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CONTENTS

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Graphs with Total Domination Number Double of the Matching Number

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Abstract — A subset S of vertices of a graph G with no isolated vertex is called a total dominating set of G if each vertex of G has at least one neighbor in the set S. The total domination number $\gamma_t(G)$ of a graph G is the minimum value of the size of a total dominating set of G. A subset M of the edges of a graph G is called a matching if no two edges of M have a common vertex. The matching number $\nu(G)$ of a graph G is the maximum value of the size of a matching in G. It can be observed that $\gamma_t(G) \leq 2\nu(G)$ holds for every graph G with no isolated vertex. This paper studies the graphs satisfying the equality and proves that $\gamma_t(G) = 2\nu(G)$ if and only if every connected component of G is either a triangle or a star.

Keywords *Domination number, matching number, total domination number*

Mathematics Subject Classification (2020) 05C69, 05C70

1. Introduction

Graphs have various parameters, such as domination number, total domination number, matching number, and the minimum size of a maximal matching, denoted by γ, γ_t, ν, and ν^*, respectively. Obtaining equalities or inequalities between those parameters and classifying the graphs satisfying a given equality or inequality are widely studied. For instance, a well-known inequality is $\gamma_t(G) \leq 2\gamma(G)$. Characterization of all the graphs G with $\gamma_t(G) = 2\gamma(G)$ is still an open problem. However, the problem is solved for trees, block graphs, and chordal graphs in [1–3]. Another example of total domination numbers is that in any connected graph with at least three vertices, the total domination number is two-thirds of the graph's order [4]. The family of graphs G satisfying $\gamma_t(G) = \frac{2|V(G)|}{3}$ is completely determined in [5].

It is well known that the inequality $\gamma(G) \leq \nu(G)$ holds for every graph G. However, the inequality $\gamma_t(G) \leq \nu(G)$ is not always true. On the other hand, $\gamma_t(G) \leq \nu(G)$ is satisfied whenever G is a d-regular graph such that $d \geq 3$ or a claw-free graph with minimum degree more than two [6]. Furthermore, the inequality is also satisfied for the connected graphs with at least four vertices in which every vertex is contained in a triangle [7]. Claw-free graphs G with $\gamma_t(G) = \nu(G)$ and $\delta(G) \geq 3$ are determined in [8], whereas trees T satisfying $\gamma_t(T) \leq \nu(T)$ are characterized in [9].

Unlike the inequality $\gamma_t(G) \leq \nu(G)$, the inequality $\gamma_t(G) \leq 2\nu(G)$ is true for every graph G which does not contain any isolated vertex. Besides, $\gamma_t(G) \leq 2\nu^*(G)$ is always valid since the set of vertices

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in a maximal matching is a total dominating set. In [10], it is shown that if $\delta(G) \geq 3$, then $\gamma_t(G) \leq 2\nu^*(G) - \delta(G) + 2$ and if $\delta(G) \leq 2$, then $\gamma_t(G) \leq 2\nu^*(G)$. In the same paper, a characterization in a constructive way for the graphs G with $\gamma_t(G) = 2\nu^*(G)$ and $\delta(G) \leq 2$ is also provided.

In this paper, we focus on graphs G with $\gamma_t(G) = 2\nu(G)$. Recall that the inequality $\gamma_t(G) \leq 2\nu^*(G)$ is true when G does not include any isolated vertex. Then, since $\nu^*(G) \leq \nu(G)$ always holds, if $\gamma_t(G) = 2\nu(G)$, then $\nu^*(G) = \nu(G)$ which implies that every maximal matching in G has the same size. A graph whose maximal matchings have the same cardinality is called equimatchable. Therefore, the set of graphs we focus on is a subfamily of equimatchable graphs. For more about equimatchable graphs, see [11–15]. Furthermore, we show that if in a graph, the total domination number is equal to double the matching number, then it is a disjoint union of triangles or stars, that is, every connected component of a graph G satisfying $\gamma_t(G) = 2\nu(G)$ is either a triangle or a star.

This paper is organized as follows: Section 2 presents some definitions and notations to be needed for the following sections. Section 3 provides the main theorem and its proof. The final section presents a discussion and conclusions.

2. Preliminaries

In this section, we present some basic definitions, notations, and some simple observations which are frequently used throughout this paper.

A graph G is formed by two sets, namely, $V(G)$ and $E(G)$. Here, $V(G)$ is a nonempty set whose elements are called vertices, and $E(G)$ is a set consisting of unordered pairs of vertices whose elements are called edges. Whenever $\{u, v\} \in E(G)$, we say that u and v are adjacent (or neighbors). Throughout this paper, if u and v are adjacent in G, then we write $uv \in E(G)$ and say uv is an edge in G.

In a graph, the set of all the neighbors of a vertex v is denoted by $N(v)$, and the number of elements in $N(v)$ is called the degree of the vertex v. In a graph G, the minimum degree is denoted by $\delta(G)$. A vertex in a graph is isolated if it has no neighbors in the graph, i.e., its degree is zero. A vertex is called a leaf if its degree is one, i.e., it has a unique neighbor in the graph, and a vertex is said to be a support vertex whenever it is adjacent to a leaf.

A triangle, denoted by C_3, is a cycle of length three. A star is a graph in which a central vertex exists such that every other vertex is adjacent to only this central vertex. Figure 1 illustrates a triangle and two stars:

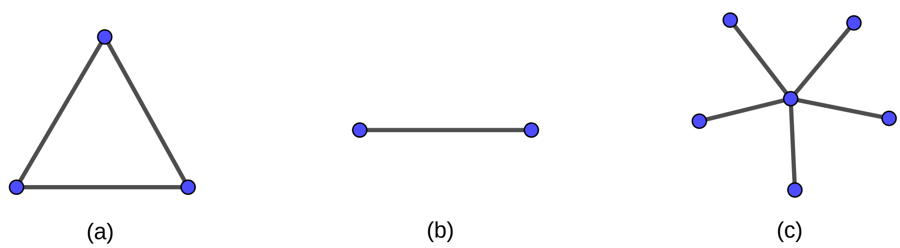

Figure 1. (a) A triangle, (b) a star with two vertices which is called K_2, and (c) a star with six vertices

A subset S of $V(G)$ is a dominating set of G if each vertex not in S has at least one neighbor belonging to S. The domination number $\gamma(G)$ of the graph G is the minimum size of a dominating set of G. If G has no isolated vertices, then a subset S of $V(G)$ is called a total dominating set of G whenever each vertex in G has at least one neighbor in S. In other words, S is a total dominating set if and only if S is a dominating set and the subgraph of G induced by S contains no isolated vertices. The total domination number of the graph G with no isolated vertices, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Notice that by definition, $\gamma(G) \leq \gamma_t(G)$. Note also that there is no total dominating set for a graph G with an isolated vertex; hence, the total domination number is undefined. Therefore, throughout this paper, we only consider graphs without isolated vertices.

If in a subset M of $E(G)$ no two edges share a common vertex, then M is a matching in G. For a matching M, the set of all the vertices serving as a vertex of an edge in M is denoted by $V(M)$. A matching is called maximal whenever it is not properly contained in another matching. The matching number of the graph G is the maximum size of a matching in G and is denoted by $\nu(G)$, $\alpha'(G)$, or $\mu(G)$. Let $\nu^*(G)$ denote the minimum cardinality of a maximal matching in G. A matching in G is maximum if its size is $\nu(G)$. Note that a maximum matching is maximal, but a maximal matching is not necessarily maximum. An example of maximal and maximum matchings is presented in Figure 2. Moreover, $\nu^*(G) \leq \nu(G)$ is always satisfied.

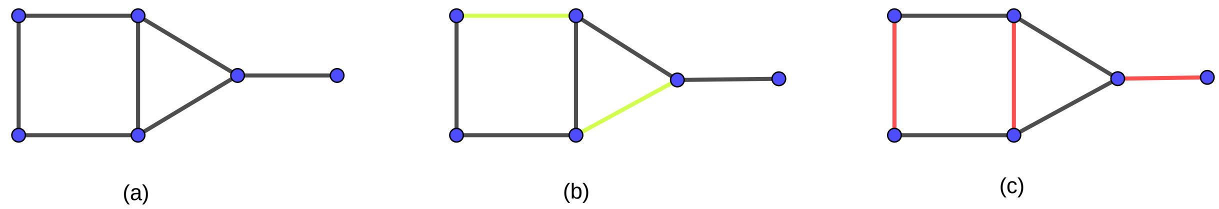

Figure 2. (a) A graph, (b) its maximal matching in yellow, and (c) its maximum matching in red

A path between vertices u and v of a graph G is a sequence of edges $v_1v_2, v_2v_3, \cdots, v_{k-1}v_k$ in G for some $k \geq 2$ where $v_1 = u$ and $v_k = v$. A graph is called connected if, for every pair of vertices, there exists a path between them. A connected component of a graph is a connected subgraph that is not contained in another connected subgraph. A subset of vertices in a graph is called independent if it has no two adjacent vertices.

Finally, we provide a simple observation frequently used in proofs: Let M be a maximal matching in a graph G. Then, since M is maximal, there is no edge in the subgraph of G induced by $V(G) \setminus V(M)$, that is, $V(G) \setminus V(M)$ is either empty or an independent set. In other words, $N(w) \subseteq V(M)$ for every $w \in V(G) \setminus V(M)$. Moreover, let $G_1, G_2, \cdots, G_n$ be all connected components of a graph G. Then,

$$\gamma_t(G) = \sum_{i=1}^{n} \gamma_t(G_i) \quad \text{and} \quad \nu(G) = \sum_{i=1}^{n} \nu(G_i)$$

As $\gamma_t(G_i) \leq 2\nu(G_i)$ is true for every $i \in \{1, 2, \cdots, n\}$, we see that $\gamma_t(G) = 2\nu(G)$ holds if and only if $\gamma_t(G_i) = 2\nu(G_i)$ is valid for every $i \in \{1, 2, \cdots, n\}$. Therefore, characterizing all the connected graphs G with $\gamma_t(G) = 2\nu(G)$ is sufficient to solve our main problem.

3. Main Result

In this section, we determine all the graphs G with $\gamma_t(G) = 2\nu(G)$. Characterizations of such graphs are presented in the following theorem.

Theorem 3.1. Let G be a graph. Then, $\gamma_t(G) = 2\nu(G)$ holds if and only if every connected component of G is a triangle or a star.

Throughout this section, we provide the proof of Theorem 3.1. We first present a lemma, which is frequently used in the rest of this section.

Lemma 3.2. Let G be a graph with $\gamma_t(G) = 2\nu(G)$, M be a maximum matching in G, and $ab \in M$. If a is not a support vertex, then a is the unique neighbor of b among the vertices in $V(M)$.

Proof. We prove the claim by contradiction. Let $S = V(M) \setminus \{a\}$ and assume that b is adjacent to a vertex c in S. An illustration of G, M, and S is given in Figure 3:

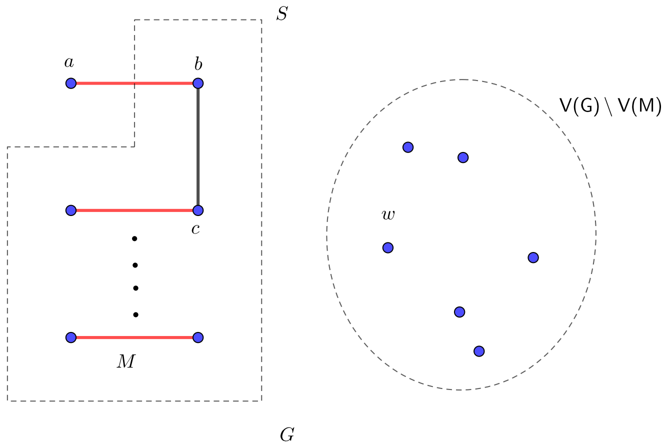

Figure 3. A graph G and a matching M, shown by red edges. The sets S and $V(G) \setminus V(M)$ consist of vertices inside the dashed polygonal region and elliptic region, respectively

Recall that for any vertex w in $V(G) \setminus V(M)$, $N(w) \subseteq V(M)$. Since a is not a support vertex, any vertex adjacent to a is not a leaf and has a neighbor other than a. Therefore, w has at least one neighbor in S. Further, it can be observed that any vertex in $V(M)$ has at least one neighbor in S. Therefore, S is a total dominating set and

$$\gamma_t(G) \leq |S| = |V(M)| - 1 = 2\nu(G) - 1$$

which contradicts with $\gamma_t(G) = 2\nu(G)$. □

We study the graphs concerning their minimum degrees. We begin with the case when the minimum degree is more than one.

Proposition 3.3. C_3 is the unique connected graph G satisfying $\gamma_t(G) = 2\nu(G)$ and $\delta(G) \geq 2$.

Proof. Let G be a connected graph satisfying the conditions $\gamma_t(G) = 2\nu(G)$ and $\delta(G) \geq 2$. We first show that $\nu(G) = 1$. Assume that $\nu(G) \geq 2$. Let $M = \{e_1, e_2, \cdots, e_k\}$ be a maximum matching where $k = \nu(G) \geq 2$. Since the minimum degree in G is at least 2, there is no leaf in G. Therefore, there is no support vertex in G either. Thus, by Lemma 3.2, any vertex in $V(M)$ has exactly one neighbor in $V(M)$. In other words, the subgraph of G induced by $V(M)$ is a disjoint union of k edges. Since G is connected, a vertex w in $V(G) \setminus V(M)$ must exist such that w has neighbors from different edges in M. Without loss of generality, suppose that $e_1 = xy$, $e_2 = zt$, and w is adjacent to y and t. Consider the edge set $M' = (M \setminus \{xy\}) \cup \{yw\}$. Then, M' matches, and because of its size, it is a maximum matching. However, as w is adjacent to t, we get a contradiction when we apply Lemma 3.2 for M', $a = y$, and $b = w$. Consequently, we see that the matching number of G is 1.

Let uv be any edge of G. Then, $\{uv\}$ is a maximum matching. Hence, since the minimum degree is two, any vertex different than u and v is a common neighbor of u and v. Thereby, G must have at least three vertices. If G has three vertices, then G has to be C_3 and thus $\gamma_t(C_3) = 2\nu(C_3) = 2$. Otherwise, let w_1 and w_2 be two distinct vertices other than u and v. Then, $\{uw_1, vw_2\}$ is a matching which yields $\nu(G) \geq 2$, a contradiction. Thereby, C_3 is the unique (up to isomorphism) connected graph G with $\gamma_t(G) = 2\nu(G)$ and $\delta(G) \geq 2$. □

We next analyze the graphs with a minimum degree of one.

Proposition 3.4. Let G be a connected graph with $\delta(G) = 1$. Then, $\gamma_t(G) = 2\nu(G)$ if and only if G is a star.

PROOF. Let G be a connected graph satisfying $\gamma_t(G) = 2\nu(G)$ and $\delta(G) = 1$. First, note that if $G = K_2$, then it is a star. Suppose that G is not K_2. Observe that K_2 is the unique connected graph containing a vertex, a leaf, and a support vertex. Thus, G has no such a vertex. Let M be a maximum matching in G and $uv \in M$. We first show that at least one of u and v is not a support vertex. Assume that u and v are support vertices. Then, u is adjacent to a leaf x, and v is adjacent to a leaf y. By the observation above, $\{x, y\} \cap \{u, v\} = \emptyset$. Moreover, since they are leaves, $x \neq y$ and none of x and y can be another vertex in $V(M)$. Then, note that $(M \setminus \{uv\}) \cup \{ux, vy\}$ is matching whose size is greater than the size of M contradicting with the fact that M is a maximum matching.

Let $v_1, v_2, \cdots, v_m$ be all the support vertices in G. For each v_i, choose a neighbor leaf u_i. Thus, $\{u_1v_1, \cdots, u_mv_m\}$ is a matching and can be extended to a maximal matching M. As $\nu^*(G) = \nu(G)$, then M is a maximum matching. Since u_i cannot be a support vertex, by Lemma 3.2, $N(v_i) \cap V(M) = \{u_i\}$ holds for every $i \in \{1, 2, \cdots, m\}$. Suppose that $M \setminus \{u_1v_1, \cdots, u_mv_m\}$ is not empty and equal to $\{x_1y_1, \cdots, x_ry_r\}$. By construction none of x_1, y_1, $\cdots$, x_r, and y_r is a support vertex and hence, by Lemma 3.2, $N(x_i) \cap V(M) = \{y_i\}$ and $N(y_i) \cap V(M) = \{x_i\}$, for every $i \in \{1, 2 \cdots, r\}$. Therefore, since G is connected and $V(G) \setminus V(M)$ is an independent set, there exists a vertex $w \in V(G) \setminus V(M)$ such that w is a common neighbor of a vertex from $\{u_1, v_1, \cdots, u_m, v_m\}$ and a vertex from $\{x_1, y_1, \cdots, x_r, y_r\}$. Note that w cannot be adjacent to some u_i since u_i is a leaf. Without loss of generality, suppose that w is adjacent to v_1 and y_1. Then, $M' = (M \setminus \{u_1v_1\}) \cup \{wv_1\}$ is a maximum matching because of its size. Applying Lemma 3.2 for M', $a = x_1$, and $b = y_1$ yields a contradiction. Consequently, $M = \{u_1v_1, \cdots, u_mv_m\}$ and $\{v_1, \cdots, v_m\}$ is an independent set.

We finally show that $m = 1$. Assume that $m \geq 2$. By similar ideas above, there exists a vertex $w \in V(G) \setminus V(M)$, which is adjacent to at least two of $v_1, \cdots, v_m$. Without loss of generality, suppose that v_1 and v_2 are neighbors of w. Then, $M' = (M \setminus \{u_1v_1\}) \cup \{wv_1\}$ is a maximum matching since $|M'| = |M| = \nu(G)$. Therefore, as v_2 and w are adjacent, we obtain a contradiction by applying Lemma 3.2 for M', $a = u_2$, and $b = v_2$. Thus, $m = 1$ and every vertex other than v_1 is a leaf and adjacent to v_1, which implies that G is a star.

Conversely, if G is a star graph, it is connected, has minimum degree one, and satisfies $\gamma_t(G) = 2\nu(G) = 2$. □

Finally, combining Propositions 3.3 and 3.4 proves Theorem 3.1.

4. Conclusion

In this paper, we have studied the graphs G whose total domination number attains the upper bound in the inequality $\gamma_t(G) \leq 2\nu(G)$. We have shown that the family of graphs whose each connected component is a triangle or a star is the set of all the graphs G satisfying $\gamma_t(G) = 2\nu(G)$. Since $\gamma_t(G) = 2\nu(G)$ implies $\nu^*(G) = \nu(G) = \frac{\gamma_t(G)}{2}$, we have obtained an extreme condition on the graphs we study, and hence, probably that is why we have not reached an interesting or large connected graph that satisfies the equality. A potential research direction is to determine all the graphs G satisfying $2\nu(G) - 1 = \gamma_t(G)$ or $2\nu(G) - 2 = \gamma_t(G)$. Notice that the method to solve the main theorem does not work. However, if $2\nu(G) - 1 = \gamma_t(G)$, then $\nu^*(G) = \nu(G)$ since $\gamma_t(G) \leq 2\nu^*(G) \leq 2\nu(G)$ and the values $\nu^*(G)$ and $\nu(G)$ are integers. Therefore, the class of graphs G with $\gamma_t(G) = 2\nu(G) - 1$ is a subfamily of equimatchable graphs as well, and hence, results on equimatchable graphs can be helpful to determine all graphs in that class. Another research direction might be to obtain an inequality involving matching and total domination numbers on various specific graph classes, such as regular, bipartite, split, and chordal graphs.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Parabolic Numbers: A New Perspective

Furkan Semih Dündar[1]

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Abstract — Thus far, many studies have been conducted on p-complex numbers. Depending on the sign of p, there are three cases: hyperbolic, dual, and elliptic. In the literature, dual numbers are called parabolic numbers, but they do not parameterize parabolas. Therefore, a number system that parameterizes parabolas is worth studying. This paper defines p as a function of the coordinate y and obtains a number system named parabolic numbers whose circles are parabolas. These parabolic numbers complete the set of number systems where circles are conic sections. Finally, this paper discusses the prospect of further research.

Keywords *Parabolic numbers, p-complex numbers, coordinate dependence*

Mathematics Subject Classification (2020) 15A66, 11H55

1. Introduction

The introduction of complex numbers in the form $z = x+iy$ with $i^2 = -1$ to generalize real numbers has had many critical applications from the fundamental theorem of algebra to advanced physics, such as the calculation of Feynman diagrams [1], needed in quantum field theory, and Bohmian interpretation of quantum mechanics [2] as well as the usual quantum mechanics [3]. Then, a generalization of complex numbers to p-complex numbers has been studied. For more details, see [4]. For p-complex numbers, p is defined via $i^2 = p$ where p can be negative, positive, or zero. These classes of number systems are called elliptic, hyperbolic, and dual, respectively. This nomenclature arises because, in these number systems, the circles, defined by the set of z where $|z|^2$ is constant, correspond to ellipses, hyperbolas, and two vertical lines. The dual numbers are also called parabolic numbers; however, they are quite distinct from our novel perspective on parabolic numbers in this study.

Elliptic numbers, to represent elliptical orbits in the central force problem of Newtonian gravity, have been studied in [5]. The case of hyperbolic orbits, however, has not yet been explored. It is feasible to extend the framework for elliptic numbers to hyperbolic numbers. What is missing is a new perspective on parabolic numbers, whose circles would correspond to parabolas. While dual numbers are also called parabolic, they do not parameterize a parabola, unlike the approach proposed in this study. Hence, the term is a new perspective. This paper introduces a number system based on hyperbolic numbers, where $p > 0$ is treated as a function of the y-coordinate. From the mathematical point of view, the results of the current study completes the list of number systems where circles are conic sections. It is hoped that the results provided here interest researchers.

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The study [6] considers l_p-complex numbers where the norm of an l_p-complex number z is given by $|z|_p \equiv (|x|^p + |y|^p)^{1/p}$ for a constant and positive number p. Here, for $p = 2$, it can be observed that the mentioned number system is the usual complex numbers. However, the number system [6] cannot parameterize parabolas since the exponent of $|x|$ and $|y|$ are the same, and their coefficients are positive, namely 1. Moreover, the distributive law does not hold unless $p = 2$ [6]. The number system defined in Section 2 named parabolic numbers has distributivity property. This is a clear advantage for parabolic numbers defined in this study.

Furthermore, in [7], the norm of a vector $r \in \mathbb{R}^3$ is defined as follows:

$$|r| \equiv \frac{|x|^{p_1}}{p_1} + \frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3}$$

for positive real numbers p_1, p_2, and p_3. Although this approach has one more dimension, it cannot describe parabolas. The reason is the same as that of the earlier work: The coefficients of $|x|$, $|y|$, and $|z|$ are positive. If, for instance, p_2 is made negative, then one has the correct sign. However, $|y|$ appears in the denominator with a positive power. Hence, a parabola can still not be parameterized even though negative p_i values are allowed.

The paper's organization is as follows: Section 2 defines generalized p-complex numbers with a coordinate dependence on p. Section 3 provides details about the properties of parabolic numbers. Section 4 offers a few ideas for applying parabolic numbers. Finally, Section 5 concludes the paper.

2. Generalized p-Complex Numbers

This section briefly mentions p-complex numbers and generalizes it by making p coordinate-dependent.

2.1. p-Complex Numbers

In the literature, p-complex numbers ($\mathbb{C}_p$) are defined via $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = p$. For $p > 0$, these are referred to as hyperbolic numbers; for $p < 0$, as elliptic numbers; and for $p = 0$, as dual numbers. These numbers systems have been named as such because the constant norm of z, defined via $|z|^2 = zz^* = (x + iy)(x - iy) = x^2 - py^2$, corresponds to hyperbola ($p > 0$), ellipses ($p < 0$), and two vertical lines in the last case ($p = 0$). The concept of squared norm, defined by $|z|^2$, varies depending on the value of p: 1) It is Lorentzian for $p > 0$, which means it may assume any sign or be zero; 2) It is Euclidean for $p < 0$; and 3) It becomes a pseudo-norm-squared for $p = 0$, which is always nonnegative. p-complex numbers and their generalizations are widely studied in the literature [4, 8–15]. For an introduction to p-complex numbers, [16] may be a good reference. Additionally, there are hypercomplex numbers, where the non-real unit u satisfies $u^2 = \alpha + u\beta$ for some $\alpha, \beta \in \mathbb{R}$ [17], a generalization of p-complex numbers. In terms of hypercomplex numbers, [18] might be a valuable source. However, none of these systems include coordinate-dependent p.

As a direct application, the idea to define $i^2 = 1$, $i \notin \mathbb{R}$ is relevant in Einstein's special theory of relativity, where space-time has Lorentzian geometry instead of a Euclidean one. Hyperbolic numbers, defined as the set of numbers $\{z = x + iy : x, y \in \mathbb{R}, i^2 = 1, i \notin \mathbb{R}\}$, can model 2D Minkowski space-time. Because the norm-square of the distance vector between two points P_1 and P_2 denoted by $\vec{v} = (t, x)$ in 2D Minkowski space-time is given as $|\vec{v}|^2 = t^2 - x^2 = |t + ix|^2$ if the norm-square is zero, then the vector $\vec{v}$ is null or light-like; if it is positive, then the vector $\vec{v}$ is time-like, and if it is negative, then the vector $\vec{v}$ is space-like. The relation of hyperbolic numbers to the special theory of relativity is also noted in [15], which cites [19]. The book [20] might also be useful for readers who would like to learn more about the relation between hyperbolic numbers and the special theory of relativity. On the contrary, if $i \in \mathbb{R}$ is assumed, then the definition of a number z in the form $z = x + iy = x \pm y$ is reduced to a summation or subtraction operation on real numbers and would not, for example, yield the space-time structure in the special theory of relativity in 2D.

2.2. Coordinate Dependent p Value

When p is constant, the set of p-complex numbers whose norm is constant cannot represent a parabola. The squared norm of a p-complex number is quadratic in x, y when $p \neq 0$, and when $p = 0$, the unit circle is not a parabola.

The approach to defining a number system in which a circle is a parabola motivates the coordinate dependence of p. This topic should be investigated in the general setting where $p = p(x, y)$. However, the goal of this study is to define parabolic numbers. To this end, i and j are defined by $i^2 = 1$ and $j^2 = p = p(y) = \frac{1}{\alpha|y|}$, for $\alpha > 0$. Hence, $j = \frac{i}{\sqrt{\alpha|y|}}$. Here, i, the hyperbolic unit, is a square root of 1 but is not a real number; it is used to express the coordinate dependence of j. The number i will be useful in expressing a parabolic number in hyperbolic form, especially in the next section.

3. Properties of Parabolic Numbers

In this section, the algebraic operations on parabolic numbers are elaborated. A parabolic number z is expressed as $z = x + jy$, where the explicit form of j is utilized to represent z as follows:

$$z = x + jy = x + i\frac{y}{\sqrt{\alpha|y|}} = x + i\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}$$

where $y = \text{sgn}(y)|y|$ and $\sigma = \text{sgn}(y)$. The value $\sigma_n = \text{sgn}(y_n)$ is defined provided y has a subscript. Here, sgn is the sign function. In the remainder of this section, the following definition is used:

$$z_n \equiv x_n + i\frac{\sigma_n\sqrt{|y_n|}}{\sqrt{\alpha}}$$

and the symbol j is omitted.

3.1. Addition

The sum of two parabolic numbers is defined as follows:

$$\begin{aligned} z_1 \oplus z_2 &= \left(x_1 + i\frac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \oplus \left(x_2 + i\frac{\sigma_2\sqrt{|y_n|}}{\sqrt{\alpha}}\right) \\ &\equiv x_1 + x_2 + i\frac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}} + i\frac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} \\ &= x_3 + i\frac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}} \end{aligned}$$

If $z_3 = z_1 \oplus z_2$, the following are obtained:

$$x_3 \equiv x_1 + x_2$$

and

$$\sigma_3\sqrt{|y_3|} \equiv \sigma_1\sqrt{|y_1|} + \sigma_2\sqrt{|y_2|}$$

It can be observed that the addition operation is closed on $\mathbb{R}^2$ and commutative and associative.

3.2. Multiplication

The multiplication of two parabolic numbers is defined as follows:

$$\begin{aligned} z_1 \otimes z_2 &= \left(x_1 + i\frac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \otimes \left(x_2 + i\frac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}}\right) \\ &= x_1x_2 + \frac{\sigma_1\sigma_2}{\alpha}\sqrt{|y_1|}\sqrt{|y_2|} + \frac{i}{\sqrt{\alpha}}(x_1\sigma_2\sqrt{|y_2|} + x_2\sigma_1\sqrt{|y_1|}) \end{aligned}$$

If $z_3 = z_1 \otimes z_2$, then the following are obtained:

$$x_3 \equiv x_1x_2 + \frac{\sigma_1\sigma_2}{\alpha}\sqrt{|y_1|}\sqrt{|y_2|} \quad \text{and} \quad \sigma_3\sqrt{|y_3|} \equiv x_1\sigma_2\sqrt{|y_2|} + x_2\sigma_1\sqrt{|y_1|}$$

It can be observed that the multiplication operation is closed on $\mathbb{R}^2$. It is obvious that the multiplication is commutative; however, more care is needed to show associativity. The expression $(z_1 \otimes z_2) \otimes z_3$ is as follows:

$$\begin{aligned} (z_1 \otimes z_2) \otimes z_3 = {} & x_1x_2x_3 + \tfrac{1}{\alpha}(x_1\sigma_2\sigma_3\sqrt{|y_2|}\sqrt{|y_3|} + x_2\sigma_1\sigma_3\sqrt{|y_1|}\sqrt{|y_3|} + x_3\sigma_1\sigma_2\sqrt{|y_1|}\sqrt{|y_2|}) \\ & + \tfrac{i}{\sqrt{\alpha}}(x_1x_2\sigma_3\sqrt{|y_3|} + x_1x_3\sigma_2\sqrt{|y_2|} + x_2x_3\sigma_1\sqrt{|y_1|}) + \tfrac{i}{\alpha^{3/2}}\sigma_1\sigma_2\sigma_3\sqrt{|y_1|}\sqrt{|y_2|}\sqrt{|y_3|} \end{aligned} \tag{3.1}$$

Using the commutativity of multiplication, $(z_1 \otimes z_2) \otimes z_3 = z_3 \otimes (z_1 \otimes z_2)$ can be written. When the indices in (3.1) are mapped via $(1, 2, 3) \mapsto (2, 3, 1)$ and it is observed that the expression is invariant, the associativity of multiplication is proven.

3.3. Complex Conjugation and Division

The complex conjugate of a parabolic number is defined by $z^* \equiv x - i\sigma\sqrt{|y|/\alpha}$, for all $z = x + i\sigma\sqrt{|y|/\alpha}$. If the norm of z is non-zero, the multiplicative inverse of z is defined as $z^{-1} \equiv \frac{z^*}{|z|^2}$, although the norm may be negative. In other words, if $|z| \neq 0$, then $\frac{1}{z} \equiv \frac{z^*}{|z|^2}$ is defined.

3.4. Distributive Property

The distributive property for three parabolic numbers is the equality $z_1 \otimes (z_2 \oplus z_3) = (z_1 \otimes z_2) \oplus (z_1 \otimes z_3)$. For $n \in \{1, 2, 3\}$, let $z_n = x_n + i\frac{\sigma_n\sqrt{|y_n|}}{\sqrt{\alpha}}$. Then, $z_2 \oplus z_3$ is calculated as follows:

$$z_2 \oplus z_3 = (x_2 + x_3) + i\left(\frac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + \frac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}}\right)$$

Hence,

$$\begin{aligned} z_1 \otimes (z_2 \oplus z_3) &= \left(x_1 + i\tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \otimes \left[(x_2 + x_3) + i\left(\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + \tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}}\right)\right] \\ &= x_1(x_2 + x_3) + \tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\left(\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + \tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}}\right) \\ &\quad + i\left[x_1\left(\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + \tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}}\right) + (x_2 + x_3)\tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right] \\ &= x_1x_2 + \tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + i\left(x_1\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + x_2\tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \\ &\quad + x_1x_3 + \tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}} + i\left(x_1\tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}} + x_3\tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \\ &= (z_1 \otimes z_2) \oplus (z_1 \otimes z_3) \end{aligned}$$

This proves the distributive property on parabolic numbers. Therefore, the equality $(z_2 \oplus z_3) \otimes z_1 = (z_2 \otimes z_1) \oplus (z_3 \otimes z_1)$ is also valid due to commutativity of the multiplication on parabolic numbers.

3.5. Euler's Formula for Parabolic Numbers

In this subsection, Euler's formula is generalized to parabolic numbers. For the case of p-complex numbers, see [16]. For a parabolic number z, the expression e^z is calculated. If z is written in hyperbolic representation, then $z = x + jy = x + i\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}$ is valid. Hence, $e^z = e^{x+i\beta}$ where $\beta = \frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}$. Because the real and imaginary part of $x + i\beta$ commutes, we have $e^z = e^x e^{i\beta}$. The expression $e^{i\beta}$ is calculated as follows:

$$e^{i\beta} = \sum_{n=0}^{\infty} \frac{(i\beta)^n}{n!} = \sum_{n=0}^{\infty} \frac{\beta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{\beta^{2n+1}}{(2n+1)!} = \cosh(\beta) + i\sinh(\beta)$$

Hence, the following is obtained:

$$e^z = e^{x+jy} = e^x\left[\cosh\left(\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}\right) + i\sinh\left(\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}\right)\right]$$

A simplification comes from the fact that $\sigma \in \{-1, 0, 1\}$, cosh is an even function and sinh is an odd function:

$$e^z = e^{x+jy} = e^x\left[\cosh\left(\frac{\sqrt{|y|}}{\sqrt{\alpha}}\right) + i\sigma\sinh\left(\frac{\sqrt{|y|}}{\sqrt{\alpha}}\right)\right]$$

The previous expression is in hyperbolic representation. Its parabolic representation can be obtained, as well. For that purpose, define $a + jb = \cosh(\beta) + i\sinh(\beta)$. It is observed that $a = \cosh(\beta)$ and $\sinh(\beta) = \frac{\sigma_b\sqrt{|b|}}{\sqrt{\alpha}}$. When the last equality is solved for b, $b = \alpha\sigma_b\sinh^2(\beta)$. From the expression $\sinh(\beta) = \frac{\sigma_b\sqrt{|b|}}{\sqrt{\alpha}}$, $\sigma_b = \sigma$. Hence,

$$\begin{aligned} e^{jy} &= \cosh(\beta) + j\alpha\sigma\sinh^2(\beta) \\ &= \cosh\left(\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}\right) + j\alpha\sigma\sinh^2\left(\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}\right) \\ &= \cosh\left(\frac{\sqrt{|y|}}{\sqrt{\alpha}}\right) + j\alpha\sigma\sinh^2\left(\frac{\sqrt{|y|}}{\sqrt{\alpha}}\right) \end{aligned}$$

3.6. Flatness of the Parabolic Number Manifold

Using the norm of $|z|^2 = x^2 - \frac{y^2}{\alpha|y|}$, for $y > 0$, (the case $y < 0$ is straightforward), the metric is defined via the line element:

$$\mathrm{d}s^2 = \mathrm{d}x^2 - \frac{\mathrm{d}y^2}{\alpha y} \tag{3.2}$$

The line element at a point (x, y) is defined as the infinitesimal distance between the points (x, y) and $(x + \mathrm{d}x, y + \mathrm{d}y)$. Hence, the Lorentzian norm-square of the number $\mathrm{d}x + j\mathrm{d}y$ is evaluated at the point (x, y). This fact justifies the line element defined in (3.2). When $\xi = 2\sqrt{\frac{y}{\alpha}}$ is defined, the line element can be written as follows:

$$\mathrm{d}s^2 = \mathrm{d}x^2 - \mathrm{d}\xi^2 \tag{3.3}$$

Thus, the Riemann tensor vanishes, and the manifold of parabolic numbers is trivially flat. Moreover, the parabolic number set is isomorphic to 2D Minkowski space-time. This is expected since there is a one-to-one map between parabolic and hyperbolic numbers. To observe this, a map defined by $y \mapsto \xi = 2\sqrt{\frac{y}{\alpha}}$ such that $y > 0$ is one-to-one. The case $y < 0$ is similar, and for $y = 0$, define $\xi = 0$, where the line element in (3.3) is that of hyperbolic numbers.

