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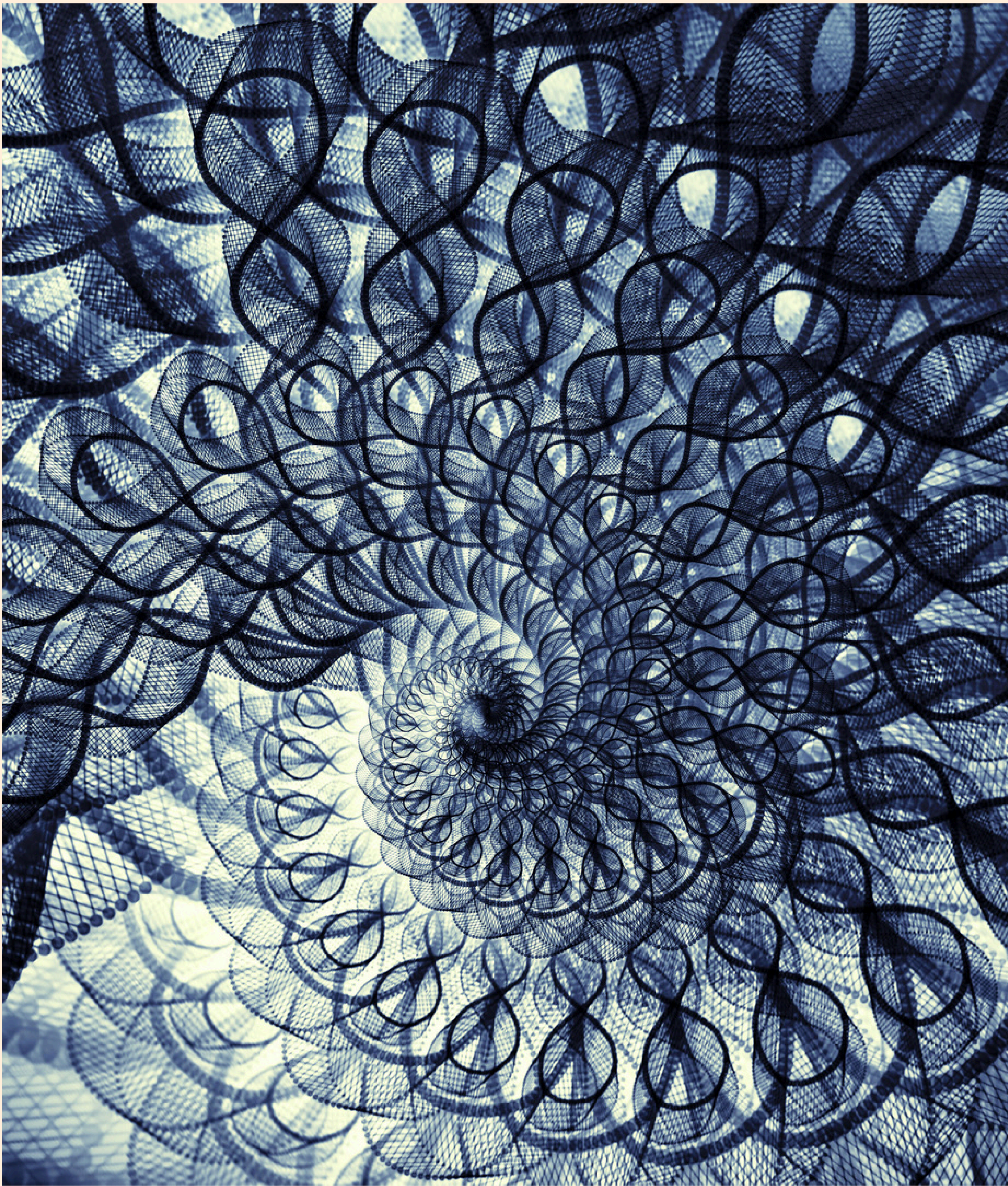

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On Absolute Tribonacci Series Spaces and Some Matrix Operators

Fadime Gökçe

Abstract
In this article, the absolute Tribonacci space $|T_\theta|_q$ is introduced as the domain of the Tribonacci matrix on ℓ_q. First, certain algebraic and topological structures such as $BK-$space, isomorphism, duals, and Schauder basis are studied. Then, some characterizations of compact and matrix operators on this space are given their norms, and Hausdorff measures of noncompactness are determined.

Keywords: *Absolute summability, Compact operator, Hausdorff measure of noncompactness, Matrix transformations, Tribonacci matrix*

AMS Subject Classification (2020): *40C05; 46B45; 40F05; 46A45*

1. Introduction

By ω, ℓ_∞, c, ℓ_q $(q > 1)$ and ℓ, we stand for the set of all sequences of complex numbers, the sequence space of all bounded, convergent sequences and also for the spaces of all q-absolutely convergent series and absolutely convergent series, respectively. Also, throughout the paper, the abbreviations HM and HMN will be used instead of "Hausdorff measure" and "Hausdorff measure of noncompactness" for brevity and $\mathbb{N} = \{0, 1, 2, 3, ...\}$. Let $\Lambda = (\lambda_{nv})$ be an arbitrary infinite matrix of complex components and U, V be two subspaces of ω. If the series

$$\Lambda_n(u) = \sum_{v=0}^{\infty} \lambda_{nv} u_v,$$

converges for all $n \in \mathbb{N}$, then, we define the Λ-transform of the sequence $u = (u_v)$ by $\Lambda(u) = (\Lambda_n(u))$. Also, it is said that Λ defines a matrix transformation from the space U into the space V, and denote it by $\Lambda \in (U, V)$ or $\Lambda : U \to V$ if $\Lambda u = (\Lambda_n(u)) \in V$ for every $u \in U$. On the other hand, the $\alpha-, \beta-, \gamma-$ duals of U are defined by

$$U^\alpha = \left\{ \epsilon \in \omega : \forall u \in U, \sum_n |\epsilon_n u_n| < \infty \right\},$$

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$$U^{\beta} = \left\{ \epsilon \in \omega : \forall u \in U, (\sum_{i=0}^{n} \epsilon_n u_n) \in c \right\},$$

$$U^{\gamma} = \left\{ \epsilon \in \omega : \forall u \in U, (\sum_{i=0}^{n} \epsilon_n u_n) \in \ell_\infty \right\}$$

respectively, and the domain of the matrix Λ in U is defined by

$$U_\Lambda = \{u = (u_n) \in \omega : \Lambda(u) \in U\}. \tag{1.1}$$

Further, if U is a complete normed space with continuous coordinates $r_m : U \to \mathbb{C}$ described by $r_m(u) = u_m$ for each $m \in \mathbb{N}$, then it is said that U is a BK-space. If there exists unique sequence of coefficients (u_k) such that, for each $u \in U$,

$$\left\| u - \sum_{k=0}^{p} u_k b_k \right\| \to 0, \ \ p \to \infty$$

then, the sequence (b_k) is called the Schauder basis for U, and it can be written $u = \sum_{k=0}^{\infty} u_k b_k$.

Assume that $1 \le q < \infty$ and $\theta = (\theta_n)$ is a sequence of positive terms and also take $\sum u_v$ as an infinite series with its nth partial sum s_n. Then, the series $\sum u_v$ is said to be summable $|\Lambda, \theta_n|_q$, $q \ge 1$, if

$$\sum_{n=0}^{\infty} \theta_n^{q-1} |\Delta \Lambda_n(s)|^q < \infty,$$

where $\Delta\Lambda_n(s) = \Lambda_n(s) - \Lambda_{n-1}(s), \ \ \Lambda_{-1}(s) = 0$ (see [1]).

It is clear that this method includes a good number of well known methods for special selections. We refer to reader [2–6]. Recently, the literature of summability theory has expanded in many respects, with many studies using both the summability methods and the absolute summability methods (see [7–19]).

On the other hand, Tribonacci numbers are the sequence of integers identified by the third order recurrence relation with initial conditions $t_0 = 1, t_1 = 1, t_2 = 2$,

$$t_j = t_{j-1} + t_{j-2} + t_{j-3}$$

$$t_{-j} = 0, j \ge 1$$

[20]. So, some of the first Tribonacci numbers can be written as follows:

$$1, 1, 2, 4, 7, 13, 24, 44, ...$$

Besides, Tribonacci numbers have the following useful properties:

$$\sum_{j=0}^{m} t_j = \frac{t_{m+2} + t_m - 1}{2}, m \ge 0,$$

$$\sum_{j=0}^{m} t_{2j} = \frac{t_{2m+1} + t_{2m} - 1}{2}, m \ge 0,$$

$$\lim_{m\to\infty} \frac{t_m}{t_{m+1}} = 0.54368901...$$

Tribonacci matrix $T = (t_{mj})$ has recently been defined by Yaying and Hazarika [19] as follows:

$$t_{mj} = \begin{cases} \frac{2t_j}{t_{m+2}+t_m-1}, & 0 \le j \le m \\ 0, & j > m \end{cases}$$

where t_m be the mth Tribonacci number for all $m \in \mathbb{N}$.

Throughout the whole paper, q^* is the conjugate of q, i.e., $1/q + 1/q^* = 1$ for $q > 1$, and $1/q^* = 0$ for $q = 1$.

Now, let remind certain lemmas which are used in the proof of our theorems.

Lemma 1.1. *[21] $\Lambda \in (\ell_q, \ell)$ iff*

$$\|\Lambda\|_{(\ell_q,\ell)} = \sup_{S\in\mathfrak{T}} \left\{ \sum_{k=0}^{\infty} \left| \sum_{n\in S}^{\infty} \lambda_{nk} \right|^{q^*} \right\}^{1/q^*}.$$

where $1 < q < \infty$ and $\mathfrak{T}$ is defined as the collection of all the finite subsets of $\mathbb{N}$.

While Lemma 1.1 introduces a condition that is very difficult to implement in applications, the following lemma, which gives the equivalent condition, will be more useful in a lot of cases.

Lemma 1.2. *[22] $\Lambda \in (\ell_q, \ell)$ iff*

$$\|\Lambda\|'_{(\ell_q,\ell)} = \left\{ \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} |\lambda_{nk}| \right)^{q^*} \right\}^{1/q^*} < \infty,$$

where $1 < q < \infty$. Moreover since

$$\|\Lambda\|_{(\ell_q,\ell)} \leq \|\Lambda\|'_{(\ell_q,\ell)} \leq 4\,\|\Lambda\|_{(\ell_q,\ell)},$$

there exists $1 \leq \eta \leq 4$ such that $\|\Lambda\|'_{(\ell_q,\ell)} = \eta\,\|\Lambda\|_{(\ell_q,\ell)}$.

Lemma 1.3. *[23] $\Lambda \in (\ell, \ell_q)$ iff*

$$\|\Lambda\|_{(\ell,\ell_q)} = \sup_{k} \left\{ \sum_{n=0}^{\infty} |\lambda_{nk}|^q \right\}^{\frac{1}{q}},$$

where $1 \leq q < \infty$.

Lemma 1.4. *[21]*

1. $\Lambda \in (\ell, c) \Leftrightarrow$ $\quad (i)\lim_{n} \lambda_{nk}$ *exists for* $k \geq 0$, $\quad (ii)\sup_{n,k} |\lambda_{nk}| < \infty$,
2. $\Lambda \in (\ell, \ell_\infty) \Leftrightarrow$ *(ii) holds,*
3. *If* $1 < q < \infty, \Lambda \in (\ell_q, c) \Leftrightarrow$ $\quad (i)$ *holds,* $\quad (iii)\sup_{n} \sum_{k=0}^{\infty} |\lambda_{nk}|^{q^*} < \infty$,
4. *If* $1 < q < \infty, \Lambda \in (\ell_q, \ell_\infty) \Leftrightarrow (iii)$ *holds.*

Let (U, d) be a metric space and $B, P \subset U$. For every $p \in P$, if there exists an $b \in B$ such that $d(p, b) < \varepsilon$ then, B is called an ε-net of P; if B is finite, then the ε-net B of P is called a finite ε-net of P. Assume that U and V are Banach spaces. If domain of a linear operator $\mathcal{S}$ is all of U and, for every bounded sequence (u_n) in U, the sequence $(\mathcal{S}(u_n))$ has a convergent subsequence in V, then the operator $\mathcal{S} : U \to V$ is called compact. $\mathcal{C}(U, V)$ determines the class of all such operators. Assume that Q defines a bounded subset of U. The HMN of Q is described by the number

$$\chi(Q) = inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon - \text{net in } U\},$$

where χ is the HMN.

Lemma 1.5. *[24] Assume that Q is a bounded subset of the normed space U where $U = \ell_q$ for $1 \leq q < \infty$ or $U = c_0$. If $R_n : U \to U$ is the operator defined by $R_n(u) = (u_0, u_1, ...u_n, 0, 0, ...)$ for all $u \in U$, then*

$$\chi(Q) = \lim_{r\to\infty} \left(\sup_{U\in Q} \|(I - R_r)(u)\| \right).$$

Assume that U, V are Banach spaces; χ_1 χ_2 are the HM on these spaces, respectively and $\mathcal{S} : U \to V$ is a linear operator. If $\mathcal{S}(Q) \subset V$ is a bounded set and there exists a positive constant ζ such that $\chi_2(\mathcal{S}(Q)) \leq \zeta\chi_1(\mathcal{S}(Q))$ for all bounded subset $Q \subset U$, then the linear operator $\mathcal{S} : U \to V$ is called (χ_1, χ_2)- bounded. If an operator $\mathcal{S}$ is (χ_1, χ_2)- bounded, then the number

$$\|\mathcal{S}\|_{(\chi_1,\chi_2)} = inf\left\{\zeta > 0 : \chi_2(\mathcal{S}(Q)) \leq \zeta\chi_1(\mathcal{S}(Q)) \text{ for all bounded set } Q \subset U\right\}$$

is called the (χ_1, χ_2)-measure noncompactness of $\mathcal{S}$. Specifically, if $\chi_1 = \chi_2 = \chi$ then we get $\|\mathcal{S}\|_{(\chi,\chi)} = \|\mathcal{S}\|_\chi$.

Lemma 1.6. *[25] Assume that U, V are Banach spaces, $\mathcal{S} \in \mathcal{B}(U, V)$ and $S_u = \{u \in U : \|u\| \leq 1\}$ is the unit sphere in U. Then,*

$$\|\mathcal{S}\|_\chi = \chi(\mathcal{S}(S_u))$$

and

$$\mathcal{S} \in \mathcal{C}(U, V) \Leftrightarrow \|\mathcal{S}\|_\chi = 0.$$

Lemma 1.7. *[26] Assume that U is a normed sequence space, $T = (t_{nv})$ is an infinite triangle, χ_T and χ denote the HMN on M_{U_T} and M_U, the collections of all bounded sets in U_T and U, respectively. Then, for all $Q \in M_{U_T}$,*

$$\chi_T(Q) = \chi(T(Q)).$$

The main purpose of the paper is to establish the space $|T_\theta|_q$ combining Tribonacci matrix given by Yaying and Hazarika [19] and the concept of absolute summability. After the introduction of the space, some inclusion relations are expressed, the $\alpha-, \beta-, \gamma-$ duals and basis of the space are constructed, and also characterizing certain matrix operators related to the space, their norms and HMN are determined.

2. Absolute Tribonacci space $|T_\theta|_q$

In this part of the paper, firstly, the absolute Tribonacci space $|T_\theta|_q$ is introduced and then, some inclusion relations, algebraic and topological structures of the space are investigated.

If we choose the Tribonacci matrix instead of Λ in (1.1), then the summabilty method $|\Lambda, \theta_n|_q$ is reduced to the absolute Tribonacci summability. To put it more clearly, take (s_n) which is a sequence of partial sum of $\sum u_v$. So, we have

$$\Lambda_n(s) = \sum_{j=0}^{n} t_{nj}s_j = \sum_{v=0}^{n} u_v \sum_{j=v}^{n} t_{nj} = \sum_{v=0}^{n} u_v \sum_{j=v}^{n} \tfrac{2t_j}{t_{n+2}+t_n-1}$$

and so, with a few calculations, we get

$$\begin{aligned}
\Delta\Lambda_n(s) &= \sum_{j=0}^{n} u_j \sum_{k=j}^{n} \tfrac{2t_k}{t_{n+2}+t_n-1} - \sum_{j=0}^{n-1} u_j \sum_{k=j}^{n-1} \tfrac{2t_k}{t_{n+1}+t_{n-1}-1} \\
&= \tfrac{2t_n}{t_{n+2}+t_n-1}u_n + \sum_{j=0}^{n-1} u_j \left(\tfrac{2t_n}{t_{n+2}+t_n-1} + \Delta\sigma_n \sum_{k=j}^{n-1} 2t_k\right) \\
&= \sum_{j=0}^{n} \phi_{nj}u_j
\end{aligned}$$

where

$$\sigma_n = \frac{1}{t_{n+2}+t_n-1},$$

$$\phi_{nj} = \begin{cases} \tfrac{2t_n}{t_{n+2}+t_n-1}, & j = n \\ \tfrac{2t_n}{t_{n+2}+t_n-1} + \Delta\sigma_n \sum_{k=j}^{n-1} 2t_k, & 0 \leq j \leq n-1 \\ 0, & j > n. \end{cases}$$

Now, we are ready to present the absolute Tribonacci space:

$$|T_\theta|_q = \left\{u \in \omega : \sum_{n=0}^{\infty} \theta_n^{q-1} \left|\sum_{j=0}^{n} \phi_{nj}u_j\right|^q < \infty\right\}.$$

Besides, it is seen immediately that

$$(F^{(q)} \circ \tilde{T})_n(u) = \theta_n^{1/q^*}(\tilde{T}_n(u) - \tilde{T}_{n-1}(u))$$

where

$$\tilde{t}_{nj} = \begin{cases} \sigma_n \sum_{v=j}^{n} 2t_v, & 0 \le j \le n \\ 0, & j > n, \end{cases} \tag{2.1}$$

$$f_{nj}^{(q)} = \begin{cases} \theta_n^{1/q^*}, & j = n \\ -\theta_n^{1/q^*}, & j = n-1 \\ 0, & j \neq n, n-1. \end{cases} \tag{2.2}$$

Taking into account the matrices $\tilde{T} = (\tilde{t}_{nk})$ and $F^{(q)} = (f_{nk}^{(q)})$ and the notation of domain, the space may be written that

$$|T_\theta|_q = (\ell_q)_{F^{(q)} \circ \tilde{T}}.$$

Also, it is known that there exists a unique inverse matrix which also is a triangle for every triangle matrix [27]. So, the matrices $\tilde{T}$ and $F^{(q)}$ have unique inverse matrices $\tilde{T}^{-1} = (\tilde{t}_{nk}^{-1})$ and $(\tilde{F}^{(q)})^{-1} = ((\tilde{f}_{nk}^{(q)})^{-1})$ given by

$$\tilde{t}_{nk}^{-1} = \begin{cases} \frac{1}{2\sigma_n t_n}, & k = n \\ -\frac{1}{2\sigma_{n-1} t_n} - \frac{1}{2\sigma_{n-1} t_{n-1}}, & k = n-1 \\ \frac{1}{2\sigma_{n-2} t_{n-1}}, & k = n-2 \\ 0, & k > n \end{cases} \tag{2.3}$$

$$(\tilde{f}_{nk}^{(q)})^{-1} = \begin{cases} \theta_k^{-1/q^*}, & 0 \le k \le n \\ 0, & k > n \end{cases} \tag{2.4}$$

respectively.

Now, to explain a relation between the natural norm of the spaces ℓ_q and the norm of $|T_\theta|_q$, we express the following theorem.

Theorem 2.1. *$|T_\theta|_q$ is BK-space with respect to the norm*

$$\|u\|_{|T_\theta|_q} = \left\| F^{(q)} \circ \tilde{T}(u) \right\|_{\ell_q},$$

where $1 \le q < \infty$.

Proof. Let $1 \le q < \infty$. It is known that ℓ_q is a BK-space. Also, since $F^{(q)} \circ \tilde{T}$ is a triangle, it is obtained immediately from Theorem 4.3.2 in [27], $|T_\theta|_q = (\ell_q)_{F^{(q)} \circ \tilde{T}}$ is a BK-space. □

Theorem 2.2. *The sequence $b^{(i)} = (b_n^{(i)})$ is a Schauder basis for the space $|T_\theta|_q$ where*

$$b_n^{(i)} = \begin{cases} \theta_i^{-1/q^*}\left(\frac{1}{2\sigma_n t_n} - \frac{1}{2\sigma_{n-1} t_n} - \frac{1}{2\sigma_{n-1} t_{n-1}} - \frac{1}{2\sigma_{n-2} t_{n-1}}\right), & i \le n-2 \\ \theta_i^{-1/q^*}\left(\frac{1}{2\sigma_n t_n} - \frac{1}{2\sigma_{n-1} t_n} - \frac{1}{2\sigma_{n-1} t_{n-1}}\right), & i = n-1 \\ \theta_i^{-1/q^*} \frac{1}{2\sigma_n t_n}, & i = n \\ 0, & i > n \end{cases}$$

$1 \le q < \infty$.

Proof. Let remind that $(e^{(i)})$ is the Schauder basis of ℓ_q. So, it is obtained from Theorem 2.3 in [28] that $b^{(i)} = (\tilde{T}_n^{-1}((F^{(q)})^{-1}(e^{(i)})))$ is a Schauder basis of the absolute space $|T_\theta|_q$. □

Theorem 2.3. *Let $1 \le q \le s < \infty$. If there exists a constant $C > 0$ such that $\theta_n \le C$ for all $n \in \mathbb{N}$, then $|T_\theta|_q \subset |T_\theta|_s$.*

Proof. Take $u \in |T_\theta|_q$. Since $\ell_q \subset \ell_s$, then $\left(\theta_n^{\frac{1}{q^*}} \sum_{j=0}^{n} \phi_{nj} u_j\right) \in \ell_s$ and also, since $\theta_n \leq C$ for all $n \in \mathbb{N}$,

$$C^{\frac{s}{q^*}-\frac{s}{s^*}} \left|\theta_n^{\frac{1}{s^*}} \sum_{j=0}^{n} \phi_{nj} u_j\right|^s \leq \left|\theta_n^{1/q^*} \sum_{j=0}^{n} \phi_{nj} u_j\right|^s$$

where q^* and s^* are the conjugate of q and s, respectively. Hence, we get that $u \in |T_\theta|_s$. This concludes the proof. □

Theorem 2.4. *The space $|T_\theta|_q$ is isomorphic to the space ℓ_q i.e., $|T_\theta|_q \cong \ell_q$ where $1 \leq q < \infty$.*

Proof. To prove the theorem, it should be shown that there exists a linear bijection between the spaces $|T_\theta|_q$ and ℓ_q where $1 \leq q < \infty$. Taking into account the transformations $\tilde{T} : |T_\theta|_q \to (\ell_q)_{F^{(q)}}, F^{(q)} : (\ell_q)_{F^{(q)}} \to \ell_q$ and the matrices corresponding to them given in (2.1) and (2.2). Since the matrices $F^{(q)}$ and $\tilde{T}$ are triangles, it is obvious that $\tilde{T}$ and $F^{(q)}$ are linear bijections and also the composite function $F^{(q)} \circ \tilde{T}$ is a linear bijective operator. Moreover,

$$\|u\|_{|T_\theta|_q} = \left\|F^{(q)} \circ \tilde{T}(u)\right\|_q,$$

i.e., the norm is preserved and so the proof is concluded. □

We define

$$D_1 = \left\{\epsilon \in \omega : \theta_v^{-1/k^*} \sum_{j=v+2}^{\infty} \epsilon_j \left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right) \text{ exist for all } v\right\},$$

$$D_2 = \left\{\epsilon \in \omega : \sup_m \left(\frac{1}{\theta_m}\left|\frac{\epsilon_m}{2\sigma_m t_m}\right|^{q^*} + \frac{1}{\theta_{m-1}}|\xi_{m-1}|^{q^*} + \sum_{v=0}^{m-2}\frac{1}{\theta_v}\left|\xi_v + \sum_{j=v+2}^{m}\epsilon_j\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right|^{q^*}\right) < \infty\right\},$$

$$D_3 = \left\{\epsilon \in \omega : \sup_{m,v}\left\{\left|\frac{\epsilon_m}{\sigma_m t_m}\right| + |\xi_{m-1}| + \left|\xi_v + \sum_{j=v+2}^{m}\epsilon_j\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right|\right\} < \infty\right\},$$

$$D_4 = \left\{\epsilon \in \omega : \sum_{v=0}^{\infty}\frac{1}{\theta_v}\left(\sum_{j=v+2}^{\infty}\left|\epsilon_j\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right| + \left|\frac{\epsilon_{v+1}}{2\sigma_{v+1}t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_v}\right| + \left|\frac{\epsilon_v}{2\sigma_v t_v}\right|\right)^{q^*} < \infty\right\},$$

$$D_5 = \left\{\epsilon \in \omega : \sup_v\left\{\sum_{j=v+2}^{\infty}\left|\epsilon_j\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right| + \left|\frac{\epsilon_{v+1}}{2\sigma_{v+1}t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_v}\right| + \left|\frac{\epsilon_v}{2\sigma_v t_v}\right|\right\} < \infty\right\},$$

$$\xi_v = \frac{\epsilon_v}{2\sigma_v t_v} + \frac{\epsilon_{v+1}}{2\sigma_{v+1}t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_{v+1}} - \frac{\epsilon_{v+1}}{2\sigma_v t_v}.$$

Similarly, when λ_{jv} is used instead of ϵ_v in above equation, the notation $\xi_v^{(j)}$ will be used instead of ξ_v.

Theorem 2.5. *Let $\theta = (\theta_n)$ be a sequence of positive numbers and $1 < q < \infty$. Then,*

(i) $\{|T_\theta|\}^\alpha = D_5$, $\left\{|T_\theta|_q\right\}^\alpha = D_4$,

(ii) $\{|T_\theta|\}^\beta = D_1 \cap D_3$, $\left\{|T_\theta|_q\right\}^\beta = D_1 \cap D_2$,

(iii) $\{|T_\theta|\}^\gamma = D_3$, $\left\{|T_\theta|_q\right\}^\gamma = D_2$.

Proof. We give the proof of (ii).

(ii) It's known that $\epsilon \in \left\{|T_\theta|_q\right\}^\beta$ iff $(\sum_{j=0}^{m}\epsilon_j u_j) \in c$ for all $u \in |T_\theta|_q$. By the inverse transformations of $\tilde{T}, F^{(q)}$, we get

$$\begin{aligned}
\sum_{j=0}^{m}\epsilon_j u_j &= \sum_{j=0}^{m}\epsilon_j\left(\frac{y_j}{2\sigma_j t_j}-\frac{y_{j-1}}{2\sigma_{j-1}t_j}-\frac{y_{j-1}}{2\sigma_{j-1}t_{j-1}}-\frac{y_{j-2}}{2\sigma_{j-2}t_{j-1}}\right)\\
&= \sum_{v=0}^{m}\sum_{j=v}^{m}\frac{\theta_v^{-1/q^*}\epsilon_j}{2\sigma_j t_j}z_v-\sum_{v=0}^{m-1}\sum_{j=v+1}^{m}\frac{\theta_v^{-1/q^*}\epsilon_j}{2\sigma_{j-1}t_j}z_v-\sum_{v=0}^{m-1}\sum_{j=v+1}^{m}\frac{\theta_v^{-1/q^*}\epsilon_j}{2\sigma_{j-1}t_{j-1}}z_v+\sum_{v=0}^{m-2}\sum_{j=v+2}^{m}\frac{\theta_v^{-1/q^*}\epsilon_j}{2\sigma_{j-2}t_{j-1}}z_v\\
&= \frac{\theta_m^{-1/q^*}\epsilon_m}{2\sigma_m t_m}z_m+\theta_{m-1}^{-1/q^*}\left(\frac{\epsilon_{m-1}}{2\sigma_{m-1}t_{m-1}}+\frac{\epsilon_m}{2\sigma_m t_m}-\frac{\epsilon_m}{2\sigma_{m-1}t_m}-\frac{\epsilon_m}{2\sigma_{m-1}t_{m-1}}\right)z_{m-1}\\
&+ \sum_{v=0}^{m-2}\theta_v^{-1/q^*}\left(\xi_v+\sum_{j=v+2}^{m}\epsilon_j\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j})-\frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right)z_v\\
&= \sum_{v=0}^{m}d_{mv}z_v \ \ (y=\tilde{T}(u), z=F^{(q)}(y))
\end{aligned}$$

where $D=(d_{mv})$ is defined by

$$d_{mv}=\begin{cases}\theta_v^{-1/q^*}\left(\xi_v+\sum_{j=v+2}^{m}\epsilon_j\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j})-\frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right), & 0\le v\le m-2\\ \theta_{m-1}^{-1/q^*}\xi_{m-1}, & v=m-1\\ \frac{\theta_m^{-1/q^*}\epsilon_m}{2\sigma_m t_m}, & v=m\\ 0, & v>m.\end{cases}$$

Therefore, $\epsilon \in \left\{|T_\theta|_q\right\}^\beta \Leftrightarrow D \in (\ell_q, c)$. Now, applying Lemma 1.4 to the matrix D, it is obtained that $\left\{|T_\theta|_q\right\}^\beta = D_1 \cap D_2$, which concludes the proof.
The proofs of other parts can be similarly verified, so there is no need for this. ☐

3. Matrix transformations

In this part of the paper, certain characterizations of matrix and compact operators on the absolute Tribonacci space $|T_\theta|_q$ are investigated and also their norms and HMN are computed.

