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RELATIONS AND INTEGRATION OF HERMITE-BASED MILNE-THOMSON AND FUBINI TYPE POLYNOMIALS

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ABSTRACT

The purpose of this paper is to present a lot of formulas for the r-parametric Hermite-based Milne-Thomson type polynomials. By applying functional equation method of generating functions, we also present a lot of relations and integral formulas incorporated the Fubini type polynomials, the r-parametric Hermite-based cosine-and sine-Milne-Thomson type Fubini polynomials, Gould-Hopper polynomials, and other special polynomials. Furthermore, we show that the special values of these results reduce to connections to previously known results.

Keywords: *Fubini type numbers and polynomials, Gould-Hopper polynomials, Hermite-based Milne-Thomson type polynomials, Trigonometric functions, Generating functions.*

1. INTRODUCTION

Special numbers and polynomials, such as, Fubini numbers, Hermite polynomials, their corresponding generating functions, and trigonometric functions, play important roles in various branches of pure and applied sciences. For instance, the Fubini numbers are used to count combinatorial problems, while the Hermite polynomials are particularly used in combinatorics, probability theory, numerical analysis, computational science, and the seismic waves of earthquake. Consequently, they have numerous applications in many disciplines, such as engineering, mathematics, physics, and other sciences. Moreover, many formulas and identities, involving some special polynomials and their parametric forms, have been examined by many authors (*cf.* [1-20]).

This paper focuses on investigating the r-parametric Hermite-based Milne-Thomson type polynomials and the Fubini type polynomials using generating function methods. From these functions and integral equations, we derive some formulas and relations involving these polynomials and the first kind Gould-Hopper polynomials. These type of polynomials have wide applications in variety of areas, especially mathematics and engineering. As a result, formulas of derived from this paper have significant potential for use in many areas such as solving mathematical modeling problems, combinatorial problems, linear differential equations, and etc.

We now begin by the following notations, definitions, and relations in order to use in the following sections.

Let

$$\mathbb{N} = \{1,2,3,\dots\}, \qquad \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and the sets of integers: $\mathbb{Z}$, real numbers: $\mathbb{R}$, complex numbers: $\mathbb{C}$, and also

$$e^t = \exp(t).$$

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The Fubini numbers, represented by $\omega_g(w)$, are defined by

$$\frac{1}{2-\exp(t)}=\sum_{w=0}^{\infty}\omega_g(w)\frac{t^w}{w!} \tag{1}$$

(*cf.* [3]).

From (1), one has

$$\omega_g(w)=\sum_{k=0}^{w}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^w$$

(*cf.* [3,10]).

The Fubini type polynomials of order z, represented by $a_w^{(z)}(x)$, are defined by

$$\frac{2^z\exp(xt)}{\left(2-\exp(t)\right)^{2z}}=\sum_{w=0}^{\infty}a_w^{(z)}(x)\frac{t^w}{w!} \tag{2}$$

(*cf.* [10]).

Setting $x=0$ yields the Fubini type numbers of order z:

$$a_w^{(z)}(0)=a_w^{(z)}.$$

When $z=1$ and $x=0$ in (2), one has

$$a_w^{(1)}(0)=a_w. \tag{3}$$

From (1) and (3), we get

$$a_w=2\sum_{k=0}^{w}\binom{w}{k}\omega_g(k)\omega_g(w-k)$$

(*cf.* [10]).

The first kind Gould-Hopper polynomials, represented by $H_w^m(x,\varphi)$, are defined by

$$\exp(xt+\varphi t^m)=\sum_{w=0}^{\infty}H_w^m(x,\varphi)\frac{t^w}{w!}, \tag{4}$$

and their explicit formula is given as follows:

$$H_w^m(x,\varphi)=\sum_{k=0}^{\left[\frac{w}{m}\right]}\frac{w!\,\varphi^k x^{w-mk}}{k!\,(w-km)!} \tag{5}$$

in which $[d]$ represents the largest integer d (*cf.* [2,4,5]).

The polynomials $H_w(\vec{u},r)$ are defined by

$$\exp\left(\sum_{k=1}^{r}u_k t^k\right)=\sum_{w=0}^{\infty}H_w(\vec{u},r)\frac{t^w}{w!}, \tag{6}$$

and their explicit formula is given as follows:

$$H_w(\vec{u},r)=\sum_{k=0}^{\left[\frac{w}{r}\right]}\frac{w!\,u_r^k H_{w-rk}(\vec{u},r-1)}{k!\,(w-kr)!},$$

where $\vec{u}=(u_1,u_2,\dots,u_r)$ (*cf.* [2,4,5,13]). Here, we note that the polynomials $H_w(\vec{u},r)$ represent generalized Hermite-Kampè de Fèriet polynomials.

The r-parametric Hermite-based Milne-Thomson type polynomials, represented by $\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)$ and $\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)$, are defined by, respectively,

$$2\big(b+f(t,p)\big)^z\exp(xt)\exp\left(\sum_{k=1}^{r}u_k t^k\right)\cos(\varphi t)=\sum_{w=0}^{\infty}\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!} \tag{7}$$

and

$$2\big(b+f(t,p)\big)^z\exp(xt)\exp\left(\sum_{k=1}^{r}u_k t^k\right)\sin(\varphi t)=\sum_{w=0}^{\infty}\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!}, \tag{8}$$

where $f(t,p)$ represents an analytic function or a meromorphic function and $p,b\in\mathbb{R}$ (*cf.* [6,13]).

When $b=0$ and $f(t,1)=\frac{2}{\left(2-\exp(t)\right)^2}$ in (7) and (8), we have

$$\frac{2^{z+1}}{\big(2-\exp(t)\big)^{2z}}\exp(xt)\exp\left(\sum_{k=1}^{r}u_k t^k\right)\cos(\varphi t)=\sum_{w=0}^{\infty}{}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)\frac{t^w}{w!} \tag{9}$$

and

$$\frac{2^{z+1}}{\big(2-\exp(t)\big)^{2z}}\exp(xt)\exp\left(\sum_{k=1}^{r}u_k t^k\right)\sin(\varphi t)=\sum_{w=0}^{\infty}{}_F\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r)\frac{t^w}{w!}, \tag{10}$$

where ${}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)$ and ${}_F\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r)$ are called r-parametric Hermite-based cosine-Milne-Thomson type Fubini polynomials of order z and r-parametric Hermite-based sine-Milne-Thomson type Fubini polynomials of order z, respectively (*cf.* [6,13]).

For $\vec{u}=(0,0,\dots,0)=\vec{0}$, and combining (9), (10) with (2), we have

$${}_F\mathfrak{h}_1\big(w,x,\varphi,z;\vec{0},r\big)=2\sum_{k=0}^{\left[\frac{w}{2}\right]}(-1)^k\binom{w}{2k}\varphi^{2k}a_{w-2k}^{(z)}(x)$$

and

$${}_F\mathfrak{h}_2\big(w,x,\varphi,z;\vec{0},r\big)=2\sum_{k=0}^{\left[\frac{w-1}{2}\right]}(-1)^k\binom{w}{2k+1}\varphi^{2k+1}a_{w-1-2k}^{(z)}(x)$$

(*cf.* [6, Theorems 3.29 and 3.38]).

When $\varphi=0$ and $\vec{u}=\vec{0}$ in (9), one has [6]:

$${}_F\mathfrak{h}_1\big(w,x,0,z;\vec{0},r\big)=2a_w^{(z)}(x).$$

2. RELATIONS FOR r-PARAMETRIC HERMITE-BASED MILNE-THOMSON TYPE POLYNOMIALS

The aim of this section is to utilize generating functions for the polynomials $\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,a,b)$ and $\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,a,b)$ in order to obtain some relations including these polynomials with the first kind Gould-Hopper polynomials, and also the polynomials ${}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)$ and ${}_F\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r)$.

Theorem 2.1. For $w \in \mathbb{N}_0$, we have

$$\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b) = \sum_{k=0}^{w} \binom{w}{k} \mathfrak{h}_1(k,0,\varphi,z;\vec{u},r-1,p,b) H_{w-k}^{r}(x,u_r). \tag{11}$$

Proof. By combining (7) with (4), we obtain

$$\sum_{w=0}^{\infty} \mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!} = \sum_{w=0}^{\infty} \mathfrak{h}_1(w,0,\varphi,z;\vec{u},r-1,p,b)\frac{t^w}{w!}\sum_{w=0}^{\infty} H_w^r(x,u_r)\frac{t^w}{w!}.$$

Therefore

$$\sum_{w=0}^{\infty} \mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!} = \sum_{w=0}^{\infty}\sum_{k=0}^{w} \binom{w}{k} \mathfrak{h}_1(k,0,\varphi,z;\vec{u},r-1,p,b) H_{w-k}^{r}(x,u_r)\frac{t^w}{w!}.$$

Matching the terms of $\frac{t^w}{w!}$ in both expressions brings us to the intended result.

Theorem 2.2. For $w \in \mathbb{N}_0$, we have

$$\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b) = \sum_{k=0}^{w} \binom{w}{k} \mathfrak{h}_2(k,0,\varphi,z;\vec{u},r-1,p,b) H_{w-k}^{r}(x,u_r). \tag{12}$$

Proof. Combining (8) with (4), we get

$$\sum_{w=0}^{\infty} \mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!} = \sum_{w=0}^{\infty} \mathfrak{h}_2(w,0,\varphi,z;\vec{u},r-1,p,b)\frac{t^w}{w!}\sum_{w=0}^{\infty} H_w^r(x,u_r)\frac{t^w}{w!}$$

and consequently

$$\sum_{w=0}^{\infty} \mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)\frac{t^w}{w!} = \sum_{w=0}^{\infty}\sum_{k=0}^{w} \binom{w}{k} \mathfrak{h}_2(k,0,\varphi,z;\vec{u},r-1,p,b) H_{w-k}^{r}(x,u_r)\frac{t^w}{w!}.$$

Matching the terms of $\frac{t^w}{w!}$ in both expressions brings us to the intended result.

For $b = 0$, putting $f(t,1) = \frac{2}{\left(2-\exp(t)\right)^2}$ in (11) and (12), yields the Corollary 2.3:

Corollary 2.3. For $w \in \mathbb{N}_0$, we have

$${}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r) = \sum_{k=0}^{w} \binom{w}{k} {}_F\mathfrak{h}_1(k,0,\varphi,z;\vec{u},r-1) H_{w-k}^{r}(x,u_r) \tag{13}$$

and

$${}_F\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r) = \sum_{k=0}^{w} \binom{w}{k} {}_F\mathfrak{h}_2(k,0,\varphi,z;\vec{u},r-1) H_{w-k}^{r}(x,u_r). \tag{14}$$

3. INTEGRAL FORMULAS FOR r-PARAMETRIC HERMITE-BASED MILNE-THOMSON AND FUBINI TYPE POLYNOMIALS

The aim of this section is to apply integral operator to the generating functions of the polynomials $\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)$ and $\mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)$ in order to give several formulas that include these polynomials, the polynomials ${}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)$, and the Fubini type polynomials.

Theorem 3.1 (*cf.* [6, Eq. (4.8)]). For $w \in \mathbb{N}_0$, we have

$$\int_c^d \mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)dx = \frac{\mathfrak{h}_1(w+1,d,\varphi,z;\vec{u},r,p,b) - \mathfrak{h}_1(w+1,c,\varphi,z;\vec{u},r,p,b)}{w+1}. \quad (15)$$

Proof. Integrating both sides of the Eq. (7), we get

$$2\big(b+f(t,p)\big)^z \exp\left(\sum_{k=1}^{r} u_k t^k\right)\cos(\varphi t)\int_c^d \exp(xt)dx = \sum_{w=0}^{\infty}\frac{t^w}{w!}\int_c^d \mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)dx.$$

After some calculations, we obtain

$$\sum_{w=0}^{\infty}\frac{t^w}{w!}\int_c^d \mathfrak{h}_1(w,x,\varphi,z;\vec{u},r,p,b)dx$$
$$= \sum_{w=0}^{\infty}\frac{\mathfrak{h}_1(w+1,d,\varphi,z;\vec{u},r,p,b)}{(w+1)!}t^w - \sum_{w=0}^{\infty}\frac{\mathfrak{h}_1(w+1,c,\varphi,z;\vec{u},r,p,b)}{(w+1)!}t^w.$$

Matching the terms of $\frac{t^w}{w!}$ in both expressions brings us to the intended result.

When $b=0$ and $f(t,1) = \frac{2}{\left(2-\exp(t)\right)^2}$ in (15) allows us to obtain the Corollary 3.2:

Corollary 3.2. For $w \in \mathbb{N}_0$, we have

$$\int_c^d {}_F\mathfrak{h}_1(w,x,\varphi,z;\vec{u},r)dx = \frac{{}_F\mathfrak{h}_1(w+1,d,\varphi,z;\vec{u},r) - {}_F\mathfrak{h}_1(w+1,c,\varphi,z;\vec{u},r)}{w+1}. \quad (16)$$

Remark 3.3. Substituting $\varphi = 0$ and $\vec{u} = \vec{0}$ into (16), and performing some calculations gives the known result:

$$\int_c^d a_w^{(z)}(x)dx = \frac{1}{w+1}\left(a_{w+1}^{(z)}(d) - a_{w+1}^{(z)}(c)\right).$$

(*cf.* [7, Eq. 20]).

Theorem 3.4 (*cf.* [6, Eq. (4.9)]). For $w \in \mathbb{N}_0$, we have

$$\int_c^d \mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)dx = \frac{\mathfrak{h}_2(w+1,d,\varphi,z;\vec{u},r,p,b) - \mathfrak{h}_2(w+1,c,\varphi,z;\vec{u},r,p,b)}{w+1}. \quad (17)$$

Proof. Integrating both sides of the Eq. (8), we get

$$2\big(b+f(t,p)\big)^z \exp\left(\sum_{k=1}^{r} u_k t^k\right)\sin(\varphi t)\int_c^d \exp(xt)dx = \sum_{w=0}^{\infty}\frac{t^w}{w!}\int_c^d \mathfrak{h}_2(w,x,\varphi,z;\vec{u},r,p,b)dx.$$

After some calculations, we obtain

$$\sum_{w=0}^{\infty}\frac{t^{w}}{w!}\int_{c}^{d}\mathfrak{h}_{2}(w,x,\varphi,z;\vec{u},r,p,b)dx$$
$$=\sum_{w=0}^{\infty}\frac{\mathfrak{h}_{2}(w+1,d,\varphi,z;\vec{u},r,p,b)}{(w+1)!}t^{w}-\sum_{w=0}^{\infty}\frac{\mathfrak{h}_{2}(w+1,c,\varphi,z;\vec{u},r,p,b)}{(w+1)!}t^{w}.$$

Matching the terms of $\frac{t^{w}}{w!}$ in both expressions brings us to the intended result.

Remark 3.5. Using integral methods with generating functions, we also presented some integral representations for these polynomials, see for detail, [8].

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PROBABILISTIC PRIMALITY TESTS AND RSA ALGORITHM

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ABSTRACT

The security of the RSA algorithm is based on the difficulty of the integer factorisation problem. Two large prime numbers are needed to construct an RSA algorithm for each user. This leads to the issue of generating large prime numbers in cryptography. In the literature, there are two main primality test methods: probabilistic and deterministic primality tests. This paper reviews the main probabilistic primality tests such as the Fermat, Slovay-Strassen and Miller-Rabin test algorithms. Then we evaluate and compare their performance based on their execution times for different sizes of inputs. We present performance analyses based on their execution times. We also review the RSA encryption algorithm that uses two sufficiently large prime numbers.

Keywords: *Cryptology, Prime numbers, Probabilistic Primality tests, RSA algorithm*

1. INTRODUCTION

The Integer Factorization Problem (IFP) is assumed as a difficult problem in mathematics for sufficiently large prime numbers. The security of the RSA algorithm is based on the difficulty of the IFP for the product of two large prime numbers. Thus, to ensure the security of the RSA algorithm, sufficiently large prime numbers must be generated. This is a challenging problem in cryptography (indeed, in number theory). In the literature, the current deterministic primality tests are not efficient for large numbers. In this context, the probabilistic primality tests are used to generate large prime numbers for the RSA algorithm.

Prime numbers were first studied in detail by the mathematicians of the Pythagorean school in ancient Greece between 500 - 300 BC. In 200 BC, Eratosthenes developed a method for finding prime numbers and named this method the "Sieve of Eratosthenes." It is a well-known fact that every natural number can be expressed as a product of the powers of prime numbers. Moreover, the number of prime numbers is infinite. In the literature, numerous scientists have studied the characterization of prime numbers and discovered significant results on prime numbers. However, any efficient deterministic primality test algorithm hasn't yet been proposed in the literature to test sufficiently large numbers.

