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# Energy Dissipation in Hilbert Envelopes on Motion Waveforms Detected in Vibrating Dynamical Systems: An Axiomatic Approach

James F. Peters[1]*, Tharaka U. Liyanage[2]

**Abstract**
This paper introduces an axiomatic approach in the theory of energy dissipation in Hilbert envelopes on motion waveforms emanating from various vibrating dynamical systems. A Hilbert envelope is a curve tangent to peak points on a motion waveform. The basic approach is to compare non-modulated vs. modulated waveforms in measuring energy loss during the vibratory motion $m(t)$ at time $t$ of a moving object such as a walker, runner, biker or the action of any spring system recorded in a video. Modulation of $m(t)$ is achieved by using Mersenne primes to adjust the frequency $\omega$ in the Fourier transform $m(t)e^{\pm j2\pi\omega t}$ on motion waveform $m(t)$, where the frequency $\omega$ is a Mersenne prime. Expenditure of energy $E_{m(t)}$ by a system is measured in terms of the area bounded by the motion $m(t)$ waveform at time $t$. Energy dissipation is measured in terms of the difference between modulated and non-modulated $m(t)$.

**Keywords:** Dissipation, Energy, Frequency, Hilbert envelope, Mersenne prime, Motion waveform, Vibrating Dynamical System, Video Frames
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## 1. Introduction

Dynamical system vibrations appear as varying oscillations in motion waveforms [1, 2]. The focus in this paper is on the detection of energy dissipation that commonly occurs in vibrating dynamical systems. For a motion waveform $m(t)$ at time $t$, the measure of motion dissipated energy is a mapping $E_{diss} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined in terms of the difference between non-modulated energy $E_{nmod}(t)$ and modulated energy $E_{mod}(t)$, i.e.,

$$\begin{aligned} E_{diss}(E_{nmod}(t), E_{mod}(t)) &= |E_{nmod}(t) - E_{mod}(t)| \\ &= |\text{non-modulated } E_m(t) - \text{modulated } E_m(t)| \end{aligned}$$

at time $t$ of a vibratory dynamical system. In this work, two forms of motion waveform energy are considered, namely, non-modulated (non-smoothing) $m(t)$ and modulated (smoothing) $m(t)$ that results from the product of $m(t)$ and the exponential

$e^{\pm j2\pi\omega t}$ introduced by Euler [3]. A formidable source of waveform energy measurement results from the Fourier transform $m(t)e^{\pm j2\pi\omega t}$ [4](see, e.g., [5]).

A non-modulated form of waveform energy $E_m(t)$ is associated with the planar area bounded by motion curve beginning at instant $t_0$ and ending instant $t_1$, namely, $E_m(t) = \int_{t_0}^{t_1} |m(t)|^2 \, dt$. In other words, system energy is identified with system waveform area, instead of the more usual energy graph [6]. Modulated system energy is measured using $E_{mod}(t) = \int_{t_0}^{t_1} \left|m(t)e^{\pm j2\pi\omega t}\right|^2 dt$.

An application of the proposed approach in measuring energy dissipation is given in terms of the Hilbert envelope on the peak points on waveforms derived from the up-and-down movements of the up-and-down movements of a walker, runner or biker recorded in a sequence of video frames. An important finding in this paper is the effective use of Mersenne primes to adjust the frequency $\omega$ of the Euler exponential to achieve waveform modulation with minimal energy dissipation (in the uniform waveform case (see Conjecture 1.i). This usage of Mersenne primes [7] of the form $2^p - 1(p$, a prime) in modulating motion waveforms first appeared in [8]. We prove that waveform energy is a characteristic, which maps to the complex plane (See Theorem 2.10. This result extends the waveform energy results in [9], [10]) as well as in [11, 12, 13].

| Symbol | Meaning |
|---|---|
| $2^A$ | Collection of subsets of a nonempty set $A$ |
| $A_i \in 2^A$ | Subset $A_i$ that is a member of $2^A$ |
| $\mathbb{C}$ | Complex plane |
| $t$ | Clock tick |
| $e^{j\omega t}$ | $cos(\omega t) + jsin(\omega t)$ [3] |
| $M$ | Mersenne prime |
| $\omega$ | Waveform Oscillation Frequency |
| $E_m(t)$ | Energy of motion waveform $m(t)$ |
| $E_{diss}$ | Energy dissipation |
| $\varphi_t : 2^A \to \mathbb{C}$ | $\varphi$ maps $2^A$ to complex plane $\mathbb{C}$ at time $t$ |
| $\varphi_t(A_i \in 2^A) \in \mathbb{C}$ | Characteristic of $\varphi(A_i \in 2^A) \in \mathbb{C}$ at time $t$. |

**Table 1.1.** Principal Symbols Used in this Paper

## 2. Preliminaries

Highly oscillatory, non-periodic waveforms provide a portrait of vibrating systems behavior. Energy dissipation (decay) is a common characteristic of every vibrating dynamical system. Included in this paper is an axiomatic basis for measuring this characteristic of dynamical systems. A **characteristic** is a mapping $\varphi_t : A_i \to \mathbb{C}$, which maps a subsystem $A_i$ in a system $A$ to a point in the complex plane $\mathbb{C}$.

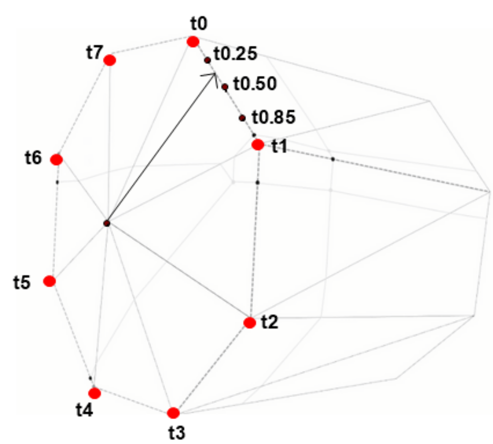


**Figure 2.1.** Morse instants clock

**Definition 2.1. (System)**
*A **system** A is a collection of interconnected components (subsystems $A_i \in 2^A$) with input-output relationships.*

**Definition 2.2. (Dynamical System)**
*A* **dynamical system** *is a time-constrained, changing physical system.*

**Definition 2.3. (Dynamical System Output Waveform)**
*The output of a* **dynamical system** *is a time-constrained sequence of discrete values.*

It has been observed that the theory of dynamical systems is a major mathematical discipline closely aligned with many areas of mathematics [14]. Energy dissipation is considered in many contexts such as heating, liquid (viscosity) and water-wave scattering. In this work, the focus is on energy decay represented by the difference between the energy of non-smooth (non-modulated) and smooth (modulated) motion waveforms. A motion waveform is a graphical portrait of the radiation emitted by moving system (e.g., walker, runner, biker) with oscillatory output.

**Axiom 1. (Instants Clock)**
*Every system has its own instants clock, which is a cyclic mechanism that is a simple closed curve with an instant hand with one end of the instant hand at the centroid of the cycle and the other end tangent to a curve point indicating an elapsed time in the motion of a vibrating system. A clock tick occurs at every instant that a system changes its state.*

**Remark 2.4. (What Euler tells us about time)**
*On an instants clock, every reading* $t \in (C)$*, a point* $t = a + jb, a, b \in \mathbb{R}$ *in the complex plane. For example,* $t_{0.25} = 0.25 + j0 = 0.25$ *in Fig. 2.1. The Morse instants clock is also called a homographic clock [15], since the tip of an instant clock* $t$*-hand moves on the circumference of a circle, where* $t$ *is a complex number [15]. For* $t$ *at the tip of a vector with radius* $r = 1$*, angle* $\theta$ *and* $a = cos\theta, b = sin\theta$ *in the complex plane, then*

$$t = a + jb = cos\theta + jsin\theta = e^{j\theta}.$$

*An instant of time viewed as an exponential is inspired by Euler [3].*

**Example 2.5.** *A sample Morse instants clock is shown in Fig. 2.1. The clock hand points to the elapsed time in the interval* $(t_{0.25} \leq t \leq t_{0.25})$ *in milliseconds (ms) after a system has begun vibrating. The clock face is a polyhedral surface in a Morse-Smale surface in a convex polyhedron in 3D Euclidean space. A Morse-Smale polyhedron is an example of a mechanical shape descriptor ideally suited as clock model because of its underlying piecewise smooth geometry. This form of an instants clock has been chosen to emphasize that the elapsed time* $t_k$ *is a real number in an instants interval* $[t_0, t_k] \in \mathbb{C}^2$ *in which* $t_k$ *is indeterminate. From a planar perspective, the proximity of sets of instants clock times is related to results given for computational proximity in the digital plane [16]. In this example, the instant hand is pointing to an elapsed time between* $t_{0.25}$ *ms and* $t_{0.50}$ *ms.*

**Definition 2.6. (Clocked Characteristic of a subsystem)**
*The clocked characteristic of a subsystem* $A_i$ *of a system A at time* $\varphi_t(A_i)$ *is a mapping* $\varphi_t : A_i \in 2^A \to \mathbb{C}$ *defined by* $\varphi_t(A_i) = a + bj \in \mathbb{C}, a, b \in \mathbb{R}, j = \sqrt{(-1)}, \varphi_t(A_i) \in \mathbb{C}$.

**Axiom 2. (Subsystem Motion Characteristic)**
*Let* $A_i \in 2^A$ *(subsystem* $A_i$ *in the collection of subsystems* $2^A$ *in system A) that emits changing radiation due to system movements (motion) and let* $t$ *be a clock tick. The motion characteristic of subsystem motion* $A_i \in 2^A$ *is a mapping* $m_t : A_i \to \mathbb{C}$ *defined by*

$$m_t(A_i) = a + bj \in \mathbb{C}, a, b \in \mathbb{R}, j = \sqrt{(-1)}, t \in \mathbb{R}.$$

*i.e., a subsystem* $A_i$ *motion characteristic of a system A is a mapping* $m_t(A_i \in 2^A) \in \mathbb{C}$ *at time* $t$*.*

**Remark 2.7.** *For the motion characteristic, we write* $dm(t)$ *when it is understood that motion is on a subset* $A_i \in 2^A$ *in a dynamical system A. Axiom 2 is consistent with the view [17] of the characteristic vector field, represented here with a planer characteristic vector field* $\xi$ *of a dynamical system with points* $p(x, y, t) \in \xi$ *that has positive complex characteristic coordinates at clock tick (time)* $t$ *such that*

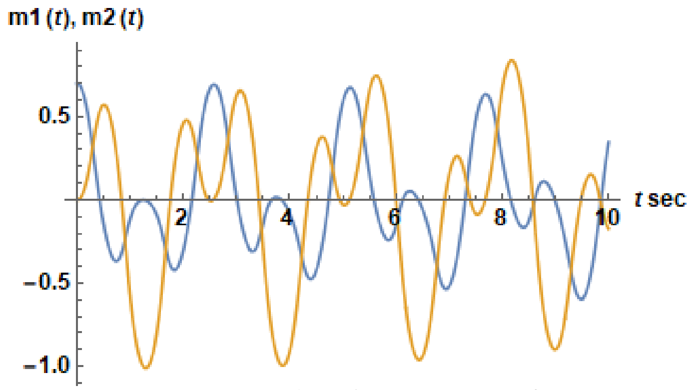


**Figure 2.2.** Sample spring system waveforms

$$\varphi_t(A_i \in 2^A) = p \in \xi = (a - jb)\frac{\partial \xi}{\partial x} + (a - jb)\frac{\partial \xi}{\partial y} + (a - jb)\frac{\partial \xi}{\partial t}, a, b \in \mathbb{R}.$$

*The 1-1 correspondence between every point p having coordinates in the Euclidean plane and points in the complex plane is lucidly introduced by D. Hilbert and S. Cohn-Vossen [18, §38, 263-265]. For an introduction to characteristic groups, see [19],[20],[21].*

**Example 2.8. Spring system vibration**
*A pair of sample sinusoidal waveforms emitted by an expanding and contracting spring system is shown in Fig. 2.2.*

Vibrating system waveform $m(t)$ modulation (smoothing) is achieved by adjusting the frequency $\omega$ in an Euler exponential $e^{\pm j2\pi\omega t}$, which is used in oscillatory waveform curve smoothing. It has been found that Mersenne primes provide an effective means an effective means of adjusting the frequency $\omega$. It has been observed by G.W. Hill [22] that Mersenne primes $M_p = 2^p - 1 = 3, 7, 31, \ldots$ for prime $p = 2, 3, 5, \ldots$ are useful in estimating variability as well as in estimating average values in sequences of discrete values.

**Axiom 3. (Waveform Energy)**
*A measure of dynamical system energy is the area of a finite planar region bounded by system waveform $m(t)$ curve at time $t$, defined by*

$$E_m(t) = \int_{t_0}^{t_1} |m(t)|^2 \, dt.$$

**Lemma 2.9.** *Dynamical system energy is time-constrained and is always limited.*

*Proof.* Let $E_m$ be the energy of a dynamical system, defined in Axiom 3. From Axiom 3, system energy always occurs in a bounded temporal interval $[t_0, t_1]$. Hence, $E_m$ is time constrained. From Axiom 1, the length of a system waveform is finite, since, from Axiom 3, system duration is finite. From Axiom 3, system energy is derived from the area of a finite, bounded region. Consequently, system energy is always finite. □

**Theorem 2.10.** *If $X$ is a dynamical system with waveform $m(t)$ at time $t$ and which changes with every clock tick, then observe*
*$1^o$ System waveform characteristic values are in the complex plane.*
*$2^o$ System energy varies with every clock tick.*
*$3^o$ System radiation characteristics are finite.*
*$4^o$ All system characteristics map to the complex plane.*
*$5^o$ Waveform energy decay is a characteristic, which maps to $\mathbb{C}$.*

*Proof.*

$1^o$ From Def. 2.6, a system characteristic is a mapping from a subsystem to the complex plane at time $t$, From Axiom 2, every waveform motion characteristic $m(t) \in \mathbb{C}$ at time $t$, which is the desired result.

$2^o$ From Lemma 2.9, system energy is time-constrained and always occurs in a bounded temporal interval. From Axiom 1, there is a new clock tick at every instant in time $t$ ms. From Axiom 3, system energy varies with every clock tick.
$3^o$ From Axiom 1, all system radiation characteristics are finite, since system duration is finite.
$4^o$ From Axiom 2, every system $A$ characteristic is a mapping from a subsystem $A_i \in 2^A$ to the complex plane, which is the desired result.
$5^o$ From the proof of step 4, the desired result follows.

□

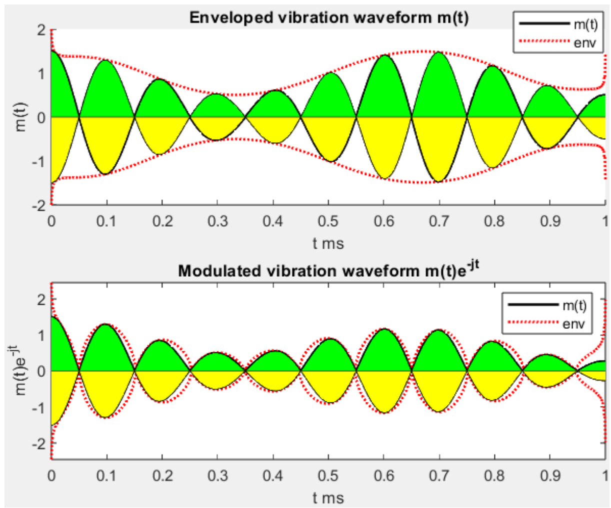


**Figure 2.3.** Hilbert envelope on modulated vibration waveform.

To obtain an approximation of system energy, a system waveform is represented by a continuous curve defined by a Hilbert envelope [23] tangent to waveform peak points, forming what known as Hilbert lobes.

A **Hilbert envelope** (denoted by $H_{env}$) is a curve that is tangent to the peak points on a waveform [24, §18.4, p. 132]. A **Hilbert envelope lobe** (denoted by $H_{env_{lobe}}$) is a tiny bounded planar region attached to single waveform peak point on a waveform envelope, defined by

$$H_{env} = \sqrt{m(t)^2 + (-m(t))^2} \text{ [23]}$$

The energy represented by a lobe $H_{env_{lobe}}$ area of a tiny planar region attached to an oscillatory motion waveform $m(t)$ is defined by

$$H_{env_{lobe}} = \int_a^b |m(t)|^2 \, dt$$

It is lobe area that provides a measure of the energy represented by a waveform segment.

The modulated vibration waveform $m(t)$ in Fig. 2.3 varies with lower peak points than the original motion waveform, depending on the choice of Mersenne prime frequency. To minimize energy loss due to modulation, a Mersenne prime is chosen for the frequency $\omega$ in an Euler exponential in $m(t)e^{\pm j\omega t}$ to obtain

**result**.$1^o$ Modulated system waveform $m(t)$ is smoother for a particular Mersenne prime frequency (i.e., the waveform oscillations are more uniform).
**result**.$2^o$ Modulated system energy loss is minimal, for a particular Mersenne prime frequency.

## 3. Application: Modulating System Waveform with Minimal Energy Dissipation

In this section, we illustrate how Mersenne primes can be used effectively to obtain the following results:

M→ $\omega$-$1^o$ Usage of a M-prime as the frequency in the Euler exponential in

$$m(t)e^{\pm j\omega Mt}$$

reduces motion $m(t)$ waveform motion energy.

M→ $\omega$-$2^o$ Energy dissipation varies in modulated vs. non-modulated waveforms for different choices of frequency $M$ in $e^{\pm j\omega Mt}$, depending on whether a waveform has uniformly or non-uniformly varying cycles around the origin.