Theorem 3.1. *Let $1\le q<\infty$, $\Lambda=(\lambda_{nj})$ be an infinite matrix of complex components for each $n,j\in\mathbb{N}$ and identify the matrix $H^{(n)}=\left(h_{mv}^{(n)}\right)$ by*

$$h_{mv}^{(n)}=\begin{cases}\xi_v^{(n)}+\sum_{j=v+2}^{m}\lambda_{nj}\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j})-\frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right), & 0\le v\le m-2\\ \xi_{m-1}^{(n)}, & v=m-1\\ \frac{\lambda_{nm}}{2\sigma_m t_m}, & v=m\\ 0, & v>m.\end{cases}$$

Moreover, let $\bar{H}=(\bar{h}_{nv})$ be a matrix whose terms is given by $\bar{h}_{nv}=\lim_m h_{mv}^{(n)}$ and $\widetilde{H}=F^{(q)}\circ\tilde{T}\circ\bar{H}$. Then, $\Lambda\in\left(|T_\theta|, |T_\theta|_q\right)$ if and only if

$$\sum_{j=v+2}^{\infty}\lambda_{nj}\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j})-\frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right) \ \textit{exists for all } v \tag{3.1}$$

$$\sup_{m,v}\left\{\left|\frac{\lambda_{nm}}{2\sigma_m t_m}\right|+\left|\xi_{m-1}^{(n)}\right|+\left|\xi_v^{(n)}+\sum_{j=v+2}^{m}\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j})-\frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\lambda_{nj}\right|\right\}<\infty, \tag{3.2}$$

$$\sup_{v}\sum_{n=0}^{\infty}\left|\widetilde{h}_{nv}\right|^{q}<\infty. \tag{3.3}$$

If $\Lambda\in\left(|T_\theta|,|T_\theta|_q\right)$*, then* Λ *is a bounded linear operator,*

$$\|\Lambda\|_{\left(|T_\theta|,|T_\theta|_q\right)}=\left\|\widetilde{H}\right\|_{(l,l_q)},$$

and

$$\|\Lambda\|_{\chi}=\lim_{v\to\infty}\left\{\sup_{r}\left(\sum_{n=v+1}^{\infty}\left|\widetilde{h}_{nr}\right|\right)^{q}\right\}^{\frac{1}{q}}.$$

Proof. $\Lambda\in\left(|T_\theta|,|T_\theta|_q\right)$ equals to $(\lambda_{nv})_{v=0}^{\infty}\in\{|T_\theta|\}^{\beta}$ and $\Lambda(u)\in|T_\theta|_q$ for all $u\in|T_\theta|$. It is easy to see from Theorem 2.5 that $(\lambda_{nv})_{v=0}^{\infty}\in\{|T_\theta|\}^{\beta}$ if and only if the conditions (3.1) and (3.2) hold. In addition to this, if a matrix $S=(s_{nv})\in(\ell,c)$, then the series $S_n(u)=\sum_{v=0}^{\infty}s_{nv}u_v$ is uniformly convergent in n, because, the remaining term of the series is uniformly tending to zero in n, since

$$\left|\sum_{v=p}^{\infty}s_{nv}u_v\right|\le\sup_{v}|s_{nv}|\sum_{v=p}^{\infty}|u_v\,|\to 0\ \ (p\to\infty).$$

So we get

$$\lim_{n}S_n\left(u\right)=\sum_{v=0}^{\infty}\lim_{n}s_{nv}u_v. \tag{3.4}$$

Considering (2.3), (2.4) and (3.4) we get immediately

$$\Lambda_n(u)=\lim_{m}\sum_{k=0}^{m}\lambda_{nk}u_k=\lim_{m}\sum_{r=0}^{m}h_{mr}^{(n)}z_r=\sum_{r=0}^{\infty}\bar{h}_{nr}z_r.$$

Moreover, according to Theorem 2.4, since there exists a linear isomorphism between $|T_\theta|_q,\ell_q$ for $1\le q<\infty$, it is written that $\Lambda(u)\in|T_\theta|_q$ for all $u\in|T_\theta|$ iff $\bar{H}\in\left(\ell,|T_\theta|_q\right)$, or equivalently, since $|T_\theta|_q=(\ell_q)_{F^{(q)}\circ\tilde{T}}$, $\widetilde{H}\in(\ell,\ell_q)$. Here, the terms of matrix $\hat{H}$ and $\widetilde{H}$ can be stated as

$$\hat{h}_{nr}=\sum_{v=0}^{n}\tilde{t}_{nv}\bar{h}_{vr}=\sum_{v=0}^{n}\sigma_n\sum_{j=v}^{n}2t_j\bar{h}_{vr},$$

$$\widetilde{h}_{nr}=\theta_r^{1/q^*}\left(\hat{h}_{nr}-\hat{h}_{n-1,r}\right),\ \ n\ge 1\text{ and }\widetilde{h}_{0r}=\bar{h}_{0r}.$$

So, if we apply Lemma 1.3 to the matrix $\widetilde{H}$, then, we get immediately the condition (3.3), and this concludes the first part of the proof.

Also, if $\Lambda\in\left(|T_\theta|,|T_\theta|_q\right)$, then, since the spaces $|T_\theta|_q$ and $|T_\theta|$ are BK-spaces, Λ determines a bounded operator. For the determination of the operator norm of Λ, take into account the isomorphisms $T:|T_\theta|_q\to(\ell_q)_{F^{(q)}}$, $F^{(q)}:(\ell_q)_{F^{(q)}}\to\ell_q$ defined as in Theorem 2.4. Then, it can be seen easily that $\Lambda=\widetilde{T}^{-1}\circ\left(F^{(q)}\right)^{-1}\circ\widetilde{H}\circ F^{(1)}\circ T$ and so,

$$\begin{aligned}\|\Lambda\|_{\left(|T_\theta|,|T_\theta|_q\right)}&=\sup_{u\neq0}\frac{\|\Lambda(u)\|_{|T_\theta|_q}}{\|u\|_{|T_\theta|}}=\sup_{u\neq0}\frac{\left\|\tilde{T}^{-1}\circ\left(F^{(q)}\right)^{-1}\circ\widetilde{H}\circ F^{(1)}\circ\tilde{T}(u)\right\|_{|T_\theta|_q}}{\|u\|_{|T_\theta|}}\\&=\sup_{z\neq0}\frac{\left\|\widetilde{H}(z)\right\|_{\ell_q}}{\|z\|_{\ell}}=\left\|\widetilde{H}\right\|_{(\ell,\ell_q)}\ (z=F^{(1)}\circ\tilde{T}(u)).\end{aligned}$$

Finally, let Q be a unique ball in $|T_\theta|$. Since $F^{(q)} \circ \tilde{T} \circ \Lambda Q = \widetilde{H} \circ F^{(1)} \circ \tilde{T} Q$, it is written that

$$\begin{aligned}
\|\Lambda\|_\chi = \chi(\Lambda Q) &= \chi\left(F^{(q)} \circ \tilde{T} \circ \Lambda Q\right) = \chi\left(\widetilde{H} \circ F^{(1)} \circ \tilde{T} Q\right) \\
&= \lim_{v\to\infty}\left(\sup_{z\in F^{(1)}(\tilde{T}(Q))}\left\|(I-R_v)\left(\widetilde{H}(z)\right)\right\|\right) \\
&= \lim_{v\to\infty}\left\{\sup_r\left(\sum_{n=v+1}^{\infty}\left|\widetilde{h}_{nr}\right|\right)^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

□

This completes the proof.

The compact operators in this class are characterized by Theorem 3.1 and Lemma 1.6. Corollary 3.1 gives us the condition:

Corollary 3.1. *Under the hypothesis of Theorem 3.1*

$$\Lambda \in \left(|T_\theta|, |T_\theta|_q\right) \text{ is compact} \Leftrightarrow \lim_{v\to\infty}\left\{\sup_r\left(\sum_{n=v+1}^{\infty}\left|\widetilde{h}_{nr}\right|\right)^q\right\}^{\frac{1}{q}} = 0.$$

Theorem 3.2. *Let $1 < q < \infty$, $\lambda = (\lambda_{nj})$ be an infinite matrix of complex components for each $n, j \in \mathbb{N}$ and $H^{(n)} = \left(h_{mv}^{(n)}\right)$ be as in Theorem 3.1. Also, describe $\bar{E} = (\bar{e}_{nv})$ by $\bar{e}_{nv} = \lim_m \theta_v^{-1/q^*} h_{mv}^{(n)}$ and $\widetilde{E} = F^{(1)} \circ \tilde{T} \circ \bar{E}$. Then, $\Lambda \in \left(|T_\theta|_q, |T_\theta|\right)$ if and only if*

$$\sum_{j=r+2}^{\infty}\lambda_{nj}\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right) \text{ exist for all } r,$$

$$\sup_m\left\{\frac{1}{\theta_m}\left|\frac{\lambda_{nm}}{2\sigma_m t_m}\right|^{q^*} + \frac{1}{\theta_{m-1}}\left|\xi_n^{(m-1)}\right|^{q^*} + \sum_{v=0}^{m-2}\frac{1}{\theta_v}\left|\xi_n^{(v)} + \sum_{j=v+2}^{m}\lambda_{nj}\sum_{v=r+2}^{m}\left(\frac{1}{2t_j}\Delta(\frac{1}{\sigma_j}) - \frac{1}{2t_{j-1}}\Delta(\frac{1}{\sigma_{j-1}})\right)\right|^{q^*}\right\} < \infty,$$

$$\sum_{r=0}^{\infty}\left(\sum_{n=0}^{\infty}|\widetilde{e}_{nr}|\right)^{q^*} < \infty.$$

Moreover, if $\Lambda \in \left(|T_\theta|_q, |T_\theta|\right)$, then Λ is a bounded linear operator,

$$\|\Lambda\|_{\left(|T_\theta|_q, |T_\theta|\right)} = \left\|\widetilde{E}\right\|_{(\ell_q, \ell)}$$

and

$$\|\Lambda\|_\chi = \frac{1}{\eta}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}|\widetilde{e}_{nr}|\right)^{q^*}\right\}^{\frac{1}{q^*}}$$

where $1 \le \eta \le 4$.

Corollary 3.2 gives us the characterization of compact operators with together Lemma 1.6 and Theorem 3.2.

Corollary 3.2. *Under the conditions of Theorem 3.2*

$$\Lambda \in \mathcal{C}\left(|T_\theta|_q, |T_\theta|\right) \Leftrightarrow \frac{1}{\eta}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}|\widetilde{e}_{nr}|\right)^{q^*}\right\}^{\frac{1}{q^*}} = 0.$$

4. Conclusion

Recently, in addition to the studies on sequence spaces obtained as the domain of some special matrices and matrix transformations related to them, new sequence spaces obtained by using the concept of absolute summability method have been introduced in the literature. In this study, the absolute Tribonacci space $|T_\theta|_q$ has been introduced as the domain of the Tribonacci matrix on l_q. Then, some algebraic and topological structure have been studied, certain characterizations of compact and matrix operators on these spaces with their norms and Hausdorff meausures of noncompactness have been given. A different perspective has been generated by including the Tribonacci sequence, which is an interesting number sequence, in the subject.

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Affiliations

FADIME GÖKÇE
ADDRESS: Pamukkale University, Dept. of Statistics, Denizli-Turkey
E-MAIL: fgokce@pau.edu.tr
ORCID ID: 0000-0003-1819-3317

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An Application to Fuss-Catalan Numbers

Cemal Çiçek

Abstract
In this paper, it was investigated how many different ways an m-rung stairs can be climbed within certain rules. It was observed that the climbing numbers of the stairs have relations with the Catalan numbers. The combinatorics problem discussed in this article is different from the ones done so far and is related not only to Catalan numbers but also to some Fuss-Catalan numbers. Some results were obtained regarding the climbing numbers. It was observed that with the initial ascent being fixed, the climbing numbers of stairs with $m, m+1, m+2, m+3$, ... rungs, where $m > 1$ is an integer, are related to respectively the some Fuss-Catalan numbers.

Keywords: *Catalan Numbers, Climbing, Fuss-Catalan Numbers, Lattice Path, Stairs*

AMS Subject Classification (2020): *05C38; 06B20*

1. Introduction

Catalan numbers appear in many combinatorics problems [1–5]. Applications of these numbers are used in some engineering fields and health sciences [3, 4, 6]. In this study, it was examined the combinatorics problem of how many different ways an m-rung stairs can be climbed within certain rules. The rules that should be applied while ascent and descending the stairs are as following. The first is that no matter how many rungs we move up, our descent should be at least one rung above the beginning of the previous ascent. The second is no matter how many rungs we move down, our ascent should be at least one rung above the beginning of the previous descent. Our last rule is that when the number of rungs on the stairs is more than 1, the first move should be at least 2 rungs up. It was observed that with the initial ascent being fixed, the climbing numbers of stairs with $m, m+1, m+2, m+3$, ... rungs, where $m > 1$ is an integer, are related to respectively the Fuss-Catalan numbers.

Definition 1.1. ([7, 8]) The generalized Fuss-Catalan numbers are integer sequence defined by

$$F_i(j,k) = \frac{k}{ij+k}\binom{ij+k}{j},\ k \geq 1,\ j \geq 1,\ \ i \geq 2. \tag{1.1}$$

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Specially, if we take $i = 2,\ k = 1$ in the formula (1.1), we have Catalan numbers C_n [3, 5, 6, 9, 10] as follows

$$F_2\left(j,1\right) = \frac{1}{2j+1}\binom{2j+1}{j} = \frac{1}{j+1}\binom{2j}{j} = C_j. \tag{1.2}$$

The Generalized Fuss-Catalan numbers are also called Raney numbers [8, 11, 12].

2. Problem statement and solution

The problem is to find the number of different climbings of an m-rung stairs within certain rules.

Notation 2.1. Let p^q be the representation of the ascent of a stairs from the p-th rung to the q-th rung.

Notation 2.2. Let $p^q r^s$ be the representation of the ascent of a stairs from the p-th rung to the q-th rung, the descent from the q-th rung to the r-th rung, and finally the ascent from the r-th to the s-th step.

Let's start with some numerical examples.

Example 2.1. Let's find the number of different climbing of a 1-rung stairs.

Solution. The number of climbs is 1.

0^1:

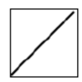


Figure 1. Climbing on a 1-rung stairs

Example 2.2. Let's find the number of different climbing of a 2-rung stairs.

Solution. The number of climbs is 1.

0^2:

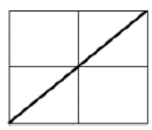


Figure 2. Climbing on a 2-rung stairs

Example 2.3. Let's find the number of different climbing of a 3-rung stairs.

Solution. The number of climbs is 2.

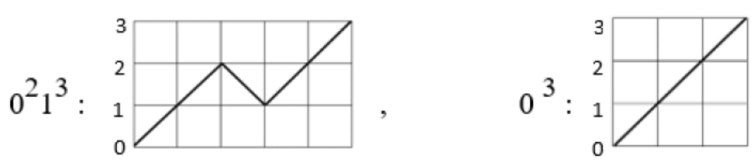


Figure 3. Climbing on a 3-rung stairs

The first one goes up from the 0-th rung to the 2-nd rung. Then go down from the 2-nd rung to the 1-st rung. So, the first climbing $0^2 1^3$ is completed by climbing from the 1-st rung to the 3-rd rung.

The second one 0^3 is completed by climbing from the 0th rung to the 3-rd rung.
In this case, according to the rules of the problem, there is no other climbing position.

Example 2.4. Let's find the number of different climbing of one 4-rung stairs.

Solution. The number of climbs is 5.

$$0^2 1^3 2^4,\ \ 0^2 1^4,\ \ 0^3 1^4,\ \ 0^3 2^4,\ \ 0^4.$$

Let's show two particular climbs with figures, the others can be done similarly.

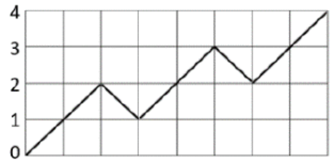
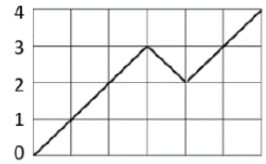
$0^2 1^3 2^4$:  , $0^3 2^4$: 

Figure 4. Two of the five climbings on a 4-rung stairs

Example 2.5. Let's find the number of different climbing of one 5-rung stairs.

Solution. The number of climbs is 14.

$$\begin{array}{lllll}
0^2 1^3 2^4 3^5, & 0^2 1^3 2^5, & 0^2 1^4 2^5, & 0^2 1^4 3^5, & 0^2 1^5, \\
0^3 1^4 2^5, & 0^3 1^4 3^5, & 0^3 1^5, & 0^3 2^4 3^5, & 0^3 2^5, \\
0^4 1^5, & 0^4 2^5, & 0^4 3^5, & & \\
0^5. & & & &
\end{array}$$

Let's show two particular climbs with figures, the others can be done similarly.

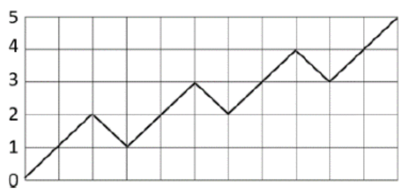
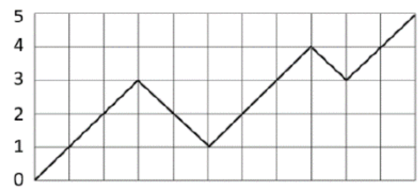
$0^2 1^3 2^4 3^5$:  , $0^3 1^4 3^5$: 

Figure 5. Two of the fourteen climbings on a 5-rung stairs

Example 2.6. Let's find the number of different climbing of a 6-rung stairs using the number of climbing of one 5-rung stairs.

Solution. Let's write the climbings of a 6-rung stairs starting at 1^3 and a 5-rung stairs starting at 0^2, respectively.

$$\begin{array}{ll}
1^3 \rightarrow & 1^3 2^4 3^5 4^6,\ 1^3 2^4 3^6,\ 1^3 2^5 3^6,\ 1^3 2^5 4^6,\ 1^3 2^6 \\
0^2 \rightarrow & 0^2 1^3 2^4 3^5,\ 0^2 1^3 2^5,\ 0^2 1^4 2^5,\ 0^2 1^4 3^5,\ 0^2 1^5
\end{array}$$

Note that here the number of different climbings starting with 1^3 in a 6-rung stairs and the number of climbings starting with 0^2 in a 5-rung stairs are equal to each other. Because in both cases, after the start the number of rungs remaining to the top of the stairs are equal. This result can be used also for other situations. We can see all the situations in the Table 1 below.

Starting Position of 5-Rung Stairs	# of climbing of 5-Rung Stairs	# of climbing of 6-Rung Stairs	Starting Position of 6-Rung Stairs
$\mathbf{0^2}$	5	5	$0^2\mathbf{1^3}$
$\mathbf{0^3}$	5	5	$0^2\mathbf{1^4}$
		5	$0^3\mathbf{1^4}$
$\mathbf{0^4}$	3	3	$0^2\mathbf{1^5}$
		3	$0^3\mathbf{1^5}$
		3	$0^4\mathbf{1^5}$

$\mathbf{0^5}$	1	1	$0^2\mathbf{1^6}$
		1	$0^3\mathbf{1^6}$
		1	$0^4\mathbf{1^6}$
$\mathbf{1^3}$	2	2	$0^3\mathbf{2^4}$
$\mathbf{1^4}$	2	2	$0^3\mathbf{2^5}$
		2	$0^4\mathbf{2^5}$
$\mathbf{1^5}$	1	1	$0^3\mathbf{2^6}$
		1	$0^4\mathbf{2^6}$
$\mathbf{2^4}$	1	1	$0^4\mathbf{3^5}$
$\mathbf{2^5}$	1	1	$0^4\mathbf{3^6}$
		1	$0^5\mathbf{3^6}$
$\mathbf{0^5}$	1	1	$0^5\mathbf{1^6}$
$\mathbf{1^5}$	1	1	$0^5\mathbf{2^6}$
$\mathbf{3^5}$	1	1	$0^5\mathbf{4^6}$
		1	0^6
		42	**Total**

Table 1. Obtaining the number of climbing of a 6-rung stairs by using the number of climbing of a 5-rung stairs

Therefore the number of different climbings of a 6-rung stairs is 42.

Remark 2.1. In general, the number of different climbings of a stairs with m-rungs starting from p^q is equal to the number of different climbings of a stairs with $(m+1)$- rungs starting from $(p+1)^{q+1}$.

Similarly, the number of different climbings of a 7-step stairs can be found using the number of different climbings of a 6-step stairs. By continuing like this, the number of different climbings up to a 13-step stairs was found and shown in the Table 2 below. Using the Table 2 we can write the following results.

	Number of Rungs of the Stairs											
Beginings	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
$\mathbf{0^2}$	1	1	2	5	14	42	132	429	1430	4862	16796	58786
$\mathbf{0^3}$	0	1	2	5	14	42	132	429	1430	4862	16796	58786
$\mathbf{0^4}$	0	0	1	3	9	28	90	297	1001	3432	11934	41990
$\mathbf{0^5}$	0	0	0	1	4	14	48	165	572	2002	7072	25194
$\mathbf{0^6}$	0	0	0	0	1	5	20	75	275	1001	3640	13260
$\mathbf{0^7}$	0	0	0	0	0	1	6	27	110	429	1638	6188
$\mathbf{0^8}$	0	0	0	0	0	0	1	7	35	154	637	2548
$\mathbf{0^9}$	0	0	0	0	0	0	0	1	8	44	208	910
$\mathbf{0^{10}}$	0	0	0	0	0	0	0	0	1	9	54	273
$\mathbf{0^{11}}$	0	0	0	0	0	0	0	0	0	1	10	65
Total	1	2	5	14	42	132	429	1430	4862	16796	58786	208012

Table 2. All climbing numbers from 2-rung stairs to 13-rung stairs

Notation 2.3. Let $N(m, 0^n)$ denotes the number of different climbs of a m-rung stairs with 0^n ascents.

For example, from the Table 2 it can be seen that $N(8, 0^6) = 20,\ N(12, 0^7) = 1638$.
Generally, denote by $N(m, k^n)$ the number of different climbs of a m-rungs stairs starting with k^n. Note that for m, n positive integers,

$$N(m, 0^n) = \begin{cases} 1, & n = m \\ 0, & n > m \end{cases}$$

Corollary 2.1. For m positive integers, $m \geq 3$, $N(m, 0^{m-1}) = m - 2$.

Proof.

$$\underbrace{0^{m-1}1^m,\ 0^{m-1}2^m,\ 0^{m-1}3^m, ..., 0^{m-1}(m-2)^m}_{(m-2)\text{ terms}}$$

Corollary 2.2. For $m, (m \geq 4)$ positive integers, $N(m, 0^{m-2}) = \frac{(m-2)(m-1)}{2} - 1$.

Proof.

$$\underbrace{0^{m-2}1^{m-1}2^m,\ 0^{m-2}1^{m-1}3^m, ..., 0^{m-2}1^{m-1}(m-2)^m,\ 0^{m-2}1^m}_{(m-2)\text{ terms}}$$

$$\underbrace{0^{m-2}2^{m-1}3^m,\ 0^{m-2}2^{m-1}4^m, ...,\ 0^{m-2}2^{m-1}(m-2)^m,\ 0^{m-2}2^m}_{(m-3)\text{ terms}}$$

$$\underbrace{0^{m-2}3^{m-1}4^m,\ 0^{m-2}3^{m-1}5^m, ...,\ 0^{m-2}3^{m-1}(m-2)^m,\ 0^{m-2}3^m}_{(m-4)\text{ terms}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\underbrace{0^{m-2}(m-3)^{m-1}(m-2)^m,\ 0^{m-2}(m-3)^{m-1}(m-1)^m,\ 0^{m-2}(m-2)^m}_{3\text{ terms}}$$

$$\underbrace{0^{m-2}(m-3)^{m-1}(m-2)^m,\ 0^{m-2}(m-3)^m}_{2\text{ terms}}$$

Therefore,
$N(m, 0^{m-2}) = (m-2) + (m-3) + (m-4) + ... + 3 + 2 = \frac{(m-2)(m-1)}{2} - 1$.

Corollary 2.3. For m positive integers, $m \geq 3$, $N(m, 0^2) = N(m, 0^3)$.

Proof. Since $m \geq 3$, the movements starting with 0^3 and 0^2 are as follows:

(i) Movements starting with 0^3 are in the form of $0^3 1^4 \ldots,\ 0^3 1^5 \ldots,\ 0^3 1^6 \ldots,\ \ldots,\ 0^3 1^{m-1} \ldots,\ 0^3 1^m \ldots$ and $0^3 2^4 \ldots,\ 0^3 2^5 \ldots,\ 0^3 2^6 \ldots,\ \ldots,\ 0^3 2^{m-1} \ldots,\ 0^3 2^m \ldots$

(ii) Movements starting with 0^2 are in the form of $0^2 1^3 2^4 \ldots,\ 0^2 1^3 2^5 \ldots,\ 0^2 1^3 2^6 \ldots,\ \ldots,\ 0^2 1^3 2^{m-1} \ldots,$ $0^2 1^3 2^m \ldots$ and $0^2 1^4 \ldots,\ 0^2 1^5 \ldots,\ 0^2 1^6 \ldots,\ \ldots,\ 0^2 1^{m-1} \ldots,\ 0^2 1^m \ldots$.

Comparing (i) and (ii), it is not difficult to see that $N(m, 0^3 1^4) = N(m, 0^2 1^4),\ N(m, 0^3 1^5) = N(m, 0^2 1^5),\ \ldots$ and $N(m, 0^3 2^4) = N(m, 0^2 1^3 2^4),\ N(m, 0^3 2^5) = N(m, 0^2 1^3 2^5), \ldots$.
Therefore we obtain

$$\begin{aligned} N(m, 0^3) =& N(m, 0^3 1^4) + N(m, 0^3 1^5) + \ldots + N(m, 0^3 1^{m-1}) + N(m, 0^3 1^m) \\ &+ N(m, 0^3 2^4) + N(m, 0^3 2^5) + \ldots + N(m, 0^3 2^{m-1}) + N(m, 0^3 2^m) \\ =& N(m, 0^2 1^4) + N(m, 0^2 1^5) + \ldots + N(m, 0^2 1^{m-1}) + N(m, 0^2 1^m) \\ &+ N(m, 0^2 1^3 2^4) + N(m, 0^2 1^3 2^5) + \ldots + N(m, 0^2 1^3 2^{m-1}) + N(m, 0^2 1^3 2^m) \\ =& N(m, 0^2) \end{aligned}$$

Hence the equality $N(m, 0^2) = N(m, 0^3)$ is valid for all $m \geq 3$.

Corollary 2.4. The number of possible climbings of a stairs with m-rungs is equal to the number of climbings of a stairs with $(m+1)$-rungs starting from 0^2.

Proof. It is sufficient to show the equality $N(m+1,0^2) = N(m,0^2)+N(m,0^3)+\ldots+N(m,0^{m-1})+N(m,0^m)$. Using Remark 2.1, we have

$$\begin{aligned} N(m+1,0^2) =& N(m+1,0^21^3)+N(m+1,0^21^4)+\ldots+N(m+1,0^21^m)+N(m+1,0^21^{m+1}) \\ &+N(m+1,1^3)+N(m+1,1^4)+\ldots+N(m+1,1^m)+N(m+1,1^{m+1}) \\ =& N(m,0^2)+N(m,0^3)+\ldots+N(m,0^{m-1})+N(m,0^m). \end{aligned}$$

Hence $N(m+1,0^2) = N(m,0^2)+N(m,0^3)+\ldots+N(m,0^{m-1})+N(m,0^m)$.