Public key cryptosystems based on prime numbers are frequently used for encryption and signature processes in real life. Sufficiently large prime numbers are required to ensure the security of certain public key cryptosystems. Thus, prime numbers are always needed in cryptography. The mystery of prime numbers, which is still not fully understood, increases interest in mathematics and computer science. Primality tests are among the first studies conducted on prime numbers.

In the main paper [6], the author aims to introduce quite modern cryptography and applications. An algorithmic approach has been emphasized with a focus on efficiency estimates. In the paper [13], a deterministic testing method has been developed to check whether an odd number is prime. In the paper

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[7], a new method for finding prime numbers has been provided and a perfect secure prime number sequence has been defined. In the paper [12], it has been observed that if Michael O. Rabin's primality test failed with a 25% probability on every composite number, factoring would be easy. The reliability of Rabin's test when used to generate a random integer that is probably prime, rather than testing a specific number for primality, is also among the topics of research.

The paper is organised as follows. In Section 2, the probabilistic primality tests, including the Fermat, Solovay-Strassen and Miller-Rabin tests are discussed. These tests allow us to determine whether an odd number is composite or prime with high probability. In Section 3, we address the security of the RSA algorithm based on two large prime numbers. In Section 4, the performance analyses of the probabilistic primality tests are provided in terms of their running time.

2. PROBABILISTIC PRIMALITY TESTS

In this section, we discuss the probabilistic primality tests such as the Fermat, Slovay-Strassen and Miller-Rabin tests.

Prime numbers and their properties were first studied in detail by mathematicians of the Pythagorean school in ancient Greece between $\mathbf{500-300}$ BCE. In $\mathbf{200}$ BCE, Eratosthenes developed the sieve method to find prime numbers, named the "Sieve of Eratosthenes." The Sieve of Eratosthenes is a method used to find prime numbers up to a certain integer. However, this method is not practical to test very large numbers, which is why it is not used in cryptography. Therefore, in cryptography, probabilistic primality tests are used to test sufficiently large prime numbers.

Probabilistic Primality Tests are used to test whether an odd number is composite or prime with high probability. The most commonly used probabilistic primality tests are the Fermat, Solovay-Strassen, and Miller-Rabin tests.

We first provide some probabilistic definitions before introducing the probabilistic primality tests. The probabilistic primality test is based on the concept of a witness and a liar.

Definition 1. [13] For a number n, if there is a number a between 1 and $n-1$ such that a confirms that n is a composite number according to the test, then a is called *witness* for the composite number n. On the other hand, there may exist a number a that says a composite number n may be prime. Such a number a is called a *liar* for a composite number n.

Note that when the liar a is used in the test, the test will incorrectly declare a composite number n to be prime. To avoid such errors, repeating the test t times (for a sufficiently large value t) will further reduce the probability of error.

2.1. Fermat's Probabilistic Primality Test

The Fermat Probabilistic Primality Test is the first test that forms the basis of probabilistic primality tests. It is based on Fermat's little theorem, which was proposed by Fermat in 1640. Fermat's little theorem can be stated as follows.

Theorem 1. [6] (Fermat's little theorem) If p is an odd prime number and if a is any integer which is not a multiple of p, then we have the congruence

$$a^{p-1} \equiv 1 \bmod p \tag{1}$$

Usually, we assume that $1 \leq a \leq p-1$. For $a=1$ and $a=p-1$, it is trivial that $a^{p-1} \equiv 1 \bmod p$. Thus, in the Fermat test, we assume that $2 \leq a \leq p-2$.

The equivalent statement of Theorem 1: if $a^{p-1} \not\equiv 1 \bmod p$ for at least one base a with $2 \leq a \leq p-2$, then p is not a prime (namely, a composite number). On the other hand, if $a^{p-1} \equiv 1 \bmod p$ for some base number a with $2 \leq a \leq p-2$, then p may still be a prime or composite number. In this case, we cannot say that p is an odd prime number, but we call p as a pseudoprime number with a base a.
The Fermat probabilistic primality test is based on Fermat's little theorem. For simplicity, we refer to the Fermat test. According to the above observation, we below define the Fermat test.

Fermat Test: Let $n \geq 3$ be an odd integer, pick randomly some number a with $2 \leq a \leq n-2$.

If the congruence $a^{n-1} \not\equiv 1 \bmod n$, then return "$n$ is composite," else return "n is pseudoprime base a".

In the Fermat test, the congruence in (1) is checked for t different values of base a with $2 \leq a \leq n-2$ to determine whether the number n is a composite or pseudoprime number with a certain error rate. The algorithm of the Fermat test is given below for an odd number n.

Algorithm 1. Fermat's Test Algorithm
Input: n and $t \in \mathbb{Z}^+$
Output: n is a composite or a pseudoprime with the error rate $E_n(t)$.
For pick randomly an integer a with $2 \leq a \leq n-2$
    $d \leftarrow \gcd(a, n)$
    if $d > 1$ **return** "composite"
    else $b \leftarrow a^{n-1} \bmod n$
    end if
    if $b \neq 1$ **return** "composite"
    end if
end for
return n is a pseudoprime with the error rate $E_n(t)$

Example 1. We verify whether 571 is composite or pseudoprime by the Fermat test.
Input: $n = 571$ and $t = 3$ iterations.
1: **For** pick randomly an integer a with $2 \leq a \leq 569$
2: For $a = 2$, $a^{n-1} = 2^{570} \equiv 1 \bmod 571$
3: For $a = 42$, $a^{n-1} = 42^{570} \equiv 1 \bmod 571$
4: For $a = 123$, $a^{n-1} = 123^{570} \equiv 1 \bmod 571$
Output: 571 is a pseudoprime number.

Definition 2. Let n be an odd composite integer and a be an integer with $1 \leq a \leq n-1$.

- An integer a with $2 \leq a \leq n-2$ is called a *Fermat witness* if $a^{n-1} \not\equiv 1 \bmod n$. In other words, an integer a approves that n is composite.
- An integer a with $1 \leq a \leq n-1$ is a *Fermat liar* for n if $a^{n-1} \equiv 1 \bmod n$.

Definition 3. (Carmichael Numbers**)** In the Fermat primality test, some composite numbers can give misleading results. These composite numbers pass the Fermat primality test for any base although they are not prime numbers. These numbers are called Carmichael numbers. Initially, in 1910, R. D. Carmichael discovered such numbers.

According to Fermat's little theorem, for n to be a prime number, for every base a, $a^n - a$ must divide a. However, there are composite Carmichael numbers that satisfy this division. Therefore, the Fermat test fails to detect Carmichael numbers.

Example 2. We verify whether 561 is a pseudoprime or composite by the Fermat test.
Input: $n = 561$ with $t = 5$ iterations.

1: For pick randomly an integer a with $2 \leq a \leq 559$
2: For $a = 13$, $a^{n-1} = 13^{560} \equiv 1$ (mod 561)
3: For $a = 29$, $a^{n-1} = 3^{560} \equiv 1$ (mod 561)
4: For $a = 52$, $a^{n-1} = 52^{560} \equiv 1$ (mod 561)
5: For $a = 76$, $a^{n-1} = 76^{560} \equiv 1$ (mod 561)
6: For $a = 125$, $a^{n-1} = 125^{560} \equiv 1$ (mod 561)
Output: 561 is a pseudoprime number

However, 561 = 3 · 11 · 17 is a composite number. The base numbers $a = 13, 29, 52, 76$ and 125 are Fermat liars for the composite number 561. Such numbers are called Carmichael numbers.

2.2. Solovay-Strassen Primality Test

The Solovay-Strassen primality test, developed by Robert Solovay and Volker Strassen, is the first probabilistic primality test used in Public Key Cryptography. This test is based on the Jacobi symbol and Euler's criterion. The Jacobi symbol is a generalization of the Legendre symbol, introduced by Jacobi in 1837.

Jacobi Symbol. [12] Given any positive odd integer $\boldsymbol{n}$ and any integer $\boldsymbol{a}$, the Jacobi symbol $\left(\frac{\boldsymbol{a}}{\boldsymbol{n}}\right)$ is defined as

$$\left(\frac{a}{n}\right) = \begin{cases} 1 \text{ if } a \text{ is a quadratic residue mod } n \\ -1 \text{ if } a \text{ is a quadratic non} - \text{residue mod } n \\ 0 \text{ if } a \text{ divides } n \end{cases}$$

Theorem 2. (Euler's Criterion) If $\boldsymbol{p}$ is an odd prime number and $\boldsymbol{a}$ is a positive integer satisfying $(\boldsymbol{a}, \boldsymbol{p}) = \mathbf{1}$, then the following congruence holds:

$$\boldsymbol{a}^{(\boldsymbol{p}-\mathbf{1})/\mathbf{2}} \equiv \left(\frac{\boldsymbol{a}}{\boldsymbol{p}}\right) \textbf{ mod } \boldsymbol{p}$$

Equivalently, if this congruence does not hold, then $\boldsymbol{p}$ is a composite number. On the other hand, if this congruence holds for at least one base $\boldsymbol{a}$, then $\boldsymbol{p}$ is pseudoprime for base $\boldsymbol{a}$.

According to these observations, the Solovay-Strassen primality test is defined as follows.

Solovay-Strassen Test [12]: Let $\boldsymbol{n}$ be an odd number and $\boldsymbol{a}$ be a number with $\mathbf{1} \leq \boldsymbol{a} \leq \boldsymbol{n} - \mathbf{1}$. If

$$\boldsymbol{a}^{(\boldsymbol{n}-\mathbf{1})/\mathbf{2}} \equiv \left(\frac{\boldsymbol{a}}{\boldsymbol{n}}\right) \textbf{ mod } \boldsymbol{n}$$

then $\boldsymbol{n}$ is called *pseudoprime* with the base $\boldsymbol{a}$. Otherwise, $\boldsymbol{n}$ is a composite number.

Definition 4. Let $\boldsymbol{n}$ be an odd composite number and $\boldsymbol{a}$ is a number in the range $\mathbf{1} \leq \boldsymbol{a} \leq \boldsymbol{n} - \mathbf{1}$.

- If $\boldsymbol{a}^{(\boldsymbol{n}-\mathbf{1})/\mathbf{2}} \not\equiv \left(\frac{\boldsymbol{a}}{\boldsymbol{n}}\right)$ **mod** $\boldsymbol{n}$, $\boldsymbol{a}$ is called an *Euler witness* of $\boldsymbol{n}$.
- If $\boldsymbol{a}^{(\boldsymbol{n}-\mathbf{1})/\mathbf{2}} \equiv \left(\frac{\boldsymbol{a}}{\boldsymbol{n}}\right)$ **mod** $\boldsymbol{n}$ although $\boldsymbol{n}$ is an odd composite number, $\boldsymbol{a}$ is called an *Euler liar* of $\boldsymbol{n}$.

If n is a prime number, the probability that a is a witness at least 50%. This test is repeated $\boldsymbol{t}$ times using $\boldsymbol{t}$ different values of $\boldsymbol{a}$. The probability of a composite number passing the test for $\boldsymbol{t}$ times are at most $\frac{\mathbf{1}}{\mathbf{2}^{\boldsymbol{t}}}$. The algorithm for this test is given below.

Algorithm 2: Solovay-Strassen Test Algorithm
Input: An odd positive integer n and $t \in \mathbb{Z}^+$
Output: n is either composite or pseudoprime with the error rate $E_n(t)$.
For pick randomly an integer a with $1 \le a \le n-1$
$d \leftarrow \gcd(a, n)$
if $d > 1$ **return** "composite"
else $b \leftarrow a^{\frac{n-1}{2}} \bmod n$
end if
if $b \neq \pm 1$ **return** "composite"
end if
$J \leftarrow \left(\frac{a}{n}\right)$
if $b \neq J \bmod n$ **return** "composite"
end if
end for
return n is pseudoprime with the error rate $E_n(t)$

Example 3. We determine if 349 is composite or pseudoprime by the Solovay-Strassen test.
Input: $n = 349,\ t = 3 \in \mathbb{Z}^+$

1: For $a = 2,\ b = -1 \leftarrow 2^{(349-1)/2} \bmod 349$
2: $J = -1 \leftarrow \left(\frac{2}{349}\right)$
1: For $a = 3,\ b = -1 \leftarrow 3^{(349-1)/2} \bmod 349$
2: $J = -1 \leftarrow \left(\frac{3}{349}\right)$
1: For $a = 5,\ b = -1 \leftarrow 5^{(349-1)/2} \bmod 349$
2: $J = -1 \leftarrow \left(\frac{5}{349}\right)$
Output: 349 is a pseudoprime number.

We finally review the Miller-Rabin probabilistic primality test, which is faster and has a lower error rate compared to the Solovay-Strassen test and the others.

2.3. Miller-Rabin Primality Test

One of the most commonly preferred techniques for testing the primality of a given large odd number is the Miller-Rabin (M-R) probabilistic primality test. This test was developed by Michael Rabin based on the idea of Gary Miller and is particularly known for its low error rate.
In the Miller-Rabin probabilistic test, to determine whether a given odd number $\boldsymbol{n}$ is prime, the first step is to find the values of $\boldsymbol{s}$ and $\boldsymbol{r}$ such that $\boldsymbol{n - 1 = 2^s\, r}$.

Theorem 3. Let $\boldsymbol{p}$ be a positive odd integer and $\boldsymbol{a}$ be a number with $\boldsymbol{1 \le a \le p-1}$. Write $\boldsymbol{p - 1 = 2^s r}$, where $\boldsymbol{r}$ is an odd integer and $\boldsymbol{s}$ is an integer. If $\boldsymbol{p}$ is an odd prime number, then the equation

$\boldsymbol{a^r \equiv 1 \bmod p}$ holds or the equation $\boldsymbol{a^{2^j r} \equiv -1 \bmod p}$ holds for any $\boldsymbol{j}$ with $\boldsymbol{0 \le j \le s-1}$.

Equivalently, if the equation $\boldsymbol{a^r \not\equiv 1 \bmod p}$ and the equation $\boldsymbol{a^{2^j r} \not\equiv -1 \bmod p}$ for every $\boldsymbol{j}$ with $\boldsymbol{0 \le j \le s-1}$, then $\boldsymbol{p}$ is a composite number. On the other hand, for an integer $\boldsymbol{a}$ in the range $\boldsymbol{1 \le a \le p-1}$, if the equation $\boldsymbol{a^r \equiv 1 \bmod p}$ holds, or if for $\boldsymbol{0 \le j \le s-1}$, the equation $\boldsymbol{a^{2^j r} = 1 \bmod p}$ holds, then $\boldsymbol{p}$ is considered as a pseudoprime for the base $\boldsymbol{a}$.

In view of the above observations, one can check whether a positive odd integer $\boldsymbol{n}$ is prime. This method is called the Miller-Rabin test.

Miller-Rabin Test [5]: Let $\boldsymbol{n}$ be a positive odd integer and $\boldsymbol{a}$ be a number with $\mathbf{1} \leq \boldsymbol{a} \leq \boldsymbol{n} - \mathbf{1}$. Write $\boldsymbol{n} - \mathbf{1} = \mathbf{2}^{\boldsymbol{s}}\boldsymbol{r}$, where $\boldsymbol{r}$ is an odd integer and $\boldsymbol{s}$ is an integer.

- If the equation $\boldsymbol{a}^{\boldsymbol{r}} \not\equiv \mathbf{1} \textbf{ mod } \boldsymbol{p}$ and the equation $\boldsymbol{a}^{\mathbf{2}^{\boldsymbol{j}}\boldsymbol{r}} \not\equiv -\mathbf{1} \textbf{ mod } \boldsymbol{p}$ for every $\boldsymbol{j}$ with $\mathbf{0} \leq \boldsymbol{j} \leq \boldsymbol{s} - \mathbf{1}$, then $\boldsymbol{p}$ is a composite number.
- If $\boldsymbol{a}^{\boldsymbol{r}} \equiv \mathbf{1} \textbf{ mod } \boldsymbol{n}$ or $\boldsymbol{a}^{\mathbf{2}^{\boldsymbol{j}}\boldsymbol{r}} \equiv -\mathbf{1} \textbf{ mod } \boldsymbol{n}$ holds for any $\boldsymbol{j}$ in the range $\mathbf{0} \leq \boldsymbol{j} \leq \boldsymbol{s} - \mathbf{1}$, then $\boldsymbol{n}$ is called a pseudoprime for the base $\boldsymbol{a}$.

The algorithm of this test is given below.

