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Research Article

Optimizing solutions with competing anisotropic (p, q)-Laplacian in hemivariational inequalities

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ABSTRACT. For differential inclusions and hemivariational inequalities driven by anisotropic differential operators, we establish the existence of generalized variational solutions and weak solutions. The main novelty consists in allowing that the driving operators might not satisfy any ellipticity condition, which is achieved for the first time in the anisotropic and nonsmooth context. The approach is based on a finite dimensional approximation process.

Keywords: Differential inclusion, hemivariational inequality, anisotropic p-Laplacian, competing operators, generalized variational solution, weak solution.

2020 Mathematics Subject Classification: 35J87, 35J92, 47J30.

1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

In this paper, we study the following differential inclusion with the Dirichlet boundary condition

$$\begin{cases} -\Delta_{\vec{p}}u + \mu\Delta_{\vec{q}}u \in \partial F(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

on a bounded domain Ω in $\mathbb{R}^N$ with $N \geq 2$ and boundary $\partial\Omega$. Here $\mu \in \mathbb{R}$ is a parameter and we have $\vec{p} = \{p_1, \cdots, p_N\}$ and $\vec{q} = \{q_1, \cdots, q_N\}$, where $1 < p_1, \cdots, p_N < \infty$, $1 < q_1, \cdots, q_N < \infty$, and $q_i < p_i$ for all $i = 1, \cdots, N$. The driving operator $-\Delta_{\vec{p}} + \mu\Delta_{\vec{q}}$ in (1.1) is formed with the anisotropic $\vec{p}$-Laplacian $\Delta_{\vec{p}}$ and the anisotropic $\vec{q}$-Laplacian $\Delta_{\vec{p}}$. We recall that the anisotropic $\vec{r}$-Laplacian with $\vec{r} = (r_1, \cdots, r_N)$ is defined as

$$\Delta_{\vec{r}} := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial(\cdot)}{\partial x_i}\right|^{r_i-2}\right)\frac{\partial(\cdot)}{\partial x_i}.$$

In (1.1), we take $\vec{r} = \vec{p}$ and $\vec{r} = \vec{q}$. For our purpose, the most relevant case of driving operator in (1.1) is the competing anisotropic operator $-\Delta_{\vec{p}} + \Delta_{\vec{q}}$. We assume that

$$\sum_{i=1}^{N} \frac{1}{p_i} > 1. \tag{1.2}$$

Set

$$p^+ := \max\{p_1, \cdots, p_N\},\ p^- := \min\{p_1, \cdots, p_N\},\ p^* := \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i} - 1},$$

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and further assume

$$p^+ < p^*. \tag{1.3}$$

In the right-hand side of inclusion (1.1), we have the generalized gradient ∂F of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ (see [9]). The multivalued expression $\partial F(u)$ means that pointwise $\partial F(u(x))$ is a subset of $\mathbb{R}$ for any $x \in \Omega$. Without loss of generality, we may suppose that $F(0) = 0$. We assume that the following condition is satisfied:

(H) There exist positive constants c_0 and c_1 with $c_1 < \lambda_{1,\vec{p}}p^-$ such that

$$|\xi| \le c_0 + c_1|t|^{p^- -1}$$

for all $t \in \mathbb{R}$ and $\xi \in \partial F(t)$, where

$$\lambda_{1,\vec{p}} := \inf_{u \in W_0^{1,\vec{p}}(\Omega), u \neq 0} \frac{\sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}}^{p^-}}{\|u\|_{L^{p^-}}^{p^-}}. \tag{1.4}$$

The definition of the generalized gradient ∂F implies that each solution $u \in W_0^{1,\vec{p}}(\Omega)$ to (1.1) is a solution of the inequality problem

$$\langle -\Delta_{\vec{p}} u, v\rangle + \mu\langle -\Delta_{\vec{q}} u, v\rangle \le \int_\Omega F^\circ(u(x); v(x))dx \tag{1.5}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$, where F° denotes the generalized directional derivative of the locally Lipschitz function F. Problem (1.5) is a hemivariational inequality in the Banach space $W_0^{1,\vec{p}}(\Omega)$. A brief presentation of the space $W_0^{1,\vec{p}}(\Omega)$ will be done in Section 2.

We are interested in two types of solutions for inclusion (1.1) and a fortiori for hemivariational inequality (1.5), namely the weak and generalized variational solutions.

Definition 1.1. *A function $u \in W_0^{1,\vec{p}}(\Omega)$ is called a weak solution to (1.1) if*

$$\langle -\Delta_{\vec{p}} u, v\rangle + \mu\langle -\Delta_{\vec{q}} u, v\rangle = \int_\Omega z(x)v(x)dx \tag{1.6}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$, with $z \in L^{\vec{p}\,'}(\Omega) \in \partial F(u)$ a.e. on Ω.

Definition 1.2. *A function $u \in W_0^{1,\vec{p}}(\Omega)$ is called a generalized variational solution to inclusion (1.1) if there exists a sequence $\{u_n\}_{n=1}^\infty \subset W_0^{1,\vec{p}}(\Omega)$ such that*

(a) $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ as $n \to \infty$;
(b) $-\Delta_{\vec{p}} u_n + \mu\Delta_{\vec{q}} u_n - z_n \rightharpoonup 0$ in $W^{-1,\vec{p}\,'}(\Omega)$ as $n \to \infty$ with $z_n \in L^{\vec{p}\,'}(\Omega)$ and $z_n \in \partial F(u_n)$ a.e. on Ω;
(c) $\lim_{n\to\infty}\langle \Delta_{\vec{p}} u_n + \mu\Delta_{\vec{q}} u_n, u_n - u\rangle = 0$.

From Definitions 1.1 and 1.2, we see that any weak solution $u \in W_0^{1,\vec{p}}(\Omega)$ to problem (1.1) is a generalized variational solution. In order to confirm this, it suffices to take $u_n = u$ in the definition of the generalized variational solution. The converse assertion is generally not valid.

Our main results are formulated as follows. Note that the part played by the parameter μ is fundamental.

Theorem 1.1. *Under the stated assumptions, there exists a generalized variational solution to problem (1.1) for every $\mu \in \mathbb{R}$. In particular, there exists a solution of the hemivariational inequality (1.5).*

Theorem 1.2. *Under the stated assumptions, if $\mu \leq 0$ then each generalized variational solution to problem (1.1) is a weak solution. Moreover, if $\mu \leq 0$, problem (1.1) admits a weak solution which is a global minimizer of the minimization problem*

$$\inf_{v\in W_0^{1,\vec{p}}(\Omega)} \left[\sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}}^{p_i} - \sum_{i=1}^{N} \frac{\mu}{q_i} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{q_i}}^{q_i} - \int_\Omega F(v(x))dx \right]. \tag{1.7}$$

The main novelty in our study is the presence of the anisotropic operator $-\Delta_{\vec{p}}u + \mu\Delta_{\vec{q}}u$ in the nonsmooth problem, which loses the ellipticity when $\mu > 0$. This extends to an anisotropic nonsmooth setting the use of competing operators considered until now in completely different situations [12, 15, 16, 17, 19]. We mention that the concept of generalized solution for equations involving competing operators and convection terms was developed in [11, 14, 15, 16, 23] (see also [1, 2, 7, 26]). In the present work, we explore the existence of generalized solutions to hemivariational solutions driven by competing anisotropic operators.

The rest of the paper, has the following structure. In Section 2, we outline the needed background of anisotropic spaces and operators and provide auxiliary results regarding the nonsmooth analysis for inclusion (1.1). In Section 3, we present our approach based on finite dimensional approximate solutions. In Sections 4 and 5, we prove Theorems 1.1 and 1.2, respectively.

2. Mathematical background and auxiliary results

The anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ is defined as the completion of the set of smooth functions with compact support $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}},$$

where $\|\cdot\|_{L^r}$ is the usual norm of the space $L^r(\Omega)$. It is separable and uniformly convex, thus a reflexive Banach space. The dual of $W_0^{1,\vec{p}}(\Omega)$ is denoted $W^{-1,\vec{p}'}(\Omega)$. The following embedding theorem can be found in [10, Theorem 1].

Theorem 2.3. *Assume that conditions (1.2) and (1.3) hold. Then for all $r \in [1, p^*]$, there is a continuous embedding $W_0^{1,\vec{p}}(\Omega) \subset L^r(\Omega)$. For $r < p^*$, the embedding is compact.*

From Theorem 2.3, we have the compact embedding

$$W_0^{1,\vec{p}}(\Omega) \subset L^{p^-}(\Omega). \tag{2.8}$$

In particular, by (2.8) we infer that there exists a constant $S_1 > 0$ such that

$$\|v\|_{L^1} \leq S_1\|v\|_{W_0^{1,\vec{p}}(\Omega)}, \quad \forall v \in W_0^{1,\vec{p}}(\Omega). \tag{2.9}$$

The quantity $\lambda_{1,\vec{p}}$ in (1.4) is finite due to the compact embedding (2.8). Since the space $W_0^{1,\vec{p}}(\Omega)$ is separable, there exists a Galerkin basis for $W_0^{1,\vec{p}}(\Omega)$, that is, a sequence of vector subspaces $\{X_n\}_{n\geq 1}$ of $W_0^{1,\vec{p}}(\Omega)$ such that

(*i*) $dim(X_n) < \infty$ for all n;
(*ii*) $X_n \subset X_{n+1}$ for all n;
(*iii*) $\overline{\cup_{n=1}^{\infty} X_n} = W_0^{1,\vec{p}}(\Omega)$.

For various aspects involving anisotropic Sobolev spaces, we refer to [3, 4, 5, 10, 13, 18, 20, 23, 24, 21, 22, 25].

We continue with a brief survey of basic elements of nonsmooth analysis that are needed in the sequel.

Given a locally Lipschitz function $F : X \to \mathbb{R}$ on a normed space X, the generalized directional derivative of F at $u \in X$ in the direction $v \in X$ is defined as

$$F^\circ(u;v) := \limsup_{w\to u, t\to 0^+} \frac{1}{t}\left(F(w+tv) - F(w)\right).$$

The generalized gradient of F at $u \in X$ is the subset of X^* given by

$$\partial F(u) := \{u^* \in X^* : \langle u^*, v\rangle \le F^\circ(u;v) \quad \text{for all } v \in X\}.$$

A case of major interest for us in connection with the resolution of problem (1.1) is when $X = \mathbb{R}$. In this case, a relevant realization of the preceding notions is as follows. Let $f \in L^\infty_{loc}(\mathbb{R})$ and its primitive $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = \int_0^t f(s)ds, \quad \forall t \in \mathbb{R} \tag{2.10}$$

which is locally Lipschitz. The explicit expression of the generalized gradient $\partial F(t)$ is $\partial F(t) = [\underline{f}(t), \overline{f}(t)]$, where

$$\underline{f}(t) = \lim_{\delta\to 0} \operatorname{ess\,inf}_{|\eta-t|<\delta} f(\eta) \text{ and } \overline{f}(t) = \lim_{\delta\to 0} \operatorname{ess\,sup}_{|\eta-t|<\delta} f(\eta)$$

for every $t \in \mathbb{R}$. With the choice in (2.10), inclusion (1.1) becomes

$$\begin{cases} -\Delta_{\vec{p}} u + \mu \Delta_{\vec{q}} u \in [\underline{f}(u), \overline{f}(u)] & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

which is important for equations with discontinuous nonlinearities (see [8]).

Now, we return to our general case of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ satisfying hypothesis (H). It follows from hypothesis (H) that the function F verifies the growth condition

$$|F(t)| \le c_0|t| + \frac{c_1}{p^-}|t|^{p^-}, \quad \forall t \in \mathbb{R}. \tag{2.11}$$

Indeed, note that $F(0) = 0$ and F is differentible almost everywhere due to Rademacher's theorem, thus

$$F(t) = \int_0^t F'(s)ds, \quad \forall t \in \mathbb{R}.$$

Since $F'(s) \in \partial F(s)$ for all $t \in \mathbb{R}$ (refer to [9, p. 32])), it turns out from hypothesis (H) that (2.11) holds true.

It is straightforward to check that the functional $\Phi : L^{p^-}(\Omega) \to \mathbb{R}$ given by

$$\Phi(v) = \int_\Omega F(v(x))dx, \quad \forall v \in L^{p^-}(\Omega) \tag{2.12}$$

is Lipschitz continuous on the bounded subsets of $L^{p^-}(\Omega)$, thus locally Lipschitz on $L^{p^-}(\Omega)$. Therefore the generalized gradient $\partial\Phi$ is well defined on $L^{p^-}(\Omega)$.

Using that the domain Ω is bounded, Hölder's inequality ensures the continuous embedding $W_0^{1,\vec{p}}(\Omega) \subset W_0^{1,\vec{q}}(\Omega)$ (note that $q_i < p_i$ for all $i = 1, \dots, N$). Then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ in (2.8) allows us to define the functional $J : W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}$ by

$$J(v) = \sum_{i=1}^N \frac{1}{p_i}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{p_i} - \sum_{i=1}^N \frac{\mu}{q_i}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^{q_i}}^{q_i} - \int_\Omega F(v(x))dx \tag{2.13}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$.

Proposition 2.1. *Assume that condition* (H) *holds. The functional* J *given by* (2.13) *is locally Lipschitz on* $W_0^{1,\vec{p}}(\Omega)$ *with the generalized gradient*

$$\partial J(v) = \sum_{i=1}^{N} \left|\frac{\partial v}{\partial x_i}\right|^{p_i-2} \frac{\partial v}{\partial x_i} - \mu \sum_{i=1}^{N} \left|\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{\partial v}{\partial x_i} - \partial\Phi(v) \tag{2.14}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$*. Moreover, the functional* J *is coercive on* $W_0^{1,\vec{p}}(\Omega)$*, which means that*

$$\lim_{\|v\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty} J(v) = +\infty. \tag{2.15}$$

Proof. The first part of the statement is a direct consequence of (2.13) and of what was said about the functional Φ introduced in (2.12).

We pass to the proof of (2.15). Hypothesis (H) in conjunction with (2.9), (1.4), (2.8), (2.13) and Hölder's inequality, leads to

$$\begin{aligned} J(v) &\geq \sum_{i=1}^{N} \frac{1}{p_i} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{p_i} - \sum_{i=1}^{N} \frac{|\mu|}{q_i} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{q_i}}^{q_i} - \int_\Omega \left(c_0|v| + \frac{c_1}{p^-}|v|^{p^-}\right) dx \\ &\geq \sum_{i=1}^{N} \frac{1}{p_i} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{p_i} - \sum_{i=1}^{N} \frac{|\mu|}{q_i} |\Omega|^{\frac{p_i-q_i}{p_i}} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{q_i} \\ &\quad -c_0 S_1 \sum_{i=1}^{N} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}} - \frac{c_1 \lambda_{1,\vec{p}}^{-1}}{p^-} \sum_{i=1}^{N} \frac{1}{p_i} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{p^-}, \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. As it was assumed that $1 < q_i < p_i$ for all $i = 1, \cdots, N$, and $c_1 < \lambda_{1,\vec{p}} p^-$, we arrive at (2.15), so the functional J is coercive. □

3. Sequence of approximate solutions

In order to simplify the notation, for any real number $r > 1$ we denote $r' := r/(r-1)$ (the Hölder conjugate of r), and we can set $\vec{p}' := (p'_1, \cdots, p'_N)$ for $\vec{p} = (p_1, \cdots, p_N)$.

As noticed in Section 2, there exists a Galerkin basis $\{X_n\}_{n\geq 1}$ for the space $W_0^{1,\vec{p}}(\Omega)$ that we now fix. We construct approximate solutions to inclusion (1.1) on each finite dimensional subspace X_n.

Proposition 3.2. *Assume that hypothesis* (H) *holds. Then, for each* n*, there exist* $u_n \in X_n$ *and* $z_n \in L^{p^{-\prime}}(\Omega)$ *with* $z_n \in \partial F(u_n)$ *almost everywhere on* Ω *such that*

$$J(u_n) = \inf_{v \in X_n} J(v) \tag{3.16}$$

and

$$\langle -\Delta_{\vec{p}} u_n, v\rangle + \mu\langle -\Delta_{\vec{q}} u_n, v\rangle - \int_\Omega z_n v dx = 0 \tag{3.17}$$

for all $v \in X_n$.

Proof. Proposition 2.1 ensures that the restriction $J|_{X_n}$ of the functional $J : W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}$ to the finite dimensional subspace X_n is locally Lipschitz and coercive. Therefore there exists $u_n \in X_n$ satisfying (3.16). We derive from (3.16) the necessary optimality condition

$$0 \in \partial\left(J|_{X_n}\right)(u_n). \tag{3.18}$$

In view of (2.14), we have that (3.18) results in (3.17). The Aubin-Clarke theorem (see [9, p. 83]) applied to the integral functional Φ on $L^{p^-}(\Omega)$ in (2.12) yields that $z_n \in \partial F(u_n)$ almost everywhere on Ω. This completes the proof. □

Corollary 3.1. *Assume that condition (H) holds. Then the sequence $\{u_n\} \subset W_0^{1,\vec{p}}(\Omega)$ constructed in Proposition 3.2 satisfies*

$$\lim_{n\to\infty} J(u_n) = \inf_{w\in W_0^{1,\vec{p}}(\Omega)} J(w). \tag{3.19}$$

Proof. Recall that $X_n \subset X_{n+1}$ for all n. Then (3.16) shows that the sequence $\{J(u_n)\}$ is nonincreasing, while the proof of Proposition 3.2 provides that is bounded from below. Hence the limit $l := \lim_{n\to\infty} J(u_n)$ exists.

Arguing by contradiction, admit that

$$l > \inf_{w\in W_0^{1,\vec{p}}(\Omega)} J(w).$$

This amounts to saying that there exists $\hat{w} \in W_0^{1,\vec{p}}(\Omega)$ such that $J(\hat{w}) < l$. Consequently, there exists a neighborhood U of $\hat{w}$ in $W_0^{1,\vec{p}}(\Omega)$ such that

$$J(w) < l \text{ for all } w \in U. \tag{3.20}$$

Since $W_0^{1,\vec{p}}(\Omega) = \overline{\cup_{n=1}^{\infty} X_n}$, there exists m such that $\tilde{w} \in U \cap X_m$. Then (3.16) and (3.20) yield

$$\min_{v\in X_m} J(v) \le J(\tilde{w}) < l \le \min_{v\in X_m} J(v).$$

The obtained contradiction proves (3.19), thus completing the proof. □

We focus on the sequence $\{u_n\}$.

Proposition 3.3. *Assume that condition (H) holds. Then the sequence $\{u_n\}$ constructed in Proposition 3.2 is bounded in $W_0^{1,\vec{p}}(\Omega)$, so there is a constant $M_1 > 0$ such that*

$$\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} \le M_1 \textit{ for all } n \ge 1. \tag{3.21}$$

Proof. Set $v = u_n$ in (3.17) (note that $u_n \in X_n$). Then, as in the proof of Proposition 2.1, we use $z_n(x) \in \partial F(u_n(x))$ for almost all $x \in \Omega$ to infer that

$$\begin{aligned}
&\sum_{i=1}^N \frac{1}{p_i}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}}^{p_i}\\
=&\mu\sum_{i=1}^N \frac{1}{q_i}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}}^{q_i} + \int_\Omega z_n u_n dx\\
\le&\sum_{i=1}^N \frac{|\mu|}{q_i}|\Omega|^{\frac{p_i-q_i}{p_i}}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}}^{q_i} + c_0 S_1\sum_{i=1}^N\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}} + \frac{c_1\lambda_{1,\vec{p}}^{-1}}{p^-}\sum_{i=1}^N\frac{1}{p_i}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}}^{p^-}.
\end{aligned}$$

Since $1 < q_i < p_i$ and $p^- \le p_i$ for all $i = 1, \ldots, N$, and $c_1 < \lambda_{1,\vec{p}}p^-$, we get the stated result. □

Corollary 3.2. *Assume that condition (H) hods. Then for the sequence $\{u_n\} \subset W_0^{1,\vec{p}}(\Omega)$ in Proposition 3.2 there is a constant $M_2 > 0$ such that*

$$\| - \Delta_{\vec{p}} u_n + \mu\Delta_{\vec{q}} u_n - z_n\|_{W^{-1,\vec{p}'}(\Omega)} \le M_2 \tag{3.22}$$

for all n, with z_n as described in Proposition 3.2.