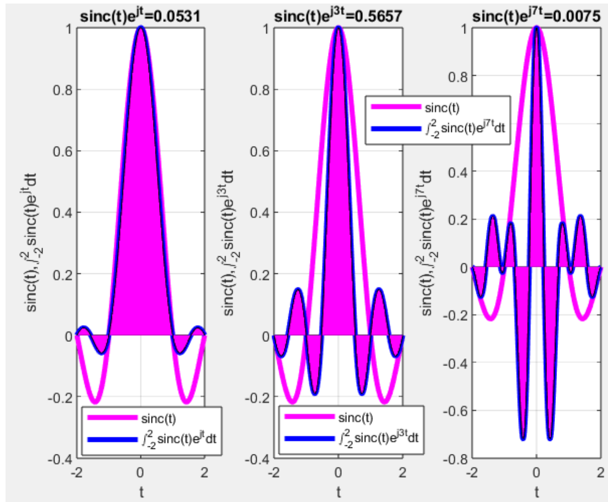


**Figure 3.1.** 3 forms of $m(t)e^{j\omega t}$

**Conjecture 1.** *The choice of a Mersenne prime $M \leq 31$ will always result in lower motion waveform peak values using $M$ as the frequency in the Euler exponential to achieve waveform modulation and minimal energy dissipation.*

There are two cases to consider: [**Partial Picture Proof**]

Case(i) Assume $m(t)$ waveform uniformly fluctuates and frequency $\omega = M = 1$ results in the lowest energy loss

*Proof.* Partial picture proof: Recall that $e^{j\omega t} = cos\omega t + jsin\omega t$, where $m(t)e^{j\omega t}$ forces the oscillation in a motion waveform to increase. Let $m(t) = sinc(t)$, introduced in 1822 by Fourier [4]. Then $m(t)$ oscillates uniformly on either side of the origin (see sample plot of *sinct* in Fig. 3.1). The area of $m(t) = \int_{-k}^{k} sinc(t)e^{j\omega t}dt, \omega \geq 1$ is always less than the area $\int_{-k}^{k} sinc(t)dt$. That is, $e^{j\omega t}$ partitions each $m(t)$ cycle into regions with smaller areas whose total area is less than the total area $\int_{-k}^{k} sinc(t)dt$. With $\omega = M = 1$, the modulated waveform energy is closest to non-modulated waveform energy, which is the desired result. □

Case(ii) Let $m(t)$ be a non-uniform waveform. We make the unproved claim that the choice of $\omega = M$, varies, i.e., $M$ is not always 1.

**Example 3.1. Sample Energy Dissipation: Non-uniform waveform Case**
*Let $m(t) = sinct$, with cycles that vary uniformly relative to the origin. This is the case in Fig. 3.1. The result for 3 choices of $M \in \{1,3,7\}$ are shown in the plots in Fig. 3.1. This leads to the following energy dissipation levels:*

$$E_{m(t)} = 0.9028 \textit{ non-modulated waveform energy}$$
$$E_{m(t)e^{jt}} = 0.1503\ M = 1, \textit{modulated waveform energy loss}$$
$$E_{m(t)e^{j3t}} = 0.3371\ M = 3, \textit{modulated waveform energy loss}$$
$$E_{m(t)e^{j7t}} = 0.8954\ M = 7, \textit{modulated waveform energy loss}$$
$$E_{m(t)e^{j31t}} = 0.9021\ M = 31, \textit{modulated waveform energy loss}$$

*The $\omega = M = 31$ case is not shown in Fig. 3.1.*

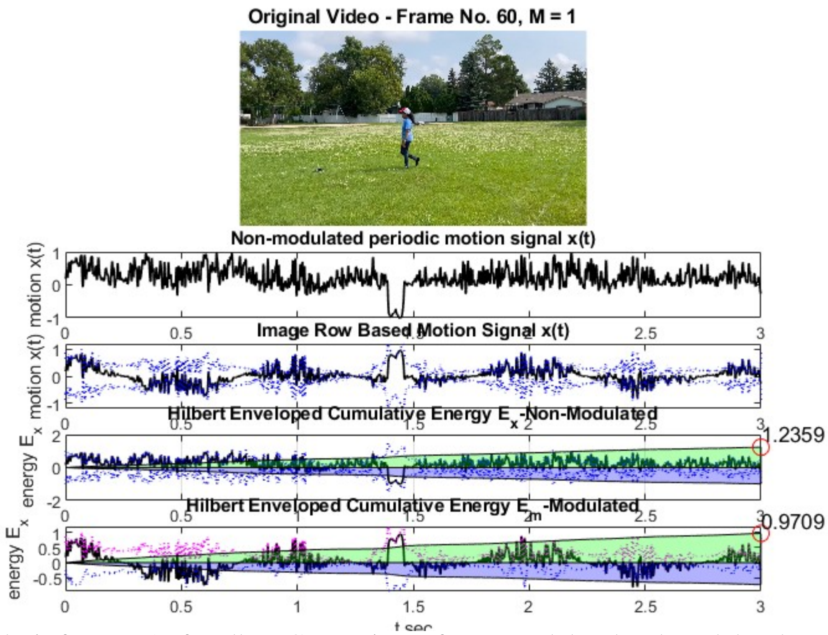


**Figure 3.2.** Energy Analysis for M = 1 of Walker: Comparison of Non-Modulated and Modulated Motion Signals of Frame = 60

Evidence of the correctness of our Conjecture for the non-uniform waveform case in the choice of the Mersenne prime to achieve minimal energy dissipation can be seen in the following two examples.

**Example 3.2. Sample Energy Dissipation for a walker waveform**
*A sample collection of non-modulated and modulated waveforms for a walker for M = 1 is shown in Fig. 3.2. In Table 2, M = 1 for the exponential frequency of a modulated waveform results in the lowest energy dissipation.*

However, if consider the choice of $M$ for the modulation frequency for a biker, this choice differs from the choice of $M = 1$ in Example 3.2.

| M | Non-Modulated Energy (Ex) | Modulated Energy (Em) | Energy Dissipation Percentage |
|---|---|---|---|
| 1 | 1.2359 | 0.9709 | 21.44% |
| 3 | 1.2359 | 0.9222 | 25.38% |
| 7 | 1.2359 | 0.9559 | 22.65% |
| 31 | 1.2359 | 0.9166 | 25.83% |

**Table 3.1.** Energy Dissipation for Walking

| M | Non-Modulated Energy (Ex) | Modulated Energy (Em) | Energy Dissipation Percentage |
|---|---|---|---|
| 1 | 0.9635 | 0.6317 | 34.43% |
| 3 | 0.9635 | 0.6729 | 30.16% |
| 7 | 0.9635 | 0.6561 | 31.90% |
| 31 | 0.9635 | 0.6396 | 33.61% |

**Table 3.2.** Energy Dissipation for Biking

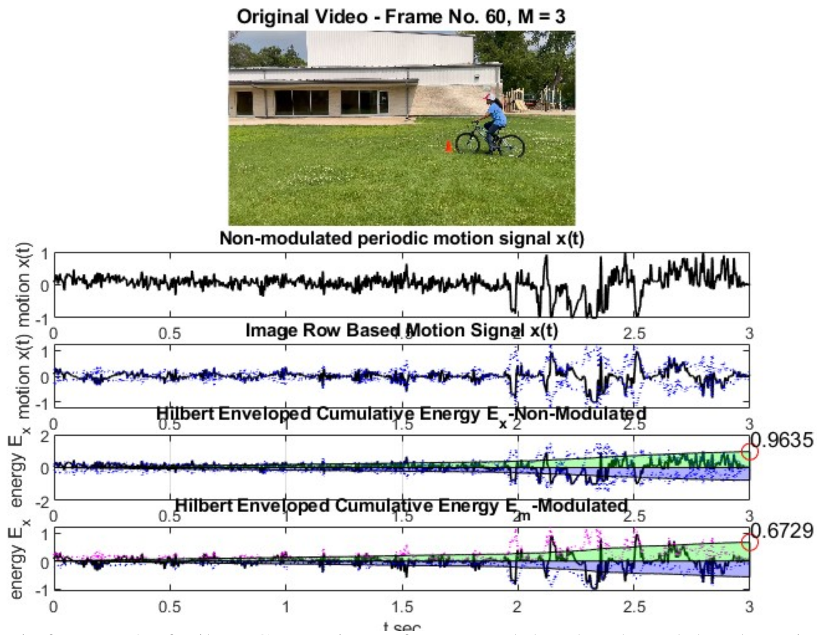


**Figure 3.3.** Energy Analysis for M = 3 of Biker: Comparison of Non-Modulated and Modulated Motion Signals of Frame = 60

**Example 3.3. Sample Energy Dissipation for a biker waveform**
*A sample collection of non-modulated and modulated waveforms for a biker for M = 3 is shown in Fig. 3.3. In Table 3, M = 3 for the exponential frequency of a modulated waveform results in the lowest energy dissipation.*

## 4. Conclusion

This paper focuses on the frequency characteristic in modulating dynamical system waveforms. The appropriate choice of Mersenne prime $M$ as the frequency $\omega$ for the Euler exponential $e^{j\omega t} \to e^{jMt}$ is considered in modulating a dynamical system waveform to obtain a smoother waveform and achieve minimal energy dissipation. It has been found that $M = 1$ is the best choice for waveforms whose cycles vary uniformly about the origin. Choice of $M \in \{1,3,7,31\}$ for the non-uniform waveforms varies, depending on how extreme the lack of self-similarity present in waveforms that vary in a chaotic fashion on either side of the origin. The appropriate choice of $M$ in modulating a non-uniform waveform is an open problem.

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# Existence and Uniqueness of Solutions for Nonlinear Fractional Differential Equations with $\mho$-Caputo Fractional Differential Equations

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**Abstract**
This paper examines the existence, uniqueness, and Ulam-Hyers stability of solutions to nonlinear $\mho$-fractional differential equations with boundary conditions with a $\mho$-Caputo fractional derivative. The acquired results for the suggested problem are validated using a novel technique and minimum assumptions about the function $f$. The analysis reduces the problem to a similar integral equation and uses Banach and Sadovskii fixed point theorems to reach the desired findings. Finally, the inquiry is demonstrated by illustrative example to validate the theoretical findings.

**Keywords:** Banach Contraction mapping, Caputo fractional derivative ,$\mho$-Caputo fractional derivative, Fixed point theorem, Fractional differential equations, Stability analysis, Ulam-Hyers stability
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## 1. Introduction

Fractional calculus has been recognized to be a useful tool for modeling many processes in economics, physics, and engineering. Since fractional derivatives (FDs) are a useful tool for characterizing memory and inherited qualities of various materials and processes, fractional-order models have actually been shown to be more applicable for a number of real-world scenarios than integer-order models. Applications where this theory is useful include material theory, transport processes, wave propagation, signal theory, economics, control theory and mechanics,. For more detail (see [1]-[5] and the references therein).That is the primary benefit of fractional differential equations (FDEs) over standard integer-order models. Basic difficulties include fractional derivatives, including Riemann-Liouville [2], Caputo [3], Hilfer [4], and Hadamard [6].

In recent years, there has been a lot of interest in FDEs, particularly boundary value issues for nonlinear FDEs, which may be used to represent and describe non-homogeneous physical processes that occur in their form. Almeida introduced the $\mho$-Caputo derivative in [2] to study FDEs in general. This is different type of FD seen in the literature. We may derive numerous well-known FDs for certain choices of $\mho$, such as the Caputo and Caputo-Hadamard FDs, which depend on a kernel. This technique also appears logical when seen through a variety of applications. Using a carefully selected "trial" function $\mho$, the $\mho$-Caputo FDs provides some control over approximating the phenomena under research. Zhang [7] used various fixed point theorems to illustrate the existence and uniqueness of outcomes for nonlinear fractional existences (P.V.Bs) using Caputo type FDs. Researchers have found unique solutions to boundary value issues for FDEs (see [8]-[13]) and other references). The importance of fractional boundary value concerns originates from the fact that they cover a wide range of dynamical systems as examples.

On the other hand, introduced the stability problem of functional equation solutions (of group homomorphisms) in a presentation at Wisconsin University in 1940 [14]. Hyers [15] provided the first answer to the topic in Banach spaces in 1941. Ulam-type stability has piqued the curiosity of numerous academics since then. Researchers became attracted to the study of stability for FDEs due to the extensive extension of the fractional calculus for more detail (see [12], [15]-[17]).

Several approaches to study FDEs have been proposed in the literature recently, based on multi-valued mappings and boundary value problems. For instance, authors [18] focused on the Caputo fractional differential inclusions with boundary conditions in a more general case, for convex-compact mappings providing critical conditions for existence and uniqueness. Based on this, Mohammadi et al. [19] studied the existence of solutions of *phi*-Caputo fractional differential inclusions by using Multi-Valued Contractions.. This method provides further evidence of the role played by contraction principles in solving non-linear inclusions.

Moreover, Kayvanloo et al. New topological techniques were introduced in [20] to prove the existence of a solution for solvability in infinite systems of Caputo-Hadamard fractional differential equations. Additionally, in [21] Mohammadi et al. provided the significance of weak Wardowski mappings and elucidates our understanding of generalized g-Caputo fractional inclusions.

Benchohra, Hamani and Ntouyas in [8] investigated the existence of solutions of the following existence for Caputo FDEs.

$$^{C}D^{\eth}_{+0}y(t) = f(t, y(t)), \qquad x \in [0, T], \qquad 0 < \eth \leq 1,$$

with boundary condition

$$ay(0) + by(T) = c,$$

where $^{C}D^{\eth}_{+0}$ is the Caputo derivative with $0 < \eth < T$, a,b are real constant such that $a + b \neq 0$, and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [22] researcher examined the existence and uniqueness of solutions for the following ℧-Caputo FDEs with boundary conditions.

$$\begin{cases} ^{C}D^{\beta,\mho(x)}_{0^+}u(x) = f(x, u(x)), \qquad x \in [0, T], \\ u(0) = u'(0) = 0, \quad u(T) = u_T. \end{cases}$$

Here $^{C}D^{\beta,\mho(x)}_{0^+}$ is the ℧-Caputo derivative with $2 < \beta < 3$, $T > 0$ , $u \in C^1[0, 1]$, and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Motivated by the works above, in this paper, we study the existence, uniqueness and stability of solutions for the following ℧-Caputo FDEs of arbitrary order with fractional boundary conditions shown below:

$$^{C}D^{\eth,\mho}_{0^+}u(x) = f(x, u(x)), \quad x \in J = [1, T], \tag{1.1}$$

with the boundary condition

$$\begin{cases} u(0) = \Omega_1, \\ Au(0) + Bu(T) = \Omega_2, \end{cases} \tag{1.2}$$

where $A, B.\Omega_1$ and $\Omega_2$ are constant, and $^{C}D^{\eth,\mho}_{0^+}$, is ℧-Caputo FDs of order $1 < \eth \leq 2$ with $f : J \times \mathbb{R} \to \mathbb{R}$, is continuous function.

## 2. Preliminaries

In this section, we give some notations and definitions.

**Definition 2.1.** *(℧-Riemann-Liouville fractional integral) [23].*
*Let $\eth > 0$, $f \in L^1(J,\mathbb{R})$, and $\mho \in C^n(J,\mathbb{R})$ such that $\mho'(\Im) > 0$ for all $x \in J$. The ℧-Riemann-Liouville fractional integral at order $\eth$ of the function $f$ is given by by*

$$I_{0^+}^{\eth} f(x) := \frac{1}{\Gamma(\eth)} \int_{0^+}^{x} [\mho(x) - \mho(\Im)]^{\eth-1} f(\Im)\mho'(\Im)d\Im, \tag{2.1}$$

**Remark 2.2.** *Note that if $\mho(x) = x$ and $\mho(x) = \log(x)$, then the equation (2.1) is reduced to the Riemann-Liouville and Hadamard fractional integrals, respectively.*

**Definition 2.3.** *(℧-Caputo fractional derivative)[23].*
*Let $\eth > 0$, $f \in L^1(J,\mathbb{R})$, and $\mho \in C^n(J,\mathbb{R})$ such that $\mho'(\Im) > 0$ for all $x \in J$. The ℧-Caputo FDs at order $\eth$ of the function $f$ is given by*

$${}^{C}D_{0^+}^{\eth,\mho} f(x) := \frac{1}{\Gamma(n-\eth)} \int_{0^+}^{x} [\mho(x) - \mho(\Im)]^{n-\eth-1} \mho'(\Im)\delta^{[n]}d\Im, \tag{2.2}$$

*where $n-1 < \eth < n$ ,$n = [a]+1$, $\delta^{[n]}(\Im) = (\frac{1}{\mho'(\Im)}\frac{d}{d\Im})^n f(\Im)$, and $[\eth]$ denotes the integer part of the real number $\eth$, and $\Gamma$ is the gamma function.*

**Remark 2.4.** *Note that if $\mho(x) = x$ and $\mho(x) = \log(x)$, then the equation (2.2) is reduced to the Caputo and Caputo-Hadamard FDs, respectively.*

**Remark 2.5.** *If $\eth \in ]0,1[$ then, we have*

$${}^{C}D_{0^+}^{\eth,\mho} f(x) = I_{0^+}^{1-\eth,\mho}\left(\frac{f'(x)}{\mho'(x)}\right) = \frac{1}{\Gamma(\eth)} \int_{0^+}^{x} [\mho(x) - \mho(\Im)]^{\eth-1} f'(\Im)d\Im.$$

**Definition 2.6.** *(Ulam-Hyers stable) [24].*
*The equation (1.1) is called* ***Ulam-Hyers stable*** *if there $\exists$ a constant $q > 0$ such that for each $\varepsilon > 0$, when $u \in C(J,\mathbb{R})$ is any solution of the inequality*

$$|{}^{C}D_{1+}^{\eth,\mho(x)} u(x) - f(x,u(x))| \le \varepsilon, \quad x \in J, \tag{2.3}$$

*then there $\exists$ another solution $w \in C(J,\mathbb{R})$ of the equation (1.1) satisfied*

$$|u(x) - w(x)| \le q\varepsilon, \quad x \in J.$$

**Definition 2.7.** *[24] The equation (1.1) is said to be* ***Ulam-Hyers-Rassias stable*** *with respect to $\rho \in C(J,\mathbb{R})$ and $b > 0$ is any constant such that for each $\varepsilon > 0$ and for each solution $u \in C(J,\mathbb{R})$ of the inequality*

$$|{}^{C}D_{1+}^{\eth,\mho(x)} u(x) - f(x,u(x))| \le \varepsilon\rho(x), \quad x \in J,$$

*then there $\exists$ a solution $w \in C(J,\mathbb{R})$ of the equation (1.1) satisfied*

$$|u(x) - w(x)| \le b\varepsilon\rho(x), \quad x \in J.$$

**Proposition 2.8.** *Let $\eth > 0$. If $f \in C^n(J,\mathbb{R})$, then we have*

1) ${}^{C}D_{0^+}^{\eth,\mho} I_{0^+}^{\eth,\mho} f(x) = f(x)$

2) $I_{0^+}^{\eth,\mho} {}^{C}D_{0^+}^{\eth,\mho} f(x) = f(x) - \sum_{r=0}^{n-1} \frac{f_{\mho}^{[r]}(0)}{r!} (\mho(x) - \mho(0)^r$

3) $I_{0^+}^{\eth,\mho}$ *is linear and bounded from $(J,\mathbb{R})$ to $(J,\mathbb{R})$.*

**Theorem 2.9.** *(The Banach Fixed Point Theorem)*
*Let $J \subset \mathbb{R}$, be a closed, not necessarily bounded, interval and $H : J \to \mathbb{R}$ a function with $H(J) \subset J$ and which satisfies for a fixed k, $0 \leq k < 1$,then the inequality*

$$|H(x) - H(y)| \leq k|x - y|$$

*for all $x, y \in J$. Then there $\exists$ exactly one fixed point of H, i.e. a $\xi \in J$, with*

$$H(\xi) = \xi.$$

In order to state Sadovskii's theorem, we define the following concepts.