Algorithm 3. Miller-Rabin Test Algorithm

Input: Positive odd integer n and $t \in \mathbb{Z}^+$

Output : n is either composite or prime with the error rate $E_n(t)$.

Write $n - 1 = 2^s r$ where r is an odd integer

for pick randomly an integer a with $1 \leq a \leq n - 1$

    $d \leftarrow$ gcd (a, n)

    if $d > 1$ **return** “composite”

    else $b \leftarrow a^r$ mod n

    end if

    if $b \neq \pm 1$

        for j from 1 to $s - 1$

            $c \leftarrow a^{2^j r}$ mod n

            if $c = 1$ **return** “composite”

            end if

        end for

        if $c \neq -1$ **return** “composite”

        end if

    end if

end for

return n is a pseudoprime with the error rate $E_n(t)$

Definition 5. Let $\boldsymbol{n}$ be an odd composite number and $\boldsymbol{a}$ is a number in the range $\mathbf{1} \leq \boldsymbol{a} \leq \boldsymbol{n} - \mathbf{1}$. Write $\boldsymbol{n} - \mathbf{1} = \mathbf{2}^{\boldsymbol{s}}\boldsymbol{r}$, where $\boldsymbol{r}$ is an odd integer and $\boldsymbol{s}$ is an integer.

- If the equation $\boldsymbol{a}^{\boldsymbol{r}} \not\equiv \mathbf{1} \textbf{ mod } \boldsymbol{p}$ and the equation $\boldsymbol{a}^{\mathbf{2}^{\boldsymbol{j}}\boldsymbol{r}} \not\equiv -\mathbf{1} \textbf{ mod } \boldsymbol{p}$ for every $\boldsymbol{j}$ with $\mathbf{0} \leq \boldsymbol{j} \leq \boldsymbol{s} - \mathbf{1}$, then then $\boldsymbol{a}$ is called a "strong witness" for $\boldsymbol{n}$.
- If $\boldsymbol{a}^{\boldsymbol{r}} \equiv \mathbf{1} \textbf{ mod } \boldsymbol{n}$ or $\boldsymbol{a}^{\mathbf{2}^{\boldsymbol{j}}\boldsymbol{r}} \equiv -\mathbf{1} \textbf{ mod } \boldsymbol{n}$ holds for any $\boldsymbol{j}$ in the range $\mathbf{0} \leq \boldsymbol{j} \leq \boldsymbol{s} - \mathbf{1}$ although $\boldsymbol{n}$ is an odd composite number, $\boldsymbol{a}$ is called a strong *liar* of $\boldsymbol{n}$.

Example 4. We apply the Miller-Rabin test to check whether 91 is prime.

Input: $n = 91$

Write $n - 1 = 90 = 2 \cdot 45$, where $s = 1,\ r = 45$

For $a = 2,\ b = a^r = 2^{45} \equiv 57$ mod 91

Since $b \neq \pm 1$ mod 91, **return** “composite”

Output: 91 is composite

Thus, $a = 2$ is a strong witness. Moreover, we test it for different base numbers.
Input: $n = 91$ and $t = 3$,
Write $n - 1 = 90 = 2 \cdot 45$, where $s = 1,\ r = 45$
For $a = 9$, $b = a^r = 9^{45} \equiv 1 \bmod 91$
For $a = 16$, $b = a^r = 16^{45} \equiv 1 \bmod 91$
For $a = 75$, $b = a^r = 75^{45} \equiv 1 \bmod 91$
Output: 91 is a pseudoprime number.

For randomly selected values of a in the range $1 \leq a \leq 90$, the result indicated that 91 is a pseudoprime number. Since 91=7*13 is not a prime number, $a = 9, a = 16, a = 75$ are strong liars.

3. RSA ALGORITHM

In this section, we review the RSA algorithm as an application of large prime numbers. Whitfield Diffie and Martin Hellman introduced public-key cryptography in 1976. Then, in 1977, Ronald Rivest, Adi Shamir and Leonard Adleman proposed the RSA cryptosystem, which became the most widely used public-key cryptography scheme [10].

In the paper [11], after defining RSA, they discuss how it can be used in the upcoming era of electronic mail. This system is based on the factorisation problem. The security of RSA relies on the difficulty of factoring a large integer that is the product of two sufficiently large prime numbers. The reliability of the algorithm is directly proportional to the size of the prime numbers used; however, due to the modular exponential nature of encryption and decryption processes, it presents time-related disadvantages. The RSA cryptosystem is the most widely used public-key cryptography scheme. Today, RSA is used in many applications such as SSL, S-HTTP, S-MIME, S/WAN, and STT. It is also used in web security certificates for credit card transactions.
In the paper [9], the measurement of the distance between the selected primes p and q for RSA is defined, and applications are provided. In the book [10] the authors explain the most important techniques of modern cryptography. In the paper [7], the author has used the perfect secure prime number sequence defined in a new method for finding prime numbers in the RSA encryption method.

3.1. The Structure of the RSA Algorithm

There are three main components in the RSA algorithm. The first step is to generate a key pair, consisting of a public key and the corresponding private key.

RSA Key Generation

1. Two distinct large prime numbers $\boldsymbol{p}$ and $\boldsymbol{q}$ are generated.
2. The value of $\boldsymbol{n} = \boldsymbol{p} \cdot \boldsymbol{q}$ is calculated.
3. The value of $\boldsymbol{\varphi(n) = (p - 1) \cdot (q - 1)}$ is calculated.
4. A random number $\boldsymbol{e}$ is selected from $\boldsymbol{1 < e < \varphi(n)}$ such that $\mathbf{gcd}\,\boldsymbol{(e, \varphi(n)) = 1}$.
5. The value of $\boldsymbol{d}$ is found such that $\boldsymbol{e \cdot d \equiv 1 \bmod (\varphi(n))}$.

The pairs $\boldsymbol{(n, e)}$ are the public parameters, and $\boldsymbol{(p, q, (\varphi(n)), d)}$ are the private parameters. The RSA modulo parameter $\boldsymbol{n}$ is always public. The parameter $\boldsymbol{e}$ is the encryption key and the parameter $\boldsymbol{d}$ is the decryption key.

Below are the steps that person A would follow for RSA encryption to encrypt a message $\boldsymbol{m}$ and send the encrypted message to person B.

RSA Encryption

- The person A obtains the person B's public key, which is the pair $(\boldsymbol{n}, \boldsymbol{e})$.
- The message $\boldsymbol{m}$ is written in the range $[\mathbf{0}; \boldsymbol{n}-\mathbf{1}]$.
- Then, $\boldsymbol{c} \equiv \boldsymbol{m}^{\boldsymbol{e}} \ \mathbf{mod}\ \boldsymbol{n}$ is calculated.
- Finally, A sends the encrypted message $\boldsymbol{c}$ to person B.

The process that person B will perform to decrypt the encrypted message $\boldsymbol{c}$ from person A is outlined below.

RSA Decryption

The person B, who wants to decrypt the encrypted message $\boldsymbol{c}$ sent by person A, uses their private key d to calculate: $\boldsymbol{m} \equiv \boldsymbol{c}^{\boldsymbol{d}}\ \mathbf{mod}\ \boldsymbol{n}$ and thus obtains the message $\boldsymbol{m}$.

3.2. Security of the RSA Algorithm

The security of the RSA algorithm derives from the difficulty of factoring large numbers. The public and private keys are functions of a pair of large prime numbers. RSA, one of the public-key encryption algorithms, uses two different keys. Plaintext encrypted with the public key can only be decrypted with the private key. The security of the RSA algorithm relies on selecting very large prime numbers. To ensure the system's security, it is crucial to generate values for p and q, and thus n, that are resistant to factorization algorithms. Therefore, the parameters p and q should be selected according to certain criteria. The selected parameters provide a security level that is proportional to the size of the n parameter [12].

4. THE PERFORMANCE ANALYSES OF THE PRIMALITY TESTS

In this section, we discuss the performance analyses of the probabilistic primality test algorithms. We implement the algorithms of the probabilistic primality tests given in **Algorithms 1,2** and **3.**

This section aims to perform and compare the performance analyses of probabilistic primality tests. When analysing the performance of these tests, criteria such as runtime, memory requirements, and the number of operations performed are considered. Among the probabilistic primality tests, the three main tests, namely Fermat, Solovay-Strassen and Miller-Rabin are compared, and it is found that the Miller-Rabin test performs better in terms of error rate and runtime. The reason for this is that the Fermat test is weak in detecting Carmichael numbers. The Solovay-Strassen test takes longer due to the increased runtime caused by Jacobi symbol calculations. Additionally, while the Solovay-Strassen test operates with an error rate of $(\mathbf{1/2})^{\boldsymbol{t}}$, the Miller-Rabin test provides more accurate results with an error rate of $(\mathbf{1/4})^{\boldsymbol{t}}$ (see in [12] for more detail).

Below, we compare the performance of the probabilistic Primality Tests (Fermat, Miller-Rabin, and Solovay-Strassen) in terms of runtime for numbers with digit lengths ranging from 2 to 10.

Fermat Test: The Fermat test runtimes for numbers with digit lengths ranging from 2 to 10 are provided in Table 1. Here, the time taken for the largest-digit number $\boldsymbol{p} = \mathbf{2147483647}$ is 7.18 seconds.

Table 1. Fermat Test runtime

FERMAT TEST		
Number of Digits	Mersenne Number	Runtime (s)
2	31	1,84
4	1023	1,84
6	262143	3,15
8	16777215	6,04
10	2147483647	7,18

Solovay-Strassen Test: The Solovay-Strassen Test runtimes for numbers with digit lengths ranging from 2 to 10 are provided in Table 2. Here, the time is taken for the largest-digit number $p = 2147483647$ is 5.62 seconds.

Table 2. Solovay-Strassen Test runtime

SOLOVAY-STRASSEN TEST		
Number of Digits	Mersenne Number	Runtime (s)
2	31	1,84
4	1023	1,84
6	262143	3,15
8	16777215	4,04
10	2147483647	5,62

Miller-Rabin Test: Miller-Rabin Test runtimes for numbers with digit lengths ranging from 2 to 10 are provided in Table 3. Here, the time is taken for the largest-digit number $p = 2147483647$ is 3.22 seconds.

Table 3. Miller-Rabin Test runtime

MILLER-RABIN TEST		
Number of Digits	Mersenne Number	Runtime (s)
2	31	1,67
4	1023	1,80
6	262143	3,00
8	16777215	3,10
10	2147483647	3,22

When we perform the performance analysis for numbers in the range of 20 to 200 digits using the Miller-Rabin test and the Solovay-Strassen test, the data shows that the Miller-Rabin test reaches the result faster.

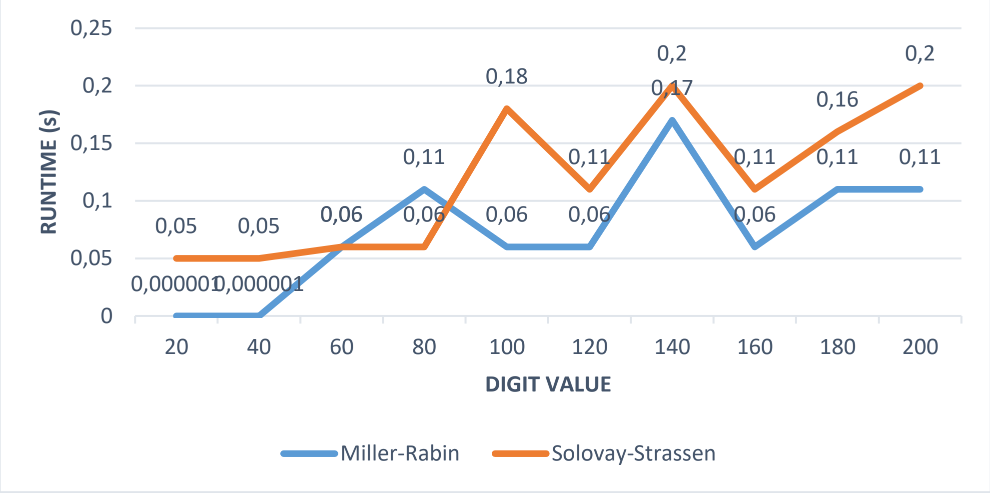

Figure 1. Comparison of the runtime of the Miller-Rabin and Solovay-Strassen tests

5.CONCLUSION

The RSA algorithm is the most popular public-key cryptosystem. This cryptosystem has both encryption and signature algorithms. The security of the RSA cryptosystem is based on the hardness of the integer factorisation problem for two sufficiently large prime numbers. To design the RSA cryptosystem for each person, two sufficiently large prime numbers are needed. Thus, finding sufficiently large prime numbers is a significant problem in the literature. To determine whether large odd numbers are prime, probabilistic primality tests such as Fermat, Solovay-Strassen and Miller-Rabin tests have been examined in this work. Moreover, performance analyses of the Fermat, Solovay-Strassen, and Miller-Rabin tests have been discussed, and their runtimes have been compared. Based on the obtained experimental results, it was concluded that the Miller-Rabin probabilistic primality test is more efficient in terms of speed and performance criteria.

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FORMULAS RELATED TO APOSTOL-GENOCCHI POLYNOMIALS AND NUMBERS

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ABSTRACT

Spline curve families have garnered significant attention in recent years due to their wide-ranging applications in fields such as mathematics, mathematical modeling, probability, statistics, and other applied sciences. Additionally, spline curves hold particular importance in computer-aided design, surface modeling, animation production, and more. In three-dimensional design, spline curve families play a crucial role. This study defines new types of spline curve families and explores their relationships with special numbers, special polynomials, and special functions. The results are expected to contribute to the theory of spline curves and extend their applications to various disciplines including mathematics, medicine, engineering, economics, and robotics.

Keywords: *Frobenius Euler numbers and polynomials, Exponential Euler Spline, Special numbers and polynomials, Generating functions, Beta-type rational functions.*

1. INTRODUCTION

The history of spline curves dates back to ancient times; however, the mathematical theory of these curves was first constructed by Isaac Schoenberg during the 1940s and 1950s. Spline curves, defined as piecewise polynomials with continuous derivatives at their knot points, are widely used in fields ranging from scientific computing and engineering to industrial design. Prominent special families of spline curves, such as cardinal splines, B-splines, exponential splines, and exponential Euler splines, are frequently employed due to their simplicity and effectiveness in curve fitting.

Among these, Bézier curves stand out as a distinguished member of the spline curve family, developed independently by Paul de Casteljau and Pierre Bézier in 1959. Bézier curves are widely used in computer-aided design (CAD), modeling applications, and the animation industry. Recent studies have further advanced the field by exploring novel spline curve types and their applications (*cf*. [1-24]).

This research contributes to the theory of spline curves by introducing new spline families, examining their mathematical properties, and demonstrating their applications in areas such as trajectory planning, velocity and acceleration analysis, and surface modeling. The implications of these findings extend beyond mathematics to practical domains, enhancing the versatility of spline curves in real-world problem-solving.

The sets of natural numbers, integers, real numbers, complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

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The Apostol- Genocchi higher-order numbers and polynomials, respectively, as follows:

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}=\sum_{n=0}^{\infty}\frac{\mathcal{G}_n^{(\alpha)}(\lambda)}{n!}t^n \tag{1}$$

and

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}e^{tx}=\sum_{n=0}^{\infty}\frac{\mathcal{G}_n^{(\alpha)}(x;\lambda)}{n!}t^n \tag{2}$$

(*cf*. [11,12,23,24]).
The Frobenius-Euler higher-order numbers and polynomials, respectively, as follows:

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha}=\sum_{n=0}^{\infty}\frac{H_n^{(\alpha)}(\lambda)}{n!}t^n \tag{3}$$

and

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha}e^{tx}=\sum_{n=0}^{\infty}\frac{H_n^{(\alpha)}(x;\lambda)}{n!}t^n \tag{4}$$

(*cf*. [2,9,10]).
The beta-type rational functions $\mathfrak{M}_{j,n}(\lambda)$ are defined by means of the following generating functions:

$$\left(\frac{\lambda}{\lambda+1}\right)^{j}e^{t(\lambda+1)}=\sum_{n=0}^{\infty}\frac{\mathfrak{M}_{j,n}(\lambda)}{n!}t^n. \tag{5}$$

The beta-type rational functions are defined by

$$\mathfrak{M}_{j,n}(\lambda)=\lambda^j(1+\lambda)^{n-j}, \tag{6}$$

where $n,j\in\mathbb{N}_0$ and $\lambda\in\mathbb{R}$, set of real numbers, (or $\mathbb{C}$, set of complex numbers) (*cf*. [17]).