If $p = p(y)$ only depends on the y variable, then the line element can be transformed via $\mathrm{d}s^2 =$

$dx^2 - p(y)dy^2$ into the form:

$$ds^2 = dx^2 - dt^2$$

where $t = \int dy\sqrt{p(y)}$, which again results in a flat manifold. However, if $p = p(x, y)$, which is not investigated in this study, then there may be curvature in the manifold, which is not the case for parabolic numbers. For example, consider the case $p(x, y) = \sin^2(x)$. Then, the manifold's Ricci scalar, the only degree of freedom in 2D, is $R = 2$. It is another problem whether $p(x, y) = \sin(x)^2$ defines a consistent number system. Consequently, the number manifold may be non-flat depending on $p(x, y)$.

4. A Few Applications

A circle in parabolic numbers is a parabola given by:

$$|z|^2 = x^2 - \frac{|y|}{\alpha} = \frac{A}{\alpha}$$

where $A \in \mathbb{R}$. Hence,

$$|y| = \alpha x^2 - A$$

For an illustration, see Figure 1 drawn with Mathematica 13.3. From Figure 1, note that when $A = 1$, there is no $y \in \mathbb{R}$ such that $|y| = x^2 - 1$, for $|x| < 1$. Hence, the domain of the parabola as a function of x is $\mathbb{R} - (-1, 1)$.

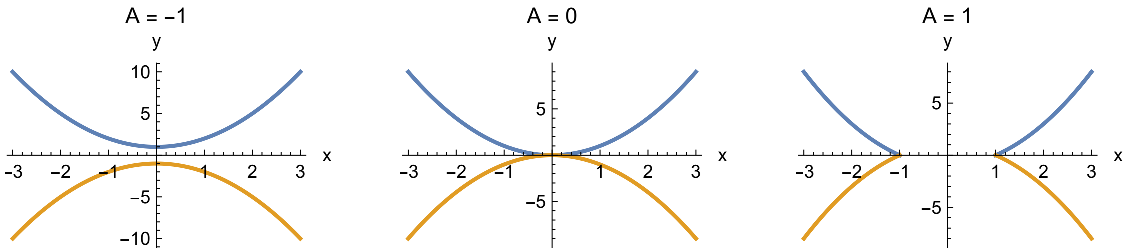

Figure 1. Some circles in parabolic numbers where $\alpha = 1$ and $A \in \{-1, 0, 1\}$

Any parabola can be expressed in this form through rotation, translation, and scaling. In the central force problem of Newtonian gravity, there are three types of trajectories: 1) Elliptic, 2) Hyperbolic, and 3) Parabolic. In [5], elliptical complex numbers where $p < 0$ and p is constant are used to model elliptic trajectories in the central force problem of Newtonian gravity. The case of hyperbolic trajectories can be approached using a similar method. Only the hyperbolic numbers where $p > 0$ are needed instead of elliptic numbers. However, it has not been studied yet. With the parabolic numbers introduced in this paper, parabolic trajectories can finally be parameterized. Another application involves projectile motion. Without air friction, the trajectory of a projectile is a parabola. Moreover, the trajectory of a charged particle under a constant electric field is also a parabola if the particle has a velocity component perpendicular to the electric field. An example of this is as follows: Consider a trajectory such as $|z|^2 = 0$. This results in $|y| = \alpha x^2$ and thus $y = -\alpha x^2$ such that $y \leq 0$. The equations of motion for an electron under constant electric field are:

$$m\ddot{x} = 0 \quad \text{and} \quad m\ddot{y} = qE$$

where $E > 0$ is the electric field and $q < 0$ is the charge of the electron. When these differential equations are integrated, the following two results are obtained:

$$x(t) = v_{0x}t + x_0$$

and

$$y(t) = \frac{1}{2}\frac{qE}{m}t^2 + v_{0y}t + y_0$$

Using the relation $y = -\alpha x^2$ and by expressing t as a function of x, the value α can be obtained as follows:

$$\alpha = -\frac{1}{2}\frac{qE}{mv_{0x}^2}$$

where $\alpha > 0$ since $q < 0$. This information determines the path's shape. Similarly, by applying the initial conditions and using specific values for q, m, and E, the position of an electron as a function of time can be determined. To illustrate, the values of the initial conditions, along with q, m, and E, can be chosen such that the numerical value of α equals 1/2 in the corresponding units. Figure 2 drawn with Mathematica 13.3 illustrates the electron's trajectory under a constant electric field, which is shown via arrows.

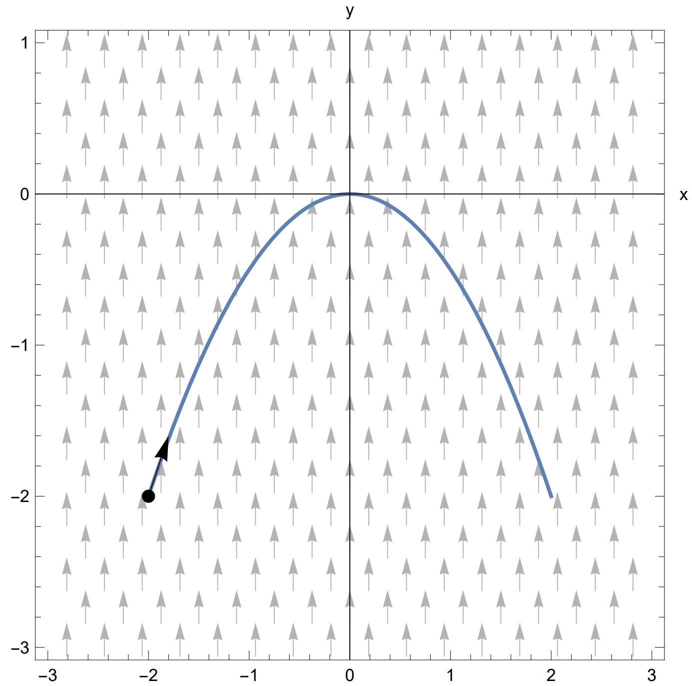

Figure 2. Trajectory of an electron under constant electric field where the numerical value of α is $\frac{1}{2}$

5. Conclusion

Elliptic numbers parameterize ellipses, and hyperbolic numbers parameterize hyperbola. However, there has not been a number system that parameterizes parabola. Through the number system introduced in this paper, parabolic numbers, a type of hyperbolic number where the imaginary unit has a specific coordinate dependence and is distinct from dual numbers, parabolas can be parameterized. The paper is the first study in the available literature considering the coordinate dependence of p. The choices of $p = p(x, y)$ in the more general setting and respective consistency relations are left to future studies.

A few other ideas that may be considered in future studies can be summarized as follows: 1) Whether a sign changing and coordinate dependent p can be consistently defined; 2) What would be the curvature the manifold on which coordinate-dependent p-complex numbers are defined; and 3) Whether it could be generalized to fours dimensions, such as modifying the quaternion algebra, where p_1, p_2, and p_3 are coordinate dependent (for more details on generalized quaternions, see [21]). The study [7] introduces the three-complex numbers system in which p_1, p_2, and p_3 are positive. In this approach, coordinate-dependent p_1, p_2, and p_3 can also be studied.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Certain Results on Extended Beta and Related Functions Using Matrix Arguments

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Abstract — In this study, we present and explore extended beta matrix functions (EBMFs) and their key properties. By utilizing the beta matrix function (BMF), we introduce novel extensions of the Gauss hypergeometric matrix function (GHMF) and Kummer hypergeometric matrix function (KHMF). We delve into their integral representations, recurrence relations, transformation properties, and differential formulas. Additionally, we investigate their statistical applications, mainly focusing on the beta distribution, and derive expressions for the mean, variance, and moment-generating functions. Furthermore, we apply EBMFs to develop the Appell matrix function (AMF) and Lauricella matrix function (LMF) and their integral forms.

Keywords *Beta matrix function, Gauss and Kummer hypergeometric matrix functions, Appell and Lauricella matrix functions*

Mathematics Subject Classification (2020) 33B15, 33E20

1. Introduction

Special matrix functions are a dynamic and intriguing area [1–14] with significant applications in mathematics and physics. When these functions are generalized from scalar to matrix arguments, they offer deeper insights and broaden the scope of their applications. Matrix versions of special functions enhance the utility of their scalar counterparts by extending their relevance to multidimensional and more complex problems. This generalization plays a crucial role in engineering, physics, statistics, and mathematics fields, providing powerful tools for addressing matrix-related challenges and advancing theoretical and practical research. Special matrix functions represent a critical extension of classical special function theory, enabling matrices to be manipulated in ways similar to numbers. This capability proves particularly valuable in applications of fields such as quantum mechanics, statistical mechanics, and signal processing, where matrices are frequently encountered.

The extended beta function is a matrix version of the classical beta function, which arises in various areas of mathematics and physics. Recent studies [1–3, 10, 11, 15] have focused on analyzing the matrix beta function and exploring its convergence regions, integral representations, and differential properties. Similarly, the extended Gauss hypergeometric and Kummer hypergeometric functions are matrix generalizations of their classical counterparts and have been the subject of considerable study in recent years [1–3, 7, 10, 15, 16]. Building on these foundational works, this paper discusses

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the extended beta matrix functions (EBMFs) and their integral representations, recurrence relations, transformation formulas, and differential properties. We also research their applications in statistics. We also define and investigate the integral representations of the extended Appell matrix function (EAMF) and the extended Lauricella matrix function (ELMF).

2. Preliminaries

Throughout this paper, the vector space of r-square matrices with complex entries is designated $\mathbb{C}^{r\times r}$. Spectrum is the set of all the eigenvalues of a matrix $\mathcal{P}\in\mathbb{C}^{r\times r}$ and represented by the symbol $\sigma(\mathcal{P})$. A matrix $\mathcal{P}$ in $\mathbb{C}^{r\times r}$ is called a positive stable matrix (PSM) if $\Re(\lambda)>0$, for all $\lambda\in\sigma(\mathcal{P})$, where $\Re(z)$ represents the real part of a complex number z.

The expression $\Gamma(\mathcal{P})$ for a PSM $\mathcal{P}$ in $\mathbb{C}^{r\times r}$ is as follows [11]:

$$\Gamma(\mathcal{P})=\int_0^\infty e^{-\ell}\ell^{\mathcal{P}-I}d\ell$$

Furthermore, if $\mathcal{P}+\kappa I$ is invertible, for all $\kappa\in\mathbb{Z}^+\cup\{0\}$, then the reciprocal gamma matrix function (GMF) is defined as [11]:

$$\Gamma^{-1}(\mathcal{P})=\mathcal{P}(\mathcal{P}+I)\cdots(\mathcal{P}+(n-1)I)\Gamma^{-1}(\mathcal{P}+nI),\quad n\geq 1$$

If $\mathcal{P}\in\mathbb{C}^{r\times r}$ is a PSM and $n\geq 0$ is an integer, then the GMF can also be defined in the form of a limit as [11]:

$$\Gamma(\mathcal{P})=\lim_{n\to\infty}(n-1)!(\mathcal{P})_n^{-1}n^{\mathcal{P}}$$

The Pochhammer symbol [12] for $\mathcal{P}\in\mathbb{C}^{r\times r}$ is defined as:

$$(\mathcal{P})_n=\begin{cases} I, & n=0\\ \mathcal{P}(\mathcal{P}+I)...(\mathcal{P}+(n-1)I), & n\geq 1\end{cases}$$

Therefore,

$$(\mathcal{P})_n=\Gamma^{-1}(\mathcal{P})\Gamma(\mathcal{P}+nI),\quad n\geq 1$$

If $\mathcal{P}$ and $\mathcal{Q}$ are PSMs in $\mathbb{C}^{r\times r}$ and $\mathcal{P}\mathcal{Q}=\mathcal{Q}\mathcal{P}$, then the beta matrix function (BMF) is defined as [11]:

$$\mathscr{B}(\mathcal{P},\mathcal{Q})=\Gamma(\mathcal{P})\Gamma(\mathcal{Q})\Gamma^{-1}(\mathcal{P}+\mathcal{Q})=\int_0^1\ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}d\ell \tag{2.1}$$

Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{H}$ be PSMs in $\mathbb{C}^{r\times r}$ and $\mathcal{H}+\kappa I$ be invertible, for all $\kappa\in\mathbb{Z}^+\cup\{0\}$. Then, the Gauss hypergeometric matrix function (GHMF) is [12]:

$${}_2\mathscr{F}_1(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\sum_{n=0}^{\infty}(\mathcal{P})_n(\mathcal{Q})_n(\mathcal{H})_n^{-1}\frac{z^n}{n!} \tag{2.2}$$

The series in (2.2) converges absolutely for $|z|<1$, and for $z=1$ if $\alpha(\mathcal{P})+\alpha(\mathcal{Q})<\beta(\mathcal{H})$, where $\alpha(\mathcal{P})=\max\{\Re(z)\mid z\in\sigma(\mathcal{P})\}$, $\beta(\mathcal{P})=\min\{\Re(z)\mid z\in\sigma(\mathcal{P})\}$, and $\beta(\mathcal{P})=-\alpha(-\mathcal{P})$.

Furthermore, if $\mathcal{Q}\mathcal{H}=\mathcal{H}\mathcal{Q}$ and $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ are PSMs, then for $|z|<1$, an integral form of (2.2) is defined as [12]:

$${}_2\mathscr{F}_1(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\left(\int_0^1(1-z\ell)^{-\mathcal{P}}\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\right)\times\Gamma^{-1}(\mathcal{Q})\Gamma^{-1}(\mathcal{H}-\mathcal{Q})\Gamma(\mathcal{H})$$

Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{A}$ be PSMs and commuting matrices in $\mathbb{C}^{r\times r}$. Then, the EBMF $\mathscr{B}(\mathcal{P},\mathcal{Q};\mathcal{A})$ is defined by Abdalla and Bakhet [2] as follows:

$$\mathscr{B}(\mathcal{P},\mathcal{Q};\mathcal{A})=\int\limits_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell(1-\ell)}\right)d\ell$$

They generalized the GHMF and Kummer hypergeometric matrix function (KHMF) using EBMF. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{H}-\mathcal{Q}$, and $\mathcal{A}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{Q}\mathcal{H}=\mathcal{H}\mathcal{Q}$, $\mathcal{H}\mathcal{A}=\mathcal{A}\mathcal{H}$, and $\mathcal{Q}\mathcal{A}=\mathcal{A}\mathcal{Q}$. The extended GHMF (EGHMF) and the extended KHMF (EKHMF) are defined as [1]:

$$\mathscr{F}^{(\mathcal{A})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\left(\sum_{m\geq 0}(\mathcal{P})_m\mathscr{B}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A})\frac{z^m}{m!}\right)\times\Gamma(\mathcal{H})\Gamma^{-1}(\mathcal{Q})\Gamma^{-1}(\mathcal{H}-\mathcal{Q})$$

and

$$\Phi^{\mathcal{A}}(\mathcal{Q};\mathcal{H};z)=\left(\sum_{m\geq 0}\mathscr{B}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A})\frac{z^m}{m!}\right)\times\Gamma(\mathcal{H})\Gamma^{-1}(\mathcal{Q})\Gamma^{-1}(\mathcal{H}-\mathcal{Q})$$

respectively.

Verma et al. [17] have introduced another extension of BMF. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs and commuting matrices in $\mathbb{C}^{r\times r}$. Then, the EBMF $\mathscr{B}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})$ is defined as [17]:

$$\mathscr{B}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})=\int\limits_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell}-\frac{\mathcal{C}}{(1-\ell)}\right)d\ell \tag{2.3}$$

Moreover, they introduced EGHMF and EKHMF by (2.3) as follows [17]:

$$\mathscr{F}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\left(\sum_{m\geq 0}(\mathcal{P})_m\mathscr{B}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times\mathscr{B}(\mathcal{Q},(\mathcal{H}-\mathcal{Q}))^{-1}$$

and

$$\Phi^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=\left(\sum_{m\geq 0}\mathscr{B}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times\mathscr{B}(\mathcal{Q},(\mathcal{H}-\mathcal{Q}))^{-1}$$

respectively.

Inspired and motivated by EBMF, GHMF, and KHMF, we introduce their extensions and discuss these extensions' integral representations, differential formulae, recurrence relations, and transformation formulae.

3. An Extension of EBMF

Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs and commuting matrices in $\mathbb{C}^{r\times r}$ and $\eta,\mu\in\mathbb{C}$. Then, we introduce an extension of EBMF (EOEBMF) $\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})$ as follows:

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})=\int\limits_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell^{\eta}}-\frac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell \tag{3.1}$$

By applying Schur decomposition [18] and substituting $\ln\ell<\ell$ and $\ln(1-\ell)<(1-\ell)$, for $0<\ell<1$, respectively, we obtain

$$\mathscr{B}(\alpha(\mathcal{P})+i-\kappa,\alpha(\mathcal{Q})+j-l;\alpha(\mathcal{A}),\alpha(\mathcal{C}))<\infty$$

Thus, an EOEBMF $\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})$ exists.

Theorem 3.1. The EOEBMF satisfies the following integral representations:

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = 2\int_0^{\pi/2} (\cos u)^{2\mathcal{P}-I}(\sin u)^{2\mathcal{Q}-I}\exp\left(-\mathcal{A}sec^{2\eta}u - \mathcal{C}csc^{2\mu}u\right) du \tag{3.2}$$

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = \int_0^{\infty} u^{\mathcal{P}-I}(1+u)^{-\mathcal{P}-\mathcal{Q}}\exp\left(-\mathcal{A}(1+u^{-1})^{\eta} - \mathcal{C}(1+u)^{\mu}\right) du \tag{3.3}$$

and

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = 2^{I-\mathcal{P}-\mathcal{Q}}\int_{-1}^{1} (1+u)^{\mathcal{P}-I}(1-u)^{\mathcal{Q}-I}\times\exp\left(-2^{\eta}\mathcal{A}(1+u)^{-\eta} - 2^{\mu}\mathcal{C}(1-u)^{-\mu}\right) du \tag{3.4}$$

Proof. Substituting $\ell = \cos^2 u$ into (3.1) yields (3.2) after minor simplifications. Similarly, substituting $\ell = \frac{u}{1+u}$ into (3.1) results in (3.3). Finally, replacing $\ell = \frac{1+u}{2}$ in (3.1) provides (3.4). □

Remark 3.2. If $\eta = \mu = 1$ in (3.2), (3.3), and (3.4), respectively, then the result in [17] is obtained.

Theorem 3.3. The EOEBMF satisfies the following properties:

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q}+I;\mathcal{A},\mathcal{C}) + \mathscr{B}_{\eta,\mu}(\mathcal{P}+I,\mathcal{Q};\mathcal{A},\mathcal{C}) = \mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) \tag{3.5}$$

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},I-\mathcal{Q};\mathcal{A},\mathcal{C}) = \sum_{n=0}^{\infty}\frac{(\mathcal{Q})_n}{n!}\mathscr{B}_{\eta,\mu}(\mathcal{P}+nI,I;\mathcal{A},\mathcal{C}) \tag{3.6}$$

and

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = \sum_{n=0}^{\infty}\mathscr{B}_{\eta,\mu}(\mathcal{P}+nI,\mathcal{Q}+I;\mathcal{A},\mathcal{C}) \tag{3.7}$$

Proof. From (3.1),

$$\begin{aligned}
\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q}+I;\mathcal{A},\mathcal{C}) + \mathscr{B}_{\eta,\mu}(\mathcal{P}+I,\mathcal{Q};\mathcal{A},\mathcal{C}) &= \int_0^1 [\ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}}]\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right) d\ell \\
&\quad + \int_0^1 [\ell^{\mathcal{P}}(1-\ell)^{\mathcal{Q}-I}]\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right) d\ell \\
&= \int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}[(1-\ell)+\ell]\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right) d\ell \\
&= \int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right) d\ell \\
&= \mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})
\end{aligned}$$

Hence, the proof of (3.5) is done. Moreover,

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},I-\mathcal{Q};\mathcal{A},\mathcal{C}) = \int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{I-\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell^{\eta}} - \frac{\mathcal{C}}{(1-\ell)^{\mu}}\right) d\ell$$

By using the relation $(1-\ell)^{-\mathcal{Q}} = \sum\limits_{n=0}^{\infty}\frac{(\mathcal{Q})_n}{n!}\ell^n$ in [12],

$$\begin{aligned}
\mathscr{B}_{\eta,\mu}(\mathcal{P},I-\mathcal{Q};\mathcal{A},\mathcal{C}) &= \int_0^1 \ell^{\mathcal{P}-I}\sum_{n=0}^{\infty}\tfrac{(\mathcal{Q})_n}{n!}\ell^n\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right) d\ell \\
&= \sum_{n=0}^{\infty}\tfrac{(\mathcal{Q})_n}{n!}\int_0^1 \ell^{\mathcal{P}+(n-1)I}\exp\left(-\tfrac{\mathcal{A}}{t^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right) d\ell \\
&= \sum_{n=0}^{\infty}\tfrac{(\mathcal{Q})_n}{n!}\mathscr{B}_{\eta,\mu}(\mathcal{P}+nI,I;\mathcal{A},\mathcal{C})
\end{aligned}$$

Thus, the proof of (3.6) is done. Similarly, by substituting its series representation for $(1-\ell)^{-I}$ in (3.1),

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = \int_0^1 (1-\ell)^{\mathcal{Q}} \sum_{n=0}^{\infty} \ell^{\mathcal{P}+(n-1)I} \exp\left(-\frac{\mathcal{A}}{\ell^\eta} - \frac{\mathcal{C}}{(1-\ell)^\mu}\right) d\ell$$

The result (3.7) is obtained by using (3.1) and altering the integration and summation orders. □

4. Application of EOEBMF

Many researchers [2, 11, 17, 19, 20] have investigated different generalizations and extensions of BMFs, showcasing their potential applications in various domains. In this section, we analyze an application of the EOEBMF in (3.1) within the realm of statistics. Specifically, we define the beta distribution and derive its mean, variance, and moment-generating function using the EOEBMF.

For $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ be commutative PSMs in $\mathbb{C}^{r\times r}$ and $\Re(\eta), \Re(\mu) > 0$. Define the beta distribution as:

$$u(\ell) = \begin{cases} [\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1} \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I} \exp\left(-\frac{\mathcal{A}}{\ell^\eta} - \frac{\mathcal{C}}{(1-\ell)^\mu}\right), & 0<\ell<1 \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

For any matrix $\mathcal{R} \in \mathbb{C}^{r\times r}$, the moment of a random variable X is as follows:

$$E\left(X^{\mathcal{R}}\right) = \mathscr{B}_{\eta,\mu}(\mathcal{P}+\mathcal{R},\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1}$$

If $\mathcal{R} = I$, then the mean of the beta distribution is as follows:

$$\rho = E\left(X^{I}\right) = \mathscr{B}_{\eta,\mu}(\mathcal{P}+I,\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1}$$

Therefore, the variance of the distribution is defined as:

$$\begin{aligned} \sigma^2 &= E(X^{2I}) - \left\{E(X^I)\right\}^2 \\ &= \mathscr{B}_{\eta,\mu}(\mathcal{P}+2I,\mathcal{Q};\mathcal{A},\mathcal{C}[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1} - \{\mathscr{B}_{\eta,\mu}(\mathcal{P}+I,\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1}\}^2 \end{aligned}$$

Besides, the moment generating matrix function of the distribution in (4.1) is as follows:

$$M(\ell) = \sum_{\kappa=0}^{\infty} \frac{\ell^\kappa}{\kappa!} E(X^{\kappa I}) = [\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1} \sum_{\kappa=0}^{\infty} \mathscr{B}_{\eta,\mu}(\mathcal{P}+\kappa I,\mathcal{Q};\mathcal{A},\mathcal{C}) \frac{\ell^\kappa}{\kappa!}$$

The cumulative distribution of (4.1) is defined as:

$$\mathscr{F}(x) = \int_0^x u(\ell) d\ell = \mathscr{B}_{x,\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1}$$

where $F(1) = I$ and $\mathscr{B}_{x,\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})$ is the incomplete BMF defined as:

$$\mathscr{B}_{x,\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = \int_0^x \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I} \exp\left(-\frac{\mathcal{A}}{\ell^\eta} - \frac{\mathcal{C}}{(1-\ell)^\mu}\right) d\ell$$

5. Graphical and Numerical Comparison of the Classical and Generalized Matrix-Variate Beta Distributions

The classical beta distribution involving the BMF in (2.1) is defined as:

$$u(\ell) = \begin{cases} [\mathscr{B}(\mathcal{P},\mathcal{Q})]^{-1} \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}, & 0<\ell<1 \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

Consider $\mathcal{P} = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 3 \end{pmatrix}$, $\mathcal{Q} = \begin{pmatrix} 3 & 0.2 \\ 0.2 & 4 \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 2 \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} 1.5 & 0.3 \\ 0.3 & 2.5 \end{pmatrix}$, and $\eta = \mu = 2$.

In Figure 1, taking $\mathcal{P}$ and $\mathcal{Q}$ matrices, compute the eigenvalues of $\mathcal{P} - I$ and $\mathcal{Q} - I$, and using in (5.1) to compute and plot the classical beta distribution over the range $0 < \ell < 1$ for 2×2 matrices.

Moreover, in Figure 2, taking $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ matrices and $\eta = \mu = 2$, compute the eigenvalues of $\mathcal{P} - I$, $\mathcal{Q} - I$, $\mathcal{A}$, and $\mathcal{C}$ and using in (4.1) to compute and plot the generalized beta distribution with parameters $\mathcal{A}$, $\mathcal{C}$, η, and μ. In Figure 3, we compare our generalized beta distribution with the classical beta distribution in matrices.

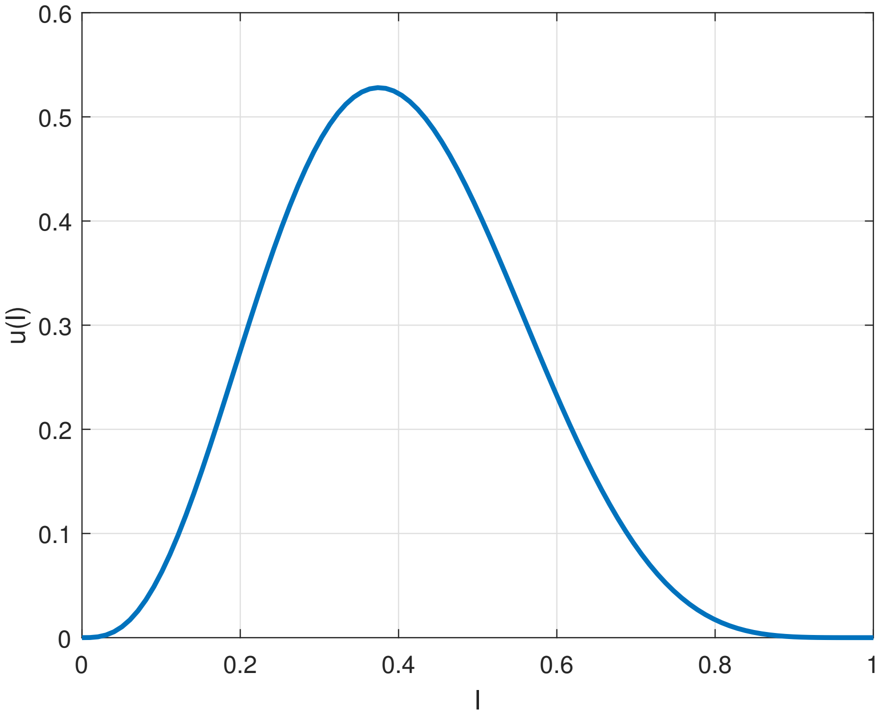

Figure 1. Classical beta distribution for 2×2 matrices $\mathcal{P}$ and $\mathcal{Q}$

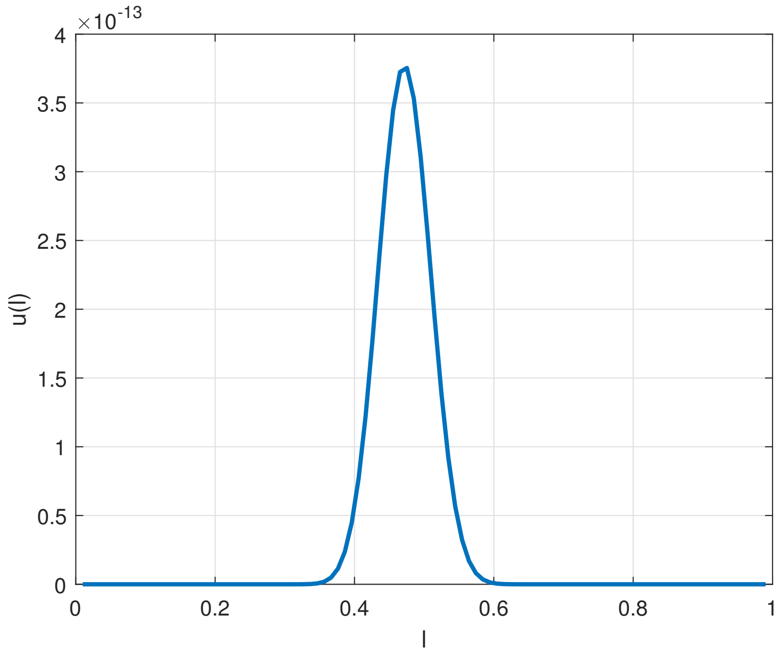

Figure 2. Generalized beta distribution with parameters $\mathcal{A}$, $\mathcal{C}$, η, and μ

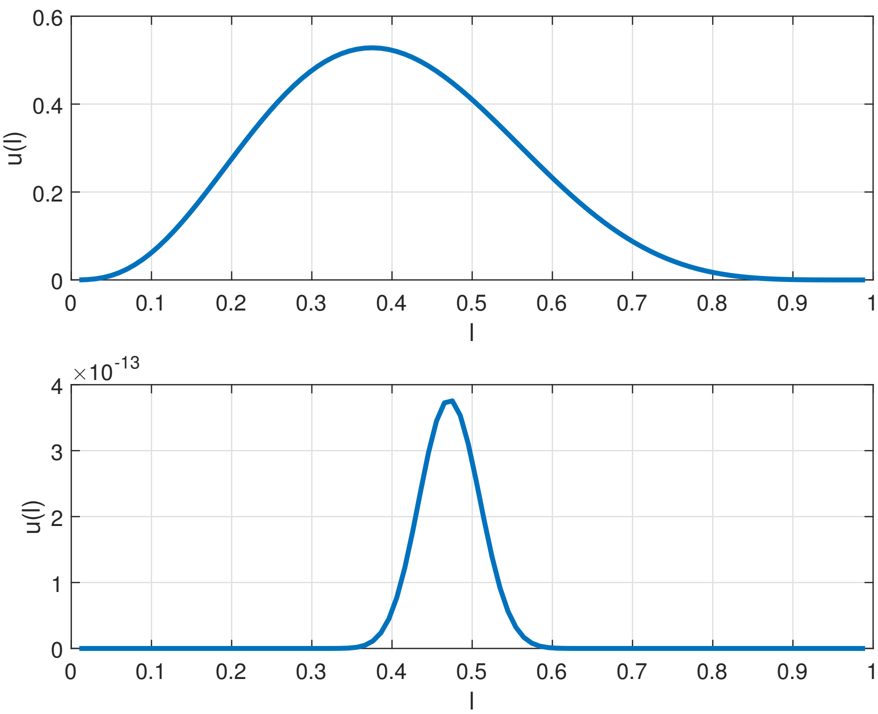

Figure 3. (a) Classical beta distribution and (b) Generalized beta distribution with exponential terms

Both distributions are normalized using a simplified approach based on the scalar beta function. In Figure 1, the distribution is closely related to the scalar classical beta distribution, generalized to matrix arguments $\mathcal{P}$ and $\mathcal{Q}$.

The simpler matrix beta distribution directly relates to random matrix theory, which has applications in signal processing, wireless communications, and finance. The simpler form is also used for matrix-variate generalizations of Bayesian analysis or weighting in optimization problems, particularly in multivariate or matrix-based Bayesian methods. However, the flexibility to model more complex real-world phenomena is restricted because it lacks additional factors like essential terms.

However, in our result, we provided the additional terms $\exp\left(-\frac{\mathcal{A}}{\ell^{\eta}} - \frac{\mathcal{C}}{(1-\ell)^{\mu}}\right)$ introduce exponential decay, which can allow for greater flexibility in fitting data or modeling more complex systems. This distribution could be used in more advanced Bayesian frameworks where the priors need to account for additional penalization or constraints, often seen in hierarchical models or models with specific tail behavior. The exponential terms can capture the behavior that decays rapidly, which is helpful in stochastic modeling, particularly in systems with non-linear dynamics or time-varying processes. In areas like financial modeling or signal processing, where matrix-valued variables may represent volatility or correlation, the exponential decay allows better control over tail risks or sensitivity. The exponential terms provide much more flexibility in controlling the shape and behavior of the distribution. This is particularly useful in real-world applications where tail behavior, constraints, or penalizations are needed. Parameters like $\mathcal{A}$, $\mathcal{C}$, η, and μ offer additional degrees of freedom for fine-tuning the distribution, making it more adaptable to complex data or phenomena.

6. EGHMF and EKHMF

The main aim of this section is to introduce extensions of GHMF and KHMF. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{H}-\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ be positive stable and commuting matrices in $\mathbb{C}^{r\times r}$. Extensions of GHMF and KHMF, i.e., EGHMF and EKHMF, are defined as follows:

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H},z)=\left(\sum_{m\geq 0}(\mathcal{P})_m\mathscr{B}_{\eta,\mu}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1} \tag{6.1}$$

and

$$\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=\left(\sum_{m\geq 0}\mathscr{B}_{\eta,\mu}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1} \tag{6.2}$$

respectively.

Theorem 6.1. For PSMs $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{H}-\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ in $\mathbb{C}^{r\times r}$, the EGHMF and EKHMF have following integral representation, respectively.