**Definition 2.10.** Let $Q$ be a bounded set in metric space $(X,d)$. The Kuratowski measure of non compactness, $\mu(Q)$, is defined as:

$$\mu(Q) = \inf\{\varepsilon : Q covered\ by\ finitely\ many\ sets\ such\ that\ the\ diameter\ of\ each\ set\ is \leq \varepsilon\}.$$

**Definition 2.11.** *Let $\theta : \Omega(\theta) \subseteq X \to X$ be a bounded and continuous operator on a Banach space $X$. Then $\theta$ is called a condensing map if $\mu(\theta(w)) < \mu(w)$ for all bounded sets $w \subset \Omega(\theta)$, where $\mu$ denotes the Kuratowski measure of non compactness.*

**Lemma 2.12.** *The map $P_1 + P_2$ is a k-set contraction with $0 \leq k < 1$, and thence also condensing, if*

1. *$P_1, P_2 : \Omega \subseteq \Xi \to \Xi$ are operators on the Banach space $\Xi$,*
2. *$P_1$ is k-contractive, that is $||P_1 x - P_1 y|| \leq k||x - y||$ for all $x, y \in \Omega$ and fixed $k \in [0,1)$*
3. *$P_2$ is compact*

**Lemma 2.13.** *(Sadovskii's fixed point theorem)*
*Assume that w be a convex, bounded and closed subset of a Banach space $\Xi$, and let $\theta : w \to w$ be a condensing map. Then $\theta$ has a fixed point.*

**Lemma 2.14.** *For any $u(x) \in C(J,\mathbb{R})$, $n-1 < \eth \leq n$, then the existence (1.1)-(1.2) has a solution*

$$\begin{aligned} u(x) =& \Omega_1 + \Omega_3 \left[\mho(x) - \mho(0)\right] - \frac{\left[\mho(x) - \mho(0)\right]}{\left[\mho(T) - \mho(0)\right]} \int_0^T \mho'(\Im) \frac{\left[\mho(T) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im \\ &+ \int_0^x \mho'(\Im) \frac{\left[\mho(x) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im \end{aligned}$$

*where $\Omega_3 = \frac{(\Omega_2 - (A+B)\Omega_1)}{B[\mho(T) - \mho(0)]}$ and $\mho(T) \neq \mho(0)$.*

*Proof.* Applying the $\mho$-fractional integral of order $\eth$ from 1 to $x$ on both sides of $\mho$-Caputo FDEs in (1.1) Which can be rewritten as follows:

$$u(x) = c_0 + c_1 \left[\mho(x) - \mho(0)\right] + \int_0^x \mho'(\Im) \frac{\left[\mho(x) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im,$$

Now, we will apply the boundary conditions(1.2) to find $c_0$ and $c_1$,

$$u(x) = \Omega_1 + c_1 \left[\mho(x) - \mho(0)\right] + \int_0^x \mho'(\Im) \frac{\left[\mho(x) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im.$$

To find $c_1$.

$$u(T) = \Omega_1 + c_{n-1} \left[\mho(T) - \mho(0)\right] + \int_0^T \mho'(\Im) \frac{\left[\mho(T) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im,$$

$$Au(0) + Bu(T) = A\Omega_1 + B\Omega_1 + Bc_{n-1} \left[\mho(T) - \mho(0)\right] + B \int_0^T \mho'(\Im) \frac{\left[\mho(T) - \mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im,$$

$$c_{n-1}=\frac{\Omega_2-(A+B)\Omega_1}{B\left[\mho(T)-\mho(0)\right]}-\frac{1}{\left[\mho(T)-\mho(0)\right]}\int_0^T\mho'(\Im)\frac{\left[\mho(T)-\mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im,$$

$$\begin{aligned}u(x)=&\Omega_1+\left[\mho(x)-\mho(0)\right]\frac{(\Omega_2-(A+B)\Omega_1)}{B\left[\mho(T)-\mho(0)\right]}-\frac{\left[\mho(x)-\mho(0)\right]}{\left[\mho(T)-\mho(0)\right]}\int_0^T\mho'(\Im)\frac{\left[\mho(T)-\mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im\\&+\int_0^x\mho'(\Im)\frac{\left[\mho(x)-\mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im,\end{aligned}$$

and this complete the poof. □

## 3. Main Results

In the sequel, we denote by $\zeta:=C(J,\mathbb{R})$ the Banach space of all continuous functions equipped with the norm

$$||u||=\sup\{|u(x)|;x\in J\}.$$

To prove the main results, we need the following assumptions:
(**H1**) There ∃ a constant $L>0$, such that $\left|f(x,u(x))\right|\leq L|u|$, for all $x\in J$ and all $u\in\mathbb{R}$
(**H2**) There ∃ a constant $k_1>0$, such that $\left|f(x,u(x))-fx,w(x))\right|\leq k_1|u-w|$. For all $x\in J$ and all $u,w\in\mathbb{R}$.

### 3.1 Existence the result for problem (1.1)

Here we apply Sadovskii's fixed point to derive the existence result for the problem (1.1)

**Theorem 3.1.** *Assume $f:J\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies (**H1**)-(**H2**). Then the existence (1.1)-(1.2) has at least one solution in $J$.*

*Proof.* We define the integer $r$, let $H_r=\{u\in\zeta:||u||\leq r\}$ be a closed bounded and convex subset of $\zeta$, where $r$ is a fixed constant. It is sufficient to show that $\Phi$ has a fixed point. We define an operator $\Phi:\zeta\to\zeta$ in a similar way in light of Lemma 2.12.:

$$\begin{aligned}\Phi(u(x))=\Omega_1+\Omega_3\left[\mho(x)-\mho(0)\right]-\frac{\left[\mho(x)-\mho(0)\right]}{\left[\mho(T)-\mho(0)\right]}\int_0^T\mho'(\Im)\frac{\left[\mho(T)-\mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im\\+\int_0^x\mho'(\Im)\frac{\left[\mho(x)-\mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im\end{aligned}$$

for all $x\in J$. We also define the operators $\Phi_1,\Phi_2:\zeta\to\zeta$ by

$$\Phi_1(u(x))=\Omega_1+\int_0^x\mho'(\Im)\frac{\left[\mho(x)-\mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im,$$

and

$$\Phi_2(u(x))=\Omega_3\left[\mho(x)-\mho(0)\right]-\frac{\left[\mho(x)-\mho(0)\right]}{\left[\mho(T)-\mho(0)\right]}\int_0^T\mho'(\Im)\frac{\left[\mho(T)-\mho(\Im)\right]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im$$

and note that,

$$\Phi(u(x))=\Phi_1(u(x))+\Phi_2(u(x))\qquad for\ all\ x\in J.$$

If the sum of operators $\Phi_1+\Phi_2$ has a fixed point, then follows that operator $\Phi$ also has one. To demonstrate that $\Phi_1+\Phi_2$ has a fixed point, the operators $\Phi_1$ and $\Phi_2$ shall be proved to meet the hypothesis of Lemma 2.13. This will be accomplished in numerous steps.

**Step 1**: $\Phi H_r\subset H_r$ Let us select

$$r_1\geq|\Omega_1+\Omega_3\left[\mho(T)-\mho(0)\right]|+|\frac{2\left[\mho(T)-\mho(0)\right]^{\eth}}{\Gamma(\eth+1)}L|$$

$$|(\Phi u)(x)| = \Big|\Omega_1 + \Omega_3\,[\mho(x) - \mho(0)] - \frac{[\mho(x) - \mho(0)]}{[\mho(T) - \mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im))d\Im + \int_0^x \mho'(\Im)\frac{[\mho(x) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im))d\Im\Big|,$$

$$\leq \Omega_1 + \Omega_3\,[\mho(T) - \mho(0)] + \frac{2\,[\mho(T) - \mho(0)]^{\eth}}{\Gamma(\eth+1)} L||u|| < r_1.$$

For all $u \in H_r$, which implies that $\Phi H_r \subset H_r$.

**Step 2**: $\Phi_2$ is compact. Consider that the operator $\Phi_2$ is uniformly limited in view of step 1. Let $t_1, t_2 \in J$, where $t_1 < t_2$ and $u \in H_r$. Then we acquire.

$$|(\Phi_2 u)(x_1) - (\Phi_2 u)(x_2)| = \Big|\Omega_3\,(\mho(x_1) - \mho(0)) - \frac{[\mho(x) - \mho(0)]}{[\mho(T) - \mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im))d\Im - \Omega_3\,[\mho(x) - \mho(0)] + \frac{[\mho(x) - \mho(0)]}{[\mho(T) - \mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im))d\Im\Big|,$$

$$\leq \Omega_3\,(\mho(x_1) - \mho(x_2)) - \frac{(\mho(x_1) - \mho(x_2))}{[\mho(T) - \mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im))d\Im,$$

$$|(\Phi_2 u)(x_1) - (\Phi_2 u)(x_2)| \leq |\Omega_3\,(\mho(x_1) - \mho(x_2))| + |\frac{(\mho(x_1) - \mho(x_2))}{[\mho(T) - \mho(0)]}\frac{[\mho(T) - \mho(0)]^{\eth}}{\Gamma(\eth+1)}|L||u||$$

which is independent of $u$ and tends to zero as $x_2 \to x_1$. Thus, $\Phi_2$ is equicontinuous. Hence, by the Arzelá-Ascoli theorem, $\Phi_2(H_r)$ is a relatively compact set.

**Step 3**: $\Phi_1$ is k-contractive. Let $u_1, u_2 \in H_r$. Then, we have

$$||(\Phi_1 u_1)(x) - (\Phi_1 u_2)(x)|| = \Big|\int_0^x \mho'(\Im)\frac{[\mho(x) - \mho(\Im)]^{\eth-1}}{\Gamma(\eth)}(f(\Im, u_1(\Im) - f(\Im, u_2(\Im))d\Im\Big|,$$

$$\leq \frac{[\mho(T) - \mho(0)]^{\eth}}{\Gamma(\eth+1)} k_1 ||u_1 - u_2||,$$

set $\lambda = \frac{[\mho(T) - \mho(0)]^{\eth}}{\Gamma(\eth+1)} k_1$ then we obtain

$$||(\Phi_1 u_1)(x) - (\Phi_1 u_2)(x)|| \leq \lambda ||u_1 - u_2||.$$

Since $\lambda < 1$ Then, $\Phi_1$ is a contractive mapping.

**Step 4**: $\phi$ is compressing. Lemma 2.12 states that $\Phi : H_r \to H_r$, with $\Phi = \Phi_1 + \Phi_2$, is a condensing map on $H_r$ due to $\Phi_1$ being continuous, a u-contraction, and $\Phi_2$ compact. Using Lemma 2.13, we may conclude that the operator $\Phi$ has a fixed point. As a result, the boundary value problem (1.1)-(1.2) has at least one solution on J. □

### 3.2 Uniqueness the result for problem (1.1)

Now we apply Banach's contraction mapping principle to prove existence and uniqueness of solutions for problems (1.1)-(1.2)

**Theorem 3.2.** Assume $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (**H1**)-(**H2**). Let $\eta = \sup_{x \in [1,T]} f(x, 0)$, if

$$|\frac{2\,[\mho(T) - \mho(0)]^{\eth}}{\Gamma(\eth+1)}| < 1.$$

Then the existence (1.1)-(1.2) has a unique solution.

*Proof.* Define the operator $\Theta : \zeta \to \zeta$ as the following

$$
\begin{aligned}
(\Theta u)(x) =& \Omega_1 + \Omega_3 \left[\mho(x) - \mho(0)\right] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]} \int_0^T \mho'(\Im) \frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im \\
&+ \int_0^x \mho'(\Im) \frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im, u(\Im)) d\Im,
\end{aligned}
$$

We have to show that $\Theta$ has a fixed point on $G_r$ which it is solution of the existence (1.1)-(1.2). Firstly we show that $\Theta H_r \subset \Theta$, The operator $\Theta$ is bounded set into the bounded sets in $\zeta$. For any $r > 0$, then for each $x \in J = [1, T]$. Then, we have

$$
\begin{aligned}
|(\Theta u)(x)| =& \Bigg| \int_0^x \mho'(\Im) \frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} (f(\Im, u(\Im)) - f(\Im, 0) + f(\Im, 0)) d\Im \\
& - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]} \int_0^T \mho'(\Im) \frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} (f(\Im, u(\Im)) - f(\Im, 0) + f(\Im, 0)) d\Im \Bigg|, \\
\\
\leq& \left| \int_0^x \mho'(\Im) \frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} (f(\Im, u(\Im)) - f(x, 0) + f(x, 0)) d\Im \right| \\
&+ \left| \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]} \int_0^T \mho'(\Im) \frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} (f(\Im, u(\Im)) - f(x, 0) + f(x, 0)) d\Im \right|, \\
\\
\leq& \left| \int_0^x \mho'(\Im) \frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} d\Im \right| (k_1 + \eta) + \left| \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]} \int_0^T \mho'(\Im) \frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} d\Im \right| (k_1 + \eta), \\
\\
\leq& \left| \frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} \right| (k_1 + \eta) + \left| \frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} \right| (k_1 + \eta), \\
\\
\leq& \left| \frac{2\,[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} \right| (k_1 + \eta).
\end{aligned}
$$

Now, let $u, u_1 \in \zeta$ and for each $x \in J$. We need to prove that $\Theta$ is contraction mapping.

$$
\begin{aligned}
|(\Theta u)(x) - (\Theta u_1)(x)| =& \Bigg| \int_0^x \mho'(\Im) \frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} (f(\Im, u(\Im)) - f(\Im, u_1(\Im)) d\Im \\
& - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]} \int_0^T \mho'(\Im) \frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} (f(\Im, u(\Im)) - f(\Im, u_1(\Im)) d\Im \Bigg|, \\
\\
\leq& \int_0^x \mho'(\Im) \frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} d\Im k_1 ||u - u_1|| + \int_0^x \mho'(\Im) \frac{[\mho(x)-\mho(\Im)]^{\eth-2}}{\Gamma(\eth-1)} d\Im k_1 ||u - u_1||, \\
\\
\leq& \left| \frac{2\,[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} \right| k_1 ||u - u_1||.
\end{aligned}
$$

If $\left|\frac{2[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}\right| < 1$. Then, $\Theta$ is a contraction mapping. Therefore, by using The Banach contraction mapping, $\Theta$ has a unique Fixed point which is a unique solution of the existence (1.1)-(1.2). □

## 4. Stability Theorems

In this section, we study stability of our result.

### 4.1 Ulam-Hyers stability

**Theorem 4.1.** Assume that the assumptions (**H2**) is hold. Then the fractional differential equation (1.1) with the boundary condition (1.2) is Ulam-Hyers stable.