Corollary 2.5. Let m,n be positive integers with $m \geq 6,\ n \geq 3$, then the equality

$$N(m-1,0^{n-1})+N(m,0^{n+1}) = N(m,0^n)$$

holds.

Proof. Firstly, let's write number of the climbs of a stairs with m-rungs starting at 0^n:

$$\begin{aligned} N(m,0^n) =& N(m,0^n1^{n+1}2^{n+2})+N(m,0^n1^{n+1}2^{n+3})+\ldots+N(m,0^n1^{n+1}2^m) \\ &+N(m,0^n1^{n+1}3^{n+2})+N(m,0^n1^{n+1}3^{n+3})+\ldots+N(m,0^n1^{n+1}3^m) \\ &+\ldots \\ &+N(m,0^n1^{n+1}n^{n+2})+N(m,0^n1^{n+1}n^{n+3})+\ldots+N(m,0^n1^{n+1}n^m) \\ &+N(m,0^n1^{n+2})+N(m,0^n1^{n+3})+\ldots+N(m,0^n1^m) \\ &+N(m,0^n2^{n+1})+N(m,0^n2^{n+2})+\ldots+N(m,0^n2^m) \\ &+N(m,0^n3^{n+1})+N(m,0^n3^{n+2})+\ldots+N(m,0^n3^m) \\ &+\ldots \\ &+N(m,0^n(n-1)^{n+1})+N(m,0^n(n-1)^{n+2})+\ldots+N(m,0^n(n-1)^m). \end{aligned}$$

Secondly, let's write number of the climbs of a stairs with m-rungs starting at 0^{n+1}:

$$\begin{aligned} N(m,0^{n+1}) =& N(m,0^{n+1}1^{n+2})+N(m,0^{n+1}1^{n+3})+\ldots+N(m,0^{n+1}1^m) \\ &+N(m,0^{n+1}2^{n+2})+N(m,0^{n+1}2^{n+3})+\ldots+N(m,0^{n+1}2^m) \\ &+N(m,0^{n+1}3^{n+2})+N(m,0^{n+1}3^{n+3})+\ldots+N(m,0^{n+1}3^m) \\ &+\ldots \\ &+N(m,0^{n+1}n^{n+2})+N(m,0^{n+1}n^{n+3})+\ldots+N(m,0^{n+1}n^m). \end{aligned}$$

Finally, let's write number of the climbs of a stairs with $(m-1)$-rungs starting at 0^{n-1}:

$$\begin{aligned} N(m-1,0^{n-1}) =& N(m-1,0^{n-1}1^n)+N(m-1,0^{n-1}1^{n+1})+\ldots+N(m-1,0^{n-1}1^{m-1}) \\ &+N(m-1,0^{n-1}2^n)+N(m-1,0^{n-1}2^{n+1})+\ldots+N(m-1,0^{n-1}2^{m-1}) \\ &+N(m-1,0^{n-1}3^n)+N(m-1,0^{n-1}3^{n+1})+\ldots+N(m-1,0^{n-1}3^{m-1}) \\ &+\ldots \\ &+N(m-1,0^{n-1}(n-2)^n)+N(m-1,0^{n-1}(n-2)^{n+1})+\ldots \\ &+N(m-1,0^{n-1}(n-2)^{m-1}). \end{aligned}$$

Since

$$
\begin{aligned}
&N(m, 0^n 1^{n+1} 2^{n+k}) = N(m, 0^{n+1} 2^{n+k}), \text{ for } k = 2, 3, \dots, (m-n), \\
&N(m, 0^n 1^{n+1} 3^{n+k}) = N(m, 0^{n+1} 3^{n+k}), \text{ for } k = 2, 3, \dots, (m-n), \\
&\dots\dots\dots\dots \\
&N(m, 0^n 1^{n+1} n^{n+k}) = N(m, 0^{n+1} n^{n+k}), \text{ for } k = 2, 3, \dots, (m-n), \\
&N(m, 0^n 1^{n+k}) = N(m, 0^{n+1} 1^{n+k}), \text{ for } k = 2, 3, \dots, (m-n), \\
&N(m, 0^n 2^{n+1+k}) = N(m-1, 0^{n-1} 1^{n+k}), \text{ for } k = 0, 1, \dots, (m-n-1), \\
&N(m, 0^n 3^{n+1+k}) = N(m-1, 0^{n-1} 2^{n+k}), \text{ for } k = 0, 1, \dots, (m-n-1), \\
&\dots\dots\dots\dots \\
&N(m, 0^n (n-1)^{n+1+k}) = N(m-1, 0^{n-1} (n-2)^{n+k}), \text{ for } k = 0, 1, \dots, (m-n-1),
\end{aligned}
$$

then we have

$$
\begin{aligned}
N(m, 0^n) =& N(m, 0^n 1^{n+1} 2^{n+k}) + N(m, 0^n 1^{n+1} 2^{n+3}) + \dots + N(m, 0^n 1^{n+1} 2^m) \\
&+ N(m, 0^n 1^{n+1} 3^{n+2}) + N(m, 0^n 1^{n+1} 3^{n+3}) + \dots + N(m, 0^n 1^{n+1} 3^m) \\
&+ \dots \\
&+ N(m, 0^n 1^{n+1} n^{n+2}) + N(m, 0^n 1^{n+1} n^{n+3}) + \dots + N(m, 0^n 1^{n+1} n^m) \\
&+ N(m, 0^n 1^{n+2}) + N(m, 0^n 1^{n+3}) + \dots + N(m, 0^n 1^m) \\
&+ N(m, 0^n 2^{n+1}) + N(m, 0^n 2^{n+2}) + \dots + N(m, 0^n 2^m) \\
&+ N(m, 0^n 3^{n+1}) + N(m, 0^n 3^{n+2}) + \dots + N(m, 0^n 3^m) \\
&+ \dots \\
&+ N(m, 0^n (n-1)^{n+1}) + N(m, 0^n (n-1)^{n+2}) + \dots + N(m, 0^n (n-1)^m) \\
=& N(m, 0^{n+1} 2^{n+2}) + N(m, 0^{n+1} 2^{n+3}) + \dots + N(m, 0^{n+1} 2^m) \\
&+ N(m, 0^{n+1} 3^{n+2}) + N(m, 0^{n+1} 3^{n+3}) + \dots + N(m, 0^{n+1} 3^m) \\
&+ \dots \\
&+ N(m, 0^{n+1} n^{n+2}) + N(m, 0^{n+1} n^{n+3}) + \dots + N(m, 0^{n+1} n^m) \\
&+ N(m, 0^{n+1} 1^{n+2}) + N(m, 0^{n+1} 1^{n+3}) + \dots + N(m, 0^{n+1} 1^m) \\
&+ N(m-1, 0^{n-1} 1^n) + N(m-1, 0^{n-1} 1^{n+1}) + \dots + N(m-1, 0^{n-1} 1^{m-1}) \\
&+ N(m-1, 0^{n-1} 2^n) + N(m-1, 0^{n-1} 2^{n+1}) + \dots + N(m-1, 0^{n-1} 2^{m-1}) \\
&+ \dots \\
&+ N(m-1, 0^{n-1} (n-2)^n) + N(m-1, 0^{n-1} (n-2)^{n+1}) + \dots + N(m-1, 0^{n-1} (n-2)^{m-1}) \\
=& \{ N(m, 0^{n+1} 1^{n+2}) + N(m, 0^{n+1} 1^{n+3}) + \dots + N(m, 0^{n+1} 1^m) \\
&+ N(m, 0^{n+1} 2^{n+2}) + N(m, 0^{n+1} 2^{n+3}) + \dots + N(m, 0^{n+1} 2^m) \\
&+ N(m, 0^{n+1} 3^{n+2}) + N(m, 0^{n+1} 3^{n+3}) + \dots + N(m, 0^{n+1} 3^m) \\
&+ \dots \\
&+ N(m, 0^{n+1} n^{n+2}) + N(m, 0^{n+1} n^{n+3}) + \dots + N(m, 0^{n+1} n^m) \} \\
&+ \{ N(m-1, 0^{n-1} 1^n) + N(m-1, 0^{n-1} 1^{n+1}) + \dots + N(m-1, 0^{n-1} 1^{m-1}) \\
&+ N(m-1, 0^{n-1} 2^n) + N(m-1, 0^{n-1} 2^{n+1}) + \dots + N(m-1, 0^{n-1} 2^{m-1}) \\
&+ \dots
\end{aligned}
$$

$$+N(m-1,0^{n-1}(n-2)^{n})+N(m-1,0^{n-1}(n-2)^{n+1})+\ldots+N(m-1,0^{n-1}(n-2)^{m-1})\}$$
$$=N(m,0^{n+1})+N(m-1,0^{n-1})$$

Therefore the equality $N(m-1,0^{n-1})+N(m,0^{n+1})=N(m,0^{n})$ holds for $m\geq 6,\ n\geq 3$.

3. Relation of the problem to some Fuss-Catalan numbers

Formula (1.2) in Definition 1.1 gives the sequence of Catalan numbers $(C_j)_{j\geq 0}=\big(1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ \ldots\big)$, which is the sequence $\big(N(m,0^2)\big)_{m\geq 2}$ in the Table 2.
If we take now $i=2,\ k=2$ in the formula (1.1), we have

$$F_2(j,2)=\frac{1}{j+1}\binom{2j+2}{j}, \tag{3.1}$$

which are the Catalan numbers $(C_{j+1})_{j\geq 0}=\big(1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ \ldots\big)$.
The sequence $(C_{j+1})_{j\geq 0}$ is the sequence $\big(N(m,0^3)\big)_{m\geq 3}$ in the Table 2.
If we take now $i=2,\ k=3$ in the formula (1.1), we obtain

$$F_2(j,3)=\frac{3}{2j+3}\binom{2j+3}{j}, \tag{3.2}$$

so the sequence $\big(F_2(j,3)\big)_{j\geq 0}=\big(1,\ 3,\ 9,\ 28,\ 90,\ 297,\ 1001,\ \ldots\big)$ is the sequence $\big(N(m,0^4)\big)_{m\geq 4}$ in the Table 2.
If we take now $i=2,\ k=4$ in the formula (1.1), we get

$$F_2(j,4)=\frac{2}{j+2}\binom{2j+4}{j}, \tag{3.3}$$

that means $\big(F_2(j,4)\big)_{j\geq 0}=\big(1,\ 4,\ 14,\ 48,\ 165,\ 572,\ 2002,\ \ldots\big)$ which is the sequence $\big(N(m,0^5)\big)_{m\geq 5}$ in the Table 2 and so on.
Generally, the sequence $\big(F_2(j,k)\big)_{j\geq 0}$ is equal to the sequence $\big(N(m,0^{k+1})\big)_{m\geq k+1},\ k\geq 1$, in the Table 2, i.e. $\big(F_2(j,k)\big)_{j\geq 0}=\big(N(m,0^{k+1})\big)_{m\geq k+1},\ k\geq 1$.

4. Conclusion

In this study, we give an application of Fuss-Catalan numbers and Catalan numbers. This application was formulated with a problem. By solving this problem some formulas related to Fuss-Catalan numbers are proved. According to that, other applications related to Fuss-Catalan numbers can be done in future studies. We believe that other formulas will be obtained with the help of this problem we presented.

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Affiliations

CEMAL ÇIÇEK
ADDRESS: Istanbul University, Dept. of Mathematics, Istanbul-Turkiye
E-MAIL: cicekc@istanbul.edu.tr
ORCID ID: 0000-0002-4855-9386

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New Asymptotic Properties for Solutions of Fractional Delay Neutral Differential Equations

Abdullah Yiğit

Abstract
In this note, we consider new asymptotic stability properties for solutions of several fractional delay neutral differential equations of a certain type. To obtain the desired properties, we use Lyapunov's direct method, which has a wide range of applications. Finally, we draw the reader's attention to some examples supporting the obtained asymptotic stability properties and their plots under different initial conditions. With this note, we extend and improve some results previously considered in the relevant literature.

Keywords: *Asymptotic stability, Continuous function, Fractional derivative, Lyapunov direct method, Neutral differential equation, Variable delay*

AMS Subject Classification (2020): *34K20; 34A08; 34K40*

1. Introduction

The subject of fractional calculus, which began with an exchange of information between two famous scientists, L' Hospital and Leibnitz, at the end of the 17th century, has spread widely in the scientific world and attracted the attention of many scientists. Control theory, model of neurons in biology, fluid mechanics, viscoelasticity, meteorology, biology, communication etc. fractional calculus modeled with differential equations in fields still maintains its currency today. This subject, which has become an important area of mathematics, physics, medicine, biology and engineering, is highly developed in terms of numerical and analytical solutions for mathematical nonlinear dynamic modeling. For this reason, this subject, which has become the focus of attention of the international academic community, has been addressed by many researchers and many studies published on this subject have taken their place in the relevant literature. We recommend that interested researchers examine the studies referred to in the bibliography and the sources in these studies.

Neutral delay differential equations, which have a wide range of applications in various fields such as applied mathematics, physics, engineering and ecology, are expressed as equations that include delays in both state variables

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and their time-dependent derivatives. Due to these wide areas of application, many studies have been conducted on these equations that have attracted the attention of scientists. We recommend that readers who want to learn this information in a broader context and learn the advantages of considering these equations in more detail examine the references in our study and the references included in them.

In 2014, Aguila-Camacho et al [1] presented a new lemma for the Caputo fractional derivative of a quadratic function, which allows the use of classic quadratic Lyapunov functions in many stability analyses of fractional order systems. Alkhazzan et al [2] investigated a new class of nonlinear fractional stochastic differential equations with fractional integrals and discussed existence, uniqueness, continuity of solutions and Ulam-Hyers stability with the help of Banach contraction theorem. Altun, investigated the asymptotic stability of Riemann-Liouville fractional neutral systems with variable delays by using the Lyapunov-Krasovskii functional in [3]. Altun and Tunç [4] discussed the asymptotic stability of a nonlinear fractional-order system with multiple variable delays. The authors proved a new result on the subject by means of Lyapunov-Krasovskii functional. Diethelm [5] introduced the Caputo derivative, which is close to the Riemann-Liouville derivative with different definitions of fractional derivatives. Graef et al [6] investigated Stability of nonlinear system of fractional-order volterra integro delay differential equations with Caputo fractional derivative. The authors of [6] proved some sufficient conditions for the stability of the zero solution of these equations with the help of Lyapunov and Razumikhin methods and gave explanatory examples of these conditions. Hristova and Tunç, obtained some new conditions for the stability of the solutions of the nonlinear Caputo fractional derivative and limited delay volterra integro-differential equations with the help of Lyapunov method in [7]. Kilbas et al [8] made an important contribution to the literature with a valuable work on the theory and applications of fractional differential equations. Krol, investigated the asymptotic properties of d-dimensional linear fractional differential equations with time delay in [9]. The author presented some necessary and sufficient conditions by using the inverse method. He also supported his work with two examples. Liu et al [10] discussed stability analysis of fractional nonlinear differential systems with Riemann-liouville derivative. The authors presented several sufficient conditions on asymptotic stability of fractional nonlinear systems. They supported their work with some examples. Moulai-Khatir discussed the asymptotic properties of some neutral delay differential equations, including the Riemann-Liouville fractional derivative by means of Lyapunov functions in [11]. He also supported his work with two examples. Podlubny [12] provided a valuable resource to the relevant literature in order to provide an overview of the solution methods of fractional differential equations and their applications. Tunç and Tunç proved some qualitative results of Caputo proportional integro differential equations [13] and volterra integro differential equations [14]. Stability analysis was performed on delayed bidirectional associative memory neural networks by Yang and Zhang [15] and on singular systems by Yiğit et al [16]. Yiğit and Tunç [17] proved the asymptotical stability of zero solution of a nonlinear fractional neutral system with unbounded delay by using Lyapunov-Krasovskii functionals. They also supported theır work with two examples. Some similar results were also obtained on the stability of certain type equations and systems by Yiğit [18], [19] and Zhang et al [20].

In this note, inspired by the above discussions and motivated by the paper of Kilbas et al [8], Moulai-Khatir [11] and Yiğit [18] and the references in these papers, we study the new asymptotic properties for solutions of fractional delay neutral differential equations. We use Lyapunov's direct method, which is widely used in practice, to obtain the properties we seek. By constructing new Lyapunov functions, we obtain three new asymptotic stability properties for three different equations. We draw the readers' attention to three examples that show the practical applicability of these properties we obtained theoretically, with their annotated solutions and graphs.

The next flow of our note is as follows. The second Section contains some definitions and lemmas. In the third Section, asymptotic stability conditions are obtained for some neutral delay differential equations. In the fourth Section, some application examples are given to show the applicability of the obtained conditions. The fifth Section is the conclusion section.

2. Preliminaries

We now present some definitions and lemmas to be used in the processes or applications for sufficient criteria to be obtained in the details of the our work.

Definition 2.1. [8] The Riemann-Liouville fractional derivative and integral of order α for a function $x(t)$ are defined as

$$
\begin{aligned}
{}_{t_0}D_t^{\alpha}x(t) =& \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\alpha-1}x(s)ds, \\
& 0 \le n-1 \le \alpha < n, n \in Z^+, \\
{}_{t_0}D_t^{-\alpha}x(t) =& \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}x(s)ds, \alpha > 0, t > t_0,
\end{aligned}
\tag{2.1}
$$

where Γ denotes the Gamma function and is defined as

$$\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1}e^{-s}ds.$$

Lemma 2.1. *[11] If $\beta > \alpha > 0$ and $x(t)$ is integrable, then*

$$ {}_{t_0}D_t^{\alpha}({}_{t_0}D_t^{-\beta}x(t)) = {}_{t_0}D_t^{\alpha-\beta}x(t) \tag{2.2}$$

is satisfied.

Lemma 2.2. *Assume that $x(t) \in R$ be a continuous and differentiable function. If the derivative of $x(t)$ is integrable, then the following relationship is satisfied as:*

$$0.5{}_{t_0}D_t^{\alpha}x^2(t) \le x(t){}_{t_0}D_t^{\alpha}x(t), \forall\alpha \in (0,1). \tag{2.3}$$

Proof. To claim inequality (2.3) is equivalent to prove only that

$$x(t){}_{t_0}D_t^{\alpha}x(t) - 0.5{}_{t_0}D_t^{\alpha}x^2(t) \ge 0, \forall\alpha \in (0,1). \tag{2.4}$$

According to Newton-Leibnitz formula, we have

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)ds = x(t_0) + {}_{t_0}D_t^{-1}x(t). \tag{2.5}$$

Substituting (2.5) into (2.1) and applying (2.2), we have

$$
\begin{aligned}
{}_{t_0}D_t^{\alpha}x(t) &= {}_{t_0}D_t^{\alpha}x(t_0) + {}_{t_0}D_t^{\alpha-1}x(t) \\
&= \frac{1}{\Gamma(1-\alpha)}[\frac{x(t_0)}{(t-t_0)^{\alpha}} + \int_{t_0}^{t}(t-s)^{-\alpha}x'(s)ds].
\end{aligned}
$$

From here, we get

$$x(t){}_{t_0}D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}[\frac{x(t)x(t_0)}{(t-t_0)^{\alpha}} + \int_{t_0}^{t}(t-s)^{-\alpha}x(t)x'(s)ds].$$

Also, a similar calculation shows that

$$0.5{}_{t_0}D_t^{\alpha}x^2(t) = \frac{1}{\Gamma(1-\alpha)}[\frac{x^2(t_0)}{2(t-t_0)^{\alpha}} + \int_{t_0}^{t}(t-s)^{-\alpha}x(s)x'(s)ds].$$

Therefore, inequality (2.4) can be rewritten as

$$\frac{1}{\Gamma(1-\alpha)}[\frac{x(t)x(t_0) - \frac{1}{2}x^2(t_0)}{(t-t_0)^{\alpha}} + \int_{t_0}^{t}(t-s)^{-\alpha}(x(t)-x(s))x'(s)ds] \ge 0. \tag{2.6}$$

Let us integrate by parts the second term of inequality (2.6), then we have

$$\int_{t_0}^{t}(t-s)^{-\alpha}(x(t)-x(s))x'(s)ds = \frac{(x(t)-x(t_0))^2}{2(t-t_0)^{\alpha}} + \frac{\alpha}{2}\int_{t_0}^{t}\frac{(x(t)-x(s))^2}{(t-s)^{\alpha+1}}ds.$$

Therefore, inequality (2.6) is reduced to the following form

$$\frac{1}{\Gamma(1-\alpha)}[\frac{x^2(t)}{2(t-t_0)^{\alpha}} + \frac{\alpha}{2}\int_{t_0}^{t}\frac{(x(t)-x(s))^2}{(t-s)^{\alpha+1}}ds] \ge 0. \tag{2.7}$$

This result shows that inequality (2.7) is clearly true. This completes the proof of Lemma 2.2. $\square$

3. Analysis of asymptotic stability conditions for fractional neutral equations

In this section, we will establish asymptotic stability criteria for some neutral equations with mixed delays. For this, we will use the Lyapunov's direct method and some inequalities. We will also give a brief evaluation of the equations we have examined at the end of this section.

Now, we describe a new fractional neutral differential equation with constant and variable delays as:

$$\begin{aligned}
{}_{t_0}D_t^{\alpha}[x(t)+ax(t-\sigma_1)+bx(t-\sigma_2)] &= -c(t)f(x(t))-d(t)g(x(t-\tau_1(t)))\\
&\quad -e(t)h(x(t-\tau_2(t)))-u(t)\int_{t-\delta_1}^{t}x(s)ds-s(t)\int_{t-\delta_2}^{t}x(s)ds,\\
{}_{t_0}D_t^{-(1-\alpha)}x(t) &= \vartheta(t), t\in[-\rho,0], \rho>0, \rho\in R,
\end{aligned} \tag{3.1}$$

for $\alpha\in(0,1)$ and for all $t\geq t_0+\rho$, where $c(t), d(t), e(t), u(t), s(t), f(x(t)), g(x(t))$ and $h(x(t))$ are continuous functions in their respective arguments, with $a+b<1$ and $f(0)=g(0)=h(0)=0$. The time variable delays $\tau_1(t)$ and $\tau_2(t)$ are continuous and differentiable functions and satisfying

$$0\leq\tau_1(t)\leq\tau_k \text{ and } \tau_1'(t)\leq\tau_K,$$
$$0\leq\tau_2(t)\leq\tau_n \text{ and } \tau_2'(t)\leq\tau_N,$$

where $\tau_k, \tau_n, \sigma_1, \sigma_2, \delta_1$ and δ_2 are real positive numbers and $\vartheta\in C([-\rho,0];R)$ with $\rho=max\{\tau_k,\tau_n,\sigma_1,\sigma_2,\delta_1,\delta_2\}$. Moreover, we assume that $f'(x(t)), g'(x(t))$ and $h'(x(t))$ are exist and continuous.

Now, we describe the operator N by:

$$N(t)=x(t)+ax(t-\sigma_1)+bx(t-\sigma_2),$$

then the equation (3.1) can be rewritten as in the form below:

$$\begin{aligned}
{}_{t_0}D_t^{\alpha}N(t) &= -c(t)f(x(t))-d(t)g(x(t-\tau_1(t)))\\
&\quad -e(t)h(x(t-\tau_2(t)))-u(t)\int_{t-\delta_1}^{t}x(s)ds-s(t)\int_{t-\delta_2}^{t}x(s)ds,\\
{}_{t_0}D_t^{-(1-\alpha)}x(t) &= \upsilon(t), t\in[-\rho,0], \rho>0, \rho\in R,
\end{aligned} \tag{3.2}$$

Before going into the details of our study, let us assume that the following sufficient criteria are met.

A. Assumptions

(A1) We assume that there exist positive numbers $c_j, d_j, e_j, u_j, s_j, f_j, g_j$ and $h_j, (j=1,2)$ and $\forall x\in R-\{0\}$, such that

i) $c_1\leq c(t)\leq c_2, d_1\leq d(t)\leq d_2, e_1\leq e(t)\leq e_2$, $u_1\leq u(t)\leq u_2, s_1\leq s(t)\leq s_2$

ii) $|f'(x)|\leq f_2, \frac{f(x)}{x}\geq f_1$

iii) $|g'(x)|\leq g_2, \frac{g(x)}{x}\geq g_1$

iv) $|h'(x)|\leq h_2, \frac{h(x)}{x}\geq h_1$

v) $2c_1f_1>\chi$

where

$$\chi=d_2+e_2+u_2+s_2+(c_2f_2^2+c_2+d_2+e_2+u_2+s_2)(a+b)+(\frac{d_2g_2^2}{1-\tau_K}+\frac{e_2h_2^2}{1-\tau_N}+\delta_1u_2+\delta_2s_2)(1+a+b).$$

Theorem 3.1. *We suppose that the assumptions (A1) are met, then the zero solution of fractional neutral differential equation (3.1) is asymptotically stable.*

Proof. Let us choose a new suitable Lyapunov function that can clearly be seen to be positive definite by

$$\begin{aligned} V(t) &= 0.5{}_{t_0}D_t^{\alpha-1}N^2(t)+\mu_1\int_{t-\sigma_1}^{t}x^2(s)ds+\mu_2\int_{t-\sigma_2}^{t}x^2(s)ds \\ &+ \lambda_1\int_{t-\tau_1(t)}^{t}x^2(s)ds+\lambda_2\int_{t-\tau_2(t)}^{t}x^2(s)ds \\ &+ \eta_1\int_{-\delta_1}^{0}\int_{t+s}^{t}x^2(\theta)d\theta ds+\eta_2\int_{-\delta_2}^{0}\int_{t+s}^{t}x^2(\theta)d\theta ds, \end{aligned}$$

where $\mu_1,\mu_2,\lambda_1,\lambda_2,\eta_1$ and η_2 are positive numbers.