Proof. For each $v \in W_0^{1,\vec{p}}(\Omega)$, by Hölder's inequality, hypothesis (H), (2.9) and (1.4), we find the estimate

$$\begin{aligned}
&|\langle -\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n, v\rangle| \\
&= \left| \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i - 2} \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \mu \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{q_i - 2} \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_\Omega z_n v dx \right| \\
&\le \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{p_i - 1} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}} + |\mu| \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{q_i}}^{q_i - 1} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{q_i}} \\
&+ \int_\Omega (c_0 + c_1 |u_n|^{p^- - 1}) |v| dx \\
&\le \left(\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{p_i - 1} + |\mu| \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{q_i}}^{q_i - 1} + c_0 S_1 + \lambda_{1,\vec{p}}^{-\frac{1}{p^-}} \|u_n\|_{L^{p^-}}^{p^- - 1} \right) \|v\|_{W_0^{1,\vec{p}}(\Omega)}.
\end{aligned}$$

This entails

$$\begin{aligned}
&\| -\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n \|_{W_0^{-1,\vec{p}\,'}(\Omega)} \\
&\le \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{p_i - 1} + |\mu| \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{q_i}}^{q_i - 1} + c_0 S_1 + \lambda_{1,\vec{p}}^{-\frac{1}{p^-}} \|u_n\|_{L^{p^-}}^{p^- - 1}.
\end{aligned} \tag{3.23}$$

By (3.23), (3.21) and Theorem 2.3, we obtain the validity of (3.22), which completes the proof. □

4. Proof of Theorem 1.1

Proposition 3.3 provides the sequence $\{u_n\} \subset W_0^{1,\vec{p}}(\Omega)$ which is bounded in $W_0^{1,\vec{p}}(\Omega)$ as demonstrated in (3.21). Therefore, thanks to the reflexivity of the space $W_0^{1,\vec{p}}(\Omega)$, up to a subsequence it holds $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ for some $u \in W_0^{1,\vec{p}}(\Omega)$. Corollary 3.2 ensures that the sequence $\{-\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n\}$ is bounded in $W^{-1,\vec{p}\,'}(\Omega)$, with $z_n \in L^{p^{-\prime}}(\Omega)$ satisfying $z_n \in \partial F(u_n)$ almost everywhere on Ω. Then along a relabeled subsequence we have $-\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n \rightharpoonup \eta$ in $W^{-1,\vec{p}\,'}(\Omega)$ for some $\eta \in W^{-1,\vec{p}\,'}(\Omega)$.

We claim that $\eta = 0$. In order to prove the claim, let $v \in \cup_{n=1}^\infty X_n$, so $v \in X_m$ for some m. Note that for each $n \ge m$, we have $v \in X_n$, which enables us to insert v in (3.17). Letting $n \to \infty$ in (3.17) renders $\langle \eta, v\rangle = 0$. Using that $\cup_{n=1}^\infty X_n$ is dense $W_0^{1,\vec{p}}(\Omega)$, we are able to conclude that $\eta = 0$. Therefore we have

$$-\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n \rightharpoonup 0 \text{ in } W^{-1,\vec{p}\,'}(\Omega). \tag{4.24}$$

Combining (3.17) and (4.24) results in

$$\lim_{n\to\infty} \left[\langle -\Delta_{\vec{p}} u_n, u_n - u\rangle + \mu \langle \Delta_{\vec{q}} u_n, u_n - u\rangle - \int_\Omega z_n (u_n - u) dx \right] = 0. \tag{4.25}$$

We stress that in the above arguments $\mu \in \mathbb{R}$ is arbitrary. We are thus in a position to assert that $u \in W_0^{1,\vec{p}}(\Omega)$ is a generalized variational solution to problem (1.1) whose sequence required in Definition 1.2 is $\{u_n\}$. As noticed before, we deduce that $u \in W_0^{1,\vec{p}}(\Omega)$ is a solution to the hemivariational inequality (1.5). The proof of Theorem 1.1 is completed.

5. Proof of Theorem 1.2

Now we assume that $\mu \le 0$. Theorem 1.1 applies producing a generalized weak solution for problem (1.1).

Let $u \in W_0^{1,\vec{p}}(\Omega)$ be a generalized weak solution to problem (1.1). According to Definition 1.2, there is a sequence $\{u_n\}$ in $W_0^{1,\vec{p}}(\Omega)$ satisfying the requirements therein. In particular, it holds (4.25). The sequence $\{z_n\}$ is bounded in $L^{p^{-\prime}}(\Omega)$ due to the Lipschitz continuity of the functional Φ on the bounded subsets of $L^{p^-}(\Omega)$ (refer to the proof of Proposition 3.2). Moreover, it is true that $u_n \to u$ in $L^{p^-}(\Omega)$ owing to the compact embedding in Theorem 2.3 for $r = p^-$. Altogether this gives

$$\lim_{n\to+\infty} \int_\Omega z_n(u_n - u)dx = 0.$$

Then (4.25) leads to

$$\lim_{n\to\infty} \langle -\Delta_{\vec{p}} u_n + \mu\Delta_{\vec{q}} u_n, u_n - u\rangle = 0. \tag{5.26}$$

Using that $\mu \le 0$ and the monotonicity of the operator $-\Delta_{\vec{q}}$ on $W_0^{1,\vec{q}}(\Omega)$, we are able to write

$$\begin{aligned} &\langle -\Delta_{\vec{p}} u_n, u_n - u\rangle \\ =&\langle -\Delta_{\vec{p}} u_n + \mu\Delta_{\vec{q}} u_n, u_n - u\rangle + \mu\langle -\Delta_{\vec{q}} u_n + \Delta_{\vec{q}} u, u_n - u\rangle + \mu\langle -\Delta_{\vec{q}} u, u_n - u\rangle \\ \le&\langle -\Delta_{\vec{p}} u_n + \mu\Delta_{\vec{q}} u_n, u_n - u\rangle + \mu\langle -\Delta_{\vec{q}} u, u_n - u\rangle. \end{aligned}$$

By (5.26) and $u_n \rightharpoonup u$ in $W_0^{1,\vec{q}}(\Omega)$, we find that

$$\limsup_{n\to\infty} \langle -\Delta_{\vec{p}} u_n, u_n - u\rangle \le 0. \tag{5.27}$$

The monotonicity of the operator $-\Delta_{\vec{p}}$ on $W_0^{1,\vec{p}}(\Omega)$ implies

$$\begin{aligned} 0 \le& \sum_{i=1}^N \int_\Omega \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx \\ =& \langle -\Delta_{\vec{p}} u_n + \Delta_{\vec{p}} u, u_n - u\rangle. \end{aligned}$$

By (5.27) and $u_n \rightharpoonup u$ in $W_0^{1,\vec{q}}(\Omega)$, we are entitled to assert that

$$\lim_{n\to\infty} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx = 0 \quad \forall\, i = 1,\dots,N$$

which yields

$$\limsup_{n\to\infty} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}} \le \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}} \qquad \forall\, i = 1,\dots,N.$$

Since the space $L^{p_i}(\Omega)$ is uniformly convex (see [6]), we infer the strong convergence $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$, thus $-\Delta_{\vec{p}} u_n \to -\Delta_{\vec{p}} u$ in $W^{-1,\vec{p}\,'}(\Omega)$ and $-\Delta_{\vec{q}} u_n \to -\Delta_{\vec{q}} u$ in $W^{-1,\vec{q}\,'}(\Omega)$.

On the other hand, taking into account that $u_n \to u$ in $L^{p^-}(\Omega)$ and $z_n \in \partial\Phi(u_n) \subset L^{p^{-\prime}}(\Omega)$, the sequence $\{z_n\}$ is bounded in $L^{p^{-\prime}}(\Omega)$, so along a subsequence $z_n \rightharpoonup z$ in $L^{p^{-\prime}}(\Omega)$ for some $z \in L^{p^{-\prime}}(\Omega)$. From [9], it is known that the generalized gradient $\partial\Phi$ is weak*-closed, so we obtain $z \in \partial\Phi(u)$. Furthermore, (4.24) ensures

$$-\Delta_{\vec{p}} u + \mu\Delta_{\vec{q}} u - z = 0 \text{ in } W^{-1,\vec{p}\,'}(\Omega).$$

Under assumption (H), the Aubin-Clarke theorem (see [9]) can be applied to the functional $\Phi : L^{p^-}(\Omega) \to \mathbb{R}$ in (2.12) establishing that $z(x) \in \partial F(u(x))$ for almost all $x \in \Omega$. Consequently,

$u \in W_0^{1,\vec{p}}(\Omega)$ satisfies (1.6), thus it is a weak solution to the inclusion problem (1.1), thereby of hemivariational inequality (1.5), too.

The last step in the proof concerns to show that $u \in W_0^{1,\vec{p}}(\Omega)$ solves the global minimization in (1.7). In view of (2.13), the global minimization in (1.7) reads as $u \in W_0^{1,\vec{p}}(\Omega)$ is a global minimizer of the functional J on $W_0^{1,\vec{p}}(\Omega)$. On the basis of the strong convergence $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$, we are allowed to pass to the limit in (3.19) finding that $\inf_{w \in W_0^{1,\vec{p}}(\Omega)} J(w)$ is achieved at $u \in W_0^{1,\vec{p}}(\Omega)$. The proof is complete.

References

[1] M. Allalou, M. El Ouaarabi and A. Raji: *On a class of nonhomogeneous anisotropic elliptic problem with variable exponents*, Rend. Circ. Mat. Palermo, II. Ser (2024).

[2] M. Bohner, G. Caristi, A. Ghobadi and Sh. Heidarkhani: *Three solutions for discrete anisotropic Kirchhoff-type problems*, Demonstr. Math., **56** (1) (2023), Article ID: 20220209.

[3] G. Bonanno, G. D'Aguì and A. Sciammetta: *Multiple solutions for a class of anisotropic $\overrightarrow{p}$-Laplacian problems*, Bound. Value Probl., **2023** (2023), Article ID: 89.

[4] B. Brandolini, F. Cîrstea: *Anisotropic elliptic equations with gradient-dependent lower order terms and L^1 data*, Math. Eng., **5** (4) (2023), 1–33.

[5] B. Brandolini, F. Cîrstea: *Boundedness of solutions to singular anisotropic elliptic equations*, Discrete Contin. Dyn. Syst. Ser. S, **17** (4) (2024), 1545–1561.

[6] H. Brezis: *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York (2011).

[7] A. Cernea: *On the solutions of a coupled system of proportional fractional differential inclusions of Hilfer type*, Modern Math. Methods, **2** (2) (2024), 80–89.

[8] K. C. Chang: *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl., **80** (1981), 102–129.

[9] F. H. Clarke: *Optimization and Nonsmooth Analysis*, John Wiley & Sons, Inc., New York, USA (1983).

[10] I. Fragalà, F. Gazzola and B. Kawohl: *Existence and nonexistence results for anisotropic quasilinear elliptic equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, **21** (5) (2004), 715–734.

[11] L. Gambera, S. A. Marano and D. Motreanu: *Quasi-linear Dirichlet systems with competing operators and convection*, J. Math. Anal. Appl., **530** (2024), Article ID: 127718.

[12] Z. Liu, R. Livrea, D. Motreanu and S. Zeng: *Variational differential inclusions without ellipticity condition*, Electron. J. Qual. Theory Differ. Equ., **43** (2020), 1–17.

[13] M. Mihăilescu, P. Pucci and V. D. Rădulescu: *Eigenvalue problems for anisotropic quasilinear elliptic equations with varaible exponent*, J. Math. Anal. Appl., **340** (2008), 687–698.

[14] D. Motreanu: *Quasilinear Dirichlet problems with competing operators and convection*, Open Math., **18** (2020), 1510–1517.

[15] D. Motreanu: *Systems of hemivariational inclusions with competing operators*, Mathematics, **12** (11) (2024), Article ID: 1766.

[16] D. Motreanu: *Hemivariational inequalities with competing operators*, Commun. Nonlinear Sci. Numer. Simulat., **130** (2024), Article ID: 107741.

[17] D. Motreanu, A. Razani: *Competing anisotropic and Finsler (p, q)-Laplacian problems*, Bound. Value Probl., **2024** (2024), Article ID: 39.

[18] D. Motreanu, E. Tornatore: *Dirichlet problems with anisotropic principal part involving unbounded coefficients*, Electron. J. Differential Equations, **2024** (11) (2024), 1–13.

[19] A. Razani: *Nonstandard competing anisotropic (p, q)-Laplacians with convolution*, Bound. Value Probl., **2022** (2022), Article ID: 87.

[20] A. Razani: *Entire weak solutions for an anisotropic equation in the Heisenberg group*, Proc. Amer. Math. Soc., **151** (11) (2023), 4771–4779.

[21] A. Razani: *Competing Kohn-Spencer Laplacian systems with convection in non-isotropic Folland-Stein space*, Complex Var. Elliptic Equ., (2024), 1–14. DOI: 10.1080/17476933.2024.2337868

[22] A. Razani, G. S. Costa and G. M. Figueiredo: *A positive solution for a weighted anisotropic p-Laplace equation involving vanishing potential*, Mediterr. J. Math., **21** (2024), Article ID: 59.

[23] A. Razani, G. M. Figueiredo: *Degenerated and competing anisotropic (p, q)-Laplacians with weights*, Appl. Anal., **102** (2023), 4471–4488.

[24] A. Razani, G. M. Figueiredo: *A positive solution for an anisotropic (p, q)-Laplacian*, Discrete Contin. Dyn. Syst. Ser. S, **16** (6) (2023), 1629–1643.
[25] A. Razani, G. M. Figueiredo: *Infinitely many solutions for an anisotropic differential inclusion on unbounded domains*, Electron. J. Qual. Theory Differ. Equ., **33** (2024), 1–17.
[26] A. Razani, E. Tornatore: *Solutions for nonhomogeneous degenerate quasilinear anisotropic problems*, Constr. Math. Anal., **7** (3) (2024), 134–149.

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Research Article

Viscosity implicit midpoint scheme for enriched nonexpansive mappings

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ABSTRACT. This article proposes and analyses a viscosity scheme for an enriched nonexpansive mapping. The scheme is incorporated with the implicit midpoint rule of stiff differential equations. We deduce some convergence properties of the scheme and establish that a sequence generated therefrom converges strongly to a fixed point of an enriched nonexpansive mapping provided such a point exists. Furthermore, we provide some examples of the implementation of the schemes with respect to certain enriched mappings and show the numerical pattern of the scheme.

Keywords: Enriched nonexpansive mapping, implicit midpoint rule, fixed point, Hilbert space, viscosity iteration.

2020 Mathematics Subject Classification: 47H09, 47H10, 47J25, 47N20, 65J15.

1. INTRODUCTION

The viscosity scheme is among the prominent iterative methods for estimating a fixed point of a nonlinear mapping through strong convergence under certain feasible control conditions. This scheme was introduced by Moudafi in [10] based upon the results of [2]. The scheme was further studied by Xu [24] in the framework of Banach spaces. The scheme uses contraction mapping to induce a nonexpansive mapping to target a particular fixed point having a unique property. For a linear space $\mathcal{H}$ and a mapping $G : \mathcal{H} \to \mathcal{H}$, the viscosity scheme generates a sequence $\{u_n\}$ recursively by

$$u_{n+1} = \beta_n f(u_n) + (1 - \beta_n)G(u_n), \quad \forall n \geq 1,$$

where $\beta_n \in (0, 1)$ and f is a contraction mapping (that is,

$$\|f(u) - f(w)\| \leq \kappa\|u - w\|$$

for some $\kappa \in [0, 1)$). It is evident, based on [10, 24], that, if G is a nonexpansive mapping and $\{\beta_n\}$ satisfies some suitable condition, then the strong convergence of the scheme $\{u_n\}$ to a fixed point of G can be achieved, where the limit point solves the variational inequality problem involving f over the set of fixed points of G. This method is further extended to nonlinear mappings that are more general than nonexpansive mappings and also to nonlinear spaces. For further details on the viscosity scheme and related concepts of fixed points, see, for example, [9, 22] and the references therein. In [5], Berinde introduced an enriched nonexpansive mapping as a generalization of nonexpansive mappings as follows:

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Let $(\mathcal{H}, \|\cdot\|)$ be a normed linear space and a mapping $G : \mathcal{H} \to \mathcal{H}$ is said to be an enriched (α-enriched) nonexpansive if there exists $\alpha \geq 0$ such that

$$\|\alpha(u-w) + Gu - Gw\| \leq (\alpha+1)\|u-w\|, \quad \forall u, w \in \mathcal{H}. \tag{1.1}$$

Later on, Berinde in [7] considered G as an α-enriched nonexpansive mapping and established that a sequence $\{u_n\}$ generated by

$$u_{n+1} = \left(1 - \frac{\delta_n}{\alpha+1}\right)(1-\beta_n)u_n + \frac{\delta_n}{\alpha+1}G\big((1-\beta_n)u_n\big), \quad \forall n \geq 1, \tag{1.2}$$

converges strongly to a fixed point of G, where $\beta_n, \delta_n \in (0,1)$ with some control conditions. The scheme in (1.2) is a modification of the scheme in [27]. For further development concerning enriched nonexpansive mappings and approximation schemes in this direction even beyond linear spaces, see, for example, [6, 14, 16, 11, 8, 15, 18] and the references therein.

On the other hand, most real-life phenomena are addressed in the form of mathematical models that result in differential equations. Some of these differential equations are difficult to solve analytically. In this regard, engineers seek a numerically generated pattern that exhibits the structure of the real solutions. Thus the emphasis is on the need for numerical approaches to solving differential equations. One of these approaches is the implicit midpoint scheme, which is very promising for handling such differential equations. This scheme is appropriate mostly for stiff equations and differential algebra equations [3, 4, 21, 20, 19].

