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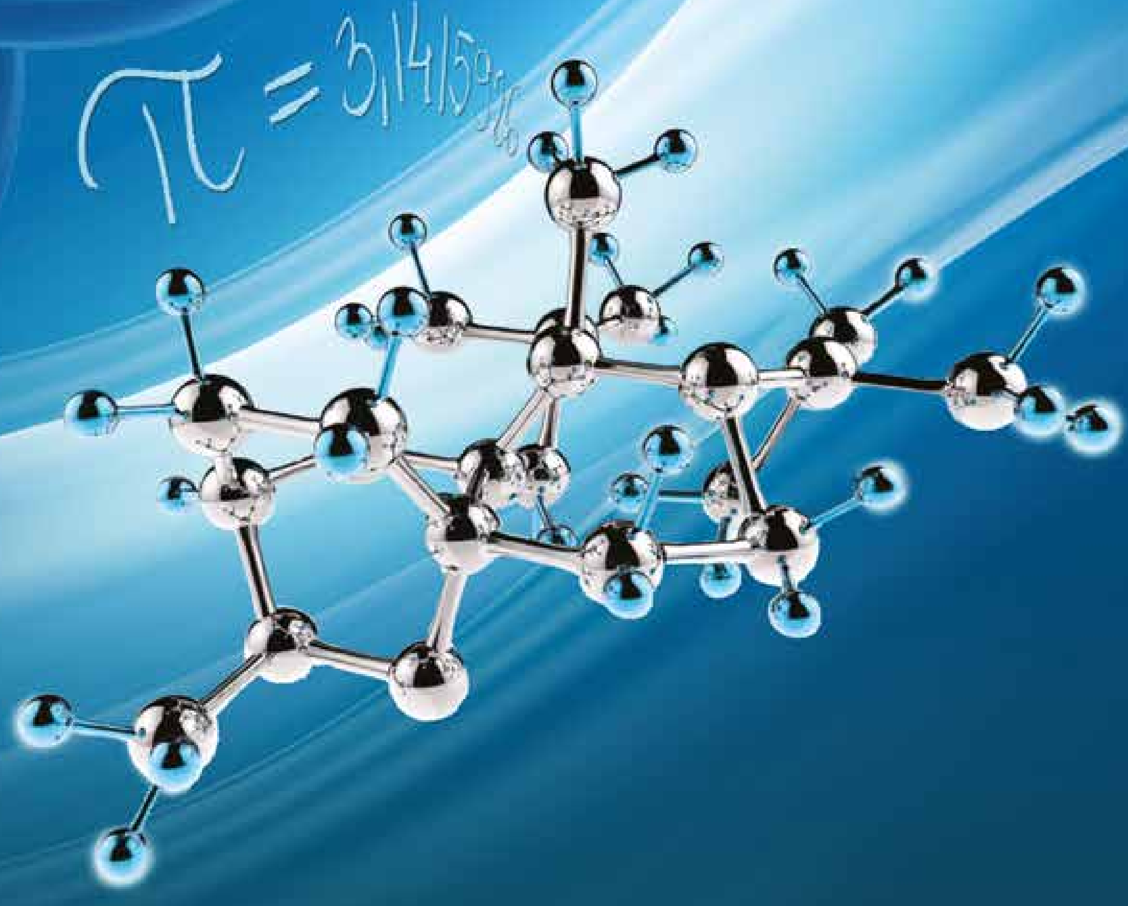

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PREFACE

Dear scientist,

I am happy to announce that Volume X - Issue II of the Eastern Anatolian Journal of Science (EAJS) has been published. This issue is composed of 6 research articles that possess some of the leading and advanced techniques of natural and applied sciences. On behalf of the owner of EAJS, I would like to thank all authors, referees, our editorial board members and section editors that provide valuable contributions for the publication of the issue.

EAJS will publish original and high-quality articles covering a wide range of topics in scientific research, dedicated to promoting high standards and excellence in the creation and dissemination of scientific knowledge. EAJS published in English is open access journal and abstracting and indexing by various international index services.

Authors are solicited to contribute to the EAJS by submitting articles that illustrate research results, projects, surveying works and industrial experiences that describe significant advances in the following areas, but are not limited to:

- Biology
- Chemistry
- Engineering
- Mathematics
- Nanoscience and Nanotechnology
- Physics

Our previous issues have an attraction in terms of scientific quality and impact factor of articles by favorable feedbacks of readers. Our editorial team lend wings to be an internationally reputable and pioneer journal of science by their outstanding scientific personality. I am hoping to work effectively with our editorial team in the future.

I'd like to express my gratitude to all authors, members of editorial board and contributing reviewers. My sincere thanks go to Prof. Dr. Abdulhalik KARABULUT, the rector of Ağrı İbrahim Çeçen University, sets the goal of being also a top-ranking university in scientific sense, for supporting and motivating us in every respect. I express my gratitude to the members of technical staff of the journal for the design and proofreading of the articles. Last but not least, my special thanks go to the respectable businessman Mr. İbrahim ÇEÇEN who unsparingly supports our university financially and emotionally, to his team and to the director and staff of IC foundation.

I invite scientists from all branches of science to contribute our journal by sending papers for publication in EAJS.

Prof. Dr. İbrahim HAN

Editor-in-Chief

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Decision Problems in Queueing Theory: A Numeric Application

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Abstract

The application of general behavioral patterns obtained from stochastic processes has always played an important role in Queuing Theory. Since the first studies in which optimization techniques were used in the decision-making process, design and control procedures have been included in studies especially in the field of statistics and operations. In the first studies where queuing systems were modeled and their operability was optimized, performance measures such as the block probability and the average waiting times in the system were considered in the decision-making process. With the availability of performance measures based on probabilistic methods in modeling queuing systems, the decision-making process has begun to be based on such measures. In this study probabilistic calculations of some performance measures of a custom queueing system is given in order to make a decision for optimum parameters of the system. In addition, a numerical example is given to illustrate the case.

Keywords: Queueing theory; Stochastic process; Decision process; Optimization

1. *Introduction*

The application of general behavioral patterns obtained from stochastic processes has always played an important role in Queuing theory. Since the first studies in which optimization techniques were used in the decision-making process, design and control procedures have been included in studies especially in the field of statistics and operations.

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On the other hand, the number of studies conducted constitutes a small part of the potential of the subject in terms of volume. Queuing Theory, which occupies a large part of the field of Stochastic processes, can be said to have started with the study conducted by A. K. Erlang (1917). The "equilibrium state of the system" behavior that inevitably occurs in most queuing systems was first studied by Polaczek (1965), in this study, the behavior of the system was analyzed and tried to be defined within a finite time interval. The first analyzed system in queuing theory is the M/M/1 system. Obtaining the equilibrium state equations of this system under some statistical assumptions and defining the limit distribution of the queue length are relatively simple and can be solved with iterative techniques. On the other hand, when the time parameter is taken into consideration, more complex mathematical calculations are needed. In this sense, the first solution methods were proposed by Bailey (1952). In addition, Ledermann and Reuter (1956) used spectral theory for solutions in their study. In the following studies, Laplace transform and techniques using Laplace transform and generating functions together were used as solution methods. Probabilistic methods were first used in the analysis of queuing systems by Kendall (1956), (1953). Stidham (1995) discussed the reasons for the inadequacy of studies on design and control in queuing theory. The decision-making process is generally based on two basic ideas: performance measures and decision problems. Decision problems are divided into two as design and control problems (T. B. Crabill et al. 1977). This study shows that a suitable process to be determined to optimize criteria such as cost, or profit is a design problem. The optimization of a control problem that we encounter in real life is dynamic. In other words, the functions of the system are in a constant change with time.

2. *Performance Measures for Decision Making*

There is a “decision making” process inherent in the solution phase of any problem we encounter in real life. In the first studies where queuing systems were modeled and the operability of the systems was optimized, performance measures such as the probability of the system being blocked and the average waiting times in the system were considered in the decision-making process. Various charts were developed for these performance measures (Bhat, 2003). Today, since the competence of data visualization and graphic software has increased greatly, these charts have become almost unnecessary. As we have mentioned before, with the availability of performance measures based on probabilistic methods in modeling queuing systems, the decision-making process has begun to be based on such measures. In addition, the simulation support provided by computers has become a multiplier power in the decision-making process. As it is known, simulations determine the accuracy and best features of the model under certain conditions and situations.

3. *Results Design Problems in Decision Making*

In design problems, regardless of the system being modeled, the main idea is to optimize the parameters in a way that will optimize the operating performance of the system. The cost function is a part of the optimization process, and this function is used to obtain the optimum values of the optimal configurations. In this sense, cost functions can be based on monetary costs or performance measures depending on the model. Therefore, the problems mentioned can also be called "economic problems". Optimization of any problem is static, and the process is carried out using predefined and specific procedures. However, when it comes to modeling queuing systems and queuing system models can enter very complex situations, the procedures inherent in static optimization may not work. Statistical and numerical procedures will be much more appropriate for such situations. If we consider the problem of determining the optimum number of customers that should be accepted for service in an $M/M/1/N$ queueing system, we will need to balance the cost of service with the cost of losing customers. Let us assume that customer arrival is Poisson with rate λ and that there are service units with rate μ. In this case, where the cost per unit of time is $F\mu$ and the gross profit per single service is B, the net profit per unit of time is calculated as follows:

$$K = \frac{\lambda B(1-\rho^N)}{1-\rho^{N+1}} - F\mu \qquad (1)$$

In the Equation (1), ρ is the traffic density of the system and is determined as $\rho = \lambda / \mu$ and N is the total number of customers in the system. If the derivative of this equation is taken with respect to the parameter μ and set equal to zero, the following equation is obtained for the maximum value of the parameter μ:

$$\rho^{N+1}\frac{N-(N+1)\rho+\rho^{N+1}}{(1-\rho^{N+1})^2} = \frac{F}{B} \qquad (2)$$

The graph obtained from this equation can be used to determine the number of customers N that should be accepted into the queue system for the varying cost parameters and service rates of the system. In the case of infinite waiting space in the queue system, the optimum value of the service rate μ is obtained with the following cost function using the standard optimization approach:

$$Y\mu + XZ = Y\mu + \frac{X}{\mu-\lambda} \qquad (3)$$

In this equation, X is the waiting cost per unit time, Y is the service cost per unit time, and Z is the average waiting time. If the derivative of this cost function is taken with respect to the parameter μ and set equal to zero:

$$\mu = \lambda + \sqrt{\frac{X}{Y}} \qquad (4)$$

is obtained. On the other hand, in systems with multiple service units, optimization is performed by trial-and-error method to determine the optimum number of service units. Let's assume that all arrivals in a three-server queue system are Poisson and the queue discipline is First come First served (FCFS). In this case, the waiting cost will be proportional to the time elapsed in the system and the service cost will be a linear function of the number of servers. Again, three basic models are used to determine the optimum values for λ, μ and s in the cost functions, namely the arrival rate per server λ, the service rate per server μ and the number of servers s: the first model finds the s value from the cost function, the second model finds the λ and s values, and the third model finds the μ and s values, respectively. Due to the multi-server structure, it should also be considered that each server should have its own waiting line if the service time is not exponential.

4. *Application: a Numeric example*

Let's assume that customers arrive to a supermarket according to a Poisson distribution with a rate of λ. After customers receive their products, they will line up in front of the cash register to pay. The time spent in this process is assumed to be exponentially distributed. Let's determine the optimum value of the number of cash registers in the supermarket under the following cost function, where the mean of the distribution of the time it takes for each customer to go to the cash register is m_1 and the second moment is m_2. (i) X_1 is the cost per unit of time of a customer waiting (ii) X_2 is the cost per unit of time of a customer paying at the cash register. Since the time spent by the customer in choosing a product is exponentially distributed, the arrival process at the checkout can also be assumed to be Poisson. When the number of checkouts is s and customers are assumed to choose checkouts randomly, the arrival rate at each checkout is assumed to be Poisson with rate λ/s. Considering the waiting time of a customer in the $M/G/1$ queue system:

$$Z_q = \left(\frac{\lambda}{s}\right) m_2/2\left(1 - \frac{\lambda m_1}{s}\right) = \frac{\lambda m_2}{2(s - \lambda m_1)} \tag{5}$$

s obtained. Let the total cost per unit time be X, then the average total cost is calculated with the following equation,

$$E(X) = \frac{\lambda m_2 X_1}{2s - 2\lambda m_1} + SX_2 \tag{6}$$

If the cost function is minimized with respect to s, It turns out that the equation minimizes $E(X)$ as given in Equation (6). For a numerical example, if $\lambda = 2$ (2 customer arrivals per minute), the checkout time will be an exponential distribution with mean $m_1 = 3$ minutes. Thus, $m_2 = 9$. In addition, $X_1 = 0.5$, $X_2 = 2.5$ are calculated and if these values are written in the equation (6), the optimal value for the number of checkouts in the supermarket is obtained as $s = 7$.

5. *Results and Discussion*

There are plenty of papers given on optimum design and Control Theory on various real-world problem. As mentioned before design and control procedures are rarely applied in stochastic process. There are several reasons for this. First, Queueing Theory has a dynamic structure, that is the parameters of the system mostly depend on time which makes it a big challenge to apply optimization and control procedure. However as shown in this study there are some probabilistic methods to calculate some important performance measures of a queueing system. On the other hand, obtaining performance characteristics of more complex queueing systems via control and design procedures can be more challenging. In this manner for more studies in the field, computer aided techniques such as simulation and computational statistical methods can be applied for the optimization and design of queueing systems.

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Coefficient Inequalities for Two New Subclasses of Bi-univalent Functions Involving Lucas-Balancing Polynomials

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Abstract

In this article, by making use of Lucas-Balancing polynomials two new subclasses of bi-univalent functions are introduced. Then we establish the bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for two new families of analytic and bi-univalent functions in the open unit disk which involve Lucas-Balancing polynomials. Furthermore, we investigate the special cases and consequences for the new family functions. In addition, the Fekete-Szegö problem is handled for the functions belonging to these new subclasses.

Keywords: Analytic and bi-univalent functions, subordination, coefficient inequality, Lucas-Balancing polynomials.

1. *Introduction*

Let A denote the class of all analytic functions of the form

$$f(z) = z + a_2 z^2 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. It is clear that the functions in A satisfy the conditions and $f(0) = 0$ and $f'(0) = 1$, known as normalization conditions. We show by $\mathcal{S}$ the subclass of A consisting of functions univalent in A.

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The Koebe one quarter theorem (see (Duren 1983)) guarantees that if $f \in \mathcal{S}$, then there exists the inverse function f^{-1} satisfying

$f^{-1}(f(z)) = z, \ (z \in \mathbb{E})$ and $f(f^{-1}(\omega)) = \omega, \ (|\omega| < r_0(f), \ r_0(f) \geq \frac{1}{4},$

where

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + +(2a_2{}^2 - a_3)\omega^3 + \cdots. \qquad (2)$$

One of the most important subclass of analytic and univalent function class on the unit disk $\mathbb{E}$ is the bi-univalent function class and is denoted by Σ. In fact, a function $f \in A$ is called bi-univalent function in $\mathbb{E}$ if both f and f^{-1} are univalent in $\mathbb{E}$. Here, we would like to remind that the problem finding an upper bound for the coefficient $|a_n|$ of the functions belonging to class Σ is still an open problem. A wide range of coefficient estimates for the functions in the class Σ can be found in the literature. For instance, Brannan and Clunie (Brannan and Clunie 1980), and Lewin (Lewin 1967), gave very important bounds on $|a_2|$, respectively. Also, Brannan and Taha (Brannan and Taha 1988), focused on some subclasses of bi-univalent functions and proved certain coefficient estimates. As mentioned above, one of the most attractive open problems in univalent function theory is to find a coefficient estimate on $|a_n|$ $(n \in \mathbb{N}, \ n \geq 3,)$ for the functions in the class Σ. Since this attraction, motivated by the works (Brannan and Clunie 1980), (Brannan and Taha 1988), (Lewin 1967), (Srivastava et al. 2010), (Buyankara et al. 2022), (Çağlar et al. 2022), (Çağlar 2019), (Çağlar et al. 2013), (Frasin et al. 2021), (Güney at al. 2018), (Güney at al. 2019), (Orhan et al 2018), (Srivastava et al. 2013), (Toklu 2019), (Toklu et al. 2019), (Zaprawa 2014), (Aktaş and Karaman 2023), (Öztürk and Aktaş 2023), (Öztürk and Aktaş 2024), (Korkmaz and Aktaş 2024), (Aktaş and Hamarat 2023), (Orhan et al. 2023), (Aktaş and Yılmaz 2022), (Yılmaz and Aktaş 2022) and references therein, the authors introduced numerous subclasses of bi-univalent functions and obtained non- sharp estimates on the initial coefficients of functions in these subclasses.

In the univalent function theory, one of the most important notions is subordination principle. Let the function $f \in A$ and $F \in A$. Then, f is called to be subordinate to F if there exists a Schwarz function ω such that
$\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = F(\omega(z))$ $(z \in \mathbb{E})$.
This subordination is shown by

$$f \prec F \text{ or } f(z) \prec F(z) \quad (z \in \mathbb{E}).$$

Especially, if the function F is univalent in $\mathbb{E}$, then subordination is equivalent to

$$f(0) = F(0), \qquad f(\mathbb{E}) \subset F(\mathbb{E}).$$

A comprehensive information about the subordination concept can be found in Monographs written by Miller and Mocanu (see (Miller et al. 2000)).