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H},z)=\int_0^1(1-z\ell)^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{\ell^\eta}-\frac{\mathcal{C}}{(1-\ell)^\mu}\right)\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\times\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1} \tag{6.3}$$

and

$$\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=\left(\int_0^1\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell^\eta}-\frac{\mathcal{C}}{(1-\ell)^\mu}\right)d\ell\right)\times\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1} \tag{6.4}$$

Proof. Using (6.1),

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H},z)=\left(\sum_{m\geq 0}(\mathcal{P})_m\mathscr{B}_{\eta,\mu}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1}$$

Using (3.1),

$$\mathscr{F}_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{P},\mathcal{Q};\mathcal{H},z)=\left(\sum_{m\geq 0}(\mathcal{P}_m)\left(\int_0^1\ell^{\mathcal{Q}+(m-1)I}(1-\ell)^{(\mathcal{H}-\mathcal{Q})-I}\exp\left(-\frac{\mathcal{A}}{\ell^\eta}-\frac{\mathcal{C}}{(1-\ell)^\mu}\right)d\ell\right)\frac{z^m}{m!}\right)\times\mathscr{B}(\mathcal{Q},(\mathcal{H}-\mathcal{Q}))^{-1}$$

Moreover, the following matrix identity is valid:

$$(1-z\ell)^{-\mathcal{P}}=\sum_{m=0}^{\infty}(\mathcal{P})_m\frac{(z\ell)^m}{m!}$$

Thus,

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H},z)=\int_0^1(1-z\ell)^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{\ell^\eta}-\frac{\mathcal{C}}{(1-\ell)^\mu}\right)\ell^{\mathcal{Q}-I}(1-\ell)^{(\mathcal{H}-\mathcal{Q})-I}d\ell\times[\mathscr{B}(\mathcal{Q},(\mathcal{H}-\mathcal{Q}))]^{-1}$$

Similarly, by (6.2), (6.4) is obtained. □

Theorem 6.2. Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{Q}\mathcal{H}=\mathcal{H}\mathcal{Q}$. Then, the following differential equations are satisfied by EGHMF and EKHMF, respectively:

$$\frac{d^n}{dz^n}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=(\mathcal{P})_n\mathscr{F}_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{P}+nI,\mathcal{Q}+nI;\mathcal{H}+nI;z)(\mathcal{Q})_n(\mathcal{H})_n^{-1}$$

and

$$\frac{d^n}{dz^n}\Phi_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{Q};\mathcal{H};z)=\Phi_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{Q}+nI;\mathcal{H}+nI;z)(\mathcal{Q})_n(\mathcal{H})_n^{-1} \tag{6.5}$$

Proof. From (6.1),

$$\begin{aligned}
\frac{d}{dz}\mathscr{F}_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{P},\mathcal{Q};\mathcal{H};z) &= \frac{d}{dz}\sum_{n=0}^{\infty}(\mathcal{P})_n\mathscr{B}_{\eta,\mu}(\mathcal{Q}+nI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}\frac{z^n}{n!} \\
&= \sum_{n=1}^{\infty}(\mathcal{P})_n\mathscr{B}_{\eta,\mu}(\mathcal{Q}+nI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}\frac{z^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty}(\mathcal{P})_{(n+1)}\mathscr{B}_{\eta,\mu}(\mathcal{Q}+(n+1)I,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}\frac{z^n}{n!} \\
&= \mathcal{P}\sum_{n=0}^{\infty}(\mathcal{P}+I)_n\mathscr{B}_{\eta,\mu}(\mathcal{Q}+(n+1)I,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q}+I,\mathcal{H}-\mathcal{Q})]^{-1}\frac{z^n}{n!}(\mathcal{Q})(\mathcal{H})^{-1} \\
&= (\mathcal{P})_1\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+I,\mathcal{Q}+I;\mathcal{H}+I;z)(\mathcal{Q})_1(\mathcal{H})_1^{-1}
\end{aligned}$$

Repeat this process n times. The differential formula appears as

$$\frac{d^n}{dz^n}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z) = (\mathcal{P})_n\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+nI,\mathcal{Q}+nI;\mathcal{H}+nI;z)(\mathcal{Q})_n(\mathcal{H})_n^{-1}$$

Similarly, (6.5) is obtained by (6.2). □

7. Transformation Formulae

In this section, we provide the transformation formulae for EGHMF and EKHMF.

Theorem 7.1. Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ be PSMs in $\mathbb{C}^{r\times r}$ and $\mathcal{Q}\mathcal{H}=\mathcal{H}\mathcal{Q}$. Then, the following formulae are satisfied by EGHMF:

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z) = (1-z)^{-\mathcal{P}}\mathscr{F}_{\mu,\eta}^{(\mathcal{A},\mathcal{C})}\left(P,H-Q;H;\frac{z}{(z-1)}\right) \tag{7.1}$$

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};1-\frac{1}{z}) = z^{\mathcal{P}}\mathscr{F}_{\mu,\eta}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{H}-\mathcal{Q};\mathcal{H};1-z) \tag{7.2}$$

and

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};\frac{z}{z+1}) = (1+z)^{\mathcal{P}}\mathscr{F}_{\mu,\eta}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{H}-\mathcal{Q};\mathcal{H};-z) \tag{7.3}$$

Proof. In (6.3), if ℓ is changed to $(1-\ell)$, then

$$\begin{aligned}
\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z) &= \int_0^1(1-z(1-\ell))^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{(1-\ell)^{\eta}}-\frac{\mathcal{C}}{\ell^{\mu}}\right)(1-\ell)^{\mathcal{Q}-I}\ell^{\mathcal{H}-\mathcal{Q}-I}d\ell[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1} \\
&= \int_0^1(1-z+z\ell)^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{(1-\ell)^{\eta}}-\frac{\mathcal{C}}{\ell^{\mu}}\right)(1-\ell)^{\mathcal{Q}-I}\ell^{\mathcal{H}-\mathcal{Q}-I}d\ell[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1} \\
&= (1-z)^{-\mathcal{P}}\int_0^1(1-\frac{z\ell}{z-1})^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{(1-\ell)^{\eta}}-\frac{\mathcal{C}}{\ell^{\mu}}\right)(1-\ell)^{\mathcal{Q}-I}\ell^{\mathcal{H}-\mathcal{Q}-I}d\ell[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1} \\
&= (1-z)^{-\mathcal{P}}\mathscr{F}_{\mu,\eta}^{(\mathcal{A},\mathcal{C})}\left(\mathcal{P},\mathcal{H}-\mathcal{Q};\mathcal{H};\frac{z}{z-1}\right)
\end{aligned}$$

To determine (7.2) and (7.3), we replace z in (7.1) with $(1-\frac{1}{z})$ and $\frac{z}{1+z}$, respectively. □

Setting $z=1$ and allowing $\mathcal{P}$ to commute with $\mathcal{Q}$ and $\mathcal{H}$ provides the link between the EGHMF and EBMF that is shown in (6.1):

$$\begin{aligned}
\mathscr{F}_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{P},\mathcal{Q};\mathcal{H},1) &= \left(\int_0^1\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{P}-\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell^{\eta}}-\frac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell\right)\times[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1} \\
&= \mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{P}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}
\end{aligned} \tag{7.4}$$

Using (7.4), we can formulate a novel generalization of Kummer's first theorem.

Theorem 7.2. Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{QH}=\mathcal{HQ}$. Then, Kummer's first theorem for new extension is provided as:

$$\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=\exp(z)\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{H}-\mathcal{Q};\mathcal{H};-z)$$

Theorem 7.3. Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{QH}=\mathcal{HQ}$. Then, EGHMF and EKHMF satisfy the following recurrence relations:

$$\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=z\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+I,\mathcal{Q}+I;\mathcal{H}+I;z)\mathcal{Q}\mathcal{H}^{-1} \tag{7.5}$$

$$\frac{d}{dz}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\frac{\mathcal{P}}{z}\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z) \tag{7.6}$$

$$\mathcal{Q}\Delta_{\mathcal{Q}}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H}+I;z)+\mathcal{H}\Delta_{\mathcal{H}}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=0 \tag{7.7}$$

and

$$\frac{d}{dz}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=\mathcal{Q}\mathcal{H}^{-1}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H}+I;z)-\Delta_{\mathcal{H}}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z) \tag{7.8}$$

where $\Delta_{\mathcal{P}}$ is the shift operator relative to $\mathcal{P}$.

PROOF. By using $\Delta_{\mathcal{P}}$ as the shift operator about $\mathcal{P}$ and the integral representation of the EGHMF (6.1),

$$\begin{aligned}\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z) &= \mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+I,\mathcal{Q};\mathcal{H};z)-\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)\\ &= \left(\int_0^1(1-z\ell)^{-\mathcal{P}-I}(1-(1-z\ell))\exp\left(-\tfrac{\mathcal{A}}{\ell^\eta}-\tfrac{\mathcal{C}}{(1-\ell)^\mu}\right)\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\right)\times[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}\end{aligned}$$

Therefore,

$$\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=z\left(\int_0^1(1-z\ell)^{-\mathcal{P}-I}\exp\left(-\tfrac{\mathcal{A}}{\ell^\eta}-\tfrac{\mathcal{C}}{(1-\ell)^\mu}\right)\ell^{\mathcal{Q}}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\right)\times[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1} \tag{7.9}$$

We can see from (6.1) that

$$\begin{aligned}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+I,\mathcal{Q}+I;\mathcal{H}+I;z) &= \left(\int_0^1(1-z\ell)^{-\mathcal{P}-I}(1-(1-z\ell))\exp\left(-\tfrac{\mathcal{A}}{\ell}-\tfrac{\mathcal{C}}{(1-\ell)}\right)\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\right)\\ &\quad\times[\mathscr{B}(\mathcal{Q}+I,\mathcal{H}-\mathcal{Q})]^{-1}\end{aligned} \tag{7.10}$$

From (7.9) and (7.10),

$$\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=z\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+I,\mathcal{Q}+I;\mathcal{H}+I;z)\mathcal{Q}\mathcal{H}^{-1}$$

Another differential recurrence relation can be found using the EGHMF's differentiation formula, as illustrated in (7.6). The results in (7.7) and (7.8) can be obtained by using the same steps as the proof in (7.5) and (7.6). □

8. EAMF and ELMF

This section extends the Appell matrix function (AMF) and Lauricella matrix function (LMF) to three variables. Specifically, we present the extended forms of the AMF, i.e., $\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w)$ and $\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w)$, and the LMF with three variables, $\mathscr{F}_{\mathcal{D}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H};z;w;v)$. These extensions are formulated using the new EBMF [7, 16, 21]. Additionally, we provide integral representations for these extended hypergeometric matrix functions.

Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}-\mathcal{P}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{P}$, $\mathcal{H}$, $\mathcal{A}$, and $\mathcal{C}$ commutes, $\mathcal{HQ}=\mathcal{QH}$, and $\mathcal{HQ}'=\mathcal{Q}'\mathcal{H}$. Then, we define an extension of EAMF as:

$$\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})=\Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\sum_{m,n\geq0}\mathscr{B}_{\eta,\mu}(\mathcal{P}+(m+n)I,\mathcal{H}-\mathcal{P};\mathcal{A},\mathcal{C})(\mathcal{Q})_m(\mathcal{Q}')_n\frac{z^m w^n}{m!n!}$$

where

$$\Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}=\Gamma(\mathcal{H})\Gamma^{-1}(\mathcal{P})\Gamma^{-1}(\mathcal{H}-\mathcal{P})$$

Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}-\mathcal{Q}$, $\mathcal{H}'-\mathcal{Q}'$, $\mathcal{A}$, and $\mathcal{C}$ in $\mathbb{C}^{r\times r}$ be commutative PSMs such that $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{A}$, and $\mathcal{C}$ commutes. We define the new extended Appell hypergeometric matrix function (EAHMF) $\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C})$ as:

$$\begin{aligned}\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C})=&\sum_{m,n\geq 0}(\mathcal{P})_{m+n}\mathscr{B}_{\eta,\mu}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\mathscr{B}_{\eta,\mu}(\mathcal{Q}'+nI,\mathcal{H}'-\mathcal{Q}';\mathcal{A},\mathcal{C})\frac{z^m w^n}{m!n!}\\ &\times\Gamma\begin{pmatrix}\mathcal{H},\mathcal{H}'\\ \mathcal{Q},\mathcal{Q}',\mathcal{H}-\mathcal{Q},\mathcal{H}'-\mathcal{Q}'\end{pmatrix}\end{aligned}\tag{8.1}$$

Suppose $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{Q}''$, $\mathcal{H}$, $\mathcal{H}-\mathcal{P}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{P}$, $\mathcal{H}$, and $\mathcal{A}$ commutes with each other, $\mathcal{H}\mathcal{Q}=\mathcal{Q}\mathcal{H}$, and $\mathcal{H}\mathcal{Q}'=\mathcal{Q}'\mathcal{H}$. Then, we define the extension of the new Lauricella hypergeometric matrix functions (LHMF) defined as:

$$\mathscr{F}_{\mathcal{D},\mathcal{A},\mathcal{C}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H},;z,w;v)=\Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\sum_{m,n,p\geq 0}\mathscr{B}_{\eta,\mu}(\mathcal{P}+(m+n+p)I,\mathcal{H}-\mathcal{P};\mathcal{A},\mathcal{C})(\mathcal{Q})_m(\mathcal{Q}')_n(\mathcal{Q}'')_p\frac{z^m w^n v^p}{m!n!p!}\tag{8.2}$$

We focus on identifying the integral representations of the three variable extensions of the AMF and the LMF. We start by representing the integral of $\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})$ determined in the following theorem.

Theorem 8.1. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}-\mathcal{P}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{P}$, $\mathcal{H}$, $\mathcal{A}$, and $\mathcal{C}$ commutes with each other, $\mathcal{H}\mathcal{Q}=\mathcal{Q}\mathcal{H}$, and $\mathcal{H}\mathcal{Q}'=\mathcal{Q}'\mathcal{H}$. Then, the EAMF $\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})$ can be presented in the integral form as:

$$\begin{aligned}\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})=&\,\Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}(1-zu)^{-\mathcal{Q}}(1-wu)^{-\mathcal{Q}'}\right.\\ &\left.\times\exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{-\mathcal{C}}{(1-u)^\mu}\right)du\right)\end{aligned}\tag{8.3}$$

Proof. Using (3.1) in the EAMF $\mathscr{F}_1^{\eta,\mu}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})$,

$$\begin{aligned}\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})=&\,\Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\sum_{m,n\geq 0}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}\exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{-\mathcal{C}}{(1-u)^\mu}\right)\right.\\ &\left.\times(\mathcal{Q})_m(\mathcal{Q}')_n\frac{(zu)^m(wu)^n}{m!n!}du\right)\end{aligned}\tag{8.4}$$

By the method discussed by Dwivedi and Sahai [21], the equality

$$(1-z)^{-\mathcal{P}}=\sum_{n=0}^{\infty}(\mathcal{P})_n\frac{z^n}{n!}\tag{8.5}$$

and (8.4),

$$\begin{aligned}\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})=&\,\Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}\exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{-\mathcal{C}}{(1-u)^\mu}\right)\right.\\ &\left.\times(1-zu)^{-\mathcal{Q}}(1-wu)^{-\mathcal{Q}'}du\right)\end{aligned}$$

□

Remark 8.2. After replacing the values $\mu=\eta=1$ in (8.3), the results described in [17] are obtained.

Theorem 8.3. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}-\mathcal{Q}$, $\mathcal{H}'-\mathcal{Q}'$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{A}$, and $\mathcal{C}$ commutes with each other. Then, the EAMF $\mathscr{F}_2^{\eta,\mu}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C})$ defined in (8.1) has the following integral representation:

$$\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C}) = \left(\int_0^1\int_0^1 (1-zu-wv)^{-\mathcal{P}} u^{\mathcal{Q}-I}(1-u)^{\mathcal{H}-\mathcal{Q}-I} v^{\mathcal{Q}'-I}(1-v)^{\mathcal{H}'-\mathcal{Q}'-I}\right.$$
$$\left.\times \exp\left(-\tfrac{\mathcal{A}}{u^\eta}-\tfrac{\mathcal{C}}{(1-u)^\mu}-\tfrac{\mathcal{A}}{v^\eta}-\tfrac{\mathcal{C}}{(1-v)^\mu}\right)dudv\right)\Gamma\begin{pmatrix}\mathcal{H},\mathcal{H}'\\ \mathcal{Q},\mathcal{Q}',\mathcal{H}-\mathcal{Q},\mathcal{H}'-\mathcal{Q}'\end{pmatrix} \tag{8.6}$$

Proof. Using (3.1) and (8.1),

$$\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C}) = \sum_{m,n\geq 0}\left(\int_0^1\int_0^1 (\mathcal{P})_{m+n}\tfrac{(zu)^m(wv)^n}{m!n!} u^{\mathcal{Q}-I}(1-u)^{\mathcal{H}-\mathcal{Q}-I} v^{\mathcal{Q}'-I}(1-v)^{\mathcal{H}'-\mathcal{Q}'-I}\right.$$
$$\left.\times \exp\left(-\tfrac{\mathcal{A}}{u^\eta}-\tfrac{\mathcal{C}}{(1-u)^\mu}-\tfrac{\mathcal{A}}{v^\eta}-\tfrac{\mathcal{C}}{(1-v)^\mu}\right)dudv\right)\Gamma\begin{pmatrix}\mathcal{H},\mathcal{H}'\\ \mathcal{Q},\mathcal{Q}',\mathcal{H}-\mathcal{Q},\mathcal{H}'-\mathcal{Q}'\end{pmatrix} \tag{8.7}$$

by the interchanging summation and integral in (8.7) via the dominated convergence theorem. Moreover, the following summation formula [22] is valid:

$$\sum_{n\geq 0} f(N)\frac{(z+w)^N}{N!} = \sum_{m,n\geq 0} f(m+n)\frac{z^m w^n}{m!n!}$$

Thus,

$$\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C}) = \left(\int_0^1\int_0^1 \sum_{N\geq 0}(\mathcal{P})_N\tfrac{(zu+wv)^N}{N!} u^{\mathcal{Q}-I}(1-u)^{\mathcal{H}-\mathcal{Q}-I} v^{\mathcal{Q}'-I}(1-v)^{\mathcal{H}'-\mathcal{Q}'-I}\right.$$
$$\left.\times \exp\left(-\tfrac{\mathcal{A}}{u^\eta}-\tfrac{\mathcal{C}}{(1-u)^\mu}-\tfrac{\mathcal{A}}{v^\eta}-\tfrac{\mathcal{C}}{(1-v)^\mu}\right)dudv\right)\Gamma\begin{pmatrix}\mathcal{H},\mathcal{H}'\\ \mathcal{Q},\mathcal{Q}',\mathcal{H}-\mathcal{Q},\mathcal{H}'-\mathcal{Q}'\end{pmatrix} \tag{8.8}$$

Using (8.5) and (8.8), (8.6) is obtained. □

Theorem 8.4. Suppose $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{Q}''$, $\mathcal{H}$, $\mathcal{H}-\mathcal{P}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{P}$, $\mathcal{H}$, and $\mathcal{A}$ commutes with each other, $\mathcal{H}\mathcal{Q}=\mathcal{Q}\mathcal{H}$, $\mathcal{H}\mathcal{Q}'=\mathcal{Q}'\mathcal{H}$, and $\mathcal{H}\mathcal{Q}''=\mathcal{Q}''\mathcal{H}$. Then, the ELMF $\mathscr{F}_{\mathcal{D},\mathcal{A},\mathcal{C}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H},;z,w;v)$ in (8.2) provides the following integral representation:

$$\mathscr{F}_{\mathcal{D},\mathcal{A},\mathcal{C}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H},;z,w;v) = \Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}\exp\left(-\tfrac{\mathcal{A}}{u^\eta}-\tfrac{\mathcal{C}}{(1-u)^\mu}\right)\right.$$
$$\left.\times(1-zu)^{-\mathcal{Q}}(1-wu)^{-\mathcal{Q}'}(1-vu)^{-\mathcal{Q}''}du\right) \tag{8.9}$$

Proof. From (3.1) and (8.2),

$$\mathscr{F}_{\mathcal{D},\mathcal{A},\mathcal{C}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H},;z,w;v) = \Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\sum_{m,n,p\geq 0}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}\exp\left(-\tfrac{\mathcal{A}}{u^\eta}-\tfrac{\mathcal{C}}{(1-u)^\mu}\right)\right.$$
$$\left.\times(\mathcal{Q})_m(\mathcal{Q}')_n(\mathcal{Q}'')_p\tfrac{(uz)^m(uw)^n(uv)^p}{m!n!p!}du\right)$$

By (8.5) and continuing in the same process as in Theorem 8.1, (8.9) is obtained. □

9. Conclusion

In conclusion, the findings presented in this paper introduce new results that can potentially extend other special matrix functions. We have developed an extension of the BMF and investigated the GHMF and KHMF, exploring their key relationships and properties. Additionally, we extended the AMF and LMF and derived their integral representations using the beta matrix function. We also highlighted significant statistical applications of the EBMF. These generalized matrix functions have wide-ranging applications, including quantum mechanics, describing the time evolution of quantum systems, multivariate statistics, modeling multivariate distributions and hypothesis testing, control theory, analyzing the stability and response of dynamic systems, and mathematical physics, solving systems of differential equations with matrix arguments. The results from this study open several promising avenues for future research. Potential directions include extending other special matrix functions, such as the Whittaker, Wright, and Fox-H matrix functions, as well as Jacobi and Laguerre matrix polynomials. Researchers could also explore special integral transforms of these extended matrix functions, including the Euler-Beta, Laplace, and k-transforms. With its exponential terms, the generalized beta distribution provides additional flexibility and could be useful in machine learning, especially in regularization and Bayesian frameworks. Researchers could explore using matrix-variate beta distributions in deep learning models for regularization, uncertainty quantification, and matrix-variate variational autoencoders.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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On Factorization and Calculation of Determinant of Block Matrices with Triangular Submatrices

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Abstract — In this paper, we consider some block matrices of dimension $nm \times nm$ whose components are triangular matrices of dimension $n \times n$. We prove that the determinant of such block matrices is determined only by the diagonal elements of their submatrices and that this determinant is expressed as the multiplication of some subdeterminants. If the components of dimension $n \times n$ are all diagonal matrices, then we prove that such a block matrix can be written as a product of simpler matrices. Besides, we investigate the eigenvalues, the adjoint, and the inverse of such block matrices.

Keywords *Block matrix, determinant, triangular matrix, trigonometric system, Wronskian, factorization*

Mathematics Subject Classification (2020) 15A15, 15A23

1. Introduction

The determinant is a scalar value corresponding to a square matrix and is denoted by $|A|$, $D(A)$, $\det A$, or $\det(A)$. Besides, it is a function that maps from square matrix spaces to complex numbers. The determinant has many uses in mathematics. For instance, it determines whether a square matrix is invertible, is used to solve a system of linear equations, helps to find the inverse of a matrix, is used to solve some boundary value problems, etc. The determinant of a square matrix can be calculated using Laplace expansion. In particular, the determinant of a matrix of dimension 3×3 can be calculated by the Sarrus rule. Calculating determinants becomes more difficult for square matrices of dimension 4×4 and larger. For more information about factorization and calculation of determinants of large block matrices, see [1–14]. Sometimes calculating a determinant is easier if there are many zeros in the entries of the considered matrix. For instance, it is easier to calculate the determinants of the following matrices B and C by hand than the determinant of the following matrix D.

$$B = \begin{pmatrix} 3 & 0 & 3 & 7 \\ 0 & -2 & 0 & 0 \\ 5 & 2 & 0 & 13 \\ 0 & 0 & 5 & -8 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 65 & 3 & 7 \\ 0 & -2 & 4 & -8 \\ 0 & 0 & 9 & 18 \\ 0 & 0 & 0 & -8 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 5 & 5 & -7 & 7 \\ 8 & 12 & -8 & 2 \\ 5 & -2 & 9 & 1 \\ 1 & -1 & 3 & -1 \end{pmatrix}$$

The more zeros, the easier it is to calculate the determinant. Indeed, the determinant of C is the easiest since C is an upper triangular matrix. The determinant of an upper or lower triangular square matrix is the product of the main diagonal entries. A similar calculation is provided for upper triangular block matrices. Let E be an upper triangular square block matrix as follows:

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$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1m} \\ 0 & E_{22} & \cdots & E_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{mm} \end{pmatrix}$$

where 0 denotes the zero matrix. Then, $|E| = \prod_{i=1}^{n} |E_{ii}|$.

Simply having many zeros does not make it easy to calculate a determinant. Both matrices B and C have six zeros. However, since matrix C is upper triangular, it is easier to calculate its determinant. That is, the location or arrangement of the zeros is also essential.

This paper considers a different arrangement of zeros in a square matrix. It presents the following type of square block matrices of dimension $nm \times nm$ whose components are upper triangular matrices of dimension $n \times n$:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}_{nm \times nm}$$

where

$$A_{ij} = \begin{pmatrix} a_{n(i-1)+1,n(j-1)+1} & a_{n(i-1)+1,n(j-1)+2} & \cdots & a_{n(i-1)+1,n(j-1)+n} \\ 0 & a_{n(i-1)+2,n(j-1)+2} & \cdots & a_{n(i-1)+2,n(j-1)+n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n(i-1)+n,n(j-1)+n} \end{pmatrix}_{n \times n} \tag{1.1}$$

Firstly, we show that the determinant of A depends only on the diagonal entries of the matrices A_{ij}. Secondly, we construct a factorization of the matrix A when all the sub-matrices A_{ij} are diagonal matrices and obtain a formula for the determinant of A. Finally, we consider the eigenvalues, adjoint, and inverse of the matrix A.

For instance, if we take $n = m = 2$, then the matrix A turns into

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & 0 & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ 0 & a_{4,2} & 0 & a_{4,4} \end{pmatrix}$$

and we show that the following equality is valid

$$|A| = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & 0 & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ 0 & a_{4,2} & 0 & a_{4,4} \end{vmatrix} = \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} \begin{vmatrix} a_{2,2} & a_{2,4} \\ a_{4,2} & a_{4,4} \end{vmatrix}$$

In this paper, we consider only the upper triangular matrices since a lower triangular matrix is the transpose of an upper triangular matrix.

2. Main Results

In this section, we delve into the detailed process of reducing the determinant of block matrices whose submatrices are triangular. This reduction is crucial for simplifying the determinant calculation of such complex matrices. We begin by analyzing the specific structure of these matrices and demonstrate how the arrangement of zeros in both the submatrices and the block matrix itself plays a fundamental role. The results presented here provide a framework for factorization and determinant computation, which will be elaborated upon in the following subsections. Our approach aims to significantly reduce the computational complexity of these calculations, offering a more efficient pathway for handling large-scale block matrices.

2.1. Reduction of Determinant

Consider the matrix of dimension $nm \times nm$ as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}$$

where the submatrices A_{ij} of dimension $n \times n$ are given by the following

$$A_{ij} = \begin{pmatrix} a_{n(i-1)+1,n(j-1)+1} & a_{n(i-1)+1,n(j-1)+2} & \cdots & a_{n(i-1)+1,n(j-1)+n} \\ 0 & a_{n(i-1)+2,n(j-1)+2} & \cdots & a_{n(i-1)+2,n(j-1)+n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n(i-1)+n,n(j-1)+n} \end{pmatrix}$$

More precisely,

$$A = \begin{pmatrix}
a_{1,1} & \cdots & a_{1,n} & a_{1,n+1} & \cdots & a_{1,2n} & \cdots & a_{1,n(m-1)+1} & \cdots & a_{1,nm} \\
0 & \cdots & a_{2,n} & 0 & \cdots & a_{2,2n} & \cdots & 0 & \cdots & a_{2,nm} \\
0 & \cdots & a_{3,n} & 0 & \cdots & a_{3,2n} & \cdots & 0 & \cdots & a_{3,nm} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{n,n} & 0 & \cdots & a_{n,2n} & \cdots & 0 & \cdots & a_{n,nm} \\
a_{n+1,1} & \cdots & a_{n+1,n} & a_{n+1,n+1} & \cdots & a_{n+1,2n} & \cdots & a_{n+1,n(m-1)+1} & \cdots & a_{n+1,nm} \\
0 & \cdots & a_{n+2,n} & 0 & \cdots & a_{n+2,2n} & \cdots & 0 & \cdots & a_{n+2,nm} \\
0 & \cdots & a_{n+3,n} & 0 & \cdots & a_{n+3,2n} & \cdots & 0 & \cdots & a_{n+3,nm} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{2n,n} & 0 & \cdots & a_{2n,2n} & \cdots & 0 & \cdots & a_{2n,nm} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n(m-1)+1,1} & \cdots & a_{n(m-1)+1,n} & a_{n(m-1)+1,n+1} & \cdots & a_{n(m-1)+1,2n} & \cdots & a_{n(m-1)+1,n(m-1)+1} & \cdots & a_{n(m-1)+1,nm} \\
0 & \cdots & a_{n(m-1)+2,n} & 0 & \cdots & a_{n(m-1)+2,2n} & \cdots & 0 & \cdots & a_{n(m-1)+2,nm} \\
0 & \cdots & a_{n(m-1)+3,n} & 0 & \cdots & a_{n(m-1)+3,2n} & \cdots & 0 & \cdots & a_{n(m-1)+3,nm} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{nm,n} & 0 & \cdots & a_{nm,2n} & \cdots & 0 & \cdots & a_{nm,nm}
\end{pmatrix} \tag{2.1}$$

Let $f_n : \mathbb{Z}^+ \to \{1, 2, \ldots, n\}$ be a function defined by

$$f_n(k) = \begin{cases} k \pmod{n}, & k \nmid m \\ n, & k \mid m \end{cases}$$

If we denote the number $f_n(k)$ by $\overline{k}$, then the following notation is true for a matrix $A = (a_{i,j})_{nm\times nm}$ of dimension $nm \times nm$:

$$\overline{i} > \overline{j} \Rightarrow a_{i,j} = 0 \tag{2.2}$$

Theorem 2.1. Let $A_{nm\times nm}$ be a matrix satisfying (2.2) and a_{i_0,j_0} be an entry of the matrix A with $\overline{i_0} < \overline{j_0}$. Then, any product including the number a_{i_0,j_0} of the following determinant formula is zero:

$$|A| = \sum_{\sigma\in S_{nm}} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} \ldots a_{n,\sigma(n)}$$

Proof. Let σ in S_{nm} be a permutation with $\sigma(i_0) = j_0$. Then, $\sigma^{nm!} = e$ where e is the identity function in S_{nm}, i.e., $e(i) = i$, for all $1 \le i \le nm$. Assume that $\overline{\sigma^k(i_0)} \le \overline{\sigma^{k+1}(i_0)}$, for all $k \in \mathbb{Z}^+$ with $1 \le k < (nm)!$. Then,

$$\overline{i_0} = \overline{e(i_0)} = \overline{\sigma^{nm!}(i_0)} \ge \overline{\sigma^{nm!-1}(i_0)} \ge \overline{\sigma^{nm!-2}(i_0)} \ge \cdots \ge \overline{\sigma(i_0)} = \overline{j_0}$$

This contradicts the assumption $\overline{i_0} < \overline{j_0}$, i.e., there exists a number k_0 such that the relations $1 \le k_0 < (nm)!$ and $\overline{\sigma^{k_0}(i_0)} > \overline{\sigma^{k_0+1}(i_0)}$ hold. If $\sigma^{k_0}(i_0)$ is denoted by α_0, then $\overline{\alpha_0} > \overline{\sigma(\alpha_0)}$ since $\sigma(\alpha_0) = \sigma\left(\sigma^{k_0}(i_0)\right) = \sigma^{k_0+1}(i_0)$. By (2.2), $a_{\alpha_0,\sigma(\alpha_0)} = 0$. Consequently,

$$a_{1,\sigma(1)} \ldots a_{i_0,\sigma(i_0)} \ldots a_{\sigma_0,\sigma(\sigma_0)} \ldots a_{n,\sigma(n)} = 0$$

□

Corollary 2.2. Let $A_{nm\times nm}$ be a matrix satisfying the condition (2.2). Then, the determinant of A depends only on the entries $a_{i,j}$ with $\overline{i} = \overline{j}$, i.e., the determinant depends only on the entries on the main diagonal in the submatrices A_{ij} of A in (1.1). The entries $a_{i,j}$ with $\overline{i} < \overline{j}$ in the submatrices A_{ij} do not affect the determinant. Therefore, when calculating the determinant of matrix A, for convenience, the entries $a_{i,j}$ with $\overline{i} < \overline{j}$ can be taken as 0. Consequently, the determinant of a matrix A as in (2.1) and the determinant of the following matrix are equal:

$$\tilde{A} = \begin{pmatrix}
a_{1,1} & \cdots & 0 & a_{1,n+1} & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{n,n} & 0 & \cdots & a_{n,2n} & \cdots & 0 & \cdots & a_{n,nm} \\
a_{n+1,1} & \cdots & 0 & a_{n+1,n+1} & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{2n,n} & 0 & \cdots & a_{2n,2n} & \cdots & 0 & \cdots & a_{2n,nm} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n(m-1)+1,1} & \cdots & 0 & a_{n(m-1)+1,n+1} & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{nm,n} & 0 & \cdots & a_{nm,2n} & \cdots & 0 & \cdots & a_{nm,nm}
\end{pmatrix} \tag{2.3}$$

2.2. Factorization

We provide in this section a method for a factorization of the matrix $\tilde{A}$ in (2.3). Note that $\tilde{A}$ can be taken as an arbitrary matrix of dimension $nm \times nm$ with the condition $\bar{i} \neq \bar{j} \Rightarrow a_{i,j} = 0$. A factorization method for matrix $\tilde{A}$ is as follows:

$$\tilde{A} = \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_n \tag{2.4}$$

where

$$\tilde{A}_1 = \begin{pmatrix}
a_{1,1} & 0 & \cdots & 0 & a_{1,n+1} & 0 & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_{n+1,1} & 0 & \cdots & 0 & a_{n+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n(m-1)+1,1} & 0 & \cdots & 0 & a_{n(m-1)+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}$$

$$\tilde{A}_2 = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & a_{2,2} & \cdots & 0 & 0 & a_{2,n+2} & \cdots & 0 & \cdots & 0 & a_{2,n(m-1)+2} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & a_{n+2,2} & \cdots & 0 & 0 & a_{n+2,n+2} & \cdots & 0 & \cdots & 0 & a_{n+2,n(m-1)+2} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & a_{n(m-1)+2,2} & \cdots & 0 & 0 & a_{n(m-1)+2,n+2} & \cdots & 0 & \cdots & 0 & a_{n(m-1)+2,n(m-1)+2} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}$$

$$\vdots$$

$$\tilde{A}_n = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{n,n} & 0 & 0 & \cdots & a_{n,2n} & \cdots & 0 & 0 & \cdots & a_{n,nm} \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{2n,n} & 0 & 0 & \cdots & a_{2n,2n} & \cdots & 0 & 0 & \cdots & a_{2n,nm} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nm,n} & 0 & 0 & \cdots & a_{nm,2n} & \cdots & 0 & 0 & \cdots & a_{nm,nm}
\end{pmatrix}$$

(2.4) can be observed by matrix multiplication.

2.3. Determinant Formula

We have proved that the determinant of A and the determinant of $\tilde{A}$ are equal in Corollary 2.2. We have factored the matrix $\tilde{A}$ in the former section. In this section, we give a formula for the determinant of the matrix A by calculating the determinants of the factors of the matrix $\tilde{A}$.