The Stirling numbers of the second kind $S_2(r,d)$ are defined by

$$\frac{(e^t-1)^d}{d!}=\sum_{r=0}^{\infty}\frac{S_2(r,d)}{r!}t^r \tag{7}$$

(*cf*. [3,5,6,8,15,16]).

Recently, we [7] defined the polynomials $u_n^{(\alpha)}(x;\lambda)$ by means of the following generating function:

$$\left(\frac{1+\lambda}{\lambda e^t+1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} \frac{u_n^{(\alpha)}(x;\lambda)}{n!} t^n. \tag{8}$$

A relation between the polynomials $u_n^{(\alpha)}(x;\lambda)$ and $H_n^{(\alpha)}(x;\lambda)$ is given by

$$u_n^{(\alpha)}(x;\lambda) = H_n^{(\alpha)}\left(x;-\frac{1}{\lambda}\right) \tag{9}$$

(*cf*. [7]).

The Apostol Euler numbers of higher order are defined by

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n^{(\alpha)}(\lambda)}{n!} t^n \tag{10}$$

(*cf*. [2,4,7,9,10,12,15,18,23]).
Substituting (10) into above equation, we have the Apostol Euler numbers of higher negative order are defined by

$$\left(\frac{\lambda e^t+1}{2}\right)^{\alpha} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n^{(-\alpha)}(\lambda)}{n!} t^n \tag{11}$$

and also

$$\mathcal{E}_n^{(\alpha)}(\lambda) = \sum_{j=1}^{n} 2^{\alpha}(-1)^j S_2(n,j)\lambda^j \frac{(\alpha)^{(j)}}{(\lambda+1)^{j+\alpha}} \tag{12}$$

(*cf*. [2,4,7,9,10,12,15,18,23]).
$(\alpha)^{(n)}$ denotes the rising factorial (the Pochhammer symbol) defined by

$$(\alpha)^{(n)} = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+n-1), & n \in \mathbb{N} \\ 1, & n=0 \end{cases} \tag{13}$$

for $(\alpha)^{(n)}$ see also [5].
Combining (8) and (10), we have

$$u_n^{(\alpha)}(x;\lambda) = \left(\frac{1+\lambda}{2}\right)^{\alpha} \mathcal{E}_n^{(\alpha)}(x;\lambda) \tag{14}$$

and

$$u_n^{(\alpha)}(\lambda) = \left(\frac{1+\lambda}{2}\right)^{\alpha} \mathcal{E}_n^{(\alpha)}(\lambda) \tag{15}$$

(*cf*. [2,4,7,9,10,12,15,18,23]).

The following integral formulas for the function $\mathfrak{M}_{j,n}(\lambda)$, as given by Simsek [17, Eqs. (18)-(19)], are presented below:

$$\int_{-1}^{0} \mathfrak{M}_{j,n}(\lambda)d\lambda = \sum_{k=0}^{n-j} (-1)^{n-k} \binom{n-j}{k} \frac{1}{n+1-k} \tag{16}$$

and

$$\int_{-1}^{0} \mathfrak{M}_{j,n}(\lambda)d\lambda = (-1)^{j} \frac{1}{(n+1)\binom{n}{j}} \tag{17}$$

(*cf.* [7,17]).
In Reference (7), the newly defined exponential splines are presented as follows:

$$Y(x;\lambda;n,\alpha) = \frac{u_n^{(\alpha)}\left(x;-\frac{1}{\lambda}\right)}{u_n^{(\alpha)}\left(-\frac{1}{\lambda}\right)} \tag{18}$$

and

$$Y(x;\lambda;n,\alpha) = \frac{H_n^{(\alpha)}(x;\lambda)}{H_n^{(\alpha)}(\lambda)} \tag{19}$$

(*cf.* [7]).

2. CALCULATION FORMULAS FOR APOSTOL-GENOCCHI NUMBERS AND POLYNOMIALS

In this section, new formulas for some combinatorial sums involving Beta-type rational functions, Apostol-Genocchi numbers of order $-\boldsymbol{m}$, and Stirling numbers of the second kind will be presented. Additionally, integral formulas using Beta-type rational integrals will be provided for certain finite combinatorial sums.

The primary focus of this study is the introduction of new spline curve families and their mathematical and practical relevance.

The relationship between Equation (12) and the Apostol-Genocchi numbers is written, and then $\alpha = -n$ is applied, the following result is obtained:

Theorem 2.1. For $d, n \in \mathbb{N}$, we have

$$\mathcal{G}_{d-n}^{(-n)}(\lambda) = \frac{2^{-n} d!}{(\mathrm{d}-\mathrm{n})!} \sum_{j=1}^{d} (-1)^{j} S_2(d,j) \lambda^{j} \frac{(-n)^{(j)}}{(\lambda+1)^{j-n}}. \tag{20}$$

The result obtained by combining the Beta-type rational function given in (6) with Theorem (2.1) is as follows:

Corollary 2.2. For $d, n \in \mathbb{N}$, we have

$$\mathcal{G}_{d-n}^{(-n)}(\lambda) = \frac{2^{-n}d!}{(\mathrm{d}-\mathrm{n})!}\sum_{j=1}^{d}\binom{n}{j}S_2(d,j)\mathrm{j}!\,\mathfrak{M}_{j,n}(\lambda). \tag{21}$$

Integrating both sides of the relation (21) for $\mathcal{G}_{d-n}^{(-n)}(\lambda)$ is taken with respect to λ from -1 to 0, the following integrals are obtained:

$$\int_{-1}^{0}\mathcal{G}_{d-n}^{(-n)}(\lambda)d\lambda = \frac{2^{-n}d!}{(\mathrm{d}-\mathrm{n})!}\sum_{j=1}^{d}\binom{n}{j}S_2(d,j)\mathrm{j}!\int_{-1}^{0}\mathfrak{M}_{j,n}(\lambda)d\lambda. \tag{22}$$

The above equation is combined with the integral formulas given by Simsek [17] for the $\mathfrak{M}_{j,n}(\lambda)$ function, the integral formulas for higher-order negative powers of Apostol-Genocchi numbers are given below:

Theorem 2.3. For $d, n \in \mathbb{N}$, we have

$$\int_{-1}^{0}\mathcal{G}_{d-n}^{(-n)}(\lambda)d\lambda = \frac{2^{-n}d!}{(\mathrm{d}-\mathrm{n})!}\sum_{j=1}^{d}\binom{n}{j}S_2(d,j)\mathrm{j}!\sum_{k=0}^{n-j}(-1)^{n-k}\binom{n-j}{k}\frac{1}{n+1-k}. \tag{23}$$

Theorem 2.4. For $d, n \in \mathbb{N}$, we have

$$\int_{-1}^{0}\mathcal{G}_{d-n}^{(-n)}(\lambda)d\lambda = \frac{2^{-n}d!}{(n+1)(\mathrm{d}-\mathrm{n})!}\sum_{j=1}^{d}(-1)^{j}j!\,S_2(d,j). \tag{24}$$

Combining (23) and (24) equation, we get

Corollary 2.5. For $d, n \in \mathbb{N}$, we have

$$(-1)^{d} = \sum_{j=1}^{d}(-1)^{j}j!\,S_2(d,j). \tag{25}$$

3. CONCLUSION

In this section, a new class of exponential Euler-type spline curves with degree n and order α will be constructed. Using the equations above, the following identities are obtained:

$$u_n^{(\alpha)}(x;\lambda) = \left(\frac{1+\lambda}{2}\right)^{\alpha} \frac{n!}{(n+\lambda)!} \mathcal{G}_{n+\alpha}^{(\alpha)}(x;\lambda) \tag{26}$$

and

$$u_n^{(\alpha)}(\lambda) = \left(\frac{1+\lambda}{2}\right)^{\alpha} \frac{n!}{(n+\lambda)!} \mathcal{G}_{n+\alpha}^{(\alpha)}(\lambda) \tag{27}$$

Using the above equations, the Apostol-Genocchi spline family will be formulated in the future. This approach will provide a deeper understanding of the relationships between Apostol-Genocchi numbers, polynomials, and spline functions, offering a new avenue for research in both number theory and combinatorics. The development of such spline families may lead to further advancements in various mathematical applications, particularly in approximation theory and the study of special numbers.

This study expands the theoretical framework of spline curves and highlights their practical utility in diverse applications. The proposed spline families not only enhance mathematical modeling capabilities but also open new avenues for research in applied sciences and engineering. Future work could focus on extending these findings to dynamic systems and real-time applications in robotics and animation.

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A CASE STUDY ON DETERMINING CONSUMER PREFERENCES FOR ROSE PRODUCTS CONSUMPTION

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ABSTRACT

Rosa damascena Mill., commonly known as the Isparta rose, is a product of economic value in the perfume, cosmetics, pharmaceutical, and food industries due to its distinctive fragrance. Rose products are not consumed directly but are offered for consumption in the form of processed value-added products such as rose oil and various derivatives. The consumption of rose products, which serve as a significant source of income for the economy of Isparta province, and the factors influencing this consumption are of great importance for the development of the sector. This study aims to identify the consumption patterns and preferences of consumers regarding rose products in Isparta, the production hub of roses and rose-based products. The data for this research were obtained through a survey conducted with 272 consumers residing in Isparta. According to the findings, the most commonly consumed rose products are Turkish delight with rose, rose water, jam, soap, cream, and cologne, whereas products such as syrup, vinegar, rose oil, room fresheners, and fresh roses are less preferred. Among daily consumed items, soap and rose water stand out, while products like rose oil are used less frequently. Key factors influencing consumers' preferences for rose products include quality, brand recognition, and natural ingredients, whereas the impact of promotions and sales campaigns was found to be limited. However, high prices of rose products and inadequate product knowledge of sales personnels were identified as significant issues in the sector.

Keywords: *Rose products, Consumer preferences, Isparta, Geographical indication products*

1. INTRODUCTION

The rose, known as a significant member of the *Rosaceae* family, has been utilized throughout history in the production of medicine and cosmetics, while also being frequently represented in poetry, religion, art, and sculpture due to its aesthetic value [18]. *Rosa damascena* Mill., commonly referred to as the "Isparta Rose" or "Pink Oil Rose," stands out among other fragrant rose species cultivated worldwide due to its distinctive, sharp, and intense fragrance, making it the most economically valuable rose species for the perfume, cosmetics, pharmaceutical, and food industries [7]. In Türkiye, rose production is carried out in two forms: cut roses (pieces) and oil roses (tons) [21]. Oil roses are predominantly used in the production of rose-based products.

It has been noted that Türkiye and Bulgaria are the leading countries in global rose production [13]. In Türkiye, 19.879 tons of oil-bearing roses were produced in 2022 on an area of 41.675 decares. Rose production in Türkiye is primarily concentrated in the provinces of Isparta, Afyonkarahisar, Denizli, and Burdur, with Isparta accounting for 85.17% of the total production [21]. However, rose products are not consumed directly; instead, they are processed into value-added products such as rose oil and its derivatives. Approximately 60-65% of the world's rose oil production originates from Isparta and its surrounding areas, while domestic consumption in Türkiye remains at around 2-5% [13]. Isparta province, serves as the central hub for rose flower production and marketing. The region's historical expertise in rose oil production, combined with its favorable climate for rose cultivation compared to other provinces, enhances its significance in the industry. In Isparta province, rose flower production is carried out on small-scale rose orchards, making it an essential income source for small family farms

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[8]. Rose flower production in the province is primarily intended for rose oil extraction [9]. The rose oil obtained from rose flowers is used to produce industrial, high-value-added rose products [7].

The indirect consumption of rose products is widespread in the cosmetics and food industries, including rose oil, rose water, cologne, perfume, deodorant, soap, cream, air fresheners, jam, Turkish delight, syrup, vinegar, tea, coffee, and similar products. However, Güneş [6] stated that the cosmetics and perfumery sector in Türkiye is underdeveloped and that a significant portion of rose oil is exported. Research and development (R&D) efforts focusing on the domestic utilization of rose oil and the exploration of new uses for rose products are crucial for enhancing Türkiye's competitiveness in the sector. While studies in the literature address the production process, costs, and marketing of rose products [1, 14], Şirikçi and Gül [3] provided significant insights into rose product consumption. However, no recent studies have been found that specifically examine consumer preferences for rose and rose products. This highlights the need for new research supported by up-to-date data, particularly on a regional basis. Hence, this study was conducted to address this gap. Identifying the demand for rose products, which are among the key income sources for Isparta's economy, and the factors influencing this demand, is considered highly important for the sector's development. Feldmann and Hamm [4] noted that consumer interest in local products has increased over the past 15 years. Similarly, Henseleit et al. [10] found that consumers perceive local foods as offering better quality and higher food safety. The Isparta Rose (Pink Oil Rose) and its products, recognized as local products, were granted Geographical Indication Certificates on 06.05.2006, and Isparta Rose Oil on 07.03.2019 [19, 20]. Having a geographical indication is a significant factor influencing consumers' preferences, as it serves as a quality indicator for the product [11]. This suggests that geographical indications guide consumers regarding the quality of such products [16].

Studies on roses and rose products generally focus on topics such as the development of oil-bearing rose production and trade [12, 14], production costs in rose product enterprises [1], attitudes, opinions, and preferences regarding cut roses [5, 17, 18], and the consumption of rose products [3]. However, no recent studies have been identified that specifically address consumer preferences for rose products in Türkiye or Isparta. This study aims to establish a general framework for rose product consumption in Isparta, reveal consumer perceptions and preferences for rose products, and thereby enhance their unique value and significance. By examining the consumption patterns and the factors influencing consumer preferences in Isparta, a production center for both roses and rose products, the study seeks to contribute to the identification of policies that can enhance the sector's competitiveness and provide suitable recommendations for its development.

2. MATERIALS AND METHODS

The study population consists of 272 consumers residing in Isparta, surveyed through face-to-face interviews. The survey included both open-ended and closed-ended questions. The sampling method employed was "Simple Random Probability Sampling Based on Population Proportions" [2, 15].

$$\frac{Np\,(1-p)}{(N-1)\,\sigma_{px}^2+p\,(1-p)} \tag{1}$$

σ_{px}^2 : Variance of the proportion
n: Sample size
N: Population size
p: Proportion (set as p = 0.5 to achieve the maximum sample size)

The population of Isparta city center was recorded as 268.595 in 2022 [22]. Accordingly, the sample size was calculated as 272 individuals, based on a 90% confidence level and a 5% margin of error.

Descriptive statistics were utilized in the study to analyse the socio-economic characteristics and preferences of consumers who use rose products in Isparta province. The analysed data were interpreted and discussed by creating cross-tables.

3. RESULT AND DISCUSSION

Table 1 presents the socio-demographic characteristics of the consumers. The average age of the respondents was 37.83 years, with an average education duration of 11.74 years. The average household size was 3.67 individuals, including an average of 1.89 women and 1.87 employed individuals per household. The average monthly household income was calculated as 60.132,73 TL, with average expenditures on food, cosmetics, and rose products amounting to 17.907,27 TL, 2.716,18 TL, and 1.396,18 TL annually, respectively. Findings on household income and expenditure data reveal that food expenditures constitute a significant share of overall expenditures (29.78%), while expenditures on rose products are relatively low. Şirikçi and Gül [3] reported an average consumer age of 36.47 years, with most respondents having postgraduate education levels.

Regarding gender distribution, 68% of the respondents were women, and 32% were men. The proportion of those with social security was 90.55%. The vast majority of consumers (94.55%) made their purchases from physical stores, while only 5.45% opted for online shopping (Table 1). This underscores the importance of physical stores for businesses selling rose products.

Table 1. Socio-demographic characteristics of consumers

	Average	**Std. Deviation**
Age	37.83	10.987
Education level (years)	11.74	4.470
Household size (individuals)	3.67	1.263
Number of women in the household (individuals)	1.89	0.960
Number of employed individuals in the household (individuals)	1.87	0.634
Average monthly household income (TL/month)	60.132,73	25.929,26
Average monthly household food expenditure (TL/month)	17.907,27	13.070,47
Average monthly household cosmetic expenditure (TL/month)	2.716,18	4.315,71
Average annual household rose products expenditure (TL/year)	1.396,18	1.010,99
	N	**%**
Gender		
Female	187	68.00
Male	88	32.00
Marital status		
Single	100	36.36
Married	175	63.64
Social security status		
Yes	249	90.55
No	26	9.45
Places of rose product purchases		
Physical store	260	94.55
Online	15	5.45

The consumption amounts and purchasing preferences of various rose products among the surveyed consumers were examined (Table 2). Turkish rose delight is the most consumed product, with an average of 0.64 kg/month, and 56.36% of consumers consume it. 91.61% of Turkish rose delight purchases are made from places selling rose products. Rose water is consumed by 44.73% of consumers, with an average of 0.54 liters per month, and 94.31% of these purchases are made from rose product-selling locations. Jam is consumed by 27.27% of consumers, with an average of 0.89 kg per month, and it is predominantly purchased from places selling rose products (92%). Among other products, cream (26.55%), cologne (25.45%), shampoo (24.36%), and perfume/deodorant (20.36%) have significant consumption rates. These products are mostly purchased from rose product-selling locations as well. The products with the lowest consumption rates are roses (4.72%), rose oil (4.72%), vinegar (1.82%), and syrup (1.45%). These findings show that consumers predominantly prefer places selling rose products, and their consumption of rose products varies depending on the product type. Food products such as Turkish rose delight and rose water reach a wider consumer base, while the consumption rates of cosmetic products like shampoo, cream, and perfume are lower.