2. Lucas-Balancing Polynomials and Its Generating Function

The notion of Balancing number was defined by Behera and Panda in (Behera et al 1999). Actually, balancing number n and its balancer r are solutions of Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

It is known that if n is a balancing number, then $8n^2 + 1$ is a perfect square and its positive square root is called a Lucas-Balancing number (Ray 2014). Recently, some properties of these new number sequences have been intensively studied and its some generalizations were defined. Interested readers can find comprehensive information regarding Lucas-Balancing numbers in (Davala and Panda 2015), (Frontczak and Baden-Württemberg 2018), (Frontczak and Baden-Württemberg 2008), (Komatsu and Panda 2016), (Keskin and Karaatlı 2012), (Ray 2014), (Ray 2015), (Ray 2018), (Patel et al. 2018) and references therein. Natural extensions of the Lucas-Balancing numbers is Lucas-Balancing polynomial and it is defined by:

Definition 1.(Frontczak 2019) Let $x \in \mathbb{C}$ and $n \geq 2$. Then, Lucas-Balancing polynomials are defined the following recurrence relation

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x), \tag{3}$$

where $C_0(x) = 1$ and

$$C_1(x) = 3x. \tag{4}$$

Using recurrence relation given by (3) we easily obtain that

$$C_2(x) = 18x^2 - 1, \tag{5}$$

$$C_3(x) = 108x^3 - 9x. \tag{6}$$

Lemma 1. (Frontczak 2019) The ordinary generating function of the Lucas-Balancing polynomials is given by

$$R(x,z) = \sum_{n=0}^{\infty} C_n(x)z^n = \frac{1-3xz}{1-6xz+z^2}. \tag{7}$$

3. New Subclasses of Bi-univalent Functions

In this subsection, we introduce some new function subclasses of analytic and bi-univalent function class Σ which is subordinate to Lucas-Balancing polynomials.

Definition 2. A function $f(z) \in \Sigma$ of the form (1) is said to be in the class $B^{C_\Sigma}(R(x,z))$ if the following conditions hold true:

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1-3xz}{1-6xz+z^2} = R(x,z) \tag{8}$$

and

$$\frac{2\omega f'(\omega)}{f(\omega)-f(-\omega)} \prec \frac{1-3x\omega}{1-6x\omega+\omega^2} = R(x,\omega), \tag{9}$$

where $z, \omega \in \mathbb{E}$, g is inverse of f and it is of the form (2).

Our second function class is bi-starlike function class $M^{C_\Sigma}(R(x,z))$ and it is defined as follows:

Definition 3. A function $f(z) \in \Sigma$ of the form (1) is said to be in the class $M^{C_\Sigma}(R(x,z))$ if the following conditions hold true:

$$\frac{2[zf'(z)]'}{[f(z)-f(-z)]'} \prec \frac{1-3xz}{1-6xz+z^2} = R(x,z) \tag{10}$$

and

$$\frac{2[\omega f'(\omega)]'}{[f(\omega)-f(-\omega)]'} \prec \frac{1-3xz}{1-6x\omega+\omega^2} = R(x,\omega), \tag{11}$$

where $z, \omega \in \mathbb{E}$, g is inverse of f and it is of the form (2).

In the present paper our main aim is to find upper bounds for the Taylor-Maclaurin coefficients of function subclasses defined by $B^{C_\Sigma}(R(x,z))$ and $M^{C_\Sigma}(R(x,z))$. A rich history for the class Σ can be found in the pioneering work (Srivastava et al. 2010), published by Srivastava et al.

4. Coefficient Estimates for the Classes $B^{C_\Sigma}(R(x,z))$ and $M^{C_\Sigma}(R(x,z))$

In this section, we present initial coefficients estimates for the function belonging to the subclasses $B^{C_\Sigma}(R(x,z))$ and $M^{C_\Sigma}(R(x,z))$, respectively.

Theorem 1. Suppose that the function $f(z) \in B^{C_\Sigma}(R(x,z))$ and $x \in \mathbb{C}\backslash\left\{\mp\frac{\sqrt{6}}{9}\right\}$. Then,

$$|a_2| \le \frac{3\sqrt{3}|x|\sqrt{|x|}}{\sqrt{2|2-27x^2|}} \tag{12}$$

and

$$|a_3| \le \frac{3|x|}{2}\left(\frac{3|x|}{2}+1\right). \tag{13}$$

Proof. Let the function $f(z) \in B^{C_\Sigma}(R(x,z))$ and $g = f^{-1}$ given by (2). In view of Definition 2, from the relations (8) and (9) we can write that

$$\frac{2zf'(z)}{f(z)-f(-z)} = R(x,\tau(z)) \tag{14}$$

and

$$\frac{2\omega f'(\omega)}{f(\omega)-f(-\omega)} = R(x,\varphi(\omega)). \tag{15}$$

Here $\tau(z) = k_1 z + k_2 z^2 + \cdots$ and $\varphi(\omega) = \varphi_1\omega + \varphi_2\omega^2 + \cdots$ are Schwarz functions such that $\tau(0) = \varphi(\omega) = 0$, $|\tau(z)| < 1$ and $|\varphi(\omega)| < 1$ for all $z, \omega \in \mathbb{E}$. On the other hand, these conditions imply

$$|\tau_j| < 1, \tag{16}$$

$$|\varphi_j| < 1 \tag{17}$$

for all $j \in \mathbb{N}$. Basic computations yield that

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + 2a_2 z + 2a_3 z^2 + \cdots \tag{18}$$

$$\frac{2\omega f'(\omega)}{f(\omega)-f(-\omega)} = 1 - 2a_2\omega + (4a_2^2 - 2a_3)\omega^2 + \cdots \tag{19}$$

$$R(x,\tau(z)) = C_0(x) + [C_1(x)k_1]z + [C_1(x)k_2 + C_2(x)k_1^2]z^2 + [C_1(x)k_3 + 2C_2(x)k_1k_2 + C_3(x)k_1^3]z^3 + \cdots \tag{20}$$

and

$$R(x,\varphi(\omega)) = C_0(x) + [C_1(x)\varphi_1]\omega + [C_1(x)\varphi_2 + C_2(x)\varphi_1^2]\omega^2 + [C_1(x)\varphi_3 + 2C_2(x)\varphi_1\varphi_2 + C_3(x)\varphi_1^3]\omega^3 + \cdots \tag{21}$$

Now, using equation (14) and comparing the coefficients of (18) and (20), we get

$$2a_2 = C_1(x)k_1, \tag{22}$$

$$2a_3 = C_1(x)k_2 + C_2(x)k_1^2. \tag{23}$$

Similarly, using equation (15) and comparing the coefficients of (19) and (21), we have

$$-2a_2 = C_1(x)\varphi_1, \tag{24}$$

$$4a_2^2 - 2a_3 = C_1(x)\varphi_2 + C_2(x)\varphi_1^2. \tag{25}$$

Now, from equations (22) and (24) we get

$$k_1 = -\varphi_1, \tag{26}$$

and

$$\frac{8a_2^2}{[C_1(x)]^2} = k_1^2 + \varphi_1^2 \tag{27}$$

Also, from the summation of the equations (23) and (25), we easily obtain that

$$4a_2^2 = C_1(x)(k_2+\varphi_2) + C_2(x)(k_1^2+\varphi_1^2), \tag{28}$$

By substituting equation (27) in equation (28) we get

$$a_2^2 = \frac{[C_1(x)]^3(k_2+\varphi_2)}{4(C_1(x))^2 - 8C_2(x)}. \tag{29}$$

Taking into account (4) and (5) in (29) we get

$$a_2^2 = \frac{27x^3(k_2+\varphi_2)}{8-108x^2}. \tag{30}$$

Now, using triangle inequality with the inequalities (16) and (17), we have

$$|a_2|^2 \le \frac{27|x|^3}{|4-54x^2|}. \tag{31}$$

Taking square root both sides of the last inequality, we have (12).

In addition, if we subtract the equation (25) from the equation (23) and consider equation (26), then we obtain

$$a_3 = \frac{C_1(x)(k_2-\varphi_2)}{4} + a_2^2. \tag{32}$$

Considering the equation (27) in (32) and a straightforward calculation yield that

$$a_3 = \frac{C_1(x)(k_2-\varphi_2)}{4} + \frac{[C_1(x)]^2(k_1^2+\varphi_1^2)}{8}. \tag{33}$$

By making use of the equation (4), and triangle inequality with the inequalities (16) and (17) in (33) we deduce the inequality (13). So, the proof is completed.

Theorem 2. Suppose that the function $f(z) \in M^{C\Sigma}(R(x,z))$ and $x \in \mathbb{C}\backslash\left\{\mp\frac{2\sqrt{2}}{\sqrt{117}}\right\}$. Then,

$$|a_2| \leq \frac{3\sqrt{3}|x|\sqrt{|x|}}{\sqrt{|2(8-117x^2)|}} \tag{34}$$

and

$$|a_3| \leq \frac{|x|}{16}(8+9|x|). \tag{35}$$

Proof. Let the function $f(z) \in M^{C\Sigma}(R(x,z))$ and $g = f^{-1}$ given by (2). In view of Definition 3, from the relations (10) and (11) we can write that

$$\frac{2[zf'(z)]'}{[f(z)-f(-z)]'} = R(x, p(z)) \tag{36}$$

and

$$\frac{2[\omega f'(\omega)]'}{[f(\omega)-f(-\omega)]'} = R(x, d(\omega)). \tag{37}$$

By virtue of the relations (32) and (33), there are two Schwarz functions $p(z) = p_1 z + p_2 z^2 + \cdots$ and $d(\omega) = d_1\omega + d_2\omega^2 + \cdots$ are Schwarz functions such that $p(0) = d(0) = 0$ and $|p(z)| < 1, |d(\omega)| < 1$ for all $z, \omega \in \mathbb{E}$. On the other hand, these conditions imply that

$$|p_j| < 1, \tag{38}$$

$$|d_j| < 1 \tag{39}$$

for all $j \in \mathbb{N}$. A straightforward calculation yields that

$$\frac{2[zf'(z)]'}{[f(z)-f(-z)]'} = 1 + 4a_2 z + 6a_3 z^2 + \cdots \tag{40}$$

and

$$\frac{2[\omega f'(\omega)]'}{[f(\omega)-f(-\omega)]'} = 1 - 4a_2\omega + (12{a_2}^2 - 6a_3)\omega^2 + \cdots. \tag{41}$$

$$R(x,p(z)) = C_0(x) + [C_1(x)p_1]z + [C_1(x)p_2 + C_2(x){p_1}^2]z^2 + [C_1(x)p_3 + 2C_2(x)p_1p_2 + C_3(x){p_1}^3]z^3 + \cdots \tag{42}$$

and

$$R(x,d(\omega)) = C_0(x) + [C_1(x)d_1]\omega + [C_1(x)d_2 + C_2(x){d_1}^2]\omega^2 + [C_1(x)d_3 + 2C_2(x)d_1d_2 + C_3(x){d_1}^3]\omega^3 + \cdots \tag{43}$$

Now, using equation (36) and comparing the coefficients of (40) and (42), we get

$$4a_2 = C_1(x)p_1, \tag{44}$$

$$6a_3 = C_1(x)p_2 + C_2(x){p_1}^2. \tag{45}$$

Similarly, using equation (37) and comparing the coefficients of (41) and (43), we have

$$-4a_2 = C_1(x)d_1, \tag{46}$$

$$12{a_2}^2 - 6a_3 = C_1(x)d_2 + C_2(x){d_1}^2. \tag{47}$$

Now, from equations (44) and (46) we get

$$p_1 = -d_1, \tag{48}$$

and

$$\frac{32{a_2}^2}{[C_1(x)]^2} = {p_1}^2 + {d_1}^2. \tag{49}$$

Also, from the summation of the equations (45) and (47), we easily obtain that

$$12{a_2}^2 = C_1(x)(p_2 + d_2) + C_2(x)({p_1}^2 + {d_1}^2), \tag{50}$$

By substituting equation (49) in equation (50) we get

$${a_2}^2 = \frac{[C_1(x)]^3(p_2+d_2)}{12(C_1(x))^2 - 32C_2(x)}. \tag{51}$$

Plugging equations (4) and (5) into (51), we get that

$${a_2}^2 = \frac{27x^3(p_2+d_2)}{4(8-117x^2)}. \tag{52}$$

Now, using triangle inequality with the inequalities (38) and (39), we have

$$|a_2|^2 \leq \frac{27|x|^3}{|2(8-117x^2)|}. \tag{53}$$

Taking square root both sides of the last inequality, we have (34).

In addition, if we subtract the equation (47) from the equation (45) and consider equation (48), then we obtain

$$a_3 = \frac{C_1(x)(p_2-d_2)}{12} + {a_2}^2. \tag{54}$$

Considering the equation (49) in (54) and a straightforward calculation yield that

$$a_3 = \frac{C_1(x)(p_2-d_2)}{12} + \frac{[C_1(x)]^2({p_1}^2+{d_1}^2)}{32}. \tag{55}$$

By making use of the equation (4), and triangle inequality with the inequalities (38) and (39) in (55), we deduce the inequality (35). So, the proof is completed.

5. Fekete-Szegö inequalities for the class $B^{C_\Sigma}(R(x,z))$ and $M^{C_\Sigma}(R(x,z))$

Our result regarding Fekete-Szegö inequality for the function class $B^{C_\Sigma}(R(x,z))$ is the following.

Theorem 3. Suppose that the function $f(z) \in B^{C_\Sigma}(R(x,z))$, $\mu \in \mathbb{R}$ and $x \in \mathbb{C} \setminus \left\{0, \mp\frac{\sqrt{6}}{9}\right\}$. Then, we have

$$|a_3 - \mu a_2{}^2| \leq \begin{cases} \frac{3}{2}|x|, & if \quad |1-\mu| \leq \frac{|2-27x^2|}{|9x^2|}, \\ \frac{27|x|^3|1-\mu|}{|4-54x^2|}, & if \quad |1-\mu| \geq \frac{|2-27x^2|}{|9x^2|}, \end{cases} \tag{56}$$

Proof. Let the function $f(z) \in B^{C_\Sigma}(R(x,z))$ and $\mu \in \mathbb{R}$. By equations (29) and (32) in Definition 2, we can write that

$$a_3 - \mu a_2{}^2 = \frac{C_1(x)(k_2-\varphi_2)}{4} + a_2{}^2 - \mu a_2{}^2$$

$$=(1-\ \mu)a_2{}^2 + \frac{C_1(x)(k_2-\varphi_2)}{4}$$

$$=(1-\ \mu)\frac{[C_1(x)]^3(k_2+\varphi_2)}{4(C_1(x))^2-8C_2(x)} + \frac{C_1(x)(k_2-\varphi_2)}{4}$$

$$= C_1(x)\left\{\left(h_1(\mu)+\frac{1}{4}\right)k_2 + \left(h_1(\mu)-\frac{1}{4}\right)\varphi_2\right\}, \tag{57}$$

where $h_1(\mu) = \frac{(1-\mu)[C_1(x)]^2}{4(C_1(x))^2-8C_2(x)}$. Now, taking modulus and using triangle inequality with (16), (17), (4) and (5) in (57), we complete the proof.

For $\mu = 1$ in Theorem 3, we obtain the following corollary.

Corollary 1. If the function $f(z) \in B^{C_\Sigma}(R(x,z))$. Then,

$$|a_3 - a_2{}^2| \leq \frac{3}{2}|x|. \tag{58}$$

Our next result regarding Fekete-Szegö inequality for the function class $M^{C_\Sigma}(R(x,z))$ is the following.

Theorem 4. Suppose that the function $f(z) \in M^{C_\Sigma}(R(x,z))$, $\mu \in \mathbb{R}$ and $x \in \mathbb{C} \setminus \left\{0, \frac{\sqrt{8}}{\sqrt{117}}\right\}$. Then, we have

$$|a_3 - \mu a_2{}^2| \leq \begin{cases} \frac{|x|}{2}, & if \quad |1-\mu| \leq \frac{|8-117x^2|}{|27x^2|}, \\ \frac{27|x|^3|1-\mu|}{|2(8-117x^2)|}, & if \quad |1-\mu| \geq \frac{|8-117x^2|}{|27x^2|}, \end{cases} \tag{59}$$

Proof. Let the function $f(z) \in M^{C_\Sigma}(R(x,z))$ and $\mu \in \mathbb{R}$. By equations (51) and (54) in definition 3, we can write that

$$a_3 - \mu a_2{}^2 = \frac{C_1(x)(p_2-d_2)}{12} + a_2{}^2 - \mu a_2{}^2$$

$$=(1-\ \mu)a_2{}^2 + \frac{C_1(x)(p_2-d_2)}{12}$$

$$= (1-\ \mu)\frac{[C_1(x)]^3(p_2+d_2)}{12(C_1(x))^2-32C_2(x)} + \frac{C_1(x)(p_2-d_2)}{12}$$

$$= C_1(x)\left\{\left(h_2(\mu)+\frac{1}{12}\right)p_2 + \left(h_2(\mu)-\frac{1}{12}\right)d_2\right\}, \tag{60}$$

where $h_2(\mu) = \frac{(1-\mu)[C_1(x)]^2}{12(C_1(x))^2-32C_2(x)}$. Now, taking modulus and using triangle inequality with (38), (39), (4) and (5) in (60), we complete the proof.

İf we take $\mu = 1$ in the Theorem 4, we have the following corollary.