In light of the fact that Lemma 2.1 and Lemma 2.2, by the time-derivative of $V(t)$ on the solution of equation (3.2), we can write the inequality given by

$$\begin{aligned} V'(t) &\le N(t){}_{t_0}D_t^{\alpha}N(t)+\mu_1x^2(t)-\mu_1x^2(t-\sigma_1)+\mu_2x^2(t)-\mu_2x^2(t-\sigma_2)+\lambda_1x^2(t)-\lambda_1(1-\tau_1'(t))x^2(t-\tau_1(t)) \\ &+ \lambda_2x^2(t)-\lambda_2(1-\tau_2'(t))x^2(t-\tau_2(t)) \\ &+ \delta_1\eta_1x^2(t)-\eta_1\int_{t-\delta_1}^{t}x^2(s)ds+\delta_2\eta_2x^2(t)-\eta_2\int_{t-\delta_2}^{t}x^2(s)ds \\ &\le (\mu_1+\mu_2+\lambda_1+\lambda_2+\delta_1\eta_1+\delta_2\eta_2)x^2(t)-\mu_1x^2(t-\sigma_1) \\ &- \mu_2x^2(t-\sigma_2)-\lambda_1(1-\tau_K)x^2(t-\tau_1(t))-\lambda_2(1-\tau_N)x^2(t-\tau_2(t)) \\ &- \eta_1\int_{t-\delta_1}^{t}x^2(s)ds-\eta_2\int_{t-\delta_2}^{t}x^2(s)ds-c(t)f(x(t))x(t) \\ &- d(t)g(x(t-\tau_1(t)))x(t)-e(t)h(x(t-\tau_2(t)))x(t) \\ &- u(t)\int_{t-\delta_1}^{t}x(s)dsx(t)-s(t)\int_{t-\delta_2}^{t}x(s)dsx(t)-ac(t)f(x(t))x(t-\sigma_1) \\ &- ad(t)g(x(t-\tau_1(t)))x(t-\sigma_1)-ae(t)h(x(t-\tau_2(t)))x(t-\sigma_1) \\ &- au(t)\int_{t-\delta_1}^{t}x(s)dsx(t-\sigma_1)-as(t)\int_{t-\delta_2}^{t}x(s)dsx(t-\sigma_1) \\ &- bc(t)f(x(t))x(t-\sigma_2)-bd(t)g(x(t-\tau_1(t)))x(t-\sigma_2) \\ &- be(t)h(x(t-\tau_2(t)))x(t-\sigma_2)-bu(t)\int_{t-\delta_1}^{t}x(s)dsx(t-\sigma_2) \\ &- bs(t)\int_{t-\delta_2}^{t}x(s)dsx(t-\sigma_2). \end{aligned}$$

With the help of the inequality $2|\varpi\nu|\le\varpi^2+\nu^2$ and the assumptions given in (A1), the following result is reached:

$$\begin{aligned} V'(t) &\le \frac{1}{2}(-2c_1f_1+d_2+e_2+c_2f_2^2(a+b)+2\mu_1+2\mu_2+2\lambda_1+2\lambda_2+2\delta_1\eta_1+2\delta_2\eta_2+u_2+s_2)x^2(t) \\ &+ \frac{1}{2}(-2\mu_1+a(c_2+d_2+e_2+u_2+s_2))x^2(t-\sigma_1) \\ &+ \frac{1}{2}(-2\mu_2+b(c_2+d_2+e_2+u_2+s_2))x^2(t-\sigma_2) \\ &+ \frac{1}{2}(-2\lambda_1(1-\tau_K)+d_2g_2^2(1+a+b))x^2(t-\tau_1(t)) \\ &+ \frac{1}{2}(-2\lambda_2(1-\tau_N)+e_2h_2^2(1+a+b))x^2(t-\tau_2(t)) \\ &+ \frac{1}{2}(-2\eta_1+u_2(1+a+b))\int_{t-\delta_1}^{t}x^2(s)ds \\ &+ \frac{1}{2}(-2\eta_2+s_2(1+a+b))\int_{t-\delta_2}^{t}x^2(s)ds. \end{aligned}$$

Let

$$
\begin{aligned}
2\mu_1 &= a(c_2+d_2+e_2+u_2+s_2), 2\mu_2 = b(c_2+d_2+e_2+u_2+s_2),\\
2\lambda_1 &= \frac{d_2g_2^2(1+a+b)}{(1-\tau_K)}, 2\lambda_2 = \frac{e_2h_2^2(1+a+b)}{(1-\tau_N)},\\
2\eta_1 &= u_2(1+a+b), 2\eta_2 = s_2(1+a+b).
\end{aligned}
$$

From here, we can deduce

$$
\begin{aligned}
V'(t) &\le \frac{1}{2}[(-2c_1f_1+d_2+e_2+u_2+s_2\\
&+ (a+b)(c_2f_2^2+c_2+d_2+e_2+u_2+s_2)\\
&+ (1+a+b)(\frac{d_2g_2^2}{1-\tau_K}+\frac{e_2h_2^2}{1-\tau_N}+\delta_1u_2+\delta_2s_2]x^2(t).
\end{aligned}
$$

Therefore, we have

$$V'(t) \le -m_0x^2(t),$$

where

$$m_0 = 2c_1f_1 - \chi > 0.$$

with

$$
\begin{aligned}
\chi &= d_2+e_2+u_2+s_2+(c_2f_2^2+c_2+d_2+e_2+u_2+s_2)(a+b)\\
&+ (\frac{d_2g_2^2}{1-\tau_K}+\frac{e_2h_2^2}{1-\tau_N}+\delta_1u_2+\delta_2s_2)(1+a+b).
\end{aligned}
$$

From here, we can deduce that the zero solution of fractional neutral differential equation (3.1) is asymptotically stable. This completes the proof. □

Moreover, if the integral terms given in system (3.1) are taken to be zero then the following neutral mixed delay equation is obtained. We define the neutral mixed delay equation as:

$$
\begin{aligned}
{}_{t_0}D_t^{\alpha}[x(t)+ax(t-\sigma_1)+bx(t-\sigma_2)] &= -c(t)f(x(t))-d(t)g(x(t-\tau_1(t)))-e(t)h(x(t-\tau_2(t))), \quad (3.3)\\
{}_{t_0}D_t^{-(1-\alpha)}x(t) &= \vartheta(t), t\in[-\rho,0], \rho>0, \rho\in R,
\end{aligned}
$$

for $\alpha \in (0,1)$ and for all $t \ge t_0+\rho$, where $c(t), d(t), e(t), f(x(t)), g(x(t))$ and $h(x(t))$ are continuous functions in their respective arguments, with $a+b<1$ and $f(0)=g(0)=h(0)=0$. The time variable delays $\tau_1(t)$ and $\tau_2(t)$ are continuous and differentiable functions and satisfying

$$
\begin{aligned}
0 \le \tau_1(t) \le \tau_k \text{ and } \tau_1'(t) \le \tau_K,\\
0 \le \tau_2(t) \le \tau_n \text{ and } \tau_2'(t) \le \tau_N,
\end{aligned}
$$

where τ_k, τ_n, σ_1 and σ_2 are real positive numbers and $\vartheta \in C([-\rho,0];R)$ with $\rho = max\{\tau_k,\tau_n,\sigma_1,\sigma_2\}$. Moreover, we assume that $f'(x(t)), g'(x(t))$ and $h'(x(t))$ are exist and continuous.

For simplicity, we describe the operator N by:

$$N(t) = x(t)+ax(t-\sigma_1)+bx(t-\sigma_2),$$

then the equation (3.3) can be rewritten as in the form below:

$$ {}_{t_0}D_t^{\alpha}N(t) = -c(t)f(x(t))-d(t)g(x(t-\tau_1(t)))-e(t)h(x(t-\tau_2(t))). \tag{3.4}$$

Before going into the details of proof of Theorem 3.2, let us assume that the following sufficient criteria are met.

B. Assumptions

(B1) We assume that there exist positive numbers c_j, d_j, e_j, f_j, g_j and h_j, $(j=1,2)$ and $\forall x \in R-\{0\}$, such that

i) $c_1 \le c(t) \le c_2, d_1 \le d(t) \le d_2, e_1 \le e(t) \le e_2$

ii) $|f'(x)| \le f_2, \frac{f(x)}{x} \ge f_1$

iii) $|g'(x)| \le g_2, \frac{g(x)}{x} \ge g_1$

iv) $|h'(x)| \le h_2, \frac{h(x)}{x} \ge h_1$

v) $2c_1 f_1 > \chi$

where

$$\chi = d_2 + e_2 + (c_2 f_2^2 + c_2 + d_2 + e_2)(a+b) + (\frac{d_2 g_2^2}{1-\tau_K} + \frac{e_2 h_2^2}{1-\tau_N})(1+a+b).$$

Theorem 3.2. *We suppose that the assumptions (B1) are met, then the zero solution of fractional neutral differential equation* (3.3) *is asymptotically stable.*

Proof. Let us choose a new suitable Lyapunov function that can clearly be seen to be positive definite by

$$\begin{aligned} V(t) &= 0.5{}_{t_0}D_t^{\alpha-1}N^2(t) + \mu_1 \int_{t-\sigma_1}^{t} x^2(s)ds + \mu_2 \int_{t-\sigma_2}^{t} x^2(s)ds \\ &+ \lambda_1 \int_{t-\tau_1(t)}^{t} x^2(s)ds + \lambda_2 \int_{t-\tau_2(t)}^{t} x^2(s)ds, \end{aligned}$$

where μ_1, μ_2, λ_1 and λ_2 are positive numbers.

In light of the fact that Lemma 2.1 and Lemma 2.2, by the time-derivative of $V(t)$ on the solution of equation (3.4), we can write the inequality given by

$$\begin{aligned} V'(t) &\le N(t){}_{t_0}D_t^{\alpha}N(t) + \mu_1 x^2(t) - \mu_1 x^2(t-\sigma_1) + \mu_2 x^2(t) \\ &- \mu_2 x^2(t-\sigma_2) + \lambda_1 x^2(t) - \lambda_1(1-\tau_1'(t))x^2(t-\tau_1(t)) \\ &+ \lambda_2 x^2(t) - \lambda_2(1-\tau_2'(t))x^2(t-\tau_2(t)) \\ &\le (\mu_1+\mu_2+\lambda_1+\lambda_2)x^2(t) - \mu_1 x^2(t-\sigma_1) \\ &- \mu_2 x^2(t-\sigma_2) - \lambda_1(1-\tau_K)x^2(t-\tau_1(t)) - \lambda_2(1-\tau_N)x^2(t-\tau_2(t)) \\ &- c(t)f(x(t))x(t) - d(t)g(x(t-\tau_1(t)))x(t) - e(t)h(x(t-\tau_2(t)))x(t) \\ &- ac(t)f(x(t))x(t-\sigma_1) - ad(t)g(x(t-\tau_1(t)))x(t-\sigma_1) \\ &- ae(t)h(x(t-\tau_2(t)))x(t-\sigma_1) - bc(t)f(x(t))x(t-\sigma_2) \\ &- bd(t)g(x(t-\tau_1(t)))x(t-\sigma_2) - be(t)h(x(t-\tau_2(t)))x(t-\sigma_2). \end{aligned}$$

With the help of the inequality $2|\varpi\nu| \le \varpi^2 + \nu^2$ and the assumptions given in (B1), the following result is reached:

$$\begin{aligned} V'(t) &\le \frac{1}{2}(-2c_1 f_1 + d_2 + e_2 + c_2 f_2^2(a+b) + 2\mu_1 + 2\mu_2 + 2\lambda_1 + 2\lambda_2)x^2(t) \\ &+ \frac{1}{2}(-2\mu_1 + a(c_2+d_2+e_2))x^2(t-\sigma_1) \\ &+ \frac{1}{2}(-2\mu_2 + b(c_2+d_2+e_2))x^2(t-\sigma_2) \\ &+ \frac{1}{2}(-2\lambda_1(1-\tau_K) + d_2 g_2^2(1+a+b))x^2(t-\tau_1(t)) \\ &+ \frac{1}{2}(-2\lambda_2(1-\tau_N) + e_2 h_2^2(1+a+b))x^2(t-\tau_2(t)). \end{aligned}$$

Let

$$\begin{aligned} 2\mu_1 &= a(c_2+d_2+e_2), 2\mu_2 = b(c_2+d_2+e_2), \\ 2\lambda_1 &= \frac{d_2 g_2^2(1+a+b)}{(1-\tau_K)}, 2\lambda_2 = \frac{e_2 h_2^2(1+a+b)}{(1-\tau_N)}. \end{aligned}$$

From here, we can deduce

$$V'(t) \leq \frac{1}{2}[(-2c_1f_1 + d_2 + e_2 + (a+b)(c_2f_2^2 + c_2 + d_2 + e_2) + (1+a+b)(\frac{d_2g_2^2}{1-\tau_K} + \frac{e_2h_2^2}{1-\tau_N}]x^2(t).$$

Therefore, we have

$$V'(t) \leq -m_1x^2(t),$$

where

$$m_1 = 2c_1f_1 - \chi > 0.$$

with

$$\chi = d_2 + e_2 + (c_2f_2^2 + c_2 + d_2 + e_2)(a+b) + (\frac{d_2g_2^2}{1-\tau_K} + \frac{e_2h_2^2}{1-\tau_N})(1+a+b).$$

From here, we can deduce that the zero solution of fractional neutral differential equation (3.3) is asymptotically stable. This completes the proof. □

Further, we define the following fractional neutral equation (3.3) with

$$e(t)h(x(t-\tau_2(t))) = 0, \tau_1(t) = \tau(t)$$

and

$$ax(t-\sigma_1) + bx(t-\sigma_2) = ax(t-\sigma),$$

$$\begin{aligned} {}_{t_0}D_t^{\alpha}[x(t) + ax(t-\sigma)] &= -c(t)f(x(t)) - d(t)g(x(t-\tau(t))), \\ {}_{t_0}D_t^{-(1-\alpha)}x(t) &= \vartheta(t), t \in [-\rho, 0], \rho > 0, \rho \in R, \end{aligned} \tag{3.5}$$

for $\alpha \in (0,1)$ and for all $t \geq t_0 + \rho$, where $c(t), d(t), f(x(t))$ and $g(x(t))$ are continuous functions in their respective arguments, with $a < 1$ and $f(0) = g(0) = 0$. The time variable delay $\tau(t)$ is continuous and differentiable function and satisfying

$$0 \leq \tau(t) \leq \tau_k \text{ and } \tau'(t) \leq \tau_K,$$

where τ_k and σ are real positive numbers and $\vartheta \in C([-\rho, 0]; R)$ with $\rho = max\{\tau_k, \sigma\}$. Moreover, we assume that $f'(x(t))$ and $g'(x(t))$ are exist and continuous.

For simplicity, we describe the operator M by:

$$M(t) = x(t) + ax(t-\sigma),$$

then the equation (3.5) can be rewritten as in the form below:

$$ {}_{t_0}D_t^{\alpha}M(t) = -c(t)f(x(t)) - d(t)g(x(t-\tau(t))). \tag{3.6}$$

Before going into the details of proof of Theorem 3.3, let us assume that the following sufficient criteria are met.

C. Assumptions

(C1) We assume that there exist positive numbers c_j, d_j, f_j and $g_j, (j = 1,2)$ and $\forall x \in R - \{0\}$, such that

i) $c_1 \leq c(t) \leq c_2, d_1 \leq d(t) \leq d_2$

ii) $|f'(x)| \leq f_2, \frac{f(x)}{x} \geq f_1$

iii) $|g'(x)| \leq g_2, \frac{g(x)}{x} \geq g_1$

iv) $2c_1f_1 > d_2 + a(c_2f_2^2 + c_2 + d_2) + \frac{d_2g_2^2(1+a)}{1-\tau_K}$

Theorem 3.3. *We suppose that the assumptions (C1) are met, then the zero solution of fractional neutral differential equation* (3.5) *is asymptotically stable.*

Proof. Let us choose a new suitable Lyapunov function that can clearly be seen to be positive definite by

$$V(t) = 0.5_{t_0}D_t^{\alpha-1}M^2(t) + \mu\int_{t-\sigma}^{t} x^2(s)ds + \lambda\int_{t-\tau(t)}^{t} x^2(s)ds,$$

where μ and λ are positive numbers.

In light of the fact that Lemma 2.1 and Lemma 2.2, by the time-derivative of $V(t)$ on the solution of equation (3.6), we can write the inequality given by

$$\begin{array}{rcl} V'(t) & \leq & M(t)_{t_0}D_t^{\alpha}M(t) + \mu x^2(t) - \mu x^2(t-\sigma) \\ & + & \lambda x^2(t) - \lambda(1-\tau_1'(t))x^2(t-\tau(t)) \\ & \leq & (\mu+\lambda)x^2(t) - \mu x^2(t-\sigma) - \lambda(1-\tau_K)x^2(t-\tau(t)) \\ & - & c(t)f(x(t))x(t) - d(t)g(x(t-\tau(t)))x(t) \\ & - & ac(t)f(x(t))x(t-\sigma) - ad(t)g(x(t-\tau(t)))x(t-\sigma). \end{array}$$

With the help of the inequality $2|\varpi\nu| \leq \varpi^2 + \nu^2$ and the assumptions given in (C1), the following result is reached:

$$\begin{array}{rcl} V'(t) & \leq & \frac{1}{2}(-2c_1f_1 + d_2 + ac_2f_2^2 + 2\mu + 2\lambda)x^2(t) \\ & + & \frac{1}{2}(-2\mu + a(c_2+d_2))x^2(t-\sigma) \\ & + & \frac{1}{2}(-2\lambda(1-\tau_K) + d_2g_2^2(1+a))x^2(t-\tau(t)). \end{array}$$

Let

$$\begin{aligned} 2\mu =& a(c_2+d_2), \\ 2\lambda =& \frac{d_2g_2^2(1+a)}{(1-\tau_K)}. \end{aligned}$$

From here, we can deduce

$$V'(t) \leq \frac{1}{2}[(-2c_1f_1 + d_2 + a(c_2f_2^2 + c_2 + d_2) + \frac{d_2g_2^2(1+a)}{1-\tau_K}]x^2(t).$$

Therefore, we have

$$V'(t) \leq -m_2x^2(t),$$

where

$$m_2 = 2c_1f_1 - d_2 - a(c_2f_2^2 + c_2 + d_2) - \frac{d_2g_2^2(1+a)}{1-\tau_K} > 0.$$

From here, we can deduce that the zero solution of fractional neutral differential equation (3.5) is asymptotically stable. This completes the proof. □

Remark 3.1. If $\tau(t) = r$ is taken, then the equation (3.5) we discussed turns into equation (1) discussed in article [11]. Similarly, if $bx(t-\sigma_2) = 0$,

$$d(t)g(x(t-\tau_1(t))) + e(t)h(x(t-\tau_2(t))) = b(t)f(x(t-r))$$

and

$$u(t)\int_{t-\delta_1}^{t} x(s)ds + s(t)\int_{t-\delta_2}^{t} x(s)ds = e(t)\int_{t-\delta}^{t} x(s)ds,$$

then the equation (3.1) we discussed turns into equation (2) discussed in article [11]. It is clear from here that the sufficient conditions we obtained include the conditions obtained in the article [11]. In addition, it should be noted that some delay terms in our study are variable dependent. This shows that our article is more general. Furthermore, in the Numerical applications section, i.e. in the next section examples that embody the sufficient conditions we have obtained theoretically and images of different initial conditions will be included.

4. Numerical applications

In this section, we will give examples and explanatory solutions showing that the sufficient conditions we have obtained for asymptotic stability are applicable in practice. We will also include graphs showing that asymptotic stability is achieved at different initial conditions with the help of MATLAB-Simulink.

Example 4.1. Let us define the following fractional delay differential equation, which is the special case of fractional neutral differential equation (3.1).

$$\begin{aligned} {}_{t_0}D_t^{\alpha}[x(t)+ax(t-\sigma_1)+bx(t-\sigma_2)] &= -c(t)f(x(t))-d(t)g(x(t-\tau_1(t))) \\ &-e(t)h(x(t-\tau_2(t)))-u(t)\int_{t-\delta_1}^{t}x(s)ds-s(t)\int_{t-\delta_2}^{t}x(s)ds. \end{aligned} \tag{4.1}$$

The values in this equation are as follows,

$$\begin{aligned} c_1 &= 8 \le c(t) = 8+\frac{1}{5+t^2} \le 8.2 = c_2, \\ d_1 &= 0.2 \le d(t) = 0.2+\frac{2}{5+t^2} \le 0.6 = d_2, \\ e_1 &= 0.3 \le e(t) = 0.3+\frac{1}{2+t^2} \le 0.8 = e_2, \\ u_1 &= 0.4 \le u(t) = 0.4+\frac{1}{10+t^2} \le 0.5 = u_2, \\ s_1 &= 0.6 \le s(t) = 0.6+\frac{1}{5+t^2} \le 0.8 = s_2, \\ a &= \frac{1}{100} < 1, b = \frac{3}{100} < 1, a+b = \frac{1}{25} < 1, \alpha \in (0,1), \\ 0 &\le \tau_1(t) = 0.15sin^2t \le 0.15 = \tau_k, \tau_1'(t) = 0.15sin2t \le 0.15 = \tau_K, \\ 0 &\le \tau_2(t) = 0.2sin^2t \le 0.2 = \tau_n, \tau_2'(t) = 0.2sin2t \le 0.2 = \tau_N, \\ f(x) &= 0.4x+\frac{x}{10+|x|}, g(x) = x+\frac{2x}{10+|x|}, h(x) = 0.7x+\frac{2x}{10+|x|}. \end{aligned}$$

It is clear that $f(0)=g(0)=h(0)=0$. Additionally, $\forall x \in R, 0 \le \frac{2}{10+|x|} \le 1$, we can deduce

$$\forall x \in R-\{0\}, \frac{f(x)}{x} \ge 0.4 = f_1, \frac{g(x)}{x} \ge 1 = g_1, \frac{h(x)}{x} \ge 0.7 = h_1.$$

Furthermore, we can get

$$\begin{aligned} |f'(x)| &= |0.4+\frac{10}{(10+|x|)^2}| \le 0.5 = f_2, \\ |g'(x)| &= |1+\frac{20}{(10+|x|)^2}| \le 1.2 = g_2, \\ |h'(x)| &= |0.7+\frac{20}{(10+|x|)^2}| \le 0.9 = h_2, \end{aligned}$$

With the help of a simple mathematical calculation, the following conclusion is reached.

$$\begin{aligned} &-2c_1f_1+d_2+e_2+u_2+s_2+(c_2f_2^2+c_2+d_2+e_2+u_2+s_2)(a+b) \\ &+(\frac{d_2g_2^2}{1-\tau_K}+\frac{e_2h_2^2}{1-\tau_N}+\delta_1u_2+\delta_2s_2)(1+a+b) = -0.91. \end{aligned}$$

From the solutions explained above, it can be seen that all criteria of Theorem 3.1. are met. Thus, the zero solution of fractional neutral differential equation (4.1) is asymptotically stable. Also, the graph showing the orbital behavior of the fractional neutral differential equation (4.1) is as follows.

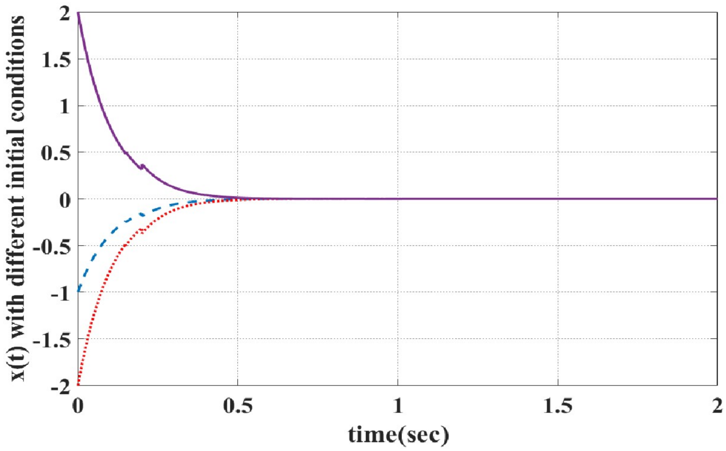

Figure 1. Orbital behavior of the fractional neutral differential equation (4.1).

Example 4.2. Let us define the following fractional delay differential equation, which is the special case of fractional neutral differential equation (3.3).

$$_{t_0}D_t^{\alpha}[x(t)+ax(t-\sigma_1)+bx(t-\sigma_2)] = -c(t)f(x(t)) - d(t)g(x(t-\tau_1(t))) - e(t)h(x(t-\tau_2(t))). \tag{4.2}$$

The values in this equation are as follows,

$$\begin{aligned}
c_1 &= 6 \le c(t) = 6 + \frac{1}{5+t^2} \le 6.2 = c_2,\\
d_1 &= 0.3 \le d(t) = 0.3 + \frac{3}{10+t^2} \le 0.6 = d_2,\\
e_1 &= 0.5 \le e(t) = 0.5 + \frac{3}{10+t^2} \le 0.8 = e_2,\\
a &= \frac{1}{100} < 1, b = \frac{3}{100} < 1, a+b = \frac{1}{25} < 1, \alpha \in (0,1),\\
0 &\le \tau_1(t) = 0.15sin^2t \le 0.15 = \tau_k, \tau_1'(t) = 0.15sin2t \le 0.15 = \tau_K,\\
0 &\le \tau_2(t) = 0.2sin^2t \le 0.2 = \tau_n, \tau_2'(t) = 0.2sin2t \le 0.2 = \tau_N,\\
f(x) &= 0.4x + \frac{x}{10+|x|}, g(x) = 0.9x + \frac{4x}{10+|x|}, h(x) = 0.7x + \frac{2x}{10+|x|}.
\end{aligned}$$

It is clear that $f(0) = g(0) = h(0) = 0$. Additionally, $\forall x \in R, 0 \le \frac{4}{10+|x|} \le 1$, we can deduce

$$\forall x \in R - \{0\}, \frac{f(x)}{x} \ge 0.4 = f_1, \frac{g(x)}{x} \ge 0.9 = g_1, \frac{h(x)}{x} \ge 0.7 = h_1.$$

Furthermore, we can get

$$\begin{aligned}
|f'(x)| &= |0.4 + \frac{10}{(10+|x|)^2}| \le 0.5 = f_2,\\
|g'(x)| &= |0.9 + \frac{40}{(10+|x|)^2}| \le 1.3 = g_2,\\
|h'(x)| &= |0.7 + \frac{20}{(10+|x|)^2}| \le 0.9 = h_2,
\end{aligned}$$

With the help of a simple mathematical calculation, the following conclusion is reached.

$$-2c_1f_1 + d_2 + e_2 + (c_2f_2^2 + c_2 + d_2 + e_2)(a+b) + (\frac{d_2g_2^2}{1-\tau_K} + \frac{e_2h_2^2}{1-\tau_N})(1+a+b) = -0.95.$$

From the solutions explained above, it can be seen that all criteria of Theorem 3.2 are met. Thus, the zero solution of fractional neutral differential equation (4.2) is asymptotically stable. Also, the graph showing the orbital behavior of the fractional neutral differential equation (4.2) is as follows.

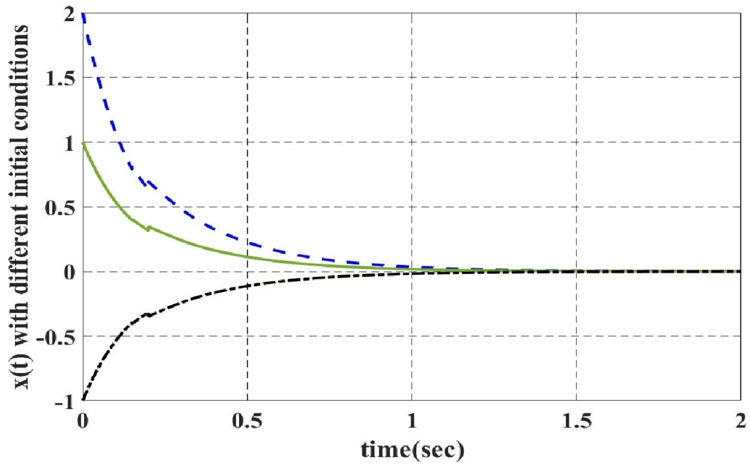

Figure 2. Orbital behavior of the fractional neutral differential equation (4.2).

Example 4.3. Let us define the following fractional delay differential equation, which is the special case of fractional neutral differential equation (3.5).

$$
{}_{t_0}D_t^{\alpha}[x(t) + ax(t-\sigma)] = -c(t)f(x(t)) - d(t)g(x(t-\tau(t))). \tag{4.3}
$$

The values in this equation are as follows,

$$
\begin{aligned}
c_1 &= 1 \leq c(t) = 1 + \frac{2}{5+t^2} \leq 1.4 = c_2, a = \frac{1}{50} < 1, \alpha \in (0,1), \\
d_1 &= 0.2 \leq d(t) = 0.2 + \frac{3}{10+t^2} \leq 0.5 = d_2, \\
0 &\leq \tau(t) = 0.15sin^2 t \leq 0.15 = \tau_k, \tau'(t) = 0.15sin2t \leq 0.15 = \tau_K, \\
f(x) &= 0.8x + \frac{4x}{10+|x|}, g(x) = 0.6x + \frac{4x}{10+|x|}.
\end{aligned}
$$

It is clear that $f(0) = g(0) = 0$. Additionally, $\forall x \in R, 0 \leq \frac{4}{10+|x|} \leq 1$, we can deduce

$$
\forall x \in R - \{0\}, \frac{f(x)}{x} \geq 0.8 = f_1, \frac{g(x)}{x} \geq 0.6 = g_1.
$$

Furthermore, we can get

$$
\begin{aligned}
|f'(x)| &= |0.8 + \frac{40}{(10+|x|)^2}| \leq 1.2 = f_2, \\
|g'(x)| &= |0.6 + \frac{40}{(10+|x|)^2}| \leq 1 = g_2,
\end{aligned}
$$

With the help of a simple mathematical calculation, the following conclusion is reached.

$$
-2c_1 f_1 + d_2 + a(c_2 f_2^2 + c_2 + d_2) + \frac{d_2 g_2^2(1+a)}{1-\tau_K} = -0.42.
$$

From the solutions explained above, it can be seen that all criteria of Theorem 3.3 are met. Thus, the zero solution of fractional neutral differential equation (4.3) is asymptotically stable. Also, the graph showing the orbital behavior of the fractional neutral differential equation (4.3) is as follows.