For a differential equation of the form

$$\begin{cases} u' = g(u), \\ u(0) = u_1, \end{cases}$$

where $g : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and smooth and the implicit midpoint scheme generates a sequence $\{u_n\}$ by solving

$$u_{n+1} = u_n + \eta g\left(\frac{u_n + u_{n+1}}{2}\right), \quad \forall n \geq 1, \tag{1.3}$$

where η is known as step size. This idea was extended in [26] to fixed point theory considering that the state of equilibrium of such differential equation reduces to a fixed point problem. Thereafter, Alghamdi et al. [1] considered a nonexpansive mapping $G : \mathcal{H} \to \mathcal{H}$ and generate $\{u_n\}$ via the implicit midpoint scheme as

$$u_{n+1} = (1-\beta_n)u_n + \beta_n G\left(\frac{u_n + u_{n+1}}{2}\right), \quad \forall n \geq 1, \tag{1.4}$$

where $\beta_n \in (0,1)$ and $u_1 \in \mathcal{H}$. The authors established that, if $\{\beta_n\}$ is such that

$$\liminf_{n\to\infty} \beta_n > 0, \quad \beta_{n+1} \leq \eta\beta_n$$

for some fixed η, then $\{u_n\}$ converges weakly to a fixed point of G. In [12], the scheme (1.4) is modified and analyzed to approximate a fixed point of an α-enriched nonexpansive mapping in the sense that $\{u_n\}$ is updated based on the equation

$$u_{n+1} = \left(1 - \frac{2\beta_n}{\alpha(2-\beta_n)+2}\right)u_n + \frac{2\beta_n}{\alpha(2-\beta_n)+2}G\left(\frac{u_n + u_{n+1}}{2}\right), \quad \forall n \geq 1, \tag{1.5}$$

where $\beta_n \in (0,1)$ for all $n \geq 1$. The authors established the weak convergence using a similar assumption as in [1]. However, the strong convergence result is more desirable in infinite dimensional spaces. In [25], Xu et al. addressed this problem for the case when G is a nonexpansive mapping by applying the viscosity technique to the scheme (1.4) and using different control conditions. The authors' scheme is as follows:

$$u_{n+1} = (1-\beta_n)f(u_n) + \beta_n G\left(\frac{u_n + u_{n+1}}{2}\right), \quad \forall n \geq 1, \tag{1.6}$$

where f is a contraction mapping.

The purpose of this work is to incorporate a contraction mapping in persuading the implicit midpoint scheme for enriched nonexpansive mappings. The proposed scheme is fashioned after (1.4), (1.5) and (1.6). We establish some convergence properties of the proposed scheme and show the strong convergence of the sequence generated therefrom to a fixed point of the mapping that also solves a variational inequality problem. It is worth noting that fixed points of enriched nonexpansive mappings have applications in many practical problems as they incorporate certain Lipschitz mappings with constants greater than 1. Finally, we give some numerical examples of the Lipschitz mappings and use them to show the explicit reduction of the scheme and the numerical implementations.

2. Preliminaries

In the sequel, unless otherwise stated, $\mathcal{E}$ stands for a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Given a mapping $G : \mathcal{E} \to \mathcal{H}$, we call a sequence $\{u_n\}$ an approximate fixed point sequence for G if

$$\|u_n - Gu_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Recall that Hilbert spaces possess Opial's property, that is, for a sequence $\{u_n\} \subset \mathcal{H}$ that converges weakly to u^*,

$$\liminf_{n\to\infty} \|u_n - u^*\| < \liminf_{n\to\infty} \|u_n - y\|, \quad \forall y \in \mathcal{H}\backslash\{u^*\}.$$

Now, we state the demiclosedness principle of an enriched nonexpansive mapping as in [12].

Lemma 2.1. *Let $G : \mathcal{E} \to \mathcal{E}$ be an α-enriched nonexpansive mapping. Suppose that $\{u_n\}$ is an approximate fixed point sequence for G and also $\{u_n\}$ weakly converges to u^*. Then u^* is a fixed point of G.*

Some identities involving two points in real Hilbert spaces are very crucial in obtaining our main results.

Lemma 2.2. *Let $u, w \in \mathcal{H}$ and $a \in \mathbb{R}$. Then, we have the following:*

(1) $\|u + w\|^2 = \|u\|^2 + \|w\|^2 + 2\langle u, w\rangle$.
(2) $\|u - w\|^2 = \|u\|^2 + \|w\|^2 - 2\langle u, w\rangle$.
(3) $\|au + (1-a)w\|^2 = a\|u\|^2 + (1-a)\|w\|^2 - a(1-a)\|u - w\|^2$.

Lemma 2.3. [23] *Let $\{\ell_n\}$ be a sequence of non-negative real numbers such that*

$$\ell_{n+1} \leq (1-\sigma_n)\ell_n + \delta_n, \quad \forall n \geq 1,$$

where $\{\sigma_n\} \subseteq (0,1)$ and $\{\delta_n\} \subseteq \mathbb{R}$. Suppose that the following conditions are satisfied

$$\text{(C1)} \sum_{n=1}^{\infty} \sigma_n = \infty; \qquad \text{(C2) either} \sum_{n=1}^{\infty} |\delta_n| < \infty \quad \text{or} \quad \limsup_{n\to\infty} \frac{\delta_n}{\sigma_n} \leq 0.$$

Then $\lim_{n\to\infty} \ell_n = 0$.

3. Viscosity Implicit Midpoint Scheme and Its Convergence

Now, we introduce the main algorithm as follows:

Algorithm 3.1. *Initialize $u_1 \in \mathcal{H}$ arbitrary and find u_{n+1} such that*

$$u_{n+1} = \frac{\alpha(1-\beta_n)}{\alpha(1+\beta_n)+2}u_n + \frac{2\beta_n(\alpha+1)}{\alpha(1+\beta_n)+2}f(u_n) + \frac{2(1-\beta_n)}{\alpha(1+\beta_n)+2}G\left(\frac{u_n+u_{n+1}}{2}\right),$$

where $\beta_n \in (0,1)$ for all $n \geq 1$, $\alpha \geq 0$ and $G : \mathcal{H} \to \mathcal{H}$ is a mapping and f is a contraction mapping with constant κ.

Remark 3.1. *It is worth noting that, for $\alpha = 0$, Algorithm 3.1 reduces to (1.6). The connection is evident since (1.1) implies that every nonexpansive mapping is 0-enriched nonexpansive.*

Remark 3.2. *It is not difficult to obtain from Algorithm 3.1 that u_{n+1} can be rewritten as follows:*

$$u_{n+1} = \frac{\alpha(1-\beta_n)}{\alpha+1}\left(\frac{u_n+u_{n+1}}{2}\right) + \beta_n f(u_n) + \frac{1-\beta_n}{1+\alpha}G\left(\frac{u_n+u_{n+1}}{2}\right). \tag{3.7}$$

Throughout this manuscript, we denote the fixed point set of a mapping G by $\mathcal{F}(G)$ and the metric projection onto a closed convex set $\mathcal{C}$ by $P_{\mathcal{C}}$.

Lemma 3.4. *Let G be an α-enriched nonexpansive mapping with $\mathcal{F}(G) \neq \emptyset$. Then $\{u_n\}$ generated through Algorithm 3.1 is bounded.*

Proof. Let $u^* \in \mathcal{F}(G)$ and set $w_n = \dfrac{u_n+u_{n+1}}{2}$. Then it follows from (3.7) and triangle inequality that

$$\begin{aligned}
\|u_{n+1}-u^*\| &= \left\|\frac{\alpha(1-\beta_n)}{\alpha+1}\left(\frac{u_n+u_{n+1}}{2}\right) + \beta_n f(u_n) + \frac{1-\beta_n}{1+\alpha}G\left(\frac{u_n+u_{n+1}}{2}\right) - u^*\right\| \\
&= \left\|(1-\beta_n)\left(\frac{\alpha}{\alpha+1}w_n + \frac{1}{1+\alpha}G(w_n) - u^*\right) + \beta_n\left(f(u_n)-u^*\right)\right\| \\
&\leq (1-\beta_n)\left\|\frac{\alpha}{\alpha+1}w_n + \frac{1}{1+\alpha}G(w_n) - u^*\right\| + \beta_n\left\|\left(f(u_n)-u^*\right)\right\| \\
&= \frac{1-\beta_n}{\alpha+1}\left\|\alpha(w_n-u^*) + G(w_n) - G(u^*)\right\| + \beta_n\left\|f(u_n)-u^*\right\|.
\end{aligned}$$

Since G is α-enriched nonexpansive mapping, we have

$$\begin{aligned}
\|u_{n+1}-u^*\| &\leq (1-\beta_n)\|w_n-u^*\| + \beta_n\|f(u_n)-u^*\| \\
&= (1-\beta_n)\left\|\frac{1}{2}(u_n-u^*) + \frac{1}{2}(u_{n+1}-u^*)\right\| + \beta_n\|f(u_n)-u^*\| \\
&\leq \frac{1-\beta_n}{2}\|u_n-u^*\| + \frac{1-\beta_n}{2}\|u_{n+1}-u^*\| + \beta_n\|f(u_n)-u^*\|.
\end{aligned}$$

This gives

$$\frac{1+\beta_n}{2}\|u_{n+1}-u^*\| \leq \frac{1-\beta_n}{2}\|u_n-u^*\| + \beta_n\|f(u_n)-u^*\|.$$

From the fact that f is contraction mapping with constant κ, we have

$$\begin{aligned}\frac{1+\beta_n}{2}\left\|u_{n+1}-u^*\right\| &\leq \frac{1-\beta_n}{2}\left\|u_n-u^*\right\|+\beta_n\left\|f\left(u_n\right)-f\left(u^*\right)\right\|+\beta_n\left\|f\left(u^*\right)-u^*\right\| \\ &\leq \frac{1-\beta_n}{2}\left\|u_n-u^*\right\|+\beta_n\kappa\left\|u_n-u^*\right\|+\beta_n\left\|f\left(u^*\right)-u^*\right\| \\ &= \frac{1-\beta_n+2\beta_n\kappa}{2}\left\|u_n-u^*\right\|+\beta_n\left\|f\left(u^*\right)-u^*\right\|.\end{aligned}$$

This implies that

$$\begin{aligned}\left\|u_{n+1}-u^*\right\| &\leq \frac{1-\beta_n+2\beta_n\kappa}{1+\beta_n}\left\|u_n-u^*\right\|+\frac{2\beta_n}{1+\beta_n}\left\|f\left(u^*\right)-u^*\right\| \\ &= \left(1-\frac{2\beta_n(1-\kappa)}{1+\beta_n}\right)\left\|u_n-u^*\right\|+\frac{2\beta_n(1-\kappa)}{1+\beta_n}\frac{\left\|f\left(u^*\right)-u^*\right\|}{1-\kappa} \\ &\leq \max\left\{\left\|u_n-u^*\right\|, \frac{\left\|f\left(u^*\right)-u^*\right\|}{1-\kappa}\right\}.\end{aligned}$$

Inductively, we obtain

$$\left\|u_{n+1}-u^*\right\| \leq \max\left\{\left\|u_1-u^*\right\|, \frac{\left\|f\left(u^*\right)-u^*\right\|}{1-\kappa}\right\}, \quad \forall \geq 1.$$

This completes the proof. □

Lemma 3.5. *Let G be an α-enriched nonexpansive mapping with $\mathcal{F}(G) \neq \emptyset$. Suppose that $\{u_n\}$ is a sequence generated through Algorithm 3.1 with $\{\beta_n\}$ satisfying the following conditions:*

$$\text{(C1) } \beta_n \to 0 \text{ as } n \to \infty \qquad \text{(C2) } \sum_{n=1}^{\infty} \beta_n = \infty \qquad \text{(C3) } \sum_{n=1}^{\infty} \left|\beta_{n+1}-\beta_n\right| < \infty.$$

Then we have the following:

$$\text{(P1) } \left\|u_{n+1}-u_n\right\| \to 0 \quad \text{as} \quad n \to \infty; \qquad \text{(P2) } \left\|u_n-G\left(u_n\right)\right\| \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. Set $w_n = \dfrac{u_n+u_{n+1}}{2}$ and G_α be the mapping defined by

$$G_\alpha(u) = \frac{\alpha}{\alpha+1}u + \frac{1}{\alpha+1}G(u), \quad \forall u \in Dom(G).$$

Then Algorithm 3.1 and (3.7) yield that

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \left\|\frac{\alpha(1-\beta_n)}{\alpha+1}(w_n)+\beta_n f(u_n)+\frac{1-\beta_n}{1+\alpha}G(w_n)-u_n\right\| \\
&= \|\beta_n f(u_n)+(1-\beta_n)G_\alpha(w_n)-u_n\| \\
&= \|\beta_n f(u_n)+(1-\beta_n)G_\alpha(w_n)-\beta_{n-1}f(u_{n-1})-(1-\beta_{n-1})G_\alpha(w_{n-1})\| \\
&= \Big\|(1-\beta_n)(G_\alpha(w_n)-G_\alpha(w_{n-1}))+(\beta_n-\beta_{n-1})(f(u_{n-1})-G_\alpha(w_{n-1})) \\
&\quad+\beta_n(f(u_n)-f(u_{n-1}))\Big\| \\
&\le (1-\beta_n)\|G_\alpha(w_n)-G_\alpha(w_{n-1})\|+|\beta_n-\beta_{n-1}|\,\|f(u_{n-1})-G_\alpha(w_{n-1})\| \\
&\quad+\beta_n\|f(u_n)-f(u_{n-1})\| \\
&= \frac{1-\beta_n}{\alpha+1}\|\alpha(w_n-w_{n-1})+G(w_n)-G(w_{n-1})\| \\
&\quad+|\beta_n-\beta_{n-1}|\,\|f(u_{n-1})-G_\alpha(w_{n-1})\|+\beta_n\|f(u_n)-f(u_{n-1})\|.
\end{aligned}$$

This and the facts that G is an α-enriched nonexpansive mapping and f is a contraction with constant κ yield

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le (1-\beta_n)\|w_n-w_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|f(u_{n-1})-G_\alpha(w_{n-1})\| \\
&\quad+\beta_n\kappa\|u_n-u_{n-1}\| \\
&= \frac{1-\beta_n}{2}\|u_{n+1}-u_{n-1}\|+|\beta_n-\beta_{n-1}|\,\|f(u_{n-1})-G_\alpha(w_{n-1})\| \\
&\quad+\beta_n\kappa\|u_n-u_{n-1}\| \\
&\le \frac{1-\beta_n}{2}\|u_{n+1}-u_n\|+\frac{1-\beta_n}{2}\|u_n-u_{n-1}\| \\
&\quad+|\beta_n-\beta_{n-1}|\,\|f(u_{n-1})-G_\alpha(w_{n-1})\|+\beta_n\kappa\|u_n-u_{n-1}\| \\
&= \frac{1-\beta_n}{2}\|u_{n+1}-u_n\|+\frac{1-\beta_n+2\beta_n\kappa}{2}\|u_n-u_{n-1}\| \\
&\quad+|\beta_n-\beta_{n-1}|\,\|f(u_{n-1})-G_\alpha(w_{n-1})\| \\
&\le \frac{1-\beta_n}{2}\|u_{n+1}-u_n\|+\frac{1-\beta_n+2\beta_n\kappa}{2}\|u_n-u_{n-1}\| \\
&\quad+\eta|\beta_n-\beta_{n-1}|,
\end{aligned}$$

where η is a positive number such that $\eta\ge\sup_{n\ge1}\|f(u_{n-1})-G_\alpha(w_{n-1})\|$. Consequently, we get

$$\frac{1+\beta_n}{2}\|u_{n+1}-u_n\|\le\frac{1-\beta_n+2\beta_n\kappa}{2}\|u_n-u_{n-1}\|+\eta|\beta_n-\beta_{n-1}|,$$

which resulted to

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le \frac{1-\beta_n+2\beta_n\kappa}{1+\beta_n}\|u_n-u_{n-1}\|+\frac{2\eta}{1+\beta_n}|\beta_n-\beta_{n-1}| \\
&= \left(1-\frac{2\beta_n(1-\kappa)}{1+\beta_n}\right)\|u_n-u_{n-1}\|+\frac{2\eta}{1+\beta_n}|\beta_n-\beta_{n-1}| \\
&\le \left(1-\frac{2\beta_n(1-\kappa)}{1+\beta_n}\right)\|u_n-u_{n-1}\|+2\eta|\beta_n-\beta_{n-1}|.
\end{aligned}$$

Thus Lemma 2.3 and the assumptions on $\{\beta_n\}$ yield the claim (P1). For Claim (P2), we start by obtaining the following inequalities:

$$\begin{aligned}\|u_n - G(u_n)\| &= (\alpha+1)\|u_n - G_\alpha(u_n)\| \\ &\leq (\alpha+1)\Big(\|u_n - u_{n+1}\| + \|u_{n+1} - G_\alpha(w_n)\| + \|G_\alpha(w_n) - G_\alpha(u_n)\| \Big) \\ &= (\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\|u_{n+1} - G_\alpha(w_n)\| \\ &+ \|\alpha(w_n - u_n) + G(w_n) - G(u_n)\|.\end{aligned}$$

This and the fact that G is an α-enriched nonexpansive mapping yield

$$\begin{aligned}\|u_n - G(u_n)\| &\leq (\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\|u_{n+1} - G_\alpha(w_n)\| \\ &+ (\alpha+1)\|w_n - u_n\| \\ &= \frac{3}{2}(\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\|u_{n+1} - G_\alpha(w_n)\| \\ &= \frac{3}{2}(\alpha+1)\|u_n - u_{n+1}\| \\ &+ (\alpha+1)\|\beta_n f(u_n) + (1-\beta_n)G_\alpha(w_n) - G_\alpha(w_n)\| \\ &= \frac{3}{2}(\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\beta_n\|f(u_n) - G_\alpha(w_n)\| \\ &\leq \frac{3}{2}(\alpha+1)\|u_n - u_{n+1}\| + (\alpha+1)\eta\beta_n.\end{aligned}$$

As $n \to +\infty$, the last inequality and Claim (P1) yield Claim (P2). This completes the proof. □

Theorem 3.2. *Let $G : \mathcal{E} \to \mathcal{E}$ be an α-enriched nonexpansive mapping with a fixed point and $f : \mathcal{E} \to \mathcal{E}$ be a contraction mapping. Suppose that $\{u_n\}$ is a sequence generated through Algorithm 3.1 with $\{\beta_n\}$ satisfying the following conditions:*

$$\text{(C1) } \beta_n \to 0 \text{ as } n \to \infty; \qquad \text{(C2) } \sum_{n=1}^{\infty} \beta_n = \infty; \qquad \text{(C3) } \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Then $\{u_n\}$ converges strongly to the unique point $u^* \in \mathcal{F}(G)$ with a minimal norm.