Corollary 2. If the function $f(z) \in M^{C_\Sigma}(R(x,z))$. Then,

$$|a_3 - a_2{}^2| \leq \frac{|x|}{2}. \tag{61}$$

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Synthesis, Characterization and Antimicrobial Activity of Copper Nanoparticles from *Lavandula Stoechas L.* by Green Synthesis Method

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Abstract

Metal nanoparticles (copper (Cu), silver (Ag), gold (Au), platinum (Pt), zinc (Zn)) have a wide antimicrobial activity against different types of microorganisms such as gram negative-gram positive bacteria and fungi and are alternatives to antibiotics. Green synthesis is particularly preferred among synthesis methods because it is simple, environmentally friendly, cost-effective, and yields products quickly. In this study, copper nanoparticles (CuNps) were synthesized using Lavandula stoechas extract as a stabilizing agent, leveraging the properties of this medicinal and aromatic plant.
The synthesized CuNps were characterized, showing that they were spherical and less than 50 nm in size. Their antibacterial activity was assessed using both broth dilution and disc diffusion methods. The minimum inhibitory concentration (MIC) values for the bacterial strains were as follows: 250 µg/mL for *Bacillus subtilis*, *Staphylococcus aureus*, *Pseudomonas aeruginosa*, and *Salmonella enteritidis*; and 500 µg/mL for *Enterococcus faecalis* and *Escherichia coli*. In the disc diffusion test, the inhibition zone diameters increased with higher CuNps concentrations across all Gram-negative and Gram-positive strains. The highest inhibition zones were recorded as 15 mm for *B. subtilis*, 16.5 mm for *S. aureus*, 14 mm for *E. faecalis*, 19.5 mm for *P. aeruginosa*, 16.5 mm for *S. enteritidis*, and 13.5 mm for *E. coli*.
In summary, this study demonstrates that CuNps can be successfully synthesised using *Lavandula stoechas* extract and exhibit significant antimicrobial properties. These findings suggest that CuNps could serve as effective alternatives to traditional antibiotics, potentially helping to address the growing issue of antibiotic resistance.

Keywords: *Lavandula stoechas L*; green synthesis; copper nanoparticles; antimicrobial activity

1. Introduction

Throughout history, scientists have battled against disease-causing microorganisms, with antibiotics becoming a key weapon against bacterial infections since the 1940s (Tenover 2006; Sengupta 2013). Despite this, infection-related morbidity and mortality remain alarmingly high (Lagedroste et al. 2019; Canlı et al. 2019). The excessive and indiscriminate use of antibiotics has led to a crisis of antibiotic resistance, marked by multidrug-resistant “superbugs” and biofilm formation (Lagedroste et al. 2019; Beyth et al. 2015). Consequently, there is an urgent need for alternative antibiotic treatments, with nanoparticles (NPs) emerging as a promising option (Lagedroste et al. 2019; Canlı et al. 2019).
Traditional antibiotics generally target bacterial cell walls, protein synthesis, or DNA replication mechanisms (Tenover 2006; Wang et al. 2017). In contrast, nanoparticles directly interact with the bacterial cell wall without entering the cells, making it difficult for bacteria to develop resistance. While the antibacterial mechanisms of NPs are not fully understood, one proposed mechanism involves metal ions from the NPs attaching to bacterial cell walls through transmembrane proteins, thereby obstructing

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transport channels and altering cell membrane structure. Once inside the cell, these ions cause cell death (Prabhu et al. 2012; Dizaj et al. 2014). Additionally, reactive oxygen species (ROS) produced by metal NPs damage essential cellular structures, including the peptidoglycan layer, cell membranes, DNA, mRNA, ribosomes, and proteins, contributing significantly to their antibacterial effects (Raffi et al. 2008; Pelgrift and Friedman 2013). Metal ions can also bind with thiol groups in enzymes, inactivating them, and they can disrupt DNA by binding to purine and pyrimidine bases, breaking hydrogen bonds and destroying DNA integrity (Jung et al. 2008; Hoseinzadeh et al. 2017; Shahzadi et al. 2018).

Copper nanoparticles (CuNps) have become popular in recent years because they have a high surface-to-volume ratio, work very well as catalysts, and kill microbes very effectively. They are also cheaper than noble metals like silver, gold, and platinum (Olajire et al. 2018). The antimicrobial activity of CuNps is attributed to the release of copper ions (Mott et al. 2007). Although various physical and chemical methods exist for NP synthesis, these methods are often costly and generate toxic by-products. Additionally, they make it difficult to precisely control NP surface chemistry, size, and structure.

Given these limitations, green synthesis has gained attention as an affordable, environmentally friendly, and non-toxic alternative. This method uses living things like plants, algae, bacteria, yeasts, and fungi to change inorganic metal ions into metal nanoparticles by using proteins and metabolites to break them down (Manikandan et al. 2017; Kumar et al. 2017). Plants, which are rich in phytochemicals like flavonoids, terpenoids, tannins, and alkaloids, are especially popular for green synthesis.

In this study, *Lavandula stoechas* L., a medicinal and aromatic plant, was chosen as the reducing and stabilizing agent for NP synthesis due to its abundance of natural polyphenols, flavonoids, glycosides, saponins, and essential oils. Using the green synthesis method, this study aims to determine the antimicrobial activity of NPs synthesized from *L. stoechas*, avoiding toxic and costly chemicals.

2. Materials and Methods

Preparation of plant extract and CuNps synthesis

CuNps were synthesized using an assisted green synthesis method with *L. stoechas* extract as a reducing and stabilizing agent, following a modified approach (Rajesh et al. 2018). Dried *L. stoechas* was washed with distilled water, and a 15 g sample was prepared in 400 ml of distilled water and incubated on a magnetic stirrer at 1000 rpm for 24 hours at room temperature. After centrifuging at 10,000 rpm and 24°C for 20 minutes, the supernatant was stored at 4°C. For CuNps synthesis, a 0.001 M copper acetate solution was added to the plant extract at a 10:1 ratio and incubated at 60–70°C for 2 hours. A color change, indicating CuNps formation, was observed. The mixture was then centrifuged, washed, and dried at 80°C for 24 hours. The dried CuNps were transferred to sterile tubes and stored in the dark at room temperature.

Characterization of CuNps

Np's size, shape, surface morphology, stability, crystallographic structure and functional groups transmission electron microscopy (TEM) (Hitachi HighTech HT 7700), scanning electron microscopy (SEM) (Zeiss Sigma 30), UV-Vis spectroscopy (UV-Vis), fourier conversion infrared spectrophotometer (FTIR) (Bruker Vertex 70v), and X-ray diffraction (XRD) (PANalytical Empyrean) has been characterized.

Antimicrobial assay

Antimicrobial activities of CuNps were tested using agar disc diffusion method and broth dilution method for *P. aeruginosa, B. subtilis, S. aureus, S. enteritidis, E. coli, E. faecalis*. Agar disc diffusion test was carried out according to Shende *et al.* (2015), the stock solution was prepared from Np as 250 µg/mL, 500 µg/mL, 750 µg/mL, 1 mg/mL and the application was made. Broth dilution test was performed according to Wiegand *et al.* (2008). Serial dilutions were made at concentrations ranging from 1000 to 1.95 µg/mL and the last tube without bacterial growth was considered as the minimum inhibitory concentration (MIC) value.

3. Results

Characterization of CuNps

TEM and SEM

TEM and SEM images of CuO NPs were given in Figure 1 and Figure 2. Both images show that the particles have different shapes and diameters. It has been determined that the shapes of CuNps's are spherical and their size are <50 nm. TEM images of CuNps have shown an organic coating layer around the Np. This layer is proof that the nanoparticles synthesized from the plant show an excellent dispersion in solution (Kahrilas et al. 2014).

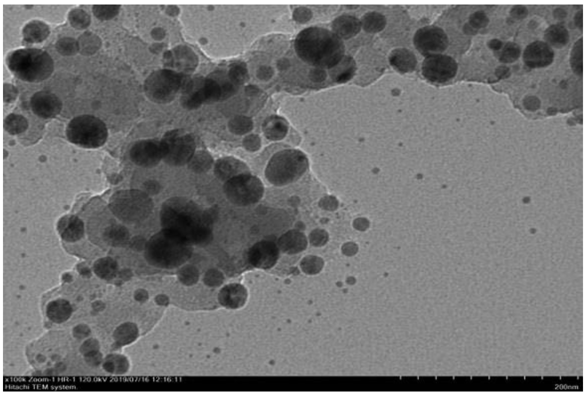

Figure 1. *TEM image of CuNps.*

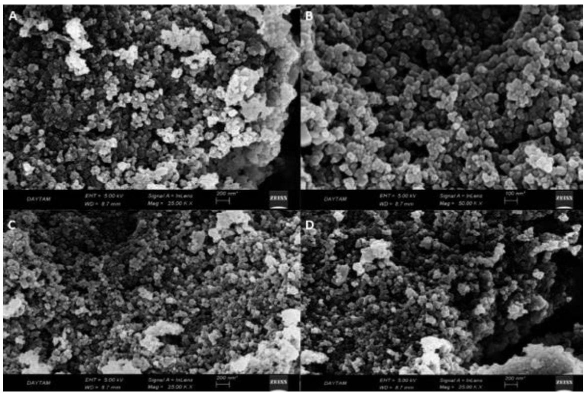

Figure 2. *SEM images of CuNps.*

UV-Vis spectroscopy

CuNps were measured at wavelength range of 200-875 nm. It shows that the maximum absorbance of the CuNps is at 310-320 nm according to the UV-Vis spectrum (Figure 3). The maximum peak value of 310-320 nm shows the reduction process and the formation of Np's. The decrease in the size of the nanoparticles leads to an increase in the UV-Vis bandwidth (Yeshchenko et al. 2012). In addition, metal nanoparticles can be agglomerated due to Van Der Waals interactions. For this reason, the absorbance values to be obtained may deviate from the expected (Hassanien et al. 2018).

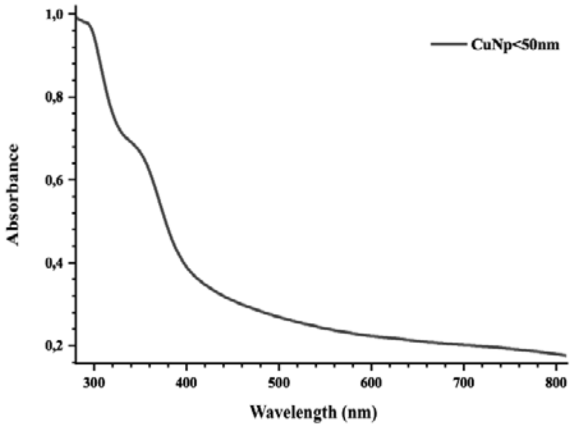

Figure 3. *UV-Vis spectrum of CuNps.*

FTIR

Nps were measured the range of 50/4000 cm^{-1} to obtain good signal to noise ratio. FTIR measurements performed to characterize the surface structure of CuNps is shown in Figure 4. FTIR spectra of CuNps have exhibited vibrations in the area of 500-600 cm^{-1}, which can be attributed to Cu vibrations that confirm the formation of CuNps's. An absorption band at 617 cm^{-1} was observed due to the vibrations of the Cu. The band at 3373 cm^{-1} corresponds to hydroxyl functional groups (Veisi et al. 2016). Also, according to the measured spectra, alkenes (C-H) at 675-781 cm^{-1}, C-O bonds at 1053 cm^{-1}, alcohol at 1219 cm^{-1}, ester, carboxylic acid, ester groups, 1412-1450 cm^{-1} aromatic ring (CH_2), aromatics at 1623-1728 cm^{-1} (C-O, C-H, C = C), alkynes at 2139 cm^{-1} (C = C), alkane stretches at 2918 cm^{-1} (C-H) and the presence of amines (NH, -OH) at 3373 cm^{-1} was confirmed by the standard IR-correlation table (Sulpizi et al. 2012; Sathish et al. 2012; Conrad et al. 2014; Save et al. 2015; Smith 2018). The emergence of these groups in the FTIR spectrum of CuNps obtained by green synthesis using *L. stoechas* confirms the presence of some metabolites such as some reducing sugars, amino acid residues, proteins, flavanones or terpenoids (Bar et al. 2009). These functional groups play a significant role in the synthesis of copper nanoparticles.

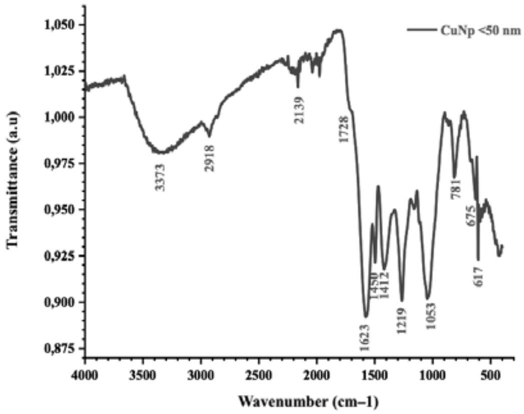

Figure 4. *FTIR spectrum of CuNps.*

XRD

The XRD pattern of the synthesized CuNps was analyzed with a step size of 0.02 in the range of 2θ between 10° and 90°. CuNps have been identified for XRD (Figure 5), which are considered as a key tool for evaluating the crystals and tertiary structures of particles at molecular levels (Sapsford et al. 2011; Li et al. 2015). Significant crystalline phases associated with CuNps with X-ray diffraction: metallic Cu, cuprite (Cu_2O) and tenorite (CuO). X-ray diffraction patterns of CuNps's were obtained in an angle of 2θ = 20–80. Diffraction peaks were observed at 36.46° (111), 42.69° (111), 50.5° (200), 61.49° (111) and 73.25° (200). The intensity of increases and decreases in the intensity of the peaks may also arise due to the presence of plant material (Berra et al. 2018).

In addition, oxidation was observed in the sample even though the samples were kept in a way that they would not get light and air. Because of the increased surface areas due to their size, the nanoparticles are much more sensitive than the bulk of the same material. So, nanoparticles that come into contact with oxygen react quickly and oxidize. It has been noted that the XRD spectrum of the oxidized CuNps (CuO Np) is 111.

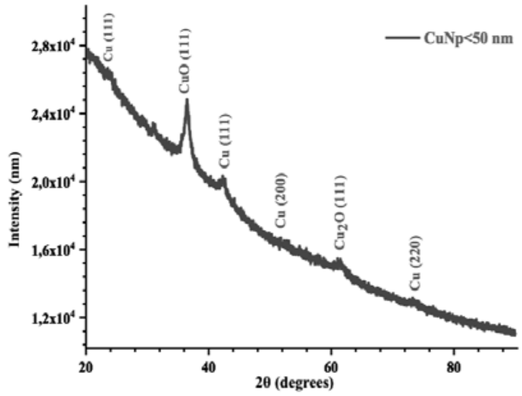

Figure 5. *XRD pattern of CuNps.*

Antimicrobial assay

In this study, the in vitro susceptibilities of CuNps synthesized from *L. stoechas* by the green synthesis method to gram-positive and gram-negative bacteria were determined by disc diffusion and broth dilution methods. While the maximum zone diameter of 19.5 mm was observed in P. aeruginosa in the disk diffusion test, CuNps at low concentrations did not form any zone diameter against *E. faecalis* and *E. coli* bacteria (Table 1). Differences in susceptibility and resistance to both gram-positive and gram-negative bacterial populations may be due to differences in cell structure, physiology, metabolism, or the degree of contact of organisms with Nps. In addition, other factors such as nanoparticle diffusion rate may affect the bacterial strain differently.

Table 1. *Antimicrobial susceptibility of CuNps to bacteria (µg / mL) by broth dilution method. "-" no bacterial reproduction, "+" bacterial reproduction.*

Bacteria	1000	500	250	125	62,5	31,25	15,62	7,81	3,90	1,95	Control
E. coli	-	-	+	+	+	+	+	+	+	+	+
P. aeruginosa	-	-	-	+	+	+	+	+	+	+	+
B. subtilis	-	-	-	+	+	+	+	+	+	+	+
S. aerous	-	-	-	+	+	+	+	+	+	+	+
S. enteridis	-	-	-	+	+	+	+	+	+	+	+
E. fecalis	-	-	+	+	+	+	+	+	+	+	+

Inhibition zone diameters (mm) formed around the discs are shown in Table 2 as a result of the agar disc diffusion test performed with the concentrations determined according to the MIC value. In addition, the graph of the inhibition zones formed by CuNps is given in Figure 6.

Table 2. *Zone diameters of CuNps in mm against bacterial strains with disc diffusion method. The zone diameter isn't formed for those indicate by "-".*

Bacteria	250 µg/mL	500 µg/mL	750 µg/mL	1000 µg/mL	$Cu(CH_3COO)_2$
B. subtilis	11.5	12.5	13	15	11
S. aureus	11	16.5	11.5	13	10
E. faecalis	-	14	-	12	9
P. *aeruginosa*	13	19.5	14	16	9
S. enteritidis	10	16.5	12	12	8
E. coli	-	13.5	-	11.5	9

The zone diameter increased as the CuNps concentration increased. The most effective stock solution in all gram negative and gram positive bacterial strains is 1000 µg / mL. Diameters of inhibition zones are seen at this concentration; 15 mm in *B. subtilis*, 16.5 mm in *S. aureus*, 14 mm in *E. faecalis*, 19.5 mm in *P. aeruginosa*, 16.5 mm in *S. enteritidis* and 13 mm in *E. coli*.

While the maximum zone diameter was observed in P. aeruginosa with 19.5 mm in agar disc diffusion test, CuNps at low concentrations applied did not create any zone diameter against *E. faecalis* and *E. coli* bacteria.

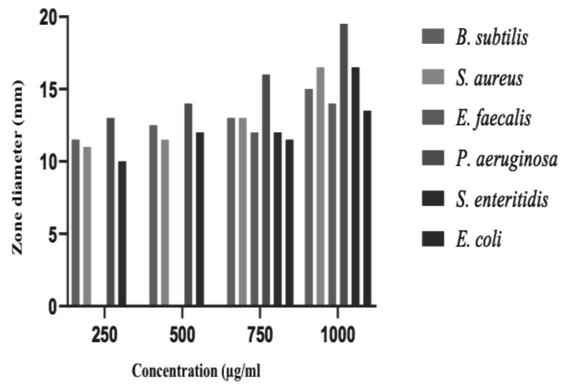

Figure 6. *Inhibition zones formed by CuNPs against various bacterial strains at different concentrations.*

4. Discussion

In recent years, traditional physical and chemical methods for synthesizing nanoparticles hazardous reducing agents and toxic organic solvents are increasingly being replaced by green synthesis techniques. This shift is due to the advantages of the green synthesis method - it is fast, clean, non-toxic, cost-effective and environmentally friendly. A preferred green synthesis approach uses plant extracts that can reduce metal ions thanks to their bioactive compounds, including flavonoids, terpenoids, tannins, alkaloids, proteins and other phytochemicals. These compounds act not only as reducing agents but also as stabilizers that limit Np growth. This green synthesis approach is easily scalable for industrial applications and offers a sustainable alternative to conventional methods due to its cost-effectiveness, low-temperature synthesis and reduced time requirements.