*Proof.* let $w \in C(J,\mathbb{R})$ be a solution of the inequality (2.3) i.e,

$$|^{C}D_{0^+}^{\eth,\mho(x)} w(x) - f(x,w(x))| \leq \varepsilon, \quad x \in J. \tag{4.1}$$

If we defined $u \in C(J,\mathbb{R})$ the unique solution of the existence (1.1)-(1.2)

When u and w being continuous functions on $J$. From lemma 2.14 we obtain

$$\begin{aligned} u(x) =& \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im \end{aligned}$$

we will take the integration of (4.1) and we obtain

$$\begin{aligned} &|w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im| \leq \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}, \end{aligned}$$

on the other hand we have

$$\begin{aligned} \Big|w(x)-u(x)\Big| =& \Big|w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im\Big|, \end{aligned}$$

$$\begin{aligned} \Big|w(x)-u(x)\Big| =& \Big|w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im + \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im \\ &- \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,w(\Im))d\Im - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} f(\Im,u(\Im))d\Im\Big|, \end{aligned}$$

$$\begin{aligned} &\leq \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} + \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} |f(\Im,w(\Im)) - f(\Im,u(\Im))|d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} |f(\Im,w(\Im)) - f(\Im,u(\Im))|d\Im, \end{aligned}$$

$$\begin{aligned} |w(x)-u(x)| \leq& \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} + \frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)} k_1 |w(x)-u(x)| \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)} |f(\Im,w(\Im)) - f(\Im,u(\Im))|d\Im, \end{aligned}$$

Let $\gamma_1 = \frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}k_1$ then, we obtain

$$|w(x)-u(x)| \leq \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)} + \frac{1}{1-\gamma_1}\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im,$$

$$|w(x)-u(x)| \leq \frac{\varepsilon\,[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}e^{\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}|w(x)-u(x)|},$$

$$|w(x)-u(x)| \leq \varepsilon C_h.$$

Where $C_h = \frac{[\mho(x)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}e^{\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}|w(x)-u(x)|}$. Hence, the solution of (1.1)-(1.2) ) is Ulam-Hyers stable. □

## 4.2 Ulam-Hyers-Rassias stability

**Theorem 4.2.** *Suppose that the assumptions (**H2**)-(**H2**) are satisfied.*
(**H3**) *The function $\rho \in C(J,\mathbb{R})$ is increasing and there $\exists\ \Lambda_\rho > 0$, such that, for each $x \in J$ we have*

$$I^{\eth}\rho(t) < \Lambda_\rho \rho(x).$$

*Then, the fractional differential equation (1.1) with the boundary condition (1.2) is Ulam-Hyers-Rassias stable with respect to $\rho$.*

*Proof.* let $w \in C(J,\mathbb{R})$ be a solution of the inequality (2.3) i.e,

$$|{}^C D_{0^+}^{\eth,\mho(x)} w(x) - f(x,w(x))| \leq \varepsilon\rho(x), \quad x \in J. \tag{4.2}$$

If we defined $u \in C(J,\mathbb{R})$ the unique solution of the existence (1.1)-(1.2)

When u and w being continuous functions on $J$. From Lemma 2.14 we obtain

$$\begin{aligned} u(x) =& \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im \end{aligned}$$

we will take the integration of (4.2) and we obtain

$$\begin{aligned} &|w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im| \leq \varepsilon\Lambda_\rho\rho(x), \end{aligned}$$

on the other hand we have

$$\begin{aligned} \Big|w(x)-u(x)\Big| = \Big|&w(x) - \Omega_1 + \Omega_3\,[\mho(x)-\mho(0)] - \frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im \\ &+ \int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im\Big|, \end{aligned}$$

$$\Big|w(x)-u(x)\Big| = \Big|w(x)-\Omega_1+\Omega_3\left[\mho(x)-\mho(0)\right]-\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im$$
$$+\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im+\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im$$
$$-\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,w(\Im))d\Im-\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im$$
$$+\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}f(\Im,u(\Im))d\Im\Big|,$$

$$\le \varepsilon\Lambda_\rho\rho(x)+\frac{[\mho(x)-\mho(0)]}{[\mho(T)-\mho(0)]}\int_0^T \mho'(\Im)\frac{[\mho(T)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im$$
$$+\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im,$$

$$|w(x)-u(x)| \le \varepsilon\Lambda_\rho\rho(x)+\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}k_1|w(x)-u(x)|+\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im.$$

Let $\gamma_1=\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}k_1$ then, we obtain

$$|w(x)-u(x)| \le \frac{\varepsilon\Lambda_\rho\rho(x)}{(1-\gamma_1)}+\frac{1}{1-\gamma_1}\int_0^x \mho'(\Im)\frac{[\mho(x)-\mho(\Im)]^{\eth-1}}{\Gamma(\eth)}|f(\Im,w(\Im))-f(\Im,u(\Im))|d\Im,$$

$$|w(x)-u(x)| \le \frac{\varepsilon\Lambda_\rho\rho(x)}{1-\gamma_1}e^{\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}|w(x)-u(x)|},$$

$$|w(x)-u(x)| \le \varepsilon C_h.$$

Here $C_h=\frac{\varepsilon\Lambda_\rho\rho(x)}{1-\gamma_1}e^{\frac{[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)(1-\gamma_1)}|w(x)-u(x)|}$ hold, this show that the solution of the existence (1.1)-(1.2) is Ulam-Hyers-Rassias stable

□

## 5. Examples

**Example 5.1.** *Take the following existence*

$$^{C}D_{0^+}^{\eth,\sqrt{x}}u(x)=\frac{\sqrt{2-x}}{10+e^x}\frac{|u(x)|}{1+|u(x)|}, \tag{5.1}$$

*with the boundary condition*

$$\begin{cases} u(0)=3, \\ 2u(0)+3u(1)=2. \end{cases} \tag{5.2}$$

*where* $\eth=\frac{3}{2},\mho(x)=\sqrt{x}$ *and* $f(x,u(x))=\frac{\sqrt{2-x}}{10+e^x}\frac{|u(x)|}{1+|u(x)|}$
*To prove Banach contraction mapping, let* $x\in J$ *and* $u,v\in\mathbb{R}$
$|f(x,u_1)-f(x,u_2)|=\frac{\sqrt{2-x}}{10+e^x}|u_1-u_2|,$

*We need to show that* $|\frac{2[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}|k_1| < 1$ *Then, the result become*

$$|(\Theta u)(x)-(\Theta v(t)| = |\frac{\sqrt{2-x}}{10+e^x}\frac{|u(x)|}{1+|u(x)|} - \frac{\sqrt{2-x}}{10+e^x}\frac{|v(x)|}{1+|v(x)|}|,$$

$$|(\Theta u)(x)-(\Theta v(t)| \leq |\frac{\sqrt{2-x}}{10+e^x}||\frac{|u(x)|}{1+|u(x)|} - \frac{|v(x)|}{1+|v(x)|}|,$$

$$\leq |\frac{\sqrt{2}}{11}||\frac{u(x)-v(x)}{(1+|u(x)|)(1+v(x))}|,$$

$$|(\Theta u)(x)-(\Theta v(t)| \leq |\frac{\sqrt{2}}{11}||u(x)-v(x)|.$$

*Thus, the assumption (**H2**) holds true with* $k_1 = \frac{\sqrt{2}}{11}$. *Moreover, we have*

$$|\frac{2\,[\mho(T)-\mho(0)]^{\eth}}{\Gamma(\eth+1)}|k_1| = |\frac{2(\sqrt{1}-\sqrt{0})^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\frac{\sqrt{2}}{11}| = 0.19342 < 1.$$

*Finally, all the conditions of Theorem 3.2 are satisfied, thus the B.V.P (5.1)-(5.2) has a unique solution on* $[0,1]$.

## 6. Conclusion

In this paper, we examined the solutions for nonlinear FDEs with boundary conditions using the parameter of ℧-Caputo derivative. The Sadovskii fixed point theorem and Banach contraction principle ensure the existence and uniqueness of solutions to nonlinear problems. Additionally, the stability of Ulam-Hyers and Ulam-Hyers-Rassias solutions for the above issues is investigated. Finally, we provide an example to show the coherence of the theoretical conclusions. In the future, one can expand the provided fractional boundary value issue to more FDs, such as the Hilfer-Hadamard FDs and Caputo-Fabrizio FDs.

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# On the Diophantine Equation $(8r^2+1)^x+(r^2-1)^y=(3r)^z$ Regarding Terai's Conjecture

Tuba Çokoksen[1]*, Murat Alan[2]

**Abstract**
This study establishes that the sole positive integer solution to the exponential Diophantine equation $(8r^2+1)^x+(r^2-1)^y=(3r)^z$ is $(x,y,z)=(1,1,2)$ for all $r>1$. The proof employs elementary techniques from number theory, a classification method, and Zsigmondy's Primitive Divisor Theorem.

**Keywords:** Diophantine equations, Primitive divisor theorem, Terai's conjecture
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## 1. Introduction

Let $p$, $q$, and $r$ be coprime positive integers greater than 1 and let us consider exponential Diophantine equation

$$p^x+q^y=r^z$$

with $x,y,z\in\mathbb{N}$. In 1956, Sierpiński demonstrated that by reformulating the Pythagorean theorem with exponential expressions as variables, the exponential Diophantine equation $3^x+4^y=5^z$ has a unique solution, $(x,y,z)=(2,2,2)$ [1]. Subsequently, Jeśmanowicz extended this idea to general Pythagorean triples, proposing that for positive integers $a$, $b$, and $c$ satisfying the exponential Diophantine equation, the only solution remains $(2,2,2)$ [2].
In 1994, Terai extended this framework by considering the equation $p^x+q^y=r^z$ for positive integers $p,q,r$ with $p,q,r\geq 2$. He conjectured that while multiple solutions may exist for some triples $(p,q,r)$, only a few specific sets of such triples yield exceptions [3]. This conjecture has been verified for numerous specific cases, including particular forms of Diophantine equations

$$(ar^2+1)^x+(br^2-1)^y=(cr)^z. \tag{1.1}$$

In this study, the following exponential Diophantine equation equation is examined

$$(8r^2+1)^x+(r^2-1)^y=(3r)^z. \tag{1.2}$$

It is important to observe that equation (1.2) serves as a special case of equation (1.1), where the condition $a+b=c^2$ is fulfilled. This research was initiated based on Terai's conjecture. Expanding on this conjecture, various specific cases of equation (1.1) have been examined, resulting in the validation of Terai's conjecture in these instances.

- [4] $(4r^2+1)^p+(5r^2-1)^q=(3r)^t$
- [5] $(r^2+1)^p+(yr^2-1)^q=(zr)^t,\ 1+y=z^2$
- [6] $(12r^2+1)^p+(13r^2-1)^q=(5r)^t$
- [7] $(xr^2+1)^p+(yr^2-1)^q=(zr)^t\ , z|r$
- [8] $(xr^2+1)^p+(yr^2-1)^q=(zr)^t\ ,\ r=\pm1 \pmod z$
- [9] $(18r^2+1)^p+(7r^2-1)^q=(5r)^t$
- [10] $((x+1)r^2+1)^p+(xr^2-1)^q=(zr)^t,\ 2x+1=z^2$
- [11] $(3xr^2-1)^p+(x(x-3)r^2+1)^q=(xr)^t$
- [12] $(4r^2+1)^p+(21r^2-1)^q=(5r)^t$
- [13] $(5xr^2-1)^p+(x(x-5)r^2+1)^q=(xr)^t$
- [14] $(3r^2+1)^p+(yr^2-1)^q=(zr)^t$
- [15] $(4r^2+1)^p+(45r^2-1)^q=(7r)^t$
- [16] $(6r^2+1)^p+(3r^2-1)^q=(3r)^t$
- [17] $(x(x-l)r^2+1)^p+(xlr^2-1)^q=(xr)^t$
- [18] $(44r^2+1)^p+(5r^2-1)^q=(7r)^t$
- [19] $(9r^2+1)^p+(16r^2-1)^q=(5r)^t$

For the Diophantine equations related to Recurrence sequences see [20], [21] and [22]. The exponential Diophantine equation (1.2), where $r$ denotes a positive integer, is analyzed, and the following theorem is established.

**Theorem 1.1.** *Let r be a positive integer. The equation* (1.2) *possesses a single positive integer solution* $(x,y,z)=(1,1,2)$ *for any* $r>1$.

The theorem's proof relies on two approaches. The initial method, leveraging [23, 24], enables the derivation of additional potential solutions for the Diophantine equations $M^2+WN^2=q^K$ and $aM^2+bN^2=q^K$ from established solutions, subject to certain conditions [25, 26]. The second method draws upon an earlier rendition of the Primitive Divisor Theorem attributed to Zsigmondy [27].

## 2. Preliminaries

Consider a positive integer $W$. The notation $h(-4W)$ denotes the class number of positive binary quadratic forms with discriminant $-4W$.

**Lemma 2.1.** *([28], Theorems 11.4.3, 12.10.1 and 12.14.3])*

$$h(-4W)<\frac{4}{\pi}\sqrt{W}\log(2e\sqrt{W}).$$

*Let* $W,W_1,W_2,q$ *be positive integers such that* $\min\{W,W_1,W_2\}>1$, $gcd(W_1,W_2)=1$, $2\not|q$ *and* $gcd(W,q)=gcd(W_1,W_2,q)=1$.

**Lemma 2.2.** *[23] Given fixed relatively prime positive integers W and q, with* $W>1$ *and q being an odd integer, the equation is considered*

$$M^2+WN^2=q^K,$$

*where* $M,N,K\in\mathbb{Z}$, $K>0$ *and* $\gcd(M,N)=1$*, has solutions* $(M,N,K)$ *then any solution to the aforementioned equation can be represented as follows*

$$M+N\sqrt{-W}=\lambda_1(M_1+\lambda_2N_1\sqrt{-W})^t,\quad K=K_1t\quad \lambda_1,\lambda_2\in\{\pm1\}$$

$M_1,N_1,K_1$ *are positive integers satisfying* $M_1^2+WN_1^2=q^{K_1}$, $\gcd(M_1,N_1)=1$ *and* $h(-4W)\equiv 0 \pmod{K_1}$.

**Lemma 2.3.** *[23] Consider relatively prime positive integers* $W_1$ *and* $W_2$*, both greater than 1. Let* $(M,N,K)$ *denote a fixed solution of the equation*

$$W_1M^2+W_2N^2=q^K. \tag{2.1}$$

Given that $K>0$, $\gcd(M,N)=1$, $2\nmid q$ and $M,N,K\in\mathbb{Z}$, there also exists a unique positive integer $s$ such that

$$s=W_1\alpha M+W_2\beta N,\qquad 0<t<q$$

where $\alpha$ and $\beta$ are integers such that $\beta M-\alpha N=1$ [[23], Lemma 1]. The positive integer $s$ is referred to as the characteristic number of the specific solution $(M,N,K)$ and is denoted by $<M,N,K>$. When $<M,N,K>=s$, it implies that $W_1M\equiv -sN \pmod q$ [[23], Lemma 6]. Let $(M_0,N_0,K_0)$ be a solution to (2.1) with $<M_0,N_0,K_0>=s_0$. Therefore, the set of all solutions $(M,N,K)$ with $<M,N,K>\equiv \pm s_0 \pmod q$ is termed a solution class of (2.1), expressed as $S(s_0)$.

**Lemma 2.4.** *[23] For each solution class* $S(s_0)$ *of* (2.1)*, a unique solution exists* $(M_1,N_1,K_1)\in S(s_0)$ *such that* $M_1$ *and* $N_1$ *are positive, and* $K_1\ge K$ *for all solutions* $(M,N,K)\in S(s_0)$*, where K spans all possible solutions. This particular solution* $(M_1,N_1,K_1)$ *is referred to as the least solution of* $S(s_0)$*. If* $(M,N,K)$ *is a solution in the set* $S(s_0)$ *then*

$$K=K_1t,\ \ 2\nmid t,\ \ t\in\ \mathbb{N},$$

$$M\sqrt{W_1}+N\sqrt{W_2}=\lambda_1\left(M_1\sqrt{W_1}+\lambda_2N_1\sqrt{-W_2}\right)^t,\ \lambda_1,\lambda_2\in\{1,-1\}.$$

**Lemma 2.5.** *[24] Let* $(M_1,N_1,K_1)$ *be the least solution of* $S(s_0)$*. If* (2.1) *has a solution* $(M,N,K)\in S(s_0)$ *satisfying* $M>0$ *and* $N=1$*, then* $N_1=1$*. Additionally, if* $(M,K)\neq(M_1,K_1)$*, in that case, at least one of the following conditions is satisfied*

*(i)* $W_1M_1^2=\frac{1}{4}(q^{K_1}\pm1),\qquad W_1=\frac{1}{4}(3q^{K_1}\pm1)$

$(M,K)=(M_1|W_1M_1^2-3W_2|,3K_1)$

*(ii)* $W_1K_1^2=\frac{1}{4}F_{3a+3\varepsilon},\quad W_2=\frac{1}{4}L_{3a},\ \ q^{K_1}=F_{3a+\varepsilon}$

$(M,K)=(M_1|W_1^2M_1^4-10W_1W_2M_1^2+5W_2^2|,\ 5K_1)$

*where a is a positive integer,* $\varepsilon\in\{1,-1\}$*, and* $F_n$ *is the n-th Fibonacci number in which each number is the sum of the two preceding ones.*

*Let* $\gamma$ *and* $\theta$ *be algebraic integers. A Lucas pair refers to a pair* $(\gamma,\theta)$ *such that* $\gamma+\theta$ *and* $\gamma\theta$ *are non-zero relatively prime integers, and* $\frac{\gamma}{\theta}$ *is not a root of unity. For any given pair* $(\gamma,\theta)$ *forming a Lucas pair, the resulting sequences of Lucas numbers are given by*

$$L_n(\gamma,\theta)=\frac{\gamma^n-\theta^n}{\gamma-\theta},\quad n=0,1,2,\ldots$$

*It's worth noting that primitive divisors of* $L_n(\gamma,\theta)$ *are prime numbers p for which* $p|L_n(\gamma,\theta)$ *and* $p\nmid(\gamma,\theta)^2L_1(\gamma,\theta)\ldots L_{n-1}(\gamma,\theta)$*. For any Lucas sequence* $L_n(\gamma,\theta)$ *determined by a finite set of parameters* $(n,\gamma,\theta)$*, if* $n\ge5$ *and* $n\neq6$*, it is guaranteed that the sequence has always a primitive divisor.*

**Lemma 2.6.** *[25] If $n > 30$, then $L_n(\gamma,\theta)$ is guaranteed to have a primitive divisor.*

**Lemma 2.7.** *[26] For $4 < n \leq 30$ and $n \neq 6$, aside from equivalence, $L_n(\gamma,\theta)$ contains a primitive divisor, except for the following pairs of parameters $(k,l)$:*

- $(1,-15),(1,-11),(1,-7),(1,5),(2,-40),(12,-76)$ *or* $(12,-1364)$ *if* $n=5$,
- $(1,-19)$ *or* $(1,-7)$ *if* $n=7$,
- $(1,-7)$ *or* $(2,-24)$ *if* $n=8$,
- $(2,-8),(5,-47)$ *or* $(5,-3)$ *if* $n=10$,
- $(1,-19),(1,-15),(1,-11),(1,-7),(1,-5)$ *or* $(2,-56)$ *if* $n=12$,
- $(1,-7)$ *if* $n=13,18$ *or* $30$.
  *where* $(\gamma,\theta)=(\frac{k+\sqrt{l}}{2},\frac{k-\sqrt{l}}{2})$.

**Lemma 2.8.** *[9] If $a,b,c$ and $r > 1$ are positive integers satisfying $a+b=c^2$, and $(x,y,z) \geq 0$ is a solution to the exponential Diophantine equation*

$$(ar^2+1)^x+(br^2-1)^y=(cr)^z,$$

*where $x$ is the larger of the two values $\{x,y\}$, In this case, the following inequalities are satisfied*

$$\left(2-\frac{\log\left(\frac{c^2}{a}\right)}{\log(cr)}\right)x < z \leq 2x.$$

*On the other hand, if $y$ is the larger value, then*

$$\left(2-\frac{\log\left(\frac{c^2r^2}{br^2-1}\right)}{\log(cr)}\right)y < z \leq 2y.$$

*In particular, when $M=\max\{x,y\} > 1$, it follows that*

$$\left(2-\frac{\log\left(\frac{c^2}{\min\{a,b-\frac{1}{r^2}\}}\right)}{\log(cr)}\right)M < z < 2M.$$

*This offers a more precise description of the range of $z$ based on $M$ and the specified parameters.*

**Proposition 2.9.** *[27] Consider $C$ and $D$ be relatively prime integers with $C > D \geq 1$. Let $\{a_n\}_{n\geq 1}$ be the sequence defined as*

$$a_n=C^n+D^n.$$

*If $n > 1$, then $a_n$ has a prime factor not dividing $a_1a_2a_3\cdots a_{n-1}$, whenever $(C,D,n) \neq (2,3,1)$.*

## 3. Proof of Theorem 1.1

### 3.1 The case $2|r$

This section demonstrates that Theorem 1.1 is valid under the condition $2 \mid r$.

