq$" is antisymmetric.*

Proposition 1 ([1]). *Every GE-algebra Y satisfies the following items.*

$$(\forall \gamma_1 \in Y)\left(\gamma_1 * 1 = 1\right). \tag{6}$$

$$(\forall \gamma_1, \varpi_2 \in Y)\left(\gamma_1 * (\gamma_1 * \varpi_2) = \gamma_1 * \varpi_2\right). \tag{7}$$

$$(\forall \gamma_1, \varpi_2 \in Y)\left(\gamma_1 \leq \varpi_2 * \gamma_1\right). \tag{8}$$

$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y)\left(\gamma_1 * (\varpi_2 * \sigma_3) \leq \varpi_2 * (\gamma_1 * \sigma_3)\right). \tag{9}$$

$$(\forall \gamma_1 \in Y)\left(1 \leq \gamma_1 \;\Rightarrow\; \gamma_1 = 1\right). \tag{10}$$

$$(\forall \gamma_1, \varpi_2 \in Y)\left(\gamma_1 \leq (\varpi_2 * \gamma_1) * \gamma_1\right). \tag{11}$$

$$(\forall \gamma_1, \varpi_2 \in Y)\left(\gamma_1 \leq (\gamma_1 * \varpi_2) * \varpi_2\right). \tag{12}$$

$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y)\left(\gamma_1 \leq \varpi_2 * \sigma_3 \;\Leftrightarrow\; \varpi_2 \leq \gamma_1 * \sigma_3\right). \tag{13}$$

If Y is transitive, then

$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y)\left(\gamma_1 \leq \varpi_2 \;\Rightarrow\; \sigma_3 * \gamma_1 \leq \sigma_3 * \varpi_2,\; \varpi_2 * \sigma_3 \leq \gamma_1 * \sigma_3\right). \tag{14}$$

$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y)\left(\gamma_1 * \varpi_2 \leq (\varpi_2 * \sigma_3) * (\gamma_1 * \sigma_3)\right). \tag{15}$$

$$(\forall \gamma_1, \varpi_2, \sigma_3 \in Y)\left(\gamma_1 \leq \varpi_2,\; \varpi_2 \leq \sigma_3 \;\Rightarrow\; \gamma_1 \leq \sigma_3\right). \tag{16}$$

Lemma 1 ([1]). *In a GE-algebra Y, the following facts are equivalent each other.*

$$(\forall \wp_3, \wp_4, \wp_5 \in Y)\left(\wp_3 * \wp_4 \leq (\wp_5 * \wp_3) * (\wp_5 * \wp_4)\right). \tag{17}$$

$$(\forall \wp_3, \wp_4, \wp_5 \in Y)\left(\wp_3 * \wp_4 \leq (\wp_4 * \wp_5) * (\wp_3 * \wp_5)\right). \tag{18}$$

Definition 3 ([1]). *A subset F of a GE-algebra Y is called a GE-filter of Y if it satisfies:*

$$1 \in F, \tag{19}$$

$$(\forall \wp_3, \wp_4 \in Y)(\wp_3 * \wp_4 \in F,\; \wp_3 \in F \;\Rightarrow\; \wp_4 \in F). \tag{20}$$

Lemma 2 ([1]). *In a GE-algebra Y, every GE-filter F of Y satisfies:*

$$(\forall \wp_3, \wp_4 \in Y)\left(\wp_3 \leq \wp_4,\; \wp_3 \in F \;\Rightarrow\; \wp_4 \in F\right). \tag{21}$$

Definition 4 ([1]). *A non-empty subset F of a GE-algebra Y is called a GE-subalgebra of Y if $\wp_3 * \wp_4 \in F$ for any $\wp_3, \wp_4 \in F$.*

3. GE-filters and ordering filters

In what follows, let Y denote a GE-algebra unless otherwise specified.

Definition 5. *A subset F of Y is called an ordering filter of Y if it satisfies (21) and*

$$(\forall \wp_3, \wp_4 \in F)(\exists \wp_5 \in F)(\wp_5 \le \wp_3,\ \wp_5 \le \wp_4). \tag{22}$$

We denote by $OF(Y)$ the set of all ordering filters of Y. It is clear that $\{1\}, Y \in OF(Y)$ and every ordering filter contains the element 1.

Example 1. *We take a GE-algebra $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5, \zeta_6\}$ with the operation table given by Table 1.*

Table 1. The binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5	ζ_6
1	1	ρ_2	ι_3	ϵ_4	ι_5	ζ_6
ρ_2	1	1	1	ϵ_4	ϵ_4	1
ι_3	1	1	1	ι_5	ι_5	ζ_6
ϵ_4	1	ρ_2	1	1	1	ζ_6
ι_5	1	1	ι_3	1	1	1
ζ_6	1	1	ι_3	ι_5	ι_5	ζ_6

Then $F_1 := \{1, \rho_2, \iota_3, \zeta_6\}$ and $F_2 := \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ are ordering filter of Y. But $F_3 := \{1, \rho_2, \iota_3, \iota_5\}$ is not an ordering filter of Y since $\iota_5 \in F_3$ and $\iota_5 \le \epsilon_4$ but $\epsilon_4 \notin F_3$. Also, $F_4 := \{1, \rho_2, \iota_3, \epsilon_4\}$ is not an ordering filter of Y since $\rho_2, \epsilon_4 \in F_4$, $\iota_5 \le \rho_2$ and $\iota_5 \le \epsilon_4$ but $\iota_5 \notin F_4$.

In general, any ordering filter may not be a GE-filter as seen in the following example.

Example 2. *The ordering filter $F_2 := \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ in Example 1 is not a GE-filter of Y since $\rho_2 * \zeta_6 = 1 \in F_2$ and $\rho_2 \in F_2$, but $\zeta_6 \notin F_2$.*

We provide conditions for an ordering filter to be a GE-filter.

Theorem 1. *In a transitive GE-algebra, every ordering filter is a GE-filter.*

Proof. Let F be an ordering filter of Y. Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 * \wp_4 \in F$ and $\wp_3 \in F$. If $\wp_3 = 1$, then $\wp_4 = 1 * \wp_4 \in F$. Suppose that $\wp_3 \ne 1$ and $\wp_4 \ne 1$. Then there exists $\wp_5 \in F$ such that $\wp_5 \le \wp_3 * \wp_4$ and $\wp_5 \le \wp_3$ by (22). Using (GE2), (2), (7) and (9), we have

$$1 = \wp_5 * (\wp_3 * \wp_4) \leq \wp_3 * (\wp_5 * \wp_4) \leq (\wp_5 * \wp_3) * (\wp_5 * (\wp_5 * \wp_4))$$
$$= (\wp_5 * \wp_3) * (\wp_5 * \wp_4) = 1 * (\wp_5 * \wp_4) = \wp_5 * \wp_4,$$

which implies from (10) and (16) that $1 = \wp_5 * \wp_4$, i.e., $\wp_5 \leq \wp_4$. Hence $\wp_4 \in F$ by (21), and hence F is a GE-filter of Y. □

Corollary 1. *Every ordering filter is a GE-filter in a belligerent GE-algebra.*

Proof. If Y is a belligerent GE-algebra, then

$$(\wp_3 * \wp_4) * ((\wp_5 * \wp_3) * (\wp_5 * \wp_4)) = (\wp_3 * \wp_4) * (\wp_5 * (\wp_3 * \wp_4))$$
$$= (\wp_3 * \wp_4) * (\wp_5 * ((\wp_3 * \wp_4) * (\wp_3 * \wp_4)))$$
$$= (\wp_3 * \wp_4) * (\wp_5 * 1) = (\wp_3 * \wp_4) * 1 = 1,$$

and so $\wp_3 * \wp_4 \leq (\wp_5 * \wp_3) * (\wp_5 * \wp_4)$ for all $\wp_3, \wp_4, \wp_5 \in Y$. Thus Y is a transitive GE-algebra, and hence every ordering filter is a GE-filter by Theorem 1. □

In the next example, we show there exists a GE-filter that is not an ordering filter.

Example 3. *We take a GE-algebra* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5, \zeta_6\}$ *in which the binary operation "$*$" is provided in Table 2.*

TABLE 2. The binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5	ζ_6
1	1	ρ_2	ι_3	ϵ_4	ι_5	ζ_6
ρ_2	1	1	1	ϵ_4	ϵ_4	ζ_6
ι_3	1	ρ_2	1	ι_5	ι_5	ζ_6
ϵ_4	1	1	ι_3	1	1	ζ_6
ι_5	1	1	1	1	1	ζ_6
ζ_6	1	ρ_2	ι_3	ϵ_4	ι_5	1

The set $F := \{1, \iota_3, \zeta_6\}$ *is a GE-filter of* Y*, but it is not an ordering filter of* Y *because there does not exist* $\wp_5 \in F$ *such that* $\wp_5 \leq \iota_3$ *and* $\wp_5 \leq \zeta_6$.

We would like to explore the conditions necessary for a GE-filter to be an ordering filter.

For every elements $\hbar_1$ and $\hbar_2$ of Y, we consider the set:

$$(Y; \hbar_2, \hbar_1) := \{\wp_3 \in Y \mid \hbar_2 \leq \hbar_1 * \wp_3\}. \tag{23}$$

It is clear that $1, \hbar_1, \hbar_2 \in (Y; \hbar_2, \hbar_1)$ and $(Y; 1, 1) = \{1\}$. If $(Y; \hbar_2, \hbar_1)$ has the least element, it will be denoted by $\hbar_2 \circledast \hbar_1$.

Definition 6 ([13]). *A GE-algebra Y is called an $\circledast$-GE-algebra if there exists $\hbar_1 \circledast \hbar_2$ for all $\hbar_1, \hbar_2 \in Y$.*

Lemma 3 ([13]). *If Y is an $\circledast$-GE-algebra, then*

$$(\forall \wp_3, \wp_4 \in Y)(\wp_3 \circledast \wp_4 \leq \wp_3,\ \wp_3 \circledast \wp_4 \leq \wp_4). \tag{24}$$

Theorem 2. *Every GE-filter is an ordering filter in an $\circledast$-GE-algebra.*

Proof. Let F be a GE-filter of an $\circledast$-GE-algebra Y, and let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 \in F$ and $\wp_3 \leq \wp_4$. Then $\wp_3 * \wp_4 = 1 \in F$, and thus $\wp_4 \in F$ by (20). Let $\wp_3, \wp_4 \in F$. Since $\wp_3 \leq \wp_4 * (\wp_3 \circledast \wp_4)$, we get $\wp_3 \circledast \wp_4 \in F$ by Lemma 2 and (20). Using Lemma 3, we can see that F is an ordering filter of Y. □

4. Left Mappings

Definition 7. *A self mapping $\eth$ on a GE-algebra Y is called a left mapping of Y if it satisfies:*

$$(\forall \wp_3, \wp_4 \in Y)(\eth(\wp_3 * \wp_4) = \wp_3 * \eth(\wp_4)). \tag{25}$$

It is clear that the identity mapping $\eth : Y \to Y$, $\wp_3 \mapsto \wp_3$, is a left mapping of Y.

Example 4. *We take a GE-algebra $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ with the Cayley table which is given in Table 3.*

Table 3. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ϵ_4	ϵ_4
ι_3	1	1	1	ι_5	ι_5
ϵ_4	1	ρ_2	ρ_2	1	1
ι_5	1	ρ_2	ι_3	1	1

Let $\eth$ be a self mapping on Y given as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4, \iota_5\}, \\ \rho_2 & \text{otherwise.} \end{cases}$$

It is easy to verify that $\eth$ is a left mapping of Y.

Proposition 2. *Given a left mapping $\eth$ of Y, we have*

(i) $\eth(1) = 1$,
(ii) $(\forall \wp_3 \in Y)\ (\wp_3 \leq \eth(\wp_3))$,
(iii) $(\forall \wp_3 \in Y)\ (\eth(\wp_3 * 1) = 1)$,
(iv) $(\forall \wp_3, \wp_4 \in Y)\ (\wp_3 \leq \wp_4 \ \Rightarrow\ \wp_3 \leq \eth(\wp_4))$.

Proof. (i) Using (GE1), (6) and (25), we get $\eth(1) = \eth(\eth(1) * 1) = \eth(1) * \eth(1) = 1$.

(ii) Using (GE1) and (i) and (25) induces $1 = \eth(1) = \eth(\wp_3 * \wp_3) = \wp_3 * \eth(\wp_3)$, that is, $\wp_3 \leq \eth(\wp_3)$ for all $\wp_3 \in Y$.

(iii) Using (6) and (i) induces $\eth(\wp_3 * 1) = \eth(1) = 1$ for all $\wp_3 \in Y$.

(iv) Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 \leq \wp_4$. Then $1 = \eth(1) = \eth(\wp_3 * \wp_4) = \wp_3 * \eth(\wp_4)$ by (25), and so $\wp_3 \leq \eth(\wp_4)$. □

Definition 8. *The GE-kernel of a left mapping* $\eth$ *of* Y *is defined to be the set:*

$$ker(\eth) := \{\wp_3 \in Y \mid \eth(\wp_3) = 1\}. \tag{26}$$

Theorem 3. *If a left mapping* $\eth$ *of* Y *is injective, then* $ker(\eth) = \{1\}$.

Proof. Suppose $\eth$ is an injective left mapping of Y and let $\wp_3 \in ker(\eth)$. Then $\eth(\wp_3) = 1 = \eth(1)$ by Proposition 2(i), and so $\wp_3 = 1$ since $\eth$ is injective. Hence $ker(\eth) = \{1\}$. □

The following example shows that the converse of Theorem 3 is not true, that is, any left mapping $\eth$ of Y with $ker(\eth) = \{1\}$ may not be injective.

Example 5. *Consider a GE-algebra* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ *with the Cayley table which is given in Table 4.*

TABLE 4. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ϵ_4	ϵ_4
ι_3	1	1	1	ι_5	ι_5
ϵ_4	1	ρ_2	ρ_2	1	1
ι_5	1	ρ_2	ρ_2	1	1

Define a self mapping $\eth$ *on* Y *as follows:*

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 = 1, \\ \rho_2 & \text{if } \wp_3 \in \{\rho_2, \iota_3\}, \\ \epsilon_4 & \text{if } \wp_3 = \epsilon_4, \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

Then $\eth$ *is a left mapping of* Y *with* $ker(\eth) = \{1\}$. *But it is not injective since* $\eth(\rho_2) = \rho_2 = \eth(\iota_3)$ *but* $\rho_2 \neq \iota_3$.

Theorem 4. *If a GE-algebra* Y *is antisymmetric and transitive, then every left mapping* $\eth$ *of* Y *with* $ker(\eth) = \{1\}$ *is injective.*

Proof. Let $\eth$ be a self mapping of a transitive and antisymmetric GE-algebra Y and $ker(\eth) = \{1\}$. Let's take $\wp_3, \wp_4 \in Y$ which satisfies $\eth(\wp_3) = \eth(y)$. Then

$\eth(\wp_3) * \eth(\wp_4) = 1$ by (GE1), and so $\eth(\eth(\wp_3) * \wp_4) = 1$ by (25), that is, $\eth(\wp_3) * \wp_4 \in ker(\eth) = \{1\}$. Hence $\eth(\wp_3) \leq \wp_4$. It follows from Proposition 2(ii) that $\wp_3 \leq \eth(\wp_3) \leq \wp_4$. Similarly, we can induce $\wp_4 \leq \wp_3$ for all $\wp_3, \wp_4 \in Y$. Hence $\wp_3 = \wp_4$, and $\eth$ is injective. □

Theorem 5. *In an antisymmetric GE-algebra, every injective left mapping is the identity mapping.*

Proof. Let $\eth$ be an injective left mapping of an antisymmetric GE-algebra Y. Then $\wp_3 \leq \eth(\wp_3)$ for all $\wp_3 \in Y$ by Proposition 2(ii). Using (GE1), (25) and Proposition 2(i) induces $\eth(1) = 1 = \eth(\wp_3) * \eth(\wp_3) = \eth(\eth(\wp_3) * \wp_3)$ for all $\wp_3 \in Y$. Since $\eth$ is injective, we have $\eth(\wp_3) * \wp_3 = 1$, i.e., $\eth(\wp_3) \leq \wp_3$. Thus $\eth(\wp_3) = \wp_3$ for all $\wp_3 \in Y$ since Y is antisymmetric. Therefore $\eth$ is the identity mapping. □

In the next example, we claim that if Y is not antisymmetric, then any injective left mapping may not be the identity mapping.

Example 6. *Consider a set* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ *with the Cayley table which is given in Table 5.*

TABLE 5. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	ι_3	ϵ_4	ι_5
ι_3	1	1	1	1	ι_5
ϵ_4	1	1	1	1	ι_5
ι_5	1	ρ_2	1	1	1

Then Y is a GE-algebra which is not antisymmetric. Define a self mapping $\eth$ on Y as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 = 1, \\ \rho_2 & \text{if } \wp_3 = \rho_2, \\ \epsilon_4 & \text{if } \wp_3 = \iota_3, \\ \iota_3 & \text{if } \wp_3 = \epsilon_4, \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

Then $\eth$ is an injective mapping of Y which is not an identity mapping of Y.

Theorem 6. *If $\eth$ is a left mapping of Y, then $ker(\eth)$ and $Im(\eth)$ are GE-subalgebras of Y.*

Proof. Let $\wp_3, \wp_4 \in ker(\eth)$. Then $\eth(\wp_3) = 1 = \eth(\wp_4)$. Hence $\eth(\wp_3 * \wp_4) = \wp_3 * \eth(\wp_4) = \wp_3 * 1 = 1$ by (6) and (25), i.e., $\wp_3 * \wp_4 \in ker(\eth)$. Thus $\ker(\eth)$ is a GE-subalgebra of Y.

Let $\hbar_1, \hbar_2 \in Im(\eth)$. Then there exist $\hbar_3, \hbar_4 \in Y$ such that $\eth(\hbar_3) = \hbar_1$ and $\eth(\hbar_4) = \hbar_2$. Now $\hbar_3 \in Y$ implies that $\eth(c) \in Y$, and so $\hbar_1 * \hbar_2 = \eth(\hbar_3) * \eth(\hbar_4) = \eth(\eth(\hbar_3) * \hbar_4) \in Im(\eth)$. Hence $Im(\eth)$ is a GE-subalgebra of Y. □

In the following example, we can see that $Im(\eth)$ is neither ordering filter nor GE-filter.

Example 7. *Let $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ be the GE-algebra in Example 5. Define a self mapping $\eth$ on Y as follows:*

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4, \iota_5\} \\ \rho_2 & \text{if } \wp_3 \in \{\rho_2, \iota_3\}. \end{cases}$$

Then $\eth$ is a left mapping of Y with $Im(\eth) = \{1, \rho_2\}$. But $Im(\eth)$ is neither an ordering filter of Y nor a GE-filter of Y since $\rho_2 \leq \iota_3$ and $\rho_2 \in Im(\eth)$ but $\iota_3 \notin Im(\eth)$.

Question 9. *If $\eth$ is a left mapping of Y, is $ker(\eth)$ a GE-filter of Y or an ordering filter of Y?*

The next example shows that the answer to Question 9 is negative.

Example 8. *1. Consider a GE-algebra $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ with the Cayley table which is given in Table 6.*

Table 6. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ϵ_4	ϵ_4
ι_3	1	1	1	ι_5	ι_5
ϵ_4	1	ρ_2	1	1	1
ι_5	1	ρ_2	ι_3	1	1

Define a self mapping $\eth$ on Y as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \iota_3\} \\ \rho_2 & \text{if } \wp_3 = \rho_2, \\ \epsilon_4 & \text{if } \wp_3 = \epsilon_4, \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

Then $\eth$ is a left mapping of Y and its kernel is $ker(\eth) = \{1, \iota_3\}$ which is not a GE-filter of Y since $\iota_3 * \rho_2 = 1 \in ker(\eth)$ and $\iota_3 \in ker(\eth)$, but $\rho_2 \notin ker(\eth)$.

2. Consider a set $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ with the Cayley table which is given in Table 7.

Then Y is a GE-algebra. Define a self mapping $\eth$ on Y as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \rho_2, \epsilon_4, \iota_5\} \\ \iota_3 & \text{if } \wp_3 = \iota_3. \end{cases}$$

TABLE 7. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ϵ_4	1
ι_3	1	ρ_2	1	ϵ_4	ρ_2
ϵ_4	1	ι_5	ι_3	1	ι_5
ι_5	1	1	1	ϵ_4	1

Then $\eth$ *is a left mapping of* Y *with* $ker(\eth) = \{1, \rho_2, \epsilon_4, \iota_5\}$. *But* $ker(\eth)$ *is not an ordering filter of* Y *since* $\rho_2 \leq \iota_3$ *and* $\rho_2 \in ker(\eth)$ *but* $\iota_3 \notin ker(\eth)$.

We explore the conditions under which a positive answer to Question 9 may come out.

Theorem 7. *If a self mapping* $\eth$ *on* Y *is an endomorphism, i.e.,* $\eth(\wp_3 * \wp_4) = \eth(\wp_3) * \eth(\wp_4)$ *for all* $\wp_3, \wp_4 \in Y$*, then* $ker(\eth)$ *is a GE-filter of* Y.

Proof. Assume that $\eth : Y \to Y$ is an endomorphism. Then $\eth(1) = \eth(\wp_3 * \wp_3) = \eth(\wp_3) * \eth(\wp_3) = 1$ for all $\wp_3 \in Y$, that is, $1 \in ker(\eth)$. Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 * \wp_4 \in ker(\eth)$ and $\wp_3 \in ker(\eth)$. Since $\eth$ is an endomorphism, it follows that

$$1 = \eth(\wp_3 * \wp_4) = \eth(\wp_3) * \eth(\wp_4) = 1 * \eth(\wp_4) = \eth(\wp_4),$$

that is $\wp_4 \in ker(\eth)$. Therefore $ker(\eth)$ is a GE-filter of Y. □

Corollary 2. *Let* $\eth$ *be a left mapping of* Y. *If* $\eth$ *is an endomorphism, then* $ker(\eth)$ *is a GE-filter of* Y.

Theorem 8. *Let* $\eth$ *be a left mapping of* Y *which is idempotent, that is,* $\eth(\eth(\wp_3)) = \eth(\wp_3)$ *for all* $\wp_3 \in Y$. *If* Y *is commutative, then* $ker(\eth)$ *is a GE-filter of* Y.

Proof. We first show the following assertion.

$$(\forall \wp_3, \wp_4 \in Y)(\wp_3 \in ker(\eth),\ \wp_3 \leq \wp_4 \ \Rightarrow\ \wp_4 \in ker(\eth)). \tag{27}$$

Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 \in ker(\eth)$ and $\wp_3 \leq \wp_4$. Then $\wp_4 = (\wp_4 * \wp_3) * \wp_3$ since Y is commutative. Hence

$$\eth(\wp_4) = \eth((\wp_4 * \wp_3) * \wp_3) = (\wp_4 * \wp_3) * \eth(\wp_3) = (\wp_4 * \wp_3) * 1 = 1,$$

and so $\wp_4 \in ker(\eth)$. It is clear that $1 \in ker(\eth)$ by Proposition 2(i). Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 * \wp_4 \in ker(\eth)$ and $\wp_3 \in ker(\eth)$. Then $1 = \eth(\wp_3 * \wp_4) = \wp_3 * \eth(\wp_4)$, and so $\wp_3 \leq \eth(\wp_4)$. It follows from (27) that $\eth(\wp_4) \in ker(\eth)$. Thus $1 = \eth(\eth(\wp_4)) = \eth(\wp_4)$ by the idempotency of $\eth$ which shows that $\wp_4 \in ker(\eth)$. Therefore $ker(\eth)$ is a GE-filter of Y. □

In Theorem 8, if Y is not commutative, then $ker(\eth)$ is not a GE-filter of Y as shown in the following example.

Example 9. *Consider a set* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ *with the Cayley table which is given in Table 8.*

TABLE 8. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	ι_5	ι_5
ι_3	1	1	1	ϵ_4	ι_5
ϵ_4	1	1	1	1	1
ι_5	1	ρ_2	ι_3	1	1

Then Y *is a GE-algebra, and it is not commutative since* $(\rho_2 * \iota_3) * \iota_3 = 1 * \iota_3 = \iota_3 \neq \rho_2 = 1 * \rho_2 = (\iota_3 * \rho_2) * \rho_2$. *Define a self mapping* $\eth$ *on* Y *as follows:*

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4, \iota_5\}, \\ \rho_2 & \text{otherwise.} \end{cases}$$

Then $\eth$ *is the idempotent left mapping of* Y, *and its kernel is* $ker(\eth) = \{1, \epsilon_4, \iota_5\}$ *which is not a GE-filter of* Y *since* $\epsilon_4 * \rho_2 = 1 \in ker(\eth)$ *and* $\epsilon_4 \in ker(\eth)$ *but* $\rho_2 \notin ker(\eth)$.

The next example shows that any left mapping may not be idempotent.

Example 10. *Consider a GE-algebra* $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ *with the Cayley table which is given in Table 9.*

TABLE 9. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	1	1	1
ι_3	1	1	1	1	1
ϵ_4	1	ρ_2	ι_3	1	1
ι_5	1	ρ_2	ι_3	1	1

Define a self mapping $\eth$ *on* Y *as follows:*

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \rho_2, \epsilon_4, \iota_5\}, \\ \rho_2 & \text{if } \wp_3 = \iota_3. \end{cases}$$

Then $\eth$ *is a left mapping of* Y. *But it is not idempotent since* $\eth(\eth(\iota_3)) = \eth(\rho_2) = 1 \neq \rho_2 = \eth(\iota_3)$.

Theorem 9. *Let ð be a left mapping of Y. If ð is idempotent, then*

$$(\forall \wp_3 \in Y)(ð(\wp_3) = \wp_3 \Leftrightarrow \wp_3 \in Im(ð)). \quad (28)$$

$$ker(ð) \cap Im(ð) = \{1\}. \quad (29)$$

Proof. Let ð be an idempotent left mapping of Y. It is clear that if $ð(\wp_3) = \wp_3$, then $\wp_3 \in Im(ð)$. Let $\wp_3 \in Im(ð)$. Then there exists $\wp_4 \in Y$ such that $ð(\wp_4) = \wp_3$. Hence $ð(\wp_3) = ð(ð(\wp_4)) = ð(\wp_4) = \wp_3$, and thus (28) is valid. If $\wp_3 \in ker(ð) \cap Im(ð)$, then $ð(\wp_3) = 1$ and $ð(\wp_4) = \wp_3$ for some $\wp_4 \in Y$. Hence $1 = ð(\wp_3) = ð(ð(\wp_4)) = ð(\wp_4) = \wp_3$, and so $ker(ð) \cap Im(ð) = \{1\}$. □

Lemma 4. *Every commutative GE-algebra Y satisfies:*

$$(\forall \wp_3, \wp_4 \in Y) (\wp_3 \leq \wp_4 \Rightarrow (\exists \hbar_1 \in Y)(\wp_4 = \hbar_1 * \wp_3)). \quad (30)$$

Proof. Let $\wp_3, \wp_4 \in Y$ be such that $\wp_3 \leq \wp_4$. Then $\wp_3 * \wp_4 = 1$ and so

$$\wp_4 = 1 * \wp_4 = (\wp_3 * \wp_4) * \wp_4 = (\wp_4 * \wp_3) * \wp_3 = \hbar_1 * \wp_3$$

where $\hbar_1 = \wp_4 * \wp_3$. □

Lemma 5. *Every GE-algebra Y satisfies:*

$$(\forall \wp_3, \wp_4 \in Y) ((\exists \hbar_1 \in Y)(\wp_4 = \hbar_1 * \wp_3) \Rightarrow \wp_3 \leq \wp_4). \quad (31)$$

Proof. Suppose that $\wp_4 = \hbar_1 * \wp_3$ for some $\hbar_1 \in Y$. Then

$$\wp_3 * \wp_4 = \wp_3 * (\hbar_1 * \wp_3) = \wp_3 * (\hbar_1 * (\wp_3 * \wp_3)) = \wp_3 * (\hbar_1 * 1) = \wp_3 * 1 = 1$$

by (GE1), (GE3) and (6). Hence $\wp_3 \leq \wp_4$. □

Proposition 3. *Let Y be a commutative GE-algebra which satisfies:*

$$(\forall \wp_3, \wp_4 \in Y)((((\wp_3 * \wp_4) * \wp_4) * \wp_4) = \wp_3 * \wp_4). \quad (32)$$

If ð is a left mapping of Y, then

$$(\forall \wp_3 \in Y)(\exists (\wp_4, \wp_5) \in ker(ð) \times Im(ð))(\wp_5 = \wp_4 * \wp_3). \quad (33)$$

Proof. Since $\wp_3 \leq ð(\wp_3)$ for all $\wp_3 \in Y$ by Proposition 2(ii), it follows from Lemma 4 that $ð(\wp_3) = \hbar_1 * \wp_3$ for some $\hbar_1 \in Y$. Hence

$$(ð(\wp_3) * \wp_3) * \wp_3 = ((\hbar_1 * \wp_3) * \wp_3) * \wp_3 = \hbar_1 * \wp_3 = ð(\wp_3)$$

by (32). If we take $\wp_5 := ð(\wp_3)$ and $\wp_4 := ð(\wp_3) * \wp_3$, then $(\wp_4, \wp_5) \in ker(ð) \times Im(ð)$ and $\wp_5 = \wp_4 * \wp_3$. □

Proposition 4. *Let ð be a left mapping of Y. If ð is idempotent, then*

$$(\forall \wp_3 \in Y)(\exists (\wp_4, \wp_5) \in ker(ð) \times Im(ð))(\wp_4 = \wp_5 * \wp_3). \quad (34)$$

Proof. Suppose that $\eth$ is an idempotent left mapping of Y. Then $\eth(\eth(\wp_3)) = \eth(\wp_3)$ for all $\wp_3 \in Y$, and so

$$\eth(\eth(\eth(\wp_3)) * \wp_3) = \eth(\eth(\wp_3)) * \eth(\wp_3) = 1.$$

Hence $\eth(\wp_3) * \wp_3 = \eth(\eth(\wp_3)) * \wp_3 \in ker(\eth)$. It follows that $\wp_5 * \wp_3 = \wp_4$ for some $\wp_4 \in ker(\eth)$ and $\wp_5 := \eth(\wp_3) \in Im(\eth)$. □

Proposition 5. *Every left mapping $\eth$ of a commutative GE-algebra satisfies the condition (34).*

Proof. Let $\eth$ be a left mapping of a commutative GE-algebra Y. Since $\wp_3 \leq \eth(\wp_3)$ for all $\wp_3 \in Y$ by Proposition 2(ii), it follows from Lemma 4 that $\eth(\wp_3) = \hbar_1 * \wp_3$ for some $\hbar_1 \in Y$. Hence

$$\eth(\eth(\wp_3)) = \eth(\hbar_1 * \wp_3) = \hbar_1 * \eth(\wp_3) = \hbar_1 * (\hbar_1 * \wp_3) = \hbar_1 * \wp_3 = \eth(\wp_3)$$

for all $\wp_3 \in Y$ by (7). Hence $\eth$ is idempotent. Using Proposition 4, we know that (34) is valid. □

Denote by $L(Y)$ and $IL(Y)$ the set of all left mappings of Y and the set of all idempotent left mappings of Y, respectively. Define an operation "$\circledast$" on $L(Y)$ by $(\eth \circledast \xi)(\wp_3) = \eth(\wp_3) * \xi(\wp_3)$ for all $\wp_3 \in Y$ and $\eth, \xi \in L(Y)$.

Proposition 6. *$L(Y)$ is closed under the function composition $\circ$, that is, if $\eth$ and ξ are left mappings of Y, then $\eth \circ \xi$ is also a left mapping of Y.*

Proof. Let $\eth, \xi \in L(Y)$ and $\wp_3, \wp_4 \in Y$. Then

$$(\eth \circ \xi)(\wp_3 * \wp_4) = \eth(\xi(\wp_3 * \wp_4)) = \eth(\wp_3 * \xi(\wp_4)) = \wp_3 * \eth(\xi(\wp_4)) = \wp_3 * (\eth \circ \xi)(\wp_4),$$

and so $\eth \circ \xi$ is a left mapping of Y. □

Theorem 10. *$(L(Y), \circ)$ is a semigroup and $IL(Y)$ is a subsemigroup of $L(Y)$.*

Proof. Straightforward. □

The following example shows that $L(Y)$ is not closed under the operation "$\circledast$", that is, there are two left mappings $\eth$ and ξ of Y such that $\eth \circledast \xi$ is not a left mapping of Y.

Example 11. *Consider a GE-algebra $Y = \{1, \rho_2, \iota_3, \epsilon_4, \iota_5\}$ with the Cayley table which is given in Table 10.*
Define self mappings $\eth$ and ξ on Y as follows:

$$\eth : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4, \iota_5\} \\ \rho_2 & \text{if } \wp_3 = \rho_2, \\ \epsilon_4 & \text{if } \wp_3 = \iota_3. \end{cases}$$

TABLE 10. Cayley table for the binary operation "$*$"

$*$	1	ρ_2	ι_3	ϵ_4	ι_5
1	1	ρ_2	ι_3	ϵ_4	ι_5
ρ_2	1	1	ι_5	1	ι_5
ι_3	1	ρ_2	1	1	1
ϵ_4	1	ρ_2	1	1	1
ι_5	1	ρ_2	1	1	1

$$\xi : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \epsilon_4\} \\ \rho_2 & \text{if } \wp_3 = \rho_2, \\ \iota_3 & \text{if } \wp_3 = \iota_3, \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

Then $\eth$ and ξ are left mappings of Y and $\eth \circledast \xi$ is given as follows:

$$\eth \circledast \xi : Y \to Y,\ \wp_3 \mapsto \begin{cases} 1 & \text{if } \wp_3 \in \{1, \rho_2, \iota_3, \epsilon_4\} \\ \iota_5 & \text{if } \wp_3 = \iota_5. \end{cases}$$

We can observe that $\eth \circledast \xi$ is not a left mapping of Y since

$$(\eth\circledast\xi)(\rho_2*\iota_3) = (\eth\circledast\xi)(\iota_5) = \iota_5 \neq 1 = \rho_2*(\epsilon_4*\iota_3) = \rho_2*(\eth(\iota_3)*(\xi(3)) = \rho_2*(\eth\circledast\xi)(\iota_3).$$

We investigate the conditions under which $L(Y)$ can be closed with respect to the operation "$\circledast$".

Theorem 11. *Let Y be a belligerent GE-algebra. For every $\eth, \xi \in L(Y)$, we have*

(i) $\eth \circledast \xi \in L(Y)$.

(ii) *If $\eth \circ \xi = \xi \circ \eth$ and ξ is idempotent, then $\eth \circledast \xi \in IL(Y)$.*

Proof. (i) For every $\wp_3, \wp_4 \in Y$, we get

$$\begin{aligned}(\eth \circledast \xi)(\wp_3 * \wp_4) &= \eth(\wp_3 * \wp_4) * \xi(\wp_3 * \wp_4) = (\wp_3 * \eth(\wp_4)) * (\wp_3 * \xi(\wp_4)) \\ &= \wp_3 * (\eth(\wp_4) * \xi(\wp_4)) = \wp_3 * (\eth \circledast \xi)(\wp_4).\end{aligned}$$

Hence $\eth \circledast \xi \in L(Y)$.

(ii) For every $\wp_3 \in Y$, we have

$$\begin{aligned}&((\eth \circledast \xi) \circ (\eth \circledast \xi))(\wp_3) = (\eth \circledast \xi)((\eth \circledast \xi)(\wp_3)) = (\eth \circledast \xi)(\eth(\wp_3) * \xi(\wp_3)) \\ &= \eth(\eth(\wp_3) * \xi(\wp_3)) * \xi(\eth(\wp_3) * \xi(\wp_3)) = (\eth(\wp_3) * \eth(\xi(\wp_3))) * (\eth(\wp_3) * \xi(\xi(\wp_3))) \\ &= (\eth(\wp_3) * \xi(\eth(\wp_3))) * (\eth(\wp_3) * \xi(\wp_3)) = \xi(\eth(\wp_3) * \eth(\wp_3)) * (\eth(\wp_3) * \xi(\wp_3)) \\ &= \xi(1) * (\eth(\wp_3) * \xi(\wp_3)) = 1 * (\eth(\wp_3) * \xi(\wp_3)) = (\eth(\wp_3) * \xi(\wp_3)) = (\eth \circledast \xi)(\wp_3).\end{aligned}$$

and thus $\eth \circledast \xi \in IL(Y)$. □

Proposition 7. *Let $\eth, \xi \in L(Y)$ satisfy $(\xi \circledast \eth)(\wp_3) = 1$ for all $\wp_3 \in Y$. If Y is antisymmetric and $\eth$ is idempotent, then $Im(\eth) \subseteq Im(\xi)$.*

Proof. If $\wp_4 \in Im(\eth)$, then $\eth(\wp_4) = \wp_4$ by (28) and hence

$$\xi(\wp_4) * \wp_4 = \xi(\wp_4) * \eth(\wp_4) = (\xi \circledast \eth)(\wp_4) = 1,$$

that is, $\xi(\wp_4) \leq \wp_4$. Since $\wp_4 \leq \xi(\wp_4)$ by Proposition 2(ii) and Y is antisymmetric, we have $\wp_4 = \xi(\wp_4) \in Im(\xi)$. Thus $Im(\eth) \subseteq Im(\xi)$. □

Theorem 12. *For every $\eth, \xi \in L(Y)$, we have*

(i) *If $\eth \circ \xi = \xi \circ \eth$, $Im(\eth) \subseteq Im(\xi)$ and ξ is idempotent, then $\xi \circledast \eth$ is constant on Y with the value 1.*

(ii) *If $\eth$ is idempotent, then $ker(\xi) \cap Im(\eth) \subseteq Im(\xi \circledast \eth)$.*

Proof. (i) Assume that $\eth \circ \xi = \xi \circ \eth$, $Im(\eth) \subseteq Im(\xi)$ and ξ is idempotent. Then Theorem 9 yields $(\xi \circ \eth)(\wp_3) = \eth(\wp_3)$ for all $\wp_3 \in Y$. Hence

$$\begin{aligned}(\xi \circledast \eth)(\wp_3) &= \xi(\wp_3) * \eth(\wp_3) = \xi(\wp_3) * (\xi \circ \eth)(\wp_3) \\ &= \xi(\wp_3) * (\eth \circ \xi)(\wp_3) = \eth(\xi(\wp_3) * \xi(\wp_3)) \\ &= \eth(1) = 1\end{aligned}$$

for all $\wp_3 \in Y$.

(ii) Suppose that $\eth$ is idempotent and let $\wp_4 \in ker(\xi) \cap Im(\eth)$. Then $\xi(\wp_4) = 1$ and $\eth(\wp_3) = \wp_4$ for some $\wp_3 \in Y$. It follows that

$$\wp_4 = \eth(\wp_3) = 1 * \eth(\eth(\wp_3)) = \xi(\wp_4) * \eth(\wp_4) = (\xi \circledast \eth)(\wp_4) \in Im(\xi \circledast \eth).$$

Thus $ker(\xi) \cap Im(\eth) \subseteq Im(\xi \circledast \eth)$.

□

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STABILITY ANALYSIS OF NEUTRAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

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Abstract. The study establishes the stability bounds of the second-order neutral Volterra integro-differential equation concerning both the right-side and initial conditions. The examples are given to show the applicability of the method and confirm the predicted theoretical analysis.

1. Introduction

Numerous scientific models and many disciplines lead to integro-differential equations (IDEs). This makes it attractive to use different methods to solve them (see, e.g. [1–5]).

IDEs are categorized by the interval of their integral terms. Volterra integro-differential equation (VIDE) is those where integration limits are variables, whereas Fredholm IDE is integration limits that only involve constants. VIDEs were first introduced by Vito Volterra in 1926, and since then many studies have been carried out on the VIDEs.

In recent years, many researchers have investigated the qualitative behaviors of solutions to these equations. For example, in [6], the authors proposed a method for obtaining sufficient conditions for the stability of solutions of systems of linear VIDEs. They give adequate criteria for the stability of the solutions of VIDE when the initial conditions are perturbed. In [7], presented some explicit criteria for the uniform asymptotic and the exponential stability of the nonlinear VIDE using spectral properties of Metzler matrices and the comparison principle. Amirali, in [8], establishes the stability inequalities for the linear nonhomogeneous Volterra

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delay integro-differential equations (VDIDEs). The author shows that the solution continuously depends on the right-side and initial data. Alahmadi et al. [9], utilize the Lyapunov functionals combined with the Laplace transform to obtain boundedness and stability results about nonlinear VIDE solutions. Yu et al. [10], are concerned with the numerical stability of Runge-Kutta methods for nonlinear neutral VDIDE. The stability analysis of exact solutions of the linear neutral VDIDE system is considered in [11]. The method authors use, shows that it preserves the delay-independent stability of exact solutions. In [12], the authors present some estimates for the exact solution of the neutral VDIDE, which show the stability of the problem for the right-side and initial condition. In [13], using the positivity of linear VIDE, the authors give an explicit criterion for the uniform asymptotic stability of positive equations. Amirali et al. [14], consider the stability inequalities which can be established for any order of derivative for high-order linear VDIDEs. Yapman et al. [15],study stability analysis of exact solution and convergence analysis of a fitted numerical method for a singularly perturbed nonlinear VIDE with delay. In [16], the authors give the stability inequalities for the following neutral VIDE with respect to the initial conditions and the right-hand side. Panda et al. [17], present stability analysis of first-order singularly perturbed VIDE.

The goal of this paper is to present the stability inequalities for the neutral second-order VIDE:

$$u''(t) + a(t)\,u(t) - \int_0^t \left[K_1(t,s)\,u''(s) + K_2(t,s)\,u(s)\right] ds = f(t),\ t \in \Omega = (0,T] \tag{1}$$

$$u(0) = A, \quad u'(0) = B, \tag{2}$$

where $a(t), f(t)\ \left(t \in \bar{\Omega} \equiv [0,T]\right)$ and $K_i(t,s),\ i = 1,2, \left((t,s) \in \bar{\Omega}^2\right)$ are the sufficiently smooth functions satisfying certain regularity conditions to be specified.

2. Stability Bounds for the Differential Problem

Here we establish stability bounds regarding the right-side and initial conditions for the problem (1)-(2).

For any function $g(t) \in C\left(\bar{\Omega}\right)$ we use $\|g\|_\infty \equiv \|g\|_{\infty,\bar{\Omega}} := \max_{\bar{\Omega}} |g(t)|$.

Theorem 1. *If $a(t), f(t) \in C\left(\bar{\Omega}\right)$, $K_1, K_2 \in C\left(\bar{\Omega}^2\right)$, then for the solution $u(t)$ of (1)-(2) holds the following inequality:*

$$\|u\|_\infty \le \alpha e^{\beta}, \tag{3}$$

where

$$\alpha = T^2\left(1 + \bar{K}_1 e^{\bar{K}_1 T} T\right)\|f\|_\infty + |A| + T\,|B|,$$

$$\beta = T^2 \left(\|a\|_\infty + \mu T\right),$$
$$\mu = \bar{K}_2 + \|a\|_\infty \bar{K}_1 e^{\bar{K}_1 T} + T\bar{K}_2 \bar{K}_1 e^{\bar{K}_1 T}$$

and

$$\bar{K}_1 = \max_{\bar{\Omega}^2} \left|K_1(t,s)\right|,$$
$$\bar{K}_2 = \max_{\bar{\Omega}^2} \left|K_2(t,s)\right|.$$

Proof. Denoting $\delta(t) = |u''(t)|$, we get

$$\delta(t) \leq \rho(t) + \int_0^t \bar{K}_1 \delta(s)\, ds,$$

where

$$\rho(t) = |f(t)| + |a(t)|\,|u(t)| + \int_0^t |K_2(t,s)|\,|u(s)|\, ds.$$

Then by Gronwall's inequality we have

$$\delta(t) \leq \rho(t) + \bar{K}_1 \int_0^t \rho(s)\, e^{\bar{K}_1(t-s)} ds.$$

Since

$$\rho(t) \leq \|a\|_\infty |u(t)| + \bar{K}_2 \int_0^t |u(s)| ds + \|f\|_\infty$$

we get

$$\begin{aligned}
|u''(t)| \leq & \|a\|_\infty |u(t)| + \|f\|_\infty + \bar{K}_2 \int_0^t |u(s)|\, ds \\
& + \bar{K}_1 e^{\bar{K}_1 T} \int_0^t [\|a\|_\infty |u(s)| + \|f\|_\infty] ds + \bar{K}_2 \bar{K}_1 e^{\bar{K}_1 T} \int_0^t \int_0^s |u(\zeta)|\, d\zeta ds \\
= & \|a\|_\infty |u(t)| + \|f\|_\infty + \left(\bar{K}_2 + \|a\|_\infty \bar{K}_1 e^{\bar{K}_1 T}\right) \int_0^t |u(s)|\, ds \\
& + \bar{K}_2 \bar{K}_1 e^{\bar{K}_1 T} \int_0^t (t-s)\, |u(s)|\, ds + \bar{K}_1 e^{\bar{K}_1 T} \int_0^t \|f\|_\infty ds
\end{aligned}$$

$$\leq \|a\|_{\infty} |u(t)| + \left(1 + \bar{K}_1 e^{\bar{K}_1 T} T\right) \|f\|_{\infty} + \mu \int_0^t |u(s)|\, ds. \tag{4}$$

Next, using the following relations which is true for any $g \in C^2$

$$g(t) = g(0) + t g'(0) + \int_0^t (t-s)\, g''(s)\, ds,$$

$$|g(t)| \leq |g(0)| + T\,|g'(0)| + T \int_0^t |g''(s)|\, ds$$

and

$$\int_0^t |u''(s)| \geq \frac{1}{T}|u(t)| - \frac{1}{T}|u(0)| - |u'(0)|,$$

the inequality (4) reduces to

$$\begin{aligned} |u(t)| \leq & T^2 \left(1 + \bar{K}_1 e^{\bar{K}_1 T} T\right) \|f\|_{\infty} + |A| + T\,|B| \\ & + T\left(\|a\|_{\infty} + \mu T\right) \int_0^t |u(s)|\, ds. \end{aligned}$$

Finally, applying the Gronwall's inequality we get

$$|u(t)| \leq \left[T^2 \left(1 + \bar{K}_1 e^{\bar{K}_1 T} T\right) \|f\|_{\infty} + |A| + T\,|B|\right] e^{Tt\left(\|a\|_{\infty} + \mu T\right)},$$

which proves Theorem (1).