In this study, copper nanoparticles (CuNps) synthesized from Lavandula stoechas using green synthesis method showed potent antimicrobial activity against both gram-positive and gram-negative bacteria. The observed differences in antibacterial effects can be attributed to various factors such as bacterial cell structure, metabolic variations and the degree of contact with nanoparticles. In particular, the thick peptidoglycan layer in gram-positive bacteria may impede the penetration of nanoparticles, potentially resulting in lower efficacy (Azam et al. 2012). Furthermore, the lipopolysaccharide structure in the outer membrane of gram-negative bacteria has been shown to allow better penetration of nanoparticles, leading to more effective results (Ruparelia et al., 2008). It was reported that CuNps synthesized and characterized using the extract of *Polyalthia longifolia* roots produced inhibition zones of 17.2 ± 0.2, 15.6 ± 0.2 and 13.7 ± 0.1 mm against *S. aureus*, *E. coli* and *C. albicans*, respectively, and can be used as an antibacterial and antifungal agent (Maulana et al. 2024). In another study evaluating the antibacterial activity of CuNps synthesized by green synthesis method against *S. aureus* and *E. Coli*, it was reported that 15.7 and 12.3 inhibition zones were formed, respectively (Maulana et al. 2023).

In the disk diffusion test, the zones of inhibition increased with higher nanoparticle concentrations in all bacterial strains. This shows the concentration-dependent microbicidal effect of CuNps. The antibacterial activity of CuNps may vary depending on microbial species, suggesting that the mechanisms of interaction of nanoparticles with bacterial cell membranes differ between bacterial species. The antibacterial activity of CuNps synthesized using *Curcuma longa* extract was tested against *B. subtilis* and *E. coli* and it was noted that the inhibition zone of *B. subtilis* was higher than that of *E. coli* (Jayarambabu et al. 2020). In another study, CuNps were synthesized using Artemisia plant, the antibacterial activity of these CuNps against *E. coli* and *B. Subtilis* was tested and similar results were obtained (Al-Khafaji et al. 2022).

The antibacterial properties of CuNps make them a promising alternative to conventional antibiotics. The global increase in antibiotic resistance has intensified the need for new and effective treatments against pathogenic bacteria (Hassan et al. 2018). In this context, CuNps synthesized via green synthesis from commonly available plants such as L. stoechas offer potential as a low-cost, eco-friendly and effective antimicrobial agent. Rajesh et al. (2018) reported that CuNps were particularly effective against multidrug-resistant bacteria, indicating that such nanoparticles may be promising in overcoming antimicrobial resistance.

In conclusion, the findings from this study suggest that *L. stoechas*-based CuNps could serve as a novel antimicrobial agent to address the antibiotic resistance crisis. Future studies should further investigate the efficacy of these nanoparticles against other pathogenic species and multidrug-resistant bacteria. Furthermore, studies on the biocompatibility and toxicological properties of these nanoparticles are crucial to ensure their safe and effective use in clinical applications.

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Development of New Al-Ni-Cr-W Alloys for Enhanced Neutron Radiation Protection

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Abstract

Neutron radiation is utilized in many applications such as nuclear therapy, nuclear power plants, material analysis, space research, and more. Neutron leaks can occur in these applications, posing hazards to staff, operators, and therapy patients. Therefore, effective neutron shielding materials are always needed. In this study, two new types of neutron shielding alloy materials were developed, consisting of aluminum, nickel, chromium, tungsten, boron carbide, manganese, molybdenum, and silicium. The chemical composition and weight ratios of the composites were determined using the Monte Carlo Simulation's GEANT4 code. Mixing and molding methods were employed in the production of the alloys. Important neutron shielding parameters, such as the effective removal cross-section, half-value layer, mean free path, and radiation protection efficiency, were theoretically determined using the GEANT4 code. Additionally, fast neutron absorption capacities were measured using an Am-Be fast neutron source and a BF_3 portable neutron detector. The results were compared with 316 LN stainless steel. All new alloy samples were determined to have better fast neutron shielding capabilities than these reference samples. It was also observed that the new alloy samples exhibit both high-temperature resistance and mechanical durability. It is suggested that these new alloy samples can be used in neutron radiation shielding applications such as nuclear reactors, radioactive waste storage, and nuclear shelters.
Keywords: Neutron, alloy, geant4.

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1. Introduction

Neutron radiation is a type of non-directly ionizing radiation released as a result of nuclear fission or fusion, and it can cause reactions in other atoms to produce new nuclides (Yue et al. 2013). Neutron radiation is commonly used in industry, diffraction and scattering experiments, material development and research applications, cosmology, oil and mineral research, and materials characterization studies. Neutron radiation does not ionize matter in the same way as electrons or protons, but when it interacts with matter, it can cause the release of ionizing radiation such as gamma rays. Neutrons do not have an electrical charge, so they can be more penetrating than gamma rays, alpha, or beta radiation, which makes shielding against them more difficult. Neutron radiation is used in Boron Neutron Capture Therapy to exterminate cancerous tissue. However, if adequate precautions are not taken, it can be hazardous to both personnel and patients. Neutrons may cause more DNA damage than other types of radiation due to their stronger interaction properties with tissues (Nouraddini-Shahabadi et al. 2024). Effective new shielding materials are always needed to protect against neutron hazards. In this regard, many new shielding materials have been developed, such as metal oxide-added glasses (Kaewkhao et al.2018), (Ekinci et al. 2024), (Alajerami et al. 2024), high-density strength alloys (Aygün et al. 2022), (Misned, et al. 2024),

stainless steels (Qi et al. 2022), (Oh et al. 2024), various chemical or organic molecules (Alaylar et al. 2021), (Aygün et al. 2020), (Aygün et al. 2024), and heavy concretes and bricks with added metals or oxides (Aygün, 2020), (Aygün and Karabulut 2018), (Makkiabadi, et al. 2024). Alloys are made by unifying two or more components with different characteristic properties. As a result, alloy materials can exhibit new properties such as high strength, temperature resistance, and corrosion resistance. Today, alloy and other composite materials are commonly used in various fields, including automotive, aerospace, aviation, military, healthcare, and construction. In recent years, alloys have also been used in nuclear technology due to their high durability. While alloy and composite materials often have high density, this property must be enhanced to withstand radiation effects (Sabhadiya, 2021).

Many alloy and composite samples were developed, and it is important to carefully consider the materials used, particularly regarding their neutron absorption capacity. Titanium (Ti) and Aluminum (Al) alloys are used in space research because they are lightweight, but they are not very durable against neutron radiation. To enhance their resistance to radiation, the radiation shielding capabilities of these alloys were improved using a vacuum plasma-sprayed method with hexagonal Boron Nitride (hBN) and titanium, incorporating 2-10 vol% of hBN. As a result, it was determined that the radiation absorption capacity increased by approximately 27% (Sukumaran et al. 2024).Fast and epithermal neutron shielding materials were produced for use with radioactive waste from nuclear reactors. The fast and epithermal neutron shielding performance was determined using the MCNP5 simulation program on an Al-Cd metal homogeneous mixture alloy. The results obtained showed that the neutron shielding performance increased by 10% in Cd-doped Al alloy samples (Heriyanto et al.2023). Shielding parameters were determined for neutron, electromagnetic, X-ray/gamma, and Bremsstrahlung radiations of Al–Li, Italma, Duralumin, Hiduminium, Magnalium, Hydronalium, Ni–Ti–Al, and Y-alloy. It was found that the Al–Ni–Ti alloy has good shielding performance compared to the other alloys against neutron, X-ray/γ radiation, Bremsstrahlung, and electromagnetic radiations (Sathish et al. 2023).

Polypropylene (PP)-based composites were developed, consisting of boron minerals such as ulexite, tincal, and colemanite. Fast neutron shielding parameters were calculated using Geant4 code, in addition to absorption experiments that were carried out. It was reported that samples with a higher content of colemanite have better shielding ability compared to other samples (Bilici et al. 2021). Epoxy resin-based composite samples were produced with the addition of lithium (LiF), chromium oxide (Cr_2O_3), and nickel oxide (NiO). To enhance temperature resistance, all samples were coated with sodium silicate paste. Neutron shielding parameters were determined both theoretically and experimentally. It was reported that these samples can be used for nuclear applications in fast neutron shielding studies (Aygün et al. 2020). Metal matrix composites have good mechanical and chemical strength, as well as high-temperature resistance; therefore, these materials can be used in nuclear technology. An Al-B_4C metal matrix composite sample was developed to determine its neutron shielding ability. It was reported that increasing the B_4C ratio in the composite leads to an increase in radiation absorption capacity (Gaylan et al. 2023).The radiation shielding capability of high entropy alloys (HEAs) such as CoNiFeCr, CoNiFeCrTi, and CoNiFeCrAl was calculated for gamma, electron, neutron, proton, and alpha radiations using the Phy-X/PSD, ESTAR (10 keV-20 MeV), and SRIM (10 keV-20 MeV) programs. It was determined that the CoNiFeCr alloy has better shielding performance than the other alloys (Sakar et al. 2023).

In this study, two new types of aluminum-based alloy samples were designed and fabricated. To evaluate their potential for nuclear applications, the neutron shielding capacities were determined through both theoretical and experimental measurements.

2. *Neutron attenuation principles*

Neutrons can interact differently with target materials through processes such as elastic or inelastic scattering, absorption, capture, or complete stopping. The probabilities of these interactions can be described by the macroscopic cross section, effective removal cross section, half-value layer, mean free path, and radiation protection efficiency.

The macroscopic cross section can be calculated as follow

$$\Sigma = \frac{\rho}{A} N_A \quad (1)$$

the units of the quantity are in cm^{-1}.

$$N_A = \frac{\rho}{A} N_0 \quad (2)$$

N_A is the number of atoms of the absorption sample per ($atom/cm^3$), N_0 is the Avogadro's number ($6.02.10^{23}$), ρ is the density of the absorption sample, A is the molecular weight of attenuation sample.

The removal cross-section, ΣR, functions similarly to a macroscopic cross-section and can be utilized to assess neutron attenuation properties; however, it does not represent the probability of neutron-nucleus interactions. This parameter reflects interactions such as fast neutron energy loss, scattering, and capture, and its value is less than that of the macroscopic cross-section (El-Khayatt. 2010).This parameter is crucial for neutron protection applications. Additionally, it can be applied to composites, alloys, and mixtures, and it can be calculated using the following method.

$$\Sigma_R = \Sigma(\Sigma_{R/\rho})_i \quad (3)$$

$$\rho_i = w_i \rho \quad (4)$$

w_i is the weight percentage and ρ_i is the density of attenuation sample i.

When neutrons traverse a barrier material, they may lose half of their number or energy. The thickness of the material at this point is referred to as the Half Value Layer (HVL), which can be calculated using the following formula.

$$HVL = ln2/\Sigma R \quad (5)$$

The mean free path represents the average distance a neutron travels before colliding with atoms in the protective material, and it can be calculated as follows.

$$\lambda = {}^{1}/_{\Sigma_R} \quad (6)$$

Neutrons exhibit particle-like characteristics as a type of radiation, making the count of incoming or passing neutrons through a protective material essential in neutron shielding studies. To assess the shielding effectiveness of a sample, the neutron transmission factor needs to be calculated, which can be done as follows:

$$NTF = \frac{I}{I_0} \quad (7)$$

I denote the number of neutrons passing through the barrier sample, while I_0 signifies the number of neutrons incident on it. The neutron Radiation Protection Efficiency (RPE) offers valuable insights into the shielding capability of the material, and it can be calculated as follows (Sayyed et al., 2019).

$$RPE = 1 - \frac{N}{N_0} \; 100\% \quad (8)$$

Where N represents the dose that passes through the barrier material, and N_0 denotes the dose incident on the barrier material.

3. *Materials and Methods*

3.1. Monte Carlo simulation code GEANT4

GEometry ANd Tracking (GEANT4) is a Monte Carlo simulation framework utilized to assess the likelihood of radiation traversing various materials and the resultant interactions. This framework allows for the design of the geometries of radiation sources and materials, as well as the detection of secondary radiation and newly generated particles following the interaction of radiation with those materials. GEANT4 finds applications in high-energy and nuclear physics, medical physics, space exploration, military fields, agriculture, mining, and various other research domains, covering energy levels from eV to TeV. This toolkit facilitates the design of novel material shapes and the development of experimental models to study the impacts of radiation on both living and non-living entities. In this research, the toolkit was employed to create new alloys, choose chemical compositions for these alloys, and evaluate radiation shielding characteristics (Wellisch, 2005). Geant4 simülation geometry is given in Figure. 1.

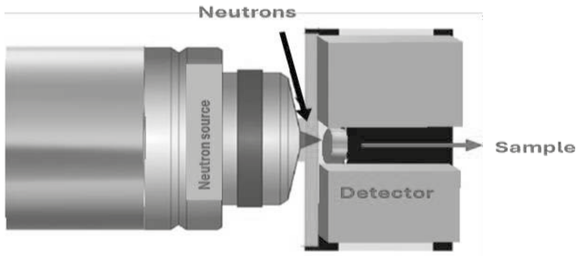

Figure. 1. Simulation geometry Geant4 3D visual

3.2. Sample preparation and experimental

Powdered forms of aluminum (Al), nickel (Ni), chromium (Cr), tungsten (W), boron carbide (B_4C), manganese (Mn), molybdenum (Mo), and silicium (Si) were blended uniformly for 20 minutes in a mixer, based on the composite ratios indicated by the Geant4 simulation results. Subsequently, this uniform mixture was subjected to cold pressing at 10 tons and 300 MPa to create pellets weighing 5g, with a thickness of 3 mm and a diameter of 1 cm, using the powder metallurgy technique. Each sample underwent tempering for one hour, where the temperature was gradually increased to 600 °C before cooling back to room temperature. The chemical compositions of these alloys are detailed in Table 1, and the produced sample images are presented in Figure. 2.

Table 1. Chemical composition ratios and density of alloy (AL) (%)

Material	**AL1** (ρ=7.63g/cm^3)	**AL2** (ρ=7.63g/cm^3)
Al	25	25
Ni	25	20
Cr	20	20
W	15	15
B_4C	15	-
Si	-	10
Mg	-	5
Mo	-	5

Al: Alloy

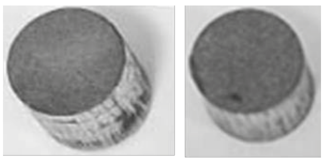

Figure. 2. Produced new alloy samples

In the dose measurement experiments, an Am-Be point fast neutron source and a BF_3 neutron detector were utilized. As shown in Figure 3, the experimental geometry was used for absorption measurements. First, the background dose (D0), which represents the dose emitted by the source, was determined. Then, each sample was placed between the source and the detector to be exposed to neutron radiation, allowing the absorbed dose measured by the detector (DD) to be recorded. Finally, the absorbed dose from the sample (DS) was calculated using the equation DS = D0 - DD.

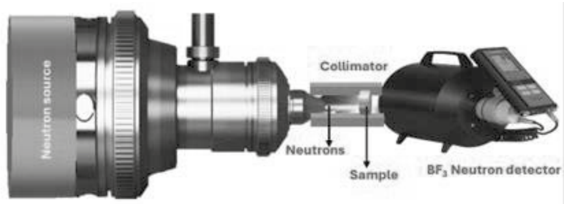

Figure 3. Neutron equivalent dose rate measurement system

4. *Results and discussion*

Aluminum alloys are both lightweight and exhibit good corrosion resistance and mechanical durability, making them preferable for nuclear applications (Sun et al. 2023). They provide effective shielding against low-energy radiation, such as α-particles and β-rays, but have limited shielding effectiveness against neutrons and gamma rays. To enhance their effectiveness for neutron shielding applications, aluminum alloys need to be reinforced with other metals. In this study, aluminum-based alloys containing various metals were designed and produced.

In Al-B_4C composites, the B_4C compound can be added to the alloy in amounts ranging from 5% to 50%. As the B_4C content increases in the alloy, the microhardness of the composite decreases, resulting in a reduction in its strength (Onaizi et al. 2024). To eliminate this disadvantage, the B_4C content has been kept constant at 15%. While studies generally focus on the shielding properties of Al alloys against thermal neutrons, this study investigates the shielding properties against fast neutrons (Jia et al. 2021).

4.1. Neutron absorption parameters

It is, of course, likely that there are differences between experimental measurements and simulation results. This is because in simulation studies, the sample is taken as completely homogeneous, and the experimental geometry is processed with exact dimensions. However, in experimental studies, it is not possible for the materials to be 100% homogeneous, and despite careful attention, some deviations in the geometry may still occur. In all studies, a margin of error of up to 10% between experimental measurements and simulation results is considered normal. However, if the shielding capacity of a material is well-predicted in simulation studies, good results are generally obtained in experimental measurements as well. In material design, the reactions of the materials to radiation can be determined in advance through simulation studies, allowing for the production of materials with the desired properties by using this prior knowledge in the manufacturing process (Kurt et al. 2020). In this study, keeping these considerations in mind, productions were made based on simulation results and experimentally verified.

The important neutron shielding parameters, such as effective removal cross-section, mean free path, half-value layer, and radiation protection efficiency, were theoretically calculated using the Geant4 code, and all results are presented in Table 2.