Theorem 2.3. Consider a block matrix A whose submatrices are triangular as in (2.1). Then, the following determinant formula is valid:

$$|A| = \prod_{k=1}^{n} \begin{vmatrix} a_{k,k} & a_{k,n+k} & \cdots & a_{k,n(m-1)+k} \\ a_{n+k,k} & a_{n+k,n+k} & \cdots & a_{n+k,n(m-1)+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+k,k} & a_{n(m-1)+k,n+k} & \cdots & a_{n(m-1)+k,n(m-1)+k} \end{vmatrix} \tag{2.5}$$

Proof. Consider the matrices $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ in (2.4). Hence,

$$|A| = \left|\tilde{A}_1\right| \left|\tilde{A}_2\right| \ldots \left|\tilde{A}_n\right|$$

The determinant of the matrix $\tilde{A}_1$ is calculated as follows: If we expand the determinant along the 2nd, 3rd, ..., nth rows, respectively, then

$$\left|\tilde{A}_1\right| = \begin{vmatrix} a_{1,1} & 0 & \cdots & 0 & a_{1,n+1} & 0 & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{n+1,1} & 0 & \cdots & 0 & a_{n+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n(m-1)+1,1} & 0 & \cdots & 0 & a_{n(m-1)+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_{1,1} & a_{1,n+1} & 0 & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & 0 & \cdots & 0 \\ a_{n+1,1} & a_{n+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n(m-1)+1,1} & a_{n(m-1)+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 \end{vmatrix}$$

We expand the last determinant along the 3rd, 4th, ..., $(n+1)$th rows, respectively:

$$\left|\tilde{A}_1\right| = \begin{vmatrix} a_{1,1} & a_{1,n+1} & a_{1,2n+1} & 0 & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & 0 & \cdots & 0 \\ a_{n+1,1} & a_{n+1,n+1} & a_{n+1,2n+1} & 0 & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & 0 & \cdots & 0 \\ a_{2n+1,1} & a_{2n+1,n+1} & a_{2n+1,2n+1} & 0 & \cdots & 0 & \cdots & a_{2n+1,n(m-1)+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n(m-1)+1,1} & a_{n(m-1)+1,n+1} & a_{n(m-1)+1,2n+1} & 0 & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 \end{vmatrix}$$

If we continue this procedure, then

$$\left|\tilde{A}_1\right| = \begin{vmatrix} a_{1,1} & a_{1,n+1} & \cdots & a_{1,n(m-1)+1} \\ a_{n+1,1} & a_{n+1,n+1} & \cdots & a_{n+1,n(m-1)+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+1,1} & a_{n(m-1)+1,n+1} & \cdots & a_{n(m-1)+1,n(m-1)+1} \end{vmatrix}$$

Similarly,

$$\left|\tilde{A}_k\right| = \begin{vmatrix} a_{k,k} & a_{k,n+k} & \cdots & a_{k,n(m-1)+k} \\ a_{n+k,k} & a_{n+k,n+k} & \cdots & a_{n+k,n(m-1)+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+k,k} & a_{n(m-1)+k,n+k} & \cdots & a_{n(m-1)+k,n(m-1)+k} \end{vmatrix}$$

□

We denote the matrix of dimension $m \times m$ on the right-hand side of the above relation by

$$\tilde{A}_k^* = \begin{pmatrix} a_{k,k} & a_{k,n+k} & \cdots & a_{k,n(m-1)+k} \\ a_{n+k,k} & a_{n+k,n+k} & \cdots & a_{n+k,n(m-1)+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+k,k} & a_{n(m-1)+k,n+k} & \cdots & a_{n(m-1)+k,n(m-1)+k} \end{pmatrix} \tag{2.6}$$

Theorem 2.3 shows that the determinants of the matrices $\tilde{A}_k$ and $\tilde{A}_k^*$ are equal.

Corollary 2.4. The matrix A of dimension $nm \times nm$ in (2.1) is invertible if and only if the matrix $\tilde{A}_k^*$, for all $1 \leq k \leq n$, of dimension $m \times m$ is invertible.

Example 2.5. Calculate the determinant of the matrices

$$\begin{pmatrix} 1 & -19 & 32 & -1 & 13 & 21 \\ 0 & -1 & -7 & 0 & -2 & 12 \\ 0 & 0 & 2 & 0 & 0 & 3 \\ -2 & 22 & -24 & 3 & 5 & -9 \\ 0 & 2 & 17 & 0 & -1 & -23 \\ 0 & 0 & -1 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 20 & -1 & 12 & 2 & 52 \\ 0 & 1 & 0 & -1 & 0 & 5 \\ 3 & 25 & -3 & 32 & 1 & 78 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 9 & 5 & 74 & 6 & 10 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{pmatrix}$$

By Theorem 2.3,

$$\begin{vmatrix} 1 & -19 & 32 & -1 & 13 & 21 \\ 0 & -1 & -7 & 0 & -2 & 12 \\ 0 & 0 & 2 & 0 & 0 & 3 \\ -2 & 22 & -24 & 3 & 5 & -9 \\ 0 & 2 & 17 & 0 & -1 & -23 \\ 0 & 0 & -1 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} \begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix} = 45$$

and

$$\begin{vmatrix} 3 & 20 & -1 & 12 & 2 & 52 \\ 0 & 1 & 0 & -1 & 0 & 5 \\ 3 & 25 & -3 & 32 & 1 & 78 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 9 & 5 & 74 & 6 & 10 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 2 \\ 3 & -3 & 1 \\ -1 & 5 & 6 \end{vmatrix} \begin{vmatrix} 1 & -1 & 5 \\ 1 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = 312$$

2.4. Wronskian of the Trigonometric System

In this section, we calculate the trigonometric system $\cos p_1 x$, $\sin p_1 x$, $\cos p_2 x$, $\sin p_2 x$, $\cdots$, $\cos p_m x$, and $\sin p_m x$ where $p_1, p_2, \cdots, p_m$ are arbitrary real constants. These $2m$ functions are the fundamental solutions of the differential equation of order $2m$ corresponding to the characteristic equation

$$\left(t^2+p_1^2\right)\left(t^2+p_2^2\right)\cdots\left(t^2+p_m^2\right)=0$$

This polynomial contains no odd terms. Therefore, the Wronskian of any fundamental solutions of the corresponding differential equation is a constant, see [15]. Then, the Wronskian can be calculated at point 0:

$$W = W\left[\cos p_1 x, \sin p_1 x, \cos p_2 x, \sin p_2 x, \ldots, \cos p_m x, \sin p_m x\right]$$

$$= \begin{vmatrix} \cos p_1 x & \sin p_1 x & \cdots & \cos p_m x & \sin p_m x \\ -p_1 \sin p_1 x & p_1 \cos p_1 x & \cdots & -p_m \sin p_m x & p_m \cos p_m x \\ -p_1^2 \cos p_1 x & -p_1^2 \sin p_1 x & \cdots & -p_m^2 \cos p_m x & -p_m^2 \sin p_m x \\ p_1^3 \sin p_1 x & -p_1^3 \cos p_1 x & \cdots & p_m^3 \sin p_m x & -p_m^3 \cos p_m x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m-1} p_1^{2m-2} \cos p_1 x & (-1)^{m-1} p_1^{2m-2} \sin p_1 x & \cdots & (-1)^{m-1} p_m^{2m-2} \cos p_m x & (-1)^{m-1} p_m^{2m-2} \sin p_m x \\ (-1)^m p_1^{2m-1} \sin p_1 x & (-1)^{m-1} p_1^{2m-1} \cos p_1 x & \cdots & (-1)^m p_m^{2m-1} \sin p_m x & (-1)^{m-1} p_m^{2m-1} \cos p_m x \end{vmatrix}$$

$$= \begin{vmatrix} \cos 0 & \sin 0 & \cdots & \cos 0 & \sin 0 \\ -p_1 \sin 0 & p_1 \cos 0 & \cdots & -p_m \sin 0 & p_m \cos 0 \\ -p_1^2 \cos 0 & -p_1^2 \sin 0 & \cdots & -p_m^2 \cos 0 & -p_m^2 \sin 0 \\ p_1^3 \sin 0 & -p_1^3 \cos 0 & \cdots & p_m^3 \sin 0 & -p_m^3 \cos 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m-1} p_1^{2m-2} \cos 0 & (-1)^{m-1} p_1^{2m-2} \sin 0 & \cdots & (-1)^{m-1} p_m^{2m-2} \cos 0 & (-1)^{m-1} p_m^{2m-2} \sin 0 \\ (-1)^m p_1^{2m-1} \sin 0 & (-1)^{m-1} p_1^{2m-1} \cos 0 & \cdots & (-1)^m p_m^{2m-1} \sin 0 & (-1)^{m-1} p_m^{2m-1} \cos 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & p_1 & 0 & p_2 & \cdots & 0 & p_m \\ -p_1^2 & 0 & -p_2^2 & 0 & \cdots & -p_m^2 & 0 \\ 0 & -p_1^3 & 0 & -p_2^3 & \cdots & 0 & -p_m^3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m-1} p_1^{2m-2} & 0 & (-1)^{m-1} p_2^{2m-2} & 0 & \cdots & (-1)^{m-1} p_m^{2m-2} & 0 \\ 0 & (-1)^{m-1} p_1^{2m-1} & 0 & (-1)^{m-1} p_2^{2m-1} & \cdots & 0 & (-1)^{m-1} p_m^{2m-1} \end{vmatrix}$$

The last determinant can be calculated by (2.5). It splits two factors:

$$W = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -p_1^2 & -p_2^2 & \cdots & -p_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{m-1} p_1^{2m-2} & (-1)^{m-1} p_2^{2m-2} & \cdots & (-1)^{m-1} p_m^{2m-2} \end{vmatrix} \begin{vmatrix} p_1 & p_2 & \cdots & p_m \\ -p_1^3 & -p_2^3 & \cdots & -p_m^3 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{m-1} p_1^{2m-1} & (-1)^{m-1} p_2^{2m-1} & \cdots & (-1)^{m-1} p_m^{2m-1} \end{vmatrix}$$

$$= \left(\prod_{k=1}^{m} p_k\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -p_1^2 & -p_2^2 & \cdots & -p_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{m-1} p_1^{2m-2} & (-1)^{m-1} p_2^{2m-2} & \cdots & (-1)^{m-1} p_m^{2m-2} \end{vmatrix}^2$$

$$= \left(\prod_{k=1}^{m} p_k\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ p_1^2 & p_2^2 & \cdots & p_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ p_1^{2m-2} & p_2^{2m-2} & \cdots & p_m^{2m-2} \end{vmatrix}^2$$

The last determinant is the Vandermonde determinant of the numbers p_1^2, p_2^2, $\cdots$, p_m^2 and it is calculated by multiplying the differences between them (for more details, see [16]). Then, the Wronskian of the trigonometric system takes the following form:

$$W = \left(\prod_{k=1}^{m} p_k\right) \left(\prod_{1\leq i<j\leq m} \left(p_j^2 - p_i^2\right)\right)^2$$

Thanks to the Theorem 2.3, the proof of the last Wronskian formula is much shorter and simpler than that in [17], provided by Kaya.

Corollary 2.6. The necessary and sufficient conditions for the linear independence of trigonometric system $\cos p_1 x$, $\sin p_1 x$, $\cos p_2 x$, $\sin p_2 x$, $\cdots$, $\cos p_m x$, and $\sin p_m x$ are the following:

i. $p_k \neq 0$, for all $k \in \overline{1,m}$

ii. $p_i \neq p_j$ and $p_i \neq -p_j$, for all $i \neq j$

Corollary 2.7. The Wronskian of the particular trigonometric system $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$, $\cos mx$, and $\sin mx$ is

$$m! \left(\prod_{1\leq i<j\leq m} \left(j^2 - i^2\right)\right)^2$$

2.5. Some Properties of Block Matrices Whose Submatrices are Triangular

This section provides some properties of block matrices whose submatrices are triangular, such as sum, product, adjoint, inverse, and eigenvalues.

Theorem 2.8. The sum and product of two matrices of type (2.1) are also of type (2.1). Besides, the adjoint matrix of a matrix of type (2.1) is also of type (2.1).

Proof. The first part of the theorem can be easily proved. Therefore, we prove the second part of the theorem. It is sufficient that the cofactor of an entry a_{i_0,j_0} with $\overline{i_0} < \overline{j_0}$ is equal to 0. According to Corollary 2.2, the determinant of a matrix as in (2.1) is independent of the variable a_{i_0,j_0} with $\overline{i_0} < \overline{j_0}$. Then, the derivative of the determinant of a matrix as in (2.1) concerning a_{i_0,j_0} is 0. On the other hand, Jacobi's formula [18] for the matrix analysis says that the cofactor of an entry in a square matrix depending on the variables a_{i_0,j_0} is the derivative of the determinant of the matrix according to the considered entry. Then, the cofactors of the entries a_{i_0,j_0} with $\overline{i_0} < \overline{j_0}$ are equal to 0. □

Corollary 2.9. If a matrix as in (2.1) has an inverse, then the inverse is also of type (2.1).

The proof is obtained from the equality:

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

Theorem 2.10. Let λ be a complex number. Then, λ is an eigenvalue of a matrix A as in (2.1) if and only if there exists a number $k \in \overline{1,n}$ such that λ is an eigenvalue of the matrix $\tilde{A}_k^*$ in (2.6).

Proof. λ is an eigenvalue of the matrix A if and only if the following relation holds:

$$|A-\lambda I| = \begin{vmatrix} a_{1,1}-\lambda & \cdots & a_{1,n} & a_{1,n+1} & \cdots & a_{1,2n} & \cdots & a_{1,n(m-1)+1} & \cdots & a_{1,nm} \\ 0 & \cdots & a_{2,n} & 0 & \cdots & a_{2,2n} & \cdots & 0 & \cdots & a_{2,nm} \\ 0 & \cdots & a_{3,n} & 0 & \cdots & a_{3,2n} & \cdots & 0 & \cdots & a_{3,nm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n,n}-\lambda & 0 & \cdots & a_{n,2n} & \cdots & 0 & \cdots & a_{n,nm} \\ a_{n+1,1} & \cdots & a_{n+1,n} & a_{n+1,n+1}-\lambda & \cdots & a_{n+1,2n} & \cdots & a_{n+1,n(m-1)+1} & \cdots & a_{n+1,nm} \\ 0 & \cdots & a_{n+2,n} & 0 & \cdots & a_{n+2,2n} & \cdots & 0 & \cdots & a_{n+2,nm} \\ 0 & \cdots & a_{n+3,n} & 0 & \cdots & a_{n+3,2n} & \cdots & 0 & \cdots & a_{n+3,nm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{2n,n} & 0 & \cdots & a_{2n,2n}-\lambda & \cdots & 0 & \cdots & a_{2n,nm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n(m-1)+1,1} & \cdots & a_{n(m-1)+1,n} & a_{n(m-1)+1,n+1} & \cdots & a_{n(m-1)+1,2n} & \cdots & a_{n(m-1)+1,n(m-1)+1}-\lambda & \cdots & a_{n(m-1)+1,nm} \\ 0 & \cdots & a_{n(m-1)+2,n} & 0 & \cdots & a_{n(m-1)+2,2n} & \cdots & 0 & \cdots & a_{n(m-1)+2,nm} \\ 0 & \cdots & a_{n(m-1)+3,n} & 0 & \cdots & a_{n(m-1)+3,2n} & \cdots & 0 & \cdots & a_{n(m-1)+3,nm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nm,n} & 0 & \cdots & a_{nm,2n} & \cdots & 0 & \cdots & a_{nm,nm}-\lambda \end{vmatrix} = 0$$

By (2.5), the last relation can be rewritten as follows:

$$|A-\lambda I| = \prod_{k=1}^{n} \begin{vmatrix} a_{k,k}-\lambda & a_{k,n+k} & \cdots & a_{k,n(m-1)+k} \\ a_{n+k,k} & a_{n+k,n+k}-\lambda & \cdots & a_{n+k,n(m-1)+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+k,k} & a_{n(m-1)+k,n+k} & \cdots & a_{n(m-1)+k,n(m-1)+k}-\lambda \end{vmatrix} = 0$$

□

3. Conclusion

This paper proves that the determinant of a large-scale block matrix whose submatrices are triangular does not need to be computed using classical and computational methods. The determinant of such matrices is equal to the product of the determinants of their special submatrices. This method greatly reduces the computational effort involved in calculating the determinant.

While the results presented in this paper significantly simplify the calculation of determinants for block matrices with triangular submatrices, several promising directions remain for future research. One area of potential exploration is the extension of these methods to non-triangular block matrices or matrices with more complex structural patterns. Additionally, investigating the applications of these findings in other branches of linear algebra, such as in solving systems of linear equations or in eigenvalue analysis, could provide further insights. Researchers may also consider applying these techniques to real-world problems in physics, engineering, or data science, where large-scale matrix computations are essential. Finally, developing more advanced computational tools and algorithms that leverage the factorization methods discussed here could contribute to faster and more efficient determinant calculations in large matrices.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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Minimal Curves on Ruled Surfaces Generated by Legendre Curves

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Abstract — In this paper, we study the conditions under which Legendre curves on ruled surfaces are classified as minimal loci. By investigating the scenario where the directive curve is a binormal vector, we establish the criteria for these curves to have minimal loci on B-scroll-ruled surfaces. Furthermore, we present illustrative examples showcasing concrete instances of minimal curves within this context. Finally, we discuss the need for further research.

Keywords *Legendre curves, minimal curves, B-scroll surfaces*

Mathematics Subject Classification (2020) 53A10, 53A04

1. Introduction

A ruled surface is one of the special surfaces represented by moving a straight line continuously along a space curve called the base curve. More explicitly, a surface $\mathcal{M}$ in $\mathbb{R}^3$ is called a ruled surface if it admits a parameterization $\Phi_{(\sigma,\gamma)} : I \times J \to \mathcal{M}$ which consists of a collection of a one-parameter family of straight lines indexed by u in the form of $\Phi_{(\sigma,\gamma)}(s,u) = \sigma(s) + u\gamma(s)$ where $s \in I$ and $u \in J$ such that I and J are open intervals in $\mathbb{R}$ [1]. Here, σ and γ are smooth mappings defined from the interval I to $\mathbb{R}^3$. Moreover, σ is the base curve or directrix, and the non-null curve γ is the director curve. The straight lines $u \to \sigma(s) + u\gamma(s)$ are the rulings. Ruled surfaces have many important applications in various fields of science and technology, such as computer-aided geometric design (CAGD), architectural designing, manufacturing technology, and robotics [2–4]. In recent years, there has been intensive research on ruled surfaces in various spaces, including Euclidean, Lorentzian, and Minkowski spaces [5–7], where important properties and characterizations are presented. In [8], two developable ruled surfaces have been introduced using the principal normal indicatrix of a regular space as a base curve for both surfaces and the tangent indicatrix and the binormal indicatrix as the director curves. Afterward, [9] has extended the work of [8] by providing the condition for a minimal locus for the related surfaces.

In [10], a moving frame of a Legendre curve in the unit tangent bundle has been introduced, and a pair of smooth functions of a Legendre curve, analogous to the curvature of a regular plane curve, has been defined. The existence and uniqueness of Legendre curves have been proved. Legendre curves on the unit tangent bundle have been provided in [11] using rotation-minimizing (RM) vector fields.

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Besides, [12] has demonstrated that any Legendre curve in TS^2 corresponds to a developable ruled surface. For recent studies on Legendre curves, we refer to [13,14]. In [15], a ruled surface's mean and Gaussian curvatures are expressed in terms of its striction and director curves.

The remainder of the present study is organized as follows: Section 2 presents fundamental concepts and properties to be used in the following sections. Section 3 introduces two main results: Theorem 3.1 and Theorem 3.2. In Theorem 3.1, we calculate the mean and Gaussian curvature of a ruled surface, generating a Legendre curve together with the director curve, whose base curve differs from those in [15]. Hence, the results obtained in [15] are simplified. In Theorem 3.2, we investigate the conditions under which Legendre curves on ruled surfaces are classified as minimal loci. Section 4 obtains the criteria for these curves to be minimal loci on B-scroll ruled surfaces by taking the directive curve as a binormal vector, as provided in Theorem 4.1. Moreover, in Theorem 4.3, we obtain the minimality condition for the developable ruled surface with a special choice of the base curve. Section 5 provides some computational examples of minimal curves. The final section is dedicated to conclusions and final remarks.

2. Preliminaries

This section presents some basic concepts of the theory of surfaces in $\mathbb{R}^3$ to be needed to prove our main results. For a detailed discussion on the related subjects, see [1,16].

Let $\Phi(s,u)$ be a regular surface with the first and second fundamental forms

$$Eds^2 + 2Fdsdu + Gdu^2 \quad \text{and} \quad Lds^2 + 2Mdsdu + Ndu^2$$

respectively. Traditionally, the mean and Gaussian curvatures of the surface are provided in the following form:

$$H = \frac{LG - 2MF + NE}{2\left(EG - F^2\right)} \quad \text{and} \quad K = \frac{LN - M^2}{EG - F^2}$$

where E, F, and G and L, M, and N are the coefficients of the first and second fundamental forms, respectively. In [15], for a ruled surface $\Phi\left(s,u\right) = \sigma(s) + u\gamma(s)$, the mean curvature is presented in the following form:

$$H = \frac{\langle \sigma'' + u\gamma'', \sigma' \times \gamma + u\left(\gamma' \times \gamma\right)\rangle - 2\left\langle \sigma', \gamma\right\rangle \left\langle \gamma', \sigma' \times \gamma\right\rangle}{2\left(EG - F^2\right)^{\frac{3}{2}}} \tag{2.1}$$

where $\sigma(s)$ is a striction curve on the surface. Moreover, the Gauss curvature is formulated as follows:

$$K = -\frac{\left\langle \gamma', \sigma' \times \gamma\right\rangle^2}{\left(\left\|\sigma' \times \gamma\right\|^2 + u^2\left\|\gamma'\right\|^2\right)\left(\left\|\sigma'\right\|^2 + u^2\left\|\gamma'\right\|^2 - \left\langle \sigma', \gamma\right\rangle^2\right)} \tag{2.2}$$

A minimal locus of a ruled surface is defined as follows:

Definition 2.1. Let $\Phi(s,u) = \sigma(s) + u\gamma(s)$ be a ruled surface. If the mean curvature of the surface $\Phi(s,u)$ along the curve $X(s) = \sigma(s) + u(s)\gamma(s)$ is zero, then the curve $X(s)$ is called the minimal curve on $\Phi(s,u)$.

Legendre curves can be provided by the following definition:

Definition 2.2. The smooth curve $\Gamma(s) = (\alpha(s), v(s)) : I \subset \mathbb{R} \to \mathcal{M}$ is called a Legendre curve in $\mathcal{M}$ if $\langle \alpha'(s), v(s)\rangle = 0$.

It can be observed that $\langle \alpha(s), v(s)\rangle = 0$ for a smooth curve $\Gamma(s) = (\alpha(s), v(s))$ in $\mathcal{M}$. Thus, we can define a new frame using the unit vector $\eta = \alpha(s) \times v(s)$ where the symbol $\times$ denotes the usual vector product in $\mathbb{R}^3$ [11]. It is obvious that

$$\langle \alpha(s), \eta(s) \rangle = \langle v(s), \eta(s) \rangle = 0$$

Hence, the following Frenet frame $\{\alpha(s), v(s), \eta(s)\}$ along $\alpha(s)$ is obtained as follows:

$$\begin{pmatrix} \eta'(s) \\ \alpha'(s) \\ v'(s) \end{pmatrix} = \begin{pmatrix} 0 & m(s) & n(s) \\ -m(s) & 0 & l(s) \\ -n(s) & -l(s) & 0 \end{pmatrix} \begin{pmatrix} \eta(s) \\ \alpha(s) \\ v(s) \end{pmatrix} \tag{2.3}$$

where

$$l(s) = \langle \alpha'(s), v(s) \rangle, \quad m(s) = -\langle \alpha'(s), \eta(s) \rangle, \quad \text{and} \quad n(s) = -\langle v'(s), \eta(s) \rangle$$

Here, the elements of the set $\{l, m, n\}$ are called the curvature functions of Γ.

If $l(s) = 0$, then the curve $\Gamma(s) = (\alpha(s), v(s))$ is Legendre in $\mathcal{M}$ with the curvature functions (m, n). Then, the Frenet frame provided in (2.3) for the Legendre condition can be provided by

$$\begin{pmatrix} \eta'(s) \\ \alpha'(s) \\ v'(s) \end{pmatrix} = \begin{pmatrix} 0 & m(s) & n(s) \\ -m(s) & 0 & 0 \\ -n(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta(s) \\ \alpha(s) \\ v(s) \end{pmatrix}$$

For the Legendre curve $(T(s), B(s))$, the definition of B-scroll is as follows:

Definition 2.3. Let $\{T, N, B\}$ be the Frenet frame of the unit speed curve $\sigma \colon I \to E^3$. The ruled surface formed by the binormal vector B along the curve σ is called the binormal scroll (briefly B-scroll). Here, the curve σ is called the base curve of the B-scroll, and the binormal vector B is called its director curve. The parametric equation of B-scroll is written as follows: $\Phi(s, u) = \sigma(s) + uB(s)$.

The notion of B-scroll surfaces has been introduced in [17]. The B-scroll's first and second fundamental forms with Cartan framed null directrix in the Minkowskian 3-space are investigated in [18]. B-scrolls in 3-dimensional Lorentzian space L^3 are studied in [19].

3. The Legendre Curves and Minimal Curves on Ruled Surface

In this study, we consider a ruled surface

$$\Phi(s, u) = \int \lambda(s)\alpha(s)ds + uv(s)$$

where $(\alpha(s), v(s))$ is a smooth Legendre curve. We present the mean curvature of this surface in the following theorem.

Theorem 3.1. Let $(\alpha(s), v(s))$ be a Legendre curve. For the Frenet frame $\{\alpha(s), v(s), \eta(s)\}$ along $\alpha(s)$, the mean and Gauss curvatures of the ruled surface $\Phi(s, u) = \int \lambda(s)\alpha(s)ds + uv(s)$ are as follows, respectively:

$$H = \frac{-n^2(s)m(s)u^2 + (\lambda'(s)n(s) - \lambda(s)n'(s))\,u - m(s)\lambda^2(s)}{2\left(\lambda^2(s) + u^2n^2\right)^{\frac{3}{2}}} \tag{3.1}$$

and

$$K = -\frac{\lambda^2 n^2}{\left(\lambda^2 + u^2n^2\right)^2}$$

Proof. Consider

$$\sigma(s) = \int \lambda(s)\alpha(s)ds, \; \gamma(s) = v(s)$$

By the formula in (2.1), the following equalities hold.

$$\langle \sigma'(s), \gamma(s) \rangle = \langle \lambda(s)\alpha(s), v(s) \rangle = 0 \tag{3.2}$$

$$\langle\sigma'(s),\gamma'(s)\rangle=\langle\lambda(s)\alpha(s),v'(s)\rangle=\langle\lambda(s)\alpha(s),-n(s)\eta(s)\rangle=0 \tag{3.3}$$

$$\sigma''(s)+u\gamma''(s)=-\left(m(s)\lambda(s)+un'(s)\right)\eta(s)+\left(\lambda'(s)-un(s)m(s)\right)\alpha(s)-un^2(s)v(s) \tag{3.4}$$

and

$$\sigma'(s)\times\gamma(s)+u\left(\gamma'(s)\times\gamma(s)\right)=\lambda(s)\eta(s)+un(s)\alpha(s) \tag{3.5}$$

Moreover, we need to compute the first fundamental forms of the surface $\Phi(s,u)$ to evaluate the mean curvature H. By the equality $\Phi(s,u)=\sigma(s)+u\gamma(s)$, $\Phi_s=\sigma'(s)+u\gamma'(s)$ and $\Phi_u=\gamma(s)$. Therefore,

$$\begin{aligned}E&=\|\Phi_s\|^2=\|\sigma'(s)+u\gamma'(s)\|^2\\&=\langle\sigma'(s)+u\gamma'(s),\sigma'(s)+u\gamma'(s)\rangle\\&=\|\sigma'\|^2+u^2\|\gamma'\|^2+2u\langle\sigma',\gamma'\rangle\end{aligned}$$

By using (3.3), the equality $E=\|\sigma'\|^2+u^2\|\gamma'\|^2$ is obtained. Besides, $G=\|\Phi_u\|^2=\|\gamma\|^2=1$. Similarly,

$$F=\langle\Phi_s,\Phi_u\rangle=\langle\sigma'(s)+u\gamma'(s),\gamma(s)\rangle=\langle\sigma'(s),\gamma(s)\rangle+u\langle\gamma'(s),\gamma(s)\rangle$$

Since $\|\gamma\|=1$, i.e., $\langle\gamma,\gamma\rangle=1$, then $\langle\gamma',\gamma\rangle=0$ which implies $F=\langle\sigma'(s),\gamma(s)\rangle$. Moreover,

$$EG-F^2=\|\sigma'\|^2+u^2\|\gamma'\|^2-\left(\langle\sigma'(s),\gamma(s)\rangle\right)^2=\lambda^2+u^2n^2$$

Using (3.2), (3.4), and (3.5) in (2.1),

$$H=\frac{-n^2(s)m(s)u^2+\left(\lambda'(s)n(s)-\lambda(s)n'(s)\right)u-m(s)\lambda^2(s)}{2\left(\lambda^2(s)+u^2n^2\right)^{\frac{3}{2}}}$$

In addition, we have the following equalities to evaluate equality (2.2):

$$\langle\gamma',\sigma'\times\gamma\rangle=\langle -n\eta,\lambda\eta\rangle=-\lambda n \tag{3.6}$$

$$\|\sigma'\times\gamma\|^2+u^2\|\gamma'\|^2=\|\lambda\eta\|^2+u^2\|-n\eta\|^2=\lambda^2+u^2n^2 \tag{3.7}$$

and

$$\|\sigma'\|^2+u^2\|\gamma'\|^2-\langle\sigma',\gamma\rangle^2=\|\lambda\alpha\|^2+u^2\|-n\eta\|^2=\lambda^2+u^2n^2 \tag{3.8}$$

If we substitute (3.6)-(3.8) into (2.2), then $K=-\frac{\lambda^2n^2}{(\lambda^2+u^2n^2)^2}$. □

In the following theorem, we provide the necessary condition for a curve to be minimal on the surface $\Phi(s,u)$.

Theorem 3.2. Let $(\alpha(s),v(s))$ be a smooth Legendre curve. Then, the curve $\beta(s)=\int\lambda(s)\alpha(s)ds+u_{1,2}v(s)$ is minimal on the ruled surface $\Phi(s,u)=\int\lambda(s)\alpha(s)ds+uv(s)$ where

$$u_{1,2}=\frac{\lambda'n-\lambda n'\pm\sqrt{\left(\lambda'n-\lambda n'\right)^2-\left(4mn\lambda\right)^2}}{2mn^2}$$

Proof. From (3.1), the condition for the curve $\beta(s)$ to be a minimal curve on the surface $\Phi(s,u)$ is as follows:

$$n^2(s)m(s)u^2+\left(\lambda(s)n'(s)-\lambda'(s)n(s)\right)u+m(s)\lambda^2(s)=0 \tag{3.9}$$

Solution of this second-order equation is

$$u_{1,2}=\frac{\lambda'n-\lambda n'\pm\sqrt{\left(\lambda'n-\lambda n'\right)^2-\left(4mn\lambda\right)^2}}{2mn^2}$$

which completes the proof. □

As a special case for $u=\lambda(s)$, we provide the following proposition:

Proposition 3.3. Let $(\alpha(s), v(s))$ be a smooth Legendre curve. Then, the curve $\beta(s)$ is minimal on the ruled surface $\Phi(s,u) = \int \lambda(s)\alpha(s)ds + uv(s)$ where

$$u = e^{\int \frac{\left(1+n^2(s)\right)m(s)+n'(s)}{n(s)}ds}$$

Proof. By (3.9), if $u = \lambda(s)$, then the following equation holds:

$$\lambda'(s)n(s) + \lambda(s)\left(-m(s) - n'(s) - n^2(s)m(s)\right) = 0$$

The solution of the equation is

$$\lambda(s) = e^{\int \frac{\left(1+n^2(s)\right)m(s)+n'(s)}{n(s)}ds}$$

Thus, for $u = \lambda(s)$, the curve $\beta(s) = \int \lambda(s)\alpha(s)ds + uv(s)$ is minimal on the ruled surface $\Phi(s,u)$. □

Specifically taking $\alpha(s) = T(s)$ and $v(s) = B(s)$, we get the following proposition.

Proposition 3.4. Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth Legendre curve with frame apparatus $\{T, N, B, \kappa, \tau\}$. Then, the curve $\beta(s) = \int \lambda(s)\alpha(s)ds + uv(s)$ is minimal on a ruled surface $\Phi(s,u) = \int \lambda(s)T(s)ds + uB(s)$ where

$$u = \lambda(s) = e^{\int \frac{1}{\tau(s)}\left(-\kappa(s)\tau^2(s)-\kappa(s)+\tau'(s)\right)ds}$$

Proof. After the necessary calculations, then the following equalities are obtained:

$$\sigma''(s) + u''(s) = \left(\lambda'(s) + u\kappa(s)\tau(s)\right)T(s) + \left(\lambda(s)\kappa(s) - u\tau'(s)\right)N(s) - u\tau^2(s)B(s)$$

and

$$\sigma'(s) \times \gamma(s) + u\left(\gamma'(s) \times \gamma(s)\right) = -u\tau(s)T(s) - \lambda(s)N(s)$$

Hence, the condition of the minimal curve on the ruled surface $\Phi(s,u)$ is equivalent to the following equation:

$$-u\tau(s)\lambda'(s) - u^2\kappa(s)\tau^2(s) - \lambda^2(s)\kappa(s) + \lambda(s)u\tau'(s) = 0$$

For $u = \lambda(s)$,

$$-u\tau(s)\lambda'(s) - u^2\kappa(s)\tau^2(s) - \lambda^2(s)\kappa(s) + \lambda(s)u\tau'(s) = 0$$

$$-\lambda(s)\lambda'(s)\tau(s) - \lambda^2(s)\kappa(s)\tau^2(s) - \lambda^2(s)\kappa(s) + \lambda^2(s)\tau'(s) = 0$$

$$-\lambda'(s)\tau(s) + \lambda(s)\left(-\kappa(s)\tau^2(s) - \kappa(s) + \tau'(s)\right) = 0 \tag{3.10}$$

After the integration of both sides of (3.10),

$$\int \frac{\lambda'(s)}{\lambda(s)}ds = \int \frac{1}{\tau(s)}\left(-\kappa(s)\tau^2(s) - \kappa(s) + \tau'(s)\right)ds$$

which yields

$$\lambda(s) = e^{\int \frac{1}{\tau(s)}\left(-\kappa(s)\tau^2(s)-\kappa(s)+\tau'(s)\right)ds}$$

Hence, for $u = \lambda(s)$, the curve $\beta(s)$ is minimal on the ruled surface $\Phi(s,u)$.

□

In particular, if $\lambda(s) = 1$, then we obtain a B-scroll ruled surface.

4. Minimal Curves on B-scroll Surfaces

In the following theorem, we investigate the condition that a curve is a minimal curve on the B-scroll surface.

Theorem 4.1. Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth Legendre curve with frame apparatus $\{T, N, B, \kappa, \tau\}$ and consider the B-scroll surface

$$\Phi(s,u) = \int T(s)ds + uB(s) = \alpha(s) + uB(s)$$

If $\sigma(s) = \alpha(s)$ and $\gamma(s) = B(s)$, then the curves $\beta(s) = \alpha(s) + u_{1,2}B(s)$ are minimal on the B-scroll ruled surface $\Phi(s,u)$ such that

$$u_{1,2} = \frac{\tau'(s) \pm \sqrt{\tau^2(s) - 4\kappa^2(s)\tau^2(s)}}{2\kappa(s)\tau^2(s)}$$

Proof. If the equalities

$$\sigma''(s) + u\gamma''(s) = u\kappa(s)\tau(s)T(s) + (\kappa(s) - u\tau'(s))N(s) - u\tau^2(s)B(s)$$

and

$$\sigma'(s) \times \gamma(s) + u(\gamma'(s) \times \gamma(s)) = -u\tau(s)T(s) - N(s)$$

are used in (2.1), then the following relation is obtained as a minimality condition:

$$-\kappa(s)\tau^2(s)u^2 + \tau'(s)u - \kappa = 0 \tag{4.1}$$

The solution of (4.1) is obtained as follows:

$$u_{1,2} = \frac{\tau'(s) \pm \sqrt{\tau^2(s) - 4\kappa^2(s)\tau^2(s)}}{2\kappa(s)\tau^2(s)}$$

Therefore the curves $\beta(s) = \alpha(s) + u_{1,2}B(s)$ are minimals on B-scroll surface. Here, if τ is arbitrary constant and $\kappa = \dfrac{1}{2}$, then $u_1 = u_2 = \frac{\tau'(s)}{\tau^2(s)}$. □

Hence, we have the following proposition.