□

3. Numerical Examples

This section includes examples that confirm the theoretical methodology.

Example 1. *Consider the following problem:*

$$u''(t) + u(t) - \int_0^t \left[\frac{t}{20} u''(s) + \left(\frac{t+s}{40}\right) u(s)\right] ds = \frac{t - \sin t}{40}, \ \ 0 < t \leq 1,$$

$$u(0) = 0, \quad u'(0) = 1.$$

The solution is given by

$$u\,(t) = \sin t.$$

Since

$$T = 1, \quad \bar{K}_1 = 0.05, \quad \bar{K}_2 = 0.05, \quad \mu = 0.1052,$$

$$\|f\|_\infty = 0.004, \quad |A| = 0, \quad |B| = 1, \quad \|a\|_\infty = 1,$$

the bound will be

$$|u\,(t)| \leq 1.0042 \times e^{1.1052t}.$$

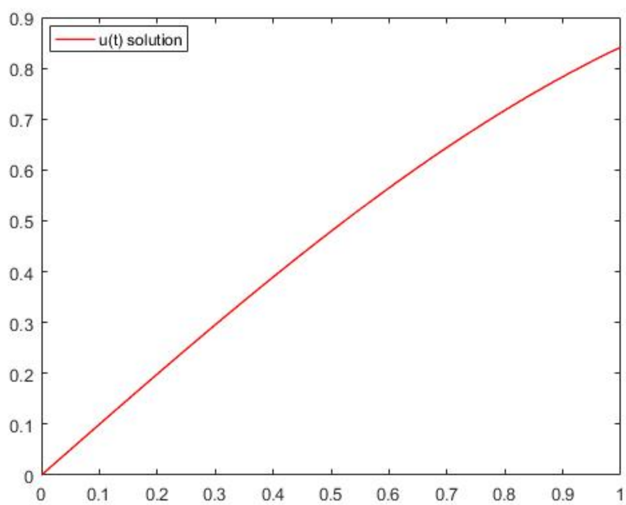

FIGURE 1. $u(t)$ solution

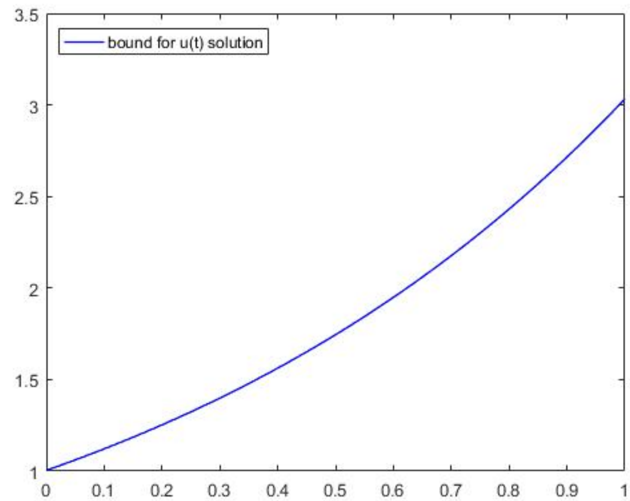

FIGURE 2. $\bar{u}(t) = 1.0042 \times exp(1.1052t)$

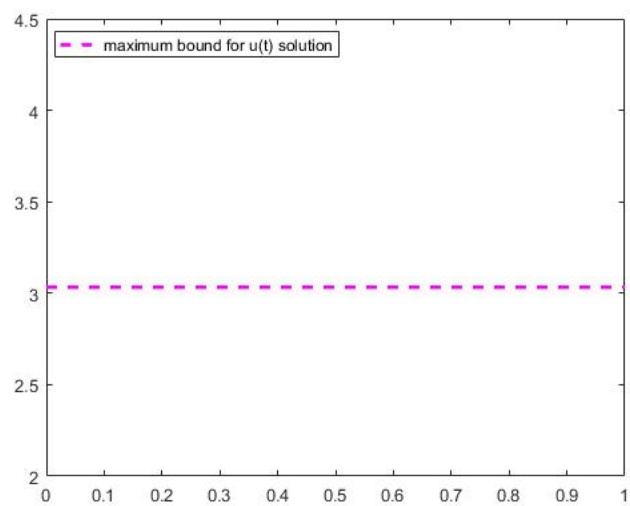

FIGURE 3. Maximum bound for the solution

Example 2. *Now give an another problem, which is defined as follows:*

$$u''(t) + \frac{t}{12}u(t) - \int_0^t \left[\sqrt{\frac{ts}{2}}u''(s) + s^2 u(s)\right] ds = \frac{t}{24}\left(t - 1 - 3t^3 + 4t^2\right), \ 0 < t \leq 0.75,$$

$$u(0) = \frac{-1}{2}, \quad u'(0) = \frac{1}{2}.$$

The solution of the problem is

$$u(t) = \frac{t-1}{2}.$$

Since

$$T = 0.75, \quad \bar{K}_1 = 0.5303, \quad \bar{K}_2 = 0.5625, \quad \mu = 0.9448,$$

$$\|f\|_\infty = 0.0229, \quad |A| = \frac{1}{2}, \quad |B| = \frac{1}{2}, \quad \|a\|_\infty = 0.0625,$$

bound for the solution $u(t)$ *will be*

$$|u(t)| \leq 0.8955 \times e^{0.5783t}.$$

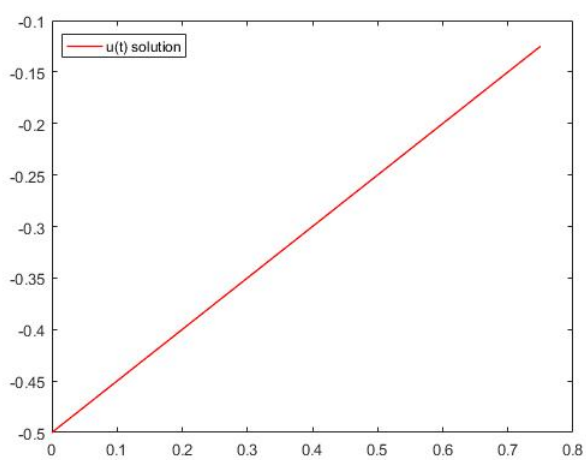

FIGURE 4. $u(t)$ solution

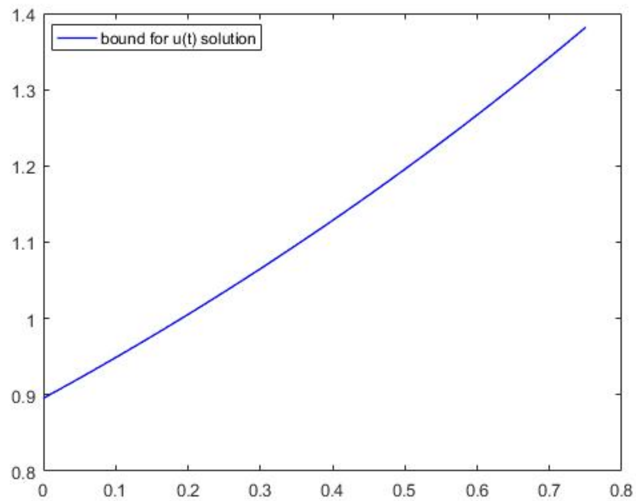

FIGURE 5. $\bar{u}(t) = 0.8955 \times exp(0.5783t)$

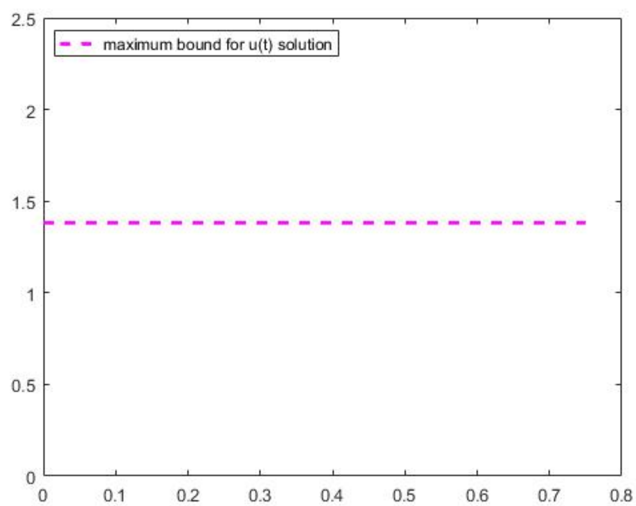

FIGURE 6. Maximum bound for the solution

Figure (1)- Figure (6) show that as t values increase, the bound of the solution expands.

4. Conclusion

This work presented the stability inequalities in respect to the right-side and initial conditions for the second-order neutral Volterra integro-differential equation. We showed that the bound of solution expressed by the inequality (3). Theoretical results are supported with examples.

Author Contribution Statements All authors contributed equally to the writing of this paper.

Declaration of Competing Interests The authors declare that they have no competing interest Author's contributions. All authors read and approved the final manuscript.

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IDEAL THEORY OF (m, n)-NEAR RINGS

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Abstract. The aim of this research work is to define and characterize a new class of n-ary algebras that we call (m, n)-near rings. We investigate the notions of i-R-groups, i-(m, n)-near field, prime ideals, primary ideals and subtractive ideals of (m, n)-near rings. We describe the concept of homomorphisms between (m, n)-near rings that preserve the (m, n)-near ring structure, and give some results in this respect.

1. Introduction

Polyadic groups were introduced in 1928 by W. Dörnte [10]. An important role in n-group theory is the paper [12], for more details see [7,11]. Then, n-ary operations are used then in the study of (m, n)-rings [5,6,13] and (m, n)-semirings [1,3,8].

Let A be a non-empty set. A map $h : A^m \longrightarrow A$ is called an m-ary operation. A non-empty set A with an m-ary operation h is called an m-ary groupoid that is denoted by (A, h). The sequence $z_i, z_{i+1}, ..., z_m$ is denoted by z_i^m where $1 \leq i \leq m$. For all $1 \leq i \leq j \leq m$, the phrase $h(z_1, z_2, ..., z_i, k_{i+1}, ..., k_j, l_{j+1}, ..., l_m)$ is represented as $h(z_1^i, k_{i+1}^j, l_{j+1}^m)$. In this case when $k_{i+1} = k_{i+2} = ... = k_j = k$, it is expressed as $h(z_1^i, k^{(j-i)}, l_{j+1}^m)$. An m-ary groupoid (A, h) is called an m-ary semigroup if h is associative; that is,

$$h(z_1^{i-1}, h(z_i^{m+i-1}), z_{m+i}^{2m-1}) = h(z_1^{j-1}, h(z_j^{m+j-1}), z_{m+j}^{2m-1}),$$

for all $z_1, z_2, ..., z_{2m-1} \in A$ where $1 \leq i \leq j \leq m$. An m-ary semigroupoid (A, h) is named an m-ary group if for all $c_1^{i-1}, c_{i+1}^n, b \in A$ exist $z_1^n \in A$, such that $h(c_1^{i-1}, z_i, c_{i+1}^n) = b$ for every $1 \leq i \leq n$. We say f is commutative if $h(z_1, z_2, ..., z_m) = h(z_{\eta(1)}, z_{\eta(2)}, ..., z_{\eta(m)})$, for every permutation η of $\{1, 2, ..., m\}$

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and $z_1, z_2, ..., z_m \in A$. An m-ary semigroup (A, h) is called a semi-abelian or $(1, m)$-commutative if $h(z, c^{(m-2)}, k) = h(k, c^{(m-2)}, z)$, for all $c, z, k \in A$.

2. (m,n)-Near Rings

We refer to [2, 4, 14], for details about near rings. In this section, we define the (m, n)-near ring and give examples for it and present definitions of α_1-(m, n)-near ring, α_2-(m, n)-near ring, R_0, R_c, constant near ring, i-zero divisor, $Z_{i,j}(R)$. We present some results in this respect.

Definition 1. *Assume that A is a non-empty set and h, k be r-ary and s-ary operations on A, respectively. In this case (A, h, k) is named an i-(r, s)-near ring, if the following conditions hold:*

1. *(A, h) is an r-ary group (not necessarily abelian),*
2. *(A, k) is an s-ary semigroup,*
3. *The s-ary operation k is i-distributive with respect to the r-ary operation h,*

where the definition of i-distributive condition is as follows: for every $c_1, c_2, ..., c_n$, $d_1, d_2..., d_m \in R$, if $i = n$, then

$$k(c_1^{n-1}, h(d_1, d_2, ..., d_m)) = h(k(c_1^{n-1}, d_1), k(c_1^{n-1}, d_2), ..., k(c_1^{n-1}, d_m)).$$

If $i = 1$ then

$$k(h(d_1, d_2, ..., d_m), c_2^n) = h(k(d_1, c_2^n), k(d_2, c_2^n), ..., k(d_m, c_2^n)).$$

If $1 < i < n$ then

$$\begin{aligned} &k(c_1^{i-1}, h(d_1, d_2, ..., d_m), c_{i+1}^n) \\ &= h(k(c_1^{i-1}, d_1, c_{i+1}^n), k(c_1^{i-1}, d_2, c_{i+1}^n), ..., k(c_1^{i-1}, d_m, c_{i+1}^n)). \end{aligned}$$

Throughout this paper, we explain i-(m, n)-near ring by (m, n)-near ring. It is clear that every (m, n)-ring [5] is an (m, n)-near ring.

Example 1. *Assume that (H, l) is an m-ary group with the identity element 0 and $N(H) = \{h : H \longrightarrow H \mid h \text{ is a function}\}$. Then $(N(H), l, \circ)$ is an $(m, 2)$-near ring, where $\circ$ is the composition of functions.*

1. *We know $(N(H), l)$ is an m-ary group (not necessarily abelian).*
2. *It is clear that $(N(H), \circ)$ is a 2-ary semigroup.*
3. *The 2-ary operation $\circ$ is 1-distributive with respect to the m-ary operation f.*

We notice that in this $(m, 2)$-near ring the 2-distributive law fails to retain. To consider this, let $d, d_j, c_i \in H, b_i \neq 0, 1 \leq j \leq m, 1 \leq i \leq 2$ and $h_{d_j} : H \longrightarrow H$, $h_{c_i} : H \longrightarrow H$ for all $g \in H$, by $h_{d_j}(g) = d_j$, $h_{c_i}(g) = c_i$. Now, for $i = 2$, we have

$$\begin{aligned} [h_{c_1} \circ (l(h_{d_1}, h_{d_2}, ..., h_{d_m}))](g) &= h_{c_1}(l((h_{d_1}(g), h_{d_2}(g), ..., h_{d_m}(g)) \\ &= h_{c_1}(l(d_1, d_2, ..., d_m)) = l(d_1, d_2, ..., d_m), \end{aligned}$$

and

$$\begin{aligned}[l(h_{c_1}\circ h_{d_1}, h_{c_1}\circ h_{d_2}, ..., h_{c_1}\circ h_{d_m})](g) &= l(h_{c_1}(h_{d_1}(g)), h_{c_1}(h_{d_2}(g)), ..., h_{c_1}(h_{d_1}(g)))\\ &= l(h_{c_1}(d_1), h_{c_1}(d_2), ..., h_{c_1}(d_n))\\ &= l(c_1^{(m)}).\end{aligned}$$

This shows that

$$[h_{c_1}\circ(l(h_{d_1}, h_{d_2}, ..., h_{d_m}))](g) \neq [l(h_{c_1}\circ h_{d_1}, h_{c_1}\circ h_{d_2}, ..., h_{c_1}\circ h_{d_m})](g).$$

For $i = 1$, we have

$$\begin{aligned}(l(h_{d_1}, h_{d_2}, ..., h_{d_m}))\circ h_{c_1}(g) &= (l(h_{d_1}, h_{d_2}, ..., h_{d_m}))(c_1)\\ &= l(h_{d_1}(c_1), h_{d_2}(c_1), ..., h_{d_m}(c_1))\\ &= l(d_1, d_2, ..., d_m),\end{aligned}$$

and

$$\begin{aligned}&[l(h_{d_1}\circ h_{d_1}, h_{d_2}\circ h_{c_1}, ..., h_{d_m}\circ h_{c_1})](g)\\ &\quad = l((h_{d_1}\circ h_{c_1})(g), (h_{d_2}\circ h_{c_1})(g), ..., (h_{d_m}\circ h_{c_1})(g))\\ &\quad = l((h_{d_1})(c_1), (h_{d_2})(c_1), ..., (h_{d_m})(c_1))\\ &\quad = l(d_1, d_2, ..., d_m).\end{aligned}$$

Hence,

$$[(l(h_{d_1}, h_{d_2}, ..., h_{d_m}))\circ h_{c_2}](g) = [l((h_{d_1}\circ h_{c_1}), (h_{d_2}\circ h_{c_1}), ..., (h_{d_m}\circ h_{c_1}))](g).$$

Therefore $N(H)$ fails to satisfy the i-distributive for $i = 2$.

Example 2. *Consider the additive group $\mathbb{Z}_{mn}$. Then $(\mathbb{Z}_{mn}, h)$ is a group, where $h(c_1, c_2, ..., c_m) = c_1 + c_2 + ... + c_m$. We define k on $\mathbb{Z}_{mn}$ by $k(c_1, c_2, ..., c_n) = c_1$, for all $c_1, c_2, ..., c_n \in \mathbb{Z}_{mn}$. It is easy to see $(\mathbb{Z}_{mn}, h, k)$ is an (m, n)-near ring. For $1 < i \leqslant n$, we have*

$$\begin{aligned}&k(c_1, c_2, ..., c_{i-1}, h(d_1, d_2, ..., d_m), c_{i+1}, ..., c_n) = c_1\\ &h(k(c_1, c_2, ..., c_{i-1}, d_1, c_{i+1}, ..., c_n), ..., k(c_1, c_2, ..., c_{i-1}, d_m, c_{i+1}, ..., c_n))\\ &\quad = h(c_1^{(m)}) = mc_1.\end{aligned}$$

If $mn = m - 1$, then $\overline{m} = \overline{1} \in \mathbb{Z}_{mn}$. Hence, for all $1 < i \leqslant n$, $(\mathbb{Z}_{mn-1}, h, k)$ is i-distributive. For $i = 1$, we have

$$\begin{aligned}&k(h(d_1, d_2, ..., d_m), c_2, ..., c_n) = h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m\\ &h(k(d_1, c_2, ..., c_n), k(d_2, d_2, ..., d_n), ..., k(d_m, c_1, ..., c_n))\\ &\quad = h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m.\end{aligned}$$

Consequently, for $i = 1$, $(\mathbb{Z}_{mn-1}, h, k)$ is 1-distributive.

Assume that A is an (m, n)-near ring. The element $e \in A$ is named an identity element if $k(e^{(i-1)}, s, e^{(n-i)}) = s$ for all $s \in A$ and $1 \leqslant i \leqslant n$.

Example 3. *We know $(\mathbb{R}, +, \cdot)$ is an (m, n)-near ring with two binary operations m-addition and n-multiplication. 1 is an identity element in $(\mathbb{R}, +, \cdot)$.*

Assume that (A, h, k) is an (m, n)-near ring. $m \in A$ is named i-cancellable, if for all $1 \leq i \leq n$, $c_i, d_i \in A$ and $k(c_1^{i-1}, m, c_i^n) = k(d_1^{i-1}, m, d_i^n)$, then $c_i = d_i$ for all $1 \leq i \leq n$. $m \neq 0$ is named an i-zero divisor, if there exist nonzero elements $c_1, c_2, ..., c_n \in R$ such that $k(c_1^{i-1}, m, c_{i+1}^n) = 0$. An (m, n)-near ring (A, h, k) is called integral near ring if it has no zero divisors. An i-(m, n)-near field is a nonempty set P together with two binary operations h and k such that (P, h) is a group (not necessarily abelian), (P, k) is a group and n-ary operation k is i-distributive with respect to the m-ary operation h.

Example 4. *Set of rational numbers with two binary operations h and k so that $k(d_1, d_2, ..., d_n) = d_1$ and $h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m$ for $d_i \in \mathbb{Q}$, $(\mathbb{Q}, h, k)$ is an (m, n)-near field.*

Definition 2. *Let (A, h, k) be an (m, n)-near ring,*

(1) *If for every $e \in A$ exists $z \in A$ such that $e = k(z^{(n-1)}, e, z^{(n-1)})$, then A is named an α_1-(m, n)-near ring.*
(2) *If for every $e \in A - \{0\}$ exists $z \in A - \{0\}$ such that $z = k(z^{(n-1)}, e, z^{(n-1)})$, then A is named an α_2-(m, n)-near ring.*

Example 5. *$(N(H), l, \circ)$ defined in Example 1 is an α_2-(m, n)-near ring.*

Example 6. *$(\mathbb{Z}_{mn}, h, k)$ defined in Example 2 is an α_2-(m, n)-near ring.*

Definition 3. *Let (A, h, k) be an (m, n)-near ring,*

(1) *A subgroup (O, h) of an m-ary group (A, h) with the property $k(O^{(n)}) \subset M$ is named an (m, n)-subnear ring of (A, h, k), It is shown by $O \leqslant N$.*
(2) *A subnear ring O of A is named i-invariant, if $h(A^{(i-1)}, O, A^{(m-i)}) \subseteq O$.*

If O is i-invariant for all $1 \leqslant i \leqslant m$, then O is named invariant.

Example 7. *The triple $(2\mathbb{Z}, h, k)$ is an (m, n)-subnear ring of the (m, n)-near ring $(\mathbb{Z}, h, k)$, that $h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m$ and $k(e_1, e_2 + ..., e_n) = e_1 \cdot e_2 \cdot ... \cdot e_n$.*

Definition 4. *Let (A, h, k) be an (m, n)-near ring and 0 is the identity element of (A, h). Then, $A_0 = \{r \in A \mid k(0^{(s-1)}, r, 0^{(n-s)}) = 0,\ 1 \leq s \leq n\}$ is called the zero symmetric part of A. In addition, $A_c = \{r \in R \mid k(0^{(s-1)}, r, 0^{(n-s)}) = r,\ 1 \leq s \leq n\}$ is named a resistant part of A. An (m, n)-near ring A is named a zero symmetric near ring if $A = A_0$. An (m, n)-near ring A is named a constant (m, n)-near ring if $A = A_c$.*

Lemma 1. *A_0 and A_c are (m, n)-subnear rings of the (m, n)-near ring (A, h, k).*

Proof. We show that A_0 is a subgroup of A. If $x_1, x_2, ..., x_m \in A_0$ then

$$k(0^{(i-1)}, x_j, 0^{(n-i)}) = 0 \text{ for } 1 \leq j \leq m \text{ and } 1 \leq i \leq n.$$

Now, we have

$$\begin{aligned} &k(0^{(i-1)}, h(x_1, x_2, ..., x_m), 0^{(n-i)}) \\ &\quad = h(k(0^{(i-1)}, x_1, 0^{(n-i)}), k(0^{(i-1)}, x_2, 0^{(n-i)}), ..., k(0^{(i-1)}, x_m, 0^{(n-i)})) = 0. \end{aligned}$$

Therefore, $h(x_1, x_2, ...x_m) \in A_0$, and so (A_0, h) is a subgroup of (A, h, k). Next, if we take $y_1, y_2, ..., y_n \in A_0$, then for all $1 \leq i \leq n$ and $1 \leq j \leq n$, we have $k(0^{(i-1)}, y_j, 0^{(n-i)}) = 0$. Then, we obtain

$$\begin{aligned}&k(0^{(n-1)}, k(y_1, y_2, ..., y_n)) = k(k(0^{(n-1)}, y_1), y_2, ..., y_n) = k(0, y_2, ..., y_n)\\&= k(k(0^{(n)}), y_2, .., y_n) = k(0, k(0^{(n-1)}, y_2), y_3, ..., y_n) = k(0, 0, y_3, ..., y_n)\\&= ... = k(0^{(n-1)}, y_n) = 0.\end{aligned}$$

Therefore, $k(y_1, y_2, ..., y_n) \in A_0$, and so $k(A_0^{(n)}) \subset A_0$. This shows that (A_0, h, k) is an (m, n)-subnear ring of (m, n)-near ring (A, h, k). We show that A_c is a subgroup of A. Let $x_1, x_2, ..., x_m \in A_0$. Then, we have $k(0^{(i-1)}, x_j, 0^{(n-i)}) = x_j$ for $1 \leq j \leq m$ and $1 \leq i \leq n$. Now, we obtain

$$\begin{aligned}&k(0^{(i-1)}, h(x_1, x_2, ...x_m), 0^{(n-i)})\\&\quad = h(k(0^{(i-1)}, x_1, 0^{(n-i)}), k(0^{(i-1)}, x_2, 0^{(n-i)}), ..., k(0^{(i-1)}, x_m, 0^{(n-i)}))\\&\quad = h(x_1, x_2, ..., x_m).\end{aligned}$$

This yields that $h(x_1, x_2, ...x_m) \in A_c$. Hence, (A_c, h) is a subgroup of (A, h, k). Next, if $y_1, ..., y_n \in A_c$, then $k(0^{(i-1)}, y_j, 0^{(n-i)}) = y_j$, for all $1 \leq i \leq n$, $1 \leq j \leq n$. This gives that $k(0^{(n-1)}, k(y_1, y_2, ..., y_n)) = k(k(0^{(n-1)}, y_1), y_2, ..., y_n) = k(y_1, y_2, ..., y_n)$. Therefore $k(y_1, y_2, ..., y_n) \in A_c$ and $k(A_c^{(n)}) \subset A_c$. Hence, (A_c, h, k) is an (m, n)-subnear ring of (m, n)-near ring (A, h, k). □

Theorem 1. *Let (A, h, k) be an (m, n)-near ring. If $r \in A_0$ is i-cancellable, then r is not an i-zero divisor.*

Proof. Suppose that $r \in A_0$ is i-cancellable and also r is an i-zero divisor, so there exist nonzero elements $d_1, d_2, ..., d_n \in A$ such that $k(d_1^{i-1}, r, d_{i+1}^n) = 0$. Since $r \in A_0$, it follows that $k(d_1^{i-1}, r, d_{i+1}^n) = 0 = k(0^{(i-1)}, r, 0^{(n-i)})$. Again, since r is i-cancellable, it follows that for all $1 \leq i \leq n$, $d_i = 0$, that it is a contradiction. □

Let (A, h, k) be an (m, n)-near ring. The center, $Z_{i,j}(A)$, is the subset of elements in A that (i, j)-commute with element of A. In the symbol, we can write:

$$\begin{aligned}Z_{i,j}(A) \quad &= \{b \in A \mid a_1, ..., a_n \in A \text{ and for } j > i,\\&\quad k(a_1^{i-1}, b, a_i^n) = k(a_1^{i-1}, a_j, a_{i+1}, ..., a_{j-1}, b, a_{j+1}^n)\}.\end{aligned}$$

Example 8. *In Example 2, for all $i, j \in 2, 3, ..., n$, we have $Z_{i,j}(A) = A$.*

Suppose that (A, h, k) is an (m, n)-near ring. If (A, k) is commutative, then A is named a commutative near ring. An element $r \in A$ is named idempotent element if $k(r^{(n)}) = r$. An element $r \in A$ is named nilpotent element if $k(r^{(n)}) = 0$.

Example 9. *In Example 2, for all $r \in \mathbb{Z}_{mn}$, we have $k(r^n) = r$, and so all elements are idempotent. Moreover, $\mathbb{Z}_{mn}$ has only one nilpotent element that is 0.*

Suppose that (A, h, k) is an (m, n)-near ring. A subset S of A is named nilpotent if $k(S^{(n)}) = 0$. A subset S of A is named nill if every element of S is a nilpotent element.

Theorem 2. *Assume that S is a subset of A. If S is nilpotent, then S is nill.*

Proof. Assume that S is nilpotent. Then $k(S^{(n)}) = 0$. This gives that $k(s^{(n)}) = 0$ for all $s \in S$. Hence, S is a nilpotent for all $s \in S$, then S is nill. $\square$

Definition 5. *Assume that (A, h, k) is an (m, n)-near ring and (W, h) be an m-group with identity element 0 of (A, h). W is named an i-A-group if there exists a mapping $l : \underbrace{W\times, ..., \times W}_{i-1} \times A \times \underbrace{W \times ... \times W}_{n-i} \to W$ the image of*

$$(r^{(i-1)}, s, r^{(n-i)}) \in \underbrace{W\times, ..., \times W}_{i-1} \times A \times \underbrace{W \times ... \times W}_{n-i} \to W,$$

for $s \in A$ and $r \in W$, is denoted by $l(r^{(i-1)}, s, r^{(n-i)}) = k(r^{(i-1)}, s, r^{(n-i)})$, satisfying the following conditions:

(1) $k(s_1^{i-1}, h(r_1, r_2, ..., r_m), s_{i+1}^n)$
$= h(k(s_1^{i-1}, r_1, s_{i+1}^n), k(s_1^{i-1}, r_2, s_{i+1}^n), ..., k(s_1^{i-1}, r_n, s_{i+1}^n))$.

(2) $k(t_1^{i-1}, k(z_1, z_2, ..., z_n), t_{i+1}^n) = k(t_1^{i-l-1}, k(t_{i-l}^{i-1}, z_1^{n-l}), z_{n-l+1}^n, t_{i+1}^n)$
$= k(t_1^{i-1}, z_1^s, k(z_{s+1}^n, t_{i+1}^{i+s}), t_{i+s+1}^n)$ *for all* $1 \le l \le i-1$ *and* $1 \le s \le n-i$,

for all $s_j, t_i \in W$ that $1 \le i, j \le n$. For all $r_i, z_t \in A$ that $1 \le i \le m$ and $1 \le t \le n$, we denote this i-A-group by $\underbrace{AA...A}_{i-1} W \underbrace{AA...A}_{n-i}$.

Example 10. *If we consider $W = \mathbb{Z}$ in Example 2, then W is an 1-$\mathbb{Z}_{mn}$-group. By taking $i = 1$ in Definition 5, the conditions of the definition are satisfied,*

$$k(h(r_1, r_2, ..., r_m), s_2^n) = h(k(r_1, s_2^n), k(r_2, s_2^n), ..., k(r_m, s_2^n)) = h(r_1, r_2, ..., r_m),$$
$$k(k(s_1, s_2, ..., s_n), t_2^n) = k(s_1^l, k(s_{l+1}^n, t_2^{1+l}), t_{2+l}^n) = s_1.$$

In Definition 5, if $k(r^{(i-1)}, g, r^{(n-i)}) = 0$ for all $g \in W$ yields $r = 0$, then W is a faithful i-A-group.

Example 11. *In Example 2, $\mathbb{Z}_{mn}$ operates faithfully on $\mathbb{Z}$.*

Assume that (A, h, k) is an (m, n)-near ring. A subgroup H of an i-A-group W is named an i-A-subgroup (written as $H \le_A W$), if it is closed under the operation of A and $k(r^{(i-1)}, h, r^{(n-i)}) \in H$ for all $r \in A$, $h \in H$. Suppose that W_1 and W_2 are two A-groups, $s : W_1 \to W_2$ is named i-A-homomorphism, if for all $l, l_1, ..., l_n \in W_1$ and for all $r \in A$, $s(h(l_1, l_2, ..., l_m)) = h(s(l_1), s(l_2), ..., s(l_m))$ and $s(k(r^{(i-1)}, l, r^{(n-i)})) = k(r^{(i-1)}, s(l), r^{(n-i)})$. If H is the kernel of an i-A-homomorphism, then it is named an i-A-normal subgroup and we write $H \trianglelefteq_A W$.

Example 12. *If $h(d_1, d_2, ..., d_m) = d_1 + d_2 + ... + d_m$, $k(d_1, d_2, ..., d_n) = d_1 \cdot d_2 \cdot ... \cdot d_n$, then $(\mathbb{R}, h, k)$ is an (m, n)-near ring and $\mathbb{Q}$ (the set of rationales) is a i-$\mathbb{R}$-subgroup of $\mathbb{R}$.*

Assume that W is an i-A-group. W is named a unitary i-A-group if A be a near ring with unity 1 so that $k(1^{(i-1)}, x, 1^{(n-i)}) = x$ for all $x \in W$.

Example 13. *If in Example 4, $d_j = 1$ for $j \in \{1, 2..., i-1, i+1, ..., n\}$, then* $k(1^{(i-1)}, x, 1^{(n-i)}) = \underbrace{1 \cdot 1 \cdot ... \cdot 1}_{i-1} \cdot x \cdot \underbrace{1 \cdot 1 \cdot ... \cdot 1}_{n-i} = x$.

Theorem 3. *In an α_1-(m,n)-near ring for every $a \in A$ exist some $s \in A$ if $n = 2i+1$, then*

(1) $k(s^{(i)}, a^{(i+1)}) = k(a^{(i+1)}, s^{(i)})$,
(2) $a = k(s^{(i)}, k(s^{(i)}, ..., k(s^{(i)}, a, s^{(i)}), ..., s^{(i)}), s^{(i)})$.

Proof. (1) Suppose that A is an α_1-(m,n)-near ring and $a \in A$. So there exists $s \in R$ such that $a = k(s^{(i-1)}, a, s^{(n-i)})$. This implies that

$$k(s^{(i)}, a^{(i+1)}) = k(s^{(i)}, a, a^{(i)}) = k(s^{(i)}, a, k(s^{(i)}, a, s^{(i)}), a^{(i-1)})$$
$$= k(k(s^{(i)}, a, s^{(i)}), a, s^{(i)}, a^{i-1}) = k(a, a, s^{(i)}, a^{(i-1)})$$
$$= k(a, a, s^{(i)}, a, k(s^{(i)}, a, s^{(i)}, a^{(i-3)})) = k(a, a, k(s^{(i)}, a, s^{(i)}), a, s^{(i)}, a^{(i-3)})$$
$$= k(a, a, a, a, s^{(i)}, a^{(i-3)}) = ... = k(a^{(i+1)}, s^{(i)}).$$

(2) We have

$$k(s^{(i)}, k(s^{(i)}, ..., k(s^{(i)}, a, s^{(i)}), ..., s^{(i)}), s^{(i)})$$
$$= k(s^{(i)}, k(s^{(i)}, a, s^{(i)}), s^{(i)}) = a.$$

□

A subnear ring M of a (m,n)-near ring A is named an α_2-subnear ring if for every $a \in M$ exists an $s \in M$ so that $n = 2i+1$, $k(s^{(i)}, a, s^{(i)}) = s$.

Theorem 4. *Suppose that A is an α_2-(m,n)-near ring. In this case*

(1) *Every invariant subgroup W of A is an α_2-subnear ring.*
(2) *Every ideal I of a zero symmetric α_2-near ring A is an α_2-subnear ring.*

Proof. (1) Take $a \in W - \{0\}$. Since A is an α_2-near ring there exists $s \in A$ such that $k(s^{(i)}, a, s^{(i)}) = s$. Now W is an invariant subgroup of A implies that $k(s^{(i)}, a, s^{(i)}) \in W$. Then $s \in W$. Consequently W is an α_2-subnear ring.

(2) Assume that I is an ideal of the zero symmetric α_2-near ring A. Let $a \in I - \{0\}$. Since A is an α_2-near ring, so there exists $s \in A - \{0\}$ so that $k(s^{(i)}, a, s^{(i)}) = s$. Now, we have $k(s^{(i)}, a, s^{(i)}) \in k((A - \{0\})^{(i)}, I - \{0\}, (A - \{0\})^{(i)}) \subseteq I - \{0\}$. The desired result now follows. □

3. Ideals and Homomorphisms of (m,n)-Near Rings

We define the notions of i-ideal, zero near ring, prime ideal, semi-symmetric, $A(S)$, k-ideal, i-N-primary and i-P-primary in the (m,n)-near rings and assert a few related theorems.

Assume that I is a non-empty subgroup of an (m,n)-near ring (A, h, k). Then I is named a normal subgroup of A if for all $a_i \in A$ and $s_1^{i-1}, s_{i+1}^m \in A$, $1 \leq i, j \leq m$, there is $b_j \in I$ that $h(s_1^{i-1}, a_i, s_{i+1}^m) = h(s_1^{j-1}, b_j, s_{j+1}^m)$.

Definition 6. *Suppose that I is a non-empty subset of an (m,n)-near ring (A,h,k). In this case I is named an ideal of A if*

(1) *I is a normal subgroup of m-ary group (A,h), (I,h) is an m-ary group,*
(2) *for every $a_1,a_2,...,a_n \in A$, $k(a_1^{i-1},I,a_{i+1}^n) \subseteq I$,*
(3) *for all $r_1,...,r_{j-1},r_{j+1},...,r_m,s_1,...,s_{j-1},s_{j+1},...,s_n \in A$ and $1 \le k \le n$, $d \in I$, there exists $l \in I$ that*

$$k(s_1^{j-1},h(r_1^{k-1},d,r_{k+1}^m),s_{j+1}^n) = h(k(s_1^{j-1},r_1,s_{j+1}^n),k(s_1^{j-1},r_2,s_{j+1}^n),...,k(s_1^{j-1},r_{k-1},s_{j+1}^n),l, (s_1^{j-1},r_{k+1},s_{j+1}^n),...,k(s_1^{j-1},r_n,s_{j+1}^n)).$$

I is named an i-ideal of A if it satisfies (1) and (2) and I is named a j-ideal of A for $j \neq i$ if it satisfies (1) and (3).

If for every $1 \le i \le n$, I is an i-ideal, then I is named an ideal of A.

Example 14. *Let $\mathbb{Z}$ and $\mathbb{Q}$ be the set of integers and the set of rational numbers, respectively. Consider two (m,n)-near rings $(\mathbb{Z},h,k)$ and $(\mathbb{Q},h,k)$, where $h(d_1,d_2,,...,d_m) = d_1+d_2+...+d_m$ and $k(d_1,d_2,...,d_n) = d_1 \cdot d_2 \cdot ... \cdot d_{n-1} \cdot d_n$. Then $\mathbb{Z}$ is an (m,n)-subnear ring of $\mathbb{Q}$, but $\mathbb{Z}$ is not an ideal of the near ring $\mathbb{Q}$.*

Remark 1. *If $J_1,J_2,...,J_n$ and $I_1,I_2,I_2,...,I_m$ are ideals of a near ring A, then*

(1) *$h(I_1,I_2,...,I_m)$ is an ideal of A,*
(2) *$J_1 \cap J_2 \cap ... \cap J_n$ is an ideal of A,*
(3) *$k(J_1,J_2,...,J_n)$ is an ideal of A.*

Assume that (A,h,k) is an (m,n)-near ring and I is an ideal. (A,h) is a group and I is a normal subgroup. The quotient group $(A/I,H,K)$ is defined. An m-ary operation h on the cosets is defined by the m-ary operation h as follows:

$$H(h(d_{1_1},d_{1_2},...,d_{1_{m-1}},I),...,h(d_{m_1},d_{m_2},...,d_{m_{m-1}},I)) = h(h(d_{1_1},d_{1_2},...,d_{1_{m-1}},h(d_{2_1},d_{2_2},...,d_{2_{m-1}},h(d_{3_1},d_{3_2},...,d_{3_{m-1}},... h(d_{(m-1)_1},d_{(m-1)_2},...,d_{(m-1)_{m-1}}h(d_{m_1},d_{m_2},...,d_{m_{m-1}},I)....)).$$

An n-ary operation k on cosets is defined by the n-ary operation k as follows:

$$K(h(d_{1_1},d_{1_2},...,d_{1_{n-1}},I),...,h(d_{n_1},d_{n_2},...,d_{n_{n-1}},I)) = h(k(h(d_{1_1},d_{1_2},...,d_{1_{n-1}},I),...,h(d_{(i-1)_1},d_{(i-1)_2},...,d_{(i-1)_{(n-1)}},I),d_{i_1}, h(d_{(i+1)_1},d_{(i+1)_2},...,d_{(i+1)_{n-1}},I))...,h(d_{n_1},d_{n_2},...,d_{n_{m-1}},I)),..., k(h(d_{1_1},d_{1_2},...,d_{1_{m-1}},I),...,h(d_{(i-1)_1},d_{(i-1)_2},...,d_{(i-1)_{m-1}},I),d_{i_{m-1}}, h(d_{(i+1)_1},d_{(i+1)_2}...,d_{(i+1)_{m-1}},I))...,h(d_{n_1},d_{n_2},...,d_{n_{m-1}},I)),I).$$

Theorem 5. *If I is an ideal in an (m,n)-near ring (A,h,k), then $(A/I,H,K)$, where the operations H and K are defined as above, has the structure of an (m,n)-near ring.*

Proof. We prove that H is well defined. Assume that

$$h(d_{i_1},d_{i_2},...,d_{i_{m-1}},I) = h(e_{i_1},e_{i_2},...,e_{i_{m-1}},I),$$

for $1 \le i \le m$. Then

$$
\begin{aligned}
&H(h(d_{1_1}, d_{1_2}, ..., d_{1_{m-1}}, I), ..., h(d_{m_1}, d_{m_2}, ..., d_{m_{m-1}}, I))\\
&= h(h(d_{1_1}, d_{1_2} ..., d_{1_{m-1}}, h(d_{2_1}, d_{2_2}, ..., d_{2_{m-1}}, h(d_{3_1}, d_{3_2}, ..., d_{3_{m-1}}, ...,\\
&\quad h(d_{(m-1)_1}, d_{(m-1)_2}, ..., d_{(m-1)_{m-1}} h(d_{m_1}, ..., d_{m_{m-1}}, I)....))\\
&= h(d_{1_1}, d_{1_2}, ..., d_{1_{m-1}}, h(d_{2_1}, d_{2_2}, ..., d_{2_{m-1}}, h(d_{3_1}, d_{3_2}, ..., d_{3_{m-1}}, ...,\\
&\quad h(d_{(m-1)_1}, d_{(m-1)_2}, ..., d_{(m-1)_{m-1}}, h(e_{m_1}, e_{m_2}, ..., e_{m_{m-1}}, I)...))\\
&= h(d_{1_1}, d_{2_2}, ..., d_{1_{m-1}}, h(d_{2_1}, d_{2_2}, ..., d_{2_{m-1}}, h(d_{3_1}, d_{3_2}, ..., d_{3_{m-1}}, ...\\
&\quad h(d_{(m-1)_1}, d_{(m-1)_2}, ..., d_{(m-1)_{m-1}}, h(I, e_{m_1}, e_{m_2}, ..., e_{m_{m-1}})...))\\
&= h(d_{1_1}, d_{1_2}, ..., h(d_{1_{(m-1)}}, h(d_{2_1}, d_{2_2}, ..., d_{2_{m-1}}, h(d_{3_1}, d_{3_2}, ..., d_{3_{m-1}}, ...,\\
&\quad h(h(d_{(m-1)_1}, d_{(m-1)_2}, ..., d_{(m-1)_{m-1}}, I), h(e_{m_1}, e_{m_2}, ..., e_{m_{m-1}})...))\\
&= h(d_{1_1}, d_{1_2}, ..., d_{1_{m-1}}, h(d_{2_1}, d_{2_2}, ..., d_{2_{m-1}},, h(d_{3_1}, d_{3_2}, ..., d_{3_{m-1}}, ...,\\
&\quad h(h(e_{(m-1)_1}, e_{(m-1)_2}, ..., e_{(m-1)_{m-1}}, I), e_{m_1}, e_{m_2}, ..., e_{m_{(m-1)}})...))\\
&= ... = h(d_{1_1}, d_{1_2}, ..., d_{1_{m-1}}, h(h(d_{2_1}, d_{2_2}, ..., d_{2_{m-1}}, I), e_{3_1}, e_{3_2}, ..., e_{3_{m-1}}), ...,\\
&\quad h(e_{(m-1)_1}, e_{(m-1)_2}, ..., e_{(m-1)_{m-1}}, h(e_{m_1}, e_{m_2}, ..., e_{m_{m-1}}, I)...))\\
&= h(d_{1_1}, d_{1_2}, ..., d_{1_{m-1}}, h(h(e_{2_1}, e_{2_2}, ..., e_{2_{m-1}}, I), e_{3_1}, e_{3_2}, ..., e_{3_{m-1}}), ...,\\
&\quad h(e_{(m-1)_1}, e_{(m-1)_2}, ..., e_{(m-1)_{m-1}}, h(e_{m_1}, e_{m_2}, ..., e_{m_{m-1}}, I)...))\\
&= h(d_{1_1}, d_{1_2}, ..., d_{1_{m-1}}, h(h(I, e_{2_1}, e_{2_2}, ..., e_{2_{m-1}}), e_{3_1}, ..., e_{3_{m-1}}), ...,\\
&\quad h(e_{(m-1)_1}, e_{(m-1)_2}, ..., e_{(m-1)_{m-1}}, h(e_{m_1}, e_{m_2}, ..., e_{m_{m-1}}, I)...))\\
&= h(h(d_{1_1}, d_{1_2}, ..., d_{1_{m-1}}, I), e_{2_1}, e_{2_2}, ..., e_{2_{m-1}}), h(e_{3_1}, e_{3_2}, ..., e_{3_{m-1}}, ...\\
&\quad h(e_{(m-1)_1}, e_{(m-1)_2}, ..., e_{(m-1)_{m-1}}, h(e_{m_1}, e_{m_2}, ..., e_{m_{m-1}}, I)...))\\
&= h(e_{1_1}, e_{1_2}, ..., e_{1_{m-1}}, h(e_{2_1}, e_{2_2}, ..., e_{2_{m-1}}, h(e_{3_1}, e_{3_2}, ..., e_{3_{m-1}}, ...,\\
&\quad h(e_{(m-1)_1}, e_{(m-1)_2}, ..., e_{(m-1)_{m-1}}, h(e_{m_1}, e_{m_2}, ..., e_{m_{m-1}}, I)...))\\
&= H(h(e_{1_1}, e_{1_2}, ..., e_{1_{m-1}}, I), ..., h(e_{m_1}, e_{m_2}, ..., e_{m_{m-1}}, I)).
\end{aligned}
$$

Since I is an ideal, then the operator k is well defined and since (A, h) is an m-ary group so $(A/I, H)$ is an m-ary group. Furthermore, since (A, k) is an n-ary semigroup, it follows that $(A/I, K)$ is an n-ary semigroup. The n-ary operation k is i-distributive with respect to the m-ary operation h. Thus, the n-ary operation k is i-distributive with respect to the m-ary operation H. □

An (m, n)- near ring (A, h, k) is named simple if A does not have non-trivial ideals. A proper ideal I of (A, h, k) is named maximal if $I \subseteq J \subseteq A$ and J is an ideal of A implies that either $I = J$ or $J = A$. A proper ideal I of an (m, n)-near ring (A, h, k) is named prime, if for every ideals $A_1, A_2, ..., A_n$ of A, $k(A_1, A_2, ..., A_n) \subseteq I$ implies $A_1 \subseteq I$ or $A_2 \subseteq I$ or ... or $A_n \subseteq I$. A proper ideal I of an (m, n)-near ring (A, h, k) is named weakly prime, if for any ideals $A_1, A_2, ..., A_n$ of A, $\{0\} \neq k(A_1, A_2, ..., A_n) \subseteq I$ implies $A_1 \subseteq I$ or $A_2 \subseteq I$ or ... or $A_n \subseteq I$. Clearly, every prime ideal is weakly prime and (0) is always weekly prime ideal of (A, h, k). An ideal I of an (m, n)-near ring (A, h, k) is named semi-symmetric if $k(\underbrace{z, z, ..., z}_{n}) \in I$, implies $k(\underbrace{\langle z\rangle, \langle z\rangle, ..., \langle z\rangle}_{n}) \subseteq I$.