**Lemma 3.1.** *If $2|r$, then $(x,y,z)=(1,1,2)$ constitutes the sole positive integer solution of the equation (1.2).*

*Proof.* For $z \leq 2$, it is evident that $(x,y,z)=(1,1,2)$ is the unique solution to equation (1.2). Thus, the assumption $z \geq 3$ is made. Considering equation (1.2) modulo $r^2$, the relation $1+(-1)^y \equiv 0 \pmod{r^2}$ holds, implying that $y$ must be odd, given that $r^2>2$. Further, reducing equation (1.2) modulo $r^3$, the following is obtained

$$1+8r^2x+(-1)+r^2y \equiv 0 \pmod{r^3},$$

$$8x+y \equiv 0 \pmod{r},$$

which results in a contradiction, since $y$ is odd and $r$ is even. Therefore, it is concluded that equation (1.2) has no positive integer solutions for $z \geq 3$. Consequently, the only positive integer solution to equation (1.2) when $r$ is even is $(1,1,2)$. The case where $r$ is odd will now be considered. □

### 3.2 The case $2 \nmid r$ where $r \equiv 0 \pmod 3$

This section demonstrates that Theorem 1.1 is valid under the condition $2 \nmid r$ where $r \equiv 0 \pmod 3$.

*Proof.* Let $(x,y,z)$ be any solution to equation (1.2). It is clear that $(x,y,z)=(1,1,2)$ constitutes a solution of (1.2). For $r>1$, examining equation (1.2) modulo $r^2$, it can be concluded, similar to the earlier scenario, that $y$ must be odd. The investigation then continues by splitting into two cases depending on the parity of $x$. First, let us assume $x$ is odd. Next, the focus turns to the Diophantine equation

$$(8r^2+1)M^2+(r^2-1)N^2=(3r)^K, \quad K>0 \quad \text{and} \quad M,N,K \in \mathbb{Z}. \tag{3.1}$$

Since $(x,y,z)$ represents any solution of equation (1.2), it follows from Lemma 2.3 that

$$(M,N,K)=\left((8r^2+1)^{\frac{x-1}{2}},(r^2-1)^{\frac{y-1}{2}},z\right) \tag{3.2}$$

is a solution of equation (3.1). Let $s=\langle(8r^2+1)^{\frac{x-1}{2}},(r^2-1)^{\frac{y-1}{2}},z\rangle$ be the characteristic number corresponding to the solution given in (3.2). From the congruence

$$(8r^2+1)^{\frac{x+1}{2}} \equiv -s(r^2-1)^{\frac{y-1}{2}} \pmod{3r},$$

it follows that $s \equiv \pm 1 \pmod{3r}$.

It is noteworthy that $(M_1,N_1,K_1)=(1,1,2)$ also satisfies equation (3.1), and let $s_0=\langle 1,1,2\rangle$ denote the characteristic number of this solution. Hence, the following holds

$$8r^2+1 \equiv -s_0 \pmod{3r} \tag{3.3}$$

$$s_0 \equiv -1 \pmod{3r}$$

Thus, it is observed by the equation (3.3) $s \equiv \pm s_0 \pmod{3r}$, indicating that the solutions $(M_1,N_1,K_1)=(1,1,2)$ and the one given in (3.2) belong to the same solution class $S(s_0)$ of equation (3.1). Furthermore, $(M,N,K)=(1,1,2)$ is clearly the least solution within $S(s_0)$. Therefore, applying Lemma 2.4, it follows that

$$z=2t, \ 2 \nmid t, \ t \in \mathbb{N},$$

$$(8r^2+1)^{\frac{x-1}{2}}\sqrt{8r^2+1}+(r^2-1)^{\frac{y-1}{2}}\sqrt{1-r^2}=\lambda_1\left(\sqrt{8r^2+1}+\lambda_2\sqrt{1-r^2}\right)^t. \tag{3.4}$$

By expanding the right-hand side of equation (3.4) and equating the coefficients of $\sqrt{1-r^2}$, the following result is obtained

$$(r^2-1)^{\frac{y-1}{2}}=\lambda_1\lambda_2\sum_{i=0}^{\frac{t-1}{2}}\binom{t}{2i+1}(8r^2+1)^{\frac{t-1}{2}-i}(r^2-1)^i \tag{3.5}$$

At this point, it is asserted that $y=1$. Suppose $y>1$. From equation (3.5), it can be deduced that

$$0 \equiv \lambda_1\lambda_2 t \cdot (8r^2+1)^{\frac{t-1}{2}} \pmod{(r^2-1)}$$

$$0 \equiv \lambda_1\lambda_2 t \cdot 9^{\frac{t-1}{2}} \pmod{(r^2-1)}.$$

This leads to a contradiction, as $2 \nmid t\cdot 9^{\frac{t-1}{2}}$ and $2 \mid (r^2-1)$. Therefore, it is concluded that $y=1$, and consequently $N=(r^2-1)^{\frac{y-1}{2}}=1$. The two conditions in Lemma 2.5 will now be verified. Given that $(M_1,N_1,K_1)=(1,1,2)$ represents the smallest solution of $S(s_0)$, Lemma 2.5 implies that either

$$8r^2+1=\frac{1}{4}((3r)^2\pm 1)$$

or

$$F_{3a+\varepsilon}=(3r)^2$$

where $\varepsilon=\pm 1$. The first equation leads to

$$4(8r^2+1)=(3^2r^2\pm 1),$$

resulting in $4\equiv \pm 1 \pmod{r^2}$, which is not possible. Moreover, since the only square Fibonacci number greater than 1 is $F_{12}=12^2$ [29], the second condition implies $3r=12$, which is also impossible due to the parity of $r$. Consequently, by Lemma 2.5, it follows that $(M,K)=((8r^2+1)^{\frac{t-1}{2}},z)=(M_1,K_1)=(1,2)$. Thus, equation (1.2) has no positive integer solutions other than $(x,y,z)=(1,1,2)$ when $x$ is odd.

Next, the case when $2|x$ is considered. From equation (1.2), the Diophantine equation

$$M^2+(r^2-1)N^2=(3r)^K, \quad \gcd(M,N)=1, \quad K>0,$$

admits the solution

$$(M,N,K)=\left((8r^2+1)^{\frac{x}{2}},(r^2-1)^{\frac{y-1}{2}},z\right).$$

Hence, by Lemma 2.2, it is concluded that

$$z=K_1 t, \quad t\in\mathbb{N}$$

$$(8r^2+1)^{\frac{x}{2}}+(r^2-1)^{\frac{y-1}{2}}\sqrt{1-r^2}=\lambda_1(M_1+\lambda_2 N_1\sqrt{1-r^2})^t \tag{3.6}$$

where $\lambda_{1,2}\in\{-1,1\}$ and $M_1,N_1,K_1$ are positive integers satisfying

$$M_1^2+(r^2-1)N_1^2=(3r)^{K_1},\ \gcd(M_1,N_1)=1 \tag{3.7}$$

$$h(-4(r^2-1))\equiv 0 \pmod{K_1}. \tag{3.8}$$

Suppose that $2|t$ and let

$$M_2+N_2\sqrt{1-r^2}=(M_1+\lambda_2 N_1\sqrt{1-r^2})^{\frac{t}{2}}. \tag{3.9}$$

By taking the norm of both sides of equation (3.8) in the field $\mathbb{Q}(\sqrt{1-r^2})$ and applying equation (3.7), the following result is obtained

$$M_2^2+(r^2-1)N_2^2=(3r)^{\frac{K_1 t}{2}}=(3r)^{\frac{z}{2}}. \tag{3.10}$$

By substituting equation (3.9) into equation (3.6), the result is obtained as follows

$$(8r^2+1)^{\frac{x}{2}}+(r^2-1)^{\frac{y-1}{2}}\sqrt{1-r^2}=\lambda_1(M_2+N_2\sqrt{1-r^2})^2$$

and therefore it follows that

$$(8r^2+1)^{\frac{x}{2}}=\lambda_1(M_2^2-N_2^2(r^2-1)), \tag{3.11}$$

$$(r^2-1)^{\frac{y-1}{2}}=2\lambda_1 M_2 N_2. \tag{3.12}$$

Since $\gcd(8r^2+1,r^2-1)=1$, it follows from equations (3.11) and (3.12) that $|M_2|=1$. Thus, $|N_2|=\frac{1}{2}(r^2-1)^{\frac{y-1}{2}}$. Substituting $|M_2|$ and $|N_2|$ into equation (3.10), the result is

$$1+\frac{1}{4}(r^2-1)^y=(3r)^{\frac{z}{2}}$$

which leads to

$$3\equiv 0 \pmod{r^2}.$$

This presents a contradiction, leading to the conclusion that $2\nmid t$. Define

$$\gamma=M_1+N_1\sqrt{1-r^2},\quad \theta=M_1-N_1\sqrt{1-r^2}.$$

By taking the complex conjugate of equation (3.6), the following relation is obtained

$$(r^2-1)^{\frac{y-1}{2}}=N_1\left|\frac{\gamma^t-\theta^t}{\gamma-\theta}\right|=N_1|L_t(\gamma,\theta)|. \tag{3.13}$$

By equation (3.7), it holds that $\gamma+\theta=2M_1$, $\gamma-\theta=2N_1\sqrt{1-r^2}$, and $\gamma\theta=(3r)^{K_1}$. Since $\gcd(M_1,N_1)=1$, the integers $\gamma+\theta=2M_1$ and $\gamma\theta=(3r)^{K_1}$ are also relatively prime, as implied by equation (3.7), and $\frac{\gamma}{\theta}\neq\pm1$, with $\gamma$ and $\theta$ being units in the ring of algebraic integers of $\mathbb{Q}(\sqrt{1-r^2})$. Consequently, $L_t(\gamma,\theta)$ forms a Lucas sequence.

From equation (3.13), it is evident that the Lucas numbers $L_t(\gamma,\theta)$ lack primitive divisors. By applying Lemma 2.6 and Lemma 2.7, it is concluded that $t\leq 30$. Furthermore, if $4<t\leq 30$ and $t\neq 6$, the parameters $(k,l)=(2M_1,4N_1^2(1-r^2))$ must match one of the parameter sets listed in Lemma 2.7. However, none of these sets align with the given parameters. Therefore, it follows that $t\leq 3$.

The case $t=3$ will be shown to be impossible. Assuming $t=3$, the right-hand side of equation (3.6) is expanded, and by equating the coefficients on both sides, it is determined that

$$(8r^2+1)^{\frac{x}{2}}=\lambda_1 M_1(M_1^2-3(r^2-1)N_1^2) \tag{3.14}$$

$$(r^2-1)^{\frac{y-1}{2}}=\lambda_1\lambda_2 N_1(3M_1^2-(r^2-1)N_1^2). \tag{3.15}$$

From equation (3.7), it is evident that $\gcd(3M_1,r^2-1)=1$. Thus, from equation (3.15), the relation $3M_1^2-(r^2-1)N_1^2=\pm1$ holds. In fact, upon considering this equation modulo 3, it can be observed that only the positive sign is feasible, and the following equation is obtained

$$3M_1^2-(r^2-1)N_1^2=1. \tag{3.16}$$

Thus, it follows that

$$|N_1|=(r^2-1)^{\frac{y-1}{2}}. \tag{3.17}$$

By substituting equation (3.17) into equation (3.14), the following result is obtained

$$(8r^2+1)^{\frac{x}{2}}=\lambda_1 M_1(M_1^2-3(r^2-1)^y) \tag{3.18}$$

By considering equations (3.16) and (3.17) modulo $3r$, it follows that $3M_1^2-(r^2-1)^y \equiv 0 \pmod{3r}$, which implies $M_1 \equiv 1 \pmod{r}$. Substituting this result into equation (3.18) yields

$$(8r^2+1)^{\frac{x}{2}}=\lambda_1 M_1(M_1^2-3(r^2-1)^y)$$

leading to

$$1 \equiv 0 \pmod{r}$$

which is evidently a contradiction. Therefore, the only possibility remaining is $t=1$. Consequently, $z=W_1t=K_1$, and according to equation (3.8), it is established that $K_1 \leq -4(r^2-1)$. Utilizing the upper bound provided by Lemma 2.1, the following result is obtained

$$z<\frac{4}{\pi}\sqrt{r^2-1}\,\log\,(2e\sqrt{r^2-1}). \tag{3.19}$$

Assume $z=3$. In this case, at least one of $x$ or $y$ must be greater than 1. If $x \geq 2$, it follows that $(3r)^3>(8r^2+1)^x \geq (8r^2+1)^2>8^2r^4$, leading to $3^3>8^2r$, which implies $64>27$, resulting in a contradiction. Similarly, if $(3r)^3>(r^2-1)^2+(8r^2+1)$, this also results in a contradiction. Thus, it can be concluded that $z \geq 4$. Examining equation (1.2) modulo $r^4$ leads to

$$(8r^2+1)^x+(r^2-1)^y \equiv 0 \pmod{r^4}$$

and hence

$$8x+y \equiv 0 \pmod{r^2}$$

$$r^2 \leq 8x+y. \tag{3.20}$$

The application of the logarithm function facilitates the straightforward derivation of the inequalities $x<z$ and $y<1.06z$. Consequently, from inequality (3.20), it follows that $r^2<9.06z$. Therefore, from the derived inequality

$$r^2<9.06z<9.06\cdot\frac{4}{\pi}\sqrt{r^2-1}\log(2e\sqrt{r^2-1}),$$

it can be concluded that $r \leq 63$. Furthermore, by consulting Lemma 2.8, the following upper bounds for $x$ and $y$ can be established

$$1.94x<\left(2-\frac{\log\left(\frac{9}{8}\right)}{\log(9)}\right)x<\left(2-\frac{\log\left(\frac{9}{8}\right)}{\log(3r)}\right)x<z \leq 2x \tag{3.21}$$

$$0.95y<\left(2-\frac{\log(10)}{\log(9)}\right)y<\left(2-\frac{\log\left(\frac{10r^2-10}{r^2-1}\right)}{\log(9)}\right)y<\left(2-\frac{\log\left(\frac{9r^2}{r^2-1}\right)}{\log(3r)}\right)y<z \leq 2y. \tag{3.22}$$

Based on equations (3.21) and (3.22), it can be concluded that equation (1.2) has no solutions in positive integers for $z \leq 6$. Assuming $z>6$, the analysis of equation (1.2) proceeds by considering it modulo $r^4$, $r^6$, and $r^8$.

1. Modulo $r^4$: By considering equation (1.2) modulo $r^4$, the following congruence is obtained

$$8r^2x+r^2y \equiv 0 \pmod{r^4}.$$

In other words,

$$8x+y \equiv 0 \pmod{r^2}. \tag{3.23}$$

2. Modulo $r^6$: Taking equation (1.2) modulo $r^6$, the following congruence is obtained

$$8r^2x+8^2r^4\frac{x(x-1)}{2}+r^2y-r^4\frac{y(y-1)}{2} \equiv 0 \pmod{r^6}.$$

Simplifying,

$$8x+8^2r^2\frac{x(x-1)}{2}+y-r^2\frac{y(y-1)}{2}\equiv 0 \pmod{r^4}. \tag{3.24}$$

3. Modulo $r^8$: Finally, taking equation (1.2) modulo $r^8$, the following congruence is obtained

$$\begin{aligned}&8r^2x+8^2r^4\frac{x(x-1)}{2}+8^3r^6\frac{x(x-1)(x-2)}{6}\\&\quad+r^2y-r^4\frac{y(y-1)}{2}+r^6\frac{y(y-1)(y-2)}{6}\equiv 0 \pmod{r^8}.\end{aligned}$$

Simplifying,

$$\begin{aligned}&8x+8^2r^2\frac{x(x-1)}{2}+8^3r^4\frac{x(x-1)(x-2)}{6}\\&\quad+y-r^4\frac{y(y-1)}{2}+r^4\frac{y(y-1)(y-2)}{6}\equiv 0 \pmod{r^6}.\end{aligned} \tag{3.25}$$

In summary, equations (3.23), (3.24), and (3.25) represent the congruence conditions derived from equation (1.2) modulo $r^2$, $r^4$, and $r^6$, respectively. Utilizing equation (3.19) alongside the conditions $x,y<z$, and the congruences (3.23), (3.24), and (3.25), a brief computer program was developed using Maple to investigate all potential solutions of equation (1.2) within the range $3\le r\le 63$. The results show that there are no positive integer solutions $(r,x,y,z)$ to equation (1.2) when $z\ge 3$. This concludes the proof. □

### 3.3 The case $r\nmid 2$ where $r\equiv\pm1 \pmod 3$

This section demonstrates that Theorem 1.1 is valid under the condition $r\nmid 2$ where $r\equiv\pm1 \pmod 3$.