Table 2. Rose product consumption quantity and purchasing preferences

	Average Consumption Amount (kg or liter/month)	Percentage of Consumers (%)	Places of Purchase[1]				
			Supermarket (%)	Grocery Store (%)	Local/Neighborhood Market (%)	Internet (%)	Places Selling Rose Products (%)
Turkish Rose Delight (kg)	0.64	56.36	3.22	0.65	0	4.52	91.61
Rose Water (liter)	0.54	44.73	0.81	0	0.81	4.07	94.31
Jam (kg)	0.89	27.27	5.34	1.33	0	1.33	92.00
Soap (kg)	0.40	27.27	8.00	0	0	8.00	84.00
Cream (kg)	0.27	26.55	0	0	0	1.37	98.63
Cologne (liter)	0.53	25.45	2.85	1.43	1.43	5.71	88.58
Shampoo (liter)	0.62	24.36	4.48	0	0	7.46	88.05
Perfume/Deodorant (liter)	0.22	20.36	3.57	0	0	5.36	91.07
Tea (kg)	0.30	17.09	4.26	0	0	0	95.74
Coffee (kg)	0.29	17.09	4.26	0	0	0	95.74
Room fragrance (liter)	0.48	6.18	5.88	0	0	5.88	88.24
Rose (pieces)	1.23	4.72	0	0	0	7.69	92.31
Rose oil (liter)	0.19	4.72	0	0	0	7.69	92.31
Vinegar (liter)	0.43	1.82	20.00	0	0	0	80.00
Syrup (liter)	0.85	1.45	0	0	0	0	100.00

[1]Only the percentage of those who consume has been calculated.

In Table 3, the purposes of rose products consumption preferences among consumers are examined. Rose is consumed entirely in relation to social needs (100%), while rose oil is consumed solely for health purposes, accounting for 100% under the safety need category. The safety need refers to the basic needs of individuals to ensure their physical, work, moral, family, and health safety, as well as special and sacred days. Rose water is preferred 85.37% for safety needs and 10.57% for physiological needs. Cologne (85.37%) and perfume/deodorant (80.36%) are also generally preferred for safety needs, particularly for special and sacred days. Cosmetic products like cream (97.26%) and soap (98.67%) are preferred to meet safety needs. Turkish rose delight and tea are consumed 73.55% and 95.74%, respectively, primarily for safety needs, with a certain proportion also for social needs. Syrup, vinegar, tea, and coffee, on the other hand, are consumed entirely in line with physiological needs. These findings indicate that the primary reason for the consumption of most rose products is the safety need. However, a significant portion of rose and rose product consumption is made to fulfill social needs. Additionally, food products (such as Turkish rose delight, jam, tea, and coffee) are mostly preferred to meet physiological needs.

Table 3. Rose products consumption preference purposes

	Consumption Preference Purposes (%)[1]		
	Physiological Needs (Breathing, eating, drinking, sleeping, healthy metabolism, etc.)	**Safety Needs (Body, work, morality, family, health, special and sacred days, etc.)**	**Social Needs (Love, friendship, etc.)**
Rose	0	0	100.00
Rose oil	0	100.00	0
Rose water	10.57	85.37	4.06
Cologne	2.86	85.71	11.43
Perfume/Deodorant	3.57	80.36	16.07
Cream	2.74	97.26	0
Soap	1.33	98.67	0
Room fragrance	5.88	82.35	11.77
Shampoo	2.99	94.02	2.99
Jam	96.00	0	4.00
Delight	73.55	0.65	25.80
Syrup	100.00	0	0
Vinegar	100.00	0	0
Tea	95.74	0	4.26
Coffee	95.74	0	4.26

[1]Only the percentage of those who consume has been calculated.

Table 4 shows the consumption frequency of rose products. It was determined that 84.62% of respondents purchase roses once a year. As for rose oil, 15.38% of consumers use it daily, while 53.85% use it once a month. Regarding rose water, 36.59% of consumers use it daily, and 43.90% use it several times a week. For rose cologne, 47.14% of consumers use it several times a week, 20% use it daily, and 11.43% use it once a week. Among consumers of rose perfume/deodorant, 37.50% use it daily, and 53.33% of those using rose cream use it daily as well. It was found that 80% of consumers use rose soap daily. As for rose syrup, 50% of consumers use it once a month, while 40% use rose vinegar several times a week, and 20% use it daily. Rose tea and rose coffee are consumed daily by a large majority of consumers (78.72% and 76.60%, respectively). The frequency of rose product consumption varies widely. Products like rose soap and rose water are consumed more frequently, while products like roses and rose oil are consumed less frequently (once a year or once a month). In another study, it was found that 27.81% of consumers consume rose products once a month, 22.52% once a week, and 19.21% daily, with the most consumed products being Turkish rose delight and rose cream [3].

Products like rose tea and coffee have become part of consumers' daily habits, while the frequency of consumption of other rose products varies depending on the purpose and type of the product. This diversity indicates that marketing strategies for rose products should be customized based on consumer behaviour and product characteristics.

Table 4. Rose products consumption frequency (%)

	Every Day	**Several Times a Week**	**Once a week**	**Once a Month**	**Once a Year**	**Only on Special Days (**e.g., Religious Holidays, Festivals)
Rose	0	0	0	15.38	84.62	0
Rose oil	15.38	0	30.77	53.85	0	0
Rose water	36.59	43.90	9.76	8.13	0.81	0.81
Cologne	20.00	47.14	11.43	18.57	2.86	0
Perfume/Deodorant	37.50	37.50	8.93	8.93	7.14	0
Cream	53.33	40.00	6.67	0	0	0
Soap	80.00	17.34	1.33	1.33	0	0
Room fragrance	23.53	29.41	41.18	5.88	0	0
Shampoo	11.94	79.10	4.48	0	4.48	0
Jam	36.00	48.00	9.33	2.67	4.00	0
Delight	9.03	29.68	23.87	27.10	9.68	0.64
Syrup	0	25.00	25.00	50.00	0	0
Vinegar	20.00	40.00	20.00	20.00	0	0
Tea	78.72	17.02	0	2.13	2.13	0
Coffee	76.60	17.02	0	4.25	2.13	0

[1]Only the percentage of those who consume has been calculated.

Table 5 shows the packaging preferences for rose products of consumers. For rose products, 50% of consumers prefer plastic packaging, while all consumers prefer glass packaging for rose oil. For rose water, 59.68% of consumers prefer plastic packaging, and 40.32% prefer glass packaging. For rose cream, 97.06% of consumers prefer plastic packaging. In the case of rose room spray, 47.06% of consumers prefer plastic packaging, 47.06% prefer glass packaging, and 5.88% prefer metal packaging. All consumers prefer plastic packaging for rose shampoo. 97.26% of consumers prefer glass packaging for rose jam, and 81.05% prefer cardboard packaging for Turkish rose delight. For rose syrup and rose vinegar, all consumers prefer glass packaging. Consumers' packaging preferences vary based on the type of product, usage purpose, and aesthetic factors. This diversity shows that packaging design plays an important role in marketing strategies, and different packaging options for each product can influence consumer preferences.

Table 5. Rose products packaging preference

	Glass	**Plastic**	**Paper**	**Cardboard**	**Metal**
Rose	25.00	50.00	0	25.00	0
Rose oil	100.00	0	0	0	0
Rose water	40.32	59.68	0	0	0
Cologne	42.86	57.14	0	0	0
Perfume/Deodorant	29.63	68.52	0	0	1.85
Cream	2.94	97.06	0	0	0
Soap	0	94.66	2.67	2.67	0
Room fragrance	47.06	47.06	0	0	5.88
Shampoo	0	100.00	0	0	0
Jam	97.26	2.74	0	0	0
Delight	0	16.99	1.96	81.05	0
Syrup	100.00	0	0	0	0
Vinegar	100.00	0	0	0	0
Tea	0	0	100.00	0	0
Coffee	0	0	100.00	0	0

The factors influencing consumers' preferences for rose products and the average scores for each factor are shown in Table 6. Product quality stands out as the most important factor with an average score of 4.68. Following quality, the factors of brand (4.20), product recognition (4.19), and price (4.17) are ranked. These factors are key elements influencing consumers' product choices. Promotional factors such as price discounts (3.99) and product + product (3.76) bundles are also considered quite important by consumers. Aesthetic factors, such as product design (3.56), appear to have a lower level of importance. Marketing strategies such as coupons (3.44), combined sales promotions (3.29), giveaways (3.13), and contests/lotteries (2.99) are identified as less influential factors. A study identified the effective factors for rose product preferences as quality, expiration date, absence of additives, and price-quality ratio [3].

These findings suggest that primary factors like quality and brand are more important than additional elements like price discounts and product design in the purchasing decision for rose products. Consumers prioritize quality while also placing significant importance on brand trust and product recognition. While packaging design and aesthetic appeal play a role in the attractiveness of products, they are of lower priority for consumers.

Table 6. Factors influencing rose products preferences

	Average	**Std. Deviation**
Quality	4.68	0.591
Brand	4.20	0.970
Product recognition	4.19	0.902
Price	4.17	0.952
Price discounts	3.99	1.211
Product + product	3.76	1.304
Product design	3.56	1.205
Gift packages	3.55	1.388
Taste panel	3.55	1.346
Extras (25-30% more)	3.46	1.346
Coupons (20% discount when purchasing the same brand)	3.44	1.364
Combined sales promotion (multiple promotions)	3.29	1.408
Giveaways	3.13	1.398
Contests/lotteries	2.99	1.472

(1. Not important at all, 2. Not important, 3. Moderately important, 4. Important, 5. Very important)

Table 7 shows the reasons why consumers prefer rose products. Naturalness is seen as the most important reason for preference. The statement "I find rose products more natural" received the highest rating with an average score of 3.99, reflecting consumers' inclination towards natural products. Easy accessibility (3.93) and product variety (3.82) are also strong factors in determining consumer preferences. While consumers think rose products are tasty (3.45), they acknowledge that their shelf life is short (3.41). Factors such as inspection (3.25) and label information (3.20) were rated lower. The statements "I prefer rose products because their price is more affordable" (2.94) and "I find the advertisements and promotions for rose products sufficient" (2.88) received lower scores. In the study by Şirikçi and Gül [3], it was noted that advertisements, promotions, fairs, and festivals related to rose products were insufficient. This issue is still observed today. Consumers' concerns about the reliability of inspection and label information indicate that improvements need to be made in these areas.

Table 7. Reasons why consumers prefer rose products

	Average	Std. Deviation
I find rose products more natural.	3.99	0.908
I can find rose products everywhere.	3.93	1.030
I find the product variety in rose products sufficient.	3.82	1.075
I believe that rose products cause less environmental harm during production.	3.63	0.921
I believe that rose products preserve traditional flavours.	3.61	0.919
I think the packaging of rose products is healthy and hygienic.	3.59	0.834
I believe that rose products are produced under hygienic conditions.	3.58	0.902
I think that rose products do not contain harmful additives.	3.53	1.015
I believe that proper storage standards are followed from the producer to the finished product.	3.47	0.860
I find rose products delicious.	3.45	1.140
I believe that rose products have a short shelf life due to their natural nature.	3.41	0.967
I think the companies producing rose products are adequately monitored by regulatory authorities.	3.25	1.059
I have no doubts about the accuracy of the label information on rose products.	3.20	1.014
I prefer rose products because they are more affordable.	2.94	1.241
I find the advertisements and promotions for rose products sufficient.	2.88	1.196
There are more promotions/product campaigns for rose products.	2.70	1.091

(1. Strongly Disagree 2. Disagree 3. Neutral 4. Agree 5. Strongly Agree)

The problems encountered by consumers in the consumption of rose products are shown in Table 8. High product prices, with an average score of 3.37, emerge as the most significant issue. This is followed by inadequate promotions (3.28). Insufficient information provided by sales staff regarding the products is considered a medium-level issue by consumers, with a score of 3.05. On the other hand, product quality issues (2.84) are perceived as a less severe problem, while harmful substance content (2.52) and not being able to access the desired product immediately (2.44) are among the least prioritized issues. These findings indicate that consumers are primarily affected by price, promotion, and information gaps when consuming rose products. While quality, content safety, and accessibility are seen as less important, improvements in these areas could enhance consumer satisfaction. Especially, price sensitivity can be addressed by increasing promotions and campaigns. Additionally, addressing the lack of promotion can help the products reach a broader audience.

Table 8. Problems faced by consumers in consuming rose products

	Average	Std. Deviaton
High product prices	3.37	1.217
Inadequate promotions	3.28	1.133
Insufficient information provided by sales staff regarding products	3.05	1.083
Quality issues in products	2.84	1.092
Presence of harmful substances	2.52	1.092
Inability to immediately access the desired product	2.44	1.235

(1. Strongly Disagree 2. Disagree 3. Neutral 4. Agree 5. Strongly Agree)

4. CONCLUSION

In this study, the consumption habits of rose products and the factors influencing these habits of consumers living in Isparta province were examined comprehensively. The findings of the study reveal various trends and preferences that hold strategic importance for the rose products market.

According to the research results, the majority of consumers prefer to make their purchases from physical stores. This highlights the importance for businesses selling rose products to improve the customer experience in-store. Among the most consumed rose products are Turkish rose delight, rose water, jam, soap, cream, and cologne, while products like syrup, vinegar, rose oil, room fragrance, and fresh rose are less preferred. It was found that consumers generally prefer to buy rose products from specialized retailers in this field.

The frequency of rose product consumption varies greatly depending on the product type and the purpose of use. For example, products like rose soap and rose water are consumed daily, while products like rose and rose oil are used more sporadically. Products like rose tea and rose coffee have become part of daily routines, while the consumption of other products is more oriented towards special needs or occasions. This diversity indicates that marketing strategies should be designed to align with consumer needs and product characteristics. Packaging preferences also vary depending on the product type and purpose of use.

In consumer preferences for rose products, factors such as quality, brand recognition, and natural ingredients are decisive, while promotions, free samples, and campaigns are less influential. Consumers prefer rose products because they find them natural, the product variety is sufficient, and they are easily accessible. However, factors such as high prices and insufficient promotion and advertising activities negatively impact consumption. High product prices and the lack of product knowledge among sales staff are significant problems in the rose products sector. To address these issues, it is recommended to diversify and increase advertising and promotional activities, and to equip sales personnel with sufficient product knowledge.

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ADVANCEMENTS IN SOLVING HIGHER-ORDER ORDINARY DIFFERENTIAL EQUATIONS VIA THE VARIATIONAL ITERATIVE METHOD

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ABSTRACT

This study presents advancements in solving higher-order ordinary differential equations (ODEs) using the Variational Iterative Method (VIM) and compares its performance with the New Iteration Method (NIM) and Adomian Decomposition Method (ADM). ODEs are critical in modeling the rate of change in various systems over time, and many do not have exact solutions, necessitating the use of numerical methods to obtain approximate results. While several iterative methods exist, a detailed comparison of VIM with other techniques, particularly for higher-order ODEs, is still lacking. This research focuses on understanding the principles and methodology of VIM and applying it to solve higher-order linear and nonlinear ODEs. The study evaluates the accuracy, convergence rate, and computational efficiency of VIM compared to NIM and ADM through the solution of third, fourth, and fifth-order differential problems. The results show that VIM outperforms NIM and ADM, with faster convergence and higher efficiency. Error analysis in Figures 1, 2, and 3 highlights the strengths and limitations of each method, providing valuable insights for researchers and practitioners in selecting the most appropriate technique for solving higher-order ODEs. These findings advance the development of iterative methods in numerical analysis and contribute to progress in the field of differential equations.