Theorem 6. *For an ideal P of an (m, n)-near ring (A, h, k), the following statements are equivalent:*

(1) *P is prime.*

(2) *If* $d_i \notin P$ *and* $1 \leqslant i \leqslant n$, *then* $k(\langle d_1\rangle, \langle d_2\rangle, ..., \langle d_n\rangle) \nsubseteq P$.

Proof. To prove (1) $\Rightarrow$ (2) assume P is a prime ideal and $d_i \notin P$ for $1 \leqslant i \leqslant n$. Then $\langle d_i\rangle \nsubseteq P$. If $k(\langle d_1\rangle, \langle d_2\rangle, ..., \langle d_n\rangle) \subseteq P$, P is a prime ideal, then $\langle d_1\rangle \subseteq P$ or $\langle d_2\rangle \subseteq P$ or ... or $\langle d_n\rangle \subseteq P$. This is a contradiction. Hence, $k(\langle d_1\rangle, \langle d_2\rangle, ..., \langle d_n\rangle) \nsubseteq P$. So (1) $\Rightarrow$ (2).

To prove (2) $\Rightarrow$ (1), suppose that $I_1, I_2, ..., I_n$ are ideals of R such that $k(I_1, I_2, ..., I_n) \subseteq P$. Assume that $I_1, I_2, ..., I_n \nsubseteq P$, Then by (2), we have $k(I_1, I_2, ..., I_n) \nsubseteq P$, that is a contradiction. Hence , $I_1 \subseteq P$ or $I_2 \subseteq P$ or ... or $I_n \subseteq P$. So, P is a prime ideal. The proof of (2) $\Rightarrow$ (1) is completed. □

An (m, n)-near ring (A, h, k) is named a zero near ring if $k(\underbrace{A, A, ..., A}_{n}) = 0$. Assume that A is an (m, n)-near ring. The intersection of all prime ideals of A is named the prime radical of A and is denoted by (A). For any proper ideal I of A, the intersection of all prime ideals of A containing I is named the prime radical of I and is denoted by $P(I)$.

Lemma 2. *Every integral* (m, n)*-near ring is prime.*

Proof. Assume that (A, h, k) is an integral (m, n)-near ring. It is enough to show (0) is a prime ideal. Let $I_1, I_2, ..., I_n$ be ideals of A such that $k(I_1, ..., I_n) \subset (0)$. If either $I_1 = (0)$ or $I_2 = (0)$ or ... or $I_n = (0)$, then there is nothing to prove. If possible, suppose that $I_1 \neq (0)$ or $I_2 \neq (0)$ or ... or $I_n \neq (0)$, then we can choose $0 \neq a_1 \in I_1$, $0 \neq a_2 \in I_2$, ..., $0 \neq a_n \in I_n$ such that $k(a_1, a_2, ..., a_n) = 0$, which is in contrast to the fact that A is integral. Therefore, either $I_1 = (0)$ or $I_2 = (0)$ or ... or $I_n = (0)$. Thus, we proved that (0) is a prime ideal of A. Hence, A is a prime (m, n)-near ring. □

Theorem 7. *If the* (m, n)*-near ring* (A, h, k) *is simple, then either* A *is prime or* A *is a zero* (m, n)*-near ring.*

Proof. Assume that A is not a zero (m, n)-near ring. Then $k(A^{(n)}) \neq (0)$. We prove that (0) is a prime ideal of A. Assume that $I_1, I_2, ..., I_n$ are ideals of A such that $k(I_1, I_2, ..., I_n) \subseteq (0)$. Since $I_1, I_2, ..., I_n$ are ideal of A and A is simple, so $I_1, I_2, ..., I_n \in \{(0), A\}$. Then $k(A^{(n)}) \subseteq k(I_1, I_2, ..., I_n) \subseteq (0)$. It is a contradiction. Hence, $I_1 = (0)$ or $I_2 = (0)$ or ... or $I_n = (0)$. Thus, (0) is a prime ideal of A. This yields that A is a prime (m, n)-near ring. □

Theorem 8. *If* I *is a semi-symmetric ideal of an* (m, n)*-near ring* (A, h, k)*, then* $P(I)$ *is completely semiprime.*

Proof. Suppose that $k(a^{(n)}) \in P(I)$. So, $k(k(a^{(n)})^{(n)}) \in I$. Because I is semi-symmetric, $\langle k(k(a^{(n)})^{(n)})\rangle \subseteq I \subseteq P(I)$, thus $a \in P(I)$. This implies that $P(I)$ is completely semiprime. □

If I is a semi-symmetric ideal of a (m,n)-near ring (A,h,k), then

$$P(I) = \{x \in A \mid k(x^{(n)}) \in I\}.$$

An (m,n)-near ring A is named semi-symmetric if $\langle 0 \rangle$ is a semi-symmetric ideal of A.

For any subset S of an (m,n)-near ring (A,h,k),

$$A(S) = \{x \in S \mid k(A^{(i-1)}, x, A^{(n-i)}) = \{0\}\}.$$

Clearly, $A(S)$ is an i-ideal of A. An ideal I of an (m,n)-near ring (A,h,k) is named subtractive or k-ideal, if $h(d_1, d_2, ..., d_m) \in I$ for any elements $d_1, d_2, ..., d_{m-1} \in I$ and $d_m \in A$, then $d_m \in I$.

Theorem 9. *Let I be a k-ideal of an (m,n)-near ring (S,h,k) with $1 \neq 0$. The following statements are equivalent:*

(1) *I is a weakly prime ideal.*
(2) *If $B_1, B_2, ..., B_n$ are ideals of S such that $\{0\} \neq k(B_1, B_2, ..., B_n) \subseteq I$, then $B_i \subseteq I$ for some $1 \leq i \leq n$.*

Proof. It is straightforward. □

Theorem 10. *Every ideal of (m,n)-near ring (S,h,k) is weakly prime if and only if for any ideals $B_1, B_2, ..., B_n$ of S, $k(B_1, B_2, ..., B_n) = B_1$ or $k(B_1, B_2, ..., B_n) = B_2$ or .... or $k(B_1, B_2, ..., B_n) = B_n$ or $k(B_1, B_2, ..., B_n) = 0$.*

Proof. Assume that every ideal of S is weakly prime. Let $B_1, B_2, ..., B_n$ be ideals of S and $k(B_1, B_2, ..., B_n) \neq S$, so $k(B_1, B_2, ..., B_n)$ is weakly prime. If $\{0\} \neq k(B_1, B_2, ..., B_n) \subseteq k(B_1, B_2, ..., B_n)$, then we have $B_1 \subseteq k(B_1, B_2, ..., B_n)$ or $B_2 \subseteq k(B_1, B_2, ..., B_n)$ or ... or $B_n \subseteq k(B_1, B_2, ..., B_n)$ (since $k(B_1, B_2, ..., B_n)$ is weakly prime ideal of S), that is, $B_1 = k(B_1, B_2, ..., B_n)$ or $B_2 = k(B_1, B_2, ..., B_n)$ or ... or $B_n = k(B_1, B_2, ..., B_n)$. If $k(B_1, B_2, ..., B_n) = S$, then $B_1 = B_2 = ... = B_n = S$ whence $S^n = S$.

Conversely, let I be any proper ideal of S and let $\{0\} \neq k(B_1, B_2, ..., B_n) \subseteq I$ for ideals $B_1, B_2, ..., B_n$ of S. Then, either $B_1 = k(B_1, B_2, ..., B_n) \subseteq I$ or $B_2 = k(B_1, B_2, ..., B_n) \subseteq I$ or ... or $B_n = k(B_1, B_2, ..., B_n) \subseteq I$. □

Lemma 3. *If P be a subtractive ideal of i-(m,n)-near ring (S,h,k) such that $2 \leq i \leq n$, then P is a weakly prime ideal but it is not a prime ideal of (m,n)-near ring S. Moreover, $k(d_1, d_2, ..., d_n) = 0$ for some $d_1, d_2, ..., d_n \notin P$, then we have $k(d_{i-1}, P^{(n-1)}) = \{0\}$.*

Proof. If $i = 2$, assume that $k(d_1, p_1^{n-1}) \neq 0$, for some $c_1, c_2, ..., c_{n-1} \in P$. Then

$$0 \neq k(d_1, h(k(1, d_2, d_3, ..., d_n), (k(1, c_1, c_2, ..., c_{n-1}))^{(m-1)}), 1^{(n-2)}) \in P.$$

Since P is a weakly prime ideal of S, it follows that $d_1 \in P$ or

$$h(k(1, d_2, d_3, ..., d_n), (k(1, c_1, c_2, ..., c_{n-1}))^{(m-1)}) \in P,$$

that is, $d_1 \in P$ or $d_2 \in P$ or ... or $d_n \in P$. It is a contradiction. Therefore $k(d_1, P^{(n-1)}) = \{0\}$. Similarly, we can show that $k(P, d_2, P^{(n-2)}) = \{0\}$.

If $3 \leq i \leq n$, suppose that $k(d_{i-1}, c_1^{n-1}) \neq 0$, for some $c_1, c_2, ..., c_{n-1} \in P$. Then, we have

$$0 \neq k(1^{i-2}, d_{i-1}, h((k(c_1^{i-2}, 1, c_i, ..., c_{n-1}))^{i-2}, k(d_1^{i-2}, 1, d_i^n), (k(c_1^{i-2}, 1, c_i^{n-1}))^{(m-i+1)}), 1^{n-i}) \in P.$$

Since P is a weakly prime ideal of S, it follows that $d_{i-1} \in P$ or

$$h((k(c_1^{i-2}, 1, c_i^{n-1}))^{(i-2)}, k(d_1^{i-2}, 1, d_i^n), (k(c_1^{i-2}, 1, c_i^{n-1}))^{(m-i+1)}) \in P,$$

that is, $d_1 \in P$ or $d_2 \in P$ or ... or $d_n \in P$. It is a contradiction. Therefore, we derive that $k(d_{i-1}, P^{(n-1)}) = \{0\}$. □

Theorem 11. *Suppose that P is a k-ideal in an i-(m, n)-near ring (S, h, k). If P is weakly prime ideal but not prime, then $P^n = \{0\}$.*

Proof. Assume that $k(c_1, c_2, ..., c_n) \neq 0$ for some $c_1, c_2, ..., c_n \in P$ and $k(d_1, d_2, ..., d_n) = 0$ for some $d_1, d_2, ..., d_n \notin P$, where P is not a prime ideal of S. Hence

$$0 \neq k(d_1^{i-2}, h(d_n, p_i^{m-1}), d_{i+1}^n) = h(k(d_1, d_2, ..., d_{i-1}, d_i, d_{i+1}, ..., d_n), (k(d_1^{i-1}, p_i, d_{i+1}^n))^{(m-1)}) \in P.$$

Hence either $d_1 \in P$ or ... or $d_{i-1} \in P$ or $d_{i+1} \in P$ or ... or $d_n \in P$ or $h(d_i, c_i^{m-1}) \in P$, thus either $d_1 \in P$ or $d_2 \in P$ or ... or $d_n \in P$, that it is a contradiction. Hence $P^n = \{0\}$. □

Corollary 1. *Assume that P is a weakly prime ideal of (m, n)-near ring (S, h, k). If P is not a prime ideal of S, then $P \subseteq Nil\ S$, where $Nil\ S$ denotes the set of all nilpotent element of S.*

A k-ideal in a commutative (m, n)-near ring (S, h, k) satisfying that $P^n = \{0\}$.

Lemma 4. *Assume that l is a homomorphism of (m, n)-near ring (S_1, h, k) onto (m, n)-near ring (S_2, h', k'). Then each of the following statements is true:*

(1) *If Y is an ideal (k-ideal) in S_1, then $l(Y)$ is an ideal (k-ideal) in S_2.*
(2) *If W is an ideal (k-ideal) in S_2, then $l^{-1}(W)$ is an ideal (k-ideal) in S_1.*

Proof. It is straightforward. □

Theorem 12. *If $l : S_1 \longrightarrow S_2$ is a homomorphism of (m, n)-near rings and P is a prime ideal in S_2, then $l^{-1}(P)$ is a prime ideal in S_1.*

Proof. By Lemma 4, $l^{-1}(P)$ is an ideal of (S_1, h, k). If $k(d_1, d_2, ..., d_n) \in l^{-1}(P)$, then $l(k(d_1, d_2, ..., d_n)) \in P$ implies $k'(l(d_1), l(d_2), ..., l(d_n)) \in P$. Hence P is a prime ideal of S_2 therefore it follows that either $l(d_1) \in P$ or $l(d_2) \in P$ or ... or $l(d_n) \in P$ and thus either $d_1 \in l^{-1}(P)$ or $d_2 \in l^{-1}(P)$ or ... or $d_n \in l^{-1}(P)$. Thus $l^{-1}(P)$ is a prime ideal of S_1. □

Theorem 13. *If (S, h, k) be an (m, n)-near ring such that $S = \langle d_1, d_2, ..., d_k\rangle$ for $k = \max\{n, m\}$, is a finitely generated ideal of S, Then each proper k-ideal A of S is included in a maximal k-ideal of S.*

Proof. Assume that β is the set of all k-ideals B of S satisfying $A \subseteq B \subset S$, that is partially ordered by inclusion. Take a chain $\{B_i \mid i \in I\}$ in β. Then $B = \bigcup B_i$ is a k-ideal of S, because if $d_1, d_2, ..., d_{n-1}, h(d_1, d_2, ..., d_n) \in B$ then by the definition of B, there is $i_1, i_2, ..., i_{n-1}, j \in I$ such that $d_1 \in B_{i_1}, d_2 \in B_{i_2}, ..., d_{n-1} \in B_{i_{n-1}}, h(d_1, d_2, ..., d_n) \in B_j$, as B_i partially ordered by inclusion, then $B_j \subseteq B_{i_1}$ or $B_{i_1} \subseteq B_j$. Without reduce totality of problem assuming $B_{i_1}, B_{i_2}, ..., B_{i_{n-1}} \subseteq B_j$. So $d_1, d_2, ..., d_{n-1}, h(d_1, d_2, ..., d_n) \in B_j$ because B_j is a k-ideal. Thus $d_n \in B_j$ and $B_j \subseteq B$ then $d_n \in B$ so B is a k-ideal and $S = \langle d_1, d_2, ..., d_k\rangle$ implies $B \neq S$ and hence $B \in \beta$. So by Zorn's lemma, β has a maximal element. □

Corollary 2. *Let (S, h, k) be an (m, n)-near ring with identity 1. Then each proper j-ideal of S is included in a maximal j-ideal of S.*

Proof. The proof is immediate by taking $S = \langle 1\rangle$. □

Lemma 5. *If C, D be two j-ideals of an (m, n)-near ring (S, h, k), then $C \cap D$ is a j-ideal.*

Proof. Let C, D be two j-ideals of S, then by definition j-ideal, C and D are subgroups of m-ary group (S, h). so $C \cap D$ is a subgroup of m-ary group (S, h). It is enough to prove for every $d_1, d_2, ..., d_n \in S$, $k(d_1^{k-1}, C \cap D, d_{k+1}^n) \subseteq C \cap D$. because C is a j-ideal, $k(d_1^{k-1}, C \cap D, d_{k+1}^n) \subseteq k(d_1^{k-1}, C, d_{k+1}^n) \subseteq C$ and because D is a j-ideal, $k(d_1^{k-1}, C \cap D, d_{k+1}^n) \subseteq k(d_1^{k-1}, D, d_{k+1}^n) \subseteq D$. therefore $k(d_1^{i-1}, C \cap D, d_{i+1}^n) \subseteq C \cap D$. □

Definition 7. *An equivalence relation ρ on an (m, n)-near ring (S, f, g) is called a congruence on S if for any $a_1, a_2, ..., a_m, b_1, b_2, ..., b_n \in S$ such that $a\rho b$, then for all $1 \leq i \leq n$ and $1 \leq j \leq m$:*

(1) $f(a_1^{j-1}, a, a_{j+1}^m)\rho f(a_1^{j-1}, b, a_{j+1}^m)$;
(2) $g(b_1^{i-1}, a, b_{i+1}^n)\rho g(b_1^{i-1}, b, b_{i+1}^n)$.

Let ρ be a congruence on an (m, n)-near ring (S, f, g). Then, the congruence class of x, S is denoted by $x\rho$ and is defined by $x\rho = \{y \in S \mid (x, y) \in \rho\}$. The set of all congruence classes of S is denoted by S/ρ.

Theorem 14. *Let (S, h, k) be an (m, n)-near ring, then $(S/\rho, h, k)$ is an (m, n)-near ring under the operations*

$$h(d_1\rho, d_2\rho, ..., d_m\rho) = h(d_1, d_2, ..., d_m)\rho,$$
$$k(d_1\rho, d_2\rho, ..., d_n\rho) = k(d_1, d_2, ..., d_n)\rho,$$

where $d_1, d_2, ..., d_m \in S$ is called quotient near ring.

Proof. Let $d_1\rho, d_2\rho, ..., d_{2m-1}\rho, e_1\rho, e_2\rho, ..., e_m\rho$ be elements of S/ρ. Then for each $1 \leq i \leq j \leq m$,

$$h(d_1\rho, d_2\rho, ..., d_{i-1}\rho, h(d_i\rho, d_{i+1}\rho, ..., d_{m+i-1}\rho), d_{m+i}\rho, d_{m+i+1}\rho, d_{2m-1}\rho)$$
$$= h(d_1\rho, d_2\rho, ..., d_{j-1}\rho, h(d_j\rho, d_{j+1}\rho, ..., d_{m+j-1}\rho), d_{m+j}\rho, d_{m+j+1}\rho, ..., d_{2m-1}\rho).$$

So, the addition is associative on S/ρ. Similarly, the multiplication is associative, too. Finally, in order to show that the right i-distributivity, we have

$$k(d_1\rho, d_2\rho, ..., d_{i-1}\rho, h(e_1\rho, e_2\rho, ..., e_m\rho), d_{i+1}\rho, d_{i+2}\rho, ..., d_n\rho)$$
$$= h(k(d_1\rho, d_2\rho, ..., d_{i-1}\rho, e_1\rho, d_{i+1}\rho, ..., d_n\rho),$$
$$k(d_1\rho, d_2\rho, ..., d_{i-1}\rho, e_2\rho, d_{i+1}\rho, ..., d_n\rho),$$
$$..., k(d_1\rho, d_2\rho, ..., d_{i-1}\rho, e_m\rho, d_{i+1}\rho, ..., d_n\rho)).$$

Therefore, we derive that S/ρ is an (m,n)-near ring. □

Lemma 6. *If (A,h,k) be an (m,n)-near ring with $1 \neq 0$. Then A has at least one j-maximal ideal.*

Proof. Since $\{0\}$ is a proper j-ideal of A, the set Δ of all proper j-ideals of A is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ, and by using Zorn's lemma to this partially ordered set, a maximal j-ideal of A is just a maximal member of the partially ordered set $(\Delta, \subseteq)$. □

Now, we define the concept of a homomorphism between (m,n)-near rings and assert some theorems in this respect.

Definition 8. *A mapping η from the (m,n)-near ring (A,h,k) into the (m,n)-near ring (A',h',k') will be named a homomorphism if for each $d_1, d_2, ..., d_m \in R$*

(1) $(k(d_1, d_2, ..., d_n))\eta = k'((d_1)\eta, (d_2)\eta, ..., (d_n)\eta)$,
(2) $(h(d_1, d_2, ..., d_m))\eta = h'((d_1)\eta, (d_2)\eta, ..., (d_m)\eta)$.

A homomorphism η from the (m,n)-near ring (A,h,k) onto the (m,n)-near ring (A',h',k') is named maximal if for each $d \in A'$ there exists $c_d \in \eta^{-1}(\{d\})$ such that $h(y, ker(\eta)^{(m-1)}) \subset h(c_d, ker(\eta)^{(m-1)})$ for each $y \in \eta^{-1}(\{d\})$ and $ker(\eta) = \{y \in A \mid y\eta = 0\}$.

Lemma 7. *Suppose that η is a homomorphism from the (m,n)-near ring (A,h,k) onto the (m,n)-near ring (A',h',k'). If η be maximal, then $ker(\eta)$ is a Q-ideal, where $Q = \{c_d\}_{d\in A'}$.*

Proof. It is clear that $\bigcup_{d\in A} h(c_d, ker(\eta)^{(m-1)}) = A$. Let c_d and c_b be different elements in Q and $d \neq b$. Let $h(c_d, ker(\eta)^{(m-1)}) \cap h(c_b, ker(\eta)^{(m-1)}) \neq \emptyset$. Thus, there exist $k_1, k_2, ..., k_{m-1}, k'_1, k'_2, ..., k'_{m-1} \in ker(\eta)$ such that $h(c_d, k_1^{m-1}) = h(c_b, {k'}_1^{m-1})$. Thus,

$$\begin{aligned} d &= h'(c_d\eta, k_1\eta, ..., k_{m-1}\eta) = (h(c_d, k_1^{m-1}))\eta = (h(c_b, {k'}_1^{m-1}))\eta \\ &= h'(c_b\eta, k'_1\eta, ..., k'_{m-1}\eta) = b. \end{aligned}$$

This is a contradiction. Hence, we derive that $ker(\eta)$ is a Q-ideal. □

Lemma 8. *Let A, A', η and Q be stated in Lemma 7, and $c_{d_1}, c_{d_2}, ..., c_{d_m}, c_{d_{m+1}}$ be elements in Q.*

(1) *If $h(h(c_{d_1}, c_{d_2}, ..., c_{d_m}), ker(\eta)^{(m-1)}) \subset h(c_{d_{m+1}}, ker(\eta)^{(m-1)})$, then $h'(d_1, d_2, ..., d_m) = d_{m+1}$.*

(2) *If $h(k(c_{d_1}, c_{d_2}, ..., c_{d_n}), ker(\eta)^{(m-1)}) \subset h(c_{d_{n+1}}, ker(\eta)^{(m-1)})$, then $k'(d_1, d_2, ..., d_n) = d_{n+1}$.*

Proof. (1) Since

$$\begin{aligned} h(c_{d_1}, c_{d_2}, ..., c_{d_m}) &\in h(h(c_{d_1}, c_{d_2}, ..., c_{d_m}), ker(\eta)^{(m-1)}) \\ &\subset h(c_{d_{m+1}}, ker(\eta)^{(m-1)}), \end{aligned}$$

it conforms that there are $k_1, k_2, ..., k_{m-1} \in ker(\eta)$ such that $h(c_{d_1}, c_{d_2}, ..., c_{d_m}) = h(c_{d_{m+1}}, k_1^{m-1})$. Thus, we get

$$\begin{aligned} h'(d_1, d_2, ..., d_m) &= h'(c_{d_1}\eta, c_{d_2}\eta, ..., c_{d_m}\eta) = (h(c_{d_1}, c_{d_2}, ..., c_{d_m}))\eta \\ &= (h(c_{d_{m+1}}, k_1^{m-1}))\eta = h'(c_{d_{m+1}}\eta, k_1\eta, ..., k_{m-1}\eta) = d_{m+1}. \end{aligned}$$

(2) We have

$k(c_{d_1}, c_{d_2}, ..., c_{d_n}) \in h(k(c_{d_1}, c_{d_2}, ..., c_{d_n}), ker(\eta)^{(m-1)}) \subseteq h(c_{d_{n+1}}, ker(\eta)^{(m-1)})$, so there exist $k_1, k_2, ..., k_{m-1} \in ker(\eta)$ such that $k(c_{d_1}, c_{d_2}, ..., c_{d_n}) = h(c_{d_{n+1}}, k_1^{m-1})$. Thus, we obtain

$$\begin{aligned} k'(d_1, d_2, ..., d_n) &= k'(c_{d_1}\eta, c_{d_2}\eta, ..., c_{d_n}\eta) = (k(c_{d_1}, c_{d_2}, ..., c_{d_n}))\eta \\ &= (h(c_{d_{n+1}}, k_1^{m-1}))\eta = h'(c_{d_{n+1}}\eta, k_1\eta, ..., k_{m-1}\eta) = d_{n+1}. \end{aligned}$$

This completes the proof. □

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ON DYNAMICS OF QUADRATIC STOCHASTIC OPERATORS GENERATED BY 3-PARTITION ON COUNTABLE STATE SPACE

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Abstract. Quadratic stochastic operator (QSO) theory has advanced significantly since the early 1920s and is still growing due to its numerous applications in a variety of fields, particularly mathematics, where QSOs have inspired mathematicians to use and integrate various mathematical knowledge and concepts to better understand their properties and behaviors. Motivated by the relationship between the number of partitions on an infinite state space and the development of the system of equations corresponding to QSOs, this work sought to investigate the dynamics of QSOs formed by three partitions. First, we define and construct the 3-partition QSOs, which result in a system of equations with three variables. We then provide the formulation of the fixed point form and discuss its behavior using Jacobian matrix analysis. Some scenarios of three-partition QSOs with three different parameters are considered to readily investigate the type of fixed point in such systems. It is demonstrated that the operators can have either an attracting or a saddle fixed point but can never be repelling. We show how the saddle fixed point behaves, by identifying a set of points known as the fixed point's stable manifold.

1. Introduction

Quadratic stochastic operator (QSO) theory has been an appealing topic among researchers in diverse knowledge areas since its establishment in the early 1920s by

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Bernstein [2] through his innovative idea on the synthesis study between Mendel's crossing law and Galton's regression law. The QSO is the simplest nonlinear operator, which refers to a complex system model and such a model is widely applied to describe a dynamical system. Proficiency of the QSO in providing a distribution of the next generation given the distribution of the current generation has led to the acknowledgment of the model as a significant analysis source of dynamical properties and modeling study in various domains running from biology to economy. Due to its immense contributions across fields, the study of QSO has been promptly developing through numerous publications, where the existing studies can be classified into two sets, namely finite and infinite state space. The most prominent QSO study on a finite state space is the study of Volterra QSO [21] due to its accessibility in applying renowned mathematical techniques such as dynamical systems theory, linear algebra, convex analysis, etc. The compelling form of the systems generated by the Volterra QSO has preceded the extension of the investigation to infinite cases [18, 19]. The noteworthy findings of the QSO study on an infinite-dimensional setting allow mathematicians to discover the properties of the operator by introducing different QSO classes on infinite state space [5–11].

Recently, researchers have conducted studies on the classes of QSO on an infinite state space. These works have incorporated the concept of measurable partitions on the state space [13–16]. The research of the dynamics of classes of quadratic stochastic operators, specifically Geometric QSO and Poisson QSO, formed by two measurable partitions on a countable state space, has been thoroughly conducted and extensively described in [13, 14, 16]. Meanwhile, in [15], the concept of measurable partitions is applied to Lebesgue QSO with nonnegative integer parameters that are specified on a continuous state space.

Currently, most studies of the classes of QSO on the countable state space focused on two measurable partitions (see [13, 14, 16]), which limits the analysis to characteristics of two distinct groups. Previous works on Geometric QSO and Poisson QSO [13,14,16] mainly discussed the regular property of such operators through the existence of fixed points, either they are attracting or repelling, since the 2-partition can be represented into a one-dimensional map. Considering the representation of 3-partition by a two-dimensional map may result to the study of an extra behavior of fixed point, namely saddle, we are motivated to extend the study to three measurable partitions to uncover additional properties of these operators. This include a whole process of constructing the QSO generated by 3-partition, followed by the representation of the operators into a system of equations. From here, we will work on the finding of the unique fixed point of the system of equations based on existing theorems and propositions. Some prominent techniques and methods will be used to examine the behavior of the fixed point.

Accordingly, this research paper will establish some forms of QSO classes created by a 3-measurable partition. These classes will be categorized and their dynamics will be further analyzed. Some examples of Geometric QSO and Poisson QSO generated by 3-partition will be demonstrated as a part of the results. Also, we aim to provide evidences of the fixed points to be saddle through an analysis on the presence of a set of points known as a stable manifold of such a saddle fixed point.

The paper is structured in the following manner. Section 2 of the paper introduces the preliminary concepts, including the definitions of QSO and measurable partitions. In Section 3, we outline the process of constructing the QSO created by the 3-partition, provide a detailed study of the dynamics of the operators, present some examples of the trajectory behavior of Geometric QSO and Poisson QSO, and lastly, discuss the behavior of saddle fixed points of such operators through the existence of the stable manifold of the fixed points.

2. Preliminaries

In this section, we provide necessary details to address the key notion of QSO and measurable partitions.

2.1. **Quadratic stochastic operators.**

The quadratic stochastic operator (QSO) has gained significant recognition as a valuable analytical tool for studying dynamical properties and modelling across various fields of study. In a thorough and methodical explanation of the dynamics of quadratic stochastic operators, Ganikhodjaev, Mukhamedov, and Rozikov [12] address the key issues in the QSO theory, including constructions, dynamics, regularity, and more.

Assume X is a state space and $\mathcal{F}$ is a σ-algebra of subsets of X. We denote $(X, \mathcal{F})$ and $S(X, \mathcal{F})$ as a measurable space and a set of all probability measures on such a measurable space, respectively. We then define a family of functions $\{P(x, y, A) : x, y \in X, A \in \mathcal{F}\}$ on $X \times X \times \mathcal{F}$ with the following conditions:

(i) for any $x, y \in X$, $P(x, y, \cdot)$ is a probability measure, where $P(x, y, \cdot) : \mathcal{F} \to [0, 1]$,
(ii) $P(x, y, A)$ is a jointly measurable function with a fixed $A \in \mathcal{F}$, and
(iii) $P(x, y, A) = P(y, x, A)$.

A QSO $V : S(X, \mathcal{F}) \to S(X, \mathcal{F})$ is defined as follows:

$$(V\mu)(A) = \int_X \int_X P(x, y, A) d\mu(x) d\mu(y) \tag{1}$$

for every $\mu \in S(X, \mathcal{F})$ and $A \in \mathcal{F}$. Note that, this operator is called a quadratic stochastic operator (see [2,4]).

Given a finite state space $X = \{1, 2, \dots\}$ and a corresponding σ-algebra $\mathcal{F}$ is a power set, $P(X)$. Then, $S(X, \mathcal{F})$ is known as an $(m-1)$-dimensional simplex with the following form:

$$S(X, \mathcal{F}) \equiv S^{m-1} = \{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R} : x_i \geq 0, i = 1, \dots, m, \sum_{i=1}^{m} x_i = 1\}.$$

Provided that the probability measure $P(i,j,\cdot)$ is a discrete measure, where $P(i,j,\{k\})$ can be written as $P_{ij,k}$ and $\sum_{k=1}^{m} P_{ij,k} = 1$, a corresponding QSO V is defined as follows:

Definition 1. *A quadratic stochastic operator V is a mapping of $V : S^{m-1} \to S^{m-1}$ for any $\mathbf{x} = (x_1, \ldots, x_m) \in S^{m-1}$ and $V\mathbf{x}$ is defined as*

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \tag{2}$$

where the coefficients $P_{ij,k}$ conform to the conditions:

$$P_{ij,k} \geq 0, P_{ij,k} = P_{ji,k}, \text{ and } \sum_{k=1}^{m} P_{ij,k} = 1 \text{ for } i,j,k = 1,\ldots,m.$$

In this work, we consider examples of QSO defined on the countable state space X. Thus, we shall provide the definition of Geometric QSO and Poisson QSO as follows:

Definition 2. *A QSO V in (2) is called a Geometric QSO if for any $i,j \in X$, where $X = \{0,1,\ldots\}$, the probability measure $P(i,j,\cdot)$ is the Geometric distribution $G_{r_{ij}}(k) = (1 - r_{ij})\, r_{ij}^{k}$ with a real parameter $r_{ij} = r_{ji}$, $0 < r_{ij} < 1$.*

Definition 3. *A QSO V in (2) is called a Poisson QSO if for any $i,j \in X$, where $X = \{0,1,\ldots\}$, the probability measure $P(i,j,\cdot)$ is the Poisson distribution $P_{\Lambda_{ij}}(k) = \exp^{-\Lambda_{ij}} \frac{\Lambda_{ij}^{k}}{k!}$ with a positive real parameter Λ_{ij} such that $\Lambda_{ij} = \Lambda_{ji}$.*

Throughout this article, the specified definitions will be used to construct the QSO. The concept of QSO generated by measurable partitions is presented in the following subsection.

2.2. **Quadratic stochastic operators generated by measurable partitions.** This subsection discusses the investigation of QSO generated by measurable partitions. The definition of measurable m-partition is provided below to serve as an overview of the concept of measurable partition that is emphasised in this study.

Definition 4. *A measurable partition of X is a partition such that each of its elements is a measurable set.*

Remark 1. *If $\mathcal{F}$ is a σ-algebra of X and A is a subset of X, then A is called measurable if A is a member of $\mathcal{F}$.*

Let $\xi = \{A_1, \ldots, A_m\}$ be a measurable m-partition of X and $\varsigma = \{B_{ij} : i,j = 1,\ldots,m\}$ be a corresponding partition of $X \times X$, where $B_{ii} = A_i \times A_i$ and $B_{ij} = (A_i \times A_j) \cup (A_j \times A_i)$ for $i \neq j$ and $i,j = 1,\ldots,m$. We choose a family of probability measures denoted by $\{\mu_{ij} : i,j = 1,\ldots,m\}$ on a measurable space $(X, \mathcal{F})$ and define a probability measure $P(x,y,A)$ with $(x,y) \in B_{ij}$ as follows:

$$P(x,y,A) = \mu_{ij}(A),$$

for any measurable set $A \in F$. Hence, for an arbitrary $\lambda \in S(X, \mathcal{F})$,

$$\begin{aligned} V\lambda(A) &= \int_X \int_X P(x,y,A)d\lambda(x)d\lambda(y) \\ &= \sum_{i,j=1}^{m} \int_{A_i} \int_{A_j} \mu_{ij}(A)d\lambda(x)d\lambda(y) \\ &= \sum_{i,j=1}^{m} \mu_{ij}(A)\lambda(A_i)\lambda(A_j). \end{aligned}$$

By a mathematical induction, it is evident that

$$\begin{aligned} V^{n+1}\lambda(A) &= \int_X \int_X P(x,y,A)dV^n\lambda(x)dV^n\lambda(y) \\ &= \sum_{i,j=1}^{m} \int_{A_i} \int_{A_j} \mu_{ij}(A)dV^n\lambda(x)dV^n\lambda(y) \\ &= \sum_{i,j=1}^{m} \mu_{ij}(A)V^n\lambda(A_i)V^n\lambda(A_j), \end{aligned}$$

with

$$V^{n+1}\lambda(A_k) = \sum_{i,j=1}^{m} \mu_{ij}(A_k)V^n\lambda(A_i)V^n\lambda(A_j) \tag{3}$$

by assuming that $\{V^n\lambda : n = 0, 1, \dots\}$ is the trajectory of the initial point λ, where $V^{n+1}\lambda = V(V^n\lambda)$ with $V^0\lambda = \lambda$.

In measure theory, it is understood that $S(X, \mathcal{F})$ is a weak compact, if X is a compact metric space. For a measurable space $(X, \mathcal{F})$, a sequence μ_n is said to converge strongly to a limit μ if

$$\lim_{n\to\infty} \mu_n(A) = \mu(A),$$

for every set $A \in \mathcal{F}$.

Definition 5. *A quadratic stochastic operator V is called a regular (weak regular), for any initial measure $\lambda \in S(X, \mathcal{F})$, where the strong limit (respectively weak limit),*

$$\lim_{n\to\infty} V^n(\lambda) = \mu,$$

exists.

Consider $x_k^{(n)} = V^n\lambda(A_k)$, where $\left(x_1^{(n)}, \dots, x_m^{(n)}\right) \in S^{m-1}$ and $P_{ij,k} = \mu_{ij}(A_k)$. Given a fact that S^{m-1} is the $(m-1)$-dimensional simplex, then the system of equations in (3) can be written as follows:

$$(W\mathbf{x})_k = \sum_{i,j=1}^{k} P_{ij,k} x_i x_j, \tag{4}$$

for all $k = 1, \ldots, m$.

The fundamental system of equations generated for the developed QSOs in this study will be the equation in (4). Upon the construction of the QSOs represented by such as system of equations, we will examine the stability of the system's fixed points and periodic points to analyse the operators' dynamics.

3. Dynamics of Quadratic Stochastic Operators Generated by 3-Partition

In this section, the construction of QSO generated by 3-partition will be demonstrated, followed by the classification of such operators for some cases and their dynamics.

First, let us define a measurable 3-partition $\xi = (A_1, A_2, A_3)$ on the state space X, where its corresponding partition on $X \times X$ is denoted by ς, where $\varsigma = (B_{11}, B_{22}, B_{33}, B_{12}, B_{13}, B_{23})$. We select a family $\{\mu_{ij} : i, j = 1, 2, 3\}$ of Geometric and Poisson distribution with a set of parameters $\{r_{11} = r_1, r_{22} = r_2, r_{33} = r_3, r_{12} = r_4, r_{13} = r_5, r_{23} = r_6\}$ and $\{\Lambda_{11} = \Lambda_1, \Lambda_{22} = \Lambda_2, \Lambda_{33} = \Lambda_3, \Lambda_{12} = \Lambda_4, \Lambda_{13} = \Lambda_5, \Lambda_{23} = \Lambda_6\}$, respectively. Subsequently, we define the probability measure $P(x, y, A)$ as follows:

$$P(x, y, A) = \mu_{ij}(A) \ if \ (x, y) \in B_{ij}, i, j = 1, 2, 3, \tag{5}$$

for any $A \in \mathcal{F}$. Then, we describe the following:

$$A(\mu) = \sum_{k \in A_1} \mu(k), B(\mu) = \sum_{k \in A_2} \mu(k), \ and \ C(\mu) = \sum_{k \in A_3} \mu(k).$$

Thus, by the family of measures (5), one can define the following operator V:

$$\begin{aligned}
V\mu\,(k) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{ij,k} \mu(i)\mu(j) \\
&= \sum_{i \in A_1} \sum_{j \in A_1} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_2} \sum_{j \in A_2} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_3} \sum_{j \in A_3} P_{ij,k} \mu(i)\mu(j) \\
&\quad \sum_{i \in A_1} \sum_{j \in A_2} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_2} \sum_{j \in A_1} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_1} \sum_{j \in A_3} P_{ij,k} \mu(i)\mu(j) \\
&\quad \sum_{i \in A_3} \sum_{j \in A_1} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_2} \sum_{j \in A_3} P_{ij,k} \mu(i)\mu(j) + \sum_{i \in A_3} \sum_{j \in A_2} P_{ij,k} \mu(i)\mu(j) \\
&= \mu_1(k) A^2(\mu) + \mu_2(k) B^2(\mu) + \mu_3(k) C^2(\mu) \\
&\quad + 2\mu_4(k) A(\mu) B(\mu) + 2\mu_5(k) A(\mu) C(\mu) + 2\mu_6(k) B(\mu) C(\mu),
\end{aligned}$$

where by a mathematical induction, it gives us

$$\begin{aligned} V^{n+1}\mu(k) &= \mu_1(k)A^2(V^n\mu) + \mu_2(k)B^2(V^n\mu) + \mu_3(k)C^2(V^n\mu) \\ &\quad + 2\mu_4(k)A(V^n\mu)B(V^n\mu) + 2\mu_5(k)A(V^n\mu)C(V^n\mu) \\ &\quad + 2\mu_6(k)B(V^n\mu)C(V^n\mu) \end{aligned} \tag{6}$$

with

$$\begin{aligned} A\left(V^{n+1}\mu(k)\right) = &\sum_{k\in A_1} \{\mu_1(k)A^2(V^n\mu) + \mu_2(k)B^2(V^n\mu) + \mu_3(k)C^2(V^n\mu) \\ &+ 2\mu_4(k)A(V^n\mu)B(V^n\mu) + 2\mu_5(k)A(V^n\mu)C(V^n\mu) \\ &+ 2\mu_6(k)B(V^n\mu)C(V^n\mu)\}, \end{aligned} \tag{7}$$

$$\begin{aligned} B\left(V^{n+1}\mu(k)\right) = &\sum_{k\in A_2} \{\mu_1(k)A^2(V^n\mu) + \mu_2(k)B^2(V^n\mu) + \mu_3(k)C^2(V^n\mu) \\ &+ 2\mu_4(k)A(V^n\mu)B(V^n\mu) + 2\mu_5(k)A(V^n\mu)C(V^n\mu) \\ &+ 2\mu_6(k)B(V^n\mu)C(V^n\mu)\}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} C\left(V^{n+1}\mu(k)\right) = &\sum_{k\in A_3} \{\mu_1(k)A^2(V^n\mu) + \mu_2(k)B^2(V^n\mu) + \mu_3(k)C^2(V^n\mu) \\ &+ 2\mu_4(k)A(V^n\mu)B(V^n\mu) + 2\mu_5(k)A(V^n\mu)C(V^n\mu) \\ &+ 2\mu_6(k)B(V^n\mu)C(V^n\mu)\}, \end{aligned} \tag{9}$$

where $n = 0, 1, \dots$.

The recurrent equations in (7), (8), and (9) are the constructed QSOs, which can be rewitten as the following system of equations:

$$\begin{aligned} (W\mathbf{x})_1 &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3, \\ (W\mathbf{x})_2 &= b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{12}x_1x_2 + 2b_{13}x_1x_3 + 2b_{23}x_2x_3, \\ (W\mathbf{x})_3 &= c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 + 2c_{12}x_1x_2 + 2c_{13}x_1x_3 + 2c_{23}x_2x_3, \end{aligned} \tag{10}$$

where

$$\begin{aligned} a_{11} = P_{11,1}, a_{22} = P_{22,1}, a_{33} = P_{33,1}, a_{12} = P_{12,1}, a_{13} = P_{13,1}, a_{23} = P_{23,1}, \\ b_{11} = P_{11,2}, b_{22} = P_{22,2}, b_{33} = P_{33,2}, b_{12} = P_{12,2}, b_{13} = P_{13,2}, b_{23} = P_{23,2}, \\ c_{11} = P_{11,3}, c_{22} = P_{22,3}, c_{33} = P_{33,3}, c_{12} = P_{12,3}, c_{13} = P_{13,3}, c_{23} = P_{23,3}, \end{aligned} \tag{11}$$

are arbitrary coefficients in $(0, 1)$. It is clear that these parameters rely on the 3-partition $\xi = \{A_1, A_2, A_3\}$. Note that $P_{ij,k} = \mu_{ij}(A_k)$, then $a_{ij} + b_{ij} + c_{ij} = 1$ for $i, j = 1, 2, 3$.

Saburov and Yusof [20] defined a QSO $Q : S^2 \to S^2$ called a positive QSO as follows:

$$Q(\mathbf{x}) = \left(\sum_{i,j=1}^{3} p_{ij}x_ix_j, \sum_{i,j=1}^{3} q_{ij}x_ix_j, \sum_{i,j=1}^{3} r_{ij}x_ix_j \right)^T, \tag{12}$$

where $p_{ij}, q_{ij}, r_{ij} > 0$ and $p_{ij} + q_{ij} + r_{ij} = 1$ with $p_{ij} = p_{ji}$, $q_{ij} = q_{ji}$, and $r_{ij} = r_{ji}$ for $1 \le i, j \le 3$.

Remark 2. *Let $p_1 \neq p_2$ and $q_1 \neq q_2$. It is apparent that two quadratic equations $x^2 + p_1x + q_1 = 0$ and $x^2 + p_2x + q_2 = 0$ have a unique common root if and only if their resultant is equal to zero, i.e.,*

$$(q_2 - q_1)^2 + p_1(q_2 - q_1)(p_1 - p_2) + q_1(p_1 - p_2)^2 = 0.$$

In this case, the only common root is $x = \dfrac{q_2 - q_1}{p_1 - p_2}$.