**Lemma 3.2.** *If $r$ is a positive odd integer such that $r\equiv\pm1 \pmod 3$, then equation (1.2) admits sole the positive integer solution $(x,y,z)=(1,1,2)$.*

*Proof.* Let $k_1$ and $k_2$ be positive integers, and consider the case where $r\equiv\pm1 \pmod 3$. In this context, equation (1.2) can be reformulated as follows

$$8r^2+1=3^{k_1}A, \qquad (8r^2+1)^x=3^{k_1x}A^x \tag{3.26}$$

$$r^2-1=3^{k_2}B, \qquad (r^2-1)^y=3^{k_2y}B^y \tag{3.27}$$

where $A,B\not\equiv 0 \pmod 3$. Then the equation (1.2) becomes

$$3^{k_1x}A^x+3^{k_2y}B^y=(3r)^z. \tag{3.28}$$

Firstly, let's consider $k_1x>k_2y$, then equation (3.28) can be written as

$$3^{k_2y}(3^{k_1x-k_2y}A^x+B^y)=3^zr^z$$

this implies that

$$k_2y=z \tag{3.29}$$

then equation (1.2) becomes

$$(8r^2+1)^x=((3r)^{k_2})^y-(r^2-1)^y.$$

Apply Proposition 2.9 , $y=1$ is found. When $y=1$ equation (3.27) turns into,

$$(r^2-1)^y=3^{k_2y}B^y=3^{k_2}B. \tag{3.30}$$

And substituting (3.29) into (3.30) with $y=1$

$$r^2=3^zB+1. \tag{3.31}$$

If $z \leq 2$, then $(x,y,z)=(1,1,2)$ is evidently the sole solution of equation (1.2). Therefore, let's assume $z=3$. Equation (1.2) becomes $(8r^2+1)^x+r^2-1=(3r)^3$. $x \geq 2$ gives $(3r)^3 > (8r^2+1)^x \geq (8r^2+1)^2 > 8^2r^4$, and hence $3^3 > 8^2r > 64$, a contradiction. Also it seen that $y=1$ and $x=1$, the equation (1.2) turns into $8r^2+1+r^2-1=(3r)^3$ also leads us a contradiction under the condition $r \equiv \pm 1 \pmod 3$. Now, consider the scenario in which $z \geq 4$. Upon taking equation (1.2) modulo $r^4$, it becomes evident that $y=1$ as a result of Proposition 2.9 [27]. Consequently, the following congruence is established.

$$8r^2x+r^2 \equiv 0 \pmod{r^4}.$$

This implies that

$$8x+1 \equiv 0 \pmod{r^2}$$

$$r^2 \leq 8x+1. \tag{3.32}$$

Substituting (3.31) into inequality (3.32), the following inequality is obtained.

$$3^zB \leq 8x. \tag{3.33}$$

Also $x$ is bounded as $x<z$. So (3.33) turns into (3.34)

$$3^zB \leq 8x < 8z$$

$$3^zB \leq 8z. \tag{3.34}$$

Consequently, it is evident that no positive integer $z$ can satisfy the condition $z \geq 4$. Similarly, upon conducting a comparable analysis in the context where $k_2y > k_1x$, it becomes clear that no positive integer $z$ can satisfy $z \geq 3$.

Finally, consider the scenario where $k_1x = k_2y$. By summing equations (3.26) and (3.27), the following relation is established.

$$9r^2 = 3^{k_1}A + 3^{k_2}B. \tag{3.35}$$

An examination of this equation will proceed based on the various cases concerning the positive integers $k_1$ and $k_2$.

#### 3.3.1 $k_1 = 2$ and $k_2 \geq 3$

In the scenario where $k_1 = 2$, it is evident that $k_2$ must be even, given that $y$ is odd. From equation (3.35), the following relationship can be established

$$2x = k_2y.$$

This implies the existence of a positive integer $k_3$ such that $2k_3 = k_2$. Substituting this into the aforementioned equation yields $x = k_3y$. Consequently, equation (1.2) can be expressed as

$$((8r^2+1)^{k_3})^y + (r^2-1)^y = (3r)^z.$$

Applying Proposition 2.9, it follows that $y=1$. Therefore, it is concluded that no solutions exist for $x>2$.

#### 3.3.2 $k_1 \geq 3$ and $k_2 = 2$

It can be expressed that

$$\frac{k_1}{k_2} = \frac{y}{x}$$

where $k_1 x = k_2 y$.Notably, since $\gcd(x,y) = 1$, if there exists an odd prime $p \geq 1$ such that $p \mid x$ and $p \mid y$, then, by Zsigmondy's Theorem, no solutions for $x$ and $y$ would exist. As a result, it follows that $x = 2$ and $k_2 = 2$, with $y$ being an odd integer. Consequently, one can derive

$$y = k_1 \geq 3 \quad \text{and} \quad x = k_2 = 2.$$

Thus, equation (3.28) transforms into

$$3^{k_1 x} A^x + 3^{k_2 y} B^y = (3r)^z.$$

This further simplifies to:

$$3^{2y}(A^2 + B^y) = (3r)^z.$$

If $3 \nmid (A^2 + B^y)$, it follows that $2y = z$. Hence, equation (1.2) can be rewritten as

$$(8r^2+1)^x = ((3r)^2)^y - (r^2-1)^y.$$

Applying Zsigmondy's Proposition, it is concluded that $y = 1$, which leads to a contradiction. Thus, it can be stated that no positive integer solutions exist for $x$ and $y$, and therefore, $z \leq 2$.
Assuming $3 \mid (A^2 + B^y)$, equations (3.26) and (3.27) can be expressed as

$$r^2 - 1 = 3^{k_2} B = 9B,$$

$$8r^2 + 1 = 3^{k_1} A.$$

Adding these two equations results in

$$9r^2 = 3^{k_1} + 9B.$$

Taking the equation modulo 3, it follows that

$$1 \equiv B \pmod 3.$$

Consequently, it becomes evident that no positive integer $A$ can satisfy the condition
$3 \mid (A^2 + B^y)$. This concludes the proof. □

## 4. Conclusion

This study investigates equation (1.1) with the parameters $(a,b,c) = (8,1,3)$, identifying the unique solution $(x,y,z) = (1,1,2)$ for $r > 1$. The findings provide additional evidence supporting Terai's Conjecture. The objective is to advance the understanding of such equations and contribute to the development of a generalized form.

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# On Some New Rhaly Sequence Spaces and Rhaly Sections in BK-Space

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**Abstract**
In this paper, we introduce some new sequence spaces and sectional subspaces related to the Rhaly matrix and BK spaces. Furthermore, we investigate their relations and identities among these subspaces and duals.

**Keywords:** AK-space, Distinguished subspaces, Matrix domain, Rhaly matrix, rK-space
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## 1. Background, Preliminaries and Notations

Let $w$ be the linear space of all complex or real valued sequences with the topology $\tau_w$ of coordinatwise convergence. A linear subspace of $w$ is called a sequence space. A sequence space $\lambda$ with a locally convex topology $\tau$ is a $K$-space if the inclusion map: $(\lambda,\tau)\to(w,\tau_w)$ is continuous. If $\tau$ is complete metrizable and locally convex, $(\lambda,\tau)$ is called FK-space. An FK-space whose topology is normable is called a BK-space. The basic properties of FK(BK-)-spaces may be found in [1, 2].

By $\ell_\infty$, $c$, $c_0$ and $\ell_p$, we denote the spaces of all bounded, convergent, null and absolutely p-summable complex sequences, respectively, where $1\le p<\infty$. The spaces $\ell_\infty$, $c$ and $c_0$ are BK-space endowed with the sup norm $\|x\|_\infty=\sup_{k\in\mathbb{N}}|x_k|$, and $\ell_p$ $(1\le p<\infty)$ is a BK-space with the norm $\|x\|_p=\left(\sum_{k=0}^\infty|x_k|^p\right)^{1/p}$, where $\mathbb{N}=\{0,1,2,3,\dots\}$.

Let $X$ and $Y$ be two sequence spaces, and $A=(a_{nk})$ be an infinite matrix of complex numbers $a_{nk}$, where $k,n\in\mathbb{N}$. Then, we say that $A$ defines a *matrix transformation* from $X$ into $Y$ and we denote it by writing $A:X\to Y$, if for every sequence $x=(x_k)\in X$ the $A$-transform $Ax=\{(Ax)_n\}$ of $x$ is in $Y$, where

$$(Ax)_n=\sum_{k=0}^\infty a_{nk}x_k \ \text{ for each } \ n\in\mathbb{N}. \tag{1.1}$$

By $(X:Y)$, we denote the class of all matrices $A$ such that $A:X\to Y$. Thus, $A\in(X:Y)$ if and only if the series on the right side of (1.1) converges for each $n\in\mathbb{N}$ and every $x\in X$, and we have $Ax\in Y$ for all $x\in X$. Also, we write $A_n=(a_{nk})_{k\in\mathbb{N}}$ for the sequence in the $n^{\text{th}}$ row of $A$.

The *domain* $X_A$ of an infinite matrix $A$ in a sequence space $X$ is defined by

$$X_A:=\left\{x=(x_k)\in w: Ax\in X\right\} \tag{1.2}$$

which is a sequence space. Depending on the choice of the matrix $A$, $X_A$ may include or be included by the original space $X$. Indeed if we choose $A = \Delta$, the backward difference matrix, then $c_\Delta \supset c$ ($bv = (\ell_1)_\Delta \supset \ell_1$) but in the case $A = \Delta^{-1} = S$, the summation matrix, $c_S = cs \subset c$ ($bs = (\ell_\infty)_S \subset \ell_\infty$), where both of two inclusions are strict. However, if we define $X = c_0 \oplus span\{z\}$ with $z = \{(-1)^k\}$, i.e., $x \in X$ if and only if $x = s + \alpha z$ for some $s \in c_0$ and some $\alpha \in \mathbb{C}$, and consider the matrix $A$ with the rows $A_n$ defined by $A_n = (-1)^n e^n$ for all $n \in \mathbb{N}$, we have $Ae = z \in X$ but $Az = e \notin X$ which gives that $z \in X \setminus X_A$ and $e \in X_A \setminus X$ where $e^k$ is a sequence whose only nonzero term is 1 in $k^{th}$ place for each $k \in \mathbb{N}$. That is to say that the sequence spaces $X_A$ and $X$ are overlap but neither contains the other. In the literature, there are many studies on the matrix domain, see for instance [3]-[19].

The *continuous dual* of a normed space $X$ is defined as the space of all bounded linear functionals on $X$ and denoted by $X'$. If A is triangle, that is $a_{nk} = 0$ if $k > n$ and $a_{nn} \neq 0$ , and $X$ is a sequence space, then $f \in X'_A$ if and only if $f = g \circ A$, $g \in X'$.

Let $(X,P)$ be a locally convex space. A set $S \subset X$ is called *fundamental* if the span of $S$ is dense in $X$. The useful results concerning with the fundamental set which are applications of Hahn-Banach Theorem as follows:

**Corollary 1.1.** *(i) $S \subset X$ is fundamental if and only if $f(S) = 0$ implies $f = 0$ for each $f \in X'$.*
*(ii) Let $S_1$ and $S_2$ be non-empty subsets of $X$. The inclusion $S_1 \subset \overline{span\{S_2\}}$ holds if and only if $f(S_2) = 0$ implies $f(S_1) = 0$ for each $f \in X'$.*

For the sequence spaces $X,Y$ and $Z$, the multiplier space $X^Y$ (or $M(X,Y)$) is defined by

$$X^Y = \{a = (a_k) \in w : \forall x \in X, x \cdot a = (x_k a_k) \in Y\},$$

and $X^{YZ} = (X^Y)^Z$. The $\beta$-, $\gamma$- and $f$-duals $X^\beta, X^\gamma$ and $X^f$ of a sequence space $X$ are defined by

$$X^\beta := X^{cs} = \left\{ a = (a_k) \in w : \left( \sum_{k=0}^{n} a_k x_k \right)_{n \in \mathbb{N}} \in c \text{ for all } x = (x_k) \in X \right\},$$

$$X^\gamma := X^{bs} = \left\{ a = (a_k) \in w : \left( \sum_{k=0}^{n} a_k x_k \right)_{n \in \mathbb{N}} \in \ell_\infty \text{ for all } x = (x_k) \in X \right\},$$

and

$$X^f := \left\{ a = (a_k) \in w : \exists f \in X', a = (f(e^k)) \right\}$$

respectively.

**Lemma 1.2.** *[2] Let $X$ be an* FK *space containing $\phi = span\{e^k\}$, and let $Y$ and $Z$ any sequence spaces. Then, the following assertions hold.*
*(i) If $\phi \subset Y \subset Z$ then $\phi \subset X^Y \subset X^Z$,*
*(ii) if $X \subset Y$ then $X^Z \supset Y^Z$ and $X^f \supset Y^f$*
*(iii) $X \subset X^{YY}$*
*(iv) $X^Y = X^{YYY}$*
*(v) $X^\beta \subset X^\gamma \subset X^f$*
*(vi) $X^f = (\overline{\phi})^f$*

Zeller in [20] introduced the theory of FK-spaces and investigated the properties of sectional convergence in [21]. Sectional boundedness in BK-spaces was studied by Sargent [22]. Given a BK-space $X \supset \phi$, we denote the $n^{th}$ section of a sequence $x \in X$ by $x^{[n]} = \sum_{k=0}^{n} x_k e^k$, and we say that $x$ has

AK-property when $\lim_{n\to\infty} \|x - x^{[n]}\|_X = 0$,
AB-property when $\sup_{n\in\mathbb{N}} \|x^{[n]}\|_X < \infty$,
AD-property when $x \in \overline{\phi}$ (closure of $\phi \subset X$),
SAK-property when $\lim_{n\to\infty} |f(x) - f(x^{[n]})| = 0$ for all $f \in X'$,
FAK-property when $(f(x^{[n]}) \in c$ for all $f \in X'$.

If one of these properties holds for every $x \in X$, then we say that the space $X$ has that property. It is trivial that AK implies AB and AD. For example, the spaces $c_0$, $cs$ and $\ell_p$ are AK and $c$, $bs$ and $\ell_\infty$ are AB but not AD-spaces, where $1 \leq p < \infty$.

The distinguished subsets of summability domains and arbitrary FK spaces have been studied by Wilansky [2], Bennett [23], and several others [24]-[38].

We denote by $\mathfrak{U}$ the set of all real sequences $u=(u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$. For a sequence $u=(u_k) \in \mathfrak{U}$, the Rhaly (or Terraced) matrix $R_u = (r_{nk}(u))$ is defined by

$$r_{nk}(u) = \begin{cases} u_n & , \quad k \leq n \\ 0 & , \quad k > n \end{cases}$$

for all $n,k \in \mathbb{N}$.

For the special case $u_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$, the Rhaly matrix $R_u$ reduces to the Cesàro matrix of order 1. For more details on this topic, see [39, 40].

In this paper, we introduce the new sequence spaces $c_0(R_u), c(R_u)$ and $\ell_\infty(R_u)$, which are the domains of the Rhaly matrix $R_u$ in the spaces $c_0, c$, and $\ell_\infty$, respectively, and study some of their properties. We also define sectional subspaces related to the Rhaly matrix in an FK space and investigate their relationships, identities and duals.

## 2. The Sequence Spaces $c_0(R_u)$, $c(R_u)$ and $\ell_\infty(R_u)$

In the present section, we introduce the sequence spaces $c_0(R_u)$, $c(R_u))$ and $\ell_\infty(R_u)$ as the domain of the matrix $R_u$ in the classical sequence spaces $c_0$, $c$ and $\ell_\infty$, respectively and examine some properties of these spaces.

Throughout the study, $y=(y_n)$ will be the $R_u$-transform of a sequence $x=(x_k)$; that is,

$$y_n = (R_u x)_n = u_n \sum_{k=0}^{n} x_k \tag{2.1}$$

for all $n \in \mathbb{N}$. Since the matrix $R_u$ is a triangle, it has an inverse. Multiplying the equality (2.1) with $1/u_n$, we have

$$\frac{1}{u_n} y_n = \sum_{k=0}^{n} x_k \tag{2.2}$$

for all $n \in \mathbb{N}$. Therefore, by using the relation (2.2) we see that

$$x_n = \frac{1}{u_n} y_n - \frac{1}{u_{n-1}} y_{n-1} \tag{2.3}$$

holds for all $n \in \mathbb{N}$, where $y_{-1} = 0$.

Now, by the equation (2.3) we have the following lemma:

**Lemma 2.1.** *The matrix $R_u$ is invertible and its inverse $(R_u)^{-1} = (r_{nk}^{-1}(u))$ defined for all $k,n \in \mathbb{N}$ by*

$$r_{nk}^{-1}(u) = \begin{cases} (-1)^{n-k} \dfrac{1}{u_n} & , \quad n-1 \leq k \leq n, \\ 0 & , \quad 0 \leq k < n-1 \ \ or \ \ k > n. \end{cases}$$

Let us introduce the sequence spaces $c_0(R_u)$, $c(R_u)$ and $\ell_\infty(R_u)$ as the set of all sequences whose $R_u$-transforms are in the classical spaces $c_0$, $c$ and $\ell_\infty$, respectively; that is

$$\begin{aligned} c_0(R_u) &:= \left\{ x=(x_k) \in w : \lim_{n\to\infty} u_n \sum_{k=0}^{n} x_k = 0 \right\}, \\ c(R_u) &:= \left\{ x=(x_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty} u_n \sum_{k=0}^{n} x_k = \alpha \right\}, \\ \ell_\infty(R_u) &:= \left\{ x=(x_k) \in w : \sup_{n\in\mathbb{N}} \left| u_n \sum_{k=0}^{n} x_k \right| < \infty \right\}. \end{aligned}$$

With the notation of (1.2), the spaces $c_0(R_u)$, $c(R_u)$ and $\ell_\infty(R_u)$ can be redefined, as follows:

$$c_0(R_u) = (c_0)_{R_u}, \quad c(R_u) = (c)_{R_u} \quad \text{and} \quad \ell_\infty(R_u) = (\ell_\infty)_{R_u}.$$

It is known from [1]-[3] and [5, 16] that if $T$ is a triangle, then the domain $X_T$ of $T$ in a normed sequence space $X$ is normed with $\|x\|_{X_T} = \|Tx\|_X$, and is linearly norm isomorphic to $X$ and $X_T$ has a basis if and only if $X$ has a basis.