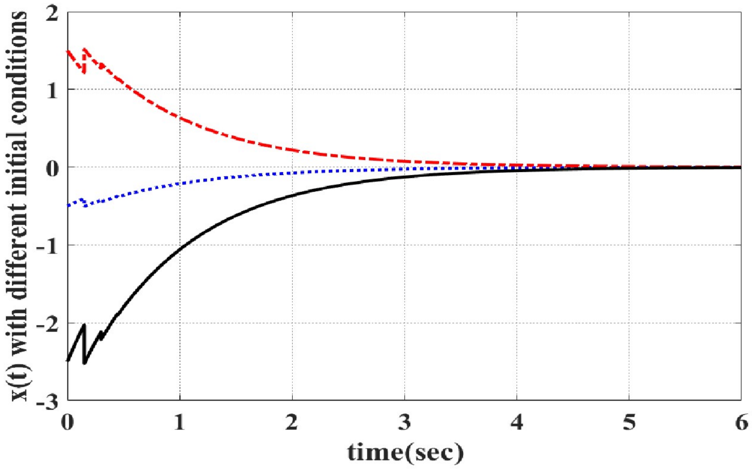

Figure 3. Orbital behavior of the fractional neutral differential equation (4.3).

Remark 4.1. When the solutions of the examples (examples 4.1, 4.2 and 4.3) given in this section are examined, the conditions that ensure stability of the zero solution of the equations discussed in a certain time interval and under different initial conditions can be easily seen. Graphs (figures 1, 2, 3) expressing these stability states are shown for different initial conditions.

In addition, it can be easily seen that the results of this study are more general when compared to the results of similar studies in the literature, especially the study we based on [11]. In this study, the time delay was taken as constant and examples showing the practical applicability of theoretical results were not supported by graphics. However, some delay terms of the equations in our study were taken as variable dependent and our examples showing the practical applicability of theoretical results were supported with graphs.

5. Conclusion

In this note, we have investigated the asymptotic stability of some fractional delay neutral differential equations of a certain type by applying three different Lyapunov functions. Also, we have obtained a new lemma of Riemann-Liouville derivative order of quadratic function. Based on the Lyapunov functions, some sufficient asymptotic stability conditions for these fractional delay neutral differential equations have been proved. Compared to the stability criteria in the relevant literature, our criteria are simple and applicable. To demonstrate the effectiveness of these criteria, we have given some examples with simulations (Figure1, Figure 2 and Figure 3). Theoretical findings, complemented by examples and graphical representations, provide meaningful insights into the orbital behavior of these equations. As a result, the obtained conditions extend and improve some criteria found in the relevant literature.

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Affiliations

ABDULLAH YİĞİT
ADDRESS: Mesut ÖZATA Anatolian High School, Governor Mithat Bey Neighborhood, Former Industrial Site, 65100, Van-Türkiye
E-MAIL: a-yigit63@hotmail.com
ORCID ID:0000-0002-0099-3095

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Hermite-Hadamard Inequalities for Generalized $\zeta-$Conformable Integrals Generated by Co-Ordinated Functions

Sümeyye Ermeydan Çiriş* and Hüseyin Yıldırım

Abstract
In this article, we introduce generalized $\zeta-$ conformable fractional integrals on co-ordinated functions and for the functions of two variables. Additionally, we derive a new Hermite-Hadamard inequality by utilizing the generalized Riemann-Liouville integrals, utilizing the generalized $\zeta-$conformable integral definition. Furthermore, we demonstrate some implications of the Hermite-Hadamard inequality and definitions introduced in this study. Consequently, we state and prove several related inequalities.

Keywords: *Co-ordinated functions, Generalized conformable integrals, Hermite-Hadamard inequality*

AMS Subject Classification (2020): *26A33; 41A55; 26D15*

****Corresponding author***

1. Introduction

Convex functions are a subject of extensive scientific research. One of the most outstanding inequality for convex functions was discovered by Hadamard in 1893. Additionally, many studies have focused on convex functions and the Hermite-Hadamard-type integral inequalities related to convex functions. Sarıkaya et al. define the general convex functions as the following inequality for $f:[\rho,\lambda]\subset\mathbb{R}\to\mathbb{R}$ on $[\rho,\lambda]$ in [1].

$$f\left(\theta\varphi\left(\tau\right)+\left(1-\theta\right)\varphi\left(\phi\right)\right)\leq\theta f\left(\varphi\left(\tau\right)\right)+\left(1-\theta\right)f\left(\varphi\left(\phi\right)\right). \tag{1.1}$$

Moreover, Cristescu defined and proved the Hermite-Hadamard-type integral inequality for general $\varphi-$convex functions in [2]. Then,

$$f\left(\frac{\varphi\left(\rho\right)+\varphi\left(\lambda\right)}{2}\right)\leq\frac{1}{\varphi\left(\rho\right)-\varphi\left(\lambda\right)}\int_{\varphi(\rho)}^{\varphi(\lambda)}f\left(z\right)dz\leq\frac{f\left(\varphi\left(\rho\right)\right)+f\left(\varphi\left(s\right)\right)}{2}. \tag{1.2}$$

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If f is a concave function, the inequality is reversed. This inequality is significant in fractional integrals and derivatives. There are many studies on the Hermite-Hadamard inequality in the literature (see, e.g., [3–5]).

Set et al. introduce $\varphi-$convex functions on co-ordinates, and they demonstrate their properties. Moreover, they obtain Hadamard type inequalities via $\varphi-$convex function on co-ordinates in [6]. We should give the following basic definition and basic theorem to use later.

Definition 1.1. [6] Let $\Delta := [\tau,\phi] \times [\theta,\mu] \subseteq [0,\infty) \times [0,\infty)$, $\tau < \phi$ and $\theta < \mu$. If $f : \Delta \to \mathbb{R}$ is said to be $\varphi-$convex on Δ for every two points $(\lambda,u), (\lambda,v), (y,u), (y,v) \in \Delta$ and $\rho, s \in [0,1]$. Then, we get

$$\begin{aligned} & f\left(\rho\varphi_1(\lambda) + (1-\rho)\varphi_1(y), s\varphi_2(u) + (1-s)\varphi_2(v)\right) \\ & \leq \rho s f\left(\varphi_1(\lambda), \varphi_2(u)\right) + \rho(1-s) f\left(\varphi_1(\lambda), \varphi_2(v)\right) \\ & + (1-\rho) s f\left(\varphi_1(y), \varphi_2(u)\right) + (1-\rho)(1-s) f\left(\varphi_1(y), \varphi_2(v)\right), \end{aligned} \tag{1.3}$$

for $\varphi_i : [\tau,\phi] \to [\theta,\mu]$, $i = 1,2$ be a continuous function. A function $f : \Delta \to \mathbb{R}$ is $\varphi-$convex function on Δ is called co-ordinated $\varphi-$convex on Δ if the partial mappings $f_{\varphi_2} : [\tau,\phi] \to \mathbb{R}$, $f_{\varphi_2}(u) = f(u,\varphi_2)$ and $f_{\varphi_1} : [\theta,\mu] \to \mathbb{R}$, $f_{\varphi_1}(v) = f(\varphi_1, v)$ are $\varphi-$convex for all $\tau \leq \varphi_2 \leq \phi$ and $\theta \leq \varphi_1 \leq \mu$.

Theorem 1.1. [6] *If $f : \Delta = [\tau,\phi] \times [\theta,\mu] \subset \mathbb{R}^2 \to \mathbb{R}$ is $\varphi-$convex on the co-ordinates on Δ with $f \in L[\Delta]$, Then, we obtain*

$$\begin{aligned} & f\left(\tfrac{\varphi(\tau)+\varphi(\phi)}{2}, \tfrac{\varphi(\theta)+\varphi(\mu)}{2}\right) \\ & \leq \tfrac{1}{(\varphi(\phi)-\varphi(\tau))(\varphi(\mu)-\varphi(\theta))} \int_{\varphi(\tau)}^{\varphi(\phi)} \int_{\varphi(\theta)}^{\varphi(\mu)} f(\rho,s)\, ds d\rho \\ & \leq \tfrac{f(\varphi(\tau),\varphi(\theta)) + f(\varphi(\tau),\varphi(\mu)) + f(\varphi(\phi),\varphi(\theta)) + f(\varphi(\phi),\varphi(\mu))}{4}. \end{aligned} \tag{1.4}$$

Furthermore, we demonstrate that the generalized $\zeta-$conformable fractional integration operator ${}_{\sigma}^{\alpha}J_{\tau+}^{\beta}$ is well-defined on $X_{\varrho}^{p}(\tau,\phi)$ for $p > \varrho$. We can write the following definition and theorem.

Definition 1.2. [7, 8] Let $\zeta(\lambda)$ be an increasing and positive monotone function on $[0,\infty)$. Furthermore, if we consider $\zeta^{'}(\lambda)$ is continuous on $[0,\infty)$ and $\zeta(0) = 0$, the space $X_{\zeta}^{p}(0,\infty)$ is the following form for $(1 \leq p < \infty)$,

$$\|f\|_{X_{\zeta}^{p}} = \left(\int_0^{\infty} |f(t)|^p \zeta^{'}(\lambda)\, dt\right)^{\frac{1}{p}} < \infty \tag{1.5}$$

and if we choose $p = \infty$,

$$\|f\|_{X_{\zeta}^{\infty}} = ess \sup_{1 \leq t < \infty} \left[f(t)\zeta^{'}(\lambda)\right]. \tag{1.6}$$

Additionally, If we take $\zeta(\lambda) = \lambda\ (1 \leq p < \infty)$ the space $X_{\zeta}^{p}(0,\infty)$, we have the $L_p[0,\infty)-$space. Moreover, if we take $\zeta(\lambda) = \frac{\lambda^{\sigma+1}}{\sigma+1}\ (1 \leq p < \infty,\ \sigma \geq 0)$ the space $X_{\zeta}^{p}(0,\infty)$, we have the $L_{p,\sigma}[0,\infty)-$space.

Definition 1.3. Let $f \in X_{\zeta}(0,\infty)$, ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0) = 0$. The left and right generalized conformable fractional integrals of order $\beta \in \mathbb{C}$, $\mathbb{R}(\beta) \geq 0$ and $\alpha > 0$,

$${}_{\zeta}^{\alpha}J_{\tau+}^{\beta} f(\lambda) = \tfrac{1}{\Gamma(\beta)} \int_{\tau}^{\lambda} \left[\tfrac{(\zeta(\lambda)-\zeta(\tau))^{\alpha} - (\zeta(\rho)-\zeta(\tau))^{\alpha}}{\alpha}\right]^{\beta-1} \tfrac{\zeta^{'}(\rho) f(\rho) d\rho}{(\zeta(\rho)-\zeta(\tau))^{1-\alpha}} \tag{1.7}$$

and

$${}_{\zeta}^{\alpha}J_{\phi-}^{\beta} f(\lambda) = \tfrac{1}{\Gamma(\beta)} \int_{\lambda}^{\phi} \left[\tfrac{(\zeta(\phi)-\zeta(\lambda))^{\alpha} - (\zeta(\phi)-\zeta(\rho))^{\alpha}}{\alpha}\right]^{\beta-1} \tfrac{\zeta^{'}(\rho) f(\rho) d\rho}{(\zeta(\phi)-\zeta(\rho))^{1-\alpha}}, \tag{1.8}$$

respectively.

Bozkurt et al. showed conformable derivatives and conformable integrals for the functions of two variables in [9]. Based on this article, we define the following the definition.

Definition 1.4. Let $f \in X_\zeta([\tau,\phi]\times[\theta,\mu])$ and $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta,\gamma \in \mathbb{C}$, $Re(\beta) > 0$, $Re(\gamma) > 0$. Meanwhile, ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0) = 0$. Then, we have the generalized $\zeta-$conformable integrals of order β,γ of $f(\rho,s)$,

$$\begin{aligned} &{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{\tau^+,\theta^+} \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_\theta^y\left[\frac{(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}-(\zeta(\rho)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\zeta(y)-\zeta(\theta))^{\alpha_2}-(\zeta(s)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \\ &\times\frac{\zeta'(\rho)}{(\zeta(\rho)-\zeta(\tau))^{1-\alpha_1}}\cdot\frac{\zeta'(s)}{(\zeta(s)-\zeta(\theta))^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho, \end{aligned} \tag{1.9}$$

$$\begin{aligned} &{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{\phi^-,\theta^+} \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_\theta^y\left[\frac{(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}-(\zeta(\phi)-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\zeta(y)-\zeta(\theta))^{\alpha_2}-(\zeta(s)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \\ &\times\frac{\zeta'(\rho)}{(\zeta(\phi)-\zeta(\rho))^{1-\alpha_1}}\cdot\frac{\zeta'(s)}{(\zeta(s)-\zeta(\theta))^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho, \end{aligned} \tag{1.10}$$

$$\begin{aligned} &{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{\tau^+,\mu^-} \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_y^\mu\left[\frac{(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}-(\zeta(\rho)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\zeta(\mu)-\zeta(y))^{\alpha_2}-(\zeta(\mu)-\zeta(s))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \\ &\times\frac{\zeta'(\rho)}{(\zeta(\rho)-\zeta(\tau))^{1-\alpha_1}}\cdot\frac{\zeta'(s)}{(\zeta(\mu)-\zeta(s))^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho, \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} &{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{\phi^-,\mu^-} \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_y^\mu\left[\frac{(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}-(\zeta(\phi)-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\zeta(\mu)-\zeta(y))^{\alpha_2}-(\zeta(\mu)-\zeta(s))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \\ &\times\frac{\zeta'(\rho)}{(\zeta(\phi)-\zeta(\rho))^{1-\alpha_1}}\cdot\frac{\zeta'(s)}{(\zeta(\mu)-\zeta(s))^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho. \end{aligned} \tag{1.12}$$

Remark 1.1. In here, when we get $\zeta(\lambda) = \frac{\lambda^{\sigma+1}}{(\sigma+1)^{\frac{1}{\alpha}}}$ in Definition 4, then we can write equations as follows,

$$\begin{aligned} &{}^{\alpha_1,\alpha_2}_{\sigma}J^{\beta,\gamma}_{\tau^+,\theta^+} \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_\theta^y\left[\frac{\left(\lambda^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}-\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\left[\frac{\left(y^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}-\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1} \\ &\times\frac{\rho^\sigma}{\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{1-\alpha_1}}\cdot\frac{s^\sigma}{\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho, \end{aligned} \tag{1.13}$$

$$\begin{aligned} &{}^{\alpha_1,\alpha_2}_{\sigma}J^{\beta,\gamma}_{\phi^-,\theta^+} \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_\theta^y\left[\frac{\left(\phi^{\sigma+1}-\lambda^{\sigma+1}\right)^{\alpha_1}-\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\left[\frac{\left(y^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}-\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1} \\ &\times\frac{\rho^\sigma}{\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{1-\alpha_1}}\cdot\frac{s^\sigma}{\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho, \end{aligned} \tag{1.14}$$

$$\begin{aligned} &{}^{\alpha_1,\alpha_2}_{\sigma}J^{\beta,\gamma}_{\tau^+,\mu^-} \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_y^\mu\left[\frac{\left(\lambda^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}-\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\left[\frac{\left(\mu^{\sigma+1}-y^{\sigma+1}\right)^{\alpha_2}-\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1} \\ &\times\frac{\rho^\sigma}{\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{1-\alpha_1}}\cdot\frac{s^\sigma}{\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho, \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} &{}^{\alpha_1,\alpha_2}_{\sigma}J^{\beta,\gamma}_{\phi^-,\mu^-} \\ &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_y^\mu\left[\frac{\left(\phi^{\sigma+1}-\lambda^{\sigma+1}\right)^{\alpha_1}-\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\left[\frac{\left(\mu^{\sigma+1}-y^{\sigma+1}\right)^{\alpha_2}-\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1} \\ &\times\frac{\rho^\sigma}{\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{1-\alpha_1}}\cdot\frac{s^\sigma}{\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho. \end{aligned} \tag{1.16}$$

Remark 1.2. If we take $\zeta(\lambda) = \lambda$ in Definition 3, then we have the following equations in [9]

$$\begin{aligned} {}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\theta^+} &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_\theta^y\left[\frac{(\lambda-\tau)^{\alpha_1}-(\rho-\tau)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(y-\theta)^{\alpha_2}-(s-\theta)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \\ &\times\frac{1}{(\rho-\tau)^{1-\alpha_1}}\cdot\frac{1}{(s-\theta)^{1-\alpha_2}}f(\rho,s)\,ds\,d\rho, \end{aligned} \tag{1.17}$$

$$
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\theta^+} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_\theta^y \left[\frac{(\phi-\lambda)^{\alpha_1}-(\phi-\rho)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(y-\theta)^{\alpha_2}-(s-\theta)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \times \frac{1}{(\phi-\rho)^{1-\alpha_1}}\cdot\frac{1}{(s-\theta)^{1-\alpha_2}} f(\rho,s)\,dsd\rho, \tag{1.18}
$$

$$
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\mu^-} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_y^\mu \left[\frac{(\lambda-\tau)^{\alpha_1}-(\rho-\tau)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\mu-y)^{\alpha_2}-(\mu-s)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \times \frac{1}{(\rho-\tau)^{1-\alpha_1}}\cdot\frac{1}{(\mu-s)^{1-\alpha_2}} f(\rho,s)\,dsd\rho, \tag{1.19}
$$

and

$$
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\mu^-} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_y^\mu \left[\frac{(\phi-\lambda)^{\alpha_1}-(\phi-\rho)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\left[\frac{(\mu-y)^{\alpha_2}-(\mu-s)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1} \times \frac{1}{(\phi-\rho)^{1-\alpha_1}}\cdot\frac{1}{(\mu-s)^{1-\alpha_2}} f(\rho,s)\,dsd\rho. \tag{1.20}
$$

Remark 1.3. [10] If we take $\zeta(\lambda)=\lambda$, $\alpha_1=1$ and $\alpha_2=1$ in Definition 3, then we get

$$
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\theta^+} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_\theta^y (\lambda-\rho)^{\beta-1}(y-s)^{\gamma-1} f(\rho,s)\,dsd\rho, \tag{1.21}
$$

$$
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\theta^+} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_\theta^y (\rho-\lambda)^{\beta-1}(y-s)^{\gamma-1} f(\rho,s)\,dsd\rho, \tag{1.22}
$$

$$
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\mu^-} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\tau^\lambda\int_y^\mu (\lambda-\rho)^{\beta-1}(s-y)^{\gamma-1} f(\rho,s)\,dsd\rho, \tag{1.23}
$$

and

$$
{}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\mu^-} = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_\lambda^\phi\int_y^\mu (\rho-\lambda)^{\beta-1}(s-y)^{\gamma-1} f(\rho,s)\,dsd\rho. \tag{1.24}
$$

Kiriş et al. studied Hermite-Hadamard inequalities for co-ordinated convex function via generalized conformable fractional integrals in [11]. Moreover, Çiriş and et al. defined generalized $\sigma-$conformable integrals by co-ordinated functions [12]. In addition, considering Definition 4, we can obtain Definition 5.

Definition 1.5. Let $f\in X_\zeta([\tau,\phi]\times[\theta,\mu])$ and $\alpha_1\neq 0$, $\alpha_2\neq 0$, $\beta,\gamma\in\mathbb{C}$, $Re(\beta)>0$, $Re(\beta)>0$. Now, ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0)=0$. In here,

$$
\left({}^{\alpha_1}_{\zeta}J^{\beta}_{\tau^+}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right) = \frac{1}{\Gamma(\beta)}\int_\tau^\lambda\left[\frac{(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}-(\zeta(\rho)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\frac{f\left(\rho,\frac{\theta+\mu}{2}\right)\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\tau))^{1-\alpha_1}},\ \lambda>\tau, \tag{1.25}
$$

$$
\left({}^{\alpha_1}_{\zeta}J^{\beta}_{\phi^-}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right) = \frac{1}{\Gamma(\beta)}\int_\lambda^\phi\left[\frac{(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}-(\zeta(\phi)-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\frac{f\left(\rho,\frac{\theta+\mu}{2}\right)\zeta'(\rho)d\rho}{(\zeta(\phi)-\zeta(\rho))^{1-\alpha_1}},\ \lambda<\phi, \tag{1.26}
$$

$$
\left({}^{\alpha_2}_{\zeta}J^{\gamma}_{\theta^+}f\right)\left(\frac{\tau+\phi}{2},y\right) = \frac{1}{\Gamma(\gamma)}\int_\theta^y\left[\frac{(\zeta(y)-\zeta(\theta))^{\alpha_2}-(\zeta(s)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\frac{f\left(\frac{\tau+\phi}{2},s\right)\zeta'(s)ds}{(\zeta(s)-\zeta(\theta))^{1-\alpha_2}},\ y>\theta, \tag{1.27}
$$

and

$$
\left({}^{\alpha_2}_{\zeta}J^{\gamma}_{\mu^-}f\right)\left(\frac{\tau+\phi}{2},y\right) = \frac{1}{\Gamma(\gamma)}\int_y^\mu\left[\frac{(\zeta(\mu)-\zeta(y))^{\alpha_2}-(\zeta(\mu)-\zeta(s))^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\frac{f\left(\frac{\tau+\phi}{2},s\right)\zeta'(s)ds}{(\zeta(\mu)-\zeta(s))^{1-\alpha_2}},\ y<\mu, \tag{1.28}
$$

we have equations.

Remark 1.4. If we take $\zeta(\lambda)=\frac{\lambda^{\sigma+1}}{(\sigma+1)^{\frac{1}{\alpha}}}$ in Definition 5, then we can write as the following,

$$
\left({}^{\alpha_1}_{\sigma}J^{\beta}_{\tau^+}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right) = \frac{1}{\Gamma(\beta)}\int_\tau^\lambda\left[\frac{\left(\lambda^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}-\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\frac{\rho^\sigma f\left(\rho,\frac{\theta+\mu}{2}\right)d\rho}{\left(\rho^{\sigma+1}-\tau^{\sigma+1}\right)^{1-\alpha_1}},\ \lambda>\tau, \tag{1.29}
$$

$$\begin{aligned}&\left({}^{\alpha_1}_{\sigma}J^{\beta}_{\phi-}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right)\\&=\frac{1}{\Gamma(\beta)}\int_{\lambda}^{\phi}\left[\frac{\left(\phi^{\sigma+1}-\lambda^{\sigma+1}\right)^{\alpha}-\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{\alpha_1}}{\alpha_1(\sigma+1)}\right]^{\beta-1}\frac{\rho^{\sigma}f\left(\rho,\frac{\theta+\mu}{2}\right)d\rho}{\left(\phi^{\sigma+1}-\rho^{\sigma+1}\right)^{1-\alpha_1}},\ \lambda<\phi,\end{aligned}\tag{1.30}$$

$$\begin{aligned}&\left({}^{\alpha_2}_{\sigma}J^{\gamma}_{\theta+}f\right)\left(\frac{\tau+\phi}{2},y\right)\\&=\frac{1}{\Gamma(\gamma)}\int_{\theta}^{y}\left[\frac{\left(y^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}-\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1}\frac{s^{\sigma}f\left(\frac{\tau+\phi}{2},s\right)ds}{\left(s^{\sigma+1}-\theta^{\sigma+1}\right)^{1-\alpha_2}},\ y>\theta,\end{aligned}\tag{1.31}$$

and

$$\begin{aligned}&\left({}^{\alpha_2}_{\sigma}J^{\gamma}_{\mu-}f\right)\left(\frac{\tau+\phi}{2},y\right)\\&=\frac{1}{\Gamma(\gamma)}\int_{y}^{\mu}\left[\frac{\left(\mu^{\sigma+1}-y^{\sigma+1}\right)^{\alpha_2}-\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{\alpha_2}}{\alpha_2(\sigma+1)}\right]^{\gamma-1}\frac{s^{\sigma}f\left(\frac{\tau+\phi}{2},s\right)ds}{\left(\mu^{\sigma+1}-s^{\sigma+1}\right)^{1-\alpha_2}},\ y<\mu.\end{aligned}\tag{1.32}$$

Remark 1.5. [11] If we get $\zeta(\lambda)=\lambda$ in Definition 4, then we obtain

$$\begin{aligned}\left({}^{\alpha_1}J^{\beta}_{\tau+}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right)&=\frac{1}{\Gamma(\beta)}\int_{\tau}^{\lambda}\left[\frac{(\lambda-\tau)^{\alpha_1}-(\rho-\tau)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\frac{f\left(\rho,\frac{\theta+\mu}{2}\right)d\rho}{(\rho-\tau)^{1-\alpha_1}},\ \lambda>\tau,\\\left({}^{\alpha_1}J^{\beta}_{\phi-}f\right)\left(\lambda,\frac{\theta+\mu}{2}\right)&=\frac{1}{\Gamma(\beta)}\int_{\lambda}^{\phi}\left[\frac{(\phi-\lambda)^{\alpha_1}-(\phi-\rho)^{\alpha_1}}{\alpha_1}\right]^{\beta-1}\frac{f\left(\rho,\frac{\theta+\mu}{2}\right)d\rho}{(\phi-\rho)^{1-\alpha_1}},\ \lambda<\phi,\\\left({}^{\alpha_2}J^{\gamma}_{\theta+}f\right)\left(\frac{\tau+\phi}{2},y\right)&=\frac{1}{\Gamma(\gamma)}\int_{\theta}^{y}\left[\frac{(y-\theta)^{\alpha_2}-(s-\theta)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\frac{f\left(\frac{\tau+\phi}{2},s\right)ds}{(s-\theta)^{1-\sigma}},\ y>\theta,\\\left({}^{\alpha_2}J^{\gamma}_{\mu-}f\right)\left(\frac{\tau+\phi}{2},y\right)&=\frac{1}{\Gamma(\gamma)}\int_{y}^{\mu}\left[\frac{(\mu-y)^{\alpha_2}-(\mu-s)^{\alpha_2}}{\alpha_2}\right]^{\gamma-1}\frac{f\left(\frac{\tau+\phi}{2},s\right)ds}{(\mu-s)^{1-\sigma}},\ y<\mu.\end{aligned}\tag{1.33}$$

In this study, we will examine Hermite-Hadamard inequalities for co-ordinated convex mappings by means of the generalized $\zeta-$conformable fractional integral operator. In addition, we are going to prove several important Theorems utilizing the Hermite-Hadamard inequality for generalized $\zeta-$conformable fractional integrals and by means of definitions which we define.

2. Hermite-Hadamard inequality

In this section, we will derive Hermite-Hadamard inequality for generalized $\zeta-$conformable fractional integrals.