Now, let us define the following constants.

$$\alpha_{11} = p_{11} - 2p_{13} + p_{33}, \alpha_{22} = p_{22} - 2p_{23} + p_{33}, \alpha_{12} = p_{12} - p_{13} - p_{23} + p_{33},$$
$$\alpha_1 = p_{13} - p_{33}, \alpha_2 = p_{23} - p_{33}, \alpha_0 = p_{33},$$
$$\beta_{11} = q_{11} - 2q_{13} + q_{33}, \beta_{22} = q_{22} - 2q_{23} + q_{33}, \beta_{12} = q_{12} - q_{13} - q_{23} + q_{33},$$
$$\beta_1 = q_{13} - q_{33}, \beta_2 = q_{23} - q_{33}, \beta_0 = q_{33},$$
$$\gamma_0 = \beta_0\alpha_{11} - \alpha_0\beta_{11}, \gamma_1 = (2\beta_2 - 1)\alpha_{11} - 2\alpha_2\beta_{11}, \gamma_2 = \alpha_{11}\beta_{22} - \alpha_{22}\beta_{11},$$
$$\delta_0 = (2\alpha_1 - 1)\beta_{11} - 2\beta_1\alpha_{11}, \delta_1 = \alpha_{12}\beta_{11} - \beta_{12}\alpha_{11}, \Delta_1 = \gamma_2\delta_0^2 - 2\gamma^1\delta_0\delta_1 + 4\gamma_0\delta_1^2,$$
$$\lambda_0 = \alpha_{11}\gamma_0^2 + (2\alpha_1 - 1)\gamma_0\delta_0 + \alpha_0\delta_0^2, \lambda_4 = \alpha_{11}\gamma_2^2 + 4\alpha_{12}\gamma_2\delta_1 + 4\alpha_{22}\delta_1^2,$$
$$\lambda_3 = 2\alpha_{11}\gamma_2\gamma_1 + 2\alpha_{12}\gamma_2\delta_0 + 4\alpha_{12}\gamma_1\delta_1 + 4\alpha_1\gamma_2\delta_1 - 2\gamma_2\delta_1 + 4\alpha_{22}\delta_1\delta_0 + 8\alpha_2\delta_1^2,$$
$$\begin{aligned}\lambda_2 &= 2\alpha_{11}\gamma_2\gamma_0 + \alpha_{11}\gamma_1^2 + 2\alpha_{12}\gamma_1\delta_0 + 4\alpha_{12}\gamma_0\delta_1 + 2\alpha_1\gamma_2\delta_0 + 4\alpha_1\gamma_1\delta_1 \\ &= \gamma_2\delta_0 - 2\gamma_1\delta_1 + \alpha_{22}\delta_0^2 + 8\alpha_2\delta_1\delta_0 + 4\alpha_0\delta_1^2,\end{aligned}$$
$$\lambda_1 = 2\alpha_{11}\gamma_1\gamma_0 + 2\alpha_{12}\gamma_0\delta_0 + 2\alpha_1\gamma_1\delta_0 + 4\alpha_1\gamma_0\delta_1 - \gamma_1\delta_0 - 2\gamma_0\delta_1 + 2\alpha_2\delta_0^2 + 4\alpha_0\delta_1\delta_0.$$

Theorem 1. *[20] Let $\alpha_{11}\beta_{11}\Delta_1 \neq 0$. The positive quadratic stochastic operator $Q : S^2 \to S^2$ has a unique fixed point (a stationary distribution) if and only if the quartic equation,*

$$\lambda_4p^4 + \lambda_3p^3 + \lambda_2p^2 + \lambda_1p + \lambda_0 = 0,$$

has a unique real root $p_0 \in (0, 1) \setminus \left\{-\frac{\delta_0}{2\delta_1}\right\}$ which satisfies $0 < P_0 < 1$ and $0 < Q_0 < 1$, where

$$P_0 = \frac{\gamma_2p_0^2 + \gamma_1p_0 + \gamma_0}{2\delta_1p_0 + p_0},$$

$$Q_0 = \frac{(\gamma_2 + 2\delta_1)p_0^2 + (\gamma_1 + \delta_0)p_0 + \gamma_0}{2\delta_1 p_0 + \delta_0}.$$

Moreover, in this case, the only fixed point (a stationary distribution) is $(P_0, p_0, 1 - Q_0)^T$.

According to Theorem 1, it signifies that the system of equations in (10) has a unique fixed point for any 3-measurable partition on the state space X. This implies that we can formulate the form of the fixed point of such a two-dimensional operator W generated by 3-partition ξ.

Suppose that the operator W in (10) has a fixed point. Then, we will have the following system of equations:

$$\begin{aligned} x_1 &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3, \\ x_2 &= b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{12}x_1x_2 + 2b_{13}x_1x_3 + 2b_{23}x_2x_3, \\ x_3 &= c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 + 2c_{12}x_1x_2 + 2c_{13}x_1x_3 + 2c_{23}x_2x_3. \end{aligned} \tag{13}$$

Since the operator W in (10) is in the same form as the operator Q in (12), we shall apply the defined constants with $a_{ij} = p_{ij}$, $b_{ij} = q_{ij}$, and $c_{ij} = r_{ij}$. Hence, the following statement may be established.

Proposition 1. *Let* $W : S^2 \to S^2$. *For the operator* W *in (10), the following statements hold true.*

(1) $|Fix(W)| = 1$,
(2) the unique fixed point $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) \in S^2$ *has the following form:*

$$\begin{aligned} x_1^* &= \frac{\gamma_2 p_0^2 + \gamma_1 p_0 + \gamma_0}{2\delta_1 p_0 + p_0}, \\ x_2^* &= p_0 \in (0,1), \\ x_3^* &= \frac{(\gamma_2 + 2\delta_1)p_0^2 + (\gamma_1 + \delta_0)p_0 + \gamma_0}{2\delta_1 p_0 + \delta_0}. \end{aligned}$$

In Lyubich's study [17], it was proven that a one-dimensional QSO may have either an attracting fixed point or a repelling fixed point that tends to a cycle of second-order depending on the value of discriminant of the following one-variable function:

$$f(x_1) = (a - 2b + c)\,x_1 + 2\,(b - c)\,x_1 + c, \tag{14}$$

where $0 \le a, b, c \le 1$ with the value of discriminant Δ of $f(x_1) = x_1$, where

$$\Delta = 4(1-a)c + (1-2b)^2, \tag{15}$$

for the system of equations as follows:

$$\begin{aligned} W(x_1) &= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2, \\ W(x_2) &= b_{11}x_1^2 + 2b_{12}x_1x_2 + b_{22}x_2^2, \end{aligned} \tag{16}$$

for $a_{11} = a$, $a_{12} = b$, $a_{22} = c$, and $a_{ij} + b_{ij} = 1$. As a result, the following assertions are established.

Theorem 2. *[17] A fixed point of the transformation (16) is a unique and belongs to the open interval* $(0,1)$*. The fixed point is attracting if* $0 < \Delta < 4$ *and is repelling if* $4 < \Delta < 5$*.*

Theorem 3. *[17] If* $0 < \Delta < 4$*, then all trajectories converge to a fixed point. If* $4 < \Delta < 5$*, then there exists a cycle of second-order and all trajectories tend to this cycle except the stationary trajectory starting with fixed point.*

Apparently, we may utilize the idea of attracting and repelling fixed points on a one-dimensional map to determine the existence of periodic points of period-2 of the system of equations in (16). Meanwhile, for the system of equations in (10), we may use the notion of non-attracting fixed points instead of repelling fixed points due to the consideration of another type of fixed point, i.e., saddle fixed point on a two-dimensional map. It is known that if a fixed point of such a system of equations is non-attracting, then there exist periodic points of period-2.

The first derivative of the quadratic function (14) with respect to one variable and its discriminant are applied to check the local behavior of the fixed point. However, the same method cannot be implied due to the multivariable functions derived from the system of equations in (10).

Definition 6. *[1] Let* $\mathbf{f} = (f_1, f_2, \ldots, f_m)$ *be a map on* $\mathbb{R}^m$*, and let* $\mathbf{x}^* \in \mathbb{R}^m$*. The Jacobian matrix of* $\mathbf{f}$ *at* $\mathbf{x}^*$*, denoted* $J(\mathbf{x}^*)$*, is the matrix*

$$J(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial f_1}{\partial x_m}(\mathbf{x}^*) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial f_m}{\partial x_m}(\mathbf{x}^*) \end{pmatrix}$$

of partial derivatives evaluated at $\mathbf{p}$*.*

Remark 3. *Given a system*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The key to solving the system is by determining the eigenvalues of $\mathbf{A}$*. To find these eigenvalues, we need to derive the characteristic polynomial of* $\mathbf{A}$*.*

$$det(\mathbf{A} - \lambda \mathbf{I}) = det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a-d)\lambda + (ad - bc).$$

Surely, $D = det(\mathbf{A}) = ad - bc$ *is the determinant of* $\mathbf{A}$*. Meanwhile, the quantity* $T = a + d$ *is the sum of the diagonal elements of the matrix* $\mathbf{A}$ *is called as the trace of* $\mathbf{A}$ *and written as* $tr(\mathbf{A})$*. It is given that the eigenvalues of* $\mathbf{A}$ *are represented by*

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

Consequently, the Jacobian matrix can be implied to investigate the local behavior of the fixed point on a two-dimensional map.

Assume that $\mathbf{x}^* = (x_1^*, x_2^*) = (P_0, p_0)$ and the multivariable functions derived from the system of equations in (10) are as follows:

$$f_1(x_1, x_2) = \alpha_{11}x_1^2 + \alpha_{22}x_2^2 + 2\alpha_{12}x_1x_2 + 2\alpha_1x_1 + 2\alpha_2x_2 + \alpha_0, \tag{17}$$

$$f_2(x_1, x_2) = \beta_{11}x_1^2 + \beta_{22}x_2^2 + 2\beta_{12}x_1x_2 + 2\beta_1x_1 + 2\beta_2x_2 + \beta_0. \tag{18}$$

The Jacobian matrix $J(\mathbf{x}^*)$ of (17) and (18) has the following representation:

$$J(x_1^*, x_2^*) = \begin{pmatrix} 2\alpha_{11}x_1^* + 2\alpha_{12}x_2^* + 2\alpha_1 & 2\alpha_{12}x_1^* + 2\alpha_{22}x_2^* + 2\alpha_2 \\ 2\beta_{11}x_1^* + 2\beta_{12}x_2^* + 2\beta_1 & 2\beta_{12}x_1^* + 2\beta_{22}x_2^* + 2\beta_2 \end{pmatrix}. \tag{19}$$

For simplicity, we will use $\alpha = 2\alpha_{11}x_1^* + 2\alpha_{12}x_2^* + 2\alpha_1$, $\beta = 2\alpha_{12}x_1^* + 2\alpha_{22}x_2^* + 2\alpha_2$, $\chi = 2\beta_{11}x_1^* + 2\beta_{12}x_2^* + 2\beta_1$, and $\delta = 2\beta_{12}x_1^* + 2\beta_{22}x_2^* + 2\beta_2$. According to Remark 3, we compute the eigenvalues of the Jacobian $J(\mathbf{x}^*)$, λ_1 and λ_2 in (19), where

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left(\alpha + \delta + \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta - \beta\chi)}\right), \\ \lambda_2 &= \frac{1}{2}\left(\alpha + \delta - \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta - \beta\chi)}\right). \end{aligned} \tag{20}$$

Definition 7. *[1] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a second-order autonomous system that has a fixed point at $\mathbf{x}^* \in \mathbb{R}^2$. Suppose that λ_1 and λ_2 be the eigenvalues of $J(\mathbf{x}^*)$. Assuming that neither λ_1 nor λ_2 lies on the boundary of the unit disk, there are three distinct characteristics of the trajectories in the neighborhood of the fixed point $\mathbf{x}^*$.*

- *(i) If $|\lambda_i| < 1$ for $i = 1, 2$, then all trajectories converge to $\mathbf{x}^*$, i.e.,$\mathbf{x}^*$ is an attracting fixed point.*
- *(ii) If $|\lambda_1| < 1$, $|\lambda_2| > 1$ or $|\lambda_1| > 1$, $|\lambda_2| < 1$, then the fixed point $\mathbf{x}^*$ is a saddle fixed point. From the stable direction that corresponds to the eigen-direction for the stable eigenvalue $|\lambda_i|$, where $|\lambda_i| < 1$ for $i = 1, 2$, as $n \to \infty$. From the unstable direction, corresponding to the eigen-direction for the unstable eigenvalue $|\lambda_i|$, where $|\lambda_i| > 1$ for $i = 1, 2$, the trajectories $\mathbf{x}^{(n)}$ move away from $\mathbf{x}^*$ as $n \to \infty$. All other trajectories follow hyperbola-like paths, i.e., at first moving closer to $\mathbf{x}^*$, and then moving away from $\mathbf{x}^*$.*
- *(iii) If $|\lambda_i| > 1$ for $i = 1, 2$, then all trajectories move away from the fixed point $\mathbf{x}^*$, so $\mathbf{x}^*$ is a repelling fixed point.*

From the Jacobian matrix in (19), one may find that $-2 < \alpha, \beta, \chi, \delta < 2$, given the fact that such coefficients are defined from the system of equations in (13). We shall let $\gamma = \alpha\delta - \beta\chi$ and $D = (\alpha+\delta)^2 - 4T$. Based on the form of eigenvalues of $J(\mathbf{x}^*)$ in (19) and Definition 6, we shall classify the eigenvalues as follows:

- (i) if $T > 0$, $(\alpha+\delta)^2 < 4T$, and $(\alpha+\delta) = \pm 2$, then the fixed point is nonhyperbolic;
- (ii) if $T > 0$, $(\alpha+\delta)^2 < 4T$, and $|\alpha+\delta| < 2$, then the fixed point is attracting;
- (iii) if $T > 0$, $(\alpha+\delta)^2 < 4T$, and $|\alpha+\delta| > 2$, then the fixed point is repelling;

(iv) if $T>0$, $(\alpha+\delta)^2>4T$, and $-2<\alpha+\delta\pm\sqrt{D}<2$, then the fixed point is attracting;
(v) if $T>0$, $(\alpha+\delta)^2>4T$, and $-2<\alpha+\delta+\sqrt{D}<2$, and $\alpha+\delta-\sqrt{D}<-2$, then the fixed point is saddle;
(vi) if $T>0$, $(\alpha+\delta)^2>4T$, and $-2<\alpha+\delta-\sqrt{D}<2$, and $\alpha+\delta+\sqrt{D}>2$, then the fixed point is saddle;
(vii) if $T=0$ and $|\alpha+\delta|<1$, then the fixed point is attracting;
(viii) if $T=0$ and $|\alpha+\delta|>1$, then the fixed point is saddle;
(ix) if $T<0$ and $-2<\alpha+\delta\pm\sqrt{D}<2$, then the fixed point is attracting;
(x) if $T<0$, $\alpha+\delta>0$, $\alpha+\delta+\sqrt{D}>2$, and $0<\alpha+\delta-\sqrt{D}<2$ then the fixed point is saddle;
(xi) if $T<0$, $\alpha+\delta>0$, $\alpha+\delta-\sqrt{D}<2$, and $-2<\alpha+\delta+\sqrt{D}<0$ then the fixed point is saddle;
(xii) if $T<0$, $\alpha+\delta>0$, $\alpha+\delta+\sqrt{D}>2$, and $\alpha+\delta-\sqrt{D}<-2$ then the fixed point is repelling;
(xiii) if $T<0$, $\alpha+\delta<0$, $\alpha+\delta+\sqrt{D}<-2$, and $\alpha+\delta-\sqrt{D}>2$ then the fixed point is repelling.

Now, we shall analyze the fixed point of the system of equations in (10) based on the given eigenvalues classification. We shall consider a case of 3-partition ξ to investigate the type of fixed point of such operators by the following conditions of the defined parameters:

(1) $\mu_{11}=\mu_{13}=\mu_{33}\neq\mu_{12}=\mu_{23}\neq\mu_{22}$,
(2) $\mu_{11}=\mu_{12}=\mu_{22}\neq\mu_{13}=\mu_{23}\neq\mu_{33}$,
(3) $\mu_{22}=\mu_{23}=\mu_{33}\neq\mu_{12}=\mu_{13}\neq\mu_{11}$.

Given such conditions, we shall obtain the following systems of equations:

$$\begin{aligned}(W_1\mathbf{x})_1&=a_{11}\left(x_1^2+2x_1x_3+x_3^2\right)+a_{22}x_2^2+2a_{12}\left(x_1x_2+x_2x_3\right),\\(W_1\mathbf{x})_2&=b_{11}\left(x_1^2+2x_1x_3+x_3^2\right)+b_{22}x_2^2+2b_{12}\left(x_1x_2+x_2x_3\right),\\(W_1\mathbf{x})_3&=c_{11}\left(x_1^2+2x_1x_3+x_3^2\right)+c_{22}x_2^2+2c_{12}\left(x_1x_2+x_2x_3\right),\end{aligned}\tag{21}$$

$$\begin{aligned}(W_2\mathbf{x})_1&=a_{22}\left(x_1^2+2x_1x_2+x_2^2\right)+a_{33}x_3^2+2a_{23}\left(x_1x_3+x_2x_3\right),\\(W_2\mathbf{x})_2&=b_{22}\left(x_1^2+2x_1x_2+x_2^2\right)+b_{33}x_3^2+2b_{23}\left(x_1x_3+x_2x_3\right),\\(W_2\mathbf{x})_3&=c_{22}\left(x_1^2+2x_1x_2+x_2^2\right)+c_{33}x_3^2+2c_{23}\left(x_1x_3+x_2x_3\right),\end{aligned}\tag{22}$$

$$\begin{aligned}(W_3\mathbf{x})_1&=a_{33}\left(x_2^2+2x_2x_3+x_3^2\right)+a_{11}x_1^2+2a_{13}\left(x_1x_2+x_1x_3\right),\\(W_3\mathbf{x})_2&=b_{33}\left(x_2^2+2x_2x_3+x_3^2\right)+b_{11}x_1^2+2b_{13}\left(x_1x_2+x_1x_3\right),\\(W_3\mathbf{x})_3&=c_{33}\left(x_2^2+2x_2x_3+x_3^2\right)+c_{11}x_1^2+2c_{13}\left(x_1x_2+x_1x_3\right).\end{aligned}\tag{23}$$

We shall denote the operators in (21), (22), and (23) as operators from class $C_1 = \{W_1, W_2, W_3\}$, identified as reducible two-dimensional QSOs due to their ability to be reduced to a one-dimensional setting.

Proposition 2. *Let $\mathbf{x}^* \in S^2$ be a fixed point of the operator W in (10) and λ_i for $i = 1, 2$ are eigenvalues of Jacobian $J(\mathbf{x}^*)$ in (19). For the operators from class C_1, the fixed point $\mathbf{x}^*$ is either attracting or saddle.*

Proof. Let us consider the first operator from class C_1, i.e., the operator W_1 in (21). Referring to the system of equations in (21) and the Jacobian $J(\mathbf{x}^*)$ in (19), we will obtain the following Jacobian matrix,

$$J(\mathbf{x}^*) = \begin{pmatrix} 0 & \beta \\ 0 & \delta \end{pmatrix}.$$

Hence, we have $T = 0$ and $D > 0$. It follows that $\lambda_1 = 0$ and $\lambda_2 = \delta$ when $\delta < 0$, while $\lambda_1 = \delta$ and $\lambda_2 = 0$ when $\delta > 0$. Apparently, if $|\delta| < 1$, then $\mathbf{x}^*$ is an attracting fixed point. Meanwhile, if $|\delta| > 1$, then $\mathbf{x}^*$ is a saddle fixed point.

Next, we shall consider the operator in (22). Considering the Jacobian $J(\mathbf{x}^*)$ in (19), we will get,

$$J(\mathbf{x}^*) = \begin{pmatrix} \alpha & \beta \\ \chi & \delta \end{pmatrix},$$

where $\alpha = \beta \neq \chi = \delta$. Consequently, $T = 0$ and when $\alpha + \delta < 0$, we have $\lambda_1 = 0$ and $\lambda_2 = \alpha + \delta$, while when $\alpha + \delta > 0$, we have $\lambda_1 = \alpha + \delta$ and $\lambda_2 = 0$. Therefore, for the operator W_2, the fixed point $\mathbf{x}^*$ is attracting if $|\alpha + \delta| < 1$, and is saddle if $|\alpha + \delta| > 1$.

Lastly, for the operator W_3 in (23), we may obtain the following Jacobian $J(\mathbf{x}^*)$, where

$$J(\mathbf{x}^*) = \begin{pmatrix} \alpha & 0 \\ \chi & 0 \end{pmatrix}.$$

This follows that $T = 0$ and $D = \alpha^2$. Subsequently, we get $\lambda_1 = 0$ and $\lambda_2 = \alpha$ when $\alpha < 0$, while when $\alpha > 0$, we have $\lambda_1 = \alpha$ and $\lambda_2 = 0$. Then, it is not difficult to verify that $\mathbf{x}^*$ is an attracting fixed point if $|\alpha| < 1$ and $\mathbf{x}^*$ is a saddle fixed point if $|\alpha| > 1$.

Thus, according to Definition 6, evidently if $|\alpha + \delta| < 1$, then $\mathbf{x}^*$ is an attracting fixed point, where $|\lambda_1| < |\lambda_2| < 1$ or $|\lambda_2| < |\lambda_1| < 1$, while if $|\alpha + \delta| > 1$, then $\mathbf{x}^*$ is a saddle fixed point, where $|\lambda_1| < 1 < |\lambda_2|$ or $|\lambda_2| < 1 < |\lambda_1|$. The analysis of the eigenvalues of the Jacobian of the operators from the class C_1 shows that for such operators, the fixed point $\mathbf{x}^*$ is either attracting or saddle as shown in condition (vii) and (viii). The proof is complete. □

In accordance with Proposition 2, one may discover that for the operator W in (10) classified under the class C_1, there exists either an attracting fixed point or a saddle fixed point for some defined partitions and parameters. Also, it is proven that for such operators, the fixed point can never be repelling.

Assume that the behavior of the operators in the class C_1 may represent the behavior of the QSO W in (10). Accordingly, we may establish the following statements.

Corollary 1. *Let $\mathbf{x}^*$ be a fixed point of the operator W in (10). Then, the fixed point $\mathbf{x}^*$ is either attracting or saddle.*

Proposition 3. *Let $\mathbf{x}^*$ be a fixed point of the operator W in (10). Then, the following statements hold true.*

(i) If the fixed point $\mathbf{x}^*$ is attracting, then the trajectory converges to that fixed point.

(ii) If the fixed point $\mathbf{x}^*$ is saddle, then there exists a second-order cycle.

We shall provide some examples using Geometric QSO and Poisson QSO to support the above statements.

Example 1. *Let $A_1 = \{0, 1, 2\}$, $A_2 = \{6, 7, \dots\}$, and $A_3 = \{3, 4, 5\}$ be the measurable 3-partition for Geometric QSO generated by 3-partition with six parameters. We define $r_1 = 0.975$, $r_2 = 0.5$,$r_3 = 0.95$, $r_4 = 0.25$, $r_5 = 0.9$ and $r_6 = 0.2$. Due to Proposition 1, the fixed point of such an operator W in (10) is as follows:*

$$\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (0.5959277932, 0.3461854580, 0.05788674882) \tag{24}$$

We also obtain the following functions, where

$$\begin{aligned} f_1(x_1, x_2) &= -0.326234375x_1^2 - 0.966375x_2^2 - 2(0.136)x_1x_2 \\ &\quad + 2(0.128375)x_1 + 2(0.849375)x_2 + 0.142625, \\ f_2(x_1, x_2) &= 0.5312781916x_1^2 + 0.7505888906x_2^2 + 2(0.2038310312)x_1x_2 \\ &\quad - 2(0.2036508906)x_1 - 2(0.7350278906)x_2 + 0.7350918906. \end{aligned} \tag{25}$$

Then, the Jacobian $J(\mathbf{x}^*)$ is as follows:

$$J(x_1^*, x_2^*) = \begin{pmatrix} -0.2262367068 & 0.8675676963 \\ 0.3670317770 & -0.7074327102 \end{pmatrix},$$

where $T = -0.1583776666$, $\alpha + \delta = -0.933669417$, $\alpha + \delta + \sqrt{D} = 0.2932165790$, and $\alpha + \delta - \sqrt{D} = -2.160555413$. These conform to the condition of a saddle fixed point as stated in (xi), in which $T < 0$, $\alpha + \delta < 0$, $0 < \alpha + \delta + \sqrt{D} < 2$, and $\alpha + \delta = \sqrt{D} < -2$. Following the Jacobian matrix, the eigenvalues are as follows:

$$\lambda_1 = 0.1466082895,$$
$$\lambda_2 = -1.080277706.$$

From this, we get $|\lambda_1| < 1 < |\lambda_2|$. Hence, $\mathbf{x}^*$ in (24) is a saddle point. This demonstrates that there exists a cycle of second-order for such an operator.

Example 2. *Let $A_1 = \{0, 1\}$, $A_2 = \{2, 3\}$, and $A_3 = \{4, 5, \dots\}$ be the measurable 3-partition for Poisson QSO generated by 3-partition with six parameters. Define $\Lambda_1 = 5.25$, $\Lambda_2 = 5.0$,$\Lambda_3 = 1.75$, $\Lambda_4 = 4.75$, $\Lambda_5 = 0.95$ and $\Lambda_6 = 1.0$. Due to Proposition 1, we shall obtain the fixed point of such an operator W in (10) as follows:*

$$\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (040777949974, 0.2535537737, 0.3386512289) \tag{26}$$

Also, the following functions are obtained:

$$\begin{aligned} f_1(x_1, x_2) &= -0.9976146574x_1^2 - 0.9532117384x_2^2 - 2(0.9622782864)x_1x_2 \\ &\quad + 2(0.2762666512)x_1 + 2(0.2578805378)x_2 + 0.4778783446, \\ f_2(x_1, x_2) &= 0.1606229184x_1^2 + 0.1554036174x_2^2 + 2(0.1984161132)x_1x_2 \\ &\quad - 2(0.1915307380)x_1 - 2(0.1760583449)x_2 + 0.4213113057. \end{aligned} \tag{27}$$

Then, the Jacobian $J(\mathbf{x}^*)$ is as follows:

$$J(x_1^*, x_2^*) = \begin{pmatrix} -0.7490898126 & -0.7524443338 \\ -0.1514407223 & -0.1114841458 \end{pmatrix},$$

where $T = -0.03043907551$, $\alpha + \delta = -0.8605739584$, $\alpha + \delta + \sqrt{D} = 0.0680507451$, and $\alpha + \delta - \sqrt{D} = -1.789198662$. These conform to the condition of a saddle fixed point as stated in (viii), in which $T < 0$ and $-2 < \alpha + \delta \pm \sqrt{D} < 2$. Consequently, the eigenvalues are as follows:

$$\lambda_1 = 0.0340253726,$$
$$\lambda_2 = -0.8945993310.$$

It is notable that $|\lambda_1| < |\lambda_2| < 1$. Hence, $\mathbf{x}^*$ in (26) is an attracting point. This shows that the trajectory of such an operator converges to this fixed point.

From the given examples, it has been demonstrated that such operators may have either an attracting or a saddle fixed point depends on the value of parameters. The discovery of non-attracting fixed point on the two-dimensional setting as a saddle fixed point is considered significant due to an initial assumption that the fixed point should be repelling based on the study of QSOs on one-dimensional simplex. Hence, in the next subsection, we shall discuss the behavior of saddle fixed point to provide a comprehensive finding on the dynamics of such operators generated by 3-partition.

3.1. **Behavior of the saddle fixed point of quadratic stochastic operators generated by 3-partition.**

Remark 4. *[1] A saddle fixed point is unstable. Most initial values near it will move away under iteration of the map. However, unlike the case of a repelling fixed point (source), not all nearby initial values will move away. The set of initial values that converge to the saddle will be called the stable manifold of the saddle.*

Definition 8. *[1] Let f be a smooth map on $\mathbb{R}^2$, and let $\mathbf{p}$ be a saddle fixed point or periodic saddle point for f. The stable manifold of $\mathbf{p}$, denoted $S(\mathbf{p})$ is the set of points $\mathbf{v}$ such that $|f^n(\mathbf{v}) - f^n(\mathbf{p})| \to 0$ as $n \to \infty$. The unstable manifold of $\mathbf{p}$, denoted $U(\mathbf{p})$, is the set of points $\mathbf{v}$ such that $|f^{-n}(\mathbf{v}) - f^{-n}(\mathbf{p})| \to 0$ as $n \to \infty$.*

From Definition 7, Remark 4, and Definition 8, the fact that the operator in (10) with a saddle fixed point is unstable, i.e., from a stable direction corresponds to the stable eigenvalue, the trajectories converge to the fixed point, while from an unstable direction corresponds to the unstable eigenvalue, the trajectories move away from such fixed point. Hence, this conforms the fact that the saddle fixed point indicates the existence of a second-order cycle of the system of equations in (10).

Verification of the saddle fixed point as the unstable fixed point of the operator in (10) and the fixed point of such an operator can never be repelling is rather ambiguous. This comes from the fact that the behavior of a repelling fixed point is quite similar to the behavior of a saddle fixed point, where all trajectories move away from the fixed point except when the initial point is the fixed point itself. Meanwhile, for a saddle fixed point, it behaves as an attractor for some trajectories and a repeller for others. Herewith, we can find a set of points $\mathbf{x} \in S^2$, where $\mathbf{x} \neq \mathbf{x}^*$ in which such points will eventually converge to the saddle fixed point.

Next, we will consider some examples of the saddle fixed point case in Example 1, where the presence of the set of points $\mathbf{x} \in S^2$, denoted by $\rho_{\mathbf{a}}$ for $n \to \infty$, where $\rho_{\mathbf{a}} = \left\{\mathbf{a} \in S^2 : \mathbf{a} \neq \mathbf{x}^*, |W^n(\mathbf{a}) - \mathbf{x}^*| \to 0\right\}$ will be provided.

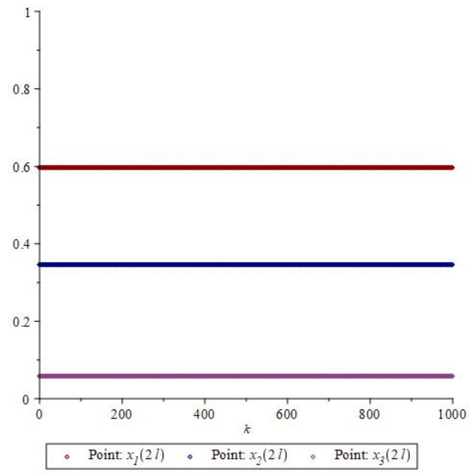
Assume that $\mathbf{a} = (x_1 + \epsilon, x_2 + \epsilon, 1 - x_1 - x_2 - 2\epsilon) = (x_1 + \epsilon, x_2 + \epsilon, x_3 + \epsilon)$, where $\epsilon = m \times 10^{-10}$ with $m = [-100, 100]$. For the operator W in (10) from Example 1, we can find the initial values near the saddle fixed point $\mathbf{x}^*$, where such an operator is regular (see Figure 1), as both even and odd number iterations of x_1 , x_2, and x_3 converge to the same value. Computationally, we obtain that when $-5.5 > m > 4.5$, the trajectories $\mathbf{x}^{(\mathbf{n})}$ approach $\mathbf{x}^*$ as $n \to \infty$.

Figure 1(a) shows Example 1, which indicates the points, x_1, x_2, and x_3 for even iterations, while Figure 1(b) displays the points of x_1, x_2, and x_3 for odd iterations. This demonstrates that both even and odd iterations of the saddle fixed point case operator will converge to the same value when we choose any initial points that belong to the stable manifold.

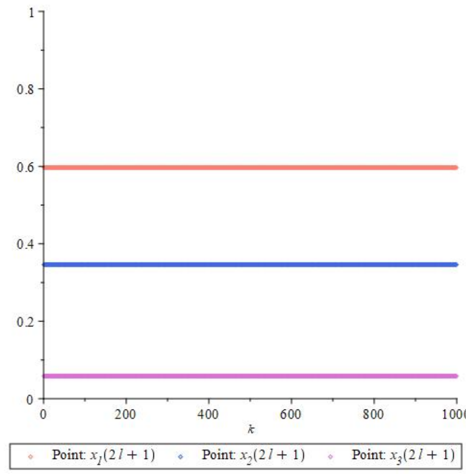
Contrarily, when we choose any initial values, which are very close to the saddle fixed point, in which $m \leq -5.5$ or $m \geq 4.5$, one can see the behavior of even and odd number iterations of all coordinates do not converge to the same values (refer Figure 2).

We use Figure 2 to illustrate the behavior of points x_1, x_2, and x_3 of the saddle fixed point case operator in Example 1 with six different colors to represent $x_i(2l)$ and $x_i(2l + 1)$ for $i = 1, 2, 3$ and $l = 0, \ldots, 500$.

In Figure 1, we show that for some initial values close to the saddle fixed point, the trajectories will eventually converge to the fixed point, indicating the existence

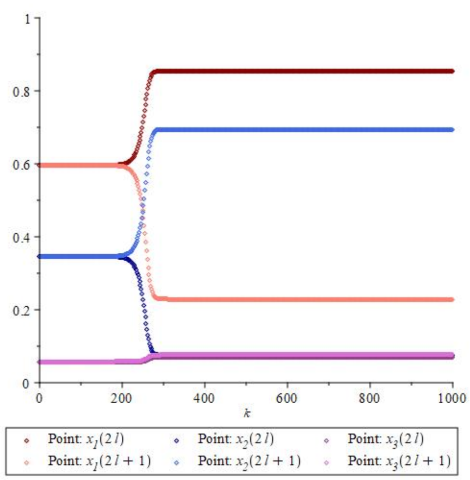


(a) $\mathbf{x}^{(0)} = \mathbf{a}$, where $m = -5.4$

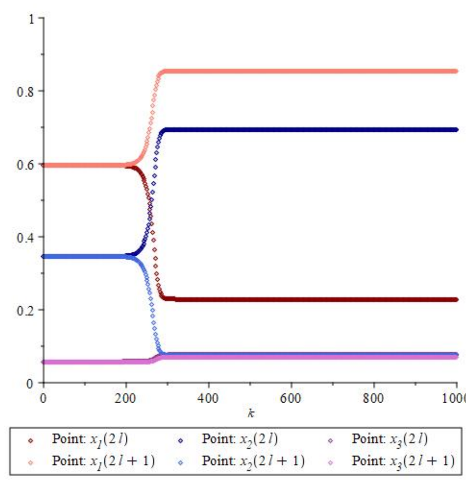


(b) $\mathbf{x}^{(0)} = \mathbf{a}$, where $m = 4.45$

FIGURE 1. Trajectory behavior of regular transformation of operator W in (10) from Example 1 for $l = 0, \dots, 500$



(a) $\mathbf{x}^{(0)} = \mathbf{a}$, where $m = -5.5$



(b) $\mathbf{x}^{(0)} = \mathbf{a}$, where $m = 4.65$

FIGURE 2. Trajectory behavior of nonregular transformation of operator W in (10) from Example 1 for $l = 0, \dots, 500$

of the set of points $\rho_{\mathbf{a}}$ known as the stable manifold of $\mathbf{x}^*$. Meanwhile, in Figure 2, it is shown that for some relatively close initial values, where $\mathbf{x}^{(\mathbf{0})} \notin \rho_{\mathbf{a}}$, the trajectories will move away from the saddle fixed point $\mathbf{x}^*$ after a number of iterations. Hence, it is evident that the saddle fixed point of the operator W behaves as an attractor for some trajectories and as a repeller for others.

With the given examples as evidence of the existence of the stable manifold of the saddle fixed point of the operator W in (10), we may establish the following statement.

Remark 5. *Let* $\rho_{\mathbf{a}} = \left\{\mathbf{a} \in S^2 : \mathbf{a} \neq \mathbf{x}^*, |W^n(\mathbf{a}) - \mathbf{x}^*| \to 0, n \to \infty\right\}$ *be the set of points that belong to the stable manifold of any saddle fixed point* $\mathbf{x}^*$ *of the operator* $W : S^2 \to S^2$ *in (10). Then, the following statements hold true.*

i If $\mathbf{x}^{(0)} \in \rho_{\mathbf{a}}$, *then the operator* W *is regular.*
ii If $\mathbf{x}^{(0)} \notin \rho_{\mathbf{a}}$, *then the operator* W *is nonregular.*

4. Conclusion

The construction of QSOs generated by 3-partition and the formulation of the fixed point form of the system of equations corresponds to such QSOs were presented throughout the paper. By implementing the analysis of the quadratic function (14) on a one-dimensional map, we can determine the existence of periodic points of period-2 through the repelling behavior of the unique fixed point. Unlike the case of one-dimensional map, where we addressed a repelling fixed point to signify the existence of the periodic points of period-2, in the case of a two-dimensional map, we used the notion of non-attracting fixed point to represent both unstable fixed points; i.e., repelling and saddle. Based on the eigenvalues of the Jacobian matrix in (19) of the system of equations (10), we classified the fixed point accordingly.

Further investigation on the dynamics of the QSOs generated by 3-partition was carried out by considering three cases of 3-partition with three parameters, where the corresponding systems of equations denoted as class C_1 can be reduced to a one-dimensional setting. These cases were then implied to explore the behavior of the fixed point through the classification of eigenvalues of the Jacobian matrix in (19), where we established Proposition 2, in which it is proven that such operators may have either an attracting or a saddle fixed point and the fixed point can never be repelling.

We provide some examples using Geometric QSO and Poisson QSO to demonstrate the behavior of the fixed point of the operators through the classification of their eigenvalues. From the obtained results, it is remarked that an attracting fixed point implies the existence of a strong limit, hence the operator is regular. Another example showed that the saddle fixed point indicates the existence of the second-order cycle, where the operator is nonregular.

To illustrate the fact that the fixed point of the operator in (10) can never be repelling, it is necessary to find a set of points denoted by $\rho_{\mathbf{a}}$ that belongs to the stable manifold of the saddle fixed point. We utilized Example 1 with a saddle fixed point and searched for the set of points $\rho_{\mathbf{a}}$. It is shown that for any saddle fixed point of the operator W in (10), there exist some relatively close initial values to the saddle fixed point, which will converge to such a fixed point, while most of the initial values will move away from it. From this, we established the statements in Remark 4.

Author Contribution Statements S. N. Karim and N. Z. A. Hamzah conceived of the presented idea. S. N. Karim developed the theory and performed the computations. N. Z. A. Hamzah verified the analytical methods and encouraged S. N. Karim to investigate the stable manifold notion and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

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NONLINEAR APPROXIMATION BY N-DIMENSIONAL SAMPLING TYPE DISCRETE OPERATORS WITH APPLICATIONS

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Abstract. In this paper, we explore N-dimensional nonlinear discrete operators, closely related to generalized sampling series. We investigate their approximation properties by using the supremum norm and employ a summability method to generalize the discrete operators. The order of convergence is studied by using suitable Lipschitz classes of uniformly continuous functions. We exemplify kernel functions that meet the necessary conditions. Additionally, in the final section of the paper, we propose an operator-based method for digital image zooming.

1. Introduction

In 1980s, the German mathematician Butzer introduces the theory of generalized sampling operators in [22] aiming to reconstruct signals that are not necessarily bandlimited (see [22,23,34]). As is well-known, these operators have numerous applications, particularly in signal theory [4,5,12–14,18,19,23,32,34]. On the other hand, the discrete operators considered in the present paper are closely associated with generalized sampling series and have significant applications, including economic forecasting, geophysics, speech processing, and others [20–22].

In [4], Angeloni and Vinti investigate the convergence problem of generalized sampling series under a φ-variational functional using one-dimensional linear discrete operators. Inspired by [4], we construct a nonlinear setting of N-dimensional discrete operators and improve upon it by using Bell-type summability methods [16,17] (which is also studied by Stieglitz in [36]) under the usual supremum

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norm (see also [6,7,38]). Our new operator is defined by

$$\mathcal{T}_{n,\upsilon}(f;\mathbf{x}) = \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} H_w\left(f\left(\mathbf{x}-\frac{\mathbf{k}}{w}\right)\right) l_{\mathbf{k},w} \quad \left(\mathbf{x}\in\mathbb{R}^N,\ n,\upsilon\in\mathbb{N}\right), \tag{1}$$

where $\mathcal{A}=\{A^{\upsilon}\}_{\upsilon\in\mathbb{N}}=\{[a_{nw}^{\upsilon}]\}_{\upsilon\in\mathbb{N}}$ is family of nonnegative regular matrices, $f:\mathbb{R}^N\to\mathbb{R}$ is bounded, $H_w:\mathbb{R}\to\mathbb{R}$, $H_w(0)=0$ and H_w is Lipschitz continuous, that is,

$$|H_w(u)-H_w(v)|\leq C\,|u-v|$$

for some $C>0$ and for every $w\in\mathbb{N}$, $u,v\in\mathbb{R}$. Here, $l_{\mathbf{k},w}:=l_{(k_1,\ldots,k_N),w}\in l^1\left(\mathbb{Z}^N\right)$ is a family of N-dimensional discrete kernels for each $w\in\mathbb{N}$.

We will prove that

$$\|\mathcal{T}_{n,\upsilon}(f)-f\|\to 0 \text{ (uniformly in } \upsilon),\ \text{ as } n\to\infty$$

for all $f\in BUC\left(\mathbb{R}^N\right)$ (the space of bounded and uniformly continuous functions on $\mathbb{R}^N$), where $\|\cdot\|$ denotes the usual supremum norm on $\mathbb{R}^N$. Then, we examine the order of convergence by means of suitable Lipschitz class of continuous fuctions. Utilizing from the relation between operators (1) and nonlinear generalized sampling operators, introduced by

$$\mathcal{S}_{n,\upsilon}(f;\mathbf{x}) = \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} H_w\left(f\left(\frac{\mathbf{k}}{w}\right)\right)\chi(w\mathbf{x}-\mathbf{k}) \quad \left(\mathbf{x}\in\mathbb{R}^N,\ n,\upsilon\in\mathbb{N}\right),$$

it is possible to show that, in some specific cases, $\mathcal{T}_{n,\upsilon}(f)$ coincides with $\mathcal{S}_{n,\upsilon}(f)$, and hence,

$$\|\mathcal{S}_{n,\upsilon}(f)-f\|\to 0 \text{ (uniformly in } \upsilon), \text{ as } n\to\infty$$

holds. Some related recent articles on multidimensional sampling operators can be found in [1,24].

For examples of $l_{\mathbf{k},w}$ that fulfill all the kernel assumptions, the reader can review the last section. Lastly, we prove that these types of discrete operators can be useful in digital zoom.

2. Preliminaries

In this section, some basic definitions, notations and kernel assumptions will be given.

Bell-type summability method is defined as follows:

Consider the following family of infinite matrices $\mathcal{A}=\{A^{\upsilon}\}_{\upsilon\in\mathbb{N}}=\{[a_{nw}^{\upsilon}]\}_{\upsilon\in\mathbb{N}}$ $(n,w\in\mathbb{N})$ with real or complex entries. For a given sequence $x=(x_w)$ and the double sequence $(\mathcal{A}x)_n^{\upsilon}$, $\mathcal{A}$-transform of x is defined by

$$(\mathcal{A}x)_n^{\upsilon} := \left\{\sum_{w=1}^{\infty} a_{nw}^{\upsilon} x_w\right\} \quad (n,\upsilon\in\mathbb{N})$$

whenever the series is convergent for all $n, \upsilon \in \mathbb{N}$. Moreover, it is called that "x is $\mathcal{A}$-summable to L" provided that

$$\lim_{n\to\infty} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} x_w = L \text{ uniformly in } \upsilon,$$

and this convergence is denoted by

$$\mathcal{A} - \lim x = L \quad \text{(see [16])}.$$

Furthermore, $\mathcal{A}$ is called regular, if $\mathcal{A} - \lim x = L$ whenever $\lim_{k\to\infty} x_k = L$ ([16, 17]). A characterization of regularity is given by Bell in [17], such that $\mathcal{A}$ is regular if and only if

a) for every fixed $w \in \mathbb{N}$, $\lim_{n\to\infty} a_{nw}^{\upsilon} = 0$ (uniformly in υ),
b) $\mathcal{A} - \lim e = 1$, where $e = (1, 1, \ldots)$
c) for every $n, \upsilon \in \mathbb{N}$, $\sum_{w=1}^{\infty} |a_{nw}^{\upsilon}| < \infty$ and there exist $N, M \in \mathbb{N}$ such that $\sup_{n\geq N, \upsilon\in\mathbb{N}} \sum_{w=1}^{\infty} |a_{nw}^{\upsilon}| \leq M$.

In the whole paper, it is supposed that $\mathcal{A}$ is regular and $a_{nw}^{\upsilon} \in \mathbb{R}_0^+$ for all $n, w, \upsilon \in \mathbb{N}$.

We should state that Bell's method has significant advantages in coping with the lack of convergence. In addition to classical convergence, by taking some definite matrices, $\mathcal{A}$-summability reduces to Cesàro summability [28], almost convergence [31], and more [29, 30]. For applications of the Bell-type summability method, we refer to [6, 8–11, 27, 33, 35, 37, 39].

Throughout the paper, the following notations and assumptions will be used.

Here are the notations:

- An N-dimensional vector $\mathbf{x} \in \mathbb{R}^N$ is denoted by $\mathbf{x} = (x_1, \ldots, x_N)$, where $|\mathbf{x}| = \sqrt{x_1^2 + \ldots + x_N^2}$.
- For a given $u_{\mathbf{k}} : \mathbb{Z}^N \to \mathbb{R}$ where $u_{\mathbf{k}} := u_{(k_1,\ldots,k_N)}$, we denote by $\|u_{\mathbf{k}}\|_{l^1}$, the l^1 norm of $u_{\mathbf{k}}$ on $\mathbb{Z}^N$, i.e., $\|u_{\mathbf{k}}\|_{l^1} := \sum \cdots \sum_{(k_1,\ldots k_N)\in\mathbb{Z}^N} |u_{\mathbf{k}}|$.
- We directly use $\sum_{\mathbf{k}\in\mathbb{Z}^N}$ instead of $\sum \cdots \sum_{(k_1,\ldots k_N)\in\mathbb{Z}^N}$.