Keywords: *Variational Iterative Method, Higher-Order equations, Comparative analysis, Iterative Technique, Numerical analysis.*

1. INTRODUCTION

Higher-Order ordinary differential equations are equations that involves the derivatives of a function with respect to one variable, where the highest derivative is of order greater than one [1]. These equations are crucial in mathematical modelling because they frequently arise in various fields of science and engineering [2, 3, 4, 5].
The general form of a higher-order differential equation with initial conditions is given by

$$f^n(x) + a_{(n-1)f^{n-1}(x)+\cdots+a_1 f'(x)+a_0 f(x)=g(x)} \quad (1)$$

With initial conditions $f(x_0) = y_0, f'(x_0)= y_0 \ldots f^{n-1}(x_0)=y_{n-1}$

These equations become difficult to solve using the traditional methods as the order increases so also the complexity. So, over the years, modified methods such as the Variational Iterative method (VIM), Adomian Decomposition Method (ADM), New Iterative Methods (NIM) and other methods has been proposed by numerous researchers to address both linear and nonlinear differential equations without requiring small parameters assumption [6, 7, 8, 9].

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The Variational Iterative Method (VIM) is an analytical technique developed for solving differential equations including Higher-Order ordinary differential equations. This method leverages on the principles of variational theory to construct correction functionals, incorporating Langrange multipliers to iteratively approximate solutions [10, 11, 12, 13].
However, despite the existence of various Iterative methods, a thorough comparison between VIM and other numerical methods for higher order is lacking.
Current studies shows a huge research gap in the comparative analysis of these elite numercial methods, recent works have only focused on VIM and ADM for lower order. This study aims to bridge that gap by expanding into VIM, NIM and ADM for complex higher-order ODEs, focusing on their computational power as well as precision in solving higher-order ODEs' [14, 15, 16, 17, 18].
The Novelty of this study lies in the need to evaluate the effectiveness and applicability of the VIM when applied to Higher-Oder ODEs and to provide a comparative analysis between VIM, NIM and ADM for higher-order differential equations in order to understand how these methods react when applied to higher-order ordinary differential equations and to compare the accuracy and efficiency of these methods, highlighting their strengths and limitations, hence guiding their optimal application in complex systems and phenomena.

2. MATERIALS AND METHOD

2.1. Description of VIM

The following authors explained the VIM method, its formula and algorithm [10, 19, 20, 21, 22, 23]
Consider the following general function equation for nonlinear equation

$$L_p(B_1, B_2, \ldots, B_p) + N_p(B_1, B_2, \cdots, B_p) = G_p \tag{2}$$

where L_p represent a linear operator and N_p is a nonlinear operator, G_p is a given function.
So, lets consider a differential equation of the form;

$$L_p(t) + N_p(t) = g_p \tag{3}$$

an initial approximation $B_{p,n+1}$ is assumed and the correction functional is constructed below;

$$B_{p,n+1} = B_{p,n} + \int_0^t \lambda_p[L_p(\tilde{B}_{1,n}(t)) + N_p(\tilde{B}_{1,n}(t)), \cdots, \tilde{B}_{m,n}(t)) - g_i(m))dt \tag{4}$$

where λ_i is a langrange multiplier to be determined using the variational theory which requires the fuctional to be stationary. The subscript m represent the mth-order approximation where B_m is taken as the bounded variation and $\delta B_m = 0$. The value of the Lagrange Multiplier is computed using the formula

$$\lambda = \frac{(-1)^m}{(m-1)!}(s-x)^{m-1} \tag{5}$$

$$L_p\left(B_p\right) + N_p\left(B_1, \cdots, B_m\right) = g_i\left(t\right), \quad i = 1, 2, \cdots, m \tag{6}$$

And t is the count of occurrencies of differentials, optimal value of λ is best determined through variational theorry, by fixing parameters defined by equation (2), the following conditions are established;

$$\lambda_i'(t)|_{t=B} = 0$$
$$\lambda_i(t) + 1|_{t=B} = 0, \quad i = 1, 2, \ ... \ , \ m$$

$$B_{i,n+1} = B_{i,n} + \int_0^t \lambda_i [L_i(B_{i,n}(t)) + N_i(B_{1,n+1}(t)), \dots \ , B_{i-1,n+1}(t), B_{i,n}(t), \dots, B_{m,n}(t)) - g_i(t)]dt \tag{7}$$

for $i = 1, 2, ..., m$ the revised value of $J_{i-1,\, n+1}$ is used to compute $J_{i,n+1}$, thus quickening the convergence of the equations.

3. NUMERICAL EXPERIMENTS

In this section, the Variational Iteration Method (VIM) is applied to solve third-, fourth-, and fifth-order ordinary differential equations (ODEs). The computations are performed using Python, and the resulting solutions are compared with those obtained from the Adomian Decomposition Method (ADM) and the New Iteration Method (NIM). To provide clear insights into the comparative performance of these methods, the results are presented in both tabulated and graphical formats.

Numerical Problem 1: Consider the third-order linear ODE below and apply the VIM[24]

$$\frac{d^3u}{dt^3} + \frac{du}{dt} = 0 \tag{8}$$

With initial conditions given as: $u(0) = 0, \quad u'(0) = 1, \ u''(0) = 2, \qquad 0 \le t \le 1$. And exact solution is provided as: $u(t) = 2(1 - cos(t)) + sin(t)$. The appropriate Langrange multiplier for third order is selected by $\lambda(k) = \frac{-(t-x)^2}{2}$ and correction functional is constructed by (9)

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{-(t-x)^2}{2}\left[u'''(t) + u'(t) - g(t)\right]dt \tag{9}$$

And the following iteration is obtained;

$$\begin{aligned}
u_0 &= t + t^2 \\
u_1 &= t + t^2 - \frac{t^3}{2} - \frac{t^4}{12} \\
u_2 &= t + t^2 - \frac{t^3}{6} - \frac{t^5}{60} \\
u_3 &= t + t^2 - \frac{t^3}{6} - \frac{t^5}{120} \\
u_4 &= t + t^2 - \frac{t^3}{6} - \frac{t^5}{120} + t^7
\end{aligned} \tag{10}$$

Numerical Problem 2: Compute the solutions of the fourth-order ODE given below using the Variational Iteration Method [25].

$$\frac{d^4u}{dt} + \frac{d^3u}{dt^3} - 10u(t) = 0 \tag{11}$$

with initial conditions $u(0)=0, u'(0)=0, u''(0)=1, u'''(0)=2, \quad 0 \le t \le 1$. Proceeding according to the method explained earlier, we select the langrange multiplier for fourth-order $\lambda(x)=\frac{-(t-x)^4}{4!}$ and the correction functional is constructed

$$u_{n+1}(t)=u_n(t)+\int_0^t \frac{-(t-x)^4}{24}\left[\frac{d^4u}{dt}+\frac{d^3u}{dt^3}-10u(t)-g(t)\right]dt \tag{12}$$

using $u_o = t+\frac{1}{2}t^2+\frac{1}{3}t^3$ from the initial conditions, the following successive estimations is obtained;

$$\begin{aligned}
u_o &= t+\frac{1}{2}t^2+\frac{1}{3}t^3 \\
u_1 &= t+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{5}{6}t^4 \\
u_2 &= t+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{5}{6}t^4+\frac{25}{12}t^5 \\
u_3 &= t+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{5}{6}t^4+\frac{25}{9}t^5+\frac{95}{12}t^6 \\
u_4 &= t+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{5}{6}t^4+\frac{179}{12}t^6+\frac{250}{63}t^7+\frac{1175}{73}t^8
\end{aligned} \tag{13}$$

Numerical Problem 3: Compute the solutions of the fifth-order nonlinear Ordinary Differential Equation given below using the VIM [24].

$$\frac{d^5u}{dt^5}-u\frac{d^3u}{dt^3}-\frac{du}{dt}=0 \tag{14}$$

with initial conditions given as: $u(0)=1, \quad u'(0)=2, \; u''(0)=1, \quad u'''(0)=0, \quad u^{iv}(0)=3, \quad 0\le t\le 1$ the exact solution is provided as: $u(t)=1+t+t^2$. Proceeding according to the method explained earlier, we select the langrange multiplier for fourth-order $\lambda(x)=\frac{-(t-x)^4}{4!}$ and the correction functional is constructed by equation (15) as

$$u_{n+1}(t)=u_n(t)+\int_0^t \frac{-(t-x)^4}{24}\left[\frac{d^5u}{dt^5}-u\frac{d^3u}{dt^3}-\frac{du}{dt}-g(t)\right]dt \tag{15}$$

using $u_o = 1+2t+\frac{t^3}{2}+\frac{t^4}{8}$ from the initial conditions, the following successive approximations is obtained;

$$
\begin{aligned}
u_0 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8} \\
u_1 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8}-\frac{t^5}{40} \\
u_2 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8}-\frac{t^5}{40}+\frac{t^6}{120} \\
u_3 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8}-\frac{t^5}{40}+\frac{t^6}{120}-\frac{t^7}{504} \\
u_4 &= 1+2t+\frac{t^3}{2}+\frac{t^4}{8}-\frac{t^5}{40}+\frac{t^6}{120}-\frac{t^7}{504}+\frac{t^8}{4052}
\end{aligned} \tag{16}
$$

The computed results are presented in the tables below for clarity and ease of comparison. These tables provide a detailed breakdown of the numerical outcomes, allowing for a straightforward evaluation of the methods applied in solving the differential equations.

Table 1. Computational Comparison for the two methods (VIM and NIM) on Problem one

X	VIM	NIM	Exact Solution
0.00	0.00000000000000	0.000000000000000	0.00000000000000
0.10	0.20965850200000	0.208189860000000	0.10982508600000
0.20	0.22000000000000	0.238553475000000	0.15385361750000
0.30	0.35800000000000	0.384851298000000	0.27848472290000
0.40	0.51210000000000	0.548342654000000	0.43729635400000
0.50	0.68047000000000	0.725428915000000	0.52426041500000
0.60	0.86132500000000	0.914657244000000	0.69397124300000
0.70	1.05290440000000	1.115535236300000	0.88453329400000
0.80	1.25345529000000	1.324632552000000	1.05394264400000
0.90	1.46121919600000	1.543127073000000	1.23010693300000
1.00	1.67442191200000	1.762839373000000	1.41086631800000

The given Table above shows the Variational Iterative Method approximate solution, New Iteration Method solution [15] and exact solution for Problem 1.

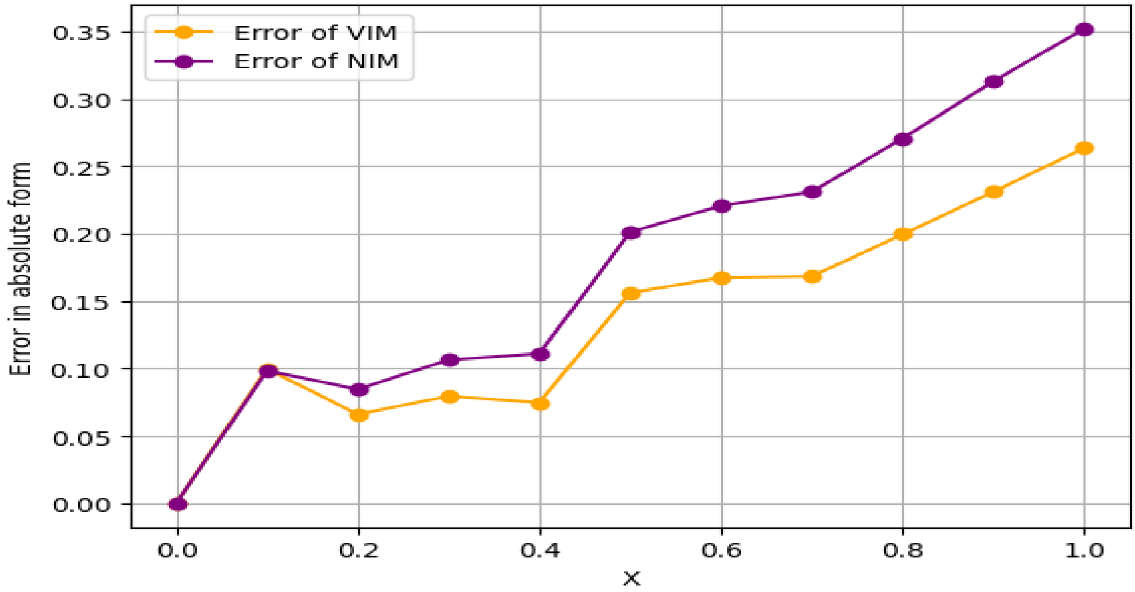

Figure 1: Plot illustrating Errors of the Compared Methods for Problem 1

The given plot shows the computed errors of the Variational Iterative and New Iteration Methods for problem one. It is observed that the error of VIM solution is lesser than that of NIM.

Table 2. Computational Comparison for the two methods (VIM and ADM) on Problem two

X	VIM	ADM	Exact Solution
0.10	0.009621383691244	0.005333333334933	0.012345678901235
0.20	0.035849143276779	0.022666667106314	0.045678901234568
0.30	0.067342991683491	0.054000012063215	0.089012345678901
0.40	0.125472524156772	0.101333462064786	0.154321098765432
0.50	0.197532855221887	0.166667484775190	0.234567890123456
0.60	0.273848076021479	0.231001507485594	0.345678901234567
0.70	0.356978715924633	0.295335530196998	0.478901234567890
0.80	0.445597930124879	0.359669552908402	0.634567890123456
0.90	0.531609842927041	0.424003575619806	0.812345678901234
1.00	0.612282933119536	0.488337598331210	1.012345678901230

The given Table above shows the Variational Iterative Method approximate solution, Adomian Decomposition Method solution and exact solution for Problem 2.

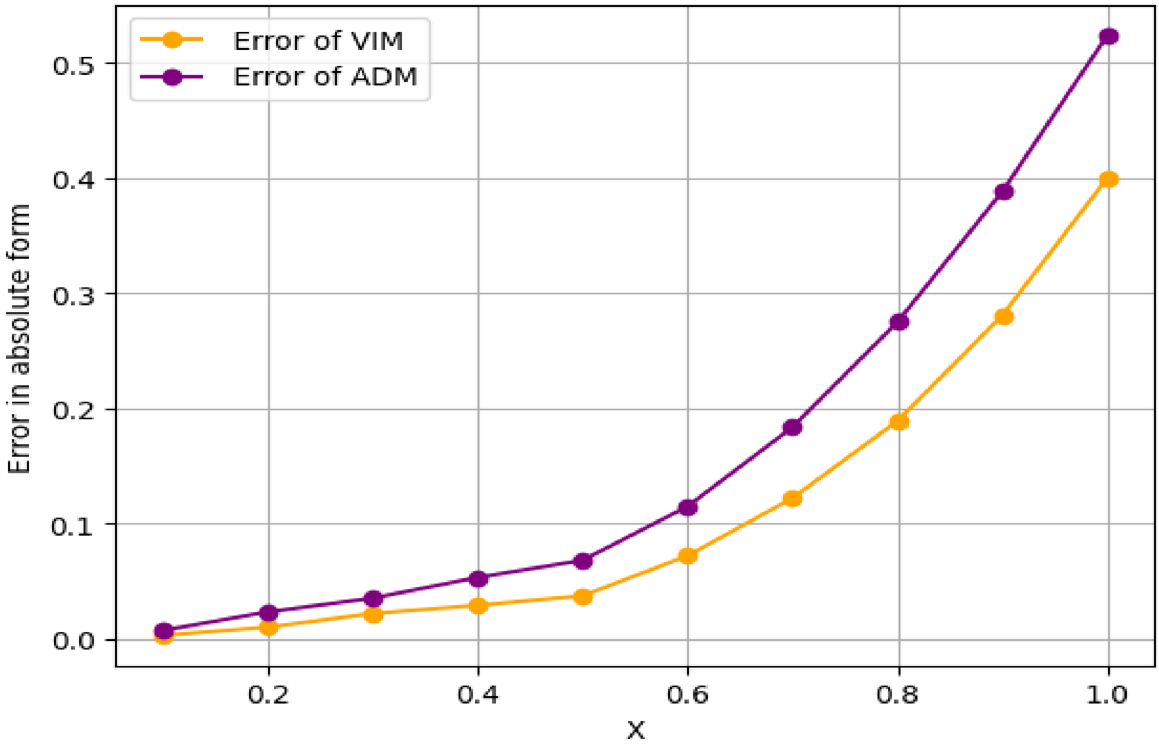

Figure 2: Plot illustrating Errors of the Compared Methods for Problem 2

The given plot shows the computed errors of the methods; Variational Iterative Method and Adomian Decomposition Method for problem four. It is observed that the error of VIM solution is lesser than the ADM.