Proof. Since f is a contraction mapping, $P_{\mathcal{F}(G)} f$ is also a contraction. Therefore, by the Banach contraction mapping, we have $u^* \in \mathcal{E}$ such that $u^* = P_{\mathcal{F}(G)} f(u^*)$. It is worth noting that the metric projection $P_{\mathcal{F}(G)}$ is well-defined since $\mathcal{F}(G)$ is nonempty closed and convex. By the properties of the metric projection, we have

$$\langle u^* - f(u^*), u^* - p\rangle \leq 0, \quad \forall p \in \mathcal{F}(G).$$

The boundedness of $\{u_n\}$ yields a subsequence $\{u_{n_k}\}$ that weakly converges to a point u^o. By the demiclosedness property of G and $(P2)$ of Lemma 2.2, we have $u^o \in \mathcal{F}(G)$. Moreover, without lost of generality, we have

$$\limsup_{n\to\infty} \langle u^* - f(u^*), u^* - u_n\rangle = \lim_{k\to\infty} \langle u^* - f(u^*), u^* - u_{n_k}\rangle.$$

Consequently, we have

$$\limsup_{n\to\infty} \langle u^* - f(u^*), u^* - u_n\rangle = \langle u^* - f(u^*), u^* - u^o\rangle \leq 0. \tag{3.8}$$

Let $w_n = \dfrac{u_n + u_{n+1}}{2}$. It follows from (3.7) and Lemma 2.2 (2) that

$$\begin{aligned}
\|u_{n+1} - u^*\|^2 &= \left\| \frac{\alpha(1-\beta_n)}{\alpha+1} \left(\frac{u_n + u_{n+1}}{2} \right) + \beta_n f(u_n) + \frac{1-\beta_n}{1+\alpha} G\left(\frac{u_n + u_{n+1}}{2} \right) - u^* \right\|^2 \\
&= \left\| (1-\beta_n) \left(\frac{\alpha}{\alpha+1} w_n + \frac{1}{1+\alpha} G(w_n) - u^* \right) + \beta_n (f(u_n) - u^*) \right\|^2 \\
&= (1-\beta_n)^2 \left\| \frac{\alpha}{\alpha+1} w_n + \frac{1}{1+\alpha} G(w_n) - u^* \right\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\beta_n(1-\beta_n) \left\langle \frac{\alpha}{\alpha+1} w_n + \frac{1}{1+\alpha} G(w_n) - u^*, f(u_n) - u^* \right\rangle \\
&= \left(\frac{1-\beta_n}{\alpha+1} \right)^2 \|\alpha(w_n - u^*) + G(w_n) - G(u^*)\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\beta_n(1-\beta_n) \langle G_\alpha(w_n) - u^*, f(u_n) - u^* \rangle .
\end{aligned}$$

As a consequence of the immediate inequality, the fact that G is α-enriched nonexpansive mapping, f is a contraction with constant κ, and the Cauchy Schwartz inequality yield that

$$\begin{aligned}
\|u_{n+1} - u^*\|^2 &\leq (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\beta_n(1-\beta_n) \langle G_\alpha(w_n) - u^*, f(u_n) - u^* \rangle \\
&\leq (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\beta_n(1-\beta_n) \langle G_\alpha(w_n) - u^*, f(u_n) - f(u^*) \rangle \\
&+ 2\beta_n(1-\beta_n) \langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle \\
&\leq (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\kappa\beta_n(1-\beta_n) \|G_\alpha(w_n) - G_\alpha(u^*)\| \|u_n - u^*\| \\
&+ 2\beta_n(1-\beta_n) \langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle \\
&= (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ \frac{2\kappa\beta_n(1-\beta_n)}{\alpha+1} \|\alpha(w_n - u^*) + G(w_n) - G(u^*)\| \|u_n - u^*\| \\
&+ 2\beta_n(1-\beta_n) \langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle \\
&\leq (1-\beta_n)^2 \|w_n - u^*\|^2 + \beta_n^2 \|(f(u_n) - u^*)\|^2 \\
&+ 2\kappa\beta_n(1-\beta_n) \|w_n - u^*\| \|u_n - u^*\| \\
&+ 2\beta_n(1-\beta_n) \langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle .
\end{aligned}$$

Now, setting

$$\theta_n = \|u_n - u^*\|$$

and

$$\phi_n = \beta_n^2 \|(f(u_n) - u^*)\|^2 + 2\beta_n(1-\beta_n) \langle G_\alpha(w_n) - u^*, f(u^*) - u^* \rangle ,$$

we get

$$(1-\beta_n)^2 \|w_n - u^*\|^2 + 2\kappa\beta_n(1-\beta_n) \|w_n - u^*\| \theta_n + \phi_n - \theta_{n+1}^2 \geq 0.$$

Solving this quadratic inequality with respect to $\|w_n - u^*\|$ yields

$$\begin{aligned}\|w_n - u^*\| &\geq \frac{-2\kappa\beta_n(1-\beta_n)\theta_n + \sqrt{4\kappa^2\beta_n^2(1-\beta_n)^2\theta_n^2 - 4\left(1-\beta_n\right)^2\left(\phi_n - \theta_{n+1}^2\right)}}{2\left(1-\beta_n\right)^2}\\ &= \frac{-\kappa\beta_n\theta_n + \sqrt{\kappa^2\beta_n^2\theta_n^2 + \theta_{n+1}^2 - \phi_n}}{1-\beta_n}.\end{aligned}$$

This implies that

$$\frac{1}{2}\|u_{n+1} - u^*\| + \frac{1}{2}\|u_n - u^*\| \geq \frac{-\kappa\beta_n\theta_n + \sqrt{\kappa^2\beta_n^2\theta_n^2 + \theta_{n+1}^2 - \phi_n}}{1-\beta_n}.$$

Thus it turns out that

$$\kappa^2\beta_n^2\theta_n^2 + \theta_{n+1}^2 - \phi_n \leq \left[\frac{1}{2}\left(1-\beta_n\right)\|u_{n+1} - u^*\| + \left(1 + (2\kappa - 1)\beta_n\right)\frac{1}{2}\|u_n - u^*\|\right]^2.$$

Thus, from the fact that $2ab \leq a^2 + b^2$ for all $a, b \in \mathbb{R}$, it follows that

$$\begin{aligned}\kappa^2\beta_n^2\theta_n^2 + \theta_{n+1}^2 - \phi_n &\leq \frac{1}{4}\left[\left(1-\beta_n\right)^2\|u_{n+1} - u^*\|^2 + \left(1 + (2\kappa-1)\beta_n\right)^2\|u_n - u^*\|^2\right]\\ &\quad + \frac{1}{2}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)\|u_{n+1} - u^*\|\,\|u_n - u^*\|\\ &\leq \frac{1}{4}\left[\left(1-\beta_n\right)^2\|u_{n+1} - u^*\|^2 + \left(1 + (2\kappa-1)\beta_n\right)^2\|u_n - u^*\|^2\right]\\ &\quad + \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)\|u_{n+1} - u^*\|^2\\ &\quad + \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)\|u_n - u^*\|^2.\end{aligned}$$

By simple calculations, we can rewrite the last inequality as follows:

$$\theta_{n+1}^2 \leq \psi_n\theta_n^2 + \varphi_n, \tag{3.9}$$

where

$$\psi_n = \frac{\frac{1}{4}\left(1 + (2\kappa-1)\beta_n\right)^2 + \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right) - \kappa^2\beta_n^2}{1 - \frac{1}{4}(1-\beta_n)^2 - \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)}$$

and

$$\varphi_n = \frac{\phi_n}{1 - \frac{1}{4}(1-\beta_n)^2 - \frac{1}{4}\left(1-\beta_n\right)\left(1 + (2\kappa-1)\beta_n\right)}.$$

Observe further that

$$\psi_n = \frac{\frac{1}{2}\left(1 + (2\kappa-1)\beta_n\right)\left(1 - (1-\kappa)\beta_n\right) - \kappa^2\beta_n^2}{1 - \frac{1}{2}(1-\beta_n)\left(1 - (1-\kappa)\beta_n\right)}$$

and

$$\varphi_n = \frac{\phi_n}{1 - \frac{1}{2}(1-\beta_n)\left(1 - (1-\kappa)\beta_n\right)}.$$

Now, we complete the proof by showing that $\theta_n \to 0$ as $n \to \infty$. For that, consider a function g defined by

$$g(t) = \frac{2(1-\kappa) - (1-\kappa)^2 t + \kappa^2 t}{1 - \frac{1}{2}(1-t)(1-(1-\kappa)t)}.$$

It can be observed that

$$g(t) = \frac{1}{t}\left[1 - \frac{\frac{1}{2}\left(1+(2\kappa-1)t\right)\left(1-(1-\kappa)t\right) - \kappa^2 t^2}{1 - \frac{1}{2}(1-t)\left(1-(1-\kappa)t\right)}\right]$$

and

$$\lim_{t\to 0} g(t) = 4(1-\kappa).$$

This implies that, for $\epsilon = 3(1-\kappa)$, there exists $\delta \in (0,1)$ such that $g(t) > \epsilon$ for all $t \in (0,\delta)$. Thus we have

$$1 - \frac{\frac{1}{2}\left(1+(2\kappa-1)t\right)\left(1-(1-\kappa)t\right) - \kappa^2 t^2}{1 - \frac{1}{2}(1-t)\left(1-(1-\kappa)t\right)} > \epsilon t \tag{3.10}$$

for all $t \in (0,\delta)$. By the assumption that $\beta_n \to 0$ as $n \to \infty$, we can have a natural number N^* such that $\beta_n < \delta$ for all $n \geq N^*$. Consequently, it follows from (3.10) that $1 - \psi_n > \epsilon\beta_n$ for all $n \geq N^*$. Thus (3.9) gives

$$\theta_{n+1}^2 \leq (1-\epsilon\beta_n)\theta_n^2 + \varphi_n \quad \forall\, n \geq N^*. \tag{3.11}$$

Moreover, we have

$$\begin{aligned}
\frac{\phi_n}{\beta_n} &= \beta_n \left\|(f(u_n) - u^*)\right\|^2 + 2(1-\beta_n)\left\langle G_\alpha(w_n) - u^*, f(u^*) - u^*\right\rangle \\
&= \beta_n \left\|(f(u_n) - u^*)\right\|^2 + 2(1-\beta_n)\left\langle G_\alpha(w_n) - u_{n+1}, f(u^*) - u^*\right\rangle \\
&+ \left\langle u_{n+1} - u^*, f(u^*) - u^*\right\rangle \\
&= \beta_n \left\|(f(u_n) - u^*)\right\|^2 + 2(1-\beta_n)\beta_n\left\langle G_\alpha(w_n) - f(u_n), f(u^*) - u^*\right\rangle \\
&+ \left\langle u_{n+1} - u^*, f(u^*) - u^*\right\rangle.
\end{aligned}$$

This, (3.8) and the assumption on $\{\beta_n\}$ yield that

$$\limsup_{n\to\infty} \frac{\phi_n}{\beta_n} \leq 0.$$

So, we have

$$\limsup_{n\to\infty} \frac{\psi_n}{\beta_n} \leq 0.$$

Finally, Lemma 2.3 and (3.11) yield that $\lim_{n\to\infty} \theta_n = 0$. This completes the proof. □

Next, we deduce the following corollary which is the main results of [25]:

Corollary 3.1. *Let $G : \mathcal{E} \to \mathcal{E}$ be a nonexpansive mapping with a fixed point and $f : \mathcal{E} \to \mathcal{E}$ be a contraction mapping. Suppose that $\{u_n\}$ is a sequence generated by (1.6) with $\{\beta_n\}$ satisfying the following conditions:*

(C1) $\beta_n \to 0$ *as* $n \to \infty$; *(C2)* $\sum_{n=1}^{\infty} \beta_n = \infty$; *(C3)* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{u_n\}$ *converges strongly to the unique point* $u^* \in \mathcal{F}(G)$ *with a minimal norm.*

Proof. When $\alpha = 0$, then Algorithm 3.1 reduces to (1.6). Consequently, Theorem 3.2 yields the proof using the fact that a nonexpansive mapping is 0-enriched nonexpansive. □

Recall that a multivalued mapping $M : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be monotone if, for every $u, w \in \mathcal{H}$, $x \in Mu$ and $y \in Mw$, we have

$$\langle u - w, x - y\rangle \geq 0.$$

Moreover, M is said to be maximal monotone if, for every $(u, x) \in \mathcal{H}$,

$$\langle x - y, u - w\rangle \geq 0$$

for every $(w, y) \in \mathrm{Graph}(M)$ implies $x \in Mu$. It is known that, if M is maximal monotone, then, for any $\xi > 0$, the mapping $(I + \xi M)^{-1}$ is single-valued, nonexpansive and

$$\mathrm{dom}\left((I + \xi M)^{-1}\right) = \mathcal{H}.$$

Furthermore, we have

$$0 \in Mu^* \quad \Longleftrightarrow \quad u \in \mathcal{F}\left((I + \xi M)^{-1}\right).$$

Corollary 3.2. *Let* $M : \mathcal{H} \to 2^{\mathcal{H}}$ *be a maximal monotone. For any* $\xi > 0$ *and* $\eta \geq 1$*, consider* $G : \mathcal{H} \to \mathcal{H}$ *defined by*

$$Gu = \eta(I + \xi M)^{-1}u - (\eta - 1)u, \quad \forall u \in \mathcal{H}.$$

Suppose that $\{u_n\}$ *is a sequence generated by* (1.6) *with* $\alpha = \eta - 1$ *and* $\{\beta_n\}$ *satisfying the following conditions:*

$$\text{(C1) } \beta_n \to 0 \ \text{ as } \ n \to \infty; \qquad \text{(C2) } \sum_{n=1}^{\infty} \beta_n = \infty; \qquad \text{(C3) } \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Then $\{u_n\}$ *converges strongly to a zero of* M*.*

Proof. Using the fact that $(I + \xi M)^{-1}$ is nonexpansive, we can deduce that G is an α-enriched nonexpansive mapping. Indeed, for all $u, w \in \mathcal{H}$, we get

$$\begin{aligned}\|\alpha(u - w) + Gu - Gw\| &= \left\|(\alpha + 1)(I + \xi M)^{-1}u - (\alpha + 1)(I + \xi M)^{-1}w\right\| \\ &= (\alpha + 1)\left\|(I + \xi M)^{-1}u - (I + \xi M)^{-1}w\right\| \\ &\leq (\alpha + 1)\|u - w\|.\end{aligned}$$

Thus Theorem 3.2 guarantees that $\{u_n\}$ converges to a fixed point of G. Let the limit point be u^*. Then we have

$$u^* = Gu^* \quad \Longleftrightarrow \quad u^* = \eta(I + \xi M)^{-1}u^* - (\eta - 1)u^* \quad \Longleftrightarrow \quad u^* = (I + \xi M)^{-1}u^*.$$

Consequently, it follows that $0 \in Mu^*$. This completes the proof. □

A particular case of the immediate corollary is the case when M is equal to the subdifferential of a convex proper and lower semi-continuous function $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$. In this regard, we have the next corollary:

Corollary 3.3. *Let* $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ *be a convex proper and lower semi-continuous function. For any* $\xi > 0$ *and* $\eta \geq 1$*, consider* $G : \mathcal{H} \to \mathcal{H}$ *defined by*

$$Gu = \eta(I + \xi \partial f)^{-1}u - (\eta - 1)u, \quad \forall u \in \mathcal{H}.$$

Suppose that $\{u_n\}$ *is a sequence generated by* (1.6) *with* $\alpha = \eta - 1$ *and* $\{\beta_n\}$ *satisfying the following conditions:*

$$\text{(C1) } \beta_n \to 0 \text{ as } n \to \infty; \qquad \text{(C2) } \sum_{n=1}^{\infty} \beta_n = \infty; \qquad \text{(C3) } \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Then $\{u_n\}$ *converges strongly to a minimizer of* f.

Proof. The proof follows from Corollary 3.2 and the fact that

$$0 \in \partial f(u^*) \quad \Longleftrightarrow \quad f(u^*) \leq f(u), \quad \forall u \in \mathcal{H}.$$

□

4. Numerical Illustrations

This part contains two numerical problems where the underlined mappings are not nonexpansive but enriched nonexpansive mappings. The purpose is to show the implementation of our method with respect to such mappings and to show the impact of the proposed scheme on handling stiff equations involving enriched nonexpansive mapping.

Example 4.1. *Consider* $\mathcal{H} = \mathbb{R}$ *endowed with the usual norm and take* $\mathcal{E} = \left[\frac{1}{2},\ 2\right]$. *Define a mapping* $G : \mathcal{E} \to \mathcal{E}$ *by* $Gu = \frac{1}{u}$, *for all* $u \in \mathcal{E}$. *Then* G *is* $\frac{3}{2}$*-enriched nonexpansive mapping with* 1 *as fixed point but not nonexpansive (see* [5]*). For this example, we set* $f : u \mapsto \frac{u+1}{2}$. *Consequently, Algorithm* 3.1 *gives*

$$u_{n+1} = \frac{\alpha(1-\beta_n)}{\alpha+1}\left(\frac{u_n + u_{n+1}}{2}\right) + \beta_n \frac{u_n + 1}{2} + \frac{1-\beta_n}{1+\alpha}\left(\frac{2}{u_n + u_{n+1}}\right).$$

Solving for u_{n+1}, *we get*

$$u_{n+1} = \frac{\tau_n u_n - \beta_n - \sqrt{(\beta_n+2)^2 u_n^2 + 2\beta_n(\beta_n+2)u_n + \beta_n^2 + 16c_n(2-\alpha c_n)}}{2(\alpha c_n - 2)} \tag{4.12}$$

for all $n \geq 1$, *where* $\tau_n = 2 - 2\alpha c_n - \beta_n$ *and* $c_n = \dfrac{1-\beta_n}{\alpha+1}$.

To show the numerical patterns of the scheme for this example, we set $\beta_n = \frac{1}{n+1}$ *and use* α *as* $3/2$. *The first few generated values when truncated to six decimal places, are shown in Table* 1. *In the table, 'IMS Alg' stands for our proposed implicit midpoint scheme which reduces to* (4.12) *and 'MKM Alg' stands for the modified Krasnosel'skiĭ-Mann scheme of Berinde* [7] *which is stated in* (1.2). *We note here that the sequence* $\{\delta_n\}$ *is considered as* $\delta_n = \frac{n}{2n+2}$ *to meet up with the assumption in* [7].