Proposition 4.2. Let $\alpha \subset E^3$ be a Salkowski curve with arbitrary τ and $\kappa = \dfrac{1}{2}$. Then, the curve $\beta(s) = \alpha(s) + \dfrac{\tau'(s)}{\tau^2(s)}B(s)$ are minimal on the B-scroll surface.

We present a theorem for a developable ruled surface whose base curves are minimal.

Theorem 4.3. Let $\gamma(s)$ be a unit speed curve. Then, the mean curvature of the developable ruled surface

$$\Phi(s,u) = \int f(s)\gamma(s)ds + u\gamma(s) \tag{4.2}$$

along its base curve, the striction curve for $u = 0$ is zero.

Proof. Consider the following Frenet frame $\{\gamma, T, S\}$ along $\gamma(s)$:

$$\begin{pmatrix} \gamma' \\ T' \\ S' \end{pmatrix} = \begin{pmatrix} 0 & m(s) & 0 \\ -m(s) & 0 & n(s) \\ 0 & -m(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ T \\ S \end{pmatrix}$$

Then,

$$\sigma'(s) = f(s)\gamma(s), \quad \sigma''(s) = f'(s)\gamma(s) + f(s)m(s)T(s)$$

$$\gamma'(s) = m(s)T(s), \quad \gamma''(s) = -m^2\gamma(s) + m'(s)T(s) + m(s)n(s)S(s)$$

and

$$\sigma'(s) \times \gamma(s) = 0, \quad \gamma'(s) \times \gamma(s) = -m(s)S(s)$$

If we substitute these equalities into formula (2.1), then the following minimality condition is obtained:

$$\begin{aligned}\langle f'(s)\gamma + f(s)m(s)T + u(-m^2\gamma(s) + m'(s)T + m(s)n(s)S), -um(s)S\rangle &= \langle m(s)n(s)S(s), -um(s)S(s)\rangle \\ &= -u^2m^2(s)n(s) \\ &= 0\end{aligned}$$

Hence, the condition is a minimal curve on the surface $\Phi(s,u)$ in the following form: $u = 0$, $m \neq 0$, and $n \neq 0$. $\square$

By using Theorem 4.3, we provide two examples for the surfaces in [8], minimal along the striction curve:

Example 4.4. Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with frame apparatus $\{T, N, B, \kappa, \tau\}$. By (4.2), if $\gamma(s) = N(s)$ and $f(s) = k(s)$, then the following ruled surface is obtained:

$$\Phi\,(s,u) = T(s) + uN(s) \tag{4.3}$$

The striction curve $T(s)$ is minimal on the ruled surface (4.3) for $u = 0$.

Example 4.5. Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with frame apparatus $\{T, N, B, \kappa, \tau\}$. By (4.2), if $\gamma(s) = N(s)$ and $f(s) = -\tau(s)$, the ruled surface $\Phi\,(s,u) = B(s) + uN(s)$ is obtained. Thus, the base curve $B(s)$ is minimal on this surface.

5. Some Computational Examples

This section provides two computational examples to illustrate the obtained results.

Example 5.1. Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth curve defined by $\alpha(s) = (\cos s, \sin s, s)$. Then, the tangent and binormal vector fields of α are as follows, respectively:

$$T(s) = (-\sin s, \cos s, 1) \quad \text{and} \quad B(s) = \frac{1}{\sqrt{2}}(\sin s, -\cos s, 1)$$

with the curvature $\kappa = \dfrac{1}{2}$ and the torsion $\tau = \dfrac{1}{2}$. The curve $\Gamma(s) = (T(s), B(s))$ is Legendre in the ruled surface $\Phi\,(s,u) = \int \lambda T(s)ds + uB(s)$. If $\lambda(s) = e^{\int \frac{1}{\tau(s)}\left(-\kappa(s)\tau^2(s) - \kappa(s) + \tau'(s)\right)ds} = e^{-\frac{5}{4}s}$, then the ruled surface $\Phi\,(s,u) = \int e^{-\frac{5}{4}s}(-\sin s, \cos s, 1) + u\frac{1}{\sqrt{2}}(\sin s, -\cos s, 1)$ is obtained (see Figure 1).

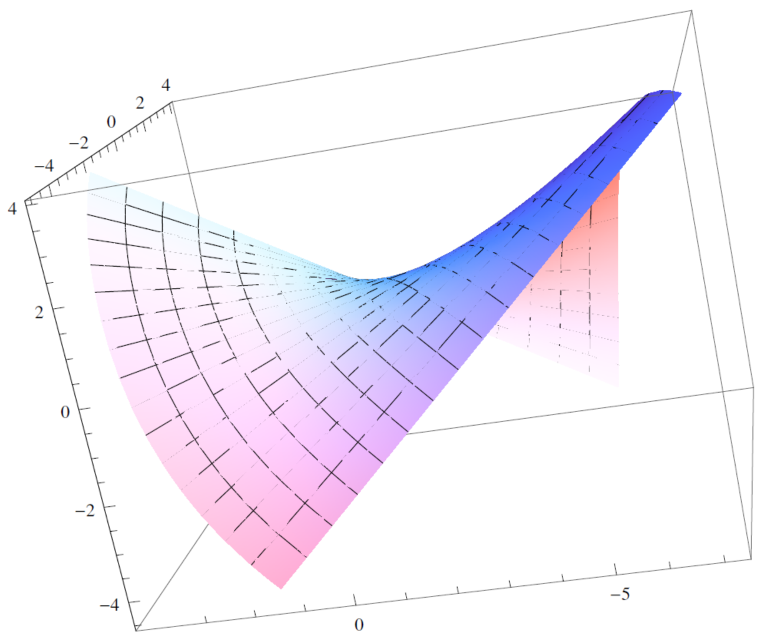

Figure 1. Ruled surface $\Phi_{(T,B)}$

Moreover, the curve

$$\beta(s) = e^{-\frac{5}{4}s}\left(\frac{16}{41}\cos s + \left(\frac{20}{41} + \frac{1}{\sqrt{2}}\right)\sin s, -\left(\frac{20}{41} + \frac{1}{\sqrt{2}}\right)\cos s + \frac{16}{41}\sin s, \left(-\frac{4}{5} + \frac{1}{\sqrt{2}}\right)\right)$$

is the minimal in the ruled surface $\Phi(s,u)$ (see Figure 2).

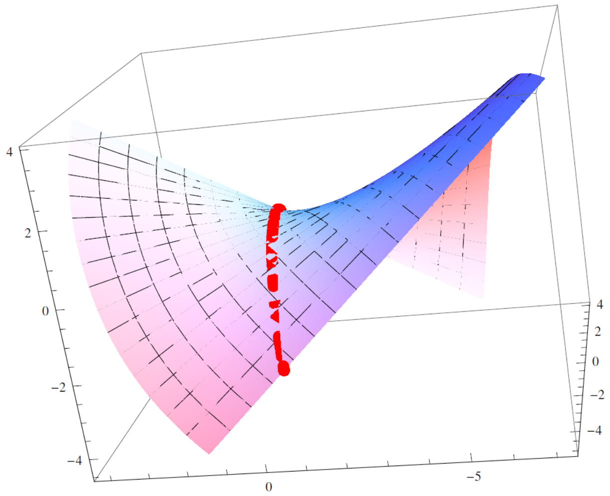

Figure 2. Minimal curve $\beta(s)$ in the red color on the ruled surface $\Phi(s,u)$

Example 5.2. Consider the smooth curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ defined by $\gamma(s) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, 1)$ and the unit vector $v(s) = \dfrac{1}{\sqrt{2}}(\cos s, \sin s, 0)$. Then, $\Gamma(s) = (\gamma(s), v(s))$ is a Legendre curve and the following Frenet frame $\{\alpha(s), v(s), \eta(s)\}$ along $\gamma(s)$ is obtained:

$$\begin{pmatrix} \eta'(s) \\ \alpha'(s) \\ v'(s) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta(s) \\ \alpha(s) \\ v(s) \end{pmatrix}$$

Hence, the ruled surface

$$\Phi(s,u) = \gamma(s) + uv(s) = -\frac{1}{\sqrt{2}}e^{-\frac{3}{2}s}\left(-\frac{6}{13}\cos s + \frac{4}{13}\sin s, -\frac{6}{13}\sin s - \frac{4}{13}\cos s, \frac{2}{3}\right) + \frac{u}{\sqrt{2}}(\cos s, \sin s, 0)$$

is obtained (see Figure 3).

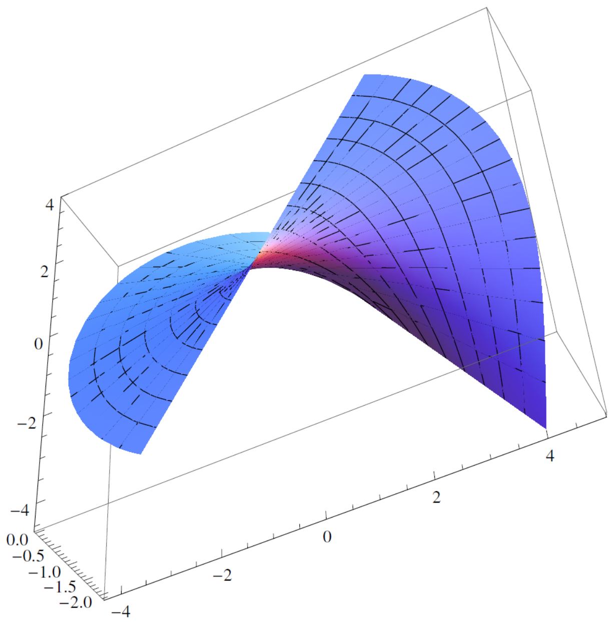

Figure 3. Ruled surface $\Phi(s,u)$

Then, the curve

$$\beta(s) = -\frac{1}{\sqrt{2}} e^{-\frac{3}{2}s} \left(-\frac{19}{13}\cos s + \frac{4}{13}\sin s, -\frac{19}{13}\sin s - \frac{4}{13}\cos s, \frac{2}{3} \right)$$

is minimal in $\Phi(s,u)$ (see Figure 4).

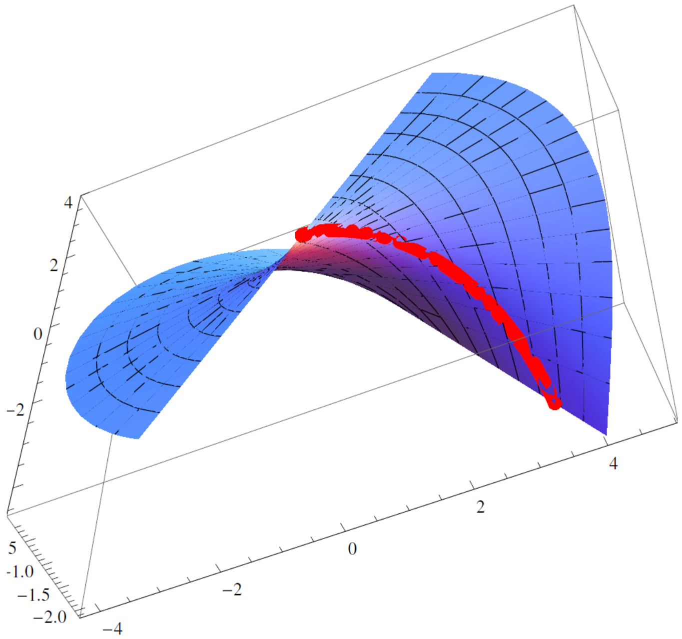

Figure 4. $\beta(s)$ in the red color on the ruled surface $\Phi(s,u)$

6. Conclusion

In this paper, we consider the ruled surfaces generated by Legendre curves and obtain the condition that these curves are minimal on them. We also study Legendre curves on B-scroll surfaces and provide the condition that a curve is minimal on these surfaces. Finally, we offer some computational examples and graphs of the related minimal curves on the ruled surfaces.

In [20], the relation between two Darboux frames of the standard curve $c(s)$ relatively to φ and Ψ was presented. It was proved that the ruled surface is minimal along its base curve if and only if the base curve is a geodesic curve on the regular surface. In this context, new results can be obtained for the ruled surfaces generated by Legendre curves considered in this paper. Furthermore, similar characterizations can be explored in Minkowskian and Lorentzian spaces.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Domination Scattering Number in Graphs

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Research Article

Abstract — Scattering number measures the stability of a graph by determining how well vertices are spread throughout the graph. However, it may not always be distinctive for different graphs, especially when comparing the same scattering numbers. In this study, we aim to provide a more nuanced and sensitive measure of stability for graphs by introducing domination scattering numbers, a new measure of graph stability. This parameter likely captures additional structural characteristics or dynamics within the graph that contribute to its stability or resilience. Moreover, we investigate the domination scattering numbers of the graphs P_n, C_n, $K_{1,n}$, $K_{m,n}$, and $P_n \times C_3$.

Keywords *Vulnerability, domination number, scattering number*

Mathematics Subject Classification (2020) 05C40, 05C69

1. Introduction

Network stability depends on nodes (processing) and links (communications or transport). Whenever a link or node is lost, the effectiveness of the network decreases. Communication networks should be stable during initial disruptions and future reconstructions. A network's stability can be measured by its cost of disruption. Analyzing the stability of a network against disruption is crucial in various fields like telecommunications, transportation, and ecology. Here are some fundamental concepts to consider [1–3]:

i. The number of non-functioning nodes in a network depends on several factors, such as the nature of the disruption. It is important to determine the number of these nodes.

ii. By analyzing how many groups still have mutual communication after a network outage, the network's topology needs to be evaluated.

iii. In terms of difficulty, connecting a network that has been disrupted varies widely based on factors such as the scale of the disruption, the nature of the network, available resources, and expertise.

Modeling a communication network as a graph is a common and effective approach to analyzing its stability and behavior. In this graph model, the following concepts are involved:

i. Vertices (Nodes): Each node in the graph represents a distinct entity within the communication network. These entities could be devices, e.g., computers, routers, or smartphones, communication endpoints, such as users or servers, or any other relevant network component.

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ii. Edges (Links): Each edge in the graph represents a communication link or connection between two nodes. These links could be physical connections, e.g., cables or wireless, or logical connections, such as virtual circuits or network paths.

By representing the considered network as a graph, various graph theory concepts and algorithms can be applied to analyze its properties, connectivity, and stability. We have some graph theoretical parameters to obtain the stability of communication networks, e.g., connectivity, integrity, toughness, and scattering number [1, 4–7, 11]. Edge versions of these graph parameters are also defined. The scattering number is handy for measuring the stability of a graph. However, it does not provide good results for some families of graphs, and the edge scattering number does not yield satisfactory results for certain graphs. In other words, these parameters are not distinctive between some families of graphs. This paper investigates a new parameter for stability, considering this situation.

If scattering numbers and dominance are thought together, then when a small group of decision-makers has effective communication links with each other, dominance in graphs can provide a valuable model for deciding what to do [12]. In essence, removing a minimum dominating set like X can trigger a cascade of adverse effects, culminating in chaos within the network. It highlights the critical role played by centralized decision-makers and effective communication channels in maintaining organizational stability and functionality [12]. The motivation of this paper is to choose the dominating set of a graph instead of the set X when calculating the scattering number. By this choice, this paper introduces a new graph parameter.

2. Preliminaries

Throughout this paper, we use the notation $w(G)$ to denote the order of the most significant component. We provide some basic definitions to be needed in the following sections.

Definition 2.1. [7] The scattering number of a noncomplete connected graph G is defined by

$$\mathrm{sc}(G) = \max\{w(G-X) - |X| \ : \ X \subset V(G) \text{ and } w(G-X) \geq 2\}$$

where the notation $|X|$ represents the cardinality of X. Moreover, a set $X \subset V(G)$ is called a scatter-set of G if $\mathrm{sc}(G) = w(G-X) - |X|$.

Some results for this parameter are provided as follows:

Theorem 2.2. [3] If G is a noncomplete connected graph of order n, then

$$2\eta(G) - n \leq \mathrm{sc}(G) \leq \eta(G) - \kappa(G)$$

where $\eta(G)$ and $\kappa(G)$ are independence number and connectivity number of the graph G, respectively.

We then present the cartesian product of two graphs.

Definition 2.3. [6, 13] Let G and H be two graphs, V_G and V_H be the sets of vertices of G and H, respectively, $V = V_G \times V_H$, and $m, n \in V$ such that $m = (m_1, m_2)$ and $n = (n_1, n_2)$. Then, the cartesian product of G and H, denoted by $G \times H$, is defined by vertices in V that m and n are adjacent in $G \times H$ if and only if $m_1 = n_1$ and the vertices m_2 and n_2 in V_H are adjacent in H or $m_2 = n_2$ and the vertices m_1 and n_1 in V_G are adjacent in G.

Theorem 2.4. [14] Let $m \geq 2$ and $n \geq 2$. Then,

$$\mathrm{sc}\left(K_{1,m} \times P_n\right) = \begin{cases} m-1, & n \text{ is even} \\ m-2, & n \text{ is odd} \end{cases}$$

For more information about scattering numbers, refer to [3, 7, 11, 14–17]. The edge version of the scattering number has been defined by Aslan [18].

Definition 2.5. [18, 19] The edge scattering number of a noncomplete connected graph G is defined by

$$\mathrm{es}(G) = \max\{w(G - X) - |X| \ : \ X \subseteq E(G) \text{ and } w(G - X) \geq 2\}$$

where the notation $|X|$ represents the cardinality of X. Moreover, a set $X \subseteq E(G)$ is called an edge scatter set (es-set) of G if $\mathrm{es}(G) = w(G - X) - |X|$.

Some results for the edge scattering number are provided as follows:

Theorem 2.6. [18] The edge-scattering number of the cycle graph C_n is 0. Moreover, the edge-scattering number of the complete bipartite graph $K_{m,n}$ is $2 - m$ where $2 \leq m \leq n$.

Theorem 2.7. [18] If $n \geq 3$ is a positive integer, then $\mathrm{es}(K_2 \times P_n) = 0$. If $n \geq 4$ is a positive integer, then $\mathrm{es}(K_2 \times C_n) = -1$.

We mention another important concept of stability.

Definition 2.8. [12] A nonempty subset $X \subset V(G)$ is called a dominating set of G if every vertex not in X is adjacent to at least one vertex in X. A dominating set is called minimal if none of its proper subsets is a dominating set. The minimum cardinality of all the dominating graph sets G is called the domination number of the graph and is denoted by $\gamma(G)$.

Any subset of vertices of a graph G is a dominating set. In other words, the subset that gives the scattering number can be a dominating set. The motivation of this paper is to use the dominating set when investigating the stability measurement.

3. Domination Scattering Number of a Graph

In this section, we first define a new parameter as stability measurement.

Definition 3.1. The domination scattering number of a noncomplete graph G is

$$\mathrm{ds}(G) = \max\{w(G - X) - |X| : w(G - X) \geq 2 \text{ and } X \text{ is a dominating set}\}$$

where the notation $|X|$ represents the cardinality of X. Moreover, a set $X \subset V(G)$ is called a domination scatter set (ds-set) of G if $\mathrm{ds}(G) = w(G - X) - |X|$.

We provide an example showing that this parameter is more distinctive than the scattering and edge scattering numbers. In other words, the stability parameter we define offers better results than other parameters for some graph families.

Consider the graphs G_1, G_2, and G_3, each having the same number of vertices. A pertinent question arises: "Can the relevance of the domination scattering number as a measure of stability in graphs be evaluated by analyzing its properties and effectiveness in distinguishing graphs based on their structural flexibility and variations in dominance?" In other words, are G_1, G_2, and G_3 distinguished by the domination scattering number?

We can find many examples of graphs that suggest that $\mathrm{ds}(G)$ is a suitable measure of stability in that it can distinguish between graphs. Consider Figure 1 as an example.

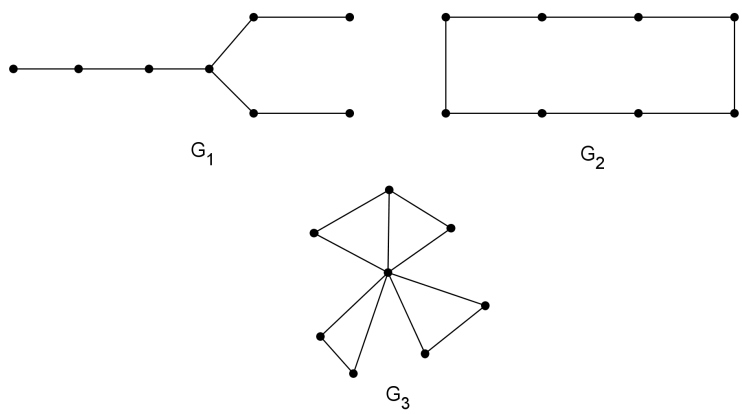

Figure 1. Graphs G_1, G_2, and G_3, each having the same number of vertices

The scattering number, edge scattering number, and domination scattering number of graphs in Figure 1 are calculated and listed in Table 1.

Table 1. Scattering, edge scattering, and domination scattering numbers of graphs in Figure 1

	sc(G)	es(G)	ds(G)
G_1	2	1	1
G_2	0	0	0
G_3	2	0	2

It can be observed from Table 1 that $\text{sc}(G_1) = \text{sc}(G_3) = 2$. Therefore, scattering numbers do not distinguish between graphs G_1 and G_3. Since $\text{ds}(G_1) \neq \text{ds}(G_3)$, the domination scattering number distinguishes between graphs G_1 and G_3. We can also say the same for the graphs G_2 and G_3. Table 1 shows $\text{es}(G_2) = \text{es}(G_3) = 0$. Therefore, edge-scattering numbers do not distinguish between graphs G_2 and G_3. However, since $\text{ds}(G_2) \neq \text{ds}(G_3)$, we say that the domination scattering number distinguishes between graphs G_2 and G_3.

Consequently, the new parameter defined in this study is more distinctive for these graphs than others. In other words, the graph parameter we defined is a suitable indicator of its stability. Therefore, we investigate which graphs the parameter we defined is better for. We provide the domination scattering number of several graphs.

3.1. Domination Scattering Number of Some Graphs

In this subsection, we provide the results obtained by the new parameter. Firstly, we start with the path graph P_n.

Theorem 3.2. Let $n \in \mathbb{Z}^+$ and $n \geq 5$. Then, $\text{ds}(P_n) = 1$.

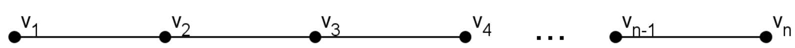

Figure 2. Path graph P_n

Proof. Let X be a dominating set of P_n and $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ (see Figure 2). From [20], since $\gamma(P_n)=\lceil \frac{n}{3} \rceil$, we have there different cases:

Case 1: Let $n \equiv 0 \pmod 3$. If we remove $|X| \geq \frac{n}{3}$ vertices, then $w(P_n - X) \leq \frac{n}{3} + 1$. Therefore,

$$\text{ds}(P_n) = \max\{w(P_n - X) - |X|\} \leq \max\left\{\frac{n}{3} + 1 - \frac{n}{3}\right\} \leq 1$$

If we choose $X^* = \{v_2, v_5, v_8, \cdots, v_{n-1}\}$ such that $|X^*| = \frac{n}{3}$ and $w(P_n - X) = \frac{n}{3} + 1$, then

$$\mathrm{ds}(P_n) = 1 \tag{3.1}$$

Case 2: Let $n \equiv 1 \pmod 3$. By removing $|X| \geq \lceil \frac{n}{3} \rceil$ vertices, we have $w(P_n - X) \leq \lceil \frac{n}{3} \rceil + 1$. Therefore,

$$\mathrm{ds}(P_n) \leq \max\left\{\left\lceil \frac{n}{3} \right\rceil + 1 - \left\lceil \frac{n}{3} \right\rceil\right\} \leq 1$$

If we choose $X^* = \{v_2, v_5, v_8, \cdots, v_{n-5}\} \cup \{v_{n-3}, v_{n-1}\}$ such that $|X^*| = \lceil \frac{n}{3} \rceil$ and $w(P_n - X) = \lceil \frac{n}{3} \rceil + 1$, then

$$\mathrm{ds}(P_n) = 1 \tag{3.2}$$

Case 3: Let $n \equiv 2 \pmod 3$. If $|X| \geq \lceil \frac{n}{3} \rceil$ vertices are removed, then $w(P_n - X) \leq \lceil \frac{n}{3} \rceil + 1$. Therefore,

$$\mathrm{ds}(P_n) \leq \max\left\{\left\lceil \frac{n}{3} \right\rceil + 1 - \left\lceil \frac{n}{3} \right\rceil\right\} \leq 1$$

Let $X^* = \{v_2, v_5, v_8, \cdots, v_{n-3}\} \cup \{v_{n-1}\}$ be a vertex cut. Then, $|X^*| = \lceil \frac{n}{3} \rceil$ and $w(P_n - X) = \lceil \frac{n}{3} \rceil + 1$. Hence,

$$\mathrm{ds}(P_n) = 1 \tag{3.3}$$

From (3.1)-(3.3), $\mathrm{ds}(P_n) = 1$. □

Theorem 3.3. Let $n \in \mathbb{Z}^+$ and $n \geq 4$. Then, $\mathrm{ds}(C_n) = 0$.

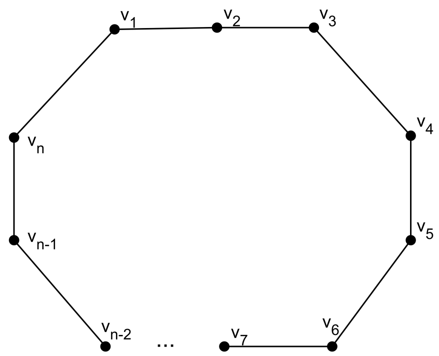

Figure 3. Cycle graph C_n

Proof. Let X be a dominating set of C_n and $V(C_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ (see Figure 3). From [20], since $\gamma(C_n)=\lfloor \frac{n+2}{3} \rfloor$, we have three different cases:

Case 1: Let $n \equiv 0 \pmod 3$. If $|X| \geq \lfloor \frac{n+2}{3} \rfloor$ vertices are removed, then $w(C_n - X) = \lfloor \frac{n+2}{3} \rfloor$ and

$$\mathrm{ds}(C_n) \leq \max\left\{\left\lfloor \frac{n+2}{3} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor\right\} \leq 0$$

Hence, if we choose $X^* = \{v_2, v_5, v_8, ..., v_{n-1}\}$, then $|X^*| = \lfloor \frac{n+2}{3} \rfloor$ and $w(C_n - X) = \lfloor \frac{n+2}{3} \rfloor$. Hence,

$$\mathrm{ds}(C_n) = 0 \tag{3.4}$$

Case 2: Let $n \equiv 1 \pmod 3$. If $|X| \geq \frac{n+2}{3}$ vertices are removed, then $w(C_n - X) \leq \frac{n+2}{3}$. Therefore,

$$\mathrm{ds}(C_n) \leq \max\left\{\frac{n+2}{3} - \frac{n+2}{3}\right\} \leq 0$$

Let $X^* = \{v_2, v_5, v_8, ..., v_{n-2}\} \cup \{v_n\}$ be a vertex cut. By the choice of $|X^*|$, we obtain $|X^*| = \frac{n+2}{3}$ and $w(C_n - X) = \frac{n+2}{3}$. Then,

$$\mathrm{ds}(C_n) = 0 \tag{3.5}$$

Case 3: Let $n \equiv 2 \pmod 3$. By removing $|X| \geq \lfloor \frac{n+2}{3} \rfloor$ vertices, we have $w(C_n - X) \leq \lfloor \frac{n+2}{3} \rfloor$. Then,

$$\mathrm{ds}(C_n) \leq \max\left\{ \left\lfloor \frac{n+2}{3} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor \right\} \leq 0$$

If we choose $X^* = \{v_2, v_5, v_8, ..., v_{n-3}\} \cup \{v_{n-1}\}$ or $X^* = \{v_2, v_5, v_8, ..., v_{n-3}\} \cup \{v_n\}$ as a vertex cut, then $|X^*| = \lfloor \frac{n+2}{3} \rfloor$ and $w(C_n - X^*)=\lfloor \frac{n+2}{3} \rfloor$. Then,

$$\mathrm{ds}(C_n) = 0 \tag{3.6}$$

From (3.4)-(3.6), $\mathrm{ds}(C_n) = 0$. □

Theorem 3.4. If n$\in \mathbb{Z}^+$ and $n \geq 2$, then $\mathrm{ds}(K_{1,n}) = n - 1$.

Proof. Let X be a dominating set of $K_{1,n}$ and v be a vertex with maximum degree. If we remove $|X| \geq$1 vertices, then $w(K_{1,n} - X) \leq n$. Then,

$$\mathrm{ds}(K_{1,n}) \leq \max\{n-1\} \leq n - 1$$

If we choose $X^* = \{v\}$ such that $|X^*| = 1$ and $w(K_{1,n} - X^*) = n$, then $\mathrm{ds}(K_{1,n}) = n - 1$. □

Theorem 3.5. If $n, m \in \mathbb{Z}^+$ and $n \geq m$, then $\mathrm{ds}(K_{m,n}) = n - m$.

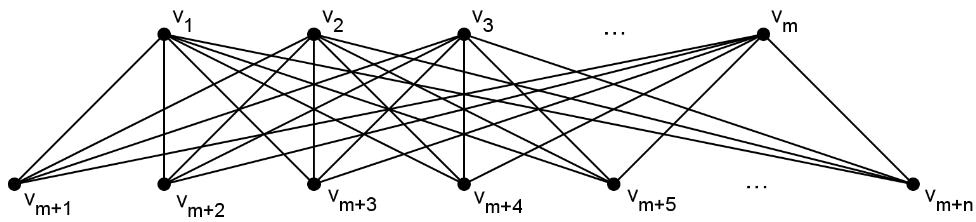

Figure 4. Complete bipartite graph $K_{m,n}$

Proof. Let X be a dominating set of $K_{m,n}$ and $V(K_{m,n}) = \{v_1, v_2, v_3, \cdots, v_{m+n-1}, v_{m+n}\}$ (see Figure 4). From [16], since $\gamma(K_{m,n}) = 2$ and $w(K_{m,n} - X) > 1$, then $|X|$ must be at least m. If we remove $|X| \geq m$ vertices, then $w(K_{m,n} - X) \leq n$. Therefore,

$$\mathrm{ds}(K_{m,n}) \leq \max\{n - m\} \leq n - m$$

If we take $X^* = \{v_1, v_2, v_3, ..., v_m\}$, then $w(K_{m,n} - X^*) = n$. Hence, $\mathrm{ds}(K_{m,n}) = n - m$. □

3.2. Cartesian Product and Domination Scattering Number

In this subsection, we provide the $\mathrm{ds}(P_n \times C_3)$ value.

Theorem 3.6. If $n \in \mathbb{Z}^+$ and $n \geq 4$, then

$$\mathrm{ds}(P_n \times C_3) = \begin{cases} 1 - \frac{2n}{3}, & n \equiv 0 \pmod 3 \\ \lceil \frac{n}{3} \rceil - n, & \text{otherwise} \end{cases}$$

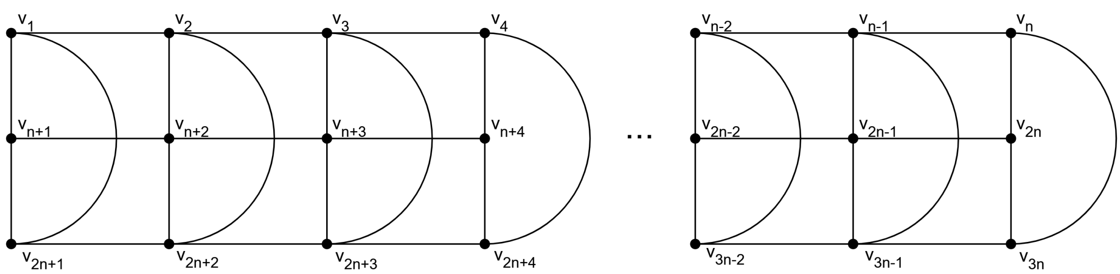

Figure 5. Graph $P_n \times C_3$

Proof. Let X be a dominating set and $V(P_n \times C_3) = \{v_1, v_2, v_3, ..., v_{3n-1}, v_{3n}\}$ (see Figure 5). From [21], since

$$\gamma(P_n \times C_3) = \begin{cases} \lceil \frac{3n}{4} \rceil + 1, & n \equiv 0 \pmod 4 \\ \lceil \frac{3n}{4} \rceil, & \text{otherwise} \end{cases}$$

then $|X|$ must be at least $\lceil \frac{3n}{4} \rceil$. Then, we consider two different cases:

Case 1: Let $n \equiv 0 \pmod 3$. If we remove $|X| = \lceil \frac{3n}{4} \rceil$ vertices, then $w((P_n \times C_3) - X) = 1$. If we remove $|X| = \lceil \frac{3n}{4} \rceil + k$ vertices such that $k \in \mathbb{Z}^+$ and $|X| < n$, then $w((P_n \times C_3) - X) \leq 1 + k$. Thus,

$$\mathrm{ds}(P_n \times C_3) \leq \max\left\{1 + k - \left(\left\lceil \frac{3n}{4} \right\rceil + k\right)\right\} \leq 1 - \left\lceil \frac{3n}{4} \right\rceil \tag{3.7}$$

If we remove $|X| \geq n$ vertices, then $w((P_n \times C_3) - X) \leq \frac{n}{3} + 1$. Thus,

$$\mathrm{ds}(P_n \times C_3) \leq \max\left\{\frac{n}{3} + 1 - n\right\} \leq 1 - \frac{2n}{3} \tag{3.8}$$

Since $1 - \left\lceil \frac{3n}{4} \right\rceil \leq 1 - \frac{2n}{3}$, for all $n \geq 4$, then $\mathrm{ds}(P_n \times C_3) \leq 1 - \frac{2n}{3}$ from (3.7) and (3.8).