Table 3. Computational Comparison for the two methods (VIM and NIM) on Problem Three

X	VIM	NIM	Exact Solution
0.10	1.0000000000	1.0000000000	1.0000000000
0.20	1.2232630000	1.2442610000	1.1234570000
0.30	1.4595960000	1.4883260000	1.2716050000
0.40	1.7094650000	1.7471440000	1.4444440000
0.50	1.9735460000	2.0417490000	1.6419750000
0.60	2.2526520000	2.4821350000	1.8641980000
0.70	2.5477310000	2.8923670000	2.1111110000
0.80	2.8598660000	3.3534780000	2.3827160000
0.90	3.1902640000	3.6453610000	2.6790120000
1.00	3.5401410000	3.9525350000	3.0000000000

The given Table above shows the Variational Iterative Method approximate solutions, New Iteration Method solutions and exact solutions for Problem three.

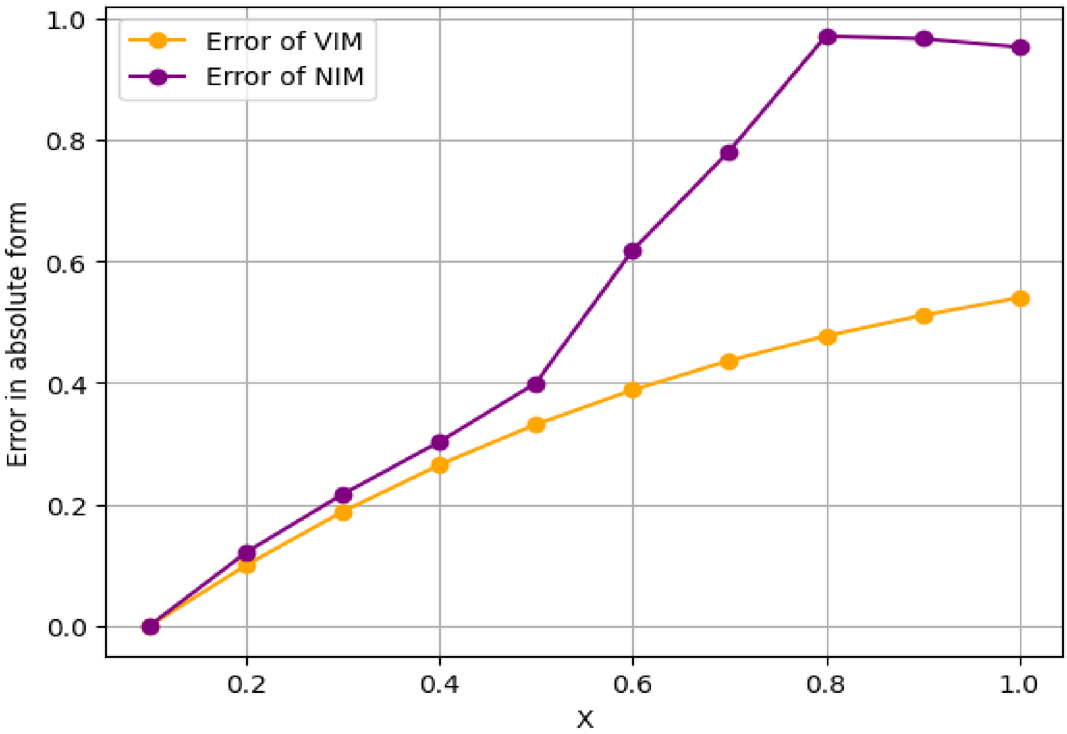

Figure 3: Plot illustrating Errors of the Compared Methods for Problem 3

The given plot shows the computed errors of the methods; Variational Iterative Method and New Iteration Method for problem three. It is observed that the error of VIM solution is lesser than that of NIM.

4. DISCUSSION OF RESULTS

Based on the criteria outlined by [26], a numerical solution is said to converge or is close to convergence if the approximate solution is closer or equal to the exact solution, if the error between the exact solution and approximate solution tends towards zero. It is based on these criteria that we can select the best method for higher-order ODE.

Comparison of Numerical Solutions (Table 1-3):

I. Concerning the comparison of numerical solutions on table 1 to 3, these tables offer great insights into the incredible solutions of VIM, NIM and ADM for problem 1 to 3.
II. The solutions gotten highlights the method's incredible performance when applied to Higher-Order ODEs.
III. Each question exposed the variations of the solution highlighting their complexities, the higher the iteration the more complex it becomes. This in turn gives us valuable information regarding how models react to different conditions.

Numerical illustration of the Error Plots (Figure 1-3)

I. The error figures show the difference between the methods and the exact solution

II. The higher the complexity of the problem, the bigger the errors become, but, on all cases, VIM outperforms the NIM and ADM as it consistently has lower errors and converges faster towards the exact solution.
III. These figures (Figure 1 to 3) aids in understanding the disparities between these methods at first glance.

Findings of the Experiments:

I. Figures (1, 2, 3) of the error analysis shows the VIM has lower errors and converges faster to the exact solution.
II. As the number of iterations increases, the error also increases highlighting the complexity of these equations towards capturing accurate solutions.
III. These observations provide valuable insights that shows areas that needs improvement

5. CONCLUSION

The Variational Iteration Method (VIM) has been successfully applied to solve three higher-order ordinary differential equation (ODE) problems, specifically of third, fourth, and fifth orders. The results obtained using VIM were compared with those from the New Iterative Method and the Adomian Decomposition Method. While all three methods produced accurate results for the ODEs, the comparison revealed that VIM outperforms the other two in terms of computational efficiency and precision. As shown in Figures (1, 2, 3), VIM proves to be superior, especially when solving higher-order ODEs. Looking ahead, we plan to extend our analysis to include additional numerical methods for higher-order ODEs. This will allow us to gain a more comprehensive understanding of the performance of various methods across different types of differential equation problems, thereby contributing to the advancement of computational mathematics and numerical analysis.

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ASEAN FOOD SECURITY: POST COVID POLICY STRATEGIES

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ABSTRACT

The food and agriculture sectors are crucial to the ASEAN economy, offering significant employment and income for a large portion of the region's population. It contributes to ensuring national and regional food security and nutrition in ASEAN. Before COVID-19, ASEAN faced unprecedented challenges such as the increase in population, climate change, geopolitical tensions and natural disasters adversely impacting the agriculture and food systems. Recently, COVID-19 caused a major disruption in the food supply chain that affected the agriculture and food system. This paper aims to summarize the impact of COVID-19 on food security in ASEAN and post-COVID-19 policy strategies in ASEAN. COVID-19 affected economic slowdown, trade restrictions policy, and market disruptions that caused price hikes, agriculture input shortage, and other dynamic factors that threaten food security and nutrition in ASEAN. The prevalence of moderate or severe food insecurity in the total population increased far above the pre-COVID-19 level in ASEAN. ASEAN has been concerned about food security issues. ASEAN prepared the policy strategies for the post-COVID-19 and future crises such as the ASEAN Comprehensive Recovery Framework, Leader’s Declarations and Regional Guidelines, AIFS Framework and SPA-FS 2021-2025, which were developed to ensure long-term food security in ASEAN. These documents encourage collaboration on rapid actions in response to crises within ASEAN and related development partners including private sectors, strengthening the function of cooperation agencies in ASEAN, accelerating the transformation of sustainable agriculture and digitalization, enhancing market connectivity, promoting investment in agricultural research and development, and facilitating access to finance.

Keywords: *Food Security; COVID-19; ASEAN; Policy Strategy; Agriculture Sector.*

1. INTRODUCTION

FAO defined food security as a condition where everyone has continuous physical, social, and economic access to enough safe and nutritious food to meet their dietary needs and preferences for an active and healthy life [8]. The World Summit on Food Security in Rome in 2009 adopted Five principles for sustainable global food security as follows, (i) principle 1: invest in country-owned plans, (ii) principle 2: foster strategic coordination at each level, (iii) principle 3: Strive for a comprehensive twin-track approach to food security that consists of direct action to immediately tackle hunger for the most vulnerable and medium and long-term sustainable agriculture, food security, nutrition and rural development programmes, (iv) principle 4: Ensure the strong role for the multilateral system by sustained improvements, (v) principle 5: Ensure sustained and substantial commitment by all partners to investment in agriculture, food security and nutrition [30].
Agriculture is the main sector for ensuring food security. The agriculture sector produces food and primary materials for industry. In this term, the agriculture sector holds a significant role in the world.
However, for the last few years, the agriculture sector has been facing unprecedented challenges that have affected food sustainability as well as food security. The increase in population, climate change,

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COVID-19, geopolitical tensions and natural disasters adversely impacted the agriculture and food systems [12].

The Association of Southeast Asian Nations (ASEAN) was established on 8 August 1967. ASEAN consists of 10 member countries, namely, Indonesia, Singapore, Malaysia, Thailand, Brunei Darussalam, Philippines, Cambodia, Lao PDR, Myanmar, and Viet Nam. ASEAN was established with the aim of accelerating economic growth, promoting regional peace and stability, promoting active collaboration and mutual assistance in response to various matters, assisting in training and research facilities, collaborating more effectively to encourage further growth in the agriculture, industry, and trade sectors, promote Southeast Asian studies, and maintain close and beneficial cooperation with existing international and regional organizations with similar aims and purposes. The ASEAN Community, anchored on three community pillars: Political-Security Community, Economic Community, and Socio-Cultural Community was launched in 2015 [27].

The food and agriculture sectors are crucial to the ASEAN economy, offering significant employment and income for a large portion of the region's population. The agriculture sector contributes to ensuring national and regional food security and nutrition in ASEAN [12]. Eight out of ten ASEAN member countries are relying heavily on this sector, contributing significantly to their economies. In Myanmar and Lao PDR, agriculture makes up over 40% of GDP. This region is a leading producer and exporter of palm oil, rubber, rice, sugar, seafood, and fruits, and holds an important position as an exporter and importer for the agricultural sector [1, 15]. In 2022, the ASEAN Agriculture, Forestry, and Fisheries (AFF) industry contributed US$354.3 billion, or 9.8%, to the region's GDP. It also provided employment share for 27.6% of ASEAN's 337.9 million workers. The share of agricultural products in AFF's total trade grew from 68.7 in 2013 to 71.4 in 2022, with a trade value of USD 447.1 billion [3].

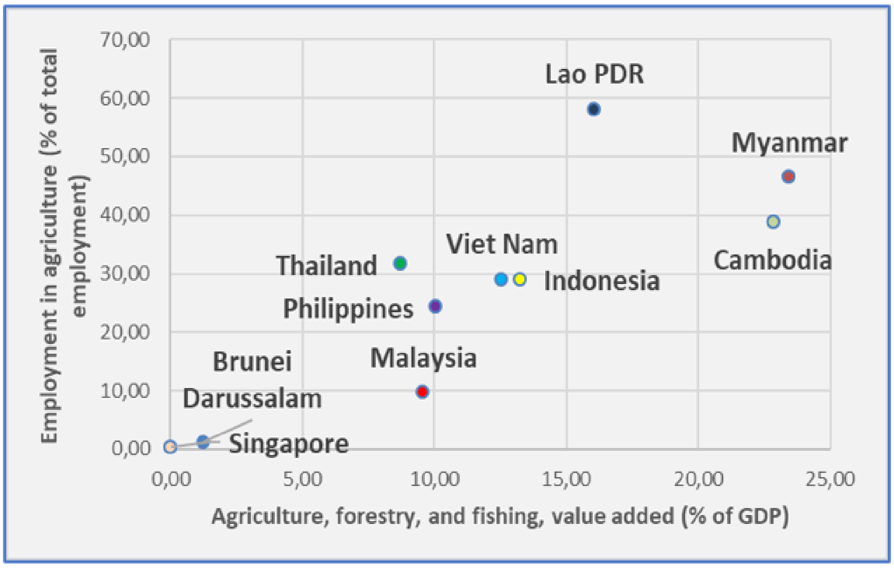

Figure 1. GDP and Employment Shares in ASEAN [7]

COVID-19 which spread rapidly worldwide in early 2020, implies food security and nutrition. Moreover, the major disruption of the food supply chain affected agriculture and food systems [9]. Global hunger, as indicated by the prevalence of undernourishment, remained significantly higher than pre-COVID-19-pandemic levels, impacting approximately 9.2 per cent of the global population in 2022, up from 7.9 per cent in 2019. About 122 million people experienced hunger in 2022 compared to 2019, prior to the global pandemic [6]. Trade restrictions affecting shortage in agriculture products. This situation causes price hikes in agricultural products. The food price index of cereals products increased 60.14% far above the price index in 2019 [10].

2. MATERIALS AND METHODS

The objective of this paper is to summarize the impact of COVID-19 on food security in ASEAN and post-COVID-19 policy strategies in ASEAN. A literature review approach to summarize the impact of COVID-19 on food security in ASEAN and the post-COVID policy strategies by ASEAN member states was used in this paper. The primary documents reviewed include the ASEAN Comprehensive Recovery Framework, ASEAN Integrated Food Security (AIFS) Framework, Strategic Plans of Action on Food Security (SPA-FS) 2021-2025, ASEAN Leaders Declaration on Strengthening Food Security in Response to Crises, ASEAN Regional Guidelines for Sustainable Agriculture in ASEAN, and ASEAN Regional Guidelines on Promoting Climate-Smart Agriculture (CSA) Practices Volume 3. These documents were chosen for their direct relevance to ASEAN's coordinated response to food security challenges before, during, and after the COVID-19 pandemic.

In addition to the policy documents, statistical data from related authorities resources, journals, and related papers are utilised to support the review. All those resources were analyzed, focusing on post-COVID food security and sustainable agriculture practices policy strategies within the region. This approach ensured a comprehensive understanding of ASEAN's efforts to strengthen regional food security and agriculture sustainability in tackling the challenges and crises.

3. RESULTS AND DISCUSSION

3.1. Impact of Covid-19 on Food Security in Asean

Before COVID-19 adversely impacted, ASEAN already being a region that highly vulnerable to climate change and natural hazards, such as heat, high humidity, and rising sea levels. Loss estimates due to climate change are higher than previously thought, exacerbated by high population densities and economic activities along extensive coastlines. Between 2008 and 2020, floods, droughts, and fires caused food crop losses amounting to $21 billion [17]. COVID-19 brings out a domino effect that could affect food security in the long term. The dynamics of COVID-19 that threaten food security and nutrition are shown in Figure 2.

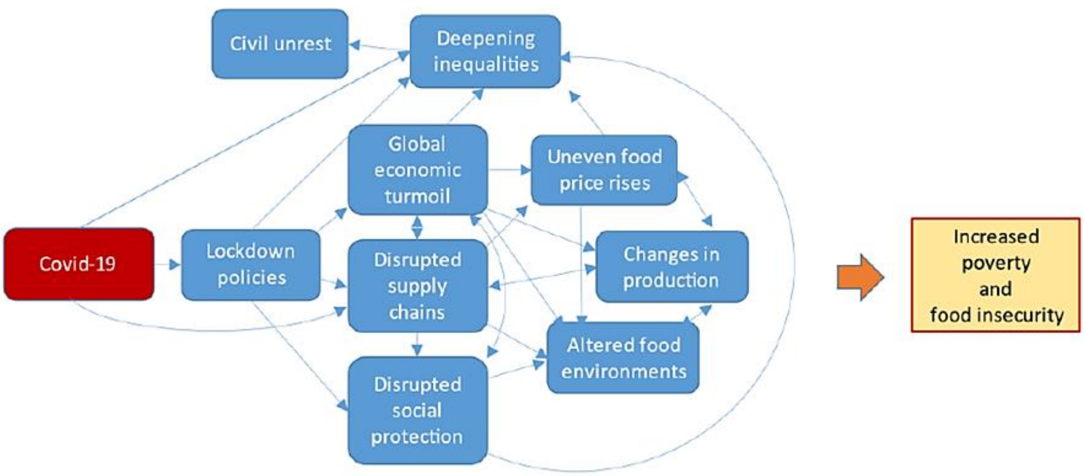

Figure 2: The Dynamics of COVID-19 that Threaten Food Security and Nutrition [11]

In ASEAN, the pandemic highlighted the necessity for resilient and sustainable food systems to ensure the availability, accessibility, and affordability of safe and nutritious food for the people. During pandemic COVID-19, the prevalence of moderate or severe food insecurity in the total population of

ASEAN (Figure 3) increased by 2.40% from the pre-COVID-19 level. This situation indicates that the domino effects of COVID-19 such as lockdown, economic slowdown, trade restrictions and health issues disrupted food access.