TABLE 1. Few numerical values of $\{u_n\}$

	Case 1		*Case 2*		*Case 3*		*Case 4*	
n	*IMS Alg*	*MKM Alg*	*IMS Alg*	*MKM Alg*	*IMS Alg*	*MKM Alg*	*IMS Alg*	*MKM Alg*
1	2	2	1.85	1.85	0.75	0.75	0.5	0.5
2	1.374738	1	1.313463	0.940608	0.927576	0.604167	0.872325	0.625
3	1.107046	0.777778	1.087951	0.756091	0.982499	0.680109	0.970009	0.681111
4	1.024461	0.752976	1.019912	0.746526	0.996253	0.72764	0.993625	0.727846
5	1.004859	0.771613	1.003945	0.769573	0.999269	0.763835	0.998758	0.763896
6	1.000887	0.79504	1.000719	0.79431	0.999867	0.792278	0.999774	0.792299
7	1.000152	0.8162	1.000124	0.815913	0.999977	0.815116	0.999961	0.815124
8	1.000025	0.834232	1.00002	0.834111	0.999996	0.833776	0.999994	0.833779
9	1.000004	0.849452	1.000003	0.849398	0.999999	0.849249	0.999999	0.849251
10	1.000001	0.862341	1	0.862317	1	0.862248	1	0.862249
11	1	0.873338	1	0.873326	1	0.873293	1	0.873294
12	1	0.882797	1	0.882791	1	0.882775	1	0.882775
13	1	0.891001	1	0.890998	1	0.89099	1	0.89099
14	1	0.898172	1	0.898171	1	0.898167	1	0.898167
15	1	0.904487	1	0.904486	1	0.904484	1	0.904484
16	1	0.910085	1	0.910084	1	0.910083	1	0.910083
17	1	0.915077	1	0.915077	1	0.915076	1	0.915076
18	1	0.919555	1	0.919555	1	0.919555	1	0.919555
19	1	0.923592	1	0.923592	1	0.923592	1	0.923592
20	1	0.92725	1	0.92725	1	0.92725	1	0.92725

Remark 4.3. *Table 1 shows that based on the Example 4.1, the proposed scheme (IMS Alg) converges faster than the modified Krasnosel'skiĭ-Mann scheme. Indeed, IMS Alg reaches the fixed point value* (1) *in less than ten loops.*

Example 4.2. *For any $\xi > 0$, consider the stiff equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = -\xi y(t), \quad y(0) = y_1 = \beta, \quad \forall t \geq 0.$$

This represents a model of a lot of physical Phenomena most of which arise through sciences and engineering. This problem has the solution

$$y(t) = \beta e^{-\xi t}, \quad y(t) \to 0 \text{ as } t \to \infty.$$

The aim of numerical methods for solving such initial value problems is primarily to exhibit the structure of the solution. So, in most cases due to the tediousness of establishing an analytical solution of stiff equations, engineers employ numerical methods to describe the solution. Since our proposed algorithm is based on the implicit midpoint rule (which is prominent in handling stiff equations), we investigate *the performance of the proposed scheme in exhibiting the structure of the solution in comparison with the modified Krasnosel'skiĭ-Mann scheme.*

Now, consider G *as a mapping such that* $u \mapsto -(\xi+1)u$. *Then* G *is not nonexpansive mapping. However* G *is* $\xi/2$*-enriched nonexpansive mapping since*

$$\begin{aligned}\left\|\frac{\xi}{2}(u-w)+Gu-Gw\right\| &= \left\|\frac{5}{2}(u-w)-(\xi+1)(u-w)\right\| \\ &= \left\|\left(\frac{\xi}{2}-\xi-1\right)(u-w)\right\| \\ &= \frac{\xi+2}{2}\|u-w\| \\ &= \left(\frac{\xi}{2}+1\right)\|u-w\|.\end{aligned}$$

For this example, we take $f : u \mapsto \dfrac{u}{5}$ *and so Algorithm 3.1 gives*

$$u_{n+1} = \frac{\alpha(1-\beta_n)}{\alpha+1}\left(\frac{u_n+u_{n+1}}{2}\right) + \frac{\beta_n}{5}u_n - (\xi+1)\frac{1-\beta_n}{1+\alpha}\left(\frac{u_n+u_{n+1}}{2}\right).$$

Solving for u_{n+1} *and substituting* $\alpha = \xi/2$, *we get*

$$u_{n+1} = \frac{7\beta_n - 5}{5(3-\beta_n)}u_n.$$

To extract numerically the strsucture of the solution using our proposed scheme and that of (1.2), *we maintain the sequence values of* $\{\beta_n\}$ *for the two algorithms as in the Example 4.1 and set* $\beta = 1$. *The measure of how far the iterate* u_n *is from the value of the exact solution* $\beta e^{-\xi(n-1)}$ *at each* n *(up to* $n = 20$*) is shown in Table 2 and Figure 1-6. In the table, the column VIMS represents in absolute value how far our proposed scheme is from the value of the exact solution. Cases 1-6 similarly show how far is the iterate* (1.2) *is to the value of the exact solution when* δ_n $(n \in \mathbb{N})$ *is set as* $\frac{1}{2}$, $\frac{n}{2n+2}$, $\frac{n}{n+100}$, $\frac{4}{5}$, $\frac{n}{5n+3}$ *and* $\frac{2n}{3n+7}$, *respectively.*

5. Conclusion Remarks

In this work, we analyzed the convergence of a viscosity implicit midpoint scheme to a fixed point of an enriched nonexpansive mapping within the setting of Hilbert spaces. We established that the sequence generated by this scheme converges strongly to a particular fixed point of the underlying mapping. We provided examples where the mappings are not nonexpansive but are instead enriched nonexpansive, and we derived the explicit form of the proposed scheme. The numerical results obtained using this scheme are reported, demonstrating the distance between the iterates of the proposed scheme and those of the exact solution, in comparison to the well-known modified Krasnosel'skiĭ-Mann scheme by Berinde [7]. Despite the computational demands, our numerical data shows that, for the example considered, the proposed scheme achieves a higher degree of numerical stability than the Krasnosel'skiĭ-Mann scheme of Berinde [7]. Given that geodesically connected spaces can be viewed as nonlinear analogs of normed linear spaces [17, 13], it would be an interesting direction for future studies to extend the analyses presented here to such settings.

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TABLE 2. Few numerical values of $\left\{\left|u_n - e^{-3/2(n-1)}\right|\right\}$

	VIMS	MKM Alg					
n		Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
1	0.950213	0.950213	0.950213	0.950213	0.950213	0.950213	0.950213
2	0.131109	0.546823	0.403966	0.266767	0.718252	0.332538	0.375395
3	0.021521	0.380174	0.228247	0.08656	0.664256	0.142341	0.20435
4	0.006226	0.308042	0.161237	0.036165	0.707768	0.075549	0.144595
5	0.001335	0.263439	0.122915	0.015374	0.80004	0.0418	0.112944
6	0.000422	0.235351	0.100118	0.007188	0.943077	0.024625	0.096086
7	0.000104	0.216107	0.08491	0.003459	1.143201	0.014917	0.086329
8	3.31E-05	0.202607	0.074303	0.001744	1.41472	0.009279	0.08094
9	8.96E-06	0.192958	0.06657	0.000905	1.778503	0.005877	0.07841
10	2.82E-06	0.186067	0.060769	0.000485	2.263781	0.003778	0.077983
11	8.09E-07	0.181234	0.056321	0.000266	2.910575	0.002458	0.079242
12	2.53E-07	0.177998	0.052857	0.00015	3.773353	0.001615	0.081977
13	7.55E-08	0.176042	0.050131	8.59E-05	4.926092	0.00107	0.086104
14	2.36E-08	0.175143	0.047975	5.04E-05	6.469265	0.000714	0.091624
15	7.20E-09	0.175143	0.04627	3.01E-05	8.53943	0.000479	0.098605
16	2.26E-09	0.175925	0.044927	1.83E-05	11.32237	0.000323	0.107172
17	7.00E-10	0.177404	0.043883	1.13E-05	15.07112	0.000219	0.117505
18	2.20E-10	0.179516	0.04309	7.13E-06	20.13071	0.000149	0.129837
19	6.90E-11	0.182215	0.04251	4.56E-06	26.97213	0.000102	0.144465
20	2.18E-11	0.185469	0.042115	2.95E-06	36.23898	6.99E-05	0.161749

Moreover, the research was only supported by King Mongkut's University of Technology Thonburi's Postdoctoral Fellowship.

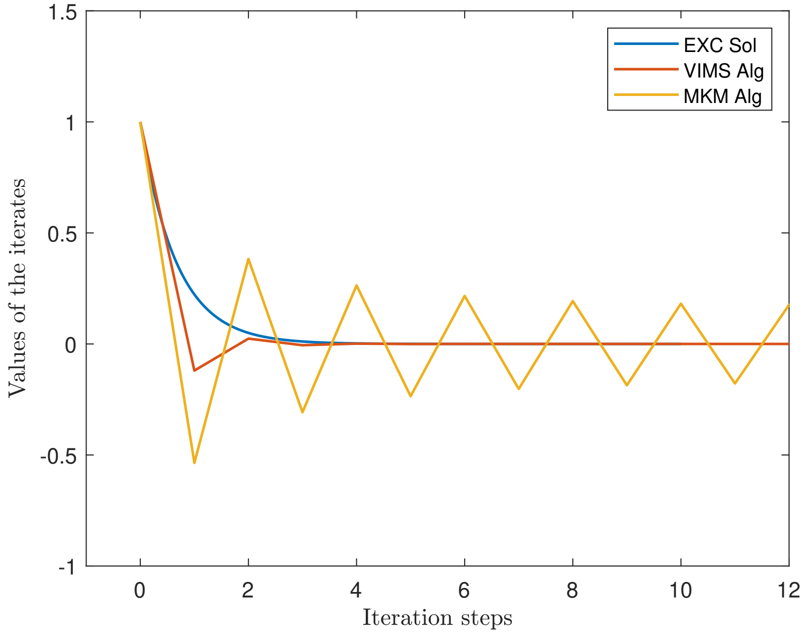

FIGURE 1. Numerical stability due to Case 1

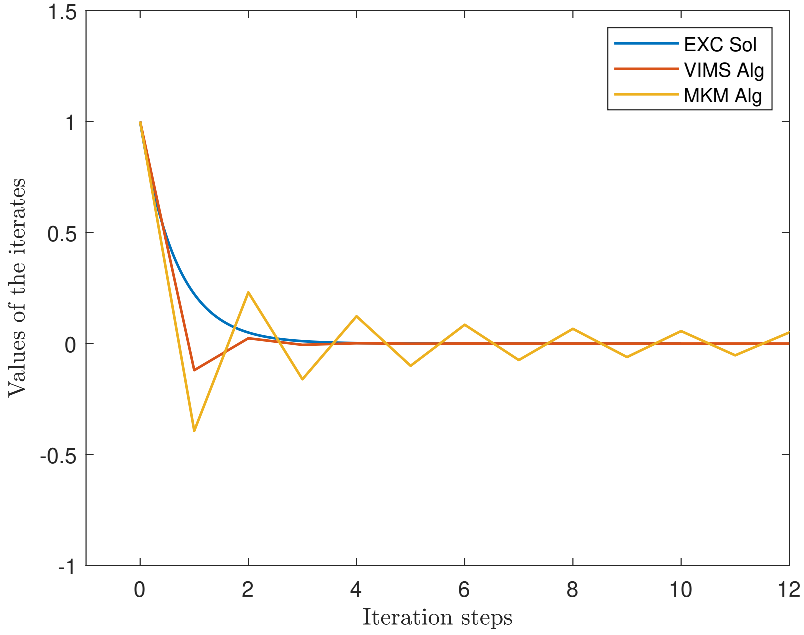

FIGURE 2. Numerical stability due to Case 2

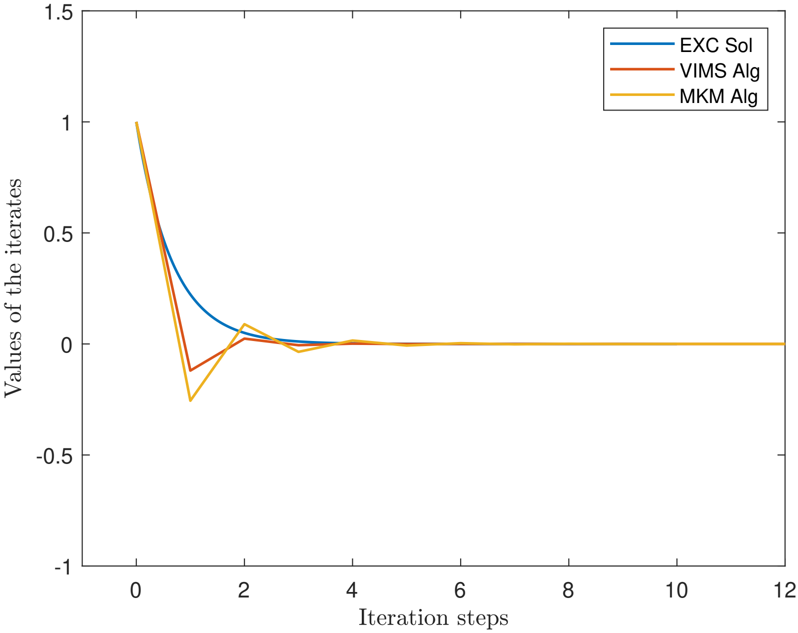

FIGURE 3. Numerical stability due to Case 3

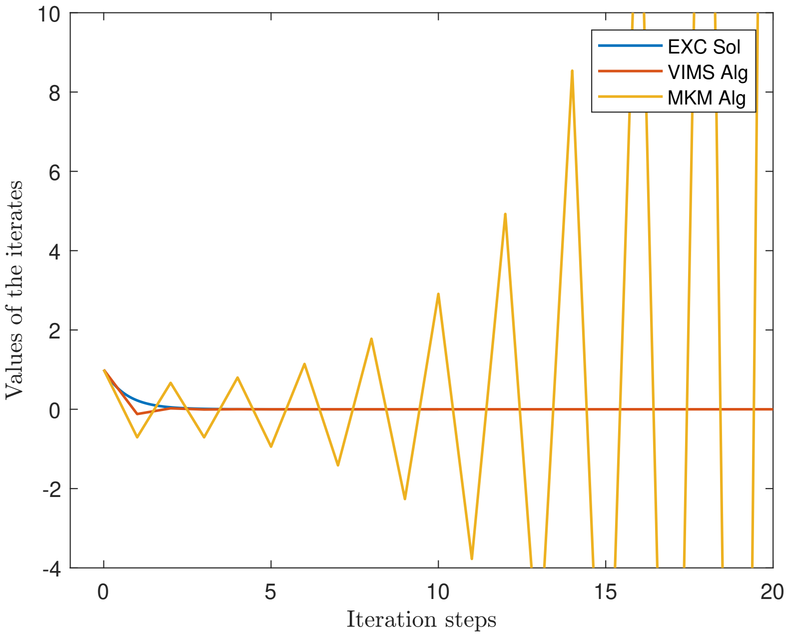

FIGURE 4. Numerical stability due to Case 4

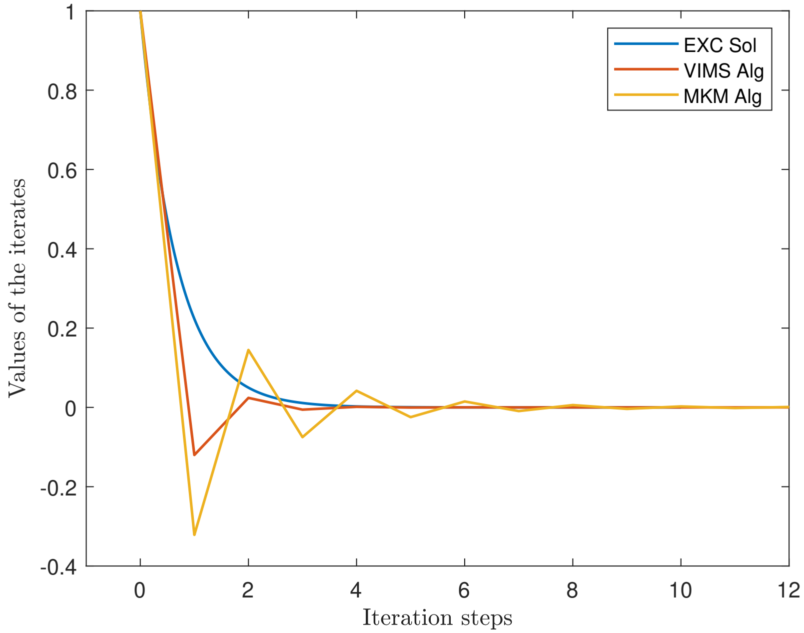

FIGURE 5. Numerical stability due to Case 5

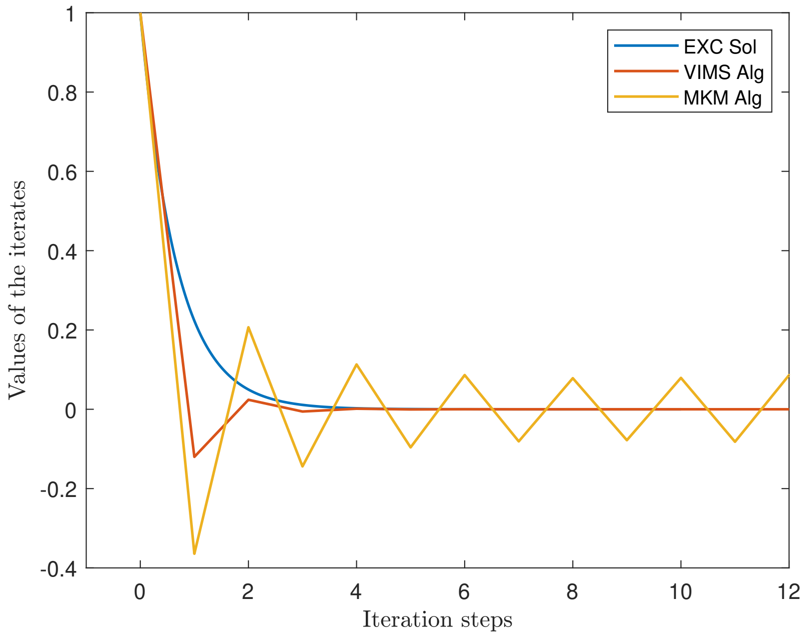

FIGURE 6. Numerical stability due to Case 6

REFERENCES

[1] M. A. Alghamdi, M. A. Alghamdi, N. Shahzad and H.-K. Xu: *The implicit midpoint rule for nonexpansive mappings*, Fixed Point Theory Appl., **2014** (2014), Article ID: 96.

[2] H. Attouch:*Viscosity solutions of minimization problems*, SIAM J. Optim., **6** (3) (1996), 769–806.

[3] W. Auzinger, R. Frank: *Asymptotic error expansions for stiff equations: an analysis for the implicit midpoint and trapezoidal rules in the strongly stiff case*, Numer. Math., **56** (5) (1989), 469–499.

[4] G. Bader, P. Deuflhard: *A semi-implicit mid-point rule for stiff systems of ordinary differential equations* Numer. Math., **41** (3) (1983), 373–398.

[5] V. Berinde: *Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces*, Carpathian J. Math., **35** (3) (2019), 293–304.

[6] V. Berinde: *Approximating fixed points of enriched nonexpansive mappings in banach spaces by using a retraction-displacement condition*, Carpathian J. Math., **36** (1) (2020), 27–34.

[7] V. Berinde: *A modified krasnosel'skiĭ–mann iterative algorithm for approximating fixed points of enriched nonexpansive mappings*, Symmetry, **14** (1) (2022), Article ID: 123.

[8] V. Berinde, M. Păcurar: *Recent developments in the fixed point theory of enriched contractive mappings. A survey*, Creat. Math. Inform., **33** (2024), 137–159.

[9] C. Izuchukwu, C. C. Okeke and F. O. Isiogugu: *A viscosity iterative technique for split variational inclusion and fixed point problems between a hilbert space and a banach space*, J. Fixed Point Theory Appl., **20** (4) (2018), 1–25.

[10] A. Moudafi: *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl., **241** (1) (2000), 46–55.

[11] S. Salisu, V. Berinde, S. Sriwongsa and P. Kumam: *On approximating fixed points of strictly pseudocontractive mappings in metric spaces*, Carpathian J. Math., **40** (2) (2024), 419–430.

[12] S. Salisu, L. Hashim, A. Y. Inuwa and A. U. Saje: *Implicit midpoint scheme for enriched nonexpansive mappings*, Nonlinear Convex Anal. Opt., **1** (2) (2022), 211–225.

[13] S. Salisu, P. Kumam and S. Sriwongsa: *Strong convergence theorems for fixed point of multi-valued mappings in Hadamard spaces*, J. Inequal. Appl., **2022** (2022), Article ID: 143.

[14] S. Salisu, P. Kumam and S. Sriwongsa: *On fixed points of enriched contractions and enriched nonexpansive mappings*, Carpathian J. Math., **39** (1) (2023), 237–254.

[15] S. Salisu, P. Kumam, S. Sriwongsa and V. Berinde: *Viscosity scheme with enriched mappings for hierarchical variational inequalities in certain geodesic spaces*, Fixed Point Theory, (in press), 2023.

[16] S. Salisu, P. Kumam, S. Sriwongsa and A. Y. Inuwa: *Enriched multi-valued nonexpansive mappings in geodesic spaces*, Rend. Circ. Mat. Palermo (2), **73** (4) (2024), 1435–1451.

[17] S. Salisu, M. S. Minjibir, P. Kumam and S. Sriwongsa: *Convergence theorems for fixed points in* $cat_p(0)$ *spaces*, J. Appl. Math. Comput., **69** (2023), 631–650.

[18] S. Salisu, S. Sriwongsa, P. Kumam and V. Berinde: *Variational inequality and proximal scheme for enriched nonexpansive mappings in cat(0) spaces*, J. Nonlinear Convex Anal., **25** (7) (2024), 1759–1776.