Theorem 2.1. *Let $f\in X_{\zeta}([\tau,\phi])$ and f is $\varphi-$convex function. ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0)=0$. We obtain the inequalities as follows utilizing the generalized $\zeta-$conformable fractional integrals for $\mathbb{R}(\beta)>0$ and $\alpha_1\in(0,1]$,*

$$\begin{aligned}&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right)\\&\leq\frac{2^{\alpha_1\beta-1}.\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_2})^{\beta}\right]}\\&\times\left[{}^{\alpha_1}_{\zeta}J^{\beta}_{(w_1)+}f\left(w_2\right)+{}^{\alpha_1}_{\zeta}J^{\beta}_{(w_4)-}f\left(w_3\right)\right]\\&\leq\frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}.\end{aligned}\tag{2.1}$$

In here, we have

$$\frac{\zeta(\tau)+\zeta(\phi)}{2}-\frac{\zeta(1)\left(\zeta(\phi)-\zeta(\tau)\right)}{2}=w_1,$$

$$\frac{\zeta(\tau)+\zeta(\phi)}{2}-\frac{\zeta(0)\left(\zeta(\phi)-\zeta(\tau)\right)}{2}=w_2,$$

$$\frac{\zeta(\tau)+\zeta(\phi)}{2}+\frac{\zeta(0)\left(\zeta(\phi)-\zeta(\tau)\right)}{2}=w_3$$

and

$$\frac{\zeta(\tau)+\zeta(\phi)}{2}+\frac{\zeta(1)\left(\zeta(\phi)-\zeta(\tau)\right)}{2}=w_4.$$

Proof. By definition of $\varphi-$convex function, we get

$$\begin{aligned} & f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right) \\ & = f\left[\frac{1}{2}\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)+\frac{1}{2}\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\right] \\ & \leq \frac{1}{2}\left[f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)+f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\right] \\ & \leq \frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}. \end{aligned}$$

Here, we can write

$$\begin{aligned} f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right) & \leq \frac{1}{2}\left[f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)\right. \\ & \quad \left.+f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\right] \\ & \leq \frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}. \end{aligned} \tag{2.2}$$

Moreover, if we multiply $\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1} \frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}$ both of inequalities in (2.2) and we integrate from 0 to 1, then we acquire

$$\begin{aligned} & f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right)\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}} \\ & \leq \frac{1}{2}\left[\int_0^1 f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}\right. \\ & \quad \left.+\int_0^1 f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}\right] \\ & \leq \frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}. \end{aligned} \tag{2.3}$$

Furthermore, we get I_1 as the following,

$$I_1=\int_0^1 f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)\right)\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}}.$$

By changing the variable with,

$$\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi)=\zeta(u), \tag{2.4}$$

we have

$$\begin{aligned} I_1 & = \int_{w_1}^{w_2}\left(\frac{1-\left(\frac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}(\zeta(u)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}\frac{f(\zeta(u))\zeta'(u)du}{(\zeta(u)-\zeta(\tau))^{1-\alpha_1}} \\ & = \frac{2^{\alpha_1\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(u)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(u))\zeta'(u)du}{(\zeta(u)-\zeta(\tau))^{1-\alpha_1}} \\ & = \frac{2^{\alpha_1\beta}\Gamma(\beta)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\cdot{}_{\zeta}^{\alpha_1}J_{w_1^+}^{\beta}f(w_2). \end{aligned}$$

At the same way, if we take I_2 as the following,

$$I_2=\int_0^1 f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)\right)\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}},$$

By changing the variable with,

$$\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi)=\zeta(u), \tag{2.5}$$

we can write

$$\begin{aligned} I_2 & = \int_{w_3}^{w_4}\left(\frac{1-\left(\frac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}(\zeta(\phi)-\zeta(u))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}\frac{f(\zeta(u))\zeta'(u)du}{(\zeta(\phi)-\zeta(u))^{1-\alpha_1}} \\ & = \frac{2^{\alpha_1\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(u))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(u))\zeta'(u)du}{(\zeta(\phi)-\zeta(u))^{1-\alpha_1}} \\ & = \frac{2^{\alpha_1\beta}\Gamma(\beta)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\cdot{}_{\zeta}^{\alpha_1}J_{w_4^-}^{\beta}f(w_3). \end{aligned}$$

Additionally, if we get I_3 as the following, then we obtain

$$\begin{aligned} I_3 & = \int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)d\rho}{(1-\zeta(\rho))^{1-\alpha_1}} \\ & = \frac{1}{\beta\alpha_1^{\beta}}\left[\left(1-(1-\zeta(1))^{\alpha_1}\right)^{\beta}-\left(1-(1-\zeta(0))^{\alpha_1}\right)^{\beta}\right]. \end{aligned}$$

If we use I_1, I_2 and I_3 in (2.3) then, we have

$$\begin{aligned}
& f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right) \\
& \leq \frac{2^{\alpha_1\beta-1}.\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
& \times\left[{}_{\zeta}^{\alpha_1}J_{(w_1)^+}^{\beta} f(w_2)+{}_{\zeta}^{\alpha_1}J_{(w_4)^-}^{\beta} f(w_3)\right] \\
& \leq \frac{f(\zeta(\tau))+f(\zeta(\phi))}{2}.
\end{aligned}$$

The proof is completed. □

If we take $\zeta(\lambda)=\lambda$ in Theorem 2, then we obtain

$$f\left(\frac{\tau+\phi}{2}\right) \leq \frac{2^{\alpha_1\beta-1}.\Gamma(\beta+1)\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}}\left[{}^{\alpha_1}J_{\tau^+}^{\beta} f\left(\frac{\tau+\phi}{2}\right)+{}^{\alpha_1}J_{\phi^-}^{\beta} f\left(\frac{\tau+\phi}{2}\right)\right] \leq \frac{f(\tau)+f(\phi)}{2},$$

which is proved in [13].

Theorem 2.2. *Let $f \in X_{\zeta}([\tau,\phi]\times[\theta,\mu])$ and f is a $\zeta-$conformable co-ordinated $\varphi-$convex function. Moreover, ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0)=0$. Additionally, we have for $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta,\gamma \in \mathbb{C}$, $Re(\beta)>0$, $Re(\gamma)>0$,*

$$\begin{aligned}
& f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\
& \times\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right) \\
& \leq \frac{1}{4}\left[\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J_{w_1^+,q_1^+}^{\beta,\gamma} f\right)(w_2,q_2)\right. \\
& \quad+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\theta)-\zeta(\mu))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J_{w_1^+,q_4^-}^{\beta,\gamma} f\right)(w_2,q_3) \\
& \quad+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\theta)-\zeta(\mu))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J_{w_4^-,q_1^+}^{\beta,\gamma} f\right)(w_3,q_2) \\
& \quad\left.+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\theta)-\zeta(\mu))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J_{w_4^-,q_4^-}^{\beta,\gamma} f\right)(w_3,q_3)\right] \\
& \leq \frac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4} \\
& \times\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right).
\end{aligned} \tag{2.6}$$

In here, we write

$$\begin{aligned}
w_1 &= \frac{\zeta(\tau)+\zeta(\phi)}{2}-\frac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}, & q_1 &= \frac{\zeta(\theta)+\zeta(\mu)}{2}-\frac{\zeta(1)(\zeta(\mu)-\zeta(\theta))}{2} \\
w_2 &= \frac{\zeta(\tau)+\zeta(\phi)}{2}-\frac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}, & q_2 &= \frac{\zeta(\theta)+\zeta(\mu)}{2}-\frac{\zeta(0)(\zeta(\mu)-\zeta(\theta))}{2} \\
w_3 &= \frac{\zeta(\tau)+\zeta(\phi)}{2}+\frac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}, & q_3 &= \frac{\zeta(\theta)+\zeta(\mu)}{2}+\frac{\zeta(0)(\zeta(\mu)-\zeta(\theta))}{2} \\
w_4 &= \frac{\zeta(\tau)+\zeta(\phi)}{2}+\frac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}, & q_4 &= \frac{\zeta(\theta)+\zeta(\mu)}{2}+\frac{\zeta(1)(\zeta(\mu)-\zeta(\theta))}{2}.
\end{aligned}$$

Proof. We can write the equality

$$\begin{aligned}
& f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\
& = f\left[\frac{1}{4}\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi),\frac{1+\zeta(s)}{2}\zeta(\theta)+\frac{1-\zeta(s)}{2}\zeta(\mu)\right)\right. \\
& \quad+\frac{1}{4}\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi),\frac{1-\zeta(s)}{2}\zeta(\theta)+\frac{1+\zeta(s)}{2}\zeta(\mu)\right) \\
& \quad+\frac{1}{4}\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi),\frac{1+\zeta(s)}{2}\zeta(\theta)+\frac{1-\zeta(s)}{2}\zeta(\mu)\right) \\
& \quad\left.+\frac{1}{4}\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi),\frac{1-\zeta(s)}{2}\zeta(\theta)+\frac{1+\zeta(s)}{2}\zeta(\mu)\right)\right].
\end{aligned}$$

By Definition 1, we have

$$\begin{aligned}
&f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\
&\leq \tfrac{1}{4}\Big[f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1-\zeta(s)}{2}\zeta(\theta)+\tfrac{1+\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+f\left(\tfrac{1-\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1+\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+f\left(\tfrac{1-\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1+\zeta(\rho)}{2}\zeta(\phi),\tfrac{1-\zeta(s)}{2}\zeta(\theta)+\tfrac{1+\zeta(s)}{2}\zeta(\mu)\right)\Big]\\
&\leq \tfrac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4}.
\end{aligned}\tag{2.7}$$

If we multiply by $\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)}{(1-\zeta(s))^{1-\alpha_2}}$ both of the inequalities in (2.7) and integrating $[0,1]\times[0,1]$ with respect to s and ρ, then we obtain

$$\begin{aligned}
&f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\
&\quad\times\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\
&\leq\tfrac{1}{4}\Big[\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\beta-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\
&\quad\times f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+\left(\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\right)\\
&\quad\times f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1-\zeta(s)}{2}\zeta(\theta)+\tfrac{1+\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+\left(\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\right)\\
&\quad\times f\left(\tfrac{1-\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1+\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&\quad+\left(\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\right)\\
&\quad\times f\left(\tfrac{1-\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1+\zeta(\rho)}{2}\zeta(\phi),\tfrac{1-\zeta(s)}{2}\zeta(\theta)+\tfrac{1+\zeta(s)}{2}\zeta(\mu)\right)\Big]\\
&\leq\tfrac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4}\\
&\quad\times\left(\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\right).
\end{aligned}\tag{2.8}$$

By changing variables,

$$\begin{aligned}
&\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi)=\zeta(u),\\
&\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)=\zeta(v)
\end{aligned}\tag{2.9}$$

we have

$$\begin{aligned}
&\int_0^1\int_0^1\left(\tfrac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\tfrac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\tfrac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\
&\times f\left(\tfrac{1+\zeta(\rho)}{2}\zeta(\tau)+\tfrac{1-\zeta(\rho)}{2}\zeta(\phi),\tfrac{1+\zeta(s)}{2}\zeta(\theta)+\tfrac{1-\zeta(s)}{2}\zeta(\mu)\right)\\
&=\int_{w_1}^{w_2}\int_{q_1}^{q_2}\left(\tfrac{1-\left(\frac{2(\zeta(u)-\zeta(\tau))}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\tfrac{1-\left(\frac{2(\zeta(v)-\zeta(\theta))}{\zeta(\mu)-\zeta(\theta)}\right)^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\\
&\times\left(\tfrac{2^{\alpha_1}(\zeta(u)-\zeta(\tau))^{\alpha_1-1}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1}}\right)\left(\tfrac{2^{\alpha_2}(\zeta(v)-\zeta(\theta))^{\alpha_2}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2}}\right)\zeta'(u)\zeta'(v)f(\zeta(u),\zeta(v))\,dudv\\
&=\left(\tfrac{2}{\zeta(\phi)-\zeta(\tau)}\right)^{\alpha_1\beta}\left(\tfrac{2}{\zeta(\mu)-\zeta(\theta)}\right)^{\alpha_2\gamma}\int_{w_1}^{w_2}\int_{q_1}^{q_2}\left(\tfrac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(u)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\\
&\times\left(\tfrac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(v)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\tfrac{f(\zeta(u),\zeta(v))\zeta'(u)\zeta'(v)dudv}{(\zeta(\phi)-\zeta(\tau))^{1-\alpha_1}(\zeta(\mu)-\zeta(\theta))^{1-\alpha_2}}\\
&=\tfrac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_1^+,q_1^+}f\right)(w_2,q_2).
\end{aligned}\tag{2.10}$$

In the same way, we have

$$\begin{aligned}&\int_0^1\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\ &\times f\left(\frac{1+\zeta(\rho)}{2}\zeta(\tau)+\frac{1-\zeta(\rho)}{2}\zeta(\phi),\frac{1-\zeta(s)}{2}\zeta(\theta)+\frac{1+\zeta(s)}{2}\zeta(\mu)\right)\\ &=\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{w_1^+,q_4^-}f\right)(w_2,q_3),\end{aligned}\tag{2.11}$$

$$\begin{aligned}&\int_0^1\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\ &\times f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi),\frac{1+\zeta(s)}{2}\zeta(\theta)+\frac{1-\zeta(s)}{2}\zeta(\mu)\right)\\ &=\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{w_4^-,q_1^+}f\right)(w_3,q_2)\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}&\int_0^1\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\ &\times f\left(\frac{1-\zeta(\rho)}{2}\zeta(\tau)+\frac{1+\zeta(\rho)}{2}\zeta(\phi),\frac{1-\zeta(s)}{2}\zeta(\theta)+\frac{1+\zeta(s)}{2}\zeta(\mu)\right)\\ &=\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{w_4^-,q_4^-}f\right)(w_3,q_3).\end{aligned}\tag{2.13}$$

By simple calculations, we have

$$\begin{aligned}&\int_0^1\int_0^1\left(\frac{1-(1-\zeta(\rho))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\rho)}{(1-\zeta(\rho))^{1-\alpha_1}}\left(\frac{1-(1-\zeta(s))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(s)dsd\rho}{(1-\zeta(s))^{1-\alpha_2}}\\ &=\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right).\end{aligned}\tag{2.14}$$

By using (2.10)-(2.14) in (2.8), we obtain

$$\begin{aligned}&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\ &\times\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right)\\ &\le\frac{1}{4}\left[\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{w_1^+,q_1^+}f\right)(w_2,q_2)\right.\\ &\quad+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{w_1^+,q_4^-}f\right)(w_2,q_3)\\ &\quad+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{w_4^-,q_1^+}f\right)(w_3,q_2)\\ &\quad+\frac{2^{\alpha_1\beta}2^{\alpha_2\gamma}\Gamma(\beta)\Gamma(\gamma)}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left({}_{\zeta}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{w_4^-,q_4^-}f\right)(w_3,q_3)\\ &\le\frac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4}\\ &\times\left(\frac{(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}}{\alpha_1^{\beta}\beta}\right)\left(\frac{(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}}{\alpha_2^{\gamma}\gamma}\right).\end{aligned}$$

The proof is completed. □

Remark 2.1. [11] If $\zeta(\lambda)=\lambda$ in Theorem 3, we obtain

$$\begin{aligned}&f\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\left(\frac{1}{\alpha_1^{\beta}\alpha_2^{\gamma}\beta\gamma}\right)\le(\frac{2^{\alpha_1\beta-1}2^{\alpha_2\gamma-1}\Gamma(\beta)\Gamma(\gamma)}{(\phi-\tau)^{\alpha_1\beta}(\mu-\theta)^{\alpha_2\gamma}}\\ &\times\left[\left({}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\theta^+}f\right)\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+\left({}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\tau^+,\mu^-}f\right)\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right.\\ &\left.+\left({}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\theta^+}f\right)\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)+\left({}^{\alpha_1,\alpha_2}J^{\beta,\gamma}_{\phi^-,\mu^-}f\right)\left(\frac{\tau+\phi}{2},\frac{\theta+\mu}{2}\right)\right]\\ &\le\frac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}\left(\frac{1}{\alpha_1^{\beta}\alpha_2^{\gamma}\beta\gamma}\right).\end{aligned}$$

Remark 2.2. By choosing $\zeta(\lambda)=\lambda$, $\alpha_1=1$ and $\alpha_2=1$ in Theorem 3, we have

$$
\begin{aligned}
f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right) &\leq \tfrac{2^{\beta-1}2^{\gamma-1}\Gamma(\beta+1)\Gamma(\gamma+1)}{(\phi-\tau)^{\beta}(\mu-\theta)^{\gamma}} \\
&\times\Big[\left({}^{1,1}_{\zeta}J^{\beta,\gamma}_{\tau^+,\theta^+}f\right)\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+\left({}^{1,1}_{\zeta}J^{\beta,\gamma}_{\tau^+,\mu^-}f\right)\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right) \\
&+\left({}^{1,1}_{\zeta}J^{\beta,\gamma}_{\phi^-,\theta^+}f\right)\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+\left({}^{1,1}_{\zeta}J^{\beta,\gamma}_{\phi^-,\mu^-}f\right)\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\Big] \\
&\leq \tfrac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}.
\end{aligned}
$$

Remark 2.3. By choosing $\zeta(\lambda)=\lambda$, $\alpha_1=1$, $\alpha_2=1$, $\beta=1$ and $\gamma=1$ in Theorem 3, we have

$$
\begin{aligned}
&f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right) \\
&\leq \tfrac{1}{(\phi-\tau)(\mu-\theta)}\int_{\tau}^{\theta}\int_{\theta}^{\mu}f(t,s)\,dsdt \\
&\leq \tfrac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}.
\end{aligned}
$$

Theorem 2.3. *Let $f:\Delta=[\tau,\phi]\times[\theta,\mu]\subset\mathbb{R}^2\to\mathbb{R}$ for $0\leq\tau<\phi$ and $0\leq\theta<\mu$. Furthermore, f is $\zeta-$conformable co-ordinated $\varphi-$convex function and $f\in X_{\zeta}(\Delta)$. ζ be an increasing and positive monotone function on $[0,\infty)$ and also derivative ζ' be continuous on $[0,\infty)$ and $\zeta(0)=0$. We can obtain as the following inequality for $\alpha_1\neq0$, $\alpha_2\neq0$, $\beta,\gamma\in\mathbb{C}$, $Re(\beta)>0$, $Re(\gamma)>0$,*

$$
\begin{aligned}
&f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\
&\leq\tfrac{2^{\alpha_1\beta-2}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
&\times\left[{}^{\alpha_1}_{\zeta}J^{\beta}_{w_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right] \\
&+\tfrac{2^{\alpha_2\gamma-2}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]} \\
&\times\left[{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right] \\
&\leq\tfrac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
&\quad\times\left[{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_1^+,q_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_1,\alpha_2}_{\zeta}J^{\beta,\gamma}_{w_1^+,q_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right] \\
&+\tfrac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
&\quad\times\left[{}^{\alpha_1,\alpha_2}_{\zeta}J^{\gamma,\beta}_{w_4^-,q_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_1,\alpha_2}_{\zeta}J^{\gamma,\beta}_{w_4-,q_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right] \\
&\leq\tfrac{2^{\alpha_1\beta-3}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\Big[{}^{\alpha_1}_{\zeta}J^{\beta}_{w_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},q_1\right)+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},q_4\right) \\
&\qquad+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},q_1\right)+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},q_4\right)\Big] \\
&+\tfrac{2^{\alpha_2\gamma-3}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\Big[{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_1^+}f\left(w_1,\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_1^+}f\left(w_4,\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\
&\qquad+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_4^-}f\left(w_1,\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_4^-}f\left(w_4,\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\Big] \\
&\leq\tfrac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{4}.
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
w_1 &= \tfrac{\zeta(\tau)+\zeta(\phi)}{2}-\tfrac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}, & q_1 &= \tfrac{\zeta(\theta)+\zeta(\mu)}{2}-\tfrac{\zeta(1)(\zeta(\mu)-\zeta(\theta))}{2} \\
w_2 &= \tfrac{\zeta(\tau)+\zeta(\phi)}{2}-\tfrac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}, & q_2 &= \tfrac{\zeta(\theta)+\zeta(\mu)}{2}-\tfrac{\zeta(0)(\zeta(\mu)-\zeta(\theta))}{2} \\
w_3 &= \tfrac{\zeta(\tau)+\zeta(\phi)}{2}+\tfrac{\zeta(0)(\zeta(\phi)-\zeta(\tau))}{2}, & q_3 &= \tfrac{\zeta(\theta)+\zeta(\mu)}{2}+\tfrac{\zeta(0)(\zeta(\mu)-\zeta(\theta))}{2} \\
w_4 &= \tfrac{\zeta(\tau)+\zeta(\phi)}{2}+\tfrac{\zeta(1)(\zeta(\phi)-\zeta(\tau))}{2}, & q_4 &= \tfrac{\zeta(\theta)+\zeta(\mu)}{2}+\tfrac{\zeta(1)(\zeta(\mu)-\zeta(\theta))}{2}.
\end{aligned}
$$

Proof. If $f:\Delta\to\mathbb{R}$ is co-ordinated $\varphi-$convex function and also $g_{\zeta(\lambda)}:[q_1,q_4]\to\mathbb{R}$, $g_{\zeta(\lambda)}(\zeta(\rho))=f(\zeta(\lambda),\zeta(\rho))$ is

$\varphi-$convex on $[q_1, q_4]$ for all $w_1 \le \zeta(\lambda) \le w_4$, then, we obtain by utilizing Theorem 2,

$$\begin{aligned}
& g_{\zeta(\lambda)}\left(\frac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\
& \le \frac{2^{\alpha_2\gamma-1}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]} \\
& \times \left[{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_1^+} g_{\zeta(\lambda)}\left(q_2\right) + {}^{\alpha_2}_{\zeta}J^{\gamma}_{q_4^-} g_{\zeta(\lambda)}\left(q_3\right)\right] \\
& \le \frac{g_{\zeta(\lambda)}\zeta(\theta)+g_{\zeta(\lambda)}\zeta(\mu)}{2}.
\end{aligned}$$

Here, we can write

$$\begin{aligned}
& f\left(\zeta\left(\lambda\right), \frac{\zeta(\theta)+\zeta(\mu)}{2}\right) \\
& \le \frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]} \\
& \quad \times \left[\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right. \\
& \quad \left. + \int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right] \\
& \le \frac{f(\zeta(\lambda),\zeta(\theta))+f(\zeta(\lambda),\zeta(\mu))}{2}.
\end{aligned} \tag{2.15}$$

If we multiply both sides of (2.15) by

$$\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\lambda)}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}},$$

and if we integrate with respect to λ on $[w_1, w_2]$, then we get

$$\begin{aligned}
& \frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f\left(\zeta(\lambda),\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}} \\
& \le \frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\alpha_2^{\gamma}\alpha_1^{\beta}.\gamma\beta}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]} \\
& \times \left[\left[\int_{w_1}^{w_2}\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\right.\right. \\
& \left. \times \frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\rho d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right] \\
& + \left[\int_{w_1}^{w_2}\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\right. \\
& \left.\left. \times \frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\rho d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right]\right] \\
& \le \frac{2^{\alpha_1\beta-2}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\theta))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right. \\
& \quad \left. + \int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\mu))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right].
\end{aligned} \tag{2.16}$$

Similarly, if we multiply both sides of (2.15) by

$$\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{\zeta'(\lambda)}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}},$$

and if we integrate with respect to λ on $[w_3, w_4]$, then we can have

$$\begin{aligned}
&\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f\left(\zeta(\lambda),\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\\
&\leq\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\alpha_2^{\gamma}\alpha_1^{\beta}.\gamma\beta}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\times\left[\left[\int_{w_3}^{w_4}\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\right.\right.\\
&\times\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\rho d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right]\\
&+\int_{w_3}^{w_4}\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\\
&\times\left.\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\rho d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right]\right]\\
&\leq\frac{2^{\alpha_1\beta-2}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left[\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\theta))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right.\\
&\qquad\left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\mu))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right].
\end{aligned}\tag{2.17}$$

If f is co-ordinated $\varphi-$convex function, then $g_{\zeta(\rho)}:[w_1,w_4]\to\mathbb{R}$, $g_{\zeta(\rho)}(\zeta(\lambda))=f(\zeta(\lambda),\zeta(\rho))$ is $\varphi-$convex function, then, by Theorem 2, we get

$$\begin{aligned}
&g_{\zeta(\rho)}\left(\frac{\zeta(\tau)+\zeta(\phi)}{2}\right)\\
&\leq\frac{2^{\alpha_1\beta-1}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\times\left[{}_{\zeta}^{\alpha_1}J_{w_1^+}^{\beta}g_{\zeta(\rho)}(w_2)+{}_{\zeta}^{\alpha_1}J_{w_4^-}^{\beta}g_{\zeta(\rho)}(w_3)\right]\\
&\leq\frac{g_{\zeta(\rho)}\zeta(\tau)+g_{\zeta(\rho)}\zeta(\phi)}{2}.
\end{aligned}$$

Here, we can write,

$$\begin{aligned}
&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\\
&\leq\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\quad\times\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right.\\
&\quad\left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right]\\
&\leq\frac{f(\zeta(\tau),\zeta(\rho))+f(\zeta(\phi),\zeta(\rho))}{2}.
\end{aligned}\tag{2.18}$$

Moreover, by multiplying both side of (2.18) by

$$\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{\zeta'(\rho)}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}},$$

and by integrating with respect to ρ on $[q_1, q_2]$, then we obtain

$$
\begin{aligned}
&\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}} \int_{q_1}^{q_2} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \frac{f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}} \\
&\leq \frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\alpha_2^{\gamma}\alpha_1^{\beta}.\gamma\beta}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
&\times \left[\left[\int_{q_1}^{q_2}\int_{w_1}^{w_2} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\right.\right. \\
&\times \left.\frac{f(\zeta(\lambda);\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\lambda d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right] \\
&+ \left[\int_{q_1}^{q_2}\int_{w_3}^{w_4} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\right. \\
&\times \left.\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\lambda d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right]\right] \\
&\leq \frac{2^{\alpha_2\gamma-2}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}} \left[\int_{q_1}^{q_2} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \frac{f(\zeta(\tau),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right. \\
&\qquad \left.+\int_{q_1}^{q_2} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \frac{f(\zeta(\phi),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right].
\end{aligned}
\tag{2.19}
$$

Furthermore, if we multiply both sides of (2.18) by

$$
\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \frac{\zeta'(\rho)}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}},
$$

and if we integrate with respect to ρ on $[q_3, q_4]$, then we get

$$
\begin{aligned}
&\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}} \int_{q_3}^{q_4} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \frac{f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}} \\
&\leq \frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\alpha_2^{\gamma}\alpha_1^{\beta}.\gamma\beta}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]} \\
&\times \left[\left[\int_{q_3}^{q_4}\int_{w_1}^{w_2} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\right.\right. \\
&\times \left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\lambda d\rho}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right] \\
&+ \left[\int_{q_3}^{q_4}\int_{w_3}^{w_4} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\right. \\
&\times \left.\left.\frac{f(\zeta(\lambda),\zeta(\rho))\zeta'(\lambda)\zeta'(\rho)d\lambda d\rho}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right]\right] \\
&\leq \frac{2^{\alpha_2\gamma-2}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}} \left[\int_{q_3}^{q_4} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \frac{f(\zeta(\tau),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_1}}\right. \\
&\qquad \left.+\int_{q_3}^{q_4} \left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1} \frac{f(\zeta(\phi),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right].
\end{aligned}
\tag{2.20}
$$

In here, if we add the inequalities (2.16)-(2.20) and divide by 2, then we have

$$\begin{aligned}
&\frac{2^{\alpha_1\beta-2}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left[{}_{\zeta}^{\alpha_1}J_{w_1^+}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_1}J_{w_4^-}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right]\\
&+\frac{2^{\alpha_2\gamma-2}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left[{}_{\zeta}^{\alpha_2}J_{q_1^+}^{\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_2}J_{q_4^-}^{\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right]\\
&\le\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]}\\
&\times\left[{}_{\zeta}^{\alpha_1,\alpha_2}J_{w_1^+,q_1^+}^{\beta,\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_1,\alpha_2}J_{w_4^-,q_4^-}^{\beta,\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right]\\
&+\frac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}.(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\times\left[{}_{\zeta}^{\alpha_1,\alpha_2}J_{w_1^+,q_1^+}^{\beta,\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_1,\alpha_2}J_{w_4-,q_4^-}^{\beta,\gamma}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right]\\
&\le\frac{2^{\alpha_1\beta-3}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}}\left[{}_{\zeta}^{\alpha_1}J_{w_1^+}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},q_1\right)+{}_{\zeta}^{\alpha_1}J_{w_1^+}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},q_4\right)\right.\\
&\qquad\left.+{}_{\zeta}^{\alpha_1}J_{w_4^-}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},q_1\right)+{}_{\zeta}^{\alpha_1}J_{w_4^-}^{\beta}f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},q_4\right)\right]\\
&+\frac{2^{\alpha_2\gamma-3}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}}\left[{}_{\zeta}^{\alpha_2}J_{q_1^+}^{\gamma}f\left(w_1,\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_2}J_{q_1^+}^{\gamma}f\left(w_4,\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right.\\
&\qquad\left.+{}_{\zeta}^{\alpha_2}J_{q_4^-}^{\gamma}f\left(w_1,\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}_{\zeta}^{\alpha_2}J_{q_4^-}^{\gamma}f\left(w_4,\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right].
\end{aligned}\tag{2.21}$$

We give some results for special circumstances, if we get $\zeta(\lambda)=\frac{\zeta(\tau)+\zeta(\phi)}{2}$ on the left side of the (2.15) inequality, then we obtain,

$$\begin{aligned}
&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\
&\le\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]}\\
&\quad\times\left[\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f\left(\zeta(\lambda),\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right.\\
&\quad\left.+\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f\left(\zeta(\lambda),\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right].
\end{aligned}\tag{2.22}$$

Similarly, if we take $\zeta(\rho)=\frac{\zeta(\theta)+\zeta(\mu)}{2}$ on the left side of the (2.18) inequality, then we get,

$$\begin{aligned}
&f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\frac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\
&\le\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\quad\times\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right.\\
&\quad\left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f\left(\frac{\zeta(\tau)+\zeta(\phi)}{2},\zeta(\rho)\right)\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right].
\end{aligned}\tag{2.23}$$

If we do the necessary calculations for (2.22) and (2.23), we can obtain

$$
\begin{aligned}
&f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\\
&\leq \frac{2^{\alpha_1\beta-2}\Gamma(\beta+1)\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\times\left[{}^{\alpha_1}_{\zeta}J^{\beta}_{w_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_1}_{\zeta}J^{\beta}_{w_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right]\\
&+\frac{2^{\alpha_2\gamma-2}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]}\\
&\times\left[{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_1^+}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)+{}^{\alpha_2}_{\zeta}J^{\gamma}_{q_4^-}f\left(\tfrac{\zeta(\tau)+\zeta(\phi)}{2},\tfrac{\zeta(\theta)+\zeta(\mu)}{2}\right)\right].
\end{aligned}\tag{2.24}
$$

The inequality in (2.24)is the first inequality of Theorem 4.