Table 2 Comparison shielding parameters in 5mm thick samples for 10^5 incident fast neutron (4.5 MeV)

Sample Code	Half value layer (cm)	Mean free path λ (cm)	Neutron transmission factor	Fast Neutron ERCS (cm^{-1})
316 LN	4.325	6.242	0.85194	0.1602
AL1	3.924	5.662	0.83828	0.1766
AL2	4.366	6.301	0.85321	0.1587

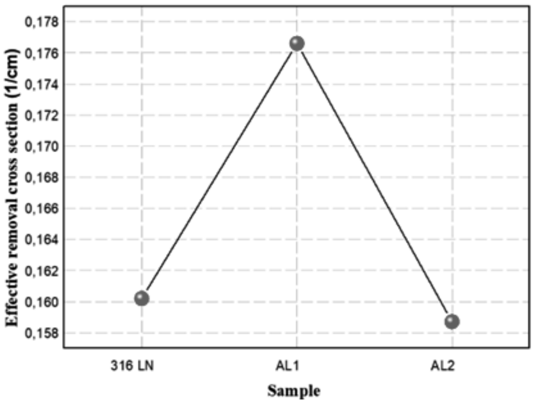

Figure. 3. Effective removal Cross Section (cm^{-1}) comparison of samples

When examining Table 2 and Figure 3, the AL1 sample has an effective removal cross-section value of 0.1766, while the AL2 sample has a value of 0.1587. In comparison, the 316 nuclear stainless steel has a value of 0.1602. According to these results, AL1 has a higher effective removal cross-section value than 316LN. A sample with a larger effective removal cross-section value indicates a higher shielding capacity (Manjunatha et al. 2019). Therefore, the AL1 sample has better shielding ability than the other samples. The AL2 sample also has good shielding capacity, but it is slightly lower than that of 316LN, although the difference between the two is small. If a sample has both a low mean free path value and a low transmission factor, it indicates a good shielding capacity for neutrons. According to Table 2, the AL1 sample has both a lower mean free path value and transmission factor compared to the reference sample 316LN. Therefore, the AL1 sample has better shielding performance than 316LN. The AL2 sample also has good shielding capacity, but it is lower than that of 316LN. Similarly, a low half-value layer (HVL) is a desirable property for shielding samples. It can be seen that the AL1 sample has a lower HVL value than 316LN, which indicates that AL1 has better shielding ability compared to 316LN. On the other hand, the AL2 sample has a higher HVL than 316LN, suggesting that AL2 has a lower shielding ability than 316LN. Based on all the theoretical results, the shielding abilities of the samples can be sorted as follows: AL1> 316LN>AL2.

4.2. Neutron absorption dose results

Table 3 gives experimental dose measurement results. As shown in Table 3, the incoming dose amount is 1.2987 µSv/h from the source. The AL1 sample has absorbed an amount of 0.5590 µSv/h, which is a ratio of 43.04%. Similarly, the AL2 sample has absorbed an amount of 0.5316 µSv/h at a ratio of 40.93%. The 316 LN sample has absorbed an amount of 0.5402 µSv/h at a ratio of 41.59%. According to these results, the AL1 sample has absorbed a greater dose than both the 316 LN stainless steel and the AL2 sample.

Al-3003 alloy was researched for its fast neutron shielding capacity, and it was determined that it has a shielding capacity of 3% (Samrah et al. 2024). However, the new alloy, Al1, has a shielding capacity of 44%, which indicates that this sample has excellent shielding ability.

However, the AL2 sample absorbed a lower dose than the 316 LN stainless steel, although the difference between the AL2 sample and the 316 LN sample is very small. These experimental measurement results indicate that all samples AL1>316 >AL2 have a relationship in terms of absorbed dose.

Table 3 Absorbed dose results of all samples

Sample	Absorbed Dose by Samples (µSv/h)	Radiation protection efficiency (%)
Background	1.2987	-
316 LN	0.5402	41.59
AL1	0.5590	43.04
AL2	0.5316	40.93

5. *Conclusions*

New types of alloys based on aluminum (Al) and nickel (Ni), with the addition of chromium (Cr), tungsten (W), boron carbide (B4C), manganese (Mn), molybdenum (Mo), and silicon (Si), were produced using the powder metallurgy method. The fast neutron shielding ability of these samples was determined both experimentally and theoretically, and the results were compared with those of 316LN stainless steel. According to the results, it is determined that both AL1 and AL2 samples exhibit good radiation shielding capacity.

It has been demonstrated that these new types of Al alloys can be effectively used against high-energy neutron leakage that may occur during the transportation and storage of radioactive waste and normal radioactive materials. Additionally, they can be used as protective shielding materials against fast neutrons in nuclear power plants, in boron neutron capture therapy applications in hospitals, and in military and space vehicles. Based on all these findings, it has been concluded that the newly designed and produced Al alloy samples are highly effective in radiation shielding and can be safely utilized in radiation protection applications.

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On the Fixed Point Property for Nonexpansive Mappings on Large Classes in Alpha-duals of Certain Difference Sequence Spaces

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Abstract

In 2000, Et and Esi introduced new type of generalized difference sequences by using the structure of Çolak's work from 1989 where he defined new types of sequence spaces while Çolak was also inspired by Kızmaz's idea about the difference operator he studied in 1981. Then, using Et and Esi's structure, Ansari and Chaudhry, in 2012, introduced a new type of generalized difference sequence spaces. Changing Ansari and Chaudhry's construction slightly, Et and Işık, in 2012, obtained a new type of generalized difference sequence spaces which have equivalent norm to that of Ansari and Chaudhry's type Banach spaces. Then, Et and Işık found α-duals of the Banach spaces they got and investigated geometric properties for them. In this study, we consider Et and Işık's work and study α-duals of their generalized difference sequence spaces. We take their study in terms of fixed point theory and find large classes of closed, bounded and convex subsets in those duals with fixed point property for nonexpansive mappings.

Keywords: Nonexpansive Mapping, Fixed Point Property, Closed Bounded Convex Set, Difference Sequences, α-duals.

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1. Introduction and Preliminaries

Fixed point theory is a central area of study in functional analysis with wide-ranging implications for optimization, nonlinear analysis, and the theory of Banach spaces. A key concept in this field is the fixed point property (FPP), which states that every nonexpansive mapping on a closed, bounded, and convex (cbc) subset of a space has a fixed point.

This property is known to hold in certain Banach spaces, such as Hilbert spaces, but fails in many classical non-reflexive Banach spaces like c_0 and ℓ^1. As a result, identifying large classes of cbc subsets within such spaces that retain the FPP has become an important line of inquiry.

The work of Goebel and Kuczumow (1979) serves as a seminal contribution in this area. They demonstrated that while ℓ^1 lacks the fixed point property in general, it is possible to identify specific large classes of cbc subsets where nonexpansive mappings do have fixed points. This discovery inspired subsequent research aimed at generalizing these results to broader classes of Banach spaces and larger families of subsets. Researchers such as Kaczor and Prus (2004) further extended these ideas by investigating affine asymptotically nonexpansive mappings on ℓ^1. However, these works often required additional assumptions, such as the affinity condition, which limited the generality of their results.

In this study, we introduce a new perspective by examining α-duals of certain generalized difference sequence spaces, which generalize the space of absolutely summable scalar sequences. Our approach differs from that of Kaczor and Prus, as we do not rely on the affinity hypothesis and instead work directly with nonexpansive mappings. Moreover, while Goebel and Kuczumow focused on ℓ^1, our work considers a more general family of sequence spaces that are isometrically isomorphic to the absolutely summable scalar sequence space but also contain a richer geometric structure. This generalization enables us to identify larger classes of cbc subsets with the FPP. Importantly, our approach extends beyond specific instances, as we are also developing our work for a general case of the space we study by taking $m \in \mathbb{N}$ arbitrarily respected to the general space.

The primary objective of this paper is to identify large classes of closed, bounded, and convex subsets in α-duals of generalized difference sequence spaces that satisfy the fixed point property for nonexpansive mappings. To achieve this, we build on concepts from Goebel and Kuczumow's analogy while introducing new methods to avoid reliance on affinity assumptions. Our findings contribute to the broader effort of understanding the fixed point theory of Banach spaces and provide a new avenue for exploring the geometric properties of generalized difference sequence spaces.

The paper is organized as follows. In Section 1, we introduce essential definitions and preliminary concepts. Section 2 presents our main results, including theorems and proofs regarding large classes of cbc subsets in α-duals with the FPP. We conclude with a discussion of the implications of our findings, highlighting potential directions for future research.

In terms of looking more deeply into the literature, we can say that researches have shown that the fixed point exists for some function classes defined on certain classes of sets in some spaces, while it cannot be found at all in others. Fixed point theory has examined how this happens or does not happen.

Then, researchers have made classifications and characterizations in this matter. In (Browder 1965a), it was proved that every Hilbert space has a property satisfying that every nonexpansive mapping defined on any closed, bounded, and convex (cbc) nonempty subset domain with the same range has a fixed point. Since that time, spaces with this property have been considered to have the fixed point property for nonexpansive mappings (fppne). Then, researchers considered looking for the spaces with the property and if the property still exists when larger classes of mappings are taken. Then also they have seen spaces failing the properties. For example, in (Browder 1965b) and (Göhde 1965) with independent studies, they saw that uniform convex Banach spaces have the fppne. Then, Kirk (1965) generalized the result for the reflexive Banach spaces with normal structure. In fact, Goebel and Kirk (1973) noticed that Kirk's result was able to extend for uniformly Lipschitz mappings and some researchers have studied estimating the Lipschitz coefficient satisfying the property for uniform Lipschitz mappings on different Banach spaces. For example, Goebel and Kirk (1990) showed that for Hilbert spaces, the best Lipschitz coefficient would be a scalar less than a number in the interval $\left[\sqrt{2}\ ,\ \frac{\pi}{2}\right]$, and Goebel and Kirk (1973) and Lim (1983) showed independently that for a Lebesgue space L^p when $2 < p < \infty$, the coefficient is smaller by a scalar larger than or equal to $\left(1+\frac{1}{2^p}\right)^{\frac{1}{p}}$ while Alspach (1981) showed that when $p = 2$, there exists a fixed point free Lipschitz mapping with Lispchitz coefficient $\sqrt{2}$ defined on a cbc subset. In fact, $\sqrt{2}$ is the smallest Lipschitz coefficient for Alspach's mapping. We need to note that, similar to the definition of the Banach spaces satisfying the fppne, if a Banach space has a property that every uniformly Lipschitz mapping defined on any cbc nonempty subset domain with the same range has a fixed point, then that Banach space has the fixed point property for uniformly Lipschitz mapping (fppul). In terms of fixed point property for uniformly Lipschitz mappings, Dowling, Lennard, and Turett (2000) showed that if a Banach space contains an isomorphic copy of ℓ^1, then it fails the fppul. It is a well-known fact by researchers that c_0 or ℓ^1 is almost isometrically embedded in every non-reflexive Banach space with an unconditional basis (Lindenstraus and Tzafriri 1977). For this reason, classical non-reflexive Banach spaces fail the fixed point property for non-expansive mappings, that is, in these spaces, there can be a closed, convex and bounded subset and a non-expansive invariant T mapping defined on that set such that T has no fixed point. This result is based on well-known theorems in literature (see for example Theorem 1.c.12 in (Lindenstraus and Tzafriri 1977) and Theorem 1.c.5 in (Lindenstraus and Tzafriri 2013)). These theorems state that for a Banach lattice or Banach space with an unconditional basis to be reflexive, it is necessary and sufficient that it does not contain any isomorphic copies of c_0 or ℓ^1. Therefore, this close relation to the reflexivity or nonreflexivity of Banach space, researchers have worked for years and questioned whether c_0 or ℓ^1 can be renormed to have a fixed point for nonexpansive mappings. Lin (2008) showed in his study that what was thought was not true and that at least ℓ^1 could be renormed to have the fixed point property for nonexpansive mappings. Then, the remaining question was if the same could have been done for c_0, but the answer still remains open. Since the researchers have considered trying to obtain the analogous results for well-known other classical nonreflexive Banach spaces, another experiment was done for Lebesgue integrable functions space $L_1[0,1]$ by Hernandes-Lineares and Maria (2012) but they were able to obtain the positive answer when they restricted the nonexpansive mappings by assuming they were affine as well. One can say that there is no doubt most research has been inspired by the ideas of the study (1979) where Goebel and Kuczumow proved that while ℓ^1 fails the fixed point property since one can easily find a cbc nonweakly compact subset there and a fixed point free invariant nonexpansive map, it is possible to find a very large class subsets in target such that invariant nonexpansive mappings defined on the members of the class have fixed points. In fact, it is easy to notice the traces of those ideas in Lin's (2008) work. Even Goebel and Kuczumow's work has inspired many other researchers to investigate if there exist more example of nonreflexive Banach spaces with large classes satisfying fixed point property. For example, in (Kaczor and Prus 2004), they wanted to generalize Goebel and Kuczumow's findings and they proved that affine asymptotically nonexpansive invariant mappings defined on a large class of cbc subsets in ℓ^1 can have fixed points. Moreover, in (Everest 2013), Kaczor and Prus' results were extended by having been found larger classes satisfying the fixed point property for affine asymptotically nonexpansive mappings. Thus, affinity condition became a tool for their works. In fact, another well-known nonreflexive Banach space, Lebesgue space $L_1[0,1]$, was studied in (Hernández-Linares Japón 2012) and in their study they obtained an analogous result to (Lin 2008) as they showed that $L_1[0,1]$ can be renormed to have the fixed point property for affine nonexpansive mappings. In this study, we will investigate some Banach spaces analogous to ℓ^1. In the present work, we study Goebel-Kuczumow analogy for α-duals of their generalized difference sequence spaces investigated by Et and Işık (2012). We prove that a very large class of closed, bounded and convex subsets in α-duals of their generalized difference sequence spaces investigated by Et and Işık has the fixed point property for nonexpansive mappings. Therefore, firstly we would like to give the definition of Cesàro sequence spaces which was defined by Shiue (1970), and next we present Kızmaz's difference sequence space definition in (Kızmaz 1981) by noting that we work on a space which is derived from his ideas' generalizations such that many researchers (see for example (Çolak 1989, Et 1996, Et and Çolak 1995, Et and Esi 2000, Orhan 1983, Tripathy et al 2005) have generalized his work as well.

In fact, we need to note that Et and Esi's (2000) work and Et and Çolak's (1995) work used a common difference sequence definition from Çolak's (1989) work.

Shiue (1970) defined the Cesàro sequence spaces by

$$\mathrm{ces}_p = \left\{ (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

such that $\ell^p \subset \mathrm{ces}_p$ and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

such that $\ell^\infty \subset ces_\infty$ where $1 \le p < \infty$. Then, from the definition of Cesàro sequence spaces, Kızmaz (1981) defined difference sequence spaces for ℓ^∞, c, and c_0 and symbolized them by $\ell^\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$, respectively. In his introduction, he defined the difference operator Δ applied to the sequence $x = (x_n)_n$ using the formula $\Delta x = (x_k - x_{k+1})_k$. In fact, he investigated Köthe-Toeplitz duals and their topological properties.

As one of the researchers generalizing his ideas, Çolak (1989) introduced firstly a generalized difference sequence space by taking an arbitrary sequence of nonzero complex values $v = (v_n)_n$ and then denoting a new difference operator by Δ_v such that for any sequence $x = (x_n)_n$, he defined the difference sequence of that $\Delta_v x = (v_k x_k - v_{k+1} x_{k+1})_k$. Then, Et and Esi (2000) generalized Çolak's difference sequence space by defining

$$\Delta_v(\ell^\infty) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v x \in \ell^\infty\},$$
$$\Delta_v(\mathrm{c}) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v x \in \mathrm{c}\},$$
$$\Delta_v(\mathrm{c}_0) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v x \in \mathrm{c}_0\}.$$

Furthermore, their m^{th} order generalized difference sequence space is given for any $m \in \mathbb{N}$ by $\Delta_v^0 x = (v_k x_k)_k$, $\Delta_v^m x = (\Delta_v^m x_k)_k = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})_k$ with $\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ for each $k \in \mathbb{N}$.

Next Bektaş, Et and Çolak (2004) obtained the Köthe-Toeplitz duals for the generalized difference sequence space of Et and Esi's. We may recall here that their

m^{th} order difference sequence space has the following norm for any $m \in \mathbb{N}$:

$$\|x\|_v^{(m)} = \sum_{k=1}^{m} |v_k x_k| + \|\Delta_v^m x\|_\infty$$

Then, the corresponding Köthe-Toeplitz dual was obtained as in (Bektaş, Et and Çolak 2004) and (Et and Esi 2000) such that it is written as below:

$$D_1^m = \{a = (a_n)_n \subset \mathbb{R} | (n^m v_n{}^{-1} a_n)_n \in \ell^1\}$$
$$= \left\{a = (a_n)_n \subset \mathbb{R}: \ \|a\|^{(m)} = \sum_{k=1}^{\infty} \frac{k^m |a_k|}{|v_k|} < \infty\right\}.$$

Note that $D_1^m \subset \ell^1$ if $k^m |v_k{}^{-1}| > 1$ for each $k, m \in \mathbb{N}$ and $\ell^1 \subset D_1^m$ if $k^m |v_k{}^{-1}| < 1$ for each $k, m \in \mathbb{N}$.