Hence, if we choose

$$X^* = \{v_2, v_5, v_8, ..., v_{n-1}\} \cup \{v_{n+2}, v_{n+5}, v_{n+8}, ..., v_{2n-1}\} \cup \{v_{2n+2}, v_{2n+5}, v_{2n+8}, ..., v_{3n-1}\}$$

then $|X^*| = n$ and $w((P_n \times C_3) - X^*) = \frac{n}{3} + 1$. Then,

$$\mathrm{ds}(P_n \times C_3) = 1 - \frac{2n}{3} \tag{3.9}$$

Case 2: Let $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. If we remove $|X| = \lceil \frac{3n}{4} \rceil$ vertices, then $w((P_n \times C_3) - X) = 1$. If we remove $|X| = \lceil \frac{3n}{4} \rceil + k$ vertices such that $k \in \mathbb{Z}^+$ and $|X| < n$, then $w((P_n \times C_3) - X) \leq 1 + k$. Thus,

$$\mathrm{ds}(P_n \times C_3) \leq \max\left\{1 + k - \left(\left\lceil \frac{3n}{4} \right\rceil + k\right)\right\} \leq 1 - \left\lceil \frac{3n}{4} \right\rceil \tag{3.10}$$

If we remove $|X| \geq n$ vertices, then $w((P_n \times C_3) - X) \leq \lceil \frac{n}{3} \rceil$. Therefore,

$$\mathrm{ds}(P_n \times C_3) \leq \max\left\{\left\lceil \frac{n}{3} \right\rceil - n\right\} \leq \left\lceil \frac{n}{3} \right\rceil - n \tag{3.11}$$

Since $1 - \lceil \frac{3n}{4} \rceil \leq \lceil \frac{n}{3} \rceil - n$, for all $n \geq 4$, then $\mathrm{ds}(P_n \times C_3) \leq \lceil \frac{n}{3} \rceil - n$ from (3.10) and (3.11).

If we choose

$$X^* = \{v_2, v_5, v_8, ..., v_{n-2}\} \cup \{v_{n+2}, v_{n+5}, v_{n+8}, ..., v_{2n-2}\} \cup \{v_{2n+2}, v_{2n+5}, v_{2n+8}, ..., v_{3n-2}\} \cup \{v_{2n}\}$$

while $n \equiv 1 \pmod 3$ and then $|X^*| = n$ and $w((P_n \times C_3) - X^*) = \lceil \frac{n}{3} \rceil$. Therefore,

$$\mathrm{ds}(P_n \times C_3) = \left\lceil \frac{n}{3} \right\rceil - n \tag{3.12}$$

If we choose

$$X^{**} = \{v_2, v_5, v_8, ..., v_{n-3}\} \cup \{v_{n+2}, v_{n+5}, v_{n+8}, ..., v_{2n-3}\} \cup \{v_{2n+2}, v_{2n+5}, v_{2n+8}, ..., v_{3n-3}\} \cup \{v_{n-1}, v_{2n}\}$$

while $n \equiv 2 \pmod 3$, then $|X^{**}| = n$ and $w((P_n \times C_3) - X^{**}) = \lceil \frac{n}{3} \rceil$. Hence,

$$\mathrm{ds}(P_n \times C_3) = \left\lceil \frac{n}{3} \right\rceil - n \tag{3.13}$$

By (3.9), (3.12), and (3.13), the results are obtained. □

4. Conclusion

The vulnerability of a communication network measures the network's resistance to the disruption of its operation after the failure of specific processors or communication links. Network designers aim to design networks with less vulnerability or more reliability. In other words, network designers care about network stability. For this reason, the vulnerability values of graphs (networks) are investigated by modeling networks with graphs. In this study, first of all, it was observed that the scattering and edge scattering numbers among the vulnerability measurements in graphs were insufficient to distinguish some graph families. Afterward, a new parameter was defined to distinguish these graph families, called the domination scattering number. The vertices removed from the graph in this parameter are also components of any dominant cluster in the graph. In this article, the domination scattering number for basic graphs is calculated. The domination scattering number of the graph $P_n \times C_3$ is also provided. In future research, the primary objective can be to obtain graphs corresponding to real-life networks using graph operations, such as the graph $P_n \times C_3$. Subsequently, the aim can be to calculate the domination scattering numbers of these graphs. However, an essential question warrants investigation: Can the domination scattering number of a graph be calculated in polynomial time? Moreover, the following questions are anticipated that obtaining answers to these questions will benefit network designers:

i. Which graph family has the smallest or largest domination scattering number?

ii. What are the relationships between the domination scattering number and other graph parameters?

iii. What are the values of the domination scattering numbers for a graph's total, line, and middle graphs?

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's doctoral dissertation supervised by the second author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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A Generalization of Source of Semiprimeness

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Abstract — This paper characterizes the semigroup ideal $\mathcal{L}_R^n(I)$ of a ring R, where I is an ideal of R, defined by $\mathcal{L}_R^0(I) = I$ and $\mathcal{L}_R^n(I) = \{a \in R \mid aRa \subseteq \mathcal{L}_R^{n-1}(I)\}$, for all $n \in \mathbb{Z}^+$, the set of all the positive integers. Moreover, it studies the basic properties of the set $\mathcal{L}_R^n(I)$ and defines n-prime ideals, n-semiprime ideals, n-prime rings, and n-semiprime rings. This study also investigates relationships between the sets $\mathcal{L}_R(I)$ and $\mathcal{L}_R^n(I)$ and exemplifies some of the related properties. It obtains the main results concerning prime rings and prime ideals by the properties of the set $\mathcal{L}_R^n(I)$.

Keywords *Source of semiprimeness, semiprime rings, semiprime ideals, prime rings, prime ideals*

Mathematics Subject Classification (2020) 16N60, 16U80

1. Introduction

Prime and semiprime ideals are essential classes of rings, especially in noncommutative rings. Therefore, many studies have been conducted on rings' prime ideals and semiprime ideals [1–4]. Additionally, numerous generalizations of these structures have been proposed by the concepts of prime and semiprime ideals [4–10]. Moreover, many studies have been undertaken on prime ideals in Noetherian rings [11–15]. Besides, prime ideals play a significant role in the theory of associative algebras [16,17]. In [18], the concept of the source of the semiprimeness of a ring R expressed by S_R has been explored through semiprime ideals, leading to the definition of new structures: The $|S_R|$-reduced rings, the $|S_R|$-domains, and the $|S_R|$-division rings. Further, several properties of these structures have been investigated. Furthermore, Karalarlıoğlu Camcı [19] has introduced the structures of $|S_R|$-semiprime and $|S_R|$-prime rings using the set S_R and analyzed the relationships between these two types of rings. The author has also researched the necessary and sufficient conditions for a ring R to be isomorphic to the subdirect sum of some of the $|S_R|$-prime rings of R and obtained a generalization related to the relationship between the prime radical $\beta(R)$ of R and S_R. In addition, Karalarlıoğlu Camcı [19] has suggested the set $\mathcal{L}_R(A) = \{a \in R : aRa \subseteq A\}$, where A is a non-empty subset of a ring R, and considered some of its basic properties, presented examples to enhance understanding of the set $\mathcal{L}_R(A)$, and investigated the relations between the sets $\mathcal{L}_R(A)$ and S_R.

This study defines the set $\mathcal{L}_R^n(I)$, a generalization of the set $\mathcal{L}_R(I)$ such that I is an ideal of a ring R, analyzes its properties, and exemplifies some of them. Moreover, this generalization proposes the definitions of n-prime ideals, n-semiprime ideals, n-prime rings, and n-semiprime rings, along with theorems and results derived from these novel notions.

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2. Preliminaries

The current section provides the following basic definitions and some properties in [18–22].

Definition 2.1. Let R be a multiplicative semigroup, $I \neq \emptyset$, and $I \subseteq R$. If $ar, ra \in I$, for all $a \in I$ and for all $r \in R$, then I is called a semigroup ideal of R.

Across this study, if R is a ring, then its multiplicative semigroup concerning the second operation of the ring R is considered for the concepts related to semigroup ideals.

Definition 2.2. Let R be a ring and I be a semigroup ideal of R. If $aRb \subseteq I$ implies $a \in I$ or $b \in I$, then I is called a semigroup prime ideal of R.

Definition 2.3. Let R be a ring and I be an ideal of R. If $aRb \subseteq I$ implies $a \in I$ or $b \in I$, then I is called a prime ideal of R.

Definition 2.4. Let R be a ring and I be a semigroup ideal of R. If $aRa \subseteq I$ implies $a \in I$, then I is called a semigroup semiprime ideal of R.

Definition 2.5. Let R be a ring and I be an ideal of R. If $aRa \subseteq I$ implies $a \in I$, then I is called a semiprime ideal of R.

Definition 2.6. Let R be a ring, $A \neq \emptyset$, and $A \subseteq R$. Then, the set $S_R(A) = \{a \in R : aAa = (0)\}$ is called the source of semiprimeness of A in R. If $A = R$, then S_R will be used instead of $S_R(R)$.

Definition 2.7. Let R be a ring. If, for all $a \in R$, $aRa \subseteq S_R$ implies $a \in S_R$, then R is called an $|S_R|$-semiprime ring, and if, for all $a, b \in R$, $aRb \subseteq S_R$ implies $a \in S_R$ or $b \in S_R$, then R is called an $|S_R|$-prime ring.

Proposition 2.8. Let R be a ring. Then, the following properties hold:

i. If I is a semigroup right (left) ideal of R, then $I \subseteq \mathcal{L}_R(I)$.

ii. If I is a semigroup right (left) ideal of R, then $\mathcal{L}_R(I)$ is a semigroup right (left) ideal of R.

iii. If I is a semigroup right (left) ideal of R, then $S_R \subseteq \mathcal{L}_R(I)$.

iv. If I is an ideal of R and $\pi : R \to R/I$ is a natural epimorphism defined by $\pi(r) = r + I$, then $\pi(\mathcal{L}_R(I)) = S_R/I$ and $\pi^{-1}(S_R/I) = \mathcal{L}_R(I)$.

v. For an ideal I of R, I is a semiprime ideal if and only if $I = \mathcal{L}_R(I)$.

3. Main Results

Let R be a ring and I be an ideal of R. In [19], the set $\mathcal{L}_R(I)$ is defined as follows:

$$\mathcal{L}_R(I) = \{a \in R : aRa \subseteq I\}$$

Motivated by this set, the following is introduced:

$$\mathcal{L}_R^0(I) = I \quad \text{and} \quad \mathcal{L}_R^n(I) = \left\{a \in R : aRa \subseteq \mathcal{L}_R^{n-1}(I)\right\}, \text{ for all } n \in \mathbb{Z}^+$$

where $\mathbb{Z}^+$ is the set of all the positive integers. Moreover, $\mathcal{L}_R^1(I)$ is denoted by $\mathcal{L}_R(I)$. Then,

$$\mathcal{L}_R(0) = \{a \in R : aRa \subseteq (0)\} \quad \text{and} \quad \mathcal{L}_R^n(0) = \left\{a \in R : aRa \subseteq \mathcal{L}_R^{n-1}(0)\right\}$$

Consider the set

$$SG_R = \{I \subseteq R : I \text{ is a semigroup ideal of } R\}$$

From Proposition 2.8, the set $\mathcal{L}_R(I) = \{a \in R : \ aRa \subseteq I\}$ is a semigroup ideal of R. Therefore, $\mathcal{L}_R(I) \in SG_R$. As a consequence,

$$\mathcal{L}_R : SG_R \to SG_R, \ \mathcal{L}_R(I) = \{a \in R : aRa \subseteq I\}$$

can be constructed. Finally, it is operationalizing as

$$\mathcal{L}_R^n(I) = \mathcal{L}_R\left(\mathcal{L}_R^{n-1}(I)\right) = \left\{a \in R : aRa \subseteq \mathcal{L}_R^{n-1}(I)\right\}$$

for all $n \in \mathbb{Z}^+$. Thus, it is noticeable from the induction that

$$\mathcal{L}_R^m\left(\mathcal{L}_R^n(I)\right) = \mathcal{L}_R^{m+n}(I)$$

for all $n, m \in \mathbb{N}$, the set of all the nonnegative integers.

Definition 3.1. Let I be an ideal of a ring R. Then, I is called an n-prime ideal if $\mathcal{L}_R^n(I)$ is a semigroup prime ideal of R.

Definition 3.2. Let I be an ideal of a ring R. Then, I is called an n-semiprime ideal if $\mathcal{L}_R^n(I)$ is a semigroup semiprime ideal of R.

Definition 3.3. Let R be a ring. Then, R is called an n-prime ring if $\mathcal{L}_R^n(0)$ is a semigroup prime ideal of R.

Definition 3.4. Let R be a ring. Then, R is called an n-semiprime ring if $\mathcal{L}_R^n(0)$ is a semigroup semiprime ideal of R.

Lemma 3.5. Let R be a ring. If P is a prime ideal of R, then P is an n-prime ideal of R.

PROOF. Let R be a ring and P be a prime ideal of R. Since $\mathcal{L}_R(P) = P$, $\mathcal{L}_R^n(P) = P$. Therefore, $\mathcal{L}_R^n(P)$ is a prime ideal of R. Thus, P is an n-prime ideal of R. □

Lemma 3.6. Let R be a ring. If P is a semiprime ideal of R, then P is an n-semiprime ideal of R.

The proof is carried out similarly to the proof of Lemma 3.5.

Example 3.7. Consider the ring $\mathbb{Z}_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$. Then, $I = \{\overline{0}, \overline{4}\}$ is an ideal of $\mathbb{Z}_8$. Thus, the set

$$\mathcal{L}_{\mathbb{Z}_8}(I) = \{a \in \mathbb{Z}_8 : a\mathbb{Z}_8 a \subseteq I\} = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$$

is a semiprime ideal of $\mathbb{Z}_8$. Thus, I is a 1-semiprime ideal of $\mathbb{Z}_8$ but not a semiprime ideal of $\mathbb{Z}_8$.

Theorem 3.8. Let R be a ring, P be an ideal of R, and A be a semigroup ideal of R such that $P \subseteq A$. Then, A/P is a semigroup ideal of the ring R/P.

PROOF. Since it follows the fact that $A/P \neq \{0 + P\}$, then

$$A/P = \{a + P : a \in A\} \subseteq R/P$$

Therefore,

$$(a + P)(r + P) = ar + P \in A/P$$

and

$$(r + P)(a + P) = ra + P \in A/P$$

for all $a \in A$ and for all $r \in R$. □

Theorem 3.9. Let R be a ring, I be an ideal of R, and $\pi : R \to R/I$ be a natural epimorphism defined by $\pi(r) = r + I$. Then, for all $n \in \mathbb{N}$,

$$\pi^{-1}\left(\mathcal{L}_{R/I}^n(0)\right) = \mathcal{L}_R^n(I)$$

Proof. For $n = 0$,

$$\pi^{-1}\left(\mathcal{L}^0_{R/I}(0)\right) = \pi^{-1}(0) = \operatorname{Ker}\pi = I = \mathcal{L}^0_R(I)$$

Let $x \in \pi^{-1}(\mathcal{L}_{R/I}(0))$, for $n = 1$. Then, $\pi(x) = x+I \in (\mathcal{L}_{R/I}(0))$. It follows that $(x+I)(r+I)(x+I) = (0+I)$, for all $r \in R$. Therefore, $xRx \subseteq I$, for all $r \in R$, because $xrx \in I$, for all $r \in R$. Hence, $x \in \mathcal{L}_R(I)$. Furthermore, if $x \in \mathcal{L}_R(I)$, then $xRx \subseteq I$. Since $xrx \in I$, for all $r \in R$, the equality $(x+I)(r+I)(x+I) = (0+I)$ holds. This requires $\pi(x) = x+I \in (\mathcal{L}_{R/I}(0))$. Thus, $x \in \pi^{-1}(\mathcal{L}_{R/I}(0))$. Hence, $\pi^{-1}(\mathcal{L}_{R/I}(0)) = \mathcal{L}_R(I)$.

Assume that

$$\pi^{-1}\left(\mathcal{L}^n_{R/I}(0)\right) = \mathcal{L}^n_R(I)$$

for an arbitrary $n \in \mathbb{N}$. Let $x \in \pi^{-1}(\mathcal{L}^{n+1}_{R/I}(0))$. Then, $\pi(x) \in (\mathcal{L}^{n+1}_{R/I}(0))$. Namely, $\pi(x)\pi(r)\pi(x) \in (\mathcal{L}^n_{R/I}(0))$. Since π is an epimorphism, $\pi(xrx) \in (\mathcal{L}^n_{R/I}(0))$, for all $r \in R$. Consequently, $xrx \in \pi^{-1}(\mathcal{L}^n_{R/I}(0)) = \mathcal{L}^n_R(I)$, for all $r \in R$. Thus, $xRx \subseteq \mathcal{L}^n_R(I)$ and hence $x \in \mathcal{L}^{n+1}_R(I)$. The converse is similar. Consequently, $\mathcal{L}^{n+1}_R(I) = \pi^{-1}(\mathcal{L}^{n+1}_{R/I}(0))$. □

Lemma 3.10. Let R be a ring, I and P be two ideals of R, and $P \subseteq I$. Then, $\mathcal{L}^n_R(I)/P = \mathcal{L}^n_{R/P}(I/P)$, for all $n \in \mathbb{N}$.

Proof. The proof is straightforward for $n = 0$.

Let $n = 1$. Since I is an ideal of R, $xr, rx \in I$, for all $x \in I$ and for all $r \in R$. Thus, $xrx \in I$ and $x \in \mathcal{L}_R(I)$. Hence, $I \subseteq \mathcal{L}_R(I)$. Moreover, let $x + P \in \mathcal{L}_R(I)/P$. Therefore, $x \in \mathcal{L}_R(I)$. Thereby, $xRx \subseteq I$. In this way, $xRx + P \subseteq I/P$. Herewith, $(x + P)(r + P)(x + P) \in I/P$. Thus, $x + P \in \mathcal{L}_{R/P}(I/P)$. As a result, $\mathcal{L}_R(I)/P \subseteq \mathcal{L}_{R/P}(I/P)$. The converse is similar. Consequently, $\mathcal{L}_R(I)/P = \mathcal{L}_{R/P}(I/P)$.

Suppose that for an arbitrary $n \in \mathbb{N}$,

$$\mathcal{L}^n_R(I)/P = \mathcal{L}^n_{R/P}(I/P)$$

Further, let $y + P \in \mathcal{L}^{n+1}_R(I)/P$. Thus, $y \in \mathcal{L}^{n+1}_R(I)$. Hence, $yRy \subseteq \mathcal{L}^n_R(I)$. Thereby, $yRy + P \subseteq \mathcal{L}^n_R(I)/P = \mathcal{L}^n_{R/P}(I/P)$. Therefore, $(y + P)(r + P)(y + P) \in \mathcal{L}^n_{R/P}(I/P)$, for all $r \in R$. In this way, $y + P \in \mathcal{L}^{n+1}_{R/P}(I/P)$. The converse is similar. In conclusion, $\mathcal{L}^{n+1}_R(I)/P = \mathcal{L}^{n+1}_{R/P}(I/P)$. □

From the aforesaid definitions and theorems, the following significant Theorem is provided.

Theorem 3.11. Let R be a ring and P be an ideal of R. Then, P is an n-prime ideal of R if and only if R/P is an n-prime ring.

Proof. Let R be a ring and P be an ideal of R.

$\Rightarrow$: Assume that P is an n-prime ideal of R. Then, $\mathcal{L}^n_R(P)$ is a semigroup prime ideal of R from Definition 3.1. Thus, R/P is an n-prime ring from Definition 3.3.

$\Leftarrow$: Let R/P is an n-prime ring. Then, $\mathcal{L}^n_{R/P}(\overline{0})$ is a semigroup prime ideal of R/P. From Lemma 3.10, $\mathcal{L}^n_{R/P}(P/P) = \mathcal{L}^n_R(P)/P$. Let $xRy \subseteq \mathcal{L}^n_R(P)$, for all $x, y \in R$. Then, $xry \in \mathcal{L}^n_R(P)$, for all $r \in R$. Hence, since $(xry) + P \in \mathcal{L}^n_R(P)/P$, $(x + P)(R/P)(y + P) \subseteq \mathcal{L}^n_R(P)/P = \mathcal{L}^n_{R/P}(P/P)$. Since $\mathcal{L}^n_{R/P}(P/P)$ is a semigroup prime ideal of R/P, $x + P \in \mathcal{L}^n_{R/P}(P/P) = \mathcal{L}^n_R(P)/P$ or $y + P \in \mathcal{L}^n_{R/P}(P/P) = \mathcal{L}^n_R(P)/P$. Namely, $x \in \mathcal{L}^n_R(P)$ or $y \in \mathcal{L}^n_R(P)$. Thence, $\mathcal{L}^n_R(P)$ is a semigroup prime ideal of R. Consequently, P is an n-prime ideal of R. □

Lemma 3.12. Let R and S be two rings, $\varphi : R \to S$ be an epimorphism, and I be an ideal of R. Then, $\mathcal{L}^n_S(\varphi(I)) = \varphi(\mathcal{L}^n_R(I))$, for all $n \in \mathbb{N}$.

Proof. Let R and S be two rings, $\varphi : R \to S$ be an epimorphism, and I be an ideal of R. Since $\varphi(I) = \varphi(I)$, then $\mathcal{L}_S^0(\varphi(I)) = \varphi(\mathcal{L}_R^0(I))$, for $n = 0$. Moreover, let $y \in \varphi(\mathcal{L}_R(I))$. Then, $y = \varphi(x)$ and $x \in \mathcal{L}_R(I)$. Thus, $ysy = \varphi(x)\varphi(r)\varphi(x) = \varphi(xrx)$, for all $r \in R$ and for all $s \in S$. Hence, $ySy \subseteq \varphi(I)$. Thereby, $y \in \mathcal{L}_S(\varphi(I))$. The other inclusion is similarly proved. Consequently, $\mathcal{L}_S(\varphi(I)) = \varphi(\mathcal{L}_R(I))$.

Assume that $\mathcal{L}_S^n(\varphi(I)) = \varphi(\mathcal{L}_R^n(I))$, for an arbitrary $n \in \mathbb{N}$. If $y \in \varphi(\mathcal{L}_R^{n+1}(I))$, then $y = \varphi(x)$ and $x \in \mathcal{L}_R^{n+1}(I)$. Thus, $ysy = \varphi(x)\varphi(r)\varphi(x) = \varphi(xrx)$, for all $r \in R$ and for all $s \in S$. Hence, $ySy \subseteq \mathcal{L}_S^n(\varphi(I))$. Thereby, $y \in \mathcal{L}_S^{n+1}(\varphi(I))$. Similarly, $\mathcal{L}_S^{n+1}(\varphi(I)) \subseteq \varphi(\mathcal{L}_R^{n+1}(I))$. Consequently, $\mathcal{L}_S^{n+1}(\varphi(I)) = \varphi(\mathcal{L}_R^{n+1}(I))$. □

Theorem 3.13. Let R and S be two rings and $\varphi : R \to S$ be an epimorphism. If $\operatorname{Ker}\varphi \subseteq P$ is an n-prime ideal of R, then $\varphi(P)$ is an n-prime ideal of S.

Proof. Let $\operatorname{Ker}\varphi \subseteq P$ be an n-prime ideal of R. Then, $\varphi(P)$ is an ideal of S. Since P is an n-prime ideal of R, $\mathcal{L}_R^n(P)$ is a semigroup prime ideal of R. Therefore, $\varphi(\mathcal{L}_R^n(P))$ is also a semigroup ideal of S. Let $a, b \in S$. Then, there exist $x, y \in R$ such that $aSb = \varphi(x)\varphi(R)\varphi(y)$. Thus, $\varphi(xRy) \subseteq \varphi(\mathcal{L}_R^n(P))$. Hence, $\varphi(xry) = \varphi(p)$ such that $p \in \mathcal{L}_R^n(P)$. Thereby, $xry - p \in \operatorname{Ker}\varphi \subseteq P$. Herewith, $xry = p + k$ such that $k \in P$ and $p \in \mathcal{L}_R^n(P)$. In this way, $xry \in \mathcal{L}_R^n(P)$. Since $\mathcal{L}_R^n(P)$ is a semigroup prime ideal of R, $x \in \mathcal{L}_R^n(P)$ or $y \in \mathcal{L}_R^n(P)$. Therefore, $a = \varphi(x) \in \varphi(\mathcal{L}_R^n(P))$ or $b = \varphi(y) \in \varphi(\mathcal{L}_R^n(P))$. Consequently, $\varphi(\mathcal{L}_R^n(P))$ is a prime ideal of S. From Lemma 3.12, since $\varphi(\mathcal{L}_R^n(P)) = \mathcal{L}_R^n(\varphi(P))$, $\mathcal{L}_R^n(\varphi(P))$ is a semigroup prime ideal of S. Thus, $\varphi(P)$ is an n-prime ideal of S. □

Theorem 3.14. Let R and S be two rings and $\varphi : R \to S$ be an epimorphism. Then, for an ideal I of S,

$$\varphi^{-1}(\mathcal{L}_S(I)) = \mathcal{L}_R\left(\varphi^{-1}(I)\right)$$

Proof. Let R and S be two rings, $\varphi : R \to S$ be an epimorphism, and I be an ideal of S. For all $x \in \varphi^{-1}(\mathcal{L}_S(I))$,

$$\varphi(x) S \varphi(x) = \varphi(x)\varphi(R)\varphi(x) = \varphi(xRx) \subseteq I$$

and

$$xRx \subseteq \varphi^{-1}(\varphi(xRx)) \subseteq \varphi^{-1}(I)$$

Therefore, $x \in \mathcal{L}_R(\varphi^{-1}(I))$ and $\varphi^{-1}(\mathcal{L}_S(I)) \subseteq \mathcal{L}_R(\varphi^{-1}(I))$. Moreover, $xRx \subseteq \varphi^{-1}(I)$, for all $x \in \mathcal{L}_R(\varphi^{-1}(I))$. Thus,

$$\varphi(xRx) = \varphi(x)\varphi(R)\varphi(x) \subseteq \varphi\left(\varphi^{-1}(I)\right) \subseteq I$$

As a result, $\varphi(x) \in \mathcal{L}_S(I)$ and $x \in \varphi^{-1}(\mathcal{L}_S(I))$. Namely, $\mathcal{L}_R(\varphi^{-1}(I)) \subseteq \varphi^{-1}(\mathcal{L}_S(I))$. □

Theorem 3.15. Let R and S be two rings and $\varphi : R \to S$ be an epimorphism. Then, for an ideal I of S,

$$\varphi^{-1}(\mathcal{L}_S^n(I)) = \mathcal{L}_R^n\left(\varphi^{-1}(I)\right), \quad \text{for all } n \in \mathbb{N}$$

Proof. Using Theorem 3.14 and the induction method, the following result is obtained:

$$\varphi^{-1}(\mathcal{L}_S^n(I)) = \mathcal{L}_R^n\left(\varphi^{-1}(I)\right), \quad \text{for all } n \in \mathbb{N}$$

□

Theorem 3.16. Let R be a ring and I be a semigroup ideal of R. Thus, for all $n \in \mathbb{N}$,

$$\mathcal{L}_R^n(I) \subseteq \mathcal{L}_R^{n+1}(I)$$

Proof. Since $\mathcal{L}_R^n(I)$ is a semigroup ideal of R, $aRa \subseteq \mathcal{L}_R^n(I)$, for all $a \in \mathcal{L}_R^n(I)$. Hence, $a \in \mathcal{L}_R^{n+1}(I)$. □

Corollary 3.17. Let R be a ring and I be a semigroup ideal of R. Then,

$$I \subseteq \mathcal{L}_R(I) \subseteq \mathcal{L}_R^2(I) \subseteq \cdots \subseteq \mathcal{L}_R^n(I) \subseteq \mathcal{L}_R^{n+1}(I) \subseteq \cdots$$

4. Conclusion

This study attempts to generalize the set $\mathcal{L}_R(I)$, expressed by $\mathcal{L}_R^n(I)$ such that I is an ideal of a ring R. In this paper, the basic properties of this set are also provided. Furthermore, adopting this generalization, it explores the definitions of n-prime ideals, n-semiprime ideals, n-prime rings, and n-semiprime rings and their properties. Moreover, the relations of this set under epimorphism are mentioned. Future studies could extend these results to different rings, utilizing the generalization of the set $\mathcal{L}_R(I)$, thereby contributing significantly to ring theory. Furthermore, this generalization paves the way for additional extensions, leading to the introduction of new definitions and the development of novel results. In addition, by utilizing the set $\mathcal{L}_R^n(I)$, researchers can define the n-prime radicals, serving as a generalization of the prime radicals of a ring R.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Ruled Surfaces of Adjoint Curve with the Modified Orthogonal Frame

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Abstract — This paper analyzes several specific ruled surfaces generated by the base curve α and its director curve or the α's adjoint curve β and its director curve, where the director curves are frame vectors of the modified orthogonal frame in E^3. Furthermore, this paper studies the flat or minimal properties of the surfaces, as well as their asymptotic and geodesic curves. Afterward, it exemplifies the theoretical results herein. Finally, this paper discusses the need for further research.

Keywords *Modified orthogonal frame, special ruled surface, adjoint curve*

Mathematics Subject Classification (2020) 53A04, 53A05

1. Introduction

The study of curve theory has long been a central topic in differential geometry research [1–4]. One intriguing aspect of this field is exploring specific curve types, such as adjoint curves, defined as the integral of the binormal vector of a curve $\alpha(s)$, parameterized by s, as mentioned in [5]. Adjoint curves have found applications in various fields, including number theory, coding theory, and algebraic geometry [6–9].

A regular curve is characterized by its curvature κ and torsion τ, which uniquely determine the curve at every point, as stated by the fundamental theorem of regular curves. However, the curvature function may vanish at certain points for analytical curves, introducing discontinuities in the principal normal and binormal vectors. This discontinuity makes the curvature function non-differentiable at those points, leading to ambiguities in the Frenet frame due to the vanishing curvature.

To address these challenges, Hord [10] and Sasai [11] introduced an alternative orthogonal frame to handle such points effectively. Sasai [12] further developed a modified orthogonal frame (MOF) for unit-speed analytic curves, offering a simple and practical solution. In this approach, the Frenet vectors are scaled by the curvature function κ, resulting in a new formulation that extends the Frenet derivative equations. This MOF has facilitated research on various frames and ruled surfaces in different spaces [13–19].

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In this paper, we focus on specific ruled surfaces whose base curves are adjoint curves of a considered curve α. The director curves of these surfaces are defined by the tangent, normal, and binormal vectors associated with the MOF in E^3, following the approach in [7]. We also provide several theorems and proofs related to these surfaces. Finally, we exemplify some of these special ruled surfaces to visualize.

2. Preliminaries

This section presents some basic notions to be needed in the following section. Throughout this paper, let $\psi(s,v)$ be a surface in Euclidean 3-space. The unit normal vector field $U(s,v)$ of the surface $\psi(s,v)$ is obtained by

$$U = \frac{\psi_s \times \psi_v}{\|\psi_s \times \psi_v\|}$$

where $\psi_s = \frac{\partial \psi}{\partial s}$ and $\psi_v = \frac{\partial \psi}{\partial v}$ are the partial derivatives of the surface $\psi(s,v)$ with respect to the parameter s and v, respectively. The first fundamental form I of the surface $\psi(s,v)$ is as follows:

$$I = g_{11}ds^2 + 2g_{12}dsdv + g_{22}dv^2$$

where $g_{11} = \langle \psi_s, \psi_s \rangle$, $g_{12} = \langle \psi_s, \psi_v \rangle$, and $g_{22} = \langle \psi_v, \psi_v \rangle$. Moreover, the second fundamental form of the surface $\psi(s,v)$ is defined as follows:

$$II = h_{11}ds^2 + 2h_{12}dsdv + h_{22}dv^2$$

where $h_{11} = \langle \psi_{ss}, U \rangle$, $h_{12} = \langle \psi_{sv}, U \rangle$, and $h_{22} = \langle \psi_{vv}, U \rangle$. The Gaussian curvature K and the mean curvature H of the surface $\psi(s,v)$ are as follows:

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} \quad \text{and} \quad H = \frac{h_{11}g_{22} - 2g_{12}h_{12} + g_{11}h_{22}}{2\left(g_{11}g_{22} - g_{12}^2\right)} \tag{2.1}$$

Theorem 2.1. [20] On a surface, asymptotic curves are defined as curves along which the normal curvature is zero. This is equivalent to the condition that the second fundamental form vanishes:

$$II = h_{11}ds^2 + 2h_{12}dsdv + h_{22}dv^2 = 0$$

where h_{11}, h_{12}, and h_{22} are the coefficients of the second fundamental form.

Theorem 2.2. [20] For a curve to be geodesic, its geodesic curvature

$$k_g = \nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

where $\nabla_{\dot{\gamma}}\dot{\gamma}$ is the covariant derivative of the tangent vector $\dot{\gamma}$ along itself.

Definition 2.3. [5] Let α be a unit speed curve in E^3 with $\tau \neq 0$. Then, the adjoint curve of α is defined by

$$\beta(s) = \int_{s_0}^{s} B_\alpha(s)ds$$

where B_α is the binormal vector of the curve α.

We express the relations between the MOF $\{T, N, B\}$ and the classical Frenet frame $\{t, n, b\}$ by

$$T = t, \; N = \kappa n, \; \text{and} \; B = \kappa b \tag{2.2}$$

where $\kappa \neq 0$ is the curvature of the curve. The MOF $\{T, N, B\}$ satisfies the following equalities:

$$< T, T >= 1, \quad < N, N >=< B, B >= \kappa^2, \quad \text{and} \quad < T, N >=< T, B >=< N, B >= 0$$

Here, the notation $<,>$ represents the inner product. Using the definitions of T, N, and B and (2.2), the following equalities hold:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \dfrac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \dfrac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{2.3}$$

and

$$\tau = \tau(s) = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}$$

is the torsion of α. Moreover, κ^2 and τ are analytic. In [1], the differentiation formula for the MOF is denoted by (2.3).

Theorem 2.4. [7] If α is a unit speed curve and β is the adjoint curve of α such that $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$ are the Frenet vectors and $\{\kappa_\alpha, \tau_\alpha\}$ and $\{\kappa_\beta, \tau_\beta\}$ are curvature and torsion of α and β, respectively, then

$$T_\beta = B_\alpha, \quad N_\beta = -N_\alpha, \quad B_\beta = T_\alpha, \quad \kappa_\beta = \tau_\alpha, \quad \text{and} \quad \tau_\beta = \kappa_\alpha$$

Theorem 2.5. [7] If α is a unit speed regular in E^3 and β is the adjoint curve of α according to the MOF with curvature such that $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$ are the MOF and $\{\kappa_\alpha, \tau_\alpha\}$ and $\{\kappa_\beta, \tau_\beta\}$ are the curvature and torsion of α and β, respectively, then

$$T_\beta = \left(\frac{1}{\kappa_\alpha}\right) B_\alpha, \quad N_\beta = -\left(\frac{\tau_\alpha}{\kappa_\alpha^2}\right) N_\alpha, \quad B_\beta = \left(\frac{\tau_\alpha}{\kappa_\alpha}\right) T_\alpha, \quad \kappa_\beta = \frac{\tau_\alpha}{\kappa_\alpha}, \quad \text{and} \quad \tau_\beta = 1$$

3. Some Special Ruled Surfaces According to the MOF

This section investigates ruled surfaces according to the MOF in E^3. It generates new special ruled surfaces to change the base curve with α and its adjoint curve β. Additionally, this section changes the director vectors of these surfaces concerning the MOF vectors T, N, and B.