Here are the assumptions:

(l_1) $\sup_{n,\upsilon\in\mathbb{N}} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \|l_{\mathbf{k},w}\|_{l^1} = A < \infty$

(l_2) $\mathcal{A} - \lim \left(\sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} \right) = 1$

(l_3) $\exists r > 0$ such that $\mathcal{A} - \lim \left(\sum_{|\mathbf{k}|\geq r} |l_{\mathbf{k},w}| \right) = 0.$

Here, conditions $(l_1) - (l_3)$ reduce to the approximate identities in [4] in the case of $\mathcal{A} = \{I\}$, where I corresponds to the identity matrix.

Due to the nonlinearity of the kernel H_w, we also require condition (2):

$$\lim_{w\to\infty} \|G_w\|_J = 0 \qquad (2)$$

(uniformly in every bounded interval $J \subset \mathbb{R}$)

where $G_w(u) := H_w(u) - u$ and $\|\cdot\|_J$ denotes the supremum norm on the bounded interval $J \subset \mathbb{R}$.

3. Approximation in Supremum Norm

First, we will investigate the well-definiteness of the operators of type (1).

Lemma 1. *Let f be a bounded function on $\mathbb{R}^N$, $f \in L^1(\mathbb{R}^N)$ and (l_1) hold. Then, $\|\mathcal{T}_{n,\upsilon}(f)\| < \infty$ for all $n, \upsilon \in \mathbb{N}$, namely, $\mathcal{T}_{n,\upsilon}$ maps from the space of bounded functions into itself.*

Proof. Using the Lipschitz property of H_w with $H_w(0) = 0$, we have

$$\begin{aligned}|\mathcal{T}_{n,\upsilon}(f;\mathbf{x})| &\le \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} \left|H_w(f(\mathbf{x}-\tfrac{\mathbf{k}}{w}))\right| |l_{\mathbf{k},w}| \\ &\le C \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} \left|f(\mathbf{x}-\tfrac{\mathbf{k}}{w})\right| |l_{\mathbf{k},w}|,\end{aligned}$$

where C is the Lipschitz constant of H_w. Since f is bounded, from (l_1)

$$\begin{aligned}|\mathcal{T}_{n,\upsilon}(f;\mathbf{x})| &\le C\|f\| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \\ &\le C\|f\| A\end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^N$. Therefore, taking supremum over all $\mathbf{x} \in \mathbb{R}^N$, we conclude that

$$\|\mathcal{T}_{n,\upsilon}(f;\mathbf{x})\| \le C\|f\| A$$

for all $n, \upsilon \in \mathbb{N}$. □

Lemma 2. *Let $f \in BUC(\mathbb{R}^N)$ and (l_1) hold. Then, $\mathcal{T}_{n,\upsilon}(f) \in BUC(\mathbb{R}^N)$ for all $n, \upsilon \in \mathbb{N}$, namely, $\mathcal{T}_{n,\upsilon}$ maps from the space of bounded and uniformly continuous functions into itself.*

Proof. By the uniform continuity of f on $\mathbb{R}^N$, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|f(\mathbf{x}-\frac{\mathbf{k}}{w}) - f(\mathbf{y}-\frac{\mathbf{k}}{w})\right| < \varepsilon$ whenever $\left|\mathbf{x}-\frac{\mathbf{k}}{w} - (\mathbf{y}-\frac{\mathbf{k}}{w})\right| = |\mathbf{x}-\mathbf{y}| < \delta$. Now, using the triangle inequality and Lipschitz property of H_w

$$\begin{aligned}|\mathcal{T}_{n,\upsilon}(f;\mathbf{x}) - \mathcal{T}_{n,\upsilon}(f;\mathbf{y})| &\le \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \left|H_w\left(f\left(\mathbf{x}-\tfrac{\mathbf{k}}{w}\right)\right) - H_w\left(f\left(\mathbf{y}-\tfrac{\mathbf{k}}{w}\right)\right)\right| \\ &\le C \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \left|f\left(\mathbf{x}-\tfrac{\mathbf{k}}{w}\right) - f\left(\mathbf{y}-\tfrac{\mathbf{k}}{w}\right)\right|\end{aligned}$$

$$< \varepsilon C \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \, ,$$

hold, where C is the Lipschitz constant of H_w. Then from (l_1)

$$|\mathcal{T}_{n,\upsilon}(f;\mathbf{x}) - \mathcal{T}_{n,\upsilon}(f;\mathbf{y})| < \varepsilon C A$$

which completes the proof. □

Our approximation theorem is as follows.

Theorem 1. *If $f \in BUC(\mathbb{R}^N)$ and $(l_1) - (l_3)$, (2) hold, then we have*

$$\lim_{n\to\infty} \|\mathcal{T}_{n,\upsilon}(f) - f\| = 0 \text{ (uniformly in } \upsilon\text{)}.$$

Proof. Adding and subtracting some suitable terms, from the triangle inequality, we obtain

$$\begin{aligned}
|\mathcal{T}_{n,\upsilon}(f;\mathbf{x}) - f(\mathbf{x})| &= \Bigg| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} \left(H_w\left(f\left(\mathbf{x}-\tfrac{\mathbf{k}}{w}\right)\right) - f\left(\mathbf{x}-\tfrac{\mathbf{k}}{w}\right)\right) \\
&\quad + \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} \left(f\left(\mathbf{x}-\tfrac{\mathbf{k}}{w}\right) - f(\mathbf{x})\right) \\
&\quad + f(\mathbf{x}) \left(\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} - 1 \right) \Bigg| \\
&\le \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \left\| H_w\left(f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)\right) - f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)\right\| \\
&\quad + \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \left\| f\left(\cdot-\tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\
&\quad + \|f\| \left| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} - 1 \right| =: A_1 + A_2 + A_3.
\end{aligned}$$

Since supremum is taken over $\mathbb{R}^N$, then we have

$$\left\| H_w\left(f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)\right) - f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)\right\| = \|H_w(f) - f\| \, .$$

Moreover, since f is bounded, then there exists an interval $J = [C_1, C_2]$ such that $C_1 \le f(\mathbf{x}) \le C_2$ and

$$|H_w(f(\mathbf{x})) - f(\mathbf{x})| \le \|G_w\|_J \qquad (3)$$

for all $\mathbf{x} \in \mathbb{R}^N$, which implies

$$\|H_w(f) - f\| \le \|G_w\|_J \, .$$

Then, from (2) for every $\varepsilon > 0$, one can find a positive number w_0 such that

$$\|H_w(f) - f\| < \varepsilon \qquad (4)$$

for all $w > w_0$. One can write A_1 as follows

$$\begin{aligned} A_1 &= \sum_{w=1}^{w_0} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \, \|H_w(f) - f\| \\ &+ \sum_{w=w_0+1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \, \|H_w(f) - f\| \\ &:= A_1^1 + A_1^2, \end{aligned}$$

from (l_1) and (4), we observe

$$A_1^2 < A\varepsilon$$

and

$$\begin{aligned} A_1^1 &\leq \sum_{w=1}^{w_0} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \, \|G_w\|_J \\ &\leq D \sum_{w=1}^{w_0} a_{nw}^{\upsilon}, \end{aligned}$$

where $D := \max_{1\leq w\leq w_0} \left\{ \|G_w\|_J \, \|l_{\mathbf{k},w}\|_{l^1} \right\}$. In A_1^1, by the regularity of $\mathcal{A}$, for each $w \in \{1, 2, \cdots, w_0\}$, there exists a $n_0 = n_0(w, \varepsilon) \in \mathbb{N}$, such that $a_{nw}^{\upsilon} < \varepsilon$ for all $n > n_0$ and $\upsilon \in \mathbb{N}$. Since $w \in \{1, 2, \cdots, w_0\}$, one can find a common $\bar{n}_0 = \bar{n}_0(\varepsilon) := \max_{w\in\{1,2,\ldots,w_0\}} \{n_0(w, \varepsilon)\}$ such that

$$a_{nw}^{\upsilon} < \varepsilon$$

and hence

$$A_1^1 < D w_0 \varepsilon$$

for all $n > \bar{n}_0$, $\upsilon \in \mathbb{N}$ and $w \in \{1, 2, \cdots, w_0\}$.

In A_2, due to the uniform continuity of f, for every $\varepsilon > 0$ there can be found a number $\delta > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon \tag{5}$$

whenever $|\mathbf{x} - \mathbf{y}| < \delta$. Besides, for a given fixed $\bar{r}$ corresponding to assumption (l_3), there exists a number $w_1 \in \mathbb{N}$ satisfying that

$$\left| \frac{\bar{r}}{w} \right| < \delta$$

for all $w > w_1$. Now, writing A_2 as follows

$$\begin{aligned} A_2 &= \sum_{w=1}^{w_1} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|<\bar{r}} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| \\ &+ \sum_{w=w_1+1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|<\bar{r}} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| \\ &+ \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|\geq\bar{r}} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| \\ &:= A_2^1 + A_2^2 + A_2^3 \end{aligned}$$

and considering $\left|\mathbf{x}-\frac{\mathbf{k}}{w}-\mathbf{x}\right|=\left|\frac{\mathbf{k}}{w}\right|<\frac{\bar{r}}{w}<\delta$ in A_2^2, one can obviously see from (5) and (l_1) that

$$A_2^2<A\varepsilon.$$

For A_2^1, using the regularity of $\mathcal{A}$, it is possible to find a number $n_1=n_1(\varepsilon)$ such that

$$A_2^1<D'w_1\varepsilon$$

for all $n>n_1$, where $D':=\max_{1\leq w\leq w_1}\left\{\sum_{|\mathbf{k}|<\bar{r}}|l_{\mathbf{k},w}|\left\|f\left(\cdot-\frac{\mathbf{k}}{w}\right)-f(\cdot)\right\|\right\}$.

For A_2^3, since f is bounded, directly from (l_3)

$$A_2^3<2\|f\|\varepsilon$$

yields for sufficiently large $n\in\mathbb{N}$.

Finally, from (l_2), we get

$$A_3<\|f\|\varepsilon$$

for sufficiently large $n\in\mathbb{N}$, which completes the proof by the arbitrariness of ε. □

4. Order of Convergence

In this section, we study the order of convergence. For this reason, we introduce the following Lipschitz class.

Let $\alpha>0$ be given. Then define $Lip_N(\alpha)$ such that:

$$Lip_N(\alpha)=\left\{f\in BUC\left(\mathbb{R}^N\right):\|f(\cdot-\mathbf{t})-f(\cdot)\|=O\left(|\mathbf{t}|^\alpha\right)\text{ as }\mathbf{t}\to\mathbf{0}\right\}.$$

Here, with the notation $f(\mathbf{t})=O(g(\mathbf{t}))$ as $\mathbf{t}\to\mathbf{0}$ we mean that one may find $\delta,R>0$ such that $|f(\mathbf{t})|\leq R|g(\mathbf{t})|$ whenever $|\mathbf{t}|<\delta$.

We require the following conditions on the kernel for the order of convergence.

Let $\alpha>0$ and $\mathcal{A}=\{[a_{nw}^{\upsilon}]\}_{\upsilon\in\mathbb{N}}$ be fixed. Then, consider the followings:

$$\left(\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{\mathbf{k}\in\mathbb{Z}^N}l_{\mathbf{k},w}-1\right)=O\left(1/n^\alpha\right)\text{ as }n\to\infty\text{ (uniformly in }\upsilon),\tag{6}$$

there exists a constant $r_0>0$ such that

$$\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{|\mathbf{k}|<r_0}\frac{|l_{\mathbf{k},w}|}{w^\alpha}=O\left(1/n^\alpha\right)\text{ as }n\to\infty\text{ (uniformly in }\upsilon),\tag{7}$$

$$\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{|\mathbf{k}|\geq r_0}|l_{\mathbf{k},w}|=O\left(1/n^\alpha\right)\text{ as }n\to\infty\text{ (uniformly in }\upsilon)\tag{8}$$

and

$$\text{for each }w\in\mathbb{N},\ a_{nw}^{\upsilon}=O\left(1/n^\alpha\right)\text{ as }n\to\infty\text{ (uniformly in }\upsilon).\tag{9}$$

Theorem 2. *Let* $\sup_{w\in\mathbb{N}} \|l_{\mathbf{k},w}\|_{l^1} = \breve{A} < \infty$. *Assume that for a fixed* $\mathcal{A} = \{[a_{nw}^{\upsilon}]\}_{\upsilon\in\mathbb{N}}$ *and* $\alpha > 0$, *(6)-(9) hold. Assume further that*

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \|G_w\|_J = O\left(1/n^{\alpha}\right) \text{ for every bounded interval } J \subset \mathbb{R} \quad \text{(uniformly in } \upsilon) \tag{10}$$

hold. If $f \in Lip_N(\alpha)$, *then*

$$\|\mathcal{T}_{n,\upsilon}(f) - f\| = O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon).$$

Proof. We know from the proof of Theorem 1 that

$$\begin{aligned}\|\mathcal{T}_{n,\upsilon}(f) - f\| &\le \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \|G_w\|_J \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \\ &+ \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ &+ \|f\| \left| \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} - 1 \right| \\ &=: B_1 + B_2 + B_3\end{aligned}$$

for some bounded interval $J \subset \mathbb{R}$. From our assumption and (10), we immediately get

$$\begin{aligned}B_1 &\le \breve{A} \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \|G_w\|_J \\ &= O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon).\end{aligned}$$

Let $\varepsilon > 0$ and $\delta > 0$ correspond to uniform continuity of f. Then, for the given fixed $r_0 > 0$, there exists a $w_2 > 0$ such that $\left|\mathbf{x} - \frac{\mathbf{k}}{w} - \mathbf{x}\right| = \left|\frac{\mathbf{k}}{w}\right| < \frac{r_0}{w} < \delta$ for all $w > w_2$. Dividing B_2 as follows,

$$\begin{aligned}B_2 &= \sum_{w=1}^{w_2} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|<r_0} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ &+ \sum_{w=w_2+1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|<r_0} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ &+ \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|\ge r_0} |l_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ &:= B_2^1 + B_2^2 + B_2^3\end{aligned}$$

then, from (9) we get

$$\begin{aligned}B_2^1 &\le D'' w_2 \max_{1\le w\le w_2} a_{nw}^{\upsilon} \\ &= O\left(1/n^{\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } \upsilon),\end{aligned}$$

where $D'' := \max_{1\leq w\leq w_2}\left\{\sum_{|\mathbf{k}|<r_0}|l_{\mathbf{k},w}|\left\|f\left(\cdot-\frac{\mathbf{k}}{w}\right)-f(\cdot)\right\|\right\}$. Seeing that $f\in Lip_N(\alpha)$, then there can be found a number $R>0$ satisfying that

$$\left\|f\left(\cdot-\tfrac{\mathbf{k}}{w}\right)-f(\cdot)\right\|\leq R\left|\tfrac{\mathbf{k}}{w}\right|^{\alpha}.$$

Thus, from (7) there holds

$$\begin{aligned}B_2^2 &\leq Rr_0^{\alpha}\sum_{w=w_2+1}^{\infty}a_{nw}^{\upsilon}\sum_{|\mathbf{k}|<r_0}|l_{\mathbf{k},w}|\frac{1}{w^{\alpha}}\\ &= O\left(1/n^{\alpha}\right)\text{ as } n\to\infty \text{ (uniformly in } \upsilon).\end{aligned}$$

For B_2^3, since f is bounded, from (8) we obtain

$$\begin{aligned}B_2^3 &\leq 2\|f\|\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{|\mathbf{k}|\geq r_0}|l_{\mathbf{k},w}|\\ &= O\left(1/n^{\alpha}\right)\text{ as } n\to\infty \text{ (uniformly in } \upsilon).\end{aligned}$$

Finally, directly from (6), the following inequality yields

$$B_3 = O\left(1/n^{\alpha}\right)\text{ as } n\to\infty \text{ (uniformly in } \upsilon).$$

□

5. Main Results

Now, using the relation between operators (1) and the generalized sampling operators, following conclusions can be obtained.

For a given $f:\mathbb{R}^N\to\mathbb{R}$, let $l_{\mathbf{k},w}\equiv\chi(\mathbf{k})$ for all $w\in\mathbb{N}$, where $\chi:\mathbb{R}^N\to\mathbb{R}$. In this particular case, (1) turns into

$$\bar{\mathcal{T}}_{n,\upsilon}(f;\mathbf{x})=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{\mathbf{k}\in\mathbb{Z}^N}H_w\left(f\left(\mathbf{x}-\tfrac{\mathbf{k}}{w}\right)\right)\chi(\mathbf{k})\ \ (\mathbf{x}\in\mathbb{R}^N),$$

which is related to $\mathcal{A}-$transform of N-dimensional nonlinear generalized sampling series (for the linear and one dimensional case, see [4,6]), that is

$$\mathcal{S}_{n,\upsilon}(f;\mathbf{x})=\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{\mathbf{k}\in\mathbb{Z}^N}H_w\left(f\left(\tfrac{\mathbf{k}}{w}\right)\right)\chi(w\mathbf{x}-\mathbf{k}),\ \mathbf{x}\in\mathbb{R}^N.$$

Now, it is not hard to see that (l_1) and (l_2) turn out to be the following assumptions:

(l_1') $\chi\in l^1\left(\mathbb{Z}^N\right)$,

(l_2') $\sum_{\mathbf{k}\in\mathbb{Z}^N}\chi(\mathbf{k})=1.$

In this case, (l_3) is not satisfied. On the other hand, instead of (2), now we may assume a more general condition (11), given by
for every bounded interval $J\subset\mathbb{R}$,

$$\mathcal{A}-\lim\|G_w\|_J=0 \tag{11}$$

in the following result.

Theorem 3. *Let $f \in BUC\left(\mathbb{R}^N\right)$. If (l_1'), (l_2') and (11) hold, then*

$$\lim_{n\to\infty} \left\|\bar{\mathcal{T}}_{n,\upsilon}(f) - f\right\| = 0 \text{ (uniformly in } \upsilon \in \mathbb{N}).$$

Proof. From the triangle inequality and (l_2'),

$$\begin{aligned}\left\|\bar{\mathcal{T}}_{n,\upsilon}(f) - f\right\| \le & \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |\chi(\mathbf{k})| \left\|H_w\left(f\left(\cdot - \tfrac{\mathbf{k}}{w}\right)\right) - f\left(\cdot - \tfrac{\mathbf{k}}{w}\right)\right\| \\ & + \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |\chi(\mathbf{k})| \left\|f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ & + \|f\| \left|\sum_{w=1}^{\infty} a_{nw}^{\upsilon} - 1\right|\end{aligned}$$

holds. Since $\left\|H_w\left(f\left(\cdot - \frac{\mathbf{k}}{w}\right)\right) - f\left(\cdot - \frac{\mathbf{k}}{w}\right)\right\| = \|H_w(f) - f\|$, then from (l_1') and (11) one can clearly see that

$$\begin{aligned}& \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} |\chi(\mathbf{k})| \left\|H_w\left(f\left(\cdot - \tfrac{\mathbf{k}}{w}\right)\right) - f\left(\cdot - \tfrac{\mathbf{k}}{w}\right)\right\| \\ & = \left(\sum_{\mathbf{k}\in\mathbb{Z}^N} |\chi(\mathbf{k})|\right)\left(\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \|H_w(f) - f\|\right) \\ & < \bar{A}\varepsilon\end{aligned}$$

for sufficiently large $n \in \mathbb{N}$, for which $\bar{A} = \|\chi\|_{l^1}$.

On the other hand, from (l_1'), for all $\varepsilon > 0$ there can be found a number $\breve{r} > 0$ such that

$$\sum_{|\mathbf{k}|\ge\breve{r}} |\chi(\mathbf{k})| < \varepsilon$$

and hence,

$$\begin{aligned}\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|\ge\breve{r}} |\chi(\mathbf{k})| \left\|f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| & < 2\|f\| \sum_{w=1}^{\infty} a_{nw}^{\upsilon}\varepsilon \\ & \le 2\|f\| M\varepsilon\end{aligned}$$

holds for sufficiently large $n \in \mathbb{N}$. Here M comes from c) in the regularity of $\mathcal{A}$. Using analogous lines of the proof of Theorem 1, one can find a number $\bar{w}_1 \in \mathbb{N}$ such that

$$\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|<\breve{r}} |\chi(\mathbf{k})| \left\|f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| < \varepsilon\left(\bar{D}\bar{w}_1 + \bar{A}M\right)$$

holds for sufficiently large $n \in \mathbb{N}$, where $\bar{D}$ is defined by

$$\bar{D} := \max_{1\le w\le\bar{w}_1} \left\{\textstyle\sum_{|\mathbf{k}|<\breve{r}} |\chi(\mathbf{k})| \left\|f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\|\right\}.$$

Finally, using the regularity of $\mathcal{A}$, we obviously see that

$$\|f\| \left| \sum_{w=1}^{\infty} a_{nw}^{v} - 1 \right| < \|f\| \varepsilon$$

for sufficiently large $n \in \mathbb{N}$. Consequently, since f is bounded, the proof is done. □

We now take into account the following Paley-Wiener spaces to prove Corollary 1 below.

For $1 \leq p \leq \infty$,

$$B_{\pi w}^{p}(\mathbb{R}) = \{f \in L^{p}(\mathbb{R}) : f \text{ has an extension to whole } \mathbb{C} \text{ s.t. } |f(z)| \leq \exp(\pi w |z|) \|f\| \text{ for every } z \in \mathbb{C}\}$$

and

$$B_{\pi w,loc}^{p}\left(\mathbb{R}^{N}\right) = \left\{f \in L^{p}\left(\mathbb{R}^{N}\right) : \text{for every fixed } (x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_N), \; f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_N) \in B_{\pi w}^{p}(\mathbb{R}) \text{ for } 1 \leq j \leq N\right\}.$$

Corollary 1. *Let* $f \in B_{\pi \hat{w},loc}^{1}\left(\mathbb{R}^{N}\right) \cap BUC\left(\mathbb{R}^{N}\right)$ *for some* $\hat{w} > 0$ *and* $\chi \in B_{\pi,loc}^{\infty}\left(\mathbb{R}^{N}\right)$*. If* (l_1'), (l_2') *and* (11) *are satisfied, then*

$$\lim_{n\to\infty} \|\mathcal{S}_{n,v}(f) - f\| = 0 \text{ (uniformly in } v \in \mathbb{N}).$$

Proof. Since $|H_w(f(\mathbf{x}))| \leq C|f(x)|$ for all $w \in \mathbb{N}$, then for every fixed $(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_N)$, $H_w(f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_N)) \in B_{\pi\hat{w}}^{1}(\mathbb{R})$ for all $f \in B_{\pi\hat{w},loc}^{1}\left(\mathbb{R}^{N}\right)$ and $j = 1, \ldots, N$. On the other hand, assuming $g(\mathbf{x}) := \chi(w\mathbf{x})$ we observe that $g \in B_{\pi w,loc}^{1}\left(\mathbb{R}^{N}\right)$ for all $w \geq \hat{w}$. Now, we can write the operators $\mathcal{S}_{n,v}(f;\mathbf{x})$ explicitly as follows

$$\begin{aligned}
&\mathcal{S}_{n,v}(f;\mathbf{x}) \\
&= \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{\mathbf{k}\in\mathbb{Z}^N} (H_w \circ f)\left(\tfrac{\mathbf{k}}{w}\right) \chi(w\mathbf{x} - \mathbf{k}) \\
&= \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{\mathbf{k}\in\mathbb{Z}^N} (H_w \circ f)\left(\tfrac{\mathbf{k}}{w}\right) g\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) \\
&= \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} (H_w \circ f)\left(\tfrac{k_1}{w}, \ldots, \tfrac{k_N}{w}\right) g\left(x_1 - \tfrac{k_1}{w}, \ldots, x_N - \tfrac{k_N}{w}\right).
\end{aligned}$$

Here, fixing the first $N-1$ terms of the previous expression and using Lemma 4.2 in [4], we get

$$\mathcal{S}_{n,v}(f;\mathbf{x}) = \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} (H_w \circ f)\left(\tfrac{k_1}{w}, \ldots, \tfrac{k_{N-1}}{w}, x_N - \tfrac{k_N}{w}\right) g\left(x_1 - \tfrac{k_1}{w}, \ldots, x_{N-1} - \tfrac{k_{N-1}}{w}, \tfrac{k_N}{w}\right).$$

Now, using Fubini-Tonelli theorem (discrete version) and applying the same process for every k_j for $j = 1, \dots, N-1$, we conclude that

$$\begin{aligned}\mathcal{S}_{n,\upsilon}(f;\mathbf{x}) &= \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} (H_w \circ f)\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) g\left(\tfrac{\mathbf{k}}{w}\right) \\ &= \sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^N} (H_w \circ f)\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) \chi(\mathbf{k}) \\ &= \bar{\mathcal{T}}_{n,\upsilon}(f;\mathbf{x}).\end{aligned}$$

Since $f \in BUC(\mathbb{R}^N)$, by Theorem 3, we complete the proof. □

Notice that, $B^1_{\pi w,loc}(\mathbb{R}^N) \subset UC_{loc}(\mathbb{R}^N)$, where $UC_{loc}(\mathbb{R}^N)$ is the space of all functions $f : \mathbb{R}^N \to \mathbb{R}$ such that for every fixed $(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_N)$, $f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_N)$ is uniformly continuous on $\mathbb{R}$ (see Proposition 4.3 in [4]).

Remark 1. *If $f \in B^p_{\pi w,loc}(\mathbb{R}^N) \cap BUC(\mathbb{R}^N)$ for $1 \le p \le 2$, then Corollary 1 is still applicable. In that case, we have to assume that $\chi \in B^q_{\pi,loc}(\mathbb{R}^N)$ to be able to apply Lemma 4.2 in [4], where $1/p + 1/q = 1$.*

Remark 2. *From Example 4.5 in [4], one can construct an N-dimensional kernel χ satisfying the conditions (l_1') and (l_2').*

Using the properties of $\mathcal{A}$-summability under suitable conditions $(l_1) - (l_3)$ and (2), the following results can easily be obtained for all $f \in BUC(\mathbb{R}^N)$:

Consider the operator $T_w(f;\mathbf{x})$, defined by

$$T_w(f;\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} H_w\left(f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right)\right) l_{\mathbf{k},w} \quad \left(\mathbf{x} \in \mathbb{R}^N,\ w \in \mathbb{N}\right).$$

Assume that $\mathcal{A} = \mathcal{F}, \{C_1\}$ and $\{I\}$, where

- $\mathcal{F}$ is the sequences of infinite matrices given by $\{[a_{nw}^{\upsilon}]\}_{\upsilon\in\mathbb{N}}$ such that $a_{nw}^{\upsilon} = 1/n$, if $\upsilon \le w \le n + \upsilon - 1$; $a_{nw}^{\upsilon} = 0$, if otherwise (see [31]),

- C_1 is the Cesàro matrix [28] such that $c_{nw} = 1/n$, if $1 \le w \le n$; $c_{nw} = 0$, if otherwise

and

- I is the identity matrix.

Then we obtain

$$\lim_{n\to\infty} \left\| \frac{T_\upsilon(f) + T_{\upsilon+1}(f) + \cdots + T_{n+\upsilon-1}(f)}{n} - f \right\| = 0 \quad (\text{uniformly in } \upsilon)$$

$$(T_w(f) \text{ is almost convergent to } f),$$

$$\lim_{n\to\infty}\left\|\frac{T_1(f)+T_2(f)+\cdots+T_n(f)}{n}-f\right\|=0$$
$$(T_w(f) \text{ is arithmetic mean convergent to } f)$$

and

$$\lim_{n\to\infty}\|T_n(f)-f\|=0$$

respectively.

Moreover, under suitable conditions, all the previous convergence methods are valid for nonlinear generalized sampling series, given by

$$S_w(f;\mathbf{x})=\sum_{\mathbf{k}\in\mathbb{Z}^N}H_w\left(f\left(\tfrac{\mathbf{k}}{w}\right)\right)\chi(w\mathbf{x}-\mathbf{k}).$$

6. Applications

In this section, we begin by providing a detailed example of the discrete kernel $l_{\mathbf{k},w}$. Following this, we explore an application of our operator in the field of digital image processing.

First, consider the 2-dimensional case and substitute the matrix $\mathcal{A}$ with matrix $\mathcal{F}$ as defined above.

Let $l_{\mathbf{k},w}$ be defined by

$$l_{\mathbf{k},w}:=\begin{cases} 2\left(\dfrac{1}{2a_w-1/2}\right)^2\frac{1}{2^{(w+1)^{|k_1|}+(w+1)^{|k2|}}}; & \text{if } w=m^2\ (m\in\mathbb{N})\\ \left(\dfrac{1}{2a_w-1/2}\right)^2\frac{1}{2^{(w+1)^{|k_1|}+(w+1)^{|k2|}}}; & \text{if } w\neq m^2\ (m\in\mathbb{N})\end{cases},$$

where

$$a_w=\sum_{k=0}^{\infty}\frac{1}{2^{(w+1)^k}}.$$

Note that, by the ratio test one can observe that a_w is finite for all $w\in\mathbb{N}$. Then, considering the following equality

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\frac{1}{2^{(w+1)^{|k|}}}&=2\sum_{k=0}^{\infty}\frac{1}{2^{(w+1)^k}}-\frac{1}{2}\\&=2a_w-1/2\end{aligned}$$

we may obtain

$$\sup_{n,v\in\mathbb{N}}\sum_{w=1}^{\infty}a_{nw}^{v}\,\|\,l_{\mathbf{k},w}\|_{l^1}=\sup_{n,v\in\mathbb{N}}\sum_{w=v}^{n+v-1}\frac{1}{n}\sum_{\mathbf{k}\in\mathbb{Z}^2}|l_{\mathbf{k},w}|$$

$$\leq \sup_{n,\upsilon\in\mathbb{N}} \sum_{w=\upsilon}^{n+\upsilon-1} \frac{2}{n}\left(\frac{1}{2a_w-1/2}\right)^2 \sum_{k_1\in\mathbb{Z}} \frac{1}{2^{(w+1)^{|k_1|}}} \sum_{k_2\in\mathbb{Z}} \frac{1}{2^{(w+2)^{|k2|}}}$$
$$= 2$$

which shows that condition (l_1) is satisfied. Furthermore, for (l_2) consider the following

$$\begin{aligned}
\left|\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} - 1\right| &= \left|\sum_{w=\upsilon}^{n+\upsilon-1} \frac{1}{n} \sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} - 1\right| \\
&= \left|\sum_{\substack{\upsilon\leq w\leq n+\upsilon-1\\ w\neq m^2}} \frac{1}{n}\sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} + \sum_{\substack{\upsilon\leq w\leq n+\upsilon-1\\ w\neq =m^2}} \frac{1}{n}\sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} - 1\right| \\
&\leq \sum_{\substack{\upsilon\leq w\leq n+\upsilon-1\\ w\neq =m^2}} \frac{1}{n}\sum_{\mathbf{k}\in\mathbb{Z}^2} l_{\mathbf{k},w} \\
&= \sum_{\substack{\upsilon\leq w\leq n+\upsilon-1\\ w\neq =m^2}} \frac{1}{n}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} 2\left(\frac{1}{2a_w-1/2}\right)^2 \frac{1}{2^{(w+1)^{|k_1|}+(w+1)^{|k2|}}} \\
&= \sum_{\substack{\upsilon\leq w\leq n+\upsilon-1\\ w\neq =m^2}} \frac{2}{n} \\
&= \frac{2}{n}\left(\sqrt{n+\upsilon-1}-\sqrt{\upsilon}+1\right) \\
&\leq \frac{4}{\sqrt{n}}
\end{aligned}$$

we obtain (l_2).

For (l_3), taking $r=1$, since

$$\{\mathbf{k}=(k_1,k_2) : |\mathbf{k}|\geq 1\} \subset \left\{\mathbf{k}=(k_1,k_2) : |k_1|\geq \frac{1}{\sqrt{2}} \text{ and } |k_2|\geq \frac{1}{\sqrt{2}}\right\},$$

we write

$$\begin{aligned}
\sum_{w=1}^{\infty} a_{nw}^{\upsilon} \sum_{|\mathbf{k}|\geq 1} |l_{\mathbf{k},w}| &= \sum_{w=\upsilon}^{n+\upsilon-1} \frac{1}{n}\sum_{|\mathbf{k}|\geq 1} |l_{\mathbf{k},w}| \\
&\leq \sum_{w=\upsilon}^{n+\upsilon-1} \frac{2}{n}\left(\frac{1}{2a_w-1/2}\right)^2 \sum_{|k_1|\geq\frac{1}{\sqrt{2}}} \frac{1}{2^{(w+1)^{|k_1|}}} \sum_{|k_2|\geq\frac{1}{\sqrt{2}}} \frac{1}{2^{(w+1)^{|k2|}}} \\
&= \sum_{w=\upsilon}^{n+\upsilon-1} \frac{2}{n}\left(\frac{1}{2a_w-1/2}\right)^2 \sum_{|k_1|\geq 1} \frac{1}{2^{(w+1)^{|k_1|}}} \sum_{|k_2|\geq 1} \frac{1}{2^{(w+1)^{|k2|}}}
\end{aligned}$$

$$= \sum_{w=\upsilon}^{n+\upsilon-1} \frac{2}{n} \left(1 - \frac{1}{4a_w - 1}\right)^2 .$$

On the other hand, since

$$\lim_{w\to\infty} \frac{1}{2^{(w+1)^{|k|}}} = \begin{cases} \frac{1}{2}; & k = 0 \\ 0; & \text{otherwise,} \end{cases}$$

by the discrete version of dominated convergence theorem, we obtain

$$\lim_{w\to\infty} 2\left(1 - \frac{1}{4a_w - 1}\right)^2 = 0$$

and by the regularity of $\mathcal{F}$, we get

$$\lim_{n\to\infty} \sum_{w=\upsilon}^{n+\upsilon-1} \frac{2}{n} \left(1 - \frac{1}{4a_w - 1}\right)^2 = 0$$

uniformly in $\upsilon \in \mathbb{N}$. Therefore, (l_3) is satisfied. However, our kernel $l_{\mathbf{k},w}$ does not adhere the classical approximate identities, since

$$\lim_{w\to\infty} \sum_{\mathbf{k}\in\mathbb{Z}^N} l_{\mathbf{k},w} = \begin{cases} 2; & \text{if } w = m^2 \ (m \in \mathbb{N}) \\ 1; & \text{if } w \neq m^2 \ (m \in \mathbb{N}) \end{cases}$$
$$\neq 1.$$

This non-fulfillment suggests that our approximation is non-trivial.

6.1. **Application on Images.** With the development of modern technology, zooming in on digital images has become common in many areas such as digital cameras, medical imaging, and mobile phones. In the literature, there are different types of zooming methods such as pixel replication, interpolation, zero-order hold method, and more. In this part of the application, we propose an operator method for zooming in on images. We should note that approximating operators can be very useful in image processing [15, 25, 26].

In classical zoom techniques, the neighborhood of a pixel is often processed. In contrast, our proposed method requires all pixel values of the zoomed image for each pixel value. Although this may reduce computational efficiency, it helps prevent issues such as loss of sharpness in the corners. Now, we apply our approximation method to zoom in on digital images.

It is known that, a grayscale $m \times m$ pixel valued digital image can be represented by a step function as follows (see [25]):

$$I(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{m} u_{ij} 1_{ij}(x, y)$$

where u_{ij} is the (i, j)'th pixel value of the given image and 1_{ij} is defined by

$$1_{ij}(x, y) = \begin{cases} 1; & \text{if } (x, y) \in (i-1, i] \times (j-1, j] \\ 0; & \text{otherwise} \end{cases} \quad (i, j = 1, 2, \ldots, m).$$

It is clear that I is a step function with compact support and therefore $I \in L^1\left(\mathbb{R}^2\right)$. Using the density of the continuous functions in $L^1\left(\mathbb{R}^2\right)$, we may approximate this image by the operator (1).

Let $\mathcal{A} = \{I\}$ and $l_{k,w}$, H_w be defined by

$$l_{\mathbf{k},w} = \left(\frac{2^w - 1}{2^w + 1}\right)^2 \frac{1}{2^{w(|k_1|+|k_2|)}}$$

and

$$H_w\left(x\right) = \begin{cases} x + \log_{10}\left(1 + \frac{x}{w}\right); & \text{if } 0 \leq x < 1 \\ x + \log_{10}\left(1 + \frac{1}{wx}\right); & \text{if } x \geq 1. \end{cases} \quad \text{(see [2,3])}$$

respectively. We assume that H_w is extended symmetrically in the odd way. One can clearly observe that $l_{k,w}$ fulfills the assumptions $(l_1) - (l_3)$ and H_w satisfies (2).

Now, consider the following image, named by "baboon" in Figure 1. We will

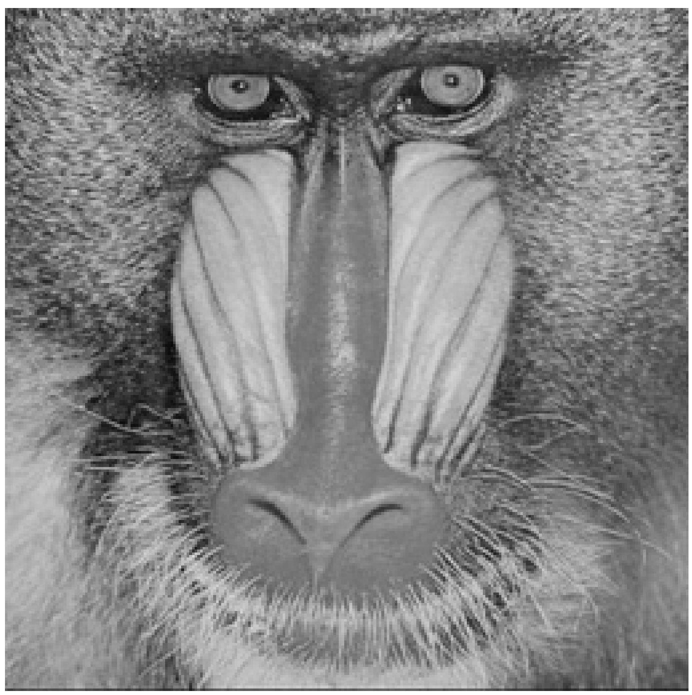

FIGURE 1. Original 256×256 pixel resolution Baboon

focus on the left eye of the baboon shown in the Figure 2. By using our nonlinear operator, we approximate the Baboon's eye for $w = 4$. By increasing the sampling rate, the following new zoomed images can be obtained (see Figure 3 and Figure 4).

Figure 2. Original 50×50 pixel resolution eye of the Baboon

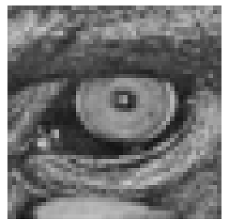

Figure 3. The eye of the Baboon zoomed in with a resolution of 100×100 pixels, obtained by nonlinear operator for $w = 4$

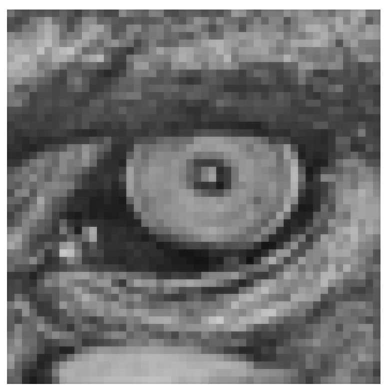

Figure 4. The eye of the Baboon zoomed in with a resolution of 200×200 pixels, obtained by nonlinear operator for $w = 4$

These new images demonstrate that our proposed method could be useful for digital image zooming. We should note that changing or scaling the kernels for different values of w may result in higher quality zoomed images.

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AN ANALYSIS ON THE SHAPE-PRESERVING CHARACTERISTICS OF $\lambda-$SCHURER OPERATORS

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Abstract. This study investigates the shape-preserving characteristics of λ-Schurer operators, a class of operators derived from a modified version of the classical Schurer bases by incorporating a shape parameter λ. The primary focus is on understanding how these operators maintain the geometric features of the functions they approximate, which is crucial in fields like computer graphics and geometric modelling. By examining the fundamental properties and the divided differences associated with λ-Schurer bases, we derive vital results that confirm the operators' capability to preserve essential shape attributes under various conditions. The findings have significant implications for the application of these operators in computational analysis and other related areas, providing a solid foundation for future research.

1. Introduction

In recent years, the study of shape-preserving approximation methods has gained significant attention due to their critical role in applications such as computer graphics, CAD modelling, and numerical analysis. Shape-preserving operators ensure that the essential geometric features of functions, such as monotonicity and convexity, are maintained during approximation [1]. Bézier bases have become particularly popular among these methods due to their ability to offer smooth and continuous approximations with limited control points [6, 11].

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In 2010, Ye et al. [12] established a new class of bases, so-called Bézier bases, based on shape parameters λ chosen from the interval $[-1, 1]$. Bézier bases are fundamental in approximation methods that aim to preserve shapes, playing a crucial role in computer graphics and geometric modelling. The recent works related to some shape parameters including λ are given as: In their exploration of the modified λ -Bernstein-polynomial, Ayman-Mursaleen et al. [7] thoroughly analyzed its approximation properties, providing valuable insights into its behavior and potential applications. Su et al. [21] conducted a rigorous analysis of the shape-preserving properties of λ -Bernstein operators, demonstrating their ability to maintain crucial geometric characteristics such as monotonicity and convexity during the approximation process. Ansari et al. [2] delved into the approximation properties of bivariate Bernstein-Kantorovich operators, extending their application by incorporating a summability method and establishing connections with related GBS operators. Kajla et al. [14] introduced the innovative Bézier-Baskakov-Beta type operators, a novel class designed to enhance shape-preserving approximation and offer improved flexibility in controlling the geometric features of the approximated function. Rao et al. [18] investigated the approximation capabilities of modified Baskakov-Durrmeyer operators, focusing on the influence of a shape parameter α on their ability to represent complex functions while preserving their fundamental geometric properties accurately. Özger et al. [17] examined the convergence behaviour of generalized blending-type Bernstein-Kantorovich operators, establishing the rate of weighted statistical convergence and providing a deeper understanding of their approximation characteristics.

Bézier bases provide a mathematical framework that ensures smoothness and continuity, making them ideal for accurately approximating complex shapes like fonts, logos, and CAD models. Bézier bases allow for precise control over curve shapes with a limited number of control points, giving designers and engineers the flexibility to fine-tune approximations while maintaining the integrity of the original shape. This ability to preserve essential features during the approximation process highlights the importance of Bézier bases in achieving visually and geometrically accurate representations. Due to all these facts these bases have become prevalent among researchers, and there have been many variations of Bézier bases inaugurated to the literature (see [4, 8, 13]).

Schurer [19] introduced a remarkable variation of the classical Bernstein operators by incorporating a nonnegative parameter ϑ, which is both linear and positive. Most recently, Özger [16] constructed a modified version of these bases, namely $\lambda-$Schurer bases, as follows: For shape parameter $\lambda \in [-1, 1]$ and integer $\vartheta \geq 0$, the $\lambda-$Schurer bases are

$$\begin{aligned}
\widehat{s}_{r,0}(\lambda;\tau) &= s_{r,0}(\tau) - \frac{\lambda}{r+\vartheta+1} s_{r+1,1}(\tau), \\
\widehat{s}_{r,p}(\lambda;\tau) &= s_{r,p}(\tau) + \lambda \left\{ \frac{r+\vartheta-2p+1}{(r+\vartheta)^2-1} s_{r+1,p}(\tau) \right. \\
&\qquad \left. - \frac{r+\vartheta-2p-1}{(r+\vartheta)^2-1} s_{r+1,p+1}(\tau) \right\}, \qquad p = 1,2,\ldots,r+\vartheta-1, \\
\widehat{s}_{r,r+\vartheta}(\lambda;\tau) &= s_{r,r+\vartheta}(\tau) - \frac{\lambda}{r+\vartheta+1} s_{r+1,r+\vartheta}(\tau),
\end{aligned} \tag{1}$$

where $s_{r,p}(\tau)$ are the fundamental Schurer bases of degree $r+\vartheta$ defined as

$$s_{r,p}(\tau) = \binom{r+\vartheta}{p} \tau^p (1-\tau)^{r+\vartheta-p}, \qquad p = 0,1,\ldots,r+\vartheta. \tag{2}$$

Then using the $\lambda-$Schurer bases given in (1), Özger established the $\lambda-$Schurer operators $S^{\lambda}_{r,\vartheta}(g;\tau) : C[0,1+\vartheta] \to C[0,1]$

$$S^{\lambda}_{r,\vartheta}(g;\tau) = \sum_{p=0}^{r+\vartheta} \widehat{s}_{r,p}(\lambda;\tau)\, g\left(\frac{p}{r}\right), \quad \tau \in [0,1], \; r \in \mathbb{N}, \tag{3}$$

for any g in $C[0,1+\vartheta]$. In [16], the statistical convergence properties of operators in (3) is examined, and an estimation for the rate of weighted A-statistical convergence is provided. Furthermore, two Voronovskaja-type theorems are established, one of which employs weighted A-statistical convergence.

Building on the foundational work of Ye et al. [12] on Bézier bases, this paper explores the $\lambda-$Schurer operators in (3), a variation introduced by Özger given in (3), which extends the classical Schurer operators by incorporating a shape parameter λ. These operators are designed to provide more flexibility in controlling the shape of the approximated function, making them a powerful tool for shape-preserving approximation. The primary objective of this study is to analyze the shape-preserving properties of these operators and to establish their effectiveness through rigorous mathematical proofs and computational analysis. The manuscript is organized as follows: Section 2 covers the fundamental concepts of fundamental Schurer bases, divided differences, as well as the notions of $0-$convex, $1-$convex, and $2-$convex functions, including the relevant relationships and results. Section 3 presents the primary theoretical, computational, and numerical results and discussions regarding the shape-preserving properties of $\lambda-$Schurer operators. In the last section, we provide an elaborate conclusion.

2. Auxiliary Results

In this section, we give the fundamental properties of the $\lambda-$Schurer bases and some essentials on the divided differences. We commence our work by providing the binomial coefficient formula as

$$\binom{r}{p} = \begin{cases} \frac{r!}{p!(r-p)!}, & 0 \leq p \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

In the next lemma, we give some basic properties of $s_{r,p}(\tau)$, such as, recursive relation, degree raising, derivative formula and endpoint interpolating properties.