Ansari and Chaudhry (2012) introduced a new type of generalized difference sequence spaces by picking an arbitrary sequence of nonzero complex values $v = (v_n)_n$ as Çolak (1989) did and next by symbolizing the new difference sequence space as $\Delta_{v,r}^m(E)$ for arbitrary $r \in \mathbb{R}, m \in \mathbb{N}$ and writing that space as below where X is any of the sequence spaces ℓ^∞, c or c_0.

$$\Delta_{v,r}^m(X) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v^m x \in X\}$$

where Ansari and Chaudhry (2012) defined the norm by

$$\|x\|_{\Delta,v}^m = \sum_{k=1}^{m} |v_k x_k| + \sup_{k \in \mathbb{N}} |k^r \Delta_v^m x_k|$$

Then, by obtaining an equivalent norm to Ansari and Chaudhry's Banach space, Et and Işık (2012) defined m^{th} order generalized type difference sequence for any $m \in \mathbb{N}$ given by

$$\Delta_{v,r}^{(m)}(X) = \{x = (x_n)_n \subset \mathbb{R} | \Delta_v^m x \in X\}$$

where the norm is as follows:

$$\|x\|_{\Delta,v}^{(m)} = \sup_{k \in \mathbb{N}} |k^r \Delta_v^m x_k|$$

Then, Et and Işık found α-duals of the Banach spaces they got and investigated geometric properties for them such that m^{th} order α-duals for their Banach spaces are written as

$$U_1^m = \{a = (a_n)_n \subset \mathbb{R} | (n^{m-r} v_n{}^{-1} a_n)_n \in \ell^1\}$$
$$= \left\{a = (a_n)_n \subset \mathbb{R}: \ \|a\|_{\sim}^{(m)} = \sum_{k=1}^{\infty} \frac{k^{m-r} |a_k|}{|v_k|} < \infty\right\}$$

Note that $U_1^m \subset \ell^1$ if $k^{m-r} |v_k{}^{-1}| > 1$ for each $k, m \in \mathbb{N}$ and $\ell^1 \subset U_1^m$ if $k^{m-r} |v_k{}^{-1}| < 1$ for each $k, m \in \mathbb{N}$.

Before starting to introduce our work and results, we can also note that recent studies have explored the fixed point property (FPP) in Banach spaces, focusing on large classes of subsets and various mappings. One notable contribution is by Tim Dalby, who in 2024 proved that uniformly nonsquare Banach spaces possess the FPP. This result provides a deeper understanding of the geometric conditions that ensure fixed points for nonexpansive mappings. Dalby's work highlights the importance of uniform nonsquareness as a sufficient condition for the FPP, offering new perspectives on the structure of Banach spaces (Dalby, 2024).

Another important development in this area is the work of Vasile Berinde and Mădălina Păcurar, published in 2021. They introduced the concept of saturated classes of contractive mappings and examined the applicability of enriched contractions. Their study provided new fixed point results for these enriched classes of mappings, broadening the scope of fixed point theorems. The authors demonstrated that these enriched contractions have unique fixed points, which can be approximated using Krasnoselskij iterative schemes. This contribution enriches the fixed point theory and extends its applicability to a wider range of contractive mappings in Banach spaces (Berinde & Păcurar, 2021).

Izhar Oppenheim's 2022 work is another significant contribution to the field. He established that higher-rank simple Lie groups, such as $SL_n(\mathbb{R})$ for n ≥ 3, and their lattices have Banach property (T) with respect to all super-reflexive Banach spaces. This result implies that these groups have the FPP for actions on super-reflexive Banach spaces. Oppenheim's findings underscore the interplay between Banach property (T) and the fixed point property, offering new insights into the algebraic and topological properties that guarantee the existence of fixed points (Oppenheim, 2022).

Research on large classes of Banach spaces with the fixed point property has also included investigations into the Prus–Szczepanik condition. Prus and Szczepanik introduced this condition to identify Banach spaces that satisfy the FPP for nonexpansive mappings. Subsequent studies have explored the relationship between the PSz condition and other geometric properties of Banach spaces, providing sufficient criteria for a space to satisfy the FPP. This line of research aims to broaden the classification of Banach spaces that support the FPP, thereby offering new tools for functional analysts (Prus & Szczepanik, 2019).

Further advancements were made by Berinde and Păcurar, who introduced the concept of enriched contractions. These contractions generalize Picard–Banach contractions and certain nonexpansive mappings. Their work demonstrated that enriched contractions possess unique fixed points, which can be approximated using Krasnoselskij iterative schemes. This approach provides a broader framework for establishing fixed points for a wide range of mappings (Berinde & Păcurar, 2021).

Additional contributions to the study of the FPP in Banach spaces include the work of Fetter Nathansky and Llorens-Fuster, who investigated the ℓ^1 sum of the van Dulst space with itself. They demonstrated that this product space retains the FPP despite lacking several known conditions that typically imply this property. This finding illustrates how new combinations of Banach spaces can yield novel insights into fixed point theory, motivating further exploration of product spaces and their fixed point properties (Nathansky & Llorens-Fuster, 2020).

Finally, Oppenheim's exploration of Banach property (T) and fixed point properties has established connections between algebraic structures and the FPP. His findings that higher-rank simple Lie groups possess Banach property (T) with respect to super-reflexive Banach spaces reveal a deeper relationship between algebra, topology, and fixed point theory. These contributions collectively highlight the ongoing effort to understand the conditions under which fixed points exist in Banach spaces and to identify large classes of sets and mappings that satisfy the FPP (Oppenheim, 2022).

Now, we would like to give some well-known and important facts that are fundamentals for our work. One may see (Goebel and Kirk 1990) as a reference.

Definition 1.1 *Consider that $(X, \|\cdot\|)$ is a Banach space and let C be a non-empty cbc subset. Let $T: C \to C$ be a mapping. We say that*

1. *T is an affine mapping if for every $t \in [0,1]$ and $a, b \in C$, $T\big((1-t)a + tb\big) = (1-t)T(a) + t\ T(b)$.*
2. *T is a nonexpansive mapping if for every $a, b \in C$,*
$$\| T(a) - T(b) \| \leq \| a - b \|.$$

Then, we will easily obtain an analogous key lemma from the below lemma in the work (Goebel and Kuczumow 1979).

Lemma 1.2 *Let $\{u_n\}$ be a sequence in ℓ^1 converging to u in weak-star topology. Then, for every $w \in \ell^1$,*
$$Q(w) = Q(u) + \|w - u\|_1$$
where
$$Q(w) = \limsup_n \|u_n - w\|_1.$$

Note that our scalar field in this study will be real numbers although Çolak (1989) considered complex values of $v = (v_n)_n$ while introducing his structure of the difference sequence which is taken as the fundamental concept in this study.

2. *Main Results*

In this section, we will present our results. As mentioned in the first section, we investigate Goebel and Kuczumow analogy for the space U_1^m for each $m \in \mathbb{N}$. We aim to show that there is a large class of cbc subsets in U_1^m such that every nonexpansive invariant mapping defined on the subsets in the class taken has a fixed point. Recall that the invariant mappings have the same domain and the range. Note that we will assume that $r \in \mathbb{R}$ is arbitrary due to the definition of the space.

First, due to isometric isomorphism, using Lemma 1.2, we will provide the straight analogous result as a lemma below which will be a key step as in the works such as (Goebel and Kuczumow 1979), and (Everest 2013) and in fact the methods in the study (Everest 2013) will be our lead in this work.

Lemma 2.1 *Fix $m \in \mathbb{N}$ and $\{u_n\}$ be a sequence in the Banach space U_1^m and assume $\{u_n\}$ converges to u in weak-star topology. Then, for every $w \in U_1^m$,*
$$Q(w) = Q(u) + \|w - u\|_{\sim}^{(m)}$$
where
$$Q(w) = \limsup_n \|u_n - w\|_{\sim}^{(m)}.$$

Then, we obtain our results by the following theorems.

Theorem 2.2 *Let $m \in \mathbb{N}$, $r \in \mathbb{R}$ and $t \in (0,1)$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence defined by $f_1 := t\ v_1\ e_1$, $f_2 :=$*

$\frac{t\ v_2}{2^{m-r}}\ e_2$, *and* $f_n := \frac{v_n}{n^{m-r}} e_n$ *for all integers* $n \geq 3$ *where the sequence* $(e_n)_{n\in\mathbb{N}}$ *is the canonical basis of both* c_0 *and* ℓ^1. *Then, consider the cbc subset* $E^{(m)} = E_t{}^{(m)}$ *of* U_1^m *by*

$$E^{(m)} := \left\{\sum_{n=1}^{\infty} \alpha_n f_n : \forall n \in \mathbb{N}, \quad \alpha_n \geq 0 \ and \ \sum_{n=1}^{\infty} \alpha_n = 1\right\}.$$

Then, $E^{(m)}$ has the fixed point property for $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mappings.

Proof. Let $m \in \mathbb{N}$, $r \in \mathbb{R}$ and $t \in (0,1)$. Let $T: E^{(m)} \to E^{(m)}$ be a $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E^{(m)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|_{\sim}^{(m)} \underset{n}{\to} 0$. Due to isometric isomorphism, U_1^m shares common geometric properties with ℓ^1 and so both U_1^m and its predual have similar fixed point theory properties to ℓ^1 and c_0, respectively. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in U_1^m, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then, we can say that we may denote the weak* closure of the set $E^{(m)}$ by

$$C^{(m)} := \overline{E^{(m)}}^{w^*} = \left\{\sum_{n=1}^{\infty} \alpha_n f_n : each\ \alpha_n \geq 0 \ and \ \sum_{n=1}^{\infty} \alpha_n \leq 1\right\}$$

and without loss of generality, we may pass to a subsequence if necessary and get a weak* limit $u \in C^{(m)}$ of $u^{(n)}$. Then, by Lemma 2.1, we have a function $r: U_1^m \to [0,\infty)$ defined by

$$Q(w) = \limsup_n \left\|u^{(n)} - w\right\|_{\sim}^{(m)}, \quad \forall w \in U_1^m$$

such that for every $w \in U_1^m$,

$$Q(w) = Q(u) + \|u - w\|_{\sim}^{(m)}.$$

Case 1. $u \in E^{(m)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{\sim}^{(m)}$ and

$$Q(Tu) = \limsup_n \left\|Tu - u^{(n)}\right\|_{\sim}^{(m)}$$
$$\leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|_{\sim}^{(m)} + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|_{\sim}^{(m)}$$
$$\leq \limsup_n \left\|u - u^{(n)}\right\|_{\sim}^{(m)} + 0$$
$$= Q(u). \qquad (1)$$

Thus, $Q(Tu) = Q(u) + \|Tu - u\|_{\sim}^{(m)} \leq r(u)$ and so $\|Tu - u\|_{\sim}^{(m)} = 0$. Therefore, $Tu = u$.

Case 2. $u \in C^{(m)} \backslash E^{(m)}$.

Then, we may find scalars satisfying $u = \sum_{n=1}^{\infty} \delta_n f_n$ such that $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\delta_n \geq 0, \forall n \in \mathbb{N}$.

Define $\xi := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\beta \in \left[\frac{-\delta_1}{\xi}, \frac{\delta_2}{\xi} + 1\right]$ define

$$h_\beta := (\delta_1 + \beta\xi) f_1 + (\delta_2 + (1-\beta)\xi) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\left\|h_\beta - u\right\|_{\sim}^{(m)} = \left\|\beta t \xi v_1 e_1 + (1-\beta)\xi \frac{t v_2 e_2}{2^{m-r}}\right\|_{\sim}^{(m)} = t|\beta|\xi + t|1-\beta|\xi.$$

$\left\|h_\beta - u\right\|_{\sim}^{(m)}$ is minimized for $\beta \in [0,1]$ and its minimum value would be $t\xi$.

Now fix $w \in E^{(m)}$. Then, we may find scalars satisfying $w = \sum_{n=1}^{\infty} \alpha_n f_n$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$ with $\alpha_n \geq 0, \ \forall n \in \mathbb{N}$. We may also write each f_k with coefficients γ_k for each $k \in \mathbb{N}$ where $\gamma_1 := t\ v_1$, $\gamma_2 := \frac{t\ v_2}{2^{m-r}}$, and $\gamma_n := \frac{v_n}{n^{m-r}}$ for all integers $n \geq 3$ such that for each $n \in \mathbb{N}$, $f_n = \gamma_n e_n$.

Then,

$$\|w - u\|_{\sim}^{(m)} = \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|_{\sim}^{(m)} = \left\|\sum_{k=1}^{\infty} (\alpha_k - \delta_k) f_k\right\|_{\sim}^{(m)} = \sum_{k=1}^{\infty} \left|(\alpha_k - \delta_k)\frac{k^{m-r}\gamma_k}{v_k}\right|.$$

Hence,

$$\|w - u\|_{\sim}^{(m)} \geq \sum_{k=1}^{\infty} t\ |\alpha_k - \delta_k| \geq t\left|\sum_{k=1}^{\infty} (\alpha_k - \delta_k)\right| = t\left|1 - \sum_{k=1}^{\infty} \delta_k\right| = t\xi.$$

Hence,

$$\|w-u\|_{\sim}^{(m)} \geq t\xi = \|h_\beta - u\|_{\sim}^{(m)}$$

and the equality is obtained if and only if $(1-t)\sum_{k=3}^{\infty} |\alpha_k - \delta_k| = 0$; that is, we have $\|w-u\|_{\sim}^{(m)} = t\xi$ if and only if $\alpha_k = \delta_k$ for every $k \geq 3$; or say, $\|w-u\|_{\sim}^{(m)} = t\xi$ if and only if $w = h_\beta$ for some $\beta \in [0,1]$.

Then, there exists a continuous function ρ: $[0,1] \rightarrow E^{(m)}$ defined by $\rho(\beta) = h_\beta$ and $\Lambda\rho([0,1])$ is a compact convex subset and so $\|w-u\|_{\sim}^{(m)}$ achieves its minimum value at $w = h_\beta$ and for any $h_\beta \in \Lambda$, we get

$$\begin{aligned} Q(h_\beta) &= Q(u) + \|h_\beta - u\|_{\sim}^{(m)} \\ &\leq Q(u) + \|Th_\beta - u\|_{\sim}^{(m)} \\ &= Q(Th_\beta) = \limsup_n \|Th_\beta - u^{(n)}\|_{\sim}^{(m)} \end{aligned}$$

then, like the inequality (1), we get

$$\begin{aligned} Q(h_\beta) &\leq \limsup_n \|Th_\beta - T(u^{(n)})\|_{\sim}^{(m)} \\ &\quad + \limsup_n \|u^{(n)} - T(u^{(n)})\|_{\sim}^{(m)} \\ &\leq \limsup_n \|h_\beta - u^{(n)}\|_{\sim}^{(m)} \\ &\quad + \limsup_n \|u^{(n)} - T(u^{(n)})\|_{\sim}^{(m)} \\ &\leq \limsup_n \|h_\beta - u^{(n)}\|_{\sim}^{(m)} + 0 \\ &= Q(h_\beta). \end{aligned}$$

Hence, $r(h_\beta) \leq Q(Th_\beta) \leq r(h_\beta)$ and so $Q(Th_\beta) = Q(h_\beta)$.

Therefore,

$$Q(u) + \|Th_\beta - u\|_{\sim}^{(m)} = Q(u) + \|h_\beta - u\|_{\sim}^{(m)}.$$

Thus, $\|Th_\beta - u\|_{\sim}^{(m)} = \|h_\beta - u\|_{\sim}^{(m)}$ and so $Th_\beta \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's (1930) fixed point theorem, easily we get the result T has a fixed point since T is continuous; thus, h_β is the unique minimizer of $\|w-u\|_{\sim}^{(m)}$: $w \in E^{(m)}$ and $Th_\beta = h_\beta$.

Therefore, $E^{(m)}$ has the fixed point property for nonexpansive mappings.

Theorem 2.3 *Let $m \in \mathbb{N}$, $r \in \mathbb{R}$ and $t \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := t\ v_1\ e_1$, $f_2 := \frac{t\ v_2}{2^{m-r}}\ e_2$, $f_3 := \frac{t\ v_3}{3^{m-r}}\ e_3$, and $f_n := \frac{v_n}{n^{m-r}} e_n$ for all integers $n \geq 4$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E^{(m)} = E_t{}^{(m)}$ of U_1^m by*

$$E^{(m)} := \left\{\sum_{n=1}^{\infty} \alpha_n f_n:\ \forall n \in \mathbb{N},\quad \alpha_n \geq 0\ and\ \sum_{n=1}^{\infty} \alpha_n = 1\right\}.$$

Then, $E^{(m)}$ has the fixed point property for $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mappings.

Proof. Let $m \in \mathbb{N}$, $r \in \mathbb{R}$ and $t \in (0,1)$. Let $T: E^{(m)} \rightarrow E^{(m)}$ be a $\|\ .\ \|_{\sim}^{(m)}$-nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $(u^{(n)})_{n\in\mathbb{N}} \in E^{(m)}$ such that $\|Tu^{(n)} - u^{(n)}\|_{\sim}^{(m)} \underset{n}{\rightarrow} 0$. Due to isometric isomorphism, U_1^m shares common geometric properties with ℓ^1 and so both U_1^m and its predual have similar fixed point theory properties to ℓ^1 and c_0, respectively. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology and c_0 is separable, in U_1^m, the unit closed ball is weak*-sequentially compact due to Banach-Alaoğlu theorem. Then, we can say that we may denote the weak* closure of the set $E^{(m)}$ by

$$C^{(m)} := \overline{E^{(m)}}^{w^*} = \left\{\sum_{n=1}^{\infty} \alpha_n f_n:\ each\ \alpha_n \geq 0\ and\ \sum_{n=1}^{\infty} \alpha_n \leq 1\right\}$$

and without loss of generality, we may pass to a subsequence if necessary and get a weak* limit $u \in C^{(m)}$ of $u^{(n)}$. Then, by Lemma 2.1, we have a function $r: U_1^m \rightarrow [0,\infty)$ defined by

$$Q(w) = \limsup_n \|u^{(n)} - w\|_{\sim}^{(m)}\ ,\quad \forall w \in U_1^m$$

such that for every $w \in U_1^m$,

$$Q(w) = Q(u) + \|u - w\|_{\sim}^{(m)}.$$

Case 1. $u \in E^{(m)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{\sim}^{(m)}$ and

$$\begin{aligned} Q(Tu) &= \limsup_n \|Tu - u^{(n)}\|_{\sim}^{(m)} \\ &\leq \limsup_n \|Tu - T(u^{(n)})\|_{\sim}^{(m)} \\ &\quad + \limsup_n \|u^{(n)} - T(u^{(n)})\|_{\sim}^{(m)} \\ &\leq \limsup_n \|u - u^{(n)}\|_{\sim}^{(m)} + 0 \\ &= Q(u). \end{aligned} \tag{2}$$

Thus, $Q(Tu) = Q(u) + \|Tu - u\|_{\sim}^{(m)} \leq r(u)$ and so $\|Tu - u\|_{\sim}^{(m)} = 0$. Therefore, $Tu = u$.