3.1. Tangent Ruled Surface with the Base Curve α

Concerning the MOF, the parameterization of the tangent ruled surface is as follows:

$$\psi_1(s, v) = \alpha(s) + vT_\alpha(s) \tag{3.1}$$

where α is the base curve. If we take the derivatives with respect to the parameter s and v of the tangent ruled surface $\psi_1(s, v)$, respectively, then

$$\begin{cases} \psi_{1_s} = T_\alpha + vN_\alpha, \quad \psi_{1_v} = T_\alpha, \\ \psi_{1_{ss}} = \kappa_\alpha^2 T_\alpha + \left(1 + v\dfrac{\kappa_\alpha'}{\kappa_\alpha}\right) N_\alpha + v\tau_\alpha B_\alpha, \quad \psi_{1_{sv}} = N_\alpha, \quad \text{and} \quad \psi_{1_{vv}} = 0 \end{cases} \tag{3.2}$$

Therefore, $g_{11} = 1 + v^2\kappa_\alpha^2$, $g_{12} = 1$, and $g_{22} = 1$ and thus $g_{11}g_{22} - g_{12}^2 = v^2\kappa_\alpha^2 \neq 0$ where g_{11}, g_{12}, and g_{22} are the coefficients of the first fundamental form of the tangent ruled surface $\psi_1(s, v)$. The unit normal vector field U_1 of the tangent ruled surface $\psi_1(s, v)$ is provided by

$$U_1 = -\frac{1}{\kappa_\alpha} B_\alpha \tag{3.3}$$

Moreover, the coefficients of the second fundamental form as follows: $h_{11} = -v\tau_\alpha\kappa_\alpha$, $h_{12} = 0$, and $h_{22} = 0$. Using (2.1), the Gaussian and mean curvatures of the tangent ruled surface $\psi_1(s, v)$ are obtained as

$$K = 0 \quad \text{and} \quad H = \frac{-\tau_\alpha}{2v\kappa_\alpha} \tag{3.4}$$

Theorem 3.1. Let $\psi_1(s,v)$ be a tangent ruled surface with the MOF in E^3. Then, the tangent ruled surface $\psi_1(s,v)$ is developable.

The proof can be readily observed from (3.4).

Theorem 3.2. Let $\psi_1(s,v)$ be a tangent ruled surface with the MOF in E^3. Then, $\psi_1(s,v)$ cannot be minimal.

The proof can be readily observed from (3.4) by $\tau_\alpha \neq 0$.

Theorem 3.3. Let $\psi_1(s,v)$ be a tangent ruled surface according to the MOF in E^3. Then, the following hold:

i. s-parameter curves of the tangent ruled surface $\psi_1(s,v)$ cannot be asymptotic.

ii. v-parameter curves of the tangent ruled surface $\psi_1(s,v)$ are asymptotic.

PROOF. By the definition of asymptotic curves, $\langle \psi_{1ss}, U\rangle = 0$ and $\langle \psi_{1vv}, U\rangle = 0$.

i. The proof is obvious since $h_{11} \neq 0$.

ii. Since $h_{22} = 0$, v-parameter curves of the $\psi_1(s,v)$ are asymptotic.

□

Theorem 3.4. Let $\psi_1(s,v)$ be a tangent ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_1(s,v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_1(s,v)$ are geodesic.

PROOF. From the definition of geodesic curves, it must be $\psi_{1ss} \times U = 0$ and $\psi_{1vv} \times U_1 = 0$ for the s and v parameter curves.

i. According to (3.2) and (3.3),

$$\psi_{1ss} \times U = -\kappa_\alpha \left(1 + v\frac{\kappa'_\alpha}{\kappa_\prime}\right) T_\alpha - v\kappa_\alpha N_\alpha$$

Since T_α and N_α are linearly independent, $\psi_{1ss} \times U = 0$ if and only if $\kappa_\alpha = 0$. However, as $\kappa_\alpha \neq 0$, $\psi_1(s,v)$ cannot be a geodesic curve.

ii. From (3.2) and (3.3), $\psi_{1vv} \times U_1 = 0$. Thus, v-parameter curves are geodesic curves.

□

3.2. Tangent Ruled Surface with the Base Curve β

Concerning the MOF, the parameterization of the tangent ruled surface with the adjoint curve β is as follows:

$$\psi_2(s,v) = \beta(s) + vT_\beta(s) \tag{3.5}$$

If we take the derivatives with respect to parameter s and v of the tangent ruled surface $\psi_2(s,v)$, respectively, then

$$\begin{cases} \psi_{2s} = -v\dfrac{\tau_\alpha}{\kappa_\alpha}N_\alpha + B_\alpha, \quad \psi_{2v} = \dfrac{1}{\kappa_\alpha}B_\alpha, \quad \psi_{2ss} = v\tau_\alpha\kappa_\alpha T_\alpha + \left(-\tau_\alpha - v\dfrac{\tau'_\alpha}{\kappa_\alpha}\right)N_\alpha + \left(\dfrac{\kappa'_\alpha}{\kappa_\alpha} - v\dfrac{\tau^2_\alpha}{\kappa^2_\alpha}\right)B_\alpha, \\ \psi_{2ss} = v\tau_\alpha\kappa_\alpha T_\alpha + \left(-\tau_\alpha - v\dfrac{\tau'_\alpha}{\kappa_\alpha}\right)N_\alpha + \left(\dfrac{\kappa'_\alpha}{\kappa_\alpha} - v\dfrac{\tau^2_\alpha}{\kappa^2_\alpha}\right)B_\alpha, \quad \psi_{2sv} = -\dfrac{\tau_\alpha}{\kappa_\alpha}N_\alpha, \quad \text{and} \quad \psi_{2vv} = 0 \end{cases} \tag{3.6}$$

Hence,

$$g_{11} = \kappa_\alpha^2 + v^2\tau_\alpha^2, \quad g_{12} = \kappa_\alpha, \quad \text{and} \quad g_{22} = 1, \quad \text{and thus} \quad g_{11}g_{22} - g_{12}^2 = v^2\kappa_\alpha^2 \neq 0 \tag{3.7}$$

The unit normal vector field U_2 of the tangent ruled surface $\psi_2(s,v)$ is provided by $U_2 = -T_\alpha$. The coefficients of the second fundamental form are as follows: $h_{11} = -v\tau_\alpha\kappa_\alpha$, $h_{12} = 0$, and $h_{22} = 0$. Using (2.1), the Gaussian and mean curvatures of the tangent ruled surface $\psi_2(s,v)$ are obtained as follows: $K = 0$ and $H = \dfrac{-\kappa_\alpha}{2v\tau_\alpha}$.

Theorem 3.5. Let $\psi_2(s,v)$ be a tangent ruled surface with the MOF in Euclidean 3-space. Then, $\psi_2(s,v)$ is a flat surface.

Theorem 3.6. Let $\psi_2(s,v)$ be a tangent ruled surface with the MOF in Euclidean 3-space. Then, $\psi_2(s,v)$ cannot be minimal.

Proof. Since $\kappa_\alpha \neq 0$, the tangent ruled surface $\psi_2(s,v)$ cannot be minimal. □

Theorem 3.7. Let $\psi_2(s,v)$ be a tangent ruled surface with the MOF in E^3. Then,

i. s-parameter curves of the tangent ruled surface $\psi_2(s,v)$ cannot be asymtotic.

ii. v-parameter curves of the tangent ruled surface $\psi_2(s,v)$ are asymptotic curves.

Proof. From the definition of asymptotic curves, $\langle \psi_{2_{ss}}, U \rangle = 0$ and $\langle \psi_{2_{vv}}, U \rangle = 0$.

i. The proof is obvious since $h_{11} \neq 0$

ii. Since $h_{22} = 0$, v-parameter curves of the $\psi_2(s,v)$ are asymptotic.

□

Theorem 3.8. Let $\psi_2(s,v)$ be a tangent ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_2(s,v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_2(s,v)$ are geodesic.

Proof. From the definition of geodesic curves $\psi_{2_{ss}} \times U_2 = 0$ and $\psi_{2_{vv}} \times U_2 = 0$ must be provided for the s and v parameter curves.

i. According to (3.6) and (3.7),

$$\psi_{2_{ss}} \times U_2 = -\left(\frac{\kappa_\alpha'}{\kappa_\alpha} - v\frac{\tau_\alpha^2}{\kappa_\alpha^2}\right) N_\alpha + \left(-v\frac{\tau_\alpha'}{\kappa_\alpha} - \tau_\alpha\right) B_\alpha$$

Since N_α and B_α are linearly independent, $\psi_{2_{ss}} \times U_2 = 0$ if and only if κ_α is a constant and $\tau_\alpha = 0$. However, because $\tau_\alpha \neq 0$, $\psi_2(s,v)$ cannot be a geodesic curve.

ii. From (3.6) and (3.7), $\psi_{2_{vv}} \times U_2 = 0$. Hence, v-parameter curves are geodesic curves.

□

3.3. Normal Ruled Surface with the Base Curve α

Concerning the MOF, the parameterization of the normal ruled surface is as follows:

$$\psi_3(s,v) = \alpha(s) + vN_\alpha(s) \tag{3.8}$$

where α is the base curve. If we take the derivatives with respect to parameter s and v of the normal ruled surface $\psi_3(s,v)$, respectively, then

$$\begin{cases} \psi_{3s} = \left(1 - v\kappa_\alpha^2\right) T_\alpha + v\dfrac{\kappa'_\alpha}{\kappa_\alpha} N_\alpha + v\tau_\alpha B_\alpha, \quad \psi_{3v} = N_\alpha, \\ \psi_{3ss} = \left(-3v\kappa'_\alpha\kappa_\alpha\right) T_\alpha + \left(1 - v\left(\kappa_\alpha^2 + \tau_\alpha^2\right) + v\dfrac{\kappa''_\alpha}{\kappa_\alpha}\right) N_\alpha + v\left(\tau'_\alpha + 2\tau_\alpha\dfrac{\kappa'_\alpha}{\kappa_\alpha}\right) B_\alpha, \\ \psi_{3sv} = -\kappa_\alpha^2 T_\alpha + \dfrac{\kappa'_\alpha}{\kappa_\alpha} N_\alpha + \tau_\alpha B_\alpha, \quad \text{and} \quad \psi_{3vv} = 0 \end{cases} \tag{3.9}$$

Thereby, $g_{11} = 1 - 2v\kappa_\alpha^2 + v^2\kappa_\alpha^2\left(\kappa_\alpha^2 + \tau_\alpha^2\right) + v^2{\kappa'_\alpha}^2$, $g_{12} = v\kappa'_\alpha\kappa_\alpha$, and $g_{22} = \kappa_\alpha^2$, and thus $g_{11}g_{22} - g_{12}^2 = \kappa_\alpha^2\left(\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2\right) \neq 0$. The unit normal vector field U_3 of the normal ruled surface $\psi_3(s,v)$ is provided by

$$U_3 = \frac{1}{\sqrt{\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2}}\left(-v\kappa_\alpha\tau_\alpha T_\alpha + \frac{\left(1 - v\kappa_\alpha^2\right)}{\kappa_\alpha} B_\alpha\right) \tag{3.10}$$

and the coefficients of the second fundamental form are obtained as:

$$h_{11} = \frac{v}{\sqrt{\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2}}\left(v\kappa_\alpha^2\left(\tau_\alpha\kappa'_\alpha - \kappa_\alpha\tau'_\alpha\right) + \left(2\kappa'_\alpha\tau_\alpha + \tau'_\alpha\kappa_\alpha\right)\right)$$

$$h_{12} = \frac{\tau_\alpha\kappa_\alpha}{\sqrt{\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2}}$$

and $h_{22} = 0$. From (2.1), the Gaussian and mean curvatures are as follows, respectively:

$$K = -\frac{\tau_\alpha^2}{\left(\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2\right)^2} \quad \text{and} \quad H = \frac{v\kappa_\alpha\left(\kappa_\alpha v\tau_\alpha\kappa'_\alpha - v\kappa_\alpha^2\tau'_\alpha + \tau'_\alpha\right)}{2\left(\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2\right)^{\frac{3}{2}}} \tag{3.11}$$

Theorem 3.9. Let $\psi_3(s,v)$ be a ruled surface in Euclidean 3-space. Then, $\psi_3(s,v)$ is not a flat surface.

PROOF. Since $\tau_\alpha \neq 0$, $\psi_3(s,v)$ cannot be flat. □

Theorem 3.10. Let $\psi_3(s,v)$ be a normal ruled surface with the MOF in Euclidean 3-space. If the curve α is a cylindirical helix, then $\psi_3(s,v)$ is minimal.

The proof is directly obtained from (3.11).

Theorem 3.11. Let $\psi_3(s,v)$ be a normal ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_3(s,v)$ are asymptotic curves if and only if the curvatures κ_α and τ_α of the curve α are constant.

ii. v-parameter curves of the $\psi_3(s,v)$ are asymptotic curves.

PROOF. From the definition of asymptotic curves, $\langle \psi_{3ss}, U_3\rangle = 0$ and $\langle \psi_{3vv}, U_3\rangle = 0$.

i. From (3.9) and (3.10),

$$h_{11} = \frac{v}{\sqrt{\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2}}\left(v\kappa_\alpha^2\left(\tau_\alpha\kappa'_\alpha - \kappa_\alpha\tau'_\alpha\right) + \left(2\kappa'_\alpha\tau_\alpha + \tau'_\alpha\kappa_\alpha\right)\right) = 0$$

Thus,

$$v\kappa_\alpha^2\left(\tau_\alpha\kappa'_\alpha - \kappa_\alpha\tau'_\alpha\right) + \left(2\kappa'_\alpha\tau_\alpha + \tau'_\alpha\kappa_\alpha\right) = 0$$

since the curvatures κ_α and τ_α of the curve α are constant.

ii. Since $h_{22} = 0$, v-parameter curves of the $\psi_3(s,v)$ are asymptotic.

□

Theorem 3.12. Let $\psi_3(s,v)$ be a ruled surface with the MOF in E^3. Then, *i.* s-parameter curves of $\psi_3(s,v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_3(s,v)$ are geodesic.

PROOF. From the definition of geodesic curves, $\psi_{3_{ss}} \times U_3 = 0$ and $\psi_{3_{vv}} \times U_3 = 0$ for the s and v parameter.

i. According to (3.9) and (3.10),

$$\begin{aligned}\psi_{3_{ss}} \times U_3 = &\left(\kappa_\alpha - v\kappa_\alpha^3 + \left(1 - v\kappa_\alpha^2\right)\left(v\kappa_\alpha'' - v\kappa_\alpha\left(\kappa_\alpha^2 + \tau_\alpha^2\right)\right)\right)T_\alpha \\ &+ \left(3\kappa_\alpha' v - 3\kappa_\alpha^2\kappa_\alpha' v^2 - 2v^2\kappa_\alpha'\tau_\alpha^2 - v^2\tau_\alpha\tau_\alpha'\kappa_\alpha\right)N_\alpha \\ &+ \left(v\tau_\alpha\kappa_\alpha - v^2\tau_\alpha\kappa_\alpha\left(\kappa_\alpha^2 + \tau_\alpha^2\right) + v^2\kappa_\alpha''\tau_\alpha\right)B_\alpha\end{aligned}$$

Since T_α, N_α, and B_α are linearly independent, $\psi_{3_{ss}} \times U = 0$ if and only if κ_α is a constant and $\tau_\alpha = 0$. However, as $\tau_\alpha \neq 0$, $\psi_3(s,v)$ cannot be a geodesic curve.

ii. From (3.9) and (3.10), $\psi_{3_{vv}} \times U = 0$. Therefore, v-parameter curves are geodesic.

□

3.4. Normal Surfaces with the Adjoint Curve β

Concerning the MOF, the parameterization of the normal ruled surface is as follows:

$$\psi_4(s,v) = \beta(s) + vN_\beta(s) \tag{3.12}$$

where β is the base curve. If we take the derivatives with respect to the parameter s and v of the normal ruled surface $\psi_4(s,v)$, then

$$\begin{cases} \psi_{4_s} = \tau_\alpha T_\alpha + v\left(\dfrac{\kappa_\alpha'\tau_\alpha}{\kappa_\alpha^3} - \dfrac{\tau_\alpha'}{\kappa_\alpha^2}\right)N_\alpha + \left(1 - v\dfrac{\tau_\alpha^2}{\kappa_\alpha^2}\right)B_\alpha \\ \psi_{4_v} = -\dfrac{\tau_\alpha}{\kappa_\alpha^2}N_\alpha \\ \psi_{4_{ss}} = v\left(2\tau_\alpha' - v\tau_\alpha\dfrac{\kappa_\alpha'}{\kappa_\alpha}\right)T_\alpha + \left(\dfrac{\kappa_\alpha'}{\kappa_\alpha} + v\left(4\dfrac{\kappa_\alpha'\tau_\alpha^2}{\kappa_\alpha^3} - 5\dfrac{\tau_\alpha\tau_\alpha'}{\kappa_\alpha^2}\right)\right)B_\alpha \\ \quad + \left(v\left(\tau_\alpha + \dfrac{\kappa_\alpha''\tau_\alpha}{\kappa_\alpha^3} - \dfrac{\tau_\alpha''}{\kappa_\alpha^2} + \dfrac{\kappa_\alpha'\tau_\alpha}{\kappa_\alpha^3} - 2\dfrac{\kappa_\alpha'^2\tau_\alpha}{\kappa_\alpha^4} + \dfrac{\kappa_\alpha'\tau_\alpha'}{\kappa_\alpha^3} + \dfrac{\tau_\alpha^3}{\kappa_\alpha^2}\right) - \tau_\alpha\right)N_\alpha \\ \psi_{4_{sv}} = \tau_\alpha T_\alpha + \left(\dfrac{\kappa_\alpha'\tau_\alpha}{\kappa_\alpha^3} - \dfrac{\tau_\alpha'}{\kappa_\alpha^2}\right)N_\alpha - \dfrac{\tau_\alpha^2}{\kappa_\alpha^2}B_\alpha \\ \psi_{4_{vv}} = 0 \end{cases} \tag{3.13}$$

Hence, $g_{11} = v^2\left(\tau_\alpha^2 + \dfrac{\kappa_\alpha'^2\tau_\alpha^2}{\kappa_\alpha^4} - 2\dfrac{\kappa_\alpha'\tau_\alpha\tau_\alpha'}{\kappa_\alpha^3} + \dfrac{\tau_\alpha'^2}{\kappa_\alpha^2} + \dfrac{\tau_\alpha^4}{\kappa_\alpha^2}\right) + \kappa_\alpha^2 - 2v\tau_\alpha^2$, $g_{12} = \dfrac{v\tau_\alpha}{\kappa_\alpha^2}\left(\tau_\alpha' - \dfrac{\kappa_\alpha'\tau_\alpha}{\kappa_\alpha}\right)$, and $g_{22} = \dfrac{\tau_\alpha^2}{\kappa_\alpha^2}$ and thus $g_{11}g_{22} - g_{12}^2 = \dfrac{\tau_\alpha^2}{\kappa_\alpha^4}\left(\left(\kappa_\alpha^2 - v\tau_\alpha^2\right)^2 + \left(v\tau_\alpha\kappa_\alpha\right)^2\right) \neq 0$. Moreover, the unit normal vector field U_4 is provided by

$$U_4 = \frac{1}{\sqrt{\left(\kappa_\alpha^2 - v\tau_\alpha^2\right)^2 + \left(v\tau_\alpha\kappa_\alpha\right)^2}}\left(\left(\kappa_\alpha^2 - v\tau_\alpha^2\right)T_\alpha - v\tau_\alpha B_\alpha\right) \tag{3.14}$$

and the coefficients of the second fundamental form are obtained as:

$$h_{11} = \frac{1}{\sqrt{\left(\kappa_\alpha^2 - v\tau_\alpha^2\right)^2 + \left(v\tau_\alpha\kappa_\alpha\right)^2}}\left(5\frac{v^2\kappa_\alpha'\tau_\alpha^3}{\kappa_\alpha} - 7v^2\tau^2\tau_\alpha' + \kappa_\alpha\kappa_\alpha'\tau_\alpha v\left(1 - v\right) + 2v\kappa_\alpha^2\tau_\alpha'\right)$$

$$h_{12} = \frac{\tau_\alpha \kappa_\alpha^2}{\sqrt{(\kappa_\alpha^2 - v\tau_\alpha^2)^2 + (v\tau_\alpha\kappa_\alpha)^2}}$$

and $h_{22} = 0$. By (2.1), the Gaussian and mean curvatures are provided as follows:

$$K = -\frac{\kappa_\alpha^8}{\left((\kappa_\alpha^2 - v\tau_\alpha^2)^2 + (v\tau_\alpha\kappa_\alpha)^2\right)^2}$$

and

$$H = \frac{\kappa_\alpha}{2\tau_\alpha^2} \frac{\left(v^2\tau_\alpha^2\left(5\kappa_\alpha'\tau_\alpha^3 - 7\kappa_\alpha\tau_\alpha'\right) + \kappa_\alpha'\tau_\alpha v\kappa_\alpha^2\left(1 - v^2 + 2\tau_\alpha^2\right) + 2v\tau_\alpha'\kappa_\alpha^3\left(1 - \tau_\alpha^2\right)\right)}{\left((\kappa_\alpha^2 - v\tau_\alpha^2)^2 + (v\tau_\alpha\kappa_\alpha)^2\right)^{\frac{3}{2}}} \tag{3.15}$$

Theorem 3.13. Let $\psi_4(s,v)$ be a normal ruled surface with the MOF in Euclidean 3-space. Then, $\psi_4(s,v)$ is not a flat surface.

Proof. Since $\kappa_\alpha \neq 0$, $\psi_4(s,v)$ cannot be flat. □

Theorem 3.14. Let $\psi_4(s,v)$ be a normal ruled surface in Euclidean 3-space. If the curve α is a cylindrical helix, then $\psi_4(s,v)$ is minimal.

Proof. Let the curve α be a cylindrical helix. Then, κ_α and τ_α are constant. By (3.15),

$$v^2\tau_\alpha^2\left(5\kappa_\alpha'\tau_\alpha^3 - 7\kappa_\alpha\tau_\alpha'\right) + \kappa_\alpha'\tau_\alpha v\kappa_\alpha^2\left(1 - v^2 + 2\tau_\alpha^2\right) + 2v\tau_\alpha'\kappa_\alpha^3\left(1 - \tau_\alpha^2\right) = 0$$

Since $\kappa_\alpha' = 0$ and $\tau_\alpha' = 0$, $H = 0$. Therefore, $\psi_4(s,v)$ is minimal. □

Theorem 3.15. Let $\psi_4(s,v)$ be a normal ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_4(s,v)$ are asymptotic curves if and only if the curvatures $\kappa_\alpha, \tau_\alpha$ of the curve α are constant, $\dfrac{\kappa_\alpha}{\tau_\alpha} = \dfrac{5v}{1-v}$, or $\dfrac{\kappa_\alpha}{\tau_\alpha} = \dfrac{7v}{2}$.

ii. v-parameter curves of the $\psi_4(s,v)$ are asymptotic curves.

Proof. From the definition of asymptotic curves, $\langle \psi_{4_{ss}}, U\rangle = 0$ and $\langle \psi_{4_{vv}}, U\rangle = 0$.

i. From (3.13) and (3.14), $h_{11} = 0$. Thus,

$$\left(5\frac{v^2\kappa_\alpha'\tau_\alpha^3}{\kappa_\alpha} - 7v^2\tau^2\tau_\alpha' + \kappa_\alpha\kappa_\alpha'\tau_\alpha v(1-v) + 2v\kappa_\alpha^2\tau_\alpha'\right) = 0$$

since the curvatures κ_α and τ_α of the curve α are constant.

ii. Since $h_{22} = 0$, v-parameter curves of the $\psi_4(s,v)$ are asymptotic.

□

Theorem 3.16. Let $\psi_4(s,v)$ be a normal ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_4(s,v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_4(s,v)$ are geodesic.

The proof is similar to the previous theorem about the normal ruled surface $\psi_3(s,v)$.

3.5. Binormal Ruled Surface with the Curve α

Concerning the MOF, the parameterization of the binormal ruled surface is as follows:

$$\psi_5(s,v) = \alpha(s) + vB_\alpha(s) \tag{3.16}$$

where the base curve α. If we take the derivatives with respect to parameter s and v of the binormal ruled surface $\psi_5(s,v)$, then $\psi_{5_s} = T_\alpha - v\tau_\alpha N_\alpha + v\frac{\kappa'_\alpha}{\kappa_\alpha}B_\alpha$, $\psi_{5_v} = B_\alpha$, $\psi_{5_{ss}} = \kappa_\alpha^2 v T_\alpha + \left(1 - v\tau'_\alpha - 2v\tau_\alpha\frac{\kappa'_\alpha}{\kappa_\alpha}\right)N_\alpha + v\left(-\tau_\alpha^2 + \frac{\kappa''_\alpha}{\kappa_\alpha}\right)B_\alpha$, $\psi_{5_{sv}} = -\tau_\alpha N_\alpha + \frac{\kappa'_\alpha}{\kappa_\alpha}B_\alpha$, and $\psi_{5_{vv}} = 0$. Hence, $g_{11} = 1 + v^2\left({\kappa'_\alpha}^2 + \tau_\alpha^2\kappa_\alpha^2\right)$, $g_{12} = v\kappa'_\alpha\kappa_\alpha$, and $g_{22} = \kappa_\alpha^2$ and thus $g_{11}g_{22} - g_{12}^2 = \kappa_\alpha^2\left(1 + (v\kappa_\alpha\tau_\alpha)^2\right) \neq 0$. Moreover, the unit normal vector field is provided by

$$U_5 = \frac{1}{\sqrt{1 + (v\kappa_\alpha\tau_\alpha)^2}}\left(-v\kappa_\alpha\tau_\alpha T_\alpha - \frac{1}{\kappa_\alpha}N_\alpha\right)$$

and the coefficients of the second fundamental form are obtained as:

$$h_{11} = \frac{1}{\sqrt{1 + (v\kappa_\alpha\tau_\alpha)^2}}\left(v\left(\kappa_\alpha\tau'_\alpha + 2\tau_\alpha\kappa'_\alpha - \kappa_\alpha^3 v\tau_\alpha\right) - \kappa_\alpha\right)$$

$$h_{12} = \frac{\tau_\alpha\kappa_\alpha}{\sqrt{1 + (v\kappa_\alpha\tau_\alpha)^2}}$$

and $h_{22} = 0$. From (2.1), the Gaussian and mean curvatures are as follows, respectively:

$$K = -\frac{\tau_\alpha^2}{\left(1 + (v\kappa_\alpha\tau_\alpha)\right)^2} \quad \text{and} \quad H = \frac{v\left(\kappa_\alpha\tau'_\alpha - \kappa_\alpha^3 v\tau_\alpha\right) - \kappa_\alpha}{2\left(1 + (v\kappa_\alpha\tau_\alpha)^2\right)^{\frac{3}{2}}} \tag{3.17}$$

Theorem 3.17. Let $\psi_5(s,v)$ be a binormal ruled surface with the MOF in Euclidean 3-space. Then, $\psi_5(s,v)$ is not a flat surface.

Proof. Since $\tau_\alpha \neq 0$, $\psi_5(s,v)$ cannot be flat. □

Theorem 3.18. Let $\psi_5(s,v)$ be a ruled surface in Euclidean 3-space. If the curve α is a cylindrical helix, then $\psi_5(s,v)$ is minimal

The result is directly obtained from (3.17).

Theorem 3.19. Let $\psi_5(s,v)$ be a ruled surface in E^3 with the MOF. Then,

i. s-parameter curves of $\psi_5(s,v)$ are asymptotic curves if and only if the curvatures κ_α and τ_α of the curve α are constant and $\tau_\alpha = \frac{1}{v^2\kappa_\alpha^2}$.

ii. v-parameter curves of $\psi_5(s,v)$ are asymptotic curves.

The proof is similar to Theorem 3.11.

Theorem 3.20. Let $\psi_5(s,v)$ be a ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_5(s,v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_5(s,v)$ are geodesic.

The proof is similar to Theorem 3.12.

3.6. Binormal Ruled Surface with the Adjoint Curve β

Concerning the MOF, the parameterization of the binormal ruled surface is as follows:

$$\psi_6(s,v) = \beta(s) + vB_\beta(s) \tag{3.18}$$

where the base curve is the adjoint curve β. If we take the derivatives with respect to the parameter s and v of the binormal ruled surface $\psi_6(s,v)$, then $\psi_{6_s} = v\left(\frac{\tau'_\alpha}{\kappa_\alpha} - \frac{\kappa'_\alpha\tau_\alpha}{\kappa_\alpha^2}\right)T_\alpha + v\frac{\tau_\alpha}{\kappa_\alpha}N_\alpha + B_\alpha$ and

$\psi_{6v} = \frac{\tau_\alpha}{\kappa_\alpha} T_\alpha$ and thus

$$\psi_{6ss} = v\left(\frac{\tau''_\alpha}{\kappa_\alpha} - 2\frac{\kappa'_\alpha\tau'_\alpha}{\kappa_\alpha^2} - \frac{\kappa''_\alpha\tau_\alpha}{\kappa_\alpha^2} + 2\frac{\kappa'^{\,2}_\alpha\tau_\alpha}{\kappa_\alpha^3} - \kappa_\alpha\tau_\alpha\right)T_\alpha$$
$$+\left(-\tau_\alpha + v\left(2\frac{\tau'_\alpha}{\kappa_\alpha} - \frac{\kappa'_\alpha\tau_\alpha}{\kappa_\alpha^2}\right)\right)N_\alpha + \left(\frac{\kappa'_\alpha}{\kappa_\alpha} + v\frac{\tau_\alpha^2}{\kappa_\alpha}\right)B_\alpha$$
$$\psi_{6sv} = \tau_\alpha T_\alpha + \left(\frac{\kappa'_\alpha\tau_\alpha}{\kappa_\alpha^3} - \frac{\tau'_\alpha}{\kappa_\alpha^2}\right)N_\alpha - \frac{\tau_\alpha^2}{\kappa_\alpha^2}B_\alpha$$

and $\psi_{6vv} = 0$ Hence, $g_{11} = \kappa_\alpha^2 + v^2\left(\tau_\alpha^2 + \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'^2\right)$, $g_{12} = \frac{v\tau_\alpha}{\kappa_\alpha}\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'$, and $g_{22} = \frac{\tau_\alpha^2}{\kappa_\alpha^2}$ and thus $g_{11}g_{22} - g_{12}^2 = \frac{\tau_\alpha^2}{\kappa_\alpha^2}\left(\kappa_\alpha^2 + v^2\tau_\alpha^2\right) \neq 0$. Moreover, the unit normal vector field U_6 is provided by

$$U_6 = \frac{1}{\sqrt{\kappa_\alpha^2 + v^2\tau_\alpha^2}}\left(N_\alpha - \frac{\tau_\alpha}{\kappa_\alpha}vB_\alpha\right)$$

and the coefficients of the second fundamental form are obtained as:

$$h_{11} = \frac{1}{\sqrt{\kappa_\alpha^2 + v^2\tau_\alpha^2}}\left(-\tau_\alpha\left(\kappa_\alpha^2 + v^2\tau_\alpha^2\right) + 2v\left(\kappa_\alpha\tau'_\alpha - \kappa'_\alpha\tau_\alpha\right)\right)$$
$$h_{12} = \frac{\tau_\alpha\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + v^2\tau_\alpha^2}}$$

and $h_{22} = 0$. By (2.1), the Gaussian and mean curvatures are as follows:

$$K = -\frac{\kappa_\alpha^4}{\left(\kappa_\alpha^2 + v^2\tau_\alpha^2\right)^2} \quad \text{and} \quad H = -\frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2 + v^2\tau_\alpha^2}} \tag{3.19}$$

Theorem 3.21. Let $\psi_6(s,v)$ be a binormal ruled surface according to the MOF in Euclidean 3-space. Then, $\psi_6(s,v)$ is not a flat surface.

Proof. Since $\kappa_\alpha \neq 0$, $\psi_6(s,v)$ cannot be flat. □

Theorem 3.22. Let $\psi_6(s,v)$ be a ruled surface in Euclidean 3-space. Then, $\psi_6(s,v)$ cannot be minimal.

The result is obtained directly from 3.19.

Theorem 3.23. Let $\psi_6(s,v)$ be a ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_6(s,v)$ cannot be asymptotic curves.

ii. v-parameter curves of $\psi_6(s,v)$ are asymptotic curves.

The proof is similar to Theorem 3.11.

Theorem 3.24. Let $\psi_6(s,v)$ be a ruled surface in E^3 with the MOF. Then,

i. s-parameter curves of $\psi_6(s,v)$ cannot be geodesic.

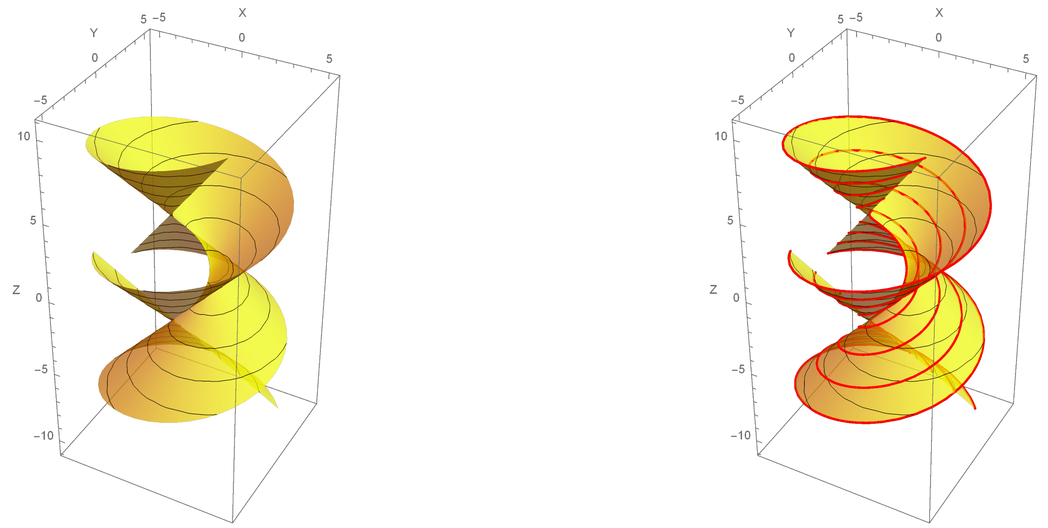
ii. v-parameter curves of $\psi_6(s,v)$ are geodesic.

The proof is similar to Theorem 3.12.

Example 3.25. Consider the curve α and and the adjoint curve β are provided by the following parametric equations, respectively:

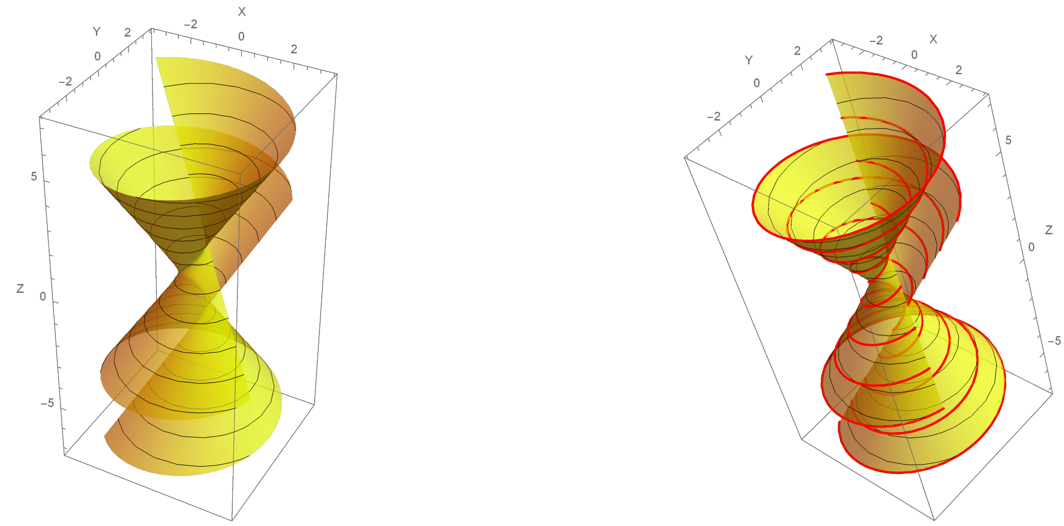
$\alpha(s) = \left(\cos\left(\frac{\sqrt{7}s}{4}\right), \sin\left(\frac{\sqrt{7}s}{4}\right), \frac{3s}{4}\right)$ and $\beta(s) = \left(-\frac{3\sqrt{7}}{16}\cos\left(\frac{\sqrt{7}}{4}s\right), -\frac{3\sqrt{7}}{16}\sin\left(\frac{\sqrt{7}}{4}s\right), -\frac{7\sqrt{7}}{64}s\right)$

According to the curves α and β, the graphs of some ruled surfaces are as in Figures 1-6.