Lemma 1. *For integer $\vartheta \geq 0$, the fundamental Schurer bases $s_{r,p}(\tau)$ in (2) satisfy the following identities:*

$$s_{r,p}(\tau) = 0 \quad \textit{if} \quad p > r + \vartheta \quad \textit{or} \quad p < 0, \tag{4}$$

$$s_{r,p}(\tau) = (1-\tau)\, s_{r-1,p}(\tau) + \tau s_{r-1,p-1}(\tau), \tag{5}$$

$$s_{r,p}(\tau) = \left(1 - \frac{p}{r+\vartheta+1}\right) s_{r+1,p}(\tau) + \left(\frac{p+1}{r+\vartheta+1}\right) s_{r+1,p+1}(\tau), \tag{6}$$

$$\frac{d}{d\tau}\left[s_{r,p}(\tau)\right] = (r+\vartheta)\left[s_{r-1,p-1}(\tau) - s_{r-1,p}(\tau)\right], \tag{7}$$

and

$$s_{r,p}(0) = \begin{cases} 0 & \textit{if} \;\; p \neq 0, \\ 1 & \textit{if} \;\; p = 0, \end{cases} \qquad s_{r,p}(1) = \begin{cases} 0 & \textit{if} \;\; p \neq r+\vartheta, \\ 1 & \textit{if} \;\; p = r+\vartheta. \end{cases} \tag{8}$$

Proof. The proof of (4) and (8) are a direct consequence of definitions of the binomial coefficient and $s_{r,p}(\tau)$ in (2), so they are omitted. To prove (5), we only apply basic algebra to the definition (2) of Schurer polynomials, which yields

$$\begin{aligned} (1-\tau)\, s_{r-1,p}(\tau) + \tau s_{r-1,p-1}(\tau) &= (1-\tau)\tbinom{r+\vartheta-1}{p}\tau^p (1-\tau)^{r+\vartheta-1-p} \\ &\quad + \tau\tbinom{r+\vartheta-1}{p-1}\tau^{p-1}(1-\tau)^{(r+\vartheta-1)-(p-1)} \\ &= \left[\tbinom{r+\vartheta-1}{p} + \tbinom{r+\vartheta-1}{p-1}\right]\tau^p (1-\tau)^{r+\vartheta-p}. \end{aligned}$$

Since $\binom{r+\vartheta-1}{p} + \binom{r+\vartheta-1}{p-1} = \binom{r+\vartheta}{p}$, we then have the desired result. Next to prove (6), we first note that

$$\begin{aligned} \tau s_{r,p}(\tau) &= \tbinom{r+\vartheta}{p}\tau^{p+1}(1-\tau)^{r+\vartheta-p} \\ &= \frac{\binom{r+\vartheta}{p}}{\binom{r+\vartheta+1}{p+1}}\tbinom{r+\vartheta+1}{p+1}\tau^{p+1}(1-\tau)^{(r+\vartheta+1)-(p+1)} \\ &= \left(\frac{p+1}{r+\vartheta+1}\right) s_{r+1,p+1}(\tau), \end{aligned} \tag{9}$$

and also

$$\begin{aligned} (1-\tau)\, s_{r,p}(\tau) &= \tbinom{r+\vartheta}{p}\tau^p(1-\tau)^{r+\vartheta+1-p} \\ &= \frac{\binom{r+\vartheta}{p}}{\binom{r+\vartheta+1}{p}}\tbinom{r+\vartheta+1}{p}\tau^p(1-\tau)^{(r+\vartheta+1)-p} \\ &= \left(1 - \frac{p}{r+\vartheta+1}\right) s_{r+1,p}(\tau). \end{aligned} \tag{10}$$

Subsequently, summation of (9) and (10) yields property (6). Lastly, by taking the derivative of $s_{r,p}(\tau)$ with respect to τ by means of basic algebra rules, we obtain the property (7) as

$$\begin{aligned}\frac{d}{d\tau}\left[s_{r,p}(\tau)\right] &= \tbinom{r+\vartheta}{p}p\tau^{p-1}(1-\tau)^{r+\vartheta-p} - \tbinom{r+\vartheta}{p}(r+\vartheta-p)\tau^{p}(1-\tau)^{r+\vartheta-1-p} \\ &= (r+\vartheta)\Big[\tfrac{(r+\vartheta-1)!}{(p-1)!(r+\vartheta-p)!}\tau^{p-1}(1-\tau)^{(r+\vartheta-1)-(p-1)} \\ &\qquad -\tfrac{(r+\vartheta-1)!}{p!(r+\vartheta-1-p)!}\tau^{p}(1-\tau)^{(r+\vartheta-1)-p}\Big] \\ &= (r+\vartheta)\left[s_{r-1,p-1}(\tau)-s_{r-1,p}(\tau)\right].\end{aligned}$$

□

The following lemma will present some auxiliary results that are essential for our main outcomes.

Lemma 2. *For $\lambda \in [-1,1]$ and integer $\vartheta \geq 0$, the $\lambda-$Schurer bases in (1) satisfy the following properties:*

$$\widehat{s}_{r,p}(\lambda;\tau) \geq 0, \tag{11}$$

$$\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}(\lambda;\tau) = 1, \tag{12}$$

$$\widehat{s}_{r,p}(\lambda;\tau) = \tilde{s}_{r,r-p}(\lambda;1-\tau). \tag{13}$$

Proof. In order to prove property (11), we first note that $s_{r,p}(\tau) \geq 0$ for all $r \in \mathbb{N}$ and $\tau \in [0,1]$ where $\vartheta \geq 0$ is integer by definition of the binomial coefficient formula. Next, we rewrite $\lambda-$Schurer bases given in (1) as

$$\begin{aligned}\widehat{s}_{r,p}(\lambda;\tau) = \frac{1}{r+\vartheta+1}\Bigg\{&\left(p+1-\lambda\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)s_{r+1,p+1}(\tau) \\ &+\left(r+\vartheta+1-p+\lambda\frac{r+\vartheta-2p+1}{r+\vartheta-1}\right)s_{r+1,p}(\tau)\Bigg\},\end{aligned}$$

by employing degree raising property (6). Since $1 \leq p \leq r+\vartheta-1$, one can easily find that $0 \leq \frac{p-1}{r+\vartheta-1} \leq 1-\frac{1}{r+\vartheta-1} \leq 1$. Then utilizing the fact $-1 \leq \lambda \leq 1$ yields $-1 \leq \lambda\left(1-\frac{2(p-1)}{r+\vartheta-1}\right) \leq 1$. Subsequently, we get

$$0 \leq r+\vartheta-p \leq r+\vartheta+1-p+\lambda\frac{r+\vartheta-2p+1}{r+\vartheta-1}. \tag{14}$$

Analogously, one can derive $-1 \leq \lambda\left(1-\frac{2p}{r+\vartheta-1}\right) \leq 1$ which implies

$$0 \leq (p+1)-1 \leq p+1-\lambda\frac{r+\vartheta-2p-1}{r+\vartheta-1}. \tag{15}$$

Hence, we have $\widehat{s}_{r,p}(\lambda;\tau) \geq 0$ by (14) and (15). The proof of partition of unity property (12) is given in [16], and symmetry property is a direct consequence of definitions (1)-(2), so they are omitted. □

The following divided differences definition and subsequent results are presented on the grounds of the pioneering work by Asher and Greif [3].

Definition 1 ([3]). *Given points $\tau_0, \tau_1, \dots, \tau_r$ with arbitrary indices $0 \leq q < p \leq r$, the divided difference of a function g with order r is defined by*

$$[\tau_0, \tau_1, \dots, \tau_r; g] = \sum_p g(\tau_p) \prod_{q \neq p} \frac{1}{(\tau_p - \tau_q)}.$$

The divided differences of g are linear and symmetric and satisfy the recursive formula

$$[\tau_0; g] = g(\tau_0)$$
$$[\tau_0, \dots \tau_r; g] = \frac{[\tau_1, \dots \tau_r; g] - [\tau_0, \dots \tau_{r-1}; g]}{\tau_r - \tau_0}.$$

By recursive formula, for $0 \leq q \leq r$, we have the following identities:

$$[\tau_q; g] = g(\tau_q),$$
$$[\tau_q, \tau_{q+1}; g] = \frac{g(\tau_{q+1}) - g(\tau_q)}{\tau_{q+1} - \tau_q},$$
$$[\tau_q, \tau_{q+1}, \tau_{q+2}; g] = \frac{[\tau_{q+1}, \tau_{q+2}; g] - [\tau_q, \tau_{q+1}; g]}{\tau_{q+2} - \tau_q}.$$

Lemma 3 ([15]). *For a fixed $r \in \mathbb{N}$, the function g is called $r-$convex if $[\tau_0, \tau_1, \dots, \tau_r; g] \geq 0$. In particular, if function g is*

- **i:** *nonnegative, then it is $0-$convex,*
- **ii:** *nondecreasing, then it is $1-$convex,*
- **iii:** *convex in the usual sense, then it is $2-$convex.*

3. Primary Results on the Shape-Preserving Characteristics of $\lambda-$Schurer Operators

This part is dedicated to the main results of the manuscript. We will present our findings on the positivity, linearity, endpoint preservation, monotonicity and convexity of $\lambda-$Schurer operators $S^{\lambda}_{r,\vartheta}(g;\tau)$. We commence our work by representing $S^{\lambda}_{r,\vartheta}(g;\tau)$ in terms of fundamental Schurer bases $s_{r,p}(\tau)$ in (2) and divided differences.

Lemma 4. *For any $\lambda \in [-1,1]$ and integer $\vartheta \geq 0$, the $\lambda-$Schurer operators in (3) can be rewritten as*

$$S^{\lambda}_{r,\vartheta}(g;\tau) = B_{r,\vartheta}(g;\tau) + \frac{\lambda}{r} \sum_{p=0}^{r+\vartheta-1} \left(\frac{r+\vartheta-2p-1}{(r+\vartheta)^2 - 1} \right) s_{r+1,p+1}(\tau) \left[\tfrac{p}{r}, \tfrac{p+1}{r}; g \right], \quad (16)$$

where $s_{r,p}(\tau)$ *are as in* (2) *and*

$$B_{r,\vartheta}(g;\tau)=\sum_{p=0}^{r+\vartheta}s_{r,p}(\tau)\left[\tfrac{p}{r};g\right],$$

are the Bernstein-Schurer operators constructed in [19].

Proof. Substitution of (1) to the expression (3) of $\lambda-$Schurer operators yields

$$\begin{aligned}S_{r,\vartheta}^{\lambda}(g;\tau)=&\left[s_{r,0}(\tau)-\frac{\lambda}{r+\vartheta+1}s_{r+1,1}(\tau)\right]g(0)\\&+\sum_{p=1}^{r+\vartheta-1}\left[s_{r,p}(\tau)+\lambda\left(\frac{r+\vartheta-2p+1}{(r+\vartheta)^2-1}s_{r+1,p}(\tau)\right.\right.\\&\left.\left.-\frac{r+\vartheta-2p-1}{(r+\vartheta)^2-1}s_{r+1,p+1}(\tau)\right)\right]g\left(\frac{p}{r}\right)\\&+\left[s_{r,r+\vartheta}(\tau)-\frac{\lambda}{r+\vartheta+1}s_{r+1,r+\vartheta}(\tau)\right]g\left(\frac{r+\vartheta}{r}\right),\end{aligned}$$

which can also be written as

$$\begin{aligned}S_{r,\vartheta}^{\lambda}(g;\tau)=&\sum_{p=0}^{r+\vartheta}s_{r,p}(\tau)g\left(\frac{p}{r}\right)-\lambda\sum_{p=0}^{r+\vartheta-1}\left(\frac{r+\vartheta-2p-1}{(r+\vartheta)^2-1}\right)s_{r+1,p+1}(\tau)g\left(\frac{p}{r}\right)\\&+\lambda\sum_{p=1}^{r+\vartheta}\left(\frac{r+\vartheta-2p+1}{(r+\vartheta)^2-1}\right)s_{r+1,p}(\tau)g\left(\frac{p}{r}\right),\end{aligned}$$

after simplifying similar terms. Reindexing the last summation in the above equation and then utilizing the notation of divided differences given in Definition 1, we obtain the desired result in (16). □

Remark 1. *In the special case* $\vartheta=0$ *and* $p\to p-1$ *in* (16)*, we get equation (6) in [21].*

Now, we are ready to present our principal conclusions on the shape-preserving properties of the $\lambda-$Schurer operators. The following theorem is on the geometric properties of $S_{r,\vartheta}^{\lambda}(g;\tau)$, such as nonnegativity, linearity and endpoint interpolation.

Theorem 1. *Let* $\lambda\in[-1,1]$, $r\in\mathbb{N}$, *and* $\vartheta\geq 0$ *integer. The* $\lambda-$*Schurer operators in* (3) *satisfy the following properties:*

- **i:** *Nonnegativity: For* $g\in C[0,1+\vartheta]$, $S_{r,\vartheta}^{\lambda}(g;\tau)\geq 0$ *whenever* $g(\tau)\geq 0$.
- **ii:** *Linearity: For* $g_1,g_2\in C[0,1+\vartheta]$ *and* $\beta_1,\beta_2\in\mathbb{R}$,

$$S_{r,\vartheta}^{\lambda}(\beta_1g_1+\beta_2g_2;\tau)=\beta_1S_{r,\vartheta}^{\lambda}(g_1;\tau)+\beta_2S_{r,\vartheta}^{\lambda}(g_2;\tau).$$

- **iii:** *Endpoint interpolation:* $S_{r,\vartheta}^{\lambda}(g;0)=[0;g]$.

Proof. We begin our work by writing $\lambda-$Schurer operators in (3) in terms of divided differences as

$$S_{r,\vartheta}^{\lambda}(g;\tau)=\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}(\lambda;\tau)\left[\tfrac{p}{r};g\right].$$

For the proof of part (**i**), assume that $g(\tau)\geq 0$. Consequently, we have $S_{r,\vartheta}^{\lambda}(g;\tau)\geq 0$ by (11) and Lemma 3. Next, by the linearity of the divided differences and summation operator, we obtain

$$\begin{aligned}S_{r,\vartheta}^{\lambda}(\beta_1 g_1+\beta_2 g_2;\tau)&=\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}(\lambda;\tau)\left[\tfrac{p}{r};\beta_1 g_1+\beta_2 g_2\right]\\&=\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}(\lambda;\tau)\left(\beta_1\left[\tfrac{p}{r};g_1\right]+\beta_2\left[\tfrac{p}{r};g_2\right]\right)\\&=\beta_1 S_{r,\vartheta}^{\lambda}(g_1;\tau)+\beta_2 S_{r,\vartheta}^{\lambda}(g_2;\tau),\end{aligned}$$

which completes the proof of part (**ii**). Lastly, for part (**iii**), substitution of (8) in (1) yields

$$\widehat{s}_{r,p}(\lambda;0)=\begin{cases}0 & \text{if } p\neq 0\\ 1 & \text{if } p=0\end{cases},$$

which consequently implies

$$S_{r,\vartheta}^{\lambda}(g;0)=\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}(\lambda;0)\left[\tfrac{p}{r};g\right]=\widehat{s}_{r,0}(\lambda;0)[0;g]+\sum_{p=1}^{r+\vartheta}\widehat{s}_{r,p}(\lambda;0)\left[\tfrac{p}{r};g\right]=[0;g].$$

□

Prior to the presentation of our primary findings on the monotonicity preservation of $\lambda-$Schurer operators, we will present the first derivative of these operators in the following lemma.

Lemma 5. *For any $\lambda\in[-1,1]$ and $g:[0,1+\vartheta]\to\mathbb{R}$, $\vartheta\geq 0$ integer, the $\lambda-$Schurer operators in (3) satisfy the following identity*

$$\begin{aligned}\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]=\frac{1}{r}\Bigg\{&\sum_{p=0}^{r+\vartheta-1}\left[r+\vartheta-p+\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]s_{r,p}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\\&+\sum_{p=0}^{r+\vartheta-1}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]s_{r,p+1}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\Bigg\}.\end{aligned}\tag{17}$$

Proof. One can differentiate equation (16)

$$\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]=(r+\vartheta)\left\{\sum_{p=1}^{r+\vartheta}s_{r-1,p-1}(\tau)\left[\tfrac{p}{r};g\right]-\sum_{p=0}^{r+\vartheta-1}s_{r-1,p}(\tau)\left[\tfrac{p}{r};g\right]\right\}$$

$$+\frac{\lambda}{r}\sum_{p=0}^{r+\vartheta-1}\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\left[s_{r,p}(\tau)-s_{r,p+1}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right],$$

by utilizing (7) and (4), respectively. Next, reindexing the summation with $s_{r-1,p-1}(\tau)$ term and then applying divided differences identity of first order yield

$$\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]=\frac{(r+\vartheta)}{r}\sum_{p=0}^{r+\vartheta-1}s_{r-1,p}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]$$
$$+\frac{\lambda}{r}\sum_{p=0}^{r+\vartheta-1}\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\left[s_{r,p}(\tau)-s_{r,p+1}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right].$$

Using property (6) implies

$$\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]=\frac{(r+\vartheta)}{r}\sum_{p=0}^{r+\vartheta-1}\left(\left(1-\tfrac{p}{r+\vartheta}\right)s_{r,p}(\tau)+\left(\tfrac{p+1}{r+\vartheta}\right)s_{r,p+1}(\tau)\right)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]$$
$$+\frac{\lambda}{r}\sum_{p=0}^{r+\vartheta-1}\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\left[s_{r,p}(\tau)-s_{r,p+1}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right],$$

and subsequently, combining the summations with similar terms produces the first derivative given in (17). □

Remark 2. *In the special case $\vartheta=0$ in (17), we obtain equation (7) in [21].*

Theorem 2 (**Monotonicity**). *If g is increasing (or decreasing) on the interval $[0,1+\vartheta]$, then so are all the corresponding $\lambda-$Schurer operators for all $\lambda\in[-1,1]$ and $r\in\mathbb{N}$.*

Proof. In order to prove that $S_{r,\vartheta}^{\lambda}(g;\tau)$ is increasing whenever g is also increasing on $[0,1+\vartheta]$, it is sufficient to show that the first derivative given in Lemma 5 is nonnegative. Firstly, for an increasing function g; i.e., $1-$convex, we have

$$\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\geq 0 \tag{18}$$

by Lemma 3. Moreover, for $0\leq p\leq r+\vartheta-1$, we have $-1\leq 1-\frac{2p}{r+\vartheta-1}\leq 1$. Since $-1\leq\lambda\leq 1$, we get $-1\leq\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)\leq 1$ which leads to

$$0\leq r+\vartheta-p-1\leq r+\vartheta-p+\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)=r+\vartheta-p+\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right), \tag{19}$$

and

$$0\leq(p+1)-1\leq p+1-\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)=p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right). \tag{20}$$

Subsequently, we obtain $\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right] \geq 0$ due to inequalities (18)-(20). Analogously, for a decreasing function g on $[0,1+\vartheta]$, we have

$$\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right] \leq 0. \tag{21}$$

Then by (19)-(21), we have $\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right] \leq 0$ which implies $S_{r,\vartheta}^{\lambda}(g;\tau)$ is also decreasing on $[0,1+\vartheta]$. Hence the proof is complete. □

Remark 3. *The $\vartheta = 0$ case is presented as Theorem 3.1 in [21].*

Lemma 6. *For any $\lambda \in [-1,1]$ and $g:[0,1+\vartheta] \to \mathbb{R}$, $\vartheta \geq 0$ integer, the $\lambda-$Schurer operators in (3) satisfy the following identity*

$$\begin{aligned}
\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right] &= \lambda\frac{(r+\vartheta)(r+\vartheta+1)}{r(r+\vartheta-1)}\left\{s_{r-1,0}(\tau)\left(-\left[0,\tfrac{1}{r};g\right]\right)\right. \\
&\qquad \left. + s_{r-1,r+\vartheta-1}(\tau)\left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\right\} \\
&+ \frac{2(r+\vartheta)}{r^2}\sum_{p=0}^{r+\vartheta-2}\left[r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right)\right] \\
&\qquad \times s_{r-1,p}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right] \\
&+ \frac{2(r+\vartheta)}{r^2}\sum_{p=0}^{r+\vartheta-2}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right] \\
&\qquad \times s_{r-1,p+1}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right].
\end{aligned} \tag{22}$$

Proof. Differentiation of the first derivative in (17) by using property (7) results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right] &= \frac{1}{r}\left\{\sum_{p=0}^{r+\vartheta-1}\left[r+\vartheta-p+\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]\right. \\
&\qquad \times (r+\vartheta)\left[s_{r-1,p-1}(\tau)-s_{r-1,p}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right] \\
&+ \sum_{p=0}^{r+\vartheta-1}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right] \\
&\qquad \left. \times (r+\vartheta)\left[s_{r-1,p}(\tau)-s_{r-1,p+1}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\right\},
\end{aligned}$$

which can also be rewritten as

$$\begin{aligned}
\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right] &= \frac{(r+\vartheta)}{r}\left\{\sum_{p=0}^{r+\vartheta-2}\left[r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right)\right]s_{r-1,p}(\tau)\left[\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]\right. \\
&- \sum_{p=0}^{r+\vartheta-1}\left[r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right)\right]s_{r-1,p}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]
\end{aligned}$$

$$+\sum_{p=-1}^{r+\vartheta-2}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]s_{r-1,p+1}(\tau)\left[\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]$$
$$\left.-\sum_{p=0}^{r+\vartheta-2}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]s_{r-1,p+1}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\right\},$$

after use of property (4) and reindexing of summations. Finally, employing the fact that

$$\left[\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]-\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]=\frac{2}{r}\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right],$$

by Definition 1 yields the desired second derivative given in (22). □

Remark 4. *In the special case $\vartheta=0$ in (22), we obtain the second derivative presented in Lemma 3.3 in [21].*

Remark 5. *To demonstrate the convexity preservation property of $\lambda-$Schurer operators $S_{r,\vartheta}^{\lambda}(g;\tau)$, it must be shown that the second derivative, as presented in Lemma 6, is nonnegative whenever the associated function g is convex. Firstly, in view of Lemma 3, we have*

$$\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]\geq 0, \tag{23}$$

for any convex function g. Secondly, for $0\leq p\leq r+\vartheta-2$, we have $0\leq\frac{2(p+1)}{r+\vartheta-1}\leq 2$ which implies $-1\leq 1-\frac{2(p+1)}{r+\vartheta-1}\leq 1$ Since $-1\leq\lambda\leq 1$, it is clear to see that $-1\leq\lambda\left(1-\frac{2(p+1)}{r+\vartheta-1}\right)\leq 1$ which leads to

$$0\leq r+\vartheta-p-2\leq r+\vartheta-p-1+\lambda\left(1-\frac{2(p+1)}{r+\vartheta-1}\right)=r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right). \tag{24}$$

In a similar fashion, for $0\leq p\leq r+\vartheta-2\leq r+\vartheta-1$ and $-1\leq\lambda\leq 1$, one can write $-1\leq-\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)\leq 1$ which implies

$$0\leq(p+1)-1\leq p+1-\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)=p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right). \tag{25}$$

Consequently, we affirm that

$$\frac{2(r+\vartheta)}{r^2}\sum_{p=0}^{r+\vartheta-2}\left[r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right)\right]$$
$$\times s_{r-1,p}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]$$
$$+\frac{2(r+\vartheta)}{r^2}\sum_{p=0}^{r+\vartheta-2}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]$$
$$\times s_{r-1,p+1}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]\geq 0,$$

due to (23)-(25). In opposition, the term

$$\lambda\frac{(r+\vartheta)(r+\vartheta+1)}{r(r+\vartheta-1)}\left\{s_{r-1,0}(\tau)\left(-\left[0,\tfrac{1}{r};g\right]\right)+s_{r-1,r+\vartheta-1}(\tau)\left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\right\}$$

may produce negative or positive values depending on the choice of shape parameter $\lambda\in[-1,1]$. *Furthermore, the monotonic behavior of function* g *will also have an effect on the determination of the sign of second derivative given in (22) since*

$$-\left[0,\tfrac{1}{r};g\right]\leq 0 \quad \text{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\geq 0, \tag{26}$$

for monotone increasing g *and*

$$-\left[0,\tfrac{1}{r};g\right]\geq 0 \quad \text{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\leq 0, \tag{27}$$

for monotone decreasing g *by Lemma 3. On the grounds of this discussion, one can expect that* $S_{r,\vartheta}^{\lambda}(g;\tau)$ *is not necessarily convex for all* $\lambda\in[-1,1]$ *and* g *on* $[0,1]$. *We verify this line of reasoning by demonstrating the following numerical examples.*

Example 1. *In this first example, we consider the monotone increasing and convex function* $g(\tau)=e^{\tau}-\log_{10}[(\tau+1)^2]$ *on* $[0,1]$, *and form Table 1 in which the intervals are given where* $\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]\geq 0$ *for different values of* λ, r, *and* ϑ.

To begin with, we have inequalities in (26) hold true since g *is monotone increasing on* $[0,1]$. *By inspecting the intervals from Table 1, one can say that* $\lambda-$*Schurer operators successfully preserve the convexity of the associated* g *function for* $\lambda>-\frac{1}{2}$ *for all* $\vartheta\geq 0$ *without loss of generality. Contrarily, it requires to utilize larger* r *values to maintain the convexity for* $-1\leq\lambda<-\frac{1}{2}$. *For instance,* $S_{r,1}^{-1}(g;\tau)$ *and* $S_{r,1}^{-7/8}(g;\tau)$ *are convex on* $[0,1]$ *for* $r\geq 14$ *and* $r\geq 9$, *respectively, when* $\vartheta=1$. *Moreover, performing calculations by taking bigger* ϑ *values definitely improves the results. For example,* $S_{r,3}^{-1}(g;\tau)$ *and* $S_{r,3}^{-7/8}(g;\tau)$ *are convex on* $[0,1]$ *for* $r\geq 6$ *and* $r\geq 2$, *respectively, when* $\vartheta=3$, *and* $S_{r,4}^{-1}(g;\tau)$ *is convex on* $[0,1]$ *for* $r\geq 2$ *when* $\vartheta=4$.

Example 2. *In this scheme, we consider* $g(\tau)=e^{-\tau}$, *which is monotone decreasing and convex on* $[0,1]$. *Similar to Example 1, we calculate the intervals when* $\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]\geq 0$ *as listed in Table 2.*

Since g *is monotone decreasing, the inequalities in (27) are satisfied. Without loss of generality, one can conclude that* $S_{r,\vartheta}^{\lambda}(g;\tau)$ *preserve the convexity of this particular function* g *for* $|\lambda|<\frac{1}{2}$. *On the other hand, the efficiency of convexity preservation decreases for* $-1\leq\lambda<-\frac{1}{2}$ *and* $\frac{1}{2}<\lambda\leq 1$. *For example,* $S_{r,1}^{-7/8}(g;\tau)$, $S_{r,1}^{-11/20}(g;\tau)$ *and* $S_{r,1}^{13/14}(g;\tau)$ *are convex on* $[0,1]$, *for* $r\geq 25$, $r\geq 5$ *and* $r\geq 9$, *respectively. Furthermore,* $S_{r,1}^{-1}(g;\tau)$ *and* $S_{r,1}^{-1}(g;\tau)$ *do not preserve the convexity on* $[0,1]$ *for* $r\leq 260$. *The results are improved if we consider bigger* ϑ *values. For example,* $S_{r,3}^{\lambda}(g;\tau)$ *is convex on* $[0,1]$, *when* $r\geq 2$, *for all* $\lambda\geq-\frac{7}{8}$ *as listed in Table 2, even though, we observe that* $S_{r,1}^{-1}(g;\tau)$ *still do not preserve the convexity*

TABLE 1. List of intervals where $S_{r,\vartheta}^{\lambda}\left(e^{\tau}-\log_{10}[(\tau+1)^2];\tau\right)$ is convex for the associated values of λ, ϑ and r.

r	$\lambda=-1$	$\lambda=-7/8$	$\lambda=-11/20$	$\lambda=-2/7$	$\lambda=3/10$	$\lambda=2/5$	$\lambda=11/16$	$\lambda=1$
				$\vartheta=1$				
2	$[0,0.772742]$	$[0,0.805152]$	$[0,0.946657]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
3	$[0,0.860669]$	$[0,0.891815]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
4	$[0,0.911864]$	$[0,0.939819]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
5	$[0,0.942694]$	$[0,0.967221]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
6	$[0,0.961928]$	$[0,0.983432]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
7	$[0,0.974392]$	$[0,0.993373]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
8	$[0,0.982752]$	$[0,0.999655]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
9	$[0,0.988529]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
10	$[0,0.992622]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
11	$[0,0.995582]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
12	$[0,0.997761]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
13	$[0,0.999387]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
14	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
				$\vartheta=3$				
2	$[0,0.989931]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
3	$[0,0.992123]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
4	$[0,0.996158]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
5	$[0,0.999500]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
6	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
				$\vartheta=4$				
2	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
3	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
4	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
5	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
6	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$

TABLE 2. List of intervals where $S_{r,\vartheta}^{\lambda}\left(e^{-\tau};\tau\right)$ is convex for the associated values of λ, ϑ and r.

r	$\lambda=-1$	$\lambda=-7/8$	$\lambda=-11/20$	$\lambda=-2/7$	$\lambda=3/10$	$\lambda=1/2$	$\lambda=13/14$	$\lambda=1$
				$\vartheta=1$				
2	[0.286817, 1]	[0.025858, 1]	[0.130190, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.792078]	[0, 0.766560]
3	[0.195798, 1]	[0.166034, 1]	[0.041414, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.929621]	[0, 0.900579]
4	[0.135709, 1]	[0.108515, 1]	[0.000300, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.973939]	[0, 0.955284]
5	[0.097872, 1]	[0.073892, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.988329]	[0, 0.974152]
6	[0.073320, 1]	[0.052226, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.994656]	[0, 0.983036]
7	[0.056714, 1]	[0.038046, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.997889]	[0, 0.987978]
8	[0.045049, 1]	[0.028389, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999692]	[0, 0.991023]
9	[0.036581, 1]	[0.021588, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.993036]
10	[0.030259, 1]	[0.016657, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.994438]
24	[0.005779, 1]	[0.000053, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999096]
25	[0.005338, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999168]
30	[0.003741, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999426]
50	[0.001369, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999796]
100	[0.000346, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999950]
200	[0.000087, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999987]
300	[0.000052, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999994]
				$\vartheta=3$				
2	[0.0196549, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
3	[0.0216047, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
4	[0.0193631, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
5	[0.0165661, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
6	[0.0140429, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
7	[0.0119356, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
8	[0.0102133, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
9	[0.0088096, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
10	[0.0302594, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
24	[0.0019536, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
25	[0.0018209, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
30	[0.0013232, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
50	[0.0005220, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
100	[0.0001399, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
200	[0.0000362, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
260	[0.0000216, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
				$\vartheta=5$				
2	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
3	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
4	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
5	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]

on $[0,1]$ *for* $r \leq 260$. *Lastly,* $S^{\lambda}_{r,5}(g;\tau)$ *is convex on* $[0,1]$, *when* $r \geq 2$, *for all* λ *values listed in Table 2.*

From the analysis presented in Remark 5 and the numerical demonstrations in Examples 1 and 2, it follows that the $\lambda-$Schurer operators may fail to maintain the convexity of associated functions with a monotonic nature for certain values of $\lambda \in [-1,1]$. To address this issue, we propose a revised result for the convexity preservation of $S^{\lambda}_{r,\vartheta}(g;\tau)$ by introducing additional conditions on the function g within the interval $[0,1+\vartheta]$.

Theorem 3 (**Convexity**). *Let* g *be a function that is nonincreasing on* $(0,\tau_0)$ *and nondecreasing* $(\tau_0,1+\vartheta)$ *for any point* $\tau_0 \in (0,1+\vartheta)$ *for* $\vartheta \geq 0$ *integer. If* g *is convex on* $[0,1]$, *then so are all the corresponding* $\lambda-$*Schurer operators for all* $\lambda \in [0,1]$ *and* $r > r_0(\tau_0)$.

Proof. Due to Remark 5, it is sufficient to establish that

$$\lambda\frac{(r+\vartheta)(r+\vartheta+1)}{r(r+\vartheta-1)}\left\{s_{r-1,0}(\tau)\left(-\left[0,\tfrac{1}{r};g\right]\right)+s_{r-1,r+\vartheta-1}(\tau)\left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\right\}\geq 0, \tag{28}$$

holds. To begin with, let $\lambda \in [0,1]$ and $\vartheta \geq 0$ be integer. Now, depending on the choice of point $\tau_0 \in (0,1+\vartheta)$, we will encounter the following cases :

Case 1: When $\tau_0 < \frac{1}{2}$, one can choose r suitably so that $\frac{1}{r} < \tau_0 < \frac{1}{2}$. Therefore, g is nonincreasing on $\left(0,\frac{1}{r}\right)$ and nondecreasing on $\left(\frac{r-1}{r},1+\vartheta\right)$, which implies

$$-\left[0,\tfrac{1}{r};g\right]=g(0)-g\left(\tfrac{1}{r}\right)\geq 0 \quad \text{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]=g\left(\tfrac{r+\vartheta}{r}\right)-g\left(\tfrac{r+\vartheta-1}{r}\right)\geq 0,$$

So inequality (28) is accurate.

Case 2: Next, we consider $\frac{1}{2} < \tau_0$ and accordingly choose r such that $\frac{1}{2} < \tau_0 < \frac{r-1}{r}$. Hence, g is nonincreasing on $\left(0,\frac{1}{r}\right)$ and nondecreasing on $\left(\frac{r-1}{r},1+\vartheta\right)$, which implies

$$-\left[0,\tfrac{1}{r};g\right]\geq 0 \quad \text{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\geq 0.$$

The inequality (28) remains valid.

Case 3: In this last scheme, we pick $\tau_0 = \frac{1}{2}$. Subsequently, it is straightforward to see that $\frac{1}{r} < \frac{1}{2} = \tau_0 < \frac{r-1}{r}$ which insinuates inequality (28) is true for all $r \geq 2$. □

Remark 6. *The* $\vartheta = 0$ *case is presented as Theorem 3.2 in [21].*

We establish the following numerical example as an implementation of the Theorem 3.

Example 3. *For this scheme, we consider the convex function* $g(\tau)=\left(\tau-\frac{1}{3}\right)^4$, *which is nonincreasing on* $\left(0,\frac{1}{3}\right)$ *and nondecreasing* $\left(\frac{1}{3},1+\vartheta\right)$ *for nonnegative integer* ϑ. *Hence, we have*

$$-\left[0,\tfrac{1}{r};g\right]\geq 0 \quad \textit{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\geq 0.$$

Next, we obtain Table 3 in which the intervals are given where $\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right] \geq 0.$

TABLE 3. List of intervals where $S_{r,\vartheta}^{\lambda}\left(\left(\tau-\frac{1}{3}\right)^4;\tau\right)$ is convex for the associated values of λ, ϑ and r.

r	$\lambda=2/15$	$\lambda=3/10$	$\lambda=4/7$	$\lambda=11/16$	$\lambda=17/20$	$\lambda=1$
			$\vartheta=1$			
2	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
3	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
4	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
5	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
6	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
			$\vartheta=3$			
2	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
3	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
4	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
5	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
6	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]

The numeric values from Table 3 confirm that $S_{r,\vartheta}^{\lambda}(g;\tau)$ *preserves the convexity of the affiliated function* $g(\tau)$ *on* $[0,1]$ *for all* $\lambda \in [0,1]$ *and integer* $\vartheta \geq 0$ *when* $r \geq 2$. *Thus, we can conclude that if the function* $g(\tau)$ *is selected according to the conditions outlined in Theorem 3, we achieve enhanced results regarding the preservation of convexity for the corresponding* $\lambda-$*Schurer operators.*

4. Conclusions and Future Work

This paper has provided a comprehensive analysis of the shape-preserving characteristics of λ-Schurer operators, highlighting their potential as a robust tool in approximation theory. The results demonstrate that these operators not only preserve the essential geometric features of the approximated functions but also offer enhanced control through the adjustable shape parameter λ. The theoretical insights and auxiliary results presented in this study contribute to a deeper understanding of shape-preserving approximation techniques and pave the way for further research into their applications in diverse fields, such as computer-aided geometric design and numerical analysis. Future studies could explore the extension of these operators to higher dimensions and their integration into practical computational tools. Moreover, we intend to further our research on the shape-preserving characteristics of the operators constructed in [5, 9, 10, 20, 22], respectively.

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THE DEPENDENCY OF THE ANALYTICAL AND NUMERICAL SOLUTION ON THE ε PARAMETER IN HYPERBOLIC AND PSEUDO-HYPERBOLIC PROBLEMS WITH INVERSE COEFFICIENTS

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Abstract. The aim of this study is to analyze the behavior of ε on the solution of an inverse coefficient nonlinear pseudo-hyperbolic equation $\omega_{tt} - \varepsilon\omega_{xxtt} - \omega_{xx} = \theta(t)f(x,t,\omega)$ with periodic boundary conditions. We also consider the inverse coefficient problem $\omega_{tt} - \omega_{xx} = \theta(t)f(x,t,\omega)$. The solution function of nonlinear pseudo-hyperbolic equation is found to be convergent to the solution function of nonlinear hyperbolic equation, when $\varepsilon \to 0$ is proved. The Fourier method was used to illustrate the theoretically relation between the inverse problems while the Finite Difference Method was used numerically. In order to get more accurate numerical solution higher precision schemes have been applied in implicit finite difference equation. The cases where $\varepsilon = 0$ and $\varepsilon \neq 0$ have been solved analytically and numerically, and compared each other.

1. Introduction

Nonlinear hyperbolic equations and nonlinear pseudo-hyperbolic equations are both types of partial differential equations (PDEs) that arise in various areas of physics and engineering. While they share some similarities, they have distinct characteristics.

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Hyperbolic equations typically describe wave phenomena, where information propagates along characteristics at finite speed [24]. In the nonlinear case, the coefficients and/or terms n the equation are nonlinear functions of the dependent variable. Examples of nonlinear hyperbolic equations include the nonlinear wave equation [21, 35], the Euler equations for compressible fluid flow [27, 39], and the nonlinear acoustics equations [30].

Pseudo-hyperbolic equations also describe wave-like behavior, but they may not exhibit strict characteristics along which information propagates. They often arise as generalizations of hyperbolic equations or in systems where certain terms introduce dispersive effects or alter the characteristics of wave propagation [19]. Nonlinear pseudo-hyperbolic equations can involve terms with mixed spatial and temporal derivatives and can exhibit dispersive or diffusive behavior alongside wave-like propagation. Examples include certain models of viscoelasticity [29], nonlinear versions of the Korteweg–de Vries equation [22], and some models in nonlinear optics.

In summary, while both types of equations describe wave-like phenomena, nonlinear hyperbolic equations typically follow characteristics along which information propagates at finite speed, while nonlinear pseudo-hyperbolic equations may exhibit dispersive effects or altered wave propagation behavior due to the presence of certain terms.

Numerous analytical techniques exist for solving differential equations. Nonetheless, detecting arbitrary functions that fulfill given boundary conditions within these equations can pose challenges. In fact, finding the general solution of partial differential equations is generally impossible except in specific scenarios. Consequently, various approaches have been devised for addressing boundary value problems. Among these, the Fourier method stands out as a well-known technique, relying on the separation of variables [5].

The study of inverse problems emerged in the 19th and 20th centuries, contributing to the resolution of numerous challenges in heat transfer, diffusion, nuclear physics, seismology. Inverse problems can be utilized with parabolic equations [6-8, 17, 28]. In addition, inverse problems can also be used for hyperbolic and/or pseudo hyperbolic equations [9, 25, 31, 32].

The present investigation employs the periodic boundary condition, which is a specific instance of the nonlocal boundary condition [1]. Periodic boundary condition is combination between Dirichlet (giving constant properties) and Neumann (giving constant flux) boundary conditions, and it generally utilizes to avoid large computational domains for numerical and analytical computation [3, 4].

For the numerical solution of one-dimensional wave equations with inverse coefficients (hyperbolic and pseudo-hyperbolic), there are several numerical methods, which are finite difference method [23, 34], finite element method [11-13], and finite volume method [10, 14-16, 20, 36, 37], available. There are many studs that solve the wave equation (hyperbolic and/or pseudo hyperbolic equations) using the finite difference method [2, 33, 38].

In the present study, we investigate an inverse problem of unknown time-dependent coefficients in the one-dimensional nonlinear hyperbolic and/or pseudo equation with periodic boundary conditions. For an analytical solution, the Fourier method is utilized to generate Fourier coefficients for the solutions, and through an iterative approach, we establish the convergence, uniqueness, and stability of the solution to the nonlinear problem. For numerical solution, implicit finite difference scheme is utilized. To achieve a more accurate solution, higher precision schemes have been employed in implicit finite difference equation. A second-order accurate time discretization is implemented, and fourth-order accurate finite difference equations are utilized for the discretization of spatial and multi-variable partial differential equations. The cases where epsilon equals 0 and epsilon not equal to 0 (different epsilon values) have been solved analytically and numerically, and compared with each other.

2. Solution of the Problems

Here, we studied mixed problems of two physical phenomena models: pseudo-hyperbolic equation (1) and hyperbolic equation (5) in the domain
$(x,t) \in \Omega\,(0 < x < \pi,\ 0 < t < T)$:

$$\tilde{\omega}_{tt} - \varepsilon\tilde{\omega}_{xxtt} - \tilde{\omega}_{xx} = \tilde{\theta}(t) f(x,t,\tilde{\omega}), \tag{1}$$

$$\begin{aligned} \tilde{\omega}(x,0,\varepsilon) &= \chi(x), \\ \tilde{\omega}_t(x,0,\varepsilon) &= \phi(x), \end{aligned} \tag{2}$$

$$\begin{aligned} \tilde{\omega}(x,0,\varepsilon) &= \chi(x), \\ \tilde{\omega}_t(x,0,\varepsilon) &= \phi(x), \end{aligned} \tag{3}$$

$$\tilde{E}(t,\varepsilon) = \int\limits_0^\pi x\tilde{\omega}(x,t,\varepsilon)dx. \tag{4}$$

The initial, boundary, and overdetermination conditions of the pseudo-hyperbolic equation are illustrated by (2), (3), and (4), respectively. Similarly, the initial and boundary conditions set for the solutions of the hyperbolic equation (5) expressed as follows:

$$\omega_{tt} - \omega_{xx} = \theta(t) f(x,t,\omega), \tag{5}$$

$$\begin{aligned} \omega(x,0) &= \chi(x), \\ \omega_t(x,0) &= \phi(x), \end{aligned} \tag{6}$$

$$\begin{aligned} \omega(0,t) &= \omega(\pi,t), \\ \omega_x(0,t) &= \omega_x(\pi,t), \end{aligned} \tag{7}$$

$$E(t) = \int_0^\pi x\omega(x,t)dx. \tag{8}$$

Equation (5) is obtained from (1) by setting $\varepsilon = 0$. Here, the equation simplifies to the standard wave equation. This describes classical wave phenomena where the speed of wave propagation is constant and there is no additional dependence on mixed spatial and temporal derivatives. Where $\varepsilon \geq 0$ is a small parameter, $\chi(x)$, $\phi(x)$ and $E(t,\varepsilon)$ are given functions on $x \in (0,\pi)$ and $t \in (0,T)$, respectively. Here, the term $\varepsilon\tilde{\omega}_{xxtt}$ introduces a damping-like or dispersive effect. This term can account for additional physical phenomena like viscosity or diffusive effects, leading to modified wave propagation characteristics. For example, it can model how waves interact with a medium that has additional resistance or how they spread out over time.
In [18, 26], the authors analyzed the dependence of the solution of direct problems on ε. In this paper, we show the dependence of the solution of inverse coefficient problems on ε; that is, the solution function $\tilde{\omega}(x,t,\varepsilon)$ of (1)-(4) is convergent to the solution function $\omega(x,t)$ of (5)-(8) as $\varepsilon \to 0$.

In mathematical physics, direct problems aim to find functions that describe physical processes, such as sound or heat propagation. Inverse problems arise when the properties of the medium are unknown and it is necessary to determine these properties based on information about the solution of the direct problem.

Definition 1. *In the inverse problem, in addition to $\omega(x,t)$, there is unknown of function included in the direct problem. This unknown pair $\{\theta(t),\, \omega(x,t)\}$ is called the solution of the inverse problem.*

Definition 2. *Banach space is a space in which there exists a set of continuous functions on $[0,T]$, denoted by $\{\omega(t)\} = \{\omega_0(t),\, \omega_{ck}(t),\, \omega_{sk}(t),\ k \in N\}$, that satisfy the norm*

$$\|\omega(t)\| = \max_{0\leq t\leq T} |\omega_0(t)| + \sum_{k=1}^{\infty} \left(\max_{0\leq t\leq T} |\omega_{ck}(t)| + \max_{0\leq t\leq T} |\omega_{sk}(t)| \right).$$

Here, we seek a general solution to (1)-(4) as in

$$\tilde{\omega}(x,t) = \frac{\tilde{\omega}_0}{2} + \sum_{k=1}^{\infty} [\tilde{\omega}_{ck}(x,t)\cos 2kx + \tilde{\omega}_{sk}(x,t)\sin 2kx].$$

The solution obtained is denoted by (9) below

$$\begin{aligned}
\tilde{\omega}(x,t,\varepsilon) =& \frac{1}{2}\left(\chi_0 + \phi_0 t + \frac{2}{\pi}\int_0^t\int_0^\pi \tilde{\theta}(\tau)(t-\tau) f(\xi,\tau,\tilde{\omega}) d\xi d\tau \right) \\
&+ \sum_{k=1}^{\infty} \Bigg(\chi_{ck}\cos\tilde{\alpha}_k t + \frac{1}{\tilde{\lambda}_k}\phi_{ck}\sin\tilde{\alpha}_k t \\
&+ \frac{1}{\tilde{\lambda}_k}\frac{2}{\pi}\int_0^t\int_0^\pi \tilde{\theta}(\tau) f(\xi,\tau,\tilde{\omega})\cos 2k\xi \sin\tilde{\lambda}_k(t-\tau) d\xi d\tau \Bigg) \cos 2kx
\end{aligned} \tag{9}$$

$$+\sum_{k=1}^{\infty}\left(\chi_{sk}\cos\tilde{\alpha}_k t+\frac{1}{\tilde{\lambda}_k}\phi_{sk}\sin\tilde{\alpha}_k t\right.$$
$$\left.+\frac{1}{\tilde{\lambda}_k}\frac{2}{\pi}\int_0^t\int_0^\pi \tilde{\theta}(\tau)f(\xi,\tau,\tilde{\omega})\sin 2k\xi\sin\tilde{\lambda}_k(t-\tau)d\xi d\tau\right)\sin 2kx,$$

$\tilde{\lambda}_k=\frac{2k}{\sqrt{1+4\varepsilon k^2}},\ k=\overline{1,\infty}$.