Case 2. $u \in C^{(m)} \backslash E^{(m)}$.

Then, we may find scalars satisfying $u = \sum_{n=1}^{\infty} \delta_n f_n$ such that $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\delta_n \geq 0, \forall n \in \mathbb{N}$.

Define $\xi := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\beta \in \left[\frac{-\delta_1}{\xi}, \frac{\delta_2}{\xi} + 1\right]$, define

$$h_\beta := \left(\delta_1 + \frac{\beta}{2}\xi\right) f_1 + \left(\delta_2 + \frac{\beta}{2}\xi\right) f_2 + (\delta_3 + (1-\beta)\xi) f_3 + \sum_{n=4}^{\infty} \delta_n f_n.$$

Then,

$$\left\|h_\beta - u\right\|_{\sim}^{(m)} = \left\| \begin{matrix} \frac{\beta}{2} t\xi v_1 e_1 + \frac{\beta}{2} t\xi \frac{v_2}{2^{m-r}} e_2 \\ +(1-\beta)\xi \frac{t\ v_3 e_3}{3^{m-r}} \end{matrix} \right\|_{\sim}^{(m)}$$

$$= t\left|\frac{\beta}{2}\right| \xi + t\left|\frac{\beta}{2}\right| \xi + t|1-\beta|\xi.$$

$\left\|h_\beta - u\right\|_{\sim}^{(m)}$ is minimized for $\beta \in [0,1]$ and its minimum value would be $t\xi$.

Now fix $w \in E^{(m)}$. Then, we may find scalars satisfying $w = \sum_{n=1}^{\infty} \alpha_n f_n$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$ with $\alpha_n \geq 0, \ \forall n \in \mathbb{N}$. We may also write each f_k with coefficients γ_k for each $k \in \mathbb{N}$ where $\gamma_1 := t\ v_1$, $\gamma_2 := \frac{t\ v_2}{2^{m-r}}$, $\gamma_3 := \frac{t\ v_3}{3^{m-r}}$, and $\gamma_n := \frac{v_n}{n^{m-r}}$ for all integers $n \geq 4$ such that for each $n \in \mathbb{N}$, $f_n = \gamma_n e_n$.

Then,

$$\|\mathrm{w} - u\|_{\sim}^{(m)} = \left\|\sum_{k=1}^{\infty} \alpha_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|_{\sim}^{(m)}$$

$$= \left\|\sum_{k=1}^{\infty} (\alpha_k - \delta_k) f_k\right\|_{\sim}^{(m)}$$

$$= \sum_{k=1}^{\infty} \left|(\alpha_k - \delta_k)\frac{\mathrm{k}^{m-r}\gamma_k}{v_k}\right|$$

$$\geq \sum_{k=1}^{\infty} t\ |\alpha_k - \delta_k|$$

$$\geq t\left|\sum_{k=1}^{\infty} (\alpha_k - \delta_k)\right|$$

$$= t\left|1 - \sum_{k=1}^{\infty} \delta_k\right|$$

$$= t\xi.$$

Hence,

$$\|\mathrm{w} - u\|_{\sim}^{(m)} \geq t\xi = \left\|h_\beta - u\right\|_{\sim}^{(m)}$$

and the equality is obtained if and only if $(1-t)\sum_{k=4}^{\infty} |\alpha_k - \delta_k| = 0$; that is, we have $\|\mathrm{w} - u\|_{\sim}^{(m)} = t\xi$ if and only if $\alpha_k = \delta_k$ for every $k \geq 4$; or say, $\|\mathrm{w} - u\|_{\sim}^{(m)} = t\xi$ if and only if $\mathrm{w} = h_\beta$ for some $\beta \in [0,1]$.

Then, there exists a continuous function ρ: $[0,1] \to E^{(m)}$ defined by $\rho(\beta) = h_\beta$ and $\Lambda\rho([0,1])$ is a compact convex subset and so $\|\mathrm{w} - u\|_{\sim}^{(m)}$ achieves its minimum value at $\mathrm{w} = h_\beta$ and for any $h_\beta \in \Lambda$, we get

$$Q(h_\beta) = Q(u) + \left\|h_\beta - u\right\|_{\sim}^{(m)}$$

$$\leq Q(u) + \left\|Th_\beta - u\right\|_{\sim}^{(m)}$$

$$= Q(Th_\beta) = \limsup_n \left\|Th_\beta - u^{(n)}\right\|_{\sim}^{(m)}$$

then same as the inequality (2), we get

$$Q(h_\beta) \leq \limsup_n \left\|Th_\beta - T(u^{(n)})\right\|_{\sim}^{(m)} + \limsup_n \left\|u^{(n)} - T(u^{(n)})\right\|_{\sim}^{(m)}$$

$$\leq \limsup_n \left\|h_\beta - u^{(n)}\right\|_{\sim}^{(m)} + \limsup_n \left\|u^{(n)} - T(u^{(n)})\right\|_{\sim}^{(m)}$$

$$\leq \limsup_n \left\|h_\beta - u^{(n)}\right\|_{\sim}^{(m)} + 0$$

$$= Q(h_\beta).$$

Hence, $r(h_\beta) \leq Q(Th_\beta) \leq r(h_\beta)$ and so $Q(Th_\beta) = Q(h_\beta)$.

Therefore,

$$Q(u) + \left\|Th_\beta - u\right\|_{\sim}^{(m)} = Q(u) + \left\|h_\beta - u\right\|_{\sim}^{(m)}.$$

Thus, $\left\|Th_\beta - u\right\|_{\sim}^{(m)} = \left\|h_\beta - u\right\|_{\sim}^{(m)}$ and so $Th_\beta \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's (1930) fixed point theorem, we can easily we get the result T has a fixed point since T is continuous. Thus, h_β is the unique minimizer of $\|\mathrm{w} - u\|_{\sim}^{(m)} \ : w \in E^{(m)}$ and $Th_\beta = h_\beta$.

Therefore, $E^{(m)}$ has the fixed point property for nonexpansive mappings.

3. Discussion

The present study introduces novel advancements in the field of fixed point theory by establishing large classes of cbc subsets in α-duals of certain generalized difference sequence spaces that satisfy the FPP for nonexpansive mappings. This work addresses a previously unexplored area, as no prior studies have examined these spaces with the goal of identifying such large classes with the FPP. Notably, while Goebel and Kuczumow (1979) achieved analogous results for the space of absolutely summable scalar sequences, our work generalizes and extends these findings to broader spaces. Our space is isometrically isomorphic to the absolutely summable scalar sequence space but incorporates a more general framework, thereby broadening the scope of applicable classes.

An essential distinction of our approach lies in the elimination of the additional affinity condition required by Kaczor and Prus (2004), as we work directly with nonexpansive mappings rather than asymptotically nonexpansive mappings. This adjustment simplifies the theoretical foundation while still achieving stronger results. Moreover, our methods are not limited to specific instances, as we are developing a more general case for arbitrary \(m\), which opens new possibilities for future research in this domain.

Recent studies have demonstrated the existence of large classes with the fixed point property under specific conditions. Our results build on this momentum, further advancing the field by identifying and characterizing classes of sets that satisfy the FPP in a broader family of Banach spaces. These results offer a valuable perspective on the geometric structure of generalized difference sequence spaces and their fixed point properties, with implications for further studies on nonexpansive mappings, Banach space theory, and related areas in functional analysis.

As has been mentioned above and in earlier sections of the study, investigating and looking for large classes of closed, bounded and convex subsets in Banach spaces alike the Banach spaces of absolutely summable scalars are center of interests for many fixed point theorists. One can investigate to get larger classes for more general spaces than those in the present study and due to isometry, that would not be hard by following the ideas of Goebel and Kuczumow's. However, trying to generalize their ideas and looking for different examples of the sets and spaces would be valuable studies.

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On the Pseudo Starlike and Pseudo Convex Bi-univalent Function Classes of Complex Order

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Abstract

In this paper, we defined a new subclass of starlike and convex bi-univalent functions and examine some geometric properties this function class. For this definition class, we gave some coefficient estimates and solve Fekete-Sezöge problem.

Keywords: Starlike function, convex function, pseudo starlike function, pseudo convex function

1. Introduction

In this section, we give some basic information which we will use in our study.

Let $H(U)$ be the class of analytic functions on the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ in the complex plane $\mathbb{C}$. By A, we will denote the class of the functions $f\in H(U)$ given by the following series expansions

$$f(z)=z+a_2z^2+a_3z^3+\cdots = z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C}. \tag{1.1}$$

The subclass of univalent functions of A is denoted by S in the literature. This class was first introduced by Köebe (Köebe 1909) and has become the core ingredient of advanced research in this field. Many mathematicians were interested coefficient estimates for this class. Within a short period, Bieberbach published a (Bieberbach 1916) paper in which the famous coefficient hypothesis was proposed. This hypothesis states that if $f\in S$ and has the series form (1.1), then $|a_n|\le n$ for each $n\ge 2$. In 1985, it was de-Branges (de-Branges 1985), who settled this long-lasting conjecture. There were a lot of papers devoted to this conjecture and its related coefficient problems (Brannan, Kirwan 1969, Janowski 1970, Sokol, Stankiewcz 1996, Sharmaet all 1996, Frasin, Aouf 2011, Sharma et all 2016, Arif et all 2019, Cho et all 2019, Kumar, Arora 2020, Mendiratta et all 2015, Bano, Raza 2020, Alotaibi et all 2020, Ullah et all 2021, Mustafa, Nezir, Kankılıç 2023a)

As is known that the function $f(z)$ is called a bi-univalent function, if itself and inverse is univalent in U and $f(U)$, respectively. The class of bi-univalent functions is denoted by Σ.

We will denote $g(w)=f^{-1}(w),\ w\in f(U)$. In this case, if $f\in\Sigma$

$$g(w)=w+A_2w^2+A_3w^3+A_4w^4+\ldots = w+\sum_{n=2}^{\infty}A_nz^n,\ w\in f(U), \tag{1.2}$$

where

$$A_2=-a_2,\ A_3=2a_2^2-a_3,\ A_4=-a_2^3+5a_2a_3-a_4,\ldots.$$

It is well known that the bi-starlike and bi-convex function classes defined on the open unit disk U are defined analytically as follows

$$S_{\Sigma}^{*}=\left\{\begin{array}{l} f\in S:\ \operatorname{Re}\left(\dfrac{zf'(z)}{f(z)}\right)>0,\ z\in U \\ \text{and } \operatorname{Re}\left(\dfrac{wg'(w)}{g(w)}\right)>0,\ w\in f(U)\end{array}\right\},$$

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$$C_{\Sigma}=\left\{\begin{array}{l} f\in S:\ \operatorname{Re}\left(\dfrac{\left(zf'(z)\right)'}{f'(z)}\right)>0,\ z\in U \\ \text{and } \operatorname{Re}\left(\dfrac{\left(zg'(w)\right)'}{g'(w)}\right)>0,\ w\in f(U) \end{array}\right\}.$$

2. Materials and Methods

It is well-known that an analytical function ω satisfying the conditions $\omega(0)=0$ and $|\omega(z)|<1$ is called Schwartz function. Let's $f,g\in H(U)$, then it is said that f is subordinate to g and denoted by $f\prec g$, if there exists a Schwartz function ω, such that $f(z)=g(\omega(z))$.
Ma and Minda using subordination terminology presented unified version of the classes $S^*(\varphi)$ and $C(\varphi)$ as follows

$$\left(S^*\vee C\right)(\varphi)=$$
$$\left\{\begin{array}{l} f\in S:\ (1-\beta)\dfrac{zf'(z)}{f(z)}+\beta\dfrac{\left(zf'(z)\right)'}{f'(z)}\prec\varphi(z), \\ z\in U \end{array}\right\},$$

$\beta\in[0,1]$,
where $\varphi(z)$ is a univalent function with $\varphi(0)=1$, $\varphi'(0)>0$ and the region $\varphi(U)$ is star-shaped about the point $\varphi(0)=1$ and symmetric with respect to real axis. Such a function has a series expansion of the following form

$$\varphi(z)=1+b_1z+b_2z^2+b_3z^3+\cdots$$
$$=1+\sum_{n=1}^{\infty}b_nz^n,\ b_1>0.$$

In the past few years, numerous subclasses of the collection S have been introduced as special choices of the classes $S^*(\varphi)$ and $C(\varphi)$ (Brannan, Kirwan 1969, Sokol, Stankiewcz 1996, Mendiratta et all 2015, Sharma et all 2016, Arif et all 2019, Shi et all 2019, Cho et all 2019, Kumar, Arora 2020, Alotaibi et all 2020, Ullah et all 2021, Mustafa, Nezir, Kankılıç 2023a, Frasin, Aouf 2011, Mustafa, Nezir, Kankılıç 2023b, Mustaf, Nezir 2023, Mustafa , Demir 2023a, Mustafa, Demir, 2023b, Mustafa, Nezir, Kankılıç 2023c, Mustafa, Nezir, Kankılıç 2023d, Mustafa, Demir 2023d).

Now, let's define some new subclass of bi-univalent functions in the open unit disk U.

Definition 2.1. For $\beta\in[0,1]$, $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$, if the following conditions are satisfied

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right]\right\}$$
$$+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}-1\right]\right\}$$
$$\prec 1+\sinh z,\ z\in U,$$
$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}-1\right]\right\}$$
$$+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}-1\right]\right\}$$
$$\prec 1+\sinh w,\ w\in f(U).$$

From the Definition 2.1, in the special values of the parameters, we obtain the following function classes.

Definition 2.2. For $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $S^*_{\Sigma,\sinh}(\lambda,\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right]\right\} \prec 1+\sinh z,\ z\in U,$$

$$\left\{1+\frac{1}{\tau}\left[\frac{w\left(g'(w)\right)^{\lambda}}{g(w)}-1\right]\right\} \prec 1+\sinh w,\ w\in f(U).$$

Definition 2.2.1. For $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $S^{*}_{\Sigma,\sinh}(\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\} \prec 1+\sinh z,\ z\in U,$$

$$\left\{1+\frac{1}{\tau}\left[\frac{wg'(w)}{g(w)}-1\right]\right\} \prec 1+\sinh w,\ w\in f(U).$$

Definition 2.2.2. For $\lambda>\frac{1}{2}$ the function $f\in\Sigma$ is said to be in the class $S^{*}_{\Sigma,\sinh}(\lambda)$, if the following conditions are satisfied

$$\frac{z\left(f'(z)\right)^{\lambda}}{f(z)} \prec 1+\sinh z,\ z\in U,$$

$$\frac{w\left(g'(w)\right)^{\lambda}}{g(w)} \prec 1+\sinh w,\ w\in f(U).$$

Definition 2.2.3. For the function $f\in\Sigma$ is said to be in the class $S^{*}_{\Sigma,\sinh}$, if the following conditions are satisfied

$$\frac{zf'(z)}{f(z)} \prec 1+\sinh z,\ z\in U,$$

$$\frac{wg'(w)}{g(w)} \prec 1+\sinh w,\ w\in f(U).$$

Definition 2.3. For $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\lambda,\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)}-1\right]\right\} \prec 1+\sinh z,\ z\in U,$$

$$\left\{1+\frac{1}{\tau}\left[\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)}-1\right]\right\} \prec 1+\sinh w,$$

$w\in f(U).$

Definition 2.3.1. For $\tau\in\mathbb{C}-\{0\}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\tau)$, if the following conditions are satisfied

$$\left\{1+\frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)}-1\right]\right\} \prec 1+\sinh z,\ z\in U,$$

$$\left\{1+\frac{1}{\tau}\left[\frac{\left(wg'(w)\right)'}{g'(w)}-1\right]\right\} \prec 1+\sinh w, w\in f(U).$$

Definition 2.3.2. For $\lambda>\frac{1}{2}$ the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}(\lambda)$, if the following conditions are satisfied

$$\frac{\left[\left(zf'(z)\right)'\right]^{\lambda}}{f'(z)} \prec 1+\sinh z,\ z\in U,$$

$$\frac{\left[\left(wg'(w)\right)'\right]^{\lambda}}{g'(w)} \prec 1+\sinh w,\ w\in f(U).$$

Definition 2.3.3. For the function $f\in\Sigma$ is said to be in the class $C_{\Sigma,\sinh}$, if the following conditions are satisfied

$$\frac{\left(zf'(z)\right)'}{f'(z)} \prec 1+\sinh z,\ z\in U,$$

$$\frac{\left(wg'(w)\right)'}{g'(w)} \prec 1+\sinh w,\ w\in f(U).$$

Definition 2.4. For $\beta \in [0,1]$ and $\tau \in \mathbb{C} - \{0\}$ the function $f \in \Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta,1,\tau)$, if the following conditions are satisfied

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}$$
$$+\beta\left\{1+\frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)}-1\right]\right\}$$
$$\prec 1+\sinh z,\ z\in U,$$
$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{wg'(w)}{g(w)}-1\right]\right\}$$
$$+\beta\left\{1+\frac{1}{\tau}\left[\frac{(wg'(w))'}{g'(w)}-1\right]\right\}$$
$$\prec 1+\sinh w,\ w\in f(U).$$

Definition 2.4.1. For $\beta \in [0,1]$ the function $f \in \Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta)$, if the following conditions are satisfied

$$(1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{(zf'(z))'}{f'(z)}\prec 1+\sinh z,\ z\in U$$
,
$$(1-\beta)\frac{w(g'(w))}{g(w)}+\beta\frac{(wg'(w))'}{g'(w)}\prec 1+\sinh w,$$
$$w\in f(U).$$

Definition 2.5. For $\beta \in [0,1]$ and $\lambda > \frac{1}{2}$ the function $f \in \Sigma$ is said to be in the class $\chi_{\Sigma,\sinh}(\beta,\lambda)$, if the following conditions are satisfied

$$(1-\beta)\frac{z(f'(z))^{\lambda}}{f(z)}+\beta\frac{\left[(zf'(z))'\right]^{\lambda}}{f'(z)}$$
$$\prec 1+\sinh z,\ z\in U,$$
$$(1-\beta)\frac{w(g'(w))^{\lambda}}{g(w)}+\beta\frac{\left[(wg'(w))'\right]^{\lambda}}{g'(w)}.$$
$$\prec 1+\sinh w,\ w\in f(U).$$

Remark 2.1. Let's point out that the classes $S^{*}_{\Sigma,\sinh}(\lambda,\tau)$ and $C_{\Sigma,\cosh}(\lambda,\tau)$ was investigated by Kankılıç and Mustafa (Kankılıç, Mustafa 2023a, Kankılıç, Mustafa 2023b).