ASEAN is off track to meet UN agreed global Sustainable Development Goals (SDGs) for zero hunger by 2030, with the COVID-19 pandemic exacerbating vulnerabilities and uncertainties [14, 29]. The Pandemic COVID-19 has revealed a significant weakness in ASEAN farming the substantial and unsustainable imports of raw materials for feed and fertilizers from another region. The price hikes for those inputs exacerbated the problem in the agriculture sector within the region [24]. The pandemic has underscored the vulnerability of supply chains, including ASEAN economies. The adverse impact of the pandemic on livelihoods, education, food security, and nutrition, along with worsened poverty, vulnerabilities, and inequalities, could ultimately reverse the region's progress in reducing poverty and hinder efforts to achieve inclusive and sustainable development in the long term [21].

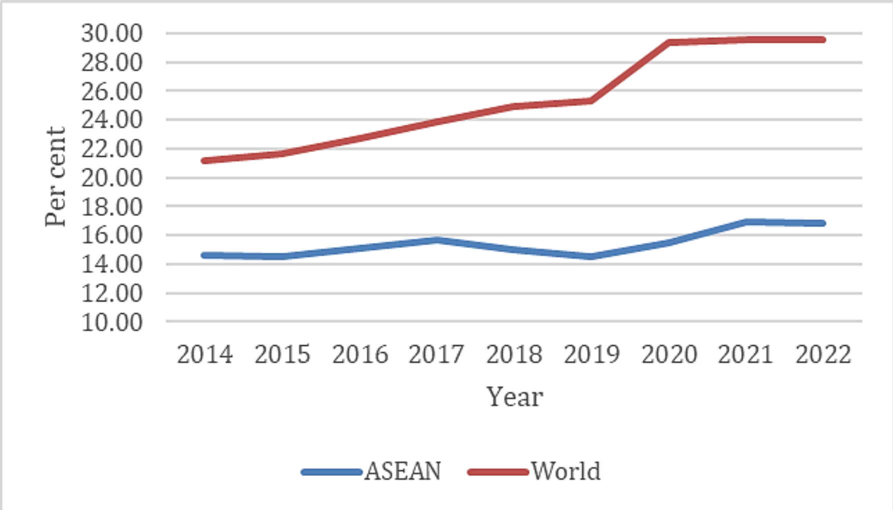

Figure 3: Prevalence of Moderate or Severe Food Insecurity In the Total Population [7]

The pandemic caused massive lockdowns around the world. This condition disrupted the supply of labor and agricultural inputs such as fertilizer. The absence of fertilizers can significantly reduce yields, while the lack of stress-resistant seeds and pesticides leaves crops more vulnerable to droughts, floods, pests, and diseases. The lockdowns also affected the disruption of market access. Disruption in market access increases the food prices, it could place food beyond the grasp of poorer populations, thus affecting their food security, Increased hunger and malnutrition are expected in many ASEAN Member States [19]. ASEAN is a net exporter of food and beverages. Brunei, Cambodia, Philippines, and Singapore are net importers while the rest are net exporters. COVID-19 directly affected the disruption of food supplies for the domestic in ASEAN. This crisis is more about logistic matters, than the food shortage [22].

3.2. Post-Covid-19 Policy Strategies in Asean

In the post-COVID-19 era, ASEAN countries are urged to enhance regional cooperation, invest in innovative solutions, and promote sustainable agriculture. These efforts aim to build resilient and secure food systems capable of withstanding future shocks and supporting the well-being of ASEAN's growing population [18]. The unexpected impact of COVID-19 highlights the necessity for a departure from "business as usual" policies toward a more forward-thinking approach that prioritizes investment in productivity, sustainability, and resilience of the global food system. The COVID-19 pandemic has emphasized the need for a strong and resilient food system that operates effectively under any conditions, ensuring that all citizens have access to an adequate and affordable food supply [24].

Developing countries are highly diverse in various aspects of food security. Therefore, different types of food-insecure nations require tailored policy interventions to enhance their nutrition status [16].

Enhancing food security has become a critical agenda for ASEAN to mitigate the impacts of future crises and build resilience. Food security integration has been prioritized in ASEAN as a collective effort to build resilience against possible crises [5].

3.2.1. ASEAN Comprehensive Recovery Framework

COVID-19 made ASEAN recognize the importance of addressing unprecedented crises requires unprecedented actions not only within the region but also beyond. In this situation, strong coordination and cooperation among ASEAN, its stakeholders and external partners becomes important. A comprehensive recovery plan is crucial. ASEAN's recovery strategies focused on five broad strategies that looked as the most impactful for the recovery process and its aftermath. The five strategies are as follows (1) Enhancing health systems; (2) Strengthening human security; (3) Maximizing the potential of intra-ASEAN market and broader economic integration; (4) Accelerating inclusive digital transformation; (5) Advancing towards a more sustainable and resilient future. All these strategies will enhance the food security and nutrition in the region. To support this strategy, the priorities will emphasize advancing sustainability across various sectors in ASEAN, with a particular focus on investment, energy, agriculture, green infrastructure, disaster management, and sustainable financing.

Under the broad strategy of strengthening human security; Ensuring food security, food safety, and nutrition becomes a key priority area to guide ASEAN’s recovery efforts. The ASEAN Ministers on Agriculture and Forestry committed to ensuring a sustainable supply of safe, nutritious, and affordable food during the pandemic while minimizing disruptions in regional food supply chains by keeping markets open and facilitating the transport of agricultural products. They also pledged to strengthen the ASEAN Food Security Information System (AFSIS) and ASEAN Plus Three Emergency Rice Reserve (APTERR) to reduce price volatility and ensure adequate food reserves and timely market information.

Under the broad strategy of advancing towards a more sustainable and resilient future, there are several actions to strengthen food security as follows improving the agriculture sector’s productivity and strengthening the food value chain approach to food safety, security and quality by implementing relevant activities such as promoting climate-smart agriculture (CSA), implementing ASEAN Good Agricultural Practices, ASEAN Good Aquaculture Practices, and CSA practices, and developing strategies for promoting Public and Private Partnership (PPP). Building high-value food industries, such as processing, packaging, and retailing, is essential for capturing more value in agriculture and raising incomes for farmers and agricultural workers [21].

3.2.2. Long-term Food Security Policy in ASEAN

ASEAN has paid attention to food security since 1979 by establishing The ASEAN Food Security Reserve Board (AFSRB). It was created to assess food security at various levels. To support AFSRB, through the collaboration between ASEAN plus Three Countries (Japan, the Republic of China, and the Republic of Korea) developed the ASEAN Food Security Information System (AFSIS) and the ASEAN Plus Three Emergency Rice Reserve (APTERR) to enhance regional food security cooperation through systematic data collection, analysis, dissemination, and other activities [28].
To ensure long-term food security and improve the livelihoods of farmers in ASEAN, ASEAN Ministers of Agriculture, Fisheries, and Forestry (AMAF) adopted the formulation and implementation of the ASEAN Integrated Food Security (AIFS) Framework, and two Strategic Plans of Action on Food Security (SPA-FS) 2015-2020 extended 2021-2025. This framework and strategy focus on a strategic set of measures based on strong commitments and ownership among all ASEAN Member States (AMS) [23]. AIFS framework plays a role as a regional umbrella for food security initiatives. In this framework,

two key mechanisms emphasised are APTERR and AFSIS. As APTERR only focused on rice, this mechanism could expand the coverage of target commodities such as maize, soybean, sugarcane, and cassava which are important to regional food security. Moreover, to provide more stable data information from AFSIS, this project could be transformed into a permanent mechanism like The APTERR Secretariat [13].

Adopted by the 42nd AMAF on 21 October 2020, ASEAN Member States agreed to jointly address the multiple challenges in food security, food safety and nutrition as follows:

- Strengthening resilience against climate change
- Addressing the food and nutritional requirements of populations, particularly those in remote, rural, and vulnerable regions
- Boosting sustainable agricultural production and productivity
- Developing sustainable approaches to the growing competition for natural resources
- Providing solutions and recommendations to tackle food and nutrition insecurity during pandemics or disease outbreaks

This framework consists of 9 strategic thrusts that consist of various programs to increase collaboration and cooperation between ASEAN countries, strengthen the functions of agencies related to food security in ASEAN, increase the promotion of sustainable agriculture, and improve food and agri-based industry to enhance food security.

3.2.3. ASEAN Leaders Declaration on Strengthening Food Security in Response to Crises

Given the background of building a rapid response to crises and strengthening sustainable food and agricultural systems for the long term, ASEAN Member States consolidated the policy recommendations and unified roles of all relevant sectors to promote food security in the region through 'ASEAN Leaders' Declaration on Strengthening Food Security in Response to Crises'. This initiative is crucial in the face of geopolitical tensions, environmental issues, climate change, and the challenging recovery post-COVID-19 pandemic. This Leaders Declaration gathered AMS committed to joint action whenever crises happened in ASEAN.

The ASEAN Leaders' Declaration consists of two sections as follows; (i) Rapid actions for food security and nutrition in response to crises and (ii) Strengthen preparedness for long-term resilience and sustainability of agri-food systems. ASEAN Leaders recognized the critical role of sustainable agriculture and food systems in ensuring food availability, accessibility, and affordability. Through the declaration, ASEAN committed to enhancing resilience against crises, such as global population growth, climate change, natural disasters, the COVID-19 pandemic, and geopolitical tensions. The declaration reaffirmed commitments to global and regional food security goals, including those outlined in the ASEAN Integrated Food Security Framework and Strategic Plan of Action on Food Security [25].

In regard to addressing food security and nutrition crises, ASEAN Leaders promise to take collective action. These include exploring and encouraging the development of local resource-based food reserves in member states, improving the productivity of agri-food systems by addressing immediate bottlenecks in inputs and logistics, and enhancing food supply chain resilience. They also committed to ensuring unimpeded trade of agri-food products, strengthening the ASEAN Plus Three Emergency Rice Reserves mechanism, and bolstering social welfare programs, including emergency food assistance. The overarching goal is to improve food security and nutrition, especially for the most vulnerable populations, while reinforcing regional cooperation and sustainability. Moreover, ASEAN is also committed to enhancing collaboration with the private sector, international organisations, and think tanks through engagement with relevant initiatives.

3.2.4. ASEAN Regional Guidelines for Sustainable Agriculture in ASEAN

The ASEAN Regional Guidelines for Sustainable Agriculture in ASEAN was adopted at the 44th Meeting of ASEAN Ministers on Agriculture and Forestry on 25 October 2022 [24]. As the document that was adopted during COVID-19 period, this guideline considered the vulnerable impact of COVID-19 on food security. The COVID-19 pandemic has exposed a critical weakness in ASEAN agriculture. ASEAN is significantly reliant on unsustainable imports of raw materials for feed and fertilizers from other regions. The guideline emphasizes the importance of enhancing agricultural input production from available resources within ASEAN, such as agriculture biomass and food waste. This approach promotes circular agriculture, reduces the cost of farming inputs, and enhances the competitiveness of ASEAN agricultural products. Moreover, the guideline also identifies policies, best practices, and incentives necessary to implement the solutions at the appropriate scope. This guideline outlined five principles and recommended 28 key strategies to boost sustainable agriculture in ASEAN.

The five key principles in guidelines balance the social, economic and environmental dimensions of sustainability as follows:

1. Improving efficiency in the use of our resources.
2. Conserving, protecting, enhancing natural ecosystems, promoting and enhancing nature resources and communities.
3. Protecting and improving rural livelihoods and social well-being.
4. Enhancing the resilience of people, communities and ecosystems, and
5. Promoting good governance of both natural and human systems [24]

The principles and key strategies serve as a foundation for crafting regional and national policies, strategies, programs, regulations, and incentives aimed at transforming agriculture to be more economically productive, viable, environmentally responsible, socially inclusive, and sustainable [12].

Through the key strategies in this guideline, ASEAN is trying to ensure food security within region as follows:

1. Exploring the widespread use of sustainable agriculture and food system within region. These initiatives include boosting aquaculture and production without expanding the land, safeguarding and restoring natural ecosystems, enhancing fish supply through better management of wild fisheries and aquaculture, and cutting greenhouse gas emissions from agriculture and food processing.
2. Facilitating funding with productive resources finance and services by providing incentives for ecosystem services. This initiative is designed to support farmers in adopting sustainable agricultural practices that enhance environmental health and strengthen long-term food security.
3. To enhance food production and develop food security in ASEAN, efforts should prioritize sustainably maximizing food output from already cleared areas, focusing on increasing productivity within the existing arable land.

Key milestone for improving food security could be to optimize the use of agricultural land while advancing farming technologies and increasing investments in agricultural infrastructure [16]. Ensuring food security in ASEAN is highlighted as a key strategy that needs to be prioritized in the mid long-term strategy and broader environment strategy [12].

3.2.5. ASEAN Regional Guidelines on Promoting Climate-Smart Agriculture (CSA) Practices Volume 3

ASEAN Regional Guidelines on Promoting Climate-Smart Agriculture (CSA) Practices Volume 3 is a practical guide that will provide guidance for AMS in planning the scaling-up of CSA programs, land use and rural development. proactively respond to the emerging impacts of climate change on social, environmental and economic aspects of food security. This guideline also adapted effectively to the challenges posed by COVID-19 pandemic.

This guideline provides several initiatives that directly respond the COVID-19 as follows:

1. Climate-Smart Village (CSV) Approach

The Climate-Smart Village (CSV) approach is used to identify, test and promote CSA in the community reflecting no one-size-fits-all approach in climate change adaptation in communities dominated by smallholder agriculture. The CSV approach naturally contributes to the local food system by using an end-to-end method that focuses on the entire process, from household consumption to market sales, demonstrating a comprehensive understanding of food systems. This approach's success is largely due to its focus on smallholder farmers. The COVID-19 pandemic helped ASEAN to understand the value of smallholder farmers to help resilience [26].

2. Direct-Seeded Rice (DSR) System Approach

COVID-19 pandemic made the labor shortage even worse in many countries. It also affected people's livelihoods including their income. DSR is one of COVID-safe strategies using drones, GPS and sensors will reduce labor requirements. DSR enables food system to sustain rice production even when facing water shortages, which are expected to be worsened in the future due to climate change. DSR is a CSA approach that can reduce labour costs and GHG emissions. DSR may reduce labour costs by 42% compared with puddled transplanted rice (PTR) and machine-transplanted rice [4].

3. Climate-Resilient Tilapia Farming Practice (CRTFP)

Aquaculture production serves as a major source of animal protein that helps AMS to achieve food and nutrition security and provides a vital economic driver in several AMS. COVID-19 impacting the availability of aquaculture input in ASEAN. The CRTFP utilise locally available resources as alternatives for these inputs. This approach integrates aquaculture technologies, agrometeorological tools, and capacity building for producers to tackle the diverse challenges climate change poses to tilapia aquaculture. It also strengthens the production aspect of aquaculture by ensuring sustainable supplies of tilapia to support subsistence, livelihoods, and greater economies of scale.

4. Low-Emission Animal Production System (LEAPS)

The livestock sector contributes to ASEAN countries' incomes and livelihoods, assets, nutrition, and food security. It also contributes to reducing poverty and enhancing food security in the region (Strategic Plan of Action for ASEAN Cooperation in Livestock, 2016-2020) [20]. It is important to develop climate-smart animal production systems by building resilience to climate change while also contributing to its mitigation. Low-Emission Animal Production System (LEAPS) comprises livestock mitigation technologies that improve sustainability and reduce environmental impact. It will optimise production in terms of livestock outputs as well as other dimensions for the economic, social and environmental well-being of communities. COVID-19 disrupted the livestock input through market closure during the lockdown. LEAPS helps the farmers use locally grown resources that meet the nutritional needs of the animals.

4.CONCLUSION

Food security is a condition where everyone has continuous physical, social, and economic access to enough safe and nutritious food to meet their dietary needs and preferences for an active and healthy life. The agriculture sector as a sector that is crucial in ASEAN holds as a main actor in food security. The COVID-19 pandemic brings out unprecedented situations in the world including this region affecting economic slowdown, trade restrictions policy, and market disruptions that caused price hikes, agriculture input shortage, and other dynamic factors that threaten food security and nutrition in ASEAN. ASEAN countries are urged to enhance regional cooperation, invest in innovative solutions, and promote sustainable agriculture. Food security integration has been prioritized in ASEAN as a collective effort to build resilience against possible crises. ASEAN exemplifies its strong commitment to enhancing food security in the region by taking rapid action through developing and implementing

Leaders' Declarations, Guidelines, and Frameworks. These documents encourage collaboration on rapid actions in response to crises within ASEAN and related development partners including private sectors, strengthening the function of food security-related agencies in ASEAN, accelerating the transformation of sustainable agriculture and digitalization, enhancing market connectivity, promoting investment in agricultural research and development, and facilitating access to finance.

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