By multiplying equation (1) by x and integrating it over the interval $[0,\pi]$, and using initial data (2) and overdetermination condition (4), we find

$$\tilde{\theta}(t)=\frac{\tilde{E''}(t)+\varepsilon\phi_t(\pi)-\varepsilon\phi_t(0)}{\int_0^\pi xf(x,t,\tilde{\omega})dx}$$
$$-\frac{\pi\sum_{k=1}^{\infty}(2k)\left\{\left(1-\varepsilon\tilde{\lambda}_k^2\right)\chi_{sk}\cos\tilde{\lambda}_k t+\left(\frac{1}{\tilde{\lambda}_k}-\varepsilon\tilde{\lambda}_k\right)\phi_{sk}\sin\tilde{\lambda}_k t+\frac{1}{\tilde{\lambda}_k}\frac{2}{\pi}\int_0^t\int_0^\pi\tilde{\theta}(\tau)f(\xi,\tau,\tilde{\omega})\sin 2k\xi\sin\tilde{\lambda}_k(t-\tau)d\xi d\tau\right\}}{\int_0^\pi xf(x,t,\tilde{\omega})dx}. \tag{10}$$

We seek a general solution to equations (5)-(8) as in

$$\omega(x,t)=\frac{\omega_0}{2}+\sum_{k=1}^{\infty}[\omega_{ck}(x,t)\cos 2kx+\omega_{sk}(x,t)\sin 2kx],$$

and we find the solution

$$\begin{aligned}\omega(x,t)=&\frac{1}{2}\left(\chi_0+\phi_0 t+\frac{2}{\pi}\int_0^t\int_0^\pi\theta(\tau)(t-\tau)f(\xi,\tau,\omega)d\xi d\tau\right)\\&+\sum_{k=1}^{\infty}\left(\chi_{ck}\cos\lambda_k t+\frac{\phi_{ck}}{2k}\sin\lambda_k t\right.\\&\quad\left.+\frac{1}{\alpha_k}\int_0^t\int_0^\pi\theta(\tau)f(\xi,\tau,\omega)\cos 2k\xi\sin\lambda_k(t-\tau)d\xi d\tau\right)\cos 2kt\\&+\sum_{k=1}^{\infty}\left(\chi_{sk}\cos\lambda_k t+\frac{\phi_{sk}}{2k}\sin\lambda_k t\right.\\&\quad\left.+\frac{1}{\alpha_k}\int_0^t\int_0^\pi\theta(\tau)f(\xi,\tau,\omega)\sin 2k\xi\sin\lambda_k(t-\tau)d\xi d\tau\right)\sin 2k,\end{aligned} \tag{11}$$

$\lambda_k = 2k,\ k = \overline{1,\infty}.$

With the same method we obtained an inverse coefficient to (5)-(8) as following;

$$\theta(t) = \frac{E''(t) - \pi \sum\limits_{k=1}^{\infty} (2k) \left(\chi_{sk} \cos \lambda_k t + \frac{\phi_{sk}}{2k} \sin \lambda_k t + \frac{1}{2k} \int_0^t \theta(\tau) f_{sk}(\tau) \sin \lambda_k (t-\tau) d\tau \right)}{\int_0^\pi x f(x,t,\omega) dx}. \tag{12}$$

3. Analysis of Convergence of Solutions

Theorem 1. *If following*
1. $E(t) \in C^2[0,T],\ \theta(t) \in C[0,T]$.
2. $\varphi(x) \in C^1[0,\pi],\ \psi(x) \in C^1[0,\pi]$.
3. The function $f(x,t,\omega)$ *be continuous to all arguments in* $\Omega \times (-\infty, \infty)$ *and satisfies the following conditions*
i) $\left| \frac{\partial^{(s)} f(x,t,\omega)}{\partial x^{(s)}} - \frac{\partial^{(s)} f(x,t,\tilde{\omega})}{\partial x^{(s)}} \right| \leq b(x,t) \left| \omega - \tilde{\omega} \right|,\ s = \overline{0,2},$
$b(x,t) \in L_2(D),\ b(x,t) \geq 0;$
ii) $f(x,t,\omega) \in C^1[0,\ \pi],\ |f(x,t,\omega)| \leq M,\ t \in [0,\ T];$
iii) $\int_0^\pi f(x,t,\omega)dx \neq 0,\ \forall t \in [0,T]$ *conditions are fulfilled, then*
$\lim\limits_{\varepsilon \to 0} \tilde{\omega}(x,t,\varepsilon) = \omega(x,t)$.

Proof. Firstly, we examine the difference of the time dependent coefficiets (10) and (12) and as follows;

$$\tilde{\theta}(t) - \theta(t) = \frac{\tilde{E''}(t) + \varepsilon \phi_t(\pi) - \varepsilon \phi_t(0)}{\int\limits_0^\pi x f(x,t,\tilde{\omega}) dx}$$
$$- \frac{\pi \sum\limits_{k=1}^{\infty} \left\{ a_k {\chi'}_{ck} \cos \tilde{\lambda}_k t + b_k \phi_{sk} \sin \tilde{\lambda}_k t + \frac{1}{\tilde{\lambda}_k} \frac{2}{\pi} \int\limits_0^t \int\limits_0^\pi \tilde{\theta}(\tau) f'(\xi,\tau,\tilde{\omega}) \cos 2k\xi \sin \tilde{\lambda}_k (t-\tau) d\xi d\tau \right\}}{\int\limits_0^\pi x f(x,t,\tilde{\omega}) dx}$$
$$- \frac{E''(t) - \pi \sum\limits_{k=1}^{\infty} \left({\chi'}_{sk} \cos \lambda_k t + \phi_{sk} \sin \lambda_k t + \frac{1}{\lambda_k} \frac{2}{\pi} \int\limits_0^t \int\limits_0^\pi \theta(\tau) f'(\xi,\tau,\omega) \cos 2k\xi \sin \lambda_k (t-\tau) d\xi d\tau \right)}{\int_0^\pi x f(x,t,\omega) dx},$$

$a_k = \frac{1}{1+4\varepsilon k^2},\ b_k = \frac{1}{\sqrt{1+4\varepsilon k^2}}$. Then we have

$$\tilde{\theta}(t) - \theta(t) = \frac{2}{\pi^2 M_*} \left(\tilde{E''}(t) - E''(t) \right) + \frac{2}{\pi^2 M_*} \left(\varepsilon \phi_t(\pi) - \varepsilon \phi_t(0) \right)$$

$$+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\chi'_{ck}\left(\cos\lambda_k t-a_k\cos\tilde{\lambda}_k t\right)+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\phi_{sk}\left(\sin\lambda_k t-b_k\sin\tilde{\lambda}_k t\right)$$
$$+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\frac{2}{\pi}\int_0^t\int_0^\pi \theta(\tau)f'(\xi,\tau,\omega)\cos 2k\xi\sin\lambda_k(t-\tau)d\xi d\tau\right.$$
$$\left.-\frac{1}{\tilde{\lambda}_k}\frac{2}{\pi}\int_0^t\int_0^\pi \tilde{\theta}(\tau)f'(\xi,\tau,\tilde{\omega})\cos 2k\xi\sin\tilde{\lambda}_k(t-\tau)d\xi d\tau\right).$$

If the absolute value of the difference is taken after adding and subtracting and making the necessary grouping, we have

$$\begin{aligned}\left|\tilde{\theta}(t)-\theta(t)\right| \leq&\frac{2}{\pi^2 M_*}\left|\tilde{E}''(t)-E''(t)\right|+\frac{2}{\pi^2 M_*}\left|\varepsilon\phi_t(\pi)-\varepsilon\phi_t(0)\right|\\
&+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\left|\chi'_{ck}\right|\left|\cos\lambda_k t-a_k\cos\tilde{\lambda}_k t\right|+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\left|\phi_{sk}\right|\left|\sin\lambda_k t-b_k\sin\tilde{\lambda}_k t\right|\\
&+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi\left[\tilde{\theta}(t)-\theta(t)\right]f'(\xi,\tau,\omega)\cos 2k\xi\sin\lambda_k(t-\tau)d\xi d\tau\right|\\
&+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\left|\frac{1}{\lambda_k}-\frac{1}{\tilde{\lambda}_k}\right|\frac{2}{\pi}\left|\int_0^t\int_0^\pi\tilde{\theta}(\tau)f'(\xi,\tau,\tilde{\omega})\cos 2k\xi\sin\tilde{\lambda}_k(t-\tau)d\xi d\tau\right|\\
&+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi\tilde{\theta}(\tau)\left[f'(\xi,\tau,\tilde{\omega})-f'(\xi,\tau,\omega)\right]\cos 2k\xi\sin\lambda_k(t-\tau)d\xi d\tau\right|\\
&+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\int_0^t\left(\frac{2}{\pi}\left|\int_0^\pi\tilde{\theta}(\tau)f'(\xi,\tau,\tilde{\omega})\cos 2k\xi d\xi\right|\right)\left|\sin\lambda_k(t-\tau)-\sin\tilde{\lambda}_k(t-\tau)\right|d\tau.\end{aligned} \tag{13}$$

From (13), the statements $\left|\tilde{E}''(t)-E''(t)\right|$, $\left|\varepsilon\phi_t(\pi)-\varepsilon\phi_t(0)\right|$, $\left|\frac{1}{\lambda_k}-\frac{1}{\tilde{\lambda}_k}\right|$, $\left|\sin\lambda_k t-b_k\sin\tilde{\lambda}_k t\right|$, $\left|\cos\lambda_k t-a_k\cos\tilde{\lambda}_k t\right|$, $\left|\sin\lambda_k(t-\tau)-\sin\tilde{\lambda}_k(t-\tau)\right|$ are bounded for k, τ and t $(0\leq\tau\leq t\leq T)$ as $\varepsilon\to 0$, also $a_k,\ b_k$ are limited. Let us denote all of these statements by $\sigma(\varepsilon)$ and we rewrite (13) as following

$$\left|\tilde{\theta}(t)-\theta(t)\right| \leq\sigma(\varepsilon)+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\frac{1}{\tilde{\lambda}_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi\left[\tilde{\theta}(\tau)-\theta(\tau)\right]f'(\xi,\tau,\omega)\cos 2k\xi\sin\lambda_k(t-\tau)d\xi d\tau\right|$$

$$+\frac{2}{\pi M_*}\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi \tilde{\theta}(\tau)\left[f'(\xi,\tau,\tilde{\omega})-f'(\xi,\tau,\omega)\right]\cos 2k\xi\sin\lambda_k(t-\tau)d\xi d\tau\right|.$$

Applying Cauchy, Bessel, Hölder inequalities to the inequality above, we have

$$\left|\tilde{\theta}(t)-\theta(t)\right|\le\frac{\sigma(\varepsilon)}{B}+\frac{2}{BM_*}\sqrt{\frac{t}{6\pi}}\left(\int_0^t\int_0^\pi\left\{\tilde{\theta}(\tau)b(\xi,\tau)\left|\tilde{\omega}-\omega\right|\right\}^2 d\xi d\tau\right)^{\frac{1}{2}}, \quad (14)$$

$B=1-\frac{4M}{\pi M_*}\sqrt{\frac{t}{\pi}\left(\frac{\pi^2}{24}+\varepsilon\right)}$.

Let us take the difference of the Fourier coefficients to examine the difference of the solutions (9) and (11),

$$\tilde{\omega}_0(t,\varepsilon)-\omega_0(t)=\frac{2}{\pi}\int_0^t\int_0^\pi\tilde{\theta}(\tau)(t-\tau)f(\zeta,\tau,\tilde{\omega})d\zeta d\tau-\frac{2}{\pi}\int_0^t\int_0^\pi\theta(\tau)(t-\tau)f(\zeta,\tau,\omega)d\zeta d\tau,$$

$$\begin{aligned}\tilde{\omega}_{ck}(t,\varepsilon)-\omega_{ck}(t)=&\sum_{k=1}^{\infty}\left(\chi_{ck}\cos\tilde{\alpha}_k t-\chi_{ck}\cos\alpha_k t\right)+\sum_{k=1}^{\infty}\left(\frac{1}{\tilde{\lambda}_k}\phi_{ck}\sin\tilde{\alpha}_k t-\frac{\phi_{ck}}{\lambda_k}\sin\alpha_k t\right)\\&+\sum_{k=1}^{\infty}\left(\frac{1}{\tilde{\lambda}_k}\frac{2}{\pi}\int_0^t\int_0^\pi\tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\cos 2k\zeta\sin\tilde{\lambda}_k(t-\tau)d\zeta d\tau\right.\\&\left.-\frac{1}{\lambda_k}\frac{2}{\pi}\int_0^t\int_0^\pi\theta(\tau)f(\zeta,\tau,\omega)\cos 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right),\end{aligned}$$

$$\begin{aligned}\tilde{\omega}_{sk}(t,\varepsilon)-\omega_{sk}(t)=&\sum_{k=1}^{\infty}\left(\chi_{sk}\cos\tilde{\alpha}_k t-\chi_{sk}\cos\alpha_k t\right)+\sum_{k=1}^{\infty}\left(\frac{1}{\tilde{\lambda}_k}\phi_{sk}\sin\tilde{\alpha}_k t-\frac{\phi_{sk}}{\lambda_k}\sin\alpha_k t\right)\\&+\sum_{k=1}^{\infty}\left(\frac{1}{\tilde{\lambda}_k}\frac{2}{\pi}\int_0^t\int_0^\pi\tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\sin 2k\zeta\sin\tilde{\lambda}_k(t-\tau)d\zeta d\tau\right.\\&\left.-\frac{1}{\lambda_k}\frac{2}{\pi}\int_0^t\int_0^\pi\theta(\tau)f(\zeta,\tau,\omega)\sin 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right).\end{aligned}$$

By adding and subtracting and taking the absolute values, we obtain

$$|\tilde{\omega}_0(t,\varepsilon)-\omega_0(t)|\le\frac{2}{\pi}\left|\int_0^t\int_0^\pi[\tilde{\theta}(\tau)-\theta(\tau)](t-\tau)f(\zeta,\tau,\omega)d\zeta d\tau\right|$$

$$+\frac{2}{\pi}\left|\int_0^t\int_0^\pi \tilde{\theta}(\tau)(t-\tau)\left[f(\zeta,\tau,\tilde{\omega})-f(\zeta,\tau,\omega)\right]d\zeta d\tau\right|,$$

$$|\tilde{\omega}_{ck}(t,\varepsilon)-\omega_{ck}(t)|\le\sum_{k=1}^{\infty}|\chi_{ck}|\left|\cos\tilde{\lambda}_k t-\cos\lambda_k t\right|+\sum_{k=1}^{\infty}|\phi_{ck}|\left|\frac{\sin\tilde{\lambda}_k t}{\tilde{\lambda}_k}-\frac{\sin\lambda_k t}{\lambda_k}\right|$$
$$+\sum_{k=1}^{\infty}\left|\frac{1}{\tilde{\lambda}_k}-\frac{1}{\lambda_k}\right|\frac{2}{\pi}\left|\int_0^t\int_0^\pi\tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\cos 2k\zeta\sin\tilde{\lambda}_k(t-\tau)d\zeta d\tau\right|$$
$$+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi\left[\tilde{\theta}(\tau)-\theta(\tau)\right]f(\zeta,\tau,\tilde{\omega})\cos 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right|$$
$$+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi\tilde{\theta}(\tau)\left[f(\zeta,\tau,\tilde{\omega})-f(\zeta,\tau,\omega)\right]\cos 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right|$$
$$+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\int_0^t\left(\frac{2}{\pi}\left|\int_0^\pi\tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\cos 2k\zeta d\zeta\right|\right)\left|\sin\tilde{\lambda}_k(t-\tau)-\sin\lambda_k(t-\tau)\right|d\tau,$$

$$|\tilde{\omega}_{sk}(t,\varepsilon)-\omega_{sk}(t)|\le\sum_{k=1}^{\infty}|\chi_{sk}|\left|\cos\tilde{\lambda}_k t-\cos\lambda_k t\right|+\sum_{k=1}^{\infty}|\phi_{sk}|\left|\frac{\sin\tilde{\lambda}_k t}{\tilde{\lambda}_k}-\frac{\sin\lambda_k t}{\lambda_k}\right|$$
$$+\sum_{k=1}^{\infty}\left|\frac{1}{\tilde{\lambda}_k}-\frac{1}{\lambda_k}\right|\frac{2}{\pi}\left|\int_0^t\int_0^\pi\tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\sin 2k\zeta\sin\tilde{\lambda}_k(t-\tau)d\zeta d\tau\right|$$
$$+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi\left[\tilde{\theta}(\tau)-\theta(\tau)\right]f(\zeta,\tau,\tilde{\omega})\sin 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right|$$
$$+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi\tilde{\theta}(\tau)\left[f(\zeta,\tau,\tilde{\omega})-f(\zeta,\tau,\omega)\right]\sin 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right|$$
$$+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\int_0^t\left(\frac{2}{\pi}\left|\int_0^\pi\tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\sin 2k\zeta d\xi\right|\right)\left|\sin\tilde{\lambda}_k(t-\tau)-\sin\lambda_k(t-\tau)\right|d\tau.$$

After that, we have

$$|\tilde{\omega}(t,\varepsilon)-\omega(t)|=\frac{|\tilde{\omega}_0(t,\varepsilon)-\omega_0(t)|}{2}+\sum_{k=1}^{\infty}\left[|\tilde{\omega}_{ck}(t,\varepsilon)-\omega_{ck}(t)|+|\tilde{\omega}_{sk}(t,\varepsilon)-\omega_{sk}(t)|\right]$$

$$
\begin{aligned}
&\leq \frac{1}{\pi}\left|\int_0^t\int_0^\pi [\tilde{\theta}(\tau)-\theta(\tau)](t-\tau)f(\zeta,\tau,\omega)d\zeta d\tau\right| \\
&+\frac{1}{\pi}\left|\int_0^t\int_0^\pi \left|\tilde{\theta}(\tau)(t-\tau)\left[f(\zeta,\tau,\tilde{\omega})-f(\zeta,\tau,\omega)\right]\right| d\zeta d\tau\right| \\
&+\sum_{k=1}^{\infty}|\chi_{ck}|\left|\cos\tilde{\lambda}_k t-\cos\lambda_k t\right|+\sum_{k=1}^{\infty}|\phi_{ck}|\left|\frac{\sin\tilde{\lambda}_k t}{\tilde{\lambda}_k}-\frac{\sin\lambda_k t}{\lambda_k}\right| \\
&+\sum_{k=1}^{\infty}|\chi_{sk}|\left|\cos\tilde{\lambda}_k t-\cos\lambda_k t\right|+\sum_{k=1}^{\infty}|\phi_{sk}|\left|\frac{\sin\tilde{\lambda}_k t}{\tilde{\lambda}_k}-\frac{\sin\lambda_k t}{\lambda_k}\right| \qquad (15)\\
&+\sum_{k=1}^{\infty}\left|\frac{1}{\tilde{\lambda}_k}-\frac{1}{\lambda_k}\right|\frac{2}{\pi}\left|\int_0^t\int_0^\pi \tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\cos 2k\zeta\sin\tilde{\lambda}_k(t-\tau)d\zeta d\tau\right| \\
&+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi \left[\tilde{\theta}(\tau)-\theta(\tau)\right] f(\zeta,\tau,\tilde{\omega})\cos 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right| \\
&+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi \tilde{\theta}(\tau)\left[f(\zeta,\tau,\tilde{\omega})-f(\zeta,\tau,\omega)\right]\cos 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right| \\
&+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\int_0^t\left(\frac{2}{\pi}\left|\int_0^\pi \tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\cos 2k\zeta d\zeta\right|\right)\left|\sin\tilde{\lambda}_k(t-\tau)-\sin\lambda_k(t-\tau)\right| d\tau \\
&+\sum_{k=1}^{\infty}\left|\frac{1}{\tilde{\lambda}_k}-\frac{1}{\lambda_k}\right|\frac{2}{\pi}\left|\int_0^t\int_0^\pi \tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\sin 2k\zeta\sin\tilde{\lambda}_k(t-\tau)d\zeta d\tau\right| \\
&+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi \left[\tilde{\theta}(\tau)-\theta(\tau)\right] f(\zeta,\tau,\tilde{\omega})\sin 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right| \\
&+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int_0^t\int_0^\pi \tilde{\theta}(\tau)\left[f(\zeta,\tau,\tilde{\omega})-f(\zeta,\tau,\omega)\right]\sin 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right| \\
&+\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\int_0^t\left(\frac{2}{\pi}\left|\int_0^\pi \tilde{\theta}(\tau)f(\zeta,\tau,\tilde{\omega})\sin 2k\zeta d\zeta\right|\right)\left|\sin\tilde{\lambda}_k(t-\tau)-\sin\lambda_k(t-\tau)\right| d\tau.
\end{aligned}
$$

The statements $\left|\tilde{E''}(t)-E''(t)\right|$, $\left|\frac{1}{\lambda_k}-\frac{1}{\tilde{\lambda}_k}\right|$, $\left|\sin\lambda_k t-b_k\sin\tilde{\lambda}_k t\right|$, $\left|\cos\lambda_k t-a_k\cos\tilde{\lambda}_k t\right|$, $\left|\sin\lambda_k(t-\tau)-\sin\tilde{\lambda}_k(t-\tau)\right|$ in the inequality (15) are bounded for k, τ and t $(0\le\tau\le t\le T)$ as $\varepsilon\to 0$. Let us denote all of these statements by $\sigma(\varepsilon)$ and we rewrite (15) as follow

$$\begin{aligned}
|\tilde{\omega}(t,\varepsilon)-\tilde{\omega}(t)| &\le \sigma(\varepsilon)+\frac{1}{\pi}\left|\int\limits_0^t\int\limits_0^\pi[\tilde{\theta}(\tau)-\theta(\tau)](t-\tau)f(\zeta,\tau,\omega)d\zeta d\tau\right| \\
&+\frac{1}{\pi}\left|\int\limits_0^t\int\limits_0^\pi\tilde{\theta}(\tau)(t-\tau)\left[f(\zeta,\tau,\tilde{\omega})-f(\zeta,\tau,\omega)\right]d\zeta d\tau\right| \\
&+\sum_{k=1}^\infty\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int\limits_0^t\int\limits_0^\pi\left[\tilde{\theta}(\tau)-\theta(\tau)\right]f(\zeta,\tau,\tilde{\omega})\cos 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right| \\
&+\sum_{k=1}^\infty\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int\limits_0^t\int\limits_0^\pi\left[\tilde{\theta}(\tau)-\theta(\tau)\right]f(\zeta,\tau,\tilde{\omega})\sin 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right| \\
&+\sum_{k=1}^\infty\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int\limits_0^t\int\limits_0^\pi\tilde{\theta}(\tau)\left[f(\zeta,\tau,\tilde{\omega})-f(\zeta,\tau,\omega)\right]\cos 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right| \\
&+\sum_{k=1}^\infty\frac{1}{\lambda_k}\frac{2}{\pi}\left|\int\limits_0^t\int\limits_0^\pi\tilde{\theta}(\tau)\left[f(\zeta,\tau,\tilde{\omega})-f(\zeta,\tau,\omega)\right]\sin 2k\zeta\sin\lambda_k(t-\tau)d\zeta d\tau\right|.
\end{aligned}$$

By applying Cauchy, Bessel, Hölder inequalities, and Lipshitz condition to the last inequality, we have

$$\begin{aligned}
&|\tilde{\omega}(t,\varepsilon)-\omega(t)|\le\sigma(\varepsilon) \qquad (16)\\
&+2\sqrt{\frac{t^3}{3\pi}}\left\{\left(\int\limits_0^t\int\limits_0^\pi\left\{\left|\tilde{\theta}(\tau)-\theta(\tau)\right|f(\xi,\tau,\omega)\right\}^2d\xi d\tau\right)^{\frac{1}{2}}+\left(\int\limits_0^t\int\limits_0^\pi\left\{\tilde{\theta}(\tau)b(\xi,\tau)\left|\tilde{\omega}-\omega\right|\right\}^2d\xi d\tau\right)^{\frac{1}{2}}\right\}\\
&+2\sqrt{\frac{\pi t}{6}}\left\{\left(\int\limits_0^t\int\limits_0^\pi\left\{\left|\tilde{\theta}(\tau)-\theta(\tau)\right|f(\xi,\tau,\tilde{\omega})\right\}^2d\xi d\tau\right)^{\frac{1}{2}}+\left(\int\limits_0^t\int\limits_0^\pi\left\{\tilde{\theta}(\tau)b(\xi,\tau)\left|\tilde{\omega}-\omega\right|\right\}^2d\xi d\tau\right)^{\frac{1}{2}}\right\}.
\end{aligned}$$

Then we use the result of the difference of the inverse coefficients (14) in (16), we have

$$|\tilde{\omega}(t,\varepsilon)-\omega(t)| \leq \left(1+\frac{CM}{B}\right)\sigma(\varepsilon)+\left(\frac{2CM}{BM_*}\sqrt{\frac{t}{6\pi}}+C\right)\left(\int_0^t\int_0^\pi \left\{\tilde{\theta}(\tau)b(\zeta,\tau)\left|\tilde{\omega}-\omega\right|\right\}^2 d\zeta d\tau\right)^{\frac{1}{2}},$$

$$C=\left(2\sqrt{\frac{t^3}{3\pi}}+2\sqrt{\frac{\pi t}{6}}\right).$$

If we take into account the inequality $(y+z)^2 \leq 2y^2+2z^2$, then

$$|\tilde{\omega}(t,\varepsilon)-\omega(t)|^2 \leq 2\left(1+\frac{CM}{B}\right)^2\sigma^2(\varepsilon) +2\left(\frac{2CM}{BM_*}\sqrt{\frac{t}{6\pi}}+C\right)^2\left(\int_0^t\int_0^\pi \left\{\tilde{\theta}(\tau)b(\zeta,\tau)\left|\tilde{\omega}-\omega\right|\right\}^2 d\zeta d\tau\right).$$

Finally, applying Gronwall inequality to the last inequality, we have

$$|\tilde{\omega}(t,\varepsilon)-\omega(t)|^2 \leq 2\left(1+\frac{CM}{B}\right)^2\sigma^2(\varepsilon) \times \exp\left\{2\left(\frac{2CM}{BM_*}\sqrt{\frac{t}{6\pi}}+C\right)^2\left(\int_0^t\int_0^\pi \left\{\tilde{\theta}(\tau)b(\zeta,\tau)\right\}^2 d\zeta d\tau\right)\right\}. \tag{17}$$

Thus, the right-hand side of (17) converges to zero as ε approaches to zero. That is,

$$\lim_{\varepsilon\to 0}\tilde{\omega}(t,\varepsilon)=\omega(t).$$

In a previous study, we looked at solutions to the problems (1)-(4) and (5)-(8) in cases $\varepsilon > 0$ and $\varepsilon = 0$, respectively. This paper investigated the convergence of the solution (9) to the solution (11) as $\varepsilon \to 0$ under the theorem conditions. The solution was therefore found to be

$$\lim_{\varepsilon\to 0}\tilde{\omega}(x,t,\varepsilon)=\omega(x,t).$$

□

4. Numerical Method

The Finite Difference Method is commonly used for solving the wave equation due to its simplicity and efficiency. It approximates derivatives using straightforward difference formulas, which is ideal for handling the second-order partial derivatives in the wave equation. Additionally, The Finite Difference Method is computationally efficient, especially for problems on structured grids, making them less resource-intensive than more complex methods like Finite Element or Spectral Methods.

We devise an iterative algorithm aimed at the linearizing the problem.

$$\frac{\partial^2 \omega^{(n)}}{\partial t^2} = \varepsilon \frac{\partial^4 \omega^{(n)}}{\partial x^2 \partial t^2} + \frac{\partial^2 \omega^{(n)}}{\partial x^2} + \theta(t) f\left(x, t, \omega^{(n-1)}\right), \tag{18}$$

$$\begin{aligned} \omega^{(n)}(x,0) &= \chi(x),\ x \in [0,\pi], \\ \omega_t^{(n)}(x,0) &= \phi(x),\ x \in [0,\pi], \end{aligned} \tag{19}$$

$$\begin{aligned} \omega^{(n)}(0,t) &= \omega^{(n)}(\pi,t),\quad t \in [0,T], \\ \omega_x^{(n)}(0,t) &= \omega_x^{(n)}(\pi,t),\ t \in [0,T]. \end{aligned} \tag{20}$$

By setting $\omega^{(n)}(x,t) = v(x,t)$ and $f\left(x,t,\omega^{(n-1)}\right) = \tilde{f}(x,t)$, we can express the problem Eqs. (18)-(20) as a linear problem.

$$\frac{\partial^2 v}{\partial t^2} = \varepsilon \frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\partial^2 v}{\partial x^2} + \theta(t) \tilde{f}(x,t),\ (x,t) \in D. \tag{21}$$

After the linearization method, implicit finite difference scheme is applied to solve the problem numerically. In Eq. (22), a second-order accurate backward finite difference scheme was used for temporal discretization. For the term with epsilon and last term in the same equation, a fourth-order accurate central differencing scheme was employed.

$$\begin{aligned} &\frac{1}{\Delta t^2}\left(2v_i^{j+3} - 5v_i^{j+2} + 4v_i^{j+1} - v_i^j\right) \\ &\quad = \frac{\varepsilon}{16\Delta x^2 \Delta t^2}\left[\left(v_{i+2}^{j+3} - 2v_i^{j+3} + v_{i-2}^{j+3}\right) - \left(2v_{i+2}^{j+1} - 4v_i^{j+1} + 2v_{i-2}^{j+1}\right)\right] \\ &\quad + \frac{\varepsilon}{16\Delta x^2 \Delta t^2}\left(v_{i+2}^{j-1} - 2v_i^{j-1} + v_{i-1}^{j-1}\right) \\ &\quad + \frac{1}{12\Delta x^2}\left(v_{i+2}^{j+3} + 16v_{i+1}^{j+3} - 30v_i^{j+3} + 16v_{i-1}^{j+3} - v_{i-2}^{j+3}\right) + s^{j+2}\tilde{f}^{j+2}. \end{aligned} \tag{22}$$

Initial condition is defined as

$$v_i^0 = \varphi_i. \tag{23}$$

Periodic boundary condition is combination of Dirichlet and Neumann boundary conditions, and it is defined as

$$v_1^j = v_{Nx}^j, \tag{24}$$

$$v_{Nx}^j = \frac{v_2^j + v_{Nx-1}^j}{2}. \tag{25}$$

The computational domain spans $[0,\pi]$ in the x-direction and $[0,T]$ in time. It's discretized into intervals such that $x_i = i\,(\Delta x - 1)$ for $i = 1,2,....,Nx$ in space, and $t_j = j\Delta t$ for $j = 1,2,....,Nt$ in time. Here Δx represents the spatial increment, calculated as π/Nx and Δt represents the time step, calculated as T/Nt Nx and Nt are two positive integers. The values v,φ and f are discretized as $v_i^j = v\,(x_i,t_j)\,,\varphi_i = \varphi\,(x_i)$ and $\tilde{f}^{j+2} = f\,(x_i,t_{j+2})$, respectively. The initial time $t=0$ denotes the initial condition. In our numerical computation $j+3$ represents the present time, $j+2$ denotes the time just before the present, $j+1$ represents the two steps before the present, and j three steps before the present.

To determine the inverse coefficient $\theta\,(t)$, we integrate Eq. (1) over the range from 0 to φ with respect to x, while incorporating Eq. (3) and Eq. (4), leading to

$$\theta\,(t) = \frac{E''\,(t) - \varepsilon\,[\pi v_{xtt}\,(\pi,t) - v_{tt}\,(\pi) + v_{tt}\,(0)] - \pi v_x\,(\pi,t)}{\int\limits_0^\pi x\tilde{f}\,(x,t)\,dx}. \tag{26}$$

The individual discretization of the elements constituting Eq. (26) using finite differences one by

$$E''\,(t) = \left[\left(2E^{j+2} - 5E^{j+1} + 4E^{j} - E^{j-1}\right)/\Delta t^2\,\right], \tag{27}$$

$$v_{tt}\,(\pi) = \left(\left(2v_{Nx}^{j+2} - 5v_{Nx}^{j+1} + 4v_{Nx}^{j} - v_{Nx}^{j}\right)/\Delta t^2\,\right), \tag{28}$$

$$v_{tt}\,(0) = \left(\left(2v_{1}^{j+2} - 5v_{1}^{j+1} + 4v_{1}^{j} - v_{1}^{j}\right)/\Delta t^2\,\right), \tag{29}$$

$$\pi v_x\,(\pi,t) = \pi\left(3v_{Nx}^{j+2} - 4v_{Nx-1}^{j+2} + 4v_{Nx-2}^{j+2}\right)/2\Delta x\,, \tag{30}$$

$$\pi v_{xtt}\,(\pi,t) = \pi\left(\left(\left(v_{i+1}^{j+2} - 2v_{i+1}^{j+1} + v_{i+1}^{j}\right) - \left(v_{i}^{j+2} - 2v_{i}^{j+1} + v_{i}^{j}\right)\right)/\Delta x\Delta t^2\,\right). \tag{31}$$

Second-order accurate backward finite difference schemes have been used for Eqs. (27)-(30). The mixed derivative used in Eq. (31) is discretized using a first-order accurate backward finite difference method.

$$\left(\tilde{f}in\right)^{j+2} = \int\limits_0^\pi x\tilde{f}\,(x,t)\,dx. \tag{32}$$

Trapezoidal rule for integration is employed to compute Eq. (30). The value of Nx utilized for numerical solutions differs from the value of Nin used for the trapezoidal rule integration.

When computing the inverse coefficient during the initial time steps, we utilize the initial value of v , yet we refrain from presenting the detailed discretization here to avoid excessive elaboration.

For the numerical solution of Eq. (22), no iterative methods were employed, a direct method was used instead. The right-hand side matrix constitutes from previous values, and it is used in direct method. The right-hand side matrix

$$rhs_i = -5u_i^{j+2} + 4u_i^{j+1} - u_i^j + \frac{\varepsilon}{8\Delta x^2}\left(u_{i+2}^{j+1} - 2u_i^{j+1} + u_{i-2}^{j+1}\right) - \frac{\varepsilon}{16\Delta x^2}\left(u_{i+2}^{j-1} - 2u_i^{j-1} - u_{i-2}^{j-1}\right) - s^{j+2}\widetilde{f}^{j+2}\Delta t^2. \quad (33)$$

5. Numerical Example

Considering inverse problem

$$f(x,t,\omega) = \varepsilon^2 + \left(4 + 4\varepsilon^3 + \varepsilon^2\right)\sin(2x),$$

$$\varphi(x) = 1 + \sin 2x, E(t) = \frac{\left(\pi^2 - \pi\right)}{2}e^{\varepsilon t}, x \in [0,\pi], t \in [0,T].$$

In that case, the problem transforms as

$$\omega_{tt} - \varepsilon\omega_{xxtt} - \omega_{xx} = \theta(t)\sin(2x)\left[\varepsilon^2 + \left(4 + 4\varepsilon^3 + \varepsilon^2\right)\sin(2x)\right]$$

$$\omega(x,0) = 1 + \sin 2x,\ x \in [0,\pi],$$

$$\omega(0,t) = \omega(\pi,t),\ \omega_x(0,t) = \omega_x(\pi,t),\ 0 \le t \le T,$$

$$\int_0^\pi x\omega(x,t)dx = \frac{\left(\pi^2 - \pi\right)}{2}e^{\varepsilon t}.$$

The analytical solution of this problem can be defined as

$$\{\theta(t), \omega(x,t)\} = \left\{e^{\varepsilon t}, (1 + \sin(2x))\, e^{\varepsilon t}\right\}.$$

5.1. Grid Independence Study, Time Step Size Determination and Validation.

Since the variation of ω over time becomes more significant for the $\varepsilon = 2$, grid independence, time step size determination, and validation studies were conducted for the $\varepsilon = 2$ case. For the grid independence study, seven different grid densities are used, these are 20, 40, 80, 160, 320, 640 and 1280. The grid independence study is repeated for five different time steps. The time steps used are in descending order: 0.01s, 0.005s, 0.0025s, 0.00125s, and 0.000625s. The grid independence studies for each time step are illustrated in Figure 1. The ω values shown in grid independence study are the maximum ω values at 1sn. The results estimated with 640 grids for all time steps are very close to those estimated with 1280 grids. Therefore, the grid number of 640 is determined as the grid independent mesh.

For the ε value of 2, the determination of the time step size for the grid independent mesh number of 640 grid is shown in Figure 2. Similarly, the ω values shown in the time step size determination study are the maximum ω values at 1s. It is observed that the omega value increases linearly, as the time step size decreases. However, it can be seen that the ω prediction for the time step sizes of 0.00125s and 0.000625s are close the each other. Therefore, the appropriate time step is determined to be 0.00125s. The result in the subsequent validation study is based on the numerical solution with 640 grid numbers and a time step size of 0.00125s.

As previously mentioned, numerical solutions are obtained by selecting 640 grid number and a time step size of 0.00125s based on the grid independence and time step size determination study. The obtained numerical solutions are compared and validated against the exact solutions. The validation study is conducted for the ε value of 2. The validation of the inverse coefficient is shown in Figure 3. In Figure 3(a), the time-dependent variation of the inverse coefficient is given as both numerical and exact solutions. Due to the exponential nature with time, the inverse coefficient increases, and the real solutions closely match the numerical solutions.

In Figure 3(b), the time-dependent variation of the real errors is observed. The real errors increase with time, although these real errors are very small. Finally, to better compare the real solution with the numerical solution, the absolute relative true error is given as a function of time in Figure 3(c). The absolute relative true errors exhibit oscillations over time, but these oscillations are on a very small scale. Overall, the average absolute relative real error is at the level of 0.192%, indicating the numerical solution for the inverse coefficient.

The validation of the omega value for $\varepsilon = 2$ is shown in Figure 4. In Figure 4(a), the numerical prediction of ω is depicted, in Figure 4(b), the values of ω obtained from the analytical solution are shown, and in Figure 4(c), the true error between these two solutions is presented as a function of time. Upon inspection of Figures 4(a) and (b), it can be observed that there is little difference between the numerical solution and the analytical solution. To better compare the two cases, the true error between the two solutions is examined, revealing that the error is minimal at the initial times and increases with time, particularly in boundary regions. However, despite this increase, the resulting real error is at the level of 0.04. This indicates that the numerical solution has been validated.

5.2. **Numerical Predictions.** After the grid independence, timed step size determination and validation studies, it has been decided to use 640 grids and time step size of 0.00125s in subsequent numerical computations. Now, numerical solutions have been computed and compared at specific interval of 0.5 ranging from $\varepsilon = 0$ to $\varepsilon = 3$.

In Figure 5, the numerical prediction of the inverse coefficients for all epsilon values is shown. The inverse coefficient is an exponential function, becoming more prominent as epsilon increases. While at zero seconds, the exponential function-based inverse coefficient takes a constant value of unity for all epsilons, its value increases as time progresses.

Figure 6 depicts the variations of ω values for all ε values considered at (a)0.5s and (b)1s. These ω values are obtained from numerical predictions. The general trend of omega values increases for all ε values from the beginning of the domain to a length of 0.79 and then decreases to approximately 2.36 length until reaching zero, after which it tends to increase again until the end of the domain. While

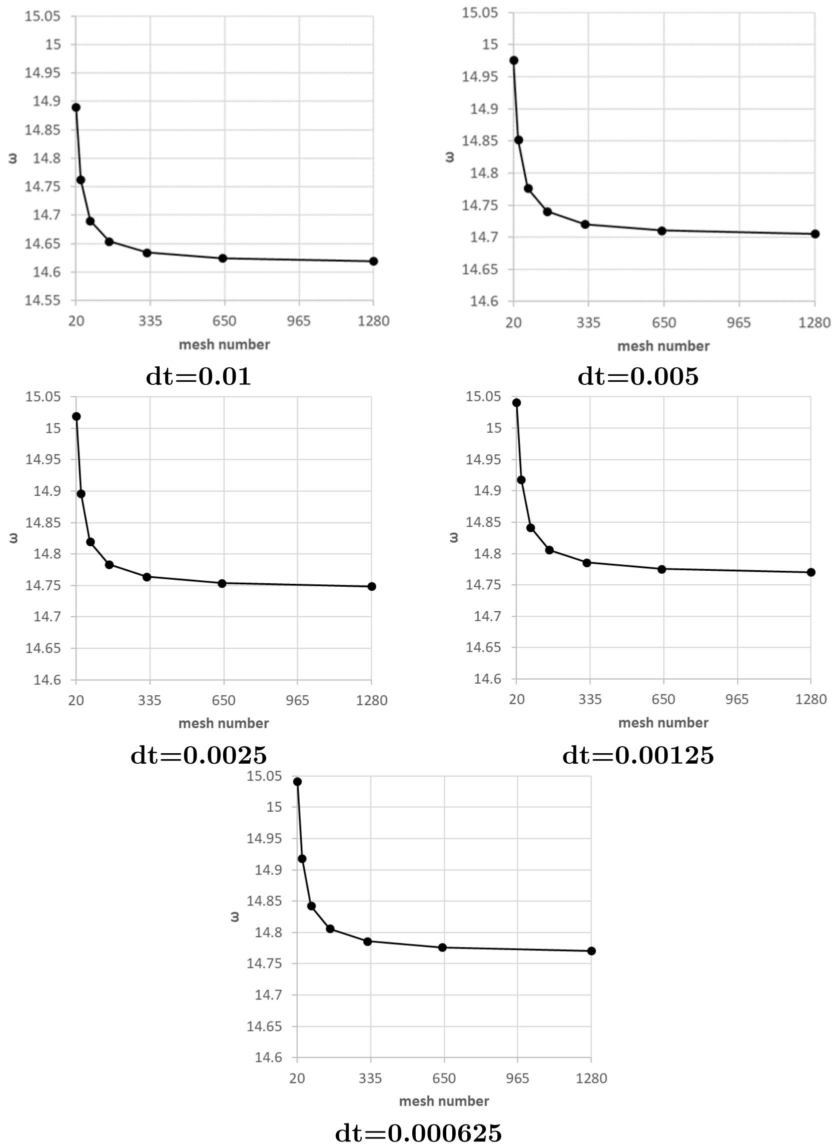

FIGURE 1. Grid independence study for $\varepsilon = 2$

the ω value obtained at $\varepsilon = 0$ is symmetric, symmetry is disrupted as ε increases. Moreover, ω values increase with both ε and time. At $t = 0.5s$, the maximum ω value for $\varepsilon = 3$ is around 9, whereas at $t = 1s$, the maximum ω value for $\varepsilon = 3$ is

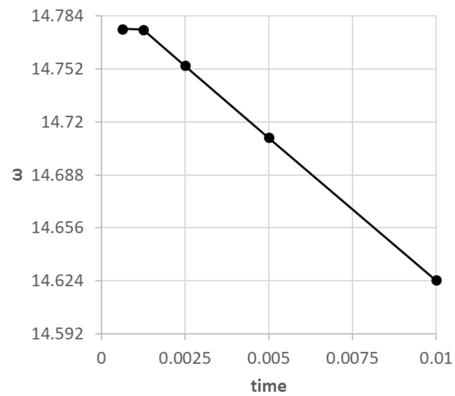

FIGURE 2. Time step size determination for $\varepsilon = 2$

approximately 40. Additionally, at $t = 1s$, when $\varepsilon = 2.5$, the maximum ω value is around 24, while it reaches approximately 41 when $\varepsilon = 3$, as mentioned earlier.

Figure 7 presents three-dimensional graphs showing the variation of ω values predicted from numerical solutions with respect to both length and time for (a) $\varepsilon = 0$, (b) $\varepsilon = 0.5$, (c) $\varepsilon = 1$, (d) $\varepsilon = 1.5$, (e) $\varepsilon = 2$, (f) $\varepsilon = 2.5$, and (g) $\varepsilon = 3$. Figure 7 transforms the lines obtained from only two times (0.5s and 1s) mentioned in Figure 6 into area plots showing all times. To ensure better comparison across all values, all graphs are drawn on the same scale. All interpretations made in Figure 6 can also be applied to Figure 7.