As a direct consequence of these facts, we have:

**Corollary 2.2.** *Let $Z \in \{c_0, c, \ell_\infty\}$. Then, the following statements hold:*

*(a) The space $Z(R_u)$ is a BK-space endowed with the norm*

$$\|x\|_{Z(R_u)} = \sup_{n\in\mathbb{N}} \left| u_n \sum_{k=0}^{n} x_k \right|.$$

*(b) The spaces $Z(R_u)$ is linearly norm isomorphic to the space $Z$.*

**Corollary 2.3.** *Define the sequence $b^{(k)}(u) = \left(b_n^{(k)}(u)\right)_{n\in\mathbb{N}}$ by*

$$b_n^{(k)}(u) := \begin{cases} (-1)^{n-k}\dfrac{1}{u_n} & , \quad k \le n \le k+1 \\ 0 & , \quad n < k \text{ or } n > k+1 \end{cases}$$

*for all $k, n \in \mathbb{N}$. Then, the following statements hold:*

*(a) The sequence $b^{(k)}(u)$ is a basis for the spaces $c_0(R_u)$ and every sequence $x \in c_0(R_u)$ has a unique representation of the form $x = \sum_{k=0}^{\infty}(R_u x)_k b^{(k)}(u)$.*

*(b) The set $\left\{\tilde{e}, b^{(k)}(u)\right\}$ is a basis for the space $c(R_u)$ and every sequence $x \in c(R_u)$ has a unique representation of the form $x = l\tilde{e} + \sum_{k=0}^{\infty}\left[(R_u x)_k - l\right] b^{(k)}(u)$, where $\tilde{e} = \left(\dfrac{1}{u_k} - \dfrac{1}{u_{k-1}}\right)$ for all $k \in \mathbb{N}$ and $(R_u x)_k \to l$, as $k \to \infty$.*

*(c) The space $\ell_\infty(R_u)$ does not have a basis.*

Since the inclusions $c_0 \subset c \subset \ell_\infty$ hold strictly, we have:

**Theorem 2.4.** *The inclusions $c_0(R_u) \subset c(R_u) \subset \ell_\infty(R_u)$ hold strictly.*

**Lemma 2.5.** *Let $X$ and $Y$ be sequence spaces, and let $A$ and $B$ be triangle matrices. Then, the inclusion $X_A \subset Y_B$ holds if and only if the matrix $BA^{-1}$ belongs to $(X,Y)$.*

*Proof.* Suppose that $X_A \subset Y_B$. Then, every $t \in X_A$ is in $Y_B$. By the definitions $Y_B$ and $X_A$, we have $Bt \in Y$ and $x = At \in X$. Since $A$ is a triangle matrix, it is invertible. From the equality $x = At$, we can obtain $t = A^{-1}x$. Hence, for each $x \in X$ the sequence $BA^{-1}x$ is in $Y$. This shows that $BA^{-1} \in (X,Y)$.

Conversely,suppose that $BA^{-1} \in (X,Y)$. Take any sequence $t \in X_A$. By the definition of $X_A$, we have $At \in X$. Since $BA^{-1} \in (X,Y)$, for $At \in X$, we have $BA^{-1}(At) \in Y$, and thus $Bt \in Y$. Therefore, $t \in Y_B$. This shows that the inclusion $X_A \subset Y_B$ holds. □

By using matrix transformations and Lemma 2.5, we can easily prove that:

**Theorem 2.6.** *The following assertions hold.*
*(a) If $(ku_k) \in \ell_\infty$ then $c_0 \subset c_0(R_u)$ and $\ell_\infty \subset \ell_\infty(R_u)$ strictly holds.*
*(b) If $(ku_k) \in c$ then $c \subset c(R_u)$ strictly holds.*
*(c) If $(ku_k) \in c_0$ then $\ell_\infty \subset c_0(R_u)$ holds.*
*(d) If $(\frac{1}{u_k} - \frac{1}{u_{k-1}}) \in \ell_\infty$ then the inclusion $c_0(R_u) \subset c_0$ and $\ell_\infty(R_u) \subset \ell_\infty$ hold.*
*(e) If $(\frac{1}{u_k} - \frac{1}{u_{k-1}}) \in c$ then the inclusion $c(R_u) \subset c$ holds.*

We shall begin with quoting the lemma due to Stieglitz and Tietz [41] which is needed in proving Theorem 2.8.

**Lemma 2.7.** *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold:*
*(a) $A \in (c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{2.4}$$

*(b) $A \in (c_0 : c)$ if and only if (2.4) and*

$$\lim_{n\to\infty} a_{nk} = \alpha_k (k \in \mathbb{N}), \tag{2.5}$$

*(c) $A \in (c : c)$ if and only if (2.4), (2.5) and*

$$\lim_{n\to\infty}\sum_k a_{nk} = \alpha.$$

*(d) $A \in (\ell_\infty : c)$ if and only if (2.4), and*

$$\lim_{n\to\infty}\sum_k |a_{nk}| = \sum_k |\lim_{n\to\infty} a_{nk}|$$

By [3, Theorem 3.1], we have:

**Theorem 2.8.** *For a sequence $u = (u_k) \in \mathfrak{U}$, let us define the sets $A_1(u)$, $A_2(u)$ and $A_3(u)$, as follows:*

$$\begin{aligned}
A_1(u) &:= \left\{a = (a_k) \in w : \left(\frac{a_k - a_{k+1}}{u_k}\right) \in \ell_1\right\},\\
A_2(u) &:= \left\{a = (a_k) \in w : \left(\frac{a_k}{u_k}\right) \in \ell_\infty\right\},\\
A_3(u) &:= \left\{a = (a_k) \in w : \left(\frac{a_k}{u_k}\right) \in c\right\},\\
A_4(u) &:= \left\{a = (a_k) \in w : \left(\frac{a_k}{u_k}\right) \in c_0\right\}.
\end{aligned}$$

*Then, the following statements hold:*

*(i)* $[c_0(R_u)]^\beta = A_1 \cap A_2$, $[c(R_u)]^\beta = A_1 \cap A_3$, $[\ell_\infty(R_u)]^\beta = A_1 \cap A_4$.

*(ii)* $[c_0(R_u)]^\gamma = [c(R_u)]^\gamma = [\ell_\infty(R_u)]^\gamma = A_1 \cap A_2$.

*Proof.* For $a = (a_n) \in w$ and $x = (x_n) \in X(R_u)$, we obtain

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k \left[\sum_{j=k-1}^{k} \frac{(-1)^{k-j}}{u_k} y_j\right]\\
&= \sum_{k=0}^{n-1}\left(\frac{a_k - a_{k+1}}{u_k}\right) y_k + \frac{a_n}{u_n} y_n \qquad (2.6)\\
&= (D_u y)_n
\end{aligned}$$

for all $n \in \mathbb{N}$, where $D_u = (d_{nk}(u))$ is defined by

$$d_{nk}(u) = \begin{cases} \dfrac{a_k - a_{k+1}}{u_k} & , \quad k < n,\\ \dfrac{a_n}{u_n} & , \quad k = n,\\ 0 & , \quad k > n \end{cases}$$

The equation (2.6) implies that $ax = (a_n x_n) \in cs$ whenever $x \in X(R_u)$ if and only if $D_u y \in c$ whenever $y \in X$. Therefore, we conclude that $a \in [X(R_u)]^\beta$ if and only if $D_u \in (X : c)$.

(i) To show that $[c_0(R_u)]^\beta = A_1 \cap A_2$, let us take $X = c_0$. It follows that $D_u \in (c_0 : c)$, which means the conditions (2.4) and (2.5) of Lemma 2.7 (b) are satisfied by the matrix $D_u$. Thus, $a = (a_k) \in A_1 \cap A_2$. Therefore, we have:

$$[c_0(A^{ru})]^\beta = A_1 \cap A_2.$$

By using the conditions of Lemma 2.7(c) and (d), the equalities $[c(R_u)]^\beta = A_1 \cap A_3$, $[\ell_\infty(R_u)]^\beta = A_1 \cap A_4$ can be proved similarly.

(ii) This is similar to the proof of Part (i) of the present theorem by using Lemma 2.7(a). To avoid the repetition of the similar statements, we omit the details.

□

## 3. Some Rhaly Subspaces of FK spaces

In this section, using sectional properties we define some new subspaces of a BK-space and give some relations between these spaces and duals.

Given a BK-space $X \supset \phi$, we define the $n^{th}$ Rhaly section of a sequence $x \in X$ as $r_x^{[n]} = u_n \sum_{k=0}^{n} x_k e^k$.

**Definition 3.1.** *Let X be a BK space contaning $\phi$. Then, a sequence $x = (x_k) \in X$ has the following properties:*

$$\begin{aligned} rK \textit{ when } & \lim_{n\to\infty} \|x - r_x^{[n]}\|_X = 0, \\ rB \textit{ when } & \sup_{n\in\mathbb{N}} \|r_x^{[n]}\|_X < \infty, \\ SrK \textit{ when } & \lim_{n\to\infty} |f(x) - f(r_x^{[n]})| = 0 \textit{ for all } f \in X', \\ FrK \textit{ when } & (f(r_x^{[n]})) \in c \textit{ for all } f \in X'. \end{aligned}$$

In connection to Definition 3.1, we can define the following distinguished subset of X;

$$\begin{aligned} X_{RS} &= \{x \in X : x \text{ has rK in X}\}, \\ &= \{x \in X : x = \lim_n u_n \sum_{k=1}^{n} \sum_{i=1}^{k} x_i e^i\} \\ X_{RW} &= \{x \in X : x \text{ has SrK in X}\}, \\ &= \{x \in X : \forall f \in X', f(x) = \lim_n u_n \sum_{k=1}^{n} \sum_{i=1}^{k} x_i f(e^i)\} \\ X_{RF^+} &= \{x \in w : x \text{ has FrK in X}\}, \\ &= \{x \in w : \left(u_n \sum_{k=1}^{n} x^{(k)}\right) \text{ is weakly Cauchy in } X\} \\ &= \{x \in X : \forall f \in X', (u_n f(e^n)) \in (c(R_u))_S\}, \\ X_{RB^+} &= \{x \in w : x \text{ has rB in X}\}, \\ &= \{x \in w : \left(u_n \sum_{k=1}^{n} x^{(k)}\right) \text{ is bounded in } X\} \\ &= \{x \in X : \forall f \in X', (u_n f(e^n)) \in (\ell_\infty(R_u))_S\}, \end{aligned}$$

and

$$X_{RF} = X_{RF^+} \cap X \text{ and } X_{RB} = X_{RB^+} \cap X,$$

where the matrix $S = (s_{nk})$ is defined as

$$s_{nk} = \begin{cases} 1 & , \quad k \le n, \\ 0 & , \quad k > n \end{cases}.$$

By definitions of $X_{RS}$, $X_{RW}$, $X_{RF^+}$ and $X_{RB^+}$ we have:

**Theorem 3.2.** *Let X be an* FK*-space containing $\phi$. Then the following inclusions hold.*

$$\phi \subset X_{RS} \subset X_{RW} \subset X_{RF} \subset X_{RB} \subset X$$

**Theorem 3.3.** *Let X be an* FK*-space containing $\phi$. Then $X_{RW} \subset \overline{\phi}$.*

*Proof.* Let $f \in X'$ with $\phi \subset Kernf$. Since for every $x \in X$ and $n \in \mathbb{N}$, $z^n = \left(a_n \sum_{k=1}^{n} x^{(k)}\right) \in \phi$, then $f(z^n) = 0$. This shows that $X_{RW} \subset Kernf$. By Corollary 1.1 (ii), we obtain the inclusion $X_{RW} \subset \overline{\phi}$. □

By definition of $X_{RF^+}$ $(X_{RB^+})$, $z \in X_{RF^+}(X_{RB^+})$ if and only if $z \cdot y \in (c(R_u))_S((\ell_\infty(R_u))_S)$ for each $y \in X^f$, we have the following theorems.

**Theorem 3.4.** *Let X be an* FK*-space containing $\phi$. Then $X_{RF} = (X^f)^{(c(R_u))_S}$.*

**Theorem 3.5.** *Let X be an* FK*-space containing $\phi$. Then $X_{RB} = (X^f)^{(\ell_\infty(R_u))_S}$*

In the study of FK-spaces, understanding the relationships between different sequence spaces and their properties is crucial. In this context, we investigate the inclusions and equalities among subspaces defined by various properties of sequences. The following results explore these relationships, focusing on the inclusion properties of different sequence spaces associated with the properties rK, SrK, FrK and rB when the inclusion $X \subset Y$ holds for FK-spaces $X$ and $Y$. These results shed light on the structure of these spaces and the behavior of the sequence spaces under certain conditions.

**Theorem 3.6.** *If $X \subset Y$ then $X_\lambda \subset Y_\lambda$ for $\lambda \in \{RS, RW, RB^+, RF^+, RB, RF\}$.*

*Proof.* For $\lambda = RS(RW)$, the continuity(weak continuity) of inclusion map $i : X \to Y$ gives the desired result.
Let $\lambda \in \{RB^+, RF^+\}$. The results follows from Theorem 3.4, 3.5 and Lemma 1.2(ii). □

**Theorem 3.7.** *If $\overline{\phi} \subset Y \subset X$, then $Y_{RB^+} = X_{RB^+}$ and $Y_{RF^+} = X_{RF^+}$.*

*Proof.* By Theorem 3.6 we have

$$\overline{\phi}_{RB^+} \subset Y_{RB^+} \subset X_{RB^+}.$$

By Theorem 3.5 and Lemma 1.2(vi) the first and the last are equal. □

**Theorem 3.8.** *Let $X$ be an* FK-*space containing $\phi$ and $X \subset X_{RB}$. Then $X_{RS} = X_{RW} = \overline{\phi}$.*

*Proof.* Since the sequence of functions $(f_n)$ defined by $f_n : X \to X$, $f_n(x) = x - u_n \sum_{k=1}^{n} x^{(k)}$ is pointwise bounded, hence equicontinuous by (7.0.2) of [2]. Since $f_n \to 0$ on $\phi$ then also $f_n \to 0$ on $\overline{\phi}$ by (7.0.3) of [2]. This is the desired conclusion. □

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# Solitons on Nearly Cosymplectic Manifold Exhibitting Schouten Van Kampen Connection

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**Abstract**
The following research investigates various types of soliton of NC (Nearly Cosymplectic) manifolds with SVK (Schouten-van Kampen) connections, which are steady, shrinking, or expanding. Further, we investigate the geometric characteristics of Ricci solitons, Yamabe solitons, $\eta$-ricci soliton etc. We also study the curvature features of the SVK connection on an NC manifold. In addition, an example is developed to demonstrate the results.

**Keywords:** Einstein manifold, Nearly cosymplectic manifold, Quasi Einstein manifold, Ricci soliton, Schouten van Kampen connection, Yamabe soliton
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## 1. Introduction

The Schouten Van Kampen connection is one of the most significant connections having the property of an affine connection [1]-[4]. The contact metric manifolds and solitons on these manifolds with respect to Schouten van Kampen connection are studied by several authors [3]-[7]. Nearly cosymplectic structures were introduced by Blair [8] and first appeared essentially as the hypersurface of nearly Kahler manifolds. A nearly cosymplectic manifold is defined as an almost contact metric manifold with a normality condition having closed 1-form $\eta$ and 2-form F [9]. Various geometrical properties of a Nearly cosymplectic manifold was investigated by Endo [2].

Hamilton introduced the idea of the Ricci flow to find out the canonical metric over a smooth manifold [10, 11]. By the introduction of Ricci flow it is easy to study manifolds with positive curvature. Perelman proved the Poincare conjecture using Ricci flow [12, 13]. The term Ricci soliton refers to the limit of the solutions of the Ricci flow. In general, an almost ricci soliton is a simplification of an Einstien metric. For a complete vector field Y on a Riemannian manifold M of dimension n, a Riemannian metric g on M is termed a nearly Ricci soliton if it satisfies

$$L_Y g + 2S + 2\alpha g = 0, \tag{1.1}$$

where $\alpha$ is a smooth function, S stands for the Ricci tensor, and L is the Lie derivative. A metric g that satisfies (1.1) is referred to as a Ricci soliton if $\alpha$ is a constant. If $\alpha > 0$, $\alpha = 0$, or $\alpha < 0$, then a Ricci soliton is expanding, steady, or shrinking, respectively. The concept of $\eta$-Ricci soliton was introduced by Cho and Kimura [9]. An almost $\eta$-Ricci soliton is a Riemannian manifold M with Riemannian metric g if for a smooth vector field Y such that,

$$L_Y g + 2S + 2\alpha g + 2\beta \eta \otimes \eta = 0, \tag{1.2}$$

for both the smooth functions, $\alpha$ and $\beta$.

If both $\alpha$ and $\beta$ are constant, the metric g is referred as a $\eta$-Ricci soliton. A Ricci soliton has constant curvature for a compact manifold of dimension two or three [2, 14].

Hamilton [11] proposed the idea of Yamabe flow. A vector field Y that is static on a Riemannian manifold M generates the Yamabe solitons, which are self-similar outcomes of the Yamabe flow and are transformed by a family of diffeomorphisms with one parameter. On a Riemannian manifold (M, g), a triplet (g, Y, $\gamma$) is said to be a nearly Yamabe soliton if [10]

$$\frac{1}{2} L_Y g = (r - \gamma) \tag{1.3}$$

where $\gamma$ is a smooth function and r is the scalar curvature of manifold (M,g). When $\gamma$ remains constant, the almost Yamabe soliton transforms into a Yamabe soliton. If $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$, respectively, then a Yamabe soliton is expanding, steady, or shrinking. Yamabe and Ricci soliton coincide for the manifold of dimension 2 but they have distinct behaviors for the manifolds of dimensions greater than 2. Furthermore, Nearly Yamabe solitons always represent Einstein manifolds. And, the Riemannian metric g becomes a Yamabe metric if the Riemannian manifold M has constant scalar curvature [4].

In this study, we investigate several forms of Ricci and Yamabe solitons over NC manifold of dimension n with an SVK connection. Section 2 gives a brief description of the NC manifold and SVK connection. Section 3 introduces the SVK connection on the NC manifold and establishes the formulas for curvature tensor, Ricci tensor, Ricci operator, and scalar curvature. In Section 4, we investigate the Ricci solitons for an NC manifold with an SVK connection. The final section investigates Yamabe solitons on an n-dimensional NC manifold with SVK connection.

## 2. Preliminaries

Consider an (2n+1) dimensional almost contact manifold with structure (M, $\phi$, $\xi$, $\eta$, g), where $\xi$ is the vector field, $\eta$ is a 1-form, g is the Riemannian Metric, and $\phi$ is a (1,1) tensor field. The following prerequisites are satisfied by this ($\phi$, $\xi$, $\eta$, g) structure [9].

$$\phi\xi = 0, \eta(\phi X) = 0, \eta(\xi) = 1,$$

$$\phi^2 X = -X + \eta(X)\xi, \eta(X) = g(X,\xi). \tag{2.1}$$

Let g be compatible i.e.

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

As $\phi$ is a skew-symmetric operator with g, as per the definition above, $\eta$ is a contact form, i.e., $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, and the bilinear form F = g(X, $\phi$Y) defines a 2-form [15].

An almost contact metric manifold with (M, $\phi$, $\xi$, $\eta$, g) is said to be a Nearly Cosymplectic manifold if

$$(\bigtriangledown_X \phi)Y + (\bigtriangledown_Y \phi)X = 0,$$

for each vector field X, Y. It is clear that this condition is the same as $(\bigtriangledown_X \phi)X = 0$.