[19] C. Schneider: *Analysis of the linearly implicit mid-point rule for differential-algebraic equations*, Electron. Trans. Numer. Anal., **1** (1993), 1–10.

[20] S. Somali: *Implicit midpoint rule to the nonlinear degenerate boundary value problems*, Int. J. Comput. Math., **79** (3) (2002), 327–332.

[21] S. Somali, S. Davulcu: *Implicit midpoint rule and extrapolation to singularly perturbed boundary value problems*, Int. J. Comput. Math., **75** (1) (2000), 117–127.

[22] Y. Song, X. Liu: *Convergence comparison of several iteration algorithms for the common fixed point problems*, Fixed Point Theory Appl., **2009** (2009), Article ID: 824374.

[23] H.-K. Xu: *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc., **66** (1) (2002), 240–256.

[24] H.-K. Xu: *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298** (1) (2004), 279–291.

[25] H.-K. Xu, M. A. Alghamdi and N. Shahzad: *The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces*, Fixed Point Theory Appl., **2015** (2015), Article ID: 41.

[26] H.-K. Xu, R. G. Ori: *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim., **22** (5-6) (2001), 767–773.

[27] Y. Yao, H. Zhou and Y.-C. Liou: *Strong convergence of a modified Krasnoselski-Mann iterative algorithm for non-expansive mappings*, J. Appl. Math. Comput., **29** (1-2) (2009), 383–389.

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Research Article

Higher order approximation of functions by modified Goodman-Sharma operators

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ABSTRACT. Here we study the approximation properties of a modified Goodman-Sharma operator recently considered by Acu and Agrawal in [1]. This operator is linear but not positive. It has the advantage of a higher order of approximation of functions compared with the Goodman-Sharma operator. We prove direct and strong converse theorems in terms of a related K-functional.

Keywords: Bernstein-Durrmeyer operator, Goodman-Sharma operator, direct theorem, strong converse theorem, K-functional.

2020 Mathematics Subject Classification: 41A35, 41A10, 41A25, 41A27, 41A17.

1. INTRODUCTION

In 1987, W. Chen and independently T. N. T. Goodman and A. Sharma presented at conferences in China and Bulgaria, respectively a new modification of the classical Bernstein operators. For $n \in \mathbb{N}$ and functions $f(x) \in C[0,1]$, they introduce the linear operator (see [5] and [9, 10]):

$$U_n(f,x) = f(0)P_{n,0}(x) + \sum_{k=1}^{n-1}\Big(\int_0^1 (n-1)P_{n-2,k-1}(t)f(t)\,dt\Big)P_{n,k}(x) + f(1)P_{n,n}(x), \tag{1.1}$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}, \qquad k = 0,\dots,n. \tag{1.2}$$

Operators of this kind were investigated by many authors (see [14], [4], [13], [11], [7, 8], [2], etc.) and are generally known as genuine Bernstein-Durrmeyer operators. Note that the operators in (1.1) are actually a limit case of Bernstein type operators with Jacobi weights studied by Berens and Xu [3]. If we set

$$u_{n,k}(f) = \begin{cases} f(0), & k = 0,\\ (n-1)\int_0^1 P_{n-2,k-1}(t)f(t)\,dt, & k = 1,\dots,n-1,\\ f(1), & k = n,\end{cases}$$

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the operators defined in (1.1) take the form

$$U_n(f,x)=\sum_{k=0}^{n}u_{n,k}(f)P_{n,k}(x)\qquad\text{or}\qquad U_nf=\sum_{k=0}^{n}u_{n,k}(f)P_{n,k}.$$

Let us denote, as usual, by

$$\varphi(x)=x(1-x)$$

the weight function which is naturally connected to the second order derivative of the Bernstein operator. Also, we set

$$\widetilde{D}f(x):=\varphi(x)f''(x) \tag{1.3}$$

and

$$\widetilde{D}^2f:=\widetilde{D}\widetilde{D}f,\qquad \widetilde{D}^{\ell+1}f:=\widetilde{D}\widetilde{D}^{\ell}f,\qquad \ell=2,3\dots .$$

Recently, Acu and Agrawal [1] studied a family of Bernstein-Durrmeyer operators, as they modify U_nf by replacing the Bernstein basis polynomials $P_{n,k}$ with linear combinations of Bernstein basis polynomials of lower degree with coefficients which are polynomials of appropriate degree. For special choice of the parameters, these operators lack the positivity but have a higher than $O(n^{-1})$ order of approximation. For example, Acu and Agrawal considered operators with $O(n^{-2})$ and $O(n^{-3})$ rate of approximation, see [1, Section 3].

The results presented in [1] inspired the authors of the current paper to explore in more depth the operators explicitly defined by

$$\widetilde{U}_n(f,x)=\sum_{k=0}^{n}u_{n,k}(f)\widetilde{P}_{n,k}(x),\qquad x\in[0,1], \tag{1.4}$$

where

$$\widetilde{P}_{n,k}(x)=P_{n,k}(x)-\frac{1}{n}\,\widetilde{D}P_{n,k}(x). \tag{1.5}$$

By defining an appropriate K-functional, we prove direct and strong converse inequality of Type B in the terminology of [6].

In order to state our main results, we need some definitions.

Let $L_\infty[0,1]$ be the space of all Lebesgue measurable and essentially bounded functions in $[0,1]$ and $AC_{loc}(0,1)$ consists of the functions absolutely continuous in any subinterval $[a,b]\subset(0,1)$. Let us set

$$W^2(\varphi)[0,1]:=\big\{g\,:\,g,g'\in AC_{loc}(0,1),\ \widetilde{D}g\in L_\infty[0,1]\big\}.$$

By $W_0^2(\varphi)[0,1]$, we denote the subspace of $W^2(\varphi)[0,1]$ of functions g satisfying the additional boundary conditions

$$\lim_{x\to0^+}\widetilde{D}g=0,\qquad \lim_{x\to1^-}\widetilde{D}g=0.$$

Henceforth, by $\|\cdot\|$ we mean the uniform norm on the interval $[0,1]$. For functions $f\in C[0,1]$ and $t>0$, we define the K-functional

$$K(f,t):=\inf\big\{\|f-g\|+t\|\widetilde{D}^2g\|\ :\ g\in W_0^2(\varphi)[0,1],\ \widetilde{D}g\in W^2(\varphi)[0,1]\big\}. \tag{1.6}$$

Here we investigate the error of approximation of functions $f\in C[0,1]$ by the modified Goodman-Sharma operator (1.4). Our main results read as follows.

Theorem 1.1. *If $n\in\mathbb{N}$, $n\ge2$, and $f\in C[0,1]$, then*

$$\|\widetilde{U}_nf-f\|\le(1+\sqrt{3})\,K\Big(f,\frac{1}{n^2}\Big).$$

Theorem 1.2. *For every function $f \in C[0,1]$ and $n \in \mathbb{N}$, $n \geq 2$, there exist constants $C, L > 0$ such that*

$$K\Big(f, \frac{1}{n^2}\Big) \leq C\,\frac{\ell^2}{n^2}\big(\|\widetilde{U}_n f - f\| + \|\widetilde{U}_\ell f - f)\|\big).$$

for all $\ell \geq Ln$.

Remark 1.1. *Another way to state Theorem 1.1 and Theorem 1.2 is the following: there exists a natural number k such that*

$$K\Big(f, \frac{1}{n^2}\Big) \sim \|\widetilde{U}_n f - f\| + \|\widetilde{U}_{kn} f - f\|.$$

The paper is organized as follows. In Section 1 state of the art is described. Preliminary and auxiliary results are presented in Section 2. Section 3 includes an estimation of the norm of the operator $\widetilde{U}_n$, a Jackson type inequality and a proof of the direct inequality in Theorem 1.1. The last Section 4 is devoted to a converse result for the modified Goodman-Sharma operator (1.4). Inequalities of the Voronovskaya type and Bernstein type for $\widetilde{U}_n$ are proved using the differential operator $\widetilde{D}$, defined in (1.3). Theorem 1.2 represents a strong converse inequality of Type B, according to Ditzian-Ivanov classification in [6]. Complete proof of the converse theorem is given.

2. Preliminaries and auxiliary results

By $B_n f$, $n \in \mathbb{N}$, we denote the Bernstein operators determined for functions f,

$$B_n(f, x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) P_{n,k}(x), \qquad x \in [0,1],$$

where $P_{n,k}$ are the Bernstein basis polynomials (1.2). The Bernstein operator central moments play important role in many applications and they are defined by

$$\mu_{n,i}(x) = B_n\big((t-x)^i, x\big) = \sum_{k=0}^{n} \Big(\frac{k}{n} - x\Big)^i P_{n,k}(x), \qquad i = 0, 1, \dots .$$

We summarize some well known useful properties of the Bernstein polynomials. Further on we assume $P_{n,k} := 0$ if $k < 0$ or $k > n$.

Proposition 2.1 (see, e.g. [12]). *(a) The following identities are valid:*

$$\sum_{k=0}^{n} k P_{n,k}(x) = nx, \qquad \sum_{k=0}^{n} (n-k) P_{n,k}(x) = n(1-x), \tag{2.7}$$

$$\sum_{k=0}^{n} k(k-1) P_{n,k}(x) = n(n-1)x^2, \tag{2.8}$$

$$\sum_{k=0}^{n} (n-k)(n-k-1) P_{n,k}(x) = n(n-1)(1-x)^2, \tag{2.9}$$

$$P'_{n,k}(x) = n\big[P_{n-1,k-1}(x) - P_{n-1,k}(x)\big], \tag{2.10}$$

$$P''_{n,k}(x) = n(n-1)\big[P_{n-2,k-2}(x) - 2P_{n-2,k-1}(x) + P_{n-2,k}(x)\big]. \tag{2.11}$$

(b) For the low-order moments $\mu_{n,i}(x)$, we have:

$$\begin{aligned}\mu_{n,0}(x) &= B_n\big((t-x)^0, x\big) = 1,\\ \mu_{n,1}(x) &= B_n\big((t-x), x\big) = 0,\\ \mu_{n,2}(x) &= B_n\big((t-x)^2, x\big) = \frac{\varphi(x)}{n},\\ \mu_{n,3}(x) &= B_n\big((t-x)^3, x\big) = \frac{(1-2x)\varphi(x)}{n^2},\\ \mu_{n,4}(x) &= B_n\big((t-x)^4, x\big) = \frac{3(n-2)\varphi^2(x)}{n^3} + \frac{\varphi(x)}{n^3}.\end{aligned}$$

The operators U_n, $\widetilde{U}_n$ and the differential operator $\widetilde{D}$ satisfy interesting properties.

Proposition 2.2. *If the operators U_n, $\widetilde{U}_n$ and the differential operator $\widetilde{D}$ are defined as in (1.1), (1.4) and (1.3), respectively, then*

(a) $\widetilde{D}U_n f = U_n \widetilde{D} f$ *for* $f \in W_0^2(\varphi)[0,1]$;
(b) $\widetilde{U}_n f = U_n\big(f - \frac{1}{n}\widetilde{D}f\big)$ *for* $f \in W_0^2(\varphi)[0,1]$;
(c) $\widetilde{D}\widetilde{U}_n f = \widetilde{U}_n \widetilde{D} f$ *for* $f \in W_0^2(\varphi)[0,1]$;
(d) $U_n\widetilde{U}_n f = \widetilde{U}_n U_n f$ *for* $f \in W_0^2(\varphi)[0,1]$;
(e) $\widetilde{U}_m\widetilde{U}_n f = \widetilde{U}_n \widetilde{U}_m f$ *for* $f \in W_0^2(\varphi)[0,1]$;
(f) $\lim_{n\to\infty} \widetilde{U}_n f = f$ *for* $f \in W^2(\varphi)[0,1]$;
(g) $\big\|\widetilde{D}U_n f\big\| \le \big\|\widetilde{D} f\big\|$ *for* $f \in W^2(\varphi)[0,1]$.

Proof. For the proof of (a), see [14, Lemma 4.2]. We have

$$\begin{aligned}\widetilde{U}_n f &= \sum_{k=0}^{n} u_{n,k}(f)\widetilde{P}_{n,k}\\ &= u_{n,0}(f)\Big(P_{n,0} - \frac{1}{n}\widetilde{D}P_{n,0}\Big) + \sum_{k=1}^{n-1} u_{n,k}(f)\Big(P_{n,k} - \frac{1}{n}\widetilde{D}P_{n,k}\Big) + u_{n,n}(f)\Big(P_{n,n} - \frac{1}{n}\widetilde{D}P_{n,n}\Big)\\ &= u_{n,0}(f)P_{n,0} + \sum_{k=1}^{n-1} u_{n,k}(f)P_{n,k} + u_{n,n}(f)P_{n,n}\\ &\quad - \frac{\varphi}{n}\Big(u_{n,0}(f)P''_{n,0} + \sum_{k=1}^{n-1} u_{n,k}(f)P''_{n,k} + u_{n,n}(f)P''_{n,n}\Big)\\ &= U_n f - \frac{1}{n}\varphi\,(U_n f)''.\end{aligned}$$

Then from (a), we obtain

$$\widetilde{U}_n f = U_n f - \frac{1}{n}\widetilde{D}U_n f = U_n f - \frac{1}{n}U_n\widetilde{D}f = U_n\Big(f - \frac{1}{n}\widetilde{D}f\Big)$$

which proves (b). Now, commutative properties (c) and (d) follow from (b) and (a):

$$\widetilde{D}\widetilde{U}_n f = \widetilde{D}U_n\Big(f - \frac{1}{n}\widetilde{D}f\Big) = U_n\Big(\widetilde{D}f - \frac{1}{n}\widetilde{D}\widetilde{D}f\Big) = \widetilde{U}_n(\widetilde{D}f),$$

and

$$U_n\widetilde{U}_n f = U_nU_n\Big(f - \frac{1}{n}\widetilde{D}f\Big) = U_nU_n f - \frac{1}{n}U_nU_n\widetilde{D}f = U_nU_n f - \frac{1}{n}U_n\widetilde{D}U_n f = \widetilde{U}_nU_n f.$$

The operators $\widetilde{U}_n$ commute in the sense of (e), since

$$\begin{aligned}\widetilde{U}_m\widetilde{U}_n f &= \widetilde{U}_m U_n\big(f - \frac{1}{n}\,\widetilde{D}f\big)\\ &= U_mU_nf - \frac{1}{n}\,U_mU_n\widetilde{D}f - \frac{1}{m}\,\widetilde{D}U_mU_nf + \frac{1}{mn}\,U_m\widetilde{D}^2U_nf\\ &= U_mU_n\Big(f - \frac{m+n}{mn}\,\widetilde{D}f + \frac{1}{mn}\,\widetilde{D}^2f\Big).\end{aligned}$$

The same expression on the right-hand side we obtain for $\widetilde{U}_n\widetilde{U}_m f$ because of properties (a), (b) and $U_mU_nf = U_nU_mf$. We recall two more properties of the operator U_n and function $f \in W^2(\varphi)[0,1]$ (see [14, eqs. (4.8), (2.4)]):

$$\|U_nf - f\| \le \frac{1}{n}\,\big\|\widetilde{D}f\big\|, \qquad (2.12)$$

$$\big\|U_n\widetilde{D}f\big\| \le \big\|\widetilde{D}f\big\|.$$

Therefore

$$\|\widetilde{U}_nf - f\| = \Big\|U_nf - \frac{1}{n}\,U_n\widetilde{D}f - f\Big\| \le \|U_nf - f\| + \frac{1}{n}\,\big\|U_n\widetilde{D}f\big\| \le \frac{2}{n}\,\|\widetilde{D}f\|,$$

hence $\lim_{n\to\infty}\|\widetilde{U}_nf - f\| = 0$, i.e. the limit (f) holds true.

From the proof of Lemma 4.2 in [14] for every $g \in W^2(\varphi)[0,1]$, we have

$$\widetilde{D}U_ng(x) = \sum_{k=1}^{n-1} P_{n,k}(x)\int_0^1 (n-1)P_{n-2,k-1}(t)\widetilde{D}g(t)\,dt,$$

From the last representation, we obtain

$$|\widetilde{D}U_ng(x)| \le \big\|\widetilde{D}g\big\|\sum_{k=1}^{n-1} P_{n,k}(x)\int_0^1 (n-1)P_{n-2,k-1}(t)\,dt \le \big\|\widetilde{D}g\big\|,$$

which proves (g). □

We now introduce a function that will prove useful in our investigations:

$$\begin{aligned}T_{n,k}(x) &:= k(k-1)\frac{1-x}{x} - 2k(n-k) + (n-k)(n-k-1)\frac{x}{1-x}\\ &= n\Big[-1 - \frac{1-2x}{\varphi(x)}\Big(\frac{k}{n} - x\Big) + \frac{n}{\varphi(x)}\Big(\frac{k}{n} - x\Big)^2\Big].\end{aligned} \qquad (2.13)$$

Observe that

$$T'_{n,k}(x) = -\frac{k(k-1)}{x^2} + \frac{(n-k)(n-k-1)}{(1-x)^2}, \qquad (2.14)$$

$$T''_{n,k}(x) = \frac{2k(k-1)}{x^3} + \frac{2(n-k)(n-k-1)}{(1-x)^3} > 0, \qquad x \in (0,1). \qquad (2.15)$$

Proposition 2.3.
(a) The following relation concerning $P_{n,k}$, $T_{n,k}$ and differential operator $\widetilde{D}$ holds:

$$\widetilde{D}P_{n,k}(x) = T_{n,k}(x)P_{n,k}(x). \qquad (2.16)$$

(b) If α is an arbitrary real number, then

$$\Phi(\alpha) := \sum_{k=0}^{n} \Big(\alpha - \frac{1}{n}\, T_{n,k}(x)\Big)^2 P_{n,k}(x) = \alpha^2 + 2 - \frac{2}{n}.$$

Proof. (a) From (2.10), (2.11) and $\varphi(x)P_{n,k}(x) = \frac{(k+1)(n-k+1)}{(n+1)(n+2)}\, P_{n+2,k+1}(x)$, it follows that

$$\begin{aligned}
\varphi(x)P''_{n,k}(x) &= n(n-1)\big[\varphi(x)P_{n-2,k-2}(x) - 2\varphi(x)P_{n-2,k-1}(x) + \varphi(x)P_{n-2,k}(x)\big]\\
&= n(n-1)\Big[\frac{(k-1)(n-k+1)}{n(n-1)}\, P_{n,k-1}(x) - 2\,\frac{k(n-k)}{n(n-1)}\, P_{n,k}(x)\\
&+ \frac{(k+1)(n-k-1)}{n(n-1)}\, P_{n,k+1}(x)\Big]\\
&= (k-1)(n-k+1)\, P_{n,k-1}(x) - 2k(n-k)\, P_{n,k}(x)\\
&+ (k+1)(n-k-1)\, P_{n,k+1}(x)\\
&= \Big[k(k-1)\frac{1-x}{x} - 2k(n-k) + (n-k)(n-k-1)\frac{x}{1-x}\Big] P_{n,k}(x)\\
&= T_{n,k}(x)P_{n,k}(x),
\end{aligned}$$

i.e. the identity (2.16).