Let P be the class of analytic functions in U satisfied the conditions $p(0)=1$ and $\mathrm{Re}(p(z))>0$, $z\in U$, which from the subordination principle easily can written

$$\mathrm{P}=\left\{p\in A:\ p(z)\prec\frac{1+z}{1-z},z\in U\right\},$$

where $p(z)$ has the series expansion of the form

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$$
$$=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U\ . \qquad (2.1)$$

The class P defined above is known as the class Caratheodory functions (Caratheodory 1907).

Now, let us present some necessary lemmas known in the literature for the proof of our main results.

Lemma 2.1 (Duren 1983). Let the function $p(z)$ belong in the class P. Then,

$|p_n|\le 2$ for each $n\in\mathbb{N}$ and $|p_n-\lambda p_kp_{n-k}|\le 2$

for $n,k\in\mathbb{N},\ n>k$ and $\lambda\in[0,1]$.

The equalities hold for

$$p(z)=\frac{1+z}{1-z}.$$

Lemma 2.2 (Duren 1983) Let the an analytic function $p(z)$ be of the form (2.1, then

$$2p_2=p_1^2+(4-p_1^2)x,$$
$$4p_3$$
$$=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2$$
$$+2(4-p_1^2)(1-|x|^2)y$$

for $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$.

In this paper, we give some coefficient estimates and examine Fekete-Szegö problem for the class $\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$. Additionally, the results obtained for specific values of the parameters in our study are compared with the results obtained in the literature.

3. *Results*

In this section, we examine the coefficient estimates problem for the function class $\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$.

Theorem 3.1. If $f\in\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$, then are provided the following inequalities

$$|a_2|\le\frac{|\tau|}{(2\lambda-1)(1+\beta)} \text{ and}$$

$$|a_3| \le |\tau|\begin{cases}\dfrac{1}{(3\lambda-1)(1+2\beta)}, & |\tau|\le\dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)},\\[2ex] \dfrac{|\tau|}{(2\lambda-1)^2(1+\beta)^2}, & |\tau|\ge\dfrac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}.\end{cases} \tag{3.1}$$

Proof. Let $f\in\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$, $\beta\in[0,1]$, $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$. Then, are Schwartz functions $\omega:U\to U,\varpi:U_{r_0}\to U_{r_0}$, such that

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{z(f'(z))^{\lambda}}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[(zf'(z))'\right]^{\lambda}}{f'(z)}-1\right]\right\},z\in U$$
$$=1+\sinh\omega(z)$$

and

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{w(g'(w))^{\lambda}}{g(w)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[(wg'(w))'\right]^{\lambda}}{g'(w)}-1\right]\right\}$$
$$=1+\sinh\varpi(w),w\in f(U). \tag{3.2}$$

Let's the functions $p,q\in P$ defined as follows:

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_2z^2+p_3z^3+\ldots$$
$$=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U,$$

$$q(w)=\frac{1+\varpi(w)}{1-\varpi(w)}=1+q_1w+q_2w^2+q_3w^3+\ldots$$
$$=1+\sum_{n=1}^{\infty}q_nw^n,\ w\in f(U). \tag{3.3}$$

From these equalities, we can write

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_1}{2}z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+\frac{1}{2}\left(p_3-p_1p_2-\frac{p_1^3}{4}\right)z^3\ldots,\ z\in U,$$
$$\varpi(w)=\frac{q(w)-1}{q(w)+1}=\frac{q_1}{2}w+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)w^2+\frac{1}{2}\left(q_3-q_1q_2-\frac{q_1^3}{4}\right)w^3\ldots,w\in f(U). \tag{3.4}$$

Then, from the (3.1) and (3.4) can written the following equalities

$$(1-\beta)\times \left\{1+\frac{1}{\tau}\left[\begin{array}{l} a_2(2\lambda-1)z \\ +\left((3\lambda-1)a_3+\left(2\lambda^2-4\lambda+1\right)a_2^2\right)z^2+\ldots \end{array}\right]\right\}$$
$$+\beta\left\{\begin{array}{l}1+ \\ \frac{1}{\tau}\left[\begin{array}{l} 2a_2(2\lambda-1)z+ \\ \left(3(3\lambda-1)a_3+\left(8\lambda^2-16\lambda+4\right)a_2^2\right)z^2+\ldots\end{array}\right]\end{array}\right\}$$
$$=1+\frac{p_1}{2}z+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)z^2+\ldots,\ z\in U,$$
$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\begin{array}{l} A_2(2\lambda-1)w \\ +\left(\begin{array}{l}(3\lambda-1)A_3 \\ +\left(2\lambda^2-4\lambda+1\right)A_2^2\end{array}\right)w^2+\ldots\end{array}\right]\right\}$$
$$+\beta\left\{1+\frac{1}{\tau}\left[\begin{array}{l} 2A_2(2\lambda-1)w \\ +\left(\begin{array}{l}3(3\lambda-1)A_3 \\ +\left(8\lambda^2-16\lambda+4\right)A_2^2\end{array}\right)w^2+\ldots\end{array}\right]\right\}$$
$$=1+\frac{q_1}{2}w+\left(\frac{q_2}{2}-\frac{q_1^2}{4}\right)w^2+\ldots,$$
$$w\in f(U). \tag{3.5}$$

Comparing the coefficients of the same degree terms on the right and left sides of the equalities (3.5), we obtain the following equalities for the coefficients a_2 and a_3 of the function f

$$\frac{1}{\tau}a_2(2\lambda-1)(1+\beta)=\frac{p_1}{2}, \tag{3.6}$$

$$\frac{1}{\tau}\left[\begin{array}{l}(3\lambda-1)(1+2\beta)a_3 \\ +\left(2\lambda^2-4\lambda+1\right)(1+3\beta)a_2^2\end{array}\right]=\frac{p_2}{2}-\frac{p_1^2}{4}, \tag{3.7}$$

$$-\frac{1}{\tau}a_2(2\lambda-1)(1+\beta)=\frac{q_1}{2}, \tag{3.8}$$

$$\frac{1}{\tau}\left[\begin{array}{l}(3\lambda-1)(1+2\beta)\left(2a_2^2-a_3\right) \\ +\left(2\lambda^2-4\lambda+1\right)(1+3\beta)a_2^2\end{array}\right]=\frac{q_2}{2}-\frac{q_1^2}{4}. \tag{3.9}$$

From the equalities (3.6) and (3.8), we can write

$$\frac{\tau p_1}{2(2\lambda-1)(1+\beta)}=a_2=-\frac{\tau q_1}{2(2\lambda-1)(1+\beta)}; \tag{3.10}$$

that is,

$$p_1=-q_1. \tag{3.11}$$

Thus, according to the Lemma 1.1, we obtain the first result of the theorem.

Using the equality (3.11), from the equalities (3.7) and (3.9) we obtain the following equality for a_3

$$a_3=\frac{\tau^2}{4(2\lambda-1)^2(1+\beta)^2}p_1^2+\frac{\tau(p_2-q_2)}{4(3\lambda-1)(1+2\beta)}. \tag{3.12}$$

From the Lemma 1.2, we can write

$$p_2-q_2=\frac{\left(4-p_1^2\right)}{2}(x-y)$$

for $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$. Substitute this expression for the p_2-q_2 difference in (3.12), we get

$$a_3= =\frac{\tau^2}{4(2\lambda-1)^2(1+\beta)^2}p_1^2+\frac{\tau\left(4-p_1^2\right)(x-y)}{8(3\lambda-1)(1+2\beta)} \tag{3.13}$$

Applying triangle inequality to the last equality, we obtain

$$|a_3| \le \frac{|\tau|^2}{4(2\lambda-1)^2(1+\beta)^2}t^2 + \frac{|\tau|}{4(3\lambda-1)(1+2\beta)}.\frac{(4-t^2)}{2}(\xi+\eta), \quad (3.14)$$
$$(\xi,\eta)\in[0,1]^2,$$

where $\xi=|x|$, $\eta=|y|$ and $t=|p_1|$.

From the inequality (3.14), we can write

$$|a_3| \le \frac{|\tau|}{4}\left[a(|\tau|,\lambda,\beta)t^2 + \frac{4}{(3\lambda-1)(1+2\beta)}\right], \quad t\in[0,2], \quad (3.15)$$

where

$$a(|\tau|,\lambda,\beta) = \left(\frac{|\tau|}{(2\lambda-1)^2(1+\beta)^2} - \frac{1}{(3\lambda-1)(1+2\beta)}\right).$$

Then, maximizing the function

$$\chi(t) = |\tau|\left(a(\lambda,\beta)t^2 + \frac{1}{(3\lambda-1)(1+2\beta)}\right),$$

we obtain the second result of theorem.

With this the proof of theorem is completed.

Taking $\beta=0$ and $\beta=1$ in the Theorem 3.1, we obtain the following results, respectively.

Corollary 3.1. If $f\in S^*_{\Sigma,\sinh}(\lambda,\tau)$, then

$$|a_2| \le \frac{|\tau|}{2\lambda-1} \text{ and}$$

$$|a_3| \le |\tau|\begin{cases}\dfrac{1}{3\lambda-1}, & |\tau| \le \dfrac{(2\lambda-1)^2}{3\lambda-1},\\[2mm] \dfrac{|\tau|}{(2\lambda-1)^2}, & |\tau| \ge \dfrac{(2\lambda-1)^2}{3\lambda-1}.\end{cases}$$

Corollary 3.2. If $f\in C_{\Sigma,\sinh}(\lambda,\tau)$, then

$$|a_2| \le \frac{|\tau|}{2(2\lambda-1)} \text{ and}$$

$$|a_3| \le |\tau|\begin{cases}\dfrac{1}{3(3\lambda-1)}, & |\tau| \le \dfrac{4(2\lambda-1)^2}{3(3\lambda-1)},\\[2mm] \dfrac{|\tau|}{4(2\lambda-1)^2}, & |\tau| \ge \dfrac{4(2\lambda-1)^2}{3(3\lambda-1)}.\end{cases}$$

Note: 3.1. In the special values of the parameters λ and τ from Corollary 3.1 and Corollary 3.2, we obtain the results for the classes $S^*_{\Sigma,\sinh}(\tau)$, $S^*_{\Sigma,\sinh}(\lambda)$, $S^*_{\Sigma,\sinh}$ and $C_{\Sigma,\sinh}(\tau)$, $C_{\Sigma,\sinh}(\lambda)$, $C_{\Sigma,\sinh}$, respectively.

4. The Fekete-Szegö problem

In this section, we focused on the solution of the Fekete-Szegö problem for the class $\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$.

Theorem 4.1. Let $f\in\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$ and $\mu\in\mathbb{C}$. Then,

$$|a_3-\mu a_2^2| \le |\tau|\begin{cases}\dfrac{1}{(3\lambda-1)(1+2\beta)} & \text{if } |1-\mu||\tau| \le l(\lambda,\beta),\\[2mm] \dfrac{|\tau||1-\mu|}{(2\lambda-1)^2(1+\beta)^2} & \text{if } |1-\mu||\tau| \ge l(\lambda,\beta),\end{cases} \quad (4.1)$$

where

$$l(\lambda,\beta)=\frac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}.$$

Obtained here result is sharp.

Proof. Let $f\in\chi_{\Sigma,\sinh}(\beta,\lambda,\tau)$, $\beta\in[0,1]$, $\lambda>\frac{1}{2}$ and $\tau\in\mathbb{C}-\{0\}$. From the equalities (3.10), (3.12) and (3.13), we can write the folloing equality for the expression $a_3-\mu a_2^2$

$$a_3-\mu a_2^2=(1-\mu)\frac{\tau^2p_1^2}{4(2\lambda-1)^2(1+\beta)^2}+\frac{\tau(4-p_1^2)}{8(3\lambda-1)(1+2\beta)}(x-y) \tag{4.2}$$

for $x,y\in\mathbb{C}$ with $|x|\le 1$ and $|y|\le 1$.

Applying triangle inequality to the equality (4.2), we obtain

$$|a_3-\mu a_2^2|\le|1-\mu|\frac{|\tau|^2t^2}{4(2\lambda-1)^2(1+\beta)^2}+\frac{|\tau|(4-t^2)}{8(3\lambda-1)(1+2\beta)}(\xi+\eta),$$

$$(\xi,\eta)\in[0,1]^2,$$

where $\xi=|x|$, $\eta=|y|$ and $t=|p_1|$.

From the last inequality, we can write

$$|a_3-\mu a_2^2|\le\frac{|\tau|}{4(2\lambda-1)^2(1+\beta)^2}\begin{bmatrix}|1-\mu||\tau|-\\ l(\lambda,\beta)\end{bmatrix}t^2+\frac{|\tau|}{(3\lambda-1)(1+2\beta)},$$

$$t\in[0,2], \tag{4.3}$$

where

$$l(\lambda,\beta)=\frac{(2\lambda-1)^2(1+\beta)^2}{(3\lambda-1)(1+2\beta)}.$$

Maximizing the expression on the right hand side of the inequality (4.3) according to the parameter t, we get

$$|a_3-\mu a_2^2|\le|\tau|\begin{cases}\dfrac{1}{(3\lambda-1)(1+2\beta)} & \text{if } |1-\mu||\tau|\le l(\lambda,\beta),\\[2ex] \dfrac{|\tau||1-\mu|}{(2\lambda-1)^2(1+\beta)^2} & \text{if } |1-\mu||\tau|\ge l(\lambda,\beta).\end{cases}$$

The result of the theorem is sharp in the case $|1-\mu||\tau|\le l(\lambda,\beta)$ for the function

$$f_1(z)=z+\frac{\sqrt{|\tau|}}{\sqrt{|1-\mu|(3\lambda-1)(1+2\beta)}}z^2+\frac{|\tau|}{|1-\mu|(3\lambda-1)(1+2\beta)}z^3,\ z\in U$$

and in the case $|1-\mu||\tau|\ge l(\lambda,\beta)$ for the function

$$f_2(z)=z+\frac{|\tau|}{(2\lambda-1)(1+\beta)}z^2+\frac{|\tau|^2}{(2\lambda-1)^2(1+\beta)^2}z^3,\ z\in U.$$

Thus, the proof of theorem is completed.

If we take $\beta=0$ and $\beta=1$ in Theorem 4.1, we obtain the following results, respectively.

Corollary 4.1. If $f\in S^*_{\Sigma,\sinh}(\lambda,\tau)$, then

$$\left|a_3-\mu a_2^2\right| \le |\tau| \begin{cases} \dfrac{1}{3\lambda-1} & \text{if } |1-\mu||\tau| \le \dfrac{2\lambda-1\ {}^{2}}{3\lambda-1}, \\ \dfrac{|\tau||1-\mu|}{\left(2\lambda-1\right)^2} & \text{if } |1-\mu||\tau| \ge \dfrac{\left(2\lambda-1\right)^2}{3\lambda-1}. \end{cases}$$

Corollary 4.2. If $f \in C_{\Sigma,\sinh}\left(\lambda,\tau\right)$, then

$$\left|a_3-\mu a_2^2\right| \le |\tau| \begin{cases} \dfrac{1}{3\left(3\lambda-1\right)} & \text{if } |1-\mu||\tau| \le \dfrac{4\left(2\lambda-1\right)^2}{3\left(3\lambda-1\right)}, \\ \dfrac{|\tau||1-\mu|}{4\left(2\lambda-1\right)^2} & \text{if } |1-\mu||\tau| \ge \dfrac{4\left(2\lambda-1\right)^2}{3\left(3\lambda-1\right)}. \end{cases}$$

Note: 4.1. In the special values of the parameters λ and τ from Corollary 4.1 and Corollary 4.2, we obtain the results for the classes $S^{*}_{\Sigma,\sinh}\left(\tau\right)$, $S^{*}_{\Sigma,\sinh}\left(\lambda\right)$, $S^{*}_{\Sigma,\sinh}$ and $C_{\Sigma,\sinh}\left(\tau\right)$, $C_{\Sigma,\sinh}\left(\lambda\right)$, $C_{\Sigma,\sinh}$, respectively.

5. *Discussion*

In this study, we defined a new subclass of starlike and convex bi-univalent functions, which we will call pseudo-starlike and pseudo-convex function class. For this definition class, we examine some geometric properties, like as coefficient and Fekete-Szegö problem.

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