The Reeb vector field $\xi$ is defined for nearly cosymplectic manifolds is killing if it fulfills the requirements $\bigtriangledown_\xi \xi = 0$ and $\bigtriangledown_\xi \eta = 0$.

Moreover, the type (1,1) tensor field H defined by [9]

$$\bigtriangledown_X \xi = HX \tag{2.3}$$

is anti-commutative with $\phi$ and skew-symmetric. Additionally, H providing

$$H\xi = 0, \eta(HX) = 0,$$

$$TraceH = 0,$$

$$\phi H = -H\phi, g(HX,Y) = g(X,HY).$$

These formulae also hold [15]-[18]

$$g((\nabla_{X\phi})Y,HZ) = \eta(Y)g(H^2X,\phi Z) - \eta(X)g(H^2Y,\phi Z),$$

$$(\nabla_X H)Y = g(H^2X,Y)\xi - \eta(Y)H^2X,$$

$$TraceH^2 = a(constant), \tag{2.4}$$

$$R(Y,Z)\xi = \eta(Y)H^2Z - \eta(Z)H^2Y,$$

$$S(X,Y) = -\lambda g(X,Y),$$

$$QX = -\lambda X,$$

$$S(X,\xi) = \lambda\eta(X), \tag{2.5}$$

where $\lambda$: M(manifold) $\rightarrow$ R(real number) is a function.

$$S(\phi Y,Z) = S(Y,\phi Z),$$

$$\phi Q = Q\phi,$$

$$S(\phi Y,\phi Z) = S(Y,Z) + \eta(Y)\eta(Z)(TraceH^2).$$

A contact metric structure ($\eta$, g) on M is $\eta$-Einstein if

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where $a$ and $b$ are constants. If $b = 0$, then the manifold M is an Einstein manifold [7].

A quasi-Einstein manifold is defined as one whose Ricci tensor S of type (0,2) is not identically zero and meets the condition [19]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.6}$$

for all vector fields $X,Y$, where $a,b$ are scalars, $b \neq 0$, and $\eta$ is a non-zero 1-form.

In the tangent bundle TM of M, there are two naturally determined distributions, U = Ker $\eta$ and V = Span $\xi$, such that TM = U$\oplus$V, U $\cap$ V= 0 and U$\perp$V. For this decomposition the SVK connection can be defined over a nearly contact metric structure.

Concerning the Levi-Civita Connection $\nabla$, the Schouten Van Kampen Connection $\widetilde{\nabla}$ on a nearly contact metric manifold is defined by [20]

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X\xi + (\nabla_X\eta)(Y)\xi. \tag{2.7}$$

## 3. Curvature Properties of NC Manifold concerning SVK Connection $\widetilde{\nabla}$

Let $\widetilde{M}$ be an NC manifold, then using (2.1), (2.3) and (2.6), in (2.7) we have

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(Y)HX - g(Y,HX)\xi. \tag{3.1}$$

Moreover,

$$\widetilde{\nabla}_X\xi = 0. \tag{3.2}$$

If R and $\widetilde{R}$ are curvature tensors with respect to $\nabla$ and $\widetilde{\nabla}$, then

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{3.3}$$

and

$$\widetilde{R}(X,Y)Z = \widetilde{\bigtriangledown}_X \widetilde{\bigtriangledown}_Y Z - \widetilde{\bigtriangledown}_Y \widetilde{\bigtriangledown}_X Z - \widetilde{\bigtriangledown}_{[X,Y]} Z. \tag{3.4}$$

By using (2.1), (2.4), (3.1), (3.2), (3.3) in (3.4), we have

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + g(Z,HX)HY - g(Z,HY)HX - X\eta(Z) + \eta(X)\eta(Z)\xi \\ &- \eta(Y)g(Z,X)\xi + Y\eta(Z) - \eta(Y)\eta(Z)\xi + \eta(X)g(Z,Y)\xi.\end{aligned}$$

Using the above equation, the Ricci tensor of NC Manifold with SVK connection can be obtained as

$$\widetilde{S}(Y,Z) = S(Y,Z) - (n-1)\eta(Z). \tag{3.5}$$

The Ricci operator $\widetilde{Q}$ for NC Manifolds with respect to the connection $\widetilde{\bigtriangledown}$ is given by

$$\widetilde{S}(Y,Z) = g(\widetilde{Q}Y,Z). \tag{3.6}$$

From equations (3.5) and (3.6), we have

$$\widetilde{Q}Y = QY - (n-1)\xi.$$

The scalar curvature with respect to the connection $\widetilde{\bigtriangledown}$ is given by

$$\widetilde{r} = r - (n-1). \tag{3.7}$$

## 4. Ricci Soliton Types on an n-dimensional NC Manifold with SVK Connection

In this section, we explore types of Ricci soliton kinds on NC manifold M having SVK connection $\widetilde{\bigtriangledown}$.

A SVK connection $\widetilde{\bigtriangledown}$ in an NC manifold $\widetilde{M}$, is said to be metric if $(\widetilde{\bigtriangledown} g) = 0$ torsion tensor $\widetilde{T} \neq 0$, where $\widetilde{T}$ is torsion tensor with respect to $\widetilde{\bigtriangledown}$.

With the help of equation (3.1), we can easily find the value

$$(\widetilde{L}_Y g)(X,Z) = g(\bigtriangledown_X Y,Z) + g(X,\bigtriangledown_Z Y) = (L_Y g)(X,Z), \tag{4.1}$$

where $L$ and $\widetilde{L}$ are Lie derivatives on NC manifold with respect to Levi-Civita connection $\bigtriangledown$ and SVK connection $\widetilde{\bigtriangledown}$ respectively.

Now, for an n-dimensional NC manifold $\widetilde{M}$ with an SVK connection $\widetilde{\bigtriangledown}$, the almost Ricci soliton is given by

$$\widetilde{L}_Y g + 2\widetilde{S} + 2\alpha g = 0. \tag{4.2}$$

Using (4.1) and (4.2), we get

$$g(\bigtriangledown_X Y,Z) + g(X,\bigtriangledown_Z Y) + 2\widetilde{S}(X,Y) + 2\alpha g(X,Y) = 0.$$

Therefore,

$$2\widetilde{S}(X,Y) = -g(\bigtriangledown_X Y,Z) - g(X,\bigtriangledown_Z Y) - 2\alpha g(X,Y).$$

By substituting Y = $\xi$ in previous equation and using (2.2) and (2.3), we have

$$\widetilde{S}(X,Z) = -\alpha g(X,Z). \tag{4.3}$$

In view of (3.5) we can write above equation as

$$S(X,Z) = -(n-1)\eta(X)\eta(Z) - g(Z,\phi X)trace\phi - \alpha g(X,Z).$$

Conversely, consider that an $n$-dimensional NC manifold $\widetilde{M}$ with respect to SVK connection $\widetilde{\bigtriangledown}$ is an Einstein manifold. For Y = $\xi$, we have

$$\widetilde{S}(X,Z) = -\lambda g(X,Z)$$

and

$$(\widetilde{L_\xi g})(X,Z)=0,$$

where $\lambda$ is a constant.

Using the above results, we can easily find the value of

$$(\widetilde{L}_\xi g)(X,Z)+2\widetilde{S}(X,Z)+2\alpha g(X,Z)=2(\alpha-\lambda)g(X,Z). \tag{4.4}$$

Therefore, it is evident from (4.4) that if $\alpha-\lambda=0$, then the manifold $\widetilde{M}$ admits a Ricci soliton. Thus, we make the following statement:

**Theorem 4.1.** *An n-dimensional NC manifold $\widetilde{M}$ admits a Ricci soliton with respect to SVK connection $\widetilde{\bigtriangledown}$ iff $\widetilde{M}$ is an Einstein manifold with respect to SVK connection $\widetilde{\bigtriangledown}$.*

**Corollary 4.2.** *A Ricci Soliton on NC manifold with SVK connection is Einstien Manifold.*

*Proof.* Put $X=\xi$ in (4.2), we have

$$(\widetilde{L}_\xi g)(X,Z)+2\widetilde{S}(X,Z)+2\alpha g(X,Z)=0.$$

Using (4.1),

$$(\widetilde{L}_\xi g)(X,Z)=0.$$

Thus

$$\widetilde{S}(X,Z)=-\alpha g(X,Z). \tag{4.5}$$

Using (4.5) and (3.6), we have

$$S(X,Z)=-(n-1)\eta(X)-\alpha g(X,Z).$$

Hence the theorem. □

**Theorem 4.3.** *The scalar curvature for an n-dimensional NC manifold $\widetilde{M}$ with SVK connection having an almost Ricci soliton, is $\widetilde{r}$ = -$\alpha n$.*

*Proof.* In view of (4.3), we have

$$\widetilde{r}=-\alpha n.$$

Hence the theorem. □

**Theorem 4.4.** *An n-dimensional NC manifold $\widetilde{M}$ with SVK connection will be an Einstien manifold if $\widetilde{M}$ with SVK connection enabling a $\eta$-Ricci soliton.*

*Proof.* Currently, based on (1.2), the $\eta$-Ricci soliton on an NCM of dimension n with SVK connection is given by

$$(\widetilde{L}_Y g)(X,Z)+2\widetilde{S}(X,Z)+2\alpha g(X,Z)+2\beta(\eta\otimes\eta)(X,Z)=0. \tag{4.6}$$

From (4.1) and (4.6), we have

$$g(\bigtriangledown_X Y,Z)+g(X,\bigtriangledown_Z Y)+2\widetilde{S}(X,Z)+2\alpha g(X,Z)+2\beta\eta(X)\eta(Z)=0. \tag{4.7}$$

Putting the values from (2.2) and (2.3) in (4.7), we get

$$\widetilde{S}(X,Z)=-\alpha g(X,Z)-\beta\eta(X)\eta(Z). \tag{4.8}$$

Hence the theorem. □

**Corollary 4.5.** *If an $\eta$-Ricci soliton on an NC manifold $\widetilde{M}$ with SVK connection is defined, then ($\widetilde{M}$, g) is Quasi Einstien.*

*Proof.* Using (1.2), (4.8) and (4.9), we can easily find the required result. □

**Theorem 4.6.** *A SVK connection on an NC manifold $\widetilde{M}$ admitting Ricci soliton is invariant iff it satisfies*

$$g(Y,HX)\eta(Z)+g(Y,HZ)\eta(X)+2(n-1)\eta(Z)=0.$$

*Proof.* Using (3.1), (4.1), (4.2), we get

$$(\widetilde{L}_Y g)(X,Z)=(L_Y g)(X,Y)-g(Y,HX)\eta(Z)-g(Y,HZ)\eta(Y). \tag{4.9}$$

By putting the values from (3.6) and (4.9) in (4.2), we obtain

$$g(Y,HX)\eta(Z)+g(Y,HZ)\eta(Y)+2(n-1)\eta(Z)=0, \tag{4.10}$$

hence the theorem. □

**Theorem 4.7.** *A Ricci soliton on NC maniold $\widetilde{M}$ with SVK connection is steady if $\lambda$ = (n-1), shrinking if $\lambda$ < (n-1) and expanding if $\lambda$ > (n-1).*

*Proof.* Using (3.5), (2.5) and (4.3), we have the required result. □

## 5. Yamabe Soliton on NC Manifold with SVK Connection

The almost Yamabe soliton on an n-dimensional NC manifold $\widetilde{M}$ with an SVK connection is studied within this segment.

We now examine an n-dimensional NC manifold that allows the SVK connection to deal with an almost Yamabe soliton, as described in (1.3). Hence, we have

$$\frac{1}{2}(\widetilde{L}_Y g)(X,Z)=(\widetilde{r}-\gamma)g(X,Z). \tag{5.1}$$

Using (3.7), (4.1) and (5.1), we have

$$\frac{1}{2}(L_Y g)(X,Z)=(r-n+1-\gamma)g(X,Z). \tag{5.2}$$

The subsequent theorem may thus be stated from (5.2).

**Theorem 5.1.** *If n = 1, then an almost Yamabe soliton (M,Y,$\gamma$,g) on an n-dimensional NC manifold $\widetilde{M}$ is invariant concerning SVK connection.*

According to (5.1) and (4.1), we have

$$\frac{1}{2}(g(\triangledown_X Y,Z)+g(X,\triangledown_Z Y))(X,Z)=(\widetilde{r}-\gamma)g(X,Z).$$

If we put $Y=\xi$ in the above equation, we obtain

$$\frac{1}{2}(g(\triangledown_X \xi,Z)+g(X,\triangledown_Z \xi))(X,Z)=(\widetilde{r}-\gamma)g(X,Z). \tag{5.3}$$

In view of (2.2) and (2.3), from (5.3), we have

$$\widetilde{r}=\gamma.$$

Thus, we may deduce the conclusion as mentioned below:

**Theorem 5.2.** *If an n-dimensional NC manifold $\widetilde{M}$ with an SVK connection, admits an almost Yamabe soliton then the scalar curvature $\widetilde{r}$ of $\widetilde{M}$ is equal to $\gamma$ iff Y and $\xi$ are pairwise collinear in TM.*

## 6. Example

Let us consider a 3-dimensional manifold $\widetilde{M} = \{(x,y,z) \in R^3\}$, where (x, y, z) represent the standard coordinates in $R^3$. Suppose

$$\tau_1 = e^{z^2}\frac{\partial}{\partial x}, \tau_2 = e^{z^2}\frac{\partial}{\partial y}, \tau_3 = \frac{\partial}{\partial z},$$

are linearly independent vector fields of $\widetilde{M}$. Then

$$[\tau_1, \tau_2] = 0,\ [\tau_2, \tau_3] = -2z\tau_2,\ [\tau_1, \tau_3] = -2z\tau_1.$$

If $g$ represent the Riemannian metric, then we have

$$g(\tau_1, \tau_1) = g(\tau_2, \tau_2) = g(\tau_3, \tau_3) = 1,$$

$$g(\tau_1, \tau_2) = g(\tau_2, \tau_3) = g(\tau_1, \tau_3) = 0.$$

Let $\eta$ be the 1-form defined by $\eta$(X) = g(X, $\tau_3$), $\forall$ X $\in \widetilde{M}$, and let $\phi$ be the (1,1) tensor field defined by

$$\phi(\tau_1) = \tau_2,\ \phi(\tau_2) = -\tau_1,\ \phi(\tau_3) = 0.$$

Using the above relations, following results holds:

$$\phi^2 X = -X + \eta(X)\xi,$$

$$\eta(\tau_3) = 1,$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

where $\xi = \tau_3$ and $X, Y$ is arbitrary vector field on $\widetilde{M}$. Hence $\widetilde{M}$ fulfills all the condition for an NC manifold.
Using the Koszul formula, we get

$$\nabla_{\tau_1}\tau_1 = 2z\tau_3,\ \nabla_{\tau_2}\tau_1 = 0,\ \nabla_{\tau_3}\tau_1 = 0,$$

$$\nabla_{\tau_1}\tau_2 = 0,\ \nabla_{\tau_2}\tau_2 = 2z\tau_3,\ \nabla_{\tau_3}\tau_2 = 0,$$

$$\nabla_{\tau_1}\tau_3 = -2z\tau_1,\ \nabla_{\tau_2}\tau_3 = -2z\tau_2,\ \nabla_{\tau_3}\tau_3 = 0,$$

and

$$\widetilde{\nabla}_{\tau_1}\tau_1 = 2z\tau_3,\ \widetilde{\nabla}_{\tau_2}\tau_1 = \tau_3,\ \widetilde{\nabla}_{\tau_3}\tau_1 = 0,$$

$$\widetilde{\nabla}_{\tau_1}\tau_2 = -\tau_3,\ \widetilde{\nabla}_{\tau_2}\tau_2 = 2z\tau_3,\ \widetilde{\nabla}_{\tau_3}\tau_2 = 0,$$

$$\widetilde{\nabla}_{\tau_1}\tau_3 = -2z\tau_1 - \tau_2,\ \widetilde{\nabla}_{\tau_2}\tau_3 = -2z\tau_2 + \tau_1,\ \widetilde{\nabla}_{\tau_3}\tau_3 = 0.$$

We can easily deduce the following identities using above results.

$$\widetilde{R}(\tau_1, \tau_2)\tau_3 = 8z\tau_3;$$

$$\widetilde{R}(\tau_2, \tau_3)\tau_3 = (2 - 4z^2)\tau_2 + 2z\tau_1;$$

$$\widetilde{R}(\tau_1, \tau_2)\tau_2 = -4z^2\tau_1 - 4z\tau_2 + \tau_1;$$

$$\widetilde{R}(\tau_1, \tau_2)\tau_1 = -4z\tau_1 - \tau_2 + 4z^2\tau_2;$$

$$\widetilde{R}(\tau_2, \tau_3)\tau_2 = (-2 + 4z^2)\tau_3;$$

$$\widetilde{R}(\tau_2, \tau_3)\tau_1 = 2z\tau_3;$$

$$\widetilde{R}(\tau_1, \tau_3)\tau_2 = -2z\tau_3;$$

$$\widetilde{R}(\tau_1, \tau_3)\tau_3 = (2 - 4z^2)\tau_1 - 2z\tau_2;$$

$\widetilde{R}(\tau_1,\tau_3)\tau_1 = (-2+4z^2)\tau_3;$

and

$\widetilde{S}(\tau_1,\tau_1) = \widetilde{S}(\tau_2,\tau_2) = 3-8z^2,\ \widetilde{S}(\tau_3,\tau_3) = 4-8z^2.$

Hence $\widetilde{r}$ = 10 - 24$z^2$. Let

$V = (x+y)e^{-z^2}z_1 + (-x+y)e^{-z^2}z_2$

and

$\sum_{i=1}^{3}(\widetilde{L}_V g)(\tau_i,\tau_i) = 4.$

Now, we put X = Y = $\tau_i$ in (4.2), summing over $i = 1,2,3$ and using above results, we get $\alpha$ = 8$z^2$ - 4, also using (4.10), we obtain
Case I: for $z^2$ = $\frac{1}{2}$,the Ricci soliton is steady.
Case II: for $z^2 \neq \frac{1}{2}$,the Ricci soliton is shrinking.

## 7. Conclusion

The study provides new insights beyond the usual Levi-Civita framework and highlights the versatility of the SVK connection as a tool for studying geometric structures with torsion. These contributions enhance the way for further research in theoretical physics and mathematics while also improving our understanding of solitons in NC manifolds.

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