Finally, if we get $\zeta(\rho)=\zeta(\theta)$ on the right-hand side of the (2.18) which we get by using the second inequality in (2.1), then we obtain

$$
\begin{aligned}
&\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\times\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\theta))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right.\\
&\left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\theta))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right]\\
&\leq\frac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\phi),\zeta(\theta))}{2}.
\end{aligned}\tag{2.25}
$$

In same way, if we take $\zeta(\rho)=\zeta(\mu)$ in (2.18), then we can write

$$
\begin{aligned}
&\frac{2^{\alpha_1\beta-1}\beta\alpha_1^{\beta}}{(\zeta(\phi)-\zeta(\tau))^{\alpha_1\beta}\left[(1-(1-\zeta(1))^{\alpha_1})^{\beta}-(1-(1-\zeta(0))^{\alpha_1})^{\beta}\right]}\\
&\times\left[\int_{w_1}^{w_2}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\lambda)-\zeta(\tau))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\mu))\zeta'(\lambda)d\lambda}{(\zeta(\lambda)-\zeta(\tau))^{1-\alpha_1}}\right.\\
&\left.+\int_{w_3}^{w_4}\left(\frac{\left(\frac{\zeta(\phi)-\zeta(\tau)}{2}\right)^{\alpha_1}-(\zeta(\phi)-\zeta(\lambda))^{\alpha_1}}{\alpha_1}\right)^{\beta-1}\frac{f(\zeta(\lambda),\zeta(\mu))\zeta'(\lambda)d\lambda}{(\zeta(\phi)-\zeta(\lambda))^{1-\alpha_1}}\right]\\
&\leq\frac{f(\zeta(\tau),\zeta(\mu))+f(\zeta(\phi),\zeta(\mu))}{2}.
\end{aligned}\tag{2.26}
$$

In a similar way, if we take $\zeta(\lambda)=\zeta(\tau)$ on the right-hand side of the (2.15) inequality, then we have

$$
\begin{aligned}
&\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]}\\
&\times\left[\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\beta-1}\frac{f(\zeta(\tau),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right.\\
&\left.+\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\beta-1}\frac{f(\zeta(\tau),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right]\\
&\leq\frac{f(\zeta(\tau),\zeta(\theta))+f(\zeta(\tau),\zeta(\mu))}{2}.
\end{aligned}\tag{2.27}
$$

Moreover, if we take $\zeta(\lambda)=\zeta(\phi)$ in (2.15), then we get

$$
\begin{aligned}
&\frac{2^{\alpha_2\gamma-1}\gamma\alpha_2^{\gamma}}{(\zeta(\mu)-\zeta(\theta))^{\alpha_2\gamma}[(1-(1-\zeta(1))^{\alpha_2})^{\gamma}-(1-(1-\zeta(0))^{\alpha_2})^{\gamma}]}\\
&\times\left[\int_{q_1}^{q_2}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\rho)-\zeta(\theta))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\phi),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\rho)-\zeta(\theta))^{1-\alpha_2}}\right.\\
&\left.+\int_{q_3}^{q_4}\left(\frac{\left(\frac{\zeta(\mu)-\zeta(\theta)}{2}\right)^{\alpha_2}-(\zeta(\mu)-\zeta(\rho))^{\alpha_2}}{\alpha_2}\right)^{\gamma-1}\frac{f(\zeta(\phi),\zeta(\rho))\zeta'(\rho)d\rho}{(\zeta(\mu)-\zeta(\rho))^{1-\alpha_2}}\right]\\
&\leq\frac{f(\zeta(\phi),\zeta(\theta))+f(\zeta(\phi),\zeta(\mu))}{2}.
\end{aligned}\tag{2.28}
$$

When we make the necessary calculations for (2.25), (2.26), (2.27) and (2.28), then we obtain the $4th$ inequality of Theorem 4. $\square$

Corollary 2.1. *[11] If $\zeta(\lambda)=\lambda$ in Theorem 4, we acquire,*

$$\begin{aligned}
&f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\\
&\leq \tfrac{2^{\alpha_1\beta-2}\Gamma(\beta+1)\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}}\left[{}^{\alpha_1}J_{\tau+}^{\beta}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+{}^{\alpha_1}J_{\phi-}^{\beta}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\right]\\
&+\tfrac{2^{\alpha_2\gamma-2}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\mu-\theta)^{\alpha_2\gamma}}\left[{}^{\alpha_2}J_{\theta+}^{\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+{}^{\alpha_2}J_{\mu-}^{\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\right]\\
&\leq \tfrac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}.(\mu-\theta)^{\alpha_2\gamma}}\\
&\times\left[{}^{\alpha_1,\alpha_2}J_{\tau^+,\theta^+}^{\beta,\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+{}^{\alpha_1,\alpha_2}J_{\tau^+,\mu^-}^{\beta,\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\right]
\end{aligned}\tag{2.29}$$

$$\begin{aligned}
&+\tfrac{2^{\alpha_2\gamma-1}.2^{\alpha_1\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)\alpha_2^{\gamma}\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}.(\mu-\theta)^{\alpha_2\gamma}}\\
&\times\left[{}^{\alpha_1,\alpha_2}J_{\phi^-,\theta^+}^{\beta,\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+{}^{\alpha_1,\alpha_2}J_{\phi^-,\mu^-}^{\beta,\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\right]\\
&\leq \tfrac{2^{\alpha_1\beta-3}\Gamma(\beta+1)\alpha_1^{\beta}}{(\phi-\tau)^{\alpha_1\beta}}\left[{}^{\alpha_1}J_{\tau+}^{\beta}f\left(\tfrac{\tau+\phi}{2},\theta\right)+{}^{\alpha_1}J_{\tau+}^{\beta}f\left(\tfrac{\tau+\phi}{2},\mu\right)\right.\\
&\qquad\left.+{}^{\alpha_1}J_{\phi-}^{\beta}f\left(\tfrac{\tau+\phi}{2},\theta\right)+{}^{\alpha_1}J_{\phi-}^{\beta}f\left(\tfrac{\tau+\phi}{2},\mu\right)\right]\\
&+\tfrac{2^{\alpha_2\gamma-3}\Gamma(\gamma+1)\alpha_2^{\gamma}}{(\mu-\theta)^{\alpha_2\gamma}}\left[{}^{\alpha_2}J_{\theta+}^{\gamma}f\left(\tau,\tfrac{\theta+\mu}{2}\right)+{}^{\alpha_2}J_{\theta+}^{\gamma}f\left(\phi,\tfrac{\theta+\mu}{2}\right)\right.\\
&\qquad\left.+{}^{\alpha_2}J_{\mu-}^{\gamma}f\left(\tau,\tfrac{\theta+\mu}{2}\right)+{}^{\alpha_2}J_{\mu-}^{\gamma}f\left(\phi,\tfrac{\theta+\mu}{2}\right)\right]\\
&\leq \tfrac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}.
\end{aligned}\tag{2.30}$$

Corollary 2.2. *By choosing $\zeta(\lambda)=\lambda$, $\alpha_1=1$ and $\alpha_2=1$ in Theorem 4, we write as the following inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\leq\tfrac{2^{\beta-2}\Gamma(\beta+1)}{(\phi-\tau)^{\beta}}\left[{}^{1}J_{\tau+}^{\beta}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+{}^{1}J_{\phi-}^{\beta}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\right]\\
&\qquad+\tfrac{2^{\gamma-2}\Gamma(\gamma+1)}{(\mu-\theta)^{\gamma}}\left[{}^{1}J_{\theta+}^{\beta}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+{}^{1}J_{\mu-}^{\beta}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\right]\\
&\leq\tfrac{2^{\gamma-1}.2^{\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)}{(\phi-\tau)^{\beta}.(\mu-\theta)^{\gamma}}\left[{}^{1,1}J_{\tau^+,\theta^+}^{\beta,\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+{}^{1,1}J_{\tau^+,\mu^-}^{\beta,\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\right]\\
&+\tfrac{2^{\gamma-1}.2^{\beta-1}.\Gamma(\gamma+1)\Gamma(\beta+1)}{(\phi-\tau)^{\beta}.(\mu-\theta)^{\gamma}}\left[{}^{1,1}J_{\phi^-,\theta^+}^{\beta,\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)+{}^{1,1}J_{\phi^-,\mu^-}^{\beta,\gamma}f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\right]\\
&\leq\tfrac{2^{\beta-3}\Gamma(\beta+1)}{(\phi-\tau)^{\beta}}\left[{}^{1}J_{\tau+}^{\beta}f\left(\tfrac{\tau+\phi}{2},\theta\right)+{}^{1}J_{\tau+}^{\beta}f\left(\tfrac{\tau+\phi}{2},\mu\right)\right.\\
&\qquad\left.+{}^{1}J_{\phi-}^{\beta}f\left(\tfrac{\tau+\phi}{2},\theta\right)+{}^{1}J_{\phi-}^{\beta}f\left(\tfrac{\tau+\phi}{2},\mu\right)\right]\\
&\qquad+\tfrac{2^{\gamma-3}\Gamma(\gamma+1)}{(\mu-\theta)^{\gamma}}\left[{}^{1}J_{\theta+}^{\gamma}f\left(\tau,\tfrac{\theta+\mu}{2}\right)+{}^{1}J_{\theta+}^{\gamma}f\left(\phi,\tfrac{\theta+\mu}{2}\right)\right.\\
&\qquad\left.+{}^{1}J_{\mu-}^{\gamma}f\left(\tau,\tfrac{\theta+\mu}{2}\right)+{}^{1}J_{\mu-}^{\gamma}f\left(\phi,\tfrac{\theta+\mu}{2}\right)\right]\\
&\leq\tfrac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}.
\end{aligned}\tag{2.31}$$

Corollary 2.3. *By choosing $\zeta(\lambda)=\lambda$, $\beta=1$, $\gamma=1$, $\alpha_1=1$ and $\alpha_2=1$ in Theorem 4, we write as the following inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&f\left(\tfrac{\tau+\phi}{2},\tfrac{\theta+\mu}{2}\right)\\
&\leq\tfrac{1}{2(\phi-\tau)}\left[\int_{\tau}^{\phi}f\left(t,\tfrac{\theta+\mu}{2}\right)dt\right]+\tfrac{1}{2(\mu-\theta)}\left[\int_{\phi}^{\mu}f\left(\tfrac{\tau+\phi}{2},s\right)ds\right]\\
&\leq\tfrac{1}{(\phi-\tau).(\mu-\theta)}\left[\int_{\tau}^{\theta}\int_{\theta}^{\mu}f(t,s)\,dsdt\right]\\
&\leq\tfrac{1}{4(\phi-\tau)}\left[{}^{1}J_{\tau+}^{1}f\left(\tfrac{\tau+\phi}{2},\theta\right)+{}^{1}J_{\tau+}^{1}f\left(\tfrac{\tau+\phi}{2},\mu\right)\right.\\
&\qquad\left.+{}^{1}J_{\phi-}^{1}f\left(\tfrac{\tau+\phi}{2},\theta\right)+{}^{1}J_{\phi-}^{1}f\left(\tfrac{\tau+\phi}{2},\mu\right)\right]\\
&+\tfrac{1}{4(\mu-\theta)}\left[{}^{1}J_{\theta+}^{1}f\left(\tau,\tfrac{\theta+\mu}{2}\right)+{}^{1}J_{\theta+}^{1}f\left(\phi,\tfrac{\theta+\mu}{2}\right)\right.\\
&\qquad\left.+{}^{1}J_{\mu-}^{1}f\left(\tau,\tfrac{\theta+\mu}{2}\right)+{}^{1}J_{\mu-}^{1}f\left(\phi,\tfrac{\theta+\mu}{2}\right)\right]\\
&\leq\tfrac{f(\tau,\theta)+f(\tau,\mu)+f(\phi,\theta)+f(\phi,\mu)}{4}.
\end{aligned}\tag{2.32}$$

3. Conclusion

There are many studies on Hermite-Hadamard inequalities and fractional integrals [14–20]. In this study, we derive the Hermite-Hadamard inequality for generalized $\zeta-$conformable fractional integrals. Moreover, we derive two distinct definitions for these integrals: one for functions with two variables and another for co-ordinated functions. Expanding on these definitions, we highlight several significant findings and illustrate their implications and applications. Furthermore, we discuss important consequences within the broader mathematical context.

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Affiliations

SÜMEYYE ERMEYDAN ÇİRİŞ
ADDRESS: Kahramanmaraş Sütçü İmam University, Dept. of Mathematics, 46000, Kahramanmaraş-Türkiye
E-MAIL: sumeyye_ermeydan@hotmail.com
ORCID ID:0009-0000-2472-5311

HÜSEYIN YILDIRIM
ADDRESS: Kahramanmaraş Sütçü İmam University, Dept. of Mathematics, 46000, Kahramanmaraş-Türkiye
E-MAIL: hyildir@ksu.edu.tr
ORCID ID:0000-0001-8855-9260

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Enhancing Generalized Interpolative Contraction Through Simulation Functions

Ekber Girgin

Abstract

In the present manuscript, we elucidate a comprehensive framework for the generalized interpolative $\alpha-(\psi,\varphi)_Z-$contractive mapping, thereby extending the foundational theoretical constructs to augment its utility within the domain of advanced mathematical analysis. The investigation encompasses a meticulous examination of fixed point results within the context of non-Archimedean modular metric spaces, which are characterized by their distinctive structural properties that diverge from those of conventional metric spaces. Moreover, we apply the results attained to substantiate the existence and uniqueness of solutions pertaining to nonlinear Fredholm integral equations. This aspect of our inquiry underscores the practical implications of our theoretical advancements and provides a rigorous framework for the resolution of complex integral equations through the principles of established contractive mappings.

Keywords: *Admissible mappings, Fredholm integral equations, Interpolative contractions, Simulation functions*

AMS Subject Classification (2020): *47H10; 54H25; 37C25*

1. Introduction

This study designates the symbol $\mathbf{N}$ to represent the set of all positive natural numbers. Additionally, the sets of positive and non-negative real numbers are represented by $\mathbf{R}^+$ and $\mathbf{R}_0^+$, respectively.

The simulation function, introduced by Khojasteh et al. [1], has emerged as an invaluable innovative control function in metric spaces. Its application in defining a $\zeta-$contraction has not only facilitated the proof of pivotal fixed point theorems but also marks a noteworthy advancement in the discipline. Following this groundbreaking work, numerous researchers have expanded and refined this concept across various abstract spaces, as evidenced in [2–6] and [7].

Recently, Karapınar [8] made significant advancements in the field of fixed point theory by modifying the classical concept of Kannan contractions. He introduced an interpolative Kannan contraction, which was designed to enhance the convergence rate of operators toward a unique fixed point. This innovation aimed to refine the existing understanding of how operators behave in mathematical spaces. However, subsequent work by Karapınar

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and Agarwal [9] revealed a critical flaw in the assumptions laid out in Karapınar's initial paper. They presented a counter-example that highlighted the problematic assumption that the fixed point must be unique. Their findings showed that it is possible for fixed points to exist without uniqueness, thereby challenging the validity of this central premise in the original theory. Following this important correction, the researchers provided a revised framework that better accommodates situations where fixed points are not unique. This development opened the door to further exploration and prompted the investigation of various results related to different types of interpolative mappings. Consequently, a plethora of results for both single-valued and multivalued mappings have been established across diverse abstract spaces [10–13].

There is extensive interest in metric fixed point theory due to its compelling structural properties and broad applications across various fields, including mathematics, computer science, and economics. Within this theoretical framework, the Banach contraction mapping theorem, first introduced by Banach in 1922, occupies a pivotal position owing to its foundational significance and versatility. This seminal work provided a robust method for establishing the existence and uniqueness of fixed points in complete metric spaces, laying the groundwork for countless subsequent research efforts aimed at expanding and refining the understanding of this profound mapping.

The Banach contraction mapping theorem has not only deepened theoretical insights but also inspired practical applications, from solving differential equations to optimization problems. Over the years, the development of this field has witnessed a notable emergence of innovative structures concerning generalized metric spaces. These generalized spaces relax some of the traditional constraints, allowing for a broader class of mappings and facilitating the exploration of fixed point theorems within varied contexts [14–19].

Among the significant advancements in this domain is the introduction of the modular metric space. This new structure, which incorporates a modular function to define distance and convergence, offers a more flexible approach to analyzing fixed points and contracts. Its unique properties enable researchers to address more complex problems that may not fit within the confines of classical metric spaces. Consequently, modular metric spaces serve as a fertile ground for further theoretical exploration and practical application, potentially leading to new discoveries in fixed point theory and beyond.

In 2010, Chistyakov [20, 21] made a significant advancement by establishing the concept of a modular metric space. This innovative framework not only extends the traditional metric space but also integrates the principles of modular linear space, paving the way for newfound research opportunities and applications in mathematical theory.

Let $\mathcal{X}$ be a nonempty set and $\Lambda : (0, \infty) \times \mathcal{X} \times \mathcal{X} \to [0, \infty]$ be a function. For the sake of brevity, we will denote the relationship as follows:

$$\Lambda_\chi(\iota, \jmath) = \Lambda(\chi, \iota, \jmath)$$

for all $\chi > 0$ and $\iota, \jmath \in \mathcal{X}$.

Definition 1.1. [20] Let $\mathcal{X}$ be nonempty set and $\Lambda : (0, \infty) \times \mathcal{X} \times \mathcal{X} \to [0, \infty]$ be a function satisfying the subsequent circumstances:

(Λ_1) $\iota = \jmath$ if and only if $\Lambda_\chi(\iota, \jmath) = 0$ for all $\chi > 0$ and and $\iota, \jmath \in \mathcal{X}$;

(Λ_2) $\Lambda_\chi(\iota, \jmath) = \Lambda_\chi(\jmath, \iota)$ for all $\chi > 0$ and $\iota, \jmath \in \mathcal{X}$;

(Λ_3) $\Lambda_{\chi+n}(\iota, \jmath) \leq \Lambda_\chi(\iota, z) + \Lambda_n(z, \jmath)$ for all $\chi, n > 0$ and $\iota, \jmath, z \in \mathcal{X}$.

Then, Λ is called modular metric in $\mathcal{X}$, and so Λ_χ is modular metric space. If the condition (Λ_1) is replaced by

(Λ_4) $\Lambda_\chi(\iota, \iota) = 0$ for all $\chi > 0$ and $\iota \in \mathcal{X}$,

then Λ is referred to as a pseudomodular metric on $\mathcal{X}$. A modular metric Λ defined on $\mathcal{X}$ is termed regular if it satisfies a weaker formulation of the condition denoted as (Λ_1).

(Λ_5) $\iota = \jmath$ if and only if $\Lambda_\chi(\iota, \jmath) = 0$ for some $\chi > 0$.

Moreover, Λ is called convex if for $\chi,\ n > 0$ and $\iota, \jmath, z \in \mathcal{X}$, the inequality holds:

(Λ_6) $\Lambda_{\chi+n}(\iota, \jmath) \leq \frac{\chi}{\chi+n}\Lambda_\chi(\iota, z) + \frac{n}{\chi+n}\Lambda_n(z, \jmath)$.

If we replace (Λ_3) by

(Λ_7) $\Lambda_{\max\{\chi,n\}}(\iota, \jmath) \leq \Lambda_\chi(\iota, z) + \Lambda_n(z, \jmath)$

for all $\chi,\ n > 0$ and $\iota, \jmath, z \in \mathfrak{X}_\Lambda$. Thus, we assert that $\mathfrak{X}_\Lambda$ represents non-Archimedean modular metric space.

Definition 1.2. [20] Let $\mathfrak{X}_\Lambda$ be a modular metric space, S be a subset of $\mathfrak{X}_\Lambda$ and $(\iota_n)_{n\in\mathbf{N}}$ be a sequence in $\mathfrak{X}_\Lambda$. Then,

(i) A sequence $(\iota_n)_{\kappa\in\mathbf{N}}$ is called $\Lambda-$convergent to $\iota \in \mathfrak{X}_\Lambda$ if and only if $\Lambda_\chi(\iota_n, \iota) \to 0$ as $n \to \infty$ for all $\chi > 0$, ι is said to be the $\Lambda-$limit of (ι_n).

(ii) A sequence $(\iota_n)_{\kappa\in\mathbf{N}}$ is called $\Lambda-$Cauchy if $\Lambda_\chi(\iota_n, \iota_p) \to 0$, as $p, n \to \infty$ for all $\chi > 0$.

(iii) A subset S is called $\Lambda-$closed if the $\Lambda-$limit of $\Lambda-$convergent sequence of S always belongs to S.

(iv) A subset S is called $\Lambda-$complete if any $\Lambda-$Cauchy sequence in S is $\Lambda-$convergent to a point of S.

(v) A subset S is called $\Lambda-$bounded if for all $\chi > 0$, we have

$$\delta_\Lambda(S) = \sup\{\Lambda_\chi(\iota, \jmath); \iota, \jmath \in S\} < \infty.$$

Definition 1.3. [20] Let $\mathfrak{X}_\Lambda$ be a modular metric space and $\coprod : \mathfrak{X}_\Lambda \to \mathfrak{X}_\Lambda$ be a mapping. It is said that $\coprod$ is a $\Lambda-$continuous when $\Lambda_\chi(\iota_n, \iota) \to 0 \Rightarrow \Lambda_\chi(\coprod \iota_n, \coprod \iota) \to 0$, as $n \to \infty$.

In recent years, the field of fixed point theory in modular metric spaces has witnessed significant developments and applications [22–25].

Khan et.al. [26] introduce the concept of altering distance function as follows.

Definition 1.4. [26] A continuous function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function if it is non-decreasing and $\varphi(r) = 0$ if and only if $r = 0$.

It is obvious that $\varphi(r) \geq 0$, for all $r \geq 0$. We denote Φ, the set of all altering distance functions.

Definition 1.5. [27] A function $\psi : [0, \infty) \to [0, \infty)$ is said to be a comparison function if it is monotonically increasing and $\psi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$.

If ψ is comparison function, then $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. The symbol Ψ denotes the set of all comparison functions.

Let $\mathfrak{X}$ be a nonempty set and $\alpha : \mathfrak{X} \times \mathfrak{X} \to \mathbf{R}$. We collect the following concepts which are necessary for our subsequent discussion.

Definition 1.6. [28] A mapping $\coprod : \mathfrak{X} \to \mathfrak{X}$ is said to be a $\alpha-$admissible if

(α_1) $\alpha(\iota, \jmath) \geq 1$ implies $\alpha(\coprod \iota, \coprod \jmath) \geq 1$, for all $\iota, \jmath \in \mathfrak{X}$.

Definition 1.7. [29] A mapping $\coprod : \mathfrak{X} \to \mathfrak{X}$ is called triangular $\alpha-$admissible if it satisfies (α_1) and

(α_2) $\alpha(\iota, z) \geq 1$ and $\alpha(z, \jmath) \geq 1$ imply $\alpha(\iota, \jmath) \geq 1$ for all $\iota, \jmath, z \in \mathfrak{X}$.

In light of the aforementioned considerations, this study aims to integrate concepts such as interpolative contraction, simulation functions, admissible mappings, and modified distance functions to establish novel fixed point theorems within non-Archimedean modular metric spaces. Furthermore, we provide a comprehensive illustration demonstrating both the existence and uniqueness of a solution for a nonlinear Fredholm integral equation.

2. Main results

Definition 2.1. Let $\mathfrak{X}_\Lambda$ be a non-Archimedean modular metric space and $\coprod : \mathfrak{X}_\Lambda \to \mathfrak{X}_\Lambda$ be a given mapping. It is said that $\coprod$ is a generalized interpolative $\alpha - (\psi, \varphi)_Z-$contractive mapping if there exist $\alpha : \mathfrak{X}_\Lambda \times \mathfrak{X}_\Lambda \to [0, \infty)$, $\psi \in \Psi$, $\varphi \in \Phi$ and $\zeta \in Z$ and $\mu_1, \mu_2 \in (0, 1)$ such that $\varphi(t) > \psi(t)$, $t > 0$ and $\mu_1 + \mu_2 < 1$ providing the subsequent inequality

$$\begin{aligned} &\zeta\left(\alpha(i,j)\,\varphi\left(\Lambda_\chi(\textstyle\coprod i, \coprod j)\right), \psi\left(\Xi(i,j)\right)\right) \geq 0, \\ &\Xi(i,j) = \Lambda_\chi(i,j)^{\mu_1}.\Lambda_\chi(i, \textstyle\coprod i)^{\mu_2}.\Lambda_\chi(j, \coprod j)^{1-\mu_1-\mu_2} \end{aligned} \tag{2.1}$$

for all $i, j \in \mathfrak{X}_\Lambda$.

Theorem 2.1. *Let $\mathcal{X}_\Lambda$ be a complete non-Archimedean modular metric space. Let $\coprod$ be a generalized interpolative $\alpha-(\psi,\varphi)_Z-$contractive mapping satisfying the following conditions:*

(i) $\coprod$ *is a triangular $\alpha-$admissible mapping,*

(ii) *there exists $i_0 \in \mathcal{X}_\Lambda$ such that $\alpha\left(i_0, \coprod i_0\right) \geq 1$,*

(iii) $\coprod$ *is a continuous mapping.*

Then, $\coprod$ admits a fixed point in $\mathcal{X}_\Lambda$.