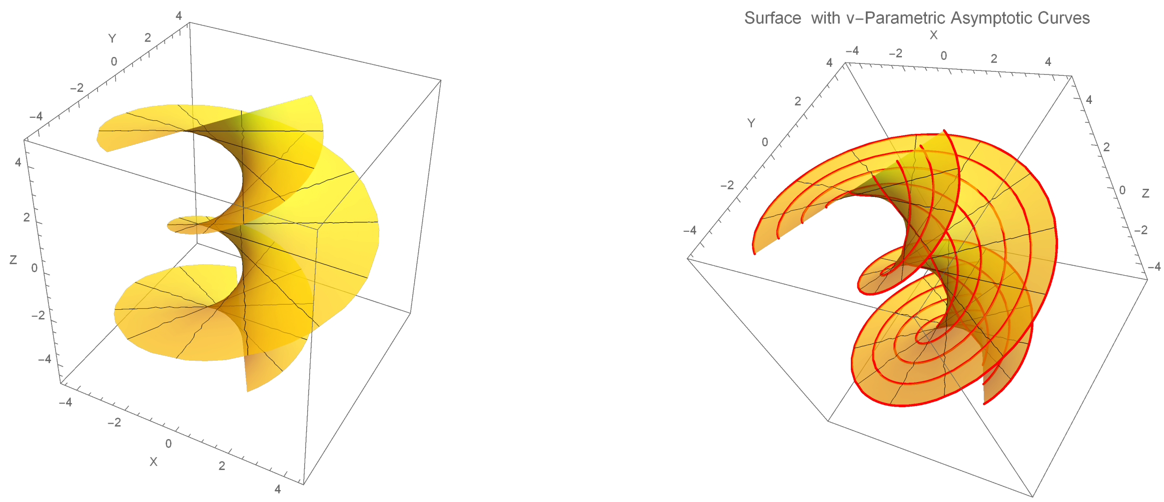
(a) Tangent ruled surface $\psi_1(s,v)$ **(b)** $v-$parameter curves of $\psi_1(s,v)$

Figure 1. Graph of the tangent ruled surface $\psi_1(s,v)$ in (3.1) whose director curve is α with the MOF



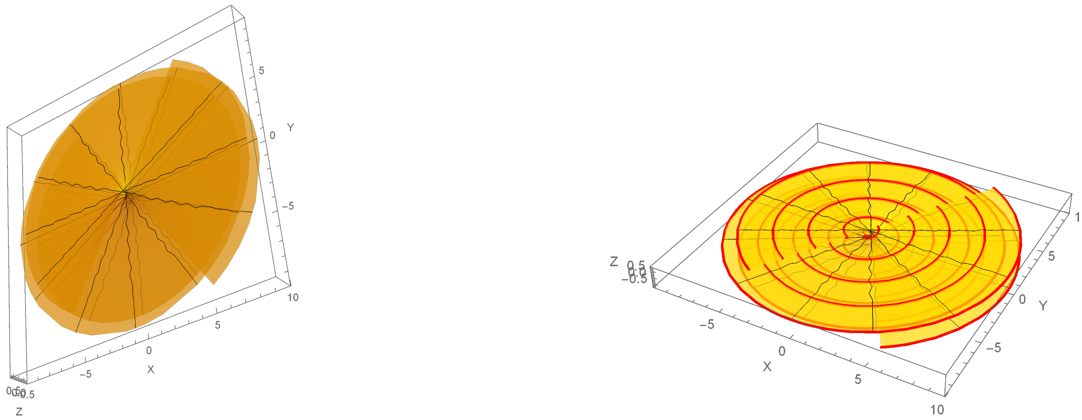
(a) Tangent ruled surface $\psi_2(s,v)$ **(b)** $v-$parameter curves of $\psi_2(s,v)$

Figure 2. Graph of the tangent ruled surface $\psi_2(s,v)$ in (3.5) whose director curve is β with the MOF



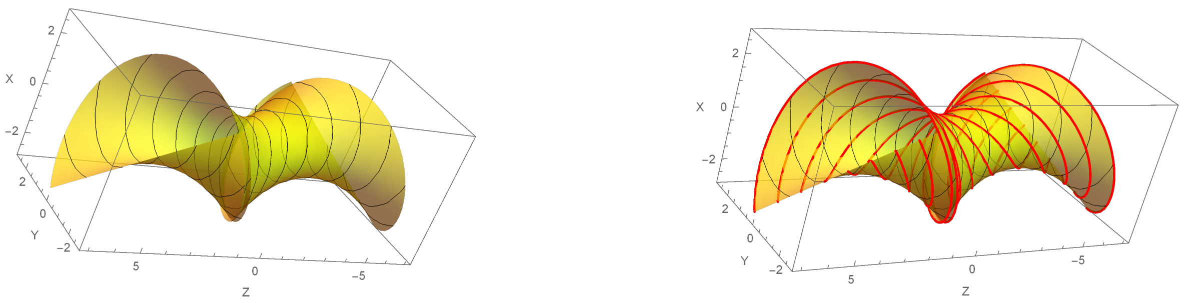
(a) Normal ruled surface $\psi_3(s,v)$ **(b)** $v-$parameter curves of $\psi_3(s,v)$

Figure 3. Graph of the normal ruled surface $\psi_3(s,v)$ in (3.8) whose the director curve is α with the MOF



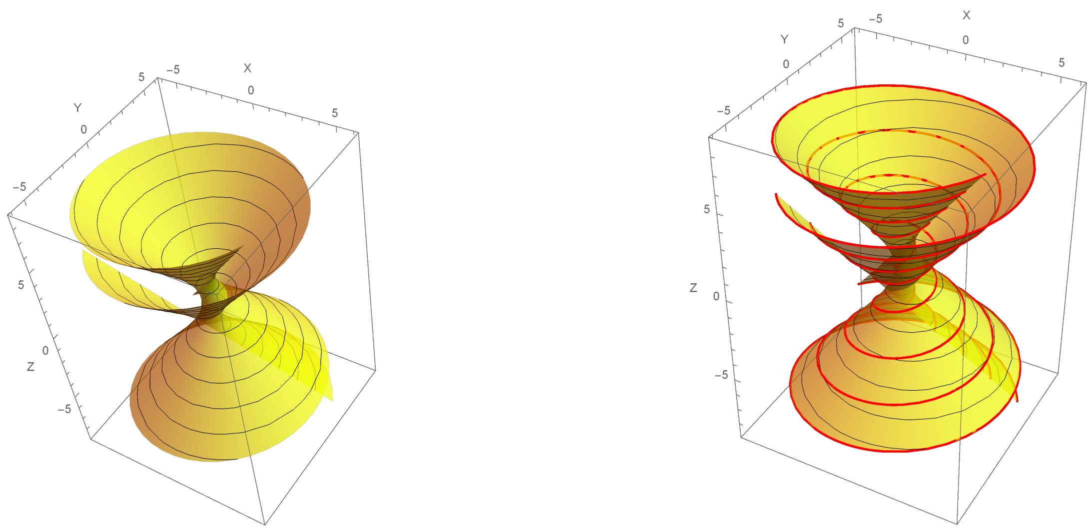
(a) Normal ruled surface $\psi_4(s,v)$ **(b)** $v-$parameter curves of $\psi_4(s,v)$

Figure 4. Graph of the tangent ruled surface $\psi_4(s,v)$ in (3.12) whose director curve is β with the MOF



(a) Binormal ruled surface $\psi_5(s,v)$ **(b)** $v-$parameter curves of $\psi_5(s,v)$

Figure 5. Graph of the binormal ruled surface $\psi_5(s,v)$ in (3.16) whose the director curve is α with the MOF



(a) binormal ruled surface $\psi_6(s,v)$ **(b)** $v-$parameter curves of $\psi_6(s,v)$

Figure 6. Graph of the binormal ruled surface $\psi_6(s,v)$ in (3.18) whose director curve is β with the MOF

4. Results

We calculated the Gaussian curvature K and the mean curvature H of some special ruled surfaces generated by the curve α and its adjoint curve β according to the MOF in E^3. While the tangent ruled surfaces are flat, the normal and binormal ruled surfaces are not flat. Even if the frame of the tangent ruled surface changes, its state of being minimal does not change, so it cannot be minimal.

We found a minimal condition for the normal and binormal ruled surfaces. Additionally, we searched s-parameter and v-parameter curves of some special ruled surfaces. Hence, we got some conditions for the s-parameter curves of some special ruled surfaces to be asymptotic and the v-parameter curves of some special ruled surfaces to be geodesic.

5. Conclusion

This study utilized the MOF to investigate the curvature characteristics and minimality of certain ruled surfaces based on a base curve and its adjoint in Euclidean 3-space. It was determined that tangent-ruled surfaces are flat, while normal and binormal surfaces are not. Additionally, only specific conditions allow for minimality in normal and binormal ruled surfaces. Future studies could explore applying these findings to different classes of ruled surfaces or extending the approach to non-Euclidean spaces. Further research may also analyze the potential applications of these geometric properties in advanced modeling, which could provide insights into mathematical physics and computer-aided geometric design.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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Statistical Convergence in L-Fuzzy Metric Spaces

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Abstract — Statistical convergence, defined in terms of the natural density of positive integers, has been studied in many different spaces, such as intuitionistic fuzzy metric spaces, partial metric spaces, and L-fuzzy normed spaces. The main goal of this study is to define statistical convergence in L-fuzzy metric spaces (L-FMSs), one of the essential tools for modeling uncertainty in everyday life. Furthermore, this paper introduces the concept of statistical Cauchy sequences and investigates its relation with statistical convergence. Then, it defines statistically complete L-FMSs and analyzes some of their basic properties. Finally, the paper inquires the need for further research.

Keywords *Statistical convergence, statistical Cauchy sequences, L-fuzzy metric spaces, completeness*

Mathematics Subject Classification (2020) 40A35, 40A05

1. Introduction

Zadeh [1] put forward the concept of fuzzy sets, which have been used in many fields, such as decision-making, artificial intelligence, weather forecasting, and probability theory, and can model problems involving uncertainty. One of the applications of these fields is fuzzy metric spaces, presented by Kramosil and Michalek [2] and Kaleva and Seikkala [3]. George and Veeramani [4] reformulated it with the help of triangular norms since this space is not Hausdorff. Gregori et al. [5] investigated convergence in fuzzy metric spaces. Atanassov [6] introduced the concept of intuitionistic fuzzy sets, a generalization of fuzzy sets. Later, Park [7] defined intuitionistic fuzzy metric spaces using fuzzy metric spaces as derived from George and Veremaani and proved some known results, such as Baire's theorem and the uniform limit theorem in the mentioned space. L-fuzzy metric spaces (L-FMSs) based on specific logical algebraic structures have been characterized by Saadati et al. [8] as a natural generalization of intuitionistic fuzzy metric spaces. Saadati [9] studied L-fuzzy topological spaces and proved that L-FMSs have many properties, such as being a normal, separable, and metrizable space. Many researchers have generalized the classical concepts of topology and functional analysis to fuzzy metric spaces. L-FMSs provided a more general framework for generalizing the classical concepts to a fuzzy setting. Motivated by this fact, in this present study, we propose statistical convergence in L-FMSs.

In 1951, the concept of statistical convergence was introduced by Fast [10] and Steinhaus [11] as a generalization of the classical convergence. This concept is dependent on the theory of natural

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densities [12]. The concept of statistical convergence was further analyzed by the authors, such as Salat [13], Fridy [14], Connor [15], and Mursaalen and Edely [16], along with many fields, such as summability theory [17], operator theory [18], and approximation theory [19]. In 2020, Li et al. [20] investigated the notion of statistical convergence in fuzzy metric spaces, Savaş [21] researched statistical convergence of double sequences, and Varol [22] defined statistical convergence in intuitionistic fuzzy metric spaces. In 2023, Özcan et al. [23, 24] researched statistical convergence of double sequences and λ-statistical convergence in these spaces, respectively.

The remainder of the present paper is organized as follows: In Section 2, we present some basic definitions and properties to be needed in the following section. In Section 3, we analyze statistical convergence in L-FMSs and then study the statistical Cauchy sequences for complete metric spaces. Furthermore, we research the relationship between these notions and obtain some results and findings. Finally, we argue that they are essential for future study in this space.

2. Preliminaries

This section presents some basic notions and properties to be required for the following sections. Throughout this study, the notations $\mathbb{N}$ and $\mathbb{R}$ represent the set of all positive integers and the set of all real numbers, respectively.

Definition 2.1. [11] A sequence (ξ_k) is referred to as statistically convergent to ξ, denoted by $st\text{-}\lim \xi_k = \xi$, if, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |\xi_k - \xi| \geq \varepsilon\}| = 0$$

where the notation $|\ \ |$ represents the cardinality of a set.

Definition 2.2. [12] The natural density of a set $A \subseteq \mathbb{N}$, denoted by $\delta(A)$, is defined by

$$\delta(A) := \lim_{n\to\infty} \frac{|\{k \leq n : k \in A\}|}{n}$$

where the notation $|\ \ |$ represents the cardinality of a set.

Definition 2.3. Let $\mathbb{L}$ be a lattice equipped with a partial order $\preceq_{\mathbb{L}}$. Then, $(\mathbb{L}, \preceq_{\mathbb{L}})$ is referred to as a complete lattice if $\sup S \in \mathbb{L}$ and $\inf S \in \mathbb{L}$, for all subsets S of $\mathbb{L}$.

Across this study, L represents the pair $(\mathbb{L}, \preceq_{\mathbb{L}})$. Moreover, the notations 1_L and 0_L denote $\sup \mathbb{L}$ and $\inf \mathbb{L}$, respectively.

Definition 2.4. [1, 25] Let $\mathbb{L}$ be a complete lattice, E be a non-empty universal set, and $\mu : E \to \mathbb{L}$ be a mapping. Then, the mapping μ is called an $\mathbb{L}$-fuzzy set on E, which for all $e \in E$, $\mu(e)$ specifies the grade of belonging of e to the $\mathbb{L}$-fuzzy set μ.

Lemma 2.5. [27] The partially ordered set $(\mathbb{L}^* \preceq_{\mathbb{L}^*})$ defined by

$$\mathbb{L}^* = \{(\alpha, \beta) : \alpha, \beta \in [0, 1] \text{ and } \alpha + \beta \leq 1\} \quad \text{and} \quad (\alpha, \beta) \preceq_{\mathbb{L}^*} (\gamma, \omega) \Leftrightarrow \alpha \leq \gamma \text{ and } \beta \geq \omega$$

is a complete lattice.

A triangular norm $T : [0, 1]^2 \to [0, 1]$ on the complete lattice $([0, 1], \leq)$ is a function that is commutative, increasing, and associative and satisfies the condition $T(1, \alpha) = \alpha$, for all $\alpha \in [0, 1]$. Using a complete lattice L, this concept have been generalized as follows:

Definition 2.6. [27] Let L be a complete lattice and $\varphi : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ be a function. Then, φ is called a triangular norm (t-norm) on L if the following are satisfied: For all $\alpha, \beta, \gamma, \omega \in \mathbb{L}$,

i. $\varphi(\beta, \alpha) = \varphi(\alpha, \beta)$

ii. $\varphi(\alpha, \varphi(\beta, \gamma)) = \varphi(\varphi(\alpha, \beta), \gamma)$

iii. $\varphi(\alpha, 1_L) = \varphi(1_L, \alpha) = \alpha$

iv. If $\gamma \preceq_{\mathbb{L}} \omega$ and $\alpha \preceq_{\mathbb{L}} \beta$, then $\varphi(\alpha, \gamma) \preceq_{\mathbb{L}} \varphi(\beta, \omega)$

Definition 2.7. [27] Let L be a complete lattice. Then, the function $\mathcal{N} : \mathbb{L} \to \mathbb{L}$ is called a negator on L if it satisfies the following conditions:

i. $\mathcal{N}$ is a decreasing function

ii. $\mathcal{N}(1_L) = 0_L$ and $\mathcal{N}(0_L) = 1_L$

In addition, $\mathcal{N}$ is referred to as an involutive negator if it provides the condition $\mathcal{N}(\mathcal{N}(\alpha)) = \alpha$, for all $\alpha \in \mathbb{L}$.

One of the well-known examples of involutive negators is the function $\mathcal{N} : [0, 1] \to [0, 1]$ defined by $\mathcal{N}(\alpha) = 1 - \alpha$ where $([0, 1], \leq)$ is a complete lattice.

Definition 2.8. [8] Let $\mathbb{X}$ be a non-empty set and φ be a continuous t-norm on L. An L-fuzzy metric is a mapping $\mu : \mathbb{X}^2 \times (0, \infty) \to \mathbb{L}$ satisfying the following conditions: For all $t, s > 0$ and for all $\alpha, \beta, \gamma \in \mathbb{X}$,

i. $\mu(\alpha, \beta, t) \succ_{\mathbb{L}} 0_L$

ii. $\mu(\alpha, \beta, t) = 1_L$ if and only if $\alpha = \beta$

iii. $\mu(\alpha, \beta, t) = \mu(\beta, \alpha, t)$

iv. $\varphi(\mu(\alpha, \beta, t), \mu(\beta, \gamma, s)) \preceq_{\mathbb{L}} \mu(\alpha, \gamma, t + s)$

v. $\mu_{\alpha\beta} : (0, \infty) \to \mathbb{L}$ is continuous

Moreover, the triple $(\mathbb{X}, \mu, \varphi)$ is called an L-fuzzy metric space (L-FMS).

Definition 2.9. [8] Let $\mathcal{N}$ be a negator on L, the triple $(\mathbb{X}, \mu, \varphi)$ be an L-FMS, $\alpha \in \mathbb{X}$, $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$, and $t > 0$. Then, the set

$$B(\alpha, \mathcal{N}(\varepsilon), t) := \{\beta \in \mathbb{X} : \mu(\alpha, \beta, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$$

is called the open ball with centre α and radius $\mathcal{N}(\varepsilon)$.

Example 2.10. [8] Let $(\mathbb{X}, d)$ be a metric space, μ be an $\mathbb{L}$-fuzzy set on $\mathbb{X}^2 \times (0, \infty)$ defined by

$$\mu(\xi, \eta, t) = \frac{ht^n}{ht^n + md(\xi, \eta)}$$

where $h, n, m > 0$, and φ be a continuous t-norm described by $\varphi(\alpha, \beta) = \alpha\beta$, for all $\alpha, \beta \in \mathbb{L}$. Then, the function μ satisfies the conditions in Definition 2.8. Thus, $(\mathbb{X}, \mu, \varphi)$ is an L-FMS.

Definition 2.11. [8] Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$.

i. (ξ_k) is called convergence to $\xi \in \mathbb{X}$, denoted by $\xi_n \xrightarrow{L} \xi$, if, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$, there exists a $k_\varepsilon \in \mathbb{N}$ such that for all $k \geq k_\varepsilon$, $\mu(\xi_k, \xi, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$.

ii. (ξ_k) is called a Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$, if, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$, there exists a $k_\varepsilon \in \mathbb{N}$ such that for all $k, n \geq k_\varepsilon$, $\mu(\xi_k, \xi_n, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$.

iii. $(\mathbb{X}, \mu, \varphi)$ is complete if and only if every Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$ is convergent.

Note 2.12. Let φ be a continuous t-norm on L and $\mathcal{N}$ be an involutive negator on L. Then, for all $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$, there exists an $\alpha \in \mathbb{L} - \{0_L, 1_L\}$ such that $\varphi(\mathcal{N}(\alpha), \mathcal{N}(\alpha)) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$.

3. Statistical Convergence in L-Fuzzy Metric Spaces

This section analyzes relations between statistical convergence and classical convergence in L-FMSs. In addition, it presents a characterization of statistical convergence with subsequences.

Definition 3.1. Let (ξ_k) be sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$. Then, (ξ_k) is referred to as statistical convergent to $\xi \in \mathbb{X}$, denoted by $\xi_k \xrightarrow{st_L} \xi$, if, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$,

$$\delta\left(\{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}\right) = 0$$

Here, the element ξ is called a statistical limit point of the sequence (ξ_k).

Example 3.2. Let μ be an $\mathbb{L}$-fuzzy set on $\mathbb{R}^2 \times (0, \infty)$ and φ be a continuous t-norm on L defined by

$$\mu(\xi, \eta, t) = \frac{t}{t + |\xi - \eta|} \quad \text{and} \quad \varphi(\alpha, \beta) = \alpha\beta$$

Then, $(\mathbb{R}, \mu, \varphi)$ is L-FMS. Consider the sequence (ξ_k) defined by

$$\xi_k = \begin{cases} 5, & \exists n \in \mathbb{N} \ni k = n^2 \\ 0, & \text{otherwise} \end{cases}$$

and the set

$$A = \{k \in \mathbb{N} : \mu(\xi_k, 0, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$$

for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$. Then, (ξ_k) is statistical convergent to 0 as $\delta(A) = 0$, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$.

Lemma 3.3. Let $(\mathbb{X}, \mu, \varphi)$ be an L-FMS. Then, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$, the following conditions are equivalent:

i. $\delta\left(\{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}\right) = 0$

ii. $\delta\left(\{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}\right) = 1$

The proof can be readily observed from Definition 3.1 and density properties.

Theorem 3.4. Let (ξ_k) be a sequence in a L-FMS $(\mathbb{X}, \mu, \varphi)$. If (ξ_k) is statistical convergent, then its statistical limit point is unique.

Proof. Let $\xi_k \xrightarrow{st_L} \ell_1$ and $\xi_k \xrightarrow{st_L} \ell_2$. From Note 2.12, for all $r \in \mathbb{L} - \{0_L, 1_L\}$, there exists an $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$ such that $\varphi\left(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)\right) \succ_{\mathbb{L}} \mathcal{N}(r)$. Assume that

$$\ell_3 \in B\left(\ell_1, \mathcal{N}(\varepsilon), \frac{t}{2}\right) \cap B\left(\ell_2, \mathcal{N}(\varepsilon), \frac{t}{2}\right)$$

Then,

$$\begin{aligned} \mu(\ell_1, \ell_2, t) &\succeq_{\mathbb{L}} \varphi\left(\mu\left(\ell_1, \ell_3, \tfrac{t}{2}\right), \mu\left(\ell_2, \ell_3, \tfrac{t}{2}\right)\right) \\ &\succeq_{\mathbb{L}} \varphi\left(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)\right) \\ &\succ_{\mathbb{L}} \mathcal{N}(r) \end{aligned}$$

which is a contradiction. Thus,

$$B\left(\ell_1, \mathcal{N}(\varepsilon), \frac{t}{2}\right) \cap B\left(\ell_2, \mathcal{N}(\varepsilon), \frac{t}{2}\right) = \emptyset$$

Hence,

$$B\left(\ell_2, \mathcal{N}(\varepsilon), \frac{t}{2}\right) \subseteq \left\{\ell_3 \in \mathbb{X} : \mu\left(\ell_1, \ell_3, \frac{t}{2}\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\}$$

Then,

$$\left\{k \in \mathbb{N} : \mu\left(\xi_k, \ell_2, \frac{t}{2}\right) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\} \subseteq \left\{k \in \mathbb{N} : \mu\left(\xi_k, \ell_1, \frac{t}{2}\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\}$$

Since

$$1 = \delta\left(\left\{k \in \mathbb{N} : \mu\left(\xi_k, \ell_2, \frac{t}{2}\right) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\}\right) \leq \delta\left(\left\{k \in \mathbb{N} : \mu\left(\xi_k, \ell_1, \frac{t}{2}\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\}\right) = 0$$

a contradiction occurs. Consequently, $\ell_1 = \ell_2$. □

Theorem 3.5. Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$ and $\xi \in \mathbb{X}$. If (ξ_k) convergent to ξ, then (ξ_k) is statistically convergent to ξ.

PROOF. Let (ξ_k) is convergent to $\xi \in \mathbb{X}$. Then, there exists a $k_\varepsilon \in \mathbb{N}$ such that $\mu(\xi_k, \xi, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$, for all $k > k_\varepsilon$, $\varepsilon \in \mathbb{L} - \{0_L\}$, and $t > 0$. Therefore, there are just a finite number of terms in the set

$$A = \{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$$

Since the property of natural density "finite subsets of natural numbers has density zero", $\delta(A) = 0$. Therefore, (ξ_k) is statistical convergent to ξ. □

The converse of the Theorem 3.5 is not always true (see Example 3.6).

Example 3.6. Consider the L-FMS in Example 3.2 and the sequence (ξ_k) defined as follows:

$$\xi_k = \begin{cases} 9, & \exists n \in \mathbb{N} \ni k = n^2 \\ 6, & \text{otherwise} \end{cases}$$

It can be observed that (ξ_k) is not convergent to 6 but statistical convergent to 6 because

$$\delta(\{k \in \mathbb{N} : \mu(\xi_k, 6, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}) = 0$$

Theorem 3.7. Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$. Then, $\xi_k \xrightarrow{st_L} \xi$ if and only if $\xi_{k_j} \xrightarrow{L} \xi$ such that $\delta(A) = 1$ where $A = \{k_j : j \in \mathbb{N}\}$.

PROOF. Assume that $\xi_k \xrightarrow{st_L} \xi$. Let

$$S_j(q) = \begin{cases} \mathbb{N}, & j = 1 \\ \{k \in \mathbb{N} : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_j)\}, & j \geq 2 \end{cases}$$

for all $q > 0$ and $j \in \mathbb{N}$. Thus, $S_{j+1}(q) \subset S_j(q)$, for all $q > 0$ and $j \in \mathbb{N}$. Since (ξ_k) is statistical convergent to ξ, then

$$\delta(S_j(q)) = 1 \tag{3.1}$$

Let $t_1 \in S_1(q)$. Since $\delta(S_2(q)) = 1$, then there is a number $t_2 \in S_2(q)$ such that $t_2 > t_1$ and

$$\frac{1}{n}\left|\{k \leq n : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_2)\}\right| > \frac{1}{2}$$

for all $n \geq t_2$. By (3.1), $\delta(S_3(q)) = 1$. Thus, there exists a $t_3 \in S_3(q)$ such that $t_3 > t_2$ and

$$\frac{1}{n}\left|\{k \leq n : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_3)\}\right| > \frac{2}{3}$$

for all $n \geq t_3$ and the procedure is continued similarly. Then, by induction, we can construct a sequence of increasing indexes of positive integers (t_j) such that $t_j \in S_j(q)$. Besides,

$$\frac{1}{n}\left|\{k \leq n : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_j)\}\right| > \frac{j-1}{j} \tag{3.2}$$

for all $n \geq t_j$ and $j \in \mathbb{N}$. Moreover, (w_j) is a decreasing sequence in $\mathbb{L} - \{0_L\}$ such that $\mathcal{N}(w_j) \to 1_L$. Suppose that

$$S := \{k \leq n : 1 < k < t_1\} \cup \left[\bigcup_{j\in\mathbb{N}} \{k \in S_j(q) : t_j \leq k < t_{j+1}\}\right]$$

Since $S_{j+1}(q) \subset S_j(q)$ and due to (3.2), $S = \{k_j : j \in \mathbb{N}\}$. Let $k > t_2$. Then, there exists a $j \in \mathbb{N}$ such that $t_j \leq k < t_{j+1}$. Hence,

$$\begin{aligned}\frac{|\{k \leq n : k \in S\}|}{n} &\geq \frac{|\{k \leq n : k \in S_j(q)\}|}{n} \\ &= \frac{|\{k \leq n : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_j)\}|}{n} \\ &> \frac{j-1}{j}\end{aligned}$$

for all $n \geq t_j$. As $n, j \to \infty$,

$$\lim_{n\to\infty} \frac{|\{k \leq n : k \in S\}|}{n} = 1$$

i.e., $\delta(S) = 1$. Let $w \in L - \{0_L\}$ and $j \in \mathbb{N}$ such that $w \succ_{\mathbb{L}} w_j$. Such a number j always exists since $w_j \to 0_{\mathbb{L}}$. Let $k \geq t_j$ and $k \in S$. Then, according to the definition of S, a number $t \geq j$ exists such that $t_j \leq k_m < t_{j+1}$ and $k_m \in S_j(q)$. Thus, for all $w \in L - \{0_L\}$ and for all $k_m \geq t_j$ such that $k_m \in S_j(q)$,

$$\mu(\xi_{k_m}, \xi, t) \succ_{\mathbb{L}} \mathcal{N}(w_j) \succ_{\mathbb{L}} \mathcal{N}(w)$$

Consequently, the sequence (ξ_{k_j}) is convergent to ξ in the L-FMS.

Conversely, suppose that the subsequence (ξ_{k_j}) is convergent to ξ in the L-FMS such that $\delta(A) = 1$ where $A = \{k_j : j \in \mathbb{N}\}$. Then, there exists $n_0 \in \mathbb{N}$ such that $\mu(\xi_{k_j}, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$, for all $k_j \geq n_0$. Therefore, $T = \{k_j \in A : \mu(\xi_{k_j}, \xi, q) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$ is a finite set, which implies that $\delta(T) = 0$. Since $\delta(A) = 1$, $\delta(\{k_j \in A : \mu(\xi_{k_j}, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}) = 1$. Because

$$\{k_j \in A : \mu(\xi_{k_j}, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\} \subseteq \{k \in \mathbb{N} : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$$

then

$$\delta(\{k \in \mathbb{N} : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}) = 1$$

Consequently, the sequence (ξ_k) is statistical convergent to ξ in the L-FMS. □

4. Completeness in L-Fuzzy Metric Spaces

This section defines statistical Cauchy sequences in an L-FMS and complete L-FMSs by them. Then, it provides the crucial relations.

Definition 4.1. Let (ξ_k) be sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$. Then, (ξ_k) is referred to as a statistically Cauchy sequence if, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$, there exists an $n \in \mathbb{N}$ such that

$$\delta\left(\{k \in \mathbb{N} : \mu\left(\xi_k, \xi_n, t\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}\right) = 0$$

Proposition 4.2. Every Cauchy sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$ is a statistical Cauchy sequence. However, the converse is not always true.

The proof is similar to the proof of Theorem 3.5.

Theorem 4.3. Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$. Then, the following conditions are equivalent:

i. (ξ_k) is a statistical Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$

ii. $\left(\xi_{k_j}\right)$ is a Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$ such that $\delta(A) = 1$ where $A = \{k_j : j \in \mathbb{N}\}$

The proof is similar to the proof of Theorem 3.7.

Theorem 4.4. Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$. If (ξ_k) is statistically convergent, then (ξ_k) is a statistically Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$.

PROOF. Let $\xi_k \xrightarrow{st_L} \xi$. From Note 2.12, for all $r \in \mathbb{L} - \{0_L, 1_L\}$, there exists an $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$ such that $\varphi(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)) \succ_{\mathbb{L}} \mathcal{N}(r)$. Since $\delta(\{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}) = 0$, $\left(\xi_{k_j}\right)$ is convergent to ξ from Theorem 3.7. Hence, there exists a $k_{j_0} \in \{k_j : j \in \mathbb{N}\}$ such that $\mu\left(\xi_{k_j}, \xi, \frac{t}{2}\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$, for all $k_j \geq k_{j_0}$. Then,

$$\mu\left(\xi_k, \xi_{k_{j_0}}, t\right) \succeq_{\mathbb{L}} \varphi\left(\mu\left(\xi_k, \xi, \frac{t}{2}\right), \mu\left(\xi, \xi_{k_{j_0}}, \frac{t}{2}\right)\right) \succeq_{\mathbb{L}} \varphi(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)) \succ_{\mathbb{L}} \mathcal{N}(r)$$

Thus, $\delta(\{k \in \mathbb{N} : \mu\left(\xi_k, \xi_{k_{j_0}}, t\right) \not\succ_{\mathbb{L}} \mathcal{N}(r)\}) = 0$. Consequently, (ξ_k) is a statistically Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$. □

Definition 4.5. Let $(\mathbb{X}, \mu, \varphi)$ be an L-FMS. If every statistical Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$ is statistical convergent, then $(\mathbb{X}, \mu, \varphi)$ is referred to as a statistically complete L-FMS.

Theorem 4.6. Let $(\mathbb{X}, \mu, \varphi)$ be an L-FMS. If $(\mathbb{X}, \mu, \varphi)$ is a statistically complete L-FMS, then it is a complete L-FMS.

PROOF. Let (ξ_k) be a Cauchy sequence in L-FMS $(\mathbb{X}, \mu, \varphi)$. From Note 2.12, for all $r \in \mathbb{L} - \{0_L, 1_L\}$, there exists an $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$ such that $\varphi(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)) \succ_{\mathbb{L}} \mathcal{N}(r)$. Thus, there exists a $K_0 \in \mathbb{N}$ such that

$$\mu\left(\xi_k, \xi_n, \frac{t}{2}\right) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$$

for all $k, n \geq K_0$. Since Proposition 4.2, it can be observed that (ξ_k) is a statistical Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$. Since $(\mathbb{X}, \mu, \varphi)$ is a statistically complete L-FSM, (ξ_k) is statistical convergence to a $\xi \in \mathbb{X}$. Therefore, by Theorem 3.7, there exists a subsequence (ξ_{k_j}) of (ξ_k) such that $\xi_{k_j} \to \xi$. Hence, there exists a $k_{j_0} \in \{k_j : j \in \mathbb{X}\}$ with $k_{j_0} \geq K_0$ such that

$$\mu\left(\xi_{k_j}, \xi, \frac{t}{2}\right) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$$

for all $k_j \geq k_{j_0}$. Therefore,

$$\begin{aligned}
\mu(\xi_k, \xi, t) &\succeq_{\mathbb{L}} \varphi\left(\mu\left(\xi_k, \xi_{k_j}, \tfrac{t}{2}\right), \mu\left(\xi_{k_j}, \xi, \tfrac{t}{2}\right)\right) \\
&\succeq_{\mathbb{L}} \varphi(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)) \\
&\succ_{\mathbb{L}} \mathcal{N}(r)
\end{aligned}$$

for all $k \geq k_{j_0} \geq K_0$. Therefore, (ξ_k) is convergent to ξ. Consequently, $(\mathbb{X}, \mu, \varphi)$ is complete. □

5. Conclusion

We were motivated to write this paper in light of Fast, Steinhaus, Zadeh, and the articles that followed these studies. In this paper, we introduced statistical convergence in L-FMSs, a generalization of convergence in L-FMSs, a mathematical tool for modeling uncertainty, and investigated the relations of essential notions. Then, we proposed the statistical Cauchy sequences and the concept of complete metric spaces with their help. Through this paper, our most significant target is stimulating authors' motivation in this critical space. In future studies, concepts such as ideal convergence, lacunary ideal convergence, and the other generalizations of statistical convergence can be analyzed in the space in question. Besides, with the help of this study, the relationship between statistical convergence and summability theory can be discussed.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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