(b) We apply the formulae for the Bernstein operator moments in Proposition 2.1 (b):

$$\begin{aligned}
\Phi(\alpha) &= \sum_{k=0}^{n} \Big[\alpha + 1 + \frac{1-2x}{\varphi(x)}\Big(\frac{k}{n} - x\Big) - \frac{n}{\varphi(x)}\Big(\frac{k}{n} - x\Big)^2\Big]^2 P_{n,k}(x)\\
&= \sum_{k=0}^{n} \Big[(\alpha+1)^2 + \frac{(1-2x)^2}{\varphi^2(x)}\Big(\frac{k}{n} - x\Big)^2 + \frac{n^2}{\varphi^2(x)}\Big(\frac{k}{n} - x\Big)^4 + \frac{2(\alpha+1)(1-2x)}{\varphi(x)}\Big(\frac{k}{n} - x\Big)\\
&- \frac{2(\alpha+1)n}{\varphi(x)}\Big(\frac{k}{n} - x\Big)^2 - \frac{2n(1-2x)}{\varphi^2(x)}\Big(\frac{k}{n} - x\Big)^3\Big] P_{n,k}(x)\\
&= (\alpha+1)^2\mu_{n,0}(x) + \frac{(1-2x)^2}{\varphi^2(x)}\, \mu_{n,2}(x) + \frac{n^2}{\varphi^2(x)}\, \mu_{n,4}(x) + \frac{2(\alpha+1)(1-2x)}{\varphi(x)}\, \mu_{n,1}(x)\\
&- \frac{2(\alpha+1)n}{\varphi(x)}\, \mu_{n,2}(x) - \frac{2n(1-2x)}{\varphi^2(x)}\, \mu_{n,3}(x)\\
&= (\alpha+1)^2\cdot 1 + \frac{(1-2x)^2}{\varphi^2(x)}\,\frac{\varphi(x)}{n} + \frac{n^2}{\varphi^2(x)}\,\frac{(3n-6)\varphi^2(x)+\varphi(x)}{n^3}\\
&+ \frac{2(\alpha+1)(1-2x)}{\varphi(x)}\cdot 0 - \frac{2(\alpha+1)n}{\varphi(x)}\,\frac{\varphi(x)}{n} - \frac{2n(1-2x)}{\varphi^2(x)}\,\frac{(1-2x)\varphi(x)}{n^2}\\
&= (\alpha+1)^2 + \frac{1-4\varphi(x)}{n\varphi(x)} + \frac{(3n-6)\varphi(x)+1}{n\varphi(x)} - 2(\alpha+1) - \frac{2(1-4\varphi(x))}{n\varphi(x)}\\
&= \alpha^2 + 2\alpha + 1 + \frac{1}{n\varphi(x)} - \frac{4}{n} + 3 - \frac{6}{n} + \frac{1}{n\varphi(x)} - 2\alpha - 2 - \frac{2}{n\varphi(x)} + \frac{8}{n}\\
&= \alpha^2 + 2 - \frac{2}{n}.
\end{aligned}$$

□

Auxiliary technical results will be useful for further estimations.

Proposition 2.4. *If* $n \in \mathbb{N}$, $n \geq 2$, *and*

$$\lambda(n) := \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)}, \qquad \theta(n) := \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)^2},$$

then

$$\frac{1}{2n^2} \leq \lambda(n) \leq \frac{1}{n^2}, \tag{2.17}$$

$$\theta(n) \leq \frac{4}{9n^3}. \tag{2.18}$$

Proof. Since $\frac{k}{k-1}\frac{n-1}{n} \leq 1$ for $k \geq n$, we have for the lower estimate of $\lambda(n)$

$$\lambda(n) \geq \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)} \cdot \frac{k}{k-1} \cdot \frac{n-1}{n} = \frac{n-1}{n} \sum_{k=n}^{\infty} \frac{1}{(k-1)k(k+1)} = \frac{n-1}{n} \cdot \frac{1}{2(n-1)n} = \frac{1}{2n^2}.$$

For the upper estimates of $\lambda(n)$ and $\theta(n)$, we obtain

$$\lambda(n) < \sum_{k=n}^{\infty} \frac{1}{(k-1)k(k+1)} = \frac{1}{2n(n-1)} \leq \frac{1}{n^2},$$

$$\theta(n) < \sum_{k=n}^{\infty} \frac{1}{(k-1)k(k+1)(k+2)} = \frac{1}{3n(n^2-1)} \leq \frac{4}{9n^3}.$$

□

3. A DIRECT THEOREM

We will first prove the next upper estimate for the norm of the operator $\widetilde{U}_n$ defined in (1.4).

Lemma 3.1. *If* $n \in \mathbb{N}$ *and* $f \in C[0,1]$, *then*

$$\|\widetilde{U}_n f\| \leq \sqrt{3}\,\|f\|, \quad \textit{i.e.} \quad \|\widetilde{U}_n\| \leq \sqrt{3}. \tag{3.19}$$

Proof. We have

$$\widetilde{P}_{n,k}(x) = P_{n,k}(x) - \frac{1}{n}\,\widetilde{D}P_{n,k}(x) = \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)P_{n,k}(x).$$

Then for $x \in [0,1]$,

$$\big|\widetilde{U}_n(f,x)\big| = \left|\sum_{k=0}^{n} u_{n,k}(f)\widetilde{P}_{n,k}(x)\right| \leq \sum_{k=0}^{n} |u_{n,k}(f)|\,\big|\widetilde{P}_{n,k}(x)\big|$$

$$\leq \|f\| \sum_{k=0}^{n} \big|\widetilde{P}_{n,k}(x)\big| = \|f\| \sum_{k=0}^{n} \Big|1 - \frac{1}{n}\,T_{n,k}(x)\Big| P_{n,k}(x).$$

Applying Cauchy inequality, we obtain

$$\big|\widetilde{U}_n(f,x)\big| \leq \|f\| \sqrt{\sum_{k=0}^{n} \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)^2 P_{n,k}(x)}\ \sqrt{\sum_{k=0}^{n} P_{n,k}(x)}.$$

Since $\sum_{k=0}^{n} P_{n,k}(x) = 1$ identically, by Proposition 2.3 (b) with $\alpha = 1$, we find

$$\big|\widetilde{U}_n(f,x)\big| \leq \sqrt{3 - \frac{2}{n}}\,\|f\| < \sqrt{3}\,\|f\|, \qquad x \in [0,1].$$

Hence, inequality (3.19) follows. □

In order to prove a direct theorem for the approximation rate for functions f by the operator $\widetilde{U}_n f$, we need a Jackson type inequality.

Lemma 3.2. *If* $n \in \mathbb{N}$, $f \in W_0^2(\varphi)[0,1]$ *and* $\widetilde{D}f \in W^2(\varphi)[0,1]$, *then*

$$\|\widetilde{U}_n f - f\| \le \frac{1}{n^2}\|\widetilde{D}^2 f\|. \tag{3.20}$$

Proof. Having in mind the relation

$$U_k f - U_{k+1} f = \frac{1}{k(k+1)}\,\widetilde{D}U_{k+1} f,$$

(see [14, Lemma 4.1]) and Proposition 2.1 (a) for $f \in W_0^2(\varphi)[0,1]$, we obtain

$$\begin{aligned}
\widetilde{U}_k f - \widetilde{U}_{k+1} f &= U_k f - \frac{1}{k}\,\widetilde{D}U_k f - U_{k+1} f + \frac{1}{k+1}\,\widetilde{D}U_{k+1} f \\
&= U_k f - U_{k+1} f + \frac{1}{k+1}\,\widetilde{D}U_{k+1} f - \frac{1}{k}\,\widetilde{D}U_k f \\
&= \Big(\frac{1}{k} - \frac{1}{k+1}\Big)\widetilde{D}U_{k+1} f + \frac{1}{k+1}\,\widetilde{D}U_{k+1} f - \frac{1}{k}\,\widetilde{D}U_k f \\
&= -\frac{1}{k}\big(\widetilde{D}U_k f - \widetilde{D}U_{k+1} f\big) \\
&= -\frac{1}{k}\big(U_k \widetilde{D} f - U_{k+1}\widetilde{D} f\big) \\
&= -\frac{1}{k}\cdot\frac{1}{k(k+1)}\,\widetilde{D}U_{k+1}\widetilde{D} f,
\end{aligned}$$

i.e.,

$$\widetilde{U}_k f - \widetilde{U}_{k+1} f = -\frac{1}{k^2(k+1)}\,\widetilde{D}U_{k+1}\widetilde{D} f. \tag{3.21}$$

Therefore for every $s > n$, we have

$$\widetilde{U}_n f - \widetilde{U}_s f = \sum_{k=n}^{s-1}\big(\widetilde{U}_k f - \widetilde{U}_{k+1} f\big) = -\sum_{k=n}^{s-1}\frac{1}{k^2(k+1)}\,\widetilde{D}U_{k+1}\widetilde{D} f.$$

Letting $s \to \infty$ and by Proposition 2.2 (a) and (f), we obtain

$$\widetilde{U}_n f - f = -\sum_{k=n}^{\infty}\frac{1}{k^2(k+1)}\,\widetilde{D}U_{k+1}\widetilde{D} f. \tag{3.22}$$

Then from Proposition 2.1 (g) for $\widetilde{D}f \in W^2(\varphi)[0,1]$

$$\|\widetilde{U}_n f - f\| \le \sum_{k=n}^{\infty}\frac{1}{k^2(k+1)}\,\big\|\widetilde{D}U_{k+1}\widetilde{D} f\big\| \le \sum_{k=n}^{\infty}\frac{1}{k^2(k+1)}\,\big\|\widetilde{D}^2 f\big\|.$$

Proposition 2.4, (2.17), yields

$$\sum_{k=n}^{\infty}\frac{1}{k^2(k+1)} \le \frac{1}{n^2}.$$

Therefore

$$\big\|\widetilde{U}_n f - f\big\| \le \frac{1}{n^2}\,\big\|\widetilde{D}^2 f\big\|.$$

□

A direct result on the approximation rate of functions $f \in C[0,1]$ by the operators (1.4) in means of the K-functional (1.6) follows immediately from both lemmas above.

Proof of Theorem 1.1. Let g be arbitrary function, such that $g \in W_0^2(\varphi)[0,1]$ and $\widetilde{D}g \in W^2(\varphi)[0,1]$. Then by Lemma 3.1 and Lemma 3.2, we obtain

$$\begin{aligned}\|\widetilde{U}_n f - f\| &\le \|\widetilde{U}_n f - \widetilde{U}_n g\| + \|\widetilde{U}_n g - g\| + \|g - f\| \\ &\le (1+\sqrt{3})\|f-g\| + \frac{1}{n^2}\,\|\widetilde{D}^2 g\| \\ &\le (1+\sqrt{3})\Big(\|f-g\| + \frac{1}{n^2}\,\|\widetilde{D}^2 g\|\Big).\end{aligned}$$

Taking infimum over all functions g with $g \in W_0^2(\varphi)[0,1]$ and $\widetilde{D}g \in W^2(\varphi)[0,1]$, we obtain

$$\|\widetilde{U}_n f - f\| \le (1+\sqrt{3})\,K\Big(f, \frac{1}{n^2}\Big).$$

□

4. A STRONG CONVERSE RESULT

First, we will prove a Voronovskaya type result for the operator $\widetilde{U}_n$.

Lemma 4.3. *If* $\lambda(n) = \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)}$, $\theta(n) = \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)^2}$ *and* $f \in C[0,1]$ *is such that* $f, \widetilde{D}f \in W_0^2(\varphi)[0,1]$ *and* $\widetilde{D}^3 f \in L_\infty[0,1]$, *then*

$$\|\widetilde{U}_n f - f + \lambda(n)\widetilde{D}^2 f\| \le \theta(n)\,\|\widetilde{D}^3 f\|. \tag{4.23}$$

Proof. We have

$$\widetilde{U}_n f - f + \lambda(n)\widetilde{D}^2 f = -\sum_{k=n}^{\infty} \frac{U_{k+1}\widetilde{D}^2 f}{k^2(k+1)} + \sum_{k=n}^{\infty} \frac{\widetilde{D}^2 f}{k^2(k+1)} = \sum_{k=n}^{\infty} \frac{\widetilde{D}^2 f - U_{k+1}\widetilde{D}^2 f}{k^2(k+1)},$$

see the proof of Lemma 3.2, eq. (3.21). Then

$$\|\widetilde{U}_n f - f + \lambda(n)\widetilde{D}^2 f\| \le \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)}\,\|\widetilde{D}^2 f - U_{k+1}\widetilde{D}^2 f\|.$$

Using (2.12) with $\widetilde{D}^2 f$ instead of f, we obtain

$$\|\widetilde{U}_n f - f + \lambda(n)\widetilde{D}^2 f\| \le \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)} \cdot \frac{1}{(k+1)}\,\|\widetilde{D}\widetilde{D}^2 f\| = \theta(n)\,\|\widetilde{D}^3 f\|.$$

□

We need an inequality of Bernstein type.

Lemma 4.4. *Let* $n \in \mathbb{N}$, $n \ge 2$ *and* $f \in C[0,1]$. *Then the following inequality holds true*

$$\|\widetilde{D}\widetilde{U}_n f\| \le \widetilde{C}\,n\|f\|, \tag{4.24}$$

where $\widetilde{C} = 6.5 + \sqrt{6}$.

Proof. Since

$$\big|\widetilde{D}\widetilde{U}_n(f,x)\big| \le \sum_{k=0}^{n} |u_{n,k}(f)|\,\big|\widetilde{D}\widetilde{P}_{n,k}(x)\big| \le \|f\| \sum_{k=0}^{n} \big|\widetilde{D}\widetilde{P}_{n,k}(x)\big|,$$

it is sufficient to find an upper estimate for the quantity

$$\sum_{k=0}^{n} \big|\widetilde{D}\widetilde{P}_{n,k}(x)\big| = \sum_{k=0}^{n} \big|\varphi(x)\widetilde{P}''_{n,k}(x)\big|.$$

Remind that, according to (2.16), we have the relation

$$\widetilde{D}P_{n,k}(x) = \varphi(x)P''_{n,k}(x) = T_{n,k}(x)P_{n,k}(x).$$

Hence

$$\widetilde{P}_{n,k}(x) = P_{n,k}(x) - \frac{1}{n}\,\widetilde{D}P_{n,k}(x) = \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)P_{n,k}(x),$$

$$\widetilde{P}''_{n,k}(x) = \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)''P_{n,k}(x) + 2\Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)'P'_{n,k}(x) + \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)P''_{n,k}(x).$$

Then,

$$\begin{aligned}
\widetilde{D}\widetilde{P}_{n,k}(x) &= \varphi(x)\widetilde{P}''_{n,k}(x)\\
&= -\frac{\varphi(x)}{n}\,T''_{n,k}(x)P_{n,k}(x) - \frac{2\varphi(x)}{n}\,T'_{n,k}(x)P'_{n,k}(x) + \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)\varphi(x)P''_{n,k}(x)\\
&= -\frac{\varphi(x)}{n}\,T''_{n,k}(x)P_{n,k}(x) - \frac{2\varphi(x)}{n}\,T'_{n,k}(x)P'_{n,k}(x) + \Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)T_{n,k}(x)P_{n,k}(x).
\end{aligned}$$

Therefore

$$\sum_{k=0}^{n} \big|\widetilde{D}\widetilde{P}_{n,k}(x)\big| \le a_n(x) + b_n(x) + c_n(x),$$

where

$$\begin{aligned}
a_n(x) &= \frac{\varphi(x)}{n} \sum_{k=0}^{n} \big|T''_{n,k}(x)\big|P_{n,k}(x),\\
b_n(x) &= \frac{2\varphi(x)}{n} \sum_{k=0}^{n} \big|T'_{n,k}(x)P'_{n,k}(x)\big|,\\
c_n(x) &= \sum_{k=0}^{n} \Big|\Big(1 - \frac{1}{n}\,T_{n,k}(x)\Big)T_{n,k}(x)\Big|P_{n,k}(x).
\end{aligned}$$

<u>1. Estimate for $a_n(x)$.</u> From (2.15) and (2.8)–(2.9),

$$\begin{aligned}
\sum_{k=0}^{n} T''_{n,k}(x)P_{n,k}(x) &= \sum_{k=0}^{n} \Big(\frac{2k(k-1)}{x^3} + \frac{2(n-k)(n-k-1)}{(1-x)^3}\Big)P_{n,k}(x)\\
&= \frac{2}{x^3} \sum_{k=0}^{n} k(k-1)P_{n,k}(x) + \frac{2}{(1-x)^3} \sum_{k=0}^{n} (n-k)(n-k-1)P_{n,k}(x)\\
&= \frac{2}{x^3}\,n(n-1)x^2 + \frac{2}{(1-x)^3}\,n(n-1)(1-x)^2\\
&= \frac{2n(n-1)}{\varphi(x)}.
\end{aligned}$$

Having in mind $T''_{n,k}(x) > 0$ in (2.15), we obtain

$$a_n(x) = \frac{\varphi(x)}{n} \sum_{k=0}^{n} T''_{n,k}(x) P_{n,k}(x) = 2(n-1). \tag{4.25}$$

2. Estimate for $b_n(x)$. Observe that

$$\sum_{k=0}^{n} \left| T'_{n,k}(x)\, P'_{n,k}(x) \right| = \sum_{k=0}^{n} \left| T'_{n,k}(1-x)\, P'_{n,k}(1-x) \right|,$$

hence, there is a symmetry of the function $b_n(x)$ in $x = \frac{1}{2}$. Therefore, it is sufficient to estimate $b_n(x)$ for $x \in \left[0, \frac{1}{2}\right]$.

We will show that in $\left[0, \frac{1}{2}\right]$ the function $b_n(x)$ has exactly $\left\lfloor \frac{n-1}{2} \right\rfloor$ local extrema h_k attained at points in intervals $\left(\frac{k-1}{n}, \frac{k}{n}\right]$, $k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor$, respectively. We will estimate all the local maxima h_k and then an estimate for $b_n(x)$ will follow immediately.

(i) First, we prove that

$$S(x) := \frac{-2\varphi(x)}{n} \sum_{k=0}^{n} T'_{n,k}(x) P'_{n,k}(x) = 4(n-1).$$

From (2.10) and (2.14),

$$\sum_{k=0}^{n} T'_{n,k}(x) P'_{n,k}(x) = n \sum_{k=0}^{n-1} \big(T'_{n,k+1}(x) - T'_{n,k}(x) \big) P_{n-1,k}(x).$$

Since

$$\begin{aligned} T'_{n,k+1}(x) - T'_{n,k}(x) &= \frac{(n-k-1)(n-k-2)}{(1-x)^2} - \frac{(k+1)k}{x^2} + \frac{k(k-1)}{x^2} - \frac{(n-k)(n-k-1)}{(1-x)^2} \\ &= -\frac{2k}{x^2} - \frac{2(n-k-1)}{(1-x)^2}, \end{aligned}$$

using (2.7) we get

$$\begin{aligned} \sum_{k=0}^{n} T'_{n,k}(x) P'_{n,k}(x) &= -\frac{2n}{x^2} \sum_{k=0}^{n-1} k P_{n-1,k}(x) - \frac{2n}{(1-x)^2} \sum_{k=0}^{n-1} (n-k-1) P_{n-1,k}(x) \\ &= -\frac{2n}{x^2} (n-1)x - \frac{2n}{(1-x)^2} (n-1)(1-x) \\ &= -\frac{2n(n-1)}{\varphi(x)}. \end{aligned}$$

Therefore,

$$S(x) = \frac{-2\varphi(x)}{n} \cdot \frac{-2n(n-1)}{\varphi(x)} = 4(n-1). \tag{4.26}$$

(ii) By (2.15), $T''_{n,k}(x) > 0$, hence $-T'_{n,k}(x)$ strictly decreases for $x \in (0,1)$.