Proof. Let $i_0 \in \mathcal{X}_\Lambda$ such that $\alpha\left(i_0, \coprod i_0\right) \geq 1$. Construct the sequence $\{i_\kappa\}$ in $\mathcal{X}_\Lambda$ by $i_{\kappa+1} = \coprod i_\kappa$, for all $\kappa \in \mathbf{N}$. If $i_{\kappa+1} = i_\kappa$, for some $\kappa \in \mathbf{N}$, then $i^* = i_\kappa$ is a fixed point for $\coprod$ and the proof completed. Hence, we presume that $i_{\kappa+1} \neq i_\kappa$, for all $\kappa \in \mathbf{N}$. Due to the fact that $\coprod$ is triangular $\alpha-$ admissible, we have:

$$\alpha\left(i_0, i_1\right) = \alpha\left(i_0, \coprod i_0\right) \geq 1 \Rightarrow \alpha\left(\coprod i_0, \coprod i_1\right) = \alpha\left(i_1, \coprod i_2\right) \geq 1.$$

By induction, we get

$$\alpha\left(i_\kappa, i_{\kappa+1}\right) \geq 1, \tag{2.2}$$

for all $\kappa \in \mathbf{N}$. Regarding (2.1), we derive that

$$\begin{aligned} 0 &\leq \zeta\left(\alpha\left(i_{\kappa-1}, i_\kappa\right)\varphi\left(\Lambda_\chi\left(\coprod i_{\kappa-1}, \coprod i_\kappa\right)\right), \psi\left(\Xi\left(i_{\kappa-1}, i_\kappa\right)\right)\right) \\ &= \zeta\left(\alpha\left(i_{\kappa-1}, i_\kappa\right)\varphi\left(\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\right), \psi\left(\Xi\left(i_{\kappa-1}, i_\kappa\right)\right)\right) \\ &< \psi\left(\Xi\left(i_{\kappa-1}, i_\kappa\right)\right) - \alpha\left(i_{\kappa-1}, i_\kappa\right)\varphi\left(\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\right), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \Xi\left(i_{\kappa-1}, i_\kappa\right) &= \Lambda_\chi(i_{\kappa-1}, i_\kappa)^{\mu_1}.\Lambda_\chi(i_{\kappa-1}, \coprod i_{\kappa-1})^{\mu_2}.\Lambda_\chi(i_\kappa, \coprod i_\kappa)^{1-\mu_1-\mu_2} \\ &= \Lambda_\chi(i_{\kappa-1}, i_\kappa)^{\mu_1}.\Lambda_\chi(i_{\kappa-1}, i_\kappa)^{\mu_2}.\Lambda_\chi(i_\kappa, i_{\kappa+1})^{1-\mu_1-\mu_2}. \end{aligned}$$

Consequently, we arrive at

$$\begin{aligned} \varphi\left(\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\right) &\leq \alpha\left(i_{\kappa-1}, i_\kappa\right)\varphi\left(\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\right) \\ &< \psi\left(\Xi\left(i_{\kappa-1}, i_\kappa\right)\right) \\ &= \psi\left(\Lambda_\chi(i_{\kappa-1}, i_\kappa)^{\mu_1}.\Lambda_\chi(i_{\kappa-1}, i_\kappa)^{\mu_2}.\Lambda_\chi(i_\kappa, i_{\kappa+1})^{1-\mu_1-\mu_2}\right). \end{aligned} \tag{2.4}$$

Suppose that $\Lambda_\chi\left(i_{\kappa-1}, i_\kappa\right) < \Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)$ for all $\kappa \in \mathbf{N}$, then from (2.4), we obtain

$$\varphi\left(\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\right) \leq \psi\left(\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\right) < \varphi\left(\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\right),$$

which causes a contradiction. Accordingly, we obtain

$$\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right) \leq \Lambda_\chi\left(i_{\kappa-1}, i_\kappa\right), \tag{2.5}$$

for all $\kappa \in \mathbf{N}$. Hence, $\{\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\}$ is a monotone decreasing sequence of positive real numbers and bounded below by zero. So, there exists $r \geq 0$ such that $\lim_{n\to\infty} \Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right) = r$. We claim that $r > 0$, otherwise from (2.3), (2.4) together with (2.5) we procure

$$\begin{aligned} 0 &\leq \zeta\left(\alpha\left(i_{\kappa-1}, i_\kappa\right)\varphi\left(\Lambda_\chi\left(\coprod i_{\kappa-1}, \coprod i_\kappa\right)\right), \psi\left(\Xi\left(i_{\kappa-1}, i_\kappa\right)\right)\right) \\ &= \zeta\left(\alpha\left(i_{\kappa-1}, i_\kappa\right)\varphi\left(\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\right), \psi\left(\Xi\left(i_{\kappa-1}, i_\kappa\right)\right)\right) \\ &< \psi\left(\Xi\left(i_{\kappa-1}, i_\kappa\right)\right) - \alpha\left(i_{\kappa-1}, i_\kappa\right)\varphi\left(\Lambda_\chi\left(i_\kappa, i_{\kappa+1}\right)\right). \end{aligned} \tag{2.6}$$

Consequently, we achieve

$$\begin{aligned}\varphi\left(\Lambda_\chi\left(i_\kappa,i_{\kappa+1}\right)\right) &\leq \alpha\left(i_{\kappa-1},i_\kappa\right)\varphi\left(\Lambda_\chi\left(i_\kappa,i_{\kappa+1}\right)\right)\\ &\leq \psi\left(\Xi\left(i_{\kappa-1},i_\kappa\right)\right)\\ &\leq \varphi\left(\Xi\left(i_{\kappa-1},i_\kappa\right)\right)\\ &\leq \varphi\left(\Lambda_\chi\left(i_{\kappa-1},i_\kappa\right)\right).\end{aligned}\tag{2.7}$$

Taking the limit as $n\to\infty$ in (2.7), we attain

$$\lim_{n\to\infty}\alpha\left(i_{\kappa-1},i_\kappa\right)\varphi\left(\Lambda_\chi\left(i_\kappa,i_{\kappa+1}\right)\right)=\lim_{n\to\infty}\psi\left(\Xi\left(i_{\kappa-1},i_\kappa\right)\right)=\varphi\left(r\right).\tag{2.8}$$

Setting $s_n=\alpha\left(i_{\kappa-1},i_\kappa\right)\varphi\left(\Lambda_\chi\left(i_\kappa,i_{\kappa+1}\right)\right),\quad t_n=\psi\left(\Xi\left(i_{\kappa-1},i_\kappa\right)\right)$ in (2.3), then by the property of simulation function and (2.8), it is yielded that

$$0\leq\limsup_{n\to\infty}\zeta\left(\alpha\left(i_{\kappa-1},i_\kappa\right)\varphi\left(\Lambda_\chi\left(i_\kappa,i_{\kappa+1}\right)\right),\psi\left(\Xi\left(i_{\kappa-1},i_\kappa\right)\right)\right)<0.$$

This is a contradiction and thus we have $\lim\limits_{n\to\infty}\Lambda_\chi\left(i_\kappa,i_{\kappa+1}\right)=0.$
Now, we show that $\{i_\kappa\}$ is a $\Lambda-$Cauchy sequence. Suppose that, there exist $\varepsilon>0$, for which one can find two sequences $\{m_\rho\}$ and $\{\kappa_\rho\}$, for all $\rho\geq1$ with $i_{m_\rho}>i_{\kappa_\rho}\geq\rho$ such that $\Lambda_\chi\left(i_{\kappa_\rho},i_{m_\rho}\right)\geq\varepsilon$. Further, we assume that m_ρ is the smallest number greater than κ_ρ, then $\Lambda_\chi\left(i_{\kappa_\rho},i_{m_\rho-1}\right)<\varepsilon$. By triangular inequality of non-Archimedean quasi modular metric space, we gain

$$\begin{aligned}\varepsilon\leq\Lambda_\chi\left(i_{\kappa_\rho},i_{m_\rho}\right)&=\Lambda_{\max\{\chi,\chi\}}\left(i_{\kappa_\rho},i_{m_\rho}\right)\\ &\leq\Lambda_\chi\left(i_{\kappa_\rho},i_{m_\rho-1}\right)+\Lambda_\chi\left(i_{m_\rho-1},i_{m_\rho}\right)\\ &<\varepsilon+\Lambda_\chi\left(i_{m_\rho-1},i_{m_\rho}\right).\end{aligned}$$

Taking the limit as $\rho\to\infty$, we get

$$\lim_{\rho\to\infty}\Lambda_\chi\left(i_{\kappa_\rho},i_{m_\rho}\right)=\varepsilon.\tag{2.9}$$

Again by triangular inequality of non-Archimedean quasi modular metric space, we have

$$\begin{aligned}\Lambda_\chi\left(i_{\kappa_\rho},i_{m_\rho}\right)&=\Lambda_{\max\{\chi,\chi\}}\left(i_{\kappa_\rho},i_{m_\rho}\right)\\ &\leq\Lambda_\chi\left(i_{\kappa_\rho},i_{\kappa_\rho+1}\right)+\Lambda_\chi\left(i_{\kappa_\rho+1},i_{m_\rho}\right)\\ &=\Lambda_\chi\left(i_{\kappa_\rho},i_{\kappa_\rho+1}\right)+\Lambda_{\max\{\chi,\chi\}}\left(i_{\kappa_\rho+1},i_{m_\rho}\right)\\ &\leq\Lambda_\chi\left(i_{\kappa_\rho},i_{\kappa_\rho+1}\right)+\Lambda_\chi\left(i_{\kappa_\rho+1},i_{m_\rho+1}\right)+\Lambda_\chi\left(i_{m_\rho+1},i_{m_\rho}\right).\end{aligned}\tag{2.10}$$

Also, we get

$$\begin{aligned}\Lambda_\chi\left(i_{\kappa_\rho+1},i_{m_\rho+1}\right)&=\Lambda_{\max\{\chi,\chi\}}\left(i_{\kappa_\rho+1},i_{m_\rho+1}\right)\\ &\leq\Lambda_\chi\left(i_{\kappa_\rho+1},i_{\kappa_\rho}\right)+\Lambda_\chi\left(i_{\kappa_\rho},i_{m_\rho+1}\right)\\ &=\Lambda_\chi\left(i_{\kappa_\rho+1},i_{\kappa_\rho}\right)+\Lambda_{\max\{\chi,\chi\}}\left(i_{\kappa_\rho},i_{m_\rho+1}\right)\\ &\leq\Lambda_\chi\left(i_{\kappa_\rho+1},i_{\kappa_\rho}\right)+\Lambda_\chi\left(i_{\kappa_\rho},i_{m_\rho}\right)+\Lambda_\chi\left(i_{m_\rho},i_{m_\rho+1}\right).\end{aligned}\tag{2.11}$$

Combining the expressions (2.10) and (2.11) and taking the limit as $\rho\to\infty$ together with (2.9), we attain

$$\lim_{\rho\to\infty}\Lambda_\chi\left(i_{\kappa_\rho+1},i_{m_\rho+1}\right)=\varepsilon.\tag{2.12}$$

As $\coprod$ is a triangular $\alpha-$admissible mapping, we obtain $\alpha\left(i_{\kappa_\rho},i_{m_\rho}\right)\geq1$, for all numbers $m_\rho,\ \kappa_\rho$ such that $m_\rho>\kappa_\rho$, where $\rho\geq1$. From (2.1), we get

$$\begin{aligned}0 &\leq \zeta\left(\alpha\left(i_{\kappa_\rho}, i_{m_\rho}\right)\varphi\left(\Lambda_\chi\left(\coprod i_{\kappa_\rho}, \coprod i_{m_\rho}\right)\right), \psi\left(\Xi\left(i_{\kappa_\rho}, i_{m_\rho}\right)\right)\right)\\ &= \zeta\left(\alpha\left(i_{\kappa_\rho}, i_{m_\rho}\right)\varphi\left(\Lambda_\chi\left(i_{\kappa_\rho+1}, i_{m_\rho+1}\right)\right), \psi\left(\Xi\left(i_{\kappa_\rho}, i_{m_\rho}\right)\right)\right)\\ &< \psi\left(\Xi\left(i_{\kappa_\rho}, i_{m_\rho}\right)\right) - \alpha\left(i_{\kappa_\rho}, i_{m_\rho}\right)\varphi\left(\Lambda_\chi\left(i_{\kappa_\rho+1}, i_{m_\rho+1}\right)\right).\end{aligned}$$

Consequently, it can be inferred that

$$\begin{aligned}\varphi\left(\Lambda_\chi\left(i_{\kappa_\rho+1}, i_{m_\rho+1}\right)\right) &\leq \alpha\left(i_{\kappa_\rho}, i_{m_\rho}\right)\varphi\left(\Lambda_\chi\left(i_{\kappa_\rho+1}, i_{m_\rho+1}\right)\right)\\ &\leq \psi\left(\Xi\left(i_{\kappa_\rho}, i_{m_\rho}\right)\right)\\ &\leq \varphi\left(\Xi\left(i_{\kappa_\rho}, i_{m_\rho}\right)\right),\end{aligned}$$

where

$$\begin{aligned}\Xi\left(i_{\kappa_\rho}, i_{m_\rho}\right) &= \Lambda_\chi\left(i_{\kappa_\rho}, i_{m_\rho}\right)^{\mu_1} . \Lambda_\chi\left(i_{\kappa_\rho}, \coprod i_{\kappa_\rho}\right)^{\mu_2} . \Lambda_\chi\left(i_{m_\rho}, \coprod i_{m_\rho}\right)^{1-\mu_1-\mu_2}\\ &= \Lambda_\chi\left(i_{\kappa_\rho}, i_{m_\rho}\right)^{\mu_1} . \Lambda_\chi\left(i_{\kappa_\rho}, i_{\kappa_\rho+1}\right)^{\mu_2} . \Lambda_\chi\left(i_{m_\rho}, i_{m_\rho+1}\right)^{1-\mu_1-\mu_2}.\end{aligned}$$

Taking the limit as $\rho \to \infty$ with (2.9), (2.10), (2.11) and (2.12), we have

$$0 \leq \varphi(\varepsilon) < \varphi(0) = 0 \quad \text{iff} \quad \varepsilon = 0.$$

This situation presents a contradiction, thereby establishing that the sequence $\{i_\kappa\}$ qualifies as a Cauchy sequence. Since $\mathcal{X}_\Lambda$ is complete non-Archimedean modular metric space, there exists $i^* \in \mathcal{X}_\Lambda$ such that $i_\kappa \to i^*$ as $\kappa \to \infty$. Based on the continuity of $\coprod$, it can be deduced that the sequence defined by $i_{\kappa+1} = \coprod i_\kappa \to \coprod i^*$ as $\kappa \to \infty$. By virtue of the uniqueness of limits, we conclude that, $i^* = \coprod i^*$, that is, i^* is a fixed point of $\coprod$. □

In the subsequent theorem, it is possible to dispense with the continuity of $\coprod$ by introducing an alternative condition.

Theorem 2.2. *Let $\mathcal{X}_\Lambda$ be a complete non-Archimedean modular metric space and $\coprod$ be a generalized interpolative $\alpha-(\psi,\varphi)_Z-$contractive mapping satisfying the following conditions:*

- (i) *$\coprod$ is a triangular $\alpha-$admissible mapping,*
- (ii) *there exists $i_0 \in \mathcal{X}_\Lambda$ such that $\alpha\left(i_0, \coprod i_0\right) \geq 1$,*
- (iii) *If $\{i_\kappa\}$ is a sequence in $\mathcal{X}_\Lambda$ such that $\alpha\left(i_\kappa, i_{\kappa+1}\right) \geq 1$ for all κ and $i_\kappa \to i \in S_\Lambda$ as $\kappa \to \infty$, then $\alpha\left(i_\kappa, i\right) \geq 1$ for all κ.*

Then, $\coprod$ admits a fixed point in $\mathcal{X}_\Lambda$.

Proof. In light of the proof of Theorem 2.1, we can conclude that $\{i_\kappa\}$ is a Cauchy sequence. Then, $i^* \in \mathcal{X}_\Lambda$ exits such that $i_{\kappa_\rho} \to i^*$ as $\rho \to \infty$. From (2.2) and the hypothesis (iii), we have

$$\alpha\left(i_{\kappa_\rho}, i^*\right) \geq 1, \tag{2.13}$$

for all ρ. From (2.1) and (2.13), we get

$$\begin{aligned}0 &\leq \zeta\left(\alpha\left(i_{\kappa_\rho}, i^*\right)\varphi\left(\Lambda_\chi\left(\coprod i_{\kappa_\rho}, \coprod i^*\right)\right), \psi\left(\Xi\left(i_{\kappa_\rho}, i^*\right)\right)\right)\\ &= \zeta\left(\alpha\left(i_{\kappa_\rho}, i^*\right)\varphi\left(\Lambda_\chi\left(i_{\kappa_\rho+1}, \coprod i^*\right)\right), \psi\left(\Xi\left(i_{\kappa_\rho}, i^*\right)\right)\right)\\ &< \psi\left(\Xi\left(i_{\kappa_\rho}, i^*\right)\right) - \alpha\left(i_{\kappa_\rho}, i^*\right)\varphi\left(\Lambda_\chi\left(i_{\kappa_\rho+1}, \coprod i^*\right)\right)\end{aligned} \tag{2.14}$$

which is equivalent to

$$\begin{aligned}\varphi\left(\Lambda_\chi\left(i_{\kappa_\rho+1}, \coprod i^*\right)\right) &\leq \alpha\left(i_{\kappa_\rho}, i^*\right)\varphi\left(\Lambda_\chi\left(i_{\kappa_\rho+1}, \coprod i^*\right)\right)\\ &< \psi\left(\Xi\left(i_{\kappa_\rho}, i^*\right)\right)\\ &< \varphi\left(\Xi\left(i_{\kappa_\rho}, i^*\right)\right),\end{aligned} \tag{2.15}$$

where

$$\Xi\left(i_{\kappa_\rho}, i^*\right) = \Lambda_\chi\left(i_{\kappa_\rho}, i^*\right)^{\mu_1} . \Lambda_\chi\left(i_{\kappa_\rho}, \coprod i_{\kappa_\rho}\right)^{\mu_2} . \Lambda_\chi(i^*, \coprod i^*)^{1-\mu_1-\mu_2}$$
$$= \Lambda_\chi\left(i_{\kappa_\rho}, i^*\right)^{\mu_1} . \Lambda_\chi\left(i_{\kappa_\rho}, i_{\kappa_\rho+1}\right)^{\mu_2} . \Lambda_\chi(i^*, \coprod i^*)^{1-\mu_1-\mu_2}. \tag{2.16}$$

Now letting $\rho \to \infty$, from the property of φ, we get $\varphi\left(\Lambda_\chi\left(i^*, \coprod i^*\right)\right) = 0$ implying $\Lambda_\chi\left(i^*, \coprod i^*\right) = 0$. This can be concluded that i^* is a fixed point of $\coprod$. □

We suggest the following hypotheses for the uniqueness of the fixed point of $\coprod$.

(U) For all $i, j \in Fix\{\coprod\}$, we get $\alpha(i, j) \geq 1$.

Theorem 2.3. *Adding the condition (U) to the hypotheses of the Theorem 2.1 (resp. Theorem 2.2), we attain the uniqueness of the fixed point of $\coprod$.*

Proof. We assume that j^* is an another fixed point of $\coprod$, that is, $\Lambda_\chi(i^*, j^*) \neq 0$. From the condition (U), we get $\alpha(i^*, j^*) \geq 1$. Owing to $\coprod$ is a generalized interpolative $\alpha - (\psi, \varphi)_Z$−contractive mapping, we derive that

$$0 \leq \zeta\left(\alpha(i^*, j^*)\varphi\left(\Lambda_\chi\left(\coprod i^*, \coprod j^*\right)\right), \psi\left(\Xi(i^*, j^*)\right)\right)$$
$$= \zeta\left(\alpha(i^*, j^*)\varphi\left(\Lambda_\chi(i^*, j^*)\right), \psi\left(\Xi(i^*, j^*)\right)\right) \tag{2.17}$$
$$< \psi\left(\Xi(i^*, j^*)\right) - \alpha(i^*, j^*)\varphi\left(\Lambda_\chi(i^*, j^*)\right),$$

which is equivalent to

$$\varphi\left(\Lambda_\chi(i^*, j^*)\right) \leq \alpha(i^*, j^*)\varphi\left(\Lambda_\chi(i^*, j^*)\right)$$
$$< \psi\left(\Xi(i^*, j^*)\right) \tag{2.18}$$
$$< \varphi\left(\Xi(i^*, j^*)\right),$$

where

$$\Xi(i^*, j^*) = \Lambda_\chi(i^*, j^*)^{\mu_1} . \Lambda_\chi\left(i^*, \coprod i^*\right)^{\mu_2} . \Lambda_\chi\left(j^*, \coprod j^*\right)^{1-\mu_1-\mu_2} = 0. \tag{2.19}$$

This results in a contradiction. Hence, $\coprod$ has a unique fixed point in $\mathcal{X}_\Lambda$. □

Example 2.1. Let $\mathcal{X}_\Lambda = \mathbf{R}$, $\Lambda_\chi(i, j) = \frac{1}{\chi}|i - j|$, for all $i, j \in \mathcal{X}_\Lambda, \chi > 0$ and $\coprod i = \frac{i}{2}$. Presume the mapping $\alpha : \mathcal{X}_\Lambda \times \mathcal{X}_\Lambda \to [0, \infty)$ is defined by

$$\alpha(i, j) = \begin{cases} 1, & i, j \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Consider the mapping as $\zeta(t, s) = s - t$, thus we get

$$\alpha(i, j)\varphi\left(\Lambda_\chi\left(\coprod i, \coprod j\right)\right) \leq \psi\left(\Xi(i, j)\right). \tag{2.20}$$

Also, if we take $\varphi(t) = \frac{t}{5}$, $\psi(t) = \frac{t}{3}$, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{3}$, $\chi = 3$ and $(i, j) \in [0, 1]$, then we demonstrate as in the figure below that the left side of inequality is less than or equal to the right side. Thus, all the hypotheses of Theorem 2.1 are satisfied, and 0 is a unique fixed point of $\coprod$.

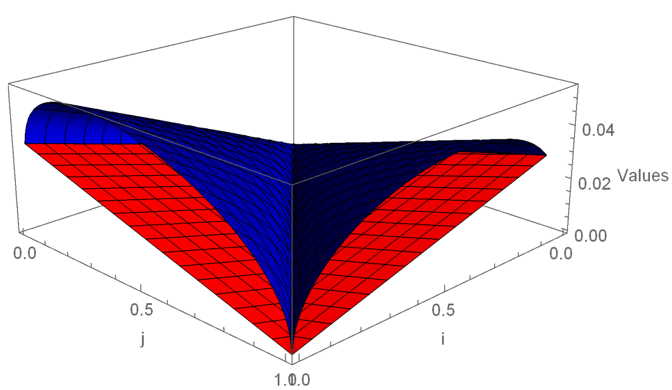

Figure 1. 3D representation of the inequality (2.20).

Corollary 2.1. *Consider $\mathcal{X}_\Lambda$ to be a complete non-Archimedean modular metric space. Presume $\coprod$ be a self mapping on $\mathcal{X}_\Lambda$ satisfying the following conditions:*

(i) *$\coprod$ is a triangular $\alpha-$admissible mapping,*

(ii) *there exists $i_0 \in \mathcal{X}_\Lambda$ such that $\alpha\left(i_0, \coprod i_0\right) \geq 1$,*

(iii) *$\coprod$ is a continuous mapping,*

(iv) *if there exist $\alpha : \mathcal{X}_\Lambda \times \mathcal{X}_\Lambda \to [0, \infty)$, $\psi \in \Psi$, $\varphi \in \Phi$ and $\mu_1, \mu_2 \in (0,1)$ such that $\varphi(t) > \psi(t)$, $t > 0$ and $\mu_1 + \mu_2 < 1$ satisfying the inequality*

$$\alpha(i,j)\,\varphi\left(\Lambda_\chi(\coprod i, \coprod j)\right) \leq \psi\left(\Xi(i,j)\right) \tag{2.21}$$

for all $i, j \in \mathcal{X}_\Lambda$.

Then, $\coprod$ admits a unique fixed point in $\mathcal{X}_\Lambda$.

Corollary 2.2. *Let $\coprod$ be a self-mapping on a complete non-Archimedean modular metric space $\mathcal{X}_\Lambda$. If there exist $\psi \in \Psi$ and $\mu_1, \mu_2 \in (0,1)$ such that $\varphi(t) > \psi(t)$, $t > 0$ and $\mu_1 + \mu_2 < 1$ satisfying the inequality*

$$\Lambda_\chi(\coprod i, \coprod j) \leq \psi\left(\Xi(i,j)\right) \tag{2.22}$$

for all $i, j \in \mathcal{X}_\Lambda$. Then, $\coprod$ admits a unique fixed point in $\mathcal{X}_\Lambda$.

3. An application to a nonlinear Fredholm integral equation

In this part, we investigate the nonlinear Fredholm integral equation in the setting of a non-Archimedean modular metric space. Let $\mathcal{X} = C[\tau, \upsilon]$ be a set of all real continuous function on $[\tau, \upsilon]$ with a non-Archimedean modular metric $\Lambda_\chi(\gamma, \delta) = \frac{1}{\chi}|\gamma - \delta| = \frac{1}{\chi}\max_{t\in[\tau,\upsilon]}|\gamma - \delta|$, for all $\gamma, \delta \in C[\tau, \upsilon]$ and $\chi \in (0,1)$. Then $\mathcal{X}_\Lambda$ is a non-Archimedean modular metric space. Now, we consider the nonlinear Fredholm integral equation

$$\iota(a) = u(a) + \frac{1}{\upsilon - \tau}\int_\tau^\upsilon K(a, b, \iota(b))db, \tag{3.1}$$

where $a, b \in [\tau, \upsilon]$. Assume that $K : [\tau, \upsilon] \times [\tau, \upsilon] \times \mathcal{X} \to R$ and $u : [\tau, \upsilon] \to R$ continuous where $u(a)$ is a given function in $\mathcal{X}$.

Theorem 3.1. *Suppose $\mathcal{X}_\Lambda$ be a complete non-Archimedean modular metric space with*

$$\Lambda_\chi(\gamma, \delta) = \frac{1}{\chi}|\gamma - \delta| = \frac{1}{\chi}\max_{t\in[\tau,\upsilon]}|\gamma - \delta|,$$

for all $\gamma, \delta \in C[\tau, \upsilon]$, $\chi \in (0,1)$ and $\coprod : \mathcal{X}_\Lambda \to \mathcal{X}_\Lambda$ be an operator defined by

$$\coprod \iota(a) = u(a) + \frac{1}{\upsilon - \tau}\int_\tau^\upsilon K(a, b, \iota(b))db. \tag{3.2}$$

If there exist $\wp \in [0,1)$, $\mu_1, \mu_2 \in (0,1)$ with $\mu_1 + \mu_2 < 1$ such that for all $\iota, \jmath \in \mathcal{X}_\Lambda$, $a, b \in [\tau, \upsilon]$ satisfying the following inequality

$$\begin{aligned} &|K(a, b, \iota(a)) - K(a, b, \jmath(a))| \leq \wp\,\Xi(\iota(a), \jmath(a)), \\ &\Xi(\iota(a), \jmath(a)) = |\iota(a) - \jmath(a)|^{\mu_1}.|\iota(a) - \coprod \iota(a)|^{\mu_2}.|\jmath(a) - \coprod \jmath(a)|^{1-\mu_1-\mu_2}. \end{aligned} \tag{3.3}$$

Then, the integral equation (3.1) has a unique solution in $\mathcal{X}_\Lambda$.

Proof. From (3.1) and (3.2), we have

$$\begin{aligned}\left|\coprod\iota(a)-\coprod\jmath(a)\right| &\leq \frac{1}{|\upsilon-\tau|}\left|\int_{\tau}^{\upsilon}K(a,b,\iota(a))\,db-\int_{\tau}^{\upsilon}K(a,b,\jmath(a))\,db\right| \\ &\leq \frac{1}{|\upsilon-\tau|}\int_{\tau}^{\upsilon}|K(a,b,\iota(a))-K(a,b,\jmath(a))|\,db \\ &\leq \frac{\wp}{|\upsilon-\tau|}\int_{\tau}^{\upsilon}\Xi(\iota(a),\jmath(a))\,db \\ &\leq \frac{\wp}{|\upsilon-\tau|}\int_{\tau}^{\upsilon}|\iota(a)-\jmath(a)|^{\mu_1}.\left|\iota(a)-\coprod\iota(a)\right|^{\mu_2}.\left|\jmath(a)-\coprod\jmath(a)\right|^{1-\mu_1-\mu_2}db.\end{aligned} \tag{3.4}$$

Taking maximum on both sides for all $a\in[\tau,\upsilon]$, we get

$$\begin{aligned}\Lambda_{\chi}\left(\coprod\iota,\coprod\jmath\right) &= \frac{1}{\chi}\max_{a\in[0,1]}\left|\coprod\iota(a)-\coprod\jmath(a)\right| \\ &\leq \frac{\wp}{|\upsilon-\tau|}\max_{a\in[\tau,\upsilon]}\int_{\tau}^{\upsilon}\frac{1}{\chi}|\iota(a)-\jmath(a)|^{\mu_1}.\frac{1}{\chi}\left|\iota(a)-\coprod\iota(a)\right|^{\mu_2}.\frac{1}{\chi}\left|\jmath(a)-\coprod\jmath(a)\right|^{1-\mu_1-\mu_2}db \\ &\leq \frac{\wp}{|\upsilon-\tau|}\max_{a\in[\tau,\upsilon]}\left[\frac{1}{\chi}|\iota(a)-\jmath(a)|^{\mu_1}.\frac{1}{\chi}\left|\iota(a)-\coprod\iota(a)\right|^{\mu_2}.\frac{1}{\chi}\left|\jmath(a)-\coprod\jmath(a)\right|^{1-\mu_1-\mu_2}\right]\int_{\tau}^{\upsilon}db \\ &= \wp\left[\Lambda_{\chi}(\iota,\jmath)^{\mu_1}.\Lambda_{\chi}(\iota,\coprod\iota)^{\mu_2}.\Lambda_{\chi}(\jmath,\coprod\jmath)^{1-\mu_1-\mu_2}\right] \\ &= \wp\Xi(\iota,\jmath).\end{aligned} \tag{3.5}$$

Thus, all the conditions of Corollary 2.2 are satisfied by setting $\psi(t)=\wp t$ for all $t>0$, where $\wp\in[0,1)$ and hence the integral equation (3.2) has a unique solution in $\mathfrak{X}_{\Lambda}$. □

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Affiliations

Ekber Girgin
Address: Sakarya University of Applied Sciences, Department of Engineering Fundamental Sciences, 54187, Sakarya-Türkiye
e-mail: ekbergirgin@subu.edu.tr
ORCID ID:0000-0002-8913-5416