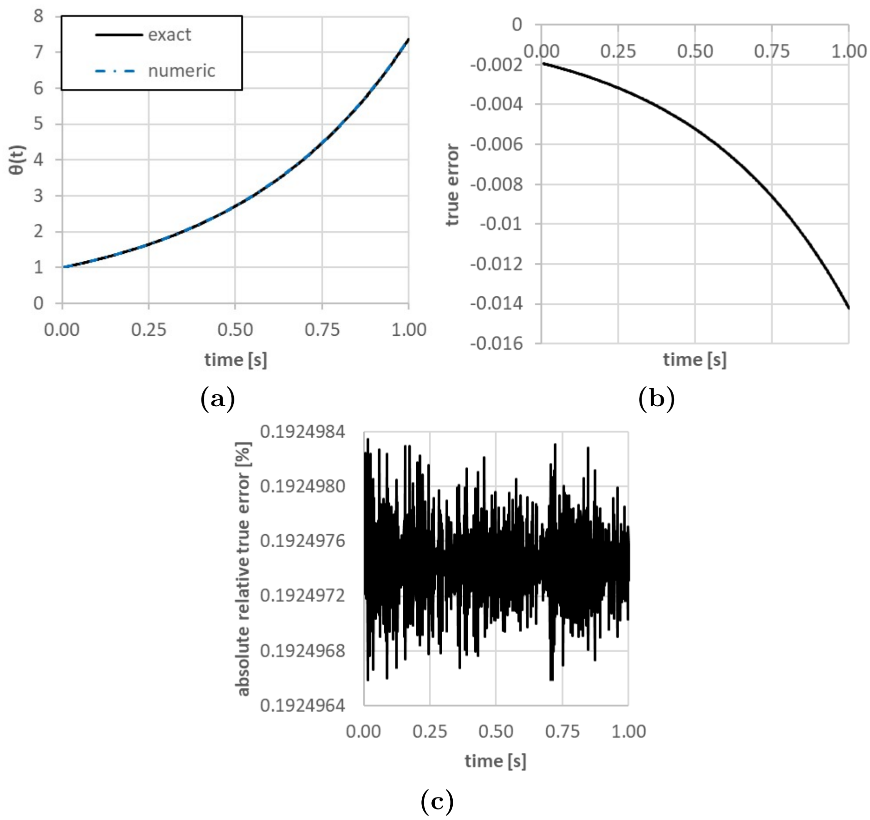

FIGURE 3. Validation of inverse coefficient for $\varepsilon = 2$

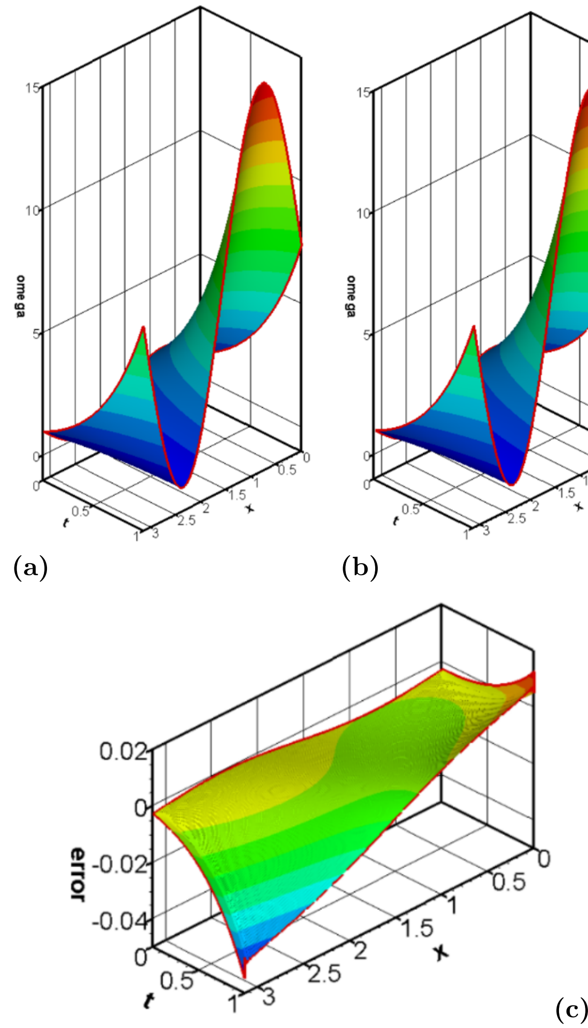

FIGURE 4. Validation of ω for $\varepsilon = 2$

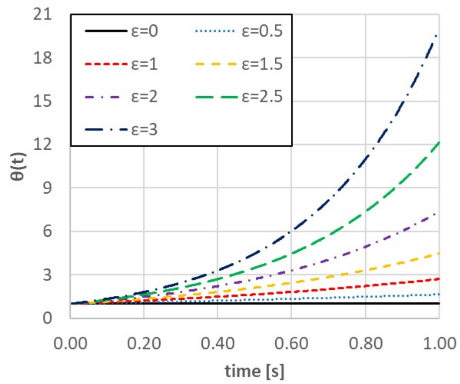

FIGURE 5. Numerical predictions of inverse coefficient for all ε

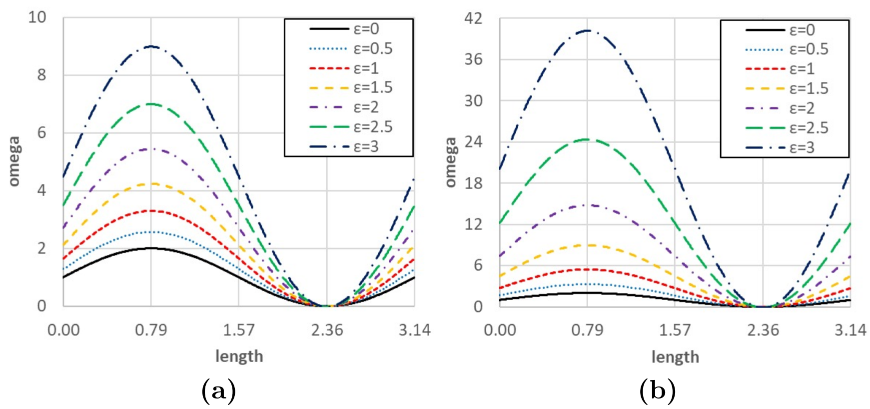

FIGURE 6. Numerical predictions of ω for all ε at(a) $t = 0.5s$ and (b)$t = 1s$

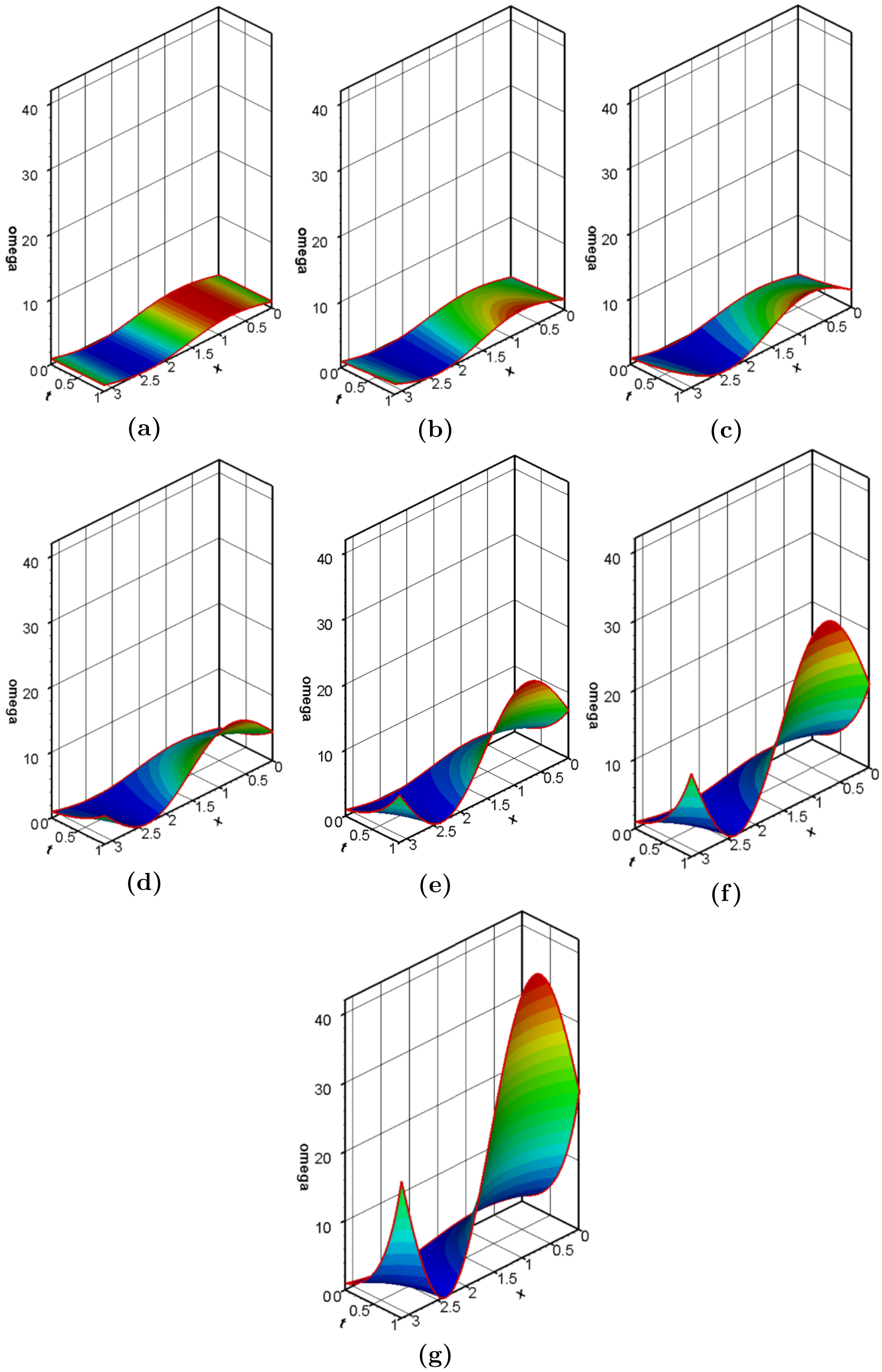

FIGURE 7. Numerical predictions of ω at all times and for (a)$\varepsilon = 0$, (b)$\varepsilon = 0.5$, (c)$\varepsilon = 1$, (d)$\varepsilon = 1.5$, (e)$\varepsilon = 2$, (f)$\varepsilon = 2.5$ and (g)$\varepsilon = 3$

6. Conclusions

Analytical and numerical investigation of one-dimensional nonlinear hyperbolic ($\varepsilon = 0$) and pseudo-hyperbolic ($\varepsilon \neq 0$) equation with periodic condition is done. This investigation contains an inverse problem of unknown unsteady coefficients. For analytical solution, the generalized Fourier method is utilized to calculate Fourier coefficients. Additionally, an iterative approach is employed to ensure convergence while assessing the uniqueness and stability of the solution for the nonlinear problem. For numerical solution, implicit finite difference equation with higher accurate schemes is applied. A second-order accurate time discretization is applied, and for the discretization of spatial and multi-variable partial differential equations, fourth-order accurate finite difference equations are implemented. The cases where ($\varepsilon = 0$) and $\varepsilon \neq 0$ (different epsilon values) have been solved analytically and numerically, and compared with each other. The main conclusions are listed below;

- In light of the grid independence and time step size determination study, 640 mesh number and 0.00125s time step size are determined. Using this mesh number and time step size, the numerical computation for the $\varepsilon = 2$ is validated against analytical results for both ω and inverse coefficient.
- In the case of $\varepsilon = 0$, the inverse coefficient does not vary with time ($\theta(t) = 1$), however, as ε and time increases, the inverse coefficient increases due to its exponential nature.
- The distribution of ω over length is symmetric at a certain time in the case of hyperbolic equation ($\varepsilon = 0$), but in the case of pseudo-hyperbolic equation ($\varepsilon \neq 0$) the distribution of ω over length is asymmetric.
- Due to periodic boundary conditions, the ω values at the boundaries of the solution domain are identical to each other, and as the ε value increases, the ω values at the boundary points also increase.
- As the time and ε value increase, the magnitude of ω oscillations increase at especially at the beginning of the solution domain.

Author Contribution Statements All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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ON THE POLAR DERIVATIVE OF LACUNARY TYPE POLYNOMIALS

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Abstract. Let $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$, where $1 \le \nu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k \le 1$. For polar derivative $D_\alpha p(z)$, it is known that for each $|\alpha| \le 1$ on $|z| = 1$,

$$|D_\alpha p(z)| \le \frac{n}{1+k^\nu}\Big\{(|\alpha| + k^\nu)\|p\|_\infty - \frac{1-|\alpha|}{k^{n-\nu}} \min_{|z|=k} |p(z)|\Big\}.$$

In this paper, we obtain the L_q mean extension and a refinement of the above and other related results for the polar derivative of polynomials.

1. Introduction

Let $\mathcal{P}_n$ be the set of polynomials of degree n with complex coefficients. If $p \in \mathcal{P}_n$, denote by

$$\|p\|_q := \Big\{\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^q d\theta\Big\}^{1/q}, \quad 0 < q < \infty,$$
$$\|p\|_\infty := \max_{|z|=1} |p(z)|.$$

For $p \in \mathcal{P}_n$, Bernstein [1], proved that

$$\|p'\|_\infty \le n\|p\|_\infty. \tag{1}$$

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In the case $q \geq 1$ the following inequality proved by Zygmund [2] and in the case $0 < q < 1$, it is due to Arestov [3],

$$\|p'\|_q \leq n\|p\|_q, \quad 0 < q < \infty. \tag{2}$$

Erdös conjectured and later Lax [4] proved that if $p(z)$ having no zeros in $|z| < 1$, then

$$\|p'\|_\infty \leq \frac{n}{2}\|p\|_\infty. \tag{3}$$

In the case that the polynomial has all its zeros in $|z| \leq 1$, Turán [5] proved that

$$\|p'\|_\infty \geq \frac{n}{2}\|p\|_\infty. \tag{4}$$

As a generalization of inequality (3), it is proved that

$$\|p'\|_q \leq \frac{n}{\|1+z\|_q}\|p\|_q \text{ , for } q > 0. \tag{5}$$

In the case $q \geq 1$ inequality (5) is proved by De-Brujin [6] and for the case $0 < q < 1$, it is due to Rahman and Schmeisser [7].
Malik [8] extended (3) and proved that if $p(z)$ does not any zeros in $|z| < k$, where $k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k}\|p\|_\infty, \tag{6}$$

whereas if $p(z)$ has all its zeros in $|z| \leq k \leq 1$, then

$$\|p'\|_\infty \geq \frac{n}{1+k}\|p\|_\infty. \tag{7}$$

It is proved by Govil and Rahman [9] that if $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then

$$\|p'\|_q \leq \frac{n}{\|k+z\|_q}\|p\|_q \text{ , for } q > 0. \tag{8}$$

The above inequalities were generalized for two class of polynomials. First class is lacunary type polynomials $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$, where $1 \leq \nu \leq n$, and second class is polynomials of the form $p(z) = a_n z^n + \sum_{j=\nu}^{n} a_{n-j} z^{n-j}$, where $1 \leq \nu \leq n$.
As a generalization of inequality (6), it was shown by Qazi [10] that if $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ and $p(z) \neq 0$ in $|z| < k, k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k^\nu}\|p\|_\infty, \tag{9}$$

Also, inequality (9) was extended by Gardner and Weems [11], they proved if $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ and $p(z) \neq 0$ in $|z| < k, k \geq 1$, then

$$\|p'\|_q \leq \frac{n}{\|k^\nu+z\|_q}\|p\|_q \text{ , for } q > 0. \tag{10}$$

On the other hand, for the class of polynomials of type $p(z) = a_n z^n + \sum_{j=\nu}^{n} a_{n-j} z^{n-j}$, where $1 \le \nu \le n$, which having all zeros in $|z| \le k \le 1$ it was proved by Aziz and Shah [12] that

$$\|p'\|_\infty \ge \frac{n}{1+k^\nu}\Big\{\|p\|_\infty + \frac{1}{k^{n-\nu}} \min_{|z|=k} |p(z)|\Big\}. \tag{11}$$

For a polynomial $p(z)$ of degree n, we define the so-called the polar derivative of $p(z)$ with respect to the point α as

$$D_\alpha p(z) := np(z) + (\alpha - z)p'(z).$$

The polar derivative $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and it is extension of the derivative $p'(z)$ by the following sense

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z)$$

Aziz and Rather [13] extended inequality (5) to the polar derivative of a polynomial and proved that if $p \in \mathcal{P}_n$ and $p(z)$ does not vanish in $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \ge 1$, and $p \ge 1$,

$$\|D_\alpha p\|_q \le n\frac{|\alpha|+1}{\|1+z\|_q}\|p\|_q \text{ , for } q \ge 1. \tag{12}$$

Inequality (10) is also generalized by Rather et al. [14] to the polar derivative of lacunary type polynomial, and specifically proved that if $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$, where $1 \le \nu \le n$ be a polynomial of degree n and $p(z) \ne 0$ for $|z| < k$ where $k \ge 1$, then

$$\|D_\alpha p\|_q \le n\frac{|\alpha|+k^\nu}{\|k^\nu + z\|_q}\|p\|_q \text{ , for } |\alpha| \ge 1 \textit{ and } q > 0. \tag{13}$$

Recently Dewan et al. [15] proved that if $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$, $1 \le \nu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k \le 1$, then for every complex number α with $|\alpha| \le 1$, on $|z| = 1$

$$|D_\alpha p(z)| \le \frac{n}{1+k^\nu}\Big\{(|\alpha|+k^\nu)\|p\|_\infty - \frac{1-|\alpha|}{k^{n-\nu}} \min_{|z|=k} |p(z)|\Big\}. \tag{14}$$

In the first theorem we obtain the L_q mean extension and a refinement of the above inequality (14), then by using of this theorem we prove the L_q mean extension for lacunary type polynomials, which proposes a generalization and refinement of inequalities (13) as well.

Theorem 1. *Let* $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$, *be a polynomial of degree* n, *has all its zeros in* $|z| \le k \le 1$, *then for every complex number* α *with* $|\alpha| \le 1$, $q > 0$, $\theta \in \mathbb{R}$ *and* $0 \le t \le 1$ *we have*

$$\Big\| |D_\alpha p(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}(1-|\alpha|)}{k^n(1+\Lambda_{\nu,t})} \Big\|_q \le \frac{n(|\alpha|+\Lambda_{\nu,t})}{\|z+\Lambda_{\nu,t}\|_q}\|p\|_q, \tag{15}$$

where $\Lambda_{\nu,t} = \frac{n(|a_n|-\frac{tm}{k^n})k^{2\nu}+\nu|a_{n-\nu}|k^{\nu-1}}{\nu|a_{n-\nu}|+n(|a_n|-\frac{tm}{k^n})k^{\nu-1}}$, *and* $m = \min\limits_{|z|=k} |p(z)|$.

Let $q \to \infty$ and choosing $t = 1$ then inequality (15) reduce to a following result.

Corollary 1. *If* $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$ *is a polynomial of degree* n*, has all its zeros in* $|z| \le k < 1$*, then for every complex number* α *with* $|\alpha| \le 1$*,*

$$\|D_\alpha p\|_\infty \le \frac{n(|\alpha| + \Lambda_\nu)}{1 + \Lambda_\nu}\|p\|_\infty - \frac{n\Lambda_\nu(1 - |\alpha|)}{k^n(1 + \Lambda_\nu)} \min_{|z|=k} |p(z)|, \tag{16}$$

where $\Lambda_\nu = \frac{n(|a_n|-\frac{m}{k^n})k^{2\nu}+\nu|a_{n-\nu}|k^{\nu-1}}{\nu|a_{n-\nu}|+n(|a_n|-\frac{m}{k^n})k^{\nu-1}}$, *and* $m = \min\limits_{|z|=k} |p(z)|$.

Remark 1. *Corollary 1 is general and a refinement for inequality (14). To see that, we must show*

$$\frac{n(|\alpha| + \Lambda_\nu)}{1 + \Lambda_\nu}\|p\|_\infty - \frac{n\Lambda_\nu(1 - |\alpha|)}{k^n(1 + \Lambda_\nu)} \min_{|z|=k} |p(z)| <$$
$$\frac{n}{1 + k^\nu}\Big\{(|\alpha| + k^\nu)\|p\|_\infty - \frac{1 - |\alpha|}{k^{n-\nu}} \min_{|z|=k} |p(z)|\Big\}.$$

Equivalently

$$\frac{(1 - |\alpha|)(k^\nu - \Lambda_\nu)}{k^n(1 + k^\nu)(1 + \Lambda_\nu)} \min_{|z|=k} |p(z)| < \frac{(1 - |\alpha|)(k^\nu - \Lambda_\nu)}{(1 + k^\nu)(1 + \Lambda_\nu)}\|p\|_\infty$$

Since $|\alpha| \le 1$ *and from (30), we have* $\Lambda_\nu \le k^\nu$*, the above inequality becomes*

$$\frac{\min_{|z|=k} |p(z)|}{k^n} < \|p\|_\infty \tag{17}$$

the inequality (17) is true by the Lemma 2, so we get the result.

If we take $\alpha = 0$ in Corollary 1, we have

Corollary 2. *If* $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$ *is a polynomial of degree* n*, has all its zeros in* $|z| \le k < 1$*, then for* $\theta \in \mathbb{R}$*, we have*

$$|np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})| \le \frac{n\Lambda_\nu}{1 + \Lambda_\nu}\{\|p\|_\infty - \frac{1}{k^n} \min_{|z|=k} |p(z)|\}. \tag{18}$$

Suppose $e^{i\theta_0}$ is such that $|p(e^{i\theta_0})| = \|p\|_\infty$, then by using the inequality $n\|p\|_\infty - |e^{i\theta_0}p'(e^{i\theta_0})| = |np(e^{i\theta_0})| - |e^{i\theta_0}p'(e^{i\theta_0})| \le |np(e^{i\theta_0}) - e^{i\theta_0}p'(e^{i\theta_0})|$ in (18), it becomes to following refinement and generalization of (11).

Corollary 3. *If* $p(z) = a_n z^n + \sum_{l=\nu}^{n} a_{n-l} z^{n-l}$ *is a polynomial of degree* n*, has all its zeros in* $|z| \le k < 1$*, then*

$$\|p'\|_\infty \ge \frac{n}{1 + \Lambda_\nu}\{\|p\|_\infty + \frac{\Lambda_\nu}{k^n} \min_{|z|=k} |p(z)|\}. \tag{19}$$

Remark 2. *Corollary 3 is general and refinement to inequality (11). To see that, we using again the method used in Remark 1, it follows that inequality (19) is better than inequality (11).*

In the second case by using Theorem 1, we can prove the following theorem that provides a refinement and generalization of (13) and related many results.

Theorem 2. *Let $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ be a polynomial of degree n, does not vanish in $|z| < k$, $k \geq 1$, then for every complex number α with $|\alpha| \geq 1$, $q > 0$, $\theta \in \mathbb{R}$ and $0 \leq t \leq 1$, we have*

$$\left\| |D_\alpha p(e^{i\theta})| + \frac{nmt(|\alpha|-1)}{1+A_{\nu,t}} \right\|_q \leq \frac{n(|\alpha| + A_{\nu,t})}{\|z + A_{\nu,t}\|_q} \|p\|_q, \tag{20}$$

where $A_{\nu,t} = \frac{n(|a_0|-tm)k^{\nu+1}+\nu|a_\nu|k^{2\nu}}{\nu|a_\nu|k^{\nu+1}+n(|a_0|-tm)}$, and $m = \min_{|z|=k} |p(z)|$.

Remark 3. *By using inequality (30) from Lemma 2, we have $A_{\nu,t} \geq k^\nu \geq 1$, resulting (20) to be a generalization and refinement of (13).*

Let $q \to \infty$ and by choosing $t = 1$, the inequality (20) reduce to a following result that recently proved by Dewan et al. [15].

Corollary 4. *If $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ be a polynomial of degree n, does not vanish in $|z| < k$, $k \geq 1$, then for every complex number α with $|\alpha| \geq 1$, we have*

$$\|D_\alpha p\|_\infty \leq \frac{n}{1+A_\nu}\{(|\alpha| + A_\nu)\|p\|_\infty - (|\alpha| - 1)\min_{|z|=k}|p(z)|\}, \tag{21}$$

where $A_\nu = \frac{n(|a_0|-m)k^{\nu+1}+\nu|a_\nu|k^{2\nu}}{\nu|a_\nu|k^{\nu+1}+n(|a_0|-m)}$, and $m = \min_{|z|=k} |p(z)|$.

By dividing both sides of (20) by $|\alpha|$ and let $|\alpha| \to \infty$, we have the following result that is an refinement and generalization of (10) .

Corollary 5. *Let $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ be a polynomial of degree n, does not vanish in $|z| < k$, $k \geq 1$, then for $q > 0$, $\theta \in \mathbb{R}$ and $0 \leq t \leq 1$, we have*

$$\left\| |p'(e^{i\theta})| + \frac{nmt}{1+A_{\nu,t}} \right\|_q \leq \frac{n}{\|z + A_{\nu,t}\|_q} \|p\|_q. \tag{22}$$

By dividing both sides of (21) by $|\alpha|$ and let $|\alpha| \to \infty$, we have

Corollary 6. *If $p(z) = a_0 + \sum_{j=\nu}^{n} a_j z^j$ be a polynomial of degree n, does not vanish in $|z| < k$, $k \geq 1$, then*

$$\|p'\|_\infty \leq \frac{n}{1+A_\nu}\{(\|p\|_\infty - \min_{|z|=k}|p(z)|\}. \tag{23}$$

Remark 4. *Inequality (23) has been studied by Gardner et al. [16].*

2. Lemmas

The following lemmas are needed for proof of the theorems. The first lemma is due to Aziz et al. [17].

Lemma 1. *Let $p(z) \in \mathcal{P}_n$ and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for each γ, $0 \leq \gamma < 2\pi$, and $q > 0$,*

$$\int_0^{2\pi}\int_0^{2\pi} |q'(e^{i\theta}) + e^{i\gamma}p'(e^{i\theta})|^q d\theta d\gamma \leq 2\pi n^q \int_0^{2\pi} |p(e^{i\theta})|^q d\theta.$$

Lemma 2. *If $p(z) = \sum_{i=0}^n a_i z^i$ is a polynomial of degree n, having all its zeros in $|z| \leq k \leq 1$, then*

$$\min_{|z|=k} |p(z)| < k^n \max_{|z|=1} |p(z)|, \tag{24}$$

and in particular $\min_{|z|=k} |p(z)| < k^n|a_n|$.

The above lemma is due to Zireh [18].

Lemma 3. *The function*

$$S(x) = \frac{nxk^{2\nu} + \nu|a_{n-\nu}|k^{\nu-1}}{nxk^{\nu-1} + \nu|a_{n-\nu}|}$$

for $k \leq 1$ is a non-increasing function of x.

Proof. The proof follows by considering the first derivative test for $S(x)$. □

The following lemma is due to Aziz and Rather [13].

Lemma 4. *If $p(z) = a_n z^n + \sum_{l=\nu}^n a_{n-l}z^{n-l}$, has all its zeros in $|z| \leq k \leq 1$, and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then on $|z| = 1$,*

$$|q'(z)| \leq L_\nu |p'(z)|, \tag{25}$$

where

$$L_\nu = \frac{n|a_n|k^{2\nu} + \nu|a_{n-\nu}|k^{\nu-1}}{\nu|a_{n-\nu}| + n|a_n|k^{\nu-1}}, \tag{26}$$

and

$$\frac{\nu}{n}\left|\frac{a_{n-\nu}}{a_n}\right| \leq k^\nu. \tag{27}$$

Lemma 5. *If $p(z) = a_n z^n + \sum_{l=\nu}^n a_{n-l}z^{n-l}$, has all its zeros in $|z| \leq k \leq 1$, and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for $0 \leq t \leq 1$ and $|z| = 1$, we have*

$$|q'(z)| \leq \Lambda_{\nu,t}|p'(z)| - \frac{nmt\Lambda_{\nu,t}}{k^n}, \tag{28}$$

where

$$\Lambda_{\nu,t} = \frac{n(\mid a_n \mid - \frac{tm}{k^n})k^{2\nu} + \nu \mid a_{n-\nu} \mid k^{\nu-1}}{\nu \mid a_{n-\nu} \mid + n(\mid a_n \mid - \frac{tm}{k^n})k^{\nu-1}} \tag{29}$$

and

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{|a_n| - \frac{tm}{k^n}} \leq k^\nu. \tag{30}$$

where $m = \min\limits_{|z|=k} |p(z)|$.

Proof. Let $m = \min\limits_{|z|=k} |p(z)|$. If $m = 0$, then inequality (28) reduce to inequality (25) in Lemma 4, which is trivial. Therefore, we suppose that the polynomial $p(z)$ having all its zeros in $|z| < k$, hence for every $\beta \in \mathbb{C}$ with $|\beta| < 1$, we have $|\frac{\beta m z^n}{k^n}| < |p(z)|$ for $|z| = k$. Now the Rouche's theorem implies that the polynomial $p(z) - \frac{\beta m z^n}{k^n}$, has all its zeros in $|z| < k < 1$. By applying Lemma 4 to the polynomial $p(z) - \frac{\beta m z^n}{k^n}$, for $|z| = 1$ we get

$$|q'(z)| \leq S_\nu |p'(z) - \frac{\beta n m z^{n-1}}{k^n}|, \tag{31}$$

where

$$S_\nu = \frac{n(|\ a_n - \frac{\beta m}{k^n}\ |)k^{2\nu} + \nu \mid a_{n-\nu} \mid k^{\nu-1}}{\nu \mid a_{n-\nu} \mid + n(|\ a_n - \frac{\beta m}{k^n}\ |)k^{\nu-1}}.$$

By applying Lemma 2 we get $|\ a_n\ | > \dfrac{m}{k^n}$, then we can substituted $|\ a_n - \frac{\beta\, m}{k^n}\ |$ by $|a_n| - \frac{|\beta| m}{k^n}$, since we have that

$$|\ a_n - \frac{\beta\, m}{k^n}\ | \geq |\ a_n\ | - \frac{|\ \beta\ |\ m}{k^n}. \tag{32}$$

By applying Lemma 3 for (32) and taking $t = |\beta|$, we get

$$S_\nu \leq \Lambda_{\nu,t}. \tag{33}$$

Combining (31) and (33), one can obtain

$$|q'(z)| \leq \Lambda_{\nu,t} |p'(z) - \frac{\beta m n z^{n-1}}{k^n}|. \tag{34}$$

Again since $|\frac{\beta m z^n}{k^n}| < |p(z)|$, by choosing the suitable argument of β, we have

$$|p'(z) - \frac{\beta m n z^{n-1}}{k^n}| = |p'(z)| - |\frac{\beta m n z^{n-1}}{k^n}|, \tag{35}$$

from (34) and (35) we get,

$$|q'(z)| \leq \Lambda_{\nu,t} |p'(z)| - \frac{n m t \Lambda_{\nu,t}}{k^n}.$$

To prove (30), we use (27) for the polynomial $p(z) - \frac{\beta m z^n}{k^n}$, as a result we have

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{|a_n - \frac{\beta m}{k^n}|} \leq k^\nu,$$

or

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{k^{\nu}} \leq |a_n - \frac{\beta m}{k^n}|. \tag{36}$$

This means $\frac{\nu}{n}\frac{|a_{n-\nu}|}{k^{\nu}}$ is lower bound for $|a_n - \frac{\beta m}{k^n}|$ for every β, it implies that $\frac{\nu}{n}\frac{|a_{n-\nu}|}{k^{\nu}}$ is less than $\min\limits_{|\beta|\leq 1}|a_n - \dfrac{\beta m}{k^n}|$, hence from (32) we have

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{k^{\nu}} \leq |a_n| - \frac{|\beta|m}{k^n}.$$

or

$$\frac{\nu}{n}\frac{|a_{n-\nu}|}{|a_n| - \frac{tm}{k^n}} \leq k^{\nu}.$$

□

The next lemma is due to Aziz et. al [19].

Lemma 6. *Let A, B, C are positive real numbers which that $B+C \leq A$, then for any real γ,*

$$|(A-C) + e^{i\gamma}(B+C)| \leq |B + e^{i\gamma}A|. \tag{37}$$

We also need the following lemma is due to Rather et al. [14].

Lemma 7. *If a, b are two non-negative real numbers which that $a \geq bc$ where $c \geq 1$, then for every $x \geq 1, q > 0$ and $0 \leq \gamma < 2\pi$*

$$(a+bx)^q \int_0^{2\pi} |c + e^{i\gamma}|^q d\gamma \leq (c+x)^q \int_0^{2\pi} |a + be^{i\gamma}|^q d\gamma \tag{38}$$

3. Proof of the theorems

Proof of the Theorem 1. By the assumptions, $p(z)$ having all its zeros in $|z| \leq k \leq 1$, therefore by Lemma 5, for $|z| = 1$, we have

$$|q'(z)| \leq \Lambda_{\nu,t}\big(|p'(z)| - \frac{nmt}{k^n}\big).$$

This inequality can be rewritten as

$$|q'(z)| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})} \leq \Lambda_{\nu,t}\{|p'(z)| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}. \tag{39}$$

Taking $A = |p'(z)|,\ B = |q'(z)|$ and $C = \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}$ in Lemma 6, and attend that $\Lambda_{\nu,t} \leq k^{\nu} \leq 1$, by (30), so $B + C \leq A - C \leq A$. Then for any real γ, we get

$$\Big|\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + e^{i\gamma}\{|q'(e^{i\theta})| + \{\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big| \leq \Big||q'(e^{i\theta})| + e^{i\gamma}|p'(e^{i\theta})|\Big|. \tag{40}$$

This implies for each $q > 0$, that

$$\begin{aligned}\int_0^{2\pi} \Big|\{| \, p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}+e^{i\gamma}\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\theta \\ \leq \int_0^{2\pi} \Big||q'(e^{i\theta})| + e^{i\gamma}|p'(e^{i\theta})|\Big|^q d\theta,\end{aligned} \tag{41}$$

From every side of (41), we integrate with respect to γ from 0 to 2π, which gives

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} \Big|\{| \, p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + e^{i\gamma}\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\theta d\gamma \\ \leq \int_0^{2\pi}\int_0^{2\pi} \Big|e^{i\gamma}|p'(e^{i\theta})| + |q'(e^{i\theta})|\Big|^q d\theta d\gamma \\ = \int_0^{2\pi}\Big\{\int_0^{2\pi} \Big|e^{i\gamma}|p'(e^{i\theta})| + |q'(e^{i\theta})|\Big|^q d\gamma\Big\} d\theta \\ = \int_0^{2\pi}\Big\{\int_0^{2\pi} \Big|e^{i\gamma}p'(e^{i\theta}) + q'(e^{i\theta})\Big|^q d\gamma\Big\} d\theta \\ = \int_0^{2\pi}\Big\{\int_0^{2\pi} \Big|e^{i\gamma}p'(e^{i\theta}) + q'(e^{i\theta})\Big|^q d\theta\Big\} d\gamma.\end{aligned}$$

From the Lemma 1 and above result, we conclude that

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} \Big|\{| \, p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + e^{i\gamma}\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\theta d\gamma \\ \leq 2\pi n^q \int_0^{2\pi} |p(e^{i\theta})|^q d\theta.\end{aligned} \tag{42}$$

For $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and using the fact that

$$|np(z) - zp'(z)| = |q'(z)| \text{ for, } |z| = 1,$$

we have

$$\begin{aligned}|D_\alpha p(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})} &= |np(e^{i\theta}) + (\alpha - e^{i\theta})p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})} \\ &\leq |np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})| + |\alpha||p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})} \\ &= |q'(e^{i\theta})| + |\alpha||p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})} \\ &= \Big\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\Big\} + |\alpha|\Big\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\Big\}\end{aligned}$$

By integrating both sides of above inequality with respect to θ from 0 to 2π, for each $q > 0$, we have

$$\int_0^{2\pi} \Big\{|D_\alpha p(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})}\Big\}^q d\theta \leq \int_0^{2\pi} \Big\{\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + |\alpha|\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big\}^q d\theta$$

Multiply both sides of above inequality by

$$\int_0^{2\pi} |e^{i\gamma} + \frac{1}{\Lambda_{\nu,t}}|^p d\gamma$$

we have

$$\Big\{\int_0^{2\pi} |e^{i\gamma} + \frac{1}{\Lambda_{\nu,t}}|^q d\gamma\Big\} \int_0^{2\pi} \Big\{|D_\alpha p(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})}\Big\}^q d\theta \leq \Big\{\int_0^{2\pi} |e^{i\gamma} + \frac{1}{\Lambda_{\nu,t}}|^q d\gamma\Big\} \int_0^{2\pi} \Big\{\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + |\alpha|\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big\}^q d\theta \tag{43}$$

By taking

$$a = |p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})},\ b = |q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})},\ c = \frac{1}{\Lambda_{\nu,t}},\ x = \frac{1}{|\alpha|},$$

the conditions of Lemma 7 are established (since the inequality (39) implies $a > bc$). Then Lemma 7 implies that for every α with $|\alpha| \leq 1$, we have

$$\Big\{(|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}) + \frac{1}{|\alpha|}(|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})})\Big\}^q \int_0^{2\pi} |e^{i\gamma} + \frac{1}{\Lambda_{\nu,t}}|^q d\gamma \leq \Big|\frac{1}{\Lambda_{\nu,t}} + \frac{1}{|\alpha|}\Big|^q \int_0^{2\pi} \Big|\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + e^{i\gamma}\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\gamma$$

Again, integrating both sides of above inequality with respect to θ, we have

$$\int_0^{2\pi} |e^{i\gamma} + \frac{1}{\Lambda_{\nu,t}}|^q d\gamma \int_0^{2\pi} \Big\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})} + \frac{1}{|\alpha|}(|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})})\Big\}^q d\theta \leq \Big|\frac{1}{\Lambda_{\nu,t}} + \frac{1}{|\alpha|}\Big|^q \int_0^{2\pi}\int_0^{2\pi} \Big|\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + e^{i\gamma}\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\gamma d\theta$$

Multiply both sides of above inequality by $|\alpha|^q$, we get

$$\int_0^{2\pi} |e^{i\gamma} + \frac{1}{\Lambda_{\nu,t}}|^q d\gamma \int_0^{2\pi} \Big\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})} + |\alpha|(|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})})\Big\}^q d\theta \leq \Big|\frac{|\alpha|}{\Lambda_{\nu,t}} + 1\Big|^q \int_0^{2\pi}\int_0^{2\pi} \Big|\{|p'(e^{i\theta})| - \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\} + e^{i\gamma}\{|q'(e^{i\theta})| + \frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\gamma d\theta \tag{44}$$

By comparing second part of (43) and first part of (44) we obtain

$$\Big\{\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^q d\gamma\Big\}\int_0^{2\pi}\Big\{|D_\alpha p(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})}\Big\}^q d\theta$$
$$\leq\Big|\frac{|\alpha|}{\Lambda_{\nu,t}}+1\Big|^q\int_0^{2\pi}\int_0^{2\pi}\Big|\{|p'(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}+e^{i\gamma}\{|q'(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}}{k^n(1+\Lambda_{\nu,t})}\}\Big|^q d\gamma d\theta \tag{45}$$

Now by comparing inequalities (45) and (42) we get

$$\Big\{\int_0^{2\pi}|e^{i\gamma}+\frac{1}{\Lambda_{\nu,t}}|^q d\gamma\Big\}\int_0^{2\pi}\Big\{|D_\alpha p(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})}\Big\}^q d\theta$$
$$\leq\Big|\frac{|\alpha|}{\Lambda_{\nu,t}}+1\Big|^q 2\pi n^q\int_0^{2\pi}|p(e^{i\theta})|^q d\theta. \tag{46}$$

Multiply both sides of above inequality by $(\Lambda_{\nu,t})^q$, we get

$$\Big\{\int_0^{2\pi}|e^{i\gamma}+\Lambda_{\nu,t}|^q d\gamma\Big\}\int_0^{2\pi}\Big\{|D_\alpha p(e^{i\theta})|-\frac{nmt\Lambda_{\nu,t}(|\alpha|-1)}{k^n(1+\Lambda_{\nu,t})})\Big\}^q d\theta$$
$$\leq\Big||\alpha|+\Lambda_{\nu,t}\Big|^q 2\pi n^q\int_0^{2\pi}|p(e^{i\theta})|^q d\theta. \tag{47}$$

Equivalently

$$\Big\{\frac{1}{2\pi}\int_0^{2\pi}|e^{i\gamma}+\Lambda_{\nu,t}|^q d\gamma\Big\}^{\frac{1}{q}}\Big\{\frac{1}{2\pi}\int_0^{2\pi}\Big||D_\alpha p(e^{i\theta})|+\frac{nmt\Lambda_{\nu,t}(1-|\alpha|)}{k^n(1+\Lambda_{\nu,t})}\Big|^q d\theta\Big\}^{\frac{1}{q}}$$
$$\leq n||\alpha|+\Lambda_{\nu,t}|\Big\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\Big\}^{\frac{1}{q}}. \tag{48}$$

This completes the proof of Theorem 1. □

Proof of the Theorem 2. By the hypothesis the polynomial $p(z)=a_0+\sum_{j=\nu}^n a_jz^j$, where $1\leq\nu\leq n$ does not any zeros in $|z|<k$, where $k\geq 1$. Therefore, the polynomial $q(z)=z^n\overline{p(\frac{1}{\bar z})}=a_0z^n+\sum_{j=\nu}^n a_jz^{n-j}$ has all its zeros in $|z|\leq\frac{1}{k}\leq 1$. By applying Theorem 1 to q(z), and replacing $\frac{1}{k}$ in equation (15), we get for every complex number α with $|\alpha|\leq 1$,

$$\Big\||D_\alpha q(e^{i\theta})|+\frac{nk^nm_1t\Lambda_{1,\nu}(1-|\alpha|)}{(1+\Lambda_{1,\nu})}\Big\|_q\leq n\frac{(|\alpha|+\Lambda_{1,\nu})}{\|\Lambda_{1,\nu}+z\|_q}\|q\|_q, \tag{49}$$

where $\Lambda_{1,\nu}=\frac{n(|a_0|-k^nm_1t)k^{-2\nu}+\nu|a_\nu|k^{1-\nu}}{\nu|a_\nu|+n(|a_0|-k^nm_1t)k^{1-\nu}}$ and $m_1=\min\limits_{|z|=\frac{1}{k}}|q(z)|$.

On the other hand

$$m_1 = \min_{|z|=\frac{1}{k}} |q(z)| = \min_{|z|=\frac{1}{k}} |z^n \overline{p(\frac{1}{\overline{z}})}| = \frac{\min_{|z|=k} |p(z)|}{k^n} = \frac{m}{k^n}.$$

Since $q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$, then $|q(e^{i\theta})| = |p(e^{i\theta})|$ and for $|D_\alpha q(e^{i\theta})|$,we have

$$|D_\alpha q(e^{i\theta})| = |nq(e^{i\theta}) + (\alpha - e^{i\theta})q'(e^{i\theta})| = |ne^{in\theta}\overline{p(e^{i\theta})} + (\alpha - e^{i\theta})q'(e^{i\theta})|$$

$$\left|ne^{in\theta}\overline{p(e^{i\theta})} + (\alpha - e^{i\theta})\Big(ne^{i(n-1)\theta}\overline{p(e^{i\theta})} - e^{i(n-2)\theta}\overline{p'(e^{i\theta})}\Big)\right| =$$

$$\left|ne^{in\theta}\overline{p(e^{i\theta})} + \Big(n\alpha e^{i(n-1)\theta}\overline{p(e^{i\theta})} - \alpha e^{i(n-2)\theta}\overline{p'(e^{i\theta})} - ne^{in\theta}\overline{p(e^{i\theta})} + e^{i(n-1)\theta}\overline{p'(e^{i\theta})}\Big)\right|$$

$$= \left|n\alpha e^{i(n-1)\theta}\overline{p(e^{i\theta})} - \alpha e^{i(n-2)\theta}\overline{p'(e^{i\theta})} + e^{i(n-1)\theta}\overline{p'(e^{i\theta})}\right|$$

$$= |\overline{\alpha e^{i(n-1)\theta}}|\left|np(e^{i\theta}) + (\frac{1}{\overline{\alpha}} - e^{i\theta})p'(e^{i\theta})\right| = |\alpha||D_{\frac{1}{\overline{\alpha}}}p(e^{i\theta})|$$

By replacing $m_1 = \frac{m}{k^n}$, $\|q\|_q = \|p\|_q$ and $|D_\alpha q(e^{i\theta})| = |\alpha||D_{\frac{1}{\overline{\alpha}}}p(e^{i\theta})|$ in (49) we get

$$\Big\||\alpha||D_{\frac{1}{\overline{\alpha}}}p(e^{i\theta})| + \frac{nmt\Lambda_{1,\nu}(1-|\alpha|)}{(1+\Lambda_{1,\nu})}\Big\|_q \leq n\frac{(|\alpha| + \Lambda_{1,\nu})}{\|\Lambda_{1,\nu} + z\|_q}\|p\|_q,$$

Or

$$|\alpha|\Big\||D_{\frac{1}{\overline{\alpha}}}p(e^{i\theta})| + \frac{nmt(\frac{1}{|\alpha|} - 1)}{\frac{1}{\Lambda_{1,\nu}} + 1}\Big\|_q \leq n\frac{|\alpha|\Lambda_{1,\nu}(\frac{1}{|\alpha|} + \frac{1}{\Lambda_{1,\nu}})}{\Lambda_{1,\nu}\|\frac{1}{\Lambda_{1,\nu}} + z\|_q}\|p\|_q, \tag{50}$$

where $\Lambda_{1,\nu} = \frac{n(|a_0|-tm)k^{-2\nu} + \nu|a_\nu|k^{1-\nu}}{\nu|a_\nu| + n(|a_0|-tm)k^{1-\nu}}$ and $m = \min\limits_{|z|=k}|p(z)|$. If we take $A_{\nu,t} = \frac{1}{\Lambda_{1,\nu}}$, $\gamma = \frac{1}{\overline{\alpha}}$, then $A_{\nu,t} \geq k^\nu \geq 1$ and $|\gamma| \geq 1$, then the inequality (50) becomes the following inequality

$$\Big\||D_\gamma(p(e^{i\theta}))| + \frac{nmt(|\gamma| - 1)}{1 + A_{\nu,t}}\Big\|_q \leq n\frac{(|\gamma| + A_{\nu,t})}{\|A_{\nu,t} + z\|_q}\|p\|_q, \tag{51}$$

where $A_{\nu,t} = \frac{n(|a_0|-tm)k^{1+\nu} + \nu|a_\nu|k^{2\nu}}{\nu|a_\nu|k^{1+\nu} + n(|a_0|-tm)}$ and $m = \min\limits_{|z|=k}|p(z)|$. This completes the proof of Theorem 2. □

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