For $k = 0, 1$, we have $-T'_{n,k}(0^+) < 0$, then $-T'_{n,k}(x) < 0$, $x \in (0,1)$, and $\varphi(x) T'_{n,1}(x)$ has its only zero in $[0,1)$ at $\xi_1 = 0$.

For $k = 2, \dots, n-2$, we have $-T'_{n,k}(0^+) > 0$, and $T'_{n,k}(x)$ has a unique simple zero at

$$\xi_k = \frac{\sqrt{\binom{k}{2}}}{\sqrt{\binom{k}{2}} + \sqrt{\binom{n-k}{2}}} \in \left(\tfrac{k-1}{n}, \tfrac{k}{n}\right).$$

For $k = n-1, n$, we have $-T'_{n,k}(x) > 0$ for $x \in (0,1)$, and $-\varphi(x)T'_{n,n}(x) = 0$ only for $\xi_n = 1$ in $(0,1]$.

(iii) For the Bernstein basis polynomials on $(0,1)$, we have

$$P'_{n,0}(x) = -n(1-x)^{n-1} < 0,$$
$$P'_{n,k}(x) = n\binom{n}{k}x^{k-1}(1-x)^{n-k-1}\big(\frac{k}{n}-x\big), \text{ and } P'_{n,k}(x) = 0 \text{ only if } x = \frac{k}{n}$$
$$P'_{n,n}(x) = nx^{n-1} > 0.$$

(iv) Now, from (ii) and (iii) for $x \in (0,1)$,

$-\varphi(x)T'_{n,0}(x)\,P'_{n,0}(x) > 0$,
$-\varphi(x)T'_{n,1}(x)\,P'_{n,1}(x) > 0$ for $x \in \big(\xi_1, \frac{1}{n}\big) = \big(0, \frac{1}{n}\big)$,
$-\varphi(x)T'_{n,k}(x)\,P'_{n,k}(x) > 0$ for $x \in \big(\frac{k-1}{n}, \xi_k\big)$, $k = 2, \dots, \lfloor\frac{n-1}{2}\rfloor$,
$-\varphi(x)T'_{n,k}(x)\,P'_{n,k}(x) < 0$ for $x \in \big(\xi_k, \frac{k}{n}\big)$, $k = 2, \dots, \lfloor\frac{n-1}{2}\rfloor$,
$-\varphi(x)T'_{n,n}(x)\,P'_{n,n}(x) > 0$.

(v) From the observations in (ii)–(iv), it follows that

$$-\varphi(x)T'_{n,k}(x)\,P'_{n,k}(x) > 0, \qquad k = 0, \dots, n$$

except

$$-\varphi(x)T'_{n,k}(x)\,P'_{n,k}(x) < 0, \qquad x \in \big(\xi_k, \tfrac{k}{n}\big), \qquad k = 1, \dots, \big\lfloor\tfrac{n-1}{2}\big\rfloor,$$
$$-\varphi(x)T'_{n,n-k}(x)\,P'_{n,n-k}(x) < 0, \quad x \in \big(\tfrac{n-k}{n}, \xi_{n-k}\big), \quad k = 1, \dots, \big\lfloor\tfrac{n-1}{2}\big\rfloor.$$

Hence,

$$\sum_{k=0}^{n}\Big|\frac{-2\varphi(x)T'_{n,k}(x)}{n}\,P'_{n,k}(x)\Big| = S(x) = 4(n-1), \quad x \in \big[0, \tfrac12\big] \setminus \bigcup_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}\big(\xi_k, \tfrac{k}{n}\big).$$

Therefore, for $k = 1, \dots, \lfloor\frac{n-1}{2}\rfloor$,

$$b_n(x) = \begin{cases} 4(n-1), & x \in \big[\frac{k-1}{n}, \xi_k\big], \\ 4(n-1) + \dfrac{2\varphi(x)}{n}\big|T'_{n,k}(x)P'_{n,k}(x)\big|, & x \in \big[\xi_k, \frac{k}{n}\big]. \end{cases} \tag{4.27}$$

Moreover,

$$b_n(x) = 4(n-1), \qquad x \in \big[\tfrac{n-2}{2n}, \tfrac{n+2}{2n}\big],\ n \text{ even, and } x \in \big[\tfrac{n-1}{2n}, \tfrac{n+1}{2n}\big],\ n \text{ odd}.$$

(vi) This means that we have to estimate the maxima of the functions

$$s_k(x) := \Big|\frac{-2\varphi(x)T'_{n,k}(x)}{n}\,P'_{n,k}(x)\Big|, \qquad x \in \big[\xi_k, \tfrac{k}{n}\big], \quad k = 1, \dots, \big\lfloor\tfrac{n-1}{2}\big\rfloor.$$

By (iv) for $k = 1$, we have:

$$s_1(x) = \frac{-2\varphi(x)T'_{n,1}(x)}{n}\,P'_{n,1}(x) = 2n(n-1)(n-2)x\Big(\frac{1}{n}-x\Big)(1-x)^{n-3}.$$

Since

$$\max_{x\in[0,1/n]} x\Big(\frac{1}{n}-x\Big) = \frac{1}{4n^2} \quad \text{and} \quad (1-x)^{n-3} \le 1,$$

we obtain

$$h_1 := \max_{x\in[0,1/n]} s_1(x) \le \frac{2n(n-1)(n-2)}{4n^2} \le \frac{n}{2}. \tag{4.28}$$

Let us fix $k \in \{2, \dots, \lfloor \frac{n-1}{2} \rfloor\}$. We estimate the local extremum

$$h_k := \max_{x \in [\xi_k, k/n]} s_k(x).$$

According to (iv), we have

$$s_k(x) = \frac{2\varphi(x)}{n} T'_{n,k}(x) P'_{n,k}(x) = \frac{2\varphi(x)}{n} T'_{n,k}(x) \binom{n}{k} x^{k-1}(1-x)^{n-k-1}\Big(\frac{k}{n} - x\Big),$$

i.e.,

$$s_k(x) = \frac{2}{n} T'_{n,k}(x) P_{n,k}(x) \Big(\frac{k}{n} - x\Big). \tag{4.29}$$

The function $T'_{n,k}(x)$ is strictly increasing in $\big[\frac{k-1}{n}, \frac{k}{n}\big]$ and change sign only at point $\xi_k = \frac{\sqrt{\binom{k}{2}}}{\sqrt{\binom{k}{2}} + \sqrt{\binom{n-k}{2}}}$. Then, for $x \in \big[\xi_k, \frac{k}{n}\big]$,

$$\max_{x \in [\xi_k, k/n]} T'_{n,k}(x) = T'_{n,k}\big(\tfrac{k}{n}\big) = -\frac{k(k-1)n^2}{k^2} + \frac{(n-k)(n-k-1)n^2}{(n-k)^2} = n^2\Big(\frac{1}{k} - \frac{1}{n-k}\Big).$$

The function $h(x) = \frac{1}{x} - \frac{1}{n-x}$ is decreasing in $\big(0, \frac{n}{2}\big)$ since $h'(x) = \big(\frac{1}{x} - \frac{1}{n-x}\big)' < 0$, hence for $k \in \{2, \dots, \lfloor \frac{n-1}{2} \rfloor\}$

$$T'_{n,k}(x) \le n^2\Big(\frac{1}{k} - \frac{1}{n-k}\Big) \le n^2\Big(\frac{1}{2} - \frac{1}{n-2}\Big) \le \frac{n^2}{2}. \tag{4.30}$$

Also, $\frac{k-1}{n} \le \xi_k \le \frac{k}{n}$ and for $x \in \big[\xi_k, \frac{k}{n}\big]$, we have $\frac{k}{n} - x \le \frac{1}{n}$. Since $0 \le P_{n,k}(x) \le 1$ in $[0,1]$, it follows from (4.29) and (4.30) that

$$h_k \le \frac{2}{n} \cdot \frac{n^2}{2} \cdot \frac{1}{n} \le 1.$$

Taking into account (4.28), for $n \ge 2$ we have

$$h_k \le h_1 \le \frac{n}{2}, \qquad k = 1, \dots, \Big\lfloor \frac{n-1}{2} \Big\rfloor. \tag{4.31}$$

Finally, for $b_n(x)$, using (4.27) and (4.31), we obtain the estimate

$$b_n(x) \le 4(n-1) + \max_{1 \le k \le \lfloor \frac{n-1}{2} \rfloor} h_k \le 4(n-1) + \frac{n}{2},$$

or

$$b_n(x) \le 4.5\, n, \qquad x \in [0,1]. \tag{4.32}$$

3. Estimate for $c_n(x)$. We apply Cauchy inequality and Proposition 2.3 (b) with $\alpha = 0$ and $\alpha = 1$:

$$\begin{aligned} c_n(x) &= \sum_{k=0}^{n} \Big| T_{n,k}(x)\Big(1 - \frac{1}{n} T_{n,k}(x)\Big)\Big| P_{n,k}(x) \\ &\le \sqrt{\sum_{k=0}^{n} T^2_{n,k}(x) P_{n,k}(x)} \sqrt{\sum_{k=0}^{n} \Big(1 - \frac{1}{n} T_{n,k}(x)\Big)^2 P_{n,k}(x)} \\ &= \sqrt{\Phi(0) n^2} \cdot \sqrt{\Phi(1)} = n\sqrt{2 - \frac{2}{n}} \cdot \sqrt{3 - \frac{2}{n}}. \end{aligned}$$

Then,

$$c_n(x) \le \sqrt{6}\, n, \qquad x \in [0,1]. \tag{4.33}$$

From (4.25), (4.32) and (4.33), we obtain

$$\sum_{k=0}^{n} \big|\widetilde{D}\widetilde{P}_{n,k}(x)\big| \le a_n(x) + b_n(x) + c_n(x) \le 2(n-1) + 4.5\, n + \sqrt{6}\, n \le \big(6.5 + \sqrt{6}\big) n.$$

Therefore

$$\big\|\widetilde{D}\widetilde{U}_n f\big\| \le \widetilde{C} n \|f\|, \qquad \widetilde{C} := 6.5 + \sqrt{6}.$$

□

Now we are ready to prove a strong converse inequality of Type B.

Proof of Theorem 1.2. We follow the approach of Ditzian and Ivanov [6].

Let $n \in \mathbb{N}$, $n \ge 2$, $f \in C[0,1]$ and $\lambda(n)$, $\theta(n)$ be defined as in Proposition 2.4. From the Voronovskaya type inequality in Lemma 4.3 for the operator $\widetilde{U}_\ell$ and function $\widetilde{U}_n^3 f$ instead of f, we have

$$\begin{aligned}
\lambda(\ell)\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\| &= \big\|\lambda(\ell)\widetilde{D}^2\widetilde{U}_n^3 f\big\| \\
&= \big\|\widetilde{U}_\ell\widetilde{U}_n^3 f - \widetilde{U}_n^3 f + \lambda(\ell)\widetilde{D}^2\widetilde{U}_n^3 f - \widetilde{U}_\ell\widetilde{U}_n^3 f + \widetilde{U}_n^3 f\big\| \\
&\le \big\|\widetilde{U}_\ell\widetilde{U}_n^3 f - \widetilde{U}_n^3 f + \lambda(\ell)\widetilde{D}^2\widetilde{U}_n^3 f\big\| + \big\|\widetilde{U}_\ell\widetilde{U}_n^3 f - \widetilde{U}_n^3 f\big\| \\
&\le \theta(\ell)\big\|\widetilde{D}^3\widetilde{U}_n^3 f\big\| + \big\|\widetilde{U}_n^3(\widetilde{U}_\ell f - f)\big\|.
\end{aligned}$$

Now, using Lemma 4.4 for the function $\widetilde{D}^2\widetilde{U}_n^2 f$ and in addition Lemma 3.1 repeatedly three times, we obtain

$$\begin{aligned}
\lambda(\ell)\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\| &\le \widetilde{C}\, n\, \theta(\ell)\big\|\widetilde{D}^2\widetilde{U}_n^2 f\big\| + 3\sqrt{3}\,\big\|\widetilde{U}_\ell f - f\big\| \\
&= \widetilde{C}\, n\, \theta(\ell)\big\|\widetilde{D}^2\widetilde{U}_n^2(f - \widetilde{U}_n f) + \widetilde{D}^2\widetilde{U}_n^3 f\big\| + 3\sqrt{3}\,\big\|\widetilde{U}_\ell f - f\big\| \\
&\le \widetilde{C}\, n\, \theta(\ell)\big\|\widetilde{D}^2\widetilde{U}_n^2(f - \widetilde{U}_n f)\big\| + \widetilde{C}\, n\, \theta(\ell)\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\| + 3\sqrt{3}\,\big\|\widetilde{U}_\ell f - f\big\|.
\end{aligned}$$

Applying the Bernstein type inequality Lemma 4.4 twice for $f - \widetilde{U}_n f$ yields

$$\lambda(\ell)\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\| \le \widetilde{C}^3 n^3 \theta(\ell)\big\|f - \widetilde{U}_n f\big\| + 3\sqrt{3}\,\big\|\widetilde{U}_\ell - f\big\| + \widetilde{C}\, n\, \theta(\ell)\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\|.$$

From inequalities (2.17) and (2.18) of Proposition 2.4, we get

$$\frac{1}{2\ell^2}\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\| \le \frac{4\widetilde{C}^3 n^3}{9\ell^3}\big\|f - \widetilde{U}_n f\big\| + 3\sqrt{3}\,\big\|\widetilde{U}_\ell - f\big\| + \frac{4\widetilde{C} n}{9\ell^3}\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\|.$$

Let us choose ℓ sufficiently large such that

$$\frac{4\widetilde{C} n}{9\ell^3} \le \frac{1}{4\ell^2}, \qquad \text{i.e.} \qquad \ell \ge \frac{16\widetilde{C}}{9}\, n.$$

If we set $L = \frac{16\widetilde{C}}{9}$, for all integers $\ell \ge Ln$ we have

$$\begin{aligned}
\frac{1}{2\ell^2}\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\| &\le \frac{4\widetilde{C}^3 n^3}{9\ell^3}\big\|f - \widetilde{U}_n f\big\| + 3\sqrt{3}\,\big\|\widetilde{U}_\ell - f\big\| + \frac{1}{4\ell^2}\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\|, \\
\frac{1}{4\ell^2}\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\| &\le \frac{4\widetilde{C}^3 n^3}{9\ell^3}\big\|f - \widetilde{U}_n f\big\| + 3\sqrt{3}\,\big\|\widetilde{U}_\ell - f\big\|,
\end{aligned}$$

$$\frac{1}{n^2}\big\|\widetilde{D}^2\widetilde{U}_n^3 f\big\| \le \widetilde{C}^2\,\big\|f - \widetilde{U}_n f\big\| + 12\sqrt{3}\,\frac{\ell^2}{n^2}\,\big\|\widetilde{U}_\ell - f\big\|. \tag{4.34}$$

By using Lemma 3.1,

$$\begin{aligned}\|f-\widetilde{U}_n^3 f\| &\le \|f-\widetilde{U}_n f\| + \|\widetilde{U}_n f-\widetilde{U}_n^2 f\| + \|\widetilde{U}_n^2 f-\widetilde{U}_n^3 f\| \\ &\le (1+\sqrt{3}+(\sqrt{3})^2)\|f-\widetilde{U}_n f\|,\end{aligned}$$

and we obtain the inequality

$$\|f-\widetilde{U}_n^3 f\| \le (4+\sqrt{3})\|f-\widetilde{U}_n f\|. \tag{4.35}$$

It remains to complete the estimation of the K-functional. Since $\widetilde{U}_n^3 f \in W_0^2(\varphi)[0,1]$, from (4.34) and (4.35) it follows

$$\begin{aligned}K\Big(f,\frac{1}{n^2}\Big) &= \inf\Big\{\|f-g\| + \frac{1}{n^2}\,\|\widetilde{D}^2 g\| \,:\, g\in W_0^2(\varphi)[0,1],\ \widetilde{D}g\in W^2(\varphi)[0,1]\Big\} \\ &\le \|f-\widetilde{U}_n^3 f\| + \frac{1}{n^2}\,\|\widetilde{D}^2\widetilde{U}_n^3 f\| \\ &\le \Big(4+\sqrt{3}+\widetilde{C}^2\Big)\|\widetilde{U}_n f-f\| + 12\sqrt{3}\,\frac{\ell^2}{n^2}\,\|\widetilde{U}_\ell f-f\|.\end{aligned}$$

Therefore,

$$K\Big(f,\frac{1}{n^2}\Big) \le C\,\frac{\ell^2}{n^2}\big(\|\widetilde{U}_n f-f\| + \|\widetilde{U}_\ell f-f)\|\big)$$

for all $\ell \ge Ln$, where $C = 4+\sqrt{3}+\widetilde{C}^2$ and $L=\frac{16\widetilde{C}}{9}$, $\widetilde{C}=6.5+\sqrt{6}$. □

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References

[1] A. M. Acu, P. Agrawal: *Better approximation of functions by genuine Bernstein-Durrmeyer type operators*, Carpathian J. Math., **35** (2) (2019), 125–136.
[2] A. M. Acu, I. Rasa: *New estimates for the differences of positive linear operators*, Numer. Algorithms, **73** (3) (2016), 775–789.
[3] H. Berens, Y. Xu: *On Bernstein-Durrmeyer polynomials with Jacobi weights*, In Approximation Theory and Functional Analysis, (Edited by C. K. Chui), pp. 25–46, Acad. Press, Boston (1991).
[4] L. Beutel, H. Gonska and D. Kacsó: *Variation-diminishing splines revised*, In Proceedings of International Symposium on Numerical Analysis and Approximation Theory, (Edited by R. Trâmbiţaş), pp. 54–75, Presa Universitară Clujeană, Cluj-Napoka (2002).
[5] W. Chen: *On the modified Bernstein-Durrmeyer operator*, Report of the Fifth Chinese Conference on Approximation Theory, Zhen Zhou (China), (1987).
[6] Z. Ditzian, K. G. Ivanov: *Strong converse inequalities*, J. Anal. Math., **61** (1993), 61–111.
[7] H. Gonska, R. Păltănea: *Simultaneous approximation by a class of Bernstein-Durrmeyer operators preserving linear functions*, Czechoslovak Math. J., **60** (135) (2010), 783–799.
[8] H. Gonska, R. Păltănea: *Quantitative convergence theorems for a class of Bernstein-Durrmeyer operators preserving linear functions*, Ukrainian Math. J., **62** (2010), 913–922.
[9] T. N. T. Goodman, A. Sharma: *A modified Bernstein-Schoenberg operator*, In Constructive Theory of Functions, Varna 1987, (Edited by Bl. Sendov et al.), pp. 166–173, Publ. House Bulg. Acad. of Sci., Sofia, (1988).
[10] T. N. T. Goodman, A. Sharma: *A Bernstein-type operator on the simplex*, Math. Balkanica (New Series), **5** (2) (1991), 129–145.
[11] K. G. Ivanov, P. E. Parvanov: *Weighted approximation by the Goodman-Sharma operators*, East J. Approx., **15** (4) (2009), 473–486.
[12] G. G. Lorentz: *Bernstein Polynomials*, Mathematical Expositions 8, University of Toronto Press, (1953).
[13] R. Păltănea: *A class of Durrmeyer type operators preserving linear functions*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity, **5** (2007), 109–117.

[14] P. E. Parvanov, B. D. Popov: *The limit case of Bernstein's operators with Jacobi weights*, Math. Balkanica (N.S.), **8** (2–3) (1994), 165–177.

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